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Atlantis Studies in Probability and Statistics
Series Editor: C.P. Tsokos

Mohammad Ahsanullah
Valery B. Nevzorov

Records via Probability Theory

Atlantis Studies in Probability and Statistics

Volume 6

Series editor

Chris P. Tsokos, Tampa, USA

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Atlantis Press
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Mohammad Ahsanullah · Valery B. Nevzorov

Records via Probability Theory

ATLANTIS PRESS

Mohammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ
USA

Valery B. Nevzorov
Department of Mathematics and Mechanics
Saint Petersburg State University
Saint Petersburg
Russia

ISSN 1879-6893 ISSN 1879-6907 (electronic)
Atlantis Studies in Probability and Statistics
ISBN 978-94-6239-135-2 ISBN 978-94-6239-136-9 (eBook)
DOI 10.2991/978-94-6239-136-9

Library of Congress Control Number: 2015942749

Printed on acid-free paper

To my parents and my wife, Masuda

Mohammad Ahsanullah

To my wife, Ludmila

Valery B. Nevzorov

Preface

"Record, Record, Record!", "The oldest Olympic Record was beaten!", "New World Record result in high-jumping!". Every day, we wait for such kind of headlines in newspapers, on the Internet, in TV announcements.

We hurry to call our friends and tell them about a new record value of the summer temperature in California, which was fixed yesterday, and about the highest level of today's earthquake somewhere in India.

We collect all editions of the Guinness Book of Records. Not only the Guinness Book but all tables of record results in various domains of human activities which are very popular among citizens all over the world.

Indeed, we regard any record result of known or unknown persons as the progress of all humankind (and, in particular, as our own progress).

It fell out that the record topic has become very popular among specialists in probability and mathematical statistics. A lot of papers (beginning with Chandler (1952)), in which various aspects of the mathematical theory of records were considered, have appeared during the past 60 years. Some monographs have been published on the records topic (see, for example, Ahsanullah (1995), Ahsanullah and Nevzorov (2001a), Arnold, Balakrishnan and Nagaraja (1998), Nevzorov (2000, 2001)).

Our readers can ask: "What is the reason for publishing a new book on records when there are so many monographs on this topic?"

The matter is that each coming year gives many new results on record times and record values. Some new record schemes have been suggested by various authors. A number of new methods for studying records have been presented in new publications. That is why we offer to our readers a book, which includes, in particular, some new materials about records. This monograph will help you to clear the situation with the modern theory of record values. You will find a lot of examples and exercises here, which will enable you to get a good experience of becoming familiar with the theoretical part of this theory.

The theory of records is connected very closely with the theory of order statistics. In some sense our book can be considered as the continuation of the Ahsanullah, Nevzorov, and Shakil (2013) edition, which provides an introduction to the world of order statistics. We suggest a lot of exercises, for which solutions are given at the end of the corresponding chapters. These exercises will help our readers to determine their level of comprehension of the material presented. Indeed, in order to understand the definitions and results given in the book, it is not necessary to read preliminarily any other book on order statistics. Some useful material from the theory of order statistics is presented in the first part of our book.

You are welcome to open the world of records for yourself!

With great pleasure we recall our friends N. Balakrishnan, P. Deheuvels, H.N. Nagaraja to whom we express our deep gratitude. The exchange of ideas and the joint work with our above-mentioned colleagues are of great importance to our scientific and publishing activities. We are also grateful to our family members for their constant encouragement and support.

We wish to express our gratitude to Prof. Chris Tsokos for valuable suggestions about the manuscript. The first author thanks Z. Karseen and K. Jones for interesting discussions at a meeting in Athens, Greece, for the publication of this book.

The work of the first author was partially supported by a summer research grant and sabbatical leave from Rider University. The work of the second author was partially supported by Saint Petersburg State University grant N 6.38.672.2013.

Mohammad Ahsanullah
Valery B. Nevzorov

Contents

Chapter 1
Introduction

Let x_1, x_2, …, x_n denote results of n participants, which were registered in some sport competition. These values can be presented in the increasing order as $x_{1,n} \le x_{2,n} \le \cdots \le x_{n-1,n} \le x_{n,n}$, where $x_{1,n} = \min\{x_1, x_2, \ldots, x_n\}$ and $x_{n,n} = \max\{x_1, x_2, \ldots, x_n\}$. In some competitions (take, for example, any running distance) $x_{1,n}$, $x_{2,n}$ and $x_{3,n}$ are correspondingly the results of the gold, silver and bronze prizewinners. For other type of competitions (say, for high jumping or long jumping) $x_{n,n}$, $x_{n-1,n}$ and $x_{n-2,n}$ are the best results. Indeed, after finishing this competition we deal with some concrete values x_1, x_2, …, x_n and $x_{1,n}$, $x_{2,n}$, …, $x_{n,n}$. Before the competition, the future results of the participants are unknown to us, and we can consider these results as random values X_1, X_2, …, X_n. Indeed, values $X_{1,n} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{n,n} = \max\{X_1, X_2, \ldots, X_n\}$, as well as other ordered values $X_{2,n} \le \cdots \le X_{n-1,n}$, are random. Up to the beginning of the competition all sport newspapers will discuss the probable realizations of random values $X_{1,n} \le \cdots \le X_{n,n}$ and the chances of a particular participant to become the winner, i.e. his/her chances to reach the result $X_{1,n}$ (or $X_{n,n}$).

This simple example shows the necessity of knowing how to work with the so-called order statistics $X_{1,n} \le \cdots \le X_{n,n}$ and their realizations $x_{1,n} \le \cdots \le x_{n,n}$.

Below some definitions connected with order statistics are given.

Let X_1, X_2, …, X_n be initial random variables. The set of the observed values $\{x_1, x_2, \ldots, x_n\}$ of random variables X_1, X_2, …, X_n is called a realization of these X's. In the most part of the book we suppose that X_1, X_2,…, X_n are independent identically distributed (i.i.d.) random variables, or simply we can say in this situation that X_1, X_2, …, X_n present n independent observations on X where X is a random variable having a certain distribution function (d.f.) F.

Then the combination $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ *denotes the variational series based on random variables* X_1, X_2, …, X_n. *If* X's *are independent and identically distributed one can say that* $X_{1,n} \le X_{2,\mathrm{n}} \le \cdots \le X_{n,n}$ *is the variational series based on a sample* X_1, X_2,…, X_n.

Elements $X_{k,n}$, $1 \le k \le n$, *are called order statistics* (*order statistics based on a sample* X_1, X_2, …, X_n*; order statistics from d.f. F; ordered observations on* X). *We*

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DOI 10.2991/978-94-6239-136-9_1

denote the observed values of $X_{1,n}$, $X_{2,n}$, …, $X_{n,n}$ *as above,* $x_{1,n}$, $x_{2,n}$, …, $x_{n,n}$, *and call them realizations of order statistics. Let us note that* $X_{1,n} = m(n) = \min\{X_1, X_2, \ldots, X_n\}$ *and* $X_{n,n} = M(n) = \max\{X_1, X_2, \ldots, X_n\}$, $n = 1, 2, \ldots$. *Rather natural is the following equality*:

$$X_{1,n} + X_{2,n} + \cdots + X_{n,n} = X_1 + X_2 + \cdots + X_n.$$

Together with the sample $X_1, X_2, \ldots, X_n$ *it is naturally to consider the empirical* (*or sample*) *distribution function*

$$F_n^*(x) = \frac{1}{n}\sum_{k=1}^{n} 1_{\{X_k \le x\}}.$$

Here $1_{\{X \le x\}}$ *is a random indicator, which equals to* 1 *if* $X \le x$ *and to* 0 *if* $X > x$.

Let us mention that $F_n^*(x)$ *can be expressed in terms of order statistics* $X_{k,n}$ *as follows:*

$$\begin{aligned} &F_n^*(x) = 0,\ \mathit{if}\ x < X_{1,n},\\ &F_n^*(x) = k/n,\ \mathit{if}\ X_{k,n} \le x < X_{k+1,n}, 1 \le k \le n-1, \end{aligned}$$

and

$$F_n^*(x) = 1,\ \mathit{if}\ x \ge X_{n,n}.$$

Usually a random sample $X_1, X_2, \ldots, X_n$ *is accompanied by the corresponding vector of ranks* $(R(1), R(2), \ldots, R(n))$, *where*

$$R(m) = \sum_{k=1}^{n} 1_{\{X_m \ge X_k\}},\ m = 1, 2, \ldots, n.$$

These ranks provide the following equalities for events:

$$\{R(m) = k\} = \{X_m = X_{k,n}\},\ m = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n.$$

Together with ranks we can use the so-called antiranks $\varDelta(1), \varDelta(2), \ldots, \varDelta(n)$, *which are defined by equalities*

$$\{\varDelta(k) = m\} = \{X_{k,n} = X_m\},\ m = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n.$$

One more type of ranks is presented by sequential ranks. For any sequence of random variables $X_1, X_2, \ldots$ *we introduce sequential ranks* $\rho(1), \rho(2), \ldots$ *as follows*:

$$\rho(m) = \sum_{k=1}^{m} 1_{\{X_m \ge X_k\}},\ m = 1, 2, \ldots.$$

Sequential rank $\rho(\mathrm{m})$ *shows the position of a new coming observation* X_m *among its predecessors* $X_1, X_2, \ldots, X_{m-1}$. *If independent random variables* $X_1, X_2, \ldots, X_m$ *have the same continuous distribution then it is possible to see that for any* $m = 1, 2, \ldots$

$$\mathrm{P}\{\rho(\mathrm{m}) = \mathrm{k}\} = \mathrm{P}\left\{\mathrm{X_m} = \mathrm{X_{k,m}}\right\} = 1/\mathrm{m},\ \mathrm{k} = 1, 2, \ldots, \mathrm{m}.$$

Here we use the fact that if X's are independent and have continuous distributions then any two of them can coincide with zero probability and the situation of symmetry, which provides that all m events $\{X_m = X_{1,m}\}, \ldots, \{X_m = X_{m,m}\}$ have the same probability.

The more complicate theory of order statistics and all types of ranks can be found in Ahsanullah and Nevzorov (2001a, 2005), Ahsanullah, Nevzorov and Shakil (2013), Arnold and Balakrishnan (1989). In Chap. 2 we will present some results for order statistics, which our reader will recall working with record times and record values.

Now let us come back to the results of the participants of some sport distance (say, 100 m running). Each year hundreds of competitions are organized, in which thousands of sportsmen run 100 m. Even the most serious lover of the field athletics cannot get and investigate all the results. Indeed, it is possible to operate with the results of participants of the Olympic Games and World championships but it is impossible to have information about participants of all these competitions.

Meantime there are the most interesting results, which can be easily found in a number of sport editions—world records, records of Olympic Games, continental and countries record values. Indeed, sport records are very popular, but record values in any domain of human activities are also interesting for millions of citizens worldwide.

Let us come back to the sequence of random variables $X_1, X_2, \ldots$. There are two classical types of record values—upper and lower records. We say that X_k is the upper record value if

$$X_k > M(k-1) = max\{X_1, X_2, \ldots, X_{k-1}\},\ k = 2, 3, \ldots,$$

and X_k is the lower record value if

$$X_k < m(k-1) = min\{X_1, X_2, \ldots, X_{k-1}\},\ k = 2, 3, \ldots.$$

In the both situations X_1 can be considered as the first record value (upper or lower). In some sense the theory of lower records is analogous to the theory of upper records. Really, if we can operate with upper records in the sequence $X_1, X_2, \ldots$, then the corresponding results can be transformed easily for the sequence $Y_1 = -X_1$, $Y_2 = -X_2, \ldots$. Really, in this situation the upper records for X's are transformed to the lower records for Y's. Hence it is enough to investigate the upper records only. The theory of upper records will be given below. Sometimes we will formulate the corresponding results for the lower records.

Usually when X's have continuous distributions it is not necessary to take into account the situations when a new coming observation coincides with the previous record value. The matter is that in this case any two X's can coincide with zero probability. For the case when X's have discrete distributions and some X's can coincide with nonzero probabilities, we will consider two options—strong records and weak records. We deal with a strong record if a new coming X is more than the last record value. In some schemes we announce as a record the observation, which is more (or even equal!) than the previous record value. In such situation we deal with the so-called weak records.

Note that the investigation of records in sequences of random variables having continuous or discrete distributions requires different methods. The most part of the book will be devoted to the situation when initial random variables have continuous d.f.'s and in one chapter (Chap. 6) we will deal with discrete distributions of X's.

Upper record values will be denoted as $X(1), X(2), \ldots, X(n), \ldots$ and lower records will be denoted as $x(1), x(2), \ldots, x(n)$.

We will use indicators ξ_k, such that $\xi_k = 1$, if X_k is an upper record value, and $\xi_k = 0$ otherwise. Note that then $N(n) = \xi_1 + \xi_2 + \cdots + \xi_n$ denote the number of the upper records among random variables $X_1, X_2, \ldots, X_n$, $n = 1, 2, \ldots$.

Together with record values we will investigate record times. We use the symbols $L(n)$ and $l(n)$, $n = 1, 2, \ldots$, correspondingly for upper and lower record times. Note that the following relations are valid for the upper record times:

$$L(1) = 1, \{L(n) = m\} = \{N(m-1) < n, N(m) = n\}, X_{L(n)} = M(L(n)) = X(n).$$

The analogous equalities are valid for lower record times $l(n)$.

If we recall sequential ranks $\rho(n)$, $n = 1, 2, \ldots$, then one can see that X_n is the upper record value if $\rho(n) = n$ and X_n is the lower record value if $\rho(n) = 1$.

Hence it is not difficult to see that the subject of record values and record times is tied very closely with the order and rank statistics.

See Chap. 2 for some useful definitions and results connected with order statistics.

Definitions, distributions and moment characteristics of record values are given in Chap. 3. In this chapter we also introduce some generalizations of the classical records—the so-called k-th record values.

Different results for record times are presented in Chap. 4.

There are some continuous distributions being the most popular in the probability theory, such as normal, uniform, exponential, logistic, Weibull. We discuss the behavior of record values in the sequences of random variables having these and some other popular types of probability distributions in Chap. 5.

Indeed, we do not forget that when we operate with records for random variables with discrete distributions we are to use a technique, which differs from the method that helps to work with records in the case of continuous distributions. The corresponding methods for discrete distributions are discussed in Chap. 6. In particular, our reader will find here some important formulae for records generated by geometrically distributed random variables. Also she/he will understand the difference between strong and weak record values.

As mentioned above, sometimes we can deal with some record values $x(1) < x(2) < \cdots < x(m)$ only (instead of all observations $x_1, x_2, \ldots, x_n$), and we must apply this set of record observations to solve some statistical problems; for example, very often we need estimating parameters of the sample distribution. This problem is discussed in Chap. 7. Some useful methods which allow us to predict the values of future records are also given there.

In one more part of the book (Chap. 8) our reader will find different characterizations of distributions by properties of record values. Some characterizations based on regression equalities for records, on independence properties of record statistics, on some moment relations for record values are presented there.

Indeed, almost all formulae for distributions and moment characteristics of $X(n)$, $L(n)$, $N(n)$ are very complicate (especially under large values n). In this situation, different limit theorems (when n tends to infinity) allow us to get the simple asymptotic expressions for these distributions and the corresponding moments. Different asymptotic results for distributions of records can be found in Chap. 9.

It was announced above that one of the reasons to write this book is to present the new results for records. The great part of these results is connected with some non-classical record schemes. We discuss the "fresh" record models in Chap. 10. For example, records in the F^{α}–scheme, records with restrictions, records with confirmation and the so-called linear draft model are presented there.

Chapter 2
Order Statistics

2.1 Order Statistics and Their Distributions

Let X_1, X_2,. . ., X_n, . . . be some random variables and let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ denote the corresponding variational series based on random variables X_1, X_2,. . ., X_n. Elements $X_{k,n}$, 1 ≤ k ≤ n, are called *order statistics*. Observed values of $X_{1,n}$, $X_{2,n}$,. . ., $X_{n,n}$. we denote $x_{1,n}$, $x_{2,n}$,. . ., $x_{n,n}$ and call a realization of order statistics. Let us note that

$$X_{1,n} = m(n) = \min\{X_1, X_2, \ldots, X_n\}$$

and

$$X_{n,n} = M(n) = \max\{X_1, X_2, \ldots, X_n\},\ n = 1, 2, \ldots.$$

In the classical theory of order statistics (see, for example, David and Nagaraja (2003), Ahsanullah et al. (2013)) the initial X's present independent and identically distributed random variables. Below we in the most part of situations also will deal with such types of random variables.

Hence consider now independent random variables X_1, X_2,. . ., X_n with some common distribution function (d.f.) F. It enables us to consider the set $\{X_1$, X_2,. . ., $X_n\}$ as a sample of size n taken from the population distribution F. The set of the observed values $\{x_1$, x_2,. . ., $x_n\}$ of random variables X_1, X_2,. . ., X_n is called a realization of the sample. In this case elements $X_{k,n}$, $1 \le k \le n$, that is order statistics based on sample

$X_{1,n} \le X_{2,n} \le \ldots \prec X_{n,n}$ can be identified as order statistics from this d.f. F.

It follows from the definition of order statistics that

$$P\{X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}\} = 1. \qquad (2.1.1)$$

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DOI 10.2991/978-94-6239-136-9_2

Moreover it is easy to show that if F is a continuous d.f. then equality (2.1.1) can be sharpened and written as

$$P\{X_{1,n} < X_{2,n} < \cdots < X_{n,n}\} = 1. \tag{2.1.2}$$

Indeed, if X's have a discrete d.f. then (2.1.2) is not true. Consider the corresponding example in Exercises 2.1.1 and 2.1.2.

Exercise 2.1.1 Let $X_1, X_2, \ldots, X_n$ be independent random variables taking two values 0 and 1 with probabilities ½. Find

$$p_n = P\{X_{1,n} = X_{2,n} = \cdots = X_{n,n}\},\ n = 1, 2, \ldots,$$

to convince yourself that $p_n \neq 0$.

Exercise 2.1.2 Let X_1, X_2, X_3 be independent random variables, having the geometric distribution, that is

$$P\{X = m\} = (1-p)p^m,\ m = 0, 1, 2, \ldots.$$

Find

$$P\{X_{1,3} < X_{2,3} < X_{3,3}\}.$$

As one sees there are two cases (F is continuous and F is discrete) which need different approach for writing distributions of order statistics.

Let us suppose below that X's are independent random variables having a common continuous d.f. F.

There are rather simple formulae for order statistics $X_{1,n}$ and $X_{n,n}$. Really,

$$\begin{aligned} F_{n,n}(x) = P\{X_{n,n} \le x\} &= P\{X_1 \le x, \ldots, X_n \le x\} \\ &= P\{X_1 \le x\} \ldots P\{X_n \le x\} = F^n(x), \end{aligned} \tag{2.1.3}$$

where

$$F(x) = P\{X_k \le x\}, k = 1, 2, \ldots, n.$$

Analogously we obtain that

$$\begin{aligned} F_{1,n}(x) = P\{X_{1,n} \le x\} &= 1 - P\{X_{1,n} > x\} \\ &= 1 - P\{X_{1\,n} > x, \ldots, X_n > x\} = 1 - (1 - F(x))^n. \end{aligned} \tag{2.1.4}$$

Let us note that (2.1.3) and (2.1.4) are valid for any d.f. F (continuous and discrete).

Exercise 2.1.3 Find the joint d.f.

$$F_{1,n,n}(x,y) = P\{X_{1,n} \le x, X_{n,n} \le y\}$$

of order statistics $X_{1,n}$ and $X_{n,n}$.

Exercise 2.1.4 Consider the case *when* $X_1, X_2, \ldots, X_n$ are independent random variables with probably different d.f.'s $F_1, F_2, \ldots, F_n$ and find in this case the d.f.'s of $m(n) = \min\{X_1, X_2, \ldots, X_n\}$ and $M(n) = \max\{X_1, X_2, \ldots, X_n\}$.

If our initial X's have probability density function (pdf) $f(x)$ then it is the following method to write the pdf $f_{k,n}(x)$ and d.f. $F_{k,n}(x)$ of $X_{k,n}$, $1 \le k \le n$. To find pdf we must consider the situation, when one of our n X's takes value x and exactly $k - 1$ of the rest X's are less than x. Thus we get easily that

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1}(1-F(x))^{n-k}f(x). \tag{2.1.5}$$

Then

$$\begin{aligned} F_{k,n}(x) &= \int_0^x f_{k,n}(y)dy = \frac{n!}{(k-1)!(n-k)!}\int_0^x (F(y))^{k-1}(1-F(y))^{n-k}f(y)dy \\ &= I_{F(x)}(k, n-k+1), \end{aligned} \tag{2.1.6}$$

where

$$I_x(a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$$

denotes the incomplete beta function.

Note that equality

$$F_{k,n}(x) = I_{F(x)}(k, n-k+1), \tag{2.1.7}$$

which is given here, was proved only for X's having some pdf $f(x)$.

What shall we do in the general situation? How to prove that (2.1.6) is valid for any d.f. F? It appears that it is enough to take into account the following two facts.

(a) If we consider the uniform $U(0,1)$ sample $U_1, U_2, \ldots, U_n$ with

$$P\{U_k \le x\} = x,\ 0 < x < 1,\ k = 1, 2, \ldots, n,$$

and the corresponding order statistics $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$, then it follows from (2.1.7) that

$$P\{U_{k,n} \le x\} = I_x(k, n-k+1), 0<x<1. \tag{2.1.8}$$

(b) Now if we compare the uniform $U_{k,n}$ order statistics and the order statistics $X_{k,n}$ from any d.f. $F(x)$, then it is known that $X_{k,n}$ can be presented as follows:

$$G(U_{k,n}) = X_{k,n}(\text{in distribution}), \tag{2.1.9}$$

where G is the inverse of F.

Hence it follows immediately from (a) and (b) that relation (2.1.7) is valid for any d.f. F.

When we want to consider joint distributions of some order statistics we must work with complicate expressions even in the case of small values of the sample size n. There are rather simple formulae in the situations when the initial random variables have some density function $f(x)$. In this case the joint density function of all order statistics $X_{1,n}$, $X_{2,n}, \ldots, X_{n,n}$ has the following form:

$$\mathrm{f}_{1,2,\ldots,n:n}(x_1, x_2, \cdots, x_n) = n! \prod_{k=1}^{n} f(x_k), \ -\infty < x_1 < x_2 < \ldots < x_n < \infty \tag{2.1.10}$$

and

$$f_{1,2,\ldots,n:n}(x_1, x_2, \ldots, x_n) = 0, \text{otherwise.}$$

It can be found from (2.1.10) that the joint pdf

$$f_{k(1),k(2),\ldots,k(r):n}(x_1, x_2, \ldots, x_r)$$

of order statistics

$$X_{k(1),n},\ X_{k(2),n}, \ldots, X_{k(r),n},$$

where

$$1 \le k(1) < k(2) < \cdots < k(r) \le n,$$

is of the form

$$f_{k(1),k(2),\ldots,k(r):n}(x_1, x_2, \ldots, x_r) = \frac{n!}{\prod_{m=1}^{r+1} (k(m) - k(m-1) - 1)!} \prod_{m=1}^{r+1} (F(x_m) - F(x_{m-1}))^{k(m)-k(m-1)-1} \prod_{m=1}^{r} f(x_m), \tag{2.1.11}$$

if $x_1 < x_2 < \cdots < x_r$, and $f_{k(1),k(2),\cdots,\ k(r):n}\ (x_1,\ x_2,\ldots,\ x_r) = 0$, otherwise.

In particular, if $r = 2$, $1 \le i < j \le n$, and $x_1 < x_2$, then the joint pdf $f_{i,j:n}\ (x_1,\ x_2)$ of two order statistics is given by the following equality:

$$f_{i,j:n}(x_1,x_2) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} (F(x_1))^{i-1}(F(x_2)-F(x_1))^{j-i-1}(1-F(x_2))^{n-j}f(x_1)f(x_2). \tag{2.1.12}$$

To understand the level of complication of the formulae for the joint distributions in the general case we suggest our reader the corresponding expression for the joint d.f. of two order statistics $X_{r,n}$ and $X_{s,n}$, $1 \le r < s \le n$ (below $x_1 < x_2$):

$$F_{r,s:n}(x_1,x_2) = \sum_{i=r}^{n} \sum_{j=\max\{0,s-i\}}^{n-i} \frac{n!}{i!j!(n-i-j)!}(F(x_1))^i(F(x_2)-F(x_1))^j(1-F(x_2))^{n-i-j}. \tag{2.1.13}$$

There are some situations when it is possible to get rather simple expressions for joint distributions of order statistics. For example, if one considers probabilities

$$P\{y_1 < X_{1,n} \le x_1,\ y_2 < X_{2,n} \le x_2, \cdots,\ y_n < X_{n,n} \le x_n\}$$

for any values

$$-\infty < y_1 < x_1 \le y_2 < x_2 \le \cdots \le y_n < x_n < \infty,$$

then it can be obtained that the corresponding equality has the form

$$P\{y_1 < X_{1,n} \le x_1,\ y_2 < X_{2,n} \le x_2, \ldots,\ y_n < X_{n,n} \le x_n\} = n! \prod_{k=1}^{n} (F(x_k) - F(y_k)). \tag{2.1.14}$$

Exercise 2.1.5 Let $X_1,\ X_2,\ldots,\ X_n$ be a sample of size n from a geometrically distributed random variable X, such that

$$P\{X = m\} = (1-p)p^m,\ m = 0,1,2,\ldots.$$

Find

$$P\{X_{1,n} \ge r,\ X_{n,n} < s\},\ r < s.$$

2.2 The Classical Representations for Order Statistics

It was mentioned above that ordered random variables $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ are dependent. Indeed there exist essentially more methods in the probability theory to work with independent variables than with dependent ones. Hence different possibilities to express dependent random variables in terms of independent values are convenient for the future investigation. There are corresponding expressions for some types of order statistics, which will be presented below. We give these results without their proof, which can be found, for example in Ahsanullah et al. (2013).

In the sequel we will use the special notation $U_{1,n} \leq \cdots \leq U_{n,n}$, $n = 1, 2, \ldots$, for the standard uniform order statistics (the case when X's have d.f. $F(x) = x$, $0 \leq x \leq 1$) and the notation $Z_{1,n} \leq \cdots \leq Z_{n,n}$, $n = 1, 2, \ldots$, for the standard exponential order statistics (here $F(x) = 1 - \exp(-x)$, $x \geq 0$).

Representation 1 *The following expression is valid for the uniform order statistics under any* $n = 1, 2, \ldots$:

$$\left(U_{1,n}, \ldots, U_{n,n}\right) \stackrel{d}{=} \left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right), \tag{2.2.1}$$

where

$$S_n = v_1 + v_2 + \cdots + v_n,\ n = 1, 2, \ldots,$$

and v_1, $v_2, \ldots$ *be independent random variables having the standard* $E(1)$ *exponential distribution.*

Remark 2.2.1 Moreover, sometimes it can come in useful that for any $n = 1, 2, \ldots$ the vector

$$\left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right)$$

and the sum S_{n+1} are independent.

Remark 2.2.2 Representation 1 can be also rewritten in the following useful form:

$$\left(U_{1,n},\ U_{2,n} - U_{1,n}, \ldots, U_{n,n} - U_{n-1,n}\right) \stackrel{d}{=} \left(\frac{v_1}{v_1 + \cdots + v_{n+1}}, \ldots, \frac{v_n}{v_1 + \cdots + v_{n+1}}\right) \tag{2.2.2}$$

Exercise 2.2.1 Show that the range

$$T(1,n) = M(n) - m(n) = U_{n,n} - U_{1,n},$$

has the same distribution as the order statistic $U_{n-1,n}$.

Representation 2 *Using the same terms as above we can write that the equality*

$$\left(U_{1,n},\ldots,U_{n,n}\right) \stackrel{d}{=} (S_1,\ldots,S_n \,|\, S_{n+1}=1), \tag{2.2.3}$$

also holds, i.e., the distribution of the vector of uniform order statistics coincides with the conditional distribution of the vector of sums $S_1,\ldots, S_n$ given that $S_{n+1} = 1$.

Representation 3 *Let W_1, $W_2,\ldots$ be independent uniformly distributed on* [0,1] *random variables. Then the next equality is true for any $n = 1, 2,\ldots$:*

$$\left(U_{1,n},U_{2,n},\ldots,U_{n,n}\right) \stackrel{d}{=} \left(W_1 W_2^{1/2}\ldots W_{n-1}^{1/(n-1)} W_n^{1/n},\ W_2^{1/2}\ldots W_{n-1}^{1/(n-1)} W_n^{1/n},\ldots,\ W_n^{1/n}\right). \tag{2.2.4}$$

Exercise 2.2.2 Show that for any $n = 2, 3,\ldots,$ ratios

$$V_k = \left(U_{k,n}/U_{k+1,n}\right)^k,\ k=1,2,\ldots,n,$$

where $U_{n+1,\,n} = 1$, are independent and have the same uniform distribution on [0,1].

Now we will show that the exponential order statistics $Z_{k,n}$ also can be presented in very convenient form.

Representation 4 *Let again ν_1, $\nu_2,\ldots,$ be independent random variables having the standard $E(1)$ exponential distribution. Then the following result holds for the exponential order statistics and any $n = 1, 2,\ldots$:*

$$\left(Z_{1,n},\,Z_{2,n},\ldots,Z_{n,n}\right) \stackrel{d}{=} \left(\frac{\nu_1}{n},\,\frac{\nu_1}{n}+\frac{\nu_2}{n-1},\ldots,\frac{\nu_1}{n}+\frac{\nu_2}{n-1}+\cdots+\frac{\nu_{n-1}}{2}+\nu_n\right). \tag{2.2.5}$$

Remark 2.2.3 It follows from (2.2.5) that normalized differences

$$(n-k+1)\left(Z_{k,n}-Z_{k-1,n}\right),\ k=1,2,\ldots,n,$$

where $Z_{0,n} = 0$, are independent and have the same exponential $E(1)$ distribution.

Exercise 2.2.3 For exponential order statistics $Z_{1,n}$, $Z_{2,n},\ldots, Z_{n,n}$ show that statistics $Z_{1,n}$ and

$$L = (n-1)Z_{n,n} - Z_{1,n} - Z_{2,n} - \cdots - Z_{n-1,n}$$

are independent.

Our reader can ask if the useful representations exist for these two distributions (standard uniform and standard exponential) only. Indeed, it is not difficult to rewrite the corresponding relations for any representative of these families of distributions. The matter is that if X's have some standard distribution (uniform or exponential, as in our situations) then any Y's which belong to the same family of distributions can be expressed via X's in the form $Y = aX + b$, where $a > 0$ and $-\infty < b < \infty$, and the corresponding order statistics $Y_{k,n}$ can be presented as

$$Y_{k,n} = aX_{k,n} + b.$$

It means that all these representations can be rewritten easily for any uniform or exponential distributions.

Moreover the given results for $U_{k,n}$ and $Z_{k,n}$ can be rewritten for order statistics from arbitrary distribution.

For any d.f. F we determine the inverse function

$$G(s) = inf\{x : F(x) \geq s\},\ 0 < s < 1. \tag{2.2.6}$$

Exercise 2.2.4 Let $F(x)$ be a continuous d.f. of a random variable X. Show that in this case

$$F(G(x)) = x,\ 0 < x < 1, \tag{2.2.7}$$

and it means that $U = F(X)$ has the uniform distribution on interval [0,1].

Remark 2.2.4 Relation (2.2.7) above is true if F is any continuous d.f. Indeed, this equality fails if F has jump points, since then the values of $F(X)$, unlike U, do not cover all interval [0,1]. But it is not difficult to show that the dual equality

$$G(F(x)) = x,\ 0 < x < 1, \tag{2.2.8}$$

holds for any x, where $F(x)$ strongly increases. Thus, presentation

$$X \stackrel{d}{=} G(U), \tag{2.2.9}$$

holds for any random variable, while the dual equality

$$F(X) \stackrel{d}{=} U \tag{2.2.10}$$

is valid for random variables with continuous distribution functions only.

Taking into account equalities (2.2.9) and (2.2.10) one can obtain the corresponding relations which allow to represent any order statistics $X_{1,n} \leq \cdots \leq X_{n,n}$ via the uniform order statistics $U_{1,n} \leq \cdots \leq U_{n,n}$.

Since d.f. F and its inverse G are monotone they don't disturb the ordering of the initial X's. It enables us to get the following presentations which correspond to equalities (2.2.9) and (2.2.10).

Representation 5 *If F is a continuous d.f. then*

$$\left(F\left(X_{1,n}\right),\ldots,\ F\left(X_{n,n}\right)\right)\overset{d}{=}\left(U_{1,n},\ldots,U_{n,n}\right). \tag{2.2.11}$$

Representation 6 *Let F be any d.f. and G be its inverse. Then* the following equality

$$\left(X_{1,n},\ldots,X_{n,n}\right)\overset{d}{=}\left(G\left(U_{1,n}\right),\ldots,G\left(U_{n,n}\right)\right), \tag{2.2.12}$$

is true.

Let now $X_{1,n} \le \cdots \le X_{n,n}$ and $Y_{1,n} \le \cdots \le Y_{n,n}$ be order statistics corresponding to an arbitrary d.f. F and a continuous d.f. H. Let also G be as above the inverse of F. Combining relations (2.2.11) for Y's and (2.2.12) for X's, one gets one more important result.

Representation 7 *The following equality ties two sets of order statistics*:

$$\left(X_{1,n},\ldots,X_{n,n}\right)\overset{d}{=}\left(G\left(H\left(Y_{1,n}\right)\right),\ldots,G\left(H\left(Y_{n,n}\right)\right)\right). \tag{2.2.13}$$

Example 2.2.1 For instance, if we compare arbitrary order statistics $X_{1,n},\ldots, X_{n,n}$ and exponential order statistics $Z_{1,n},\ldots, Z_{n,n}$, then

$$H(x) = 1 - \exp(-x),\ x > 0,$$

and (2.2.13) can be rewritten as

$$\left(X_{1,n},\ldots,X_{n,n}\right)\overset{d}{=}\left(G\left(1-\exp\left(-Z_{1,n}\right)\right),\ldots,G\left(1-\exp\left(-Z_{n,n}\right)\right)\right). \tag{2.2.14}$$

Remark 2.2.5 Indeed, the results analogous to (2.2.13) are valid for any monotone increasing function $R(x)$ (no necessity to suppose that R is a distribution function). Namely, if

$$X_k = R(Y_k),\ k = 1, 2,\ldots, n,$$

then the following relation is true:

$$\left(X_{1,n},\ldots,X_{n,n}\right)\overset{d}{=}\left(R\left(Y_{1,n}\right),\ldots,R\left(Y_{n,n}\right)\right). \tag{2.2.15}$$

If R is a monotone decreasing function, then the transformation $R(Y)$ changes the ordering of the original Y's and we have the following equality:

$$\left(X_{1,n},\ldots,X_{n,n}\right)\overset{d}{=}\left(R\left(Y_{n,n}\right),\ldots,R\left(Y_{1,n}\right)\right). \tag{2.2.16}$$

Thus, equalities (2.2.12) and (2.2.14) give opportunity to express any order statistics via independent terms. The following example gives one of the possible presentations.

Example 2.2.2 One can express distributions of arbitrary order statistics $X_{k,n}$ (related to some d.f. F) as follows. If G is the inverse of F and ν_1, ν_2,... are independent exponentially $E(1)$ distributed random variables, then

$$X_{k,n} \stackrel{d}{=} G\left(\frac{\nu_1 + \cdots + \nu_k}{\nu_1 + \cdots + \nu_{n+1}}\right) \stackrel{d}{=} G\left(1 - \exp\left(-\left(\frac{\nu_1}{n} + \frac{\nu_2}{n-1} + \cdots + \frac{\nu_k}{n-k+1}\right)\right)\right),$$
$$k = 1, 2, \ldots, n. \tag{2.2.17}$$

Exercise 2.2.5 Let $X_{1,n}, \ldots, X_{n,n}$ be order statistics corresponding to the distribution with the density

$$f(x) = ax^{a-1}, 0 < x < 1, \; a > 0. \tag{2.2.18}$$

Express the product $X_{r,n}$, $X_{s,n}$, $1 \le r < s \le n$, in terms of independent uniformly distributed random variables.

2.3 Moment Characteristics of Order Statistics

Let us recall equality (2.1.7), which gives the expression for distribution functions $F_{k,n}(x)$ of order statistics $X_{k,n}$:

$$F_{k,n}(x) = I_{F(x)}(k, n-k+1),$$

where

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt.$$

Taking into account this expression one can write immediately that formula for moments

$$\mu_{k:n}^{(r)} = E\left(X_{k,n}\right)^r$$

of order statistics $X_{k,n}$, related to a population with a d.f. F, is given as follows:

$$\begin{aligned}\mu_{k:n}^{(r)} &= \int_{-\infty}^{\infty} x^r dF_{k:n}(x) \\ &= \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} x^r (F(x))^{k-1}(1-F(x))^{n-k} dF(x)\end{aligned}. \tag{2.3.1}$$

If F is continuous then this equality can be expressed as

$$\mu_{k:n}^{(r)} = \frac{n!}{(k-1)!(n-k)!} \int_0^1 (G(u))^r u^{k-1}(1-u)^{n-k} du, \tag{2.3.2}$$

where $G(u)$ is the inverse of F. For distributions with probability density function f the RHS of (2.3.2) coincides with

$$\frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} x^r (F(x))^{k-1}(1-F(x))^{n-k} f(x) dx. \tag{2.3.3}$$

Similar relations are valid for joint (product) moments of order statistics. For the sake of simplicity we consider joint moments

$$\mu_{k(1),k(2):n}^{(r(1),r(2))} = E\left(X_{k(1),n}\right)^{r(1)} \left(X_{k(2),n}\right)^{r(2)}, 1 \le k(1) < k(2) \le n,$$

of two order statistics only.

For absolutely continuous distributions one gets that

$$\begin{aligned}\mu_{k(1),k(2):n}^{(r(1),r(2))} = c(r(1), r(2), n) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{r(1)} y^{r(2)} (F(x))^{r(1)-1} (F(y) - F(x))^{r(2)-r(1)-1} \\ (1-F(y))^{n-r(2)} f(x) f(y) dx dy,\end{aligned} \tag{2.3.4}$$

where

$$c(r(1), r(2), n) = \frac{n!}{(r(1)-1)!(r(2)-r(1)-1)!(n-r(2))!}. \tag{2.3.5}$$

In the general case (for any d.f. F) we have the following expression:

$$\mu_{k(1),k(2):n}^{(r(1),r(2))} = c(r(1), r(2), n) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{r(1)} y^{r(2)} (F(x))^{r(1)-1} (F(y) - F(x))^{r(2)-r(1)-1} (1 - F(y))^{n-r(2)} dF(x) dF(y), \tag{2.3.6}$$

where coefficients $c(r(1),\ r(2),n)$ are defined in (2.3.5). Below we will use the following notations also:

$$\mu_{k:n} = EX_{k.n}$$

will be applied for the sake of simplicity instead of $\mu_{k:n}^{(1)}$; $\mu_{k(1),k(2):n} = EX_{k(1),n}\, X_{k(2),n}$ will change $\mu_{k(1),k(2):n}^{(1,1)}$;

$$Var(X_{k,n}) = \mu_{k:n}^{(2)} - (\mu_{k:n})^2$$

will denote the variance of $X_{k,n}$;

$$cov(X_{r,n}, X_{s,n}) = \mu_{r,s:n} - \mu_{r:n}\mu_{s:n}$$

will be used for the covariance between $X_{r,n}$ and $X_{s,n}$.

It is interesting to find the corresponding conditions, which provide the existence of different moments of order statistics.

Example 2.3.1 Let there exist the population moment $\alpha_r = EX^r$, i.e.,

$$E|X|^r = \int_{-\infty}^{\infty} |x|^r dF(x) < \infty. \tag{2.3.7}$$

Then due to (2.3.1) we obtain that

$$\begin{aligned} E|X_{k,n}|^r &\leq \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} |x|^r (F(x))^{k-1} (1 - F(x))^{n-k} dF(x) \\ &\leq \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} |x|^r dF(x) = \frac{n!}{(k-1)!(n-k)!} E|X|^r < \infty. \end{aligned} \tag{2.3.8}$$

It follows from (2.3.8) that the existence of the moment α_r implies the existence of all moments

$$E\left|X_{k,n}\right|^r, 1 \le k \le n,\ n = 1, 2, \ldots$$

of order statistics $X_{k,n}$.

Remark 2.3.1 It is not difficult to prove that if

$$E|X|^r = \infty$$

for some r, then for any n = 1, 2,... there exists such order statistic $X_{k,n}$ that

$$E\left|X_{k,n}\right|^r = \infty.$$

Remark 2.3.2 Since

$$P\{X_{s,n} \le X_{k,n} \le X_{m,n}\} = 1$$

for any $s < k < m$, one has the evident inequality

$$E\left|X_{k,n}\right|^r \le E\left(\left|X_{s,n}\right|^r + \left|X_{m,n}\right|^r\right).$$

Hence, if $E|X_{k,n}|^r = \infty$, then at least one of equalities $E|X_{s,n}|^r = \infty$ or $E|X_{m,n}|^r = \infty$ is valid. It means that if $E|X_{k,n}|^r = \infty$, then at least one of the following relations hold:

$$E\left|X_{1,n}\right|^r = E\left|X_{2,n}\right|^r = \ldots = E\left|X_{k-1,n}\right|^r = \infty$$

or

$$E\left|X_{k+1,n}\right|^r = E\left|X_{k+2,n}\right|^r = \ldots = E\left|X_{n,n}\right|^r = \infty.$$

Exercise 2.3.1 Let X's have the Cauchy distribution with the density function

$$f(x) = \frac{1}{\pi(1 + x^2)}.$$

Show that for any r = 1, 2,..., relation

$$E\left|X_{k,n}\right|^r < \infty$$

holds if $r < k < n - r + 1$.

Remark 2.3.3 The more general situation than one given in Exercise 2.3.1 was considered by Sen (1959).

Let $E|X|^{\alpha} < \infty$. It appears that then moments $\mu_{k:n}^{(r)}$ exist for all k such that

$$r/\alpha \leq k \leq (n - r + 1)/\alpha.$$

Some useful relations for moments one can obtain from the evident identity

$$X_{1,n} + \cdots + X_{n,n} = X_1 + \cdots + X_n \tag{2.3.9}$$

and related equalities. For instance, the simplest corollary of (2.3.9) is as follows:

$$E\big(X_{1,n} + \cdots + X_{n,n}\big) = E(X_1 + \cdots + X_n) = nEX. \tag{2.3.10}$$

Natural generalizations of (2.3.10) have the form

$$g\left(\sum_{k=1}^{n} h\big(X_{k,n}\big)\right) = g\left(\sum_{k=1}^{n} h(X_k)\right), \tag{2.3.11}$$

where $g(x)$ and $h(x)$ are arbitrary functions.

Example 2.3.2 The corresponding equalities based on (2.3.11) can be useful in some situations:

$$E\left(\sum_{k=1}^{n} X_{k,n}^{m}\right)^{r} = E\left(\sum_{k=1}^{n} X_{k}^{m}\right)^{r}, \; m = 1, 2, \ldots, \; r = 1, 2, \ldots. \tag{2.3.12}$$

If $r = 1$ we get from (2.3.12) that the equality

$$\sum_{k=1}^{n} EX_{k,n}^{m} = \sum_{k=1}^{n} EX_{k}^{m} = nEX^{m} \tag{2.3.13}$$

holds for any m provided that the corresponding moment EX^m exists.

It is not difficult to understand that the representations of the uniform and exponential order statistics via sums or products of independent random variables given above allow us to find rather easily single and joint moments of these statistics.

Consider the case of the uniform $U([0,1])$ distribution and the corresponding order statistics $U_{k,n}$. From Representation 1 and Remark 2.2.1 we know that

$$U_{k,n} \stackrel{d}{=} \frac{S_k}{S_{n+1}}$$

and S_k/S_{n+1} is independent on the sum $S_{n+1} = v_1 + v_2 + \ldots + v_{n+1}$,where $v_1, v_2, \ldots$ be independent random variables having the standard $E(1)$ exponential distribution.

Then

$$E\left(U_{k,n}\right)^{\alpha} = E\left(\frac{S_k}{S_{n+1}}\right)^{\alpha}.$$

Now due to the independence of S_k/S_{n+1} and S_{n+1} we have the following relation:

$$E(S_k)^{\alpha} = E\left(\frac{S_k}{S_{n+1}}S_{n+1}\right)^{\alpha} = E\left(\frac{S_k}{S_{n+1}}\right)^{\alpha} E(S_{n+1})^{\alpha}.$$

Thus,

$$E\left(\frac{S_k}{S_{n+1}}\right)^{\alpha} = E(S_k)^{\alpha}/E(S_{n+1})^{\alpha}.$$

Now we must recall that S_m has gamma distribution with parameter m and hence

$$E(S_m)^{\alpha} = \frac{1}{(m-1)!}\int_0^{\infty} x^{a+m-1}e^{-x}dx = \Gamma(\alpha+m)/\Gamma(m), m = 1, 2, \ldots.$$

Here $\Gamma(s)$ denotes gamma function with parameter s.

Finally,

$$E\left(U_{k,n}\right)^{\alpha} = \frac{\Gamma(\alpha+k)\Gamma(n+1)}{\Gamma(k)\Gamma(\alpha+n+1)} = \frac{n!\Gamma(\alpha+k)}{(k-1)!\Gamma(\alpha+n+1)} \tag{2.3.14}$$

Note that (2.3.14) is true for any $\alpha > -k$.

Now we can use Representation 3 to get joint (product) moments of the uniform order statistics. It follows from this representation, where $W_1, W_2, \ldots$ are independent uniformly $U([0,1])$ distributed random variables, that

$$\begin{aligned} EU_{r,n}U_{s,n} &= E\left(W_r^{1/r}W_{r+1}^{1/(r+1)}\ldots W_n^{1/n}W_s^{1/s}W_{s+1}^{1/(s+1)}\ldots W_n^{1/n}\right) \\ &= E\left(W_r^{1/r}\right)E\left(W_{r+1}^{1/(r+1)}\right)\ldots E\left(W_s^{2/s}\right)E(W_{s+1})^{2/(s+1)}\ldots E\left(W_n^{2/n}\right) \\ &= \prod_{k=r}^{s-1}\frac{1}{(1+1/k)}\prod_{k=s}^{n}\frac{1}{(1+2/k)} = \frac{r(s+1)}{(n+1)(n+2)}. \end{aligned} \tag{2.3.15}$$

for any $1 \le r < s \le n$.

Remark 2.3.4 Indeed, Representation 3 allows us to get also more complicate joint moments of any order for any number of the uniform order statistics. Taking into account equality (2.3.14) we can fix the following important partial cases:

$$EU_{k,n} = \frac{k}{n+1}, \; 1 \le k \le n; \tag{2.3.16}$$

$$E\left(1/U_{k,n}\right) = \frac{n}{k-1}, \; 2 \le k \le n; \tag{2.3.17}$$

$$E\left(U_{k,n}\right)^2 = \frac{k(k+1)}{(n+1)(n+2)}, \; 1 \le k \le n. \tag{2.3.18}$$

In general form, for $r = 1, 2, \ldots$, we have

$$E\left(U_{k,n}\right)^r = \frac{k(k+1)\ldots(k+r-1)}{(n+1)(n+2)\ldots(n+r)}, \; 1 \le k \le n. \tag{2.3.19}$$

It follows from (2.3.16) and (2.3.18) that

$$Var\left(U_{k,n}\right) = \frac{k(n-k+1)}{(n+1)^2(n+2)}, \; 1 \le k \le n. \tag{2.3.20}$$

Taking into account equality (2.3.15) one gets that

$$Cov\left(U_{r,n}, U_{s,n}\right) = EU_{r,n}U_{s,n} - EU_{r,n}EU_{s,n} = \frac{r(n-s+1)}{(n+1)^2(n+2)}, \; r \le s. \tag{2.3.21}$$

Now from (2.3.20) and (2.3.21) we obtain the corresponding expression for correlation coefficients of the uniform order statistics:

$$\rho\left(U_{r,n}, U_{s,n}\right) = \left(\frac{r(n-s+1)}{s(n-r+1)}\right)^{1/2}, \; r \le s. \tag{2.3.22}$$

Exercise 2.3.2 Find the variance of $1/U_{k,n}$.

Exercise 2.3.3 Find product moments

$$E(U_{r,n}^{\alpha} U_{s,n}^{\beta}), \alpha \ge 0, \beta \ge 0.$$

Let us consider now order statistics $Z_{1,n} \le Z_{2,n} \le \cdots \le Z_{n,n}$, $n = 1, 2, \ldots$, which correspond to the standard $E(1)$ exponential distribution with d.f.

$$H(x) = 1 - \exp(-x),\, x > 0.$$

Let us recall Representation 4:

$$\left(Z_{1,n},\, Z_{2,n}, \ldots, Z_{n,n}\right) \stackrel{d}{=} \left(\frac{v_1}{n}, \frac{v_1}{n} + \frac{v_2}{n-1}, \ldots, \frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_{n-1}}{2} + v_n\right),$$

where $v_1,\ v_2, \ldots,\ v_n$ are independent random variables having the standard $E(1)$ exponential distribution.

One immediately gets applying this result that

$$EZ_{k,n} = E\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_k}{n-k+1}\right) = \sum_{r=1}^{k} \frac{1}{n-r+1} \tag{2.3.23}$$

and

$$Var\left(Z_{k,n}\right) = \sum_{r=1}^{k} \mathrm{Var}\left(\frac{v_r}{n-r+1}\right) = \sum_{r=1}^{k} \frac{1}{(n-r+1)^2}. \tag{2.3.24}$$

Exercise 2.3.4 Applying Representation 4 find now covariances between order statistics $Z_{r,n}$ and $Z_{s,n}$.

Calculations of the integral

$$\frac{n!}{(k-1)!(n-k)!} \int_0^\infty x^\alpha (1-e^{-x})^{k-1} e^{-x(n-k+1)} dx$$

also allow us to get moments

$$E\left(Z_{k,n}\right)^\alpha,\ k = 1, 2, \ldots, n.$$

We obtain after the corresponding calculations that

$$E\left(Z_{k,n}\right)^\alpha = \frac{n!}{(k-1)!(n-k)!} \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \Gamma(\alpha+1)/(n-k+r+1)^{(\alpha+1)}. \tag{2.3.25}$$

For instance, if $k = 1$, then

$$E\left(Z_{1,n}\right)^\alpha = \Gamma(\alpha+1)/\mathrm{n}^\alpha, \alpha > -1. \tag{2.3.26}$$

For $k = 2$ and $\alpha > -1$ we have

$$E\left(Z_{2,n}\right)^{\alpha}= n(n-1)\Gamma(\alpha+1)\{(n-1)^{-(\alpha+1)}-n^{-(\alpha+1)}\}. \tag{2.3.27}$$

Indeed, one of the most important distributions in the probability theory is normal. Let now $X_1, X_2, \ldots$ be independent random variables having the standard $N(0,1)$ normal distribution and $X_{1,n} \le \cdots \le X_{n,n}$ be the corresponding normal order statistics.

One can write immediately that

$$EX_{k,n}^r = \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} x^r \Phi^{k-1}(x)(1-\Phi(x))^{n-k}\varphi(x)dx, \tag{2.3.28}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)$$

and

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt.$$

There are some effective numerical methods to compute integrals (2.3.28). Unlike the uniform and exponential cases moments of the normal order statistics have the explicit expressions for small sample sizes n only.

Example 2.3.3 Consider the case $n = 2$. We get that

$$\begin{aligned} EX_{2,2} &= 2\int_{-\infty}^{\infty} x\Phi(x)\varphi(x)dx = -2\int_{-\infty}^{\infty} \Phi(x)d(\varphi(x)) \\ &= 2\int_{-\infty}^{\infty} \varphi^2(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left(-x^2\right)dx = \frac{1}{\sqrt{\pi}}. \end{aligned}$$

From the identity

$$E\left(X_{1,2}+X_{2,2}\right) = E(X_1+X_2) = 0$$

we obtain now that

$$EX_{1,2} = -EX_{2,2} = -\frac{1}{\sqrt{\pi}}.$$

Remark 2.3.5 If we have two samples $X_1, X_2,\ldots, X_n$ (from the standard $N(0,1)$ normal distribution) and $Y_1, Y_2,\ldots, Y_n$ (from the normal $N(a,\sigma^2)$ distribution with expectation a and variance σ^2, $\sigma > 0$), then evidently

$$EY_{k,n} = a + \sigma X_{k,n}.$$

Exercise 2.3.5 Let $X_{1,3} \le X_{2,3} \le X_{3,3}$ be the order statistics corresponding to the standard normal distribution. Find expectations $EX_{k,3}$ and variances $Var(X_{k,3})$, $k = 1, 2, 3$.

The explicit expressions for moments of the normal order statistics are rather complicated, although the normal distribution possesses a number of useful properties, which can simplify the computational schemes in some situations.

Example 2.3.4 A lot of statistical procedures for the normal distribution are based on the independence property of vector

$$(X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$$

and the sample mean

$$\bar{X} = (X_1 + X_2 + \cdots + X_n)/n.$$

This property yields also the independence of vector

$$(X_{1,n} - \bar{X}, X_{2,n} - \bar{X}, \ldots, X_{n,n} - \bar{X})$$

and the sample mean $\bar{X}$.

Let $X_1, X_2,\ldots, X_n$ be a sample from the standard normal distribution. We see then that

$$E(X_{k,n} - \bar{X})\bar{X} = E(X_{k,n} - \bar{X})E\bar{X} = 0 \tag{2.3.29}$$

and we obtain the following results:

$$EX_{k,n}\bar{X} = E(\bar{X}^2) = Var\,\bar{X} = 1/n,\ k = 1, 2, \ldots, n, \tag{2.3.30}$$

and hence

$$\sum_{m=1}^{n} EX_{k,n}X_m = 1, \tag{2.3.31}$$

as well as

$$\sum_{m=1}^{n} EX_{k,n}X_{m,n} = 1. \tag{2.3.32}$$

As corollaries of (2.3.31) one gets that

$$E\left(X_{k,n}X_m\right) = 1/n \tag{2.3.33}$$

and

$$cov\left(X_{k,n}, X_m\right) = E\left(X_{k,n}X_m\right) - EX_{k,n}EX_m = 1/n \tag{2.3.34}$$

for any $k = 1, 2,\ldots, n$, $m = 1, 2,\ldots, n$ and $n = 1, 2,\ldots$.

Note that there are different tables (see, for example, Teichroew (1956) or Tietjen et al. (1977)) which give expected values and some other moments of order statistics for samples of large sizes from the normal distribution.

2.4 Extremes

For any person who wants to study records it is necessary to know the theory of the extreme order statistics.

Order statistics $X_{k(n),n}$ are said to be extreme if $k = k(n)$ or $n - k(n) + 1$ is fixed, as $n \to \infty$. The most popular are maximal order statistics $X_{n,n}$ and minimal order statistics $X_{1,n}$.

It is not difficult to see that if we have two sequences $Y_1 = -X_1$, $Y_2 = -X_2,\ldots, Y_n = -X_n$ then the following equalities hold for any $k = 1, 2,\ldots, n$:

$$X_{n-k+1,n} \stackrel{d}{=} -Y_{k,n}, \tag{2.4.1}$$

and, in particular, in this case

$$X_{n,n} \stackrel{d}{=} -Y_{1,n}. \tag{2.4.2}$$

Due to these relationships between maximal and minimal order statistics we can study one type of them only, say, maximal ones.

Very often we need to know asymptotic distributions of $X_{1,n}$ and $X_{n,n}$, as $n \to \infty$.

Consider a sequence of order statistics $M(n) = X_{n,n}$, $n = 1, 2,\ldots$. Let $F(x)$ be the distribution function of X and

$$\beta = \sup\{x : F(x) < 1\}$$

be the right end point of the support of X. If $\beta = \infty$, then for any finite x one gets that $F(x) < 1$ and hence

$$P\{M(n) \le x\} = (F(x))^n \to 0, \text{as}\, n \to \infty.$$

It means that $M(n)$ converges to infinity. In the case, when $\beta < \infty$, we need to distinguish two situations. If

$$P\{X = \beta\} = p > 0,$$

then

$$P\{M(n) = \beta\} = 1 - P\{M(n) < \beta\} = 1 - P^n(X < \beta) = 1 - (1-p)^n$$

and

$$P\{M(n) = \beta\} \to 1, \ \text{as}\, n \to \infty. \tag{2.4.3}$$

If $P\{X = \beta\} = 0$, then we get that $P\{M(n) < \beta\} = 1$ for any n and $M(n) \to \beta$ in distribution. Thus, we see that in all situations $M(n) \to \beta$ in distribution. This result can be sharpened if we consider the asymptotic distributions of the centered and normalized order statistics $X_{n,n}$. Indeed, if $\beta < \infty$ and $P\{X = \beta\} > 0$, then relation (2.4.3) gives completed information on $M(n)$ and in this case any centering and normalizing can not improve our knowledge about the asymptotic behavior of $M(n)$. We have another situation if $P\{X = \beta\} = 0$. In this case one can try to solve the following problem: if there exist any centering (a_n) and norming (b_n) constants, such that the sequence $(M(n) - a_n)/b_n$ converges to some nondegenerate distribution? Let us solve firstly the following exercises.

Exercise 2.4.1 Consider the exponential distribution with d.f. $F(x) = 1 - \exp(-x)$, $x \ge 0$. Show that in this case $M(n) - \log n$ *converges* to some nondegenerate distribution with d.f.

$$H_0(x) = \exp(-\exp(-x)). \tag{2.4.4}$$

Exercise 2.4.2 Consider X's with d.f.

$$F(x) = 1 - (-x)^\alpha, -1 < x < 0, \ \alpha > 0,$$

and prove that the asymptotic distribution of $n^{1/\alpha}M(n)$ has the following form:

$$H_{1,\alpha}(x) = \exp(-(-x)^\alpha), -\infty < x \le 0,$$

and

$$H_{1,\alpha}(x) = 1,\ x > 0. \tag{2.4.5}$$

Exercise 2.4.3 Let X's have Pareto distribution with d.f.

$$F(x) = 1 - x^{-\alpha},\ x > 1, \alpha > 0.$$

Prove that the asymptotic d.f. of $M(n)/n^{1/\alpha}$ is of the form:

$$H_{2,\alpha}(x) = 0,\ x<0,$$

and

$$H_{2,\alpha}(x) = \exp\{-x^{-\alpha}\},\ x \geq 0. \tag{2.4.6}$$

Remark 2.4.1 Changing suitably the normalized constants a_n and b_n for maximal values considered in Exercises 2.4.1–2.4.3, one gets that any d.f. of the form $H_0(a + bx)$, $H_{1,\alpha}(a + bx)$, $H_{2,\alpha}(a + bx)$, where $\alpha > 0$, $b > 0$ and a are arbitrary constants, can serve as the limit distribution for

$$(M(n) - a_n)/b_n.$$

Considering two d.f.'s, $H(d + cx)$ and $H(a + bx)$, where $b > 0$ and $c > 0$, we say that these d.f.'s belong to the same type of distributions. Any distribution of the given type can be obtained from other distribution of the same type by some linear transformation.

Usually one of distributions, say $H(x)$, having the most simplest (or convenient) form, is chosen to represent all the distributions of the given type, which we call then H-type. As basic for their own types, we suggested above the following distributions:

$$\begin{aligned}
H_0(x) &= \exp(-\exp(-x));\\
H_{1,\alpha}(x) &= \exp\left(-(-x)^{\alpha}\right),\ -\infty < x \leq 0,\ \text{ and } H_{1,\alpha}(x) = 1, x > 0;\\
H_{2,\alpha}(x) &= 0, x < 0, \text{ and } H_{2,\alpha}(x) = \exp\{-x^{-\alpha}\},\ x \geq 0,
\end{aligned}$$

where $\alpha > 0$.

Very often one can find that the types of distributions based on $H_0(x)$, $H_{1,\alpha}(x)$ and $H_{2,\alpha}(x)$ are named correspondingly as *Gumbel*, *Frechet* and *Weibull* types of the limiting extreme value distributions.

Note also that any two of d.f.'s $H_{1,\alpha}$ and $H_{1,\beta}$, $\alpha \neq \beta$, do not belong to the same type, as well as d.f.'s $H_{2,\alpha}$ and $H_{2,\beta}$, $\alpha \neq \beta$.

There are considered above three situations when we can get d.f.'s (2.4.4)–(2.4.6) as the limit d.f.'s for the normalized order statistics $X_{n,n}$.

It is surprising that there are none of other non-degenerate distributions, besides of $H_0(x)$-, $H_{1,\alpha}(x)$- and $H_{2,\alpha}(x)$—types, which would be limit for the suitably centering and norming maximal values.

Remark 2.4.2 It appeared that the set of all possible non-degenerate limit distributions for maximal values includes only the types generated by d.f.'s H_0, $H_{1,\alpha}$ and $H_{2,\alpha}$ only. Now it is important for us to know what d.f.'s F belong to the domains of attraction ($D(H_0)$, $D(H_{1,\alpha})$ and $D(H_{2,\alpha})$) of the corresponding limit laws. We write that $F \in D(H)$, if the suitably normalized maximal values $M(n)$, based on X's with a common d.f. F, have the limit d.f. H. For instance, it follows (from the results of Exercises 2.4.1–2.4.3 and Remark 2.4.1) that if

$$F(x) = 1 - \exp(-x),\ x > 0,$$

then $F \in D(H_0)$;
if X's are the uniformly $U([a,b])$ distributed random variables with

$$F(x) = (x - a)/(b - a),\ a < x < b,$$

then $F \in D(H_{1,1})$
and if

$$F(x) = 1 - x^{-\alpha},\ x > 1$$

(Pareto distribution), then $F \in D(H_{2,\alpha})$.

There are necessary and sufficient conditions for F to belong $D(H_0)$, $D(H_{1,\alpha})$ and $D(H_{2,\alpha})$ but the form of these conditions is rather cumbersome. Hence, simple sufficient conditions are more interesting for us. We present below some of them.

