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Yu Feng
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# Robust Control of Linear Descriptor Systems

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# Robust Control of Linear Descriptor Systems

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# Preface

Differential algebraic equations arise naturally in many significant applications, for example in mechanical body motion, chemical processing, power generation, network fluid flow, aircraft guidance. Such equations also exist in economy, for instance, the famous input–output Leontief model and its several important extensions, or in ecology and growth population, for instance, the Leslie growth population model and backward population projections.

Such systems may not be regular, referred to as singular, and often considered as a generalization of conventional state-space systems. The fact that these systems include algebraic equations enables them to encompass static constraints. These constraints may be intrinsic, such as some physical or behavior constraints, or induced by the analysis and/or the control approach. It is common to call these systems *descriptor systems* since they allow keeping the physical significance of state variables. Moreover, in their irregular form, they can also model systems with an impulsive (e.g. a derivative action) or non-causal behavior.

Since the 1970s, an abundant literature has shown the advantages of the generic specificity of descriptor systems. One issue of great importance is that unique solutions to initial value problems consistent with the system may not exist and the system may not be controllable or observable. A critical aspect of control system design is therefore to ensure regularity of the system. Likewise, many research works dealt with those new aspects that are inexistent for regular state-space systems. The frequency domain and time domain approaches are both discussed to give an overall picture of the status of the theory in the 1980s.

Twenty years later a remarkable number of results have been produced particularly on some extensions of state-space control theory to descriptor systems. Many control issues are tackled, such as pole placement, optimal control, observer synthesis, filtering and fault detection, control design under $H_2$ or $H_\infty$ constraints, and their LMI-based solutions. This reading key is valid even today. In the same vein, many recent papers consider the open problems in this theme associated with the discrete-time control design case or state-space controllers design for descriptor systems, fault estimation, and fault-tolerant control.

This book develops some original results about linear descriptor systems through the two following different visions. The first one is generalizing some existing results from classical state-space case to descriptor systems, such as dilated LMI characterizations and performance control under regulation constraints. This part of the book is somehow in the continuity of the work mentioned before. Second, a different contribution of descriptor systems is highlighted. A new path is taken by considering these systems as a powerful tool for conceiving new control laws, understanding and deciphering some controller's architecture or even homogenizing different existing ways to obtain some new or known results for state-space systems. To emphasize this concern, for instance, the comprehensive control problem for continuous-time descriptor systems is an example of the descriptor framework used in order to transform a nonstandard control problem with unstable and nonproper weights into a traditional stabilization control problem. In another register, an exact solution is derived for the sensitivity constrained linear optimal control also by using the descriptor framework.

On new developments for descriptor systems or on the use of the descriptor approach to solve some extended or constrained control problems, this book brings a new brick to the knowledge edifice of descriptor systems. It also represents a synthesis of a work that lasts for a decade.

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# Notations

| | |
|---|---|
| $\mathbb{C}$ | Filed of complex numbers |
| $\mathbb{R}$ | Filed of real numbers |
| $\mathbb{R}^n$ | Space of $n$-dimensional real vectors |
| $\mathbb{R}^{n\times m}$ | Space of $n \times m$ real matrices |
| $\in$ | 'Belongs to' |
| $\triangleq$ | 'Defined as' |
| $\times$ | Inner product |
| $\circ$ | Element wise multiplication of vectors |
| $\oplus$ | Sum of vector spaces |
| $\otimes$ | Kronecker product |
| $\mathcal{L}[\cdot]$ | Laplace transform of an argument |
| $\mathcal{F}_l(\cdot,\cdot)$ | Lower linear fractional transformation |
| $\mathrm{rank}(\cdot)$ | Rank of a matrix |
| $\det(\cdot)$ | Determinant of a matrix |
| $\deg(\cdot)$ | Degree of a polynomial |
| $\mathrm{Re}(\cdot)$ | Real part of a complex number |
| $\lambda_{min}(\cdot)$ | Minimum eigenvalue of a real matrix |
| $\lambda_{max}(\cdot)$ | Maximum eigenvalue of a real matrix |
| $\alpha(\cdot,\cdot)$ | Generalized spectral abscissa of a matrix |
| $\alpha(\cdot)$ | Spectral abscissa of a matrix |
| $\rho(\cdot,\cdot)$ | Generalized spectral radius of a matrix |
| $\rho(\cdot)$ | Spectral radius of a matrix |
| $\sigma_{\max}(\cdot)$ | Maximum singular value of a matrix |
| $\mathrm{vec}(\cdot)$ | Ordered stack of the columns of a matrix from left to right starting with the first column |
| $I_n$ | Identity matrix of the size $n \times n$ |
| $0_{n\times m}$ | Zero matrix of the size $n \times m$ |
| $X^\top$ | Transpose of matrix $X$ |
| $X^{-1}$ | Inverse of matrix $X$ |
| $X^*$ | Conjugate transpose of matrix $X$ |

| | |
|---|---|
| $X^{\dagger}$ | Moore–Penrose inverse of matrix $X$ |
| $X^{\perp}$ | Any matrix satisfying $X^{\perp}X = 0$ and $X^{\perp}(X)^{\top}>0$ |
| $\mathrm{Im}(X)$ | Range space of matrix $X$ |
| $\mathrm{Ker}(X)$ | Kernel (null) space of matrix $X$ |
| $\mathrm{diag}(X_1, \ldots, X_m)$ | Block diagonal matrix with blocks $X_1, \ldots, X_m$ |
| $X \geq ( > )0$ | $X$ is real symmetric positive semi-definite (positive definite) |
| $\mathrm{He}\{X\}$ | $X^{\top} + X$ |
| $\bullet$ | Off-diagonal blocks of a symmetric matrix represented blockwise, e.g. $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ \bullet & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & \bullet \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}$ |
| $\ell_2$ | Space of square integrable functions on $[0, \infty)$ |
| $RH_{\infty}$ | Set of all rational proper stable transfer matrices |
| $RH_2$ | Set of strictly proper and real rational stable transfer matrices |
| $\left[\begin{array}{c\|c} A - sE & B \\ \hline C & D \end{array}\right]$ | Descriptor system associated with system data $(E, A, B, C, D)$ |
| $\\| \cdot \\|_2$ | $\ell_2$ norm |
| $\\|G\\|_{\infty}$ | $H_{\infty}$ norm of transfer function $G$ |
| $M_g$, $M_{\phi}$, $M_r$, $M_m$ | Gain margin, phase margin, delay margin, modulus margin |

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# Chapter 1
# Introduction

## 1.1 Differential Algebraic Equations

A classical dynamic system is, in systems and control theory, often considered as a set of *ordinary differential equations* (ODEs), which describe relations among the system variables, usually known as state variables. For the most general purpose of system analysis, the first-order system as follows is widely adopted

$$\boldsymbol{F}\left(\dot{x}(t), x(t)\right) = 0, \tag{1.1}$$

where $\boldsymbol{F}$ and $x$ are vector value functions. The form (1.1) contains not only differential equations, but also a set of algebraic constraints. It is referred to as *differential algebraic equations* (DAEs). For control engineering, it is usually assumed that the ODEs can be expressed in an explicit way

$$\dot{x}(t) = \boldsymbol{f}\left(x(t)\right), \tag{1.2}$$

where $\boldsymbol{f}$ is a vector value function. A set of ODEs of the form (1.2) is generally referred to as state-space description. This representation has been the predominant tool in systems and control theory.

One notes that a state-space system model is obtained on the assumption that the plant is governed by the causality principle. However, in certain cases, the state in the past may depend on its state in the future, which violates the causality assumption. There are practical situations in which:

(i) physical variables can not be chosen as state variables in a natural way to meet the form (1.2),
(ii) physical senses of variables or coefficients are lost after transformation to (1.2).

Even in the area of signal processing where significant results have also expressed the filter in the state-space form, the limitations of the use of the state-space system model have been recognized by some scholars. As pointed out in [HCW07], the



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analysis of the rounding effect of a specific coefficient in a particular realization can become very difficult after transformation to the state-space form. Moreover, many realizations require the computation of intermediate variables that cannot be expressed in the state-space form.

Here, an example is given to show the limitations of the use of state-space systems. Let us consider an economic process where $n$ interrelated production sectors are involved [Lue77b]. The relationship between the production levels of the sectors can be described by the *Leontieff Model* of the form

$$x(k) = Ax(k) + Ex(k+1) - Ex(k) + u(k), \tag{1.3}$$

where $x(k) \in \mathbb{R}^n$ is the vector of production level of the sectors at time $k$. $Ax(k)$ is interpreted as the capital required as direct input for production of $x$, and an element $a_{ij}$ of $A$ called the *flow coefficient matrix* indicates the amount of product $i$ needed to produce one unit of product $j$. $Ex(k+1) - Ex(k)$ stands for the stocked capital for producing $x$ in the next time period, and a coefficient $e_{ij}$ of $E$ called the *stock coefficient matrix* indicates the amount of product $i$ that has to be in stock for producing one unit of product $j$ in the next time period. Moreover, $u(k)$ is the demanded production level. The type of econometric model shown in (1.3) was first studied by Leontieff and both continuous-time and discrete-time cases were considered in [Leo53].

The stock coefficient matrix $E$ is, in general, quite sparse, and most of its entries are zero, which means that $E$ is often singular. The singularity of $E$ can be explained by the fact that the production of one sector does not generally require the capital in stock from all the other sectors. In addition, in many cases, there are few sectors offering capital in stock to other sectors. The Eq. (1.3) can be rewritten as

$$Ex(k+1) = (I - A + E)x(k) - u(k), \tag{1.4}$$

which is similar to, but not exactly the representation given in (1.2). If the matrix $E$ is invertible, we can left-multiply the above equation by $E^{-1}$, and then a state-space model is obtained. For the case with $E$ being singular, it is clear that this economic process cannot be represented by a state-space model via simply inverting the matrix $E$. In fact, whether this transformation can be done depends upon the properties of the matrix pencil $(E, I - A + E)$.

## 1.2 Descriptor Systems

Let us decompose the DAEs (1.1) into two parts

$$\begin{cases} \dot{x}(t) = \phi\left(x(t)\right), \\ \quad 0 = \varphi\left(x(t)\right), \end{cases} \tag{1.5}$$

where $\phi$ and $\varphi$ are both vector value functions. Compared with the form (1.2), the DAEs in (1.5) contain not only differential equations, but also the algebraic constraints.

For a linear time-invariant system, the second equation in (1.5) concerns the static properties and impulsive behaviors of the system, which do not arise in a state-space system. Thanks to this add-on, the systems for which the writing of (1.2) is impossible or undesirable can be represented by the DAEs (1.5).

Dynamic systems of the form (1.5) have different nomenclature depending upon the research fields. For example, control theorists and mathematicians call them *singular systems* [Dai89, Lew86, Ail89, Cob84, XL06] due to the fact that the matrix on the derivative of the state ("generalized state" is more appropriate in this case), such as $E$ in (1.4), is generally singular. They sometimes use the name *generalized (extended) state-space systems* [Ail87, VLK81, Cam84, HFA86] since the form (1.5) can be viewed as a generalization of conventional state-space systems. In the engineering economic systems community, the terminology *descriptor system* [Lue77a, BL87, HM99b, WYC06] is most frequently adopted for the raison that the form (1.5) offers a fairly natural description of system properties; while numerical analysts like to call this representation *differential algebraic equations* [BCP96, GSG+07, KM06], probably due to its original form. Besides, in circuit theory, the form (1.5) is named a *semistate system* [ND89, RS86] because it describes "almost state" of the underlying system. Sometimes the term *implicit system* [SGGG03, IS01] is also used by some researchers to mention systems of form (1.5). Throughout this book, the name descriptor system is used and we will focus on the LTI dynamic systems.

Descriptor systems defined by DAEs do not evidently belong to the class of ODEs since an ODE does not include any algebraic constraints. Hence, descriptor systems contain conventional state-space systems as a special case and behave much more powerful in terms of system modeling. Compared with state-space systems, they can not only preserve the structure of physical systems, but also describe static constraints and impulsive behaviors. Such systems arise in real systems, for instance, large-scaled systems networks [Lue77a, SL73], circuits [ND89, ZHL03], boundary control systems [Pan90], power systems [Sto79], economic systems [Lue77a, Lue77b], chemical processes [KD95], mechanical engineering systems [HW79, SGGG03], robotics [MG89] and aircraft modeling [SL91].

## 1.3 Review on Analysis and Control for Descriptor Systems

As descriptor systems describe an important class of systems of both theoretical and practical significance, they have been a subject of research for many years. The history of studying descriptor systems dates back to the 1860s. The foundation for the study of linear descriptor systems was laid by Weierstrass. In the seminal work [Wei67], he developed the theory of elementary divisor for systems of the form

$$E\dot{x} = Ax + Bu. \tag{1.6}$$

His results were restricted in the regular case, that is, the determinant $sE - A$ is not identically zero. Then, by using the notion of minimal indices, Kronecker extended this theory to general cases where $|sE - A| = 0$ or $E$ and $A$ are rectangular [Kro90].

Foundational research of descriptor systems in the system theoretic context began from the 1970s. The 1970 and 1980 s were characterized as the development of basic yet essential results for descriptor systems, such as, the structure of matrix pencils, impulsive behavior, solvability, controllability, reachability, observability and system equivalence. From the beginning of the 1990s, scholars began to generalize the classic control issues to descriptor systems, for both continuous-time and discrete-time settings. Let us list briefly the main research outcomes of linear descriptor systems as follows.

- controllability and observability [Cob84, Lew85, YS81, Ail87, CP85, Hou04]
- system equivalence [BS00, HFA86, VLK81, ZST87]
- canonical form [Glu90, HS89, LL94, VK97]
- regularity and regularization [BKM97, CH99, KLX03, BGMN94, WS99]
- Admissibility and admissibilization [Var95, Kat96, Tor96, XY99, XL04, SC04, XL06, FY16]
- linear quadratic optimal control [Cob83, BL87, Wan04]
- pole assignment [Ail89, FKN86, GF95, DP98, VK03, YW03, RBCM08]
- generalized Lyapunov and Riccati equations [Ben87, ZLZ99, ZLX02, TMK95, IT02, SMA95]
- positive real lemma [FJ04, WC96, ZLX02, LC03, Mas06, Mas07, CT08, Fen15]
- $H_2$ analysis and control synthesis [ILU00, IT02, TK98, ZMC02, ZHL03, FYC12a]
- $H_\infty$ analysis and control synthesis [MKOS97, KK97, Mas07, RA99, UI99, ZXS08, WYC06, FYC12b, FY13, DYH14]
- observer design [Dai88, DZH96, Dar14, HM99a, MH93, Gao05, WDF07]
- generalized Sylvester equation [KW89, GLAM92, Dua96, CdS05, Dar06, WH14]
- output regulation problem [LD96, IK05, FYC13]
- model reduction problem [XL03, Sty04, WLZX04, WSL06, LZD07]
- controller implementation via a descriptor framework [HCW07, HCW10, FCH11]
- toolbox for analysis and synthesis of descriptor systems [Var00, VKAV08]

On the other hand, descriptor systems bring extra complexities to system analysis and controller design synthesis, though they are much more natural and powerful than conventional state-space systems. Roughly speaking, given a descriptor system (1.6), the major difficulties of studying such a system are rooted in the analysis of the matrix pair $(E, A)$ instead of a single matrix $A$ for the state-space case. Two new concepts called regularity and impulsiveness (respectively causality for the discrete-time setting) which are not necessarily considered for the state-space case need to be taken into account. For instance, in order to stabilize a descriptor system, the closed-loop system must be stable, as well as regular and impulsive-free. The latter two are intrinsic properties of conventional state-space systems. Another example is the indefiniteness of the Lyapunov matrix for descriptor systems. As known, in the state-space setting, a desirable Lyapunov matrix should be symmetric and positive definite. However, this is not the case within the descriptor framework. A desirable Lyapunov matrix associated with a descriptor system is indefinite, only the part related to the image of the matrix $E$ is supposed to be positive definite. Furthermore, contrary to state-space systems, the Lyapunov matrix within the descriptor framework is not unique, and it is possible to find several Lyapunov matrices for a single descriptor system. The requirement of uniqueness is no longer imposed on the Lyapunov matrix, but on the product of $E^\top$ and the Lyapunov matrix. Therefore, some control problems that have been successfully solved within the state-space framework are still open and deserve further investigation for descriptor systems.

## 1.4 Highlights of the Book

The remainder of this book is organized in the following and the key results are highlighted accordingly.

Chapter 2 recalls the basic concepts concerning linear time-invariant descriptor systems, for both continuous-time and discrete-time settings, which will be used subsequently in this book. Fundamental definitions and results of descriptor systems, such as regularity, admissibility, equivalent realizations, system decomposition, temporal response, controllability, observability, and duality are reviewed.

Chapter 3 discusses dilated LMI characterizations of descriptor systems for both the continuous-time and discrete-time settings. Conditions with regard to admissibility, $H_2$ performance, and dissipativity are systematically explored through reciprocal application of the projection lemma. Moreover, by the use of auxiliary matrices and a positive scalar, a novel bounded real lemma for discrete-time descriptor systems is derived. Based on this, a numerically efficient and reliable design procedure for state feedback $H_\infty$ controller design is given.

Chapter 4 is concerned with a nonstandard multi-objective output control problem for continuous-time descriptor systems. In this problem, an output signal is to be regulated with the presence of an infinite-energy exo-system; while a specified dissipative performance from a finite external disturbance to a tracking signal has also to be satisfied. It is shown that the regulation constraint is satisfied if and only if a generalized Sylvester equation is solvable. In addition, every controller achieving regulation objective admits a specific structure. Furthermore, using this structure, additional dissipative performance specification is satisfied by solving a set of LMIs.

Chapter 5 considers the comprehensive control problem for continuous-time descriptor systems. Systems and their weights are all described within the descriptor framework. Comprehensive admissibility issue under this circumstance is addressed in terms of two generalized Sylvester equations. A parametrization of all observer-based controllers achieving comprehensive admissibility is also given. In order to further clarify the impact of weighting filters on resulting controllers, a structured controller is explicitly conducted and it is shown that the comprehensive admissibility control problem can be transformed into a standard admissibility control problem without weights for an augmented system.

Chapter 6 tackles the $H_2$ and $H_\infty$ performance control subjects to unstable and nonproper weighting filters. The so-called quasi-admissible solution is defined instead of the conventional admissible solution for generalized algebraic Riccati equations (GAREs). Optimal comprehensive $H_2$ controller is conducted from quasi-admissible solutions to the two GAREs, together with solutions to the two generalized Sylvester equations. As for the comprehensive $H_\infty$ control problem, the necessary and sufficient conditions for the existence of a comprehensive $H_\infty$ controller are deduced and a set of controllers is explicitly parameterized in terms of two generalized Sylvester equations and two GARE together with a spectral radius condition.

Chapter 7 is dedicated to a synthesis method of parametric sensitivity constrained linear quadratic (SCLQ) controller for uncertain LTI systems. System sensitivity to parameter variation is handled through an additional quadratic trajectory parametric sensitivity term in the standard LQ criterion. A descriptor system approach is adopted to establish the relationship between a singular LQ control and the SCLQ control, and the solvability condition is conducted in terms of a nonstandard Riccati equation. Moreover, multiple parametric SCLQ control synthesis is addressed to cover the whole parametric uncertainty while degrading as less as possible the intrinsic robustness properties of each local LQ controller. An adequate particle swarm optimization (PSO) based algorithm is presented.

Chapter 8 contains concluding remarks that summarize the main achievements of this book.

Appendices A and B contain useful results for generalized Sylvester equations and GAREs associated with descriptor systems. Numerical algorithms for solving these equations are also presented.

# Chapter 2
# Linear Time-Invariant Descriptor Systems

In this chapter, we recall some basic concepts concerning linear time-invariant descriptor systems for both continuous-time and discrete-time settings, which will be used subsequently. Descriptor systems offer a powerful tool for system modelling since they allow to describe a system by both dynamic equations and algebraic constraints. We give here a quick reminder of the fundamental definitions and results of descriptor systems, such as regularity, admissibility, equivalent realizations, system decomposition, temporal response, controllability, observability, and duality. Most of the results presented in this chapter can be found in [Ros74, Cob84, Lew85, VLK81, YS81, Ail87, CP85, Hou04, Dai89, Lew86, IT02, XY99, XL04, Mar03].

## 2.1 Introduction

Let us recall the first-order DAE discussed in the previous chapter

$$\boldsymbol{F}\left(\dot{x}(t), x(t)\right) = 0, \tag{2.1}$$

where $\boldsymbol{F}$ and $x$ are vector value functions. Representing the Jacobians as

$$E \triangleq \frac{\partial \boldsymbol{F}}{\partial \dot{x}(t)}, \quad A \triangleq -\frac{\partial \boldsymbol{F}}{\partial x(t)}, \tag{2.2}$$

we can write

$$E d\dot{x}(t) = A d\dot{x}(t) + \left( d\boldsymbol{F} - \frac{\partial \boldsymbol{F}}{\partial t} dt \right). \tag{2.3}$$



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As mentioned before, if the matrix $E$ is not singular, i.e. $|E| \neq 0$, we can convert this system into a conventional state-space system by left-multiplying the two sides by $E^{-1}$. On the other hand, if $E$ is singular, this conversion is no longer possible. In the parts to follow, we omit the time index $t$ for continuous-time descriptor systems to simplify the writing.

A linear time-invariant version of (2.1) including a control input $u(t)$ and a measurement output $y(t)$ is written as

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the descriptor variable, measurement and control input vector, respectively. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular, i.e. $\text{rank}(E) = r \leq n$.

Note that the form (2.4) can be used without loss of generality. In the case where the feedthrough matrix from $u$ to $y$ is not null, we can introduce an extra descriptor variable to render the $D$ matrix zero. For example, consider the following system

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{2.5}$$

which can be equivalently represented by

$$\begin{aligned} \underbrace{\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} &= \underbrace{\begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ I \end{bmatrix}}_{\mathcal{B}} u, \\ y &= \underbrace{\begin{bmatrix} C & D \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} x \\ \zeta \end{bmatrix}. \end{aligned} \tag{2.6}$$

By introducing the auxiliary variable $\zeta$, this system is rewritten as the form of (2.4).

A descriptor system $G$ associated with the system data $(E, A, B, C, D)$ can also be represented by the form of

$$G(s) := \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]. \tag{2.7}$$

## 2.2 Regularity

One of the basic notations of descriptor systems is *regularity* or *solvability*. If a descriptor system is regular, then it has a unique solution for any initial condition and any continuous input function [VLK81, Cob83].

**Definition 2.1** (*Regularity*) The descriptor system in (2.4) is said to be regular if $\det(sE - A) \not\equiv 0$ for all $s \in \mathbb{C}$.

This definition is the same as the one called solvability used by Yip and Sincovec in [YS81]. To illustrate the physical mean of regularity, let us examine the Laplace transformed version of $E\dot{x} = Ax + Bu$ as follows

$$sE\mathcal{L}[x] - Ex(0) = A\mathcal{L}[x] + B\mathcal{L}[u], \tag{2.8}$$

which can be arranged as

$$(sE - A)\mathcal{L}[x] = B\mathcal{L}[u] + Ex(0). \tag{2.9}$$

It is observed that, if the system is regular, then

$$\mathcal{L}[x] = (sE - A)^{-1}(B\mathcal{L}[u] + Ex(0)), \tag{2.10}$$

which guarantees the existence and uniqueness of $\mathcal{L}[x]$ for any initial condition and any continuous input function. On the other hand, if the system is not regular, or equivalently, the matrix $sE - A$ is of rank deficiency, there exists a nonzero vector $\theta(s)$ such that

$$(sE - A)\theta(s) \equiv 0. \tag{2.11}$$

Consequently, one can state that, if the system has a solution denoted $\mathcal{L}[x]$, then $\mathcal{L}[x] + \alpha\theta(s)$ is also a solution for any $\alpha$. It is clear that a solution to this system is not unique, and it is also obvious that there may be no solution to this system. The following characterizations of regularity are given in [YS81].

**Lemma 2.1** (Regularity) *The following statements are equivalent.*

*(a)* $(E, A)$ *is regular.*
*(b)* *If* $X_0$ *is the null space of* $A$ *and* $X_i = \{x : Ax \in EX_{i-1}\}$ *then* $\mathrm{Ker}(E) \bigcap X_i = 0$ *for* $i = 0, 1, 2, \ldots$.
*(c)* *If* $Y_0$ *is the null space of* $A^T$ *and* $Y_i = \{x : A^T x \in E^T Y_{i-1}\}$ *then* $\mathrm{Ker}(E^T) \bigcap Y_i = 0$ *for* $i = 0, 1, 2, \ldots$.
*(d)* *The matrix*

$$G(n) = \left.\begin{bmatrix} E & 0 & \cdots & 0 \\ A & E & \cdots & 0 \\ 0 & A & \cdots & 0 \\ & & & E \\ 0 & & \cdots & A \end{bmatrix}\right\} n+1 \tag{2.12}$$

*has full column rank for* $n = 1, 2, \ldots$.

*(e) The matrix*

$$F(n) = \underbrace{\begin{bmatrix} E & A & 0 & \cdots & 0 \\ 0 & E & A & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & E & A \end{bmatrix}}_{n+1} \tag{2.13}$$

*has full row rank for* $n = 1, 2, \ldots$.

*(f) There exist nonsingular matrices M and N such that* $E\dot{x} = Ax + Bu$ *is decomposed into possibly two subsystems: a subsystem with only a state variable, and an algebraic-like subsystem, i.e.* $MENN^{-1}\dot{x} = MANN^{-1}x + MBu$ *has one of the following forms.*

*(i)*

$$\begin{aligned} \dot{x}_1 &= E_1 x_1 + B_1 u, \\ E_2 \dot{x}_2 &= x_2 + B_2 u, \quad E_2^k = 0, \quad E_2^{k-1} \neq 0. \end{aligned} \tag{2.14}$$

*In this case, both E and A are singular, or A is nonsingular and E is singular but not nilpotent, i.e.,* $E^k \neq 0$ *for all positive integers k.*

*(ii)*

$$\dot{x}_1 = E_1 x_1 + B_1 u. \tag{2.15}$$

*In this case, E is nonsingular.*

*(iii)*

$$E_2 \dot{x}_2 = x_2 + B_2 u, \quad E_2^k = 0, \quad E_2^{k-1} \neq 0. \tag{2.16}$$

*In this case, A is nonsingular and E is nilpotent.*

*In all cases,*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N^{-1} x, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = MB, \tag{2.17}$$

*and the exact solution is*

$$\begin{aligned} x_1(t) &= e^{E_1 t} x_{10} + \int_0^t e^{(t-\tau)E_1} B_1 u(\tau) d\tau, \\ x_2(t) &= -\sum_{i=0}^{k-1} E_2^i B_2 u^{(i)}(t), \end{aligned} \tag{2.18}$$

*where* $x_{10}$ *is the transformed initial condition, i.e.,* $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = N^{-1} x_0$.

Among these equivalent statements, the easiest one to characterize regularity for a given descriptor system is the condition (d) or its dual version (e). For the sake of avoiding computing a matrix with huge dimension, Luenberger proposed the so-called shuffle algorithm which requires manipulations only on the rows and columns of the matrix $[E \ A]$ [Lue78]. For convenience, we usually check regularity directly from its definition, that is, $sE - A \not\equiv 0$ for all $s \in \mathbb{C}$. In addition, if the descriptor system (2.4) is regular, $(sE - A)^{-1}$ is a rational matrix and we can further define its transfer function as

$$G(s) = C(sE - A)^{-1}B. \tag{2.19}$$

## 2.3 Equivalent Realizations and System Decomposition

To model a physical system, one has to choose a set of states which are related to the same performance, such as acceleration, velocity, position, temperature and mass. The choice of these states is in general not unique, and this fact leads to a set of different models (realizations) which yield the same input–output relationship for a given system. Consequently, it is of great interest to determine the relation of equivalence among these different representations.

**Definition 2.2** (*Restricted System Equivalence*) Reference [Ros74] Consider two descriptor systems $G$ and $\bar{G}$ given by

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u}, \\ \bar{y} &= \bar{C}\bar{x}. \end{aligned} \tag{2.21}$$

The two systems $G$ and $\bar{G}$ are termed restricted system equivalent if there exist nonsingular matrices $M$ and $N$ such that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array}\right] \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} s\bar{E} - \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right]. \tag{2.22}$$

**Definition 2.3** (*Strong Equivalence*) Reference [VLK81] Consider two descriptor systems $G$ and $\bar{G}$ given in (2.20) and (2.21), respectively. The two systems are termed strongly equivalent if there exist nonsingular matrices $M$, $N$ and two matrices $Q$, $R$ such that

$$\begin{aligned}&\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\left[\begin{array}{c|c} sE-A & B \\ \hline C & 0 \end{array}\right]\begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} s\bar{E}-\bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right], \\ &QE = ER = 0.\end{aligned} \tag{2.23}$$

Note that in book the term "equivalence" means "restricted system equivalence."

Among many equivalent representations, there are two particular realizations of great importance for system analysis and control. They are referred to as the Kronecker-Weierstrass form [Wei67, Kro90] and the singular value decomposition (SVD) [BL87] form.

**Lemma 2.2** (Kronecker-Weierstrass Decomposition) *Reference [Dai89] The descriptor system* (2.4) *is regular if and only if there exist nonsingular matrices* $M_1$ *and* $N_1$ *such that*

$$M_1 E N_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{bmatrix}, \quad M_1 A N_1 = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{2.24}$$

*where* $n_1 + n_2 = n$ *and* $\mathcal{N}$ *is a nilpotent matrix.*

The form (2.24) is referred to as the Kronecker-Weierstrass decomposition. This form can be viewed as an equivalent condition for regularity, and is also referred to by some scholars as *slow–fast decomposition* [Cob84]. The subsystem related to $\mathcal{A}$ is called the slow subsystem, while what is related to $\mathcal{N}$ is called the fast subsystem. Although the Kronecker-Weierstrass decomposition divides the systems into two parts which may bring simplicity for analysis, the use of this decomposition requires that the underlying system is regular. If the regularity of the system is not known, then this form cannot be applied. Moreover, the Kronecker-Weierstrass decomposition is numerically unreliable, especially in the case where the order of the system is relatively large.

Another decomposition that does not depend upon the regularity of systems is called the SVD form. This form can be obtained via a singular value decomposition on $E$ and followed by scaling of the bases. Under the SVD form, the pair $(E, A)$ is decomposed by two nonsingular matrices $M_2$ and $N_2$ as

$$M_2 E N_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 A N_2 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \tag{2.25}$$

The SVD form was discussed by Bender and Laub for using it to examine general system properties and to derive a linear-quadratic regulator for continuous-time descriptor systems [BL87]. Similar to the Kronecker-Weierstrass decomposition, $M_2$ and $N_2$ for SVD form are in general not unique.

## 2.4 Temporal Response

Assume that the descriptor system (2.4) is regular. According to the Kronecker-Weierstrass decomposition, there exist matrices $M_1$ and $N_1$ such that

$$\begin{aligned}\dot{x}_1 &= \mathcal{A}x_1 + B_1 u,\\ y_1 &= C_1 x_1,\end{aligned} \tag{2.26}$$

$$\begin{aligned}\mathcal{N}\dot{x}_2 &= x_2 + B_2 u,\\ y_2 &= C_2 x_2,\end{aligned} \tag{2.27}$$

where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N_1^{-1}x, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M_1 B, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} = C N_1.$$

Suppose that $h$ is the degree of the nilpotent matrix $\mathcal{N}$, that is, $\mathcal{N}^{h-1} \neq 0$ and $\mathcal{N}^h = 0$. The subsystem (2.26) is a normal state-space system, whose temporal response for a given input $u(t)$ and initial condition $x_{10}$ can be written as

$$y_1(t) = C_1 e^{\mathcal{A}t} x_{10} + C_1 \int_0^t e^{\mathcal{A}(t-\tau)} B_1 u(\tau) d\tau. \tag{2.28}$$

Then we consider the subsystem (2.27). Suppose that $u(t) \in \mathcal{C}^{h-1}$, where $\mathcal{C}^{h-1}$ stands for the set of $h-1$ times continuously differentiable functions. Then we have the following relations

$$\begin{aligned}
\mathcal{N}\dot{x}_2(t) &= x_2(t) + B_2 u(t),\\
\mathcal{N}^2 \dot{x}_2^{(2)}(t) &= \mathcal{N}\dot{x}_2(t) + \mathcal{N}B_2\dot{u}(t),\\
&\cdots,\\
\mathcal{N}^k \dot{x}_2^{(k)}(t) &= \mathcal{N}^{k-1}x_2^{(k-1)}(t) + \mathcal{N}^{k-1}B_2 u^{(k-1)}(t),\\
&\cdots,\\
\mathcal{N}^{h-1}\dot{x}_2^{(h-1)}(t) &= \mathcal{N}^{h-2}x_2^{(h-2)}(t) + \mathcal{N}^{h-2}B_2 u^{(h-2)}(t),\\
0 &= \mathcal{N}^{h-1}x_2^{(h-1)}(t) + \mathcal{N}^{h-1}B_2 u^{(h-1)}(t).
\end{aligned} \tag{2.29}$$

Hence, the expression of $x_2(t)$ can be obtained as

$$\begin{aligned}
x_2(t) &= \mathcal{N}\dot{x}_2 - B_2 u(t),\\
x_2(t) &= \mathcal{N}^2 \dot{x}_2^{(2)}(t) - B_2 u(t) - \mathcal{N}B_2\dot{u}(t),\\
&\cdots,\\
x_2(t) &= -\textstyle\sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(t),
\end{aligned} \tag{2.30}$$

which gives

$$y_2(t) = -\sum_{k=0}^{h-1} C_2 \mathcal{N}^k B_2 u^{(k)}(t). \tag{2.31}$$

Hence, the temporal response $y(t)$ of the descriptor system (2.4) is

$$y(t) = C_1 \left( e^{\mathcal{A}t} x_{10} + \int_0^t e^{\mathcal{A}(t-\tau)} B_1 u(\tau) d\tau \right) - \sum_{k=0}^{h-1} C_2 \mathcal{N}^k B_2 u^{(k)}(t). \tag{2.32}$$

It is observed that the response of the subsystem (2.26) depends on the matrix $\mathcal{A}$, initial condition $x_{10}$, as well as the input $u(t)$; while the response of the subsystem (2.27) depends only on the derivative of the input $u(t)$ on time $t$. That is why we also call these two subsystems slow subsystem and fast subsystem, respectively. If $t \to 0^+$, then we can deduce the following constraint on the initial condition

$$x(0^+) = N_1 \begin{bmatrix} I \\ 0 \end{bmatrix} x_{10} - N_1 \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(0^+). \tag{2.33}$$

Any initial condition satisfying (2.33) is called an *admissible condition*. From this point of view, only one initial condition is allowed and hence only one solution is allowed for each choice of $u(t)$. In [VLK81, Cob83], the authors used the theory of distributions and generalized this viewpoint to allow arbitrary initial conditions. Under this theory, for the fast subsystem, we have

$$x_2(t) = -\sum_{k=1}^{h-1} \delta^{(k-1)} \mathcal{N}^k x_{20} - \sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(t), \tag{2.34}$$

where $\delta$ is the Dirac delta. As pointed out in [Cob81], the form of (2.34) suggests that in any conventional sense the dynamics of the overall system are concentrated in the slow subsystem in (2.26). With the theory of distributions, we can represent the systems whose initial conditions are not admissible or those who contain "jump" behaviors. For example, when we switch an electrical circuit on, there will be a jump in the current or voltage at this moment. For these cases, the first term of (2.34) can transform the system into an admissible state.

## 2.5 Admissibility

Stability is a fundamental concept for state-space systems, which can be characterized, by one of the definitions, that the system has no poles located in the right-hand plane including the imaginary axis. Under the descriptor framework, a similar yet more complicated concept called *admissibility* plays the same role as stability for state-space systems.

**Definition 2.4** (*Admissibility*) References [Dai89, Lew86]

(a) The descriptor system (2.4) is said to be regular if $\det(sE - A)$ is not identically null;
(b) The descriptor system (2.4) is said to be impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$;
(c) The descriptor system (2.4) is said to be stable if all the roots of $\det(sE - A) = 0$ have negative real parts;
(d) The descriptor system (2.4) is said to be admissible if it is regular, impulse-free, and stable.

From the definition, the admissibility of a descriptor system concerns stability, as well as regularity and impulsiveness. The latter two are intrinsic properties of conventional state-space systems and are not necessarily considered in the state-space case. Furthermore, it can be deduced that if a descriptor system is impulse-free, then it is regular.

Now we give some equivalent conditions for admissibility.

**Lemma 2.3** *Reference [Dai89] Suppose that the descriptor system* (2.4) *is regular and there exist nonsingular matrices* $M_1$ *and* $N_1$ *such that the Kronecker and Weierstrass form* (2.24) *holds. Then*

*(i) this system is said to be impulse-free if and only if* $\mathcal{N} = 0$*;*
*(ii) this system is said to be stable if and only if* $\alpha(\mathcal{A}) < 0$*;*
*(iii) this system is said to be admissible if and only if* $\mathcal{N} = 0$ *and* $\alpha(\mathcal{A}) < 0$.

**Lemma 2.4** *Reference [Dai89] Consider the descriptor system* (2.4) *and suppose that there exist nonsingular matrices* $M_2$ *and* $N_2$ *such that the SVD form* (2.25) *holds. Then*

*(i) this system is said to be impulse-free if and only if* $|A_4| \neq 0$*;*
*(ii) this system is said to be admissible if and only if* $|A_4| \neq 0$ *and* $\alpha(A_1 - A_2 A_4^{-1} A_3) < 0$.

Furthermore, if the descriptor system is regular and the matrices $M_1$ and $N_1$ exist to render it Kronecker-Weierstrass form, then the transfer function of this system can be written as

$$G(s) = C_1(sI - \mathcal{A})^{-1} B_1 + C_2(s\mathcal{N} - I)^{-1} B_2. \tag{2.35}$$

For an impulse-free system, that is, $\mathcal{N} = 0$, we have

$$G(s) = C_1(sI - \mathcal{A})^{-1} B_1 - C_2 B_2. \tag{2.36}$$

It is noted that the term $C_2(s\mathcal{N} - I)^{-1} B_2$ leads to polynomial terms of $s$ if both $B_2$ and $C_2$ are nonzero. Hence the impulse-free assumption guarantees the properness of the transfer function. The converse statement is, however, not true. Clearly, if either $B_2$ or $C_2$ vanishes, the transfer function is still proper, even if the system is impulsive. Hence, given a stable transfer function $G(s)$ and its corresponding system data $(E, A, B, C, (D))$, the admissibility of this system can not be concluded.

Now we discuss briefly the issue of generalized eigenvalues of a matrix pencil. The theory mentioned here has been reported in the literature, for instance see [GvL96, BDD+00].

Consider a matrix pencil $\lambda E - A$, where $E$ and $A$ are both real $n \times n$ matrices, and $\lambda$ is a scalar. First, we assure that this pencil is regular, that is, $|\lambda E - A| \not\equiv 0$ for all $\lambda$. The generalized eigenvalues are defined as those $\lambda$ for which

$$|\lambda E - A| = 0. \tag{2.37}$$

**Definition 2.5** (*Infinite Generalized Eigenvectors*) Reference [BL87]

1. Grade 1 infinite generalized eigenvectors of the pencil $(sE - A)$ satisfy

$$Ev_i^1 = 0. \tag{2.38}$$

2. Grade $k$ $(k \geq 2)$ infinite generalized eigenvectors of the pencil $(sE - A)$ corresponding to the $i^{th}$ grade 1 infinite generalized eigenvectors satisfy

$$Ev_i^{k+1} = Av_i^k. \tag{2.39}$$

Moreover, the finite generalized eigenvalues of $sE - A$ are called the finite dynamic modes. The infinite generalized eigenvalues of $sE - A$ with the grade 1 infinite generalized eigenvectors determine the static modes, while the infinite generalized eigenvalues with the grade $k$ $(k \geq 2)$ infinite generalized eigenvectors are the impulsive modes.

Let $q$ be the degree of the polynomial $|\lambda E - A|$. One can state that the matrix pencil $\lambda E - A$ has $q$ finite generalized eigenvalues and $n - q$ infinite generalized eigenvalues where the number of static modes is $n - r$ and the number of impulsive modes is $r - q$.

## 2.6 Controllability

In this section, we introduce controllability for descriptor systems in a way that reduces to the state-space definition when $E = I$. We suppose that the descriptor system (2.4) is regular and it is transformed into the Kronecker-Weierstrass form as follows

$$\begin{aligned} \theta_s :\ \ \dot{x}_1 &= \mathcal{A}x_1 + B_1 u, \quad y_1 = C_1 x_1, \\ \theta_f : \mathcal{N}\dot{x}_2 &= x_2 + B_2 u, \quad y_2 = C_2 x_2, \\ y &= y_1 + y_2, \end{aligned} \tag{2.40}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ and $\mathcal{N}$ is a nilpotent matrix with degree $h$.

Let us define

- $\mathcal{C}_p^i$ be the $i$ times piecewise continuously differentiable maps on $\mathbb{R}$ with range depending on context;
- $\mathcal{I}$ be the set of admissible initial conditions, that is,

$$\mathcal{I} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \in \mathbb{R}^{n_1}, x_2 = -\sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(0), u \in \mathcal{C}_m^{h-1} \right\}; \tag{2.41}$$

- $\langle X, Y \rangle = \beta + X\beta + X^2\beta + \cdots + X^{n-1}\beta$, where $X$ is a square matrix, $n$ is the order of $X$, the product $XY$ is well defined and $\beta = \operatorname{Im}(Y)$.

**Definition 2.6** (*Reachable State*) Reference [YS81] A state $x_r$ is reachable from a state $x_0$ if there exists $u(t) \in \mathcal{C}_m^{h-1}$ such that $x(t_r) = x_r$ for some $t_r > 0$.

