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Normal and Student's *t* Distributions and Their Applications

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To my wife, Masuda

M. Ahsanullah

To my late parents, Mr. Abdur Rahim Bhuiyan and Mrs. Ayesha Bhuiyan, and my wife, Orchi

B. M. Golam Kibria

To my late parents, Mr. M. A. Rab and Mrs. Humaira Khatoon, and my wife, Nausheen

M. Shakil

Preface

The normal and Student's t distributions are two of the most important continuous probability distributions, and are widely used in statistics and other fields of sciences. The distributions of the sum, product, and ratio of two independent random variables arise in many fields of research, for example, biology, computer science, control theory, economics, engineering, genetics, hydrology, medicine, number theory, statistics, physics, psychology, reliability, risk management, etc. This has increased the need to explore more statistical results on the sum, product, and ratio of independent random variables. The aim of this book is to study the *Normal and Student's t Distributions and Their Applications*. First, the distributions of the sum, product, and ratio of two independent normal random variables, which play an important role in many areas of research, are presented, and some of the available results are surveyed. The distributions of the sum, product, and ratio of independent Student's t random variables, which are of interest in many areas of statistics, are then discussed. The distributions of the sum, product, and ratio of independent random variables belonging to different families are also of considerable importance and one of the current areas of research interest. This book introduces and develops some new results on the distributions of the sum of the normal and Student's t random variables. Some properties of these distributions are also discussed. A new symmetric distribution has been derived by taking the product of the probability density functions of the normal and Student's t distributions. Some characteristics of the new distributions are presented. Before a particular probability distribution model is applied to fit the real-world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in probability and statistics. We have also provided some characterizations of the family of normal and Student's t distributions.

We hope the findings of the book will be useful for the advanced undergraduate and graduate students, and practitioners in various fields of sciences.

As a preparation to study this book, the readers are assumed to have knowledge of calculus and linear algebra. In addition, they need to have taken first courses in probability and statistical theory.

We wish to express our gratitude to Dr. Chris Tsokos for his valuable suggestions and comments about the manuscript, which certainly improved the quality and presentation of the book. The first author thanks Z. Karssen and K. Jones of

Atlantis Press for the interesting discussions at a meeting in Athens, Greece, for the publication of this book. Summer research grant and sabbatical leave from Rider University enabled the first author to complete his part of the work. Part of the book is from the independent study of the third author with Dr. Kibria. The book was partially written while the second author was on sabbatical in 2010–2011, and he gratefully acknowledges the excellent research facilities of Florida International University. The third author is grateful to Miami Dade College for all the support, including STEM grants. Last but not least, the authors would like to express their deep regret for any error or omission or misprint or mistake, which is very likely to occur in any textbook of this type. We have endeavored our best that our book be typo free (which is impossible but our intention). All suggestions in this regard for improvement in the future are welcome, and will be highly appreciated and gratefully acknowledged.

Contents

Chapter 1
Introduction

The normal and Student's t distributions are two of the most important distributions in statistics. These distributions have been extensively studied and used by many researchers since their discoveries. This book reviews the normal and Student's t distributions, and their applications. These studies involve some preliminaries on random variables and distribution functions, which are defined below (in Sect. 1.1), (for details, see Lukacs 1972, Dudewicz and Mishra 1988, Rohatgi and Saleh 2001, Severini 2005, and Mukhopadhyay 2006, among others). Some special functions and mathematical results will also be needed, which, for the sake of completeness, are given below (in Sect. 1.2), (for details, see Abramowitz and Stegun 1970, Lebedev 1972, Prudnikov et al. 1986, and Gradshteyn and Ryzhik 2000, among others).

1.1 Some Preliminaries on Random Variables and Distributions

Definition 1.1.1 (Random Variable): Let $(\Omega,\ T,\ P)$ be a probability space, where $\Omega = \{w\}$ is a set of simple events, T is a σ-algebra of events, and P is a probability measure defined on $(\Omega,\ T)$. Let B be an element of the Borel σ-algebra of subsets of the real line R. A random variable $X = X(w)$ is defined as a finite single-valued function $X : \Omega \to R$ such that $X^{-1}(B) = \{w : X\ (w) \in B\} \in T$, $\forall$ Borel set $B \in R$. Thus, a random variable X is a real-valued function with domain Ω, that is, $X(w) \in R = \{y : -\infty < y < +\infty\}, \ \forall\, w \in \Omega$.

Definition 1.1.2 (Cumulative Distribution Function): Let $B = (-\infty,\ x]$ in the above definition 1.1. Then the cumulative distribution function (cdf) or distribution function (df) of the random variable $X = X\ (w)$ is defined by $F_X\,(x) = P\ [X \le x], \forall x \in (-\infty,\ +\infty)$, with the following properties:

(i) $F_X\,(x)$ is a non-decreasing function of x.
(ii) $F_X\,(-\infty) = 0, \quad F_X\,(+\,\infty) = 1.$
(iii) $F_X\,(x)$ is right continuous.

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Definition 1.1.3 (Absolutely Continuous Distribution Function): The distribution function $F_X(x)$ of a random variable X is said to be absolutely continuous (with respect to Lebesgue measure) if $\exists$ a function $f_X(x) \geq 0$ such that $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$.

Definition 1.1.4 (Probability Density Function): The function $f_X(x)$ in the above definition 1.1.3 is called the probability density function (pdf) or density function of the random variable X if it satisfies the following condition:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Definition 1.1.5 (Moments): If a random variable X has an absolutely continuous (with respect to Lebesgue measure) distribution with a pdf $f_X(x)$, then the nth moment about zero and the nth central moment of X are respectively defined by the following expressions:

$$\alpha_n = E\left(X^n\right) = \int_{-\infty}^{\infty} x^n f_X(x)dx,$$

when

$$E|X|^n = \int_{-\infty}^{\infty} |x|^n f_X(x)dx < \infty.$$

and

$$\beta_n = E[X - E(X)]^n = \int_{-\infty}^{\infty} (x - E(X))^n f_X(x)dx,$$

when

$$E|X - E(X)|^n = \int_{-\infty}^{\infty} |x - E(X)|^n f_X(x)dx < \infty.$$

Note that, in the above definitions, $\alpha_1 = E(X)$, and $\beta_2 = E[X - E(X)]^2$ are respectively called the expected value (or mean or mathematical expectation) and variance of the random variable X.

Definition 1.1.6 (Entropy): An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. As proposed by Shannon (1948), if a random variable X has an absolutely continuous distribution with a pdf $f_X(x)$, then the entropy of X is defined as

$$H_X[f_X(X)] = E[-\ln(f_X(X)] = -\int_{-\infty}^{\infty} f_X(x) \ln [f_X(x)]\, dx.$$

1.2 Some Useful Mathematical Results

The following special functions and mathematical results will be useful in our analyses.