Theorem 2.4.1 *Let d.f. F have positive derivative F' for all $x > x_0$. If the following relation is valid for some $\alpha > 0$:*

$$xF'(x)/(1 - F(x)) \to \alpha, \tag{2.4.7}$$

as $x \to \infty$, then $F \in D(H_{2,\alpha})$. The centering, a_n, and normalizing, b_n, constants can be taken to satisfy relations $a_n = 0$ and $F(b_n) = 1 - 1/n$.

Theorem 2.4.2 *Let d.f. F have positive derivative F' for x in some interval (x_1, x_0) and $F'(x) = 0$ for $x > x_0$. If*

$$(x_0 - x)F'(x)/(1 - F(x)) \to \alpha, x \to x_0, \tag{2.4.8}$$

then $F \in D(H_{1,\alpha})$. The centering, a_n, and normalizing, b_n, constants can be taken to satisfy relations $a_n = x_0$ and $F(x_0 - b_n) = 1 - 1/n$.

Theorem 2.4.3 *Let d.f. F have negative second derivative $F''(x)$ for x in some interval (x_1, x_0), and let $F'(x) = 0$ for $x > x_0$. If*

$$F''(x)\,(1 - F(x))/(F'(x))^2 = -1, \tag{2.4.9}$$

then $F \in D(H_0)$. The centering, a_n, and normalizing, b_n, constants can be taken to satisfy relations $F(a_n) = 1 - 1/n$ and $b_n = h(a_n)$, where

$$h(x) = (1 - F(x))/F'(x).$$

Exercise 2.4.4 Let

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$$

(the Cauchy distribution). Prove that in this situation

$$F \in D\big(H_{2,1}\big).$$

What normalizing constants, a_n and b_n, can be taken in this case?

Exercise 2.4.5 Let

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-t^2/2\right)dt.$$

Show that

$$F \in D(H_0)$$

and the normalizing constants can be taken as follows:

$$a_n = (2\log n - \log\log n - \log 4\pi)^{1/2} \text{and}\, b_n = (2\log n)^{1/2}.$$

Exercise 2.4.6 Use Theorem 2.4.2 and find the limit distribution and the corresponding normalizing constants for gamma distribution with p.d.f.

$$f(x) = x^{\alpha-1}\exp(-x)/\Gamma(\alpha), x > 0, \alpha > 0.$$

Returning to relations (2.4.1) and (2.4.2) one can find the possible types of limit distributions for minimal values

$$m(n) = \min\{X_1, \ldots, X_n\}.$$

It appears that the corresponding set of non-degenerate asymptotic d.f.'s for the suitably normalized minimal values are defined by the following basic d.f.'s:

$$L_0(x) = 1 - \exp(-\exp(x)); \tag{2.4.10}$$

$$L_{1,\alpha}(x) = 0, x < 0, \text{ and } L_{1,\alpha}(x) = 1 - \exp\left(-x^{\alpha}\right), 0 \le x < \infty; \tag{2.4.11}$$

$$L_{2,\alpha}(x) = 1 - \exp\{-(-x)^{-\alpha}\}, x < 0, \text{ and } L_{2,\alpha}(x) = 1,\ x \ge 0, \tag{2.4.12}$$

where $\alpha > 0$.

Above we considered the situation with the asymptotic behavior of extremes $X_{n,n}$ and $X_{1,n}$. Analogous methods can be applied to investigate the possible asymptotic distributions of the kth extremes—order statistics $X_{n-k+1,n}$ and $X_{k,n}$, when $k = 2, 3, \ldots$ is some fixed number and n tends to infinity. The following results are valid in these situations.

Theorem 2.4.4 *Let random variables $X_1, X_2, \ldots$ be independent and have a common d.f. F and $X_{n-k+1,n}$, $n = k, k + 1, \ldots$, be the $(n - k + 1)$th order statistics. If for some normalizing constants a_n and b_n,*

$$P\{(X_{n,n}-a_n)/b_n < x\} \to H(x)$$

in distribution, as $n \to \infty$, then the limit relation

$$P\{(X_{n-k+1,n}-a_n)/b_n < x\} \to H(x)\sum_{j=0}^{k-1}(-\log H(x))^j/j! \qquad (2.4.13)$$

holds for any x, as $n \to \infty$.

Theorem 2.4.5 *Let random variables $X_1, X_2, \ldots$ be independent and have a common d.f. F and $X_{k,n}$, $n = k, k + 1, \ldots$, be the kth order statistics. If for some normalizing constants a_n and b_n,*

$$P\{(X_{1,n} - a_n)/b_n < x\} \to L(x),\ n \to \infty,$$

in distribution, then the limit relation

$$P\{(X_{k,n} - a_n)/b_n < x\} \to L(x)\sum_{j=0}^{k-1}(-\log L(x))^j/j! \qquad (2.4.14)$$

is valid for any x, as $n \to \infty$.

2.5 Order Statistics and Ranks

In Chap. 1 we mentioned that a random sample $X_1, X_2, \ldots, X_n$ is accompanied by the corresponding vector of ranks $(R(1), R(2), \ldots, R(n))$, where

$$R(m) = \sum_{k=1}^{n} 1_{\{X_m \geq X_k\}}, m = 1, 2, \ldots, n. \qquad (2.5.1)$$

The following relations are valid for ranks and order statistics:

$$\{R(m) = k\} = \{X_m = X_{k,n}\},\ m = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n. \tag{2.5.2}$$

If X's are proposed to have a continuous distribution then different X's can coincide with zero probability and (2.5.1) can be rewritten in the following form:

$$R(m) = 1 + \sum_{k=1}^{n} 1_{\{X_m > X_k\}}. \tag{2.5.3}$$

These ranks show the location of X's among the elements of the variational series

$$X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}.$$

We can write also that

$$X_m = X_{R(m),n},\ m = 1, 2, \ldots, n. \tag{2.5.4}$$

Consider the case when X's have a continuous distribution. Then any rank $R(m)$, $m = 1, 2, \ldots, n$, has the discrete uniform distribution on set $\{1, 2, \ldots, n\}$. Really, we know that random variables $X_1, X_2, \ldots, X_n$ taken from a continuous distribution have no coincidences with probability one. Hence, realizations $(r(1), \ldots, r(n))$ of the corresponding vector of ranks $(R(1), R(2), \ldots, R(n))$ represent all permutations of values $1, 2, \ldots, n$. Any realization $(r(1), \ldots, r(n))$ corresponds to the event

$$(X_{\delta(1)} < X_{\delta(2)} < \cdots < X_{\delta(n)}),$$

where $\delta(r(k)) = k$. Taking into account the symmetry of the sample $X_1, X_2, \ldots, X_n$, one obtains that events

$$(X_{\delta(1)} < X_{\delta(2)} < \cdots < X_{\delta(n)})$$

have the same probabilities for any permutations $(\delta(1), \ldots, \delta(n))$ of numbers $1, 2, \ldots, n$. Hence

$$\begin{aligned} &P\{R(1) = r(1), R(2) = r(2), \ldots, R(n) = r(n)\} \\ &= \mathrm{P}\{(\mathrm{X}_{\delta(1)} < \mathrm{X}_{\delta(2)} < \cdots < \mathrm{X}_{\delta(\mathrm{n})})\} = 1/\mathrm{n}! \end{aligned} \tag{2.5.5}$$

for any permutation $(r(1), \ldots, r(n))$ of numbers $1, 2, \ldots, n$. It follows now from (2.5.5) that

$$P\{R(m) = k\} = 1/n$$

for any fixed $m = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.

Indeed, ranks $R(1), R(2), \ldots, R(n)$ are dependent random variables for any $n = 2, 3, \ldots$. This dependence is approved by the evident equality

$$R(1) + R(2) + \cdots + R(n) = 1 + 2 + \cdots + n = n(n+1)/2,$$

but it follows from (2.5.5) that ranks present exchangeable random variables: for any permutation $(\alpha(1), \alpha(2), \ldots, \alpha(n))$ of numbers $1, 2, \ldots, n$ vectors $(R(\alpha(1)), \ldots, R(\alpha(n)))$ and $(R(1), \ldots, R(n))$ have the same distributions.

Exercise 2.5.1 Find expectations and variances of $R(k)$, $1 \le k \le n$, covariances $Cov(R(k), R(m))$ and the correlation coefficients $\rho(R(k), R(m))$ between $R(k)$ and $R(m)$, $1 \le k < m \le n$.

Above we mentioned that any realization $(r(1), \ldots, r(n))$ of $(R(1), R(2), \ldots, R(n))$ corresponds to the event

$$(X_{\delta(1)} < X_{\delta(2)} < \cdots < X_{\delta(n)}),$$

where $\delta(r(k)) = k$. Here $\delta(k)$ denotes the index of X, the rank of which for this realization takes on the value k. For different realizations of the vector $(R(1), R(2), \ldots, R(n))$, $\delta(k)$ can take on different values from the set $\{1, 2, \ldots, n\}$ and we really deal with new random variables, with realizations as $\delta(r(1)), \delta(r(2)), \ldots, \delta(r(n))$.

Let as above $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from a continuous distribution and $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ be the corresponding order statistics. We introduce random variables $\Delta(1), \Delta(2), \ldots, \Delta(n)$, which satisfy the following equalities:

$$\{\Delta(m) = k\} = \{X_{m,n} = X_k\}, m = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n. \tag{2.5.6}$$

These random variables are said to be *antiranks*.

The same arguments, which we used for ranks, show that any realization $(\delta(1), \delta(2), \ldots, \delta(n))$ of the vector $(\Delta(1), \Delta(2), \ldots, \Delta(n))$ is a permutation of numbers $1, 2, \ldots, n$ and all $n!$ such realizations have equal probabilities, $1/n!$ each.

Indeed, vectors of antiranks are tied closely with the corresponding order statistics and vectors of ranks. In fact, for any k and m equalities

$$\{\Delta(k) = m\} = \{X_{k,n} = X_m\} = \{R(m) = k\} \tag{2.5.7}$$

hold with probability one. One can write also the following relations for ranks and antiranks:

$$\Delta(R(m)) = m \tag{2.5.8}$$

and

$$R(\Delta(m)) = m, \tag{2.5.9}$$

which hold with probability one for any $m = 1, 2,\ldots, n$.

Exercise 2.5.2 Find the joint distribution of $\Delta(1)$ and $R(1)$.

While ranks and antiranks are associated with some random sample $X_1, X_2,\ldots, X_n$ of some fixed size n, there are rank statistics (the so-called sequential ranks), which characterize a sequence of random variables $X_1, X_2, \ldots$.

Let $X_1, X_2, \ldots$ be independent random variables, having continuous (not necessary identical) distributions. Random variables $\rho(1), \rho(2),\ldots$ given by equalities:

$$\rho(m) = \sum_{k=1}^{m} 1_{\{X_m \geq X_k\}}, m = 1, 2, \ldots, \tag{2.5.10}$$

are said to be *sequential ranks*.

Any sequential rank $\rho(m)$ shows the position of a new coming observation X_m among its predecessors $X_1, X_2,\ldots, X_{m-1.}$ For instance, if $\rho(m) = 1$, then X_m is less than $X_{1,\ m-1}$ and it means that

$$X_m = X_{1,m}.$$

In general, $\rho(m) = k$ implies that

$$X_m = X_{k,m}.$$

It is not difficult to see that $\rho(m)$ can take on the values $1, 2,\ldots, m$. If independent random variables $X_1, X_2,\ldots, X_m$ have the same continuous distribution then the standard arguments used above enable us to see that for any $m = 1, 2,\ldots,$

$$P\{\rho(m) = k\} = P\{X_m = X_{k,m}\} = 1/m,\ k = 1, 2, \ldots, m. \tag{2.5.11}$$

Exercise 2.5.3 Let $X_1, X_2, \ldots$ be independent random variables with a common continuous d.f. F. Prove that the corresponding sequential ranks $\rho(1), \rho(2),\ldots$ are independent.

Note that if $\rho(m) = m$ than it means that $X_m = X_{m,m}$ and

$$X_m > \max\{X_1, X_2, \ldots, X_{m-1}\}. \tag{2.5.12}$$

Some later it will stay clear that (2.5.12) defines that X_m is the upper record value in the sequence X_1, X_2, ….

2.6 Exercises (solutions)

Exercise 2.1.1 (*solution*) There are two situations here when $X_{1,\mathrm{n}}$ and $X_{n,n}$ can be equal: all X's are equal 0 and all X's are equal 1. Hence

$$p_n = P\{X_1 = X_2 = \ldots = X_n = 0\} + P\{X_1 = X_2 = \ldots = X_n = 1\} = 1/2^n + 1/2^n \\ = 1/2^{n-1}$$

Exercise 2.1.2 (*solution*) In this situation the necessary probability can be written as follows:

$$\begin{aligned} P\{X_{1,3} < X_{2,3} < X_{3,3}\} &= 1 - P\{X_1 = X_2 \neq X_3\} - P\{X_1 = X_3 \neq X_2\} - P\{X_2 = X_3 \neq X_1\} - P\{X_1 = X_2 = X_3\} \\ &= 1 - 3P\{X_1 = X_2 \neq X_3\} - P\{X_1 = X_2 = X_3\} \\ &= 1 - 3(P\{X_1 = X_2\} - P\{X_1 = X_2 = X_3\}) - P\{X_1 = X_2 = X_3\} \\ &= 1 - 3P\{X_1 = X_2\} + 2P\{X_1 = X_2 = X_3\}. \end{aligned}$$

It is not difficult to find that

$$P\{X_1 = X_2\} = (1-p)^2/\left(1-p^2\right)$$

and

$$P\{X_1 = X_2 = X_3\} = (1-p)^3/\left(1-p^3\right).$$

Thus one gets the final expression:

$$P\left\{X_{1,3} < X_{2,3} < X_{3,3}\right\} = 6p^3/(1+p)\left(1+p+p^2\right).$$

Exercise 2.1.3 (*solution*) Indeed, if x ≥ y, then

$$P\left\{X_{1,n} \leq x, X_{n,n} \leq y\right\} = P\left\{X_{n,n} \leq y\right\} = F^n(y).$$

If x < y, then

$$P\{X_{1,n} \le x, X_{n,n} \le y\} = P\{X_{n,n} \le y\} - P\{X_{1,n} > x, X_{n,n} \le y\}$$
$$= F^n(y) - (F(y) - F(x))^n.$$

Exercise 2.1.4 (*solution*) It is easy to see that

$$P\{m(n) \le x\} = 1 - P\{m(n) > x\} = 1 - P\{X_1 > x\}\ldots P\{X_n > x\}$$
$$= 1 - (1 - F_1(x))\ldots(1 - F_n(x))$$

and

$$P\{M(n) \le x\} = P\{X_1 \le x\}\ldots P\{X_n \le x\} = F_1(x)\ldots F_n(x).$$

Exercise 2.1.5 (*solution*) We see that

$$P\{Y \ge r,\, Z < s\} = P\{r \le X_k < s,\, k = 1, 2, \ldots, n\}$$
$$= (P\{r \le X < s\})^n = (P\{X \ge r\} - P\{X \ge s\})^n = (p^r - p^s)^n.$$

Exercise 2.2.1 (*solution*) It follows from (2.2.2) that

$$T(1, n) = U_{n,n} - U_{1,n} \stackrel{d}{=} \frac{v_2 + \cdots + v_n}{v_1 + \cdots + v_{n+1}}.$$

Indeed the ratio $(v_2 + \cdots + v_n)/(v_1 + \cdots + v_{n+1})$ coincides in distribution with the ratio $(v_1 + \cdots + v_{n-1})/(v_1 + \cdots + v_{n+1})$. Hence

$$T(1, n) \stackrel{d}{=} \frac{v_1 + \cdots + v_{n-1}}{v_1 + \cdots + v_{n+1}} \stackrel{d}{=} U_{n-1,n}$$

Exercise 2.2.2 (*solution*) The necessary statement follows immediately from relation (2.2.4), which allows us to get the equality

$$(V_1, V_2, \ldots, V_n) \stackrel{d}{=} (W_1, W_2, \ldots, W_n),$$

where W's are independent uniformly distributed random variables.

Exercise 2.2.3 (*solution*) It follows from (2.2.5) that

$$(Z_{1,n},\, L) \stackrel{d}{=} \left(\frac{v_1}{n}, \sum_{k=2}^{n} b_k n_k\right),$$

where

$$b_k = (k-1)/(n-k+1), k = 2, 3, \ldots.$$

Since v_k, $k = 1, 2, \ldots, n$, are independent random variables,

$$\frac{v_1}{n} \text{ and } \sum_{k=2}^{n} b_k n_k$$

are also independent. Hence, so are $Z_{1,n}$ and L.

Exercise 2.2.4 (*hint*) It is enough to use definitions of d.f. $F(x)$ and its inverse function $G(x)$.

Exercise 2.2.5 (*solution*) The corresponding d.f. and its inverse are given here by equalities

$$F(x) = x^a,\ 0 < x < 1.$$

and

$$G(x) = x^{1/a}, 0 < x < 1.$$

It is possible to express $X_{r,n}$ and $X_{s,n}$ via the uniform order statistics as follows:

$$X_{r,n} \stackrel{d}{=} \left(U_{r,n}\right)^{1/a} \text{ and } X_{s,n} \stackrel{d}{=} \left(U_{s,n}\right)^{1/a}.$$

Hence

$$X_{r,n} X_{s,n} \stackrel{d}{=} \left(U_{r,n} U_{s,n}\right)^{1/a}.$$

Now it follows from presentation (2.2.4) that

$$X_{r,n} X_{s,n} \stackrel{d}{=} W_r^{1/ra} W_{r+1}^{1/(r+1)a} \ldots W_{s-1}^{1/(s-1)a} W_s^{2/sa} W_{s+1}^{2/(s+1)a} \ldots W_n^{2/na},$$

$1 \le r < s \le n$, where $W_1, W_2, \ldots, W_n$ are independent random variables with the common uniform on [0,1] distribution.

Exercise 2.3.1 (*hint*) Prove that

$$G(x) \sim \frac{1}{\pi(1-x)}, x \to 1,$$
$$G(x) \sim -\frac{1}{\pi x}, x \to 0,$$

and use equality (2.3.2) to provide the necessary statement.

Exercise 2.3.2 (*answer*)

$$Var(1/U_{k,n}) = \frac{n(n-k+1)}{(k-1)^2(k-2)}, 3 \le k \le n.$$

Exercise 2.3.3 (*answer*)

$$E\left(U_{r,n}^{\alpha} U_{s,n}^{\beta}\right) = \frac{n!\Gamma(r+\alpha)\Gamma(s+\alpha+\beta)}{(r-1)!\Gamma(s+\alpha)\Gamma(n+1+\alpha+\beta)}.$$

Exercise 2.3.4 (*solution*) Let $r \le s$. Due to (2.2.5),

$$\begin{aligned}
Cov(Z_{r,n}, Z_{s,n}) &= Cov\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}, \frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_s}{n-s+1}\right) \\
&= Cov\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}, \frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}\right) \\
&\quad + Cov\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}, \frac{v_{r+1}}{n-r} + \cdots + \frac{v_s}{n-s+1}\right).
\end{aligned}$$

Since sums

$$\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}$$

and

$$\frac{v_{r+1}}{n-r} + \cdots + \frac{v_s}{n-s+1}$$

are independent, we get that

$$\begin{aligned}
Cov(Z_{r,n}, Z_{s,n}) &= Cov\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}, \frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}\right) \\
&= Var\left(\frac{v_1}{n} + \frac{v_2}{n-1} + \cdots + \frac{v_r}{n-r+1}\right) = Var Z_{r,n} = \sum_{k=1}^{r} \frac{1}{(n-k+1)^2}.
\end{aligned}$$

Exercise 2.3.5 (*solution*) From symmetry of $N(0,1)$ distribution (note that X has the same distribution as $-X$) it follows that $EX_{2,3} = 0$ and $EX_{1,3} = -EX_{3,3}$. Thus, we need to find $EX_{3,3}$ only. One sees that

$$EX_{3,3} = 3 \int_{-\infty}^{\infty} x\Phi^2(x)\varphi(x)dx = 3 \int_{-\infty}^{\infty} \Phi^2(x)d\varphi(x)$$
$$= 6 \int_{-\infty}^{\infty} \Phi(x)\varphi^2(x)dx.$$

Consider

$$I(a) = \int_{-\infty}^{\infty} \Phi(ax)\varphi^2(x)dx.$$

We obtain that

$$I(0) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi^2(x)dx = \frac{1}{4\pi} \int_{-\infty}^{\infty} \exp\left(-x^2\right)dx = 1/(4\sqrt{\pi})$$

and

$$I'(a) = \int_{-\infty}^{\infty} x\varphi(ax)\varphi^2(x)dx = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} x\exp\left\{-x^2\left(a^2+2\right)\right\}dx = 0.$$

It means that

$$I(a) = 1/(4\sqrt{\pi})$$

and, in particular,

$$\int_{-\infty}^{\infty} \Phi(ax)\varphi^2(x)dx = I(1) = 1/(4\sqrt{\pi}).$$

Finally, we have that

$$EX_{3,3} = 6\,I(1) = \frac{3}{2\sqrt{\pi}}.$$

Due to the symmetry of N(0,1) distribution, one obtains also that

$$E\left(X_{1,3}\right)^2 = E\left(X_{3,3}\right)^2$$

and

$$VarX_{1,3} = Var\ X_{3,3}.$$

Now

$$\begin{aligned}
E(X_{3,3})^2 &= 3\int_{-\infty}^{\infty} x^2\Phi^2(x)\varphi(x)dx = -3\int_{-\infty}^{\infty} x\Phi^2(x)d\varphi(x)\\
&= 3\int_{-\infty}^{\infty}\varphi(x)d(x\Phi^2(x)) = 3\int_{-\infty}^{\infty}\varphi(x)\Phi^2(x)dx + 6\int_{-\infty}^{\infty} x\varphi^2(x)\Phi(x)dx\\
&= \int_{-\infty}^{\infty} d(\Phi^3(x)) + \frac{3}{\pi}\int_{-\infty}^{\infty} x\exp(-x^2)\Phi(x)dx = 1 - \frac{3}{2\pi}\int_{-\infty}^{\infty}\Phi(x)d(\exp(-x^2))\\
&= 1 + \frac{3}{2\pi}\int_{-\infty}^{\infty}\exp(-x^2)\varphi(x)dx = 1 + \frac{3}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\exp(-3x^2/2)dx = 1 + \frac{\sqrt{3}}{2\pi}.
\end{aligned}$$

Taking into account that

$$EX_{3,3} = \frac{3}{2\sqrt{\pi}},$$

we obtain that

$$Var\left(X_{3,3}\right) = E\left(X_{3,3}\right)^2 - \left(EX_{3,3}\right)^2 = 1 + \frac{\sqrt{3}}{2\pi} - \frac{9}{4\pi}.$$

Further,

$$\begin{aligned}
E\left(X_{2,3}\right)^2 &= 6\int_{-\infty}^{\infty} x^2\Phi(x)(1-\Phi(x))\varphi(x)dx\\
&= 6\int_{-\infty}^{\infty} x^2\Phi(x)\varphi(x)dx - 6\int_{-\infty}^{\infty} x^2\Phi^2(x)\varphi(x)dx\\
&= 6\int_{-\infty}^{\infty} x^2\Phi(x)\varphi(x)dx - 2E\left(X_{3,3}\right)^2.
\end{aligned}$$

Now we obtain that

$$\begin{aligned}\int\limits_{-\infty}^{\infty} x^2\Phi(x)\varphi(x)dx = &-\int\limits_{-\infty}^{\infty} x\Phi(x)d\varphi(x)\\ &= \int\limits_{-\infty}^{\infty} \varphi(x)d(x\Phi(x)) = \int\limits_{-\infty}^{\infty} \varphi(x)\Phi(x)dx + \int\limits_{-\infty}^{\infty} x\varphi^2(x)dx\\ &= \frac{1}{2}\int\limits_{-\infty}^{\infty} d\left(\Phi^2(x)\right) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} x\exp\left(-x^2\right)dx = \frac{1}{2}.\end{aligned}$$

Hence,

$$E\left(X_{2,3}\right)^2 = 3 - 2E\left(X_{3,3}\right)^2 = 3 - 2\left(1 + \frac{\sqrt{3}}{2\pi}\right) = 1 - \frac{\sqrt{3}}{\pi}$$

and

$$Var\left(X_{2,3}\right) = E\left(X_{2,3}\right)^2 - \left(EX_{2,3}\right)^2 = 1 - \frac{\sqrt{3}}{\pi},$$

so far as $EX_{2,3} = 0$. Thus,

$$\begin{aligned}E\left(X_{1,3}\right)^2 &= E\left(X_{3,3}\right)^2 = 1 + \frac{\sqrt{3}}{2\pi},\\ E\left(X_{2,3}\right)^2 &= 1 - \frac{\sqrt{3}}{\pi},\\ Var\left(X_{1,3}\right) &= Var\left(X_{3,3}\right) = 1 + \frac{\sqrt{3}}{2\pi} - \frac{9}{4\pi}\end{aligned}$$

and

$$Var\left(X_{2,3}\right) = E\left(X_{2,3}\right)^2 - \left(EX_{2,3}\right)^2 = 1 - \frac{\sqrt{3}}{\pi}.$$

Exercise 2.4.1 (*solution*) It is evident that

$$\begin{aligned}P(M(n) - \log n < x) = (1 - \exp(-x - \log n))^n = (1 - \exp(-x)/n)^n\\ \to \exp(-\exp(-x)),\end{aligned}$$

as $n \to \infty$.

Exercise 2.4.2 (*solution*) In this case

$$P\{n^{1/\alpha}M(n) < x\} = (1 - (-x/\, n^{1/\alpha})^{\alpha})^{n} = (1 - (-x)^{\alpha}/n)^{n} \to \exp(-(-x)^{\alpha})$$
$$= H_{2,\alpha}(x),$$

as $n \to \infty$, for any x, $-\infty < x \le 0$.

Exercise 2.4.3 (*solution*) One can see that

$$P\{M(n)/n^{1/\alpha} < x\} = (1 - (x\,n^{1/\alpha})^{-\alpha})^{n} = (1 - x^{-\alpha}/n)^{n} \to \exp\{-x^{-\alpha}\},$$

as $n \to \infty$, for any $x \ge 0$.

Exercise 2.4.4 (*hint and answer*) In this case

$$(1 - F(x)) \sim 1/\pi x \text{ and } F'(x) \sim 1/\pi x^2, \ as\, x \to \infty.$$

Use the statement of Theorem 2.4.1 to see that $\alpha = 1$, centering constants $a_n = 0$, and normalizing constants b_n can be taken from the equality $F(b_{n)} = 1 - 1/n$, that is possible to take $b_n = n/\pi$.

Exercise 2.4.5 (*hint*) In this situation it is easy to show that

$$1 - F(x) \sim F'(x)/x = \frac{1}{x\sqrt{2\pi}}\exp\left(-x^2/2\right),$$
$$F''(x) \sim -xF'(x),\ x \to \infty,$$

and then to use the statement of Theorem 2.4.3. To find constants a_n one needs to solve the equation

$$1/n = 1 - F(a_n) \sim \frac{1}{a_n\sqrt{2\pi}}\exp\left(-(a_n)^2/2\right)$$

or simply to check that the sequence

$$a_n = (2\log n - \log\log n - \log\, 4\pi)^{1/2}$$

satisfies this equation. Theorem 2.4.3 recommends to take $b_n \sim h(a_n)$, where

$$h(x) = 1 - F(x)/F'(x).$$

In this case one can obtain that $b_n \sim (2\log n)^{1/2}$ suits us.

Exercise 2.4.6 (*answer*) It follows from Theorem 2.4.2 that in this case

$$F \in D(H_1),\ a_n = \log n, b_n = 1.$$

Exercise 2.5.1 (*answers*) Here

$$ER(k) = (n+1)/2, Var\,R(k) = (n-1)^2/12,\ 1 \le k \le n,$$
$$Cov(R(k), R(m)) = -(n-1)/12,\ 1 \le k < m \le n,$$

and

$$\rho(R(k), R(m)) = -1/(n-1),\ 1 \le k < m \le n.$$

Exercise 2.5.2 (*answers*) $P\{\varDelta(1) = 1,\ R(1) = 1\} = 1/n$,

$$P\{\varDelta(1) = 1, R(1) = s\} = P\{\varDelta(1) = s, R(1) = 1\} = 0,\ if\ s \ne 1,$$

and

$$P\{\varDelta(1) = m, R(1) = s\} = 1/n(n-1), if\ s \ne 1,\ m \ne 1.$$

Exercise 2.5.3 (*solution*) Since

$$P\{\rho(m) = k\} = 1/m, k = 1, 2, \ldots m,$$

it is enough to show that for any $n = 1, 2, \ldots$, and any a(k), taking on values $1, 2, \ldots, k$, $1 \le k \le n$,

$$P\{\rho(1) = a(1), \rho(2) = a(2), \ldots, \rho(n) = a(n)\} = 1/n!.$$

Fix n and consider ranks $R(1), R(2), \ldots, R(n)$. It is not difficult to see that a set $\{a(1), a(2), \ldots, a(n)\}$ uniquely determines values $r(1), r(2), \ldots, r(n)$ of $R(1), R(2), \ldots, R(n)$. In fact, $r(n) = a(n)$. Further,

$$r(n-1) = a(n-1), \text{if a(n)} > \text{a(n}-1), \text{ and } r(n-1)$$
$$= a(n-1) + 1, \text{if } a(\text{n}) \le a(\text{n}-1).$$

The value of $R(n-2)$ is analogously determined by values $a(n)$, $a(n-1)$ and $a(n-2)$ and so on. Hence, each of $n!$ events

$$\{\rho(1) = a(1), \rho(2) = a(2), \ldots, \rho(n) = a(n)\}$$

coincides with one of $n!$ events

$$\{R(1) = r(1), R(2) = r(2), \ldots, R(n) = r(n)\}.$$

For instance,

$$\{\rho(1) = 1, \rho(2) = 1, \ldots, \rho(n) = 1\} = \{R(1) = n, R(2) = n-1, \ldots, R(n) = 1\}.$$

Since

$$P\{R(1) = r(1), R(2) = r(2), \ldots, R(n) = r(n)\} = 1/n!$$

for any permutation $(r(1), r(2), \ldots, r(n))$ of the numbers $1, 2, \ldots, n$, one gets that

$$P\{\rho(1) = a(1), \rho(2) = a(2), \ldots, \rho(n) = a(n)\} = 1/n!$$

for any set $\{a(1), a(2), \ldots, a(n)\}$, where $1 \le a(k) \le k$, $k = 1, 2, \ldots, n$.

Chapter 3
Record Times

3.1 Introduction

> Very close to order statistics are the so-called record times and record values. Beginning from the Chandler's (1952) pioneer paper records became very popular in the probability theory and statistics.

Let $X_1, X_2, \ldots$ be a sequence of random variables and $X_{1,n} \leq \cdots \leq X_{n,n}$, $n = 1, 2, \ldots$, be the corresponding order statistics. Let us consider the increasing sequence of the sequential maximal values

$$M(1) \leq M(2) \leq \cdots \leq M(n-1) \leq M(n) \leq \cdots \leq \cdots$$

and fix the times, when signs of the strong inequality appear in this sequence. Such times correspond to the situations, when $M(n) > M(n-1)$. It means that $X_n > M(n-1)$. The random variable X_n, which is more than all previous X's, is called the *upper record value*. Correctly speaking we deal here with the *strong upper record*. The matter is that sometimes (say, in some sport competitions) the repetition of the previous record value also is considered as a record. It means that in such cases we have situations when $X_n = M(n-1)$ for some n. In this situation we deal with the so-called *weak record value*. In reality these types of records are different only if distribution functions of X's, have discontinuity points. If some X_n is a record value, then the corresponding index $\boldsymbol{n}$ is named as an *upper record time*.

One more situation is symmetrical to the previous. It is the case when one considers a sequence of minimal values $m(1) \geq m(2) \geq \cdots \geq m(n-1) \geq m(n) \geq \cdots$. The appearance of the sign of strong inequality in this sequence signifies the appearance of the *strong lower record*. Analogously, if $X_n = m(n-1)$ for some n, one deals with a *weak lower record value*. Note that in all options X_1 is taken as the first (upper or lower, strong or weak) record value.

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_3

We mentioned above that these two constructions (upper and lower records) are symmetrical in some sense. Really, if together with some sequence X_1, X_2,… one considers the sequence $Y_1 = -X_1, Y_2 = -X_2, \ldots, Y_n = -X_n, \ldots$, then it becomes evident that the lower record times for Y's coincide with the corresponding upper record times for X's and the lower record values $Y(r)$ and the upper record values $X(r)$ satisfy the equality $Y(r) = -X(r)$, $r = 1, 2, \ldots$. It means that it is enough to study in details only one of these record types. Below we will deal as a rule with the upper record times and the upper record values.

Denote $X(1) < X(2) < \cdots$ the corresponding record values (strong upper record values) in the sequence X_1, X_2,⋯ and let $1 = L(1) < L(2) < \cdots$ be the corresponding record times. Introduce also record indicators $\xi_n, n = 1, 2, \ldots$, which take values 0 and 1, and mark the appearance of record values, that is $\xi_n = 1$, if $X_n > M(n-1)$, and $\xi_n = 0$, otherwise. As we agreed above, $L(1) = 1, X(1) = X_1$ and $\xi_1 = 1$. Note that $\xi_{L(n)} = 1, n = 1, 2, \ldots$.

3.2 Definitions of Record Values and Record Times

As it was mentioned above we will deal as a rule with the upper records. Let $X_1, X_2, \ldots$ be a sequence of random variables and $X_{1,n} \leq X_{2,n} \leq X_{n,n}, n = 1, 2, \ldots$, be the corresponding order statistics. For any $n = 1, 2, \ldots$ we have also that $X_{n,n} = M(n) = \max\{X_1, X_2, \ldots, X_n\}$. One can define the classical *upper record times* $L(n)$ and *upper record values* $X(n)$ as follows:

$$L(1) = 1,\; X(1) = X_1$$

and then

$$L(n+1) = \min\{j : X_j > X(n)\}, X(n+1) = X_{L(n+1)}, n = 1, 2, \ldots \tag{3.2.1}$$

One can use the following alternative definitions:

$$L(1) = 1, L(n+1) = \min\{j :\, X_j > M(L(n))\},\; n = 1, 2, \ldots,$$

and

$$X(n) = M(L(n)), n = 1, 2, \ldots.$$

Using the sign ≥ in (3.2.1) instead of > we introduce *weak upper records*, when any repetition of the previous record value is also considered as a new record.

One more definition—the definition of the so-called *inter-record times* $\Delta(n), n = 1, 2, \ldots$, is closely connected with the record times. These random variables for the strong upper records are defined as follows:

$$\Delta(1) = L(1) = 1, \Delta(n) = L(n) - L(n-1),\ n = 2, 3, \ldots.$$

Indeed, one immediately has the next equality:

$$L(n) = \Delta(1) + \Delta(2) + \cdots + \Delta(n), n = 1, 2, \ldots.$$

Analogously it is possible to define inter-record times for lower records.

The *k*th records are a natural extension of the classical records. The *k*th *record times* $L(n,k)$ and the *k*th *record values* $X(n,k)$ for any $k = 1, 2, \ldots$ are defined as follows:

$$L(1,k) = k, L(n+1,k) = \min\{j > L(n,k) : X_j > X_{j-k,j-1}\},\ n = 1, 2, \ldots, \tag{3.2.2}$$

and

$$X(n,k) = X_{L(n,k)-k+1,\, L(n,k),}\, n = 1, 2, \ldots. \tag{3.2.3}$$

To be precise, (3.2.2) and (3.2.3) define the kth upper record times and the kth upper record values respectively. Indeed, if we change in (3.2.2) equality $X_j > X_{j-k,j-1}$ by relation $X_j \geq X_{j-k,j-1}$ we get the definitions of the *k*th *weak record times* $L(n,\ k)$ and the *k*th *weak record values* $X(n,k)$.

If $k = 1$ then definitions of *k*th record values $X(n,k)$ and *k*th record times $L(n, k)$ coincide with the definitions of $X(n)$ and $L(n)$ given in (3.2.1).

Exercise 3.2.1 Give the definitions of lower record times $l(n), n = 1, 2, \ldots,$ and lower record values $x(n), n = 1, 2, \ldots,$ and the corresponding definitions of the kth lower record times $l(n,k)$ and the kth lower record values $x(n,k)$.

Remark 3.2.1 Note that the theory of record values is very close to the theory of extremal random variables. Really,

$$X(n) = X_{L(n),L(n)} = M(L(n)),\ n = 1, 2, \ldots \tag{3.2.4}$$

and

$$x(n) = X_{l(n),l(n)} = m(l(n)),\ n = 1, 2, \ldots. \tag{3.2.5}$$

Analogously,

$$X(n,k) = X_{L(n,k)-k+1,\, L(n,k),}\, n = 1, 2, \ldots \tag{3.2.6}$$

and

$$x(n,k) = X_{k,\,l(n,k),}\, n = 1,2,\ldots. \tag{3.2.7}$$

Hence the asymptotic distributions of record values (upper and lower) are close in some sense to the corresponding distributions of maximal and minimal order statistics.

3.3 Record Indicators

Let us introduce record indicators ξ_n, $n = 1, 2,\ldots$, which take values 0 and 1 and mark the appearance of the strong upper record values, that is, $\xi_n = 1$, if $X_n > M(n-1)$, and $\xi_n = 0$, otherwise. As we agreed above, $L(1) = 1$, $X(1) = X_1$ and hence $\xi_1 = 1$. Note that $\xi_{L(n)} = 1$, $n = 1, 2,\ldots$.

Indeed, the corresponding indicators can be introduced analogously for the lower records, as well as for the kth records (upper and lower). Consider the classical case when the initial X's are independent and have the same continuous distribution function $F(x)$. In this case the equality $\xi_n = 1$ corresponds to the event $\{M(n) = X_n\}$. For continuous $F(x)$ some of X's can coincide with the zero probability. In this situation events $\{M(n) = X_1\}$, $\{M(n) = X_2\},\ldots$, $\{M(n) = X_n\}$ must have equal probabilities and we get immediately that

$$P\{\xi_n = 1\} = P\{M(n) = X_n\} = 1/n, n = 1,2,\ldots. \tag{3.3.1}$$

Note also that in this situation

$$P\{\xi_1 = 1, \xi_2 = 1,\ldots,\xi_n = 1\} = P\{X_1 < X_2 < \cdots < X_n\} = 1/n!,\ n = 1,2,\ldots. \tag{3.3.2}$$

Exercise 3.3.1 Based on equalities (3.3.1), (3.3.2) and the analogous equalities for any set of indicators show that if the initial X's are independent and have the same continuous distribution function F then ξ_1, $\xi_2,\ldots$ are independent random variables.

Moreover, the following result (see Exercise 3.3.2) is also valid for record indicators.

Exercise 3.3.2 Show that if X's are independent and have the same continuous distribution function F then for any $n = 1, 2,\ldots$ record indicators ξ_1, $\xi_2,\ldots$, ξ_n and maximal value $M(n)$ are independent random variables.

Now let $N(n)$ denote the number of the strong upper record values among the random variables X_1, $X_2,\ldots,X_n$. It is evident that $N(n)$ can be expressed as follows:

$$N(n) = \xi_1 + \xi_2 + \cdots + \xi_n,\ n = 1, 2, \ldots. \tag{3.3.3}$$

Any information about the distribution of $N(n)$ helps us to investigate the distribution properties of record times $L(n)$. Really, the following evident equalities are valid:

$$P\{L(n) > m\} = P\{N(m) < n\} \tag{3.3.4}$$

and

$$P\{L(n) = m\} = P\{N(m-1) = n-1, N(m) = n\}. \tag{3.3.5}$$

Hence any results for record indicators and for their sums $N(n)$ are very important.

Note that record indicators can be expressed also via sequential ranks of random variables X_1, X_2,…. The definition of these ranks $\rho(1)$, $\rho(2)$,… was given by equality (2.5.10). It was shown earlier that if X's are independent random variables having the same continuous distribution function then $\rho(1)$, $\rho(2)$,… are also independent and

$$P\{\rho(m) = k\} = 1/m,\ k = 1, 2, \ldots, m.$$

It is evident, that events $\{\xi_n = 1\}$ and $\{\rho_n = n\}$ coincide for any n = 1, 2,…. Hence we have one more way to find distributions of random indicators:

$$P\{\xi_n = 1\} = 1 - P\{\xi_n = 0\} = P\{\rho_n = n\} = 1/n,\ n = 1, 2,\ \ldots.$$

Immediately we get that $E\xi_n = 1/n$ and $Var\,\xi_n = (n-1)/n^2, n = 1, 2, \ldots.$ Hence

$$EN(n) = 1 + 1/2 + \cdots + 1/n \tag{3.3.6}$$

and

$$Var\,(N(n)) = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k^2}\right),\ n = 1, 2, \ldots. \tag{3.3.7}$$

Note that

$$EN(n) \sim \log n, \quad n \to \infty, \tag{3.3.8}$$

and

$$Var\,(N(n)) \sim \log n, \quad n \to \infty. \tag{3.3.9}$$

Expression (3.3.3) implies that the generating function $P_n(s)$ of $N(n)$ satisfies the following equalities:

$$\begin{aligned} P_n(s) = \ & Es^{N(n)} = \prod_{j=1}^{n} Es^{\xi_j} \\ & = \prod_{j=1}^{n} (1 + (s-1)/j) = s(1+s)(2+s)\cdots(n-1+s)/n! \end{aligned} \tag{3.3.10}$$

and

$$P_n(-s) = (-1)^n s(s-1)\cdots(s-n+1)/n!. \tag{3.3.11}$$

The expression (3.3.11) enables us to use Stirling numbers of the first kind, which are defined by equalities

$$x(x-1)\cdots(x-n+1) = \sum_{k \geq 0} S_n^k x^k. \tag{3.3.12}$$

Exercise 3.3.3 Show (taking into account (3.3.10)–(3.3.12)) that

$$P\{N(n) = k\} = (-1)^k S_n^k/n! = |S_n^k|/n!,\ n = 1, 2, \ldots,\ k = 1, 2, \ldots, n. \tag{3.3.13}$$

Above (see (3.2.2)) the so-called kth record times $L(n, k)$ were defined for any $k = 1, 2, \ldots$. Indeed, the corresponding indicators $\xi_n(k)$ of the kth records also can be defined as follows:

$$\xi_n(k) = 1,\ \text{if}\ X_n > X_{n-k,n-1},\ \text{and}\ \xi_n(k) = 0, \quad \text{otherwise},\ n = k, k+1, \ldots. \tag{3.3.14}$$

Exercise 3.3.4 Give the alternative definition of indicators $\xi_n(k)$ via sequential ranks $\rho(1), \rho(2), \ldots$ and prove that

$$P\{\xi_n(k) = 1\} = k/n,\ n = k,\ k+1, \ldots. \tag{3.3.15}$$

Indeed, if we consider now the numbers $N(n, k)$ of the kth records among random variables $X_1, X_2, \ldots, X_n$, then the following equality can be used:

$$N(n,k) = \xi_k(k) + \xi_{k+1}(k) + \cdots + \xi_n(k),\ n = k, k+1, \ldots,\ k = 1, 2, \ldots. \tag{3.3.16}$$

One more important situation in the classical record theory is connected with sequences of independent identically distributed random variables having a discrete distribution. Without loss of generality, we can suppose that X's take nonnegative integer values. For discrete distributions we introduce another type of record indicators.

Let $\eta_n = 1$ if n is a record value in the sequence $X_1, X_2, \ldots$, that is, there exists such $m = 1, 2,\ldots$ that $X(m) = n$, and $\eta_n = 0$ otherwise (compare with indicators ξ_n!). Analogously, for any $k = 1, 2,\ldots$ we can introduce indicators $\eta_n(k)$ for the kth record values: $\eta_n(k) = 1$, if n is a kth record value in the sequence $X_1, X_2,\ldots$, and $\eta_n(k) = 0$ otherwise. The following result is valid for such type of indicators.

Theorem 3.3.1 *Let $X, X_1, X_2,\ldots$ be a sequence of independent identically distributed random variables taking values 0, 1, 2,… with probabilities $p_n = P\{X = n\} > 0$, $n = 0, 1, 2,\ldots$. Then for any fixed $k = 1, 2,\ldots$ indicators $\eta_n(k)$, $n = 0, 1, 2,\ldots$, are independent and*

$$P\{\eta_n(k) = 1\} = 1 - P\{\eta_n(k) = 0\} = (p_n/P\{X \geq n\})^k,\ n = 0, 1, 2, \ldots. \tag{3.3.17}$$

Exercise 3.3.5 Prove Theorem 3.3.1 for the simplest partial case $k = 1$ and show that record indicators $\eta_0, \eta_1, \eta_2,\ldots$ are independent and

$$P\{\eta_n = 1\} = 1 - P\{\eta_n = 0\} = p_n/P\{X \geq n\},\ n = 0, 1, 2, \ldots. \tag{3.3.18}$$

It is easy to see that under conditions of Theorem 3.3.1 one can express distributions of the kth record values for discrete random variables via distributions of sums of independent indicators:

$$P\{X(n,k) > m\} = P\{\eta_0(k) + \cdots + \eta_m(k) < n\},\ m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots, \tag{3.3.19}$$

and, in particular, under k = 1 we have equality

$$P\{X(n) > m\} = P\{\eta_0 + \cdots + \eta_m < n\},\ m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots. \tag{3.3.20}$$

Exercise 3.3.6 Consider the case, when X's have the geometric distribution with some parameter $0 < p < 1$, that is,

$$P\{X_j = n\} = (1-p)p^n,\ n = 0, 1, 2, \ldots,$$

for $j = 1, 2,\ldots$, and show that in this situation the sum $(\eta_0 + \cdots + \eta_m)$ of record indicators has the binomial $B(m + 1, q)$-distribution with a parameter $q = (1 - p)$.

It was mentioned above that for discrete distributions it is useful to introduce weak records together with classical (strong) record values. Weak records may arise, for example, in some sport competitions when any athlete who repeats the previous record achievement is also declared as a record-holder. If we consider X's having a common discrete distribution it is useful to introduce one more type of random variables which generalize the concept of record indicators $\eta_0, \eta_1, \eta_2, \ldots$. We define now random variables $\mu_0, \mu_1, \mu_2, \ldots$, where μ_n denotes the number of those weak records in the sequence $X_1, X_2, \ldots$ that are equal to n. The following result is valid.

Theorem 3.3.2 *Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables taking values 0, 1, 2,... with probabilities $p_n = P\{X = n\} > 0$, $n = 0, 1, 2, \ldots$. Then random variables $\mu_0, \mu_1, \mu_2, \ldots$ are independent and*

$$P\{\mu_n = m\} = (1 - r(n))\,(r(n))^m,\ n = 0, 1, 2, \ldots;\ m = 0, 1, 2, \ldots, \tag{3.3.21}$$

where

$$r(n) = p_n/P\{X \geq n\}.$$

Exercise 3.3.7 Show that equality (3.3.21) is valid for the partial case, when X's have the geometric distribution with some parameter $0 < p < 1$, that is, consider the situation with probabilities $p_n = P\{X_j = n\} = (1-p)p^n$, $n = 0, 1, 2, \ldots$, for $j = 1, 2, \ldots$.

Let $X_\omega(1), X_\omega(2), \ldots$ denote the weak (upper) record values in the sequence $X_1, X_2, \ldots$. Then for any $n = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ the following relation is valid:

$$P\{X_\omega(n) > m\} = P\{\mu_0 + \mu_{1+\cdots+}\mu_m < n\}.$$

Thus we see that there are some very convenient representations of record values, record times, numbers of records, which allow us to impress these record statistics in terms of sums of independent random variables.

3.4 Limit Theorems for Numbers of Records

Above we got some results for numbers of records $N(n)$ among random variables $X_1, X_2, \ldots, X_n$. For example, it was shown (see (3.3.13)) that the corresponding distributions are expressed via Stirling numbers of the first kind:

$$P\{N(n) = k\} = (-1)^k S_n^k/n!\,,\ n = 1, 2, \ldots,\ k = 1, 2, \ldots.$$

The expression given here is not very convenient to work with it under large values n. Hence in such situations it is better to know the simple asymptotic formulae for these probabilities. The presentation of $N(n)$ via the independent indicators helps us to solve this problem. Applying to $N(n)$ the classical limit theorems for sums of independent random variables gives us the possibility to obtain the following statements which describe the asymptotic behavior of $N(n)$ under the assumption that $n \to \infty$. The asymptotic results given below are formulated for the independent X's having any joint continuous distribution function F. One can get these limit (as $n \to \infty$) theorems immediately, taking into account relations (3.3.3), (3.3.8), (3.3.9).

(a) *Central Limit theorem*:

$$\sup_x |P\{N(n) - \log n < x\sqrt{\log n}\} - \Phi(x)| \to 0, \tag{3.4.1}$$

$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt$ *being the distribution function of the standard normal law.*

(b) *Uniform estimate in Central Limit theorem*:

$$\sup_x |P\{N(n) - \log n < x\sqrt{\log n}\} - \Phi(x)| \le C/\sqrt{\log n},\ n = 1, 2, \ldots, \tag{3.4.2}$$

C being some absolute constant.

(c) Strong Law of Large Numbers:

$$P\{\lim(N(n)/\log n) = 1\} = 1. \tag{3.4.3}$$

(d) *Laws of Iterative Logarithm:*

$$P\{limsup \frac{N(n) - \log n}{(2\log n\ \ \log\log\log n)^{1/2}} = 1\} = 1 \tag{3.4.4}$$

and

$$P\{liminf \frac{N(n) - \log n}{(2\log n\ \ \log\log\log n)^{1/2}} = -1\} = 1. \tag{3.4.5}$$

Exercise 3.4.1 Generalize relations (3.4.1) and (3.4.2) for numbers $N(n, k)$ of the kth record values in the sequences of independent random variables X_1, X_2,…, having a joint continuous distribution function $F(x)$.

3.5 Distributions of Record Times

Let us consider now the upper record times $L(n)$. These random variables are rather close to numbers of records $N(n)$. Really,

$$N(L(n)) = n,\ n = 1, 2, \ldots,$$

and recalling relations (3.3.4) and (3.3.5) one can write that

$$P\{L(n) > m\} = P\{N(m) < n\},\ n = 1, 2, \ldots,\ m = 1, 2, \ldots,$$

and

$$\begin{aligned} P\{L(n) = m\} &= P\{N(m-1) = n-1,\ \xi_m = 1\} \\ &= P\{N(m-1) = n-1\}/m,\ 1 \le n \le m. \end{aligned} \tag{3.5.1}$$

Since (see (3.3.13))

$$P\{N(n) = k\} = (-1)^k S_n^k / n! = |S_n^k| \,/\, n!,\ n = 1, 2, \ldots,\ k = 1, 2, \ldots,$$

we get the following expression (in terms of Stirling numbers of the first kind) for record times:

$$P\{L(n) = m\} = |S_{m-1}^{n-1}|/m!,\ n = 1, 2, \ldots,\ m = n, n+1, \ldots. \tag{3.5.2}$$

Based on properties of Stirling numbers Westcott (1977a) showed that

$$P\{L(n) = m\} \sim (\log m)^{n-2}/m^2(n-2)! \tag{3.5.3}$$

as $m \to \infty$.

Relations (3.3.5) and (3.5.1) help us to find generating functions

$$Q_n(s) = Es^{L(n)},\ n = 1, 2, \ldots,$$

of record times. Since $P\{L(1) = 1\} = 1$, it is clear that

$$Q_1(s) = s.$$

For $n = 2, 3, \ldots$, $|s| < 1$ and $|z| < 1$, one gets equalities

$$Q_n(s) = \sum_{m=1}^{\infty} P\{L(n) = m\} s^m = \sum_{m=1}^{\infty} \frac{1}{m} P\{N(m-1) = n-1\} s^m$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} Q_n(s)z^n &= \sum_{m=1}^{\infty} \frac{s^m}{m} \sum_{n=1}^{\infty} P\{N(m-1) = n-1\} z^n \\ &= z\sum_{m=1}^{\infty} \frac{s^m}{m} P_{m-1}(z) = z\sum_{m=1}^{\infty} \frac{s^m}{m!} z(1+z)\cdots(m-2+z) \\ &= -\frac{z}{1-z} \sum_{m=1}^{\infty} \frac{(-s)^m}{m!}(1-z)(-z)(-1-z)\cdots(2-m-z).\end{aligned} \tag{3.5.4}$$

Note that

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{(-s)^m}{m!}(1-z)(-z)(-1-z)\cdots(2-m-z) = (1-\mathrm{s})^{1-z} \\ &= (1-s)\,\exp\{-z\,\log(1-s)\}\end{aligned} \tag{3.5.5}$$

and then (3.5.4) and (3.5.5) imply that

$$(1-z)\sum_{n=1}^{\infty} Q_n(s)z^{n-1} = -(1-s)\exp\{-z\log(1-s)\} + 1. \tag{3.5.6}$$

Transforming the LHS of (3.5.6) in

$$\sum_{n=0}^{\infty} Q_{n+1}\,(s)^n - \sum_{n=1}^{\infty} Q_n(s)z^n = s + \sum_{n=1}^{\infty} (Q_{n+1}(s) - \,Q_n(s))z^n,$$

allows us to come to the following equality:

$$1 - s - \sum_{n=1}^{\infty} (Q_{n+1}(s) - \,Q_n(s))z^n = \,(1-s)\sum_{n=0}^{\infty} \frac{(-\log(1-s))^n}{n!} z^n. \tag{3.5.7}$$

Let us denote that

$$R_n(s) = Q_n(s) - Q_{n+1}(s),\; n = 1, 2, \ldots.$$

Then one can get from (3.5.7) that

$$R_n(s) = \,(1-s)\frac{(-\log(1-s))^n}{n!},\; n = 1, 2, \ldots.$$

Hence

$$Q_n(s) = Q_1(s) - (R_1(s) + \cdots + R_{n-1}(s)) = s - (1-s)\sum_{r=1}^{n-1} \frac{(-\log(1-s))^r}{r!}$$
$$= 1 - (1-s)\sum_{r=0}^{n-1} \frac{(-\log(1-s))^r}{r!}. \tag{3.5.8}$$

Exercise 3.5.1 Prove that equality (3.5.8) can be rewritten as

$$Q_n(s) = \frac{1}{(n-1)!} \int_0^{-\log(1-s)} v^{n-1} \exp(-v) dv. \tag{3.5.9}$$

The independence property of record indicators enables us to get joint distributions of record times.

Theorem 3.5.3 *For any n = 1, 2,... and any integers 1 = j(1) < j(2) < ··· < j(n) the following equality holds:*

$$P\{L(1) = 1, L(2) = j(2), \ldots, L(n) = j(n)\}$$
$$= 1/(j(2)-1)(j(3)-1)\ldots(j(n)-1)j(n). \tag{3.5.10}$$

Proof Evidently, the event on the left side of (3.5.10) coincides with the event

$$A_n = \{\xi_2 = 0, \ldots, \xi_{j(2)-1} = 0, \xi_{j(2)} = 1, \xi_{j(2)+1} = 0, \ldots,$$
$$\xi_{j(3)-1} = 0, \xi_{j(3)} = 1, \ldots, \xi_{j(n-1)-1} = 0, \xi_{j(n-1)}$$
$$= 1, \xi_{j(n-1)+1} = 0 \ldots, \xi_{j(n)-1} = 0, \xi_{j(n)} = 1\}.$$

The independence property of record indicators allows us to obtain now that

$$P\{A_n\} = P\{\xi_2 = 0\}\ldots P\{\xi_{j(2)-1} = 0\}\, P\{\xi_{j(2)} = 1\}\, P\{\xi_{j(2)+1} = 0\}\ldots P\{\xi_{j(3)-1} = 0\}$$
$$P\{\xi_{j(3)} = 1\}\ldots P\{\xi_{i(n-1)-1} = 0\}\, P\{\xi_{j(n-1)} = 1\}\, P\{\xi_{j(n-1)+1} = 0\}\ldots P\{\xi_{j(n)-1} = 0\}\, P\{\xi_{i(n)} = 1\}$$
$$= \frac{1}{j(n)} \prod_{t=2}^{n} \frac{1}{j(t)-1},$$

and this expression coincides with the LHS of (3.5.10).

Corollary 3.5.1 *Indeed, one can get from* (3.5.10) *the following form (compare with* (3.5.2)) *for distributions of record times:*

$$P\{L(n)=m\} = \sum_{1<j(2)<\cdots<j(n-1)<m} (1/(j(2)-1)(j(3)-1)\cdots(j(n-1)-1)(m-1)m),\ m=n,\ n+1,\ldots. \tag{3.5.11}$$

In particular, for n = 2 we obtain that

$$P\{L(2)=m\} = 1/(m-1)m, m=2,3,\ldots, \tag{3.5.12}$$

and hence

$$P\{L(2)>m\} = 1/m,\ m=1,2,\ldots. \tag{3.5.13}$$

It is interesting to investigate the dependence structure of record times.

Taking into account equality (3.5.10) one gets immediately that for any $1 = j(1)<j(2)<\cdots<j(n-1)<j(n)<m$, the following equality holds:

$$P\{L(n+1)=m|L(n)=j(n),\ L(n-1)=j(n-1),\ldots,\ L(2)=j(2),\ L(1)=1\} = j(n)/m(m-1). \tag{3.5.14}$$

Also we obtain that

$$P\{L(n+1)=m|L(n)=j\} = j/m(m-1) 1 \le j<m. \tag{3.5.15}$$

Remark 3.5.1 Thus it follows from equalities (3.5.14) and (3.5.15) that the sequence of record times $L(1)$, $L(2)$,… forms a Markov chain. Recall that we consider the situation when the initial *random variables* X_1, X_2,... *are independent* and have a joint continuous distribution function.

Williams (1973) proved that the following presentation for record times is valid:

$$L(1)=1,\ L(n+1)=\ [L(n)\exp(W_n)]+1,\ n=1,2,\ldots, \tag{3.5.16}$$

where W_1, W_2,... are independent random variables having the standard exponential distribution and $[x]$ denotes the entire part of x. Evidently that this presentation can be rewritten as follows:

$$L(1)=1,\ L(n+1)=[L(n)/U_n]+1,\ n=1,2,\ldots, \tag{3.5.17}$$

where independent random variables U_1,U_2,… are uniformly distributed on [0, 1].

To prove (3.5.17) it suffices to show that $L(1)$, $L(2)$,…, defined by relations (3.5.17), also form a Markov chain with transition probabilities given by (3.5.15).

By construction, $L(n)$, given in (3.5.17), depends on U_1, $U_2,\ldots,U_{n-1}$ only and does not depend on U_n. Then

$$P\{L(n+1)=m|L(n)=j\} = P\{[L(n)/U_n]+1=m|L(n)=j\} = P\{[j/U_n]+1=m|L(n)=j\}$$
$$= P\{[j/U_n]+1=m\} = P\{j/m<U_n \le j/(m-1)\} = j/m(m-1).$$

Thus, presentation (3.5.17), as well as result (3.5.16), is valid.

A very interesting result for record times was obtained in Galambos and Seneta (1975). The integer-valued random variables $T(2)$, $T(3),\ldots$, which were defined by relations

$$T(n)-1<L(n)/L(n-1)\le T(n),\ n=2,3,\ldots, \tag{3.5.18}$$

were considered there. It was shown that $T(2)$, $T(3),\ldots$ are independent and

$$P\{T(n)=j\} = 1/j(j-1),\ j=2,3,\ldots,\ n=2,3,\ldots. \tag{3.5.19}$$

Exercise 3.5.2 Prove that equality (3.5.19) is valid.

3.6 Moment Characteristics of Record Times

Above the distributions of record times were investigated. Some problems appear with moments of these random variables. It follows from (3.5.12) that $EL(2) = \infty$. Hence $EL(n) = \infty$ for any $n = 2, 3,\ldots$, and

$$E(L(n))^{\alpha} = \infty,\ n=2,3,\ldots,\quad \text{if } \alpha \ge 1.$$

In this situation the following expression for logarithmic moment may be useful:

$$E\log L(n) = n - C - 2^{-(n+1)} + O(3^{-n}),\ n\to\infty, \tag{3.6.1}$$

where $C = 0.5772\ldots$ is Euler's constant.

Note also that the next analogous expression is valid for moments $E(L(n))^{1-\beta}$, $\beta > 0$:

$$E(L(n))^{1-\beta} = \frac{1}{\Gamma(\beta)}\{\beta^{-n} + \frac{\beta-1}{2}(\beta+1)^{-n} + O((\beta+2)^{-n})\},\ n\to\infty. \tag{3.6.2}$$

3.7 Exercises (solutions)

Exercise 3.2.1 (*solution*) The definitions of lower record times $l(n)$, $n = 1, 2,\ldots$, and lower record values $x(n)$, $n = 1, 2,\ldots$, are given as follows (compare with the corresponding definitions of upper record times and upper record values):

$l(1) = 1,\ x(1) = X_1$

and

$$l(n+1) = \min\{j:\ X_j < x(n)\}, x(n+1) = X_{l(n+1)},\ n = 1, 2, \ldots. \qquad (3.7.1)$$

Indeed, the following alternative definitions also can be used:

$$l(1) = 1,\ l(n+1) = \min\{j:\ X_j < m(l(n))\},\ n = 1, 2, \ldots,$$

and

$$x(n) = m(l(n)),\ n = 1, 2, \ldots.$$

Using the sign $\le$ in (3.7.1) instead of $<$ we get the definitions of the weak lower record times and the weak lower record values.