**Lemma 2.5** *Reference [YS81] Let $\mathcal{R}(0)$ be the set of reachable states from $x_0 = 0$. Then,*

$$\mathcal{R}(0) = \langle \mathcal{A}, B_1 \rangle \oplus \langle \mathcal{N}, B_2 \rangle. \tag{2.42}$$

**Lemma 2.6** *Reference [YS81] Let $\mathcal{R}(x)$ be the set of reachable states from $x \in \mathcal{I}$. Then the complete set of reachable states $\mathcal{R}$ is*

$$\mathcal{R} = \bigcup_{x \in \mathcal{I}} \mathcal{R}(x) = \mathbb{R}^{n_1} \oplus \langle \mathcal{N}, B_2 \rangle. \tag{2.43}$$

We can adopt the conventional definition of controllability for descriptor systems.

**Definition 2.7** (*$\mathcal{C}$-controllability*) The descriptor system (2.40) is said to be completely controllable ($\mathcal{C}$-controllable) if one can reach any state from any initial state.

Within the descriptor framework, we also define two different types of controllability as follows.

**Definition 2.8** (*$\mathcal{R}$-controllability*) The descriptor system (2.40) is said to be controllable within the set of reachable states ($\mathcal{R}$-controllable) if, from any initial state $x_0 \in \mathcal{I}$, there exists $u(t) \in \mathcal{C}_m^{h-1}$ such that $x(t_f) \in \mathcal{R}$ for any $t_f > 0$.

Note that, for state-space systems, $\mathcal{C}$-controllability and $\mathcal{R}$-controllability are equivalent. This is, however, not the case for descriptor systems.

**Definition 2.9** (*Imp-controllability*) Reference [Cob84] The descriptor system (2.40) is said to be impulse controllable (Imp-controllable) if for every $w \in \mathbb{R}^{n_2}$ there exists $u(t) \in \mathcal{C}_m^{h-1}$ such that the fast subsystem $\theta_f$ satisfies

$$x_2(t_f) = \sum_{k=1}^{h-1} \delta_{t_f}^{(k-1)} \mathcal{N}^k w, \quad \forall t_f > 0. \tag{2.44}$$

**Theorem 2.1** (Regarding $\mathcal{C}$-controllability) *References [YS81, Cob84, Dai89, Lew86]*

*(1) The following statements are equivalent.*

*(1i) The descriptor system (2.40) is $\mathcal{C}$-controllable.*
*(1ii) $\theta_s$ and $\theta_f$ are both controllable.*
*(1iii)* $\langle \mathcal{A}, B_1 \rangle \oplus \langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_1+n_2}$.
*(1iv)* $\text{rank}\,([sE - A\ \ B]) = n$, *for a finite* $s \in \mathbb{R}$ *and* $rank\,([E\ \ B]) = n$.
*(1v)* $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$ *and* $\text{Im}(E) \oplus \text{Im}(B) = \mathbb{R}^n$.
*(1vi) The matrix $\mathfrak{C}$ is full row rank,*

$$\mathfrak{C} = \begin{bmatrix} -A & & & & B & & & \\ E & -A & & & & B & & \\ & E & \ddots & & & & B & \\ & & \ddots & -A & & & & \ddots \\ & & & E & & & & & B \end{bmatrix}$$

*(2) The following statements are equivalent.*

*(2i) $\theta_s$ is controllable.*
*(2ii) The descriptor system (2.40) is $\mathcal{R}$-controllable.*
*(2iii)* $\langle \mathcal{A}, B_1 \rangle = \mathbb{R}^{n_1}$.
*(2iv)* $\text{rank}\,([sE - A\ \ B]) = n$, *for a finite* $s \in \mathbb{R}$.
*(2v)* $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$.

*(3) The following statements are equivalent.*

*(3i) $\theta_f$ is controllable.*
*(3ii)* $\langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_2}$.
*(3iii)* $\text{rank}\,([E\ \ B]) = n$.
*(3iv)* $\text{Im}(E) \oplus \text{Im}(B) = \mathbb{R}^n$.
*(3v)* $\text{Im}(\mathcal{N}) \oplus \text{Im}(B_2) = \mathbb{R}^{n_2}$.
*(3vi) The rows of $B_2$ corresponding to the bottom rows of all Jordan blocks of $\mathcal{N}$ are linearly independent.*
*(3vii)* $v^\top (s\mathcal{N} - I)^{-1} B_2 = 0$ *for constant vector* $v$ *implies that* $v = 0$.

**Theorem 2.2** (Regarding $\mathcal{R}$-controllability) *References [YS81, Cob84, Dai89] The following statements are equivalent.*

1. *The descriptor system (2.40) is $\mathcal{R}$-controllable.*
2. *$\theta_s$ is controllable.*
3. $\langle \mathcal{A}, B_1 \rangle = \mathbb{R}^{n_1}$.
4. $\text{rank}\,([sE - A\ \ B]) = n$, *for a finite* $s \in \mathbb{R}$.
5. $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$.

**Theorem 2.3** (Regarding Imp-controllability) *References [Cob84, Dai89, Lew86] The following statements are equivalent.*

1. *The descriptor system* (2.40) *is Imp-controllable.*
2. $\theta_f$ *is Imp-controllable.*
3. $\mathrm{Ker}(\mathcal{N}) \oplus \langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_2}$.
4. $\mathrm{Im}(\mathcal{N}) = \langle \mathcal{N}, B_2 \rangle$.
5. $\mathrm{Im}(\mathcal{N}) \oplus \mathrm{Ker}(\mathcal{N}) \oplus \mathrm{Im}(B_2) = \mathbb{R}^{n_2}$.
6. $\mathrm{rank}\left(\begin{bmatrix} A & E & B \\ E & 0 & 0 \end{bmatrix}\right) = n + r$.
7. *The rows of* $B_2$ *corresponding to the bottom rows of the nontrivial Jordan blocks of* $\mathcal{N}$ *are linearly independent.*
8. $v^\top \mathcal{N}(s\mathcal{N} - I)^{-1} B_2 = 0$ *for constant vector* $v$ *implies that* $v = 0$.

It is observed that the conditions characterizing $\mathcal{R}$-controllability are only concerned with the slow subsystem $\theta_s$. The response of the fast subsystem depends only on $u(t)$ and its derivatives. Any reachable state of $\theta_f$ $w \in \langle \mathcal{A}, B_1 \rangle$ can be written as $w = \sum_{k=0}^{h-1} \eta_k \mathcal{N}^k B_2$. Then it is easy to find an input $u(t)$ satisfying $u^{(k)}(t_f) = \eta_k$, for $k = 0, 1, \ldots, h-1$ (for example $u(t) = \sum_{k=0}^{h-1} \eta_k (t - t_f)^k / k!$) which leads to $x_2(t_f) = w$. Hence, the fast subsystem has no impact on $\mathcal{R}$-controllability.

Imp-controllability guarantees the ability to generate a maximal set of impulses, at each instant, in the following sense: suppose $E$ and $A$ are given but $B$ and $u$ are allowed to vary over all values.

## 2.7 Observability

In this section, we introduce observability for descriptor systems in a way that allows for a set of results analogous to the last section. Similarly, the concepts, that is, $\mathcal{C}$-observability, $\mathcal{R}$-observability, and Imp-observability are defined.

**Definition 2.10** (*$\mathcal{C}$-observability*) The descriptor system (2.40) is said to be completely observable ($\mathcal{C}$-observability) if knowledge of $u(t)$ and $y(t)$ for $t \in [0, \infty]$ is sufficient to determine the initial condition $x_0$.

**Definition 2.11** (*$\mathcal{R}$-observability*) The descriptor system (2.40) is said to be observable within the set of reachable states ($\mathcal{R}$-observable) if, for $t \geq 0$, $x(t) \in \mathcal{I}$ can be computed from $u(\tau)$ and $y(\tau)$ for any $\tau \in [0, t]$.

**Definition 2.12** (*Imp-observability*) The descriptor system (2.40) is said to be impulse observable (Imp-observable) if, for every $w \in \mathbb{R}^{n_2}$, knowledge of $y(t)$ for $t \in [0, \infty]$ to determine $x_2(t)$.

$$x_2(t) = \sum_{k=1}^{h-1} \delta_{t_f}^{(k-1)} \mathcal{N}^k w. \tag{2.45}$$

**Theorem 2.4** (Regarding $\mathcal{C}$-observability) *References [YS81, Cob84, Dai89, Lew86]*

*(1) The following statements are equivalent.*

*(1i) The descriptor system (2.40) is $\mathcal{C}$-observable.*
*(1ii) $\theta_s$ and $\theta_f$ are both observable.*
*(1iii)* $\langle \mathcal{A}^\top, C_1^\top \rangle \oplus \langle \mathcal{N}^\top, C_2^\top \rangle = \mathbb{R}^{n_1+n_2}$.
*(1iv)* $\operatorname{rank}\left([sE^\top - A^\top \; C^\top]\right) = n$, *for a finite* $s \in \mathbb{R}$ *and* $\operatorname{rank}\left([E^\top \; C^\top]\right) = n$.
*(1v)* $\operatorname{Ker}(\lambda E - A) \bigcap \operatorname{Ker}(C) = \{0\}$ *and* $\operatorname{Ker}(E) \bigcap \operatorname{Ker}(C) = \{0\}$.
*(1vi) The matrix $\mathfrak{O}$ is full row rank,*

$$\mathfrak{O} = \begin{bmatrix} -A^\top & & & & C^\top & & & \\ E^\top & -A^\top & & & & C^\top & & \\ & E^\top & \ddots & & & & C^\top & \\ & & \ddots & -A^\top & & & & \ddots & \\ & & & E^\top & & & & & C^\top \end{bmatrix}.$$

*(2) The following statements are equivalent.*

*(2i) $\theta_s$ is observable.*
*(2ii) The descriptor system (2.40) is $\mathcal{R}$-observable.*
*(2iii)* $\langle \mathcal{A}^\top, C_1^\top \rangle = \mathbb{R}^{n_1}$.
*(2iv)* $\operatorname{rank}\left([sE^\top - A^\top \; C^\top]\right) = n$, *for a finite* $s \in \mathbb{R}$.
*(2v)* $\operatorname{Ker}(\lambda E - A) \bigcap \operatorname{Ker}(C) = \{0\}$.

*(3) The following statements are equivalent.*

*(3i) $\theta_f$ is observable.*
*(3ii)* $\langle \mathcal{N}^\top, C_2^\top \rangle = \mathbb{R}^{n_2}$.
*(3iii)* $\operatorname{rank}\left([E^\top \; C^\top]\right) = n$.
*(3iv)* $\operatorname{Ker}(E) \bigcap Ker(C) = 0$.
*(3v)* $\operatorname{Ker}(\mathcal{N}) \bigcap \operatorname{Ker}(C_2) = \{0\}$.
*(3vi) The rows of $C_2^\top$ corresponding to the bottom rows of all Jordan blocks of $\mathcal{N}^\top$ are linearly independent.*
*(3vii)* $C_2(s\mathcal{N} - I)^{-1}v = 0$ *for constant vector $v$ implies that* $v = 0$.

**Theorem 2.5** (Regarding $\mathcal{R}$-observability) *References [YS81, Cob84, Dai89] The following statements are equivalent.*

1. *The descriptor system (2.40) is $\mathcal{R}$-observable.*
2. *$\theta_s$ is observable.*
3. $\langle \mathcal{A}^\top, C_1^\top \rangle = \mathbb{R}^{n_1}$.
4. $\operatorname{rank}\left([sE^\top - A^\top \; C^\top]\right) = n$, *for a finite* $s \in \mathbb{R}$.
5. $\operatorname{Ker}(\lambda E - A) \bigcap \operatorname{Ker}(C) = \{0\}$.

**Theorem 2.6** (Regarding Imp-observability) *References [Cob84, Dai89, Lew86] The following statements are equivalent.*

1. *The descriptor system* (2.40) *is Imp-observable.*
2. $\theta_f$ *is Imp-observable.*
3. $\text{Im}(\mathcal{N}^\top) \bigcap \text{Ker}\left(\langle \mathcal{N}^\top, C_2^\top \rangle\right) = \{0\}$.
4. $\text{Ker}(\mathcal{N}^\top) = \mathcal{N}\text{Ker}\left(\langle \mathcal{N}^\top, C_2^\top \rangle\right)$.
5. $\text{Ker}(\mathcal{N}) \bigcap Im(\mathcal{N}) \bigcap \text{Ker}(C_2) = \{0\}$.
6. $\text{rank}\left(\begin{bmatrix} A^\top & E^\top & C^\top \\ E^\top & 0 & 0 \end{bmatrix}\right) = n + r$.
7. *The rows of* $C_2^\top$ *corresponding to the bottom rows of the nontrivial Jordan blocks of* $\mathcal{N}^\top$ *are linearly independent.*
8. $C_2(s\mathcal{N} - I)^{-1}\mathcal{N}v = 0$ *for constant vector* $v$ *implies that* $v = 0$.

Similar to $\mathcal{R}$-controllability, the characterizations for evaluating $\mathcal{R}$-observability are only concerned with the slow subsystem $\theta_s$.

## 2.8 Duality

As known, there is a strong sense of symmetry between controllability and observability for the state-space setting. We now extend this idea to descriptor systems. Corresponding to (2.4), we define the dual system $\bar{\theta}$

$$\begin{aligned} E^\top \dot{x} &= A^\top x + C^\top u, \\ y &= B^\top x. \end{aligned} \tag{2.46}$$

Then we have the following statements.

**Theorem 2.7** (Duality)

1. *The descriptor system* (2.4) *is* $\mathcal{C}$*-controllable (*$\mathcal{C}$*-observable) if and only if the system* (2.46) *is* $\mathcal{C}$*-observable (*$\mathcal{C}$*-controllable).*
2. *The descriptor system* (2.4) *is* $\mathcal{R}$*-controllable (*$\mathcal{R}$*-observable) if and only if the system* (2.46) *is* $\mathcal{R}$*-observable (*$\mathcal{R}$*-controllable).*
3. *The descriptor system* (2.4) *is Imp-controllable (Imp-observable) if and only if the system* (2.46) *is Imp-observable (Imp-controllable).*

## 2.9 Discrete-Time Descriptor Systems

Consider the following linear time-invariant discrete-time descriptor system:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{2.47}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the descriptor variable and control input vector, respectively. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular, i.e., $\text{rank}(E) = r \leq n$.

The aforementioned notations for continuous-time descriptor systems can be adopted directly for its discrete-time counterpart. The only two differences between continuous-time and discrete-time settings are impulsiveness and stability. For discrete-time descriptor systems, we use the term *causality* instead of impulsiveness; while the discrete-time descriptor system (2.47) is said to be stable if $\rho(E, A) < 1$. Moreover, the definition of the transfer function of a regular discrete-time descriptor system is the same as that defined in the continuous-time setting, except for the use of the shift operator $z$ instead of the Laplace operator $s$. Interested readers may referred to [Dai89, XL06] for a comprehensive discussion of discrete-time descriptor systems.

## 2.10 Conclusion

This chapter recalls some basic concepts for linear time-invariant descriptor systems. Some fundamental and important results, such as regularity, admissibility, equivalent realizations, system decomposition and temporal response, are reviewed. The definitions of controllability and observability are also presented. Compared with state-space systems, for a descriptor system, three types of controllability are involved, that is, $\mathcal{C}$-controllability, $\mathcal{R}$-controllability, and Imp-controllability. This is also the case for observability. In addition, the duality between controllability and observability is stated.

# Chapter 3
# Dilated Linear Matrix Inequalities

The history of the use of linear matrix inequalities (LMIs) in the context of systems and control dates back more than 120 years. This story probably began in about 1890, when Aleksandr Mikhailovich Lyapunov published his fundamental work on the stability of motion. Lyapunov showed that the differential equations of the form

$$\dot{x}(t) = Ax(t) \tag{3.1}$$

are stable if and only if there exists a positive definite matrix $P$ such that

$$A^\top P + PA < 0. \tag{3.2}$$

This statement is now called Lyapunov theory and the requirement $P > 0$ together with (3.2) is what we now call Lyapunov inequality on $P$ that is commonly referred to as a Lyapunov matrix. The expression (3.2) might be the most well-known LMI to control theorists, and can be solved analytically by solving a set of linear equations. In the early 1980s, it is observed that many LMIs arising in systems and control theory can be formulated as convex optimization problems, which can be reliably solved by computer, even if for many of them no analytical solutions have been found.

Over the past two decades, LMI-based techniques [IS94, GA94, CG96, SGC97] have been widely employed as an important tool in system analysis and controller design synthesis because of its efficient and reliable solvability through convex optimization algorithms and powerful numerical supports of LMI toolboxes available in popular application software [GNLC95]. This method benefits not only from simplifying in a wide sense the necessity of certain cumbersome material of Riccati (Riccati-like) equations when the classical approaches are used, but also from its capability of gaining access to a vast panorama of control problems. Stability and performance specifications, such as, eigenvalue assignment, $H_2$ and $H_\infty$ control,



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multiobjective design problems and linear parameter-varying (LPV) synthesis, can be interpreted into LMIs [BGFB94, SGC97, MOS98].

However, the conservatism of the LMI formulations emerges when handling some "tedious" control problems. For instance, while using standard LMIs for solving a multiobjective control design problem, a common Lyapunov matrix is imposed on all equations involved to render the synthesis problem convex. This restriction inherently causes conservatism into design procedure. In order to reduce this conservatism, a new characterization named the dilated (extended/enhanced) LMI was first introduced in [GdOH98] for continuous-time state-space systems. From then on, tremendous investigations have been launched to explore new dilated LMI characterizations, and constructive results have been reported in the literature for analysis and controller design synthesis in both discrete-time and continuous-time settings [ATB01, BBdOG99, EH04, EH05, dOBG99, dOGB99, dOGH99, dOGB02, PABB00, Xie08, PDSV09]. Generally speaking, the advantages of these dilated LMIs over the standard ones can be resumed as follows.

- The dilated LMIs do not involve product terms of the Lyapunov matrix and the system matrix $A$. This separation enables the use of parameter-dependent Lyapunov functions for robust system analysis and controller synthesis [ATB01, dOBG99, dOGB02, PABB00];
- No indefinite quadratic terms of the system matrix $A$;
- Auxiliary (slack) variables are introduced that means more decision variables are involved. This fact might reduce the conservatism in robust analysis and controller synthesis.

Same enthusiasm has been witnessed for descriptor systems and the resulting dilated LMIs have also been studied in [XL06, Yag10, Seb07, Seb08, FYC10, Bar11, FY13]. In the current chapter, we explore dilated LMIs with regard to admissibility and performance specifications for linear descriptor systems.

## 3.1 Uniformed Methodology

Consider the LTI descriptor system $\Sigma(\lambda)$ as follows:

$$\Sigma(\lambda) : \begin{cases} E\sigma x = Ax + Bw, \\ z = Cx + Dw, \end{cases} \tag{3.3}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, and $w \in \mathbb{R}^m$ are the descriptor variable, the controlled output, and disturbance input belonging to $\ell_2[0\ +\infty)$, respectively. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular, i.e., $\text{rank}(E) = r \leq n$; and $A$, $B$, $C$, and $D$ are all known real constant matrices with appropriate dimensions. For the continuous-time, $\sigma x = \frac{dx}{dt}$ and $\lambda = s$, while in the discrete-time, $\sigma$ stands for the shift operator $q$ and $\lambda = z$.

If $(E, A)$ is regular, then the transfer matrix of (3.3) is written as

$$G(\lambda) = C(\lambda E - A)^{-1}B + D. \tag{3.4}$$

Throughout this chapter, let us define matrices $U, V \in \mathbb{R}^{n\times(n-r)}$ with full column rank satisfying $E^\top U = 0$ and $EV = 0$, respectively. We also define matrices $E_L, E_R \in \mathbb{R}^{n\times(n-r)}$ with full column rank satisfying $E_L^\top U = 0$ and $E_R^\top V = 0$. Hence, $E$ can be decomposed as $E = E_L \Delta E_R^\top$, with $\Delta \in \mathbb{R}^{r\times r}$ being nonsingular.

The following lemma plays an essential role to conduct the main results of this chapter.

**Lemma 3.1** (Projection Lemma) *Reference [BGFB94] Consider a symmetric matrix $\Xi \in \mathbb{R}^{n\times n}$ and two matrices $\Psi \in \mathbb{R}^{n\times m}$ and $\Upsilon \in \mathbb{R}^{k\times n}$ with* $\operatorname{rank}(\Psi) < n$ *and* $\operatorname{rank}(\Upsilon) < n$*, respectively. There exists an unstructured matrix $\Theta$ satisfying*

$$\Xi + \Upsilon^\top \Theta^\top \Psi + \Psi^\top \Theta \Upsilon < 0, \tag{3.5}$$

*if and only if the following inequalities with respect to $\Theta$ are satisfied:*

$$N_\Psi^\top \Xi N_\Psi < 0, \quad N_\Upsilon^\top \Xi N_\Upsilon < 0, \tag{3.6}$$

*where $N_\Psi$ and $N_\Upsilon$ are any matrices whose columns form a basis of the nullspaces of $\Psi$ and $\Upsilon$, respectively.*

In the parts to follow, standard LMI characterizations with respect to admissibility, $H_2$ and dissipativity are reformed into quadratic forms, as the first inequality in (3.6), where $N_\Psi$ is related to the system data. Then the dilated LMI conditions can be derived by applying Projection Lemma. Four types of dilated LMIs are explored, according to different constructions of $N_\Upsilon$:

I: $N_\Upsilon = 0$. In this case, $\Upsilon = I$.
II: Choose $N_\Upsilon$ such that the second inequality of (3.6) is equivalent to positive definiteness of (partial entries of $P$) $E_L^\top P E_L$.
III: Choose $N_\Upsilon$ such that a trivial inequality is introduced.
IV: Combine II and III.

With aforementioned choices of $N_\Upsilon$, the resulting dilated LMIs are denoted as Characterizations I, II, III, IV.

## 3.2 Admissibility Analysis

**Lemma 3.2** (Standard LMI) *References [IT02, XL04] The continuous-time (respectively discrete-time) descriptor system (3.3) is admissible if and only if there exist matrices $P \in \mathbb{R}^{n\times n} > 0$ and $Q \in \mathbb{R}^{(n-r)\times n}$ such that*

$$\begin{bmatrix} I \\ A \end{bmatrix}^\top \begin{bmatrix} 0 & (PE+UQ)^\top \\ \bullet & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0, \tag{3.7}$$

$$\text{respectively} \begin{bmatrix} I \\ A \end{bmatrix}^\top \begin{bmatrix} -E^\top PE & Q^\top U^\top \\ \bullet & P \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0. \tag{3.8}$$

The LMIs (3.7) and (3.8) are already in the quadratic form with $N_\Psi = \begin{bmatrix} I & A^\top \end{bmatrix}^\top$. According to the strategy I, setting $N_\Upsilon = 0$ leads to the following results, which were reported in [XL06].

**Theorem 3.1** (Characterization I) *The continuous-time (respectively discrete-time) descriptor system (3.3) is admissible if and only if there exist matrices $P \in \mathbb{R}^{n\times n} > 0$, $Q \in \mathbb{R}^{(n-r)\times n}$, and $\Theta_1, \Theta_2 \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} 0 & (PE+UQ)^\top \\ \bullet & 0 \end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \end{bmatrix}\begin{bmatrix} A & -I \end{bmatrix}\right\} < 0, \tag{3.9}$$

$$\text{respectively} \begin{bmatrix} -E^\top PE & Q^\top U^\top \\ \bullet & P \end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \end{bmatrix}\begin{bmatrix} A & -I \end{bmatrix}\right\} < 0. \tag{3.10}$$

In Characterization I, one note that the matrix $P$ is positive definite. Here $N_\Upsilon$ can be chosen such that the second inequality in (3.6) is equivalent either to $-2\epsilon E_L^\top PE_L < 0$ for the continuous-time case with an arbitrary positive $\epsilon$, or to $-E_L^\top PE_L < 0$ for the discrete-time setting. Hence, the following choices are made.

For continuous-time descriptor systems, set

$$N_\Upsilon = \begin{bmatrix} -\epsilon E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ E_L \end{bmatrix} \to \Upsilon = \left[\begin{array}{c:c} E & \epsilon I \\ \hdashline V^\top & 0_{(n-r)\times n} \end{array}\right]. \tag{3.11}$$

**Theorem 3.2** (Characterization II) *For a continuous-time descriptor system, the LMI condition (3.7) is equivalent to*

$$\begin{bmatrix} 0 & (PE+UQ)^\top \\ \bullet & 0 \end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix} E^\top\Theta_1^\top + V\Theta_2^\top \\ \epsilon\Theta_1^\top \end{bmatrix}\begin{bmatrix} A & -I \end{bmatrix}\right\} < 0, \tag{3.12}$$

*where $\epsilon$ is an arbitrary positive scalar, and $\Theta_1 \in \mathbb{R}^{n\times n}$ and $\Theta_2 \in \mathbb{R}^{n\times(n-r)}$ are auxiliary matrices.*

For discrete-time descriptor systems, choose

$$N_\Upsilon = \begin{bmatrix} E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ 0 \end{bmatrix} \to \Upsilon = \left[\begin{array}{c:c} V^\top & 0_{(n-r)\times n} \\ \hdashline 0 & I \end{array}\right]. \tag{3.13}$$

**Theorem 3.3** (Characterization II) *For a discrete-time descriptor system, the LMI condition (3.8) is equivalent to*

$$\begin{bmatrix} -E^\top PE & Q^\top U^\top \\ \bullet & P \end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix} V\Theta_1^\top \\ \Theta_2^\top \end{bmatrix}\begin{bmatrix} A & -I \end{bmatrix}\right\} < 0, \tag{3.14}$$

*where* $\Theta_1 \in \mathbb{R}^{n\times(n-r)}$ *and* $\Theta_2 \in \mathbb{R}^{n\times n}$ *are auxiliary matrices.*

In the case where $E = I$, in other words, the state-space systems, the matrices $U$, $V$ $Q$, $\Theta_1$ in (3.14) and $\Theta_2$ in (3.12) become empty matrices with the number of columns being 0. Since the multiplication of two empty matrices with compatible dimensions is a zero matrix [NH93], the present results cover the existing dilated LMIs [BBdOG99, CM04, EH04, GdOH98, dOBG99, dOGH99, PABB00] state-space systems.

## 3.3 $H_2$ Performance

For the sake of simplicity, we assume, in this section, that the direct gain from disturbance to output is null, i.e., $D = 0$ and consider the standard assumption $\text{Ker}(E) \subseteq \text{Ker}(C)$ [ILU00, TK97].

For a continuous-time admissible and strictly proper descriptor systems, the $H_2$ norm is defined as

$$\|G(s)\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} trace\{G(j\omega)^* G(j\omega)\}d\omega\right]^{\frac{1}{2}}, \tag{3.15}$$

where $G(j\omega)$ stands for the frequency response function of the transfer function given in (3.4). For discrete-time descriptor systems the integral bounds are $-\pi$ and $\pi$; while the frequency response function is written as $G(e^{j\omega})$.

The standard LMI of $H_2$ performance for the continuous-time descriptor systems is available in [ILU00], and its counterpart for discrete-time descriptor systems can be derived by following the similar thread.

**Lemma 3.3** (Standard LMI) *Given* $\gamma_2 > 0$, *the continuous-time (respectively discrete-time) descriptor system* (3.3) *with* $D = 0$ *is admissible and satisfies* $\|G(s)\|_2 < \gamma_2$ *(respectively* $\|G(z)\|_2 < \gamma_2$*), if and only if there exist matrices* $P \in \mathbb{R}^{n\times n} > 0$, $Q \in \mathbb{R}^{(n-r)\times n}$ *and* $Z \in \mathbb{R}^{m\times m}$ *such that*

$$\begin{bmatrix} I_n & 0 \\ A & 0 \\ 0 & I_p \end{bmatrix}^\top \begin{bmatrix} 0 & (PE+UQ)^\top & C^\top \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & 0 \\ 0 & I_p \end{bmatrix} < 0, \tag{3.16}$$

$$\begin{bmatrix} Z & B^\top P \\ \bullet & P \end{bmatrix} > 0, \quad trace(Z) < \gamma_2^2; \tag{3.17}$$

$$\text{respectively} \quad \begin{bmatrix} I_n & 0 \\ A & 0 \\ 0 & I_p \end{bmatrix}^\top \begin{bmatrix} -E^\top PE & Q^\top U^\top & C^\top \\ \bullet & P & 0 \\ \bullet & \bullet & -I \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & 0 \\ 0 & I_p \end{bmatrix} < 0, \tag{3.18}$$

$$\begin{bmatrix} Z & B^\top P \\ \bullet & P \end{bmatrix} > 0, \quad trace(Z) < \gamma_2^2. \tag{3.19}$$

The conditions (3.16) and (3.18) are already in the form of (3.6) with $N_\Psi = \begin{bmatrix} I & 0 \\ A & 0 \\ 0 & I \end{bmatrix}$ and $N_\Upsilon = 0$. Consequently, $\Psi$ and $\Upsilon$ are given as $\Psi = \begin{bmatrix} A & -I & 0 \end{bmatrix}$ and $\Upsilon = I$.

**Theorem 3.4** (Characterization I) *The LMI (3.16) (respectively (3.18)) is equivalent to*

$$\begin{bmatrix} 0 & (PE+UQ)^\top & C^\top \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & 0 \end{bmatrix} \right\} < 0, \tag{3.20}$$

$$\text{respectively} \begin{bmatrix} -E^\top PE & Q^\top U^\top & C^\top \\ \bullet & P & 0 \\ \bullet & \bullet & -I \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & 0 \end{bmatrix} \right\} < 0, \tag{3.21}$$

*with* $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n+p)}$.

The second characterization is derived based on the fact that $E_L^\top PE_L > 0$. Similar to the admissibility case, for continuous-time descriptor systems, set

$$N_\Upsilon = \begin{bmatrix} -\epsilon E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ E_L \\ 0 \end{bmatrix} \rightarrow \Upsilon = \left[\begin{array}{c:c:c} E & \epsilon I & 0 \\ \hdashline V^\top & 0_{(n-r)\times n} & 0 \\ \hdashline 0 & 0 & I_p \end{array}\right], \tag{3.22}$$

with an arbitrary $\epsilon > 0$.

**Theorem 3.5** (Characterization II) *Consider the continuous-time descriptor system (3.3). The LMI condition (3.16) is equivalent to*

$$\begin{bmatrix} 0 & (PE+UQ)^\top & C^\top \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} E^\top\Theta_1^\top + V\Theta_2^\top \\ \epsilon\Theta_1^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & 0 \end{bmatrix} \right\} < 0, \tag{3.23}$$

*where $\epsilon$ is an arbitrary positive scalar and $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n-r+p)}$, with $\Theta_2 \in \mathbb{R}^{n\times(n-r)}$, is an auxiliary matrix.*

Similarly, for the discrete-time descriptor systems, choose

$$N_\Upsilon = \begin{bmatrix} E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ 0_{n\times r} \\ 0_{p\times r} \end{bmatrix} \rightarrow \Upsilon = \left[\begin{array}{c:c:c} V^\top & 0_{(n-r)\times n} & 0 \\ \hdashline 0 & I & 0 \\ \hdashline 0 & 0 & I_p \end{array}\right]. \tag{3.24}$$

**Theorem 3.6** (Characterization II) *Consider the discrete-time descriptor system* (3.3). *The LMI condition* (3.18) *is equivalent to*

$$\begin{bmatrix} -E^\top PE & Q^\top U^\top & C^\top \\ \bullet & P & 0 \\ \bullet & \bullet & -I \end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix} V\Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix}\begin{bmatrix} A & -I & 0 \end{bmatrix}\right\} < 0, \tag{3.25}$$

*where* $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n-r+p)}$, *with* $\Theta_1 \in \mathbb{R}^{n\times(n-r)}$, *is an auxiliary matrix.*

In order to conduct the third characterization, we introduce a trivial inequality. From the standard LMIs (3.16) and (3.18), the following construction of $N_\Upsilon$

$$N_\Upsilon = \begin{bmatrix} 0 \\ 0 \\ I_p \end{bmatrix} \to \Upsilon = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \tag{3.26}$$

yields $-I_p < 0$.

**Theorem 3.7** (Characterization III) *Enforcing* $\Theta_3 = 0$ *in* (3.20) *and* (3.21) *does not introduce conservatism.*

Moreover, the forth dilated LMI characterization is derived by combining the second and third strategies. To this end, the inequality $\begin{bmatrix} -2\epsilon E_L^\top PE_L & 0 \\ 0 & -I_p \end{bmatrix} < 0$ is introduced, with an arbitrary $\epsilon > 0$ for the continuous-time setting and $\begin{bmatrix} -E_L^\top PE_L & 0 \\ 0 & -I_p \end{bmatrix} < 0$ for the discrete-time setting. Hence, the following choices are made.

For continuous-time descriptor systems, set

$$N_\Upsilon = \left[\begin{array}{c:c} -\epsilon E_R(E_R^\top E_R)^{-1}\Delta^{-1} & 0 \\ E_L & 0 \\ 0 & I_p \end{array}\right] \to \Upsilon = \left[\begin{array}{c:c:c} E & \epsilon I & 0 \\ \hdashline V^\top & 0 & 0_{(n-r)\times p} \end{array}\right]. \tag{3.27}$$

**Theorem 3.8** (Characterization IV) *Consider the continuous-time descriptor system* (3.3). *Enforcing* $\Theta_3 = 0$ *in* (3.23) *does not introduce conservatism.*

For discrete-time descriptor systems, choose

$$N_\Upsilon = \left[\begin{array}{c:c} E_R(E_R^\top E_R)^{-1}\Delta^{-1} & 0 \\ 0 & 0 \\ 0 & I_p \end{array}\right] \to \Upsilon = \left[\begin{array}{c:c:c} V^\top & 0 & 0_{(n-r)\times p} \\ \hdashline 0 & I_n & 0 \end{array}\right]. \tag{3.28}$$

**Theorem 3.9** (Characterization IV) *Consider the discrete-time descriptor system* (3.3). *Enforcing* $\Theta_3 = 0$ *in* (3.25) *does not introduce conservatism.*

If $E = I$, then the present results cover the existing dilated LMIs for state-space systems [PABB00, dOGB99, dOGB02, EH04, PDSV09].

## 3.4 Dissipativity

Dissipativity [Wil72a, Wil72b] plays an important role in systems and control theory both for theoretical considerations as well as from a practical point of view. Roughly speaking, a dissipative system is characterized by the property that at any time the amount of energy which the system can supply to its environment cannot exceed the amount of energy that has been supplied to it. Many important control issues can be formulated as dissipativity with quadratic supply functions, for instance, positive realness, bounded realness, and circle criterion. It is known that strict dissipativity of a descriptor system can be described by a frequency-domain inequality (FDI) of the form

$$\begin{bmatrix} I \\ G(\Omega) \end{bmatrix}^* S \begin{bmatrix} I \\ G(\Omega) \end{bmatrix} < 0, \tag{3.29}$$

where

$$S \triangleq \begin{bmatrix} S_1 & S_2 \\ \bullet & S_3 \end{bmatrix}, \quad S_1 = S_1^\top \in \mathbb{R}^{m\times m}, \quad S_3 = S_3^\top \in \mathbb{R}^{p\times p}, \tag{3.30}$$

$\Omega = j\omega$ for the continuous-time case, $\Omega = e^{j\omega}$ for the discrete-time setting, and $\omega \in \mathbb{R} \cup \{\infty\}$. We make in this section the classical assumption that $S_3 \geq 0$ [SW09]. Let us denote

$$M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}. \tag{3.31}$$

Then, the FDI (3.29) is reformed as

$$\begin{bmatrix} (\Omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (\Omega E - A)^{-1}B \\ I \end{bmatrix} < 0. \tag{3.32}$$

As special cases, positive realness of $G(\lambda)$ indicates $S = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$, while for $\gamma_\infty > 0$, $\|G\|_\infty < \gamma_\infty$ (bounded realness) implies $S = \begin{bmatrix} -\gamma_\infty^2 I & 0 \\ 0 & I \end{bmatrix}$.

The standard LMI conditions for the FDI (3.29) together with admissibility are given as follows.

**Lemma 3.4** (Standard LMI) *Reference [XL06] The continuous-time (respectively discrete-time) descriptor system (3.3) is admissible and strictly dissipative, if and only if there exist matrices $P \in \mathbb{R}^{n\times n} > 0$, $Q \in \mathbb{R}^{(n-r)\times n}$ and $R \in \mathbb{R}^{(n-r)\times m}$ such that*

$$M + \boldsymbol{He}\left\{\begin{bmatrix}(PE+UQ)^\top \\ R^\top U^\top\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\right\} < 0, \tag{3.33}$$

$$\text{respectively} \;\; M + \begin{bmatrix}A^\top PA - E^\top PE & A^\top PB \\ \bullet & B^\top PB\end{bmatrix} + \boldsymbol{He}\left\{\begin{bmatrix}Q^\top U^\top \\ R^\top U^\top\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\right\} < 0. \tag{3.34}$$

We reform the above two inequalities into the quadratic from as, respectively,

$$\begin{aligned}
&M + \mathbf{He}\left\{\begin{bmatrix}(PE+UQ)^\top \\ R^\top U^\top\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\right\} \\
&= \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix}^\top \begin{bmatrix}I & 0 & 0 \\ 0 & I & 0 \\ \hdashline I & 0 & 0 \\ 0 & 0 & I\end{bmatrix}^\top \left[\begin{array}{cc:cc}0 & (PE+UQ)^\top & 0 & 0 \\ \bullet & 0 & 0 & UR \\ \hdashline \bullet & \bullet & M_1 & M_2 \\ \bullet & \bullet & \bullet & M_3\end{array}\right] \begin{bmatrix}I & 0 & 0 \\ 0 & I & 0 \\ \hdashline I & 0 & 0 \\ 0 & 0 & I\end{bmatrix} \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix} \\
&= \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix}^\top \begin{bmatrix}M_1 & (PE+UQ)^\top & M_2 \\ \bullet & 0 & UR \\ \bullet & \bullet & M_3\end{bmatrix} \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix} < 0,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&M + \begin{bmatrix}A^\top PA - E^\top PE & A^\top PB \\ \bullet & B^\top PB\end{bmatrix} + \mathbf{He}\left\{\begin{bmatrix}Q^\top U^\top \\ R^\top U^\top\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\right\} \\
&= \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix}^\top \begin{bmatrix}I & 0 & 0 \\ 0 & I & 0 \\ \hdashline I & 0 & 0 \\ 0 & 0 & I\end{bmatrix}^\top \left[\begin{array}{cc:cc}-E^\top PE & Q^\top U^\top & 0 & 0 \\ \bullet & P & 0 & UR \\ \hdashline \bullet & \bullet & M_1 & M_2 \\ \bullet & \bullet & \bullet & M_3\end{array}\right] \begin{bmatrix}I & 0 & 0 \\ 0 & I & 0 \\ \hdashline I & 0 & 0 \\ 0 & 0 & I\end{bmatrix} \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix} \\
&= \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix}^\top \begin{bmatrix}M_1 - E^\top PE & Q^\top U^\top & M_2 \\ \bullet & P & UR \\ \bullet & \bullet & M_3\end{bmatrix} \begin{bmatrix}I & 0 \\ A & B \\ 0 & I\end{bmatrix} < 0,
\end{aligned} \tag{3.36}$$

where $M \triangleq \begin{bmatrix}M_1 & M_2 \\ \bullet & M_3\end{bmatrix}$, $M_1 \in \mathbb{R}^{n\times n}$.

By the first strategy, the following statement can be made.

**Theorem 3.10** (Characterization I) *The inequality (3.33) (respectively (3.34)) is equivalent to*

$$\begin{bmatrix} M_1 & (PE+UQ)^\top & M_2 \\ \bullet & 0 & UR \\ \bullet & \bullet & M_3 \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \right\} < 0, \quad (3.37)$$

respectively
$$\begin{bmatrix} M_1 - E^\top PE & Q^\top U^\top & M_2 \\ \bullet & P & UR \\ \bullet & \bullet & M_3 \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \right\} < 0, \quad (3.38)$$

*with* $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n+m)}$.

For continuous-time descriptor systems, set

$$N_\Upsilon = \begin{bmatrix} -\epsilon E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ E_L \\ 0 \end{bmatrix} \rightarrow \Upsilon = \left[\begin{array}{c:c:c} E & \epsilon I & 0 \\ \hdashline V^\top & 0_{(n-r)\times n} & 0 \\ \hdashline 0 & 0 & I_m \end{array}\right]. \quad (3.39)$$

This choice yields

$$\epsilon^2 \Gamma M_1 \Gamma^\top - \epsilon E_L^\top P E_L < 0, \quad (3.40)$$

with $\Gamma \triangleq \Delta^{-\top}(E_R^\top E_R)^{-1}E_R^\top$. It is observed that in order to keep the equivalence, the positive scalar $\epsilon$ cannot be chosen arbitrarily, in contrast with the cases of admissibility and $H_2$ performance, but must be viewed as a decision variable.

**Theorem 3.11** (Characterization II) *Consider the continuous-time descriptor system* (3.3). *The LMI condition* (3.33) *is equivalent to*

$$\begin{bmatrix} M_1 & (PE+UQ)^\top & M_2 \\ \bullet & 0 & UR \\ \bullet & \bullet & M_3 \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} E^\top\Theta_1^\top + V\Theta_2^\top \\ \epsilon\Theta_1^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \right\} < 0, \quad (3.41)$$

*where* $\epsilon > 0$ *and* $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n-r+m)}$, *with* $\Theta_2 \in \mathbb{R}^{n\times(n-r)}$, *is an auxiliary matrix.*

For the discrete-time case, choose

$$N_\Upsilon = \begin{bmatrix} E_R(E_R^\top E_R)^{-1}\Delta^{-1} \\ 0_{n\times r} \\ 0_{m\times r} \end{bmatrix} \rightarrow \Upsilon = \left[\begin{array}{c:c:c} V^\top & 0_{(n-r)\times n} & 0 \\ \hdashline 0 & I & 0 \\ \hdashline 0 & 0 & I_m \end{array}\right]. \quad (3.42)$$

This choice leads to

$$\Gamma M_1 \Gamma^\top - E_L^\top P E_L < 0. \quad (3.43)$$

This condition is in general more strict than $-E_L^\top P E_L < 0$. Consequently, for discrete-time descriptor systems, equivalent Characterization II is impossible to derive. However, for positive realness where $M_1 = 0$, (3.43) is not conservative.