(a) **Special Functions**

(i) The series

$${}_pF_q\left(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z\right) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha_1)_k\ (\alpha_2)_k \cdots \left(\alpha_p\right)_k}{(\beta_1)_k\ (\beta_2)_k \cdots \left(\beta_q\right)_k} \frac{z^k}{k!} \right\},$$

is called a generalized hypergeometric series of order (p, q), where $(\alpha)_k$ and $(\beta)_k$ represent Pochhammer symbols and

$$(x)_k = x(x-1)......(x-k+1), \text{ and } (\mathrm{x})_0 = 1.$$

(ii) For $p = 1$ and $q = 2$, we have generalized hypergeometric function ${}_1F_2$ of order $(1, 2)$, given by ${}_1F_2(\alpha_1; \beta_1, \beta_2; z) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha_1)_k}{(\beta_1)_k (\beta_2)_k} \frac{z^k}{k!} \right\}$.

(iii) For $p = 2$ and $q = 1$, the series given by ${}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) \equiv F(\beta, \alpha; \gamma; z) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha)_k (\beta)_k z^k}{(\gamma)_k k!} \right\}$ is called generalized hypergeometric function ${}_2F_1$ of order $(2, 1)$. Also, we have $F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F\left(\beta, \gamma-\alpha; \gamma; \frac{z}{z-1}\right)$.

(iv) For $p = 2$ and $q = 0$, the function defined by

$$\begin{aligned} \psi(\alpha, \gamma; z) &\equiv z^{-\alpha} {}_2F_0\left(\alpha, 1+\alpha-\gamma; \frac{-1}{z}\right) \equiv U(\alpha, \gamma, z) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt, \quad Re(\alpha) > 0 \end{aligned}, \tag{1.1}$$

is called Kummer's function. Here $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(z) > 0$, (see Prudnikov et al., 1986, volume 3, equation 7.2.2.7, page 435, or Abramowitz and Stegun, 1970, equation 13.2.5, page 505).

(v) The function defined by

$$\Phi_1 (a, b; c; w, z) = \sum_{k,\, l=0}^{\infty} \frac{(a)_{k+l}\, (b)_k}{(c)_{k+l}} \frac{w^k\, z^l}{(k\,!)\,(l\,!)}$$
$$= \Gamma\begin{bmatrix} c \\ a,\ c-a \end{bmatrix} \int_0^1 u^{a-1}\,(1-u)^{c-a-1}(1-u\,w)^{-b}\, e^{u\,z}\, du, \tag{1.2}$$

where Re (a), Re $(c-a) > 0, |w| < 1$, and $(a)_k$ denote the Pochhammer symbol, is called a generalized (or confluent) hypergeometric function of two variables.

(vi) The integrals given by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt,\ \alpha > 0,\ \gamma(\alpha, x) = \int_0^{x} t^{\alpha-1} e^{-t} dt \text{ , and } \Gamma(\alpha,\ x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt,\ \alpha > 0,$$

are called (complete) gamma, incomplete gamma and complementary incomplete gamma functions, respectively. Note that $\Gamma(\alpha,\ x) + \gamma(\alpha, x) = \Gamma(\alpha)$.

(vii) The functions defined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$, and $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du = 1 - erf(x)$, are called error and complementary error functions respectively.

(viii) The function defined by $B(p,\ q) = \int_0^1 t^p (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, $p > 0,\ q > 0$, is known as beta function (or Euler's function of the first kind).

(ix) **Struve Function,** $H_\nu(x)$**:** It is defined as

$$H_\nu(x) = \frac{2x^{\nu+1}}{\sqrt{\pi}2^{\nu+1}\Gamma(\nu+3/2)} \sum_{k=0}^{\infty} \frac{1}{(3/2)_k(\nu+3/2)_k} \left(-\frac{x^2}{4}\right)^k.$$

(x) **Bessel Function of the First Kind,** $J_\nu(x)$**:** It is defined as

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k\ (k!)} \left(-\frac{x^2}{4}\right)^k.$$

(xi) **Bessel Function of the Second Kind,** $Y_\nu(x)$**:** It is defined as

$$Y_\nu(x) = \frac{\cos(\nu\pi)\ J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

with $Y_0(.)$ interpreted as the limit $Y_0(x) = \lim_{\nu\to 0} Y_\nu(x)$.

(xii) **Modified Bessel Function of the First Kind,** $I_\nu(x)$**:** It is defined as

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k\ (k!)} \left(\frac{x^2}{4}\right)^k.$$

(xiii) **Modified Bessel Function of the second kind** (note we need later to find the pdf of the product of two independent normal random variables). Modified Bessel function of the Second Kind, K_α(x) is defined as

$$K_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha x) - J_{-\alpha}(x)}{\text{Sin}(\alpha x)},$$

where

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)} \alpha F_1(\alpha+1, -\frac{x^2}{4}).$$

In the case of integer order n, the function is defined by the limit as non-integer α tends to n. For n = 0

$$K_0(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos tx}{\sqrt{1+t^2}} dt$$

(xiv) **Modified Bessel Function of the Third Kind,** $K_\nu(x)$**:** It is defined as

$$K_\nu(x) = \frac{\pi\ \{I_{-\nu}(x) - I_\nu(x)\}}{2\sin(\nu\pi)},$$

with $K_0(.)$ interpreted as the limit $K_0(x) = \lim_{\nu\to 0} K_\nu(x)$.

(xv) **Meijer** G**-Function:** It is defined as

$$G_{p,q}^{m,n}\left(x\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right) = \frac{1}{2\pi i}\int_L \frac{x^{-t}\Gamma(b_1+t)\cdots\Gamma(b_m+t)\Gamma(1-a_1-t)\cdots\Gamma(1-a_n-t)}{\Gamma(a_{n+1}+t)\cdots\Gamma(a_p+t)\Gamma(1-b_{m+1}-t)\cdots\Gamma(1-b_q-t)} dt,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial and L denotes an integration path (for details on Meijer G-Function, see, Gradshteyn and Ryzhik (2000), Sect. 9.3, Page 1068).

(xvi) $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}(n!)}$, where $n > 0$ is an integer.

(xvii) For negative values, gamma function can be defined as

$$\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5.\ldots(2n-1)}, \text{ where } n > 0 \text{ is an integer.}$$

The organization of this book is as follows. In Chap. 2 , some basic ideas, definitions and some detailed properties along with applications of the normal distributions have been presented. Some basic ideas, definitions and some detailed properties along with applications of the Student's *t* distributions have been presented in Chap. 3. Chapter 4 presents the distributions of the sum, product and ratio of normal random variables. In Chap. 5, sum, product and ratio for Student's *t* random variables have been given. Chapter 6 discusses the sum, product and ratio for random variables X and Y having the normal and Student's *t* distributions respectively and distributed independently of each other. In Chap. 7, a new symmetric distribution and its properties have been presented by taking the product of the probability density functions of the normal and Student's t distributions for some continuous random variable X. The characterizations of normal distributions are presented in Chap. 8. In Chap. 9, we presented the characterizations of Student's *t* distribution. Some concluding remarks and some future research on the sum, product and ration of two random variables are provided in Chap. 10.