The kth lower record times $l(n, k)$ and the kth lower record values $x(n, k)$ for any $k = 1, 2,\ldots$ are defined as follows:

$$l(1,k) = k, l(n+1,k) = \min\{j > l(n,k):\ X_j < X_{k,j-1}\},\ n = 1, 2, \ldots,$$

and

$$x(n,k) = X_{k,\,l(n,k),} n = 1, 2, \ldots.$$

Exercise 3.3.1 (*solution*) It is enough to get that in this situation the following equalities are valid for any $n = 1, 2,\ldots$, $k = 1, 2,\ldots,n$ and any configurations $1 < r(1) < r(2) < \cdots < r(k) \le n$:

$$\begin{aligned} &P\{\xi_{r(1)} = 1, \ldots, \xi_{r(k)} = 1\} \\ &= P\{X_{r(1)} > M(r(1)-1), \ldots, X_{r(k)} > M(r(k)-1)\} = 1/r(1)r(2)\ldots r(k). \end{aligned} \qquad (3.7.2)$$

Note that in the partial case, when $k = n$, $r(1) = 1$, $r(2) = 2,\ldots,r(n) = n$, this equality coincides with (3.3.2). The simplest way to prove (3.7.2) is to use sequential ranks $\rho(m) = \sum_{k=1}^{m} 1_{\{X_m \ge X_k\}}, m = 1, 2, \ldots.$

Equality (3.7.2) in our case can be rewritten in the terms of $\rho(m)$ as follows:

$$P\{\xi_{r(1)} = 1, \ldots, \xi_{r(k)} = 1\} = P\{\rho(r(1)) = r(1), \ldots, \rho(r(k)) = r(k)\}. \quad (3.7.3)$$

From the independence property of sequential ranks (see, for example, Exercise 2.5.3 above) one gets that

$$P\{\xi_{r(1)} = 1, \ldots, \xi_{r(k)} = 1\} = P\{\rho(r(1)) = r(1)\} \ldots P\{\rho(r(k)) = r(k)\}. \quad (3.7.4)$$

We know also that

$$P\{\rho(n) = m\} = 1/n$$

for any $m = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$.

Hence (3.7.2) immediately follows from (3.7.4).

Exercise 3.3.2 (*solution*) Since the probability integral transformation does not affect the ordering of our random variables and preserves the distributions of record indicators it is sufficient to prove the necessary result for $F(x) = x$, $0 < x < 1$. In this situation

$$P\{M(n) < x\} = x^n, \; 0 < \mathrm{x} < 1,$$

and for any $k = 1, 2, \ldots, n$ and $1 \leq r(1) < r(2) \leq \cdots \leq r(k) \leq n$ we have the following necessary relations:

$$\begin{aligned} &P\{\xi_{r(1)} = 1, \xi_{r(2)} = 1, \ldots, \xi_{r(k)} = 1, \, M(n) < x\} \\ &\quad = x^{n-r(k)} \int_0^x v_1^{r(1)-1} dv_1 \int_{v_1}^x v_2^{r(2)-1} dv_2 \ldots \int_{v_{r-1}}^x v_k^{r(k)-1} dv_k = x^n / r(1) r(2) \cdots r(n) \\ &\quad = P\{M(n) < x\} P\{\xi_{r(1)} = 1\} P\{\xi_{r(2)} = 1\} \ldots P\{\xi_{r(k)} = 1\}. \end{aligned}$$

Exercise 3.3.3 (*solution*) From (3.3.10) and (3.3.11) one gets the expression for the following generating function

$$E(-s)^{N(n)} = \prod_{j=1}^{n} E(-s)^{\xi_j} = (-1)^n s(s-1) \cdots (s-n+1)/n!\,.$$

As we know, Stirling numbers of the first kind S_n^k are defined by equalities

$$s(s-1) \cdots (s-n+1) = \sum_{k \geq 0} S_n^k s^k.$$

Hence it follows immediately that

$$P\{N(n)=k\}=(-1)^k S_n^k/n! \; = \; |S_n^k|/n!,\ n=1,2,\ldots,\ k=1,2,\ldots,n.$$

Exercise 3.3.4 (*solution*) It is evident that the event $\{\xi_n(k)=1\}$ can be expressed in the terms of sequential ranks as $\{\rho(n)\geq n-k+1\}$. Recalling that

$$P\{\rho(n)=m\}=1/n,\ m=1,2,\ldots,n,$$

one immediately obtains that

$$P\{\xi_n(k)=1\}=k/n,\ n=k,\ k+1,\ldots.$$

Exercise 3.3.5 (*solution*) We get immediately that

$$\begin{aligned}P\{\eta_n=1\}&=P\{X_1=n\}+P\{X_1<n,X_2=n\}+P\{X_1<n,X_2<n,X_3=n\}+\cdots\\&=p_n+P\{X<n\}p_n+P^2\{X<n\}p_n+\cdots=p_n/(1-P\{X<n\})=p_n/P\{X\geq n\}.\end{aligned}$$

Analogous (but more complicated) proof allows to obtain for any $r=2,3,\ldots$ and $0\leq k(1)<k(2)<\cdots<k(r)$ the following equalities:

$$P\Big\{\eta_{k(1)}=1,\ \eta_{k(2)}=1,\ldots,\ \eta_{k(r)}=1\Big\}=\prod_{m=1}^{r}(p_{k(m)}/P\{X\geq k(m)\}).$$

These equalities show that the indicators $\eta_0,\ \eta_1,\ \eta_2,\ldots$ are independent.

Exercise 3.3.6 (*solution*) It follows from the previous exercise that record indicators $\eta_0,\ \eta_1,\ \eta_2,\ldots$ are independent, take one of two values 0 and 1 and in the considered case

$$P\{\eta_k=1\}=1-P\{\eta_k=0\}=(1-p),\ k=0,1,2,\ldots.$$

Hence the sum of $(m+1)$ two-valued independent identically distributed random variables has the binomial $B(m+1,\ 1-p)$ distribution.

Exercise 3.3.7 (*solution*) For this geometric distribution we have that

$$r(n)=p_n/P\{X\geq n\}=(1-p),\ n=0,1,2,\ldots.$$

Consider $n=0$. Then $r(0)=(1-p)$, $1-r(0)=p$. In this case one gets for $m=0,\ 1,\ 2,\ldots$ that

$$\begin{aligned}&P\{\mu_0=m\}=P\{X_1=0,X_2=0,\ldots,X_m=0,X_{m+1}>0\}=(1-p)^m P\{X_{m+1}>0\}\\&=(1-p)^m p=(1-r(0))(r(0))^m.\end{aligned}$$

If $n > 0$, we can take off all X's, which are less than n, from the sequence X_1, X_2, ... It means that now we begin to work with the new sequence Y_1, Y_2,…, where Y's are independent and

$$P\{Y = k\} = P\{X = k\,|X \geq n\} = (1-p)p^{k-n},\ k = n, n+1, \ldots.$$

Evidently, one can write now that

$$\begin{aligned} P\{\mu_n = m\} &= P\{Y_1 = n, Y_2 = n, \ldots, Y_m = n, Y_{m+1} > n\} = (1-p)^m p \\ &= (r(n))^m (1 - r(n)). \end{aligned}$$

Exercise 3.4.1 (*answers*) For numbers $N(n,\ k)$ of the kth record values in the sequences of independent random variables the following relations are valid:

$$\sup_x |P\{N(n,k) - k \log n < x\sqrt{k \log n}\} - \Phi(x)| \to 0, n \to \infty,$$

and

$$\sup_x |P\{N(n) - k \log n < x\sqrt{k \log n}\} - \Phi(x)| \leq C(k)/\sqrt{\log n},\ n = 1, 2, \ldots,$$

where

$$\Phi\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt$$

and the constant $C(k)$ depends on k only.

Exercise 3.5.1 (*hint*) It is enough to differentiate the RHS of (3.5.8) and the RHS of (3.5.9) and to compare them.

Exercise 3.5.2 (*hint*) Use the fact that

$$P\{T(n) = k\} = P\{(k-1)L(n+1) < L(n) \leq kL(n+1)\}$$

and recall that

$$P\{L(n+1) = m|L(n) = j\} = j/m(m-1), 1 \leq j < m,$$

as it was shown in (3.5.15).

Chapter 4
Record Values

4.1 Introduction

In the previous chapter together with record times we defined record values. Indeed, there are different types of these random variables as well as different models of record times. One can work with upper or lower, strong or weak record values. If $L(n)$, $n = 1, 2, \ldots$, are the corresponding record times then $X_{L(n)}$, $n = 1, 2, \ldots$ present the associate record values. We will study below the most popular model of the upper record values. It was shown in Chap. 3 that the classical upper record times $L(n)$ and upper record values $X(\mathrm{n})$ are defined as follows:

$$L(1) = 1,\ X(1) = X_1$$

and then

$$L(n+1) = \min\{j : X_j > X(n)\},\ X(n+1) = X_{L(n+1)},\ n = 1, 2, \ldots. \tag{4.1.1}$$

Indeed it is possible to use the following alternative definition for upper record values:

$$X(n) = M(L(n)) = \max\{X_1, X_2, \ldots, X_{L(n)}\} = X_{L(n),L(n)},\ n = 1, 2, \ldots. \tag{4.1.2}$$

Analogously we can define the lower record times $l(n)$ and lower record values $x(n)$:

$$l(1) = 1,\ l(n+1) = \min\{j : X_j < m(l(n))\},\ n = 1, 2, \ldots, \tag{4.1.3}$$

and

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_4

$$x(n) = m(l(n)) = \min\{X_1, X_2, \ldots, X_{l(n)}\} = X_{l(n),l(n)},\ n = 1, 2, \ldots. \tag{4.1.4}$$

It was mentioned earlier that there is a rather simple correspondence between upper and lower records. Really, if we consider (together with our initial X's) a new sequence of random variables $Y_n = -X_n$, $n = 1, 2,\ldots$, then the lower record values y (n) in the sequence of Y's can be expressed via upper record values $X(n)$ as $y(n) = -X(n)$, $n = 1, 2,\ldots$. Indeed, if to change signs of strong equalities > and < in (4.1.1) and (4.1.3) by signs ≥ and ≤, one will deal (instead of strong records) with weak upper and weak lower records accordingly.

4.2 Exact Distributions of Record Values

Let us consider the strong upper record values $X(n)$, $n = 1, 2,\ldots$, which are based on the sequence of i.i.d. random variables $X_1, X_2,\ldots$, having a continuous distribution function $F(x)$. Using the definition $X(n) = M(L(n))$, one can write that

$$\begin{aligned} P\{X(n) < x\} &= \sum_{m=n}^{\infty} P\{M(L(n)) < x | L(n) = m\} P\{L(n) = m\} \\ &= \sum_{m=n}^{\infty} P\{M(m) < x | L(n) = m\} P\{L(n) = m\}. \end{aligned} \tag{4.2.1}$$

Since

$$\{L(n) = m\} = \{\xi_1 + \xi_2 + \cdots + \xi_{m-1} = n - 1,\ \xi_m = 1\}$$

and the vector of record indicators $(\xi_1, \xi_2,\ldots, \xi_{m-1}, \xi_m)$ and maxima $M(m)$ are independent (see Exercise 3.2.2), we obtain that for any $m = 1, 2,\ldots$ the event $\{L(n) = m\}$ and the random variable $M(m)$ are independent. Hence the RHS of (4.2.1) can be transformed as follows:

$$\begin{aligned} \sum_{m=n}^{\infty} P\{M(m) < x | L(n) = m\} P\{L(n) = m\} &= \sum_{m=n}^{\infty} P\{M(m) < x\} P\{L(n) = m\} \\ &= \sum_{m=n}^{\infty} F^m(x) P\{L(n) = m\} = E(F(x))^{L(n)}. \end{aligned} \tag{4.2.2}$$

Finally we get the relation

$$P\{X(n) < x\} = E(F(x))^{L(n)} = Q_n(F(x)), \tag{4.2.3}$$

where the corresponding expression for the generating function $Q_n(s)$ is given in (3.5.9):

$$Q_n(s) = \frac{1}{(n-1)!} \int_0^{-\log(1-s)} v^{n-1} \exp(-v) dv.$$

Thus, we obtain finally that

$$P\{X(n) < x\} = \frac{1}{(n-1)!} \int_0^{-\log(1-F(x))} v^{n-1} \exp(-v) dv, \ -\infty < x < \infty, \ n = 1, 2, \ldots. \tag{4.2.4}$$

Exercise 4.2.1 Consider the case, when independent X_1, X_2,… have the standard E(1)-exponential distribution with d.f. $F(x) = 1 - \exp(-x)$, $x \geq 0$, and prove that in this situation the following relation in distribution holds for $X(n)$:

$$X(n) \stackrel{d}{=} X_1 + \cdots + X_n, \ n = 1, 2, \ldots. \tag{4.2.5}$$

The joint distribution functions of record values have a rather complicate form. Say, in the simplest case ($n = 2$) the following equalities hold:

$$\begin{aligned} P\{X(1) < x_1, \ X(2) < x_2\} &= \sum_{m=2}^{\infty} P\{X_1 < x_1, \ \max\{X_2, \ldots, X_{m-1}\} \leq X_1, \ X_1 < X_m < x_2\} \\ &= \sum_{m=2}^{\infty} \int_{-\infty}^{\min(x_1,x_2)} F^{m-2}(u) \ (F(x_2) - F(u)) \ dF(u) \ = \int_{-\infty}^{\min(x_1,x_2)} ((F(x_2) - F(u))/(1 - F(u)))dF(u) \\ &= \int_0^{F(\min(x_1,x_2))} ((F(x_2) - u)/(1 - u))du \ = \ (1 - F(x_2))\log(1 - F(\min(x_1, x_2))) + F(\min(x_1, x_2)). \end{aligned}$$

It means that

$$P\{X(1) < x_1, \ X(2) < x_2\} = (1 - F(x_2))\log(1 - F(x_1)) + F(x_1), \quad \text{if } x_1 < x_2,$$

and

$$P\{X(1) < x_1, \ X(2) < x_2\} = (1 - F(x_2)) \log(1 - F(x_2)) + F(x_2),$$

otherwise.

The general expression (where $n = 2, 3,\ldots$) for the joint distribution functions of the record values corresponding to any continuous distribution function F is given as follows:

$$P\{X(1)<x_1, X(2)<x_2, \ldots, X(n)<x_n\} = \int \cdots \int \prod_{j=1}^{n-1} \frac{dF(u_j)}{1-F(u_j)} dF(u_n), \quad (4.2.6)$$

where integration on the RHS of (4.2.6) holds over the set

$$B = \{u_j < x_j,\ j = 1, 2, \ldots, n,\ -\infty < u_1 < \cdots < u_n < \infty\}.$$

A more simple expression is valid for the joint density functions of the record values. Suppose that F is an absolutely continuous distribution function with a density function f. Let us denote $R(x) = f(x)/(1 - F(x))$. Then the joint density function of record values $X(1), X(2),\ldots, X(n)$ is given as

$$f_{1,\,2,\ldots,\,n}(x_1, x_2, \ldots, x_n) = R(x_1)R(x_2)\ldots R(x_{n-1})f(x_n),\ \text{if } x_1 < x_2 < \cdots < x_n, \quad (4.2.7)$$

and $f_{1,\,2,\ldots,\,n}(x_1, x_2,\ldots, x_n) = 0$, otherwise.

Exercise 4.2.2 Write the expressions for the joint density functions of record values for the cases when X's have the uniform $U([0, 1])$ and the exponential $E(1)$ distributions.

Let us consider now the sequence of independent exponentially $E(1)$—distributed random variables. For our convenience we denote these values as $Z_1, Z_2,\ldots$ and the corresponding exponential record values as $Z_1 = Z(1) < Z(2) < \cdots < Z(n) < \cdots$. Taking into account the result obtained in Exercise 4.2.2 we get that the joint probability density function of $Z(1), Z(2),\ldots, Z(n)$ has the following form:

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = \exp\{-x_n\},\ \text{if } 0 < x_1 < x_2 < \cdots < x_n, \quad (4.2.8)$$

and $f_{1,\,2,\,\ldots,\,n}(x_1, x_2,\ldots, x_n) = 0$, otherwise.

Exercise 4.2.3 Prove that the same (as (4.2.8)) joint probability density functions have sums $S_k = \nu_1 + \nu_2 + \cdots + \nu_k$, $k = 1, 2,\ldots, n$, of independent exponentially $E(1)$-distributed random variables $\nu_1, \nu_2, \ldots$.

Comparing (4.2.8) and the result, formulated in Exercise (4.2.3), one gets the next result, which in some sense generalizes relation (4.2.5).

Representation 4.2.1 *For any $n = 1, 2,\ldots$ the following equality in distribution is valid*:

$$\{Z(1), Z(2), \ldots, Z(n)\} \stackrel{d}{=} \{S_1, S_2, \ldots, S_n\}, \tag{4.2.9}$$

where

$$S_k = v_1 + v_2 + \cdots + v_k,\ k = 1, 2, \ldots, n,$$

and $v_1, v_2, \ldots$ *are independent random variables having the exponential* $E(1)$-*distribution with the probability density function* $f(x) = \exp(-x)$, $x \geq 0$.

Corollary 4.2.1 *It follows from* (4.2.9) *that the inter-record values* $Z(1)$, $Z(2) - Z(1), \ldots, Z(n) - Z(n-1), \ldots$ *are mutually independent and have the standard exponential distribution.*

Are there such kind of simple representations as given in (4.2.9) for other distributions of X's? It is well-known that if a random variable X has a continuous distribution function F, then the transformation $U = F(X)$ produces the uniformly $U([0, 1])$-distributed random variable U. This transformation does not change the order of X's and the vector $\{U(1), U(2), \ldots, U(n)\}$ of records in a sequence U_1, U_2, …, where U's are independent $U([0,1])$-distributed random variables, coincides in distribution with the vector $\{F(X(1)), F(X(2)), \ldots, F(X(n))\}$.

Representation 4.2.2 *Let* F *be a continuous distribution function and* G *be the inverse of* F. *Then the following equality is also valid for any* $n = 1, 2, \ldots$

$$\{X(1), X(2), \ldots, X(n)\} \stackrel{d}{=} \{G(U(1)),\ G(U(2)), \ldots, G(U(n))\}. \tag{4.2.10}$$

Now let record values $X(1) < X(2) < \cdots$ and $Y(1) < Y(2) < \cdots$ correspond to X's and Y's with continuous distribution functions F and H respectively. Then it follows from (4.2.10) that for any $n = 1, 2, \ldots$, the next relation holds in distribution:

$$\{X(1),\ X(2), \ldots, X(n)\} \stackrel{d}{=} \{G(H(Y(1))),\ G(H(Y(2))), \ldots,\ G(H(Y(n)))\}, \tag{4.2.11}$$

where G is the inverse function of F.

Combining (4.2.9) and (4.2.11) we come to the next equality:

$$\{X(1),\ X(2), \ldots,\ X(n)\} \stackrel{d}{=} \{H(v_1),\ H(v_1 + v_2),\ \ldots, H(v_1 + v_2 + \cdots + v_n)\},\ n = 1, 2, \ldots, \tag{4.2.12}$$

where $H(x) = G(1 - \exp(-x))$, G is the inverse of F and $v_1, v_2, \ldots$ are independent exponentially $E(1)$-distributed random variables.

Taking into account Representations 4.2.1 and 4.2.2 one can mention that the next relation for the uniform record values will be very convenient.

Representation 4.2.3 *Let* U_1, U_2,… *be independent* $U([0, 1])$-*distributed random variables and* $U(1) < U(2) < \ldots$ *be the corresponding record values*. *Then the following equality* in distribution is valid for any $n = 1, 2, \ldots$

$$\{U(1), U(2), \ldots, U(n)\} \stackrel{d}{=} \{1 - U_1,\ 1 - U_1U_2, \ldots, 1 - U_1U_2\ldots U_n\}. \quad (4.2.13)$$

Exercise 4.2.4 Prove that equality (4.2.13) is true.

Corollary 4.2.2 *From* (4.2.13) *it is possible for the lower record values* $u(1) > u(2) > \cdots > u(n) > \cdots$, *based on the uniformly* $U([0, 1])$-*distributed random variables, to get one simple representation. Since in this situation*

$$\{u(1), u(2), \ldots, u(n)\} \stackrel{d}{=} \{1 - U(1),\ 1 - U(2), \ldots, 1 - U(n)\},$$

one obtains that for any $n = 1, 2, \ldots$ *the following equality holds*:

$$\{u(1), u(2), \ldots, u(n)\} \stackrel{d}{=} \{U_1,\ U_1U_2, \ldots, U_1U_2, \ldots, U_n\}. \quad (4.2.14)$$

It was mentioned above that we consider in general the records for X's having continuous distribution functions, but it is useful to mention here how to study record values based on some discrete random variables.

Let now X_1, X_2,… be independent identically distributed random variables taking nonnegative integer values and $X(1) < X(2) < \ldots$ be the corresponding strong record values.

For discrete distributions we introduce record indicators η_n, $n = 0, 1, 2, \ldots$, such that $\eta_n = 1$, if n is a record value in the sequence $X_1, X_2, \ldots$, and $\eta_n = 0$, otherwise. It was shown (see Theorem 3.3.1 and Exercise 3.3.5) that these record indicators are independent and

$$P\{\eta_n = 1\} = 1 - P\{\eta_n = 0\} = p_n / P\{X \geq n\},\ n = 0, 1, 2, \ldots.$$

Now one can get the simple expressions for distributions of record values $X(n)$ based on such type of discrete X's. Really, in this situation it is possible to obtain the following equalities:

$$\begin{aligned} P\{X(n) = m\} &= P\{\eta_0 + \eta_1 + \cdots + \eta_{m-1} = n - 1,\ \eta_m = 1\} \\ &= P\{\eta_0 + \eta_1 + \cdots + \eta_{m-1} = n - 1\} p_m / P\{X \geq m\}, \end{aligned} \quad (4.2.15)$$

and

$$P\{X(n) > m\} = P\{\eta_0 + \eta_1 + \cdots + \eta_{m-1} < n\}, \quad (4.2.16)$$

which are valid for any $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$.

4.3 Distributions of Conditional Record Values

Let us consider the conditional distribution

$$\varphi(x|x_1, x_2, \ldots, x_n) = P\{X(n+1) > x|X(1) = x_1,\ X(2) = x_2, \ldots, X(n) = x_n\},\ x_1 < x_2 < \ldots < x_n < x.$$

The following result will be useful for us.

Exercise 4.3.1 Show that if $L(n)$ is the nth record time in the sequence X_1, X_2,…, then random variables $X_{L(n)+1}$, $X_{L(n)+2}$,… are independent, have the same distribution function F(x), as the initial X_1, X_2,…, and these elements don't depend on the X_1, X_2,…,$X_{L(n)}$.

Taking into account the assumption of Exercise 4.3.1 and denoting $Y_1 = X_{L(n)+1}$, $Y_2 = X_{L(n)+2}$,…, one gets that

$$\begin{aligned}\varphi(x|x_1, x_2, \ldots, x_n) &= P\{Y_1 > x\} + P\{Y_1 \le x_n, Y_2 > x\} + \cdots \\ &\quad + P\{Y_1 \le x_n, Y_2 \le x_n, \ldots, Y_{r-1} \le x_n, Y_r > x\} + \cdots \\ &= (1 - F(x)) + F(x_n)(1 - F(x)) + \cdots \\ &= (1 - \mathrm{F(x)})/(1 - \mathrm{F(x_n)}),\ \mathrm{x} > \mathrm{x_n}. \end{aligned} \tag{4.3.1}$$

In the case, when there exists the density function $f(x)$ *of* X's, one can write the corresponding conditional density function $f_{n+1}(x|\ x_1, x_2,\ldots, x_n)$ of $X(n + 1)$ given that $X(1) = x_1$, $X(2) = x_2$,…, $X(n) = x_n$. It has the following form:

$$f_{n+1}(x|x_1, x_2, \ldots, x_n) = f(x)/(1 - F(x_n)),\ x > x_n. \tag{4.3.2}$$

The result of Exercise 4.3.1 implies that the sequence of record values $X(1) < X(2) < \ldots$ forms a Markov chain.

4.4 Moments of Record Values

Let us consider exponential record values $Z(1) < Z(2) < \ldots$ based on the independent random variables Z_1, Z_2,… having the exponential $E(1)$-distribution. From (4.2.9) we know that for any n = 1, 2,…

$$\{Z(1), Z(2), \ldots, Z(n)\} \stackrel{d}{=} \{S_1, S_2, \ldots, S_n\},$$

where $S_k = v_1 + v_2 + \cdots + v_k$, k = 1, 2,…, n and $v, v_1, v_2, \ldots$ are independent random variables also having the exponential $E(1)$-distribution.

It follows immediately from this representation and from the independence property of $v_1, v_2, \ldots$ that

$$EZ(n) = ES_n = nEv = n, \tag{4.4.1}$$

$$VarZ(n) = VarS_n = nVarv = n, \tag{4.4.2}$$

$$Cov(Z(m), Z(n)) = cov(S_m,\ S_n) = cov(S_m,\ S_m) = Var(S_m) = m, \text{ if } m \le n, \tag{4.4.3}$$

and the correlation coefficients are given as follows:

$$\rho(Z(m), Z(n)) = (m/n)^{1/2}, \text{ if } m \le n. \tag{4.4.4}$$

Exercise 4.4.1 Use equalities (4.2.13) and (4.2.14) to find expectations $EU(n)$ and $Eu(n)$, variances $Var\ U(n)$ and $Var\ u(n)$, covariances $Cov\ (U(m), U(n))$ and Cov(u (m),$u(n)$) of the upper and lower uniform record values.

From (4.2.12) for any X's with continuous d.f. $F(x)$ we have the following equalities in distribution:

$$\{X(1),\ X(2), \ldots, X(n)\} \overset{d}{=} \{H(v_1),\ H(v_1 + v_2), \ldots, H(v_1 + v_2 + \cdots + v_n)\},$$
$$n = 1, 2, \ldots,$$

where $H(x) = G(1 - \exp(-x))$, G is the inverse of F and $v_1, v_2, \ldots$, as above, are independent exponentially $E(1)$-distributed random variables. Note that sums

$$S_n = v_1 + v_2 + \cdots + v_n,\ n = 1, 2, \ldots,$$

have the Gamma distributions with parameters 1, 2,…, correspondingly. It allows us to write that

$$EX(n) = E\,H(v_1 + v_2 + \cdots + v_n) = \int_0^\infty H(x)\exp(-x)x^{n-1}dx/(n-1)!$$
$$= \int_0^\infty G(1 - \exp(-x))\exp(-x)x^{n-1}dx/(n-1)! = \int_{-\infty}^\infty u(-\ln(1 - F(u)))^{n-1}dF(u)/(n-1)!. \tag{4.4.5}$$

Similarly one gets that

$$EX^2(n) = \int_{-\infty}^\infty u^2(-\ln(1 - F(u)))^{n-1}dF(u)/(n-1)!. \tag{4.4.6}$$

Exercise 4.4.2 Show that equality (4.4.6) is valid and write the expression for variance of $X(n)$, $n = 1, 2, \ldots$.

Exercise 4.4.3 Generalize expressions (4.4.5) and (4.4.6) and write equalities for moments $EX^r(n)$ for $r = 3, 4, \ldots$ and $n = 1, 2, \ldots$.

The analogous method gives, in particular, the following equalities for the joint moments $E\,X(m)\,X(n)$.

Let $m \le n$. Then we can get that

$$E\,X(m)X(n) = \int_0^{\infty}\int_0^{\infty} H(u)H(u+v)\exp(-(u+v))u^{m-1}v^{n-m-1}dudv/(m-1)!(n-m-1)!. \quad (4.4.7)$$

Based on equalities (4.4.5)−(4.4.7) it is possible to write the corresponding formulae for covariances $cov(X(m), X(n))$ and correlation coefficients

$\rho(X(m), X(n))$ of any two record values $X(m)$ and $X(n)$. Indeed we don't forget every time that it is necessary to check preliminary the existence of the second moments of these record values.

Remark 4.4.1 It is interesting to know (see, for example, Nevzorov (2001), Lecture 22) that for any $m \le n$ and for any continuous d.f., such that

$$\int_{-\infty}^{\infty} u^2(-\ln(1-F(u)))^{m-1}dF(u) < \infty, \quad \int_{-\infty}^{\infty} u^2(-\ln(1-F(u)))^{n-1}dF(u) < \infty, \quad (4.4.8)$$

the following inequality for the correlation coefficients holds:

$$\rho(X(m), X(n)) \le (m/n)^{1/2}. \quad (4.4.9)$$

Thus, we obtain, recalling (4.4.4), that the maximal value of $\rho(X(m), X(n))$ is attained for the exponential distributions and it is equal to $(m/n)^{1/2}$ if $m \le n$.

4.5 Joint Distributions of Record Values and Record Times

Above we discussed separately distributions and properties of record times and record values. There are some useful results connected with the joint distributions of these random variables. Lower we consider different results connected with the joint distributions of record values $X(n)$, record times $L(n)$ and inter-record times $\Delta(1) = L(1) = 1$, $\Delta(n) = L(n) - L(n-1)$, $n = 2, 3, \ldots$

The joint distributions of the sets of record times and record values for any n, any values $x_1, x_2, \ldots, x_n$ and $1 = k(1) < k(2) < \cdots < k(n)$ are given as follows:

$$P\{X(1) < x_1, X(2) < x_2, \ldots, X(n) < x_n,\ L(1) = 1,\ L(2) = k(2), \ldots, L(n) = k(n)\}$$
$$= \int\limits_{-\infty}^{x_1} \cdots \int\limits_{-\infty}^{x_n} h(v_1, v_2, \ldots, v_n) dF(v_1) dF(v_2) \ldots dF(v_n), \tag{4.5.1}$$

where

$$h(v_1, v_2, \ldots, v_n) = \prod_{r=1}^{n-1} F^{k(r+1)-k(r)-1}(v_r),$$

if $-\infty < v_1 < v_2 < \cdots < v_n < \infty$, and $h(v_1, v_2, \ldots, v_n) = 0$, otherwise.

In the case when X's have a density function $f(x)$, one can consider the function $f(x_1, x_2, \ldots, x_n, k(1), k(2), \ldots, k(n))$, which presents a density function with respect to the record values and a probability distribution with respect to our discrete record times. To get $f(x_1, x_2, \ldots, x_n, k(1), k(2), \ldots, k(n))$ one must differentiate (4.5.1) n times with respect to $x_1, x_2, \ldots, x_n$. These differentiating allows to obtain rather simple equality:

$$f(x_1, x_2, \ldots, x_n,\ k(1), k(2), \ldots, k(n))$$
$$= (F(x_1))^{k(2)-k(1)-1} (F(x_2))^{k(3)-k(2)-1} \ldots (F(x_n))^{k(n)-k(n-1)-1} f(x_1) f(x_2) \ldots f(x_n), \tag{4.5.2}$$

$$\text{if} \quad -\infty < x_1 < x_2 < \ldots < x_n < \infty,\ 1 = k(1) < k(2) < \ldots < k(n)$$

and

$$f(x_1, x_2, \ldots, x_n, k(1), k(2), \ldots, k(n)) = 0,\ \text{otherwise.}$$

Exercise 4.5.1 Make the substitution $m(r) = k(r) - k(r-1)$, $r = 2, 3, \ldots, n$, in (4.5.2) and prove that the corresponding joint "density-distribution" function $h(x_1, x_2, \ldots, x_n, m(1), m(2), \ldots, m(n))$ for record values $X(1), X(2), \ldots, X(n)$ and inter-record times $\Delta(1), \Delta(2), \ldots, \Delta(n)$ has the form

$$h(x_1, x_2, \ldots, x_n,\ m(1), m(2), \ldots, m(n))$$
$$= (F(x_1))^{m(2)-1} (F(x_2))^{m(3)-1} \ldots (F(x_n))^{m(n)-1} f(x_1) f(x_2) \ldots f(x_n), \tag{4.5.3}$$

$$\textit{if} \quad -\infty < x_1 < x_2 < \ldots < x_n < \infty,\ m(1) = 1,\ m(2) > 0, \ldots, m(n) > 0,$$

and

$$h(x_1, x_2, \ldots, x_n, m(1), m(2), \ldots, m(n)) = 0,\ \textit{otherwise}.$$

Recalling (see (4.2.7)) that the joint density function of $X(1)$, $X(2)$,…,$X(n)$ has the form

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = R(x_1)R(x_2)\ldots R(x_{n-1})f(x_n),\ \textit{if}\ x_1 < x_2 < \ldots < x_n,$$

where $R(x) = f(x)/(1 - F(x))$, we can obtain from (4.5.3) the expression for the conditional probabilities of the inter-record times:

$$\begin{aligned} &P\{\Delta(1) = 1, \Delta(2) = m(2), \ldots, \Delta(n) = m(n) | X(1) = x_1,\ X(2) = x_2, \ldots,\ X(n) = x_n\} \\ &\quad = (F(x_1))^{m(2)-1}(1 - F(x_1))(F(x_2))^{m(3)-1}(1 - F(x_2))\ldots(F(x_{n-1}))^{m(n)-1}(1 - F(x_{n-1})). \end{aligned} \tag{4.5.4}$$

Remark 4.5.1 Thus, it was shown that inter-record times $\Delta(1)$, $\Delta(2)$,…, $\Delta(n)$ are conditionally independent given the fixed record values $X(1)$, $X(2)$,…, $X(n)$ and for any $r = 2, 3, \ldots$, $m = 1, 2, \ldots$, $n = 2, 3, \ldots$ the following equality holds:

$$P\{\Delta(r) = m | X(1) = x_1, X(2) = x_2, \ldots, X(n) = x_n\} = (1 - F(x_{r-1}))(F(x_{r-1}))^{m-1}. \tag{4.5.5}$$

4.6 *K*th Record Values

As it was mentioned above the *k*th records are a natural extension of the classical records. Recall that *k*th record times $L(n, k)$ and the *k*th record values $X(n, k)$ for any $k = 1, 2, \ldots$ are defined as follows:

$$L(1, k) = k,\ L(n + 1, k) = \min\{j > L(n, k) : X_j > X_{j-k,j-1}\},\ n = 1, 2, \ldots,$$

and

$$X(n, k) = X_{L(n,k)\ -\ k+1,\ L(n,k)}, n = 1, 2, \ldots$$

Some results describing the probability structure of the *k*th record values and their relationships with order statistics were given in Deheuvels (1984b). We are going to consider here only the case when the parent distribution function F is continuous, although some of the results discussed below are true under more general conditions. Note that relations for the kth records are rather close to the corresponding results for the usual (when $k = 1$) records.

Let us denote as $Z(1, k) < Z(2, k) < \ldots$ the kth record values for the case of the standard exponential distribution with d.f. $F(x) = 1 - \exp(-x)$, $x \geq 0$.

Representation 4.6.1 *For any $n = 1, 2, \ldots$ and any $k = 1, 2, \ldots$ the following equality in distribution is valid*:

$$\{Z(1,k), Z(2,k), \ldots, Z(n,k)\} \stackrel{d}{=} \{S_1/k, S_2/k, \ldots, S_n/k\}, \tag{4.6.1}$$

where $S_k = \nu_1 + \nu_2 + \ldots + \nu_k$, $k = 1, 2, \ldots, n$, and *$\nu_1, \nu_2, \ldots$ are independent random variables having the exponential $E(1)$-distribution.*

Indeed if $k = 1$, one gets Representation (4.2.1) for the usual exponential record values, as a partial case of (4.6.1).

Exercise 4.6.1 Describe (as it was done in Corollary 4.2.1) the structure of the kth inter-record values $Z(1, k)$, $Z(2, k) - Z(1, k), \ldots, Z(n, k) - Z(n - 1, k)$.

Exercise 4.6.2 Find moments $EZ(n, k)$, $VarZ(n, k)$ and $cov(Z(m, k), Z(n, k))$.

Taking into account (4.6.1) it is possible to investigate more general situations. Let now kth record values $X(1, k) < X(2, k) < \ldots$ correspond to X's with a continuous distribution function F. Then it follows from (4.6.1) that for any $n = 1, 2, \ldots$, the next relation holds in distribution:

$$\begin{aligned}&\{X(1,k), X(2,k), \ldots, X(n,k)\}\\ &\stackrel{d}{=}\{H(\nu_1/k), H((\nu_1+\nu_2)/k), \ldots, H((\nu_1+\nu_2+\cdots+\nu_n)/k)\},\end{aligned} \tag{4.6.2}$$

where $H(x) = G(1 - \exp(-x))$, G is the inverse of F and $\nu_1, \nu_2, \ldots$, as above, are independent $E(1)$-distributed random variables.

It is not difficult to find that for any continuous d.f. F, any $k \geq 1$, $n \geq 1$, and $x > u$,

$$P\{X(n+1,k) > x | X(n,k) = u\} = ((1 - F(x))/(1 - F(u)))^k. \tag{4.6.3}$$

Really, it follows from (4.6.2) that

$$\begin{aligned}P\{X(n+1) > x | X(n,k) = u\} &= P\{H((\nu_1+\nu_2+\cdots+\nu_{n+1})/k)\\ &\qquad > x | H((\nu_1+\nu_2+\cdots+\nu_n)/k) = u)\}\\ &= P\{(\nu_1+\nu_2+\cdots+\nu_{n+1})\\ &\qquad > -k\log(1-F(x)) | (\nu_1+\nu_2+\cdots+\nu_n)\\ &= -k\log(1-F(u))\}\\ &= P\{\nu_{n+1} > -k\log(1-F(x))\\ &\qquad + k\log(1-F(u)) | (\nu_1+\nu_2+\cdots+\nu_n)\\ &= -k\log(1-F(u))\}.\end{aligned}$$

Since v_{n+1} and the sum $v_1 + v_2 + \cdots + v_n$ are independent one gets that

$$P\{X(n+1) > x | X(n,k) = u\}$$
$$= P\{v_{n+1} > -k\log(1 - F(x)) + k\log(1 - F(u))\} = ((1 - F(x))/(1 - F(u)))^k.$$

Exercise 4.6.3 Consider two sequences of independent random variables:

$X_1, X_2, \ldots$ with a continuous d.f. F and $Y_1 = \min\{X_1, \ldots, X_k\}$, $Y_2 = \min\{X_{k+1}, \ldots, X_{2k}\}, \ldots$, having d.f. $T(x) = 1 - (1 - F(x))^k$. Let also $X(n, k)$ be the kth record values based on $X_1, X_2, \ldots$, and $Y(n, 1)$ be the usual ($k = 1$) record values constructed with the help of the sequence $Y_1, Y_2, \ldots$.

Prove that for any $k = 1, 2, \ldots$ and any $n = 1, 2, \ldots$ the following equality in distribution is valid:

$$X(n,k) \stackrel{d}{=} Y(n,1) \tag{4.6.4}$$

This correspondence between the kth record values $X(n, \mathrm{k})$ and the usual records $Y(n, 1)$ helps us to get immediately distribution functions for $X(n, k)$. From relation (4.2.3) we know that

$$P\{X(n,1) < x\} = \frac{1}{(n-1)!} \int_0^{-\log(1-F(x))} u^{n-1} \mathrm{e}^{-u} du. \tag{4.6.5}$$

The only we need now to obtain $P\{Y(n, 1) < x\}$ is to write $T(x) = 1 - (1 - F(x))^k$ in (4.6.5) instead of F(x). Hence, for any $k \geq 1$ and $n \geq 1$ the following result is valid:

$$P\{X(n,k) < x\} = \frac{1}{(n-1)!} \int_0^{-k\log(1-F(x))} u^{n-1} \mathrm{e}^{-u} du. \tag{4.6.6}$$

4.7 Exercises (solutions)

Exercise 4.2.1 (*hint*) It is enough to consider (4.2.4) with $F(x) = 1 - \exp(-x)$, $x \geq 0$, and to see that in this case the RHS of (4.2.4) corresponds to the *Gamma*(n)-distribution. Then it will be enough to recall that the sum $X_1 + \cdots + X_n$ of independent $E(1)$-distributed X's has the same *Gamma*-distribution.

Exercise 4.2.2 (*answers*)

(a) It follows from (4.2.7) for the uniformly $U([0,1])$-distributed X's that

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = 1/(1-x_1)(1-x_2)\ldots(1-x_{n-1}),\ \text{if}\ 0<x_1<x_2<\ldots<x_n<1,$$

and

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = 0,\ \text{otherwise}.$$

(b) The joint density function of record values for the exponentially $E(1)$-distributed X's was given in (4.2.8):

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = \exp\{-x_n\},\ \text{if}\ 0<x_1<x_2<\ldots<x_n,$$

and

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = 0,\ \text{otherwise}.$$

Exercise 4.2.3 (*solution*) It is enough to recall that the joint density function $g\ (u_1, u_2,\ldots,u_n)$ of independent $E(1)$—distributed random variables $v_1, v_2, \ldots v_n$ is given as

$$g(u_1,u_2,\ldots,u_n) = \exp\{-(u_1+u_2+\cdots+u_n)\},\ u_1\geq 0,\ u_1\geq 0,\ldots,\ u_n\geq 0. \tag{4.7.1}$$

Changing $u_1, u_2,\ldots,u_n$ in (4.7.1) by $x_1 = u_1$, $x_2 = u_1 + u_2,\ldots,x_n = u_1 + u_2 + \ldots + u_n$ one immediately gets that

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = \exp\{-x_n\},\ \textit{if}\ 0<x_1<x_2<\cdots<x_n.$$

Exercise 4.2.4 (*solution*) In this situation equality (4.2.12) must be considered with $H(x) = 1 - \exp(-x)$, $x \geq 0$. Then

$$\{U(1),\ U(2),\ldots,U(n)\} \stackrel{d}{=} \{1-\exp(-v_1),\ 1-\exp(-(v_1+v_2)),\ldots,\ 1-\exp(-(v_1+v_2+\cdots+v_n))\}.$$

It is not difficult to see that random variables $\exp(-v_1), \exp(-v_2),\ldots,\exp(-v_n)$ are independent and have the same uniform $U([0,1])$ distribution.

Exercise 4.3.1 (*hint*) Evidently, the sequence $Y_1 = X_{L(n)+1}$, $Y_2 = X_{L(n)+2},\ldots$, under condition that $L(n) = m$, coincides in distribution with $X_{m+1}, X_{m+2},\ldots$ and these Y's do not depend on $X_1, X_2,\ldots,X_m$. Moreover, for any m the event $C_{n,\,m} = \{L(n) = m\}$ is determined by random variables $X_1, X_2,\ldots,X_m$ only and does not depend on X_{m+1},

$X_{m+2}, \ldots$. Hence now if to consider arbitrary event B, generated by record values $X(1)$, $X(2), \ldots, X(n)$ (in particular, event $\{L(n) = m\}$), and any event

$$A = \{ Y_1 < x_1, Y_2 < x_2, \ldots, Y_k < x_k \},\ k = 1, 2, \ldots,$$

one can get that

$$P\{AB\} = F(x_1)F(x_2)\ldots F(x_k)P\{B\},$$

which proves the necessary assertion.

Exercise 4.4.1 (*hints and answers*) It is evident that

$$U(n) \stackrel{d}{=} 1 - u(n),\ n = 1, 2, \ldots$$

Hence

$$Eu(n) = 1 - EU(n),\ Var\ u(n) = VarU(n),\ Cov(u(m), u(n)) = Cov(U(m), U(n)).$$

From (4.2.14) one immediately gets that

$$Eu(n) = (EU)^n = 1/2^n,\ Eu^2(n) = (EU^2)^n = 1/3^n,\ Var\ u(n) = 1/3^n - 1/4^n,\ n = 1, 2, \ldots,$$
$$Eu(m)u(n) = (EU^2)^m (EU)^{n-m} = 1/3^m 2^{n-m},\ m \le n,$$
$$\mathrm{Cov}(\mathrm{u(m), u(n)}) = 1/3^{\mathrm{m}} 2^{\mathrm{n-m}} - 1/2^{\mathrm{n+m}},\ \mathrm{m} \le \mathrm{n}.$$

Exercise 4.4.2 (*solution*) Analogously to (4.4.5) one gets that

$$EX^2(n) = EH^2(v_1 + v_2 + \cdots + v_n) = \int_0^\infty H^2(x) \exp(-x) x^{n-1} dx/(n-1)!$$
$$= \int_{-\infty}^{\infty} u^2(-\ln(1 - F(u)))^{n-1} dF(u)/(n-1)!.$$

Thus,

$$Var\ X(n) = \int_{-\infty}^{\infty} u^2(-\ln(1 - F(u)))^{n-1} dF(u)/(n-1)!$$
$$- \left(\int_{-\infty}^{\infty} u(-\ln(1 - F(u)))^{n-1} dF(u)/(n-1)!)^2 \right).$$

Exercise 4.4.3 (*answer*)

$$\mathrm{EX^r(n)} = \mathrm{E\ H^r}(v_1 + v_2 + \cdots + v_n) = \int_0^\infty H^r(x)\exp(-x)x^{n-1}dx/(\mathrm{n}-1)!$$

$$= \int_{-\infty}^{\infty} u^r(-\ln(1-F(u)))^{n-1}dF(u)/(\mathrm{n}-1)!,\ \mathrm{r} = 1,2,\ldots,\ \mathrm{n} = 1.2,\ldots.$$

Exercise 4.5.1 (*hint*) The substitution $m(r) = k(r) - k(r-1)$, $r = 2, 3, \ldots, n$, in (4.5.2) immediately allows to obtain equality (4.5.3).

Exercise 4.6.1 (*answer*) It follows from (4.6.1) that for any $k = 1, 2, \ldots$ the inter-record values $Z(1, k)$, $Z(2, k) - Z(1, k), \ldots, Z(n, k) - Z(n-1, k), \ldots$ are independent and have the exponential $E(1/k)$-distribution with the density function

$$f(x) = k\exp(-kx),\ x \geq 0.$$

Exercise 4.6.2 (*answers*) Immediately one gets from (4.6.1) that

$$EZ(n,k) = n/k, \quad VarZ(n,k) = n/k^2, \quad Cov(Z(m,k), Z(n,k)) = \min(m,n)/k^2,$$

and the correlation coefficients do not depend on k and have the form

$$\rho(Z(m,k), Z(n.k)) = (m/n)^{1/2},\ if\ m \leq n,\ k = 1,2,\ldots$$

Exercise 4.6.3 (*solution*) It is enough to observe that the function $W(x)$, inverse to $T(x)$, has the form $W(x) = G(1 - (1-x)^{1/k})$, where $G(x)$ is the inverse function to $F(x)$. Then the application of relation (4.6.2) with $k = 1$ and

$$H(x) = W(1 - \exp(-x)) = G(1 - \exp(-x/k))$$

allows to get the equality

$$Y(n,1) \stackrel{d}{=} H((v_1 + v_2 + \cdots + v_n)/k),\ n = 1,2,\ldots,$$

which means that $Y(n, 1)$ and $Z(n, k)$ have the same distribution.

Chapter 5
Record Values of Some Well Known Distributions

5.1 Exponential Distribution

5.1.1 Introduction

A continuous random variable X is said to be exponentially distributed with parameters μ and σ, $\sigma > 0$, if its pdf is of the following form

$$\begin{aligned} f(x) &= \sigma^{-1}\exp\left(-\sigma^{-1}(x-\mu)\right), \quad -\infty < \mu < x < \infty,\ \sigma > 0, \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{5.1.1.1}$$

The corresponding distribution function F(x) and the hazard rate r(x) of the rv X are respectively

$$F(x) = 1 - \exp\left(-\sigma^{-1}(x-\mu)\right), -\infty < \mu < x < \infty, \sigma > 0$$

and

$$r(x) = f(x)/(1-F(x)) = \sigma^{-1}. \tag{5.1.1.2}$$

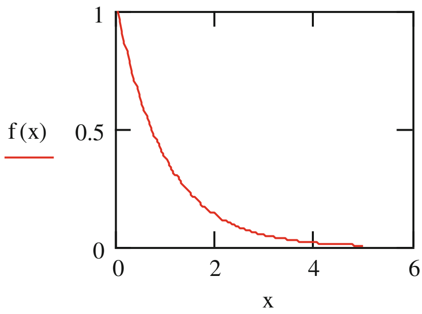
We will denote the exponential distribution with the pdf as given in (5.1.1.1) as $E(\mu, \sigma)$. The graph of the pdf of E(0, 1) is given in Fig. 5.1.

The exponential distribution possesses the memory less property i.e. an item whose lifetime is exponentially distributed, the residual life does not depend on the past life. In terms of probability, we can write

$$P[X > s + t | X > t] = P[X > s] \tag{5.1.1.3}$$

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_5

Fig. 5.1 Pdf of E(0, 1)



In terms of the distribution function we can write (5.1.1.3) as

$$1 - F(s+t) = [1 - F(s)]\,[1 - F(t)]. \tag{5.1.1.4}$$

This property is utilized in many characterizations of the exponential distribution.

5.1.2 Distribution of Record Values

Using $R(x) = -\log(1 - F(x)) = \sigma^{-1}(x - \mu)$, we have the pdf $f_n(x)$ of the nth upper record X(n) as

$$\begin{aligned} f_n(x) &= \frac{\sigma^{-n}}{\Gamma(n)}(x-\mu)^{n-1}\exp\left(-\sigma^{-1}(x-\mu)\right),\ -\infty<\mu<x<\infty,\ \sigma>0 \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{5.1.2.1}$$

The corresponding d.f. $F_n(x)$ is

$$F_n(x) = 1 - \sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{x-\mu}{\sigma}\right)^k e^{-\frac{x-\mu}{\sigma}}.$$

The joint pdf of $X(m)$ and X(n), $n > m$ is

$$\begin{aligned} f_{m,n}(x,y) &= \frac{\sigma^{-n}}{\Gamma(m)}\cdot\frac{(x-\mu)^{m-1}}{\Gamma(n-m)}(y-x)^{n-m-1}\exp\left(-\sigma^{-1}(x-\mu)\right), \\ &\quad -\infty<\mu\le x<y<\infty, \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{5.1.2.2}$$

It is easy to see that $X(n) - X(n-1)$ and $X(m) - X(m-1)$ are identically distributed for $1 < m < n < \infty$. It can be shown that $X(m) \underline{\underline{d}} X(m-1) + U,\ \ m > 1$ where U is independent of X(m) and $X(m-1)$ and is identically distributed as X_i's.

The conditional pdf of $X(n)||X(m) = x$ is

$$f(y|X(m) = x) = \frac{\sigma^{m-n}}{\Gamma(n-m)}(y-x)^{n-m-1}\exp(-\sigma^{-1}(y-x)),\ -\infty < \mu \leq x < y < \infty,$$
$$= 0, \quad \text{otherwise.} \tag{5.1.2.3}$$

It can be shown that if μ = 0, then $X(n) - X(m)$ is identically distributed as $X(n-m), n > m$.

If we take μ = 0 and σ = 1 and W_n = X(1) + X(2) +⋯+ X(n), then, the characteristic function of $\phi_n(t)$ of W_n can be written as

$$\phi_n(t) = \frac{1}{1-it}\cdot\frac{1}{1-2it}\cdots\frac{1}{1-nit}. \tag{5.1.2.4}$$

Inverting (5.1.2.4), we obtain the pdf $f_W(w)$ of W_n as

$$f_W(w) = \sum_{j=1}^{n}\frac{1}{\Gamma(j)}\cdot\frac{(-1)^{n-j}}{\Gamma(n-j+1)}\cdot e^{-w/j}j^{n-2}. \tag{5.1.2.5}$$

Theorem 5.1.2.1 *Let* X_j, i = 1, 2,… *be independently and exponentially distributed with* μ = 0*and* σ = 1. *Suppose* $\xi_i = \frac{X(i)}{X(i+1)}$ i = 1, 2, m − 1 *then* ξ_i*'s are independent.*

Proof The joint pdf of X(1), X(2),…,X(m) is

$$f(x_1, x_2, \ldots, x_m) = e^{-x_m}, 0 < x_1 < x_2 < \cdots < x_m < \infty.$$

Let us use the transformation

$$\xi_0 = X(1), \text{ and } \xi_i = \frac{X(i)}{X(i+1)}, \quad i = 2, \ldots, m-1.$$

The Jacobian of the transformation

$$J = \left|\frac{\partial(X(1), X(2), \ldots, X(m))}{\partial(\xi_o, \xi_1, \ldots, \xi_{m-1})}\right| = \frac{\xi_o^{m-1}}{\xi_1^m\,\xi_2^{m-1}\ldots\xi_{m-1}^2}$$

We can write the pdf of ξ_i, i = 0, 1,……,m − 1, as

$$f(e_o, e_1, \ldots, e_{m-1}) = \frac{e_o^{m-1}}{e_1^m e_2^{m-1}\ldots e_{m-1}^2}e^{-\left(\frac{e_o}{e_1\cdots e_{m-1}}\right)}.$$

Now integrating the above expression with respect to e_o, we obtain the joint pdf of ξ_i, i = 1,……, m − 1, as

$$f(e_1, \ldots, e_{m-1}) = \Gamma(\mathrm{m}) e_2 \ldots e_{m-1}^{m-2}.$$

Thus ξ_i, i = 1, 2,…, m − 1 are independent and

$$\mathrm{P}(\xi_k \le \mathrm{x}) = \mathrm{x}^{\mathrm{k}}, \quad 1 \le \mathrm{k} \le \mathrm{m} - 1.$$

Exercise 5.1.2.1 Let $W_k = (\xi_k)^k$, k = 1, 2,…, m − 1, then W_1, W_2,…., W_{m-1} are independent and identically distributed as uniform (over the interval (0, 1)) random variable.

5.1.3 Moments

Without any loss of generality we will consider in this section the standard exponential distribution, E(0, 1), with pdf f(x) = exp(−x), $0 \le x < \infty$, in which case we have f(x) = 1 − F(x).

For 1 ≤ m < n,

$$\begin{aligned} &E(X((n))^p (X(m)))^q \\ &= \int_0^\infty \int_0^u \frac{1}{\Gamma(m)} \cdot \frac{1}{\Gamma(n-m)} u^q e^{-x} v^{m+p-1} (u-v)^{n-m-1} dv du \end{aligned}$$

Substituting tu = v and simplifying we get

$$\begin{aligned} &E\left(X_{U(n)}^p X_{U(m)}^q\right) \\ &= \int_0^\infty \int_0^1 \frac{1}{\Gamma(m)} \frac{1}{\Gamma(n-m)} u^{n+p+q-1} e^{-x} t^{m+p-1} (1-t)^{n-m-1}\, dt du \\ &= \frac{\Gamma(m+p)\Gamma(n+p+q)}{\Gamma(m)\Gamma(n+p)} \end{aligned} \tag{5.1.3.1}$$

It can easily be shown that if S_n = X(1) + X(2) +⋯+ X(n) then

$$\mathrm{E}(\mathrm{S_n}) = \frac{n(n+1)}{2} \text{ and } \mathrm{Var}(\mathrm{S_n}) = \frac{n(n+1)(2n+1)}{6}.$$

Some simple recurrence relations satisfied by single and product moments of record values are given in the following theorems.

Theorem 5.1.3.1 *For* n > 1 *and* r = 0, 1, 2,…

$$E\Big(X(n))^{r+1}\Big) = E\Big((X(n-1))^{r+1}\Big) + (r+1)E(X(n))^{r}) \tag{5.1.3.2}$$

and consequently, for 0 < m < n − 1 *we can write*

$$E(X(n))^{r+1} = E(X(m))^{r+1} + (r+1)\sum_{i=r+11}^{n} E(X(i))^{r} \tag{5.1.3.3}$$

with $E\Big((X(0)^{r+1}\Big) = 0$ and $E\Big(X(n)^{0}\Big) = 1.$

Proof For n $\geq$ 1 and r = 0, 1,…, we have

$$\begin{aligned} E(X(n))^{r}) &= \frac{1}{\Gamma(n)}\int_0^\infty x^r\{R(x)\}^{n-1}f(x)dx \\ &= \frac{1}{\Gamma(n)}\int_0^\infty x^r\{R(x)\}^{n-1}\{1-F(x)\}dx, \end{aligned}$$

since f(x) = 1 − F(x).

Upon integrating by parts treating x^r for integration and the rest of the integrand for differentiation, we obtain

$$\begin{aligned} E(X(n))^{r} &= \frac{1}{(r=1)\Gamma(n)}\int_0^\infty x^r(R(x))^{n-1}f(x)dx \\ &= \frac{1}{r+1}\left[\int_0^\infty x^{r+1}\frac{1}{\Gamma(n)}\{R(x)\}^{n-1}f(x)dx \right. \\ &\qquad \left. - \int_0^\infty x^{r+1}\frac{1}{\Gamma(n-1)}\{R(x)\}^{n-2}f(x)dx\right] \\ &= E(X(n)^{r+1} - E\Big(X(n-1)^{r+1}\Big), \end{aligned}$$

which, when rewritten, gives the recurrence relation in (5.1.3.2). By repeatedly applying the recurrence relation (5.1.3.2), we simply derive the recurrence relation in (5.1.3.3).

Remark 5.1.3.1 The recurrence relation in (5.1.3.2) can be used in a simple way to compute all the simple moments of all record values. Once again, using the property that f(y) = 1 − F(y), we can derive some simple recurrence relations for the product moments of record values.

Balakrishnan and Ahsanullah (1995) proved the following recurrence relations of the product moments given in Theorem 5.1.3.2.

Theorem 5.1.3.2 *For* m > 1 and r, s = 0, 1, 2,…

$$E((X(m))^r(X(m+1))^{s+1}) = E\left(X(m))^{r+s+1}\right) + (s+1)E((X(m))^r(X(m+1))^s) \tag{5.1.3.4}$$

and for 1 < m < n − 2, r, s = 0, 1, 2,…

$$E((X(m))^r(X(n))^{s+1}) = E((X(m))^r(X(n-1))^{s+1})(s+1)E((X(m))^r(X(n))^s) \tag{5.1.3.5}$$

Remark 5.1.3.2 By repeated application of the recurrence relation in (5.1.3.5), with the help of the relation in (5.1.3.4), we obtain for n $\geq$ m + 1 that

$$E((X(m))^r(X(n))^{s+1}) = E((X(m))^{r+s+1} + (s+1)\sum_{p=m+1}^{n} E((X(m))^r(X(n))^p) \tag{5.1.3.6}$$

Exercise 5.1.3.2 If X(n) is the nth upper record from an exponential distribution with pdf $f(x) = e^{-x}$, $x \geq 0$, then $X(n)\underline{\underline{d}}X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent and identically distributed with pdf $f(x) = e^{-x}$, $x \geq 0$.

Exercise 5.1.3.3 How that the Shannon entropy E_n of X(n) of E(0, σ)is

$$E_n = n + \log\Gamma(n) - \ln\sigma - (n-1)\psi(n),$$

where $\psi(n)$ is the digamma function, $\psi(n) = \frac{d}{dn}\ln\,\Gamma(n)$.

5.2 Generalized Extreme Value Distributions

5.2.1 Introduction

A random variable X is said to have the generalized extreme value distribution if its distribution function F(x) is of the following form:

$$\mathrm{F(x)} = \exp[-\{1-\gamma\,\sigma^{-1}(\mathrm{x}\ -\mu)\}^{1/\gamma}] \tag{5.2.1.1}$$

where σ > 0, γ ≠ 0 and

$$\begin{aligned} &x < \mu + \sigma\gamma^{-1}, \quad \text{for } \gamma > 0 \\ &x > \mu + \sigma\gamma^{-1}, \quad \text{for } \gamma < 0. \end{aligned} \tag{5.2.1.2}$$

If γ = 0 then

$$F(x) = \exp[-\exp\{(x-\mu)/\sigma\}], \sigma > 0, -\infty < x < \infty. \tag{5.2.1.3}$$

We will write X ∈ GEV(μ, σ, γ) if X has the d.f. as given in (5.2.1.1).
Since
$\lim_{\gamma\to 0}\{1-\gamma\,\sigma^{-1}(x-\mu)\}^{1/\gamma} = \exp\{-\sigma^{-1}(x-\mu)\}$, we can take

$$\lim_{\gamma\to 0} GEV(\mu, \sigma, \gamma) = GEV(\mu, \sigma, 0).$$

The pdf of GEV(μ, σ, γ) is

$$\begin{aligned} f(x) &= \sigma^{-1}\{1-\gamma\,\sigma^{-1}(x-\mu)\}^{(1-\gamma)/\gamma}\exp[-\{1-\gamma\,\sigma^{-1}(x-\mu)\}^{1/\gamma}], \gamma \neq 0 \\ &x < 1/\gamma, \quad \text{for } \gamma > 0, \\ &x > 1/\gamma, \quad \text{for } \gamma < 0, \end{aligned}$$

and
$f(x) = e^{-x}\ \exp(-e^{-x})$, for γ = 0, for all x.
Figures 5.2 and 5.3 give the pdfs of GEV(0, 1, 1/2) and GEV(0, 1, 0).
The extreme value distribution for γ = 0, is also known Gumbel distribution.
The largest order statistic $X_{n,n}$ when properly standardized tends to one of the following three types of limiting distribution functions as n → ∞.

(1) Type 1: (Gumbel) $F(x) = \exp(-e^{-x})$, for all x,
(2) Type 2: (Frechet) $F(x) = \exp(-x^{-\delta})$, x > 0, δ > 0
(3) Type 3: (Weibull) $F(x) = \exp(-(-x)^{\delta})$, x < 0, δ > 0.

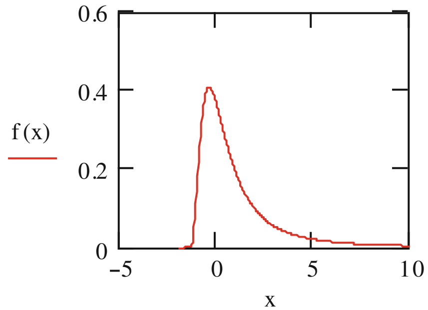
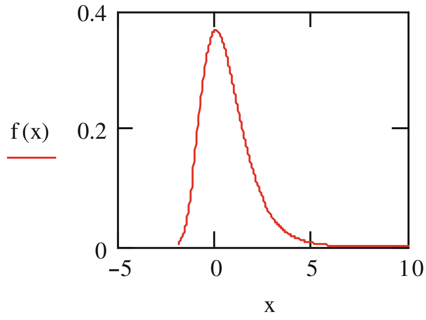


Fig. 5.2 Pdf of GEV(0, 1, 1/2)

Fig. 5.3 Pdf of GEV(0, 1, 0)



Since the smallest order statistic $X_{1,n} = Y_{n,n}$, where $Y = -X$, $X_{1,n}$ when properly standardized will also converge to one of the above three limiting distributions. Gumbel (1958) has given various applications of these distributions. The Type 1 (Gumbel distribution) is the limiting distribution of $X_{n,n}$ when F(x) is normal, log normal, logistic, gamma etc. The generalized extreme value distribution (5.2.1.1) has been discussed by Jenkinson (1955). It includes as special case the above three well known extreme value distributions.

The type 2 and type 3 distributions can be transformed to Type 1 distribution by the transformations $V_2 = \text{llog } X$ and $V_3 = -\log X$ respectively.

These distributions were originally introduced by Fisher and Tippet (1928). Extreme value distributions have been used in the analysis of data concerning floods, extreme sea levels and air pollution Excercises; for details see Gumbel (1958), Horwitz (1980), Jenkinson (1955) and Roberts (1979).