**Theorem 3.12** (Characterization II for Positive Realness) *Consider the discrete-time descriptor system* (3.3)*. Suppose* $M_1 = 0$*, then the LMI condition* (3.34) *is equivalent to*

$$\begin{bmatrix} -E^\top P E & Q^\top U^\top & M_2 \\ \bullet & P & UR \\ \bullet & \bullet & M_3 \end{bmatrix} + \boldsymbol{He}\left\{ \begin{bmatrix} V\Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \right\} < 0, \tag{3.44}$$

*where* $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix} \in \mathbb{R}^{n\times(2n-r+m)}$*, with* $\Theta_1 \in \mathbb{R}^{n\times(n-r)}$*, is an auxiliary matrix.*

Unlike the case of $H_2$ performance, the third strategy cannot be applied without assumption on the realization of systems. To obtain the corresponding dilation, we assume that $M_3 < 0$.

Similarly, set

$$N_\Upsilon = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \rightarrow \Upsilon = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}. \tag{3.45}$$

**Theorem 3.13** (Characterization III) *Assume* $M_3 < 0$*. Enforcing* $\Theta_3 = 0$ *in* (3.37) *and* (3.38) *does not introduce conservatism.*

Combining the second and the third strategies yields the forth dilated LMI formulations for dissipativity together with admissibility. For continuous-time descriptor systems, set

$$N_\Upsilon = \left[\begin{array}{c:c} -\epsilon E_R (E_R^\top E_R)^{-1} \Delta^{-1} & 0 \\ E_L & 0 \\ 0 & I_m \end{array}\right] \rightarrow \Upsilon = \left[\begin{array}{c:c:c} E & \epsilon I & 0 \\ \hdashline V^\top & 0_{(n-r)\times n} & 0 \end{array}\right] \tag{3.46}$$

with $\epsilon > 0$. This choice of $N_\Upsilon$ implies that

$$\begin{aligned} &\left[\begin{array}{c:c} \epsilon^2 \Gamma M_1 \Gamma^\top - 2\epsilon E_L^\top P E_L & -\epsilon \Gamma M_2 \\ \hdashline \bullet & M_3 \end{array}\right] \\ &= \begin{bmatrix} -\Gamma & 0 \\ 0 & I_m \end{bmatrix} \left[\begin{array}{c:c} \epsilon^2 M_1 - 2\epsilon E^\top P E & \epsilon M_2 \\ \hdashline \bullet & M_3 \end{array}\right] \begin{bmatrix} -\Gamma & 0 \\ 0 & I_m \end{bmatrix}^\top < 0, \end{aligned}$$

where $\Gamma \triangleq \Delta^{-\top} (E_R^\top E_R)^{-1} E_R^\top$. This inequality is equivalent to

$$\epsilon^2 \Gamma M_1 \Gamma^\top - 2\epsilon E_L^\top P E_L < 0, \quad M_3 < 0,$$
$$2\epsilon E_L^\top P E_L > \epsilon^2 \Gamma (M_1 - M_2 M_3^{-1} M_2^\top) \Gamma^\top.$$

Since $M_1 \geq 0$ ($S_3 \geq 0$), the choice of $N_\Upsilon$ in (3.46) does not introduce conservatism, provided that $\epsilon > 0$ is a decision variable.

**Theorem 3.14** (Characterization IV) *Consider the continuous-time descriptor system* (3.3). *Assume that* $M_3 < 0$. *Enforcing* $\Theta_3 = 0$ *in* (3.41) *does not introduce conservatism.*

For the discrete-time case, choosing

$$N_\Upsilon = \left[\begin{array}{c:c} E_R(E_R^\top E_R)^{-1}\Delta^{-1} & 0 \\ 0 & 0 \\ 0 & I_m \end{array}\right] \rightarrow \Upsilon = \left[\begin{array}{c:c:c} V^\top & 0 & 0_{(n-r)\times m} \\ \hdashline 0 & I_n & 0 \end{array}\right], \tag{3.47}$$

leads to

$$\left[\begin{array}{c:c} \Gamma M_1 \Gamma^\top - E_L^\top P E_L & \Gamma M_2 \\ \hdashline \bullet & M_3 \end{array}\right] < 0,$$

which is only a sufficient condition for the existence of (3.34). Hence, nonc conservative Characterization IV is impossible to derive for discrete-time descriptor systems.

The positive scalar $\epsilon$ is introduced for continuous-time descriptor systems. The way to treat $\epsilon$ can be resumed as for admissibility and $H_2$ performance, $\epsilon$ can be any arbitrary positive scalar, while it must be considered as an additional decision variable for dissipativity. These choices coincide with the existing results for state-space systems [EH04, PDSV09, Xie08].

## 3.5 Robust Analysis

In this section, the benefit of using dilated LMI formulations deduced previously is illustrated through robust analysis for a class of affinely parameter-dependent descriptor systems. Let us consider the following parameter-dependent descriptor system whose coefficient matrices are affine functions of a time-invariant uncertain parameter vector $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_j]$

$$\begin{cases} E\dot{x} = A(\boldsymbol{\delta})\,x + B_1 w + B u, \\ z = C_1 w, \end{cases} \tag{3.48}$$

where $\delta_i \in [\delta_i^l, \ \delta_i^u]_{i=1,\ldots,j}$. $\boldsymbol{\delta}$ is supposed to be contained in the hype-rectangle $\Pi = \{\theta_1, \ldots, \theta_{2^j}\}$.

We consider here the problem of robust $H_\infty$ performance analysis. Let $\gamma_\infty^{opt}$ be the worst-case $H_\infty$ norm with respect to $\boldsymbol{\delta}$ contained in the hype-rectangle $\Pi$. The use of standard LMIs requests parameter-dependent Lyapunov matrix and results in a non-convex problem. To render it convex, a constant Lyapunov matrix is imposed over the entire hype-rectangle. This treatment is obviously conservative. Denote $\gamma_\infty^{sta}$ as the optimal $H_\infty$ norm obtained by the standard LMI approach with a constant Lyapunov matrix, and we have $\gamma_\infty^{opt} \leq \gamma_\infty^{sta}$.

The dilated LMIs enable the use of parameter-dependent Lyapunov matrices for robust analysis. For example, using Characterization I gives

$$\min_{P_i, Q_i, R_i, \Theta_1, \Theta_2, \Theta_3, \gamma_\infty} \quad \gamma_\infty \tag{3.49}$$

$$\text{subject to} \quad P_i > 0, \quad \forall i \in \{1, \ldots, 2^j\} \tag{3.50}$$

$$\begin{bmatrix} C_1^\top C_1 & (P_i E + U Q_i)^\top & 0 \\ \bullet & 0 & U R_i \\ \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} + \boldsymbol{He} \left\{ \begin{bmatrix} \Theta_1^\top \\ \Theta_2^\top \\ \Theta_3^\top \end{bmatrix} \begin{bmatrix} A & -I & B_1 \end{bmatrix} \right\} < 0. \tag{3.51}$$

Although the auxiliary matrices $\Theta_1$, $\Theta_2$, and $\Theta_3$ are still chosen as parameter-independent for convexity, compared with the standard LMI, the resulting estimated value, denoted $\gamma_\infty^I$, is in general smaller than $\gamma_\infty^{sta}$. Other dilations can equally be used for robust $H_\infty$ performance estimation. Denote the corresponding estimations as $\gamma_\infty^{II}$, $\gamma_\infty^{III}$, and $\gamma_\infty^{IV}$, respectively. The following relations hold:

$$\gamma_\infty^{opt} \leq \gamma_\infty^{I} \leq \gamma_\infty^{II} \leq \gamma_\infty^{IV} \leq \gamma_\infty^{sta}, \tag{3.52}$$

$$\gamma_\infty^{opt} \leq \gamma_\infty^{I} \leq \gamma_\infty^{III} \leq \gamma_\infty^{IV} \leq \gamma_\infty^{sta}. \tag{3.53}$$

## 3.6 Discrete-Time State Feedback $H_\infty$ Control

State feedback $H_\infty$ control for discrete-time descriptor systems in general leads to a nonlinear programming problem. Directly adopting the existing results in the literature ends with a nonlinear matrix inequality problem for controller synthesis. Dilated LMIs are appealed to address this problem. However, as it is shown previously, Characterization IV that contains only one single auxiliary matrix and is suitable for controller synthesis is impossible to derive. While the use of Characterizations I, II, and III that contain more than one auxiliary matrices indicates to impose certain structure on auxiliary matrices. This treatment of course gives a numerically tractable solution, but the resulting design process is conservative and no general conclusion can be found for the induced conservatism.

In this section, we present a numerically efficient and reliable controller design process for state feedback $H_\infty$ control of discrete-time descriptor systems through a new dilated LMI characterization.

Consider the following discrete-time descriptor system:

$$\begin{cases} \bar{E}\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}_1 w(k) + \bar{B}u(k), \\ y(k) = \bar{C}\bar{x}(k), \end{cases} \tag{3.54}$$

where $\bar{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^q$, and $u \in \mathbb{R}^m$ are the descriptor variable, controlled output, disturbance, and control input vectors, respectively. The matrix $\bar{E} \in \mathbb{R}^{n\times n}$ may be singular, i.e., $\mathrm{rank}(\bar{E}) = r \le n$; the matrices $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B}_1 \in \mathbb{R}^{n\times q}$, $\bar{B} \in \mathbb{R}^{n\times m}$, and $\bar{C} \in \mathbb{R}^{p\times n}$ are constant.

It is known that for the matrix pair $(\bar{E}, \bar{A})$, there exist nonsingular matrices $M$ and $N$ rendering this pair to the following SVD form:

$$E := M\bar{E}N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A := M\bar{A}N = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{3.55}$$

where $A_1 \in \mathbb{R}^{r\times r}$.

Now let us consider a state feedback control law as

$$u(k) = \bar{F}\bar{x}(k), \quad \bar{F} \in \mathbb{R}^{m\times n}. \tag{3.56}$$

Applying this controller to (3.54) yields the following closed-loop system:

$$\begin{cases} \bar{E}\bar{x}(k+1) = (\bar{A} + \bar{B}\bar{F})\bar{x}(k) + \bar{B}_1 w(k), \\ y(k) = \bar{C}\bar{x}(k). \end{cases} \tag{3.57}$$

**Problem 3.1** (*State Feedback $H_\infty$ Control Problem*) Given $\gamma > 0$, the state feedback $H_\infty$ control problem for the discrete-time descriptor system (3.54) is to find the state feedback controller (3.56) such that the closed-loop system (3.57) is admissible and $\|G_{yw}(z)\|_\infty < \gamma$, where $G_{yw}(z) = \bar{C}(z\bar{E} - \bar{A} - \bar{B}\bar{F})^{-1}\bar{B}_1$.

For simplicity of arguments, we consider, in rest of this section, the SVD form of the descriptor system (3.54), that is

$$\begin{cases} Ex(k+1) = Ax(k) + B_1 w(k) + Bu(k), \\ y(k) = Cx(k), \end{cases} \tag{3.58}$$

where $x(k) = N\bar{x}(k)$ and

$$B_1 = M\bar{B}_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, C = \bar{C}N = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, B = M\bar{B}, \tag{3.59}$$

compatible with the decompositions of the matrices $E$ and $A$ in (3.55). Consider a state feedback controller $u(k) = Fx(k)$, where $F \in \mathbb{R}^{m\times n}$. Then, the resulting closed-loop system is

$$\begin{cases} Ex(k+1) = (A+BF)x(k) + B_1 w(k), \\ \quad y(k) = Cx(k). \end{cases} \tag{3.60}$$

**Lemma 3.5** *Problem 3.1 is solvable, if and only if there exists F such that the closed-loop system (3.60) is admissible and $\|C(zI - A - BF)^{-1}B_1\|_\infty < \gamma$. Moreover, $\bar{F} = FN^{-1}$.*

*Proof* Note that the systems (3.57) and (3.60) have identical zeros and poles. Moreover, the transfer function $G_{yw}(z)$ can be rewritten as

$$\begin{aligned} G_{yw}(z) =& \bar{C}NN^{-1}(z\bar{E} - \bar{A} - \bar{B}\bar{F})^{-1}M^{-1}M\bar{B}_1 \\ =& \bar{C}N(zM\bar{E}N - M\bar{A}N - M\bar{B}\bar{F}N)^{-1}M\bar{B}_1 \\ =& C(zE - A - BF)^{-1}B_1. \end{aligned}$$

□

A new bounded real lemma is given in the following theorem for the system (3.58) with $u(k) = 0$.

**Theorem 3.15** *Given $\gamma > 0$ and $u(k) = 0$, the descriptor system (3.58) is admissible and $\|G_{yw}(z)\|_\infty < \gamma$, where $G_{yw}(z) = C(zE - A)^{-1}B_1$, if and only if there exist matrices $P \in \mathbb{R}^{r\times r} > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$ and a sufficiently large scalar $\alpha > 0$, such that*

$$\begin{bmatrix} \Phi_1 & \bullet & \bullet & \bullet & \bullet \\ A^\top\Gamma^\top & \Phi_3 & \bullet & \bullet & \bullet \\ B_1^\top\Gamma^\top & B_1^\top\Pi^\top & -\gamma^2 I_q & \bullet & \bullet \\ \Phi_2 & \Gamma A & \Gamma B_1 & -Q - Q^\top & \bullet \\ 0 & \Phi_4 & \Phi_5 & 0 & -I_p \end{bmatrix} < 0, \tag{3.61}$$

*where*

$$\begin{aligned} &\Phi_1 = -\frac{1}{2}Q - \frac{1}{2}Q^\top, \quad \Phi_2 = P - Q - \frac{1}{2}Q^\top, \\ &\Phi_3 = \Pi A + A^\top\Pi^\top - \Theta, \quad \Phi_4 = C + \alpha C\Pi A, \quad \Gamma = \begin{bmatrix} Q & R \end{bmatrix}, \\ &\Phi_5 = \alpha C\Pi B_1, \quad \Theta = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}. \end{aligned}$$

The following two lemmas are needed for the proof of Theorem 3.15.

**Lemma 3.6** *Reference [XL06] Let*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix},$$

*where $\Omega_{11}$, $\Omega_{12}$, $\Omega_{21}$, and $\Omega_{22}$ are any real matrices with appropriate dimensions such that $\Omega_{22}$ is invertible and $\Omega + \Omega^\top < 0$. Then we have*

$$\Omega_{11} + \Omega_{11}^\top - \Omega_{12}\Omega_{22}^{-1}\Omega_{21} - \Omega_{21}^\top \Omega_{22}^{-\top}\Omega_{12}^\top < 0.$$

**Lemma 3.7** *Reference [FY13] Consider $\gamma > 0$ and the discrete-time state-space system*

$$\begin{cases} x(k+1) = \mathcal{A}x(k) + \mathcal{B}v(k), \\ y(k) = \mathcal{C}x(k) + \mathcal{D}v(k), \end{cases} \tag{3.62}$$

*where the matrices $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$, and $\mathcal{D} \in \mathbb{R}^{p\times q}$ are constant. The system (3.62) is stable and $\|G_{yv}(z)\|_\infty < \gamma$, where $G_{yv}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$, if and only if there exist matrices $X \in \mathbb{R}^{n\times n} > 0$ and $Y \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^\top & Y\mathcal{A} & Y\mathcal{B} & X - \frac{1}{2}Y - Y^\top \\ \bullet & \mathcal{C}^\top\mathcal{C} - X & \mathcal{C}^\top\mathcal{D} & \mathcal{A}^\top Y^\top \\ \bullet & \bullet & \mathcal{D}^\top\mathcal{D} - \gamma^2 I_q & \mathcal{B}^\top Y^\top \\ \bullet & \bullet & \bullet & -Y - Y^\top \end{bmatrix} < 0. \tag{3.63}$$

*Proof of Theorem 3.15* Sufficiency: Suppose that the inequality (3.61) holds. We prove that the system is admissible and its $H_\infty$ norm is bounded by $\gamma$. With the decompositions (3.55), there holds

$$\begin{bmatrix} \Delta_{11} & \bullet & \bullet & \bullet & \bullet & \bullet \\ \Delta_{21} & -P & \bullet & \bullet & \bullet & \bullet \\ \Delta_{31} & SA_3 & \Delta_{33} & \bullet & \bullet & \bullet \\ \Delta_{41} & 0 & B_{12}^\top S^\top & -\gamma^2 I_q & \bullet & \bullet \\ \Delta_{51} & \Delta_{52} & \Delta_{53} & \Delta_{54} & \Delta_{55} & \bullet \\ 0 & \Delta_{62} & \Delta_{63} & \Delta_{64} & 0 & -I_p \end{bmatrix} < 0,$$

where

$$\begin{aligned} &\Delta_{11} = -\tfrac{1}{2}Q - \tfrac{1}{2}Q^\top, && \Delta_{21} = A_1^\top Q^\top + A_3^\top R^\top, \\ &\Delta_{31} = A_2^\top Q^\top + A_4^\top R^\top, && \Delta_{33} = SA_4 + A_4^\top S^\top, \\ &\Delta_{41} = B_{11}^\top Q^\top + B_{12}^\top R^\top, && \Delta_{51} = P - \tfrac{1}{2}Q^\top - Q, \\ &\Delta_{52} = QA_1 + RA_3, && \Delta_{53} = QA_2 + RA_4, \\ &\Delta_{54} = QB_{11} + RB_{12}, && \Delta_{55} = -Q - Q^\top, \\ &\Delta_{62} = C_1 + \alpha C_2 SA_3, && \Delta_{63} = C_2 + \alpha C_2 SA_4, \\ &\Delta_{64} = \alpha C_2 SB_{12}. \end{aligned}$$

Left and right multiplying this inequality by

$$T = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & I_{n-r} & 0 & 0 & 0 \end{bmatrix} \tag{3.64}$$

and its transpose, respectively, give

$$W + W^\top < 0, \tag{3.65}$$

with

$$W = \begin{bmatrix} -\frac{1}{2}Q & 0 & 0 & 0 & 0 & 0 \\ W_{21} & -\frac{1}{2}P & 0 & W_{24} & W_{25} & A_3^\top S^\top \\ W_{31} & 0 & -\frac{\gamma^2}{2}I_q & W_{34} & W_{35} & B_{12}^\top S^\top \\ W_{41} & 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}I_p & 0 \\ W_{61} & 0 & SB_{12} & W_{64} & W_{65} & A_4^\top S^\top \end{bmatrix},$$

where

$$\begin{aligned} W_{21} &= A_1^\top Q^\top + A_3^\top R^\top, & W_{24} &= A_1^\top Q^\top + A_3^\top R^\top, \\ W_{25} &= C_1^\top + \alpha A_3^\top S^\top C_2^\top, & W_{31} &= B_{11}^\top Q^\top + B_{12}^\top R^\top, \\ W_{34} &= B_{11}^\top Q^\top + B_{12}^\top R^\top, & W_{35} &= \alpha B_{12}^\top S^\top C_2^\top, \\ W_{41} &= P - \tfrac{1}{2}Q^\top - Q, & W_{61} &= A_2^\top Q^\top + A_4^\top R^\top, \\ W_{64} &= A_2^\top Q^\top + A_4^\top R^\top, & W_{65} &= C_2^\top + \alpha A_4^\top S^\top C_2^\top. \end{aligned}$$

The (6, 6) entry of the above inequality reads to $A_4^\top S^\top + SA_4 < 0$. Hence, one can claim that the matrices $A_4$ and $S$ are both nonsingular. Hence, the system (3.58) is causal and can be transformed into a state-space system associated with the realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, with

$$\begin{aligned} \tilde{A} &= A_1 - A_2A_4^{-1}A_3, \; \tilde{B} = B_{11} - A_2A_4^{-1}B_{12}, \\ \tilde{C} &= C_1 - C_2A_4^{-1}A_3, \; \tilde{D} = -C_2A_4^{-1}B_{12}. \end{aligned} \tag{3.66}$$

Now, it suffices to prove that the matrix $\tilde{A}$ is stable and $\|\tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\|_\infty < \gamma$. Since $SA_4$ is invertible, applying Lemma 3.6 to the inequality (3.65) yields

$$\mathcal{M} < 0, \tag{3.67}$$

where

$$\mathcal{M} = \begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^\top & \bullet & \bullet & \bullet & \bullet \\ \tilde{A}^\top Q^\top & -P & \bullet & \bullet & \bullet \\ \tilde{B}^\top Q^\top & 0 & -\gamma^2 I_q & \bullet & \bullet \\ P - \frac{1}{2}Q^\top - Q & Q\tilde{A} & Q\tilde{B} & -Q - Q^\top & \bullet \\ 0 & \tilde{C} & \tilde{D} & 0 & -I_p \end{bmatrix}.$$

According to Schur complement and Lemma 3.7, it is shown that $\tilde{A}$ is stable and $\|\tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\|_\infty < \gamma$.

Necessity: Suppose that the descriptor system (3.58) is admissible and $\|G_{yw}\|_\infty < \gamma$. We prove that the condition (3.61) holds. The assumption of admissibility indicates that the matrix $A_4$ is invertible and the system (3.58) can be rewritten as the state-space system associated with the realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given in (3.66). Using Lemma 3.7, together with Schur complement, gives (3.67). Since this inequality is strict, there exists a small positive scalar $\epsilon$ such that

$$\mathcal{M} + \epsilon \begin{bmatrix} 0 \\ A_3^\top A_4^{-1} \\ B_{12}^\top A_4^{-1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ A_3^\top A_4^{-1} \\ B_{12}^\top A_4^{-1} \\ 0 \\ 0 \end{bmatrix}^\top < 0.$$

Set $S = -\frac{1}{2}\epsilon A_4^{-1}$. Then, the above inequality can be rewritten as

$$\mathcal{M} - \begin{bmatrix} 0 \\ A_3^\top S^\top \\ B_{12}^\top S^\top \\ 0 \\ 0 \end{bmatrix} (A_4^\top S^\top + S A_4)^{-1} \begin{bmatrix} 0 \\ A_3^\top S^\top \\ B_{12}^\top S^\top \\ 0 \\ 0 \end{bmatrix}^\top < 0.$$

Using Schur complement, and setting $R = -Q A_2 A_4^{-1}$ and $\alpha = \frac{2}{\epsilon}$ yield

$$\bar{W} + \bar{W}^\top < 0,$$

where

$$\bar{W} = \begin{bmatrix} -\frac{1}{2}Q & 0 & 0 & 0 & 0 & 0 \\ \bar{W}_{21} & -\frac{1}{2}P & 0 & \bar{W}_{24} & \bar{W}_{25} & A_3^\top S^\top \\ \bar{W}_{31} & 0 & -\frac{\gamma^2}{2} I_q & \bar{W}_{34} & \bar{W}_{35} & B_{12}^\top S^\top \\ \bar{W}_{41} & 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}I_p & 0 \\ 0 & 0 & S B_{12} & 0 & 0 & A_4^\top S^\top \end{bmatrix},$$

with

$$\begin{aligned}
\bar{W}_{21} &= A_1^\top Q^\top + A_3^\top R^\top, \quad \bar{W}_{24} = A_1^\top Q^\top + A_3^\top R^\top, \\
\bar{W}_{25} &= C_1^\top + \alpha A_3^\top S^\top C_2^\top, \ \bar{W}_{31} = B_{11}^\top Q^\top + B_{12}^\top R^\top, \\
\bar{W}_{34} &= B_{11}^\top Q^\top + B_{12}^\top R^\top, \quad \bar{W}_{35} = \alpha B_{12}^\top S^\top C_2^\top, \\
\bar{W}_{41} &= P - \tfrac{1}{2} Q^\top - Q.
\end{aligned}$$

Moreover, with the aforementioned choices of $R$, $S$, and $\alpha$, one has

$$QA_2 + RA_4 = 0, \quad C_2 + \alpha C_2 S A_4 = 0.$$

Replacing the (6, 1), (6, 4), and (6, 5) entries of $\bar{W}$ with $(QA_2 + RA_4)^\top$, $(QA_2 + RA_4)^\top$, and $(C_2 + \alpha C_2 S A_4)^\top$, respectively, we have (3.65). Then, by left and right multiplying (3.65) by $T^{-1}$ given in (3.64) and its transpose, respectively, there holds the inequality (3.61). That ends the proof. □

It is worth noting that Theorem 3.15 is relied on the SVD form of the descriptor system, but the feasibility of (3.61) does not depend on a particular choice of SVD decomposition. Consider an arbitrary transformation matrix pair $(M_1, N_1)$. It is easy to see that, if the condition (3.61) is feasible for the system (3.58), then it is also feasible for the SVD-based realization associated with $(M_1, N_1)$.

Now we are in a position to give a solution to state feedback controller synthesis.

**Theorem 3.16** *Problem 3.1 is solvable, if and only if there exist matrices $P \in \mathbb{R}^{r\times r} > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $Z \in \mathbb{R}^{n\times m}$ and a sufficiently large scalar $\alpha > 0$ such that*

$$\begin{bmatrix} \Lambda_1 & \bullet & \bullet & \bullet & \bullet \\ \Lambda_2 & \Lambda_5 & \bullet & \bullet & \bullet \\ \Lambda_3 & \Lambda_6 & -\gamma^2 I_p & \bullet & \bullet \\ \Lambda_4 & \Lambda_2^\top & \Lambda_3^\top & -Q - Q^\top & \bullet \\ 0 & \Lambda_7 & \Lambda_8 & 0 & -I_q \end{bmatrix} < 0, \tag{3.68}$$

*where*

$$\begin{aligned}
&\Lambda_1 = -\frac{1}{2} Q - \frac{1}{2} Q^\top, \quad \Lambda_2 = A\Gamma^\top + BZ^\top \Omega^\top, \quad \Lambda_3 = C\Gamma^\top, \\
&\Lambda_4 = P - Q - \frac{1}{2} Q^\top, \quad \Lambda_8 = \alpha B_1^\top \Pi C^\top, \\
&\Lambda_5 = \Pi A^\top + A\Pi^\top + \Phi Z B^\top + B Z^\top \Phi^\top - \Theta, \quad \Lambda_6 = C\Pi^\top, \\
&\Lambda_7 = B_1^\top + \alpha B_1^\top \Pi A^\top + \alpha B_1^\top \Phi^\top Z B^\top, \quad \Gamma = \begin{bmatrix} Q & R \end{bmatrix}, \\
&\Theta = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad \Omega = \begin{bmatrix} I_r & 0 \end{bmatrix}.
\end{aligned}$$

*Moreover, the feedback gain $\bar{F}$ is given as*

$$\bar{F} = Z^\top \begin{bmatrix} Q^{-\top} & 0 \\ -S^{-\top} R^\top Q^{-\top} & S^{-\top} \end{bmatrix} N^{-1}. \tag{3.69}$$

*Proof* According to Lemma 3.5, the solvability of Problem 3.1 is equivalent to the existence of a feedback gain $F$ such that the closed-loop system (3.60) is admissible and the $H_\infty$ norm of its transfer function from $w(k)$ to $y(k)$ is less than $\gamma$. Moreover, $\bar{F} = FN^{-1}$.

Assume that the inequality (3.68) holds. Then, the (1, 1) entry implies that $Q$ is invertible. And $S$ can also be assumed to be invertible without loss of generality. Replacing $Z$ with $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F^\top$ in (3.68) and straightforward calculation give (3.61) associated with the dual system of (3.60). Hence (3.60) is admissible and the $H_\infty$ performance of its transfer function from $w(k)$ to $y(k)$ is satisfied.

Conversely, if Problem 3.1 is solvable, then the condition (3.61) holds for the system (3.60). Hence, it also holds for its dual system. By the linear change of variables $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F^\top = Z$, which implies

$$\begin{bmatrix} Q & R \end{bmatrix} F^\top = \begin{bmatrix} I_r & 0 \end{bmatrix} Z, \quad \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F^\top = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z,$$

we have (3.68). Moreover, as pointed out before, $Q$ and $S$ are both invertible. Therefore, the feedback gain $F$ for the closed-loop system (3.60) is $F = Z^\top \begin{bmatrix} Q^{-\top} & 0 \\ -S^{-\top} R^\top Q^{-\top} & S^{-\top} \end{bmatrix}$. Hence $\bar{F}$ is given by (3.69). □

Following the same thread, we can also conduct a dilated LMI characterization and give a numerically efficient and reliable controller design process for discrete-time state feedback positive realness control. Relevant results have been reported in [Fen15].

*Example 3.1* Let the descriptor system (3.54) be given as follows:

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top, \quad \bar{C} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

It is easy to see that the system is noncausal. In addition, the system has a pole at 2.6180, and hence it is not stable either. The purpose is to find a state feedback $H_\infty$ controller with $\gamma = 2$. Using Theorem 3.16, together with $\alpha = 1$, gives

$$\bar{F} = \begin{bmatrix} 7.3827 & -20.4128 & 1.9135 \end{bmatrix}.$$

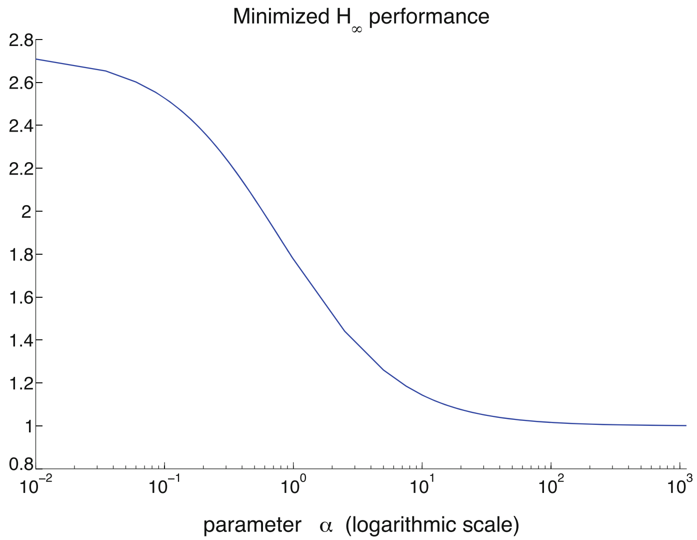


**Fig. 3.1** Minimized $H_\infty$ performance with respect to different $\alpha$

Moreover, Fig. 3.1 plots the minimized $H_\infty$ performance achieved by Theorem 3.16 with respect to different values of $\alpha$. It is observed that when $\alpha$ is selected to be relatively large, conservatism is significantly reduced.

## 3.7 Conclusion

In this chapter, we have discussed dilated LMI characterizations for descriptor systems for both the continuous-time and discrete-time settings. By reciprocal application of the projection lemma, dilated LMI conditions for admissibility, $H_2$ performance and dissipativity are conducted. These dilated LMIs are denoted as Characterization I, II, III, IV. As in the state-space case, among all, Characterization I has the largest number of decision variables and is in general suitable for robust analysis, while Characterization IV holds the smallest number of decision variables. Moreover, relied on the use of auxiliary matrices and a positive scalar, a novel necessary and sufficient condition for the bounded real lemma for discrete-time descriptor systems is derived, and a numerically efficient and reliable design procedure for state feedback $H_\infty$ controller design is given.

# Chapter 4
# Dissipative Control Under Output Regulation

Relied on the internal model principle [FSW74, Won85, Hua04], exact asymptotic regulation objective is fulfilled using a structured controller with a exosystem. Extensions of this scheme have been considered by integrating other performance requirements. Such multiobjective problems have been extensively investigated in the literature [SSS00a, SSS00b, KS08]. Moreover, the regulation problem has been expanded to descriptor systems. In [Dai89], a solution to this problem was given in terms of a set of nonlinear matrix equations depending on system parameters and some other parameters. In [LD96], a more clear-cut solution was obtained through a generalized Sylvester equation.

In this chapter, we focus on the design of a measurement output feedback controller to solve the problem of output regulation for descriptor systems. A generalized Sylvester equation is proposed to achieve internal stabilization subject to asymptotic regulation constraints, and a specific structure of the resulting controller is deduced. Then, based on this structure, the additional dissipative performance objective is further handled using an LMI-based approach.

## 4.1 Problem Formulation

Consider the descriptor system as follows:

$$(G):\begin{bmatrix} E\dot{x} \\ \hline e \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_w & B_d & B \\ \hline C_e & D_{ew} & D_{ed} & D_{eu} \\ C_z & D_{zw} & D_{zd} & D_{zu} \\ C & D_{yw} & D_{yd} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ d \\ u \end{bmatrix}, \tag{4.1}$$

where $x \in \mathbb{R}^n$, $e \in \mathbb{R}^{q_e}$, $z \in \mathbb{R}^{q_z}$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^{n_w}$, $d \in \mathbb{R}^{m_d}$, and $u \in \mathbb{R}^m$ are the state, tracking error, controlled output, measurement, exogenous disturbance,



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external disturbance, and control input, respectively. The matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B_w \in \mathbb{R}^{n\times n_w}$, $B_d \in \mathbb{R}^{n\times m_d}$, $B \in \mathbb{R}^{n\times m}$, $C_e \in \mathbb{R}^{q_e\times n}$, $D_{ew} \in \mathbb{R}^{q_e\times n_w}$, $D_{ed} \in \mathbb{R}^{q_e\times m_d}$, $D_{eu} \in \mathbb{R}^{q_e\times m}$, $C_z \in \mathbb{R}^{q_z\times n}$, $D_{zw} \in \mathbb{R}^{q_z\times n_w}$, $D_{zd} \in \mathbb{R}^{q_z\times m_d}$, $D_{zu} \in \mathbb{R}^{q_z\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_{yw} \in \mathbb{R}^{p\times n_w}$, and $D_{yd} \in \mathbb{R}^{p\times m_d}$ are constant.

We assume here that the exogenous disturbance $w$ is generated by the linear autonomous system $G_w$ referred to as the exosystem as follows:

$$\dot{w} = A_w w, \quad A_w \in \mathbb{R}^{n_w\times n_w}.$$

Denote the new descriptor variable as $\zeta^\top = [x^\top \ w^\top]$. Then, the descriptor system $G$ is rewritten as

$$(G):\left[\begin{array}{cc|cc} A-sE & B_w & B_d & B \\ 0 & A_w - sI & 0 & 0 \\ \hline C_e & D_{ew} & D_{ed} & D_{eu} \\ C_z & D_{zw} & D_{zd} & D_{zu} \\ C & D_{yw} & D_{yd} & 0 \end{array}\right]. \tag{4.2}$$

The following assumptions are made:

(A.1) $(E, A, B)$ is finite dynamics stabilizable and impulse controllable;

(A.2) $\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A & B_w \\ 0 & A_w \end{bmatrix}, \begin{bmatrix} C & D_{yw} \end{bmatrix}\right)$ is finite dynamics detectable and impulse observable;

(A.3) $\sigma(A_w) \in \overline{\mathbb{C}^+} \triangleq \{\lambda \in \mathbb{C} | Re(\lambda) \geq 0\}$.

Consider the following measurement feedback controller:

$$(\Sigma_c):\begin{cases} E_K\dot{\xi} = A_K\xi + B_K y, \\ u = C_K\xi + D_K y, \end{cases} \tag{4.3}$$

where $E_K \in \mathbb{R}^{n_k\times n_k}$, $A_K \in \mathbb{R}^{n_k\times n_k}$, $B_K \in \mathbb{R}^{n_k\times p}$, $C_K \in \mathbb{R}^{m\times n_k}$, and $D_K \in \mathbb{R}^{m\times p}$. The matrix $E_K$ may be singular, i.e., $\text{rank}(E_K) = r_k \leq n_k$.

The closed-loop system is denoted by $G \times \Sigma_c$, and the transform matrix from the external disturbance $d$ to the controlled output $z$ of $G \times \Sigma_c$ is denoted by $T_{zd}(G \times \Sigma_c)$.

**Problem 4.1** (*Asymptotic Output Regulation Problem*) The problem of asymptotic output regulation is to find a controller $\Sigma_c$ such that the following conditions hold.

C.1 In the absence of the disturbances $w$ and $d$, the closed-loop system $G \times \Sigma_c$ is internally stable.

C.2 The closed-loop system $G \times \Sigma_c$ satisfies $\lim_{t\to\infty} e(t) = 0$ for any $d \in \ell_2$, and for all $x(0) \in \mathbb{R}^n$ and $w(0) \in \mathbb{R}^{n_w}$.

Note that Assumptions (A.1)–(A.3) are consistent with the standard assumptions in the regulator theory for state-space systems [SSS00a, SSS00b]. For state-space systems, the assumptions associated with the impulse controllability and observability vanish. Assumption (A.1) together with another assumption that $(E, A, C)$ is finite dynamics detectable and impulsive observable is essential to the existence of an internally stabilizing controller. The Condition (A.3) can be assumed without loss of generality, since any asymptotically stable modes of $G_w$ decay to zero and thus do not affect the regulation objective.

## 4.2 Extended Regulator Theory

In this section, the well-known regulator theory for state-space systems is extended to linear descriptor systems and a structured dynamical controller achieving asymptotic output regulation is exhibited.

**Lemma 4.1** *Consider the plant G in* (4.2). *There exists a controller* $\Sigma_c$ *in* (4.3) *such that Problem 4.1 is solvable, if and only if there exist two matrices* $T \in \mathbb{R}^{n \times n_w}$ *and* $\Pi \in \mathbb{R}^{m \times n_w}$ *such that the following generalized Sylvester equation holds:*

$$\begin{aligned} B\Pi &= AT - B_w - ETA_w, \\ D_{eu}\Pi &= C_e T - D_{ew}. \end{aligned} \tag{4.4}$$

*Proof* Necessity: Suppose that there exists a controller $\Sigma_c$ solving Problem 4.1. According to [FYC11], there exist two matrices $T$ and $T_K$ such that the following equations hold:

$$0 = D_{ew} + D_{eu}D_K D_{yw} - (C_e T + D_{eu}D_K CT + D_{eu}C_K T_K), \tag{4.5}$$

$$ETA_w = AT + BD_K CT + BC_K T_K - B_w - BD_K D_{yw}, \tag{4.6}$$

$$E_K T_K A_w = A_K T_K + B_K(CT - D_{yw}). \tag{4.7}$$

By setting $\Pi = D_K(D_{yw} - CT) - C_K T_K$, (4.5) and (4.6) lead to (4.4).