Chapter 2
Normal Distribution

The normal distribution is one of the most important continuous probability distributions, and is widely used in statistics and other fields of sciences. In this chapter, we present some basic ideas, definitions, and properties of normal distribution, (for details, see, for example, Whittaker and Robinson (1967), Feller (1968, 1971), Patel et al. (1976), Patel and Read (1982), Johnson et al. (1994), Evans et al. (2000), Balakrishnan and Nevzorov (2003), and Kapadia et al. (2005), among others).

2.1 Normal Distribution

The normal distribution describes a family of continuous probability distributions, having the same general shape, and differing in their location (that is, the mean or average) and scale parameters (that is, the standard deviation). The graph of its probability density function is a symmetric and bell-shaped curve. The development of the general theories of the normal distributions began with the work of de Moivre (1733, 1738) in his studies of approximations to certain binomial distributions for large positive integer $n > 0$. Further developments continued with the contributions of Legendre (1805), Gauss (1809), Laplace (1812), Bessel (1818, 1838), Bravais (1846), Airy (1861), Galton (1875, 1889), Helmert (1876), Tchebyshev (1890), Edgeworth (1883, 1892, 1905), Pearson (1896), Markov (1899, 1900), Lyapunov (1901), Charlier (1905), and Fisher (1930, 1931), among others. For further discussions on the history of the normal distribution and its development, readers are referred to Pearson (1967), Patel and Read (1982), Johnson et al. (1994), and Stigler (1999), and references therein. Also, see Wiper et al. (2005), for recent developments. The normal distribution plays a vital role in many applied problems of biology, economics, engineering, financial risk management, genetics, hydrology, mechanics, medicine, number theory, statistics, physics, psychology, reliability, etc., and has been has been extensively studied, both from theoretical and applications point of view, by many researchers, since its inception.

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2.1.1 Definition (Normal Distribution)

A continuous random variable X is said to have a normal distribution, with mean μ and variance σ^2, that is, $X \sim N(\mu,\ \sigma^2)$, if its pdf $f_X(x)$ and cdf $F_X(x) = P(X \le x)$ are, respectively, given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty, \tag{2.1}$$

and

$$\begin{aligned} F_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-(y-\mu)^2/2\sigma^2} dy \\ &= \frac{1}{2}\left[1 + erf\left(\frac{x - \mu}{\sigma\sqrt{2}}\right)\right], \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0, \end{aligned} \tag{2.2}$$

where $erf(.)$ denotes error function, and μ and σ are location and scale parameters, respectively.

2.1.2 Definition (Standard Normal Distribution)

A normal distribution with $\mu = 0$ and $\sigma = 1$, that is, $X \sim N(0, 1)$, is called the standard normal distribution. The pdf $f_X(x)$ and cdf $F_X(x)$ of $X \sim N(0, 1)$ are, respectively, given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ -\infty < x < \infty, \tag{2.3}$$

and

$$\begin{aligned} F_X(x) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-t^2/2} dt, \quad -\infty < x < \infty, \\ &= \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right], \quad -\infty < x < \infty. \end{aligned} \tag{2.4}$$

Note that if $Z \sim N(0,\ 1)$ and $X = \mu + \sigma Z$, then $X \sim N(\mu, \sigma^2)$, and conversely if $X \sim N(\mu,\ \sigma^2)$ and $Z = (X - \mu)/\ \sigma$, then $Z \sim N(0,\ 1)$. Thus, the pdf of any general $X \sim N(\mu,\ \sigma^2)$ can easily be obtained from the pdf of $Z \sim N(0,\ 1)$, by using the simple location and scale transformation, that is, $X = \mu + \sigma\ Z$. To describe the shapes of the normal distribution, the plots of the pdf (2.1) and cdf (2.2),

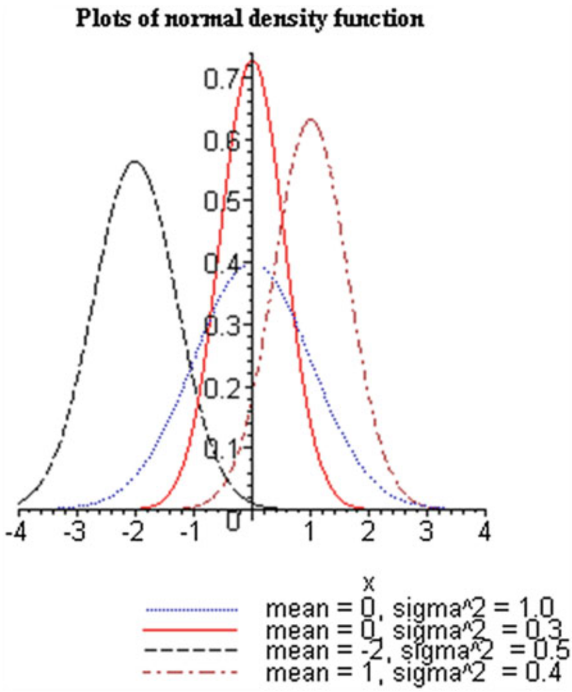


Fig. 2.1 Plots of the normal pdf, for different values of μ and σ^2

for different values of μ and σ^2, are provided in Figs. 2.1 and 2.2, respectively, by using Maple 10. The effects of the parameters, μ and σ^2, can easily be seen from these graphs. Similar plots can be drawn for other values of the parameters. It is clear from Fig. 2.1 that the graph of the pdf $f_X(x)$ of a normal random variable, $X \sim N(\mu, \ \sigma^2)$, is symmetric about mean, μ, that is $f_X(\mu + x) = f_X(\mu - x)$, $-\infty < x < \infty$.

2.1.3 Some Properties of the Normal Distribution

This section discusses the mode, moment generating function, cumulants, moments, mean, variance, coefficients of skewness and kurtosis, and entropy of the normal distribution, $N(\mu, \ \sigma^2)$. For detailed derivations of these, see, for example, Kendall and Stuart (1958), Lukacs (1972), Dudewicz and Mishra (1988), Johnson et al. (1994), Rohatgi and Saleh (2001), Balakrishnan and Nevzorov (2003), Kapadia et al. (2005), and Mukhopadhyay (2006), among others.