For a given set of n observations, let $X_{1,n} < \cdots < X_{n,n}$ be the associated order statistics. Suppose that $P\{a_n (X_{n,n} - b_n) < x\} \to G(x)$ as $n \to \infty$ for some suitable constants a_n and b_n. Then it is known (see Leadbetter et al. 1983, p. 33) that

$$P\{a_n(X_{n-m,n} - b_n) \le x\} \xrightarrow{d} G(x) \sum_{s=0}^{m} \frac{[-\ln G(x)]^s}{\Gamma(s+1)}.$$

We have already seen that the right hand side of the above expression is the d.f. of the mth lower record value from the distribution function G(x).

Thus the limiting distribution of the $(n - m + 1)$th order statistic (m finite) as $n \to \infty$ from the generalized extreme value distribution is the same as the mth lower record value from the generalized extreme value distribution. In this chapter we will study the lower record values of GEV (μ, σ, γ).

5.2.2 Distributional Properties

If $X \in \text{GEV}(\mu, \sigma, \gamma)$, then we can write for $\gamma \neq 0$, the pdf $f_{(m)}(x)$ of the mth lower record value x(n) as

$$\mathrm{f}_{(\mathrm{m})}(\mathrm{x}) = \{1 - \gamma\sigma^{-1}(x-\mu)\}^{(m-1)/\gamma} f_m^*(x) \tag{5.2.2.1}$$

where

$$f_m^*(x) = \frac{\{1-\gamma\,\sigma^{-1}(x-\mu)\}^{(1-\gamma)/\gamma}}{\sigma(m-1)!}\exp\{-(1-\gamma\sigma^{-1}(x-\mu))\}^{1/\gamma}$$

and for $\gamma = 0$,

$$\mathrm{f}_{(\mathrm{m})}(\mathrm{x}) = \frac{e^{-m\sigma^{-1}(x-\mu)}}{\sigma(m-1)!}\exp\{-e^{-\sigma^{-1}(x-\mu)}\}, \mathrm{m} = 1, 2, \ldots \tag{5.2.2.2}$$

From (5.2.2.1) and (5.2.2.2) it can be shown that

$$\mathrm{x(m)} \underline{\underline{d}}\, \mu + \sigma\gamma^{-1}\{1 - (\mathrm{W}_1 + \cdots + W_m)^\gamma\},\ \text{for } \gamma \neq 0 \tag{5.2.2.3}$$

$$\mathrm{X(m)} \underline{\underline{d}}\, X - \sigma(W_1 + \frac{W_2}{2} + \cdots + \frac{W_{m-1}}{m-1}),\ \text{for } \gamma = 0, \tag{5.2.2.4}$$

where W_1, $W_2, \ldots, W_m$ are independently distributed as exponential random variables with unit mean and $\underline{\underline{d}}$ denotes the equality in distribution. It can easily be shown that.

5.2.3 Moments

$$\begin{aligned}
\mathrm{E(x(m))} &= \mu + \sigma\gamma^{-1}\{1 - \Gamma(\mathrm{m}\ +\gamma)/\Gamma(\mathrm{m})\}.\\
\mathrm{Var(x(m))} &= \sigma^2\gamma^{-2}[\mathrm{E}(\mathrm{W}_1 + \cdots + \mathrm{W}_\mathrm{m})^{2\gamma} - \{\mathrm{E}(\mathrm{W}_1 + \cdots + \mathrm{W}_\mathrm{m})\gamma\}^2]\\
&= \sigma^2\gamma^{-2}\left[\frac{\Gamma(m+2\gamma)}{\Gamma(m)} - \left\{\frac{\Gamma(m+\gamma)}{\Gamma(m)}\right\}^2\right].
\end{aligned}$$

For r < m

$$\begin{aligned}
\gamma^2\sigma^{-2}\mathrm{Cov(x(r)x(m))} &= \mathrm{E}(\sum_{\mathrm{j=1}}^{\mathrm{r}}\mathrm{W_j})^\gamma(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{W_j})^\gamma - \mathrm{E}(\sum_{\mathrm{j=1}}^{\mathrm{r}}\mathrm{W_j})^\gamma\mathrm{E}(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{W_j})^\gamma\\
&= \int_0^\infty\int_0^\infty u^\gamma(u+v)^\gamma\frac{e^{-u}u^{r-1}}{\Gamma(r)}\frac{e^{-v}v^{m-r-1}}{\Gamma(m-r)}du\,dv\\
&= \frac{\Gamma(r+\gamma)\Gamma(r+2\gamma)}{\Gamma(r)\Gamma(r+\gamma)} - \frac{\Gamma(r+\gamma)\Gamma(m+\gamma)}{\Gamma(r)\Gamma(m)},
\end{aligned}$$

since u and v are independent. We can write for r < m

$$\mathrm{Cov}\{x(r)x(m)\} = \sigma_o^2 a_r b_m,$$

where

$$a_r = \frac{\Gamma(r+\gamma)}{\Gamma(r)}, \quad b_m = \frac{\Gamma(m+2\gamma)}{\Gamma(m+\gamma)} - \frac{\Gamma(m+\gamma)}{\Gamma(m)} \quad \text{and } \sigma_o^2 = \frac{\sigma^2}{\gamma^2}.$$

Using (5.2.2.4) we obtain for γ = 0,

$$\begin{aligned} \mathrm{E}(x(r)) &= \mu + \upsilon_r^* \sigma \\ \mathrm{Var}(x(r)) &= \sigma^2 V_{r,r}^*, \mathrm{r} = 1, 2, \ldots \\ \mathrm{Cov}(x(r)x(m)) &= \mathrm{Var}(x(m)), \mathrm{r} < \mathrm{m}, \end{aligned}$$

with

$$\begin{aligned} \upsilon_1^* &= \upsilon \\ \upsilon_j^* &= \upsilon_{j-1}^* - (j-1)^{-1}, j \geq 2, \\ V_{1,1}^* &= \pi^2/6, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_{j,j}^* &= V_{j-1,j-1}^* - (j-1)^{-2}, j \geq 2 \end{aligned}$$

Here υ (=0.57722...) is the Euler's constant.

For X ∈ GEV (μ, σ, 0), the joint pdf $f_Y(y)$ of Y = H(x(m + 1))/H(x(m)) is

$$f_Y(y) = \mathrm{my}^{\mathrm{m}-1}, \; 0 < \mathrm{y} < \infty \tag{5.2.3.1}$$

Thus $(Y)^m$ is distributed as uniform over the interval (0, 1). Consequently m [−log H(x(m)) + log H(x(m + 1)) is distributed as exponential distribution with mean unity. Tables 5.1 and 5.2 give the values of $E(X_L(n))$ and $Var(X_L(n))$ for some selected values of n and γ.

Table 5.1 Expected values of $X_{L(n)}$ γ

n/γ	0	0.5	1.0	1.5
5	−1.5061	−2.3619	−4.0000	−5.6301
10	−2.2518	−4.2460	−9.0000	−21.1944
15	−2.6743	5.6817	−14.0000	−39.0221
20	−2.9705	−6.8886	−19.0000	−60.0718
25	−3.1987	−7.9501	−24.0000	−83.9094
30	−3.3844	−8.9089	−29.0000	−110.2405

Table 5.2 Variances of $X_{L(n)}$

n\γ	γ = 0	γ = 0.5	γ = 1.0	γ = 1.5
5	0.2213	0.9738	5.0000	29.3843
10	0.1052	0.9872	10.0000	108.7898
15	0.0689	0.9915	15.0000	238.1350
20	0.0513	0.9937	20.0000	415.5101
25	0.4080	0.9950	25.0000	646.8852
30	0.0339	0.9958	30.0000	926.2602

We will derive the recurrence relation for the moments of standardized extreme value distribution, GEV(0, 1, γ). The distribution function of the standard generalized extreme value distribution can be written as

$$F(x) = e^{-\{1-\gamma x\}^{1/\gamma}},\ x < 1/\gamma,\ \text{when}\ \gamma > 0,$$
$$\mathrm{x} > 1/\gamma,\ \text{when}\ \gamma < 0$$

and

$$F(x) = e^{-e^{-x}}, -\infty < x < \infty,\ \text{when}\ \gamma = 0$$

The corresponding pdfs are

$$f(x) = e^{-\{1-\gamma x\}^{1/\gamma}}\{1-\gamma x\}^{\frac{1}{\gamma}-1},\ \text{when x}\ < 1/\gamma,\ \text{when}\ \gamma > 0$$
$$\mathrm{x} > 1/\gamma\ \text{when}\ \gamma < 0$$

$$= e^{-e^{-x}}e^{-x}, -\infty < x < \infty, \gamma = 0$$

Note that for $\gamma \neq 0$

$$f(x) = F(x).\{1-\gamma x\}^{\frac{1}{\gamma}-1}$$
$$\Rightarrow \{1-\gamma x\}f(x) = F(x)\{1-\gamma x\}^{1/\gamma} = -F(x)\ln F(x) \tag{5.2.3.2}$$

(a) Let us consider the case $\gamma \neq 0$

Let x(1), x(2),…. be the sequence of lower record values from the above generalized extreme value distribution when $\gamma \neq 0$. Then, the pdf $f_n(x)$ of x(n)) (n > 1) is given by

$$f_n(x) = \frac{1}{\Gamma(n)}\{-\log F(x)\}^{n-1}f(x),\ x < 1/\gamma,\ \text{when}\ \gamma > 0, \quad \mathrm{x}\ > 1/\gamma,\ \text{when}\ \gamma < 0 \tag{5.2.3.3}$$

Theorem 5.2.3.1 *For* n = 1, 2,… and r = 0, 1, 2,…,

$$E(x(n+1))^{r+1} = \left\{1 + \frac{\gamma(r+1)}{n}\right\} E(x(n))^{r+1} - \frac{r+1}{n} E(x(n))^{r} \tag{5.2.3.4}$$

Proof For n > 1 and r = 0, 1,

$$\begin{aligned} E(x(n))^{r} - \gamma E(x(n))^{r-1} &= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} (x^{r} - \gamma x^{r+1}) \{-\log F(x)\}^{n-1} f(x) dx \\ &= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} x^{r} \{-\log F(x)\}^{n} F(x) dx \end{aligned}$$

Upon integrating by parts, we obtain

$$\begin{aligned} E((x(n))^{r}) - \gamma E(x(n))^{r-1} &= \frac{1}{\Gamma(n)(r+1)} \int_{-\infty}^{\infty} n x^{r+1} \{-\log F(x)\}^{n-1} f(x) dx \\ &\quad - \int_{-\infty}^{\infty} x^{r+1} \{-\log F(x)\}^{n} f(x) dx \\ &= \frac{n}{(r+1)} \left\{ E(x(n))^{r+1} - E(x(n+1))^{r+1} \right\}. \end{aligned}$$

The relation in (5.2.3.4) is derived upon rewriting the above equation.

Remark 5.2.3.1 By starting with the first k raw moments of the generalized extreme value distribution, Theorem 1 will enable one to determine the first k raw moments of all the lower record values.

Corollary 5.2.3.1

For n > 1,

$$E(x(n+1)) = \left(1 + \frac{\gamma}{n}\right) E(x(n)) - \frac{1}{n} \tag{5.2.3.5}$$

(b) Case γ = 0.

When γ = 0, the generalized extreme value distribution is also known as the Gumbel distribution or type I extreme value distribution. Let us consider the standard type I extreme value distribution with the pdf as

$$\mathrm{f(x)} = e^{-e^{-x}} e^{-x}, -\infty < \mathrm{x} < \infty, \tag{5.2.3.6}$$

and the d.f. as

$$F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty, \tag{5.2.3.7}$$

It is easy to see from (5.2.3.6) and (5.2.3.7) that for the standard type I extreme value distribution

$$f(x) = F(x)\{-\log F(x)\}, \quad -\infty < x < \infty. \tag{5.2.3.8}$$

We can make use of this property of the type I extreme distribution to derive some recurrence relations for the single and product moments of lower record values.

Exercise 5.2.3.1

$$\begin{aligned} &\text{For } n \geq 1 \text{ and } r = 0, 1, 2, \ldots \\ &E(x(n+1))^{r+1} = E(x(n))^{r+1} - \frac{r+1}{n} E(x(n))^{r} \end{aligned} \tag{5.2.3.9}$$

5.3 Generalized Pareto Distribution

5.3.1 Introduction

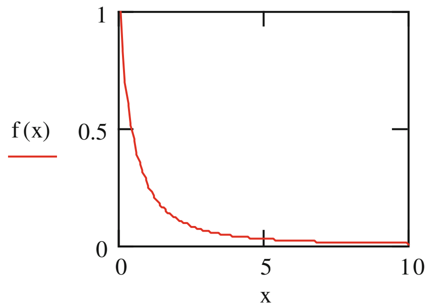
A random variable (rv) X is said to have the generalized Pareto distribution if its probability density function (pdf) is of the following form:

$$\begin{aligned} f(x) = \frac{1}{\sigma}\left(1 + \beta\left(\frac{x-\mu}{\sigma}\right)\right)^{-(1+\beta^{-1})} \\ &x \geq \mu, \text{ for } \beta > 0, \\ &\mu < x < \mu - \sigma/\beta, \text{ for } \beta < 0, \\ &= \frac{1}{\sigma} e^{-(x-\mu)\sigma^{-1}}, \ x \geq \mu \text{ for } \beta = 0, \\ &= 0, \text{ otherwise}, \\ &\text{for } \sigma > 0. \end{aligned} \tag{5.3.1.1}$$

We will say that $X \in GP(\mu, \sigma, \beta)$ if X has the pdf as given in (5.3.1.1). For $\beta > 0$, $GP(\mu, \sigma, \beta)$ is known as Pareto type II or Lomax distribution. For $\beta = -1$, $GP(\mu, \sigma, \beta)$ coincide with the uniform distribution on $(\mu, \mu + \sigma)$. Figure 5.4 gives the pdf of GP(0, 1, 1).

The generalized Pareto distribution was introduced by Pickands (1975). Some of its applications include its uses in the analysis of the extreme events, in the modeling of large insurance claims and to describe the annual maximum flood at river

Fig. 5.4 Pdf of GP(0, 1, 1)



gauging station. GP(μ, σ, β) has finite variance if $\beta < 1/2$. GP(μ, σ, β) for $\beta = 0$ is the exponential distribution which is discussed here. In this chapter we will take $\beta \neq 0$.

If $X \in$ GP(μ, σ, β), then

$$\mathrm{r(x)} = \frac{1}{\sigma}[1 + \beta(x - \mu)\sigma^{-1}]^{-1}$$

and

$$\mathrm{R(x)} = \frac{1}{\beta}\log[1 + \beta(x - \mu)\sigma^{-1}].$$

The hazard rate r(x) is monotonically increasing (decreasing) in x if $\beta > (<)\ 0$ and r(x) is constant for $\beta = 0$.

5.3.2 Distributional Properties

We will consider the upper record values from GP(μ, σ, β). The pdf $f_n(x)$ of the nth upper record X(n) is

$$f_n(x) = \frac{1}{\Gamma(n)}\left\{\frac{1}{\beta}\log\left[1 + \frac{\beta(x-\mu)}{\sigma}\right]\right\}^{n-1}\frac{1}{\sigma}\left[1 + \frac{\beta(x-\mu)}{\sigma}\right]^{-(1+\beta^{-1})}$$
$$\mu < \mathrm{x} < \infty,\ \beta > 0 \tag{5.3.2.1}$$

We can write

$$\mathrm{X(n)} \underline{\underline{d}}\, \mu - \frac{\sigma}{\beta} + \frac{\sigma}{\beta}\prod_{i=1}^{n} U_i, \tag{5.3.2.2}$$

where U_1, U_2, U_n are independent and identically distributed as

$$\begin{aligned} P[Ui < x] &= 1 - (x)^{-1/\beta}, x \geq 1,\ \beta > 0, \\ &= (x)^{-1/\beta}, 0 < x < 1, \beta < 0. \end{aligned} \tag{5.3.2.3}$$

5.3.3 Moments

Substituting $\frac{1}{\beta}\log\left[1 + \frac{\beta(x-\mu)}{\sigma}\right] = t$, we have

$$\begin{aligned} E(X(n)) &= \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}\left[\mu + \frac{\sigma}{\beta}(e^{\beta t} - 1)\right]e^{-t}dt \\ &= \mu + \frac{\sigma}{\beta}\{(1-\beta)^{-n} - 1\}, if\ \beta < 1 \end{aligned}$$

Similarly

$$\begin{aligned} E(X(n))^2 &= \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}[\mu + \frac{\sigma}{\beta}(e^{\beta t} - 1)]^2 e^{-t}dt \\ &= (\mu - \frac{\sigma}{\beta})^2 + 2(\mu - \frac{\sigma}{\beta})\frac{\sigma}{\beta}(1-\beta)^{-n} + \frac{\sigma^2}{\beta^2}(1-2\beta)^{-n},\ \text{if } \beta < 1/2 \\ Var(X(n)) &= \sigma^2\beta^{-2}\{(1-2\beta)^{-n} - (1-\beta)^{2n},\ \beta < 1/2, \end{aligned}$$

and

$$Cov(X(m)X(n)) a_m b_n \sigma^2\beta^{-2},\quad m < n, \tag{5.3.3.1}$$

where

$$\begin{aligned} a_m &= (1-\beta)^m\{(1-2\beta)^{-m} - (1-\beta)^{-2m}, \beta < 1/2 \\ b_n &= (1-\beta)^{-n}, \beta \neq 1 \end{aligned}$$

Example 5.3.3.1 If $\beta = -1$, i.e. when X is distributed uniformly in the interval $(\mu, \mu + \sigma)$, then

$$\begin{aligned} E(X(n)) &= \mu + (1 - 2^{-n})\sigma \\ Var(X(n)) &= (3^{-n} - 4^{-n})\sigma^2 \\ Cov(X(m)X(n)) &= 2^{m-n} Var(X_{(m)}),\ m < n. \end{aligned}$$

Example 5.3.3.2 For the classical Pareto distribution with the cumulative distribution function F(x), where

$$\begin{aligned}
&\mathrm{F(x)}1-\left(\frac{\theta}{x}\right)^{v},\ 0<\theta\leq x<\infty,\ v>0,\\
&\mathrm{E(X(n))}=\theta\Big(\frac{v}{v-1}\Big)^{n},\ \text{if } v>1\\
&\mathrm{Var(X(n))}=\theta^{2}\left[\Big(\frac{v}{v-2}\Big)^{n}-\Big(\frac{v}{v-1}\Big)^{2n}\right],\ \text{if } v>2\\
&\mathrm{Cov(X(m)X(n))}=\theta^{2}\Big(\frac{v}{v-1}\Big)^{n-m}\mathrm{Var(X(m))},\ \mathrm{m<n}.
\end{aligned}$$

We will consider in this section without any loss of generality μ = 0 and σ = 1 i.e.

$$\begin{aligned}
f(x)&=\beta(1+\beta x)^{-(1+\beta^{-1})},x\geq 0,\text{for } \beta>0,\\
&\quad 0<x<-\beta^{-1}\ \text{ for }\ \beta<0\\
&=\ \mathrm{e}^{-\mathrm{x}},x\geq 0\,\text{for } \beta=0,\\
&=\ 0,\ \text{otherwise.}
\end{aligned} \tag{5.3.3.2}$$

It is also known that (5.3.3.2) is also the pdf of Lomax distribution.

In this section some recurrence relations satisfied by the single and product moments of upper record values from the generalized Pareto distribution. These recurrence relations will enable one to obtain all the single and product moments of all record values in a simple recursive manner. It is shown here that the results for the exponential distribution proved in Chap. 2 can be deduced from the results established in this section by letting the shape parameter β tend to 0.

We will derive some recurrence relations between moments and product moments of the record values. These results are given by Balakrishnan and Ahsanullah (1994). (a) Relations for single moments

First of all, we may note that for the generalized Pareto distribution in (5.3.3.2)

$$f(x)(1+\beta x)=\beta(1-F(x)). \tag{5.3.3.3}$$

The relation in (5.3.3.3) will be exploited in this section to derive recurrence relations for the moments of record values from the generalized Pareto distribution.

Let X(1), X(2),……. be the sequence of upper record values from (5.3.3.2). For convenience, we shall also take X(0) = 0. The pdf of X(n), n = 1, 2,…, is given by

$$\mathrm{f_n}(x)=\frac{1}{(n-1)!}\{-\log(1-\mathrm{F}(x))\}^{n-1}\mathrm{f}(x) \tag{5.3.3.4}$$

For the existence of the (r + 1)th moment (r + 1) β must be less than 1.

Theorem 5.3.3.1 *For* n > 1 *and* r = 0, 1, 2,….,

$$\mathrm{E(X(n))^{r+1}} = \frac{1}{1-(r+1)\beta}\left\{(r+1)\mathrm{E}\left(X^{r}_{U(n)}\right) + \mathrm{E(X^{r+1}_{U(n-1)}}\right\}$$
$$\text{for } \beta < (r+1)^{-1}. \tag{5.3.3.5}$$

Proof For $n \geq 1$ and r = 0, 1, 2,…., we have from (5.3.3.3)

$$\begin{aligned}\mathrm{E(X(n)^r)} + \beta\mathrm{E}(\mathrm{X}(n)^{r+1}) &= \int (x^r + \beta x^{r+1})\mathrm{f_n}(x)\mathrm{dx} \\ &= \int_x x^r \frac{1}{(n-1)!}\{-\log(1-\mathrm{F(x)})\}^{n-1}(! + \beta x)\mathrm{f(x)dx} \\ &= \frac{1}{(n-1)!}\int_x x^r\{-\log(1-\mathrm{F(x)})\}^{n-1}(1-\mathrm{F}(x))\mathrm{dx}\end{aligned}$$

Integrating now by parts treating x^r for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned}&\mathrm{E(X(n)^r)} + \beta\,\mathrm{E}(\mathrm{X}(n)^{r+1}) \\ &= \frac{1}{(r+1)(n-1)!}\left[-(n-1)\int_x x^{r+1}\{-\log(1-\mathrm{F(x)})\}^{n-2}\mathrm{f(x)dx}\right. \\ &\qquad \left. + \int_x \mathrm{x^{r+1}}\{-\log(1-\mathrm{F(x)})\}^{n-2}\mathrm{f(x)dx}\right] \\ &\frac{1}{r+1}\left[\mathrm{E}\left((\mathrm{X(n)^{r+1}}\right) - \mathrm{E}\left(\mathrm{X(n-1)^{r+1}}\right)\right]\end{aligned}$$

The relation in (5.3.3.5) is derived simply by rewriting the above equation.

Remark 5.3.3.1 The recurrence relation in Theorem 5.3.3.1 can be used in a simple recursive manner to compute all the single moments of all record values. By setting r = 0 in (5.3.3.5), we get the relation

$$E(X(n)) = \frac{1}{1-\beta}\{1 + E(X(n-1))\},\ n \geq 2, \beta < 1. \tag{5.3.3.6}$$

Repeated application of (5.3.3.6) will readily yield 0.

$$E\left(X_{U(n)}\right) = \frac{1}{1-\beta} + \frac{1}{(1-\beta)^2} + \cdots + \frac{1}{(1-\beta)^{n-1}} = \frac{1}{\beta}\left[\frac{1}{(1-\beta)^n} - 1\right]$$

an expression given by Ahsanullah (1992).

Balakrishnan and Ahsanullah (1994a) proved the following recurrence relation of the product moment.

Theorem 5.3.3.2 *For* m > 1, r, s = 0, 1, 2,….,

$$E\Big((X(m))^r(X(m+1))^{s+1}\Big) = \frac{1}{1-(s+1)\beta}\Big[(s+1)E\Big((X(m))^r(X(m+1))^s\Big) + E(X(m))^{r+s+1}\Big] \tag{5.3.3.7}$$

for $\beta < \frac{1}{s+1}$; *for* $1 \le m \le n-2$ *and* $r, s = 0, 1, 2, \ldots,$

Exercise 5.3.3.1 Show that

$$Cov(X(m)X(n)) = \frac{1}{(1-\beta)^{n-m}} Var(X(m)). \tag{5.3.3.8}$$

5.4 Logistic Distribution

5.4.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. rvs from the standard logistic distribution with pdf and d.f., respectively as

$$\begin{aligned} f(x) &= \frac{e^{-x}}{(1+e^{-x})^2} \\ \mathrm{F(x)} &= \frac{1}{1+e^{-x}}, -\infty < x < \infty. \end{aligned} \tag{5.4.1.1}$$

The Fig. 5.5 gives the pdf of the logistic distribution as given by (7.1.1). The pdf is symmetric around zero.

Since, X(n) $\underline{\underline{d}}$ − x(n) all n $\geq$ 1. We will consider here the upper record values.

5.4.2 Moments

The pdf $f_n(x)$ of the nth upper record value is given by

$$f_n(x) = \frac{1}{\Gamma(n)}(\log(1+e^x))^{n-1}\frac{e^{-x}}{(1+e^{-x})^2}. -\infty < x < \infty.$$

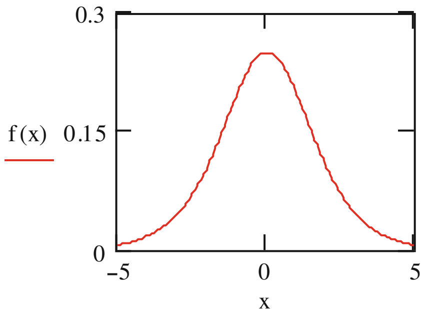


Fig. 5.5 pdf of X_i's

Let $\mu_n = E(X(n))$, $\mu_{m,n} = E(X(m)X(n))$, then

$$\mu_n = \mu_{n-1} + \zeta(n), n > 1, \mu_1 = 0,$$
$$\mu_{m,n} = nE\left(X_{U(m)}\right) + mE\left(X_{U(n)}\right) + m\zeta(m+1) + m\zeta(n+1) - m(n+1)$$
$$+ \sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n-m}} \sum_{l=1}^{\infty} \frac{1}{l(l+1+k)^m},$$

where ζ (.) is the Riemann zeta function.

Proof

$$\mu_1 = E(X_1) = \int_{-\infty}^{\infty} \frac{xe^{-x}}{(1+e^{-x})^2} dx = 0.$$

The conditional pdf of $X(n)|X(n-1) = y$, for $n > 1$, is

$$f_{n,n-1,c}(x|X_{U(n-1)} = y) = \frac{e^{-x}(1+e^y)}{(1+e^{-x})^2}, -\infty < y < x < \infty \tag{5.4.2.1}$$

The conditional expectation of $X(n)|X(n-1) = y$ is given by

$$E\left(X_{U(n)}|X_{U(n-1)} = y\right) = \int_y^{\infty} \frac{x e^{-x}(1+e^y)}{(1+e^{-x})^2} dx$$
$$= y + (1+e^y)\ln(1+e^{-y})$$

Thus

$$E(X(n) = E(X(n-1)) + \int_{-\infty}^{\infty} (1+e^y)\ln(1+e^{-y}) \frac{\{y+\ln(1+e^{-y})\}^{n-2}}{\Gamma(n-1)} \frac{e^{-y}}{(1+e^{-y})^2} dy \tag{5.4.2.2}$$

Substituting $t = e^{-y}(1+e^{-y})^{-1}$ and simplifying, we get

$$\begin{aligned} \mathrm{E}\big(\mathrm{X}_{\mathrm{U(n)}}\big) &= \mathrm{EX}_{\mathrm{U(n-1)}} + \int_o^1 \frac{(-\ln t)(-\ln(1-t))^{n-2}}{(1-t)\Gamma(n-1)} dt \\ &\mathrm{E}\big(\mathrm{X}_{\mathrm{U(n-1)}}\big) + \zeta(\mathrm{n}) \end{aligned} \tag{5.4.2.3}$$

where ζ(n) is the Riemann zeta function.

Since $\mu_1 = 0$, we have

$$\mu_n = \sum_{i=2}^{n} \zeta(\mathrm{i}), \mathrm{n} \geq 2. \tag{5.4.2.4}$$

The expected values of the first ten upper record values are given in Table 5.3.

$$\begin{aligned} \mu_{n,n} = E(X(n))^2 &= \int_{-\infty}^{\infty} x^2 f_n(x) dx \\ &= \int_{-\infty}^{\infty} x^2 \frac{(\log(1+e^x))^{n-1}}{\Gamma(n)} \frac{e^{-x}}{(1+e^{-x})^2} dx \\ &= \int_o^1 [\log t - \log(1-t)]^2 \frac{\{-\log(1-t)\}^{n-1}}{\Gamma(n)} dt \\ &= \int_0^1 \frac{\{-\log(1-t)\}^{n+1}}{\Gamma(n)} dt - 2\int_o^1 \frac{(\log t)\{-\log(1-t)\}^n}{\Gamma(n)} dt \\ &\quad + \int_0^1 \frac{\{\log t\}^2 \{-\log(1-t)\}^{n-1}}{\Gamma(n)} dt \end{aligned}$$

Table 5.3 $\mathrm{E}(_{\mathrm{XU(n)}})$, $1 \leq \mathrm{n} \leq 10$

n	$\mathrm{E}(\mathrm{X}_{\mathrm{U(n)}})$
1	0.0000
2	1.6449
3	2.8470
4	3.9293
5	4.9662
6	5.9836
7	6.9919
8	7.9960
9	8.9980
10	9.9990

Now

$$\int_0^\infty \frac{\{-l\log(1-t)\}^{n+1}}{\Gamma(n)}dt = n(n+1), -\int_0^\infty \frac{\ln t\{-\ln(1-t)\}^n}{\Gamma(n)}dt]$$
$$= \mathrm{n}\left[\mathrm{E}\left(X_{U(n+1)}\right) - (\mathrm{n}\ +1)\right]$$

and

$$\int_0^1 \frac{\{\log t\}^2\{-\log 1-t\}^{n-1}}{\Gamma(n)}dt = \int_0^\infty \left\{\sum_{k=1}^\infty \frac{e^{-tk}}{k}\right\}^2 \frac{1}{\Gamma(n)}t^{n-1}e^{-t}dt$$
$$= \sum_{j=2}^\infty B_j \int_0^\infty e^{-jt}\frac{1}{\Gamma(n)}t^{n-1}e^{-t}dt = \sum_{j+2}^\infty \frac{B_j}{(j+1)^n},$$

where $\mathrm{B_j}$ is the coefficient of $\mathrm{e^{-jt}}$ in $\left\{\sum_{k=1}^\infty \frac{e^{-kt}}{k}\right\}^2$.

Thus

$$\mu_{\mathrm{n,n}} = \mathrm{n(n+1) + 2n[E(X(n-1)) - (n+1)]} + \sum_{j+2}^\infty \frac{B_j}{(j+1)^n}$$
$$= \mathrm{2nE(X(n-1))} - n(n+1) + \sum_{l=2}^\infty \frac{B_l}{(l+1)^n},$$
$$= \mathrm{2n}\sum_{i=2}^{n+1}\zeta(i) - n(n+1) + \sum_{l=2}^\infty \frac{B_l}{(l+1)^n},$$

where

$$\mathrm{B_i} = \frac{1}{i}\left(1+\frac{1}{2}+\cdots+\frac{1}{i-1}\right), i \geq 2.$$

$$\mathrm{Var(X(n))} = \mathrm{2n}\sum_{i=2}^{n+1}\zeta(i) - n(n+1) + \sum_{l=2}^\infty \frac{B_l}{(l+1)^n} - \left(\sum_{i=2}^n \xi(i)\right)^2,$$

$$\mu_{\mathrm{m,n}} = \int_{-\infty}^\infty \int_y^\infty xyf_{m,n}(y,x)dydx$$
$$= \iint\limits_{-\infty<y<x<\infty} \frac{(-\ln \bar{F}(y))^{m-1}}{\Gamma(m)\Gamma(n-m)}[\log \bar{F}(x) - \log \bar{F}(y)]^{n-m-1} r(y)f(x)dydx,$$

where $r(y) = \frac{f(y)}{\bar{F}(y)}$, $\bar{F}(y) = 1-\mathrm{F(y)}$ and f(y) and F(y) are given by (7.1.1) and (7.1.2). Substituting $v = \frac{1}{1+e^{-x}}$ and $u = \frac{1}{1+e^{-y}}$ and simplification, we get

$$\mu_{m,n} = \int_o^1 \int_u^1 \frac{\{-\log(1-u)\}^{m-1}}{\Gamma(m)\Gamma(n-m)}[-\log(1-v)+\log(1-u)]^{n-m-1}$$
$$[\log u - \log(1-u)][\log nv - \log(1-v)]\frac{1}{1-u}dvdu$$

Let $\mathrm{I_u} = \int_u^1 [-\log(1-v)+\log(1-u)]^{n-m-1}[\log v - \log(1-v)]dv$
Substituting w = log (1 − v) + log(1 − u), we have

$$\begin{aligned}\mathrm{I_u} &= \int_o^\infty w^{n-m-1}[\log\{1-(1-u)e^{-w}\} - \{w + \log 1 - u\}](1-u)e^{-w}dw \\ &= (1-u)\left[\sum_{k=1}^\infty \int_o^\infty (-1)(1-u)^k \frac{e^{-kw}}{k} e^{-w} w^{n-m-1} dw \right. \\ &\qquad \left. + \Gamma(n-m+1) - \Gamma(n-m)\log(1-u)\right] \\ &= \Gamma(n-m)(1-u)\left[-\sum_{k=1}^\infty \frac{(1-u)^k}{k(k+1)^{n-m}} + (n-m) - \log(1-u)\right]\end{aligned}$$

Upon substituting $\mathrm{I_u}$, we have

$$\begin{aligned}\mu_{m,n} &= \int_o^1 \frac{\{-\log(1-u)\}^{m-1}}{\Gamma(m)} n - m - \log(1-u) - \sum_{k=1}^\infty \frac{(1-u)^k}{k(k+1)^{n-m}}[\log u - \log(1-u)]du \\ &= \mathrm{(n-m)E(X(m)) + m\,E(X(m-1)) - m}\sum_{k=1}^\infty \frac{1}{k(k+1)^{n+1}} \\ &\quad + \sum_{k=1}^\infty \frac{1}{k(k+1)^{n-m}} \sum_{l=1}^\infty \frac{1}{l(l+1+k)^m} \\ &= \mathrm{nE(X(m)) + m\,E(X(n)) + m\zeta(m+1) + m\zeta(n+1) - m(n+1)} \\ &\quad + \sum_{k=1}^\infty \frac{1}{k(k+1)^{n-m}} \sum_{l=1}^\infty \frac{1}{l(l+1+k)^m} \\ &\mathrm{Cov(X(m)X(n))} = \mu_{\mathrm{m,n}} - \mathrm{E(X(m))E(X(n))}.\end{aligned}$$

Table 5.4 gives the variances and covariance of X(m)X(n), for $1 \leq m \leq n \leq 10$.

5.5 Normal Distribution

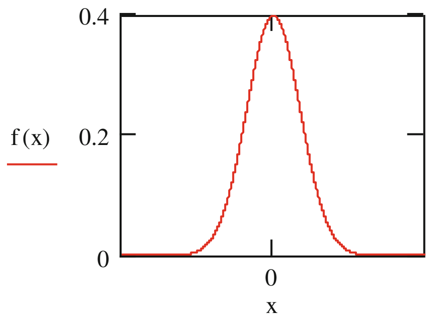
5.5.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables from standard normal distribution (N(0, 1)) with pdf

Table 5.4 Variances and covariances of the upper record values

m/n	1	2	3	4	5	6	7	8	9	10
1	3.2899									
2	2.4426	2.9882								
3	1.9701	2.6887	3.5414							
4	1.7913	2.5310	3.3885	4.3096						
5	1.7139	2.4636	3.3132	4.2258	5.1779					
6	1.6782	2.4327	3.2788	4.1853	5.1311	6.1016				
7	1.6612	2.4181	3.2625	4.1660	5.1084	6.0754	7.0576			
8	1.6530	2.4110	3.2546	4.1567	5.0974	6.0625	7.0429	8.0323		
9	1.6489	2.4075	3.2508	4.1522	5.0920	6.0563	7.0356	8.0241	9.0180	
10	1.6469	2.4058	3.2489	4.1500	5.0893	6.0532	7.0321	8.0200	9.0134	10.0100

Fig. 5.6 Pdf of X_i's



$$\phi(X) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < X < \infty. \tag{5.5.1.1}$$

and d.f.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

The graph of $\phi(x)$ is given in Fig. 5.6.
The pdf of the nth upper record X(n) is

$$f_n(x) = \frac{1}{\Gamma(n)} (-\log \Phi(x))^{n-1} \phi(x) - \infty, \; x < \infty, n \geq 1.$$

5.5.2 Moments

Theorem 5.5.2.1 *Let* $\mu_n = \mathrm{E(X(n))}$, $\mu_{n,n} = E(X(n))^2$ *and* $\mu_{\mathrm{m,\,n}} = \mathrm{E(X(m)X(n))}$, then

$$\mu_{n,n} = 1 + \mu_{\mathrm{n-1,n}},$$
$$\mu_n = \frac{1}{\Gamma(n-1)}\int_{-\infty}^{\infty}\{-\log(1-\Phi(\mathrm{X}))\}^{n-2}(1-\Phi(\mathrm{X}))^{-1}(\phi(\mathrm{X}))^2\mathrm{dx}$$

and

$$\mu_{m,n} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^1\int_{-1}^1\{\Phi^{-1}(u)\}\{\Phi^{-1}(v)\}$$
$$\frac{\{-\log(1-u)\}^{m-1}}{1-u}\log(1-u)-\log(1-v)\}^{n-m-1}dudv.$$

Proof

$$\mu_{n,n} = \frac{1}{\Gamma(n)}\int_{-\infty}^{\infty} x^2[-\log n\{1-\Phi(x)\}]^{n-1}\phi(\mathrm{x})\mathrm{dx}$$
$$= -\frac{1}{\Gamma(n)}\int_{-\infty}^{\infty} x[-\log\{1-\Phi(x)\}]^{n-1}\mathrm{d}\phi(\mathrm{x})$$

Since—x $\phi(\mathrm{x})$ dx = $\mathrm{d}\phi(\mathrm{x})$. Integrating the above expression by parts and simplifying, we have

$$\mu_{n,n} = \frac{1}{\Gamma(n)}\int_{-\infty}^{\infty}[-\log\{1-\Phi(x)\}]^{n-1}(\phi(\mathrm{x}))\mathrm{dx}$$
$$+\frac{1}{\Gamma(n-1)}\int_{-\infty}^{\infty} x[-\log n\{1-\Phi(x)\}]^{n-2}\frac{1}{1-\Phi(x)}(\phi(\mathrm{x}))^2\mathrm{dx}$$
$$= 1+\frac{1}{\Gamma(n-1)}\int_{-\infty}^{\infty} x[-\log n\{1-\Phi(x)\}]^{n-2}\frac{1}{1-\Phi(x)}(\phi(\mathrm{x}))^2\mathrm{dx}.$$

We can write

$$\phi(\mathrm{x}) = \int_x^{\infty} -\phi'(\mathrm{y})\mathrm{dy} = \int_x^{\infty} \mathrm{y}\phi(\mathrm{x})\mathrm{dy}, \text{ thus}$$

$$\mu_{n,n}$$
$$= 1+\frac{1}{\Gamma(n-1)}\int_{-\infty}^{\infty}\int_x^{\infty} xy[-\log n\{1-\Phi(x)\}]^{n-2}\frac{1}{1-\Phi(x)}\phi(\mathrm{x})\phi(\mathrm{y})\mathrm{dx\ dy}.$$
$$= 1+\mu_{\mathrm{n-1,\,n}}.$$

$$\mu_{m,n}\int_{-\infty}^{\infty} c\int_0^{y}\int_0 \frac{y}{1-\Phi(y)}\frac{x}{1-\Phi(x)}[\log\{1-\Phi(x)\}-\{\log 1-\Phi(y)\}]^{n-m-1}\phi(\mathrm{x}).\phi(\mathrm{y})\mathrm{dxdy}$$

On simplification, we get

$$\mu_{m,n} = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^1 \int_o^{\Phi^{-1}} \{\Phi^{-1}(u)\}\{\Phi^{-1}(v)\}$$
$$\frac{\{-\ln(1-u)\}^{m-1}}{1-u} \{\log(1-u) - \log(1-v)\}^{n-m-1} dudv$$

The means, variances and covariances of the upper record values were obtained by numerical methods. The variances and covariances of the lower record values are the same. The means of the lower record values are the negatives of the upper record values. The following tables give the means, variances and covariance of $X_{U(m)}$ and X(n) for $1 \leq m \leq n \leq 10$ (Tables 5.5 and 5.6).

Table 5.5 Mean of X(n)

n	E\(X(n, n))
1	0.0000
2	0.9032
3	1.4990
4	1.9687
5	2.3667
6	2.7174
7	3.0339
8	3.3244
9	3.5942
10	3.8471

Table 5.6 Variances and Covariances of the upper record values X(m) and X(m), m < n

m/n	1	2	3	4	5	6	7	8	9	10
1	1.0000									
2	0.5956	0.7799								
3	0.4534	0.5953	0.7022							
4	0.3775	0.4964	0.5859	0.6611						
5	0.3292	0.4331	0.5115	0.5753	0.6353					
6	0.2951	0.3885	0.4589	0.5181	0.5702	0.6174				
7	0.2696	0.3550	0.4194	0.4735	0.5212	0.5643	0.6014			
8	0.2495	0.3286	0.3883	0.4385	0.4827	0.5226	0.5595	0.5938		
9	0.2332	0.3073	0.3631	0.4100	0.4514	0.4888	0.5233	0.5554	0.5856	
10	0.2197	0.2895	0.3421	0.3864	0.4253	0.4606	0.4931	0.5234	0.5519	0.5788

5.6 Power Function Distribution

5.6.1 Introduction

A random variable (rv) X is said to have the three parameter power function distribution if its probability density function (pdf) is of the following form:

$$\begin{aligned} f(x,\alpha,\beta,\gamma) &= \gamma\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}, \text{ for } \alpha < x < \alpha+\beta, \beta > 0, \gamma > 0, \\ &= 0, \text{ otherwise.} \end{aligned} \tag{5.6.1.1}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (5.6.1.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (5.6.1.1) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha+\beta)^{\gamma}$, the Y has the uniform distribution in (0, 1). The following Figs. 5.7 and 5.8 of f(x) and g(x) are the graphs of the pdfs of PF (α, β, γ) for α = 0, β = 1, γ = 1/2 and α = 0, β = 1, γ = 4 respectively.

5.6.2 Distributional Properties

The joint pdf of X(1), X(2),…,X(m) can be written as

Fig. 5.7 Pdf of GP(0, 1, 1/2)

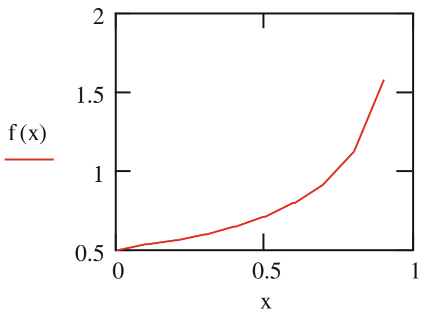
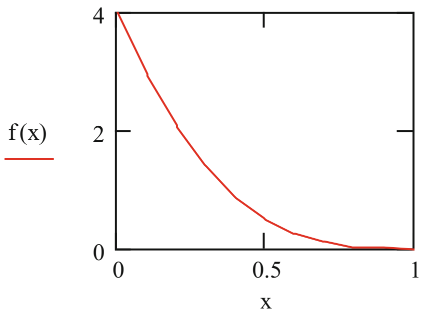


Fig. 5.8 Pdf of GP(0, 1, 4)



$$f(r_1, r_2, \ldots, r_m) = \beta^{-m\gamma}\gamma(\alpha+\beta-r_m)\prod_{i=1}^{m-1}(\alpha+\beta-r_i),$$
$$\alpha < r_1 < \cdots < r_m < \alpha+\beta$$
$$= 0, \text{ otherwise.} \tag{5.6.2.1}$$

The pdf of X(m) is

$$f_m(x) = \frac{1}{\Gamma(m)}\gamma^n\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}(\log\beta - \log(\alpha+\beta-x))^{m-1},$$
$$\alpha < x < \alpha+\beta,$$
$$= 0, \text{ otherwise.} \tag{5.6.2.2}$$

Now

$$\begin{aligned}
\mathrm{EX(m)} &= \int_\alpha^{\alpha+\beta} x f_m(x)dx, \text{ letting } \alpha+\beta-x = \beta e^{-t/\gamma}, \\
&= \int_o^\infty \frac{1}{\Gamma(m)} t^{m-1} e^{-t}(\alpha+\beta-\beta e^{-t/\gamma})dt. \\
&= \int_o^\infty \frac{1}{\Gamma(m)} t^{m-1} e^{-t}(\alpha+\beta-\beta e^{-t/\gamma})dt. \\
&\alpha+\beta-\beta\left(\frac{\gamma}{\gamma+1}\right)^m.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{E(X(m))}^2 &= \int_o^\infty \frac{1}{\Gamma(m)} t^{m-1} e^{-t}(\alpha+\beta -^{-t/\gamma})^2 dt \\
&= (\alpha+\beta)^2 - 2\beta(\alpha+\beta)\left(\frac{\gamma}{\gamma+1}\right)^m + \beta^2\left(\frac{\gamma}{\gamma+1}\right)^m
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var(X(m))} &= \mathrm{E(X(m))}^2 - \{\,\mathrm{E(X(m))}\}^2 \\
&= \beta^2\left\{\left(\frac{\gamma}{\gamma+2}\right)^m - \left(\frac{\gamma}{\gamma+1}\right)^{2m}\right\}.
\end{aligned} \tag{5.6.2.3}$$

The joint pdf of X(m) and X(n) is

$$f_{m,n}(x,y) = c_{mn}\gamma^n\beta^{-\gamma}(\alpha+\beta-y)^{\gamma-1}(\log\beta - \log(\alpha+\beta-x)^{m-1}(\log\beta - \log\,(\alpha+\beta-x)^{n-m-1}$$
$$\text{for } \alpha < \mathrm{x} < \mathrm{y} < \alpha+\beta,$$
$$= 0, \text{ otherwise,} \tag{5.6.2.4}$$

where

$$c_{mn} = \frac{1}{\Gamma(m)\Gamma(n-m)}, \text{n} > \text{m} > 1.$$

The conditional pdf of X(n)|X(m) = x is

$$\begin{aligned} &f(y|X_{U(m)} = x) \\ &= \frac{\gamma^{n-m}}{\Gamma(n-m)} \left(\frac{\alpha+\beta-y}{\alpha+\beta-x}\right)^{\gamma-1} \left\{\log\frac{\alpha+\beta-x}{\alpha+\beta-y}\right\}^{n-m-1}, \\ &\qquad \alpha < x < y < \alpha+\beta. \end{aligned}$$

The conditional expectation of X(m)|X(m) = x is

$$\begin{aligned} E(y|X(m) = x) \\ &= \int_{\alpha}^{\alpha+\beta} \frac{\gamma^{n-m}}{\Gamma(n-m)} \left(\frac{\alpha+\beta-y}{\alpha+\beta-x}\right)^{\gamma-1} \left\{\log\frac{\alpha+\beta-x}{\alpha+\beta-y}\right\}^{n-m-1} dy \\ &= \int_0^{\infty} \frac{\gamma^{n-m}}{\Gamma(n-m)} (\alpha+\beta-(\alpha+\beta-xe^{-t}))t^{n-m-1}e^{-\gamma t}dt \\ &= \alpha+\beta-(\alpha+\beta-x)\left(\frac{\gamma}{\gamma+1}\right)^{n-m}. \end{aligned}$$

For m < n,

$$\text{Cov}(\text{X(m)}, \text{X(n)}) = \left(\frac{\gamma}{\gamma+1}\right)^{n-m} Var(X(m)).$$

The correlation coefficient $\rho_{m.n}$ of X(m) and X(n) is given by

$$\rho_{m,n} = \sqrt{\left(\frac{\gamma^2+2\gamma+1}{\gamma^2+\gamma}\right) - 1} \Big/ \left[\sqrt{\left(\frac{\gamma^2+2\gamma+1}{\gamma^2+\gamma}\right)^n - 1}\right] \tag{5.6.2.5}$$

As $\gamma \to \infty$, $\rho_{m,n} \to \sqrt{\frac{m}{n}}$.

Exercise 5.6.2.1 Show that if X has BF(0, β, χ), then

$$X(n) \underline{\underline{d}} \, \beta\left(1 - \left(\frac{\beta - X_1}{\beta}\right)^{\gamma} \left(\frac{\beta - X_2}{\beta}\right)^{\gamma} \cdots \left(\frac{\beta - X_n}{\beta}\right)^{\gamma}\right)$$

where $X_1X_2,\ldots,X_n$ are i.i.d with BF(0,β, χ).

5.6.3 Recurrence Relation Between Moments

The following theorem gives a recurrence relation of the single moments. We will assume without any loss of generality $\alpha = 0$ and $\beta = 1$.

Theorem 5.6.3.1 *For* n ≥ 2, *and* r = 0, 1, 2,……

$$E(X(n))^{r+1} = \frac{r+1}{\gamma+r+1} E(X(n))^{r} + \frac{\gamma}{\gamma+r+1} E(X(n-1)^{r+1}) \tag{5.6.3.1}$$

Proof We have

$$\lambda(1-F(x)) = (1-x)f(x) \tag{5.6.3.2}$$

Using (5.6.2.2) for n > 2 and r = 0, 1,….,

$$\begin{aligned}
E\Big(X(n)^{r^r} - E(X(n))^{r+1}\Big) &= \frac{1}{\Gamma(n)} \int_o^1 \big(x^r - x^{r+1}\big)(-\gamma \log(1-\ \mathrm{x}))^{\mathrm{n}-1}\, \gamma(1-\mathrm{x})^{\lambda-1} dx \\
&= \frac{\gamma}{(n-1)!} \int_0^1 x^r \{-\gamma \log(1-x)\}^{n-1} \{1-\mathrm{x}^{\gamma}\} dx \\
&= \frac{\gamma}{r+1} \left[\int_0^1 x^{r+1} \frac{1}{(n-1)!} \{-\gamma \log(1-x)\}^{n-1} \gamma (1-x)^{\gamma-1} dx \right. \\
&\qquad \left. - \int_0^1 x^{r+1} \frac{1}{(n-2)!} \{-\log(1-F(x))\}^{n-2} f(x) dx \right] \\
&= \frac{\gamma}{r+1} \left[E\Big(X(n)^{r+1}\Big) - E(X(n-1))^{r+1} \right]
\end{aligned}$$

where the last but one step follows by integrating by parts. The recurrence relation in Eq. (5.6.3.4) is derived simply by rewriting the above equation.

Corollary 5.6.3.1 *By repeatedly applying the recurrence relation in* (5.6.2.1)*, we get for* n > 2, 1 < m < n − 1 *and* r = 0, 1, 2,….

$$\begin{aligned}
E\Big(X(n)^{r+1}\Big) &= \left(\frac{r+1}{\gamma+r-1}\right) \sum_{p=0}^{n-m-1} \left(\frac{\gamma}{\gamma+r+1}\right)^{p} E(X(n-m))^{r}) \\
&\quad + \left(\frac{\gamma}{\gamma+r+1}\right)^{n-m} E(X(m))^{r+1}
\end{aligned} \tag{5.6.3.3}$$

Corollary 5.6.3.2 *Write* (r + 1)(−p) = 1 *for* p = 0 *and* =(r + 1) r… (r − p + 2), *for* p > 1. *By repeated application of the recurrence relation in* (5.2.4)*, we obtain for* n > 2, r = 0, 1, 2,…

$$E(X(n))^{r+1} = \gamma \sum_{p=0}^{r+1} \frac{(r+1)^{(p)}}{(\gamma+1+p)^{(p+1)}} E(X(n-1))^{r=1-p} \tag{5.6.3.4}$$

Next, we have the joint density function of X(m) and X(n), 1 < m < n, as

$$\begin{aligned} \mathrm{f}_{\mathrm{m,n}}(x,y) = {} & \frac{1}{(m-1)!(n-m-1)!} \{-\log(1-F(x))\}^{m-1} \frac{f(x)}{1-F(x)} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} f(y), \end{aligned} \tag{5.6.3.5}$$

Once again, upon using the relation in (5.6.3.2), we can derive the following recurrence relations for the product moments of record values.

Theorem 5.6.3.2 *For* m > 1 *and* r, s = 0, 1, 2,...

$$\begin{aligned} E\Big((X(m))^r (X(m+1))^{s+1}\Big) = {} & \frac{s+1}{\gamma+s+1} E((X(m))^r (X(n)^s)^s) \\ & + \frac{\gamma}{\gamma+s+1} E(X(m))^{r+s=1} \end{aligned} \tag{5.6.2.6}$$

and for 1 < m < n − 2 *and* r, s = 0, 1, 2,...,

$$\begin{aligned} E\Big(X(m)^r (X(n))^{s+1}\Big) = {} & \frac{s+1}{\gamma+s+1} E((X(m))^r (X(n)^{ss})) \\ & + \frac{\gamma}{\gamma+s+1} E\Big((X(m))^r (Xn-1)^{s+1}\Big) \end{aligned} \tag{5.6.2.7}$$

Proof For 1 < m < n − 1 and r, s = 0, 1, 2…

$$\begin{aligned} & E((X(m))^r (X(n))^s - (X(m))^r (X(n))^{s+1}) \\ & \quad = \iint\limits_{0 \le x < y < 1} (x^r y^s - x^r y^{s+1}) f_{m,n}(x,y)\,dy\,dx \\ & \quad = \frac{1}{(m-1)!(n-m-1)!} \int_o^1 x^r \{-\log(1-F(x))\}^{m-1} \cdot \frac{f(x)}{1-F(x)} I(x)\,dx \\ I(x) & = \int_x^1 y^s (1-y) \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} f(y)\,dy \\ & = \gamma \int_x^1 y^s \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} \{1-F(y)\}\,dy \\ & = \frac{\gamma}{s+1} \left[\int_x^1 y^{s+1} \ \mathrm{f(y)dy} \ - \mathrm{x}^{\mathrm{s}+1} \{1-F(x)\} \right], \text{ for n = m + 2} \\ & = \frac{\gamma}{s+1} \left[\int_x^1 y^{s+1} \{-l \log(1-F(y)) + \log(1-F(x))\}^{n-m-1} f(y)\,dy \right. \\ & \quad \left. -(\mathrm{n-m-1}) \int_x^1 y^{s+1} \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-2} f(y)\,dy \right], \\ & \quad \text{for } \mathrm{n} > \mathrm{m} + 2. \end{aligned} \tag{5.6.2.8}$$

The last two equations are derived by integrating by parts. We obtain on simplification for n = m + 1 that

$$E((X(m))^r(X(m+1))^s - E(X(m))^r(X(m+1))^{s+1})$$
$$= \frac{\gamma}{s+1}\left\{E(X(m))^r(X(n))^{s+1} - E(X(m))^{r+s+1}\right\},$$

and when n − m ≥ 2 that

$$E((X(m))^r(X(n))^s - E(X(m))^r(X(n))^{s+1})$$
$$= \frac{\gamma}{s+1}\left\{E(X(m))^r(X(n))^{s+1} - E((X(m))^r(X(n-1))^{s+1}\right\}, \tag{5.6.2.9}$$

The recurrence relations in (5.6.2.7) and (5.6.2.8) are derived simply by rewriting the above equations.

Corollary 5.6.3.3 *For* m > 1

$$Cov(X(m)X(m+1)) = \frac{\gamma}{\gamma+1}Var(X(m)) \tag{5.6.2.10}$$

$$Cov(X(m)X(n)) = \frac{\gamma}{\gamma+1}Cov(X(m)X(n-1)) \tag{5.6.2.11}$$

Consequently, for 1 < m < n − 1

$$Cov(X(m)X(n)) = \left(\frac{\gamma}{\gamma+1}\right)^{n-m} Var\,X(m) \tag{5.6.2.12}$$

5.7 Rayleigh Distribution

5.7.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Rayleigh distribution with pdf as

$$\mathrm{f(x)} = xe^{-x^2/2},\ \ \mathrm{x} > 0 \tag{5.7.1.1}$$

and d.f. as

$$\mathrm{F(x)} = 1 - e^{-x^2/2}, \mathrm{x} > 0. \tag{5.7.1.2}$$

We say X € RH(0, 1) if the pdf of X is given by (7.3.1)’ Fig. 7.3.1 gives the graph of f(x).

5.7.2 Distributional Property

The pdf of nth upper record value $f_n(x)$ of the nth upper record value is given by

$$f_n(x) = \frac{1}{2^{n-1}\Gamma(n)} x^{2n-1} e^{-\frac{x^2}{2}}, x \geq 0, n = 1, 2, \ldots$$

$r(x) = \frac{f(x)}{1-F(x)} = x$. The mode of X(n) is at $\sqrt{2n-1}$.

Theorem 5.7.2.1 *Let* $\mu_n = E(X(n))$, $V_{n,n} = Var(X(n))$ *and* $V_{m,n} = Cov(X(m)X(n))$ then

$$\mu_n = \sqrt{2}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}, V_{n,n} = 2\left[n - \left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{and}$$

$$V_{m,n} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{ for } 1 < m < n.$$

Proof

$$\begin{aligned}
\mu_n &= \frac{1}{\Gamma(n)} \int_o^\infty x\{-\log 1 - F(x)\}^{n-1} f(x)dx \\
&= \frac{1}{\Gamma(n)} \int_o^\infty x\left(\frac{x^2}{2}\right)^{n-1} e^{-x^2/2} x dx \\
&= \frac{1}{\Gamma(n)} \sqrt{2} \int_o^\infty u^{1/2} u^{n-1} e^{-u} du \\
&= \sqrt{2}\frac{\Gamma(n+1/2)}{\Gamma(n)}.
\end{aligned}$$

Similarly it can be shown that

$$\mu_n^2 = E(X_{U(n)}^2) = 2\frac{\Gamma(n+1)}{\Gamma(n)} = 2n$$

$$\begin{aligned}
\mu_{m,n} &= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_o^\infty \int_o^y xy\left(\frac{x^2}{2}\right)^{m-1} x\left(\frac{y^2}{2} - \frac{x^2}{2}\right)^{n-m-1} ye^{-y^2/2} dxdy \\
&= \frac{1}{\Gamma(m)\Gamma(n-m)} \frac{1}{2^{m-1}} \int_o^\infty y\left(\frac{y^2}{2}\right)^{n-m-1} ye^{-y^2/2} I_y dy,
\end{aligned}$$

where

$$\mathrm{I_y} = \int_o^y (x^2)^m \left(1 - \frac{x^2}{y^2}\right)^{n-m-1} dx = \frac{1}{2} y^{2m+!} B(m+1/2,\, n-m),$$

with

$$\mathrm{B(a, b)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

On simplification we get

$$\mathrm{V_{n,n}} = 2\left[n - \left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{and}$$

$$\mathrm{V_{m,n}} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{ for } 1 < \mathrm{m} < \mathrm{n}.$$

$$= \left[\frac{\Gamma(m+1/2}{\Gamma(m)}\right]\left[\frac{\Gamma(n)}{\Gamma(n+1/2}\right] V_{n,n}$$

Table 5.7 gives the variances and covariances of $X_{U(m)}$ and $X_{U(n)}$ for $1 \le m \le n \le 10$.

The correlation coefficient $\rho_{m,n}$ between X(m) and X(n) is

$$\rho_{m,n} = \frac{\Gamma(n)}{\Gamma(m)} \cdot \frac{\Gamma(m+1/2)}{\Gamma(n+1/2)} \cdot \sqrt{\frac{V_{n,n}}{V_{m,m}}}$$

$$= \frac{\Gamma(m+\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \cdot \frac{\Gamma(n)}{\Gamma(m)} \cdot \frac{\sqrt{n - \left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right)^2}}{\sqrt{m - \left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(n)}\right)^2}}, 1 < \mathrm{m} < \mathrm{n}.$$

Exercise 5.7.2.1 Show that the Shannon entropy $H_n(X)$ of X(n) is

$$H_n(X) = \ln(\Gamma(n) - \ln 2 + \left(\frac{1}{2} - n\right)\Psi(n) + n.$$

where $\Psi(n)$ is digamma function.

Table 5.7 Variances and Covariances of upper record values X(m) and X(n), m > n

m/n	1	2	3	4	5	6	7	8	9	10
	0.429204									
	0.310472	0.465708								
	0.254757	0.382135	0.477669							
	0.221026	0.331539	0.414424	0.483494						
	0.197860	0.296791	0.370988	0.432820	0.486922					
	0.180706	0.271059	0.338823	0.395294	0.444705	0.489176				
	0.167348	0.251023	0.313778	0.366075	0.411834	0.453018	0.490769			
	0.156569	0.234854	0.293567	0.342495	0.385307	0.423837	0.459157	0.491954		
	0.147634	0.221450	0.276813	0.322948	0.363317	0.369649	0.432953	0.463878	0.492870	
	0.140070	0.210150	0.26632	0.306404	0.344704	0.39714	0.410772	0.440113	0.467620	0.493599

5.8 Uniform Distribution

5.8.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables from a uniform distribution with the following pdf

$$f(x) = \frac{1}{\theta_1 - \theta_2}, \; \theta_1 < x < \theta_2 \tag{5.8.1.1}$$

and d.f.

$$F(x) = \frac{x - \theta_1}{\theta_2 - \theta_1}, \quad \theta_1 < x < \theta_2. \tag{5.8.1.2}$$

We will say $X \in U(\theta_1, \theta_2)$ if the pdf of X is as given in (5.8.1.1).

5.8.2 Distributional Property

n

The pdf $f_n(x)$ of X(n) can be written as

$$f_n(x) = \frac{1}{\Gamma(n)} \frac{1}{\theta_2 - \theta_1} \left\{ \log \frac{\theta_2 - \theta_1}{\theta_2 - x} \right\}^{n-1}, \; \theta_1 < x < \theta_2 \tag{5.8.2.1}$$

Theorem 5.8.2.1 *Let* $\varsigma_1 = \frac{\theta_2 - X_{U(1)}}{\theta_2 - \theta_1}, \varsigma_i = \frac{\theta_2 - X_{U(i)}}{\theta_2 - X_{U(i-1)}}, i = 2, \ldots, n$, then $\varsigma_1, \ldots, \varsigma_n$ *are i.i.d.* U(0, 1).