Sufficiency: We suppose that the condition (4.4) holds. In order to prove the sufficiency, it suffices to construct the controller $\Sigma_c$ solving Problem 4.1. To this end, consider the following system $\tilde{G}$:

$$(\tilde{G}) : \begin{cases} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} A & B\Pi \\ 0 & A_w \end{bmatrix} \tilde{x} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} d + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} u_c, \\ e = \begin{bmatrix} C_e & D_{eu}\Pi \end{bmatrix} \tilde{x} + D_{ed}d + \begin{bmatrix} D_{eu} & 0 \end{bmatrix} u_c, \\ y_c = \begin{bmatrix} C & CT - D_{yw} \end{bmatrix} \tilde{x} + D_{yd}d. \end{cases} \tag{4.8}$$

According to the Assumptions (A.1) and (A.2) together with some appropriate basis transformation, it is easy to show that $\tilde{G}$ is finite dynamics stabilizable and impulsive controllable, and finite dynamics detectable and impulsive observable. Hence, there exists an internally stabilizing controller $\tilde{\Sigma}_c$ for $\tilde{G}$. Let us suppose that $\tilde{\Sigma}_c$ admits the following form:

$$(\tilde{\Sigma}_c): \begin{cases} \tilde{E}_k \dot{x}_c = \tilde{A}_k x_c + \tilde{B}_k y_c, \\ u_c = \begin{bmatrix} \tilde{C}_{k1} \\ \tilde{C}_{k2} \end{bmatrix} x_c + \begin{bmatrix} \tilde{D}_{k1} \\ \tilde{D}_{k2} \end{bmatrix} y_c, \end{cases} \tag{4.9}$$

where $\tilde{E}_k \in \mathbb{R}^{(n_k-n_w)\times(n_k-n_w)}$, $\tilde{A}_k \in \mathbb{R}^{(n_k-n_w)\times(n_k-n_w)}$, $\tilde{B}_k \in \mathbb{R}^{(n_k-n_w)\times p}$, $\tilde{C}_{k1} \in \mathbb{R}^{n_w\times(n_k-n_w)}$, $\tilde{C}_{k2} \in \mathbb{R}^{m\times(n_k-n_w)}$, $\tilde{D}_{k1} \in \mathbb{R}^{n_w\times p}$, and $\tilde{D}_{k2} \in \mathbb{R}^{m\times p}$. Then, we can construct the controller $\Sigma_c$ as follows:

$$\begin{cases} \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k \end{bmatrix} \dot{\xi} = \begin{bmatrix} A_w + \tilde{D}_{k2}(CT - D_{yw}) & \tilde{C}_{k2} \\ \tilde{B}_k(CT - D_{yw}) & \tilde{A}_k \end{bmatrix} \xi + \begin{bmatrix} \tilde{D}_{k2} \\ \tilde{B}_k \end{bmatrix} y, \\ u = \begin{bmatrix} \Pi + \tilde{D}_{k1}(CT - D_{yw}) & \tilde{C}_{k1} \end{bmatrix} \xi + \tilde{D}_{k1} y. \end{cases} \tag{4.10}$$

Hence, the resulting closed-loop system formed by $G$ and the controller (4.10) is given by

$$(G_{CL}): \left[\begin{array}{c|c} A_c - sE_c & B_c \\ \hline C_c & D_c \end{array}\right],$$

where

$$A_c = \begin{bmatrix} A + B\tilde{D}_{k1}C & B_w + B\tilde{D}_{k1}D_{yw} & B\Pi + B\tilde{D}_{k1}\Upsilon & B\tilde{C}_{k1} \\ 0 & A_w & 0 & 0 \\ \tilde{D}_{k2}C & \tilde{D}_{k2}D_{yw} & A_w + \tilde{D}_{k2}\Upsilon & \tilde{C}_{k2} \\ \tilde{B}_k C & \tilde{B}_k D_{yw} & \tilde{B}_k \Upsilon & \tilde{A}_k \end{bmatrix},$$

$$E_c = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & 0 & I_{n_w} & 0 \\ 0 & 0 & 0 & \tilde{E}_k \end{bmatrix}, \quad B_c = \begin{bmatrix} B_d + B\tilde{D}_{k1}D_{yd} \\ 0 \\ \tilde{D}_{k2}D_{yd} \\ \tilde{B}_k D_{yd} \end{bmatrix},$$

$$C_c = \begin{bmatrix} C_e + D_{eu}\tilde{D}_{k1}C & D_{ew} + D_{eu}\tilde{D}_{k1}D_{yw} & D_{eu}(\Pi + \tilde{D}_{k1}\Upsilon) & D_{eu}\tilde{C}_{k1} \end{bmatrix},$$

$$D_c = D_{ed} + D_{eu}\tilde{D}_{k1}D_{yd}, \quad \Upsilon = CT - D_{yw}.$$

Using the following two transformation matrices

$$M = \begin{bmatrix} I_n & ET & 0 & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & -I_{n_w} & I_{n_w} & 0 \\ 0 & 0 & 0 & I_{n_k-n_w} \end{bmatrix}, \quad N = \begin{bmatrix} I_n & -T & 0 & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & I_{n_w} & I_{n_w} & 0 \\ 0 & 0 & 0 & I_{n_k-n_w} \end{bmatrix},$$

together with the generalized Sylvester equation (4.4), yield

$$MA_cN = \begin{bmatrix} A + B\tilde{D}_{k1}C & \Phi_1 & B\Pi + B\tilde{D}_{k1}\Upsilon & B\tilde{C}_{k1} \\ 0 & A_w & 0 & 0 \\ \tilde{D}_{k2}C & \Phi_2 & A_w + \tilde{D}_{k2}\Upsilon & \tilde{C}_{k2} \\ \tilde{B}_kC & \Phi_3 & \tilde{B}_k\Upsilon & \tilde{A}_k \end{bmatrix}, \quad ME_cN = E_c,$$

$$MB_c = B_c, \quad C_cN = \begin{bmatrix} C_e + D_{eu}\tilde{D}_{k1}C & \Phi_4 & D_{eu}(\Pi + \tilde{D}_{k1}\Upsilon) & D_{eu}\tilde{C}_{k1} \end{bmatrix},$$

where

$$\begin{aligned} \Phi_1 &= B_w + B\tilde{D}_{k1}D_{yw} + B\Pi + B\tilde{D}_{k1}\Upsilon - (A + B\tilde{D}_{k1}C)T + ETA_w = 0, \\ \Phi_2 &= \tilde{D}_{k2}D_{yw} + A_w + \tilde{D}_{k2}\Upsilon - \tilde{D}_{k2}CT - A_w = 0, \\ \Phi_3 &= \tilde{B}_kD_{yw} + \tilde{B}_k\Upsilon - \tilde{B}_kCT = 0, \\ \Phi_4 &= D_{ew} + D_{eu}\tilde{D}_{k1}D_{yw} + D_{eu}\Pi + D_{eu}\tilde{D}_{k1}\Upsilon - C_eT + D_{eu}\tilde{D}_{k1}CT = 0. \end{aligned}$$

Therefore, the closed-loop system $G_{CL}$ is equivalent to

$$\begin{cases} \begin{bmatrix} E & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{E}_k \end{bmatrix} \dot{x}_{cl} = \begin{bmatrix} A + B\tilde{D}_{k1}C & B\Pi + B\tilde{D}_{k1}\Upsilon & B\tilde{C}_{k1} \\ \tilde{D}_{k2}C & A_w + \tilde{D}_{k2}\Upsilon & \tilde{C}_{k2} \\ \tilde{B}_kC & \tilde{B}_k\Upsilon & \tilde{A}_k \end{bmatrix} x_{cl} \\ \qquad\qquad + \begin{bmatrix} B_d + B\tilde{D}_{k1}D_{yd} \\ \tilde{D}_{k2}D_{yd} \\ \tilde{B}_kD_{yd} \end{bmatrix} d, \\ e = \begin{bmatrix} C_e + D_{eu}\tilde{D}_{k1}C & D_{eu}(\Pi + \tilde{D}_{k1}\Upsilon) & D_{eu}\tilde{C}_{k1} \end{bmatrix} x_{cl} + (D_{ed} + D_{eu}\tilde{D}_{k1}D_{yd})d, \end{cases}$$

which is nothing else but the feedback system formed by the controller $\tilde{\Sigma}_c$ in (4.9) and the systemt $\tilde{G}$ in (4.8). Since $\tilde{\Sigma}_c$ internally stabilizes $\tilde{G}$ and $d \in \ell_2$, the closed-loop system $G_{CL}$ is internally stable and $\lim_{t\to\infty} e(t) = 0$. □

**Lemma 4.2** *Any controller solving Problem 4.1 contains a copy of the dynamics of $A_w$ that are unobservable with respect to $CT - D_{yw}$.*

*Proof* For simplicity, we assume that the eigenvalues of $A_w$ are distinct. If $A_w$ have multiple eigenvalues, then the same argument used in [SSS00a] can be applied. It has shown that if the asymptotic output regulation problem is solved, then there exist matrices $T$ and $T_K$ satisfying

$$\begin{aligned} E_KT_KA_w &= A_KT_K + B_K(CT - D_{yw}), \\ \Pi &= -C_KT_K - D_K(CT - D_{yw}). \end{aligned}$$

Denote $\lambda \in \overline{\mathbb{C}^+}$ and $v$ as an arbitrary unobservable eigenvalue of $A_w$ and its associated eigenvector, respectively. Hence, we have $(CT - D_{yw})v = 0$. Then, there holds $A_KT_Kv = E_KT_KA_wv = \lambda E_KT_Kv$. Now, we prove by contradiction that $T_Kv \neq 0$. According to the finite dynamics stabilizability assumption made in (A.2), we have

$$\text{rank}\left(\begin{bmatrix} A-\lambda E & B_w \\ 0 & A_w-\lambda I \\ C & D_{yw} \end{bmatrix}\right) = n + n_w.$$

Suppose $T_K v = 0$, there holds

$$\begin{bmatrix} A-\lambda E & B_w \\ 0 & A_w-\lambda I \\ C & D_{yw} \end{bmatrix}\begin{bmatrix} Tv \\ -v \end{bmatrix} = \begin{bmatrix} (AT - B_w - \lambda ET)v \\ (\lambda I - A_w)v \\ (CT - D_{yw})v \end{bmatrix}$$
$$= \begin{bmatrix} (B\Pi + ETA_w - \lambda ET)v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -BC_K T_K v + ET(A_w - \lambda E_w)v \\ 0 \\ 0 \end{bmatrix} = 0,$$

which is contradictory to Assumption (A.2). Hence $T_K v \neq 0$. According to the definition of generalized eigenvalues of a matrix pencil [GvL96], one can conclude that the unobservable eigenvalues of $(A_w, CT - D_{yw})$ are also eigenvalues of the controller. □

## 4.3 Dissipativity with Output Regulation

In this section, we further address a multiobjective problem by adding an additional dissipative performance in Problem 4.1. Before formulating the problem, we present a quick reminder of dissipativity. Let us consider the following descriptor system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bv(t), \\ z(t) = Cx(t) + Dv(t), \end{cases} \tag{4.11}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$, and $v(t) \in \mathbb{R}^m$. The matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are constant. Moreover, the matrix $E$ may be singular. If the descriptor system is regular, then the transfer function is defined as

$$H(s) = C(sE - A)^{-1}B + D. \tag{4.12}$$

Moreover, let $S = S^\top \in \mathbb{R}^{(m+p)\times(m+p)}$ and the supply function $s(v(t), z(t)) \in \mathbb{R}_m \times \mathbb{R}_p \to \mathbb{R}$ be a mapping with the following quadratic form of $v(t)$ and $z(t)$:

$$s(v(t), z(t)) = \begin{bmatrix} v(t) \\ z(t) \end{bmatrix}^\top S \begin{bmatrix} v(t) \\ z(t) \end{bmatrix}, \tag{4.13}$$

where

$$S \triangleq \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}, \quad S_{11} \in \mathbb{R}^{m\times m}. \tag{4.14}$$

**Definition 4.1** (*Strict Dissipativity*) The descriptor system (4.11) is said to be strictly dissipative with respect to $s(v(t), z(t))$, if it is impulse-free and there exists $\epsilon > 0$, such that for any $v(t) \in \ell_2$ the following inequality holds, provided $x(0) = 0$:

$$\int_0^\infty s\big(v(t), z(t)\big)dt \le -\epsilon^2 \int_0^\infty v(t)^\top v(t)dt. \tag{4.15}$$

The multiobjective control problem under consideration is defined as follows.

**Problem 4.2** (*Asymptotic Output Regulation with Dissipative Performance Constraints*) The problem of asymptotic output regulation with dissipative performance constraints is to find a controller $\Sigma_c$ such that the following conditions are satisfied:

C.1 In the absence of the disturbances $w$ and $d$, the closed-loop system $G \times \Sigma_c$ is internally stable;
C.2 The solution of the closed-loop system $G \times \Sigma_c$ satisfies $\lim_{t\to\infty} e(t) = 0$ for any $d \in \ell_2$, and for all $x(0) \in \mathbb{R}^n$ and $w(0) \in \mathbb{R}^{n_w}$;
C.3 Given a specific $s(d, z)$, $T_{zd}(G \times \Sigma_c)$ is strictly dissipative and admissible.

The Conditions C.1 and C.2 stand for the standard output regulation constraints. In addition, the third condition means that at any time the amount of energy which the closed-loop system can conceivably supply to its environment cannot exceed the amount of energy that has been supplied to it. From the previous discussion, it is observed that, with the parameters of the controller (4.10), we still have some degree of freedom, that is $\tilde{E}_k$, $\tilde{A}_k$, $\tilde{B}_k$, $\tilde{C}_{k1}$, $\tilde{C}_{k2}$, $\tilde{D}_{k1}$, and $\tilde{D}_{k2}$, to address the additional dissipative performance requirement.

Suppose that the generalized Sylvester equation (4.4) holds. Then, the closed-loop system formed by $G$ in (4.2) and the $\Sigma_c$ in (4.10) can be written by

$$\left[\begin{array}{ccc|c} A + B\tilde{D}_{k1}C - sE & B\Pi + B\tilde{D}_{k1}\Upsilon & B\tilde{C}_{k1} & B_d + B\tilde{D}_{k1}D_{yd} \\ \tilde{D}_{k2}C & A_w + \tilde{D}_{k2}\Upsilon - sI & \tilde{C}_{k2} & \tilde{D}_{k2}D_{yd} \\ \tilde{B}_kC & \tilde{B}_k\Gamma & \tilde{A}_k - s\tilde{E}_k & \tilde{B}_kD_{yd} \\ \hline C_e + D_{eu}\tilde{D}_{k1}C & D_{eu}(\Pi + \tilde{D}_{k1}\Upsilon) & D_{eu}\tilde{C}_{k1} & D_{ed} + D_{eu}\tilde{D}_{k1}D_{yd} \\ C_z + D_{zu}\tilde{D}_{k1}C & D_{zu}(\Pi + \tilde{D}_{k1}\Upsilon) & D_{zu}\tilde{C}_{k1} & D_{zd} + D_{zu}\tilde{D}_{k1}D_{zd} \end{array}\right],$$

where $\Upsilon = CT - D_{yw}$. This system can be further rewritten as the closed-loop system formed by the controller $\tilde{\Sigma}_c$ in (4.9) and the following system $\tilde{\mathcal{G}}$

$$(\tilde{\mathcal{G}}) : \begin{cases} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \dot{\tilde{\zeta}} = \begin{bmatrix} A & B\Pi \\ 0 & A_w \end{bmatrix} \tilde{\zeta} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} d + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} u_c, \\ e = \begin{bmatrix} C_e & D_{eu}\Pi \end{bmatrix} \tilde{\zeta} + D_{ed}d + \begin{bmatrix} D_{eu} & 0 \end{bmatrix} u_c, \\ z = \begin{bmatrix} C_z & D_{zu}\Pi \end{bmatrix} \tilde{\zeta} + D_{zd}d + \begin{bmatrix} D_{zu} & 0 \end{bmatrix} u_c, \\ y_c = \begin{bmatrix} C & CT - D_{yw} \end{bmatrix} \tilde{\zeta} + D_{yd}d. \end{cases} \tag{4.16}$$

We have proved that the regulation constraint is achieved, if $\tilde{\Sigma}_c$ internally stabilizes $\tilde{\mathcal{G}}$. Hence, the multiobjective problem formulated in Problem 4.2 for $G$ is reduced into an unconstrained control problem for the associated system $\tilde{\mathcal{G}}$, which can be solved using the standard LMI-based techniques [Mas07, XL06]. However, it is worthy of noting that direct application of these results may give a conservative solution, since the realization of $\tilde{\mathcal{G}}$ depends on a solution to the generalized Sylvester equation (4.4) and this solution is in general not unique. This fact means that the underlying controller synthesis may result in a nonlinear matrix inequality problem. In order to render the controller synthesis convex, let us consider the following coordinate transformation matrices:

$$M = \begin{bmatrix} I & ET \\ 0 & I \end{bmatrix}, \quad N = \begin{bmatrix} I & -T \\ 0 & I \end{bmatrix},$$

where $T$ is an arbitrary solution to (4.4). Under this transformation, an alternative representation of $\tilde{\mathcal{G}}$ (with $e$ being removed) is obtained as

$$(\tilde{\mathcal{G}}) : \begin{cases} \mathcal{E}\dot{\bar{\zeta}} = \mathcal{A}\bar{\zeta} + \mathcal{B}_d d + \mathcal{B}(T)u_c, \\ z = \mathcal{C}_z(T, \Pi)\bar{\zeta} + D_{zd} d + \mathcal{D}_{zu} u_c, \\ y_c = \mathcal{C}\bar{\zeta} + D_{yd} d, \end{cases} \tag{4.17}$$

where

$$\mathcal{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} A & -B_w \\ 0 & A_w \end{bmatrix}, \ \mathcal{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \ \mathcal{B}(T) = \begin{bmatrix} B & ET \\ 0 & I \end{bmatrix},$$
$$\mathcal{C}_z(T, \Pi) = \begin{bmatrix} C_z & D_{zu}\Pi - C_z T \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} C & -D_{yw} \end{bmatrix}, \ \mathcal{D}_{zu} = \begin{bmatrix} D_{zu} & 0 \end{bmatrix}.$$

Then, we have the following lemma that transforms Problem 4.2 into an unconstrained control problem.

**Lemma 4.3** *Consider the system $G$ in (4.2) and a specific matrix $S$ in (4.14). Then, Problem 4.2 is solved by the controller (4.10), if and only if there exists the controller $\tilde{\Sigma}_c$ (4.9), such that the closed-loop system in (4.17) is admissible and strictly dissipative.*

*Proof* Straightforward. □

Before presenting a solution to Problem 4.2, the following standard assumption is made [Mas07].

(A.4) For $S$ defined in (4.14), $S_{22} \geq 0$ and $[S_{12}^\top \ \ S_{22}^\top]^\top$ has full column rank.

We also define the following matrices:

$$\Lambda \triangleq \left[ \begin{bmatrix} S_{12} \\ \Gamma \end{bmatrix} \left( \begin{bmatrix} S_{12} \\ \Gamma \end{bmatrix}^\perp \right)^\top \right], \tag{4.18}$$

$$[\Psi_1 \;\; \Psi_2] \triangleq [I \;\; 0]\Lambda^{-\top}, \tag{4.19}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ * & H_{22} \end{bmatrix} \triangleq \Lambda^{-1} \begin{bmatrix} S_{11} & 0 \\ 0 & -I \end{bmatrix} \Lambda^{-\top}, \tag{4.20}$$

where $\Gamma$ is a matrix satisfying $S_{22} = \Gamma^\top \Gamma$.

**Theorem 4.1** *Consider the system $\tilde{\mathcal{G}}$ in (4.17) and a specific matrix $S$ in (4.14). Then, there exists a controller $\tilde{\Sigma}_c$ in (4.9) with $\operatorname{rank}(\tilde{E}_k) = r + n_w$, such that the closed-loop system formed by $\tilde{\Sigma}_c$ and $\tilde{\mathcal{G}}$ is admissible and strictly dissipative, if and only if there exist matrices $T \in \mathbb{R}^{n \times n_w}$, $\Pi \in \mathbb{R}^{m \times n_w}$, $P \in \mathbb{R}^{(n+n_w)\times(n+n_w)}$, $Q \in \mathbb{R}^{(n+n_w)\times(n+n_w)}$, $U \in \mathbb{R}^{(n+n_w)\times m_d}$, and $V \in \mathbb{R}^{m_d \times (n+n_w)}$ satisfying the generalized Sylvester equation (4.4), as well as*

$$\begin{bmatrix} \mathcal{E}^\top & 0 \\ 0 & \mathcal{E} \end{bmatrix} \begin{bmatrix} P & M^{-\top} \\ N & Q \end{bmatrix} = \begin{bmatrix} P^\top & N^\top \\ M^{-1} & Q^\top \end{bmatrix} \begin{bmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^\top \end{bmatrix} \geq 0, \tag{4.21}$$

$$\mathcal{E}^\top U = 0, \tag{4.22}$$

$$\mathcal{E} V^\top = 0, \tag{4.23}$$

$$\Upsilon(T, \Pi, P, U) < 0, \tag{4.24}$$

$$\Xi(Q, V) < 0, \tag{4.25}$$

*where*

$$\Upsilon(T, \Pi, P, U) = \begin{bmatrix} \mathcal{N}_o & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{He}\{\mathcal{A}^\top P\} & P^\top \mathcal{B}_d + \mathcal{A}^\top U + \mathcal{C}_z(T,\Pi)^\top S_{12}^\top & \mathcal{C}_z(T,\Pi)^\top \Gamma^\top \\ * & \boldsymbol{He}\{U^\top \mathcal{B}_d + S_{12} D_{zd}\} - S_{11} & D_{zd}^\top \Gamma^\top \\ * & * & -I \end{bmatrix} \begin{bmatrix} \mathcal{N}_o & 0 \\ 0 & I \end{bmatrix},$$

$$\Xi(Q, V) = \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{He}\{\mathcal{Q}A\} & (B_d + A\mathcal{V}^\top)\Psi_1 + \mathcal{Q}^\top C_z^\top & (B_d + A\mathcal{V}^\top)\Psi_2 \\ * & \boldsymbol{He}\{(C_z\mathcal{V}^\top + D_{zd})\Psi_1\} + H_{11} & (C_z\mathcal{V}^\top + D_{zd})\Psi_2 + H_{12} \\ * & * & H_{22} \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix},$$

$$M = \begin{bmatrix} I & ET \\ 0 & I \end{bmatrix}, \quad N = \begin{bmatrix} I & -T \\ 0 & I \end{bmatrix}, \quad \mathcal{N}_o = \begin{bmatrix} \mathcal{C} & D_{yd} \end{bmatrix}^\perp, \quad N_c = \begin{bmatrix} B \\ D_{zu} \end{bmatrix}^\perp,$$

$$\mathcal{Q} = \begin{bmatrix} I & 0 \end{bmatrix} Q \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathcal{V} = V \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

*Proof* According to [Mas07], the dissipative performance and admissibility control problem for $\tilde{\mathcal{G}}$ is solvable, if and only if there exist matrices $P$, $\bar{Q}$, $U$, and $\bar{V}$ such that (4.22) and (4.24) hold, as well as

$$\begin{bmatrix} \mathcal{E}^\top & 0 \\ 0 & \mathcal{E} \end{bmatrix} \begin{bmatrix} P & I \\ I & \bar{Q} \end{bmatrix} = \begin{bmatrix} P^\top & I \\ I & \bar{Q}^\top \end{bmatrix} \begin{bmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^\top \end{bmatrix} \geq 0, \tag{4.26}$$

$$\mathcal{E}\bar{V}^\top = 0, \tag{4.27}$$

$$\begin{bmatrix} \mathcal{N}_c & 0 \\ 0 & I \end{bmatrix}^\top \Theta \begin{bmatrix} \mathcal{N}_c & 0 \\ 0 & I \end{bmatrix} < 0, \tag{4.28}$$

where $\mathcal{N}_c = \begin{bmatrix} \mathcal{B}(T) \\ \mathcal{D}_{zu} \end{bmatrix}^\perp$, and

$$\Theta = \begin{bmatrix} \mathbf{He}\{\mathcal{A}\bar{Q}\} & (\mathcal{A}\bar{V}^\top + \mathcal{B}_d)\Psi_1 + \bar{Q}^\top \mathcal{C}_z(T,\Pi)^\top & (\mathcal{A}\bar{V}^\top + \mathcal{B}_d)\Psi_2 \\ * & \mathbf{He}\{(\mathcal{C}_z(T,\Pi)\bar{V}^\top + D_{zd})\Psi_1\} + H_{11} & (\mathcal{C}_z(T,\Pi)\bar{V}^\top + D_{zd})\Psi_2 + H_{12} \\ * & * & H_{22} \end{bmatrix}.$$

Note that the inequality (4.28) involves the products of the decision variables ($T$, $\Pi$, $\bar{Q}$, and $\bar{V}$), and hence it is not linear. Now, we prove that (4.28) is equivalent to the condition (4.25), and the conditions (4.26) and (4.27) are equivalent to the conditions (4.21) and (4.23), respectively. To this end, left- and right-multiplying the condition (4.26) by $\begin{bmatrix} I & 0 \\ 0 & M^{-1} \end{bmatrix}$ and its transpose lead to

$$\begin{bmatrix} \mathcal{E}^\top P & \mathcal{E}^\top M^{-\top} \\ M^{-1}\mathcal{E} & M^{-1}\mathcal{E}\bar{Q}M^{-\top} \end{bmatrix} = \begin{bmatrix} P^\top \mathcal{E} & \mathcal{E}^\top M^{-\top} \\ M^{-1}\mathcal{E} & M^{-1}\bar{Q}^\top \mathcal{E}^\top M^{-\top} \end{bmatrix} \geq 0,$$

which can be further rewritten as

$$\begin{bmatrix} \mathcal{E}^\top P & \mathcal{E}^\top M^{-\top} \\ M^{-1}\mathcal{E}N^{-1}N & M^{-1}\mathcal{E}N^{-1}N\bar{Q}M^{-\top} \end{bmatrix} = \begin{bmatrix} P^\top \mathcal{E} & N^\top N^{-\top}\mathcal{E}^\top M^{-\top} \\ M^{-1}\mathcal{E} & M^{-1}\bar{Q}^\top N^\top N^{-\top}\mathcal{E}^\top M^{-\top} \end{bmatrix} \geq 0.$$

Note that $M^{-1}\mathcal{E}N^{-1} = \mathcal{E}$ and set $Q = N\bar{Q}M^{-\top}$. Then, the above inequality yields (4.21). Furthermore, there holds $\mathcal{E}\bar{V} = M^{-1}\mathcal{E}\bar{V} = M^{-1}\mathcal{E}N^{-1}N\bar{V} = \mathcal{E}N\bar{V} = 0$. Hence, choosing $V^\top = N\bar{V}^\top$, the equality (4.27) leads to (4.23).

Then, we prove that this inequity (4.28) can be reduced into (4.25). Note that

$$\begin{bmatrix} \mathcal{B}(T)^\top & \mathcal{D}_{zu}^\top \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}^\top & \begin{bmatrix} D_{zu} & 0 \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} M^\top & 0 \\ 0 & I \end{bmatrix}.$$

Hence,

$$\mathcal{N}_c = \begin{bmatrix} M^{-\top} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & I \\ D_{zu} & 0 \end{bmatrix}^\perp = \begin{bmatrix} M^{-\top} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N_{c1} \\ 0 \\ N_{c2} \end{bmatrix},$$

where $\begin{bmatrix} N_{c1} \\ N_{c2} \end{bmatrix} = \begin{bmatrix} B \\ D_{zu} \end{bmatrix}^\perp$. Hence, the condition (4.28) is equivalent to

$$\begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix} \Theta \begin{bmatrix} M^{-\top} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} < 0.$$

Set $Q = N\bar{Q}M^{-\top}$ and $V^\top = N\bar{V}^\top$. Using the partitions of $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$, $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ compatible with the realization of $\tilde{\mathcal{G}}$, the above inequality can be rewritten as

$$\begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Delta_1 & * & \Delta_2 + \Delta_3\Psi_1 & \Delta_3\Psi_2 \\ \bullet & * & * & * \\ \bullet & \bullet & \mathbf{He}\{\Delta_4\Psi_1\} + H_{11} & \Delta_4\Psi_2 + H_{12} \\ \bullet & \bullet & \bullet & H_{22} \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Delta_1 & \Delta_2 + \Delta_3\Psi_1 & \Delta_3\Psi_2 \\ \bullet & \mathbf{He}\{\Delta_4\Psi_1\} + H_{11} & \Delta_4\Psi_2 + H_{12} \\ \bullet & \bullet & H_{22} \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} < 0,$$

where '$*$' stands for the elements irrelative to the discussion and

$$\Delta_1 = \mathbf{He}\{AQ_{11} + B\Pi Q_{21}\}, \quad \Delta_2 = Q_{11}^\top C_z^\top + Q_{21}^\top \Pi^\top D_{zu}^\top,$$
$$\Delta_3 = AV_1^\top + B\Pi V_2^\top + B_d, \quad \Delta_4 = C_z V_1^\top + D_{zu}\Pi V_2^\top + D_{zd}.$$

This inequality is equivalent to

$$\begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \mathbf{He}\{B\Pi Q_{21}\} & Q_{21}^\top \Pi^\top D_{zu}^\top + B\Pi V_2^\top \Psi_1 & B\Pi V_2^\top \Psi_2 \\ \bullet & \mathbf{He}\{D_{zu}\Pi V_2^\top \Psi_1\} & D_{zu}\Pi V_2^\top \Psi_2 \\ \bullet & \bullet & 0 \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}$$
$$+ \Xi(Q, \mathcal{V}) < 0.$$

Since $N_{c1}^\top B + N_{c2}^\top D_{zu} = 0$, the first term of the above inequality is zero. Hence, (4.25) holds. □

We propose the following algorithm to solve Problem 4.2.

**Algorithm 4.1** 1. Given $S$, solve the generalized Sylvester equation (4.4) and the LMIs (4.21)–(4.25), and denote the solution as $(T_s, \Pi_s, P_s, Q_s, U_s, V_s)$;
2. With $(P_s, Q_s, U_s, V_s)$, construct the controller $\tilde{\Sigma}_c$ given in (4.9) by the algorithm proposed in [Mas07];
3. Obtain a realization of the controller $\Sigma_c$ by (4.10).

Note that the obtained controller $\Sigma_c$ is proper. Since the LMI-based approach leads to a controller $\tilde{\Sigma}_c$ such that $(\tilde{E}_k, \tilde{A}_k)$ is impulsive-free, if necessary, adding a small perturbation to the solution, $\tilde{\Sigma}_c$ is nonsingular. Hence, according to (4.10), the resulting controller $\Sigma_c$ is also nonsingular, since the exosystem is proper. This fact implies that Algorithm 4.1 gives a systematic way to obtain proper (state-space) controllers solving the defined multiobjective control problem for descriptor systems, even if original plants are singular. This property is also revealed via a numerical example.

*Example 4.1* Consider the descriptor system $G$ shown in (4.1) as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$B_d = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^\top,$$
$$C = C_e = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^\top, \quad D_{yw} = D_{ew} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^\top,$$
$$D_{yd} = D_{ed} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D_{eu} = 0,$$
$$D_{zw} = D_{zd} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{zu} = 1.$$

Suppose that the exogenous disturbance $w$ is a sinusoidal disturbance as

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w.$$

We attempt to find the worst-case energy gain, represented by $\gamma$, from $d$ to $z$, under asymptotic output regulation constraint. For this case, we set $S = diag\{-\gamma^2 I, I\}$ and this specific $S$ implies $\Gamma = I$, $\Psi_1 = 0$, $\Psi_2 = I$, $H_{11} = -I$, $H_{12} = 0$, and $H_{22} = -\gamma^2 I$ for (4.18)–(4.20). The semidefinite programming solver SeDuMi [Stu99] in MATLAB® is used for controller construction.

By Theorem 4.1, the following solution is obtained:

$$T = \begin{bmatrix} 0.6667 & 0 \\ -1 & -0.6667 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -1.3333 \\ -0.6667 \end{bmatrix}^\top, \quad W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.8787 & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} 4.0319 & -0.4237 & -0.0414 & -0.5328 & 0 \\ -0.4237 & 3.8990 & 0.4803 & 1.1723 & 0 \\ -0.04143 & 0.4803 & 1.7101 & -0.0626 & 0 \\ -0.5328 & 1.1723 & -0.0626 & 1.2823 & 0 \\ 1.0318 & 0.6681 & 1.1420 & -0.9740 & 0.9486 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.2611 & 1 \end{bmatrix}^\top,$$

$$Y = \begin{bmatrix} 1.4774 & -0.4532 & 0.5431 & -0.5605 & -0.2310 \\ -0.4532 & 2.2845 & -0.0595 & -0.6628 & 2.6477 \\ 0.5431 & -0.0595 & 1.9823 & 0.1497 & 0 \\ -0.5605 & -0.6628 & 0.1497 & 2.5431 & 0 \\ 0 & 0 & 0 & 0 & 0.4430 \end{bmatrix}.$$

Hence, the controller $\tilde{\Sigma}_c$ in (4.9) is

$$\begin{cases} \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix} \dot{x}_c = \begin{bmatrix} -30.6240 & 29.0841 & 73.6053 & -25.6846 & 20.2507 \\ 1.7273 & -6.3351 & -19.5730 & 11.5056 & -9.7048 \\ -0.2495 & 0.3634 & -1.8633 & 0.9937 & -0.6653 \\ 1.5034 & -1.6601 & -3.7462 & -0.8105 & -0.0575 \\ 0.2732 & -1.0323 & -0.7171 & -0.3969 & 0.6471 \end{bmatrix} x_c \\ \qquad + \begin{bmatrix} 7.0890 \\ -6.3114 \\ 0.0035 \\ 0.0550 \\ -0.2164 \end{bmatrix} y_c, \\ u_c = \begin{bmatrix} 0.2147 & 0.0440 & -0.0878 & 0.6611 & -0.3227 \\ -4.2560 & 5.2260 & 12.6593 & -5.2216 & 4.1972 \\ 6.2838 & -7.6980 & -10.9353 & 0 & 0.9947 \end{bmatrix} x_c + \begin{bmatrix} -0.5955 \\ 2.1098 \\ -0.3187 \end{bmatrix} y_c. \end{cases}$$

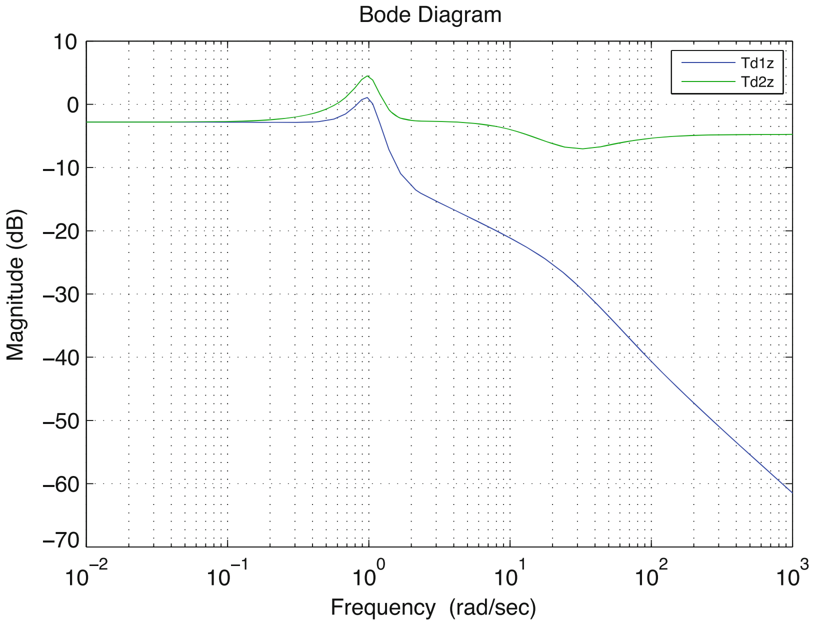


**Fig. 4.1** Singular value plot of $T_{zd}(s)$

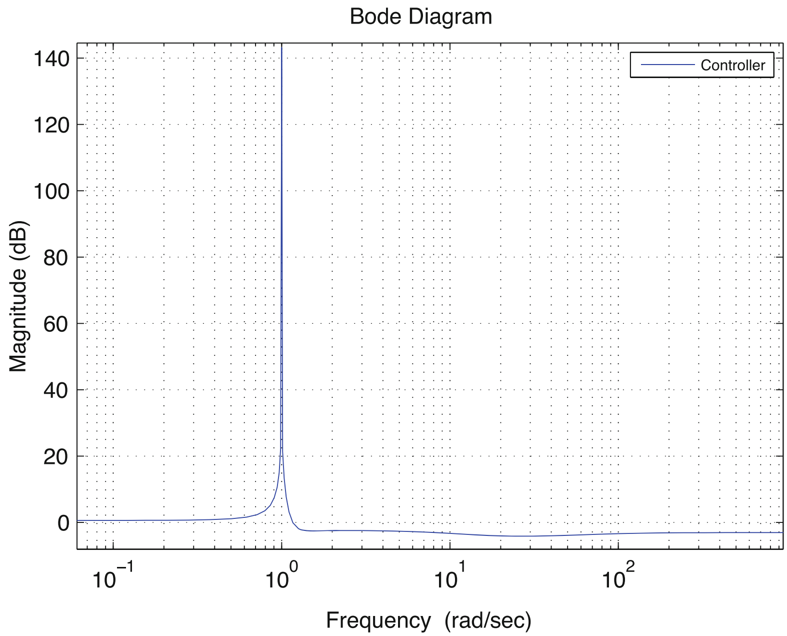


**Fig. 4.2** Controller performance

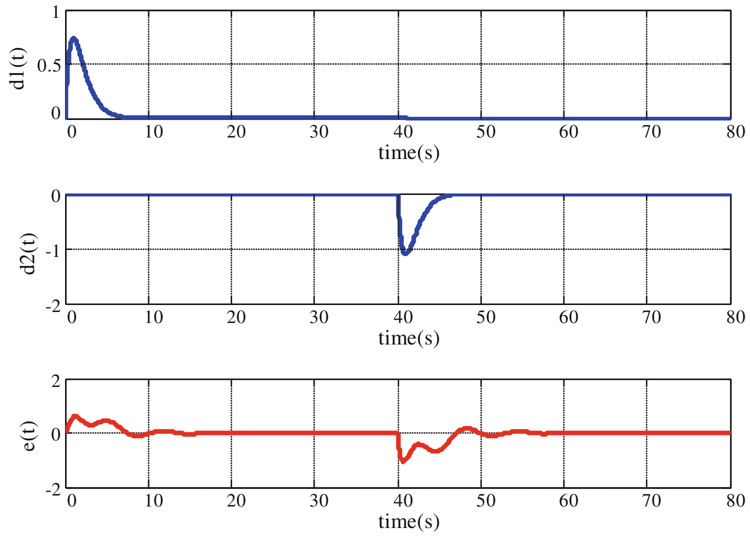


**Fig. 4.3** Asymptotic output regulation

According to the structure (4.10), the resulting controller $\Sigma_c$ is

$$\Sigma_c = \frac{-0.70335(s+38.97)(s+15.46)(s^2+2.852s+2.427)(s^2+0.3527s+1.22)}{(s+50.98)(s+10.72)(s^2+2.738s+2.144)(s^2+1)},$$

and the minimal value of $\gamma$ is 1.765. The singular value plot of $T_{zd}(s)$ is exhibited in Fig. 4.1. Moreover, it is observed that $y = e$, referred to as the error feedback case, where $C = C_e$, $D_{yw} = D_{ew}$, and $D_{eu} = 0$. Hence, we have $CT - D_{yw} = 0$. According to Lemma 4.2, the controller must contain a copy of the whole exosystem. This fact is verified, since the controller contains two complex poles $\{\pm jw\}$ and Fig. 4.2 clearly reveals the copy of the exosystem in the controller.

Let us consider an external disturbance signal as follows: $d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \in \ell_2$, where $d_1(t) = 2te^{-t}\Gamma(t)$ and $d_2(t) = -3te^{-t}\Gamma(t-40)$ are shown in Fig. 4.3. The third curve shows that the asymptotic output regulation constraint is achieved.

## 4.4 Conclusion

This chapter is focused on the design of a measurement output dynamic controller for the problem of performance control subject to asymptotic output regulation for continuous-time descriptor systems. It is shown that the asymptotic regulation objective is achieved if and only if a generalized Sylvester equation admits a solution, and a specific structured controller that contains a copy of the dynamics of the exosystem in a certain way is exhibited. Based on this conducted structure, the additional dissipative performance objective is transformed into a standard performance control problem associated with some augmented descriptor system, and the underlying controller synthesis is further addressed using a nonconservative LMI-based approach.

# Chapter 5
# Admissibility with Unstable and Nonproper Weights

In systems and control theory, many problems require the definition of a standard model consisting of physical plant, disturbances and reference signals, and control objectives. For example, one approach of controlling the longitudinal motion of fighter airplanes is to optimize certain weighted closed-loop transfer matrix with weighting filters that are in general interpreted as the models of reference input and disturbances [CS92, Kwa02]. In this context, it is well known that the use of stable state-space weighting filters is restrictive, since the exo-system models and weights are generally unstable, or even nonproper [HZK92, Mei95, SSS00a, Che02]. Using, for instance, integral or derivative weights introduce potentially some unstabilizable or undetectable finite dynamics, even uncontrollable or unobservable impulsive elements, in the standard model.

In this circumstance, the stabilization issue is quite different from that of the general setting. The conventional internal stability cannot be achieved in general, since some weights are unstable, or even nonproper. Instead, the so-called *extended stability* or *comprehensive stability* is introduced in the literature. This concept may be regarded as a generalization of internal stability and is highly related to practical concerns, for instance, the regulator and servomechanism problems [LM94].

In this chapter, we discuss the output feedback admissibility control problem for continuous-time descriptor systems with input and output weighs. Systems and their weighting filters are all described within the descriptor framework. Hence, it is possible to take into account not only unstable weights, but nonproper weights as well. Necessary and sufficient conditions for the existence of an observer-based stabilizing controller are given in terms of generalized Sylvester equations, and a Youla parameterization of the class of stabilizing controllers is also formulated. Moreover, in order to further clarify the effects of the weights on the underlying controller, a specifically structured output feedback controller is conducted in terms of the dynamics of the weighs. It is shown that determining suitable controllers for a given descriptor system with the presence of unstable and nonproper weights requires



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solving an admissibility problem for an augmented system explicitly constructed in this chapter.

## 5.1 Why Unstable and Nonproper Weights

For many problems, control specifications are usually interpreted by weighting filters. For example, the use of weights having a pole at the origin is in general appealed to achieve perfect rejection (and/or tracking) of constant disturbances (and/or references).

Let us take the $H_\infty$ control problem to show the importance of the use of unstable and nonproper weights. Examine the mixed sensitivity problem [Kwa93, Mei95] represented in Fig. 5.1, where $G$ stands for the given plant, $K$ is the controller to be determined, and $W_1$, $W_2$ and $W_2$ are input and output weighting filters. Figure 5.1 yields the following transfer matrix:

$$T_{zw} = \begin{bmatrix} W_2(I+GK)^{-1}W_1 \\ W_3K(I+GK)^{-1}W_1 \end{bmatrix}. \tag{5.1}$$

For the mixed sensitivity problem, the weights should be appropriately chosen such that stabilizing controllers with which the $H_\infty$ norm of $T_{zw}$ is bounded by certain prescribed bound $\gamma$ make the closed-loop system behave well. For this problem, standard procedure is available under MATLAB routines through transforming it into a standard $H_\infty$ control problem [BDG+91, CS92].

A desirable choice for the weighting filter $W_2$ is to choose $W_2$ having a pole at the origin, since the $\|T_{zw}\|_\infty$ is finite only if the sensitivity, that is, $(I+GK)^{-1}$ has a zero at the origin. This fact indicates that the underlying stabilizing making $\|T_{zw}\|_\infty < \gamma$ achieves either perfect rejection of constant disturbances or tracking of constant references. Another well-known fact which can explain this choice is that

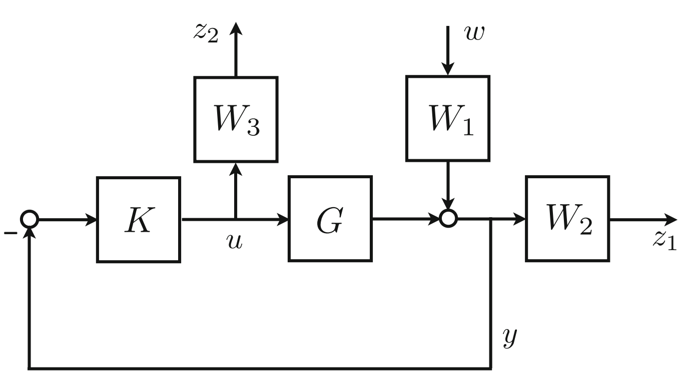


**Fig. 5.1** A mixed sensitivity configuration

if the plant $G$ does not have a pole at the origin, then any desirable controller $K$ has integration action. Moreover, in order to avoid undesirable high-frequency noise sensitivity and limited robustness, it is also often desirable to select a nonproper weight $W_3$. In particular, $\|W_3\|_\infty$ should be large outside the desirable closed-loop bandwidth due to the fact that this choice ensures that the controller is small outside the closed-loop bandwidth.

## 5.2 Existing Approaches

The importance of the use of unstable and nonproper weights has been highlighted. The control objective for such nonstandard problems is different from the conventional ones, since the overall weighted system cannot be internally stabilized due to the presence of these either unstabilizable or undetectable weights.