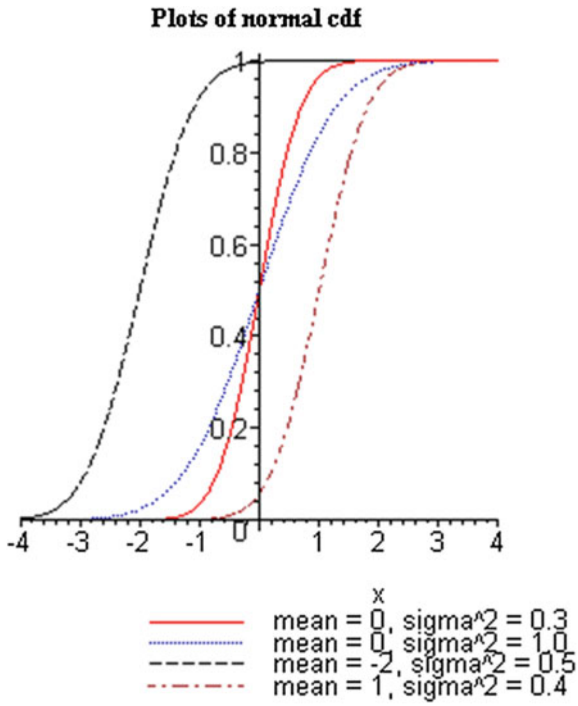


Fig. 2.2 Plots of the normal cdf for different values of μ and σ^2

2.1.3.1 Mode

The mode or modal value is that value of X for which the normal probability density function $f_X(x)$ defined by (2.1) is maximum. Now, differentiating with respect to x Eq. (2.1), we have

$$f_X'(x) = -\sqrt{\frac{2}{\pi}}\left[\frac{(x-\mu)\, e^{-(x-\mu)^2/2\sigma^2}}{\sigma^3}\right],$$

which, when equated to 0, easily gives the mode to be $x = \mu$, which is the mean, that is, the location parameter of the normal distribution. It can be easily seen that $f_X''(x) < 0$. Consequently, the maximum value of the normal probability density function $f_X(x)$ from (2.1) is easily obtained as $f_X(\mu) = \frac{1}{\sigma\sqrt{2\pi}}$. Since $f'(x) = 0$ has one root, the normal probability density function (2.1) is unimodal.

2.1.3.2 Cumulants

The cumulants k_r of a random variable X are defined via the cumulant generating function

$$g(t) = \sum_{r=1}^{\infty} k_r \frac{t^r}{r!}, \text{ where g(t)} = \ln\left(\mathrm{E}(\mathrm{e}^{\mathrm{tX}})\right).$$

For some integer $r > 0$, the rth cumulant of a normal random variable X having the pdf (2.1) is given by

$$\kappa_r = \begin{cases} \mu, & when\ r = 1; \\ \sigma^2, & when\ r = 2; \\ 0, & when\ r > 2 \end{cases}$$

2.1.3.3 Moment Generating Function

The moment generating function of a normal random variable X having the pdf (2.1) is given by (see, for example, Kendall and Stuart (1958), among others)

$$M_X(t) = E\left(e^{tX}\right) = e^{t\mu + \frac{1}{2}t^2\sigma^2}.$$

2.1.3.4 Moments

For some integer $r > 0$, the rth moment about the mean of a normal random variable X having the pdf (2.1) is given by

$$E\left(X^r\right) = \mu_r = \begin{cases} \frac{\sigma^r\ (r!)}{2^{\frac{r}{2}}\ [(r/2)!]}, & \text{for r even;} \\ 0, & \text{for r odd} \end{cases} \quad (2.5)$$

We can write $\mu_r = \sigma^r(r!!)$, where $m!! = 1.3.5\ldots(m-1)$ for m even.

2.1.3.5 Mean, Variance, and Coefficients of Skewness and Kurtosis

From (2.5), the mean, variance, and coefficients of skewness and kurtosis of a normal random variable $X \sim N(\mu,\ \sigma^2)$ having the pdf (2.1) are easily obtained as follows:

(i) **Mean:** $\alpha_1 = E(X) = \mu$;
(ii) **Variance:** $Var(X) = \sigma^2, \quad \sigma > 0$;
(iii) **Coefficient of Skewness:** $\gamma_1(X) = \frac{\mu_3}{\mu_2^{3/2}} = 0$;
(iv) **Coefficient of Kurtosis:** $\gamma_2(X) = \frac{\mu_4}{\mu_2^2} = 3$.

where μ_r has been defined in Eq. (2.5).

Since the coefficient of kurtosis, that is, $\gamma_2(X) = 3$, it follows that the normal distributions are mesokurtic distributions.

2.1.3.6 Median, Mean Deviation, and Coefficient of Variation of $X \sim N(\mu, \sigma^2)$

These are given by

(i) **Median:** μ

(ii) **Mean Deviation:** $\left(\frac{2\sigma^2}{\pi}\right)^{\frac{1}{2}}$

(iii) **Coefficient of Variation:** $\frac{\sigma}{\mu}$

2.1.3.7 Characteristic Function

The characteristic function of a normal random variable $X \sim N(\mu, \sigma^2)$ having the pdf (2.1) is given by (see, for example, Patel et al. (1976), among others)

$$\phi_X(t) = E\left(e^{itX}\right) = e^{it\mu - \frac{1}{2}t^2\sigma^2}, \ \mathrm{i} = \sqrt{-1}.$$

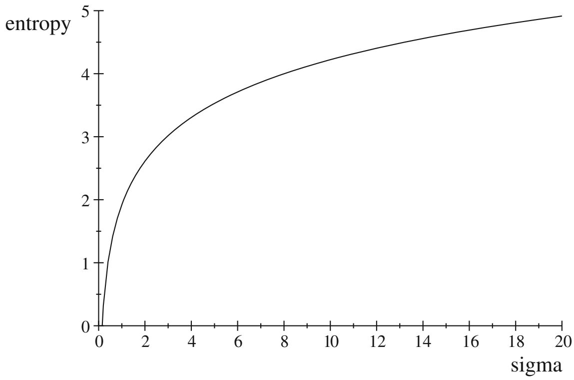
2.1.3.8 Entropy

For some $\sigma > 0$, entropy of a random variable X having the pdf (2.1) is easily given by

$$\begin{aligned} H_X[f_X(x)] &= E[-\ln(f_X(X)] \\ &= -\int_{-\infty}^{\infty} f_X(x) \ln [f_X(x)] \, dx, \\ &= \ln\left(\sqrt{2\pi e}\sigma\right) \end{aligned}$$

(see, for example, Lazo and Rathie (1978), Jones (1979), Kapur (1993), and Suhir (1997), among others). It can be easily seen that $\frac{d(H_X[f_X(x)])}{d\sigma} > 0$, and $\frac{d^2(H_X[f_X(x)])}{d\sigma^2} < 0, \forall\sigma > 0, \forall\mu$. It follows that that the entropy of a random variable X having the normal pdf (2.1) is a monotonic increasing concave function of $\sigma > 0, \ \forall\mu$. The possible shape of the entropy for different values of the parameter σ is provided below in Fig. 2.3, by using Maple 10. The effects of the parameter σ on entropy can easily be seen from the graph. Similar plots can be drawn for others values of the parameter σ.

Fig. 2.3 Plot of entropy



2.1.4 Percentiles

This section computes the percentiles of the normal distribution, by using Maple 10. For any $p(0 < p < 1)$, the $(100p)th$ percentile (also called the quantile of order p) of $N(\mu,\ \sigma^2)$ with the pdf $f_X(x)$ is a number z_p such that the area under $f_X(x)$ to the left of z_p is p. That is, z_p is any root of the equation

$$\Phi(z_p) = \int_{-\infty}^{z_p} f_X(u)du = p.$$

Using the Maple program, the percentiles z_p of $N(\mu,\ \sigma^2)$ are computed for some selected values of p for the given values of μ and , which are provided in Table 2.1, when $\mu = 0$ and $\sigma = 1$. Table 2.1 gives the percentile values of z_p for $p \geq 0.5$. For $p < 0.5$, use $1 - Z_{1-p}$.