Proof The joint pdf of X(1), X(2),…,X(n) can be written as

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{\theta_2 - x_1} \cdot \frac{1}{\theta_2 - x_2} \cdots \frac{1}{\theta_2 - x_{n-1}} \frac{1}{\theta_2 - \theta_1},$$
$$\theta_1 < x_1 < x_2 < \ldots < x_n < \theta_2.$$

Let

$$X(1) = \theta_2 - (\theta_2 - \theta_1)\xi_1$$
$$X(2) = \theta_2 - (\theta_2 - \theta_1)\xi_1\xi_2$$
$$X(i) = \theta_2 - (\theta_2 - \theta_1)\xi_1\xi_2\ldots\xi_i, \; i = 3, \ldots, m$$

The Jacobian of the transformation is

$$J = \frac{\partial(X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})}{\partial(\xi_1, \xi_2, \ldots, \xi_n)} = (\theta_2 - \theta_1)^n \xi_1^{n-1} \xi_1^{n-2} \ldots \xi_{n-1}.$$

Thus s

$$\mathrm{f}(\mathrm{e}_1, \mathrm{e}_1, \ldots, \mathrm{e}_n) = 1, 0 < \mathrm{e}_i < 1, \mathrm{i} = 1, 2, \ldots, \mathrm{n}.$$

Using r the Theorem 5.8.2.1, it can be shown that

$$\begin{aligned} \mathrm{EX(n)} &= 2^{-n}\theta_1 + (1 - 2^{-n})\theta_2 \\ \mathrm{Var(X(n))} &= (3^{-n} - 4^{-n})(\theta_2 - \theta_1)^2. \end{aligned} \tag{5.8.2.2}$$

The joint pdf of X(m) and X(n) is

$$\mathrm{f}_{m,n}(x, y) = \frac{1}{\Gamma(m)} \frac{1}{\Gamma(n-m)} \frac{1}{\theta_2 - \theta_1} \frac{1}{\theta_2 - x} \left\{ \log n \frac{\theta_2 - \theta_1}{\theta_2 - x} \right\}^{m-1} \left\{ \log \frac{\theta_2 - \theta_1}{\theta_2 - y} \right\}^{n-m-1},$$

for $\theta_1 < x < y < \theta_2$

(5.8.2.3)

Thus, it follows that

$$\mathrm{E(X(n)|X(m) = x)} = 2^{m-n}x + (1 - 2^{m-n})\theta_2$$

and

$$\mathrm{Cov(XM)X(n)} = 2^{m-n}\mathrm{Var(X(m))}. \tag{5.8.2.4}$$

The following table gives the correlation coefficient between X(m) and X(n) for $1 \leq m \leq n \leq 10$ (Table 5.8).

For fixed m, $\rho_{m,n}$ decreases as n increases. For fixed n, $\rho_{m,n}$ increases as n increases.

Exercise 5.8.2.1 If X is distributed as uniform on (0, 1), then log(1 − x(n − 1)) and log(1 − X(n))/(1 − X(n − 1)) are independent and log((1 − X(n))/1 − X(n − 1) is distributed as E(0, 1).

Table 5.8 Correlation coefficients of upper recordvalues X(n) and X(m), m < n

	n									
m	1	2	3	4	5	6	7	8	9	10
1	1									
2	0.654654	1								
3	0.493197	0.753371	1							
4	0.392792	0.600000	0.796421	1						
5	0.322045	0.491932	0.652975	0.819887	1					
6	0.268647	0.410365	0.544705	0.683941	0.834189	1				
7	226603	0.346142	0.459457	0.576903	0.703637	0.843498	1			
8	0.192571	0.294157	0.390454	0.490261	0.597962	0.716818	0.849816	1		
9	0.164499	0.251277	0.333537	0.418795	0.510796	0.612326	0.725936	0.854228	1	
10	0.141037	0.215437	0.285964	0.359062	0.413794	0.524989	0.622395	0.857369	1	

5.9 Weibull Distribution

5.9.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Weibull distribution with pdf

$$f(x) = x^{\gamma-1} e^{-x^{\gamma}/\gamma}, x > 0, \gamma > 0, \tag{5.9.1.1}$$

and g.f.

$$F(x) = 1 - e^{-\frac{1}{\gamma}x^{\gamma}}, x > 0, \gamma > 0, \tag{5.9.1.2}$$

The pdf of the Weibull distribution as given by (5.9.1.1) becomes identical with the pdf of Raleigh distribution as given in (7.1.1) for $\gamma = 2$. The pdf of Weibull distribution for $\gamma = 1$ coincides with that of E(0, 1). Figure 5.9 gives the pdf of Weibull distribution for $\gamma = 1.5$ (Fig. 5.10).

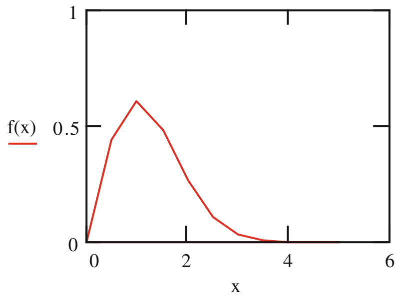


Fig. 5.9 Pdf of RH(0, 1)

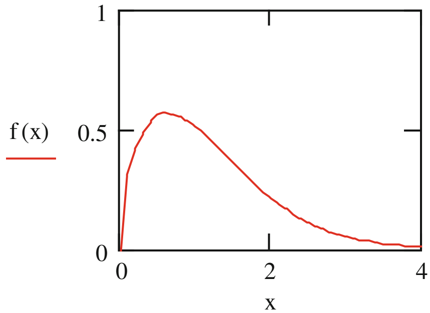


Fig. 5.10 Pdf of X when $\gamma = 1.5$

5.9.2 Distributional Property

Theorem 5.9.2.1 *Let* i = 1, 2,…,m − 1, $\xi_i = \frac{X(i)}{X(i+1)}$ *then* ξ_i, i = 1, 2,…, m − 1, *are independent.*

Proof The joint pdf of X(1), X(2),…,X(n) is

$$\mathrm{f}(x_1, x_2, \ldots, x_m) = \gamma^m (x_1 x_2 \ldots x_m)^{\gamma-1} e^{-x_m^{\gamma}},\ 0 < x_1 < x_2 < \cdots < x_m < \infty.$$

Let us use the transformation

$$\xi_o = X_{U(1)},\ \text{and}\ \xi_i = \frac{X(i)}{X(i+1)},\ \mathrm{i} = 2,\ \ldots,\ \mathrm{m} - 1.$$

The Jacobian of the transformation

$$J = \left| \frac{\partial(X(1),\ X(2), \ldots, X(n))}{\partial(\xi_o\ ,\ \xi_1, \ldots, \xi_{m-1})} \right| = \frac{\xi_o^{m-1}}{\xi_1^m\ \xi_2^{m-1}\ \ldots \xi_{m-1}^2}$$

We can write the pdf of ξ_i, i = 0, 1,…,m − 1, as

$$\mathrm{f}(e_o, e_1, \ldots, e_{m-1}) = \frac{\gamma^m e_o^{m\gamma-1}}{e_1^{(m-1)\gamma+1} e_2^{(m-2)\gamma+1} \ldots e_{m-1}^{\gamma+1}} e^{-\frac{1}{\gamma}\left(\frac{e_0}{e_1 e_2 \cdots e_{m-1}}\right)^{\gamma}}.$$

Now integrating the above expression with respect to e_o, we obtain the joint pdf of ξ_i, i = 1,…,m − 1, as

$$\mathrm{f}(e_1, \ldots, e_{m-1}) = \Gamma(\mathrm{m})\, \gamma^{\mathrm{m}-1}\, e_1^{\gamma-1}\, e_2^{2\gamma-1} \cdots e_{m-1}^{(m-1)\gamma-1}.$$

Thus ξ_i, i = 1, 2,…, m − 1 are independent and

$$\mathrm{P}(\xi_k < \mathrm{x}\ ) = \mathrm{x}^{\mathrm{k}\gamma},\ 1 \le \mathrm{k} \le \mathrm{m}$$

We have already seen similar results for ratios of the record values of the exponential distribution.

Theorem 5.9.2.2 *Let* μ_n = E(X(n)), $V_{n,n}$ = Var(X(n))) *and* $V_{m,n}$ = Cov(X(m)X(n)), *then*

$$\mu_{\mathrm{n}=} = \gamma^{\frac{1}{\gamma}} \frac{\Gamma(n + \frac{1}{\gamma})}{\Gamma(n)},\ \mathrm{V}_{\mathrm{n,n}=} = \gamma^{2/\gamma} \left\{ \frac{\Gamma(n + \frac{2}{\gamma})}{\Gamma(n)} - \left(\frac{\Gamma(n + \frac{1}{\gamma})}{\Gamma(n)} \right)^2 \right\}$$

and

$$\mathrm{V_{m,n}} = \frac{\Gamma(m+\frac{1}{\gamma})}{\Gamma(m)}.\gamma^{2/\gamma}\left\{\frac{\Gamma(n+\frac{2}{\gamma})}{\Gamma(n+\frac{1}{\gamma})} - \frac{\Gamma(n+\frac{1}{\gamma})}{\Gamma(n)}\right\}, \text{ for} 1 \le \mathrm{m} \le \mathrm{n}.$$

Proof The pdf $f_n(x)$ of X(n) can be written as

$$\mathrm{f_n(x)} = \frac{1}{\Gamma(n)}\left\{\frac{x^{\gamma n-1}}{\gamma^{n-1}}\right\}e^{-x^{\gamma}/\gamma}, x > 0,\ \gamma > 0. \tag{5.9.2.1}$$

$$E(X_{U(n)}) = \int_o^\infty \frac{1}{\Gamma(n)}\frac{x^{n\gamma}}{\gamma^{n-1}}e^{-(1/\gamma)x^\gamma}dx,$$

Substituting $t = \frac{x^\gamma}{\gamma}$, we obtain

$$\begin{aligned} E(X(m)) &= \int_o^\infty \frac{\gamma^{1/\gamma}}{\Gamma(n)}e^{-t}(t)^{n+\frac{1}{\gamma}-1}dt \\ &= \gamma^{\frac{1}{\gamma}}\frac{\Gamma(n+\frac{1}{\gamma})}{\Gamma(n)} \end{aligned}$$

Similarly

$$E(X(n))^2 = \int_o^\infty \frac{1}{\Gamma(n)}\frac{x^{n\gamma+1}}{\gamma^{n-1}}e^{-(1/\gamma)x^\gamma}dx,$$

Substituting $t = \frac{x^\gamma}{\gamma}$, we get

$$\begin{aligned} E(X(n))^2 &= \int_o^\infty \frac{\gamma^{2/\gamma}}{\Gamma(n)}e^{-t}(t)^{n+\frac{2}{\gamma}-1}dt \\ &= \gamma^{2/\gamma}\frac{\Gamma(n+\frac{2}{\gamma})}{\Gamma(n)} \end{aligned}$$

The joint pdf of X(m) and X(n) can be written as

$$f_{m,n}(x,y) = \frac{x^{\gamma-1}}{\Gamma(m)\Gamma(n-m)}\left(\frac{y^\gamma - x^\gamma}{\gamma}\right)^{n-m-1}y^{\gamma-1}e^{-(1/\gamma)x^\gamma}$$
for $0 < \mathrm{x} < \mathrm{y} < \infty$ and $\mathrm{m} < \mathrm{n}$,

$$\begin{aligned} &\mathrm{E(X(m))(X(n))} \\ &= \int_o^\infty \int_0^y \frac{xy}{\Gamma(m)\Gamma(n-m)}\cdot\frac{x^{m\gamma-1}}{\gamma^{n-2}}[y^\gamma - x^\gamma]^{n-m-1}y^{\gamma-1}e^{-(1/\gamma)y^\gamma}dydx \\ &= \int_o^\infty \frac{y}{\Gamma(m)\Gamma(n-m)}\cdot\frac{I_y}{\gamma^{n-2}}y^{\gamma-1}e^{-(1/\gamma)y^\gamma}dy, \end{aligned}$$

where

$$\begin{aligned}\mathrm{I_y} &= \int_o^y x^{m\gamma}(y^\gamma - x^\gamma)^{n-m-1}dx \\ &= \int_o^1 \frac{y^{n\gamma-\gamma+1}}{\gamma} t^{m+\frac{1}{\gamma}-1}(! - t)^{n-m-1}dt \\ &= \frac{y^{n\gamma-\gamma+1}}{\gamma} B\left(m + \frac{1}{\gamma}, n - m\right)\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{E(X(m))(X(n))} &= \int_o^\infty \frac{B(m+\frac{1}{\gamma}, n-m)}{\Gamma(m)\Gamma(n-m)} \frac{y^{n\gamma+1}}{\gamma^{n-1}} e^{-(1/\gamma)y^\gamma} dy \\ &= \frac{\Gamma(m+\frac{1}{\gamma})\Gamma(n+\frac{2}{\gamma})}{\Gamma(m)\Gamma(n-m)}.\gamma^{2/\gamma} \\ \mathrm{Var\ (X(n))} &= \gamma^{2/\gamma}\left\{\frac{\Gamma(n+\frac{2}{\gamma})}{\Gamma(n)} - \left(\frac{\Gamma(n+\frac{1}{\gamma})}{\Gamma(n)}\right)^2\right\}\end{aligned}$$

and

$$\mathrm{Cov(X(m)X(n))} = \frac{\Gamma(m+\frac{1}{\gamma})}{\Gamma(m)}.\gamma^{2/\gamma}\left\{\frac{\Gamma(n+\frac{2}{\gamma})}{\Gamma(n+\frac{1}{\gamma})} - \frac{\Gamma(n+\frac{1}{\gamma})}{\Gamma(n)}\right\},\ 1 \le \mathrm{m} \le \mathrm{n}.$$

Exercise 5.9.2.1 Show that $(X(i))^\gamma - (X(i-1))^\gamma$, i ≥ 1 with X(0) = 0 is identically distributed.

5.10 Exercises (solutions)

Exercise 5.1.2.1 (*hint*) The joint pdf of ξ_i, i = 1,……, m − 1, is

$$\mathrm{f}(e_1, \ldots, e_{m-1}) = \Gamma(\mathrm{m})\ e_2 e_3^2 \cdots e_{m-1}^{m-2} \cdots$$

Use the transformation $W_k = (\xi_k)^k$, k = 1, 2,…, m − 1,

Exercise 5.1.3.1 (*solution* (*hint*)) Using (5.1.3.1) with p = 1 and q = 0, we obtain

$$\mathrm{E(X(1))} = \frac{\Gamma(m+1)}{\Gamma(m)} = m.$$

Putting p = 1 and q = 1, we obtain (X(m)X(n)) $-\frac{\Gamma(m+1)\Gamma(n+2)}{\Gamma(m)\Gamma(n+1)} - m(n\_1), m<n$

Putting p = 2 and q = o, we obtain $E(X(m))^2 = \frac{\Gamma(m+2)}{\Gamma(m)} = m(m+1)$

Exercise 5.1.3.2 (*solution*) G(x) $= F^{-1}(x) = -\ln(1-x)$, $G(1-e^{-x}) =$ x. Thus by Representation 4, 3,

$$X(n) \underline{\underline{d}} X_1 + X_2 + \cdots + X_n$$

Exercise 5.1.3.3 (*solution*)

$$\begin{aligned}
\mathrm{E_N} &= \int_0 - \ln\left(\frac{x^{n-1}}{\Gamma(n)\sigma^n} e^{-\frac{x}{\sigma}}\right) \frac{x^{n-1}}{\Gamma(n)\sigma^n} e^{-\frac{x}{\sigma}} dx \\
&= \int_0^\infty (\ln\Gamma(n) + n\ln\sigma + \frac{x}{\sigma} - (n-1)\ln x) \frac{x^{n-1}}{\Gamma(n)\sigma^n} e^{-\frac{x}{\sigma}} dx \\
&= \ln\Gamma(n) - \ln\sigma - (n-1)\psi(n),
\end{aligned}$$

Exercise 5.2.3.1 (*solution*) Let us consider n ≥ 1 and r = 0, 1, 2,…,

$$\begin{aligned}
E(x(n))^r &= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} x^r \{-\ln F(x)\}^{n-1} f(x)\, dx \\
&= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} x^r \{-\ln F(x)\}^n F(x) dx
\end{aligned}$$

Upon integrating by parts treating x^r for integration and the rest of the integrand for differentiation. We simply obtain

$$\begin{aligned}
E(x(n))^r &= \frac{1}{(r+1)\Gamma(n)} n \int_{-\infty}^{\infty} x^{r+1} \{-\ln F(x)\}^{n-1} f(x) dx \\
&\quad - \int_{-\infty}^{\infty} x^{r+1} \{-\ln F(x)\}^n f(x) dx \\
&= \frac{n}{r+1} \int_{-\infty}^{\infty} \frac{x^{r+1}}{\Gamma(n)} \{-\ln F(x)\}^{n-1} f(x) dx \\
&\quad - \int_{-\infty}^{\infty} \frac{x^{r+1}}{\Gamma(n+1)} - \ln F(x)\}^{n-1} f(x) dx \\
&= \frac{n}{r+1} \{E(x(n))^{r+1} - E(X(n+1))\}^{r+1}
\end{aligned}$$

Upon rewriting the above equation, we obtain the recurrence relation.

Problem 5.3.3.1 (*solution*) The recurrence relations presented in Theorem 5.3.3.2 can be used in a simple recursive manner to compute all the product moments of all record values. It is known that the generalized Pareto distribution in (5.3.3.1) has finite variance if β < ½. In Theorem 5.3.3.2 setting r = 1 and s = 0 we get

$$E(X(m)X(m+1)) = \frac{1}{1-\beta}\left[E((X(m))) + E\left((X(m))^2\right)\right]$$

We know $\mathrm{E(X(m+1))} = \frac{1}{1-\beta}(1+E(X(m)))$
Thus

$$\begin{aligned}\mathrm{E(X(m)X(m+1)) - E(X(m))E(X(m+1))} &= E(X(m)X(m+1)) \\ &= \frac{1}{1-\beta}\left[E((X(m))) + E\left((X(m))^2\right)\right] \\ &\quad - \mathrm{E(X(m))}\frac{1}{1-\beta}(1+E(X(m))),\end{aligned}$$

On simplification

$$\begin{aligned}Cov(X(m))X(m+1) &= \frac{1}{1-\beta}Var\left(X_{U(m)}\right). \\ Cov(X(m)X(n)) &= \frac{1}{1-\beta}Cov(X(m)X(n-1)) \\ &= \frac{1}{(1-\beta)^{n-m}}Var(X(m)).\end{aligned}$$

Problem 5.6.2. 1 (*solution*)

$$F^{-1}(x) = \phi(1-(1-u)^{1/\gamma}) \text{ and } -\ln(1-F(x)) = -\ln\left(\frac{\beta-x}{\chi}\right)^{\gamma}.$$

$$\begin{aligned}X(\mathrm{N}) &= F^{-1}\left(1 - e^{-\sum_{i=1}^{n}\ln(1-F(X_i))}\right) \\ &= \beta\left[1 - \prod_{i=1}^{n}\left(\frac{\beta - X_i}{\beta}\right)\right].\end{aligned}$$

where $\mathrm{X_i}$ are i.i.d with PF(0, β, γ).

Exercise 5.7.2.1 (*solution*)

$$\begin{aligned}H_n(x) &= -\int_0^{\infty}\ln f_n(x)f_n(x)dx \\ &\quad -\int_0^{\infty}\ln\left[\frac{2}{\Gamma(n)} - x^{2n-1}e^{-x^2}\right]\frac{x^{2(n-1)}}{\Gamma(n)}2xe^{-x^2}dx \\ &= \ln\frac{\Gamma(n)}{2} - \left(n-\frac{1}{2}\right)\Psi(n) + n\end{aligned}$$

Exercise 5.8.2.1 (*solution*) The joint pdf $f_{n,n-1}(x, y)$ of X(n − 1)and X(n) is

$$f_{n-1,n}(x,y) = \frac{(-\ln(1-x))^{n-1}}{\Gamma(n)}\frac{1}{1-x}, \mathbf{0 < X < Y < 1}$$

Using the transformation U = −log(1 − X(n − 1)), V = log((! − X(n))/(1 − X (n − 1)), we obtain the joint pdf of $f(u,v)$ as

$$f(u,v) = \frac{u^{n-1}}{\Gamma(n)}\left(e^{-(u+v)}\right), < u, v < \infty.$$

Thus U and V are independent and V is distributed as E(0, 1).

Exercise 5.9.2.1 (*solution*) We have $U = X^{\gamma}$ is distributed as (0, 1). The result follows by Exercise 5.1.3.2.

Chapter 6
Records of Discrete Distributions

6.1 Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence independent and identically distributed random variables taking values on $0, 1, 2, \ldots$ such that $F(n) < 1$ for all $n = 0, 1, 2, \ldots$ We define the upper record times, U(n) as $U(1) = 1, U(n+1(= \min\{j > U(n), X_j > X_{U(n)}\}, n = 1, 2, \ldots.$ The nth upper record value is defined as $X_d(n)$. Let $p_k = P(X_1 = k)$, $P(k) = \sum_{j=0}^{k} p(j)$, $k \geq 0$ and $\bar{P}(k) = 1 - P(k)$ with and $P(\infty) = 1$. The joint probability mass function (pmf) of the $X_d(1)_), Xd(2), \ldots Xd(n)_)$ is defined as

$$
\begin{aligned}
P_{1,2,\ldots,n}(x_1, x_2, \ldots x_n) &= P\Big(X_d(1)1_) = x_1,\ X_d(2)_) = x_2, \ldots., X_d(n) = x_n\Big) \\
&= \tfrac{p(x_1)}{\bar{P}(x_1)} \cdot \tfrac{p(x_2)}{\bar{P}(x_2)} \cdot \ldots \cdot \tfrac{p(x_{n-1})}{\bar{P}(x_{n-1})} p(x_n) \\
&\quad 0 \leq x_1 < x_2 < \cdots < x_n < \infty, \\
&= 0, \ \text{otherwise.}
\end{aligned}
\tag{6.1.1}
$$

The marginal pmf's of the upper record values are given as

$$
\begin{aligned}
p_1(x_1) &= P(X_d(1) = x_1) = p(x_1),\ x_1 = 0, 1, 2, \ldots, \\
p_2(x_2) &= P(X_d(2) = x_2) = R_1(x_2)\, p(x_2),
\end{aligned}
$$

where

$$
R_1(k) = \sum_{0 \leq x_1 < x_2} B(x_1), B(x) = \frac{p(x)}{\bar{P}(x)}, x_2 = 1, 2, \ldots
$$

$$
P_n(x_n) = P\Big(X_d(n)_) = x_n\Big) = R_{n-1}(x_n)\, p(x_n),
$$

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_6

where

$$R_{n-1}(x_n) = \sum_{0 \le x_1 < x_2 < \ldots < x_{n-1} < x_n} B(x_1)\, B(x_2) \ldots B(x_{n-1}),\, x_n = n-1, n, \ldots \quad (6.1.2)$$

The joint pmf of $X_d(m)_)$ and $X_d(n)$, m < n is given by

$$\begin{aligned} P_{m,n}(x_m, x_n) &= P\Big(X_d(m) = x_m, X_d(n)_) = x_m\Big) \\ &= R_{m-1}(x_m) A(x_m) R_{m+1,n}(x_m, x_n)\, p(x_n),\, m \le x_m \le x_n - n + m < \infty, \end{aligned} \quad (6.1.3)$$

where

$$\begin{aligned} R_{m+1,n}(x, y) &= \sum_{x_m < x_{m+1} < x_{m+2} < \ldots < x_n} B(x_{m+1}) \ldots\ldots B(x_{n-1}),\ \ m < n-1 \\ &= 1 \ \text{if}\ \ m = n-1 \end{aligned}$$

The conditional pmf of $X_d(n)$ given $X_d(m) = x_m$ is given by

$$P_{n|m}(x_n | X_d(m) = x_m) = R_{m,n}(x_m, x_n)\, \frac{p(x_n)}{\vec{P}(x_m)},\, x_m \le x_n - n + m < \infty. \quad (6.1.4)$$

$$P_{n\,|\,n-1}(x_n | X_d(n-1) = x_{n-1}) = \frac{p(x_n)}{\vec{P}(x_{n-1})},\, x_{n-1} < x_n. \quad (6.1.5)$$

Using (6.1.1) and (6.1.4) it follows that the sequence of upper record values $X_d(1)$, $X_d(2)\ldots$ forms a Markov chain. Let I_n = 1 if n is a record value i.e. $X_d(m)$ = n, for m = 1, 2,... and In = 0 if it is not a record. The following Theorem is due to Shorrock (1972).

Theorem 6.1.1 *The random variables* I_1, $I_{2,}$ ... *are mutually independent and*

$$P(I_n = 1) = P\{X = n\, | X \ge n) = \frac{P(X = n)}{P(X \ge n)},\, n = 0, 1, 2, \ldots.$$

Proof

$$\begin{aligned} P(I_n = \ n) &= P(X_1 = n) + P(X_1 < n, X_2 = n) + P(X_1 < n, X_2 < n,\ X_3 = n) + \cdots \\ &= P(X_1 = n)\big(P(X < n) + P^2(X < n) + \cdots\big) \\ &= \frac{P(X = n)}{1 - P(X < n)} = \frac{P(X = n)}{P(X \ge n)}. \end{aligned} \quad (6.1.6)$$

Let $0 \le \alpha(1) < \alpha(2)$, then$P(I_{\alpha(1)} = 1, I_{\alpha(2)} = 1) = \sum_m P(I_{\alpha(1)} = 1, I_{\alpha(2)} = 1),$ $t(1) = m)$, t(1) is time when α(1) occurs.

$$
\begin{aligned}
\sum_m \sum_{r=1}^{\infty} P\{I_{\alpha(1)} = 1, t(1) = m, X_{m+1} > \alpha(1), \sum_m P\{I_{\alpha(1)} = 1, t(k) = m\} \sum_{r=1}^{\infty} \mathrm{P}(\mathrm{X_{m+1}} > \alpha(1)) \\
\ldots.X_{m+r-1} > \alpha(1), X_{m+r} = \alpha(2)\} \\
\mathrm{P}(\mathrm{X_{m+r-1}} > \alpha(1))\mathrm{P}(\mathrm{X_{m+r}} = \alpha(2)) \\
= \sum_m P\{I_{\alpha(1)} = 1, t(k) = m\} \frac{P(X = \alpha(2))}{P(X \ge \alpha(1))} \\
= P(I_{\alpha(1)} = 1) P(I_{\alpha(2)} = 1), \text{by } (6.2.4).
\end{aligned}
\tag{6.1.7}
$$

By iteration the independence of $I_1, I_2, \ldots$ follows.

The following result was proved by Aliev and Ahsanullah (2002).

Theorem 6.1.2 *Let* $X_1, X_2, \ldots$ *be a sequence of independent and identically distributed random variables taking values on* $0, 1, 2, \ldots$ *with common distribution function* F *such that* $F(n) < 1$ *for all* n *and* $E(X_i^2) < \infty$. *Suppose that* $\{B_k, k = 0, 1, \ldots |$ *be a sequence of numbers such that* $2 + 2B_{n+1} - B_s - B_{s+2} \ge 0$. *If there exits* F(x) *such that* $E\{(X_d(2) - X_d(1))^2 | X_d(1) = s\} = B_s$, $s = 0, 1, 2$.

Then F(x) is unique.

Proof From (6.1.5), we obtain

$$
E\left\{(X_d(2) - X_d(1))^2 | X_d(1) = s\right\} = \frac{\sum_{j=1}^{\infty} j^2_{p_{s+j}}}{\sum_{j=1}^{\infty} p_{s+j}}, \qquad p_j = P(X = J),\ j = 0, 1, 2, \ldots \tag{6.1.8}
$$

Thus the condition $E\left\{(X_d(2) - X_d(1))^2 | X_d(1) = s\right\} = B_s$ implies

$$
\sum_{j=1}^{\infty} j^2_{p_{s+j}} = B_s \sum_{j=1}^{\infty} p_{s+j}, \quad \text{for } s > 0, \tag{6.1.9}
$$

Writing s = s + 1 in (6.1.9), we get from (6.1.8) and (6.1.9),

$$
\sum_{j=1}^{\infty} (j - 1)^2_{p_{s+j}} = B_{s+1} \sum_{j=1}^{\infty} p_{s+j+1} \tag{6.1.10}
$$

Subtracting (6.1.10) from (6.1.9) we obtain

$$\sum_{j=1}^{\infty}(2\mathrm{j}-1)_{\mathrm{p_{s+j}}} = B_s\sum_{j=1}^{\infty}\mathrm{p_{s+j}} - B_{s+1}\sum_{j=1}^{\infty}p_{s+1+j} \tag{6.1.11}$$

Now substituting s = s + 1 in (6.1.11), we will have

$$\sum_{j=1}^{\infty}(2\mathrm{j}-1)_{\mathrm{p_{s+1+j}}} = B_{s+1}\sum_{j=1}^{\infty}\mathrm{p_{s+1+j}} - B_{s+2}\sum_{j=1}^{\infty}p_{s+2+j} \tag{6.1.12}$$

Subtracting (6.1.12) from (6.1.11) and on simplification, we obtain

$$\begin{aligned}\mathrm{P_{s+2}} &= \frac{1+2\mathrm{B_{s+1}}-B_{s+2}}{\mathrm{B_{s+2}}}\mathrm{p_{s+1}}\\ &\quad -\frac{2+2\mathrm{B_{s+1}}-B_s-B_{s+2}}{\mathrm{B_{s+2}}}(1-\mathrm{p_0}-\mathrm{p_1}-\cdots\mathrm{p_s})\end{aligned} \tag{6.1.13}$$

Since the coefficients $-\frac{2+2\mathrm{B_{s+1}}-B_s-B_{s+2}}{\mathrm{B_{s+2}}}$ and $\frac{1+2\mathrm{B_{s+1}}-B_{s+2}}{\mathrm{B_{s+2}}}$ are positive, it means that P = s + 2 is increasing (decreasing) if ps + 1 increases(decreases) for all s ≥ 0. It means that for any p = 0, all probabilities $\mathrm{p_2}$, $\mathrm{p_3}$,... increases when $\mathrm{p_1}$ increases. Together with the condition $\sum_{i=0}^{\infty}p_i = 1$ we conclude that for any given $\mathrm{p_0}$, we have only one F(x) which satisfy (6.1.1).

6.2 Geometric Distribution

A discrete random variable X is said to have geometric distribution if its probability mass function (pmf) is of the following form:

$$\begin{aligned}\mathrm{p(k)} = \mathrm{P(X=k)} &= \mathrm{pq^k}\ -\ 1,\\ &\quad 0<\mathrm{p}<1, \mathrm{q}=1-\mathrm{p}, \mathrm{k}\in\mathrm{A_0}\\ &= 0,\ \text{otherwise},\end{aligned} \tag{6.2.1}$$

where $\mathrm{A_n}$ = is the set of integers $\mathrm{n}+1, \mathrm{n}+2, \ldots$, and n ≥ 0. We say X ∈ GE(p), if the pmf of X is as given in (6.2.1). For k > 0, we define r(k) = P[X = k| X ≥ k].

We choose to distinguish between GE(p) and the larger class of distributions having geometric tail (GET). We write X ∈ GET(s, p) if the pmf of X is as follows:

$$\begin{aligned}\mathrm{p(k)} = \mathrm{P[X=k]} &= \mathrm{cq^{k-1}}, \mathrm{q}=1-\mathrm{p},\ \mathrm{k}\in\mathrm{As},\\ &= 0, \text{otherwise},\end{aligned} \tag{6.2.2}$$

where c is such that $\sum_{k=s+1}^{\infty} p(k) = 1$. If s = 0, then GET(s, p) = GE(p) with c = p. The geometric distribution like the exponential distribution possesses the memory less property i.e.

$$\bar{p}(r + s) = \bar{p}(r)\,\bar{p}(s), \tag{6.2.3}$$

where r and s are positive integers and $\bar{p}(j) = \sum_{k=j+1}^{\infty} p(k)$.

Geometric distribution is said to a discrete analogue of the exponential distribution.

If $X \in GE(p)$, then $\bar{P}(x) = q^x$ and $p(x) = pq^{x-1}$, for $x \in A_o$. Then substituting the values of $\bar{P}(x_i)$ and $p(x_i)$ in (6.2.1), we get pmf of $X_d(1), X_d(2), \ldots X_d(m)$

$$\begin{aligned} p(x_1, x_2, \ldots, x_m) &= p^n q^{x_m - m}, 1 \le x_1 < x_2 < \ldots < x_m < \infty \\ &= 0, \text{ otherwise.} \end{aligned} \tag{6.2.4}$$

The conditional pmf of $X_{U(n)} \mid X_{U(n-1)} = x_{n-1}$ is

$$\begin{aligned} P\Big(X_d(n) = x_n | X_d(n-1)_{)=} x_{n-1}\Big) &= pq^{x_n - x_{n-1} - 1}, n-1 \le x_{n-1} < x_n < \infty, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Thus $X_d(n)_) - X_d(n-1)$ is independent of $X_d(n-1)$ and $X_d(n)_) - X_d(n-1) \in GE(p)$, n = 2, 3,….

Let

$$\begin{aligned} V_1 &= X_d(1) \\ V_2 &= X_d(2 - X_d(1) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_n &= X_d(n) - X_d(n-1) \end{aligned}$$

Then V_i's are independent and $V_i \in GE(p)$.

We have

$$X_d(n) = V_1 + V_2 + \cdots + V_n. \tag{6.2.5}$$

It is known that if X. ∈G E(p), then

$$E(s^x) = \sum_{x=1}^{\infty} s^x pq^{x-1} = \frac{ps}{1 - qs}. \tag{6.2.6}$$

Using (6.2.6), We obtain

$$E(s^{X_d(n)} = E\left(s^{V_1+V_2+\cdots V_n}\right) = \left(\frac{ps}{1-qs}\right)^n \tag{6.2.7}$$

The coefficient of s^x in $\left(\frac{ps}{1-qs}\right)^n$ is $\binom{x-1}{n-1}p^n q^{x-n}$, $x \geq n$.
Thus the marginal pmf of $X_d(m)$ can be written as

$$\begin{aligned} p_m(x) = p[X_d(m) = x] &= \binom{x-1}{m-1} p^m q^{x-m}, x \in A_{m-1}, m \geq 1 \\ &= 0, \text{ otherwise.} \end{aligned} \tag{6.2.8}$$

We see that $X_d(m)$ has a negative binomial distribution with parameters m and p. We can write

$$X_d(n)|X_d(m)_) = x_m \overset{d}{=} V_{m+1} + \cdots + V_n + x_m, \; n > m.$$

and

$$E(s^{X_d(n)}|X_d(m) = x) = s^{x_m}\left(\frac{ps}{1-qs}\right)^{n-m}.$$

The coefficient of s^y in $s^{x_m}\left(\frac{ps}{1-qs}\right)^{n-m}$ is $\binom{y-x_m-1}{n-m-1} p^{n-m} q^{y-x_m-n+m}$,
Thus we obtain the conditional pmf of $X_d(n)$ given $X_d(m)$ as

$$P\Big(X_d(n)_) = x_n|X_d(m)_) = \; x_m\Big) = \binom{x_n - x_m - 1}{n-m-1} p^{n-m} q^{x_n - x_m - n+m},$$
$$0 < m \leq, x_m \leq x_n - n + m < \infty.$$

But we know that the marginal pmf of $X_d(m)$ is

$$p_m(x) = p[X_d(m) = x] = \binom{x-1}{m-1} p^m q^{x-m}, \; x \in A_{m-1}, \; m \geq 1$$

Thus the joint pmf of $X_d(m)$ and $X_d(n)_($ is

$$\begin{aligned} p_{m,n}(x,y) = P[X_d(m) = x, X_d(n) = y) = (array{*}20cx - 1m - 1)\binom{y-x-1}{n-m-1} p^n q^{y-n} \\ m \leq x < y - n + m < \infty \\ = 0, \text{ otherwise.} \end{aligned}$$

We have seen that

$$P[X_d(n+1) - X_d(n) = u | X_d(n) = y] = pq^{u-1},\ u \in A_o$$
$$= P[X_k = u]$$

Thus

$$E[(X_d(n+1) - X_d(n))^2 | X_d(n) = y] = \sum_{u=1}^{\infty} u^2 pq^{u-1}$$
$$= \tfrac{1+q}{p^2}.$$

Thus $E[(X_d(n+1) - X_d(n))^2 | X_d(n) = y]$ is independent of y. If fact it can be shown that $E[(X_d(n+1) - X_d(n))^k | X_d(n) = y]$, for any positive integer k, is independent of y.

Since $X_d(m) \stackrel{d}{=} V_1 + V_2 + \ldots. + V_m$, where $V_1, \ldots, V_m$ are independent and identically distributed as GE(p), using this property, we get

$$E(X_d(m) = np^{-1}, \mathrm{Var}\Big(X_d(m)_)\Big) = np^{-2}q. \tag{6.2.9}$$

$$\mathrm{Cov}(X_d(n)X_d(m)) = \mathrm{Var}(X_d(m)) = (n-m)p^{-2}q \tag{6.2.10}$$

The following Theorem is due to Srivastava (1978).

Theorem 6.2.1 *Suppose* F(x) *is the distribution function of the sequence of i.i.d. random variables* $\{X_n,\ n \geq 1\}$ *with positive mass function only at* 1, 2,... *Then* $P[X_d(2) - X_d(1) = 1 | X_d(1) = i] = P[X_d(2) - X_d(1) = 1|1]$ for $i = 1, 2, \ldots,$ *if and only if* X_n *has the geometric distribution with pmf as given by*

$$pj = P[\ X = j] = c\ p(1-p)^{j-2}, j = 2, 3, \ldots. \tag{6.2.11}$$

and

$$p_1 = 1 - \sum_{j=2}^{\infty} p_j = 1 - c, 0 < p < 1, 0 < c < 1.$$

Proof We give here the original proof of Srivastasva.

$$P[X_d(2)_) - X_d(1) = 1 | X_d(1) = i]]$$
$$= P\Big[X_d(2)_) = i+1, X_d(1) = i\Big] / P\big[X_{U(1)=i}\big].$$
$$= \tfrac{p_{i+1}}{1-(p_1+\cdots+p_i)}$$

Since this conditional probability is independent of i, we must have

$$\begin{aligned}&\mathrm{p} = \tfrac{p_{i+1}}{1-(p_1+\cdots+p_i)}, \text{for j} = 1, 2, \ldots\\ &\text{For j} = 1,\ \text{we get}\, p_2 = p(1-p_1) = cp, \text{say}.\\ &\text{For j} = 2, \text{we get}\, p_3 = p(1-p_1-p_2) = (1-p_1)p(1-p) = cp(1-p)\end{aligned}$$

Similarly for j = k, we have

$$p_k = cp(1-p)^{j-2},\ \mathrm{k} > 3. \tag{6.2.12}$$

If pj is as given by (6.2.11) then

$$\begin{aligned}&\mathrm{P}\left[\, \mathrm{X}_{\mathrm{U}(2)} - \mathrm{X}_{\mathrm{U}(1)} = 1 | \mathrm{X}_{\mathrm{U}(1)} = \mathrm{i}\right]\\ &= \mathrm{P}\left[\, \mathrm{X}_{\mathrm{U}(2)} - \mathrm{X}_{\mathrm{U}(1)} = 1\right].\end{aligned}$$

A generalization of the Theorem 8.2.1 is the following theorem.

Theorem 6.2.2 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed discrete random variables with common distribution function* F. *Suppose* X *is concentrated on the positive integers and* $a = \sup\{x | F(x) < 1\} = \infty$. *Then* $X_n \in \mathrm{GET}(n,p)$ *for some fixed* n, $n \geq 1$, *if and only* $X_d(n+1) - X_d(n)$ *and* $X_d(n)$ *are independent.*

Proof The 'if' part follows immediately from the from Eq. (6.2.5), so we need to established the 'only if' part. For $x \in$ Ao, let

$$\begin{aligned}\mathrm{c(u)} &= \mathrm{P}\left[\mathrm{X_d(n+1)} - \mathrm{X_d(n)} = \mathrm{u} | \mathrm{X}_{\mathrm{U(n)}} = \mathrm{x}\right]\\ &= \frac{p(u+x)}{\bar{p}(x)}\\ &= \frac{\bar{p}(u+x-1) - \bar{p}(u+x)}{\bar{p}(x)}, u \in A_o, x \in A_n.\end{aligned} \tag{6.2.13}$$

Summing both sides of (6.2.13) with respect to u from 1 to u_o, and writing

$$c_1(u_o) = \sum_{u=1}^{u_o} c(u) \text{ and } \mathrm{c_o} = 1 - c_1(u_o). \tag{6.2.14}$$

On simplification, we get

$$\bar{p}\,(x + u_o) = c_o\,(u_o)\,\bar{p}\,(x),\ \ u \in A_o, x \in A_n \tag{6.2.15}$$

The general solution of (6.2.15) is

$$\bar{p}\,(x) = c\,p^x,\ x \in A_n \tag{6.2.16}$$

where c is independent of p.

Using the boundary condition $\bar{p}\ (\infty)\ = 0$, we get

$$\bar{p}(x) = c\,p^x, x \in A_n, 0 < \mathrm{p} < 1, \mathrm{x} \in \mathrm{A_n}.$$

We have already seen that

$$\begin{aligned}\mathrm{P[X_d(n+1)X_d(n) = u|X_d(n) = y] = pq^{u-1}, u \in Ao} \\ \mathrm{= P[\ X_k = u].}\end{aligned}$$

Does the above condition characterize the geometric distribution? As an answer to that question we have the following theorem.

Theorem 6.2.3 *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of independent and identically distributed random variables with common distribution function* F. *Suppose* X *is concentrated on the positive integers with* $a = \sup\ \{x|F(x) < 1\} = \infty$. *Further if* $\mathrm{P[X_d(n+1) - X_d(n) = u|\ X_d(n) = y) = P\ [\ X_1 =\ u]}$ *for two fixed* $\mathrm{y \in A_{n-1}}$, y_1, y_2 *relatively prime and all* $u \in Ao$, *then* $X \in GET(n,p)$.

Proof Suppose that

$$\mathrm{P[X_d(n+1) - X_d(n) = u|X_d(n) = y) = P[\ X_1\ = u]} \tag{6.2.17}$$

then

$$\mathrm{P[P[X_d(n+1) - X_d(n) = u|X_d(n) = y)]} = \frac{p(u+y)}{\bar{p}(y)} = \mathrm{p(u)} \tag{6.2.18}$$

for two relatively prime $y_1, y_2 \in A_{n-1}$ and all $u \in A_o$. Summing (6.2.18) with respect to u from $u_o + 1$ to ∞, we get

$$\frac{\bar{p}\ (u_o\ +\ y)}{\bar{p}\ (y)}\ =\ \bar{p}(u_o), \tag{6.2.19}$$

for two relatively prime y_1 , $y_2 \in A_{n-1}$ and all $u \in Ao$. The general solution of (6.2.19) is

$$\bar{p}\ (x)\ =\ c\,p^x\ , x \in\ A_n,$$

and since $\bar{p}\ (\infty)\ =\ 0$, we must have

$$\bar{p}\ (x)\ =\ c\,p^x, 0\ <\ p\ <\ 1,\quad x\ \in\ A_n. \tag{6.2.20}$$

Srivastava (1979) gave a characterization of the geometric distribution using the condition $E(X_d(2)|X_d(1) = y) = \alpha + y$ Ahsanullah and Holland (1984) proved the following theorem which is a generalization of Srivastava's result.

Exercise 6.2.1 Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed discrete random variables with common distribution function F. Suppose X is concentrated on the positive integers with $a = \sup\{x|F(x) < 1\} = \infty$. Further suppose $E(X_d(n+1))^2 < \infty$. If $E(X_d(n+1)|X_d(n) = y) = y + p^{-1}$ for all $y \in A_{n-1}$, then $X_1 \in GET(n,p)$ and $0 < p < 1$.

6.3 Weak Records

Vervaat (1973) introduced the concept of weak records of discrete distribution. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables taking values on 0, 1,… with distribution function F such that $F(n) < 1$ for any n. The weak record times $U_w(n)$ and weak upper record values $X_{dw}(n)$ are defined as follows:

$$\begin{aligned} U_w(1) &= 1 \\ U_w(n+1) &= \min\{j > \ L_w(n), X_j > \max(X_1, X_2, \ldots X_{j-1})\} \end{aligned}$$

and the corresponding weak upper record value is defined as $X_{dw}(n+1)$. If in the above expression if we replace $\geq$ by $>$, then we obtain record times and record values instead of weak record times and weak record values.

The joint pmf of $X_{dw}(1), X_{dw}(2), \ldots, X_{dw}(n)$ is given by

$$P_{w,1,2,\ldots,n(x_1,x_2,\ldots,x_n)} = \left(\prod_{i=1}^{n-1} \frac{p(x_i)}{\vec{P}(x_i - 1)}\right) p(x_n) \quad \text{for } 0 < x_1 < x_2 < \cdots x_n < \infty. \tag{6.3.1}$$

For any m > 1 and n > m, we can write

$$\begin{aligned} &P(X_{dw}(n) = x_n, \ldots . X_{dw}(m+1) = x_{m+1} | X_{dw}(m) = x_m, \ldots . \\ &\quad X_{dw}(1) = x_1) \\ &\quad \left(\prod_{i=m}^{n-1} \frac{p(x_i)}{\vec{P}(x_i - 1)}\right) \frac{p(x_n)}{\vec{F}(x_m - 1)} \end{aligned} \tag{6.3.2}$$

It follows easily from (6.3.1) and (6.3.2) that the weak records, $X_{dw}(1)$, $X_{dw}(2), \ldots$., form a Markov chain.

The marginal pmf's of the upper weak records are given by

$$\begin{aligned} P(X_{dw}(1) = x_1) &= P_{w,1}(x_1) = p(x_1), x_1 = 0, 1, 2, \ldots . \\ P(X_{dw}(2) = x_2) &= P_{w,2}(x_2) = R_{w,1}(x_2) p(x_2), x_2 = 0, 1, 2, \ldots . \end{aligned}$$

where

$$\mathrm{R_{w,1}(x_2)} = \sum_{0 \le x_1 \le x_2} \frac{p(x_1)}{\vec{P}(x_1 - 1)} \tag{6.3.3}$$

$$\mathrm{P(X}_{dw)}(n) = x_n) = \mathrm{P_{w,n}(x_n) = R_{w,n-1}(x_n)p(x_n)},$$

where

$$\mathrm{R_{w,n-1}(x_n)} = \sum_{0 \le x_1 \le x_2 \le \ldots x_{n-1}} \prod_{i=1}^{n-1} \frac{p(x_i)}{\vec{P}(x_i - 1)} p(x_n) \tag{6.3.4}$$

The joint pmf of $X_{dw}\,(m)$ and $X_{dw},(m)$ m < n, m < n is given by

$$\mathrm{P_{w,m,n}(x_m, x_n) = R_{w,m}(x_m)A_w(x_m)R_{wm,+1,n}(x_m, x_n)p(x_n)},$$
$$\mathrm{m \le x_{m \le} x_n - n \ + m < \infty},$$

where

$$\mathrm{R_{w,m,n}(x, y)} = \sum_{x_m \le x_{m+1} \le x_{m+2} \ldots \le \cdot n} A_w(x_{m+1}) \ldots \ldots A_w(x_{n-1}),$$
$$\mathrm{m < n - 1 = 1}$$
$$\mathrm{if\ \ m = n - 1},$$

$$\text{and}\, \mathrm{A_w(x)} = \frac{p(x)}{\bar{P}(x-1)}.$$

The conditional pmf of $X_{dw}(n)$ given $X_{dw},(m)$ is given by

$$\begin{aligned} &\mathrm{P_{w,n|m}(X_{dw}}(n) = \mathrm{x_n|X_{dw}}(m) = \ \mathrm{x_m}) \\ &\qquad = \mathrm{R_{w,m+1,n}}(x_m, x_n)\ \frac{p(x_n)}{P(x_m - 1)}, \\ &\qquad\qquad \text{for} \le \mathrm{x_m \le x_n} < \infty. \end{aligned} \tag{6.4.5}$$

Thus the pmf of $X_{U_w(n)}$ given $X_{U_w(n-1)}$ is

$$\begin{aligned} &\mathrm{P_{w,n|n-1}}(X_{dw}(n) = x_n | X_{dw}(n-1) = x_{n-1} \\ &\qquad = \frac{p(x_n)}{\bar{p}(x_{n-1})} \end{aligned} \tag{6.3.6}$$

Theorem 6.3.1 *Let* {X_i, i = 1, 2,…} *be sequence of independent and identically distributed random variables taking values on* 0, 1, 2,..,n, n ≤ ∞, *with distribution F such that* F(n) < ∞ *for* n < ∞ *and* $E(X_1 \ln(1 + X_1) < \infty$. *The for some continuous function* ψ, *the condition* $E(\psi(X_{dw}(n)| \; X_{dw}(n-1)=j)$ = g(j) *determines the distribution.*

Proof We have from (6.3.6)

$$P(X_{dw}(n) = y\,|X_{dw}(n-1) = x) \;=\frac{p_y}{q_x}, \tag{6.3.7}$$

where p_y = P(X = y) and q_x = P(X ≥ x)

Now

$$E(\psi(X_{dw}(n))\,|X_{dw}(n-1) = j) \;= \frac{1}{q_j}\sum_{k=j}^{N}\psi(k)p_k.$$

Using the condition as given in the theorem, we can write the above expression as

$$g(j)\;q_j = \sum_{k=j}^{n}\psi(k)p_k \tag{6.3.8}$$

Taking first order difference, we obtain from (6.3.8)

$$\mathrm{g(j)q_j - g(j+1)q_{j+1}} = \psi(j)p_j \tag{6.3.9}$$

Thus

$$\mathrm{g(j)q_j \;-\; g(j+1)\Big(q_j \;-\; p_j\Big)} = \psi(j)p_j \tag{6.3.10}$$

i.e.

$$p_j = \frac{g(j+1) - g(j)}{g(j+1)\;-\psi(j)}q_j \tag{6.3.11}$$

Since $q_j = \frac{q_j}{q_{j-1}}\cdot\frac{q_{j-1}}{q_{j-2}}\cdots\frac{q_1}{q_o}$, $q_0 = 1$, we have from (6.3.11)

$$p_j \;= \frac{g(j+1)\;-g(j)}{g(j+1)\;-\;\psi(j)}\prod_{k=0}^{j-1}\left(\frac{q_{k+1}}{q_k}\right) \tag{6.3.12}$$

From (6.3.10)

$$\mathrm{g(j)\;q_j - g(j+1)q_{j+1}} = \psi(j)\,(q_j - q_{j+1})$$

i.e.

$$\frac{q_{j+1}}{q_j} = \frac{g(j) - \psi(j)}{g(j+1) - \psi(j)} \tag{6.3.13}$$

From (6.3.13), we can write

$$p_j = \frac{g(j+1) - g(j)}{g(j+1) - \psi(j)} \prod_{k=0}^{j-1} \left(\frac{g(k) - \psi(k)}{g(k+1) - \psi(k)} \right) \tag{6.3.14}$$

6.3.1 Geometric Distribution

Suppose X_1, X_2,.... be a sequence of independent and identically random variables with p(k) = pq^k and $\vec{P}(k-1) = q^k$, $k = 0, 1, 2, \ldots$

Here $R_{w,1(}x_2) = \sum_{1 \le x_1 \le x_2} \frac{p(x_1)}{\vec{P}(x_1 - 1)} = x_2 p.$

Thus

$$P_{w,2}(k) = R_{w,1}(k)p(k) = kp^2q^k, k = 0, 1, 2, \ldots.$$

Since

$$P_{w,n|n-1}(x_{n|}\, X_{dw}(n-1) = x_m) = \frac{p(x_n)}{\vec{P}(x_{n-1} - 1)} = pq^{x_n - x_m},$$
$$x_n > x_{n-1}$$

we obtain

$$\begin{aligned} P_{w,3}(x_3) &= \sum_{x_2=0}^{x_3} x_2 p^2 q^{x_2} p q^{x_3 - x_2} \\ &= \sum_{x_2=0}^{x_3} x_2 p^3 q^{x_3} \\ &= \tfrac{x_3(x_3+1)}{2} p^3 q^{x_3}, x_3 = 0, 1, 2, , \ldots \end{aligned}$$

By induction it can be proved that

$$\begin{aligned} P_{w,n}(x_n) &= \frac{x_n(x_n+1)\ldots(x_n+n-2)}{(n-1)!} p^n q^{x_n} \\ &= \tbinom{x_n+n-2}{n-1} p^n q^{x_n} n \ge 2 \text{ and } x_n = 0, 1, 2, \ldots \\ &E(X_{dw}(n)|X_{dw}(n-1) = x_{n-1}) \\ &= \sum_{x_n = x_{n-1}}^{\infty} x_n\, pq^{x_n - x_{n-1}} = x_{n-1} + \frac{q}{p}. \end{aligned}$$

The conditional expectation of $X_{dw}(n)\,|X_{dw}(n-2)=x$ is

$$
\begin{aligned}
&E(X_{dw}(n)|X_{dw}(n-2)=x_{n-2})\\
&=\sum_{x=x_{n-2}}^{\infty} x(x-x_{n-2}+1)p^2q^{x-x_{n-2}}\\
&=x_{n-2}+\sum_{x=x_{n-2}}^{\infty}[(x-x_{n-2})\,(x-x_{n-2}+1)]p^2q^{x-x_{n-2}}\\
&=\mathrm{x}_{\mathrm{n-2}}+\frac{2q}{p}
\end{aligned}
$$

Wesolowski and Ahsanullah proved the following characterization theorem using conditional expectation.

Theorem 6.3.2 *Let* $\{X_i,\ i=1,2,\ldots\}$ *be sequence of independent and identically distributed random variables taking values on* 0, 1, 2,.., n, $n \le \infty$, *with* cdf F; *Then* $E(X_{dw}(n+1)|X_{dw}(n)=x)=x+b$, *where* b *is a constant characterized the geometric distribution.*

We have seen that

$$
\begin{aligned}
\mathrm{P}[\ \mathrm{X_d(n+1)})\ -\ \mathrm{X_d(n)}=\mathrm{u}|\mathrm{X_d(n)}=\mathrm{y}] &= \mathrm{pq^{u-1}},\ \mathrm{u}\in \mathrm{A_o}\\
&=\mathrm{P}[\ \mathrm{X_k}=\mathrm{u}].
\end{aligned}
$$

Thus

$$
\begin{aligned}
E[(X_d(n+1)-X_d(n))^2|X_d(n)=y] &= \sum_{u=1}^{\infty} u^2pq^{u-1}\\
&=\frac{1+q}{p^2}.
\end{aligned}
$$

Thus $E[(X_d\,(n+1)-X_d\,(n))^2|\,X_d\,(n)\ =\ y]$ is independent of y. If fact it can be shown that $E[(X_d\,(n+1)-X_d\,(n)\,)^k|\,X_d\,(n)\ =\ y]$, for any positive integer k, is independent of y.

Exercise 6.3.1 Suppose $\{X_n,\ n>1|$ be a sequence of i.i.d. random variables with cdf F(x). We assume X_n is concentrated on 0, 1, 2,. . . and F(x). 0 for all x > 0. Then $X\in GE(p)$ if and only if $X_{dw}(2)\underline{\underline{d}}X_1+X_2$, where X_1 and X_2 are independent copies of X_n.

Ahsanullah and Aliev (2011) extended the result of the above exercise to $X_{dw}(n)$. They prove that under the same assumptions of the above exercise the condition $X_{dw}(n)\underline{\underline{d}}X_1+X_2+\cdots+X_n$ where $X_1, X_2,\ldots, X_n$ are independent copies of X_n characterizes GE(p).

Ahsanullah and Hijab proved that the following theorem.

Theorem 6.3.3 *Suppose* $\{X_n, n > 1|$ *be a sequence of i.i.d. random variables with* cdf F(x). *We assume* X_n *is concentrated on* 0, 1, 2,.... *Then* $X \in GE(p)$ if and only if $X_{dw}(n+1) \underline{\underline{d}} Xdw(n) + W$, *where* W *has the same distribution as* X's.

Example 6.3.2 Beta—Binomial distribution.

Suppose X_1, X_2,... be a sequence of independent and identically random variables having Beta—Binomial distribution, BB(N, β) with pmf as

$$p(j) = P(X = j) = \binom{\beta + N - j - 1}{N - j}\binom{\beta + N}{N}^{-1}, j = 0, 1, 2, \ldots, N \tag{6.3.19}$$

and

$$\begin{aligned}\vec{P}(k-1) = q_k &= \sum_{j=k}^{N} p_j,\\ &= \sum_{j=k}^{N}\binom{\beta + N - j - 1}{N - j}\binom{\beta + N}{N}^{-1}\\ &= \sum_{j=0}^{N-k}\binom{\beta + N - j - 1}{j}\binom{\beta + N}{N}^{-1}\\ &= \binom{\beta + N - k}{N - k}\binom{\beta + N}{N}^{-1}.\end{aligned}$$

Here

$$\begin{aligned}R_{w,1(}x_2) = \sum_{1 \le x_1 \le x_2}\frac{p(x_1)}{\vec{P}(x_1 - 1)} &= \sum_{1 \le x_1 \le x_2}\frac{\binom{\beta+N-x_1-1}{N-x_1}}{\binom{\beta+N-x_1}{N-x_1}}\\ &= \sum_{1 \le x_1 \le x_2}\frac{\beta}{N + n - x_1}\end{aligned}$$

Thus

$$\begin{aligned}p(X_{dw}(2) = j) &= R_{w,1}(j)\ p(j)\\ &= \sum_{0 \le x_1 \le j}\frac{\beta}{N + n - x_1}\binom{\beta+N-j-1}{N-j}\binom{\beta+N}{N}^{-1}\\ &\qquad j = k, \ldots, N\end{aligned}$$

$$\begin{aligned}&E(X_{dw}(n)|X_{dw}(n-2) = m)\\ &= \frac{1}{q_m}\sum_{r=m}^{N}\frac{p_r}{q_r}\sum_{k=r}^{N} kp_k\\ &= \binom{\beta+N-m}{N-m}^{-1}\sum_{r=m}^{N}\frac{\beta}{\beta + N - r}\sum_{k=r}^{N} k\binom{\beta+N-k-1}{N-k}\end{aligned}$$

Now

$$
\begin{aligned}
&\sum_{k=r}^{N} k\binom{\beta+N-k-1}{N-k} \\
&= \sum_{i=0}^{N-r} (N-i)\binom{\beta+i-1}{i} \\
&= N\sum_{i=0}^{N-r} \binom{\beta+i-1}{i} - \sum_{i=0}^{N-r} i\binom{\beta+i-1}{i} \\
&= N\binom{\beta+N-r}{n-r} - \binom{\beta+N-r}{N-r-1} \\
&= N\binom{\beta+N-r}{N-r} - \frac{(N-r)\beta}{\beta+1}\binom{\beta+N-r}{N-r} \\
&= \frac{N+r\beta}{\beta+1}\binom{\beta+N-r}{N-r}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathrm{E}(\mathrm{X}_{\mathrm{dw}}(n)|\mathrm{X}_{\mathrm{dw}}(n-2)=m) \\
&= \frac{\beta}{\beta+1}\binom{\beta+N-m}{n-m}^{-1}\sum_{r=m}^{n}\frac{N+r\beta}{\beta+N-r}\binom{\beta+N-r}{N-r}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sum_{r=m}^{N}\frac{N+r\beta}{\beta+N-r}\binom{\beta+N-r}{N-r} + \sum_{r=m}^{N}\frac{N+r\beta}{\beta}\binom{\beta+N-r-1}{N-r} \\
&= \frac{N}{\beta}\binom{\beta+N-m}{N-m} - \sum_{r=m}^{N} r\binom{\beta+N-r-1}{N-r}.
\end{aligned}
$$

It can easily be shown that

$$
\begin{aligned}
\sum_{r=m}^{N} r\binom{\beta+N-r-1}{N-r} &= \sum_{t=0}^{N-m}(N-t)\binom{\beta+t-1}{t} \\
&= N\binom{\beta+N-m}{N-m} - \sum_{t=0}^{N-m} t\binom{\beta+t-1}{t} \\
&= \mathrm{N}\binom{\beta+N-m}{N-m} - \frac{(N-m)\beta}{\beta+1}\binom{\beta+N-m}{N-m} \\
&= \frac{N+m\beta}{\beta+1}\binom{\beta+N-m}{N-m}.
\end{aligned}
$$

Hence

$$\begin{aligned}&\mathrm{E}(\mathrm{X}_{\mathrm{dw}}(n)|\mathrm{X}_{\mathrm{dw}}(n-2)| = m)\\&=\left(\frac{\beta}{\beta+1}\right)\left(\frac{N}{\beta}+\frac{N+m\beta}{\beta+1}\right)\\&=\left(\frac{\beta}{\beta+1}\right)^2 m+\left(\frac{2\beta+1}{(\beta+1)^2}\right)N.\end{aligned}$$

Exercise 6.3.2 Suppose $X_1, X_2, \ldots$ be a sequence of independent and identically random variables having Beta Binomial distribution, BNB(β, γ) with pmf as

$$\mathrm{p(j)} = \mathrm{P(X = j)} = \frac{\gamma}{\gamma+j}\binom{\beta}{\gamma}\binom{\beta+j+1}{\gamma+j}^{-1}, \mathrm{j} = 0, 1, 2, \ldots, \tag{6.3.20}$$

where $\beta > \gamma > 0$.