To illustrate this situation, consider an asymptotic tracking problem depicted in Fig. 5.2, where $w$ is a step reference. The input–output relation is given by

$$T_{ew} = \frac{1}{s}(I + GK)^{-1}.$$

It is observed that the dynamic of the integrator is not stabilizable by the controller. Hence, the internal stability of the weighted closed-loop system cannot be achieved. However, the weighting filter $1/s$ stands for specifications that will not be realized in real devices, and we are only interested in the internal stability of the feedback system formed by $G$ and $K$. Therefore, this asymptotic tracking problem can still be solved by finding a controller internally stabilizing $G$ and making $T_{ew}$ stable as well.

To handle such nonstandard problems, several techniques exist in the literature. Here, we take the mixed sensitivity problem presented in Fig. 5.1 as an example to give a brief reminder of the existing approaches [Mei95].

Method 1: One method is to treat these undesirable elements by slight perturbation to render the problem standard [CS92]. For example, one takes $W_2(s) = 1/(s + 0.0001)$ instead of $W_2(s) = 1/s$. Similarly, one can also replace $W_3(s) = s$ with $W_3(s) = s/(1 + 0.0001s)$. This treatment is obviously an approximation and is widely used. The disadvantage of this approach is that it is vulnerable to the

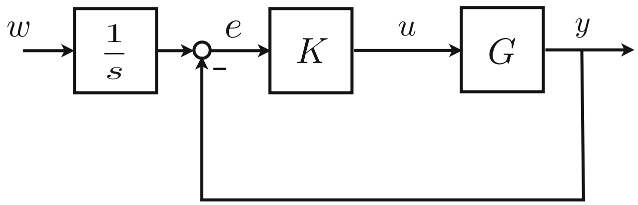


**Fig. 5.2** Asymptotic tracking problem

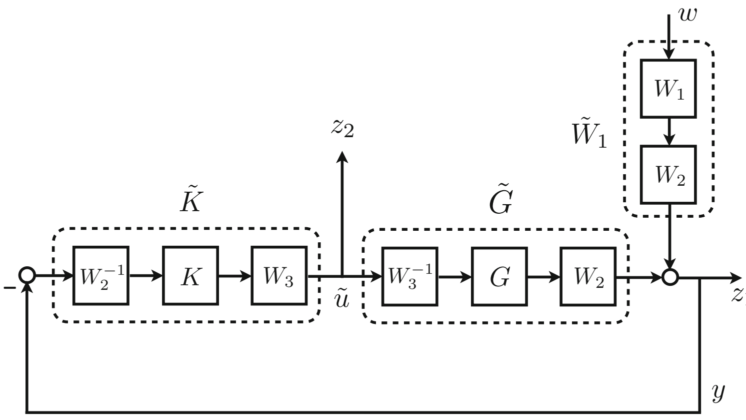


**Fig. 5.3** Modified mixed sensitivity configuration

troubles related with lightly damped poles and may lead to higher order and nonstrictly proper controllers.

Method 2: Another method is the so-called polynomial method [Kwa93], which copes naturally with nonproper weighting filters. This theory can, however, not be applied to cases where $W_2$ has imaginary poles.

Method 3: The third method involves plant augmentations as well as philosophically similar "plant state tapping" techniques [Kra92, Mei95]. Let us call it here the filter absorption method. Figure 5.3 shows how to absorb the weights into the loop. For the modified problem, the controller $\tilde{K}$ can be constructed, and the corresponding controller $K$ is given by $K = W_3^{-1}\tilde{K}W_2$. This method is easy to explain and not difficult to implement. Note that if there exists an unstable pole-zero cancellation in the modified plant, that is, $\tilde{G} = W_2 G W_3^{-1}$, then the stability properties of the original loop and the modified loop are not the same. In other words, the weights $W_2$ and $W_3$ must be appropriately chosen. Moreover, this method obviously requires a pretreatment to absorb the weights into the loop.

Method 4: The theory of mode cancellation or comprehensive stabilization [LM94, LM95, LZM97, MXA00] has been proposed for solving these nonstandard problems. Roughly speaking, the main idea is to make, respectively, the unstabilizable and undetectable elements unobservable and uncontrollable by feedback in the underlying closed loop. However, this theory does not allow nonproper weights.

## 5.3 Comprehensive Admissibility

We first define the comprehensive admissibility control problem for linear continuous-time descriptor systems. The "comprehensive" term indicates here that the desirable

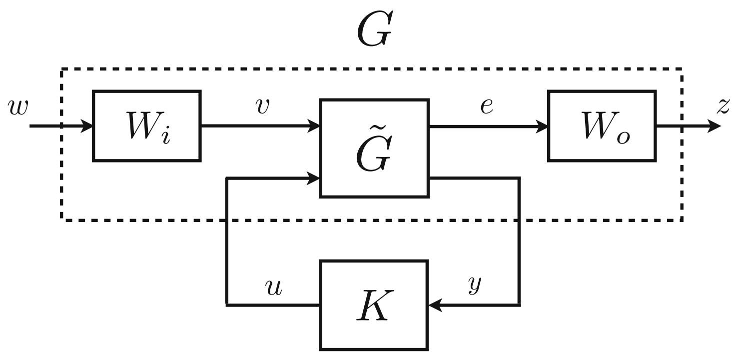


**Fig. 5.4** Comprehensive admissibility control problem

controller can and must stabilize a part of the overall closed-loop system. The problem setup is depicted in Fig. 5.4 and the physical plant $\widetilde{G}(s)$ is given by

$$\begin{bmatrix} e(s) \\ y(s) \end{bmatrix} = \widetilde{G} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} \widetilde{G}_{ev} & \widetilde{G}_{eu} \\ \widetilde{G}_{yv} & \widetilde{G}_{yu} \end{bmatrix} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} \tag{5.2}$$

where $e \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, $v \in \mathbb{R}^l$, and $u \in \mathbb{R}^m$ are the controlled output, measurement, disturbance input, and control input vector, respectively. The system (5.2) can be rewritten as

$$\widetilde{G} = \left[ \begin{array}{c|cc} A_g - sE_g & B_{g1} & B_{g2} \\ \hline C_{g1} & D_{g11} & D_{g12} \\ C_{g2} & D_{g21} & D_{g22} \end{array} \right]$$

where $E_g \in \mathbb{R}^{n_g \times n_g}$, $A_g \in \mathbb{R}^{n_g \times n_g}$, $B_{g1} \in \mathbb{R}^{n_g \times l}$, $B_{g2} \in \mathbb{R}^{n_g \times m}$, $C_{g1} \in \mathbb{R}^{q \times n_g}$, $C_{g2} \in \mathbb{R}^{p \times n_g}$, $D_{g11} \in \mathbb{R}^{q \times l}$, $D_{g12} \in \mathbb{R}^{q \times m}$, $D_{g21} \in \mathbb{R}^{p \times l}$ and $D_{g22} \in \mathbb{R}^{p \times m}$ are known real constant matrices. The matrix $E_g$ may be singular, i.e., $\mathrm{rank}(E_g) = r_g \leq n_g$.

Suppose that the input weight $W_i$ and the output weight $W_o$ are both descriptor systems described as

$$W_i = \left[ \begin{array}{c|c} A_i - sE_i & B_i \\ \hline C_i & D_i \end{array} \right], \quad W_o = \left[ \begin{array}{c|c} A_o - sE_o & B_o \\ \hline C_o & D_o \end{array} \right],$$

where $E_i \in \mathbb{R}^{n_i \times n_i}$, $E_o \in \mathbb{R}^{n_o \times n_o}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $A_o \in \mathbb{R}^{n_o \times n_o}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $B_o \in \mathbb{R}^{n_o \times q}$, $C_i \in \mathbb{R}^{l \times n_i}$, $C_o \in \mathbb{R}^{p_o \times n_o}$, $D_i \in \mathbb{R}^{l \times m_i}$ and $D_o \in \mathbb{R}^{p_o \times q}$ are known real constant matrices. The matrices $E_i$ and $E_o$ may be singular, i.e., $\mathrm{rank}(E_i) = r_i \leq n_i$ and $\mathrm{rank}(E_o) = r_o \leq n_o$.

For the sake of simplicity, $W_i$ and $W_o$ are assumed to have only unstable and impulsive modes. This assumption can be made without loss of generality, since the stable and static modes of the weights decay to zero eventually and do not affect the admissibility of the closed-loop system.

The resulting overall weighted system $G$ is written as

$$G = \left[\begin{array}{ccc|cc} A_o - sE_o & B_oC_{g1} & B_oD_{g11}C_i & B_oD_{g11}D_i & B_oD_{g12} \\ 0 & A_g - sE_g & B_{g1}C_i & B_{g1}D_i & B_{g2} \\ 0 & 0 & A_i - sE_i & B_i & 0 \\ \hline C_o & D_oC_{g1} & D_oD_{g11}C_i & D_oD_{g11}D_i & D_oD_{g12} \\ 0 & C_{g2} & D_{g21}C_i & D_{g21}D_i & D_{g22} \end{array}\right]. \tag{5.3}$$

Moreover, we denote in the sequel $G$ as

$$G = \left[\begin{array}{c|cc} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \triangleq \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}. \tag{5.4}$$

**Definition 5.1** (*Comprehensive Admissibility*) The feedback system $\mathcal{F}_l(G, K)$ is said to be comprehensively admissible if $\mathcal{F}_l(\widetilde{G}, K)$ is internally stable and the closed-loop system defined by

$$T_{zw} = \mathcal{F}_l(G, K) = G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \tag{5.5}$$

is admissible.

**Problem 5.1** (*Comprehensive Admissibility Control Problem*) The comprehensive admissibility control problem for the system $G$ in (5.4) is to find a controller $K$ such that the overall feedback system formed by $G$ and $K$ is comprehensively admissible.

It will be shown that comprehensive admissibility is achieved if and only if two generalized Sylvester equations admit solutions. Additional performance objectives, such as $H_2$ and $H_\infty$ conditions, are further tackled by solving two generalized algebraic Riccati equations (GAREs). Moreover, the well-known output regulation problem addressed in the last chapter can be regarded as a special case of the comprehensive admissibility control problem by removing the output weighting filters.

## 5.4 Observer-Based Solution

In this section, we start with state feedback case and follow its dual case to conduct an observer-based solution to Problem 5.1.

### 5.4.1 State Feedback

Let us partition $G$ in (5.3) with regard to the input weight as follows

$$G = \left[\begin{array}{cc|cc} \bar{A}_{11} - s\bar{E} & \bar{A}_{12} & \bar{B}_{11} & \bar{B}_{12} \\ 0 & A_i - sE_i & \bar{B}_{21} & 0 \\ \hline \bar{C}_{11} & \bar{C}_{12} & D_{11} & D_{12} \\ \bar{C}_{21} & \bar{C}_{22} & D_{21} & D_{22} \end{array}\right], \tag{5.6}$$

where

$$\begin{cases} \bar{E} = \begin{bmatrix} E_o & 0 \\ 0 & E_g \end{bmatrix}, \quad \bar{A}_{11} = \begin{bmatrix} A_o & B_o C_{g1} \\ 0 & A_g \end{bmatrix}, \\ \bar{A}_{12} = \begin{bmatrix} B_o D_{g11} C_i \\ B_{g1} C_i \end{bmatrix}, \quad \bar{B}_{11} = \begin{bmatrix} B_o D_{g11} D_i \\ B_{g1} D_i \end{bmatrix}, \\ \bar{B}_{12} = \begin{bmatrix} B_o D_{g12} \\ B_{g2} \end{bmatrix}, \quad \bar{B}_{21} = B_i, \quad \bar{C}_{11} = \begin{bmatrix} C_o & D_o C_{g1} \end{bmatrix}, \\ \bar{C}_{12} = D_o D_{g11} C_i, \quad \bar{C}_{21} = \begin{bmatrix} 0 & C_{g2} \end{bmatrix}, \quad \bar{C}_{22} = D_{g21} C_i. \end{cases} \tag{5.7}$$

**Assumption 5.1**

(A1) $(\bar{E}, \bar{A}_{11}, \bar{B}_{12})$ is finite dynamics stabilizable and impulse controllable.

The following result gives a necessary and sufficient condition for the existence of a static feedback gain $F \in \mathbb{R}^{m\times(n_g+n_o+n_i)}$ such that the system given by $\left[\begin{array}{c|c} A_F - sE_F & B_F \\ \hline C_F & D_F \end{array}\right]$ with

$$\begin{cases} E_F = \begin{bmatrix} \bar{E} & 0 \\ 0 & E_i \end{bmatrix}, \quad A_F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & A_i \end{bmatrix} + \begin{bmatrix} \bar{B}_{12} \\ 0 \end{bmatrix} F, \\ B_F = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix}, \quad C_F = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \end{bmatrix} + D_{12} F, \quad D_F = D_{11}, \end{cases} \tag{5.8}$$

is comprehensively admissible.

**Lemma 5.1** *There exists a static feedback gain $F$ such that (5.8) is comprehensively admissible, if and only if there exist matrices $X^i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $Y_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$ and $F_a \in \mathbb{R}^{m\times n_i}$ such that the following generalized Sylvester equation holds*

$$\begin{aligned} \bar{B}_{12} F_a &= \bar{A}_{11} Y_i - \bar{A}_{12} - X_i A_i, \\ D_{12} F_a &= \bar{C}_{11} Y_i - \bar{C}_{12}, \\ 0 &= \bar{E} Y_i - X_i E_i. \end{aligned} \tag{5.9}$$

*Moreover, the feedback gain $F$ is given by*

$$F = \begin{bmatrix} F_1 & F_a + F_1 Y_i \end{bmatrix}, \tag{5.10}$$

*where $F_1 \in \mathbb{R}^{m\times(n_g+n_o)}$ is such that $(\bar{E}, \bar{A}_{11} + \bar{B}_{12}F_1)$ is admissible.*

*Proof* (Sufficiency) Suppose that there exist matrices $X_i$, $Y_i$, and $F_a$ satisfying (5.9). Since $(\bar{E}, \bar{A}_{11}, \bar{B}_{12})$ is finite dynamic stabilizable and impulse controllable, a static feedback gain $F_1$ such that $(\bar{E}, \bar{A}_{11} + \bar{B}_{12}F_1)$ is admissible exists. Replacing $F$ in (5.8) with (5.10) gives

$$\begin{cases} E_F = \begin{bmatrix} \bar{E} & 0 \\ 0 & E_i \end{bmatrix}, \quad A_F = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12}F_1 & \bar{A}_{12} + \bar{B}_{12}(F_a + F_1 Y_i) \\ 0 & A_i \end{bmatrix}, \\ B_F = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix}, \quad C_F = \begin{bmatrix} \bar{C}_{11} + D_{12}F_1 & \bar{C}_{12} + D_{12}(F_a + F_1 Y_i) \end{bmatrix}. \end{cases}$$

Then, let us introduce two nonsingular matrices $M$ and $N$

$$M = \begin{bmatrix} I_{n_g+n_o} & X_i \\ 0 & I_{n_i} \end{bmatrix}, \quad N = \begin{bmatrix} I_{n_g+n_o} & -Y_i \\ 0 & I_{n_i} \end{bmatrix}. \tag{5.11}$$

From (5.9), there holds

$$\begin{cases} ME_FN = \begin{bmatrix} \bar{E} & 0 \\ 0 & E_i \end{bmatrix}, \quad MA_FN = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12}F_1 & 0 \\ 0 & A_i \end{bmatrix}, \\ MB_F = \begin{bmatrix} \bar{B}_{11} + X_i\bar{B}_{21} \\ \bar{B}_{21} \end{bmatrix}, \quad C_FN = \begin{bmatrix} \bar{C}_{11} + D_{12}F_1 & 0 \end{bmatrix}, \end{cases}$$

which implies that (5.8) is equivalent to

$$\left[\begin{array}{c|c} \bar{A}_{11} + \bar{B}_{12}F_1 - s\bar{E} & \bar{B}_{11} + X_i\bar{B}_{21} \\ \hline \bar{C}_{11} + D_{12}F_1 & D_{11} \end{array}\right].$$

Since $F_1$ is such that $(\bar{E}, \bar{A}_{11} + \bar{B}_{12}F_1)$ is admissible, comprehensive admissibility is achieved.

(Necessity) Suppose that there exists $F = [F_1 \;\; F_2]$ such that (5.8) is comprehensively admissible. Let $\bar{Z} \in \mathbb{R}^{(n_g+n_o)\times(n_g+n_o-r_g-r_o)}$ be any full column rank matrix satisfying $\bar{E}\bar{Z} = 0$, and $Z_i \in \mathbb{R}^{(n_i-r_i)\times n_i}$ be any full row rank matrix satisfying $Z_iE_i = 0$. Then, using the transformation matrices in (5.11) together with $X_i = \bar{E}T + UZ_i$ and $Y_i = TE_i - \bar{Z}V$ with $T \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $U \in \mathbb{R}^{(n_g+n_o)\times(n_i-r_i)}$ and $V \in \mathbb{R}^{(n_g+n_o-r_g-r_o)\times n_i}$ yields

$$\begin{cases} ME_F N = \begin{bmatrix} \bar{E} & 0 \\ 0 & E_i \end{bmatrix}, \quad MB_F = \begin{bmatrix} \bar{B}_{11} + X_i \bar{B}_{21} \\ \bar{B}_{21} \end{bmatrix}, \\ MA_F N = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} F_1 & \bar{B}_{12} F_a + \bar{A}_{12} + X_i A_i - \bar{A}_{11} Y_i \\ 0 & A_i \end{bmatrix}, \\ C_F N = \begin{bmatrix} \bar{C}_{11} + D_{12} F_1 & D_{12} F_a + \bar{C}_{12} - \bar{C}_{11} Y_i \end{bmatrix}, \\ FN = \begin{bmatrix} F_1 & F_a \end{bmatrix}, \end{cases} \tag{5.12}$$

with $F_a = F_2 - F_1 Y_i$. Since the system (5.12) is comprehensively admissible, $\bar{B}_{12} F_a + \bar{A}_{12} + X_i A_i - \bar{A}_{11} Y_i = 0$. Otherwise, the admissibility of $\mathcal{F}_l(\widetilde{G}, K)$ cannot be satisfied. Moreover, the closed-loop system can be written as

$$\begin{cases} \bar{E}\dot{\zeta}_1 = (\bar{A}_{11} + \bar{B}_{12} F_1)\zeta_1 + (\bar{B}_{11} + X_i \bar{B}_{21})w, \\ E_i \dot{\zeta}_2 = A_i \zeta_2 + \bar{B}_{21} w, \\ y = (\bar{C}_{11} + D_{12} F_1)\zeta_1 + (D_{12} F_a + \bar{C}_{12} - \bar{C}_{11} Y_i)\zeta_2 + D_{11} w. \end{cases}$$

Note that $(E_i, A_i)$ is unstable or nonproper. Hence, there holds $D_{12} F_a + \bar{C}_{12} - \bar{C}_{11} Y_i = 0$. Otherwise, the signal $y$ does not converge, which is contradictory to the fact that the overall closed-loop system is admissible. □

### 5.4.2 Estimation

Let us now partition $G$ in (5.3) with respect to the output weight as follows:

$$G = \left[\begin{array}{cc|cc} A_o - sE_o & \hat{A}_{12} & \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{A}_{22} - s\hat{E} & \hat{B}_{21} & \hat{B}_{22} \\ \hline \hat{C}_{11} & \hat{C}_{12} & D_{11} & D_{12} \\ 0 & \hat{C}_{22} & D_{21} & D_{22} \end{array}\right], \tag{5.13}$$

with

$$\begin{cases} \hat{E} = \begin{bmatrix} E_g & 0 \\ 0 & E_i \end{bmatrix}, \quad \hat{A}_{12} = \begin{bmatrix} B_o C_{g1} & B_o D_{g11} C_i \end{bmatrix}, \\ \hat{A}_{22} = \begin{bmatrix} A_g & B_{g1} C_i \\ 0 & A_i \end{bmatrix}, \quad \hat{B}_{11} = B_o D_{g11} D_i, \quad \hat{B}_{12} = B_o D_{g12}, \\ \hat{B}_{21} = \begin{bmatrix} B_{g1} D_i \\ B_i \end{bmatrix}, \quad \hat{B}_{22} = \begin{bmatrix} B_{g2} \\ 0 \end{bmatrix}, \quad \hat{C}_{11} = C_o, \\ \hat{C}_{12} = \begin{bmatrix} D_o C_{g1} & D_o D_{g11} C_i \end{bmatrix}, \quad \hat{C}_{22} = \begin{bmatrix} C_{g2} & D_{g21} C_i \end{bmatrix}. \end{cases} \tag{5.14}$$

**Assumption 5.2**

(A2) $(\hat{E}, \hat{A}_{22}, \hat{C}_{22})$ is finite dynamics detectable and impulse observable.

The following result gives a necessary and sufficient condition for the existence of a static estimation gain $L \in \mathbb{R}^{(n_g+n_i+n_o)\times p}$ such that the system given by $\left[\begin{array}{c|c} A_L - sE_L & B_L \\ \hline C_L & D_L \end{array}\right]$ with

$$\begin{cases} E_L = \begin{bmatrix} E_o & 0 \\ 0 & \hat{E} \end{bmatrix}, \quad A_L = \begin{bmatrix} A_o & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} + L \begin{bmatrix} 0 & \hat{C}_{22} \end{bmatrix}, \\ B_L = \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} + LD_{21}, \quad C_L = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \end{bmatrix}, \quad D_L = D_{11}, \end{cases} \tag{5.15}$$

is comprehensively admissible.

**Lemma 5.2** *There exists a static estimation gain* $L$ *such that* (5.15) *is comprehensively admissible if and only if there exist matrices* $X_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$, $Y_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$ *and* $L_a \in \mathbb{R}^{n_o\times p}$ *such that the following generalized Sylvester equation holds*

$$\begin{aligned} L_a\hat{C}_{22} &= A_oY_o - \hat{A}_{12} - X_o\hat{A}_{22}, \\ L_aD_{21} &= -\hat{B}_{11} - X_o\hat{B}_{21}, \\ 0 &= E_oY_o - X_o\hat{E}. \end{aligned} \tag{5.16}$$

*Moreover, the estimation gain* $L$ *is given by*

$$L = \begin{bmatrix} L_a - X_oL_2 \\ L_2 \end{bmatrix} \tag{5.17}$$

*where* $L_2$ *is such that* $(\hat{E}, \hat{A}_{22} + L_2\hat{C}_{22})$ *is admissible.*

*Proof* Duality of Lemma 5.1.

The Eq. (5.9) is introduced to make the dynamics of $(E_i, A_i)$, which are neither finite dynamics stabilizable nor impulse controllable, unobservable by feedback in the underlying closed-loop system. Similarly, the Eq. (5.16) is used to make the dynamics of $(E_o, A_o)$, which are neither finite dynamics detectable nor impulse observable, uncontrollable by feedback in the underlying closed-loop system. Moreover, when $E_g = I_{n_g}$, $E_i = I_{n_i}$ and $E_o = I_{n_o}$, the generalized Sylvester equations (5.9) and (5.16) cover the regulator equations reported in the literature [Fra77, LM94, SSS00b, Che02] for state-space systems.

The solvability of the Eqs. (5.9) and (5.16) is essential for the problem under consideration. Note that (5.9) can be reformed as

$$-\begin{bmatrix} I \\ 0 \end{bmatrix} X_i A_i + \begin{bmatrix} \bar{A}_{11} & -\bar{B}_{12} \\ \bar{C}_{11} & -D_{12} \end{bmatrix} \begin{bmatrix} Y_i \\ F_a \end{bmatrix} = \begin{bmatrix} \bar{A}_{12} \\ \bar{C}_{12} \end{bmatrix},$$
$$X_i E_i + \begin{bmatrix} \bar{E} & 0 \end{bmatrix} \begin{bmatrix} Y_i \\ F_a \end{bmatrix} = 0.$$

Furthermore, this equation can be further written as a linear system of equations

$$\begin{bmatrix} I \otimes \begin{bmatrix} \bar{A}_{11} & -\bar{B}_{12} \\ \bar{C}_{11} & -D_{12} \end{bmatrix} & A_i \otimes \begin{bmatrix} -I \\ 0 \end{bmatrix} \\ I \otimes \begin{bmatrix} \bar{E} & 0 \end{bmatrix} & E_i \otimes I \end{bmatrix} \begin{bmatrix} Col\left(\begin{bmatrix} Y_i \\ F_a \end{bmatrix}\right) \\ Col(X_i) \end{bmatrix} = \begin{bmatrix} Col\left(\begin{bmatrix} \bar{A}_{12} \\ \bar{C}_{12} \end{bmatrix}\right) \\ 0 \end{bmatrix}.$$

So the solution can be obtained through a linear program. The same discussion also holds for the Eq. (5.16). A more comprehensive discussion about Sylvester equations is given in Appendix A.

Now, we are in a position to state the condition to existence of an observer-based controller solving Problem 5.1.

**Theorem 5.1** *Consider the generalized weighted plant $G$ in (5.4) and suppose that Assumptions (A1) and (A2) hold. There exists an observer-based controller $K$ such that Problem 5.1 is solved if and only if the generalized Sylvester equations (5.9) and (5.16) hold. Moreover, the controller $K$ is given by*

$$K = \left[\begin{array}{c|c} A + B_2 F + L C_2 - sE & -L \\ \hline F & 0 \end{array}\right], \tag{5.18}$$

*where $F$ and $L$ are given in (5.10) and (5.17), respectively.*

*Proof* Straightforward through Lemmas 5.1 and 5.2. □

## 5.5 Youla Parameterization

Based on the observer-based solution given previously, we provide a Youla parameterization of all controllers achieving the specification of comprehensive admissibility. Consider an output feedback dynamic controller $K$ of the form

$$K = \left[\begin{array}{c|c} A_K - sE_K & B_K \\ \hline C_K & D_K \end{array}\right]. \tag{5.19}$$

**Lemma 5.3** *Consider the overall weighted plant $G$ in (5.4). Suppose that Assumptions (A1) and (A2) hold. The controller $K$ in (5.19) stabilizing $\widetilde{G}$ solves Problem 5.1 if and only if there exist matrices $X_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $Y_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $F_a \in \mathbb{R}^{m\times n_i}$, $X_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$, $Y_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$, $L_a \in \mathbb{R}^{n_o\times p}$, $X_K^i \in \mathbb{R}^{n_k\times n_i}$, $Y_K^i \in \mathbb{R}^{n_k\times n_i}$, $X_K^o \in \mathbb{R}^{n_o\times n_k}$, and $Y_K^o \in \mathbb{R}^{n_o\times n_k}$ such that the two generalized Sylvester equations (5.9) and (5.16) hold, as well as*

*(i)*

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} Y_K^i \\ \bar{C}_{21} Y_i - \bar{C}_{22} \end{bmatrix} - \begin{bmatrix} X_K^i \\ 0 \end{bmatrix} A_i = \begin{bmatrix} 0 \\ -F_a \end{bmatrix}, \qquad E_K Y_K^i = X_K^i E_i, \tag{5.20}$$

*(ii)*

$$\begin{bmatrix} X_K^o & X_o \hat{B}_{22} + \hat{B}_{12} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} - A_o \begin{bmatrix} Y_K^o & 0 \end{bmatrix} = \begin{bmatrix} 0 & -L_a \end{bmatrix}, \qquad X_K^o E_K = E_o Y_K^o. \tag{5.21}$$

*Proof* (i) By following the same thread of the proof of Lemma 5.1, it is observed that the controller $K$ achieves comprehensive admissibility for $G$ if and only if there exist matrices $X_i$, $Y_i$, $F_a$, $X_K^i$ and $Y_K^i$ such that

$$\begin{aligned} 0 &= D_{12}(D_K \bar{C}_{22} - D_K \bar{C}_{21} Y_i - C_K Y_K^i) - \bar{C}_{11} Y_i + \bar{C}_{12}, \\ X_i A_i &= \bar{A}_{11} Y_i + \bar{B}_{12}(D_K \bar{C}_{21} Y_i - D_K \bar{C}_{22} + C_K Y_K^i) - \bar{A}_{12}, \\ X_i E_i &= \bar{E} Y_i, \\ X_K^i A_i &= A_K Y_K^i + B_K(\bar{C}_{21} Y_i - \bar{C}_{22}), \\ X_K^i E_i &= E_K Y_K^i. \end{aligned}$$

By setting $F_a = D_K \bar{C}_{22} - D_K \bar{C}_{21} Y_i - C_K Y_K^i$, the above matrix equations lead to (5.9) and (5.20).

(ii) By duality and Lemma 5.2. □

The following theorem gives a complete observer-based solution to Problem 5.1 and can be viewed as an extension of the conventional Youla parameterization to the nonstandard case.

**Theorem 5.2** *Suppose that Assumptions (A1) and (A2) hold. All controllers solving Problem 5.1 can be parameterized as*

$$\mathbf{K} = \mathcal{F}_l(J, Q), \quad \forall Q \in RH_\infty, \tag{5.22}$$

*with*

$$J = \left[\begin{array}{c|cc} A + B_2 F + L C_2 - sE & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right], \tag{5.23}$$

*where $F$ and $L$ are defined in (5.10) and (5.17), respectively.*

*Proof* (Sufficiency) We show that $\mathbf{K} = \mathcal{F}_l(J, Q)$ achieves comprehensive admissibility. Consider $Q \in RH_\infty$ of the form

$$Q = \left[\begin{array}{c|c} A_Q - sE_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right].$$

Then the resulting closed-loop system is written as

$$\mathcal{F}_l(G, \mathcal{F}_l(J, Q)) = \left[\begin{array}{ccc|c} A + B_2F - sE & B_2C_Q & B_2F - B_2D_QC_2 & B_1 + B_2D_QD_{21} \\ 0 & A_Q - sE_Q & -B_QC_2 & B_QD_{21} \\ 0 & 0 & A + LC_2 - sE & -B_1 - LD_{21} \\ \hline C_1 + D_{21}F & D_{12}C_Q & D_{21}F - D_{12}D_QC_2 & D_{11} + D_{11}D_QD_{21} \end{array}\right].$$

According to the choice of $F$, all unstable and impulsive modes of the pair $(E, A + B_2F)$ are unobservable from the output $z$. Similarly, by the choice of $L$, all unstable and impulsive modes of the pair $(E, A + LC_2)$ are uncontrollable from the input $w$. Moreover, $(E_Q, A_Q)$ is admissible. Hence, the closed-loop system $\mathcal{F}_l(G, \mathcal{F}_l(J, Q))$ is comprehensively admissible.

(Necessity) We show that for any controller $\mathcal{K}$ solving Problem 5.1, there exists a $\mathcal{Q} \in RH_\infty$ such that $\mathcal{K} = \mathcal{F}_l(J, \mathcal{Q})$. To this end, we define

$$\mathcal{J} = \left[\begin{array}{c|cc} A - sE & -L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right],$$

and $\mathcal{Q} = \mathcal{F}_l(\mathcal{J}, \mathcal{K})$. Note that the system $\mathcal{F}_l(G, \mathcal{K})$ is comprehensively admissible, since $\mathcal{K}$ solving Problem 5.1. Then, according to Lemma 5.3, the conditions (5.9) and (5.20) hold. Note that $G$ and $\mathcal{J}$ share the same $E$, $A$ and $B_2$ matrices, which indicates that the first and third equations of (5.9) also hold with respect to the system data of $\mathcal{J}$. Moreover, we have $F = [F_1 \;\; F_a + F_1Y_i]$. Then, $F_a = F_a + F_1Y_i - F_1Y_i$, which is nothing else but the second equation of (5.9). Hence, the generalized Sylvester equation (5.9) holds for $\mathcal{J}$. Besides, $G$ and $\mathcal{J}$ share the same $C_2$ matrix. Hence, the condition (5.20) also holds with respect to the system data of $\mathcal{J}$.

With the same thread, we can also prove that the conditions (5.16) and (5.21) hold with respect to $\mathcal{J}$. Therefore, by Lemma 5.3, the controller $\mathcal{K}$ achieves comprehensive admissibility for $\mathcal{J}$, which means that $\mathcal{Q}$ is comprehensively admissible. Moreover, there holds

$$\mathcal{F}_l(J, \mathcal{Q}) = \mathcal{F}_l(J, \mathcal{F}_l(\mathcal{J}, \mathcal{K})) = \mathcal{F}_l(\mathcal{F}_l(J, \mathcal{J}), \mathcal{K}) = \mathcal{F}_l(J_1, \mathcal{K}),$$

with

$$J_1 = \left[\begin{array}{cc|cc} A + LC_2 - sE & B_2F & -L & B_2 \\ 0 & A - sE & -L & B_2 \\ \hline F & -F & 0 & I \\ -C_2 & C_2 & I & 0 \end{array}\right] = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Hence, $\mathcal{F}_l(J, \mathcal{Q}) = \mathcal{F}_l(J_1, \mathcal{K}) = \mathcal{K}$. □

Note that no constraint has been imposed on the Youla parameter $Q$ so far, multiobjective control synthesis with certain additional performance objectives may be dealt with by using a usual projection of the Youla parameter on some orthonormal basis such as Laguerre or Kautz basis [Mä91, Wah94] and convex optimization tools.

*Example 5.1*

Consider the example as shown in Fig. 5.4. with $W_i = \frac{1}{s}$ and $W_o = s$. Let $\widetilde{G}$ be given as

$$\widetilde{G} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 1 & -1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{array}\right] \right\},$$

and

$$W_i = \left\{ 1, \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \right\}, \quad W_o = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \\ \hline 1 & 0 & 0 \end{array}\right] \right\}.$$

Solving the Eq. (5.9) gives

$$X_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad Y_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \quad F_a = 1.$$

The gain $F_1$, such that $(\bar{E}, \bar{A}_{11} + \bar{B}_{12}F_1)$ is admissible, can be calculated by solving a feasibility problem under an LMI constraint. Hence, we obtain

$$F = \left[\begin{array}{ccccc|c} 0.4283 & 5.9326 & -34.8241 & -1.5169 & 35.0731 & 34.5 \end{array}\right].$$

Similarly, applying Lemma 5.2 leads to

$$X_o = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Y_o = \begin{bmatrix} 1 & -1 & 1 & 4 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad L_a = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$L = \begin{bmatrix} -13.3445 & -0.7713 | -7.1723 & 1.1157 & -0.2287 & 0.7628 \end{bmatrix}^\top.$$

Then, according to Theorem 5.2, we can parameterize all controllers solving Problem 5.1 for the given plant. For this purpose, we arbitrarily choose $Q \in RH_\infty$, for instance, taking

$$Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \\ \hline 0 & 1 & 1 \end{array} \right] \right\}.$$

which yields the following controller $K$

$$E_K = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$A_K = \begin{bmatrix} 1 & 0 & -13.3 & -13.3 & -13.3 & -26.6 & 0 & 0 \\ -0.43 & -4.93 & 34.05 & 2.746 & -34.8 & -32.1 & 0 & -1 \\ 0.428 & 5.933 & -42 & -9.69 & 26.90 & 17.21 & 0 & 1 \\ 0.428 & 5.933 & -34.7 & -0.4 & 35.19 & 34.79 & 0 & 1 \\ 0.428 & 5.933 & -36.1 & -2.75 & 34.84 & 32.1 & 0 & 1 \\ 0 & 0 & 0.763 & 0.763 & 0.763 & 1.526 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 13.35 \\ -0.23 \\ 8.172 \\ -0.12 \\ 1.229 \\ -0.76 \\ 1 \\ 0 \end{bmatrix}, \quad C_K = \begin{bmatrix} 0.428 \\ 5.933 \\ -35.8 \\ -2.52 \\ 34.07 \\ 32.56 \\ 0 \\ 1 \end{bmatrix}^\top, \quad D_K = 1.$$

Then, the resulting closed-loop system is obtained

$$T_{zw} = \frac{-2s(s+68.13)(s-0.3456)(s^2-12.73s+111.6)}{(s+11.26)(s+2.65)(s+0.9641)(s+0.4852)(s^2+6.288s+10.38)}.$$

Moreover, using Lemma 5.3 together with the controller's data gives

$$X_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad Y_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \quad X_o = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ -0.606 & -1 \\ 0 & 0 \end{bmatrix}^\top, \quad Y_o = \begin{bmatrix} 1.921 & 2 \\ -0.079 & 0 \\ -0.684 & 0 \\ -2.159 & 0 \end{bmatrix}^\top,$$

$$F_a = 1, \quad L_a = \begin{bmatrix} -0.08 & 1 \end{bmatrix}^\top,$$

$$X_K^i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad Y_K^i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad X_K^o = \begin{bmatrix} 1 & 0 \\ 0.88 & 1 \\ -2 & 0 \\ 0 & 0 \\ 1.406 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0.54 & 0 \end{bmatrix}^\top, \quad Y_K^o = \begin{bmatrix} 0.97 & 0 \\ 0.421 & 1 \\ -0.122 & -2 \\ 1.07 & 0 \\ 1.208 & 0 \\ 1.259 & 0 \\ 0 & 0 \\ 0.46 & 0 \end{bmatrix}^\top.$$

Hence, the obtained dynamic controller $K$ satisfies the characterization given in Lemma 5.3.

## 5.6 Structured Controllers

So far, we have developed an observer-based solution to the nonstandard problem with unstable and nonproper weights. Additional performance objective control with the presence of weights can be further addressed based on the previous results, which will be reported in the next chapter. However, the effects of the weights on the underlying controller's structure have not yet been clarified. The main purpose of this section is to explicitly exhibit the structure of the resulting controller. It will be shown that the comprehensive admissibility control problem can be transformed into a problem without weights for an augmented system explicitly constructed from the data of the given system.

**Theorem 5.3** *Consider the partitions* (5.6) *and* (5.13)*. Problem 5.1 is solvable, if and only if there exist matrices* $X_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$*,* $Y_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$*,* $F_a \in \mathbb{R}^{m\times n_i}$*,* $X_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$*,* $Y_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$*, and* $L_o \in \mathbb{R}^{n_o\times p}$ *such that the generalized Sylvester equations* (5.9) *and* (5.16) *hold. Moreover, a desired output feedback controller is given by*

$$K = \left[\begin{array}{c|c} A_K - sE_K & B_K \\ \hline C_K & D_{k1}^1 \end{array}\right], \tag{5.24}$$

*with*

$$E_K = \begin{bmatrix} E_i & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & E_o \end{bmatrix}, \quad A_K = \begin{bmatrix} A_i + D_{k1}^2\Gamma_i & C_k^2 & D_{k2}^2 \\ B_{k1}\Gamma_i & A_k & B_{k2} \\ \Omega_o\Gamma_i & -\Gamma_o C_k & A_o - \Gamma_o D_{k2} \end{bmatrix},$$

$$B_K = \begin{bmatrix} D_{k1}^2 \\ B_{k1} \\ \Omega_o \end{bmatrix}, \quad C_K^T = \begin{bmatrix} \Omega_i^T \\ \left(C_k^1\right)^T \\ \left(D_{k2}^1\right)^T \end{bmatrix}, \quad \Gamma_i = \bar{C}_{21}Y_i - \bar{C}_{22},$$

$$\Gamma_o = X_o\tilde{B}_{22} + \tilde{B}_{12}, \quad \Omega_i = F_a + D_{k1}^1\Gamma_i, \quad \Omega_o = L_o - \Gamma_o D_{k1},$$

$$\tilde{B}_{12} = \begin{bmatrix} \hat{B}_{12} & X_i^1 \end{bmatrix}, \quad \tilde{B}_{22} = \left[\begin{array}{c|c} \hat{B}_{22} & \begin{matrix} X_i^2 \\ -I_{n_i} \end{matrix} \end{array}\right], \quad X_i^1 = \begin{bmatrix} I_{n_o} & 0 \end{bmatrix} X_i,$$

$$X_i^2 = \begin{bmatrix} 0 & I_{n_g} \end{bmatrix} X_i, \quad C_k = \begin{bmatrix} C_k^1 \\ C_k^2 \end{bmatrix}, \quad D_{k1} = \begin{bmatrix} D_{k1}^1 \\ D_{k1}^2 \end{bmatrix}, \quad D_{k2} = \begin{bmatrix} D_{k2}^1 \\ D_{k2}^2 \end{bmatrix},$$

*where $E_k$, $A_k$, $B_{k1}$, $B_{k2}$, $C_k^1$, $C_k^2$, $D_{k1}^1$, $D_{k1}^2$, $D_{k2}^1$, and $D_{k2}^2$ are parameters of the controller $\mathcal{K}$ given by*

$$\mathcal{K} = \left[\begin{array}{c|cc} A_k - sE_k & B_{k1} & B_{k2} \\ \hline C_k^1 & D_{k1}^1 & D_{k2}^1 \\ C_k^2 & D_{k1}^2 & D_{k2}^2 \end{array}\right] \tag{5.25}$$

*internally stabilizing the system $\mathcal{G}$ as follows:*

$$\mathcal{G} = \left[\begin{array}{c|cc} A - sE & B_1 & \left[\begin{array}{c|c} B_2 & \begin{matrix} X_i \\ I_{n_i} \end{matrix} \end{array}\right] \\ \hline C_1 & D_{11} & \begin{bmatrix} D_{12} & 0 \end{bmatrix} \\ \left[\begin{array}{cc} \multicolumn{2}{c}{C_2} \\ \hline -I_{n_o} & -Y_o \end{array}\right] & \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}\right]. \tag{5.26}$$

*Proof* Necessity: Straightforward from Lemma 5.3.