Table 2.1 Percentiles of $N(0,\ 1)$

p	z_p
0.5	0.0000000000
0.6	0.2533471031
0.7	0.5244005127
0.75	0.6744897502
0.8	0.8416212336
0.9	1.281551566
0.95	1.644853627
0.975	1.959963985
0.99	2.326347874
0.995	2.575829304
0.9975	2.807033768
0.999	3.090232306

Suppose $X_1, X_2, \ldots X_n$ are n independent N (0, 1) random variables and $M(n) = \max(X_1, X_2, \ldots X_n)$. It is known (see Ahsanullah and Kirmani (2008) p.15 and Ahsanullah and Nevzorov (2001) p.92) that

$P(M(n) \leq a_n + b_n x) \rightarrow e^{-e^{-x}}$, for all x as n $\rightarrow \infty$.

where $a_n = \beta_n - \frac{D_n}{2\beta_n}$, $D_n = \ln \ln n + \ln 4\pi$, $\beta_n = (2 \ln n)^{1/2}$, $b_n - (2 \ln n)^{-1/2}$.

2.2 Different Forms of Normal Distribution

This section presents different forms of normal distribution and some of their important properties, (for details, see, for example, Whittaker and Robinson (1967), Feller (1968, 1971), Patel et al. (1976), Patel and Read (1982), Johnson et al. (1994), Evans et al. (2000), Balakrishnan and Nevzorov (2003), and Kapadia et al. (2005), among others).

2.2.1 Generalized Normal Distribution

Following Nadarajah (2005a), a continuous random variable X is said to have a generalized normal distribution, with mean μ and variance $\frac{\sigma^2 \Gamma\left(\frac{3}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}$, where $s > 0$, that is, $X \sim N\left(\mu, \frac{\sigma^2 \Gamma\left(\frac{3}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}\right)$, if its pdf $f_X(x)$ and cdf $F_X(x) = P(X \leq x)$ are, respectively, given by

$$f_X(x) = \frac{s}{2\sigma \Gamma\left(\frac{1}{s}\right)} e^{-\left|\frac{x-\mu}{\sigma}\right|^s}, \tag{2.6}$$

and

$$F_X(x) = \begin{cases} \frac{\Gamma\left(\frac{1}{s}, \left(\frac{\mu - x}{\sigma}\right)^s\right)}{2\Gamma\left(\frac{1}{s}\right)}, & if \ x \leq \mu \\ 1 - \frac{\Gamma\left(\frac{1}{s}, \left(\frac{x - \mu}{\sigma}\right)^s\right)}{2\Gamma\left(\frac{1}{s}\right)}, & if \ x > \mu \end{cases} \tag{2.7}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $s > 0$, and $\Gamma(a, x)$ denotes complementary incomplete gamma function defined by $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$. It is easy to see that the Eq. (2.6) reduces to the normal distribution for $s = 2$, and Laplace distribution for $s = 1$. Further, note that if has the pdf given by (2.6), then the pdf of the standardized random variable $Z = (X - \mu)/\sigma$ is given by

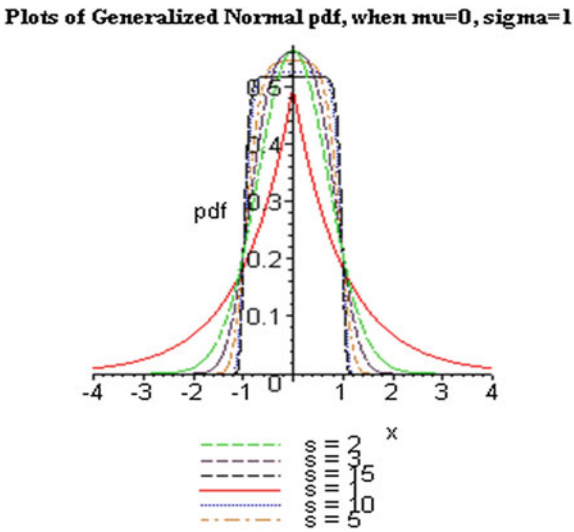


Fig. 2.4 Plots of the generalized normal pdf for different values of s

$$f_Z(z) = \frac{s}{2\Gamma\left(\frac{1}{s}\right)} e^{-|z|^s} \tag{2.8}$$

To describe the shapes of the generalized normal distribution, the plots of the pdf (2.6), for $\mu = 0, \sigma = 1$, and different values of s, are provided in Fig. 2.4 by using Maple 10. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters. It is clear from Fig. 2.4 that the graph of the pdf $f_X(x)$ of the generalized normal random variable is symmetric about mean, μ, that is

$$f_X(\mu + x) = f_X(\mu - x), \quad -\infty < x < \infty.$$

2.2.1.1 Some Properties of the Generalized Normal Distribution

This section discusses the mode, moments, mean, median, mean deviation, variance, and entropy of the generalized normal distribution. For detailed derivations of these, see Nadarajah (2005).

2.2.1.2 Mode

It is easy to see that the mode or modal value of x for which the generalized normal probability density function $f_X(x)$ defined by (2.6) is maximum, is given by $x = \mu$, and the maximum value of the generalized normal probability density function (2.6) is given by $f_X(\mu) = \frac{s}{2\sigma\Gamma\left(\frac{1}{s}\right)}$. Clearly, the generalized normal probability density function (2.6) is unimodal.

2.2.1.3 Moments

(i) For some integer $r > 0$, the rth moment of the generalized standard normal random variable Z having the pdf (2.8) is given by

$$E\left(Z^r\right) = \frac{1+(-1)^r}{2\Gamma\left(\frac{1}{s}\right)}\Gamma\left(\frac{r+1}{s}\right) \tag{2.9}$$

(i) For some integer $n > 0$, the nth moment and the nth central moment of the generalized normal random variable X having the pdf (2.6) are respectively given by the Eqs. (2.10) and (2.11) below:

$$E\left(X^n\right) = \frac{(\mu^n)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sigma}{\mu}\right)^k\left[1+(-1)^k\right]\Gamma\left(\frac{k+1}{s}\right)}{2\Gamma\left(\frac{1}{s}\right)} \tag{2.10}$$

and

$$E\left[(X-\mu)^n\right] = \frac{(\sigma^n)\left[1+(-1)^n\right]\Gamma\left(\frac{n+1}{s}\right)}{2\Gamma\left(\frac{1}{s}\right)} \tag{2.11}$$

2.2.1.4 Mean, Variance, Coefficients of Skewness and Kurtosis, Median and Mean Deviation

From the expressions (2.10) and (2.11), the mean, variance, coefficients of skewness and kurtosis, median and mean deviation of the generalized normal random variable X having the pdf (2.6) are easily obtained as follows:

(i) Mean: $\alpha_1 = E(X) = \mu$;

(ii) Variance: $Var(X) = \beta_2 = \frac{\sigma^2\Gamma\left(\frac{3}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}, \quad \sigma > 0,\ s > 0$;

(iii) Coefficient of Skewness: $\gamma_1(X) = \frac{\beta_3}{\beta_2^{3/2}} = 0$;

(iv) Coefficient of Kurtosis: $\gamma_2(X) = \frac{\beta_4}{\beta_2^2} = \frac{\Gamma\left(\frac{1}{s}\right)\Gamma\left(\frac{5}{s}\right)}{\left[\Gamma\left(\frac{3}{s}\right)\right]^2}, \quad s > 0$;

(**v**) **Median** (X): μ;

(**vi**) **Mean Deviation**: $E\,|X - \mu| = \dfrac{\sigma\,\Gamma\left(\frac{2}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}, \quad s > 0.$

2.2.1.5 Renyi and Shannon Entropies, and Song's Measure of the Shape of the Generalized Normal Distribution

These are easily obtained as follows, (for details, see, for example, Nadarajah (2005), among others).

(i) **Renyi Entropy:** Following Renyi (1961), for some reals $\gamma > 0$, $\gamma \neq 1$, the entropy of the generalized normal random variable X having the pdf (2.6) is given by

$$\Im_R(\gamma) = \frac{1}{1 - \gamma} \ln \int_{-\infty}^{+\infty} [f_X(X)]^{\gamma}\, dx$$
$$= \frac{\ln(\gamma)}{s(\gamma - 1)} - \ln\left[\frac{s}{2\sigma\Gamma\left(\frac{1}{s}\right)}\right], \quad \sigma > 0,\ s > 0,\ \gamma > 0,\ \gamma \neq 1.$$

(ii) **Shannon Entropy:** Following Shannon (1948), the entropy of the generalized normal random variable X having the pdf (2.6) is given by

$$H_X[f_X(X)] = E[-\ln(f_X(X)] = -\int_{-\infty}^{\infty} f_X(x)\,\ln\,[f_X(x)]\,dx,$$

which is the particular case of Renyi entropy as obtained in (i) above for $\gamma \to 1$. Thus, in the limit when $\gamma \to 1$ and using L'Hospital's rule, Shannon entropy is easily obtained from the expression for Renyi entropy in (i) above as follows:

$$H_X[f_X(X)] = \frac{1}{s} - \ln\left[\frac{s}{2\sigma\Gamma\left(\frac{1}{s}\right)}\right], \sigma > 0,\ s > 0.$$

(iii) **Song's Measure of the Shape of a Distribution:** Following Song (2001), the gradient of the Renyi entropy is given by

$$\Im'_R(\gamma) = \frac{d}{d\gamma}\left[\Im_R(\gamma)\right] = \frac{1}{s}\left\{\frac{1}{\gamma(\gamma - 1)} - \frac{\ln(\gamma)}{(\gamma - 1)^2}\right\} \tag{2.12}$$

which is related to the log likelihood by

$$\Im_R'(1) = -\frac{1}{2}Var[\ln f(X)].$$

Thus, in the limit when $\gamma \to 1$ and using L'Hospital's rule, Song's measure of the shape of the distribution of the generalized normal random variable X having the pdf (2.6) is readily obtained from the Eq. (2.12) as follows:

$$-2\,\Im_R'(1) = \frac{1}{s},$$

which can be used in comparing the shapes of various densities and measuring heaviness of tails, similar to the measure of kurtosis.

2.2.2 Half Normal Distribution

Statistical methods dealing with the properties and applications of the half-normal distribution have been extensively used by many researchers in diverse areas of applications, particularly when the data are truncated from below (that is, left truncated,) or truncated from above (that is, right truncated), among them Dobzhansky and Wright (1947), Meeusen and van den Broeck (1977), Haberle (1991), Altman (1993), Buckland et al. (1993) , Chou and Liu (1998), Klugman et al. (1998), Bland and Altman (1999), Bland (2005), Goldar and Misra (2001), Lawless (2003), Pewsey (2002, 2004), Chen and Wang (2004) and Wiper et al. (2005), Babbit et al. (2006), Coffey et al. (2007), Barranco-Chamorro et al. (2007), and Cooray and Ananda (2008), are notable. A continuous random variable X is said to have a (general) half-normal distribution, with parameters μ (location) and σ (scale), that is, $X|\mu, \sigma \sim HN(\mu, \sigma)$, if its pdf $f_X(x)$ and cdf $F_X(x) = P(X \le x)$ are, respectively, given by

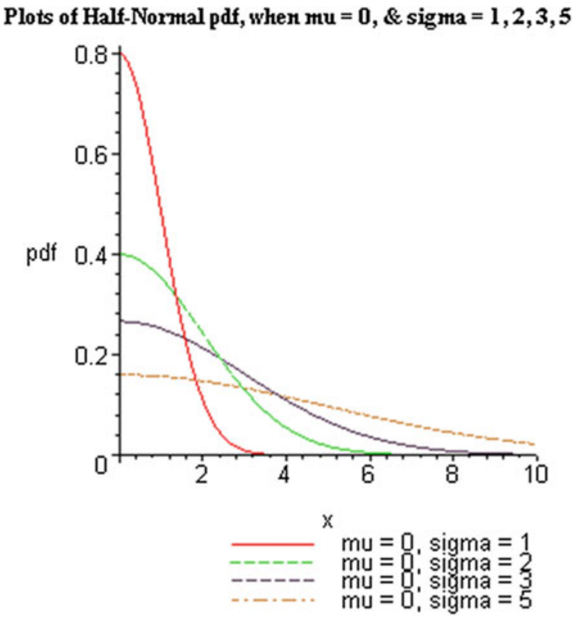
$$f_X(x|\mu, \sigma) = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \tag{2.13}$$

and

$$F_X(x) = erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) \tag{2.14}$$

where $x \ge \mu$, $-\infty < \mu < \infty$, $\sigma > 0$, and $erf(.)$ denotes error function, (for details on half-normal distribution and its applications, see, for example, Altman (1993), Chou and Liu (1998), Bland and Altman (1999), McLaughlin (1999), Wiper et al. (2005), and references therein). Clearly, $X = \mu + \sigma|Z|$, where $Z \sim N(0, 1)$ has a standard normal distribution. On the other hand, the random variable $X = \mu - \sigma|Z|$ follows a negative (general) half- normal distribution. In particular, if $X \sim N\left(0, \sigma^2\right)$, then it is easy to see that the absolute value $|X|$ follows a half-normal distribution, with its pdf $f_{|X|}(x)$ given by

Fig. 2.5 Plots of the half-normal pdf



$$f_{|X|}(x) = \begin{cases} \frac{2}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases} \tag{2.15}$$

By taking $\sigma^2 = \frac{\pi}{2\theta^2}$ in the Eq. (2.15), more convenient expressions for the pdf and cdf of the half-normal distribution are obtained as follows

$$f_{|X|}(x) = \begin{cases} \frac{2\theta}{\pi} e^{-\left(\frac{x\theta}{\sqrt{\pi}}\right)^2} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases} \tag{2.16}$$

and

$$F_{|X|}(x) = erf\left(\frac{\theta x}{\sqrt{\pi}}\right) \tag{2.17}$$

which are implemented in *Mathematica* software as HalfNormalDistribution[theta], see Weisstein (2007). To describe the shapes of the half-normal distribution, the plots of the pdf (2.13) for different values of the parameters μ and σ are provided in Fig. 2.5 by using Maple 10. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

2.2.3 Some Properties of the Half-Normal Distribution

This section discusses some important properties of the half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$.