Show that

$$\mathrm{E}(\mathrm{X}_{\mathrm{wU(n)}}|\,\mathrm{X}_{\mathrm{wU(n-2)}} = m)$$

6.4 Exercises (solutions)

Exercise 6.2.1 (*solution*) Suppose $\mathrm{E}(\mathrm{X_d(n + 1)}|\mathrm{X_d(n)} = \mathrm{y}) = \mathrm{y} + \mathrm{p}^{-1}$ for all $\mathrm{y} \in \mathrm{A}_{\mathrm{n}-1}$, then we have

$$\mathrm{P}\left[\mathrm{X}_{\mathrm{U(n+1)}} = \mathrm{u} + \mathrm{y}|\mathrm{X}_{\mathrm{U(n)}} = \mathrm{y}\right] = \frac{p(u+y)}{\bar{p}(y)}, \mathrm{y} \in \mathrm{A}_{\mathrm{n}-1},$$

and hence

$$\begin{aligned}\mathrm{E}\left[\mathrm{X}_{\mathrm{U(n+1)}}|\mathrm{X}_{\mathrm{U(n)}} = \mathrm{y}\right] &= \sum_{u=y}^{\infty}(u+y)\frac{p(u+y)}{\bar{p}(y)}\\&= \mathrm{p}^{-1} + \mathrm{y},\end{aligned}$$

which implies that

$$\sum_{u=y}^{\infty} u\frac{p(u+y)}{\bar{p}(y)} = \mathrm{p}^{-1}.$$

Thus we have

$$p^{-1} = \frac{\sum_{u=1}^{\infty} u[\bar{p}(u+y-1) - \bar{p}(u+y)}{\bar{p}(y)}$$
$$= \sum_{j=o}^{\infty} \frac{\bar{p}(y+i)}{\bar{p}(y)}$$
$$= \frac{J(y)}{J(y+1) - J(y+1)},$$

where $\mathrm{J(y)} = \sum_{i=o}^{\infty} \bar{p}(y+i)$. Thus

$$\mathrm{J(y+1) = q\ J(y), q = 1 - p} \tag{6.4.1}$$

for all $\mathrm{y} \in \mathrm{A_{n-1}}$.

The general solution of (6.4.1) with the boundary condition, J(n) = θ, 0 < θ < 1, is

$$\mathrm{J(x)} = \theta\, \mathrm{q^{x-n,}x \in A_{n-1}}.$$

Therefore

$$\bar{p}\ (x) = \mathrm{J(x) - J(x+1)} = \theta\, \mathrm{pq^{x-n}, x \in A_{n-1}}.$$

i.e. $\mathrm{x} \in$ GET(n,p).

Exercise 6.3.1 (*solution*) We have already seen that if $\mathrm{X \in GE(p)}$ then $\mathrm{X_{dw}(2)} \underline{\underline{d}}\, \mathrm{X_1 + X_2}$.

We will prove here the only if condition.

$$\mathrm{P(X_{dw}(2)} = \sum_{0 \le x \le x} \frac{p(x_i)}{\bar{P}(x_i)} p(x)$$
$$\mathrm{P(X_1 + X_2 - x)} = \sum_{u=0}^{x} p(x-u)p(u),$$

Hence

$$\sum_{0 \le x \le x} \frac{p(x_i)}{\bar{P}(x_i)} p(x) = \sum_{u=0}^{x} p(x-u)p(u),$$

Putting x = 1, we obtain

$$\left(\frac{p(0)}{\bar{P}(0)}+\frac{p(1)}{\bar{P}(1)}\right)p(1)=2p(0)p(1)$$

Let p(0) = p, then
P(1) = pq, q = 1 − p.
For x = 2, we have

$$\left(\frac{p(0)}{\bar{P}(0)}+\frac{p(1)}{\bar{P}(1)}+\frac{p(2)}{\bar{P}(2)}\right)p(2)=2p(0)p(2)+(p(1))^2$$

Substituting p(0) = p. p(1) = q in the above equation, we obtain $\mathrm{p}(2)=\mathrm{p}q^2$. Suppose $\mathrm{p}(\mathrm{u})=pq^u$ for u ≤ x, then we have for u = x + 1.

Exercise 6.3.2 (*solution*)

$$\begin{aligned}\vec{P}(k-1)=q_k&=\sum_{j=k}^{\infty}p_j\\&=\sum_{j=k}^{\infty}\frac{\gamma}{\gamma+j}\binom{\beta}{\gamma}\binom{\beta+j+1}{\gamma+j}^{-1},\\&=\sum_{i=0}^{\infty}\frac{\gamma}{\gamma+k+i}\binom{\beta}{\gamma}\binom{\beta+k+i+1}{\gamma+k+i}^{-1}.\\&=\frac{\gamma}{\gamma+k}\binom{\beta}{\gamma}\binom{\beta+k}{\gamma+k}^{-1},\ \mathrm{k}=0,1,2,\ldots\end{aligned}$$

Here $\mathrm{R_{w,1}(j)}=\sum_{0\le x_1\le j}\frac{p(x_1)}{q_{x_1}}=\sum_{0\le x_1\le j}\frac{\binom{\beta+x_1}{\gamma+x_1}}{\binom{\beta+x_1+1}{\gamma+x_1}}$

$$=\sum_{0\le x_1\le j}\frac{\beta-\gamma+1}{\beta+x_1+1}$$

Thus

$$\begin{aligned}P(X_{dw}(2)=j)&=\mathrm{R_{w,1}(j)p(j)}\\=\sum_{0\le x_1\le j}\frac{\beta-\gamma+1}{\beta+x_1+1}\frac{\gamma}{\gamma+j}\binom{\beta}{\gamma}\binom{\beta+j+1}{\gamma+j}^{-1}&\\ \mathrm{j}=0,1,\ldots,&\end{aligned}$$

Now

$$\begin{aligned}&E(X_{dw}(n)|X_{dw}(n-2)| = m)\\&= \frac{1}{q_m}\sum_{r=m}^{\infty}\frac{p_r}{q_r}\sum_{k=r}^{\infty}kp_k\end{aligned}$$

We have

$$\begin{aligned}\sum_{k=r}^{\infty}kp_k &= \sum_{j=0}^{\infty}\frac{\gamma(r+j)}{\gamma+r+j}\binom{\beta}{\gamma}\binom{\beta+r+j+1}{\gamma+r+j}\\&= \frac{\gamma(r-1)}{\gamma+r}\binom{\beta}{\gamma}\binom{\beta+r}{\gamma+r}^{-1}\sum_{j=0}^{\infty}\frac{\gamma+r}{\gamma+r+j}\binom{\beta+r}{\gamma+r}\binom{\beta+r+j+1}{\gamma+r+j}^{-1}\\&= \frac{\gamma(r-1)}{\gamma+r}\binom{\beta}{\gamma}\binom{\beta+r}{\gamma+r}^{-1}+\frac{\gamma}{\gamma+r}\binom{\beta}{\gamma}\binom{\beta+r-1}{\gamma+r}^{-1}\\&= \frac{\gamma[r(\beta-\gamma-1)+\gamma]}{(\gamma+r)(\beta-\gamma)}\binom{\beta}{\gamma}\binom{\beta+r}{\gamma+r}^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}&E(X_{dw}(n)|X_{dw}(n-2) = m)\\&= \frac{1}{q_m}\sum_{r=m}^{\infty}\frac{(\beta-\gamma+1)\gamma[r(\beta-\gamma-1)+\gamma]}{(\beta+r+1)(\gamma+r)(\beta-\gamma)}\binom{\beta}{\gamma}\binom{\beta+r}{\gamma+r}^{-1}\\&= \binom{\beta}{\gamma}\frac{1}{q_m}\left[\frac{\beta-\gamma+1}{\beta-\gamma}\sum_{r=m}^{\infty}\frac{\gamma r}{\gamma+r}\binom{\beta+r+1}{\gamma+r}^{-1}\right.\\&\left.+\frac{\gamma}{\beta-\gamma}\sum_{r=m}^{\infty}\frac{\gamma}{\gamma+r}\binom{\beta+r+1}{\gamma+r}^{-1}\right]\\&\frac{1}{q_m}\frac{(\beta-\gamma+1)[m(\beta-\gamma+1)+\gamma]}{(\beta-\gamma)^2}\ \mathrm{q_m}\end{aligned}$$

On simplifying we the obtain

$$\begin{aligned}E(X_{wU(n)}|X_{wU(n-2)} = m) &= \frac{(\beta-\gamma+1)[m(\beta-\gamma+1)+\gamma]}{(\beta-\gamma)^2}+\frac{\gamma}{\beta-\gamma}\\&= \frac{(\beta-\gamma+1)^2m}{(\beta-\gamma)^2}+\frac{\gamma\{2(\beta-\gamma)+1\}}{(\beta-\gamma)^2}.\end{aligned}$$

Chapter 7
Estimation of Parameters and Predictions of Records

7.1 Exponential Distribution

We will consider here the two parameter exponential distribution with pdf f(x) as given by

$$\begin{aligned} f(x) &= \frac{1}{\sigma}\exp(-\sigma^{-1}(x-\mu)),\ -\infty<\mu<x<\infty,\ \sigma>0, \\ &= 0,\ \text{otherwise.} \end{aligned} \tag{7.1.1}$$

7.1.1 Minimum Variance Linear Unbiased Estimates (MVLUE) of *μ* and *σ*

Suppose that $X(1), X(2), \ldots, X(m)$ are the m (upper) record values from $E(\mu, \sigma)$ with pdf as given in (7.1.1)

Let

$$\begin{aligned} Y_i &= \sigma^{-1}(X(i)-\mu), i = 1, 2, \ldots, m, \text{ then} \\ E(Y_i) &= i = Var(Y_i), i = 1, 2, \ldots, m, \end{aligned}$$

and $Cov(Y_i, Y_j) = \min(i, j)$.

Let

$$\begin{aligned} X &= (X(1), X(2), \ldots, X(m)), \text{ then} \\ E(X) &= \mu L + \sigma\delta \\ Var(X) &= \sigma^2 V, \end{aligned}$$

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_7

where

$$L' = (1,1,\ldots,1)', \delta' = (1,2,\ldots m)'$$
$$V = \left(V_{ij}\right),\ V_{ij} = \min(i,j), \qquad i,j = 1,2,\ldots,m.$$

The inverse $V^{-1}(= V^{ij})$ can be expressed as

$$V^{ij} = \begin{matrix} 2 & \mathit{if}\ i = j = 1,2,\ldots,m-1 \\ 1 & \mathit{if}\ i = j = m \\ -1 & \mathit{if}\ |i-j| = 1,\ i,j = 1,2,\ldots,m \\ 0 & \mathit{otherwise}. \end{matrix}$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L'V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = \left(L'V^{-1}L\right)\left(\delta' V^{-1}\delta\right) - \left(L'V^{-1}\delta\right)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 L'V^{-1}\delta/\Delta$$
$$Var(\hat{\sigma}) = \sigma^2 L'V^{-1}L/\Delta$$
$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2 L'V^{-1}\delta/\Delta.$$

It can be shown that

$$L'V^{-1} = (1,0,0,\ldots,0), \delta' V^{-1} = (0,0,0,\ldots,1),$$
$$\delta' \mathrm{V}^{-1}\delta = \mathrm{m} \quad \text{and} \quad \Delta = \mathrm{m} - 1.$$

On simplification, we get

$$\begin{aligned} \hat{\mu} &= (mX(1) - X(m))/(m-1) \\ \hat{\sigma} &= (X(m) - X(1))/(m-1) \end{aligned} \tag{7.1.1.1}$$

with

$$\begin{aligned} &Var(\hat{\mu}) = m\sigma^2/(m-1), \quad Var(\hat{\sigma}) = \sigma^2/(m-1) \ \text{ and} \\ &Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2/(m-1). \end{aligned} \tag{7.1.1.2}$$

Exercise 7.1.1.1 If μ = 0, then the MVLUE $\hat{\sigma}_0$ of ϖ is

$$\hat{\sigma}_0 = \frac{X(m)}{m}.$$

7.1.2 Best Linear Invariant Estimators

The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIE) $\tilde{\mu}$ $\tilde{\sigma}$ of μ and σ are

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1 + E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

where

$\hat{\mu}$ and $\hat{\sigma}$ are *MVLUE* of μ and σ and

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$MSE(\tilde{\mu}) = \sigma^2\left(E_{11} - E_{12}^2(! + E_{22})^{-1}\right) \text{ and}$$
$$MSE(\tilde{\sigma}) = \sigma^2 E_{22}(1 + E_{22})^{-1}$$

We have

$$E(\tilde{\mu} - \mu)(\tilde{\sigma} - \sigma) = \sigma^2 E_{12}(1 + E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} from (7.1.1.2), we obtain

$$\hat{\mu} = ((m + 1)X(1) - X(m))/m,$$
$$\hat{\sigma} = (X(m)) - X(1))/m$$
$$Var(\tilde{\mu}) = \frac{m + 1}{m}\sigma^2 \text{ and } \quad Var(\hat{\sigma}) = \frac{m - 1}{m^2}\sigma^2$$

7.1.3 Maximum Likelihood Estimate

The log likelihood equation based on the m upper X(1), X(2),..., X(m) can be written as $\ln \mathrm{L} = -\mathrm{mln}\sigma - \frac{1}{\sigma}(X(m) - \mu), \mu < X(1) < X(2) \cdots < X(m) < \infty$. The maximum likelihood estimate $\hat{\mu}_{ml}$ and $\hat{\sigma}_{\mathrm{ml}}$ of μ and σ are respectively.

$$\hat{\mu}_{ml} = \mathrm{X}(1)$$

and

$$\begin{aligned}
\hat{\sigma}_{ml} &= \frac{1}{m}(X(m) - X(1)) \\
E(\hat{\mu}_{ml}) &= \mu + \sigma, Var(\hat{\mu}_{ml}) = \sigma^2, \\
E(\hat{\sigma}_{ml}) &= \frac{(m-1)\sigma}{m}, Var(\hat{\theta}_{ml}) = \frac{(m-1)\sigma^2}{m^2} \\
&\text{and}\, Cov(\hat{\mu}_{ml}\hat{\sigma}_{mj}) = 0
\end{aligned}$$

Exercise 7.1.3.1 Show that in the case of one parameter exponential with $F(x) = 1 - e^{=x/\sigma}$, $\mathrm{x} \geq 0$, $\sigma > 0$. The maximum likelihood estimate σ^*_{ml} of σ based on m upper records $\mathrm{X}(1), \mathrm{X}(3), \ldots, \mathrm{X}(\mathrm{m})$ is

$$\begin{aligned}
\sigma^*_{ml} &= \frac{\mathrm{x(m)}}{\mathrm{m}} \text{ with} \\
E(\sigma^*_{ml}) &= \sigma + \frac{\mu}{m} \text{ and } Var(\sigma^*_{ml}) = \frac{\sigma^2}{m}.
\end{aligned}$$

7.1.4 Prediction of Record Values

We will predict the sth upper record value based on the first m record values for s > m.

Let $W' = (W_1, W_2, \ldots, W_m)$, where

$$\sigma^2 W_{ij} Cov(X(i), X(j)), i = 1, \ldots, m \quad \text{and} \quad \alpha^* = \sigma^{-1} E(X(s) - \mu).$$

The best linear unbiased predictor of X(s) is $\hat{X}(s)$ where

$$\hat{X}(s) = \hat{\mu} + \hat{\sigma}\alpha^* + W'V^{-1}(X - \hat{\mu}L - \hat{\sigma}\delta),$$

where $\hat{\mu}, \hat{\sigma}$ are the MVLUE of μ, σ respectively. It can be shown that $W'V^{-1}(X - \hat{\mu}L - \hat{\sigma}\delta) = 0$.

$$\begin{aligned} \hat{X}(s) &= ((s-1)X(m) + (m-s)X(1))/(m-1) \\ E\big(\hat{X}(s)\big) &= \mu + s\sigma \\ Var\left(\hat{X}(s)\right) &= \sigma^2\left(m + s^2 - 2s\right)/(m-1). \end{aligned} \tag{7.1.4.1}$$

Let $\tilde{X}(s)$ be the best linear invariant predictor of $X(s)$. Then it can be shown that

$$\tilde{X}(s) = \hat{X}(s) - C_{12}(1 + E_{22})^{-1}\hat{\sigma}, \tag{7.1.4.2}$$

where

$$C_{12}\sigma^2 = Cov\big(\hat{\sigma}, \big(L - W'V^{-1}L\big)\hat{\mu} + \big(\alpha^* - W'V^{-1}\delta\big)\hat{\sigma}\big)$$

and

$$\sigma^2 E_{22} = Var(\hat{\sigma}).$$

On simplification we get

$$\begin{aligned} \tilde{X}(s) &= \frac{m-s}{m}X(1) + \frac{s}{m}X(m) \\ E\left(\tilde{X}(s)\right) &= \mu + \left(\frac{ms + m - s}{m}\right)\sigma \\ Var\big(\tilde{X}(s)\big) &= \sigma^2\big(m^2 + ms^2 - s^2\big)/m^2. \end{aligned}$$

It is well known that the best (unrestricted) least squares predictor $\bar{\tilde{X}}$ of X(s) is

$$\begin{aligned} \hat{\bar{X}}(s) &= E(X(s)|X(1), \ldots, X(m)) \\ &= X(m) + (s-m)\sigma. \end{aligned} \tag{7.1.4.3}$$

But $\bar{X}^{\wedge}{}_{U(s)}$ depends on the unknown parameter σ. If we substitute the minimum variance linear unbiased estimate $\hat{\sigma}$ for σ, then $\hat{\bar{X}}(s)$ becomes equal to $\hat{X}(s)$. Now

$$\begin{aligned} E(\hat{\bar{X}}(s)) &= \mu + s\sigma = E(X(s)) \\ Var(\hat{\bar{X}}(s)) &= m\sigma^2 \end{aligned}$$

7.2 Generalized Extreme Value Distribution

We will consider here the generalized extreme value distribution having d.f. F(x) as

$$F(x) = \exp[-\{1 - \gamma\sigma^{-1}(x - \mu)\}1/\gamma] \tag{7.2.1}$$

where σ > 0, γ ≠ 0,

$$x < \mu + \sigma\gamma^{-1}, \text{ for } \gamma > 0,$$

and

$$x > \mu + \sigma\gamma^{-1}, \text{ for } \gamma < 0.$$

If γ = 0 then

$$F(x) = \exp[-\exp -\{-(x - \mu)/\sigma\}], \sigma > 0, -\infty < x < \infty. \tag{7.2.2}$$

We will write $X \in GEV(\mu, \sigma, \gamma)$ if X has the d.f. as given in (7.2.1). Since

$$\lim_{\gamma \to 0}\left\{1 - \gamma\sigma^{-1}(x - \mu)\right\}^{1/\gamma} = \exp\left\{-\sigma^{-1}(x - \mu)\right\}, \text{ we can take}$$

$$\lim_{\gamma \to 0} GEV(\mu,\sigma,\gamma) = GEV(\mu,\sigma,0).$$

The density function of $GEV(\mu, \sigma, \gamma)$ is

$$\begin{aligned} f(x) &= \sigma^{-1}\left\{1 - \gamma\sigma^{-1}(x - \mu)\right\}^{\frac{1-\gamma}{\gamma}} \exp\left[-\left\{1 - \gamma\sigma^{-1}(x - \mu)\right\}^{1/\gamma}\right], \\ & x > 1/\gamma, \gamma > 0, \\ & x < 1/\gamma, \text{ for } \gamma < 0, \\ \text{and } f(x) &= e^{-\left(\frac{x-\mu}{\sigma}\right)} \exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right), \quad \text{for } \gamma = 0. \end{aligned}$$

7.2.1 *Minimum Variance Linear Unbiased Estimate of μ and σ for Known γ*

Suppose γ ≠ 0 but γ is known. Let $x(1)$, x(2),… x(m) be the m (lower) record values from $GEV(\mu, \sigma, \gamma)$ with pdf as given in (7.2.1). Then the MVLUE $\hat{\mu}$ *and* $\hat{\sigma}_o$ of μ and σ_o respectively, based on m upper record values, x(1), X(2), … x(m), are:

$$\hat{\mu} = D^{-1}\{x(m)(1'V^{-1}\alpha) - \alpha_m 1'V^{=1}R\Big|$$
$$\hat{\sigma} = -D^{-1}\{x(m)(1'V^{-1}1) - 1'V^{-1}R\Big\}$$

where

$$D = \Gamma(m+1)\left(\frac{1'V^{-1}1}{\Gamma(m)} - -\frac{1}{b_m}\right), V - (V_{ij}) - (a_i b_j), V^{-1} = (V^{ij})$$
$$\mathrm{a_i} = \frac{\Gamma(i+\gamma)}{\Gamma(i)}, b_i = \frac{\Gamma(i+2\gamma)}{\Gamma(i+\gamma)} - \frac{\Gamma(i+\gamma)}{\Gamma(i)} \text{ and } \sigma_0^2 = \frac{\sigma^2}{\gamma^2}.$$
$$V^{11} = \frac{(1+\gamma)^2}{\gamma^2}\frac{1}{\Gamma(1+2\gamma)}$$
$$V^{ii} = \frac{\Gamma(i)}{\Gamma(i+2\gamma)} \cdot \frac{1}{\gamma^2}\Big\{(i+\gamma)^2 + (i-1)(i-1+2\gamma)\Big\}, i = 2,\ldots,m-1,$$
$$V^{ij} = V^{ji} = \frac{i+\gamma}{\gamma^2}\frac{\Gamma(i+1)}{\Gamma(i+2\gamma)}, j = i+1,\ i = 1,\ldots,m-1$$
$$V^{ij} = 0, \text{if } |\mathrm{i}-\mathrm{j}| > 1, \text{ here}$$
$$1' = (1,\ldots,1),$$
$$\mathrm{R}' = (\mathrm{x}(1), \mathrm{x}(2), \ldots \mathrm{x}(\mathrm{m}), \alpha' = (\alpha_1,\ldots,\alpha_m),$$
$$\alpha_i = 1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)}, \mathrm{i} = 1,2,\ldots,\mathrm{m},$$
$$Var(\hat{\mu}) = \sigma_0^2\left\{b_m\left(1'V^{-1}1\right) - 2 + \frac{\Gamma(m+\gamma)}{\Gamma(m)}\right\}\Big/D$$
$$Var(\hat{\sigma}_0) = \sigma_0^2 b_m\left\{\left(1'V^{-1}1\right)\right\}/D$$

and

$$Cov(\hat{\mu}, \hat{\sigma}_0) = \hat{\sigma}_0^2\left\{b_m\left(1'V^{-1}1\right) - 1\right\}/D.$$

Let R = (x(1), x(2), …, x(m)). Then we can write

$$\mathrm{E(R)} = \mu\underline{1} + \sigma_0\alpha$$
$$\mathrm{Var(R)} = \sigma_0^2\,\mathrm{V},$$

where

$$\alpha' = (\alpha,\ldots,\alpha_m), \quad \alpha_i = 1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)},$$
$$1' = (1,\ldots,1), \quad \mathrm{V} = \left\{\mathrm{V_{ij}}\right\}, \mathrm{V_{ij}} = \mathrm{a_i\,b_j}, \quad 1 \le \mathrm{i},\ \mathrm{j} \le \mathrm{m}$$
$$\mathrm{a_i} = \frac{\Gamma(i+\gamma)}{\Gamma(i)}, \quad b_i = \frac{\Gamma(i+2\gamma)}{\Gamma(i+\gamma)} - \frac{\Gamma(i+\gamma)}{\Gamma(i)} \text{ and } \sigma_0^2 = \frac{\sigma^2}{\gamma^2}.$$

Let $\mathrm{V}^{-1} = (V^{ij})$, then

$$
\begin{aligned}
V^{11} &= \frac{a_2}{a_1(a_2 b_1 - a_1 b_2)} = \frac{1}{\gamma^2}\frac{\Gamma(1+\gamma)^2}{\Gamma(1+2\gamma)} \\
V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})} \\
&= \frac{\Gamma(i)}{\gamma^2\Gamma(i+2\gamma)}\left\{(i+\gamma)^2 + (i-1)(i-1+2\gamma)\right\}, \mathrm{i} = 2, \ldots, \ \mathrm{m}-1 \\
V^{mm} &= \frac{b_{m-1}}{b_m}\frac{1}{a_m b_{m-1} - a_{m-1}b_m} = \frac{b_{m-1}}{b_m}\frac{m+1-\gamma}{\gamma^2}\frac{\Gamma(m)}{\Gamma(m-1+\gamma)}, \\
V^{ij} &= V^{ij} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -\frac{i+\gamma}{\gamma^2}\frac{\Gamma(i+1)}{\Gamma(i+2\gamma)} \\
&\mathrm{j} = \mathrm{i}+1, \ \mathrm{i} = 1, 2, \ldots, \ \mathrm{m}-1,
\end{aligned}
$$

and

$$V^{ij} = 0, \ \text{if} \ \ |\mathrm{i}-\mathrm{j}| > 1.$$

It follows from the method of Lloyd (1952) that the MVLUE of μ and σo based on the observed value r of R are, respectively,

$$
\begin{aligned}
\hat{\mu} &= \alpha' V^{-1}(1'\alpha - \alpha 1')V^{-1}R/\Delta| \\
\hat{\sigma} &= 1'V^{-1}(1\alpha' - \alpha 1')R/\Delta
\end{aligned}
$$

where

$$\Delta = \left\{1' \ \mathrm{V}^{-1}1\right\}\left\{\alpha' \ \mathrm{V}^{-1}\alpha\right\} - \left\{1' \ \mathrm{V}^{-1}\alpha\right\}^2$$

and

$$
\begin{aligned}
Var(\hat{\mu}) &= \sigma_o^2\left(\alpha' V^{-1}\alpha\right)/\Delta \\
Var(\hat{\sigma}_o) &= \sigma_o^2\left(1'V^{-1}\right)/\Delta \\
Cov(\hat{\mu}, \hat{\sigma}_o) &= \sigma_o^2\left(1'V^{-1}\alpha\right)/\Delta
\end{aligned}
$$

It can be shown that, upon simplification,

$$
\begin{aligned}
1'\mathrm{V}^{-1}\alpha &= 1'\mathrm{V}^{-1}1 - 1/\mathrm{b_m} \\
\alpha'\mathrm{V}^{-1}\alpha &= 1'\mathrm{V}^{-1}1 - 1/\mathrm{b_m} + \mathrm{a_m}/\mathrm{b_m} \\
\alpha'\mathrm{V}^{-1}\mathrm{R} &= 1'\mathrm{V}^{-1}\mathrm{R} - \mathrm{x(m)}/\mathrm{b_m},
\end{aligned}
$$

and

$$\Delta = \Gamma(m+\gamma)\left\{\frac{1'V^{-1}1}{\mathrm{b_m}\Gamma(m)}\right\} - \frac{1}{b_m^2}.$$

Suppose γ = 0.

Then the MVLUE $\hat{\mu}$, $\hat{\sigma}_o$ of μ and σ respectively based on the lower record values x(1), x(2), …, x(m) are

$$\hat{\mu} = x(m) - v_m^*\hat{\sigma}$$
$$\hat{\sigma} = (m-1)^{-1}\sum_{i=1}^{m-1} x(i) - x(m)$$

Their corresponding variances and covariance are

$$Var(\hat{\mu}) = \sigma^2\left\{\left(\upsilon_m^*\right)^2(m-1)^{-1} + V_{mm}^*\right\}$$
$$Var(\hat{\sigma}) = \sigma^2/(m-1), \text{ and}$$

where

$$v_m^* = E(x(m)) \text{ and } \mathrm{v_m^*} = Var(x(m)).$$

Proof For γ = 0, we know (see Sect. 5.2.3).

$$\mathrm{E(x(r))} = \mu + \upsilon_r^*\sigma$$
$$\mathrm{Var(x(r))} = \sigma^2 V_{r,r'}^*,$$
$$\mathrm{Cov\,(x(r)\,x(j) = Var\;(x(j)), \ldots 1 \le j \le J,\; j = 1, 2, \ldots, \ldots}$$

with

$$\upsilon_r^* = \upsilon$$
$$\upsilon_j^* = \upsilon_{j-1}^* - (j-1)^{-1},\, j \ge 2,$$
$$V_{1,1}^* = \frac{\pi^2}{6},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$V_{j,j}^* = v_{j-1,j-1}^* - (j-1)^{-1},\, j \ge 2,$$

where υ is the Euler's constant.

Let $\Omega = V^{-1} = (V^{ij})$, then

$$
\begin{aligned}
\mathrm{V}^{\mathrm{ii}} &= \mathrm{i}^2 + (\mathrm{i}-1)^2,\ \mathrm{i} = 1, 2, \ldots, \mathrm{m}-1 \\
\mathrm{V}^{\mathrm{ij}} &= -\min\left(\mathrm{i}^2, \mathrm{j}^2\right),\ \mathrm{i} \neq \mathrm{j}, |\mathrm{i}-\mathrm{j}| = 1 \\
&= 0,\ \text{if}\ |\mathrm{i}-\mathrm{j}| > 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}^{\mathrm{mm}} &= (\mathrm{m}-1)^2 + 1/. \\
1'\,\mathrm{V}^{-1} &= (0, 0, \ldots, 1/V^*_{mm})0 \\
\alpha'\,\mathrm{V}^{-1} &= \left(1, 1, \ldots, \alpha_{\mathrm{m}}/V^*_{mm} - (\mathrm{m}-1)\right) \\
\alpha'\,\mathrm{V}^{-1}1 &= \alpha_{\mathrm{m}}/V^*_{mm}, \\
\alpha'\,\mathrm{V}^{-1}\alpha &= (\alpha_{\mathrm{m}})^2/V^*_{mm} + \mathrm{m} - 1
\end{aligned}
$$

and

$$
\Delta = (\mathrm{m}-1)/V^*_{mm}.
$$

Substituting these values in the expression of $\hat{\mu}\, and\, \hat{\alpha}$, where

$$
\begin{aligned}
\hat{\mu} &= \alpha' V^{-1}(1'\alpha - \alpha 1')V^{-1}R/\Delta \\
\hat{\sigma} &= 1'V^{-1}(1\alpha' - \alpha 1')R/\Delta.
\end{aligned}
$$

On simplification the results follow.

7.2.2 Best Linear Invariant Estimates (BLIE)

Suppose $\gamma \neq 0$ but γ is known. Then the best linear invariant (best in the sense of minimum mean squared error and invariant with respect to the location parameter μ) estimators $\tilde{\mu}\, and\, \tilde{\sigma}_o$ of μ and σ_o are respectively

$$
\begin{aligned}
\tilde{\mu} &= \hat{\mu} - c_1\hat{\sigma}_o \\
\tilde{\sigma} &= c_2\hat{\sigma}_o
\end{aligned}
$$

where

$$
c_1 = \frac{b_m\{(1'V^{-1})b_m - 1\}}{\{\Gamma(m+2\gamma)/\Gamma(m+\gamma)\{b_m(1'V^{-1}1) - 1\}}
$$

and

$$c_2 = \frac{D}{D + b_m(1'V^{-1}1)}.$$

Proof The BLIE $\tilde{\mu}\, and\, \tilde{\sigma}_o$ of μ and σ_o are:

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}_o\left\{E_{12}(1 + E_{22})^{-1}\right\},$$

and

$$\tilde{\sigma}_o = \hat{\sigma}_o(1 + E_{22})^{-1},$$

where

$$\sigma_o = \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

defines the covariance matrix of the MVLUEs of $\tilde{\mu}\, and\, \tilde{\sigma}_o$. The mean squares errors (MSE) of $\tilde{\mu}\, and\, \tilde{\sigma}_o$ are:

$$\begin{aligned} MSE(\tilde{\mu}) &= \sigma_o^2\left\{E_{11} - E_{12}^2(1 + E_{22})^{-1}\right\}, \\ MSE(\tilde{\sigma}_o) &= \sigma_o^2 E_{22}(1 + E_{22})^{-1}, \\ E(\tilde{\mu} - \mu)(\tilde{\sigma}_o - \sigma) &= \sigma_o^2 E_{12}(1 + E_{22})^{-1} \end{aligned}$$

Substituting the values of E_{11}, E_{12} and E_{22}, the results follow on simplification. Suppose γ = 0. Then the BLIE $\tilde{\mu}\, and\, \tilde{\sigma}$ of μ and σ are:

$$\begin{aligned} \tilde{\mu} &= \hat{\mu} - \upsilon_m^* \hat{\sigma}/m \\ \tilde{\sigma} &= \hat{\sigma}(m - 1)/m \\ MSE(\tilde{\mu}) &= \sigma^2\left[V_{mm}^* + \left(\upsilon_m^*\right)^2/m\right] \end{aligned}$$

and

$$MSE(\tilde{\sigma}) = \sigma^2/m,$$

where $\hat{\mu}\, and\, \hat{\sigma}$ are the MVLUE of μ and σ when γ = 0.

Proof We know

$$Var(\hat{\mu}) = \sigma^2\left\{\left(\upsilon_m^*\right)^2(m-1)^{-1} + V_{mm}^*\right\}$$
$$Var(\hat{\sigma}) = \sigma^2/(m-1), \text{ and}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = \hat{\sigma}\upsilon_m^*/(m-1),$$

Since $1 + \mathrm{E}_{22} = \frac{m}{m-1}$, on simplification, we get the results.

7.2.3 Maximum Likelihood Estimates (MLE)

Suppose $\gamma \neq 0$. But γ is known. Then we can write the log likelihood function L based on the record values x(1), x(2),…, x(m) are:

$$\log L = \sum_{k=1}^{m-1} \frac{f(x(k))}{F(x(k))} + \ln f(x(m)) \tag{7.2.3.1}$$

Differentiating (7.2.3.1) with respect to μ and equating to zero, we obtain

$$\begin{aligned}(-1+\gamma^{-1})\sum_{i=1}^{m}\gamma\sigma^{-1}(1-\gamma\sigma^{-1}(x(i)-\mu))^{-1} \\ +\gamma\sigma^{-1}\mu)(1-\gamma\sigma^{-1}(x(m)-\mu))^{-1+\gamma^{-1}.3.2} = 0\end{aligned} \tag{7.2.3.2}$$

Differentiating (7.2.3.1) with respect to σ and equating to zero, we obtain

$$\begin{aligned}-m\sigma^{-1} + (-1+\gamma^{-1})\sum_{i=1}^{m}\gamma(x(i)_i-\mu)\sigma^{-2}(1+\gamma\sigma^{-1}(x_i-\mu))^{-1} \\ -\gamma\sigma^{-2}(x(m)-\mu)(1-\gamma\sigma^{-1}(x(m)-\mu))^{-1+\gamma^{-1}} = 0.\end{aligned} \tag{7.2.3.3}$$

From (7.2.3.2) and (7.2.3.3), we obtain the maximum likelihood estimators $\widehat{\mu}_l \, and \, \widehat{\sigma}_l$ of μ and σ assuming γ is known and not equal to zero.

Exercise 7.2.3.1 Show that if $\gamma = 0$, then the maximum likelihood estimates of σ and μ are respectively

$$\hat{\sigma}_0^* = \bar{x} - x(m)$$

and

$$\hat{\mu}_0^* = x(m) + \hat{\sigma}_0^* \ln m$$

7.3 Generalized Pareto Distribution

We will consider the generalized Pareto distribution with the following pdf f(x)

$$
\begin{aligned}
\mathrm{f(x)} &= \frac{1}{\sigma}\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)\right)^{-\left(1+\beta^{-1}\right)} \\
&\quad x \geq \mu, \text{ for } \beta > 0, \\
&\quad \mu < x < \mu - \sigma/\beta, \text{ for } \beta < 0, \\
&= \frac{1}{\sigma} e^{-(x-\mu)\sigma^{-1}}, \; x \geq \mu \text{ for } \beta = 0, \; \sigma > 0 \\
&= 0, \text{otherwise},
\end{aligned} \tag{7.3.1}
$$

7.3.1 Minimum Variance Linear Unbiased Estimator of *μ* and *σ* When *β* Is Known

Theorem 7.3.1.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ *and* $\hat{\sigma}$ *of* μ *and* σ *based on the observed upper record values* X(1), X(2),…, X(m)

$$
\begin{aligned}
\hat{\mu} =& X(1)_1 - (1-\beta)^{-1}\hat{\sigma} \\
\hat{\sigma} =& (1-\beta)(\beta - D^{-1}(1-2\beta)^3 X(1) + D^{-1}\beta(1-\beta)\sum_{i=2}^{m-1}(1-2\beta)^{i+1}X(i) \\
& + D^{-1}(1-\beta)^2(! - 2\beta)^{m+1}X(m)
\end{aligned}
$$

where

$$
\mathrm{D} = \sum_{I=2}^{m}(1-2\beta)^{i+1} \text{ and } \beta < 1/2.
$$

Proof We assume GP(μ, σ, β) with β ≠ 0 and with finite variance. Let R be the m × 1 vector corresponding to X(i), i = 1, 2,…, m, then we can write

$$
\mathrm{E(R)} = \mu\,\mathrm{L} + \sigma\,\delta
$$

where

$$
\begin{aligned}
&\mathrm{R}' = (\mathrm{X}(1), \mathrm{X}(2), \ldots, \mathrm{X}(m)) \\
&\mathrm{L}' = (1, 1, \ldots, 1), \quad \delta' = (\alpha 1, \alpha 2, \ldots, \alpha_m) \\
&\alpha_i = \beta^{-1}(1-\beta)^{-i},
\end{aligned}
$$

and

$$\alpha_i = \beta^{-1}(1-\beta)^{-i},\ i = 1, 2, \ldots, m.$$

We can write

$V(R) = \sigma^2 V$, $V = (V_{i,j})$, $V_{i,j} = \beta^{-2} a_i b_j$, $1 \le i \le j \le m$ and $V_{i,j} = V_{j,i}$. The inverse V^{-1} ($=V_{i,j}$) can be expressed as

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(1-2\beta)^{i+1}(1-\beta), \quad i = 1, 2, \ldots, m-1,$$

$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, \quad i = 1, 2, \ldots, n, \quad V^{i,j} = 0, \text{ for } |i-j| > 1,$$

where $a_o = 0 = b_{n+1}$ *and* $b_o = 1 = a_{n+1}$.

On simplification, we obtain

$$V^{i,i} = (1-2\beta)^i (2 - 4\beta + 2\beta^2),\ i = 1, 2, \ldots, m-1$$

and

$$V^{m,m} = (1-2\beta)^m (1-\beta).$$

The minimum variance linear unbiased estimators (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ are respectively based on the upper record values are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}R/\Delta,$$

and

$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}R/\Delta,$$

where

$$\Delta = \left(L'\ V^{-1}L\right)(\delta' V^{-1}\delta) - (\ L'\ V^{-1}\delta)^2$$

On substituting the values for δ and V^{-1} and subsequent simplification, it can be shown that

$$\hat{\mu} = X(1) - \hat{\sigma}(1-\beta)^{-1} \quad \textit{and}$$

$$\hat{\sigma} = (1-\beta)(\beta - D^{-1}(1-2\beta)^3 X(1) r_1) + D^{-1}\beta(1-\beta)\sum_{i=2}^{m}(1-2\beta)^{i+1}X(i)_i$$

where

$$D = \sum_{i=2}^{m} (1-2\beta)^{i+1}.$$

The corresponding variances and the covariance of the estimates are

$$\begin{aligned} Var(\hat{\mu}) &= \sigma^2 \frac{T}{D} \\ Var(\hat{\sigma}) &= \sigma^2 \frac{\beta T - (1-2\beta)}{D} \\ Cov(\hat{\mu}, \hat{\sigma}) &= \sigma^2 \frac{\{(1-2\beta)^2 + \beta^2 T\}}{D} \end{aligned}$$

and

$$T = \sum_{i=2}^{m} (1-2\beta)^{i}.$$

Exercise 7.3.1.1 Find the MVLUE $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ based on n upper record values X(1), X(2),…, X(n) of the Pareto Type II (Lomax) distribution with pdf f(x) *as*

$$f(x) = \frac{v}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-(v+1)}, \quad x > \mu, \quad \sigma > 0 \text{ and } v > 0,$$

7.3.2 Best Linear Invariant Estimators (BLIE)

Theorem 7.3.2.1 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators* $\tilde{\mu}, \tilde{\sigma}$ *of* μ and σ *are respectively*

$$\begin{aligned} \tilde{\mu} &= \hat{\mu} - \frac{\beta T - (1-2\beta)}{T(1-\beta)^2}\hat{\sigma} \quad \text{and} \\ \tilde{\sigma} &= \frac{D}{T(1-\beta)^2}\hat{\sigma}, \text{ where} \\ D &= \sum_{i=2}^{m} (1-2\beta)^{i+1}, \quad T = \sum_{i=1}^{m} (1-2\beta)^{i} \end{aligned}$$

and $\hat{\mu}$ *and* $\hat{\sigma}$ *are MVLUE of* μ *and* σ.

Proof The BLIE $\tilde{\mu}$ *and* $\tilde{\sigma}$ can be written as

$$\hat{\mu} = \hat{\mu} - \frac{E_{12}}{1 + E_{22}} \hat{\sigma}$$

and

$$\tilde{\sigma} = \frac{1}{1 + E_{22}} \hat{\sigma},$$

where

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\mu}$ *and* $\tilde{\sigma}$ are

$$MSE(\tilde{\mu}) = \sigma^2 \left(E_{11} - \frac{E_{12}^2}{1 + E_{22}} \right),$$
$$MSE(\tilde{\sigma}) = \sigma^2 \left(\frac{E_{22}}{1 + E_{22}} \right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of β, T and D, we get the result.

7.3.3 Estimator of β for Known μ and σ

A Moment Estimator of β.

We have seen that for μ = 0 and σ = 1.

$E(X(m)) = \beta^{-1} \{(1 - \beta)^{-m} - 1\}$. Thus

$$E(\bar{X}) = E\{(X(1) + X(2) + \cdots + X(m)\}/m\} = \frac{1}{m\beta^2}\{(1 - \beta)^{-m} - 1\} - \frac{1}{\beta}$$
$$= \frac{X(m) - m}{m\beta}$$

Thus we can take $\tilde{\beta}$ as an estimator of β where

$$\tilde{\beta} = \frac{X(m) - m}{X(1) + X(2) + \cdots + X(m)}, \ \ for \ \ X(1) + X(2) + \cdots + X(m) \neq 0$$

7.4 Power Function Distribution

We will consider the following pdf f(x) of power function distribution

$$f(x,\alpha,\beta,\gamma) = \gamma\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}, \quad \text{for } \alpha<x<\alpha+\beta, \beta>0, \gamma>0, \qquad (7.4.1)$$
$$= 0, \text{ otherwise.}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (7.4.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (7.4.1) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha + \beta)^{\gamma}$, the Y has the uniform distribution in (0, 1).

7.4.1 The Minimum Variance Linear Unbiased Estimate of α and β When γ Is Known and γ ≠ 0

Suppose X(1), X(2),..., X(m) be the first m upper records from this distribution.

Let

$$W_k = c_k\left(X(k) - \frac{\gamma}{\gamma+1}X(k(k-1)\right), \quad k = 1,2,\ldots,m$$

with X(0) = 0, and $c_k = (\gamma+1)\left(\frac{\gamma+2}{\gamma}\right)^{k/2}, \quad k = 1,2,\ldots,m.$

Now

$$E(W_1) = \left(\frac{\gamma+2}{\gamma}\right)^{1/2}\{(\gamma+1)\alpha+\beta\},$$
$$E(W_k) = \left(\frac{\gamma+2}{\gamma}\right)^{k/2}(\alpha+\beta), \quad k = 2,3,\ldots,m.$$
$$\mathrm{Var}(W_k) = \beta^2, \quad k = 1,2,\ldots,m$$
$$\mathrm{Cov}(W_i W_j) = 0, \quad i \neq j,\ 1 \leq i, j \leq m.$$

Let $W' = (W_1, W_2, \ldots, W_n)$, then $E(W) = X\theta$, where

$$X = \begin{bmatrix} \left((\gamma+2/\gamma)^{1/2}\right)_{(\gamma+1)} & ((\gamma+2)/\gamma)^{1/2} \\ (\gamma+2)/\gamma & (\gamma+2)/\gamma \\ . & . \\ \left((\gamma+2/\gamma)^{n/2}\right) & \left((\gamma+2/\gamma)^{n/2}\right) \end{bmatrix}, \quad \theta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We can write X′X as

$$X'X = \begin{pmatrix} (\gamma+2)^2 + T & \gamma+2+T \\ \gamma+2+T & T \end{pmatrix}$$

$$\mathrm{T} = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^k$$

$$X'X^{-1} = D_o^{-1} \begin{pmatrix} T & -(\gamma+2+T \\ -(\gamma+2+T) & (\gamma+2)^2+T \end{pmatrix}$$

$$D_o = (\gamma+2)(\gamma T - \gamma - 2)$$

$$X'W = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$V_1 = (\gamma(\gamma+2))^{1/2} W_1 + V_2$$

$$V_2 = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^{k/2} W_k$$

Theorem 7.4.1.1 *The minimum variance linear unbiased estimates of* α *and* β *respectively based on* $Y_1, \ldots, Y_n$ *(assuming* γ *as known and* $\gamma \neq 0$ *are)*

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (X'X)^{-1} X'W$$

On simplification, we get

$$\hat{\alpha} = \frac{1}{D_o}\left[\left(\gamma(\gamma+2)^{1/2} W_1 - \sum_{k=1}^{n} ((\gamma+2)/\gamma)^{k/2} W_k\right)\right]$$

$$\hat{\beta} = \frac{1}{D_o}\left[-(T+\gamma+2)(\gamma(\gamma+2))^{-1/2} W_1 + (\gamma+2)(\gamma+1)\sum_{k=1}^{n} ((\gamma+2)/\gamma)^{k/2} W_k\right]$$

The variances and covariance of are given by

$$\mathrm{Var}(\hat{\alpha}) = \beta^2 T D_o^{-1},$$
$$\mathrm{Var}\left(\hat{\beta}\right) = \beta^2\left((\gamma+2)^2 + T\right) D_o^{-1}$$

and

$$\mathrm{Cov}\left(\hat{\alpha}, \hat{\beta}\right) = -\beta^2(\gamma+2+T) D_o^{-1}$$

7.4.2 Minimum Variance Linear Invariance Estimators

Theorem 7.4.2.1 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* α*) estimators* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β *are respectively*

$$\tilde{\alpha} = \hat{\alpha} - \frac{\gamma + 2 + T}{(\gamma+1)\{(\gamma+1)T - (\gamma+2)\}}\hat{\beta}$$
$$\text{and } \tilde{\beta} = \frac{D_o}{(\gamma+1)\{(\gamma+1)T - (\gamma+2)\}}\hat{\beta}$$

where

$$D_o = (\gamma+2)\{\gamma T - (\gamma+2)\}, \quad T = \sum_{i=1}^{m}\left(\frac{\gamma+2}{\gamma}\right)^{\mathrm{i}}$$

and $\hat{\alpha}$ *and* $\hat{\beta}$ *are MVLUEs of* α *and* β.

Proof The BLIE $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β can be written as

$$\tilde{\alpha} = \hat{\alpha}\frac{E_{12}}{1+E_{22}}\hat{\beta}$$

and

$$\tilde{\beta}\frac{1}{1+E_{22}}\hat{\beta},$$

where

$$\begin{pmatrix} Var(\hat{\alpha}) & Cov\left(\hat{\alpha},\hat{\beta}\right) \\ Cov\left(\hat{\alpha},\hat{\beta}\right) & Var\left(\hat{\beta}\right) \end{pmatrix} = \gamma^2\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β are

$$MSE(\tilde{\alpha}) = \gamma^2\left(E_{11} - \frac{E_{12}^2}{1+E_{22}}\right),$$
$$MSE\left(\tilde{\beta}\right) = \gamma^2\left(\frac{E_{22}}{1+E_{22}}\right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of γ, we get the results.

7.4.3 Maximum Estimator of β for Known μ and σ

Without any loss of generality we will assume μ = 0 and σ = 1. The log likelihood function can be written as

$$\log L = m\log\gamma - \sum_{i=1}^{m}\frac{1}{1-x(i)} + \gamma\log(1-x(m))$$

Differentiating with respect γ and equating to zero, we get $\breve{\gamma}$ as the maximum likelihood estimator of γ as

$$\breve{\gamma} = \frac{m}{\ln(1-x(m))} -$$

A moment Estimator of γ.

Taking α = 0 and β = 1, we get $\mathrm{E(X(i))} = \left(\frac{\gamma}{\lambda+1}\right)^{i} - 1$ and

$$\mathrm{E(X(1)+(X(2)))+\cdots+X(m)} = \gamma\left\{\left(\frac{\gamma}{\gamma+1}\right)^{m} - 1\right\} - m.$$

Thus we can a moment estimator based on the m record values X(1), X(2),.., X (m) is

$$\ddot{\lambda} = \frac{X(1)+\cdots+X(m)+m}{x(m)}.$$

7.5 Rayleigh Distribution

Let {Xn, n ≥ 1} be a sequence of i.i.d random variables from standard Rayleigh distribution with pdf

$$\mathrm{f(x)} = x e^{-x^2/2},\ \mathrm{x} > 0 \tag{7.5.1}$$

and d.f.

$$\mathrm{F(x)} = 1 - e^{-x^2/2},\ \mathrm{x} > 0. \tag{7.5.2}$$

We say X€ RH(0, 1) if the pdf of X is given by (7.5.1)

Theorem 7.5.1 *Let* $\mu_n = E(X_{U(n)})$, $V_{n,n} = Var(X_{U(n)})$ *and* $V_{m,n} = Cov(X_{U(m)}\, X_{U(n)})$, *then*

$$\mu_n = \sqrt{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}, V_{n,n} = 2\left[n - \left(\frac{\Gamma(n+1/2)^2}{\Gamma(n)}\right)\right] \text{ and}$$

$$V_{m,n} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{ for } 1 \le m \le n.$$

Proof

$$\begin{aligned}\mu_n &= \frac{1}{\Gamma(n)}\int_o^\infty x\{-\mathrm{In}(1-F(x)\}^{n-1}f(x)dx\\ &= \frac{1}{\Gamma(n)}\int_o^\infty x\left(\frac{x^2}{2}\right)^{n-1}e^{-x^2/2}xdx\\ &= \frac{1}{\Gamma(n)}\sqrt{2}\int_o^\infty u^{1/2}u^{n-1}e^{-u}du\\ &= \sqrt{2}\frac{\Gamma(n+1/2)}{\Gamma(n)}.\end{aligned}$$

Similarly it can be shown that

$$\mu_n^2 = E\left(X_{U(n)}^2\right) = 2\frac{\Gamma(n+1)}{\Gamma(n)} = 2n$$

$$\begin{aligned}\mu_{m,n} &= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_o^\infty\int_o^y xy\left(\frac{x^2}{2}\right)^{m-1}x\left(\frac{y^2}{2}-\frac{x^2}{2}\right)^{n-m-1}y\,e^{-y^2/2}dxdy\\ &= \frac{1}{\Gamma(m)\Gamma(n-m)}\frac{1}{2^{m-1}}\int_o^\infty y\left(\frac{y^2}{2}\right)^{n-m-1}y\,e^{-y^2/2}I_y dy,\end{aligned}$$

where

$$\begin{aligned}I_y &= \int_o^y \left(x^2\right)^m\left(1-\frac{x^2}{y^2}\right)^{n-m-1}dx\\ &= \frac{1}{2}y^{2m+!}B(m+1/2, n-m),\end{aligned}$$

with

$$\mathrm{B(a,b)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

On simplification we get

$$V_{\mathrm{n,n}} = 2\left[n - \left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{and}$$

$$\mathrm{V_{m,n}} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{for } 1 \leq \mathrm{m} \leq \mathrm{n}.$$

$$= \left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n)}{\Gamma(n+1/2)}\right] V_{n,n}$$

We will consider the estimation of μ and σ based on the m upper record values X(1), X(2),…, X(m) of the two parameter Rayleigh distribution with the pdf

$$\mathrm{f(x, \mu, \sigma)} = \frac{x-\mu}{\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}}, \ \mu < x < \infty, \sigma > 0$$

7.5.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 7.5.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ *and* $\hat{\sigma}$ *of* μ *and* σ *based on the* X(1), X(2),…, X(m) *are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{ and } \hat{\sigma} \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{3}{2}\frac{\alpha_m b_m}{D}, \ c_i = \frac{1}{2i}\frac{\alpha_m b_m}{D}, \ i = 2, 3, \ldots, m-1,$$

$$c_m = 1 - \frac{\alpha_m b_m}{2D}\left[3 + \sum_{i=2}^{m-1}\frac{1}{i}\right], \ \ d_1 = -\frac{3}{2}\frac{b_m}{D}, \ \ d_i = -\frac{1}{2i}\frac{b_m}{D}, \ i = 2, 3, \ldots, m-1,$$

$$d_m = \frac{1}{2}\frac{b_m}{D}\left\{3 + \sum_{i=2}^{m-1}\frac{1}{i}\right\},$$

where

$$\mathrm{D} = \alpha_{\mathrm{m}} \mathrm{b}_{\mathrm{m}} \mathrm{T} - 1,\ \mathrm{T} = \left[\frac{3}{2} + \sum_{i=2}^{m-1} \frac{1}{2i} + (2m-1)\left(\frac{b_{m-1}}{b_m} - 1\right)\right]$$
$$\alpha_{\mathrm{k}} = \sqrt{2}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)} = \mathrm{a}_{\mathrm{k}} \text{ and } b_k = \sqrt{2}\left\{\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)}\right\},$$
$$\mathrm{k} = 1, 2, \ldots, \mathrm{m}.$$

Proof Let R be the m × 1 vector corresponding to X(k), ki = 1, 2,…, m, then we have

$$\mathrm{E(R)} = \mu \mathrm{L} + \sigma \delta$$

where

$$\mathrm{R}' = (\mathrm{X}(1), \mathrm{X}(2), \ldots, \mathrm{X}(\mathrm{m}))$$
$$\mathrm{L}' = (1, 1, \ldots, 1),\quad \delta' = (\alpha_1, \alpha_2, \ldots, \alpha_{\mathrm{m}})$$
$$\alpha_{\mathrm{i}} = \sqrt{2}\frac{\Gamma(i+1/2)}{\Gamma(i)},\quad \mathrm{i} = 1, 2, \ldots, \mathrm{m}.$$

We can write

$$\mathrm{V(R)} = \sigma^2 \mathrm{V},\ \mathrm{V} = (\mathrm{V}_{\mathrm{i,j}}),\ \mathrm{V}_{\mathrm{i,j}} = \mathrm{a}_{\mathrm{i}}\, \mathrm{b}_{\mathrm{j}},\ 1 \le \mathrm{i} \le \mathrm{j} \le \mathrm{m} \text{ and } \mathrm{V}_{\mathrm{i,j}} = \mathrm{V}_{\mathrm{j,i}}.$$

The inverse V^{-1} (=$\mathrm{V}_{\mathrm{i,j}}$) can be expressed as

$$\mathrm{V}^{\mathrm{i+1,i}} = \mathrm{V}^{\mathrm{i,i+1}} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(2\mathrm{i}+1),\ \mathrm{i} = 1, 2, \ldots, \mathrm{m}-1,$$
$$\mathrm{V}^{\mathrm{i,i}} = -\frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})},\ \mathrm{i} = 1, 2, \ldots, n,$$
$$\mathrm{V}^{\mathrm{i,j}} = 0,\ \text{for} |\mathrm{i}-\mathrm{j}| > 1,$$

where $a_o = 0 = b_{n+1}$ *and* $b_o = 1 = a_{n+1}$.

On simplification, we obtain

$$\mathrm{V}^{\mathrm{i,i}} = \frac{8i^2+1}{2i},\ \mathrm{i} = 1, 2, \ldots, \mathrm{m}-1,$$

and

$$\mathrm{V}^{\mathrm{m,m}} = (2\mathrm{m}-1)\frac{b_{m-1}}{b_m}.$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}$, $\hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = -L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = \left(L'V^{-1}L\right)\left(\delta' V^{-1}\delta\right) - \left(L'V^{-1}\delta\right)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 L'V^{-1}\delta/\Delta,$$
$$Var(\hat{\sigma}) = \sigma^2 L'V^{-1}L/\Delta$$
$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2 L'V^{-1}\delta/\Delta.$$

On simplification, we obtain the MVLUE $\hat{\mu}, \hat{\sigma}$ of μ and σ. The corresponding variances and the covariance of the estimates are

$$\begin{aligned} Var(\hat{\mu}) &= \sigma^2\frac{\alpha_n b_n}{D} \\ Var(\hat{\sigma}) &= \sigma^2\frac{b_n^2 T}{D} \\ Cov(\hat{\mu},\hat{\sigma}) &= -\sigma^2\frac{b_n}{D}. \end{aligned} \tag{7.5.1.1}$$

7.5.2 Best Linear Invariant Estimators (BLIEs) of μ and σ

Theorem 7.5.2.1 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators (BLIEs)* $\hat{\mu}$ $\tilde{\sigma}$ *of* μ *and* σ *are*

$$\tilde{\mu} = \tilde{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

where $\hat{\mu}$ *and* $\hat{\sigma}$ *are MVLUEs of* μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of these estimators are

$$\mathrm{MSE}(\tilde{\mu}) = \sigma^2 \left(\mathrm{E}_{11} - \mathrm{E}_{12}^2 (!1 + \mathrm{E}_{22})^{-1} \right)$$

and

$$MSE(\tilde{\sigma}) = \sigma^2 E_{22} (1 + E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} *from* (7.3.4)*, we obtain*

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma} \left(\frac{b_m}{D + b_m^2 T} \right)$$

and

$$\tilde{\sigma} = \hat{\sigma} \frac{D}{D + b_m^2 T}.$$

Exercise 7.5.2.1 Show that if μ = 0, then MVLUE of **σ** based on upper records X (1), X(2),…, X(m) is

$$\hat{\sigma} = cX(m)$$

where

$$c = \frac{\sigma}{E(X(m))} = \frac{1}{\sqrt{2}} \frac{\Gamma(m)}{\Gamma\left(m + \frac{1}{\gamma}\right)}$$

Exercise 7.5.2.2 Show that the minimum variance linear unbiased predictor $\hat{X}(s)$ of X(s) of X(s) based on X(1), X(2),…, X(m), s > m $\hat{X}(s) = \hat{\mu} + \alpha_{s\hat{\sigma}}$ where $\hat{\mu}$ and σ are the MVLUEs of μ and σ, Respectively.

7.6 Uniform Distribution

7.6.1 Two Parameter Uniform Distribution

Let $\{X_n, n > 1\}$ be a sequence of i.i.d. random variables from a uniform distribution with the following pdf

$$\mathrm{f}(\mathrm{x}) = \frac{1}{\theta_1 - \theta_2}, \ \theta_1 < x < \theta_2 \tag{7.6.1.1}$$

and d.f.

$$\mathrm{F}(\mathrm{x}) = \frac{x - \theta_1}{\theta_2 - \theta_1}, \ \theta_1 < x < \theta_2. \tag{7.6.1.2}$$

We will say $X \in U(\theta_1, \theta_2)$ if the pdf of X is as given in (7.6.1.1).

The pdf $f_n(x)$ of X(n) can be written as

$$\mathrm{f_n}(\mathrm{x}) = \frac{1}{\Gamma(n)} \frac{1}{\theta_2 + \theta_1} \left\{ \mathrm{In} \frac{\theta_2 - \theta_1}{\theta_2 - x} \right\}^{n-1}, \ \theta_1 < x < \theta_2 \tag{7.6.1.3}$$

$$\begin{aligned} \mathrm{E(X(m))} &= 2^{-n}\theta_1 + (1 - 2^{-n})\theta_2 \\ \mathrm{Var((X(m))} &= (3^{-n} - 4^{-n})(\theta_2 - \theta_1)^2. \end{aligned} \tag{7.6.1.4}$$

The joint pdf of X(m) and X(n) is

$$\begin{aligned} \mathrm{f_{m,n}}(\mathrm{x}, \mathrm{y}) = {} & \frac{1}{\Gamma(m)} \frac{1}{\Gamma(n-m)} \frac{1}{\theta_2 - \theta_1} \frac{1}{\theta_2 - x} \left\{ \mathrm{In} \frac{\theta_2 - \theta_1}{\theta_2 - x} \right\}^{m-1} \left\{ \mathrm{In} \frac{\theta_2 - \theta_1}{\theta_2 - y} \right\}^{n-m-1}, \\ & \text{for } \theta_1 < x < y < \theta_2 \end{aligned} \tag{7.6.1.5}$$

We have t

$$\begin{aligned} & \mathrm{E(X(n) \| X(m) = y_m)} = 2^{m-n} y_m + (1 - 2^{m-n})\theta_2 \\ & \text{and} \\ & \mathrm{Cov(X(m)X(n))} = 2^{m-n} \mathrm{Var}\left(\mathrm{XU_{(m)}}\right). \end{aligned} \tag{7.6.1.6}$$

7.6.2 Minimum Variance Linear Unbiased Estimate of θ_1 and θ_2

We will consider here the estimation of θ_1 and θ_2 based on m upper record values X(1), X(2),.., X(m).