Sufficiency: We prove that under (5.9) and (5.16), the controller (5.24) solves Problem 5.1. The proof is divided in three steps. First, we show that the uncontrollable dynamics of $W_i$ are canceled in the closed-loop system. For simplicity, we denote the controller (5.24) as

$$K = \left[\begin{array}{cc|c} A_i + D_{k1}^2\Gamma_i - sE_i & \bar{C}_{k2} & D_{k1}^2 \\ \bar{B}_k\Gamma_i & \bar{A}_k - s\bar{E}_k & \bar{B}_k \\ \hline \Omega_i & \bar{C}_{k1} & D_{k1}^1 \end{array}\right],$$

where

$$\bar{E}_k = \begin{bmatrix} E_k & 0 \\ 0 & E_o \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} A_k & B_{k2} \\ -\Gamma_o C_k & A_o - \Gamma_o D_{k2} \end{bmatrix}, \tag{5.27}$$

$$\bar{B}_k = \begin{bmatrix} B_{k1}, \\ \Omega_o \end{bmatrix}, \quad (\bar{C}_{k1})^T = \begin{bmatrix} (C_k^1)^T \\ (D_{k2}^1)^T \end{bmatrix}, \quad (\bar{C}_{k2})^T = \begin{bmatrix} (C_k^2)^T \\ (D_{k2}^2)^T \end{bmatrix}. \tag{5.28}$$

Then, the closed-loop system is given by

$$\mathcal{F}_l(G, K) = \left[\begin{array}{c|c} A_c - sE_c & B_c \\ \hline C_c & D_c \end{array}\right],$$

where

$$A_c = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} D_{k1}^1 \bar{C}_{21} & \Phi & \bar{B}_{12}\Omega_i & \bar{B}_{12}\bar{C}_{k1} \\ 0 & A_i & 0 & 0 \\ D_{k1}^2 \bar{C}_{21} & D_{k1}^2 \bar{C}_{22} & A_i + D_{k1}^2 \Gamma_i & \bar{C}_{k2} \\ \bar{B}_k \bar{C}_{21} & \bar{B}_k \bar{C}_{22} & \bar{B}_k \Gamma_i & \bar{A}_k \end{bmatrix},$$

$$E_c = \begin{bmatrix} \bar{E} & 0 & 0 & 0 \\ 0 & E_i & 0 & 0 \\ 0 & 0 & E_i & 0 \\ 0 & 0 & 0 & \bar{E}_k \end{bmatrix}, \quad B_c = \begin{bmatrix} \bar{B}_{11} + \bar{B}_{12} D_{k1}^1 D_{21} \\ B_i \\ D_{k1}^2 D_{21} \\ \bar{B}_k D_{21} \end{bmatrix},$$

$$C_c = \begin{bmatrix} \bar{C}_{11} + D_{12} D_{k1}^1 \bar{C}_{21} & \bar{C}_{12} + D_{12} D_{k1}^1 \bar{C}_{22} & D_{12}\Omega_i & D_{12}\bar{C}_{k1} \end{bmatrix},$$

$$D_c = D_{11} + D_{12} D_{k1}^1 D_{21}, \quad \Phi = \bar{A}_{12} + \bar{B}_{12} D_{k1}^1 \bar{C}_{22}.$$

Using the two transformation matrices

$$M_1 = \begin{bmatrix} I & X_i & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad N_1 = \begin{bmatrix} I & Y_i & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

together with (5.9), gives

$$\mathcal{F}_l(G, K) = \left[\begin{array}{c|c} \bar{A}_c - s\bar{E}_c & \bar{B}_c \\ \hline \bar{C}_c & D_c \end{array}\right], \tag{5.29}$$

where

$$\bar{A}_c = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} D_{k1}^1 \bar{C}_{21} & \bar{B}_{12}\Omega_i & \bar{B}_{12}\bar{C}_{k1} \\ D_{k1}^2 \bar{C}_{21} & A_i + D_{k1}^2 \Gamma_i & \bar{C}_{k2} \\ \bar{B}_k \bar{C}_{21} & \bar{B}_k \Gamma_i & \bar{A}_k \end{bmatrix},$$

$$\bar{E}_c = \begin{bmatrix} \bar{E} & 0 & 0 \\ 0 & E_i & 0 \\ 0 & 0 & \bar{E}_k \end{bmatrix}, \; B_c = \begin{bmatrix} \bar{B}_{11} + \bar{B}_{12} D_{k1}^1 D_{21} + X_i B_i \\ D_{k1}^2 D_{21} - B_i \\ \bar{B}_k D_{21} \end{bmatrix},$$
$$\bar{C}_c = \begin{bmatrix} \bar{C}_{11} + D_{12} D_{k1}^1 \bar{C}_{21} & D_{12}\Omega_i & D_{12}\bar{C}_{k1} \end{bmatrix}.$$

Therefore, the uncontrollable dynamics of $W_i$ are canceled in the closed-loop system.

Second, we prove that the unobservable dynamics of $W_o$ are also canceled in the closed-loop system. Consider further the closed-loop system (5.29), which can be written as $\mathcal{F}_l(\bar{G}, \bar{K})$, where

$$\bar{G} = \left[\begin{array}{cc|cc} \bar{A}_{11} - s\bar{E} & \bar{B}_{12}\Pi_i & \bar{B}_{11} + X_i B_i & \begin{bmatrix} \bar{B}_{12} & 0 \\ 0 & I_{n_i} \end{bmatrix} \\ 0 & A_i - sE_i & -B_i & \\ \hline \bar{C}_{11} & D_{12}\Pi_i & D_{11} & \begin{bmatrix} D_{12} & 0 \end{bmatrix} \\ \bar{C}_{21} & \Gamma_i & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right],$$
$$\bar{K} = \left[\begin{array}{c|c} \bar{A}_k - s\bar{E}_k & \bar{B}_k \\ \hline \bar{C}_{k1} & D_{k1}^1 \\ \bar{C}_{k2} & D_{k1}^2 \end{array}\right].$$

With the Eq. (5.9) and two transformation matrices

$$M_2 = \begin{bmatrix} I_{n_g+n_o} & X_i \\ 0 & -I_{n_i} \end{bmatrix}, \quad N_2 = \begin{bmatrix} I_{n_g+n_o} & Y_i \\ 0 & -I_{n_i} \end{bmatrix},$$

the system $\bar{G}$ is represented alternatively by $\hat{G}$ as

$$\hat{G} = \left[\begin{array}{c|cc} \begin{bmatrix} A_o & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} - s\begin{bmatrix} E_o & 0 \\ 0 & \hat{E} \end{bmatrix} & \begin{matrix} \hat{B}_{11} & \tilde{B}_{12} \\ \hat{B}_{21} & \tilde{B}_{22} \end{matrix} \\ \hline \begin{matrix} C_o & \hat{C}_{12} \\ 0 & \hat{C}_{22} \end{matrix} & \begin{matrix} D_{11} & \tilde{D}_{12} \\ D_{21} & 0 \end{matrix} \end{array}\right],$$

where $\tilde{B}_{12}$ and $\tilde{B}_{22}$ are given in (5.24), $\tilde{D}_{12} = \begin{bmatrix} D_{12} & 0 \end{bmatrix}$, and the other data are given in the partition (5.13). Proceeding with the controller $\bar{K}$ specified in (5.27) and (5.28) gives

$$\mathcal{F}_l(\hat{G}, \bar{K}) = \left[\begin{array}{c|c} M_2\bar{A}_c N_2 - sM_2\bar{E}_c N_2 & M_2\bar{B}_c \\ \hline \bar{C}_c N_2 & D_c \end{array}\right],$$

where

$$M_2\bar{A}_cN_2 = \begin{bmatrix} A_o & \hat{A}_{12} + \tilde{B}_{12}D_{k1}\hat{C}_{22} & \tilde{B}_{12}C_k & \tilde{B}_{12}D_{k2} \\ 0 & \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} \\ 0 & B_{k1}\hat{C}_{22} & A_k & B_{k2} \\ 0 & \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} \end{bmatrix},$$

$$M_2\bar{E}_cN_2 = \begin{bmatrix} E_o & 0 & 0 & 0 \\ 0 & \hat{E} & 0 & 0 \\ 0 & 0 & E_k & 0 \\ 0 & 0 & 0 & E_o \end{bmatrix}, \; M_2\bar{B}_2 = \begin{bmatrix} \hat{B}_{11} + \tilde{B}_{12}D_{k1}D_{21} \\ \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \end{bmatrix},$$

$$\bar{C}_cN_2 = \begin{bmatrix} C_o & \hat{C}_{12} + \hat{D}_{12}D_{k1}\hat{C}_{22} & \hat{D}_{12}C_k & \hat{D}_{12}D_{k2} \end{bmatrix}.$$

Using the two transformation matrices

$$M_3 = \begin{bmatrix} I & X_o & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad N_3 = \begin{bmatrix} I & -Y_i & 0 & -I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

together with (5.16), leads to

$$\mathcal{F}_l(\hat{G}, \bar{K}) = \left[\begin{array}{c|c} \hat{A}_c - s\hat{E}_c & \hat{B}_c \\ \hline \hat{C}_c & D_c \end{array}\right], \tag{5.30}$$

where

$$\hat{A}_c = \begin{bmatrix} \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} \\ B_{k1}\hat{C}_{22} & A_k & B_{k2} \\ \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} \end{bmatrix},$$

$$\hat{E}_c = \begin{bmatrix} \hat{E} & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & E_o \end{bmatrix}, \quad \hat{B}_c = \begin{bmatrix} \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \end{bmatrix},$$

$$\hat{C}_c = \begin{bmatrix} \hat{C}_{12} + \hat{D}_{12}D_{k1}\hat{C}_{22} - C_oY_o & \hat{D}_{12}C_k & \hat{D}_{12}D_{k2} - C_o \end{bmatrix}.$$

Therefore, the unobservable dynamics of $W_o$ are canceled in the closed-loop system.

Finally, it suffices to show that the closed-loop system is admissible. It is observed that the closed-loop system (5.30) can be written as $\mathcal{F}_l(\bar{\mathcal{G}}, \mathcal{K})$, where $\mathcal{K}$ is given in (5.25) and

$$\bar{\mathcal{G}} = \left[\begin{array}{cc|cc} \hat{A}_{22} - s\hat{E} & 0 & \hat{B}_{21} & \tilde{B}_{22} \\ \Pi_o\hat{C}_{22} & A_o - sE_o & \Pi_oD_{21} & -\Gamma_o \\ \hline \hat{C}_{12} - C_oY_o & -C_o & D_{11} & \hat{D}_{12} \\ \multicolumn{2}{c|}{\begin{bmatrix} \hat{C}_{22} & 0 \\ 0 & I_{n_o} \end{bmatrix}} & \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right].$$

Using the Eq. (5.16) and two transformation matrices

$$M_4 = \begin{bmatrix} I_{n_g+n_i} & 0 \\ -X_o & -I_{n_o} \end{bmatrix}, \quad N_4 = \begin{bmatrix} I_{n_g+n_i} & 0 \\ -Y_o & -I_{n_o} \end{bmatrix},$$

the system $\bar{\mathcal{G}}$ is represented alternatively by $\mathcal{G}$ in (5.26). Under Assumptions (A1) and (A2), it is shown that the system $\mathcal{G}$ is both finite dynamics stabilizable and impulse controllable, and finite dynamics detectable and impulse observable. Therefore, a stabilizing controller $\mathcal{K}$ (5.25) always exists for $\mathcal{G}$. This ends the proof. □

Figure 5.5 exhibits explicitly the structure of the controller (5.24) and clearly shows the impact of weighting filters on the resulting controller. With this structured controller, the admissibility problem subject to unstable and nonproper weights is transformed into an admissible problem without weights for the augmented system (5.26). Free parameters of (5.25) serve as extra degrees of freedom and pave a way to additional performance objective control with weights.

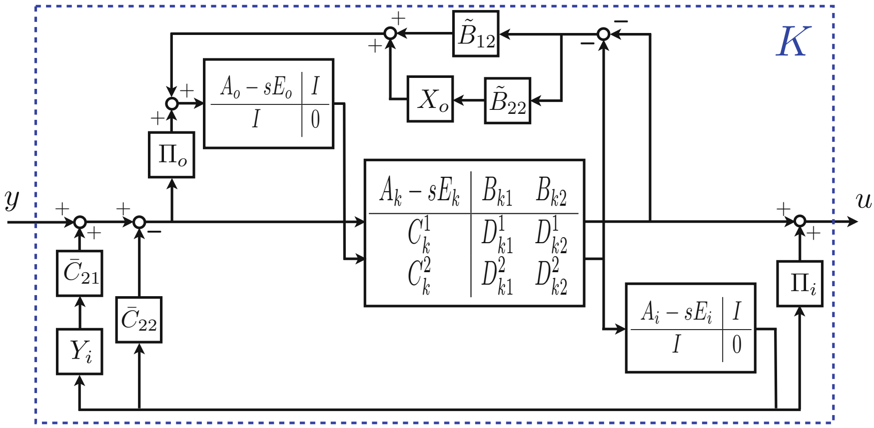


**Fig. 5.5** Structured controller

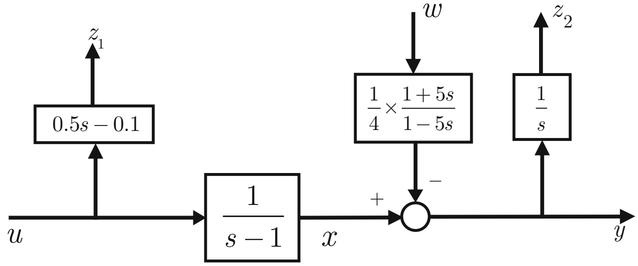


**Fig. 5.6** Weighted system

*Example 5.2*
Now, let us present an example to illustrate Theorem 5.3. Figure 5.6 depicts a weighted system that yields the transfer matrix from $w$ to $(z_1^T, z_2^T)^T$ as

$$T_{zw} = \begin{bmatrix} -W_1K(I-HK)^{-1}W_3 \\ -W_2(I-HK)^{-1}W_3 \end{bmatrix},$$

where $W_1 = 0.5s - 0.1$, $W_2 = 1/s$, $W_3 = (1+5s)/4(1-5s)$, $H = 1/(s-1)$ and $K$ is the controller to be design. It is clear that the output weights $W_1$ and $W_2$ are unobservable, and the input weight $W_3$ is uncontrollable. Hence, it is impossible to render the closed-loop system internally stable. The objective here is to find an output feedback controller $K$ such that the resulting closed-loop system is comprehensively admissible. Note that the weighted system $G$ can be represented as

$$E = \begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&5 \end{bmatrix}, \quad A = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&-1 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ 0 \\ -0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0&0.5&0&0&0 \\ 0&0&1&0&0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0&0&0&1&-1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \quad D_{21} = 0.25.$$

Clearly, the Assumptions (A1)–(A2) hold. Solving the two generalized Sylvester equations yields

$$X_i = \begin{bmatrix} 0.16 \\ 0 \\ 0 \\ -0.2 \end{bmatrix}, \quad Y_i = \begin{bmatrix} 0.8 \\ 0.16 \\ 0 \\ -1 \end{bmatrix} \quad F_a = -0.8,$$

$$X_o = Y_o = \begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{bmatrix}, \quad L_o = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Hence, the comprehensive admissibility problem for the system in Fig. 5.6 amounts to finding a controller $\mathcal{K}$ of the form

$$\mathcal{K} := \left[\begin{array}{c|cc} A_k - sE_k & B_{k1} & B_{k2} \\ \hline C_k^1 & D_{k1}^1 & D_{k2}^1 \\ C_k^2 & D_{k1}^2 & D_{k2}^2 \end{array}\right]$$

to internally stabilizes the following augmented system $\mathcal{G}$

$$\mathcal{G} := \left[\begin{array}{c|cc} A - sE & B_1 & \mathcal{B}_2 \\ \hline C_1 & D_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & 0 \end{array}\right],$$

where

$$\mathcal{B}_2 = \begin{bmatrix} 0 & 0.16 \\ -1 & 0 \\ 0 & 0 \\ 1 & -0.2 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathcal{D}_{12} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{21} = \begin{bmatrix} 0.25 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

other system data are given previously. It is easy to see that the system $\mathcal{G}$ is both finite dynamics stabilizable and impulse controllable, and finite dynamics detectable and impulse observable. Therefore, a stabilizing controller $\mathcal{K}$ always exists. The parameters in $\mathcal{K}$ can be further appealed for the mixed sensitivity problem through some existing results, if $H_\infty$ performance is imposed on the transfer matrix $T_{zw}$.

## 5.7 Conclusion

This chapter addresses the admissibility control problem subject to input–output unstable and nonproper weighting filters for continuous-time descriptor systems. By introducing a new concept called comprehensive admissibility, necessary and sufficient conditions for this nonstandard problem are given in terms of the two generalized Sylvester equations. A set of observer-based controllers is characterized trough Youla parameterization. In order to further clarify the impact of weighting filters on resulting controllers, a structured controller is explicitly conducted and it proves that the comprehensive admissibility control problem can be transformed into a standard admissibility control problem with regard to an augmented system without weights.

# Chapter 6
# Performance with Unstable and Nonproper Weights

The last chapter addresses the stabilization subject to input-output unstable and nonproper weighting filters for continuous-time descriptor systems. Systems and weights are all described within the descriptor framework to take into consideration not only unstable weights, but also nonproper weights. The so-called comprehensive admissibility is appealed to tackle the stabilization issue within this circumstance. Both observer-based controllers and general output feedback controllers solving this nonstandard problem are parameterized and it has been shown that the solvability conditions are relied on the two generalized Sylvester equations.

In this chapter, we attempt to further explore additional $H_2$ and $H_\infty$ performance objectives subject to comprehensive admissibility. It will be shown that performance objectives can be achieved by solving two generalized algebraic Riccati equations (GAREs). As the overall weighted system is neither wholly stabilizable nor detectable, the underlying GAREs have no admissible solutions. Similar to the definition of comprehensive admissibility, the concept of so-called quasi-admissible solution will be adopted. It is observed that the quasi-admissible solutions to the GAREs are formed by admissible solutions to two reduced GAREs and solutions to the two generalized Sylvester equations. Thus, solvability conditions to the problem of performance control under comprehensive admissibility are characterized in terms of quasi-admissible solutions to the underlying GAREs.

## 6.1 Problem Formulation

The same setup used in the last chapter is adopted here. Consider the interconnection depicted in Fig. 6.1, where $\widetilde{G}(s)$ is written by

$$\begin{bmatrix} e(s) \\ y(s) \end{bmatrix} = \widetilde{G} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} \widetilde{G}_{ev} & \widetilde{G}_{eu} \\ \widetilde{G}_{yv} & \widetilde{G}_{yu} \end{bmatrix} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} \tag{6.1}$$



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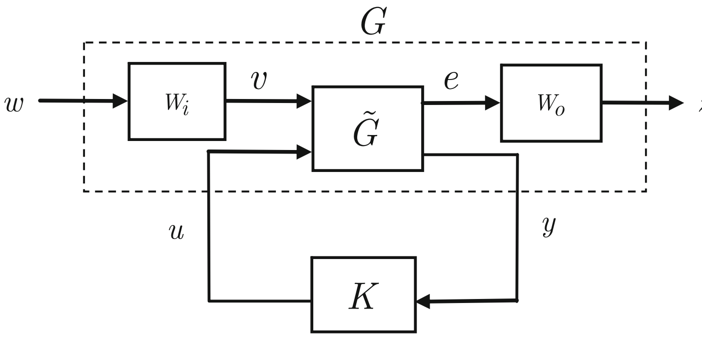


**Fig. 6.1** Comprehensive control problem

where $e \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, $v \in \mathbb{R}^l$, and $u \in \mathbb{R}^m$ are the controlled output, measurement, disturbance input, and control input, respectively. The system (6.1) can be rewritten as

$$\widetilde{G} = \left[\begin{array}{c|cc} A_g - sE_g & B_{g1} & B_{g2} \\ \hline C_{g1} & D_{g11} & D_{g12} \\ C_{g2} & D_{g21} & D_{g22} \end{array}\right]$$

where $E_g \in \mathbb{R}^{n_g \times n_g}$, $A_g \in \mathbb{R}^{n_g \times n_g}$, $B_{g1} \in \mathbb{R}^{n_g \times l}$, $B_{g2} \in \mathbb{R}^{n_g \times m}$, $C_{g1} \in \mathbb{R}^{q \times n_g}$, $C_{g2} \in \mathbb{R}^{p \times n_g}$, $D_{g11} \in \mathbb{R}^{q \times l}$, $D_{g12} \in \mathbb{R}^{q \times m}$, $D_{g21} \in \mathbb{R}^{p \times l}$, and $D_{g22} \in \mathbb{R}^{p \times m}$ are known as real constant matrices. The matrix $E_g$ may be singular, i.e., $\text{rank}(E_g) = r_g \leq n_g$.

Suppose that the input weight $W_i$ and the output weight $W_o$ are both descriptor systems described as

$$W_i = \left[\begin{array}{c|c} A_i - sE_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} A_o - sE_o & B_o \\ \hline C_o & D_o \end{array}\right],$$

where $E_i \in \mathbb{R}^{n_i \times n_i}$, $E_o \in \mathbb{R}^{n_o \times n_o}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $A_o \in \mathbb{R}^{n_o \times n_o}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $B_o \in \mathbb{R}^{n_o \times q}$, $C_i \in \mathbb{R}^{l \times n_i}$, $C_o \in \mathbb{R}^{p_o \times n_o}$, $D_i \in \mathbb{R}^{l \times m_i}$, and $D_o \in \mathbb{R}^{p_o \times q}$ are known real constant matrices. The matrices $E_i$ and $E_o$ may be singular, i.e., $\text{rank}(E_i) = r_i \leq n_i$ and $\text{rank}(E_o) = r_o \leq n_o$.

Then the resulting overall weighted plant $G$ is written as

$$G = \left[\begin{array}{ccc|cc} A_o - sE_o & B_oC_{g1} & B_oD_{g11}C_i & B_oD_{g11}D_i & B_oD_{g12} \\ 0 & A_g - sE_g & B_{g1}C_i & B_{g1}D_i & B_{g2} \\ 0 & 0 & A_i - sE_i & B_i & 0 \\ \hline C_o & D_oC_{g1} & D_oD_{g11}C_i & D_oD_{g11}D_i & D_oD_{g12} \\ 0 & C_{g2} & D_{g21}C_i & D_{g21}D_i & D_{g22} \end{array}\right]. \tag{6.2}$$

Moreover, we denote in the sequel $G$ as

$$G = \left[\begin{array}{c|cc} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \triangleq \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}. \tag{6.3}$$

Let us recall the definition of comprehensive admissibility as follows.

**Definition 6.1** (*Comprehensive Admissibility*) The feedback system $\mathcal{F}_l(G, K)$ is said to be comprehensively admissible if $\mathcal{F}_l(\widetilde{G}, K)$ is internally stable and the closed-loop system defined as

$$T_{zw} = \mathcal{F}_l(G, K) = G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \tag{6.4}$$

is admissible.

Throughout this chapter we make the following assumptions.

**(A0)**. The weights $W_i$ and $W_o$ contain only unstable and impulsive modes.
**(A1)**. $D_{12}^{\top}D_{12} > 0$, $D_{21}D_{21}^{\top} > 0$, and $D_{22} = 0$.
**(A2)**. $(E, A)$ is regular.

Assumption (A0) is made without loss of generality since the stable and static modes of the weights are a mere consequence of internal stabilization. (A1) is made here to conduct the optimal controller in terms of GAREs. Moreover, this assumption causes no loss of generality within the descriptor framework. If it does not hold, an equivalent realization satisfying this assumption can always be obtained.

Let us consider two partitions of $G$ with regard to $W_i$ and $W_o$, respectively.

Partition 1:

$$G = \left[\begin{array}{cc|cc} \bar{A}_{11} - s\bar{E} & \bar{A}_{12} & \bar{B}_{11} & \bar{B}_{12} \\ 0 & A_i - sE_i & \bar{B}_{21} & 0 \\ \hline \bar{C}_{11} & \bar{C}_{12} & D_{11} & D_{12} \\ \bar{C}_{21} & \bar{C}_{22} & D_{21} & D_{22} \end{array}\right], \tag{6.5}$$

where

$$\begin{cases} \bar{E} = \begin{bmatrix} E_o & 0 \\ 0 & E_g \end{bmatrix}, \quad \bar{A}_{11} = \begin{bmatrix} A_o & B_oC_{g1} \\ 0 & A_g \end{bmatrix}, \\ \bar{A}_{12} = \begin{bmatrix} B_oD_{g11}C_i \\ B_{g1}C_i \end{bmatrix}, \quad \bar{B}_{11} = \begin{bmatrix} B_oD_{g11}D_i \\ B_{g1}D_i \end{bmatrix}, \\ \bar{B}_{12} = \begin{bmatrix} B_oD_{g12} \\ B_{g2} \end{bmatrix}, \quad \bar{B}_{21} = B_i, \quad \bar{C}_{11} = \begin{bmatrix} C_o & D_oC_{g1} \end{bmatrix}, \\ \bar{C}_{12} = D_oD_{g11}C_i, \quad \bar{C}_{21} = \begin{bmatrix} 0 & C_{g2} \end{bmatrix}, \quad \bar{C}_{22} = D_{g21}C_i. \end{cases} \tag{6.6}$$

Partition 2:

$$G = \left[\begin{array}{cc|cc} A_o - sE_o & \hat{A}_{12} & \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{A}_{22} - s\hat{E} & \hat{B}_{21} & \hat{B}_{22} \\ \hline \hat{C}_{11} & \hat{C}_{12} & D_{11} & D_{12} \\ 0 & \hat{C}_{22} & D_{21} & D_{22} \end{array}\right], \tag{6.7}$$

with

$$\begin{cases} \hat{E} = \begin{bmatrix} E_g & 0 \\ 0 & E_i \end{bmatrix}, \quad \hat{A}_{12} = \begin{bmatrix} B_o C_{g1} & B_o D_{g11} C_i \end{bmatrix}, \\ \hat{A}_{22} = \begin{bmatrix} A_g & B_{g1} C_i \\ 0 & A_i \end{bmatrix}, \quad \hat{B}_{11} = B_o D_{g11} D_i, \quad \hat{B}_{12} = B_o D_{g12}, \\ \hat{B}_{21} = \begin{bmatrix} B_{g1} D_i \\ B_i \end{bmatrix}, \quad \hat{B}_{22} = \begin{bmatrix} B_{g2} \\ 0 \end{bmatrix}, \quad \hat{C}_{11} = C_o, \\ \hat{C}_{12} = \begin{bmatrix} D_o C_{g1} & D_o D_{g11} C_i \end{bmatrix}, \quad \hat{C}_{22} = \begin{bmatrix} C_{g2} & D_{g21} C_i \end{bmatrix}. \end{cases} \tag{6.8}$$

## 6.2 Comprehensive $H_2$ Control

First, we define the problem of comprehensive $H_2$ control as follows.

**Problem 6.1** (*Comprehensive $H_2$ Control Problem*) The comprehensive $H_2$ control problem for the system $G$ in (6.3) is to find a controller $K$ such that $\mathcal{F}_l(G, K)$ is comprehensively admissible and the $H_2$ norm of the closed-loop $T_{zw}$ in (6.4) is minimized.

We introduce the concept of quasi-admissible solution to GAREs for the nonstandard problem defined previously. Relied on this concept, a solution to Problem 6.1 is presented.

### *6.2.1 Quasi-admissible Solutions to GAREs*

It is well known that the standard $H_2$ control problem for continuous-time descriptor systems is closely related to the following two GAREs.

$$\begin{cases} E^\top X = X^\top E, \\ A^\top X + X^\top A + C_1^\top C_1 \\ -(C_1^\top D_{12} + X^\top B_2)(D_{12}^\top D_{12})^{-1}(D_{12}^\top C_1 + B_2^\top X) = 0; \end{cases} \tag{6.9}$$

$$\begin{cases} EY = Y^\top E^\top, \\ AY + Y^\top A^\top + B_1 B_1^\top \\ -(B_1 D_{21}^\top + Y^\top C_2^\top)(D_{21} D_{21}^\top)^{-1}(D_{21} B_1^\top + C_2 Y) = 0. \end{cases} \tag{6.10}$$

Solvability condition to the standard $H_2$ control problem is based on the following assumptions.

**(A3)**. $(E, A, B_2)$ is finite dynamics stabilizable and impulse controllable.

**(A4)**. $\begin{bmatrix} A - sE & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has no invariant zeros on the imaginary axis including infinity;

**(A5)**. $(E, A, C_2)$ is finite dynamics detectable and impulse observable.

**(A6)**. $\begin{bmatrix} A - sE & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has no invariant zeros on the imaginary axis including infinity.

The conditions (A3) and (A5) guarantee the existence of a controller $K$ internally stabilizing the closed-loop system $\mathcal{F}_l(G, K)$. Moreover, it is known that, under Assumptions (A1) and (A2), the GARE (6.9) has an admissible solution if the conditions (A3) and (A4) hold. Similarly, the GARE (6.10) has an admissible solution if the conditions (A5) and (A6) hold. Admissible solutions to the two GAREs form the optimal $H_2$ controller. A more comprehensive discussion about GARE is given in Appendix B. However, as for Problem 6.1, (A3) and (A5) are obviously not fulfilled due to the presence of unstable and nonproper weights $W_i$ and $W_o$. In order to solve such nonstandard GAREs, we relax conventional admissible solutions and define the following quasi-admissible solutions.

**Definition 6.2** (*Quasi-admissible Solution*) Let $X$ and $Y$ be solutions to the GAREs (6.9) and (6.10), respectively. Define

$$F = -(D_{12}^\top D_{12})^{-1}(D_{12}^\top C_1 + B_2^\top X), \tag{6.11}$$

$$L = -(B_1 D_{21}^\top + Y^\top C_2^\top)(D_{21} D_{21}^\top)^{-1}. \tag{6.12}$$

**(i)** A solution $X$ to the GARE (6.9) is called a quasi-admissible solution if $E^\top X \geq 0$ and the closed-loop system defined by $\left[\begin{array}{c|c} A + B_2 F - sE & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right]$ is admissible.

**(ii)** A solution $Y$ to the GARE (6.10) is called a quasi-admissible solution if $EY \geq 0$ and the closed-loop system defined by $\left[\begin{array}{c|c} A + LC_2 - sE & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right]$ is admissible.

Also, let us introduce a relaxed version of the aforementioned standard assumptions that give sufficient conditions for the existence of quasi-admissible solutions to the GAREs (6.9) and (6.10).

**(A3')** $(\bar{E}, \bar{A}_{11}, \bar{B}_{12})$ is finite dynamics stabilizable and impulse controllable.

**(A4')** $\begin{bmatrix} \bar{A}_{11} - s\bar{E} & \bar{B}_{12} \\ \bar{C}_{11} & D_{12} \end{bmatrix}$ has no invariant zeros on the imaginary axis including infinity.

**(A5')** $(\hat{E}, \hat{A}_{22}, \hat{C}_{22})$ is finite dynamics detectable and impulse observable.

**(A6')** $\begin{bmatrix} \hat{A}_{22} - s\hat{E} & \hat{B}_{21} \\ \hat{C}_{22} & D_{21} \end{bmatrix}$ has no invariant zeros on the imaginary axis including infinity.

The following theorem gives a condition to the existence of a quasi-admissible solution to the GARE (6.9).

**Theorem 6.1** *Under Assumptions (A0)–(A2), (A3'), and (A4'), if there exist matrices* $U_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $V_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, *and* $F_a \in \mathbb{R}^{m\times n_i}$ *such that the following generalized Sylvester equation holds*

$$\begin{cases} \bar{B}_{12}F_a = \bar{A}_{11}V_i - \bar{A}_{12} - U_iA_i, \\ D_{12}F_a = \bar{C}_{11}V_i - \bar{C}_{12}, \\ \bar{E}V_i = U_iE_i, \end{cases} \tag{6.13}$$

*then the GARE* (6.9) *with regard to the partition* (6.5) *admits a quasi-admissible solution. Moreover, the quasi-admissible solution* $X$ *is given by*

$$X = \begin{bmatrix} I & U_i \end{bmatrix}^\top X_c \begin{bmatrix} I & V_i \end{bmatrix}, \tag{6.14}$$

*where* $X_c$ *is an admissible solution to the GARE*

$$\begin{cases} \bar{E}^\top X_c = X_c^\top \bar{E}, \\ \bar{A}_{11}^\top X_c + X_c^\top \bar{A}_{11} + \bar{C}_{11}^\top \bar{C}_{11} \\ -(\bar{C}_{11}^\top D_{12} + X_c^\top \bar{B}_{12})(D_{12}^\top D_{12})^{-1}(D_{12}^\top \bar{C}_{11} + \bar{B}_{12}^\top X_c) = 0, \end{cases} \tag{6.15}$$

*Proof* Under Assumptions (A3') and (A4'), the GARE (6.15) has an admissible solution $X_c$. Using matrices $U_i$ and $V_i$ satisfying (6.13), there holds

$$\begin{aligned} E^\top X &= \begin{bmatrix} \bar{E}^\top \\ E_i^\top U_i^\top \end{bmatrix} X_c \begin{bmatrix} I & V_i \end{bmatrix} \\ &= \begin{bmatrix} \bar{E}^\top \\ V_i^\top \bar{E}^\top \end{bmatrix} X_c \begin{bmatrix} I & V_i \end{bmatrix} \\ &= \begin{bmatrix} I & V_i \end{bmatrix}^\top \bar{E}^\top X_c \begin{bmatrix} I & V_i \end{bmatrix} \\ &\geq 0, \end{aligned}$$

since $X_c$ is an admissible solution to (6.15), i.e., $\bar{E}^\top X_c \geq 0$. The first equation of (6.15), that is $\bar{E}^\top X_c = X_c^\top \bar{E}$, implies

$$\begin{aligned} E^\top X &= \begin{bmatrix} I & V_i \end{bmatrix}^\top X_c^\top \begin{bmatrix} \bar{E} & \bar{E} V_i \end{bmatrix} \\ &= \begin{bmatrix} I & V_i \end{bmatrix}^\top X_c^\top \begin{bmatrix} \bar{E} & U_i E_i \end{bmatrix} \\ &= \begin{bmatrix} I & V_i \end{bmatrix}^\top X_c^\top \begin{bmatrix} I & U_i \end{bmatrix} \begin{bmatrix} \bar{E} & 0 \\ 0 & E_i \end{bmatrix} \\ &= X^\top E. \end{aligned}$$

Moreover, using the partition shown (6.5) gives

$$\begin{aligned} &A^\top X + X^\top A + C_1^\top C_1 - (C_1^\top D_{12} + X^\top B_2)(D_{12}^\top D_{12})^{-1}(D_{12}^\top C_1 + B_2^\top X) \\ = &\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Pi_{11} =& \bar{A}_{11}^\top X_c + X_c^\top \bar{A}_{11} + \bar{C}_{11}^\top \bar{C}_{11} \\ &- (\bar{C}_{11}^\top D_{12} + X_c^\top \bar{B}_{12})(D_{12}^\top D_{12})^{-1}(D_{12}^\top \bar{C}_{11} + \bar{B}_{12}^\top X_c) \\ =& 0. \end{aligned}$$

By (6.13), it is observed that $\Pi_{12} = \Pi_{11} V_i$. Hence $\Pi_{12} = 0$. In addition, we have $\Pi_{22} = V_i^\top \Pi_{11} V_i = 0$. Hence, $X$ given in (6.14) satisfies the GARE (6.9). Now, it suffices to prove that the closed-loop system $T = \left[\begin{array}{c|c} A + B_2 F - sE & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right]$ is admissible. To this end, let us define

$$F_c = -(D_{12}^\top D_{12})^{-1}(D_{12}^\top \bar{C}_{11} + \bar{B}_{12}^\top X_c).$$

Substituting $X$ in (6.11) leads to

$$F = \begin{bmatrix} F_c & F_a + F_c V_i \end{bmatrix}.$$

Define two transformation matrices $M$ and $N$ as follows

$$M = \begin{bmatrix} I & U_i \\ 0 & I \end{bmatrix}, \quad N = \begin{bmatrix} I & -V_i \\ 0 & I \end{bmatrix}.$$

Then, it is easy to see that an alternative representation of $T$ is

$$T = \left[\begin{array}{c|c} \bar{A}_{11} + \bar{B}_{12} F_c - s\bar{E} & \bar{B}_{12} \\ \hline \bar{C}_{11} + \bar{D}_{12} F_c & D_{12} \end{array}\right].$$

Since $F_c$ is the stabilizing gain of the GARE (6.15), the closed-loop system $T$ is admissible. This ends the proof. □

A quasi-admissible solution to the GARE (6.10) is characterized in the following theorem.

**Theorem 6.2** *Under Assumptions (A0)–(A2), (A5'), and (A6'), if there exist matrices* $U_o \in \mathbb{R}^{n_o \times (n_g+n_i)}$, $V_o \in \mathbb{R}^{n_o \times (n_g+n_i)}$, *and* $L_a \in \mathbb{R}^{n_o \times p}$ *such that the following generalized Sylvester equation holds*

$$\begin{cases} L_a \hat{C}_{22} = A_o V_o - \hat{A}_{12} - U_o \hat{A}_{22}, \\ L_a D_{21} = -U_o \hat{B}_{21} - \hat{B}_{11}, \\ U_o \hat{E} \;\; = E_o V_o, \end{cases} \tag{6.16}$$

*then the GARE* (6.10) *with regard to the partition* (6.7) *admits a quasi-admissible solution. Moreover, the quasi-admissible solution* $Y$ *is given by*

$$Y = \begin{bmatrix} V_o^\top & I \end{bmatrix}^\top Y_o \begin{bmatrix} -U_o^\top & I \end{bmatrix}, \tag{6.17}$$

*where* $Y_o$ *is the admissible solution to the GARE*

$$\begin{cases} \hat{E} Y_o = Y_o^\top \hat{E}^\top, \\ \hat{A}_{22} Y_o + Y_o^\top \hat{A}_{22}^\top + \hat{B}_{21} \hat{B}_{21}^\top \\ -(\hat{B}_{21} D_{21}^\top + Y_o^\top \hat{C}_{22}^\top)(D_{21} D_{21}^\top)^{-1}(D_{21} \hat{B}_{21}^\top + \hat{C}_{22} Y_o) = 0. \end{cases} \tag{6.18}$$

*Proof* By the same thread of Theorem 6.2. □

## 6.2.2 Optimal Controller

Based on quasi-admissible solutions to the GAREs, we provide an explicit form of the optimal comprehensive $H_2$ controller and also prove the optimality of the resulting closed-loop system.

**Lemma 6.1** *Suppose that there exist quasi-admissible solutions* $X$ *and* $Y$ *to the two GAREs* (6.9) *and* (6.10), *respectively. Consider the following systems*

$$T_1 := \left[\begin{array}{cc|c} A_F - sE & -B_2 F & B_1 \\ 0 & A_L - sE & B_L \\ \hline C_F & -D_{12} F & D_{11} \end{array}\right], \tag{6.19}$$

$$T_2 := \left[\begin{array}{c|c} A_F - sE & B_2 \\ \hline C_F & D_{12} \end{array}\right], \tag{6.20}$$

$$T_3 := \left[\begin{array}{c|c} A_L - sE & B_L \\ \hline C_2 & D_{21} \end{array}\right], \tag{6.21}$$

*where* $A_F = A + B_2 F$, $A_L = A + L C_2$, $C_F = C_1 + D_{12} F$, $B_L = B_1 + L D_{21}$, *and* $F$, $L$ *are defined in* (6.11) *and* (6.12), *respectively. Then,* $T_1$, $T_2$, *and* $T_3$ *are admissible.*

*Proof* Suppose that the GAREs (6.9) and (6.10) admit quasi-admissible solutions $X$ and $Y$ given in (6.14) and (6.17), respectively. Let $X_c$ and $Y_o$ be the admissible solutions to the GAREs (6.15) and (6.18), respectively. Then define $F_c$ and $L_o$ as

$$\begin{aligned} F_c &= -(D_{12}^\top D_{12})^{-1}(D_{12}^\top \bar{C}_{11} + \bar{B}_{12}^\top X_c), \\ L_o &= -(\hat{B}_{21} D_{21}^\top + Y_o^\top \hat{C}_{22}^\top)(D_{21} D_{21}^\top)^{-1}. \end{aligned}$$

Substituting $X$ and $Y$ in (6.11) and (6.12), respectively, yields

$$F = \begin{bmatrix} F_c & F_a + F_c V_i, \end{bmatrix}, \quad L = \begin{bmatrix} L_a - U_o L_o \\ L_o \end{bmatrix}.$$

Define two transformation matrices $\mathcal{M}_1$ and $\mathcal{N}_1$ as follows:

$$\mathcal{M}_1 = \begin{bmatrix} I & U_i \\ 0 & I \end{bmatrix}, \quad \mathcal{N}_1 = \begin{bmatrix} I & -V_i \\ 0 & I \end{bmatrix}.$$

Then, an alternative representation of $T_2$ is given as

$$T_2 = \left[\begin{array}{c|c} \bar{A}_{11} + \bar{B}_{12} F_c - s\bar{E} & \bar{B}_{12} \\ \hline \bar{C}_{11} + \bar{D}_{12} F_c & D_{12} \end{array}\right]. \tag{6.22}$$

Since $F_c$ is the admissible gain of the GARE (6.15), $(\bar{E}, \bar{A}_{11} + \bar{B}_{12} F_c)$ is admissible. Hence, the system $T_2$ is admissible. By the same thread, with the following transformation matrices

$$\mathcal{M}_2 = \begin{bmatrix} I & U_o \\ 0 & I \end{bmatrix}, \quad \mathcal{N}_2 = \begin{bmatrix} I & -V_o \\ 0 & I \end{bmatrix},$$

we can obtain an alternative representation of $T_3$

$$T_3 = \left[\begin{array}{c|c} \hat{A}_{22} + L_o \hat{C}_{22} - s\hat{E} & \hat{B}_{21} + L_o \hat{D}_{21} \\ \hline \hat{C}_{22} & D_{21} \end{array}\right]. \tag{6.23}$$

Since $L_o$ is the admissible gain of the GARE (6.18), the system $T_3$ is admissible. Finally, let $\begin{bmatrix} \hat{F}_1 & \hat{F}_2 \end{bmatrix} = F$ be a partition compatible with (6.7). The admissibility of $T_1$ can be conducted through the following transformation matrices

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} \mathcal{N}_1 & 0 \\ 0 & \mathcal{N}_2 \end{bmatrix}. \tag{6.24}$$

Hence, $T_1$ can be equivalently rewritten as

$$T_1 = \left[\begin{array}{cc|c} \bar{A}_{11} + \bar{B}_{12}F_c - s\bar{E} & \bar{B}_{12}\left(\hat{F}_1 V_o - \hat{F}_2\right) & \bar{B}_{11} + U_i\bar{B}_{21} \\ 0 & \hat{A}_{22} + L_o\hat{C}_{22} - s\bar{E} & \hat{B}_{21} + L_o\hat{D}_{21} \\ \hline \bar{C}_{11} + \bar{D}_{12}F_c & D_{12}\left(\hat{F}_1 V_o - \hat{F}_2\right) & D_{11} \end{array}\right]. \tag{6.25}$$

Thus, it is clear that the system $T_1$ is admissible. □

The following equation

$$T_1(\infty) - T_2(\infty)\Theta T_3(\infty) = 0, \quad \Theta \in \mathbb{R}^{m\times p}. \tag{6.26}$$

is a necessary and sufficient condition for the existence of a stabilizing controller $K$ such that the underlying closed loop is strictly proper. Note that the realizations of $T_i(s)$, $(i = 1, 2, 3)$ are regular and impulse-free, so the underlying transfer matrices can be computed. Then, $T_i(\infty)$ can be calculated by taking $s$ to infinity for $T_i(s)$.