2.2.3.1 Special Cases

The half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$ is a special case of the Amoroso, central chi, two parameter chi, generalized gamma, generalized Rayleigh, truncated normal, and folded normal distributions (for details, see, for example, Amoroso (1925), Patel and Read (1982), and Johnson et al. (1994), among others). It also arises as a limiting distribution of three parameter skew-normal class of distributions introduced by Azzalini (1985).

2.2.3.2 Characteristic Property

If $X \sim N(\mu, \sigma)$ is folded (to the right) about its mean, μ, then the resulting distribution is half-normal, $X|\mu, \sigma \sim HN(\mu, \sigma)$.

2.2.3.3 Mode

It is easy to see that the mode or modal value of x for which the half-normal probability density function $f_X(x)$ defined by (2.13) is maximum, is given at $x = \mu$, and the maximum value of the half-normal probability density function (2.13) is given by $f_X(\mu) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}$. Clearly, the half-normal probability density function (2.13) is unimodal.

2.2.3.4 Moments

(i) kth **Moment of the Standardized Half-Normal Random Variable:** If the half-normal random variable X has the pdf given by the Eq. (2.13), then the standardized half-normal random variable $|Z| = \frac{X-\mu}{\sigma} \sim HN(0, 1)$ will have the pdf given by

$$f_{|Z|}(z) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} & if\ z \geq 0 \\ 0 & if\ z < 0 \end{cases} \tag{2.18}$$

For some integer $k > 0$, and using the following integral formula (see Prudnikov et al. Vol. 1, 1986, Eq. 2.3.18.2, p. 346, or Gradshteyn and Ryzhik

1980, Eq. 3.381.4, p. 317)

$$\int_0^{\infty} t^{\alpha - 1} e^{-\rho t^{\mu}} dt = \frac{1}{\mu} \rho^{-\frac{\alpha}{\mu}} \Gamma\left(\frac{\alpha}{\mu}\right), \quad \text{where } \mu, \ \operatorname{Re}\alpha, \ \operatorname{Re}\rho > 0,$$

the kth moment of the standardized half-normal random variable Z having the pdf (2.18) is easily given by

$$E\left(Z^k\right) = \frac{1}{\sqrt{\pi}} 2^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right), \tag{2.19}$$

where $\Gamma(.)$ denotes gamma function.

(ii) **Moment of the Half-Normal Random Variable:** For some integer $n > 0$, the nth moment (about the origin) of the half-normal random variable X having the pdf (2.13) is easily obtained as

$$\mu'_n = E\left(X^n\right) = E\left[(\mu + z\sigma)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k E\left(Z^k\right)$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} \binom{n}{k} 2^{\frac{k}{2}} \mu^{n-k} \sigma^k \Gamma\left(\frac{k+1}{2}\right) \tag{2.20}$$

From the above Eq. (2.20), the first four moments of the half-normal random variable X are easily given by

$$\mu'_1 = E[X] = \mu + \sigma\sqrt{\frac{2}{\pi}}, \tag{2.21}$$

$$\mu'_2 = E\left[X^2\right] = \mu^2 + 2\sqrt{\frac{2}{\pi}}\mu\sigma + \sigma^2, \tag{2.22}$$

$$\mu'_3 = E\left[X^3\right] = \mu^3 + 3\sqrt{\frac{2}{\pi}}\mu^2\sigma + 3\mu\sigma^2 + 2\sqrt{\frac{2}{\pi}}\sigma^3, \tag{2.23}$$

and

$$\mu'_4 = E\left[X^4\right] = \mu^4 + 4\sqrt{\frac{2}{\pi}}\mu^3\sigma + 6\mu^2\sigma^2 + 8\sqrt{\frac{2}{\pi}}\mu\sigma^3 + 3\sigma^4. \tag{2.24}$$

(iii) **Central Moment of the Half-Normal Random Variable:** For some integer $n > 0$, the nth central moment (about the mean $\mu'_1 = E(X)$) of the half-normal random variable X having the pdf (2.13) can be easily obtained using the formula

$$\mu_n = E\left[\left(X - \mu_1'\right)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \left(-\mu_1'\right)^{n-k} E\left(X^k\right), \tag{2.25}$$

where $E\left(X^k\right) = \mu_k'$ denotes the kth moment, given by the Eq. (2.20), of the half-normal random variable X having the pdf (2.13).

Thus, from the above Eq. (2.25), the first three central moments of the half-normal random variable X are easily obtained as

$$\mu_2 = E\left[\left(X - \mu_1'\right)^2\right] = \mu_2' - \left(\mu_1'\right)^2 = \frac{\sigma^2(\pi - 2)}{\pi},$$

$$\mu_3 = \beta_3 = E\left[\left(X - \mu_1'\right)^3\right] \tag{2.26}$$

$$= \mu_3' - 3\mu_1'\mu_2' + 2\left(\mu_1'\right)^3 = \sqrt{\frac{2}{\pi}}\frac{\sigma^3(4 - \pi)}{\pi}, \tag{2.27}$$

and

$$\mu_4 = \beta_4 = E\left[\left(X - \mu_1'\right)^4\right] = \mu_4' - 4\mu_1'\mu_3' + 6\left(\mu_1'\right)^2\mu_2' - 3\left(\mu_1'\right)^4$$

$$= \frac{\sigma^4\left(3\pi^2 - 4\pi - 12\right)}{\pi^2}. \tag{2.28}$$

2.2.3.5 Mean, Variance, and Coefficients of Skewness and Kurtosis

These are easily obtained as follows:

(**i**) **Mean** : $\alpha_1 = E(X) = \mu + \sigma\sqrt{\frac{2}{\pi}}$;

(**ii**) **Variance** : $Var(X) = \mu_2 = \sigma^2\left(1 - \frac{2}{\pi}\right), \quad \sigma > 0$;

(**iii**) **Coefficient of Skewness** : $\gamma_1(X) = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\sqrt{2}(4 - \pi)}{\sqrt{(\pi - 2)^3}} \approx 0.9953$;

(**iv**) **Coefficient of Kurtosis** : $\gamma_2(X) = \frac{\mu_4}{\mu_2^2} = \frac{8(\pi - 3)}{(\pi - 2)^2} \approx 0.7614$;

2.2.3.6 Median (i.e., 50th Percentile or Second Quartile), and First and Third Quartiles

These are derived as follows. For any $p(0 < p < 1)$, the $(100p)th$ percentile (also called the quantile of order p) of the half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$, with the pdf $f_X(x)$ given by (2.13), is a number z_p such that the area under $f_X(x)$ to the left of z_p is p. That is, z_p is any root of the equation

$$F(z_p) = \int_{-\infty}^{z_p} f_X(t)dt = p. \tag{2.29}$$

For $p = 0.50$, we have the 50th percentile, that is, $z_{0.50}$, which is called the median (or the second quartile) of the half-normal distribution. For $p = 0.25$ and $p = 0.75$, we have the 25th and 75th percentiles respectively.