Consider the following transformation

$$\begin{aligned} W_1 &= X_{U(1)} \\ W_i &= (3)^{(i-1)/2}\left(X_{U(i)} - \frac{1}{2}X_{U(i-1)}\right), \quad i = 2, 3, \ldots, m \end{aligned} \tag{7.6.2.1}$$

It can easily be verified that

$$\begin{aligned} \mathrm{E}(\mathrm{W}_1) &= \frac{\theta_1 + \theta_2}{2}, \\ W_k &= \frac{\theta_2}{2}, k = 2, 3, \ldots, m. \\ \mathrm{Var}(\mathrm{W}_\mathrm{i}) &= \frac{\sigma^2}{12}, \; i = 1, 2, \ldots, m \end{aligned} \tag{7.6.2.2}$$

and

$$\mathrm{Cov}\left(\mathrm{W}_\mathrm{i}, \mathrm{W}_\mathrm{j}\right) = 0, \mathrm{i} \neq \mathrm{j}.$$

Let W′ = (W1, W2,…., W_m), then E(W) = H θ, where

$$\mathrm{H} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2^i}(3)^{\frac{1}{2}} \\ & \cdots \\ 0 & \frac{1}{2^n}(3)^{(n-1)/2} \end{bmatrix}, \; \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

We have

$$(H'H)^{-1} = \frac{32}{3(3^{m-1} - !)} \begin{bmatrix} \frac{3^m - 1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Thus, expressing W’s in terms of the X(1), X(2),.., X(m) we obtain

$$\hat{\theta}_1 = 2X(1) - \hat{\theta}_2$$

and

$$\hat{\theta}_1 = \frac{4}{3(3^{m-1}-1)}(3^{m-1}X(m) - \frac{3^{m-2}}{2}x(m-1) - \cdots - \frac{3}{2}X(2) - \frac{3}{2}X(1)).$$

The variances covariance of these estimates are

$$Var\left(\hat{\theta}\right) = \frac{1}{9}\frac{3^m - 1}{3^{m-1}-1}(\theta_2 - \theta_1)^2,$$
$$Var\left(\hat{\theta}_2\right) = \frac{2}{9}\frac{1}{3^{n-1}-1}(\theta_2 - \theta_1)^2$$

and

$$\text{Cov}\left(\hat{\theta}_2 - \hat{\theta}_1\right) = \frac{2}{9}\frac{1}{3^{m-1}-1}(\theta_2 - \theta_1)^2.$$

The generalized variance $\hat{\Sigma}\left(\hat{\Sigma} = \text{var}\theta_1 \cdot \text{var}\theta_2 - (\text{cov}(\theta_1\theta_2))^2\right)$ is

$$\frac{2}{27}\cdot\frac{1}{3^{n-1}-1}(\theta_2 - \theta_1)^2.$$

Exercise 7.6.2.1 Suppose X(1), X(2),…, X(m) are m upper record values from a one parameter uniform distribution with pdf $f_U(u)$ as $f_U(u) = \frac{1}{\theta}, 0 < x < \theta, \theta > 0$. Then the MVLUE $\hat{\theta}$ of θ is

$$\hat{\theta} = \frac{2}{3^n - 1}(2.3^{n-1}X(n) - 3^{n-2}X(n-1) - 3^{n-3}X(n-2) - \cdots - X(1))$$

Proof Let X″ = (X(1), X(2),…, X(m)); We have E(X′) = $\delta\theta$ and

$$\text{Var(X)} = \theta_2 V.\ \text{V} + (\text{V}_{\text{ij}})$$

where $\delta' = (\delta_1, \delta_2, \ldots, \delta_m),\quad \delta_i = 1 - \frac{1}{2^i},\quad i = 1, 2, \ldots m$

$$\text{V}_{\text{ii}} = \frac{1}{3^i} - \frac{1}{4^i},\ i = 1, 2, \ldots, m \text{ and}$$

Let $V = \left(\text{V}_{\text{ij}}\right)$,, then,

$$V_{ii} = \frac{1}{3^i} - \frac{1}{4^i},\quad \text{i} = 1, 2, \ldots, \text{m}$$

$$V_{ij} = 2^{i-j}\left(\frac{1}{3^i} - \frac{1}{4}\right),\quad i < j < m.$$

Let $V^{-1} - (V^{ij})$, then $V^{ii} = 73^{i}$, i = 1, 2,.., m − 1.

$V^{mm} = 4.3^{\ m}$, $V^{ii+1} = -2,\ 3^{i+1} = V^{i+1i}$, and $V^{ij} = 0$ for |i − j|.
The MVLUE $\hat{\sigma}$ of σ **is**

$$\begin{aligned}\hat{\sigma} &= \frac{\delta' V^{-1} X}{\delta' V^{-1} \delta} \\ &= \frac{2}{3^m - 1}\left(2.3^{m-1} X(m) - 3^{m-2} X(m-1), \ldots - -X(1)\right) \\ Var(\hat{\sigma}) &= \frac{2\sigma^2}{3(3^n - 1)}.\end{aligned}$$

7.6.3 One Parameter Uniform Distribution

Suppose γ = 1 and α = 0, i.e. when X is distributed uniformly in the interval (0, β), We have in this case the pdf $f_n(x)$ of X(n) as

$$f_n(x) = \frac{1}{\Gamma(n)}\left[\text{In}\frac{\beta}{x}\right]^{n-1},\ 0 < x < \beta. \tag{7.6.3.1}$$

Using (7.6.3.1), we obtain

$$\begin{aligned}\mathrm{E(X(n))} &= (1 - 2^{-n})\beta. \\ \mathrm{Var(X(n))} &= (3^{-n} - 4^{-n})\beta^2\end{aligned}$$

The joint pdf of X(m) and X(n), n > m is

$$f_{m,n}(x, y) = \frac{1}{\Gamma(m)}\frac{1}{\Gamma(n-m)}\frac{1}{\beta}\frac{1}{\beta - x}\left[\ln\frac{\beta}{\beta - x}\right]^{m-1}\left[\ln\frac{\beta}{\beta - y}\right]^{n-m-1}, \tag{7.6.3.2}$$
$$\mathrm{n > m > 0,\ 0 < x < y < \beta.}$$

It follows from (7.6.6) that

$$\mathrm{E(X(n)|X(m) = x_m) = 2^{m-n} x_m + (1 - 2^{m-n})\beta.}$$

and

$$\mathrm{Cov(X(n)X(m)) = 2^{m-n} Var(X(m)),\ \ m < n,\ 1 \le m \le n}$$

The correlation coefficient $\rho_{m,n}$ of X(m) and X(n) − s

$$\rho_{m,n} = \left(\left(\frac{4}{3}\right)^m - 1\right)^{\frac{1}{2}} \left(\left(\frac{4}{3}\right)^n - 1\right)^{\frac{1}{2}},\ m < n$$

7.6.3.1 Minimum Variance Unbiased Estimator of β

We will consider here n upper records X(1), X(2),…, X(n).

Using the following transformation

$$\begin{aligned} \mathrm{W}_1 &= \mathrm{X}(1) \\ \mathrm{W}_i &= 3^{\frac{i-1}{2}}(X(i) - \frac{1}{2}X^* i - 1)),\ i = 2, \ldots, n \end{aligned}$$

We have

$$\begin{aligned} \mathrm{E}(\mathrm{W_i}) &= (1/2)(3)^{(\mathrm{i}-1)/2}\beta \\ \mathrm{Var}(\mathrm{W_i}) &= \frac{\beta^2}{12}, \\ \mathrm{Cov}\left(\mathrm{W}_i, \mathrm{W}_j\right) &= 0,\ \mathrm{i} \neq \mathrm{j}, \mathrm{i}, \mathrm{j} = 1, 2, \ldots, \mathrm{n}. \end{aligned}$$

Let

$$\mathrm{X}' = \left(\frac{1}{2}, \frac{1}{2}(3)^{1/2}, \frac{1}{2}(3), \ldots, \frac{1}{2}(3)^{n-1}\right)$$

and

$$\mathrm{W}' = (\mathrm{W}_1, \mathrm{W}_2, \ldots, \mathrm{Wn}),$$

then minimum variance linear unbiased estimator $\hat{\beta}$ of β based on the first n record values is

$$\begin{aligned} \hat{\beta} &= (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{W} \\ &= \frac{4}{3^n - 1}\left(\sum_{i=1}^{n} (3)^{(i-1)/2} W_i\right) \\ &= \frac{4}{3^n - 1}\left(3^{n-1}X(n) - \frac{3^{n-2}}{2}X(n-1) - \frac{3^{n-3}}{2}X(n-2) - \cdots - \frac{1}{2}X(1)\right) \end{aligned}$$

Since $\mathrm{X}'\mathrm{X} = \frac{3^n - 1}{8}$ and $\mathrm{Var}(W_i) = \frac{\beta^2}{12}$, we have

$$Var\left(\hat{\beta}\right) = (X'X)^{-1}\frac{\beta^2}{12}$$
$$= \frac{2\beta^2}{3(3^n - 1)}$$

7.6.3.2 Minimum Mean Square Estimate of β

If we drop the condition of unbiasedness, then the estimator $\tilde{\beta}$, where

$$\tilde{\beta} = \frac{3(3^n) - 1}{3^{n+1} - 1}\hat{\beta}$$

has minimum mean squared error.

$$\text{Bias of } \tilde{\beta} = \mathrm{E}(\tilde{\beta}) - \beta = -\frac{2}{3^{n+1} - 1}\beta$$

and

$$\mathrm{MSE}(\hat{\beta}) = \frac{2\beta^2}{3^{n+1} - 1}$$

Exercise 7.6.3.1 Find the maximum likelihood estimate of β.

7.6.3.3 Prediction of Record Values

Writing

$$\mathrm{Y}_{n+s} = Y_{n+s}\frac{1}{2}Y_{n+s-1} + \frac{1}{2}\left(Y_{n+s-1}\frac{1}{2}Y_{n+s-2}\right) + \cdots + \frac{1}{2}Y_{n+s-2}\left(Y_2 - \frac{1}{2}Y_1\right)$$
$$+ \frac{1}{2^{n+s-1}}Y_1,$$

it can be shown that

$$\mathrm{Cov}(\ \mathrm{Y}_{\mathrm{n+s}}, \mathrm{W}_{\mathrm{i}}) =, \ \mathrm{i} = 1, 2, \ldots, \mathrm{n}.$$

It can be shown that the best linear unbiased predictor (BLUP) of Y_{n+s} is $\hat{Y}_{n+s}$, where

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\hat{\beta} + c'V^{-1}(W - X\hat{\beta})),$$

where

$$c' = (c_1, c_2, \ldots, c_n), V^{-1} = (X'X)^{-1} \text{ and } c_i \mathrm{Var}(W_i) = \mathrm{Cov}(Y_{n+s}, W_i),\ s \geq 1.$$

Thus

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\hat{\beta} + \frac{8}{3^n - 1}\left[\sum_{i=1}^{n} \frac{1}{2^{n+s-i}} \cdot \frac{W}{3^{(i-1)/2}} - \frac{\hat{\beta}}{2^s}\left(1 - \frac{1}{2^n}\right)\right]$$

The best linear (unrestricted) least square predictor of Y_{n+s} is $\tilde{Y}_{n+s}$, where

$$\begin{aligned}\overleftrightarrow{Y}_{r+s} &= E(Y_{n+s}|Y_1, Y_2, \ldots, Y_n)\\ &= \frac{Y_n}{2^s} + \left(1 + \frac{1}{2^s}\right)\beta,\end{aligned}$$

Substituting $\hat{\beta}$ for β, we get the best linear least squares predictor as

$$\frac{Y_n}{2^s} + \left(1 - \frac{1}{2^s}\right) \cdot \frac{4}{3^n - 1}\left(3^{n-1}y_n - \frac{1}{2}(3)^{n-2}y_{n-1} - \cdots - \frac{1}{2}y_1\right).$$

7.7 Weibull Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Weibull distribution with pdf

$$f(x) = x^{\gamma-1}e^{-x^{\gamma}/\gamma},\ x > 0, \gamma > 0, \tag{7.7.1}$$

and d.f.

$$F(x) = 1 - e^{\frac{1}{\gamma}x^{\gamma}},\ x > 0,\ \gamma > 0, \tag{7.7.2}$$

Let $\mu_n = E(X(n)$, $Vn, n = \mathrm{Var}(X(n))$ and $V_{mn} = \mathrm{Cov}(X(m)X(n))$, $m < n$, then

$$\mu_n = \gamma^{\frac{1}{\gamma}} \frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}, \quad V_{n,n} == \gamma^{\frac{2}{\gamma}} \left\{ \frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma(n)} - \left(\frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)} \right)^2 \right\}.$$

and

$$V_{m,n} = \frac{\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma(m)} \cdot \gamma^{2/\gamma} \left\{ \frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma(n+\frac{1}{\gamma})} - \frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)} \right\}, \text{ for } 1 < m < n.$$

We will consider the following pdf $f(x, \mu, \sigma)$, for Weibull distribution,

$$f(x, \mu, \sigma) = \frac{(x-\mu)^{\gamma-1}}{\sigma^\gamma} e^{-\frac{1}{\gamma}\left(\frac{x-\mu}{\sigma}\right)^\gamma} - \infty < \mu < x < \infty, \sigma > 0.,$$

7.7.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 7.7.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ *and* $\hat{\sigma}$ *of* μ *and* σ *based on the record values* X(1), X(2),.., X(n) *are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{ and } \hat{\sigma} = \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{\alpha_m b_m}{D} \frac{(\gamma+1)\gamma^{2/\gamma}}{\Gamma\left(1+\frac{1}{\gamma}\right)}, \; c_i = \frac{\alpha_m b_m}{D} \gamma^{-2/\gamma} (\gamma - !) \frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, \; i = 2, 3, \ldots, m-1,$$

$$c_m = 1 - \frac{\alpha_m b_m}{D} \gamma^{-2/\gamma} \left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1) \sum_{i=2}^{m-1} \frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)} \right],$$

$$d_1 = -\frac{b_m(\gamma+1)\gamma^{-2/\gamma}}{D},$$

$$d_i = -\frac{b_m}{D}(\gamma-1)\gamma^{-2/\gamma}, \frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, \; i = 2, 3, \ldots, m-1,$$

$$d_m = \frac{b_m}{D} \gamma^{-2/\gamma} \left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1) \sum_{i=2}^{m-1} \frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)} \right],$$

where

$$D = \alpha_m b_m T - 1,$$

$$T = \gamma^{-2/\gamma}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)} + \frac{\Gamma(m)}{\Gamma\left(m+\frac{2}{\gamma}\right)}(m\gamma-\gamma+1)(m\gamma-\gamma+2)\left(\frac{b_{m-1}}{b_m}-1\right)\right]$$

$$\alpha_m = \gamma^{1/\gamma}\frac{\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma(m)} \text{ and } b_m = \gamma^{1/\gamma}\left\{\frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma\left(n+\frac{2}{\gamma}\right)} - \frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}\right\}.$$

We can write

$$V(R) = \sigma^2 V,\ V = (V_{i,j}),\ V_{i,j} = a_i\, b_j, 1 \le i \le j \le m \text{ and } V_{i,j} = V_{j,i}.$$

The inverse V^{-1} (= $V^{i,j}$) *can be expressed as*

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}}$$
$$= -\gamma^{-2/\gamma} i\gamma(i\gamma+1)\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)},\ i = 1,2,\ldots,m-1,$$
$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i-1})},\ i = 1,2,\ldots,n, V^{i,j}=0, \text{ for } |i-j| > 1,$$

where $a_o = 0 = b_{n+1}$ and $b_o = 1 = a_{n+1}$.

On simplification, we obtain

$$V^{i,i} = \gamma^{-2/\gamma}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\left[\gamma^2\left(2i^2-2i+1\right)+\gamma(4i-2)+1\right],\ i = 1,2,\ldots,m-1$$
$$V^{m,m} = \gamma^{-2/\gamma}\frac{\Gamma(n)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\frac{b_{n-1}}{b_n}[(n\gamma-\gamma+1)(n\gamma-\gamma+2)].$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ *of* μ *and* σ *respectively are*

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = -L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = \left(L'V^{-1}L\right)\left(\delta'V^{-1}\delta\right) - \left(L'V^{-1}\delta\right)^2,$$
$$\mathrm{X}' = (\ \mathrm{X}(1), \mathrm{X}(2), \ldots \mathrm{X}(\mathrm{n})).$$

and

$$Var(\hat{\mu}) = \sigma^2 L'V^{-1}\delta/\Delta$$
$$Var(\hat{\sigma}) = \sigma^2 L'V^{-1}L/\Delta$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 L'V^{-1}\delta/\Delta.$$

On simplification, we obtain the MVLUEs $\hat{\mu}, \hat{\sigma}$ *of* μ *and* σ*. The corresponding variances and the covariance of the estimates are*

$$\begin{aligned} Var(\hat{\mu}) &= \sigma^2 \frac{\alpha_n b_n}{D} \\ Var(\hat{\sigma}) &= \sigma^2 \frac{b_n^2 T}{D} \\ Cov(\hat{\mu}, \hat{\sigma}) &= -\sigma^2 \frac{b_n}{D} \end{aligned} \tag{7.7.4}$$

Best Linear Invariant Estimators (BLIEs) of μ *and* σ.

Theorem 7.7.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators (BLIEs)* $\tilde{\mu}\ \tilde{\sigma}$ *of* μ *and* σ *are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1 + E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

where $\hat{\mu}$ *and* $\hat{\sigma}$ *are MVLUEs of* μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$\mathrm{MSE}(\tilde{\mu}) = \sigma^2\left(\mathrm{E}_{11} - \mathrm{E}_{12}^2(1 + \mathrm{E}_{22})^{-1}\right)$$

and

$$MSE(\tilde{\sigma}) = \sigma^2\left(\mathrm{E}_{22}(1 + \mathrm{E}_{22})^{-1}\right).$$

Using the values of E_{11}, E_{12} and E_{22} *from* (7.7.4), *we obtain*

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma}\left(\frac{b_m}{D + b_m^2 T}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}\left(\frac{D}{D + b_m^2 T}\right).$$

Exercise 7.7.2.1 Show that if $\mu = 0$, then MVLUE estimator σ based on the record values X(1), X(2),…, X(m) for known v is

$$\bar{\sigma} = c_0 X(m),$$

7.8 Exercises (solutions)

Exercise 7.1.1.1 (*hint*) X(m) is the sufficient statistics and E(X(m) = $m\sigma$.

Exercise 7.1.3.1 (*solution*) The log likelihood function logL is

$$\ln L = m \ln \sigma - \frac{x(m)}{\sigma}.$$

Thus $\sigma^*_{ml} = \frac{\mathrm{x(m)}}{\mathrm{m}}$.

Exercise 7.2.3.1 (*hint*)

$$\log nL = -m \log \sigma - \sum_{i=1}^{m} \frac{x(i) - \mu}{\sigma} - \exp(-\sigma^{-1}(x(m) - \mu)$$

The solutions of the equations as given in (7.2.3.4) will give the MLE of μ and σ as

$$\hat{\sigma}^*_0 = \bar{x} - x(m)$$

and

$$\hat{\mu}_0^* = x(m) + \hat{\sigma}_0^* \ln m$$

where

$$\bar{x} = (x(1) + x(2) + \cdots + x(m))/m$$

Exercise 7.3.1.1 (*solution*) We have

$$\mathrm{E(x(1))} = \mu - \sigma + \sigma\left(\frac{v}{v-1}\right)^n, \mathrm{n} > 1,$$
$$\mathrm{Var(x(1))} = \sigma^2\left\{\left(\frac{v}{v-2}\right)^n - \left(\frac{v}{v-2}\right)^{2n}\right\}, \ \mathrm{n} > 2$$

From the joint pdf of $X_{U(m)}$ and $X_{U(n)}$, it can be derived that the

$$\mathrm{Cov}(X_{U(m)}, X_{U(n)}) = \left(\frac{v}{v-1}\right)^{n-m} Var(X_{U(m)}).$$

Let V_{ij} =Cov(X(i)X(j)), V = (V_{ij})and $\Omega = (V)^{-1} = (V^{i,j})$, i,j = 1,2,. . ., n, then

$$\begin{aligned}
V^{i,i} &= (2v^2 - 4v + 1)c^i, i = 1, 2, \ ..., \ n - 1 \\
V^{i+1,i} &= -(v^2 - 3v + 2)c^i = V^{i,i+1}, i = 1, 2, \ldots, n - 1 \\
V^{n,n} &= (v^2 - 2v + 1)c^n \\
V_{i,j} &= 0, \ |i - j| > 1 \\
\text{and } c &= \frac{v}{v-2}.
\end{aligned}$$

Using Lloyd's (1952) method we obtain the minimum variance linear unbiased estimator (MVLUE), as

$$\hat{\mu} = \sum_{j=1}^{n} p_j X(i)$$

and

$$\hat{\sigma} = \sum_{j=1}^{n} q_j X(i),$$

where

$$
\begin{aligned}
p_1 &= (v(v-1)T_1 - (v-2))/(v(vT_1 - v + 2)) \\
p_j &= -(\frac{v-2}{v})^j/(vT_1 - v + 2),\ j = 2, \ldots, n-1 \\
p_n &= \left\{ (v-1)T_1 - \frac{(v-1)^2(v-2)}{v} \right\} \Big/ (vT_1 - v + 2) \\
q_1 &= p_1 \frac{(v-2)^2}{vT_1 - v + 2} \\
q_j &= (v-1)p_j,\ j = 2, \ldots, n
\end{aligned}
$$

and

$$T_1 = \sum_{j=1}^{n} \left(\frac{v-2}{v} \right)^j.$$

Further we have

$$
\begin{aligned}
Var(\hat{\mu}) &= \sigma^2 \frac{T_1}{T_2} \\
Var(\hat{\sigma}) &= \sigma^2 \frac{T_1 + (v-2)^2}{T_2} \\
Cov(\hat{\mu}, \hat{\sigma}) &= -\sigma^2 \frac{T_1 + v + 2}{T_2}
\end{aligned}
$$

where

$$\mathrm{T}_2 = (v-2)(v\mathrm{T}_1 - v + 2)$$

Exercise 7.5.2.1 (*hint*) If **if** $\mu = 0$, then X(m) is the sufficient statistics for $\boldsymbol{\sigma}$ **and**

$$E(X(m)) = \frac{\sqrt{2}\Gamma\left(m + \frac{1}{\gamma}\right)}{\Gamma(m)}$$

Exercise 7.6.1.1 (*solution*) If **if** $\mu = 0$, X(m) is the sufficient statistics for $\boldsymbol{\sigma}$ **and**

$$E(X(m)) = \frac{\sqrt{2}\Gamma(m + \frac{1}{\gamma})}{\Gamma(m)}$$

Exercise 7.6.4.1 (*solution*) The log likelihood function logL is

$$\log L = \log \beta - \sum_{i=1}^{N} \log(\beta - X(i))$$

Thus the maximum likelihood estimate $\hat{\beta}_{ml}$ of β is

$$\hat{\beta}_{ml} = X(n)$$
$$E(\hat{\beta}_{ml}) = (1 - 2^{-n})\beta$$

and

$$Var(\hat{\beta}_{ml}) = \left(\frac{1}{3^n} - \frac{1}{4^n}\right)\beta^2$$

Exercise 7.7.1.1 Solution (*hint*) X(m) is a sufficient statistic for σ, and $E(X(m)) = \frac{\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma(m)}$ /Hence $c_0 = \frac{\Gamma(m)}{\Gamma\left(m+\frac{1}{\gamma}\right)}$

Chapter 8
Characterizations of Distributions

8.1 Characterizations Using Conditional Expectations

Suppose $\{X_i, i = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables with d.f. F(x) and pdf f(x). We assume $E(X_i)$ exists. Let X(n), n ≥ 1 be the corresponding upper records. We have the following theorem for the determine F(x) based on the conditional expectation.

Theorem 8.1.1 *The condition*

$$E(\psi(X(k+s)|X(k)=z)=g(z)$$

where k, s ≥ 1 *and* ψ(x) *is a continuous function, determines the distribution* F(x) *uniquely*

Proof

$$E(\psi(X(k+s)|X(k)=z)=\int\limits_z^\infty \frac{\psi(x)(R(x)-R(z))^{s-1}}{\overline{F}(z)}f(x)dx \tag{8.1.1}$$

where $R(x) = -\ln \bar{F}(x)$.

Case s = 1

Using the Eq. (8.1.1), we obtain

$$\int\limits_z^\infty \psi(x)f(x)dx = g(z)\overline{F}(z) \tag{8.1.2}$$

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_8

Differentiating both sides of (8.1.2) with respect to z and simplifying, we obtain

$$r(z) = \frac{f(z)}{\overline{F}(z)} = \frac{g'(z)}{g(z) - \psi(z)} \tag{8.1.3}$$

where r(z) is the failure rate of the function. Hence the result.

If $\psi(x) = x$ and $\mathrm{g(x)} = a\mathrm{x} + \mathrm{b},\ a,\ \mathrm{b} \geq 0$, then

$$r(x) = \frac{a}{(a-1)x + b} \tag{8.1.4}$$

If $a \neq 1$, then $F(x) - 1 - ((a-1)x + b)^{-\frac{a}{a-1}}$, which is the power function distribution for $a < 1$ and the Pareto distribution with >1. For $a = 1$, (8.1.4) will give exponential distribution. Nagaraja (1977) gave the following characterization theorem.

Theorem 8.1.2 *Let* F *be a continuous cumulative distribution function. If, for some constants* a *and* b, $\mathrm{E(X(n)|X(n-1) = x) = ax + b}$, *then except for a change of location and scale*,

(i) $F(x) = 1-(-x)^{\theta}$, $x < 0$, if $0 < a < 1$
(ii) $F(x) = 1-e^{-x}$, $x \geq 0$, if $a = 1$
(iii) $F(x) = 1-x^{\theta}$, $x > 1$ if $a > 1$,
where $\theta = a/(1-a)$. *Here* $a > 0$.

Proof of Theorem 8.1.1 for s = 2
In this case, we obtain

$$\int_z^z \psi(x)(R(x) - R(z)f(x)dx = g(z)\overline{F}(z) \tag{8.1.5}$$

Differentiating both sides of the equation with respect to z, e obtain

$$-\int_z^\infty \psi(x)f(z)dx = g'(z)\frac{(\overline{F}(z))^2}{f(z)} - g(z)\overline{F}(z) \tag{8.1.6}$$

Differentiating both sides of (8.1.6) with respect to z and using the relation $\frac{f'(z)}{f(z)} = \frac{r'(z)}{r(z)} - r(z)$ we obtain on simplification

$$g'(z)\frac{r'(z)}{r(z)} + 2g'(z)r(z) = g''(z) + (r(z))^2(g(z) - \psi(z)) \tag{8.1.7}$$

Thus r′(z) is expressed in terms of r(z) and known functions. The solution of r(x) is unique (for details see Gupta and Ahsanullah (2004a)).

Putting $\psi(x) = x$ and g(x) = ax + b, we obtain from (8.1.7)

$$a\frac{r'(z)}{r(z)} + 2ar(z) = (r(z))^2((a-1)a+b) \tag{8.1.8}$$

The solution of (8.1.8) is

$$\mathrm{r(x)} = \frac{\mathrm{a} + \sqrt{\mathrm{a}}}{(\mathrm{a}-1)\mathrm{x}+\mathrm{b}}.$$

Thus X will have (i) exponentially distributed if a = 1, (ii) power function distribution if a < 1 and (iii) Pareto distribution if a > 1.

Ahsanullah and Wesolowski (1998) extended the result Theorem 8.1.2 for non adjacent record values. Their result is given in the following theorem.

Theorem 8.1.3

$$\textit{If}\ \mathrm{E}\,(\mathrm{X(n+2)|X(n)} = \mathrm{a\,X(n) + b.u} > 1.$$

where the constants a *and* b, *then if*:

(a) a = 1 *then* X_i *has the exponential* distribution,
(b) <1 , *then* X_I *has the power function distribution*
(c) a > 1 X_I *has the Pareto distribution*

Proof of Theorem 8.1.1 for s > 2 In this case, the problem becomes more complicated because of the nature of the resulting differential equation Lopez-Blazquez and Moreno-Rebollo (1997) also gave characterizations of distributions by using the following linear property

$$E(X(k)|X(k+s)=z) = az+b, 1 \le k < s \le n,$$

Raqab (2002) and Wu (2004) considered this problem for non-adjacent record values under some stringent smoothness assumptions on the distribution function F(.). Dembinska and Wesolowski (2000) characterized the distribution by means of the relation

$$E(X(s+k)||X(k)=z) = az+b,\ \text{for } 1<\mathrm{k}<\mathrm{s}<\mathrm{n}.$$

They used a result of Rao and Shanbhag (1994) which deals with the solution of extended version of integrated Cauchy functional equation. It can be pointed out earlier that Rao and Shanbhag's result is applicable only when the conditional expectation is a linear function.

Bairamov et al. (2005) gave the following characterization.

Theorem 8.1.4 *Let* X *be an absolutely continuous random variable with d.f.* F(x) *with* F(0) = 0 *and* F(x) > 0 *for all* x > 0 *and pdf* f(x), *then*

(a) *for 1 ≤ k ≤ n – 1,*

$$E((X(n)|X(n-k)=x),X(n+1)=y)\_=\frac{u+kv}{k+1},0<u<v<\infty$$

If and only if

$$F(x)=1-e^{\lambda x},x\geq 0,\lambda>0,$$

(b) for 2 ≤*k* ≤ *n–1,*

$$E((X(n)|X(n-k+1)=x),X(n+2)=y)\_$$
$$=\frac{2u+(k-1)v}{k+1},0<u<v<\infty$$

If and only if

$F(x)$ = 1 – $e^{=\lambda x}$, x ≥ 0, λ > 0, *Yanev et al. (2007) extended these results for general cases of nonadjacent record values. Under the conditions of Theorem* 8.1.4, *Akhundov and Nevzorov (2008) proved that*

$$E\left(\frac{X(2)\_X(3)+\cdots+X(n)}{n-1}|X(1)=u,X(n+1)=v\right)=\frac{u+v}{2}$$

characterizes the exponential distribution under mild condition on F(x).

Exercise 8.1.1 Let *X* be an absolutely continuous random variable with d.f. *F(x)* with F(0) = 0 and F(x) > 0 for all x > 0 and pdf *f(x)* Suppose h(x) is a monotonic and differentiable function with respect to x. We assume that $\lim_{\to 0}\frac{F(x)}{x}=\lambda,\lambda>0$ and $\lim_{\to 0}h(x)\overline{F}(x)=0$. Then for two values o r, s−1th. Then, for two values of *r* and *s*, 1 ≤ *r* < *s* − 1 < *n*, $(h(X(s)-h(X(r))|X(r))=x)=\frac{s-r}{\lambda}$ if and I If and only if

$$F(x)=1-e^{-\lambda x},x\geq 0,\lambda>0.$$

8.2 Characterization by Independence Property of Record Statistics

Tata (1969) presented a characterization of the exponential distribution by the independence of the random variables X(2) and X(2) − X(1) given in the following theorem.

Theorem 8.2.1 *Let* $\{X_n, n \geq 1\}$ *be an i.i.d. sequence of non-negative continuous random variables with d.f.* F(x) *and pdf* f(x). *We assume* F(0) = 0 *and* F(x) > 0 *for all* x > 0. Then for X_n *to have the d.f.*, $F(x) = 1 - e^{-x/\sigma}$, *x* ≥ 0, *σ* > 0, *it is necessary and sufficient that* X(2) − X(1) *and* X(1) *are independent.*

Proof The necessary condition is easy to establish, We give here the proof of the sufficiency condition. The property of the independence of X(2) − X(1) and X(1) will lead to the functional equation

$$\overline{F}(0)\overline{F}(x+y) = \overline{F}(x)\,\overline{F}(y), \quad 0 < x, y < \infty. \tag{8.2.1}$$

The continuous solution of this functional equation with the boundary condition F(0) = 0, is

$$\overline{F}(x) = e^{-x\sigma^{-1}},\ \mathrm{x}\ > 0, \sigma > 0.$$

The following generalization theorem was given by Ahsanullah (1079)

Theorem 8.2.2 *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of i.i.d. random variables with common distribution function* F *which is absolutely continuous with pdf f. Assume that* F(0) = 0 *and* F(x) > 0 *for all* x > 0. *Then* X_n *to have the d.f.,* $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0$, $\sigma > 0$, *it is necessary and sufficient that* X(n) − X(n−1) *and* X(n−1) *are independent.*

Proof It is easy to establish that if X_n has the d.f., $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0$, $\sigma > 0$, then X(n) – X(n–10) and X(n–1) are independent. Suppose that X(n + 1) − X(n) and X (n), n ≥ 1, are independent. Now the joint pdf f(z, u) of Z = X(n−1)(−X(n) and U = $X(n)_1$ can be written as

$$\begin{aligned} \mathrm{f(z,u)} &= \frac{[R(u)]^{n-1}}{\Gamma(n)} r(u) f(u+z),\ 0<\mathrm{u},\ \mathrm{z}<;\infty. \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{8.2.2}$$

But the pdf f_n (u) of X(n) can be written as

$$\begin{aligned} \mathrm{F_{n-1}(u)} &= \frac{[R(u)]^{n-1}}{\Gamma(n)} f(u), 0<u<\infty, \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{8.2.3}$$

Since Z and U are independent, we get from (8.2.2) and (8.2.3)

$$\frac{f(u+z)}{\overline{F}(u)} = g(z), \tag{8.2.4}$$

where g(z) is the pdf of u. Integrating (8.3.4) with respect z from 0 to z_1, we obtain on simplification

$$\overline{F}(u) - \overline{F}(u+z_1) = \overline{F}(u) G(z_1). \tag{8.2.5}$$

Since $G(z_1) = \int_0^{z_1} g(z)dz$. Now $u \to 0^+$ and using the boundary condition $\overline{F}(0) = 1$, we see that $G(z_1) = F(z_1)$. Hence we get from (8.2.5)

$$\overline{F}(u+z_1) = \overline{F}(u)\overline{F}(z_1). \tag{8.2.6}$$

The only continuous solution of (8.2.6) with the boundary condition F(0) = 0, is

$$\overline{F}(x) = e^{-\sigma^{-1}x}\text{ , x} \geq 0 \tag{8.2.7}$$

where σ is an arbitrary positive real number.

The following theorem (Theorem 8.2.3) is a generalization of the Theorem 8.2.2.

Theorem 8.2.3 *Let* $\{X_n, n \geq 1\}$ *be independent and identically distributed with common distribution function* F *which is absolutely continuous and* F(0) = 0 *and* F(x) < 1 *for all* x > 0. *Then* X_n *has the d.f.*, $F(x) = 1 - e^{-\sigma x}$, $x \geq 0$, $\sigma > 0$, *it is necessary and sufficient that are independent* X(n) − X(m) *and* X(m).

Proof The necessary condition is easy to establish. To proof the sufficient condition, we need the following lemma.

For proof of sufficiency we need the following lemma.

Lemma 8.2.1 *Let* F(x) *be an absolutely continuous function and* $\overline{F}(x) > 0$, *for all* x > 0. *Suppose that* $\overline{F}(u+v)(\overline{F}(v))^{-1} = \exp\{-\mathrm{q(u,v)}\}$ *and* h(u,v) = {q(u,v)}r exp {−q(u,v)} $\frac{\partial}{\partial u}$ q(u,v), *for* r ≥ 0. *Further if* h(u,v) ≠ 0, *and* $\frac{\partial}{\partial u}$ q(u,v) ≠ 0 *for any positive* u *and* v. *If* h(u,v) *is independent of* v, *then* q(u,v) *is a function of* u *only*.

We refer to Ahsanullah () for the proof of the lemma.

Proof of the sufficiency pf Theorem 8.2.4.

The conditional pdf of Z = X(n) = X(m) given V(m) = x is

$$\begin{aligned} \mathrm{f(z|X(m) = x)} &= \frac{1}{\Gamma(n-m)}[R(z+x) \\ &\quad - R(x)]^{n-m-1}\frac{f(z+x)}{\overline{F}(x)},\ 0 < \mathrm{z} < \infty, 0 < \mathrm{x} < \infty. \end{aligned}$$

Since Z and X(m) are independent, we will have for all z > 0,

$$(R(z+x) - R(x))^{n-m-1}\frac{f(z+x)}{\overline{F}(x)} \tag{8.2.8}$$

as independent of x. Now let

$$R(z+x) - R(x) = -\ln\frac{\overline{F}(z+x)}{\overline{F}(x)} = q(z,x),\ \text{say.}$$

Writing (8.2.9) in terms of q(z,x), we get

$$[q(z,x)]^{n-m-1}\exp\{-q(z,x)\}\frac{\partial}{\partial z}q(z,x), \tag{8.2.9}$$

as independent of x. Hence by the Lemma 8.2.1, we have

$$-\ln\left\{\overline{F}(z+x)(\overline{F}(x))^{-1}\right\}=q(z+x)=c(z), \tag{8.2.10}$$

where c(z) is a function of z only. Thus

$$\overline{F}(z+x)(\overline{F}(x))^{-1}=c_1(z), \tag{8.2.11}$$

and $c_1(z)$ is a function of z only.

The relation (8.2.11) is true for all z ≥ 0 and any arbitrary fixed positive number x. The continuous solution of (8.2.11) with the boundary conditions, $\overline{F}(0)=1$ and $\overline{F}(\infty)=0$ is

$$\overline{F}(x)=\exp(-x\,\sigma^{-1}), \tag{8.2.12}$$

for x ≥ 0 and any arbitrary positive real number σ. The assumption of absolute continuity of F(x) in the Theorem can be replaced by the continuity of F(x).

Cheng (2007) gave an interesting characterization of the Pareto distribution. Unfortunately the statement and the proof of the theorem were wrong. Here we will give a correct statement and proof of his theorem.

Theorem 8.2.4 *Let* $\{X_n,\ n \geq 1\}$ *be independent and identically distributed with common distribution function* F *which is absolutely continuous and* F(1) = 0 *and* F(x) < 1 *for all* x > 1. *Then* X_n has the d.f., $F(x) = 1 - x^{-\theta}$, $x \geq 1$, $\theta > 0$, *it is necessary and sufficient that* $\frac{X(n)}{X(n+1)-X(n)\_}$ *and* X(m), n ≥ 1 *are independent*.

Proof If $F(x)=1-x^{-\theta}, x\geq 1, \quad \theta > 0$, then the joint pdf $f_{n,n+1}(x,y)$ of X(n) and X(n + 1) is

$$f_{n,n+1}(x,y)=\frac{1}{\Gamma(n)}\frac{\theta^{n+1}(\ln x)^{n-1}}{xy^{\theta+1}},\ 1<x<y<\infty, \theta>0.$$

Using the transformation, U = X(n) and V = $\frac{X(n)}{X(n+1)-X(n)\_}$. The joint pdf_{UV} f(u,v) cab be written as

$$f_{W,V}(w,\ v)=\frac{1}{\Gamma(n)}\frac{\theta^{n+1}(\ln u)^{n-1}}{u^{\theta+3}}\left(\frac{v}{1+v}\right)^{\theta+1},1<u,v<\infty,\ \theta>0. \tag{8.2.13}$$

Thus U and are independent.

The proof of sufficiency

The joint pdf of W and V can be written as

$$f_{W,V}(u,v) = \frac{(R(u))^{n-1}}{\Gamma(n)} r(u) f\left(\frac{1+v}{v}u\right)\frac{u}{V^2}, 1 < u, v < \infty, \tag{8.2.14}$$

where R(x) = –lnx, r(x) = $\frac{d}{dx}R(x)$.

We have the pdf $f_U(u)$ od U as $f_U(u) = \frac{(R(u))^{n-1}}{\Gamma(n)} f(u)$. Since U and V are independent, we must the pdf $f_V(v)$ of V as

$$f_V(v) = f\left(\frac{1+v}{v}u\right)\frac{w}{V^2}\frac{1}{1-F(u)}, 0 < v < \infty. \tag{8.2.15}$$

Integrating the above pdf from v_0 to ∞, we obtain

$$1 - \mathrm{F}(\mathrm{v}_0) = \frac{1 - F\left(\frac{1+v_0}{v_0}u\right)}{1 - F(u)} \tag{8.2.16}$$

Since $F(v_0)$ is independent of U, we must have

$$\frac{1 - F\left(\frac{1+v_0}{v_0}u\right)}{1 - F(u)} = \mathrm{G}(\mathrm{v}_0) \tag{8.2.17}$$

where $G(v_0)$ is independent of u

Letting u → 1, we obtain $G(v_0) = 1 - F\left(\frac{1+v_0}{v_0}\right)$.

We can rewrite (8.2.17) as

$$1 - F\left(\frac{1+v_0}{v_0}u\right) = \left(1 - F\left(\frac{1+v_0}{v_0}\right)(1 - F(u)\right) \tag{8.2.18}$$

Since the above equation is true all u ≥ 1 and almost all v_0 ≥ 1, we must have $F(x) = 1 - x^{\beta}$. Since F(1) = 0 and F($F(\infty)$) = 0, we must have

$$\mathrm{F}(\mathrm{x}) = 1 - x^{-\theta}, \ \mathrm{x} \geq 1 \,\mathrm{and}\, \theta > 0.$$

Example 8.2.1 Let $\{X_n, n \geq 1\}$ be independent and identically distributed with common distribution function F which is absolutely continuous and F(1) = 0 and F (x) < 1 for all x > 1. Then X_n has the d.f., $F(x) = 1 - x^{-\theta}$, $x \geq 1$, $\theta > 0$, it is necessary and sufficient that $\frac{X(n+1)-X(n)}{X(n)}$ and X(n),n ≥ 1 are independent.

The following theorem is a generalization of Chang's (2007) result.

Theorem 8.2.5 *Let* $\{X_n, n \geq 1\}$ *be independent and identically distributed with common distribution function* F *which is absolutely continuous and* F(1) = 0 *and* F(x) < 1 *for all* x > 0. *Then* X_n *has the d.f.,* $F(x) = 1 - x^{-\theta}$, $x \geq 1$, $\theta > 0$, *it is necessary and sufficient that* $\frac{X(m)}{X(n)-X(m)\_}$, 1 < m < n and X(m) *are independent*.

Proof The joint pdf $f_{m,n}$ (x,y) of X(n) and X(m) is

$$f_{m,n}(x,y) = \frac{(R(x))^{m-1}}{\Gamma(m)} \frac{(R(y) - R(x))^{n-m-1}}{\Gamma(n-m)} r(x)f(y), \tag{8.2.19}$$

We have $F(x) = 1 - x^{-\theta}$, $R(x) = \theta \ln x$, $r(x) = \frac{\theta}{x}$, thus we obtain

$$f_{m,n}(x,y) = \frac{(\theta \ln x)^{m-1}}{\Gamma(m)} \frac{(\ln y - \ln x)^{n-m-1}}{\Gamma(n-m)} \frac{1}{xy^{\theta+1}}. \tag{8.2.20}$$

where $1 \leq x < y < \infty, \theta > 0$.

Using the transformation U = X(m) and $V = \frac{X(m)}{X(n)-X(m)\_}$, we obtain the pdf $f_{U,V}$ (u,v) of U and V as

$$f_{U,V}(u,v) = \frac{\theta^n (\ln u)^{n-1}}{\Gamma(n)} \frac{(\ln(\frac{1+v}{v}))^{n-m-1}}{\Gamma(n-m)} \frac{v^{\theta-1}}{u^{\varsigma\theta+1}(1+v)^{\theta+1}}$$

Thus X(m) and $\frac{X(m)}{X(n)-X(m)\_}$ are independent.

Proof of sufficiency.

Using U = X(m) and $V = \frac{X(m)}{X(n)-X(m)\_}$, we can obtain the pdf $f_{U,V \text{ of}}$ U and V from (8.2.19) as

$$f_{U,V}(u,v) = \frac{(Ru)^{m-1}}{\Gamma(m)} \frac{(R(\frac{u(1+v)}{v}) - R(u))^{n-m-1}}{\Gamma(n-m)} r(u)f(\frac{u(1+v)}{v}), \tag{8.2.21}$$

We can write the conditional pdf $f_{V|U}(v|u)$ of V|U as

$$f_{V|UV}(v|u) = \frac{(R(\frac{u(1+v)}{v}) - R(u))^{n-m-1}}{\Gamma(n-m)} \frac{uf(\frac{u(1+v}{v}),}{v^2\overline{F}(u)}, 1 < u < \infty,\ 0 < v < \infty. \tag{8.2.22}$$

Using the relation R(x) = −ln $\overline{F}(x)$, we obtain from (8.2.22) that

$$f_{V|U}(v|u) = \frac{(-\ln(\frac{\overline{F}(\frac{u(1+v)}{v})}{\overline{F}(u)}))^{n-m-1}}{\Gamma(n-m)} \frac{d}{dv}(\frac{\overline{F}(\frac{u(1+v)}{v})}{\overline{F}(u)}), 1 < u < \infty, 0 < v < \infty. \tag{8.2.23}$$

Since V and U are independent, we must have $\frac{\overline{F}(\frac{u(1+v)}{v})}{\overline{F}(u)}$ independent of U.

Let

$$\frac{\overline{F}(\frac{u(1+v)}{v})}{\overline{F}(u)} = \mathrm{G(v)}, \tag{8.2.24}$$

Letting $u \to 1$, we obtain

$$\overline{F}\left(\frac{u(1+v)}{v}\right) = \overline{F}(u)\overline{F}\left(\frac{1+v}{v}\right) \tag{8.2.25}$$

For all u, 1 < u < ∞ and all v, 0 < v < ∞.

The continuous solution of (8.2.24) with the boundary condition F(0) 0 and F (∞) = 1 is

$$\mathrm{F(x)} = 1 - x^{-\theta}, x \geq 1 \text{ and } \theta \ > 0.$$

The following exercise is proved under the assumption of monotone hazard rate. We will say F belongs to the class C_2 if r(x) is either monotone increasing or decreasing.

Exercise 8.2.1 If X_k, k ≥ 1 has an absolutely continuous distribution function F with pdf f and F(0) = 0. If $I_{n,n+1}$ and $I_{n-1,n}$, n ≥ 1, are identically distributed and F belongs to C_2, then X_k has the d.f. $\mathrm{F(x)} = 1 - e^{-\varsigma x x}$, $x \geq 0, \sigma > 0$. $k \geq 1$.

8.3 Characterizations Based on Identical Distribution and Moment Properties

If F is the distribution function of a non-negative random variable, we will call F is "new better than used" (NBU) if for x, y ≥ 0, $\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y)$, and F is "new worse than used" (NWU) if for x, y ≥ 0, $\overline{F}(x+y) \geq \overline{F}(x)\overline{F}(y)$. We will say F belongs to the class C_1 if either F is NBU or NWU. We will say F belongs to the class C_2 if the hazard rate $\mathrm{r(x)} = \frac{f(x)}{1-F(x)}$ increases monotonically increases or decreases for all x.

Theorem 8.3.1 Let X_n, n ≥ 1 *be a sequence of i.i.d. random variables which has absolutely continuous distribution function* F *with pdf f and* F(0) = 0. *Assume that* F (x) < 1 *for all* x > 0. *If* X_n *belongs to the class* C_1 *and* $I_{n-1,n}$ = X(n) − X(n–1), n > 1., *has an identical distribution with* X_k, k ≥ 1, *then* X_k *has the d.f.* $\mathrm{F(x)} = 1 - e^{-\varsigma x x}$, $x \geq 0$, $\sigma > 0$.

Proof The if condition is easy to establish. We will proof here the only if condition.

The pdf $f_{n-n,n}$ of $I_{n-1,n}$ can be written as

$$
\begin{aligned}
F_{n=1,n(x.y)} &= \int_0^\infty \frac{[R(u)]^{n-1}}{\Gamma(n)} r(u)f(u+z)du, \quad z \geq 0 \\
&= 0, \quad \text{otherwise.}
\end{aligned} \tag{8.3.1}
$$

By the assumption of the identical distribution of $I_{n-1,n}$ and X_k, we must have

$$\int_0^\infty [R(u)]^{n-1} \frac{r(u)}{\Gamma(n)} f(u+z)du = f(z), \text{ for all z } > \text{o}. \tag{8.3.2}$$

Substituting

$$\int_0^\infty [R(u)]^{n-1} f(u)du = \Gamma(n), \tag{8.3.3}$$

we have

$$\int_0^\infty [R(u)]^{n-1} r(u)f(u+z)du = f(z) \int_0^\infty [R(u)]^{n-1} f(u)du, \text{ z } > 0. \tag{8.3.4}$$

Thus

$$\int_0^\infty [R(u)]^{n-1} f(u)[f(u+z)(\overline{F}(u))^{-1} - f(z)]du = 0, z > 0. \tag{8.3.5}$$

Integrating the above expression with respect to z from z_1 to ∞, we get from (8.3.5)

$$\int_0^\infty [R(u)]^{n-1} f(u)[\overline{F}(u+z_1)(\overline{F}(u))^{-1} - \overline{F}(z_1)]du = 0, \text{ z}_1 > 0. \tag{8.3.6}$$

If F(x) is NBU, then (8.3.6) is true if

$$\overline{F}(u+z_1)(\overline{F}(u))^{-1} = \overline{F}(z_1), \text{ z}_1 > 0. \tag{8.3.7}$$

The only continuous solution of (8.3.7) with the boundary conditions $\overline{F}(0) = 1$ and $\overline{F}(\infty) = 0$ is $\overline{F}(x) = \exp(0, \sigma^{-1})$, where σ is an arbitrary real positive number.

Similarly, if F is NWU then (8.3.6) is true if (8.3.7) is satisfied and X_k has the d.f. $F(x) = 1 - e^{-\varsigma x x}$, $x \geq 0$, $\sigma > 0$. $k > 1$.

Theorem 8.3.2 *Let* X_n, $n \geq 1$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function* F(x) *with* f(x) *as the corresponding density function. If* $F \in C_2$ *and for some fixed* n,m, $1 \leq m < n < \infty$, $I_{m,n} = X(n - m - 1)$, *then* X_k *has the d.f.* $F(x) = 1 - e^{-\varsigma x x}$, $x \geq 0, \sigma > 0$, $k \geq 1$.

Proof The pdfs $f_1(x)$ of R_{n-m} and $f_2(x)$ of $I_{m,n}$ (= $R_n - R_m$) can be written as

$$f_1(x) = \frac{1}{\Gamma(n-m)}[R(x)]^{n-m-1} f(x), \text{ for } 0 < \mathrm{x} < \infty, \tag{8.3.8}$$

and

$$f_2(x) = \int_0^\infty \frac{[R(x)]^{m-1}}{\Gamma(m)} \frac{[R(x+u) - R(x)]^{n-m-1}}{\Gamma(n-m)} r(u) f(u+x) du, \; 0 < \mathrm{x} < \infty. \tag{8.3.9}$$

Integrating (8.3.8) and (8.3.9) with respect to x from 0 to x_0, we get

$$F_1(x_0) = 1 - g_1(x_0), \tag{8.3.10}$$

where

$$g_1(x_0) = \sum_{j=1}^{n-m} \frac{[R(x_0)]}{\Gamma(j)} e^{-R(x_0)},$$

and

$$3F_2(x_0) = 1 - g_2(x_{0,u}), \tag{8.3.11}$$

where

$$g_2(x_0, u) = \sum_{j=1}^{n-m} \frac{[R(u+x_0) - R(u)]^{j-1}}{\Gamma(j)} \exp\{-(R(u+x_0) - R(u))\}.$$

Now equating (8.3.10) and (8.3.11), we get

$$\int_0^\infty \frac{[R(y)]^{m-1}}{\Gamma(m)} f(u)[g_2(u, x_0) - g_1(x_0)] du = 0, \; x_0 > 0. \tag{8.3.12}$$

Now $g_2(x_0, 0) = g_1(0)$ and

$$0 = \frac{[R(u) - R(u)]^{n-m-1}}{\Gamma(n-m)} \exp\{-(R(u + x_0) - R(u)\}[r(x_0) - r(u + x_0)].$$

Thus if $F \in C_2$, then (8.3.12) is true if

$$\mathrm{r(u + x_0) = r(u)} \tag{8.3.13}$$

for almost all u and any fixed $x_0 \geq 0$. Hence X_k has the d.f. $F(x) = 1 - e^{-\varsigma xx}$, $x \geq 0$, $\sigma > 0$. $k \geq 1$. Here σ is an arbitrary positive real number. Substituting m = n − 1, we get $I_{n-1,n} \stackrel{d}{=} X1$ as a characteristic property of the exponential distribution.

Theorem 8.3.3 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function* F(x) *and the corresponding density function* f(x). *If* F *belongs to* C_2 *and for some* m, m > 1, X(n) *and* X(n−1) + U *are identically distributed, where* U *is independent of* X(n) *and* X(n−1) *is distributed as* X_n'*s, then* X_k *has the d.f.* $F(x) = 1 - e^{-\varsigma xx}$, $x \geq 0$, $\sigma > 0$. $k \geq 1$.

Proof The pdf $f_m(x)$ of R_m, m ≥ 1, can be written as

$$\begin{aligned} fm(y) &= \frac{[R(y)]^m}{\Gamma(m+1)} f(y), \quad 0 < \mathrm{y} < \infty, \\ &= \frac{d}{dy}\left(-\bar{F}(y)\int_0^y \frac{[R(x)^{m-1}]}{\Gamma(m)} r(x)dx + \int_0^y \frac{[R(x)^m]}{\Gamma(m)} f(x)dx\right), \end{aligned} \tag{8.3.14}$$

The pdf $f_2(y)$ of X(n−1) + U can be written as

$$\begin{aligned} f_2(y) &= \int_0^y \frac{[R(x)^{m-1}]}{\Gamma(m)} f(y-x)f(x)dy \\ &= \frac{d}{dy}\left(-\frac{[R(x)^{m-1}]}{\Gamma(m)}\bar{F}(y-x)f(x)\,dx + \int_0^y \frac{[R(x)^{m-1}]}{\Gamma(m)} f(x)dx\right). \end{aligned} \tag{8.3.15}$$

Equating (8.3.14) and (8.3.15), we get on simplification

$$\int_0^y \frac{[R(x)^{m-1}]}{\Gamma(m-1)} f(x)H_1(x,y)\,dx = 0, \tag{8.3.16}$$

where $H_1(x,y) = \bar{F}(y-x) - \bar{F}(y)\bar{F}(x))^{-1}, \quad 0 < x < y < \infty$. Since F $\in$ C_1, therefore for (8.3.16) to be true, we must have

$$H_1(x, y) = 0, \tag{8.3.17}$$

for almost all x, $0 < x < y < \infty$.

This implies that

$$\bar{F}(y-x)\bar{F}(x) = \bar{F}(y), \tag{8.3.18}$$

for almost all x, $0 < x < y < \infty$. The only continuous solution of (8.3.18) with the boundary conditions $\bar{F}(0) = 1$, and $\bar{F}(\infty) = 0$, is

$$\bar{F}(x) = e^{-x\sigma^{-1}}, \tag{8.3.19}$$

where σ is an arbitrary positive number.

Remark 8.3.1 The Theorem 8.3.4 can be used to obtain the following known results of a two parameter exponential distribution $(\bar{F}(x) = \exp\{-\sigma^{-1}(x-\mu)\})$.

$$\begin{aligned} E(X(n)) &= \mu + n\,\sigma \\ Var(X(n)) &= n\ \sigma^2 \\ Cov(X(m)X(n)) &= m\ \sigma^2,\ m < n. \end{aligned}$$

Theorem 8.3.4 *Let* $X_1, X_2, \ldots, X_m, \ldots$ *be independent and identically distributed random variables with probability density function* f(x), $x \geq 0$ *and* m *is an integer valued random variable independent of* X's *and* $P(m = k) = p(1-p)^{k-1}$, k = 1, 2, …, *and* $0 < p < 1$. *Then the following two properties are equivalent*:

(a) X'*s are distributed as* E(0, σ), *where* σ *is a positive real number*

(b) p $\sum_{j=1}^{m} X_j \overset{d}{=} I_{n-1,n}$, *for some fixed n*, $n \geq 2$, $X_j \in c_2$ *and* $E(X_j) < \infty$.

Proof It is easy to verify (a) ⇒ (b). We will prove here that (b) ⇒ (a). Let $\phi_1(t)$ be the characteristic function of of $I_{n-1,n}$ then

$$\begin{aligned} \phi_1(t) &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} e^{itx} [R(u)]^{n-1} r(u) f(u+x) du\, dx \\ &= 1 + it \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} e^{itx} [R(u)]^{n-1} r(u) \bar{F}(u+x) du\, dx \end{aligned} \tag{8.3.20}$$

The characteristic function p ϕ_2 (t) of p $\sum_{j=1}^{m} X_j$ can be written as

$$\begin{aligned}\Phi_2(t) &= E(e^{itp\sum_{j=1}^{m} X_j}) \\ &= \sum_{k=1}^{\infty} [\Phi(tp)]^k p(1-p)^{k-1}, \\ &= \mathrm{p}(\Phi(\mathrm{t\ p}))(1-\mathrm{q}\Phi(\mathrm{p\ t}))^{-1}, \mathrm{q} = 1-\mathrm{p},\end{aligned} \tag{8.3.21}$$

where Φ(t) is the characteristic function of X's.

Equating (8.3.20) and (8.3.21), we get on simplification

$$\frac{\Phi(pt)-1}{1-q\Phi(pt)}\frac{1}{it}\int_0^{\infty}\int_0^{\infty}\frac{1}{\Gamma(n)}e^{itx}[R(u)]^{n-1}r(u)\bar{F}(u+x)du\,dx \tag{8.3.22}$$

Now taking limit of both sides of (8.3.22) as t goes to zero, we have

$$\frac{\Phi'(0)}{i} = \int_0^{\infty}\int_0^{\infty}\frac{1}{\Gamma(n)}[R(u)]^{n-1}r(u)\bar{F}(u+x)du\,dx. \tag{8.3.23}$$

Writing

$$\frac{\Phi'(0)}{i} = \int_0^{\infty}\bar{F}(x)dx, \text{ we get from} \tag{8.3.24}$$

$$\int_0^{\infty}\int_0^{\infty}(R(u))^{n-1}r(u)\{\bar{F}(u+x)-\bar{F}(u)\bar{F}(x)\}du\,dx = 0. \tag{8.3.25}$$

Since X's belong to C_1, we must have

$$\bar{F}(u+x) = \bar{F}(x)\bar{F}(u), \tag{8.3.26}$$

for almost all x, u, 0 < u, x < ∞ The only continuous solution of (8.3.26) with the boundary condition $\bar{F}(0) = 1$ and $\bar{F}(\infty) = 0$, is

$$\bar{F}(x) = \exp(-x\sigma^{-1}),\ x \geq 0, \tag{8.3.27}$$

where σ is an arbitrary positive real number.

We will prove the following characterization theorem under the assumption of the finite first moment.

Exercise 8.3.1 Let X_n, $n \geq 1$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function F(x) and the corresponding density function f(x). Let $a = \inf\{x|F(x) > 0\} = 0$, $F(x) < 1$ for all $x > 0$. If F belongs to the class C_1 and $E(X_k)$, $k \geq 1$ is finite, then $X_k \in E(0,\sigma)$, if and only if for some fixed n, $n > 1$, $E(I_{n-1,n}) = E(X_k)$.

The following theorem uses the property of homoscedasticity but does not use NBU or NWU property.

Exercise 8.3.2 Let X_n, $n \geq 1$ be a sequence of independent and identically distributed random variables with common distribution function F which is absolutely continuous and $\inf\{x|\ F(x) > 0\} = 0$ and $E(X_n^2) < \infty$. Then X_k, $k \geq 1$ has the exponential distribution if and only if $\mathrm{Var}(I_{n-1,n}|X(n-1) = x) = b$ for all x, where b is a positive constant independent of X and $I_{n-1,n}$.

Exercise 8.3.3 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non negative random variables with continuous distribution function F (x) and the corresponding density function f(x). Let $a = \inf\{x|F(x) = 0\} = 0$, $F(x) < 1$ for all $x > 0$ and F belongs to class C_2. Then $X_n \in E(0,\sigma)$, if and only if for some fixed n, $n \geq 1$, the hazard rate r_1 of $I_{n-1,n}$ = the hazard rate r of X_k.

Suppose that the random variables $\{X_j, j = 1,2,\ldots\}$ are from Gumbel distribution with d.f. F(x) as follows:

$$F(x) = e^{-e^{-x}}, -,, -\infty < x < \infty$$

It is known that (see Ahsanullah and Holland (1994), p. 475) that

$$X(n) \overset{d}{-} X - \left(W_1 + \frac{W_2}{2} + \cdots + \frac{W_{m-1}}{m-1} + \frac{W_m}{m} \right),\ m \geq 1$$

where $W_1, W_2, \ldots, W_{m-1}, W_m$ are independently distributed as negative exponential with $F(w) = 1 - e^{-w}$, $w > 0$. $X(1) = X$. Thus $S_{(m)} = m(X(m-1) - X(m)$, $m = 2,\ldots,$ are identically distributed as exponential. Similarly if we consider the upper records from the distribution, $F(x) = e^{-e^{x}}, -\infty\ < x\ < \infty$, then for any $m \geq 1$, $S_m = m(X(m-1)-X(m)$, $m = 2,\ldots.$ are identically distributed as exponential distribution. It can be shown that for one fixed m, $S_{(m)}$ or S_m distributed as exponential does not characterize the exponential distribution.

Arnold and Villasenor (1997) raised the question suppose that S_1 and $2\ S_2$ are i.i. d. exponential with unit mean, can we consider that X_j's are (possibly translated) Gumbel variables? Here, we will prove that for a fixed $m > 1$, the condition $X(n) \overset{d}{=} X(n-1) + \frac{W}{m}$, where W is distributed as exponential distribution with mean unity characterizes the Gumbel distribution.

Theorem 8.3.5 *Let* $\{X_j, j = 1,\ldots,\}$ *be a sequence of independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue* *measure) distribution function* F(x). *Then the following two statements are identical*.

(a) $F(x) = e^{-e^{-x}}, -\infty < x < \infty$,

(b) *For a fixed* m > 1, *the condition* $X(m) \stackrel{d}{=} X(m-1) + \frac{W}{m}$, *where W is distributed as negative exponential mean unity*.

Proof It is enough to show that (b) ⇒ (a). Suppose that for a fixed m > 1, $X(m) \stackrel{d}{=} X(m-1) + \frac{W}{m}$, then

$$\begin{aligned} F_{(m)}(x) &= \int_{-\infty}^{x} P(W \le m(x-y) f_{(m+1)}(y)dy \\ &= \int_{-\infty}^{x} [1 - e^{-m(x-y)}] f_{(m+1)}(y)dy \qquad (8.3.28) \\ &= \mathrm{F}_{(\mathrm{m}+1)}(\mathrm{x}) - \int_{-\infty}^{x} e^{-m(x-y)} f_{(m+1)}(y)dy. \end{aligned}$$

Thus

$$e^{mx}[F_{(m+1)}(x) - F_{(m)}(x)] = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy \qquad (8.3.29)$$

Using the relation

$$e^{mx} \frac{F(x)[H(x)]^m}{\Gamma(m+1)} = e^{H(x)} \sum_{j=0}^{m} \frac{[H(x)]^j}{m!)},$$

we obtain

$$e^{mx} \frac{F(x)(H(x))^m}{\Gamma(m+1)} = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy \qquad (8.3.30)$$

Taking the derivatives of both sides of (8.3.30), we obtain

$$\frac{d}{dx}\left[e^{mx} \frac{(H(x))^m}{\Gamma(m+1)} F(x)\right] = e^{mx} f_{(m+1)}(x) \qquad (8.3.31)$$

This implies that

$$\frac{d}{dx}\left[e^{mx}\frac{H^m(x)}{\Gamma(m+1)}\right]F(x) = 0. \tag{8.3.32}$$

Thus

$$\frac{d}{dx}\left[e^{mx}\frac{(H(x))^m}{\Gamma(m+1)}\right] = 0. \tag{8.3.33}$$

Hence

$$\mathrm{H(x)} = \mathrm{c\ e^{-x}},\ -\infty < \mathrm{x} < \infty \tag{8.3.34}$$

Thus

$$\mathrm{F(x)} = e^{-ce^{-x}},\ \infty < \mathrm{x} < \infty. \tag{8.3.35}$$

Since F(x) is a distribution function we must have c as positive. Assuming F (0) = e^{-1}, we obtain

$$\mathrm{F(x)} = \mathrm{e}^{-e^{-x}},\ -\infty < x < \infty. \tag{8.3.36}$$

Ahsanullah and Malov (2004) proved the following characterization theorem.