Now, we are in a position to give a solution to Problem 6.1.

**Theorem 6.3** *Suppose that Assumptions (A0)–(A2), (A3')–(A6') hold, and the two generalized Sylvester equations* (6.13) *and* (6.16) *admit solutions. For a given G in* (6.3)*, there exists an optimal comprehensive $H_2$ controller if and only if the following conditions hold*

- **(I)** *Equation* (6.26) *with $T_1$, $T_2$, and $T_3$ defined in* (6.25)*,* (6.22)*, and* (6.23) *admits a solution $\Theta$.*
- **(II)** *$(E, A + B_2F + LC_2 + B_2\Theta C_2)$ is regular.*

*Moreover, the resulting optimal comprehensive $H_2$ controller is given by*

$$K = \left[\begin{array}{c|c} A + B_2F + LC_2 + B_2\Theta C_2 - sE & -B_2\Theta - L \\ \hline F + \Theta C_2 & -\Theta \end{array}\right], \tag{6.27}$$

*where $F$ and $L$ are defined in* (6.11) *and* (6.12)*, respectively.*

*Proof* Under the conditions (I) and (II), there holds

$$\|\mathcal{F}_l(G, K)\|_2 = \|\mathcal{F}_l\left(G, \mathcal{F}_l(J, \Theta)\right)\|_2 = \|\mathcal{F}_l\left(\mathcal{F}_l(G, J), \Theta\right)\|_2,$$

where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 - sE & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$

Using the same transformation matrices $\mathcal{M}$ and $\mathcal{N}$ in (6.24) leads to

$$\|\mathcal{F}_l(G, K)\|_2 = \|T_1(s) - T_2(s)\Theta T_3(s)\|_2.$$

According to Theorem 11 in [TK98], $\|T_1(s) - T_2(s)\Theta T_3(s)\|_2$ is minimal. □

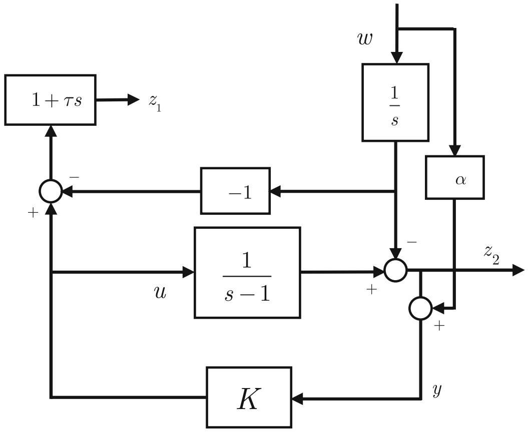


**Fig. 6.2** Numerical example for comprehensive $H_2$ control

*Example 6.1* Consider the overall weighted system in Fig. 6.2 with $\alpha = 0.1$ and $\tau = 0.5$. It consists of an unstable system with an additive disturbance described by an integral model and a nonproper weight on the control deviation to get roll-off at high frequency. The input-output relationship is given by

$$\begin{bmatrix} z_1(s) \\ z_2(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} \frac{0.5(s+2)}{s} & 0.5(s+2) \\ -\frac{1}{s} & \frac{1}{s-1} \\ \frac{0.1(s-10)}{s} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}.$$

Equivalently, this system can also be represented as (6.3) by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}^\top, \quad C_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}^\top,$$

$$D_{11} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}^\top, \quad D_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, \quad D_{21} = 0.1.$$

It is shown that the related assumptions hold. Applying Theorems 6.1 and 6.2, and solving the generalized Sylvester equations and GAREs lead to

$$U_i = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad V_i = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad F_a = -1,$$

$$U_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_a = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.5661 & 0 & 1.7732 & -1.2071 \\ 4.9055 & 1 & 9.3351 & -4.4297 \\ 1.7732 & 0 & 5.7887 & -4.0154 \\ -1.2071 & 0 & -4.0154 & 2.8083 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.42 & 2.2 \\ 0 & 0 & 2.2 & 2.2 \end{bmatrix},$$

$$F = \begin{bmatrix} 3.1322 & 0.5 & 3.5465 & -1.4142 \end{bmatrix}, \quad L^\top = \begin{bmatrix} 0 & 0 & -22 & -10 \end{bmatrix}.$$

Moreover, solving the equation (6.26) gives $\Theta = -5$. Hence, the resulting optimal $H_2$ controller is

$$K = \frac{-10(s + 11.53)(s + 0.2453)}{s(s + 9.694)(s + 5.571)}.$$

The four poles of the underlying closed-loop system are $\{-1.632 \pm 0.4057i, -1, -10\}$, and the $H_2$ norm of the closed-loop system is $\|T_{zw}\|_2 = 4.0318$.

Note that $K$ has, as expected, a relative degree of 1 and an integral action. Nearly the same $H_2$ norm can be obtained through pre-stabilizing the integral disturbance model and incorporating the filter $\frac{1}{1+\tau s}$ as proposed in [CS92]. It is also shown that the unstable model of the disturbance is hidden in the closed-loop system.

## 6.3 Comprehensive $H_\infty$ Control

In this section, we present a solution to the $H_\infty$ performance control under unstable and nonproper weights. The existence of a solution to this problem is characterized in terms of two generalized Sylvester equations and two GAREs together with a spectral radius condition. We adopt the notations defined previously.

**Problem 6.2** (*Comprehensive $H_\infty$ Control Problem*) Given $\gamma > 0$, the comprehensive $H_\infty$ control problem for the system $G$ in (6.3) is to find a controller $K$ such that $\mathcal{F}_l(G, K)$ is comprehensively admissible and the $H_\infty$ norm of the closed-loop $T_{zw}$ satisfies $\|T_{zw}\|_\infty < \gamma$.

**Theorem 6.4** *Suppose that Assumptions (A0)–(A2), (A3')–(A6') hold. Then Problem 6.2 is solvable, if and only if the following conditions hold.*

*(i) There exist matrices $U_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $V_i \in \mathbb{R}^{(n_g+n_o)\times n_i}$, $F_a \in \mathbb{R}^{m\times n_i}$, $U_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$, $V_o \in \mathbb{R}^{n_o\times(n_g+n_i)}$ and $L_a \in \mathbb{R}^{n_o\times p}$ such that the following two generalized Sylvester equations hold*

$$\begin{cases} \bar{B}_{12}F_a &= \bar{A}_{11}V_i - \bar{A}_{12} - U_iA_i, \\ D_{12}F_a &= \bar{C}_{11}V_i - \bar{C}_{12}, \\ \bar{E}V_i &= U_iE_i. \end{cases} \tag{6.28}$$

$$\begin{cases} L_a\hat{C}_{22} &= A_oV_o - \hat{A}_{12} - U_o\hat{A}_{22}, \\ L_aD_{21} &= -U_o\hat{B}_{21} - \hat{B}_{11}, \\ U_o\hat{E} &= E_oV_o. \end{cases} \tag{6.29}$$

*(ii) There exists an admissible solution $X_c$ to the GARE*

$$\begin{cases} \bar{E}^\top X_c = X_c^\top \bar{E} \geq 0, \\ (\bar{A}_{11} - \bar{B}_{12}(D_{12}^\top D_{12})^{-1}D_{12}^\top \bar{C}_{11})^\top X_c + X_c^\top(\bar{A}_{11} - \bar{B}_{12}(D_{12}^\top D_{12})^{-1}D_{12}^\top \bar{C}_{11}) \\ +\bar{C}_{11}^\top(I - D_{12}(D_{12}^\top D_{12})^{-1}D_{12}^\top)\bar{C}_{11} \\ +X_c^\top\left(\frac{1}{\gamma^2}\mathcal{B}\mathcal{B}^\top - \bar{B}_{12}(D_{12}^\top D_{12})^{-1}\bar{B}_{12}^\top\right)X_c = 0, \end{cases} \tag{6.30}$$

*where $\mathcal{B} = \bar{B}_{11} + U_iB_i$.*

*(iii) There exists an admissible solution $Y_o$ to the GARE*

$$\begin{cases} \hat{E}Y_o = Y_o^\top \hat{E}^\top \geq 0, \\ (\hat{A}_{22} - \hat{B}_{21}D_{21}^\top(D_{21}D_{21}^\top)^{-1}\hat{C}_{22})Y_o + Y_o^\top(\hat{A}_{22} - \hat{B}_{21}D_{21}^\top(D_{21}D_{21}^\top)^{-1}\hat{C}_{22})^\top \\ +\hat{B}_{21}(I - D_{21}^\top(D_{21}D_{21}^\top)^{-1}D_{21})\hat{B}_{21}^\top \\ +Y_o^\top\left(\frac{1}{\gamma^2}\mathcal{C}^\top\mathcal{C} - \hat{C}_{22}^\top(D_{21}D_{21}^\top)^{-1}\hat{C}_{22}\right)Y_o = 0, \end{cases} \tag{6.31}$$

*where $\mathcal{C} = C_oV_o - \hat{C}_{12}$.*

*(iv) The spectral radius $\rho(YX) < \gamma^2$, where*

$$X = \begin{bmatrix} I & U_i \end{bmatrix}^\top X_c \begin{bmatrix} I & V_i \end{bmatrix}, \quad Y = \begin{bmatrix} -V_o^\top & I \end{bmatrix}^\top Y_o \begin{bmatrix} -U_o^\top & I \end{bmatrix}.$$

*Moreover, the set of comprehensive $H_\infty$ controllers is parameterized by*

$$K_\infty = \mathcal{F}_l(J_\infty, Q_\infty), \quad \|Q_\infty\|_\infty < \gamma, \tag{6.32}$$

$$J_\infty = \left[\begin{array}{c|cc} A_\infty - sE & B_{1\infty} & B_{2\infty} \\ \hline C_{1\infty} & 0 & (D_{12}^\top D_{12})^{-1}D_{12}^\top \\ C_{2\infty} & D_{21}^\top(D_{21}D_{21}^\top)^{-1} & 0 \end{array}\right], \tag{6.33}$$

*where*

$$\begin{aligned}
A_\infty &= A + B_2C_{1\infty} - B_{1\infty}C_2 + \tfrac{1}{\gamma^2}(B_1 - B_{1\infty}D_{21})B_1^\top X,\\
B_{1\infty} &= Z^\top(C_2 + \tfrac{1}{\gamma^2}D_{21}B_1^\top X)^\top(D_{21}D_{21}^\top)^{-1} + B_1D_{21}^\top(D_{21}D_{21}^\top)^{-1},\\
B_{2\infty} &= (B_2 - Z^\top C_{1\infty}^\top)(D_{12}^\top D_{12})^{-1}D_{12}^\top,\\
C_{1\infty} &= -(D_{12}^\top D_{12})^{-1}(B_2^\top X + D_{12}^\top C_1),\\
C_{2\infty} &= -D_{21}^\top(D_{21}D_{21}^\top)^{-1}(C_2 + \tfrac{1}{\gamma^2}D_{21}B_1^\top X),\\
Z &= (I - \tfrac{1}{\gamma^2}YX)^{-1}Y.
\end{aligned}$$

To prove Theorem 6.4, we need Lemma 5.3 and Theorem 5.3 given in Chap. 5. For the purpose of convenience, we summarize the related results in the following lemma.

**Lemma 6.2** *Given G in (6.3), suppose that Assumptions (A0), (A3'), and (A4') hold.*

*(i) There exists a controller such that the closed-loop system is comprehensively admissible if and only if there exist $U_i$, $V_i$, $F_a$, $U_o$, $V_o$, and $L_o$ satisfying the generalized Sylvester equations (6.28) and (6.29).*

*(ii) All controllers making G comprehensively admissible is parameterized as*

$$K = \mathcal{F}_l(J, Q), \quad Q \in RH_\infty, \tag{6.34}$$

$$J := \left[\begin{array}{c|cc} A + B_2F + LC_2 - sE & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right], \tag{6.35}$$

*where*

$$F = \begin{bmatrix} F_c & F_a + F_cV_i \end{bmatrix}, \quad L = \begin{bmatrix} L_a - U_oL_o \\ L_o \end{bmatrix},$$

*with $F_c$ and $L_o$ such that $(\bar{E}, \bar{A}_{11} + \bar{B}_{12}F_c)$ and $(\hat{E}, \hat{A}_{22} + L_o\hat{C}_{22})$ are admissible, respectively.*

*Proof of Theorem* 6.4. Necessity: If there exists a controller $K$ solving Problem 6.2, according to Lemma 6.2, the generalized Sylvester equations (6.28) and (6.29) hold. Moreover, there exist admissible solutions $X_c$ and $Y_o$ to the GAREs (6.30) and (6.31), respectively, since Assumptions (A0)–(A2), (A3')–(A6') hold. Now we prove that the condition related to the spectral radius holds. With the parametrization of the controller $K$ given in Lemma 6.2, there holds

$$\mathcal{F}_l(G, K) = \mathcal{F}_l(G, \mathcal{F}_l(J, Q)) = \mathcal{F}_l(G_J, Q).$$

where,

$$G_J = \mathcal{F}_l(G, J) = \left[\begin{array}{cc|cc} A + B_2F - sE & B_2F & B_1 & B_2 \\ 0 & A + LC_2 - sE & -B_1 - LD_{21} & 0 \\ \hline C_1 + D_{12}F & D_{12}F & D_{11} & D_{12} \\ 0 & -C_2 & D_{21} & 0 \end{array}\right].$$

With $U_i$, $V_i$, $U_o$, and $V_o$ being solutions to (6.28) and (6.29), we define

$$M_1 = \left[\begin{array}{cc} I & U_i \\ \hline 0 & I \end{array}\right] \triangleq \left[\begin{array}{c} M_{11} \\ \hline M_{12} \end{array}\right], \quad N_1 = \begin{bmatrix} I & -V_i \\ 0 & I \end{bmatrix},$$
$$M_2 = \begin{bmatrix} I & U_o \\ 0 & I \end{bmatrix}, \quad N_2 = \left[\begin{array}{c|c} I & -V_o \\ 0 & I \end{array}\right] \triangleq \left[\begin{array}{c|c} N_{21} & N_{22} \end{array}\right].$$

Left- and right-multiplying $G_J$ by diag$(M_1, M_2)$ and diag$(N_1, N_2)$, respectively, lead to an alternative representation of $G_J$ as

$$G_J = \left[\begin{array}{cc|cc} \bar{A}_{11} + \bar{B}_{12}F_c - s\bar{E} & \bar{B}_{12}FN_{22} & \bar{B}_{11} + U_i\bar{B}_{21} & \bar{B}_{12} \\ 0 & \hat{A}_{22} + L_o\hat{C}_{22} - s\hat{E} & -\hat{B}_{21} - L_oD_{21} & 0 \\ \hline \bar{C}_1 + D_{12}F_c & D_{12}FN_{22} & D_{11} & D_{12} \\ 0 & -\hat{C}_{22} & D_{21} & 0 \end{array}\right]. \quad (6.36)$$

Since the system $G_J$ satisfies the standard assumptions [WYC06], the standard $H_\infty$ control problem associated with $G_J$ is solvable. From the realization in (6.36), we form an admissible solution to the underlying GARE as $X_J = diag(X_c, 0)$, where $X_c$ is an admissible solution to the GARE (6.30). Furthermore, let us define

$$\Xi = \begin{bmatrix} I & M_{11}\begin{bmatrix} -U_o \\ I \end{bmatrix} \\ 0 & I \end{bmatrix}, \quad \Pi = \begin{bmatrix} I & -\begin{bmatrix} I & V_i \end{bmatrix} N_{22} \\ 0 & I \end{bmatrix}.$$

By $\Xi$ and $\Pi$, the system $G_J$ can be represented alternatively as

$$G_J = \left[\begin{array}{cc|cc} \bar{A}_{11} + \bar{B}_{12}F_c - s\bar{E} & M_{11}L\hat{C}_{22} & -M_{11}LD_{21} & \bar{B}_{12} \\ 0 & \hat{A}_{22} + L_o\hat{C}_{22} - s\hat{E} & -\hat{B}_{21} - L_oD_{21} & 0 \\ \hline \bar{C}_1 + D_{12}F_c & \hat{C}_{12} - \hat{C}_{11}V_o & D_{11} & D_{12} \\ 0 & -\hat{C}_{22} & D_{21} & 0 \end{array}\right].$$

With this realization, we define $\hat{Y}_J = diag(0, Y_o)$, where $Y_o$ is an admissible solution to the GARE (6.31). Then, an admissible solution $Y_J$ to the underlying GARE with respect to this realization is formed by $Y_J = \Pi\hat{Y}_J\Xi^{-\top}$. Hence,

$$\rho(Y_JX_J) = \rho(\mathcal{N}_{11}N_{22}Y_o\mathcal{M}_{21}^\top M_{11}^\top X_o) = \rho(YX) < \gamma^2.$$

Sufficiency: Suppose that the conditions (i)–(iv) hold. We prove that there exists a controller $K$ solving Problem 6.2 and the controller $K$ can be parameterized by Theorem 6.4. To this end, define

$$F_c = -(D_{12}^\top D_{12})^{-1}(\bar{B}_{12}^\top X_c + D_{12}^\top \bar{C}_{11}),$$
$$L_o = -(Y_o^\top \hat{C}_{22}^\top + \hat{B}_{21} D_{21}^\top)(D_{21} D_{21}^\top)^{-1}.$$

Note that $X_c$ and $Y_o$ are admissible solutions to the GAREs (6.30) and (6.31), respectively. Hence, $(\bar{E}, \bar{A}_c)$, and $(\hat{E}, \hat{A}_o)$ are both admissible, where

$$\bar{A}_c = \bar{A}_{11} - \bar{B}_{12}(D_{12}^\top D_{12})^{-1} D_{12}^\top \bar{C}_{11} + \left(\frac{1}{\gamma^2}\mathcal{B}\mathcal{B}^\top - \bar{B}_{12}(D_{12}^\top D_{12})^{-1}\bar{B}_{12}^\top\right) X_c,$$
$$\hat{A}_o = \hat{A}_{22} - \hat{B}_{21} D_{21}^\top (D_{21} D_{21}^\top)^{-1}\hat{C}_{22} + Y_o^\top \left(\frac{1}{\gamma^2}\mathcal{C}^\top\mathcal{C} - \hat{C}_{22}^\top (D_{21} D_{21}^\top)^{-1}\hat{C}_{22}\right).$$

Furthermore, define

$$F_X = -(D_{12}^\top D_{12})^{-1}(B_2^\top X + D_{12}^\top C_1),$$
$$L_Y = -(Y^\top C_2^\top + B_1 D_{21}^\top)(D_{21} D_{21}^\top)^{-1}.$$

After simple computation, there holds

$$F_X = \begin{bmatrix} F_c & F_a + F_c V_i \end{bmatrix}, \quad L_Y = \begin{bmatrix} L_a - U_o L_o \\ L_o \end{bmatrix}.$$

Hence, $F_X$ and $L_Y$ satisfy Lemma 6.2. Then, we consider the system $G_J$ defined in (6.36), where $J$ is given in (6.35) with $F = F_X$ and $L = L_Y$. It has been shown that $X_J$ and $Y_J$ are admissible solutions to the underlying GAREs with respect to $G_J$, and $\rho(Y_J X_J) < \gamma^2$. Therefore, the standard $H_\infty$ control problem associated with $G_J$ is solvable and the set of $H_\infty$ controllers is parameterized by

$$\mathcal{K} = \mathcal{F}_l(\mathcal{J}, \mathcal{Q}), \quad \|\mathcal{Q}\|_\infty < \gamma,$$

where $\mathcal{J}$ is computed from (6.33) with respect to $X_J$ and $Y_J$. Then, we construct

$$K = \mathcal{F}_l(J, \mathcal{K}) = \mathcal{F}_l(\mathcal{F}_l(J, \mathcal{J}), \mathcal{Q}).$$

Hence,

$$\mathcal{F}_l(G, K) = \mathcal{F}_l(\mathcal{F}_l(G, J), \mathcal{K}) = \mathcal{F}_l(G_J, \mathcal{K}),$$

which indicates that $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$. Note that $G_J$ is admissible and $G_{J_{22}}$ induced from $G_J$ is zero. Hence, its $H_\infty$ controller $\mathcal{K}$ is admissible. According to Lemma 6.2, comprehensive admissibility for $\mathcal{F}_l(G, K)$ is achieved. Therefore, Problem 6.2 is

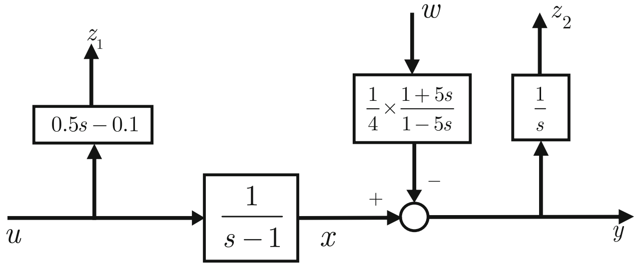


**Fig. 6.3** A mixed sensitivity problem

solvable. In addition, after straightforward computation, there holds

$$\mathcal{J} = \mathcal{F}_l(\Upsilon J^{-1}\Upsilon, J_\infty), \quad \Upsilon = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

□

*Example 6.2* Consider the mixed sensitivity problem in Fig. 6.3 that yields the transfer matrix from $w$ to $(z_1^T, z_2^T)^T$ as

$$T_{zw} = \begin{bmatrix} -W_1K(I - HK)^{-1}W_3 \\ -W_2(I - HK)^{-1}W_3 \end{bmatrix},$$

where $W_1 = 0.5s - 0.1$, $W_2 = 1/s$, $W_3 = (1 + 5s)/4(1 - 5s)$, $H = 1/(s - 1)$, and $K$ is the controller to be design. The objective here is to find $K$ such that $\|T_{zw}\|_\infty < 1$. This overall weighted system can be represented as in (6.3) by

$$E = \begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&5 \end{bmatrix}, \quad A = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&-1 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ -0.25 \\ 0 \\ -0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0&0.5&0&0&0 \\ 0&0&1&0&0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0&0&0&1&-1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \quad D_{21} = 0.25.$$

Solving the generalized Sylvester equations (6.28) and (6.29) gives

$$U_i = \begin{bmatrix} 0.16 \\ 0 \\ 0 \\ -0.2 \end{bmatrix}, \quad V_i = \begin{bmatrix} 0.8 \\ 0.16 \\ 0 \\ -1 \end{bmatrix}, \quad F_a = -0.8,$$

$$U_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_a = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Then, solving the GAREs (6.30) and (6.31) yields

$$X_c = \begin{bmatrix} 1.0466 & 0 & 0.5923 & 1.9480 \\ 2.1573 & 0 & 2.3153 & 5.3837 \\ 0.5923 & 0 & 1.8517 & 2.1968 \\ 1.9480 & 0 & 2.1968 & 4.9941 \end{bmatrix}, \quad Y_o = \begin{bmatrix} 0.2813 & 0.4688 \\ 0.0938 & 0.1563 \end{bmatrix},$$

which leads to

$$X = \begin{bmatrix} 1.05 & 0 & 0.60 & 1.94 & -1.11 \\ 2.16 & 0 & 2.3153 & 5.38 & -3.66 \\ 0.59 & 0 & 1.85 & 2.20 & -1.72 \\ 1.95 & 0 & 2.20 & 4.99 & -3.44 \\ -0.22 & 0 & -0.34 & -0.69 & 0.51 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.28 & 0.47 \\ 0 & 0 & 0 & 0.09 & 0.16 \end{bmatrix}.$$

Therefore, the central $H_\infty$ controller with $Q_\infty = 0$ is

$$K_\infty = \frac{759.796(s^2 + 0.09234s + 0.02681)}{s(1 - 5s)(s + 18.51)(s + 4.379)}$$

and the $H_\infty$ norm of the closed-loop system is $\|T_{zw}\|_\infty = 0.9972$. As expected, the controller contains poles at $s = 0$ and $s = 0.2$ for a perfect rejection of constant and unbounded disturbance specified by the weights $W_2$ and $W_3$. At the same time, the weight $W_1$ allows the control input to be divergent for counteracting the disturbance $W_3$.

## 6.4 Conclusion

In this chapter, we have discussed the performance control problem subject to comprehensive admissibility requirement. Two typical performance indices that is $H_2$ and $H_\infty$ norms are adopted. Due to the use of unstable and nonproper weighting filters, the so-called quasi-admissible solution is defined instead of conventional admissible solution to the underlying GAREs. It has been shown that the optimal comprehensive $H_2$ controller is conducted from quasi-admissible solutions to two GAREs, together with solutions to two generalized Sylvester equations that are closely related to comprehensive admissibility. As for the comprehensive $H_\infty$ control problem, the necessary and sufficient conditions to the existence of a comprehensive $H_\infty$ controller are deduced and a set of controllers is also explicitly parameterized in terms of two generalized Sylvester equations and two GARE together with a spectral radius condition.

It is worthy of noting that the solutions presented in the current paper require assumptions concerning the invariant zeros. These assumptions are indeed not necessary and can be removed through the use of the structured controller explicitly given in Theorem 5.3 in Chap. 5. With the structured controller, the comprehensive admissibility problem is transformed into a standard problem for an augmented system without weights. Hence, additional performance control objectives can be directly tackled for this augmented system through LMI-based approaches in the literature.

# Chapter 7
# Parametric Sensitivity Constrained LQ Control

Finding a parametric sensitivity constrained linear quadratic controller by including a quadratic trajectory sensitivity to the standard quadratic cost functional is still of major importance from a practical point of view. Since the seminal works of Kreindler [Kre69] and Newmann [New70], the system sensitivity to parameter variations has been handled in various ways through criterion sensitivity, closed-loop eigenvalues sensitivity or trajectory sensitivity measures [FN77, KB88]. Sensitivity to parameter variation remains a relevant controller and filter design criterion as attested by the large number of references on the subject. See, for instance, [AM13, LP00, Apk11, FCH11]. When the bounds on parameter deviation are not a priori known, it is still of interest to reduce the potential performance degradation due to uncertain parameter deviation with respect to some nominal values [GY14].

Moreover, such a problem can pave the way to a potentially new parametric sensitivity constrained $H_2$ control design owing to the well-known superposition principle. Even if many attempts have been carried out in the literature to solve this problem in the $H_2$ context [YC04, YC05], existing methods are still either computationally unwieldy or suffer from an augmentation of the controller order. In this chapter, we consider the problem of parametric sensitivity constrained linear quadratic (SCLQ) control for uncertain linear time-invariant systems by the use of descriptor systems. System sensitivity to parameter variation is handled through an additional quadratic trajectory parametric sensitivity in the standard LQ criterion to be minimized. The main purpose here is to find a suboptimal linear quadratic control law that takes explicitly into account the parametric uncertainties.

In a first step, it is shown that the SCLQ problem leads to a singular infinite horizon LQ optimal control problem, that is the matrix, conventionally denoted by $R$ weighting the input in the cost function is only positive semi-definite. Thus, relying on a descriptor system approach [BL87, ITS03], the link between the SCLQ control problem and a nonstandard Riccati equation with a pseudo-inverse of the weighting

 
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matrix $R$ is explicitly exhibited. Then, a new solution to the SCLQ problem is conducted based on a Lur'e matrix equations formulation of the underlying nonstandard Riccati equation.

Furthermore, we focus, in the last part of this chapter, on how to design a parsimonious partition of the uncertainty simultaneously with a set of SCLQ controllers to improve the total insensitivity to parametric variations while preserving, as far as possible, the classical robustness margins of the standard LQ controllers.

## 7.1 SCLQ Control Problem

Consider the uncertain linear system given by

$$\dot{x} = A(\theta)x + B(\theta)u, \quad x(0) = x_0, \tag{7.1}$$

where $A(\theta) \in \mathbb{R}^{n\times n}$ and $B(\theta) \in \mathbb{R}^{n\times m}$ are matrix functions of a time-invariant parameter vector $\theta = \left[\theta_1, \ldots, \theta_q\right] \in \mathbb{R}^q$.

We focus on a parameter dependence such that $A(\theta)$ and $B(\theta)$ are matrix functions with all entries of class $C^{n_\theta}$, $n_\theta \geq 1$. The system given by (7.1) is assumed to be controllable. Let us also define the trajectory sensitivity $x_\theta = \dfrac{\partial x}{\partial \theta}$ due to parametric deviation from a nominal value $\theta = \theta^0$. To simplify the presentation, only the first-order derivative term is considered in the sequel, although the results presented in this chapter can be easily extended to the higher order derivative case.

**Problem 7.1** (*SCLQ control problem*) The SCLQ control problem is to find a control law $u$ that minimizes the following linear quadratic cost functional

$$J_{SC} = \int_0^\infty x^\top Qx + u^\top Ru + x_\theta^\top Q_\theta x_\theta dt, \tag{7.2}$$

where $Q$ and $Q_\theta$ are positive semi-definite, and $R$ is positive definite.

The trajectory sensitivity function, when differentiating the equations in (7.1) with respect to $\theta$ is described by the following state-space equation

$$\dot{x}_\theta = A_\theta x + (I_q \otimes A)x_\theta + B_\theta u + (I_q \otimes B)u_\theta, \quad x_\theta(0) = 0, \tag{7.3}$$

with $A_\theta = \left.\dfrac{\partial A(\theta)}{\partial \theta}\right|_{\theta=\theta^0}$, $B_\theta = \left.\dfrac{\partial B(\theta)}{\partial \theta}\right|_{\theta=\theta^0}$ and $u_\theta = \dfrac{\partial u}{\partial \theta}$.

To solve the SCLQ problem, Fleming and Newmann used a full state feedback control law of the form [FN77]

$$u = Kx + Fx_\theta. \tag{7.4}$$

Clearly, in order to implement such a control law the trajectory sensitivity vector $x_\theta$ has to be simulated. Partially differentiating (7.4) with respect to $\theta$ gives

$$u_\theta = K x_\theta + F \frac{\partial x_\theta}{\partial \theta}. \tag{7.5}$$

Then, substituting (7.4) and (7.5) into (7.3) leads to

$$\dot{x}_\theta = (A_\theta + B_\theta K)\,x + (I_q \otimes B)F \frac{\partial x_\theta}{\partial \theta} + \left[(I_q \otimes A) + (I_q \otimes B)K + B_\theta F\right] x_\theta. \tag{7.6}$$

Thus, Fleming and Newmann proposed to neglect the second-derivative term $\frac{\partial x_\theta}{\partial \theta}$ in order to implement the control law (7.5). Unfortunately, neither optimality nor robustness of the resulting dynamic state feedback control was discussed due to this approximation. In fact, the real control law in this case is of the following form

$$u = \left\{ F \left[ s I_{nq} - (I_q \otimes A) + (I_q \otimes B)K + B_\theta F \right]^{-1} (A_\theta + B_\theta K) + K \right\} x. \tag{7.7}$$

Authors in [KB88] adopted a structured control law as follows

$$u = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\theta \end{bmatrix}. \tag{7.8}$$

Hence, the SCLQ problem can be formulated as an optimal structured constrained LQ problem that is to find a state feedback gain $K \in \mathbb{R}^{n \times m}$ such that the structured control law given by

$$\begin{bmatrix} u \\ u_\theta \end{bmatrix} = (I_{q+1} \otimes K) \begin{bmatrix} x \\ x_\theta \end{bmatrix} \tag{7.9}$$

minimizes an approximated objective function of the form

$$J = \int_0^\infty \left( \bar{x}^\top Q \bar{x} + \bar{u}^\top R \bar{u} \right) dt, \quad \bar{R} = \operatorname{diag}(R, \varepsilon I_m), \quad \bar{Q} = \operatorname{diag}(Q, Q_\theta), \tag{7.10}$$

with $\bar{x} = \begin{bmatrix} x \\ x_\theta \end{bmatrix}$, $\bar{u} = \begin{bmatrix} u \\ u_\theta \end{bmatrix}$, and $\varepsilon > 0$ being small enough, under the following constraints

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u}, \quad \bar{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ \bar{A} &= \begin{bmatrix} A & 0 \\ A_\theta & (I_q \otimes A) \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & 0 \\ B_\theta & (I_q \otimes B) \end{bmatrix}. \end{aligned} \tag{7.11}$$

Note that the cost functional $J$ is a standard LQ cost functional since the matrix $\bar{R} > 0$. This problem can be formulated as an optimization problem of a linear objective under some bilinear matrix inequality (BMI) constraints. However, the underlying optimization control problem is hard to be addressed due to the type of structure imposed by (7.9). One can note that the difficulty is accentuated in the case where higher order trajectory sensitivity is considered.

## 7.2 New Solution to SCLQ Control Problem

In this section, we investigate the singular LQ control problem through the SCLQ problem associated with the cost functional (7.2). Characterizing the solution to this singular LQ problem will lead to a new formulation of the SCLQ problem where the structure constraint (7.9) is simplified. Thereby, based on a connection to the Lur'e matrix equations, an alternative solution to the SCLQ problem is conducted.

Consider the singular LQ problem that is to find a control law minimizing the following objective function

$$\begin{aligned} J_s &= \int_0^\infty \left(\bar{x}^\top \bar{Q}\bar{x} + \bar{u}^\top \bar{R}\bar{u}\right) dt, \\ \bar{R} &= \operatorname{diag}(R, 0_m), \quad \bar{Q} = \operatorname{diag}(Q, Q_\theta). \end{aligned} \tag{7.12}$$

The following theorem gives a solution to this singular LQ problem that is closely related to the SCLQ problem defined previously.

**Theorem 7.1** *Let the matrix $X^* > 0$ be the maximal solution to the following non-standard Riccati equation*

$$\bar{A}^\top X + X\bar{A} - X\bar{B}\bar{R}^\dagger\bar{B}^\top X + \bar{Q} = 0, \tag{7.13}$$

*where $X \in \mathbb{R}^{n(q+1)\times n(q+1)} > 0$ is an unknown matrix. Then, all solutions to the singular LQ problem are given by*

$$\bar{u}^*(t) = -\bar{R}^\dagger\bar{B}^\top X^*\bar{x}(t) + V^\top\tilde{u}(t), \tag{7.14}$$

*where $\tilde{u} \in \ell_2$ is an arbitrary function and $V = \begin{bmatrix} 0 & I_m \end{bmatrix}$. Moreover, the optimal cost is given by $J_s^* = \bar{x}_0^\top X^*\bar{x}_0$.*

*Proof* Consider the following augmented descriptor system

$$(\Sigma): \begin{cases} E_a \begin{bmatrix} \dot{\bar{x}} \\ \dot{\xi} \end{bmatrix} = A_a \begin{bmatrix} \bar{x} \\ \xi \end{bmatrix} + (E_a\bar{x}_0)\, w + B_a\bar{u}, \\ z = C_a \begin{bmatrix} \bar{x} \\ \xi \end{bmatrix} + D_a\bar{u}, \end{cases} \tag{7.15}$$

where

$$E_a = \begin{bmatrix} I_{n(q+1)} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_a = \begin{bmatrix} \bar{A} & 0 \\ 0 & -I \end{bmatrix}, \quad B_a = \begin{bmatrix} \bar{B} \\ V \end{bmatrix},$$

$$C_a = \begin{bmatrix} \text{diag}\left(\bar{Q}^{1/2}, 0\right) \\ V \end{bmatrix}, \quad D_a = \begin{bmatrix} 0 \\ \text{diag}\left(R^{1/2}, -I\right) \end{bmatrix},$$

and $w$ is a virtual exogenous input. Thus, there holds

$$\begin{aligned}\int_0^\infty z^\top z dt &= \int_0^\infty \left[\bar{x}^\top \bar{Q}^{1/2} \;\; u^\top R^{1/2} \;\; 0\right] \begin{bmatrix} \bar{Q}^{1/2}\bar{x} \\ R^{1/2}u \\ 0 \end{bmatrix} dt \\ &= \int_0^\infty \bar{x}^\top \bar{Q}\bar{x} + u^\top R u dt = \int_0^\infty \bar{x}^\top \bar{Q}\bar{x} + \bar{u}^\top \bar{R}\bar{u} dt \\ &= J_s.\end{aligned}$$

According to Lemma 10 in [ITS03], the descriptor system $\Sigma$ (7.15) satisfies the following condition $\left[\bar{x}_0^\top \;\; 0\right] \text{Ker}\left(E_a^\top\right) = \{0\}$. Hence, the following static feedback gain

$$K^* = -\left(D_a^\top D_a\right)^{-1}\left(D_a^\top C_a + B_a^\top P\right) \tag{7.16}$$

minimizes $J_s$ with the matrix $P$ being a stabilizing solution to the following GARE

$$\begin{cases} E_a^\top P = P^\top E_a, \\ A_a^\top P + P^\top A_a + C_a^\top C_a - \left(C_a^\top D_a + P^\top B_a\right)\left(D_a^\top D_a\right)^{-1}\left(D_a^\top C_a + B_a^\top P\right) = 0. \end{cases} \tag{7.17}$$

Note that the sufficient solvability conditions [TK98] for the GARE (7.17) obviously hold for the descriptor system $\Sigma$. Let us now partition the matrix $P$ as

$$P = \begin{bmatrix} X & P_1 \\ P_2 & P_3 \end{bmatrix}, \quad P_3 \in \mathbb{R}^m.$$

Thus, the first equation of (7.17) leads to $X = X^\top$ and $P_1 = 0$. The second one gives

$$\bar{A}^\top X + X\bar{A} + \bar{Q} - \left(X\bar{B} + P_2^\top V\right)(\text{diag}(R, -I))^{-1}\left(X\bar{B} + P_2^\top V\right)^\top = 0, \tag{7.18}$$

$$P_2 + \left(I + P_3^\top\right) V (\text{diag}(R, -I))^{-1}\left(X\bar{B} + P_2^\top V\right)^\top = 0, \tag{7.19}$$

$$P_3 + P_3^\top + I + \left(I + P_3^\top\right) V (\text{diag}(R, -I))^{-1} V^\top (I + P_3) = 0. \tag{7.20}$$

From the Eq. (7.20), it follows that $P_3 = 0$. Hence, substituting $P_3 = 0$ into (7.19) yields

$$V\bar{B}^\top X = 0. \tag{7.21}$$

Note that the following equality holds

$$(X\bar{B})\,(\mathrm{diag}(R, I))^{-1}\,(X\bar{B})^\top = (X\bar{B})\,(\mathrm{diag}(R, I))^{-1}\,(X\bar{B})^\top + (X\bar{B})V^\top V(X\bar{B})^\top.$$

Therefore, the Eq. (7.18) leads to (7.13). Furthermore, according to (7.16) the optimal state feedback gain is given by

$$K^* = -\begin{bmatrix} R^{-1} & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \bar{B}^\top X^* & V^\top \end{bmatrix}.$$

Thus, by (7.21) there holds

$$\begin{aligned}
\bar{u}\,(t) &= K^*\begin{bmatrix} \bar{x} \\ \xi \end{bmatrix} \\
&= \begin{bmatrix} \left(\begin{bmatrix} -R^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ V \end{bmatrix}\right) \bar{B}^\top X^* & V^\top \end{bmatrix}\begin{bmatrix} \bar{x} \\ \xi \end{bmatrix} \\
&= \begin{bmatrix} -\bar{R}^\dagger \bar{B}^\top X^* & V^\top \end{bmatrix}\begin{bmatrix} \bar{x} \\ \xi \end{bmatrix} \\
&= -\bar{R}^\dagger \bar{B}^\top X^* \bar{x} + V^\top \xi.
\end{aligned}$$

Moreover, on account of the structure of the matrix $\bar{R}$, a control law given in (7.14) is optimal. Finally, according to Theorem 11 in [ITS03], the optimal cost in this case is given by $J_s^* = \bar{x}_0^\top E_a^\top P E_a \bar{x}_0 = \bar{x}_0^\top X^* \bar{x}_0$. □

Theorem 7.1 characterizes all optimal solutions of the singular LQ problem. Since this singular LQ problem can be viewed as a special case of the defined SCLQ control problem, a new solution to the latter is thus conducted. Nevertheless, neither the nonstandard Riccati Eq. (7.13) nor the GARE (7.17) is helpful to solve numerically the SCLQ problem because of the structured constraint (7.8). The following result provides a new LMI formulation of a suboptimal SCLQ control problem relying on the Lur'e matrix equations. Interested readers may refer, for instance, to [Rei11] and references therein for more information concerning the Lur'e matrix equations.

**Theorem 7.2** *Suppose that the following matrix equations are solved for* $X \in \mathbb{R}^{n(q+1)\times n(q+1)}$ *and* $K_0 \in \mathbb{R}^{m\times n}$

$$\begin{cases} \bar{A}^\top X + X\bar{A} + \bar{Q} = \mathrm{diag}\left(K_0^\top K_0, 0_{qn}\right), \\ X\bar{B} = \mathrm{diag}\left(K_0^\top R^{1/2}, 0_{qn\times m}\right), \\ X = X^\top > 0. \end{cases} \tag{7.22}$$

*Then* $u^* = -R^{-1/2}K_0 x$ *is an optimal solution to the SCLQ problem.*

*Proof* According to Theorem 7.1, the SCLQ problem admits $\bar{u}^*$ given by (7.14) as an optimal solution with $X^* > 0$ being the maximal solution to the nonstandard Riccati Eq. (7.13). Thus, $X^*$ is also a maximal solution to the following Lur'e equations

$$\begin{cases} \bar{A}^\top X + X\bar{A} + \bar{Q} = K_1^\top K_1, \\ X\bar{B} = K_1^\top L_1, \\ \bar{R} = L_1^\top L_1, \end{cases} \tag{7.23}$$

where $(K_1, L_1) \in \mathbb{R}^{p\times(q+1)n} \times \mathbb{R}^{p\times m}$ with $p \leq m$. Moreover, since $\bar{R} = \text{diag}\,(R, 0_m)$ and $R > 0$, we have $L_1 = \begin{bmatrix} R^{1/2} & 0 \end{bmatrix}$.