Theorem 8.3.6 *Let* X_1, X_2, …, *be a sequence of independent and identically distributed r.v.'s with distribution function* F(x). *If* $X(m) \stackrel{d}{=} X(m-2) + \frac{W_1}{m} + \frac{W_2}{m-1}$, m > 2, *for twice differentiable* F(x), *where* W_1 *and* W_2 *are independent as exponential distribution with unit mean then* $\mathrm{F(x)} = 1 - \mathrm{e}^{-e^{-x}},\ -\infty < \mathrm{x} < \infty$.

Ahsanullah and Kirmani (1991) gave the following result.

Exercise 8.3.5 Suppose $\{X_n, n \le 1\}$ is a sequence of I.i.d. random variable with h continuous d.f. F(x) with F(0) = 0 and X(x) > 0 for all x > 0. We assume $\lim \to 0 \frac{F(x)}{x} = \lambda, \lambda\ > 0$. We define the random variable N such that N = min{i > 1, $X_i < X_1$}. Then the random variables NX_N and X_1 are identically distributed if and only if $\mathrm{F(x)} = 1 - \mathrm{e}^{-\lambda x}, x \geq 0$.

8.4 Exercises (solutions)

Exercise 8.2.1 (*solution*)

$$\mathrm{P}\left(\mathrm{I}_{n-1,n} > \ z\right) = \int_0^\infty [R(u)]^{n-1}\frac{r(u)}{\Gamma(n)}\bar{F}(u+z)du, \text{for all } z > 0,$$
$$= 0, \qquad \text{otherwise.}$$

Since $\mathrm{I}_{n,n+1}$ and $\mathrm{I}_{n-1,n}$ are identically distributed, we get using the above equation,

$$\int_0^\infty [R(u)]^n r(u)\bar{F}(u+z)du = n\int_0^\infty [R(u)]^{n-1} r(u)\bar{F}(u+z)du, z \ > 0.$$

Substituting the identity

$$n\int_0^\infty [R(u)]^{n-1} r(u)\bar{F}(u+z)du = \int_0^\infty [R(u)]^n f(u+z)du$$

in (8.3.8), we get on simplification

$$n\int_0^\infty [R(u)]^{n-1} r(u)\bar{F}(u+z)\left[1-\frac{r(u+z)}{r(u)}\right]du = 0, \ z \ > 0.$$

Thus if $F \in C_2$, then the above equation is true: if for almost all u and any fixed $z > 0$,

$$r(u+z) = r(u).$$

The constant hazard rate is the well known characterization property of the exponential distribution.

Exercise 8.3.1 (*solution*) If $X_k \in E(0, \sigma)$, then it can easily be seen that $E(I_{n-1,n}) = E(X_k)$. Suppose that for some foxed n, n > 1, $E(I_{n-1,n}) = E(X_k)$, then we must have

$$\int_0^\infty \int_0^\infty \frac{[R(u)]^{n-1}}{\Gamma(n)} f(u)\bar{F}(u))^{-1}dudx = \int_0^\infty \bar{F}(u)du$$

But we know

$$\Gamma(n) = \int_0^\infty [R(u)]^{n-1} f(u)du$$

0 m simplifying, we obtain

$$\bar{F}(u+z) = \bar{F}(u)\bar{F}(z), \tag{8.3.37}$$

for all u,z, 0 < u, z < ∞. Now the continuous solution of (8.3.3) with the boundary conditions $\bar{F}(0) = 1$ and $F(\infty) = 0$, is $\bar{F}(x) = \exp\{-x\sigma^{-1}\}$, where σ is an arbitrary real number.

Exercise 8.3.2 (*solution*) The "if" condition is easy to establish. We will prove here the "only if" condition. Now

$$\mathrm{b} = \mathrm{E}(\mathrm{I}_{n-1,n}^2|\mathrm{X(n-1)}| - \left[\mathrm{E}(\mathrm{I}_{\mathrm{n-1,n}}|\mathrm{X(n-1) = x})\right]^2.$$

Also

$$\mathrm{E}(\mathrm{I}_{n-1,n}^2|\mathrm{X(n-1) = x})| = \int_0^\infty z^2[\bar{F}(x)]^{-1} d\bar{F}(z+x) = 2\int_0^\infty z[\bar{F}(x)]^{-1}\bar{F}(z+x)dz$$

and

$$\mathrm{E}(\mathrm{I}_{\mathrm{n-1,\,n}}|\mathrm{X(n-1) = x}) = \int_0^\infty z(\bar{F}(x))^{-1} d\bar{F}(z+x) = \int_0^\infty (\bar{F}(x))^{-1}\bar{F}(z+x)dz$$

Substituting G(x) = $\int_0^\infty z\bar{F}(z+x)dz$ and denoting $\mathrm{G}^{(\mathrm{r})}(\mathrm{x})$ as the rth derivative of G(x), we have on simplification

$$G^{(1)}(x) = \int_0^\infty \bar{F}(z+x)dz,\ G^{(2)}(x) = \bar{F}(x)\ and\ G^{(3)}(x) = -f(x).$$

Writing in terms of G(x) and G(r)(x), we obtain

$$\mathrm{2G(x)\{G(r)(x)\} - 1 - \left\{\ G^{(1)}(x)\left(G^{(2)}(x)\right) - 1\right\}2 = b,\ \text{for all } x > 0.}$$

Differentiating the above equation with respect to x and simplifying, we obtain

$$2G^{(3)}(x)\left\{G^{(2)}(x)\right\}^{-3} - \left[\left(G^{(1)}(x)\right)^2 - G(x)\ G^{(2)}(x)\right] = 0$$

Since $G^{(3)}$ (x) ≠ 0 for all x > 0, we must have

$$\left\{G^{(1)}(x)\right\}^2 - G(x)\ G^{(2)}(x) = 0,$$

i.e.

$$\frac{d}{dx}\{G(x)\left(G^{(1)}(x)\right)^{-1}\} = 0, \text{ for all } x > 0.$$

The solution of the above equation is

$$G(x) = a\ e^{-cx},\ x > 0$$

where a and c are arbitrary constants. Hence

$$\bar{F}(x) = G^{(2)}(x) = \ ac^2e^{-cx},\ x > 0.$$

Since F(x) is a distribution function with F(0) = 0, it follows that

$$\bar{F}(x) = e^{-x\sigma^{-1}},$$

where σ is an arbitrary real positive number.

Exercise 8.3.4 (*solution*) If $X_k \in E(0, \sigma)$, then it can easily be shown that $r_1 = r$. Suppose $r_1 = r$, then we can write the joint pdf of X(n) and $X(n–1)_{-1}$ as

$$f_{n-1,n}(x, y) = \frac{1}{\Gamma(n)}[R(x)]^{n-1}r(x)f(y),\ 0 < x < y < \infty,$$
$$= 0,\ \text{otherwise}.$$

Substituting $I_{n-1,n} = X(n) - X(n-1)_1$ and $U = X(n-1)$, we get the pdf of $I_{n-1,n}$ and U as

$$f_1^*(z, u) = \frac{1}{\Gamma(n)}[R(u)]^{n-1}r(u)f(u + z),\quad 0 < x < y < \infty,$$
$$= 0,\quad \text{otherwise}.$$

Thus we can write

$$r_1(z) = \frac{\int_0^\infty [R(u)]^{n-1} r(u) f(u+z) du}{\int_0^\infty [R(u)]^{n-1} r(u) \bar{F}(u+z) du}.$$

for all z ≥ 0. Since $r_1(z) = r(z)$ for all z, we must have

$$\frac{\int\limits_0^\infty [R(u)]^{n-1} r(u) f(u+z) du}{\int\limits_0^\infty [R(u)]^{n-1} r(u) \bar{F}(u+z) du} = \frac{f(z)}{\bar{F}(z)}$$

for all z ≥ 0. Now simplifying the above equation, we obtain

$$\int\limits_0^\infty [R(u)]^{n-1} r(u) \bar{F}(z) \bar{F}(u+z)[r(u+z) - r(z)] du = 0$$

for all z ≥ 0. Since F belongs to class C_2, for the above equation to be true, we must have

$$r(u+z) = r(u)$$

for all z ≥ 0 and almost al u, u ≥ 0.

Exercise 8.3.5 (*solution*) To proof the exercise we need the following two lemmas.

Lemma 8.3.1 *Suppose* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with d. f.* F *and* F(0) = 0. *Let* N *is the r.v. defined as* $N = \min\{i > 1: X_i < X_1\}$. *It can easily be shown that* $P(N = n) = \frac{1}{n(n-1)}$, $n = 2, 3, \ldots,$

Lemma 8.3.2

$$P(NX_N > x) = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} (\bar{F}(\frac{x}{n}))^n, \text{ for all } x \geq 0.$$

Proof

$$
\begin{aligned}
P(NX_N > x) &= \sum_{n=2}^{\infty} P(NX_N > x, N > n) \\
&= \sum_{n=2}^{\infty} \int_{-\infty}^{\infty} P(nX_n > x, X_n < y, X_i > y)|\mathrm{X}_1 = \mathrm{y})\mathrm{d}\ \mathrm{f(y)},\ \text{for all i} = 2, 3, \ldots, n-1, \\
&= \sum_{n=2}^{\infty} \int_{\frac{x}{n}}^{\infty} P(\frac{x}{n} < x_n < y)(P(x_i > y))^{n-2} dF(y) \\
&= \sum_{n=2}^{\infty} \int_{\frac{x}{n}}^{\infty} \{\bar{F}(\frac{x}{n}) - \bar{F}(y)\}\{\bar{F}(y)\}^{n-2} dF(y) \\
&= \sum_{n=2}^{\infty} \left[\bar{F}(\frac{x}{n}) \frac{[\bar{F}(x)]^{n-1}}{n-1} - \frac{[\bar{F}(\frac{x}{n})]^n}{n} \right] \\
&= \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \left[\bar{F}(\frac{x}{n}) \right]^n .
\end{aligned}
$$

Exercise 8.3.1 (*solution*) Define $\mathrm{u(x)} = -\frac{\ln \bar{F}(x)}{x}$, $\mathrm{x} > 0;\ \mathrm{u(0)} = \mathrm{u(0+)}$ and suppose that $NX_N \stackrel{d}{=} X_1$.

Then

$$
\sum_{n=2}^{\infty} \frac{1}{n(n-1)} e^{-xn(x/n)} = e^{-xn(x)},\ \mathrm{x} > 0.
$$

We shall show that the above holds iff u(x) is a constant, i.e. given any T > 0

$$
\min_{x \in [0,T]} U(x) = \min_{x \in [0,T]} U(x).
$$

Let

$$
\begin{aligned}
a_0 &= \min_{x \in [0,T]} U(x), x_0 = \inf\{x \in [0,T] | u(x) = a_0\}, \\
a_1 &= \min_{x \in [0,T]} U(x), x_1 = \inf\{x \in [0,T] | u(x) = a_1\}.
\end{aligned}
$$

It is obvious that (8.3.25) will be proved if we show that $x_0 = 0 = x_1$. By continuity of u, $x_0 \in [0,T]$ and $u(x_0) = a_0$.

Hence

$$u(x_0) \le u(x_0/n) \ \text{for all } n > 1.$$

If equality holds for all n ≥ 2, then $u(x_0) = u(0)$ which by definition of $x_0 = 0$. Suppose now that $x_0 > 0$ (so that $x_0/n \neq x_0$ for all n. Then, the strict inequality must hold for at least one value of n > 1 ≥ 2. Now

$$e^{-x_0 u(x_0)} - \sum_{n=2}^{\infty} \frac{1}{n(n-1)} e^{-x_0 u(x_0/n)} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} e^{-x_0 u(x_0)} - e^{-x_0 u(x_0/n)}\} > 0,$$

which contradicts. Therefore $x_0 = 0$. Similarly $x_1 = 0$. Thus $NX_N \overset{d}{=} X_1 \Rightarrow u(x) \equiv \text{constant}$.

Chapter 9
Asymptotic Distributions of Records

9.1 Limit Behavior of Record Times

Above (see Chap. 3) it was described the asymptotic (as n $\to \infty$) behavior of numbers of records $N(n)$ among random variables X_1, $X_2, \ldots$. X_n. Don't forget that we suppose that X's are independent and have the same continuous distribution function F. It appears that applying to $N(n)$ classical limit theorems for sums of independent random variables immediately allows to obtain the following statements (see relations (3.4.1)–(3.4.5)):

$$\sup_x |P\{N(n) - \log n < x\sqrt{\log n}\} - \Phi(x)| \to 0, \tag{9.1.1}$$

where

$$\Phi(\mathrm{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt;$$

$$\sup_x |P\{N(n) - \log n < x\sqrt{\log n}\} - \Phi(x)| \le C/\sqrt{\log n}, n = 1, 2, \ldots,$$

C being some absolute constant;

$$P\{\lim(N(n)/\log n) = 1\} = 1;$$

$$P\left\{limsup \frac{N(n) - \log n}{(2\log n \log\log\log n)^{1/2}} = 1\right\} = 1$$

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Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_9

and

$$P\left\{ liminf \frac{N(n) - \log n}{(2 \log n \log \log \log n)^{1/2}} = -1 \right\} = 1.$$

To study the asymptotic behavior of record times $L(n)$ one needs to recall that random variables $L(n)$ and $N(m)$ are tied by the following equality:

$$P\{L(n) > m\} = P\{N(m) < n\}, \tag{9.1.2}$$

which holds for any $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$.

Due to this equality the most part of the limit theorems given for $N(n)$, were overworked (see, for example, Renyi (1962)) into the corresponding theorems for record times $L(n)$.

(a) *Central Limit Theorem* (*CLT*) *for* $L(n)$.

It appears that in this case *CLT* is formulated not for $L(n)$ but for its logarithm $\log L(n)$:

$$\sup_x |P\{\log L(n) - n < x\sqrt{n}\} - \Phi(x)| \to 0, n \to \infty. \tag{9.1.3}$$

Note that

$$P\{\log L(n) - n \leq x\sqrt{n}\} = P\{L(n) \leq \exp\{n + x\sqrt{n}\}\}.$$

For the sake of simplicity we will suppose that

$$R(n,x) = \exp\{n + x\sqrt{n}\}$$

is the entire number.

Then we get from (9.1.2) that

$$\begin{aligned} P\{\log L(n) - n \leq x\sqrt{n}\} &= 1 - P\{L(n) > R(n,x)\} = 1 - P\{N(R(n,x)) < n\} \\ &= 1 - P\{N(R(n,x)) - \log(R(n,x)) < n - \log(R(n,x))\} \\ &= 1 - P\{N(R(n,x)) - \log(R(n,x)) < -x\sqrt{n}\} \\ &= 1 - P\{N(R(n,x)) - \log(R(n,x)) < -x(\log R(n,x))^{1/2}(n^{1/2}/(\log R(n,x))^{1/2})\}. \end{aligned}$$

By noticing that $(n^{1/2}/(\log R(n,x))^{1/2}) \to 1$ for any fixed x, as $n \to \infty$, we see that

$$P\{N(R(n,x)) - \log(R(n,x)) < -x(\log R(n,x))^{1/2}(n^{1/2}/(\log R(n,x))^{1/2})\}$$

behaves as

$$P\{N(R(n,x)) - \log(R(n,x)) < -x(\log R(n,x))^{1/2}\}.$$

Now, taking into account relation (9.1.1), one gets that these probabilities are approximated by $\Phi\,(-x)$. Finally it appears that

$$1 - P\{N(R(n,x)) - \log(R(n,x)) < n - \log(R(n,x))\}.$$

behaves asymptotically as $1 - \Phi\,(-x) = \Phi\,(x)$. It proves relation (9.1.3). The same approach based on equality (9.1.2) allowed Renyi (1962) to work over the corresponding limit theorems for $N(n)$ into the following results for record times $L(n)$:

(b) *Strong Law of Large Numbers*:

$$P\{\lim \log L(n)/n = 1\} = 1; n \to \infty. \tag{9.1.4}$$

(c) *Law of Iterative Logarithm*:

$$P\left\{ limsup \frac{\log L(n) - n}{(2n\ \log\log n)^{1/2}} = 1 \right\} = 1, \quad n \to \infty, \tag{9.1.5}$$

and

$$P\left\{ liminf \frac{\log L(n) - n}{(2n\ \log\log n)^{1/2}} = -1 \right\} = 1, \quad n \to \infty. \tag{9.1.6}$$

Let us recall now the kth record times $L(n, k)$ and the numbers $N(n, k)$ of the kth records among random variables $X_1, X_2, \ldots, X_n$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$, which were determined in Chap. 3.

Exercise 9.1.1 Write the generalization of equality (9.1.2), which will tie distributions of $N(n, k)$ and $L(n, k)$.

Exercise 9.1.2 It was proved (see Exercise 3.4.1) that for r.v.'s $N(n, k)$ the following relation is valid:

$$\sup_x |P\{N(n,k) - k\log n < x\sqrt{k\log n}\} - \Phi(x)| \to 0,\ n \to \infty, \tag{9.1.7}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \exp(-t^2/2)dt.$

Based on (9.1.7) and on the result of Exercise (9.1.1) get the corresponding generalization of relation (9.1.3) for the kth record times $L(n, k)$.

Remark 9.1.1 It is interesting to recognize that relations (9.1.3)−(9.1.6) stay true if to take there inter-record times $\Delta(n) = L(n) - L(n-1), n = 2, 3, \ldots$, instead of $L(n)$. These surprising results are presented in Neuts (1967), Holmes and Strawderman (1969), Strawderman and Holmes (1970).

There are some simple limit results for the ratios of the record times

$$T_n = L(n+1)/L(n), n = 1, 2, \ldots$$

One of them is formulated in the following exercise.

Exercise 9.1.3 Prove that for any $x > 1$

$$P\{T_n > x\} \to 1/x, \quad n \to \infty. \tag{9.1.8}$$

9.2 Limit Behavior of Record Values

Since $X(n) = \max\{X_1, \ldots, X_{L(n)}\} = M(L(n))$, the limit distributions of record values must be close to the analogous distributions of maximal order statistics. As it is known (see Chap. 2) there are three types of asymptotic distributions for the suitable centering and normalizing maxima

$$(M(n) - b(n))/a(n).$$

For the sake of simplicity denote the standard representatives of this distribution types as

$$\Lambda(x) = \exp(-\exp(-x)),$$
$$\Phi_\alpha(x) = \begin{cases} 0, & if \quad x < 0, \\ \exp(-x^{-\alpha}), & if \quad x > 0 \end{cases}; \quad \alpha > 0,$$

and

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & if \quad x < 0, \\ 1, & if \quad x > 0 \end{cases}, \alpha > 0.$$

Indeed, under the corresponding **random** centering and normalizing we will obtain the same limit distributions for random variables

$$(X(n) - b(L(n)))/a(L(n)).$$

The following question arises: *what types of asymptotic distributions can one get for nonrandomly normalized record values*? That is, what types of the limit distribution functions $T(x)$, where

$$T(x) = \lim P\{X(n) - B(n) < xA(n)\}, n \to \infty, \tag{9.2.1}$$

can be obtained under the corresponding choice of the normalizing and centering constants $A(n) > 0$ and $B(n)$?

Let us consider firstly the situation when $X_1, X_2, \ldots$ have the standard exponential ***E*(1)-**distribution. In Exercise 2.4.1 it was obtained that in this case

$$P\{X_{n,n} - \log n < x\} \to \Lambda(x), \quad n \to \infty. \tag{9.2.2}$$

Hence, we also get that

$$\lim P\{X(n) - \log L(n) < x\} = \Lambda(x), \quad n \to \infty. \tag{9.2.3}$$

In the next exercise let us consider the asymptotic distribution for these $X(n)$, when they are **nonrandomly** normalized. Below, as usual, we will denote the distribution function of the standard $N(0, 1)$ normal distribution as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt.$$

Exercise 9.2.1 Taking into account Representation 4.2.1 for the exponential record values $X(n)$ show that

$$P\{X(n) - n < xn^{1/2}\} \to \Phi(x), \quad n \to \infty. \tag{9.2.4}$$

Now let us consider the general situation.

Tata (1969) (see also Resnick (1973a)) proved that all possible limit distribution functions $T(x)$ in (9.2.1) have (up to linear transformations) the form

$$T(x) = \Phi(g_k(x)), \ k = 1, 2, 3, \tag{9.2.5}$$

where

$$g_1(x) = x;$$
$$g_2(x) = \gamma \log x, \ \gamma > 0, \textit{if } x > 0, \ \textit{and } g_2(x) = -\infty, \ \textit{if } x < 0;$$
$$g_3(x) = -\gamma \log(-x), \ \gamma > 0, \textit{if } x < 0, \ \textit{and } g_3(x) = \infty, \ \textit{if } x > 0.$$

Remark 9.2.1 We see that in Exercise 9.2.1 the limit distribution function $T(x)$ can be expressed as

$$T(x) = \Phi(g_1(x)).$$

Let us recall now the kth record times $X(n, k)$, which present the natural generalizations of the classical record values $X(n) = X(n, 1)$. For the case, when X's have the standard $E(1)$-distribution, there is Representation 4.6.1, from which it follows that in this situation

$$X(n,k) \stackrel{d}{=} (v_1 + v_2 + \cdots + v_n)/k, \quad k = 1, 2, \ldots, n$$

where v_1, v_2,…, are independent random variables having the exponential $E(1)$-distribution. From here one gets for the exponential X's that

$$X(n,k) \stackrel{d}{=} X(n,1)/k = X(n)/k \tag{9.2.6}$$

Exercise 9.2.2 Formulate and prove the analogue of Exercise 9.2.1 for the kth records generating by the standard exponential $E(1)$-distribution.

Remark 9.2.2 In Chap. 4 (see Exercise 4.6.3) we considered two sequences of independent random variables: $X_1, X_2, \ldots$ with a continuous distribution function F, and $Y_1 = \min\{X_1, \ldots, X_k\}, Y_2 = \min\{X_{k+1}, \ldots, X_{2k}\}, \ldots,$ having distribution function $T(x) = 1 - (1 - F(x))^k$. There were introduced the kth record values $X(n, k)$ based on $X_1, X_2, \ldots$, and the usual ($k = 1$) record values $Y(n)$, constructed with the help of the sequence $Y_1, Y_2, \ldots$. It was shown that for any $k = 1, 2, \ldots$ and any $n = 1, 2, \ldots$ the following equality in distribution is valid:

$$X(n,k) \stackrel{d}{=} Y(n,1). \tag{9.2.7}$$

It follows from (9.2.7) that for any $k = 2, 3, \ldots$ the set of all possible limit distributions for the suitably normalized kth record values also coincides with set (9.2.5) of the limit distributions for the classical ($k = 1$) record values.

9.3 Asymptotic Behavior of Discrete Records

Let us discuss some problems connected with record values in the sequences of discrete X's. Practically, without loss of generality, we can deal only with sequences of independent identically distributed $X, X_1, X_2, \ldots$, taking values 0, 1, 2, … with positive probabilities

$$p_n = P\{X = n\}, n = 0, 1, 2, \ldots$$

Denote

$$q_n = P\{X \geq n\}, n = 0, 1, 2, \ldots$$

and consider record values $X(n)$ in the sequence X_1, X_2,….

Let us introduce also random indicators η_n, $n = 0, 1, 2,\ldots$, such that $\eta_n = 1$, if n is a record value in our sequence, and $\eta_n = 0$, if the sequence of records $X(1)$, $X(2),\ldots$ does not contain this value. As it was showed in Shorrock (1972b), these indicators are independent. Moreover the probabilities

$$r_n = P\{\eta_n = 1\} = 1 - P\{\eta_n = 0\}$$

are expressed as

$$r_n = p_n/q_n,\ n = 0, 1, 2, \ldots.$$

It is evident that in this situation

$$P\{X(n) > m\} = P\{\eta_0 + \eta_1 + \ldots + \eta_m < n\}, m = 0, 1, 2, \ldots, n = 1, 2, \ldots. \tag{9.3.1}$$

Indeed, on the RHS of (9.3.1) one can see a sequence of independent indicators with expectations $E\eta_n = r_n$ and variances $Var\eta_n = r_n(1 - r_n), n = 0, 1, 2\ldots$ Hence, under some specific conditions on probabilities r_0, r_1,… one obtains the asymptotical normality of sums

$$(\eta_0 + \eta_1 + \ldots + \eta_m - a_m)/b_m, m \to \infty,$$

where

$$a_m = r_0 + r_1 + \ldots + r_m,$$
$$b_m = (r_0(1 - r_0) + r_1(1 - r_1) + \ldots + r_m(1 - r_m))^{1/2}, m = 0, 1, 2, \ldots, \tag{9.3.2}$$

and then this result allows to prove the normality (under $n \to \infty$) of the suitably centered and normalized record values $X(n)$. Indeed, since indicators η_n are bounded random variables, the condition $b_m \to \infty$ provides the asymptotical normality of sums $(\eta_0 + \eta_1 + \ldots + \eta_m - a_m)/b_m$. Then it is possible to state the asymptotical normality of $(X(n)-A(n))/B(n)$ under the suitable choice of constants $A(n)$ and $B(n)$. The matter is that really in the general situation one can't express $A(n)$ and $B(n)$ via the initial probabilities r_0, r_1,…. Hence the case of the geometric distributions is rather interesting and important because it allows to present the limit distribution of $X(n)$ in the simplest form.

Let us consider the situation of the geometric distribution with probabilities

$$p_n = P\{X = n\} = (1-p)p^{n-1}, n = 1, 2, \ldots. \tag{9.3.3}$$

In this case

$$r_n = (1-p), n = 1, 2, \ldots.$$

Exercise 9.3.1 Show that for X's having geometric distribution (9.3.3) the following asymptotical relation holds for any $-\infty < x < \infty$:

$$P\{(1-p)X(n) - n < x(np)^{1/2}\} \to \Phi(x), n \to \infty. \tag{9.3.4}$$

9.4 Exercises (solutions)

Exercise 9.1.1 (*answer*) For any $k = 1, 2, \ldots$, $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ the following equality holds:

$$P\{L(n,k) > m\} = P\{N(m,k) < n\}.$$

Exercise 9.1.2 (*answer*) For any $k = 1, 2, \ldots$ the following limit relation is valid:

$$\sup_x |P\{k \log L(n,k) - n < x\sqrt{n}\} - \Phi(x)| \to 0, \quad n \to \infty.$$

Exercise 9.1.3 (*solution*) We see that

$$P\{T_n > x\} = P\{L(n+1) > xL(n)\} = \sum_{i=n}^{\infty} P\{L(n+1) > xL(n) | L(n) = i\} P\{L(n) = i\}$$
$$= \sum_{i=n}^{\infty} P\{L(n+1) > [xi] | L(n) = i\} P\{L(n) = i\}.$$

Here $[xi]$ denotes the entire part of xi. It follows from (3.5.15) that

$$P\{L(n+1) > m | L(n) = i)\} = i/m, m = i, i+1, \ldots.$$

Thus,

$$P\{T_n > x\} = \sum_{i=n}^{\infty} (i/[xi])P\{L(n) = i\}.$$

Taking into account relations

$$1/\mathrm{x} \le \mathrm{i}/[\mathrm{xi}] < 1/\mathrm{x} + 1/\mathrm{x}[\mathrm{xi}]$$

and

$$\sum_{i=n}^{\infty} \mathrm{P}\{\mathrm{L}(\mathrm{n}) = \mathrm{i}\} = 1,$$

one gets that

$$1/x \le \sum_{i=n}^{\infty} (i/[xi])P\{L(n) = i\} < 1/x + \sum_{i=n}^{\infty} (1/x[xi])\ P\{L(n) = i\} < 1/x + 1/x[xn], \tag{9.4.1}$$

and it follows now from (9.4.1) that

$$P\{T_n > x\} \to 1/x, \quad n \to \infty.$$

Exercise 9.2.1 (*hint*) In this situation (see Representation 4.2.1)

$$X(n) \overset{d}{=} S_n = v_1 + \cdots + v_n,$$

where v_1, $v_2, \ldots$ are independent random variables, and

$$\mathrm{E}v_k = Var v_k = 1,\ k = 1,\ 2, \ldots$$

Hence it is enough to apply the classical Central Limit Theorem for independent identically distributed summands and to get that

$$P\{X(n) - n < xn^{1/2}\} \to \Phi(x), \quad n \to \infty.$$

Exercise 9.2.2 (*hint and answer*) It is enough to use equality (9.2.6) and the corresponding result for $X(n,\ 1) = X(n)$, which was obtained in the previous Exercise 9.2.1. Then the final relation will have the following form:

$$P\{kX(n,k) - n < xn^{1/2}\} \to \Phi(x), n \to \infty.$$

Exercise 9.3.1 (*solution*) It follows from (9.3.1) that in this case

$$P\{X(n) > m\} = P\{\eta_1 + \cdots + \eta_m < n\}, m = 1, 2, \ldots, n = 1, 2, \ldots,$$

and

$$r_n = P\{\eta_n = 1\} = 1 - P\{\eta_n = 0\} = p_n/q_n = (1-p), n = 1, 2, \ldots$$

Applying the classical Central Limit Theorem for independent identically distributed random variables having expectations $(1 - p)$ and variances $p(1 - p)$ we obtain that

$$P\left\{(\eta_1 + \cdots + \eta_m - m(1-p))/(mp(1-p))^{1/2} < x\right\} \to \Phi(x),\ n \to \infty \quad (9.4.2)$$

Denote

$$a(n,\ x) = (n + x(np)^{1/2})/(1-p).$$

We need to find the limit (under $n \to \infty$) expression for

$$P\{X(n) \le a(n, x)\} = 1 - P\{\eta_1 + \ldots + \eta_{[a(n,x)]} < n\}$$

Substituting $[a(n,\ x)]$ instead of m to the LHS of (9.4.2) one gets that

$$P\{\eta_1 + \ldots + \eta_{[a(n,x)]} < n\} \to \Phi(-x), n \to \infty.$$

Thus,

$$P\{X(n) \le a(n, x)\} \to 1 - \Phi(-x) = \Phi(x), n \to \infty.$$

Chapter 10
Nonclassical Record Schemes

10.1 Records in the F^{α}-Scheme

Really the first record model for sequences of non-identically distributed X's was suggested by Yang (1975). In this scheme records are considered in the sequence

$$Y_k = \max\{X_{k,1}, \ldots, X_{k,n(k)}\}, k = 1, 2, \ldots,$$

where $\{X_{k,j}\}$, $j = 1, 2, \ldots, n(k)$, $k = 1, 2, \ldots$, are i.i.d. random variables with a common continuous distribution function F. Indeed here any Y_k, $k = 1, 2, \ldots$, has a specific distribution function $F_k(x) = F^{n(k)}(x)$, where $n(k)$ is the corresponding integer value. This partial model initiated the appearance of the more general the so-called ***F^{α}*-scheme** (see, for details, Nevzorov (1981, 1985) and books Nevzorov (2000, 2001), Ahsanullah and Nevzorov (2001a)), where independent r.v.'s X_k, $k = 1, 2, \ldots$, have distribution functions

$$F_k(x) = F^{\alpha(k)}(x), \alpha(k) > 0, \ k = 1, 2, \ldots, \tag{10.1.1}$$

and F is any continuous distribution function.

In this generalization of Yang's model coefficients $\alpha(k)$ can take any positive values. The most important property of this scheme is that the record indicators ξ_n, $n = 1, 2, \ldots$ (which mark the appearance of the strong upper record values), defined in Chap. 3, save the independence property and the following equalities are valid:

$$P\{\xi_n = 1\} = \alpha(n)/S(n), n = 1, 2, \ldots. \tag{10.1.2}$$

where $S(n) = (\alpha(1) + \ldots + \alpha(n))$.

Exercise 10.1.1 Let independent random variables $X_1, X_2, \ldots$ have distribution functions of the form (10.1.1). Let $\xi_n = 1$ if X_n is a record value in the sequence X_1,

M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9_10

$X_2,\ldots$ and $\xi_n = 0$, otherwise. Show that for any $n = 2, 3,\ldots$ indicators $\xi_1, \xi_2,\ldots, \xi_n$ are independent.

Really, the joint distributions of record indicators in the F^{α}-scheme are given by the following equality:

$$P\{\xi_{k(1)} = 1,\ \xi_{k(2)} = 1, \ldots, \xi_{k(n)} = 1\} = \prod_{m=1}^{n} \frac{\alpha(k(m))}{S(k(m))},\ 1 \le k(1) < k(2) < \ldots < k(n), \tag{10.1.3}$$

It is interesting also to consider the relationship between record indicators in the F^{α}-scheme and the corresponding maximal values $M(n) = \max\{X_1, X_2,\ldots, X_n\}$, $n = 1, 2,\ldots$.

Exercise 10.1.2 Show that for any $n = 1, 2,\ldots$ random indicators $\xi_1, \xi_2,\ldots, \xi_n$ and maximal value $M(n)$ are independent.

The independence property of record indicators provided the possibility to obtain for the F^{α}-scheme a lot of results for record times and record values, which are analogous to the corresponding results for the "classical" records.

For example, it follows from (10.1.2) that

$$\mathrm{E}\xi_n = p_n \text{ and } Var\xi_n = p_n(1 - p_n),\ n = 1, 2, \ldots,$$

where

$$p_n = \mathrm{P}\{\xi_n = 1\} = \alpha(n)/S(n).$$

Hence it is easy to investigate (under some conditions on probabilities p_n) the limit behavior of numbers of records $N(n)$ and record values $L(n)$.

Exercise 10.1.3 Formulate for the numbers of records $N(n)$ and the record times $L(n)$ in the F^{α}-scheme the results analogous to relations (9.1.1) and (9.1.3), which are valid for the classical records.

As to the asymptotic behavior of the record values in the F^{α}-scheme, it was proved (see, for example, Nevzorov (1995)), that the set of all possible asymptotic distributions of the suitably normalized record values $X(n)$ (under some rather mild restrictions on coefficients $\alpha(1), \alpha(2),\ldots$) consists of the same three (given in Chap. 9.2) limit distributions

$$T(x) = \Phi(g_k(x)),\ k = 1, 2, 3,$$

where

$$g_1(x) = x;$$
$$g_2(x) = \gamma \log x,\ \gamma > 0,\ \textit{if}\ x > 0,\ \text{and}\ g_2(x) = -\infty,\ \textit{if}\ x < 0;$$
$$g_3(x) = -\gamma \log(-x),\ \gamma > 0,\ \textit{if}\ x < 0,\ \text{and}\ g_3(x) = \infty,\ \textit{if}\ x > 0.$$

10.2 Linear Draft Model

One more simplest way to get a nonstationary record scheme is to take a sequence of independent X's with any common distribution function F and then construct a new sequence $Y_n = X_n + c(n)$, $n = 1, 2, \ldots$, where constants $c(n)$ provide a non-random trend. The most natural in this construction is the situation when $c(n) = cn$, $n = 1, 2, \ldots$, where c is some constant. Indeed, it is clear that if $c < 0$ then the number of records in the sequence $Y_1, Y_2, \ldots$ is finite with probability 1. If $c = 0$ one gets the well-known classical record model. Hence really this scheme is interesting for $c > 0$. This model is called the **Linear Draft** record model. This record scheme and some its generalizations were investigated in many papers. See, for example, the works by Foster and Teichroew (1955), Ballerini and Resnick (1985, 1987), de Haan and Verkade (1987), Smith (1988), Nagaraja (1994a).

Let us consider the sequence

$$Y_n = X_n + cn,\ n = 1, 2, \ldots,$$

where $c > 0$. In this case the distributions of record indicators ξ_n are given as

$$p_n = \mathrm{P}\{\xi_n = 1\} = 1 - \mathrm{P}\{\xi_n = 0\} = \int_{-\infty}^{\infty} \prod_{j=1}^{n-1} F(x + cj)\, dF(x). \qquad (10.2.1)$$

Exercise 10.2.1 Let now $F(x)$ in this model be the Gumbel distribution function:

$$F(x) = e^{-e^{-x}},\ -\infty < x < \infty. \qquad (10.2.2)$$

Find for this case probabilities p_n, $n = 1, 2, \ldots$.

It is interesting that only for the case, when X's in the Linear Draft model have the Gumbel distribution, record indicators $\xi_1, \xi_2, \ldots$ are independent. Note also that in this case

$$P\{Y_n < x\} = (F(x))^{\alpha(n)}, \qquad (10.2.3)$$

where $\alpha(n) = \exp(cn)$, $n = 1, 2, \ldots.$, that is we deal here with the partial case of F^{α}-scheme.

It is not difficult to see that probabilities p_n, given in (10.2.1), decrease: $p_1 > p_2 > \ldots.$ Hence there exists the limit

$$p = \lim_{n\to\infty} p_n.$$

This limit p is named the *limiting record rate*. For example, Ballerini and Resnick (1985) showed that if $c = 1$ and $F(x) = \Phi(x)$, then $p = 0.72506\ldots.$

It was shown (see Nagaraja (1994a)) that if $c > 0$, then p is positive if and only if

$$\int_0^\infty (1 - F(x))dx < \infty.$$

Note also that the limiting record rate $\boldsymbol{p}$ in this model appears in some limit theorems for the numbers of records $N(n)$ (see, Ballerini and Resnick (1985)). For example, the following relations are valid for the Linear Draft model:

$$E(N(n)/n) \to p \text{ and } E(N(n)/n - p)^2 \to 0, \text{ as } n \to \infty.$$

10.3 δ-Exceedance Record Scheme

This scheme was suggested by Balakrishnan et al. (1997). A new observation X_j is called *δ-exceedance upper record value* if it is larger than the previous record by a prefixed positive quantity $\delta > 0$. In this model $L(1) = 1$ and

$$L(n+1) = \min\left\{j | j > L(n),\ X_j > X_{L(n)} + \delta\right\}, n = 1, 2, \ldots.$$

The sequence $\{X_{L(n)}, n = 1, 2, \ldots.\}$ forms the ***δ-exceedance* upper record** scheme. If we have the sequence of i.i.d. X's having some continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$, then the joint pdf of the first m upper records $X(1), X(2), \ldots, X(m)$ is given in this scheme by equalities

$$f_{X(1),X(2),\ldots,X(m)}(x_1, \ldots, x_m) = \frac{f(x_1)}{1 - F(x_1 + \delta)} \cdots \frac{f(x_{m-1})}{1 - F(x_{m-1} + \delta)} f(x_m), x_j > x_{j-1} + \delta,$$
$$j = 2, 3, \ldots, m. \tag{10.3.1}$$

Exercise 10.3.1 Consider the exponential $E(1)$ distribution with $F(x) = 1 - \exp(-x)$, $x \geq 0$ and write the expression for the joint pdf of $X(1), X(2),\ldots, X(m)$. Show that the marginal pdf of $X(m)$, m $= 1, 2, \ldots$, is given as

$$f_{X(m)}(x) = \frac{1}{\Gamma(m)}(x - m\delta)^{m-1}e^{-(x-m\delta)}, x > m\delta. \tag{10.3.2}$$

We can see from (10.3.2) that $X(m) - m\delta$ has the gamma distribution. Moreover, in this situation the form of the joint pdf allows us to state that for any $m = 1, 2,\ldots$ the vector $\{X(1), X(2),\ldots, X(m)\}$ have the same distribution as the vector $\{Y_1, Y_1 + Y_2,\ldots, Y_1 + Y_2 + \ldots + Y_m\}$, where Y_i's are independent and have the exponential distribution with $F(x) = 1 - e^{-(x-\delta)}$, $x > \delta$. From here it follows that in this situation inter record values $X(m) - X(m - 1)$, $m = 2, 3,\ldots$, are independent and have the same exponential distribution.

10.4 Records with Restrictions I

The next object of our consideration is the so-called ***records with restrictions***, which are rather close in some sense to the previous *δ-exceedance* record scheme. Consider the sequence of independent identically distributed random variables X_1, X_2, ... and fixed some positive constant C. We take $X(1) = X_1$ and $L(1) = 1$ as the first record value and the first upper record time correspondingly. The next record times $L(n)$ and record values $X(n)$ are defined as follows:

$$\mathrm{L(n) = \min\{j > L(n-1) \,:\, X(n-1) < X_j \leq X(n-1) + C\},\ X(n) = X_{L(n)},\ n = 2, 3, \ldots.} \tag{10.4.1}$$

It means that the new coming observation is ignored as a record, if it exceeds essentially the previous record value.

Suppose that the initial r.v.'s have a common pdf $f(x)$. It is not difficult to get the expression for the conditional pdf $f_n(x_n|x_1, x_2,\ldots, x_{n-1})$, that is the expression for the pdf of $X(n)$ under condition that $X(1) = x_1$, $X(2) = x_2,\ldots, X_{n-1} = x_{n-1}$, where $0 < x_j - x_{j-1} \leq C$, $j = 2, 3,\ldots, n-1$, are fixed. One obtains in this case that

$$f_n(x_n|x_1, x_2, \ldots, x_{n-1}) = f(x_n)/(F(x_{n-1} + C) - F(x_{n-1})), \tag{10.4.2}$$

if $x_{n-1} < x_n \leq x_{n-1} + C$, and $f_n(x_n|x_1, x_2,\ldots, x_{n-1}) = 0$, otherwise.

These equalities allow us to find the corresponding relations for probability density functions $f_{n-1}(x)$ and $f_n(x)$ of record values $X(n-1)$ and $X(n)$:

$$f_n(x) = f(x)\int_{x-C}^{x}(f_{n-1}(u)/(F(u + C) - F(u)))du,\ \mathrm{n} = 2, 3, \ldots. \tag{10.4.3}$$

From (10.4.3) one gets that the joint pdf $f_n(x_1, x_2,\ldots, x_n)$ of record values $X(1), X(2),\ldots, X(n)$ has the following form:

$$f_n(x_1,x_2,\ldots,x_n) = f(x_1)\frac{f(x_2)}{(F(x_1+C)-F(x_1))}\cdots\frac{f(x_n)}{(F(x_{n-1}+C)-F(x_{n-1}))}, \tag{10.4.4}$$

$$\textit{if } x_1<x_2\leq x_1+C, x_2<x_3\leq x_2+C, \ldots, x_{n-1}<x_n\leq x_{n-1}+C,$$

and $f_n(x_1, x_2,\ldots, x_n) = 0$, otherwise.

These relations can be simplified essentially for the case of exponential distributions of X's.

Exercise 10.4.1 Let us consider the sequence of X's having the standard exponential $E(1)$-distribution function $F(x) = \max\{0, 1 - \exp(-x)\}$. Define also $X(0) = X_0 = 0$. Apply (10.4.4) to obtain the joint pdf of record values $X(1), X(2),\ldots, X(n)$ with some restriction value $C > 0$.

Show also that in this case the joint probability density function $g(v_1, v_2,\ldots, v_n)$ of the differences $V_1 = X(1) - X(0)$, $V_2 = X(2) - X(1),\ldots, X_n = X(n) - X(n - 1)$ has the form

$$g(v_1,v_2,\ldots,v_n) = \exp\{-(v_1+v_2+\ldots+v_n)\}/(1-\exp(-C))^n, \; 0\leq v_1\leq C,\ldots,0\leq v_n\leq C. \tag{10.4.5}$$

Expression (10.4.5) allows us to state that inter record values $V_1, V_2, \ldots, V_n$ are independent and identically distributed with pdf $g(x)$ given as follows:

$$g(x) = \exp(-x)/(1-\exp(-C)), \text{ if } 0\leq x\leq C, \tag{10.4.6}$$

and $g(x) = 0$, otherwise.

It means that in this situation vectors $\{X(1), X(2),\ldots, X(n)\}$ and $\{V_1, V_1 + V_2,\ldots, V_1 + V_2 + \ldots + V_n\}$, where independent r.v.'s $V_1, V_2,\ldots, V_n$ have pdf (10.4.6), are identically distributed. Thus exponential record value $X(n)$ is presented as the sum of n independent identically distributed random variables.

Note that the means and the variances of $V_1, V_2,\ldots$ are given as

$$a(c) = EV_k = (1-(1+C)e^{-C}/(1-e^{-C}), k = 1,2,\ldots, \tag{10.4.7}$$

and

$$\sigma^2(c) = VarV_k = (1-(2+C^2)e^{-C}+e^{-2C})/(1-e^{-C})^2, k = 1,2,\ldots. \tag{10.4.8}$$

It is possible now to obtain, in particular, different limit theorems for the exponential record values with restrictions. For example, it follows immediately that in this case random variables $(X(n) - na(c))/\sigma(c)n^{1/2}$, where $a(c)$ and $\sigma(c)$ are defined in (10.4.7) and (10.4.8), tend, as $n \to \infty$, to the standard normal distribution.

10.5 Records with Restrictions II

The next scheme is very close to the previous record models. Now we also fix some constant $C > 0$, which determines the acceptable rate of exceeding of the previous record value. The discrepancy of these two schemes is the following. In the second case we do not ignore the observations, which are essentially greater than the previous record value $X(n)$. In this situation we simply determine the new record value $X(n + 1)$ as $X(n) + C$. In the case when the new coming observation belongs to the interval $(X(n), X(n) + C]$ it is announced as the record value $X(n + 1)$.

If we again consider the initial independent $E(1)$—distributed random variables (with pdf $f(x) = \exp(-x)$, $x \geq 0$) $X_1, X_2, \ldots$ and denote the records with such type of restrictions as $0 = X(0, C) < X(1, C) < X(2, C) < \ldots$, then arguments analogous to those, which are given in the previous model, show that differences (inter record values)

$$W_k(C) = X(k, C) - X(k-1, C),\ k = 1, 2, \ldots,$$

are also independent. In this case one gets that

$$\mathrm{P}\{W_k(C) < x\} = 1 - \exp(-x), \text{if } 0 \leq x \leq C, \text{ and } P\{V_k < x\} = 1, \text{if } x > C. \tag{10.5.1}$$

In particular,

$$P\{W_k(C) = C\} = \exp(-C), k = 1, 2, \ldots.$$

One can see that in this record scheme for any $n = 1, 2, \ldots$ the distribution of the vector of record values $\{X(1, C), X(2, C), \ldots, X(n, C)\}$ coincides with the distribution of the vector

$$W_1(C), W_1(C) + W_2(C), \ldots, W_1(C) + W_2(C) + \ldots + W_n(C),$$

where $W_1(C), W_2(C), \ldots$ are independent random variables having the same distribution function (10.5.1).

Exercise 10.5.1 Find expressions for expectations and variances of summands $W_1(C)$, $W_2(C), \ldots$ and formulate the Central Limit Theorem for the exponential record values $X(n, C)$.

10.6 Records with Confirmation

One more nonclassical record scheme—the so called "**records with confirmation**" (confirmed records), was considered in Saghatelyan (2008) and Nevzorov and Saghatelyan (2009). Let us fix some $k = 1, 2,\ldots$. In the simple option of this scheme to determine a new record value $X(n + 1, k)$ one must wait for appearance of k observations which will be greater than the previous record value $X(n, k)$ and only after this event it is possible to determine $X(n + 1, k)$ as the maximal value among these k observations. This scheme is useful in the situation when among X'*s*, which have some fixed distribution, can appear an observation having another distribution (the contamination of the original sample). Taking into account this situation we need to find the distributions of the corresponding record times $L(n, k)$, $n = 1, 2,\ldots$. Note that $L(1, k) = k$.

Consider a random sample $X_1, X_2, \ldots, X_n$ of size n from a population with a continuous distribution function $F(x)$. It is easy to find that the distribution of the minimal number $R(n)$ of additional observations, which are needed to get the first value exceeding $X_{n,n} = \max\{X_1,\ldots, X_n\}$, is given as follows:

$$P\{R(n) > m\} = n/(n+m). \tag{10.6.1}$$

Let now $R(n, k)$ denote the number of observations, which are needed to get exactly k values exceeding $X_{n,n}$.

Exercise 10.6.1 Find the distribution of $R(n, k)$, $n = 1, 2,\ldots$, $k = 1, 2,\ldots$.

Let us consider now more complicate situations, when a new record will coincide with the mth ($1 \le m \le k$) in order observation taken from k random variables, which are greater than the previous record value. The reason why we consider here the case when m can be less than k is rather simple. For example, imagine that we know that the given sample can be contaminated with some outliers and we expect that these outliers are presented in the sample by the "top" observations. Hence it is naturally to delete these extraneous observations from the further consideration.

As above the most interesting case here is connected again with the exponential distribution. The following result is valid for the corresponding exponential record values $X(n, k, m)$.

Theorem 10.6.1 *Let $F(x) = \max\{0,1 - \exp(-x)\}$, and $1 \le m \le k$, defined above, are fixed. Then for any $n = 1, 2,\ldots$ the following equality holds:*

$$X(n+1,k,m) \overset{d}{=} X(n,k,m) + X_1^{(n)}/k + \ldots + X_m^{(n)}/(k-m+1), \tag{10.6.2}$$

where $X(n,k,m)$, $X_1^{(n)}, \ldots, X_m^{(n)}$ are independent and $P\{ X_j^{(n)} < x\} = \max\{0,1 - \exp(-x)\}$, $1 \le j \le m$.

Corollary 10.6.1 *It follows from Theorem 10.6.1, that*

$$X(n,k,m) \overset{d}{=} \mu_1/k + \mu_2/(k-1) + \ldots + \mu_m/(k-m+1), \qquad (10.6.3)$$

where μ_1, μ_2,... *are independent Gamma*(n)*-distributed r.v.'s.*

From this theorem, which is formulated for the exponential record values, it can be obtained (using the standard methods from the theory of records) that a set of all possible types of the limit (as $n \to \infty$) distributions $G(x)$ (under any fixed k = 1, 2,... and m = 1, 2,..., k) for the records with confirmations $X(n, k, m)$ coincides with the corresponding set for the classical records:

$$G_i(x) = \Phi(-ln(-ln\, H_i(x))),\ i = 1,2,3,$$

where

$$H_1(x) = \exp\{-\exp(-x)\}, -\infty < x < \infty;$$
$$H_2(x) = 0, \text{if } x < 0, \text{ and } H_2(x) = \exp\{-x^{-\delta}\}, \text{ if } x > 0, \delta > 0;$$
$$H_3(x) = \exp\{-(-x)^{\delta}\}, \text{if } x < 0, \delta > 0, \text{ and } H_3(x) = 1, \ if\ \delta > 0;$$

and $\Phi(x)$ is the normal $N(0, 1)$-distribution function.

Indeed, the centering and normalizing constants in this situation will differ from the corresponding constants for the classical records and will depend on k and m.

10.7 The Record Scheme of Balabekyan-Nevzorov

One more nonstationary record model was suggested by Balabekyan and Nevzorov (1986). Consider the following situation. Let m athletes of different skill have in succession n starts each. In this case the distribution functions, which correspond to their results $X_1, X_2, \ldots, X_m, X_{m+1}, \ldots, X_{2m}, \ldots, X_{(n-1)m+1}, \ldots, X_{nm}$, form a sequence

$$F_1(x), F_2(x), \ldots, F_m(x), F_1(x), \ldots, F_m(x), \ldots, F_1(x), \ldots, F_m(x),$$

that is, a group of m different distribution functions is repeated n times. Hence one can see that this record scheme contains at the same time elements of stationarity and nonstationarity.

Let $N(nm)$ be the number of records in a sequence $X_1, X_2, \ldots, X_{nm}$. It appears that comparing $N(nm)$ with the number of the classical records in the sequence $Y_1 = \max\{X_1, X_2, \ldots, X_m\}, \ldots, Y_n = \max\{X_{m(n-1)+1}, \ldots, X_{mn}\}$ allows us to obtain the following result.

Theorem 10.7.2 *Let $F_1, F_2, \ldots, F_m$ be continuous distribution functions. Then*

$$\sup_x \left| P\left\{ N(nm) - \log n < x(\log n)^{1/2} \right\} - \Phi(x) \right| \to 0,\ n \to \infty. \tag{10.7.1}$$

Remark 10.7.1 Note that in this theorem m need not to have some fixed value. It may be permitted to increase to a certain degree with n, like

$$m = m(n) = o\left((\log n)^{1/2} \right),\ n \to \infty.$$

10.8 Exercises (solutions)

Exercise 10.1.1 (*solution*) Denote

$$S(n) = \alpha(1) + \alpha(2) + \ldots + \alpha(n),\ M(n) = \max\{X_1, X_2, \ldots, X_n\}$$

and

$$G_n(x) = P\{M(n) < x\} = F_1(x)F_2(x)\ldots F_n(x) = (F(x))^{S(n)}, n = 1, 2, \ldots.$$

One can see that in this situation for any $n = 2, 3, \ldots$ the following equality holds:

$$\begin{aligned} P\{\xi_n = 1\} &= \int_{-\infty}^{\infty} G_{n-1}(x) dF_n(x) = \int_{-\infty}^{\infty} (F(x))^{S(n-1)} d\left(F^{\alpha(n)}(x) \right) \\ &= \int_0^1 x^{S(n-1)} \mathrm{d}\left(\mathrm{x}^{\alpha(n)} \right) = \alpha(n)/S(n). \end{aligned} \tag{10.8.1}$$

Now in order to get the independence of the indicators it suffices to obtain that for any $1 \le k(1) < k(2) < \ldots < k(r)$, $r = 2, 3, \ldots$, we have relations

$$P\left\{ \xi_{k(1)} = 1,\ \xi_{k(2)} = 1, \ldots, \xi_{k(r)} = 1 \right\} = \prod_{m=1}^{r} \frac{\alpha(k(m))}{S(\alpha(k(m)))}. \tag{10.8.2}$$

Recall that the probability integral transformation does not change the order of random variables and hence it does not change the values of the record indicators. Hence for simplicity (without loss of generality) we can take $F(x) = x$, $0 < x < 1$, and $F_k(x) = x^{\alpha(k)}$, $k = 1, 2, \ldots$. Then, taking into account that

$$\begin{aligned}&P\{\xi_{k(1)}=1,\ \xi_{k(2)}=1,\ldots,\ \xi_{k(r)}=1\}\\ =&P\{X_{k(1)}>M(k(1)-1),\ X_{k(2)}>M(k(2)-1),\ldots,\ X_{k(r)}>M(k(r)-1)\},\end{aligned}$$

one gets the relation

$$\begin{aligned}&P\{\xi_{k(1)}=1,\ \xi_{k(2)}=1,\ldots,\ \xi_{k(r)}=1\}\\ &=\int_0^1 v_1^{S(k(1)-1)}d(v_1^{\alpha(k(1))})\int_{v_1}^1 v_2^{S(k(2)-1)-S(k(1))}d(v_2^{\alpha(k(2))})\ldots\int_{v_{r-1}}^1 v_r^{S(k(r)-1)-S(k(r-1))}d(v_r^{\alpha(k(r))}).\end{aligned} \tag{10.8.3}$$

Accurately continue the integration in (10.8.3.) we get relation (10.8.2.).

Exercise 10.1.2 (*solution*) This case is very close to the situation in Exercise 10.1.1. Instead of (10.8.2.) one can verify that for any 1 ≤ k(1) < k(2) < … < k (r) ≤ n, r = 2, 3,…, we have relations

$$P\{\xi_{k(1)}=1,\ \xi_{k(2)}=1,\ldots,\ \xi_{k(r)}=1,\ M(n)<x\}=\prod_{m=1}^{r}\frac{\alpha(k(m))}{S(\alpha(k(m)))}F(x)^{S(n)}. \tag{10.8.4}$$

Indeed, again it is possible to take $F(x) = x$, $0 < x < 1$. Then the LHS of (10.8.4) can be written as

$$\begin{aligned}&P\{\xi_{k(1)}=1,\xi_{k(2)}=1,\ldots,\xi_{k(r)}=1,M(n)<x\}\\ &=\int_0^x v_1^{S(k(1)-1)}d\left(v_1^{\alpha(k(1))}\right)\int_{v_1}^x v_2^{S(k(2)-1)-S(k(1))}d\left(v_2^{\alpha(k(2))}\right)\cdots\int_{v_{r-1}}^x v_r^{S(k(r)-1)-S(k(r-1))}d\left(v_r^{\alpha(k(r))}\right)x^{S(n)}\end{aligned} \tag{10.8.5}$$

Continue the integration in (10.8.5) one gets the necessary equality

$$P\{\xi_{k(1)}=1,\ \xi_{k(2)}=1,\ldots,\ \xi_{k(r)}=1,\ M(n)<x\}=\prod_{m=1}^{r}\frac{\alpha(k(m))}{S(\alpha(k(m)))}x^{S(n)}.$$

Exercise 10.1.3 (*answer*) Denote

$$p_n=P\{\xi_n=1\}=1-P\{\xi_n=0\}=\alpha(n)/S(n),\ n=1,2,\ldots.$$

Then

$$A(n) = EN(n) = \sum_{k=1}^{n} p_k \text{ and } B_n = Var\, N(n) = A(n) - \sum_{k=1}^{n} p_k^2.$$

In this case, if $B_n \to \infty$, then

$$\sup_x |P\{N(n) - A(n) < x\sqrt{B(n)}\} - \Phi(x)| \to 0,\ n \to \infty. \tag{10.8.6}$$

Let additionally the following relation holds:

$$\sum_{k=1}^{n} p_k^2 = o\left(\sum_{k=1}^{n} p_k\right),\ n \to \infty.$$

Then (10.8.6) can be simplified as follows:

$$\sup_x |P\left\{(N(n) - \log S(n)) < x(\log S(n))^{1/2}\right\} - \Phi(x)| \to 0, n \to \infty. \tag{10.8.7}$$

Taking into account the equality

$$P\{L(n) > m\} = P\{N(m) < n\},$$

it is possible to obtain from (10.8.7) that if

$$A(n) \to \infty,\ p_n = \alpha(n)/S(n) \to 0$$

and

$$\sum_{k=1}^{n} p_k^2 = o\left(\sum_{k=1}^{n} p_k\right), n \to \infty,$$

then

$$\sup_x |\mathrm{P}\left\{(\log \mathrm{S}(\mathrm{L}(\mathrm{n})) - \mathrm{n}) < \mathrm{x}\mathrm{n}^{1/2}\right\} - \Phi(x)| \to 0,\ n \to \infty. \tag{10.8.8}$$

The proof of relations (10.8.6)–(10.8.8) and some related results concerning the Central Limit Theorem for $N(n)$ and $L(n)$ can be found in Nevzorov (1986c, 1995).

Exercise 10.2.1 (*solution*) It is enough to see that

$$F_n(x) = P\{Y_n < x\} = F(x - cn) = (F(x))^{\alpha(n)},\ n = 1, 2, \ldots,$$

where $\alpha(n) = \exp(cn)$. Thus random variables $Y_1, Y_2, \ldots$ in this case present the F^{α}-scheme. Hence, the distributions of record indicators ξ_n are given as

$$\begin{aligned} p_n &= P\{\xi_n = 1\} = 1 - P\{\xi_n = 0\} = \exp(cn)/(\exp(c) + \exp(2c) + \ldots + \exp(nc)) \\ &= (1 - \exp(c))\exp(c(n-1))/(1 - \exp(nc)), \ n = 1, 2, \ldots. \end{aligned}$$

Exercise 10.3.1 (*solution*) Taking into account the general expression (10.3.1) and substituting $f(x) = \exp(-x)$ and $1 - F(x) = \exp(-x)$, $x > 0$, in (10.3.1) we get that

$$f_{X(1),\ldots,X(m)}(x_1, \ldots, x_m) = \exp(\delta(m-1))\exp(-x_m), x_1 > 0, x_j > x_{j-1} + \delta, j = 2, 3, \ldots, m. \tag{10.8.9}$$

By fixing any $x_m > m\delta$ and integrating the RHS of (10.8.9) over the domain $\{x_1 > 0, x_2 > x_1 + \delta, \ldots, x_{m-2} + \delta < x_{m-1} < x_m - \delta\}$, one obtains now that

$$f_{X(m)}(x_m) = \frac{1}{\Gamma(m)}(x_m - m\delta)^{m-1} e^{-(x_m - m\delta)}, x_m > m\delta.$$

Exercise 10.4.1 (*answer*) Substituting $f(x) = \exp(-x)$ and $1 - F(x) = \exp(-x)$, $x > 0$, in (10.4.4) one immediately gets that in this partial case

$$f_n(x_1, x_2, \ldots, x_n) = \exp(-x_n)/(1 - \exp(-c))^n, \tag{10.8.10}$$

$$\text{if } x_1 < x_2 \leq x_1 + C, \ x_2 < x_3 \leq x_2 + C, \ldots, \ x_{n-1} < x_n \leq x_{n-1} + C,$$

and $f_n(x_1, x_2, \ldots, x_n) = 0$, otherwise.

It immediately follows from (10.8.10) that the joint probability density function of the differences

$$V_1 = X(1) - X(0), V_2 = X(2) - X(1), \ldots, X_n = X(n) - X(n-1)$$

has the form (10.4.5).

Exercise 10.5.1 (*answers*) In this case

$$\begin{aligned} a(C) &= EW_k(C) = 1 - \exp(-C), \\ \sigma^2(C) &= VarW_k(C) = 1 - 2C\exp(-C) - \exp(-2C), \ k = 1, 2, \ldots. \end{aligned} \tag{10.8.11}$$

Since $W_k(C)$, $k = 1, 2, \ldots$ are independent identically distributed and restricted random values one can state that the following asymptotical relation, where normalizing coefficients are defined in (10.8.11), holds *as* $n \to \infty$:

$$\sup_x |(P(X(n, C) - na(c)) < x\sigma(c)n^{1/2} - \Phi(x)| \to 0.$$

Exercise 10.6.1 (*answer*) For any n = 1, 2,. . ., k = 1, 2,. . ., the following equality holds:

$$P\{R(n,k) > m\} = 1 - m!(n+m-k)!/(m-k)!(n+m)!, \ m \geq k.$$

Compare it with (10.6.1), where the corresponding relation for $R(n)$ = $R(n,1)$ *is given.*

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M. Ahsanullah and V.B. Nevzorov, *Records via Probability Theory*,
Atlantis Studies in Probability and Statistics 6,
DOI 10.2991/978-94-6239-136-9