The solution to (7.23) implies that $K_1\bar{x} + L_1\bar{u} = K_1\bar{x} + R^{1/2}u = 0$. Note that a sufficient condition for obtaining a structured state feedback gain of the form (7.8) is $K_1 = \begin{bmatrix} K_0 & 0 \end{bmatrix}$. Hence, it is easy to see that (7.23) reduces to (7.22). The control law $u^* = -R^{-1/2}K_0 x$ is an optimal solution to the SCLQ problem since the choice $\tilde{u} = -\left[I_{q+1} \otimes \left(R^{-1/2}K_0\right)\right] x_\theta$ can be made without loss of the optimality according to Theorem 7.1. □

The condition (7.22) may be regarded as a necessary and sufficient condition to the existence of a structured solution to the optimal SCLQ problem. If the Eq. (7.22) does not admit a solution, then it is possible to find a suboptimal SCLQ controller. In fact, suppose that the symmetric matrix $X^*$ is the maximal solution to the Lur'e matrix Eq. (7.23). According to the LMI formulation of the LQ problem [BGFB94], solving the following optimization problem with a linear objective subject to LMI/LME (linear matrix equality) constraints

$$\begin{aligned} &\max_{X, K_0} \quad \bar{x}_0^\top X^* \bar{x}_0, \\ &\bar{A}^\top X + X\bar{A} + \bar{Q} \geq \text{diag}\left(K_0^\top K_0, 0_{qn}\right), \\ &X\bar{B} = \text{diag}\left(K_0^\top R^{1/2} 0_{qn\times m}\right), \\ &X = X^\top > 0, \end{aligned} \tag{7.24}$$

leads to a suboptimal SCLQ controller of the form $u^* = -R^{-1/2}K_0 x$.

The Lur'e equations allow formulating a suboptimal SCLQ problem as an optimization problem of linear objective under LMI/LME constraints while taking explicitly into account the structured constraints (7.8).

## 7.3 Multiple SCLQ Controller Design

In order to have a reduced number of tuning parameters, we use, hereafter, the well-known finite time controllability Gramian and some additional sensitivity reduction parameters associated with each parameter $\theta_i, i \in \{1, \ldots, q\}$. We adopt the following weighting matrices

$$\begin{aligned} &R = T_c \int_0^{T_c} \left(e^{At}B\right)\left(e^{At}B\right)^\top dt, \quad T_c > 0, \\ &Q = I_n, \\ &Q_\theta = \text{diag}\left(\sigma_1, \ldots, \sigma_q\right) \otimes I_n, \quad \sigma_i > 0. \end{aligned} \tag{7.25}$$

The set of tuning parameters $\sigma_i$ has a direct effect on the sensitivity reduction from a nominal value $\theta = \theta_0$ and the extent of the parametric area.

### 7.3.1 Problem Formulation

Assume that $N$ balls of predetermined, sufficiently small, radiuses $\delta_{\theta^j}$ centered on $\theta = \theta^j$, $j \in \{1, \ldots, N\}$, and given by

$$\Theta^{\theta^j}\left(\delta_{\theta^j}, \mathbb{R}^q\right) = \left\{\theta \in \mathbb{R}^q / \left\|\theta - \theta^j\right\|_2 \leq \delta_{\theta^j}\right\} \tag{7.26}$$

are uniformly distributed in the search space (i.e., parameter set). As an initialization step, it is supposed that

$$\forall j \in \{1, \ldots, N\}, 0 < \delta_{\theta_j} < \left(\varepsilon/2\right)\left\|\theta^j - \theta^{j+1}\right\|_2,$$

for some $0 < \varepsilon < 1$. For each nominal value, namely $\theta = \theta^j$, an SCLQ controller, minimizing a criterion $J^j$ of the form (7.12), is synthesized by means of the following set of tuning parameters $T_c$, $\sigma^j_{i\in\{1,\ldots,q\}} \in \mathbb{R}$ where $T_c$ is fixed and $\sigma^j_{i\in\{1,\ldots,q\}} = \delta_{\theta^j}$.

**Problem 7.2** (*Multiple SCLQ Control Problem*) The multiple SCLQ control problem is to determine a set of radiuses $\delta_{\theta^j}$ and tuning parameters $\sigma_i^j$, $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, N\}$ that solve the following optimization problem

$$\begin{aligned}
&\max_{\delta^j>0,\sigma_i^j,i\in\{1,\ldots,q\},j\in\{1,\ldots,N\}} \quad \sum_{j=1}^{N} \delta_{\theta^j} \\
&\max_j \quad M_g^j < \bar{M}_g \\
&\max_j \quad M_\phi^j < \bar{M}_\phi \\
&\max_j \quad M_r^j < \bar{M}_r \\
&\max_j \quad M_m^j < \bar{M}_m \\
&\sigma_{m_i}^j \geq \alpha\delta_{\theta^j}
\end{aligned} \tag{7.27}$$

where $\bar{M}_g$, $\bar{M}_\phi$, $\bar{M}_r$, and $\bar{M}_m$ are some predetermined upper bounds on different margins, $\alpha > 1$, $\sigma^j_{m_i} = \min_i \sigma_i^j$, and $M_g^j$, $M_\phi^j$, $M_r^j$, $M_m^j$ are the margins obtained with the SCLQ controllers minimizing criterions $J^j$ and synthesized by means of $T_c$ and $\sigma^j_{i\in\{1,\ldots,q\}}$.

The number of decision variables is $N(1+q)$. The margin constraints can be considered entirely or in part. In fact, in some cases only the constraints on $M_\phi$ and $M_r$ are needed, since $M_g$ and $M_m$ are slightly degraded. To reduce the number of decision variables, it is possible to substitute the last constraint in (7.27) by an equality constraint of the form $\forall i \in \{1, \ldots, q\}$, $\sigma_i^j = \alpha \delta_{\theta^j}$ with some given $\alpha > 1$. In other words, it is possible to limit the decision variables to the radiuses of the $N$ small balls given by (7.26). Moreover, the parameter $\alpha$ is used to reduce or expand the space search for the parametric sensitivity reduction tuning parameters and from thence it has a direct impact on the global computation time.

One can note that maximizing the criterion $\sum_{j=1}^{N} \delta_{\theta^j}$ implies a maximization of the sensitivity tuning parameters $\sigma_i^j$, $i \in \{1, \ldots, q\}$ which is, indeed, needed for the sensitivity reduction. In opposition, it is also the margin constraints in (7.27). Moreover, the sample generation problem consisting of generating real vector samples $\theta^j \in {}^q$, uniformly distributed in the search space $\Theta$, can be reduced to multiple random vector generation for which the technique reported in [CDT99] can be used.

### 7.3.2 PSO-Based Algorithm

Firstly introduced in [EK95], particle swarm optimization (PSO) is inspired by the social behavior, for instance, bird flocking or fish schooling. Let us consider the following optimization problem

$$\min_{x \in \Lambda} \quad f(x), \tag{7.28}$$

where particles are moving in the search space $\Lambda$.

$x_p^k$ and $v_p^k$ denote the position and velocity of particle $p$ at iteration $k$, respectively. Each particle is able to remember where it has found its best position, which is defined as $b_p^k = \arg\min(f(x))$, $x \in \{b_p^{k-1}, x_p^k\}$. Moreover, $V(x_p^k) \subset \{1, 2, \ldots, P\}$ denotes the set of "co-particles" of particle $p$ at iteration and $g_p^k = \arg\min f(x)$, $x \in \{b_i^k, i \in V(x_p^k)\}$ denotes the best position found by the co-particles of particle $p$ until iteration $k$.

The particles move in $\Lambda$ according to the following transition equations

$$\begin{aligned} v_p^{k+1} &= c_0.v_p^k + c_1 \circ (b_p^k - x_p^k) + c_2 \circ (g_p^k - x_p^k), \\ x_p^{k+1} &= x_p^k + v_p^{k+1}. \end{aligned} \tag{7.29}$$

In this equation $c_0$ is the inertia factor and $c_1$ and $c_2$ are random numbers in $[0, \bar{c}_1]$ and $[0, \bar{c}_2]$, respectively. To guarantee the convergence of the PSO algorithm, the choice of parameters $(c_0, c_1, c_2)$ is central [Tre03]. It is well known that, in case of a large number of decision variables, the PSO algorithm may suffer from undesirable convergence to local minima. This may be the case when dealing with the multiple SCLQ control synthesis. To overcome this difficulty, some recent PSO modified versions have been proposed. The underlying idea is to modify the rules (7.29) so as to bring a random movement toward the best particle. Particularly, a step is considered as a success if the best value found by the particles is improved and a failure otherwise. Then the number of consecutive successes and failures is used in the modified transition rule.

Following the lines of the algorithm presented in [dBE02], the new transition rules are considered as follows

$$\begin{aligned} v_p^{k+1} &= c_0.v_p^k + (g_p^k - x_p^k) + \rho^k(1 - 2r_{[0,1]}), \\ x_p^{k+1} &= x_p^k + v_p^{k+1}. \end{aligned} \tag{7.30}$$

where $r_{[0,1]}$ denotes a random vector in [0, 1]. The value of $\rho^k$ is updated at each iteration according to the following

$$\rho^{k+1} = \begin{cases} 2\rho^k & \text{if} \quad nb\_success > s_c, \\ 0.5\rho^k & \text{if} \quad nb\_failure > f_c, \\ \rho^k & \text{otherwise}, \end{cases} \tag{7.31}$$

where $nb\_success$ is the number of consecutive successes, $nb\_failure$ the number of consecutive failures, and $s_c$ and $f_c$ are some additional tuning parameters.

The following algorithm sketches in few lines the proposed method for solving the multiple SCLQ control problem.

**Algorithm 7.1**

Step 0 : Fix $T_c$, $\alpha > 1$, $\varepsilon < 1$, $N$, and a maximum iteration number $\bar{k}$.

Step 1 : Initialization

Generate $N$ real vector samples $\theta^j \in \mathbb{R}^q$, uniformly distributed in $\Theta$. Choose randomly $N$ parameters $\delta_{\theta^j}$ such that

$$0 < \delta_{\theta^j} < \frac{\varepsilon}{2}\left\|\theta^j - \theta^{j+1}\right\|_2.$$

Fix $\sigma^j_{i\in\{1,\ldots,q\}} = \delta_{\theta^j}$, $j \in \{1, \ldots, N\}$.

Step 2 : PSO Optimization

Associate the particle positions in the perturbed PSO algorithm, with the transition rule given by (31), to

$$x_p = \left[\delta^1, \ldots, \delta^N, \sigma_1^1, \ldots, \sigma_1^N, \ldots, \sigma_q^1, \ldots, \sigma_q^N\right].$$

Optimize (7.27) until a stopping criterion is verified or $\bar{k}$ is reached.

Step 3 : Uncertainty Set Covering Test

Generate randomly $N^{2q}$ points in $\Theta$. If all these points belong to $\bigcup_j \Theta^{\theta^j}$, then stop. Else, set $N = N + 1$ and go to Step 1.

*Example 7.1* Consider a second order system with a rational parametric dependence given by

$$\dot{x} = \begin{bmatrix} \theta^3 - 1 & 0 \\ 1 & \frac{-1}{\theta} \end{bmatrix} x + \begin{bmatrix} 1 \\ 2\theta \end{bmatrix} u, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\theta_0 = 1$. The optimal control law for the standard LQ problem with $Q = I_2$ and $R = 1$ is

$$u^* = -\begin{bmatrix} 0.8572 & 0.5571 \end{bmatrix} x,$$

when $\theta = \theta_0$. Here, we consider an SCLQ problem with a first and a second derivative of the trajectory sensitivity such as

$$J_{SC} = \int_0^\infty x^\top Q x + u^\top R u + x_\theta^\top Q_\theta x_\theta + x_{\theta\theta}^\top Q_{\theta\theta} x_{\theta\theta} dt,$$

with $x_{\theta\theta} = \dfrac{\partial x_\theta}{\partial \theta}$. The objective is to find a structured state feedback gain of the form

$$u = \begin{bmatrix} K & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\theta \\ x_{\theta\theta} \end{bmatrix}$$

minimizing the following objective function

$$J = \int_0^\infty \left(\bar{x}^\top \bar{Q} \bar{x} + \bar{u}^\top \bar{R} \bar{u}\right) dt, \quad \bar{x}^\top = \begin{bmatrix} x^\top & x_\theta^\top & x_{\theta\theta}^\top \end{bmatrix}, \quad \bar{u}^\top = \begin{bmatrix} u^\top & u_\theta^\top & u_{\theta\theta}^\top \end{bmatrix},$$
$$\bar{R} = \mathrm{diag}\,(R, 0)\,, \quad \bar{Q} = \mathrm{diag}\,(Q, Q_\theta, Q_{\theta\theta})\,,$$

under the following constraints

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}, \quad \bar{x}_0^\top = \begin{bmatrix} x_0^\top & 0_{1\times 2n} \end{bmatrix},$$
$$\bar{A} = \begin{bmatrix} A & 0 & 0 \\ A_\theta & (I_q \otimes A) & 0 \\ A_{\theta\theta} & 2A_\theta & (I_q \otimes A) \end{bmatrix}, \quad A_{\theta\theta} = \frac{\partial A_\theta}{\partial \theta},$$
$$\bar{B} = \begin{bmatrix} B & 0 & 0 \\ B_\theta & (I_q \otimes B) & 0 \\ B_{\theta\theta} & 2B_\theta & (I_q \otimes B) \end{bmatrix}, \quad B_{\theta\theta} = \frac{\partial B_\theta}{\partial \theta}.$$

Solving the associated LMI/LME problem, with $Q = I_2$, $Q_\theta = Q_{\theta\theta} = 0.1I_2$, and $R = 1$ gives

$$u^* = \begin{bmatrix} -1.5950 & 0.0098 \end{bmatrix} x.$$

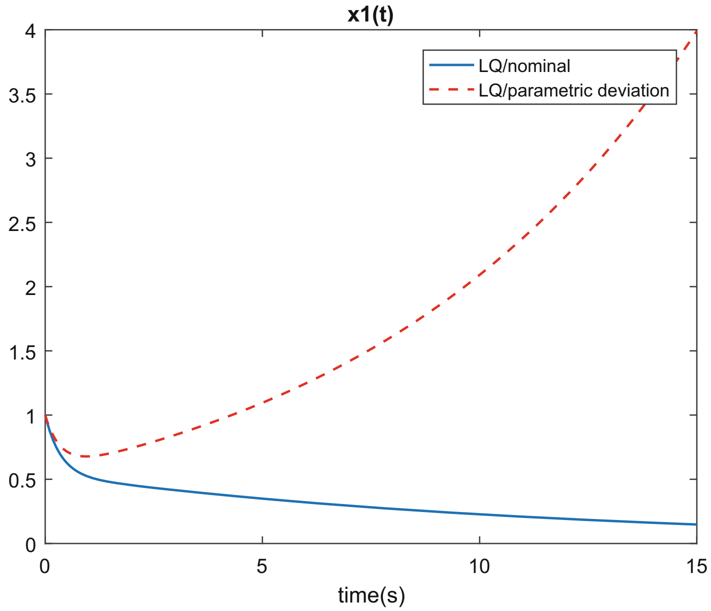


**Fig. 7.1** Trajectory of $x_1(t)$ with LQ controller

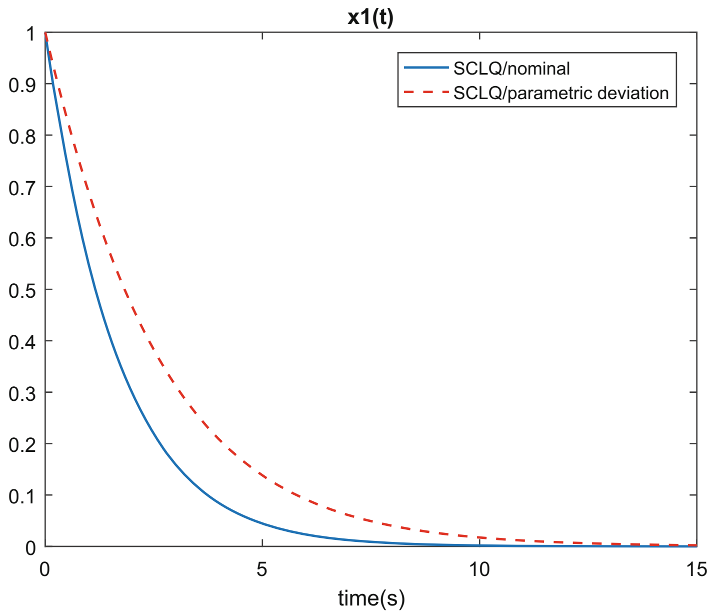


**Fig. 7.2** Trajectory of $x_1(t)$ with SCLQ controller

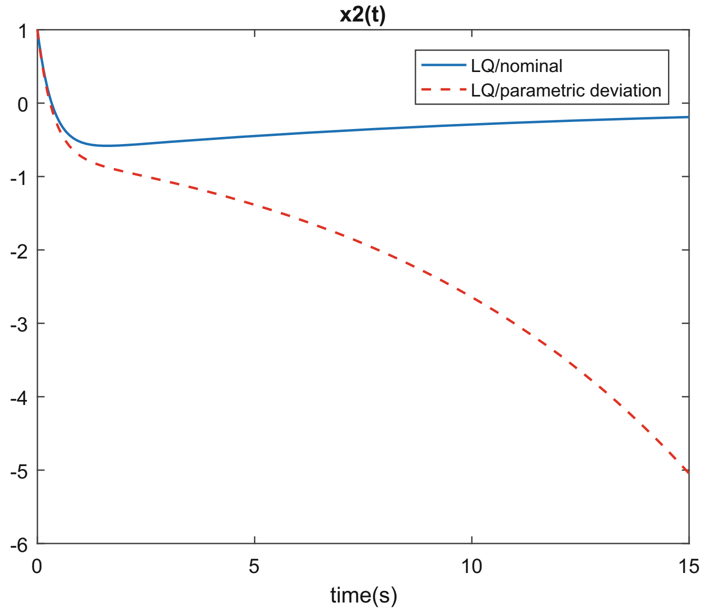


**Fig. 7.3** Trajectory of $x_2(t)$ with LQ controller

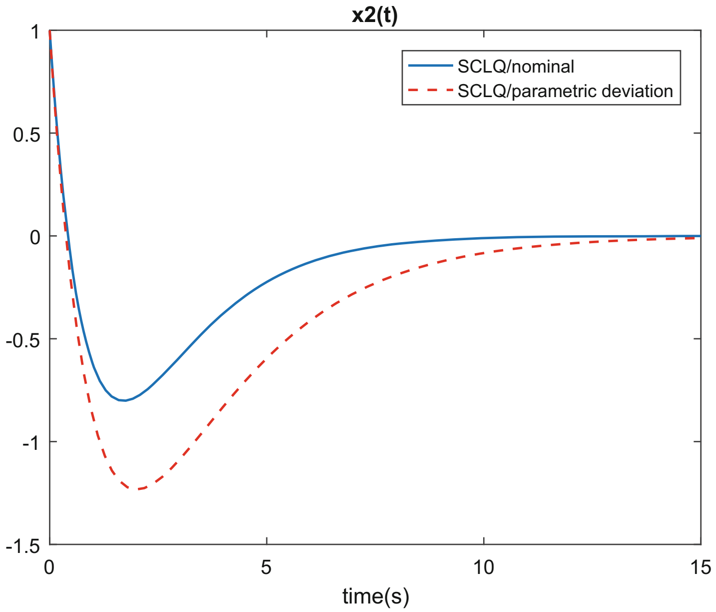


**Fig. 7.4** Trajectory of $x_2(t)$ with SCLQ controller

Figures 7.1, 7.2, 7.3 and 7.4 show performance comparison between the standard LQ and SCLQ controllers for $\theta = \theta_0$ (corresponding to solid lines in Figs. 7.3 and 7.4) and $\theta = 1.2$ (corresponding to dashed lines in Figs. 7.3 and 7.4). It is observed that the standard LQ controller leads to instability when $\theta = 1.2$; while the closed-loop system with the SCLQ controller is stable and only deviates slightly from the nominal value $\theta = \theta_0$ case.

## 7.4 Conclusion

This chapter shows a new necessary and sufficient condition for the existence of the optimal SCLQ controller, through solving a singular LQ control problem within the descriptor system framework. A suboptimal parametric SCLQ controller is then obtained by means of a computationally tractable optimization problem subject to some LMI/LME constraints.

Furthermore, a new synthesis method for multiple parametric SCLQ controllers is proposed. These controllers are designed to cover the entire parametric uncertainty set while degrading as less as possible the intrinsic robustness properties of each local linear quadratic controller. An adequate PSO-based algorithm is presented to find the best distribution of the local design regions simultaneously with the set of the sensitivity reduction tuning parameters.

# Chapter 8
# Concluding Remarks

Composed of eight chapters, this book develops some original results about dynamic LTI descriptor systems by following two different visions. The first one consists on generalizing some results from the classical state-space case to linear descriptor systems. The second approach uses the descriptor framework in order to solve, exactly or numerically, some robust and optimal control problems known in the literature and unsolved yet.

After recalling some basic concepts and fundamental results for LTI descriptor systems, dilated LMIs, also referred to as extended LMIs, for both the continuous-time and discrete-time settings are given. These conditions are unifying with respect to the existing tremendous LMI conditions with some auxiliary variables. Moreover, relied on the use of an additional positive scalar, a necessary and sufficient condition for the bounded real lemma for discrete-time descriptor systems is presented. A numerically efficient and reliable design procedure for state feedback $H_\infty$ controller design is given. Furthermore, the design of a measurement output feedback controller solving the problem of regulation constraints for descriptor systems is also discussed. A solvability condition is conducted in terms of a generalized Sylvester equation, together with a specific structured controller, and the additional dissipative performance objective is further tackled using an LMI-based approach.

In addition, on the control design aspects, this book reports a certain number of results associated with the use of input-output unstable and nonproper weighting filters for continuous-time descriptor systems. Stabilization, $H_2$ control and $H_\infty$ control under a new concept called comprehensive admissibility are dealt with. In addition to necessary and sufficient conditions, the structure of the resulting controller is explicitly conducted, which indicates that the comprehensive admissibility control problem can be transformed into a standard admissibility control problem for some augmented system without weights.

Finally, when reconsidering the well-known optimal control problem under parametric sensitivity constraints, one heads to a singular optimal control problem. A necessary and sufficient condition to the existence of optimal sensitivity constrained



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LQ controller is derived thanks to an augmented descriptor system. A numerically efficient and reliable design LMI/LME procedure is proposed for suboptimal cases.

Numerically, a certain number of tools, such as the generalized Sylvester equations, generalized algebraic Riccati equations, particle swarm optimization, are recalled and adopted to derive solutions to the examples used throughout the book.

Authors hope to pave the way to some promising and original developments on the use of the descriptor approach for solving multiobjective control problems or on generalizations of existing results to this broader class of systems.

# Appendix A
# Generalized Sylvester Equation

## A.1 Sylvester Matrix Equation

Many problems in systems and control theory are related to solvability of Sylvester equations. It is well known that these equations have important applications in stability analysis, observer design, output regulation problems and eigenvalue assignment [Tsu88, Doo84, FKKN85, Dua93, WH14]. In this appendix, we present some existing results for generalized Sylvester equations associated with continuous-time descriptor systems.

A matrix equation of interest in control theory is of the form,

$$\sum_{i=1}^{k} A_i X S_i = R, \tag{A.1}$$

where $A_i$, $S_i$ and $R$ are given matrices and $X$ is an unknown. In [Hau94], Hautus provided a detailed discussion on such equations while recalling historical origins of them.

A well known example of the linear matrix equation (A.1) is what is referred to as the Sylvester equation,

$$AX - XS = R, \tag{A.2}$$

where $A$ and $S$ are square matrices. As proved by Sylvester [Syl84], the Eq. (A.2) is universally solvable[1] if and only if the matrices $A$ and $S$ have no eigenvalues in common. A result for the general equation (A.1), in the same spirit as that of the Sylvester equation, is still not known. Thus, researchers restrict themselves to some special cases. For example, the authors in [Chu87, HG89, GLAM92] considered the solvability for the matrix equation of the form,

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[1]Equation (A.1) is said to be universally solvable if it has a solution for every $R$.



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$$AXB - CXD = E. \tag{A.3}$$

It is proved that the Eq. (A.3) has a unique solution if and only if the matrix pencils $A - \lambda C$ and $D - \lambda B$ are regular and the spectrum of one is disjoint from the negative of the spectrum of the other.

The generalized Sylvester equation of the form

$$\begin{aligned} AX - YB &= C, \\ DX - YE &= F, \end{aligned} \tag{A.4}$$

has also been studied in the literature, e.g. see [Ste73, KW89, Wim94]. It is shown that in the case where the matrices of (A.4) are real and $A$, $B$, $D$ and $E$ are square, the generalized Sylvester equation has a unique solution if and only if the polynomials $\det(A - sB)$ and $\det(D - sE)$ are coprime [Ste73]. With these assumptions, the authors in [KW89] deduced a solution of (A.4) by applying generalized Schur methods. Moreover, Wimmer extended Roth's equivalence theorem [Rot52] to a pair of Sylvester equations and concluded the following statement for the consistency of (A.4) without assumptions.

**Theorem A.1** ([Wim94]) *The Eq. A.4 has a solution X and Y if and only if there exist nonsingular matrices R and S with appropriate dimensions such that*

$$S\left[\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} - \lambda \begin{bmatrix} D & F \\ 0 & E \end{bmatrix}\right] = \left[\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \lambda \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}\right] R. \tag{A.5}$$

The above theorem can also be interpreted as the polynomial matrices $\begin{bmatrix} A - \lambda D & C - \lambda F \\ 0 & B - \lambda E \end{bmatrix}$ and $\begin{bmatrix} A - \lambda D & 0 \\ 0 & B - \lambda E \end{bmatrix}$ are unimodularly equivalent.

Within the descriptor framework, the generalized Sylvester equations have also been received wide attention from scholars. In [Dua96], Duan considered the generalized Sylvester matrix equation of the form

$$AV + BW = EVC, \tag{A.6}$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times r}$, $C \in \mathbb{C}^{p \times p}$, $E \in \mathbb{C}^{m \times n}$ $(p \leq n)$ are known, and $V \in \mathbb{C}^{n \times p}$ and $W \in \mathbb{C}^{r \times p}$ are to be determined. This equation is directly related to the eigenstructure assignment and observer design for linear descriptor systems. Based on the Smith canonical form of the matrix $[A - \lambda E \;\; B]$, the author provided a simple, direct complete and explicit parametric solution of (A.6) for the matrix $C$ in the Jordan form with arbitrary eigenvalues.

Moreover, combined with some rank and regional pole placement constraints, the following problem was investigated in [CdS05, Dar06].

**Problem A.1** *Consider a linear descriptor system represented by*

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{A.7}$$

*where* $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ *and* $u \in \mathbb{R}^m$ *are the descriptor variable, measured output and control input, respectively. The matrix* $E \in \mathbb{R}^{n\times n}$ *is such that* $\mathrm{rank}(E) = r < n$ *and* $p < r$*. Let* $\mathcal{D}$ *be a region in the open left-half complex plane,* $\mathcal{D} \subseteq \mathbb{C}^-$*, symmetric with respect to the real axis. Find matrices* $T \in \mathbb{R}^{(r-p)\times n}$*,* $Z \in \mathbb{R}^{(r-p)\times p}$ *and* $H \in \mathbb{R}^{(r-p)\times(r-p)}$ *such that*

$$TA - HTE = -ZC, \quad \sigma(H) \subset \mathcal{D}, \tag{A.8}$$

*under the rank constraint*

$$\mathrm{rank}\left(\begin{bmatrix} TE \\ LA \\ C \end{bmatrix}\right) = n, \tag{A.9}$$

*where* $L \in \mathbb{R}^{(r-p)\times n}$ *is any full row rank matrix satisfying* $LE = 0$*.*

The main motivation of solving this problem is directly concerned with the design of a reduced-order observer of minimal order $r - p$ under the form

$$\begin{aligned} \dot{z}(t) &= Hz(t) + TBu(t) - Zy(t), \\ \hat{x}(t) &= Sz(t) + \bar{N}\bar{y}(t) + Ny(t), \end{aligned} \tag{A.10}$$

where $z \in \mathbb{R}^{(r-p)}$ is the state of the observer and $\bar{y} \in \mathbb{R}^{(n-r)}$ is a fictitious output. As shown in [CdS05], if Problem A.1 is solved for some matrices $T$, $Z$ and $H$ and if we compute the matrices $S$, $\bar{N}$ and $N$ satisfying

$$\begin{bmatrix} S & \bar{N} & N \end{bmatrix} \begin{bmatrix} TE \\ LA \\ C \end{bmatrix} = I, \tag{A.11}$$

then, the corresponding minimal order observer given by (A.10) is such that

**(i)** the observer state verifies

$$\lim_{t\to\infty} [z(t) - TEx(t)] = 0, \quad \forall z(0), \quad Ex(0); \tag{A.12}$$

**(ii)** for $\bar{y}(t) = -LBu(t)$, the estimated state $\hat{x}(t)$ satisfies

$$\lim_{t\to\infty} [x(t) - \hat{x}(t)] = 0, \quad \forall x(0), \quad \hat{x}(0). \tag{A.13}$$

Note that for $L = 0$, Problem A.1 reduces to finding matrices $T \in \mathbb{R}^{(n-p)\times n}$, $Z \in \mathbb{R}^{(n-p)\times p}$ and $H \in \mathbb{R}^{(n-p)\times(n-p)}$ such that

$$TA - HTE = -ZC, \quad \sigma(H) \subset \mathcal{D}, \tag{A.14}$$

under the rank constraint

$$\text{rank}\left(\begin{bmatrix} TE \\ C \end{bmatrix}\right) = n. \tag{A.15}$$

These are indeed the conditions for the reduced-order observer design with order $n - p$, see for example [DB95, DZH96, Var95].

The solvability of Problem A.1 was deduced in terms of the concept of $\mathcal{D}$-strong detectability.

**Definition A.1** (*$\mathcal{D}$-strong Detectability*) The descriptor system (A.7) is $\mathcal{D}$-strongly detectable if and only if the following conditions are satisfied

**(1)** rank $\left(\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}\right) = n, \forall \lambda \in \mathbb{C}, \lambda \notin \mathcal{D},$

**(2)** rank $\left(\begin{bmatrix} E \\ LA \\ C \end{bmatrix}\right) = n.$

**Theorem A.2** ([CdS05, Dar06]) *There exist matrices $T \in \mathbb{R}^{(r-p)\times n}$, $Z \in \mathbb{R}^{(r-p)\times p}$ and $H \in \mathbb{R}^{(r-p)\times(r-p)}$ with $\sigma(H) \subset \mathcal{D} \subseteq \mathbb{C}^-$ solving Problem A.1, if and only if the descriptor system (A.7) is $\mathcal{D}$-strongly detectable and*

$$\text{rank}\left(\begin{bmatrix} LA \\ C \end{bmatrix}\right) = n - r + p. \tag{A.16}$$

## A.2 Considered Generalized Sylvester Equation

For a general case, we define the following matrix equation

$$\sum_{1\le i\le f, 1\le j\le k} \Phi_{ij}\Theta_j\Psi_{ij} = P_i, \tag{A.17}$$

where $\Phi_{ij}$, $\Psi_{ij}$ and $P_i$ are constant matrices with appropriate dimensions, and $\Theta_j$ is the matrix variable. It is worth pointing out that (A.17) can be regarded as a generalized Sylvester equation , which covers the aforementioned generalized Sylvester equations reported in [KW89, Chu87, HG89, GLAM92, Dua96].

For example, the Sylvester equation (A.3) can be obtained by setting $f = 1, k = 2$, $\Phi_{11} = A, \Phi_{12} = C, \Psi_{11} = -B, \Psi_{12} = D, P_1 = E$, and $\Theta_1 = \Theta_2 = X$ in (A.17); the

Eq. (A.4) can be viewed as (A.17) with $f = k = 2$, $\Phi_{11} = A$, $\Psi_{11} = I$, $\Phi_{12} = -I$, $\Psi_{12} = B$, $\Phi_{21} = D$, $\Psi_{21} = I$, $\Phi_{22} = -I$, $\Psi_{22} = E$, $P_1 = C$, $P_2 = F$, $\Theta_1 = X$ and $\Theta_2 = Y$; while the generalized Sylvester equation (A.6) can be regarded as (A.17) with $f = 1$, $k = 2$, $\Phi_{11} = \begin{bmatrix} A & B \end{bmatrix}$, $\Phi_{12} = \begin{bmatrix} E & 0 \end{bmatrix}$, $\Psi_{11} = I$, $\Psi_{12} = C$, $P_1 = 0$, and $\Theta_1 = \Theta_2 = \begin{bmatrix} V \\ W \end{bmatrix}$.

Now, we discuss briefly the solvability of a special case of (A.17). According to the properties of the Kronecker product, we have the following relationship

$$AXB = (B^\top \otimes A)\text{vec}(X). \tag{A.18}$$

Then, the matrix equation (A.17) can be written as

$$(\mathbf{N}^\top \otimes \mathbf{M})\text{vec}\left(\text{diag}(I_f \otimes \Theta_1, \ldots, I_f \otimes \Theta_k)\right) = \text{vec}(\mathbf{L}) \tag{A.19}$$

where

$$\mathbf{M} = \left[\, \text{diag}(\Phi_{11}, \ldots, \Phi_{f1}) \middle| \cdots \middle| \text{diag}(\Phi_{1k}, \ldots, \Phi_{fk}) \,\right], \tag{A.20}$$

$$\mathbf{N} = \left[\, \text{diag}(\Psi_{11}^\top, \ldots, \Psi_{f1}^\top) \middle| \cdots \middle| \text{diag}(\Psi_{1k}^\top, \ldots, \Psi_{fk}^\top) \,\right]^\top, \tag{A.21}$$

$$\mathbf{L} = \left[\, P_1^\top \cdots P_k^\top \,\right]^\top. \tag{A.22}$$

Now, we focus on the special case of the generalized Sylvester equation (A.17). Let us consider the following matrix equation

$$\begin{aligned} AXB - CYD &= E, \\ FXG - HYJ &= K, \end{aligned} \tag{A.23}$$

where $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $J$ and $K$ are known matrices with appropriate dimensions, and $X$ and $Y$ are the matrix variables to determine.

Using the Kronecker product, (A.23) can be written by

$$\begin{bmatrix} B^\top \otimes A & D^\top \otimes C \\ G^\top \otimes F & J^\top \otimes H \end{bmatrix} \begin{bmatrix} \text{vec}(X) \\ \text{vec}(Y) \end{bmatrix} = \begin{bmatrix} \text{vec}(E) \\ \text{vec}(K) \end{bmatrix}. \tag{A.24}$$

Clearly, the solution to this equation can easily be obtained through a linear program.

# Appendix B
# Generalized Algebraic Riccati Equation

In mathematics, a Riccati equation is named after the Italian mathematician Count Jacopo Francesco Riccati and is referred to any ODE that is quadratic in the unknown function. In systems and control theory, the term "Riccati equation" is used to refer to matrix equations with an analogous quadratic term that occur in both continuous-time and discrete-time linear quadratic Gaussian (LQG) control problems. The steady state version of these is referred to as the algebraic Riccati equation (ARE). The ARE is either of the following matrix equations: the continuous-time algebraic Riccati equation (CARE)

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0, \tag{B.1}$$

or the discrete-time algebraic Riccati equation (DARE)

$$A^\top PA - (A^\top PB)(R + B^\top PB)^{-1}(B^\top PA) + Q = P, \tag{B.2}$$

where $P \in \mathbb{R}^{n\times n}$ is the unknown symmetric matrix and $A$, $B$, $Q$, $R$ are known real coefficient matrices with appropriate dimensions. The ARE determines the solution of two of the most fundamental problems in control theory, namely, the infinite horizon time-invariant linear quadratic regulator (LQR) problem and the infinite horizon time-invariant LQG control problem. Comprehensive studies on the AREs in both continuous-time and discrete-time settings have been reported in the literature, for example, see [WAL84, LR95].

In this section, we study the continuous-time generalized algebraic Riccati equation (GARE) of the form

$$\begin{aligned} &E^\top P = P^\top E, \\ &A^\top P + P^\top A - (P^\top B + S)R^{-1}(P^\top B + S)^\top + Q = 0, \end{aligned} \tag{B.3}$$

where $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{m\times m}$ and $\text{rank}(E) = r \le n$. The above GARE is associated with the following continuous-time descriptor system

 
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$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{B.4}$$

Katayama and Minamino, in [KM92], proposed an asymmetric GARE to study the optimal LQR problem for descriptor systems. The GARE (B.3) can be viewed as the symmetric version of the one proposed by Katayama and Minamino. The authors in [WFC93] also used a symmetric GARE for the robustness properties of the LQR problem within the descriptor framework.

**Definition B.1** (*Admissible Solution*) A solution $P$ to the GARE (B.3) is called an admissible solution if $\big(E, A - BR^{-1}(B^\top P + S^\top)\big)$ is regular, impulsive and stable as well as $E^\top P \geq 0$.

It is noted that the admissible solution $P$ might not be unique, but $E^\top P$ is unique.

The $H_2$ and $H_\infty$ control problems for descriptor systems reported in [TK98, TMK94, KK97] are directly related to admissible solutions to the underlying GAREs. For example, for the $H_2$ control problem, we set

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \tag{B.5}$$

Furthermore, the $H_\infty$ control problem needs to solve the following $H_\infty$-like Riccati equation

$$\begin{aligned} &E^\top P = P^\top E, \\ &A^\top P + P^\top A + P^\top(\gamma^{-2} B_1 B_1^\top - B_2 B_2^\top)P + C^\top C = 0, \end{aligned} \tag{B.6}$$

where $\gamma \in \mathbb{R}^+$. The second equation can be rewritten in the GARE format as

$$A^\top P + P^\top A + P^\top \underbrace{\begin{bmatrix} B_1 & B_2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \gamma^{-2} & 0 \\ 0 & -I \end{bmatrix}}_{R^{-1}} \begin{bmatrix} B_1^\top \\ B_2^\top \end{bmatrix} P + \underbrace{C^\top C}_{Q} = 0. \tag{B.7}$$

The following assumptions are generally made for the existence of an admissible solution to the GARE.

**Assumption B.1**

**(A1)**. $(E, A)$ is regular;
**(A2)**. $D^\top D > 0$;
**(A3)**. $(E, A, B)$ is finite dynamics stabilizable and impulse controllable;
**(A4)**. $\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$ has no invariant zeros on the imaginary axis including infinity;

These assumptions are quite standard and coincide with the classical assumptions for conventional state-space systems [ZDG96]. Note that (A2) is widely made for state-space systems to guarantee a regular problem, i.e. one without zeros at infinity.

However, descriptor systems can still have zeroes at infinity, even if $D$ is full column rank. This condition is made here in order to deduce the controller in terms of the GARE. Moreover, Assumption (A2) can be made without loss of generality within the descriptor framework. If it does not hold, an equivalent realization satisfying this assumption can always be obtained [MKOS97, TK98]. Assumption (A3) is obviously essential to the existence of an admissible solution. Assumption (A4) is made to guarantee a regular problem. Moreover, for the state-space case where $E = I$, (A1) always holds.

Based on the generalized eigenvalue problem (GEP), numerical methods for solving the GARE (B.3) have been reported in [TMK94, TK98, KK97, KM92, WYC98]. Now we recall these processes. To this end, let us construct the Hamiltonian pencil of the form

$$H - \lambda J = \begin{bmatrix} A & 0 & B \\ -Q & -A^\top & -S \\ S^\top & B^\top & R \end{bmatrix} - \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{B.8}$$

with $\lambda \in \mathbb{C}$. Under Assumptions (A1)–(A4), $(J, H)$ is regular, impulse-free and has no finite dynamic modes on the imaginary axis including infinity. In addition, this matrix pencil contains $r$ stable finite eigenvalues, $r$ unstable eigenvalues, and $2n + m - 2r$ infinite eigenvalues. Let $\Lambda = [\Lambda_1^\top \ \Lambda_2^\top \ \Lambda_3^\top]^\top \in \mathbb{C}^{(2n+m)\times n}$ be the matrix consisting of the generalized eigenvectors and the generalized principal vectors related to the stable finite eigenvalues. We have

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix} \Delta = \begin{bmatrix} A & 0 & B \\ -Q & -A^\top & -S \\ S^\top & B^\top & R \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix}, \tag{B.9}$$

where $\Delta \in \mathbb{C}^{r\times r}$ is a Jordan form with all eigenvalues in the open left-half complex plane.

According to [TMK94], any admissible solution $P$ to the GARE (B.3) is given by

$$P = \begin{bmatrix} \Lambda_2 & W_2 P_r \end{bmatrix} \begin{bmatrix} \Lambda_1 & W_1 \end{bmatrix}^{-1}, \tag{B.10}$$

where $P_r$ satisfies

$$\begin{aligned} &A_r^\top P_r + P_r^\top A_r - (P_r^\top B_r + S_r) R^{-1} (P_r^\top B_r + S_r)^\top + Q_r = 0, \\ &A_r = W_2^\top A W_1, \quad B_r = W_2^\top B, \quad Q_r = W_1^\top Q W_1, \quad S_r = W_1^\top S, \end{aligned} \tag{B.11}$$

and $W_1 \in \mathbb{R}^{n\times(n-r)}$, $W_2 \in \mathbb{R}^{n\times(n-r)}$ are any full column rank matrices satisfying $EW_1 = 0$ and $E^\top W_2 = 0$, respectively.

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