2.2.3.7 Derivation of Median(X)

Let m denote the median of the half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$, that is, let $m = z_{0.50}$. Then, from the Eq. (2.29), it follows that

$$0.50 = F(z_{0.50}) = \int_{-\infty}^{z_{0.50}} f_X(t)dt = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\int_{-\infty}^{z_{0.50}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt. \tag{2.30}$$

Substituting $\frac{t-\mu}{\sqrt{2}\,\sigma} = u$ in the Eq. (2.30), using the definition of error function, and solving for $z_{0.50}$, it is easy to see that

$$\begin{aligned} m = \text{Median}(X) = z_{0.50} &= \mu + \left(\sqrt{2}\right) erf^{-1}(0.50)\,\sigma \\ &= \mu + (\sqrt{2})(0.476936)\sigma \\ &\approx \mu + 0.6745\sigma, \quad \sigma > 0, \end{aligned}$$

where $erf^{-1}[0.50] = 0.476936$ has been obtained by using *Mathematica*. Note that the inverse error function is implemented in *Mathematica* as a *Built-in Symbol*, Inverse Erf[s], which gives the inverse error function obtained as the solution for z in $s = erf(z)$. Further, for details on Error and Inverse Error Functions, see, for example, Abramowitz and Stegun (1972, pp. 297–309), Gradshteyn and Ryzhik (1980), Prudnikov et al., Vol. 2 (1986), and Weisstein (2007), among others.

2.2.3.8 First and Third Quartiles

Let Q_1 and Q_3 denote the first and third quartiles of $X \sim HN(\mu, \sigma)$, that is, let $Q_1 = z_{0.25}$ and $Q_3 = z_{0.75}$. Then following the technique of the derivation of the Median(X) as in 2.2.3.7, one easily gets the Q_1 and Q_3 as follows.

(i) First Quartile: $Q_1 = \mu - 0.3186\sigma, \quad \sigma > 0;$

(ii) Third Quartile: $Q_3 = \mu + 1.150\sigma, \quad \sigma > 0.$

2.2.3.9 Mean Deviations

Following Stuart and Ord, Vol. 1, p. 52, (1994), the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, denoted as δ_1 and δ_2, respectively, and are defined as follows:

$$\textbf{(i)}\ \delta_1 = \int_{-\infty}^{+\infty} |x - E(X)| f(x) dx,$$

$$\textbf{(ii)}\ \delta_2 = \int_{-\infty}^{+\infty} |x - M(X)| f(x) dx.$$

Derivations of δ_1 and δ_2 for the Half-Normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$: To derive these, we first prove the following Lemma.

Lemma 2.2.1: Let $\delta = \frac{\omega-\mu}{\sigma}$. Then

$$\int_{\mu}^{\infty} \frac{1}{\sigma}|x-\omega|\sqrt{\frac{2}{\pi}} e^{-(1/2)(\frac{x-\mu}{\sigma})^2} dx$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left(-1 - \delta\sqrt{\frac{\pi}{2}} + e^{-\frac{\delta^2}{2}} + \delta\sqrt{\frac{\pi}{2}} erf\left(\frac{\delta}{\sqrt{2}}\right)\right),$$

where $erf(z) = \int_0^z \frac{2}{\sqrt{\pi}} e^{-t^2} dt$ denotes the error function.

Proof: We have

$$\int_{\mu}^{\infty} \frac{1}{\sigma}|x-\omega|\sqrt{\frac{2}{\pi}}e^{-(1/2)(\frac{x-\mu}{\sigma})^2}dx$$

$$= \int_{\mu}^{\infty} \frac{|x-\mu-(\omega-\mu|}{\sigma}\sqrt{\frac{2}{\pi}}e^{-(1/2)(\frac{x-\mu}{\sigma})^2}dx$$

$$= \sigma\int_{0}^{\infty} |u-\delta|\sqrt{\frac{2}{\pi}}\, e^{-(1/2)u^2}du,$$

Substituting $\frac{x\ -\ \mu}{\sigma} = u,\ and\ \delta{=}\frac{\omega{-}\mu}{\sigma}$

$$= \sigma\int_{0}^{\delta} (\delta-u)\sqrt{\frac{2}{\pi}}e^{-(1/2)u^2}du + \sigma\int_{\delta}^{\infty} (u-\delta)\sqrt{\frac{2}{\pi}}e^{-(1/2)u^2}du$$

$$= \frac{\sigma}{\sqrt{\pi}}\left(\delta\sqrt{\pi}erf(\frac{\delta}{\sqrt{2}}) + \sqrt{2}e^{-\frac{\delta^2}{2}} - \sqrt{2}\right)$$

$$+ \frac{\sigma}{\sqrt{\pi}}\left(\delta\sqrt{\pi}erf(\frac{\delta}{\sqrt{2}}) + \sqrt{2}e^{-\frac{\delta^2}{2}} - \delta\sqrt{\pi}\right)$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left(-1 - \delta\sqrt{\frac{\pi}{2}} + e^{-\frac{\delta^2}{2}} + \delta\sqrt{\frac{\pi}{2}}erf(\frac{\delta}{\sqrt{2}})\right).$$

This completes the proof of Lemma. □

Theorem 2.1: For $X|\mu,\ \sigma\ \sim\ HN(\mu,\ \sigma)$, the mean deviation, δ_1, about the mean, μ_1, is given by

$$\delta_1 = E|X - \mu_1| = \int_0^{\infty} |x - \mu_1|\, f(x)dx$$

$$= 2\sigma\sqrt{\frac{2}{\pi}}\left(-1 + e^{-\pi^{-1}} + erf(\pi^{-1/2})\right) \tag{2.31}$$

Proof: We have

$$\delta_1 = \int_0^{\infty} |x - \mu_1|\, f(x)dx$$

From Eq. (2.21), the mean of $X|\mu,\ \sigma\ \sim\ HN(\mu,\ \sigma)$ is given by

$$\mu_1 = E\ [X] = \mu + \sigma\sqrt{\frac{2}{\pi}}.$$

Taking $\omega = \mu_1$, we have

$$\delta = \frac{\omega - \mu}{\sigma} = \sqrt{\frac{2}{\pi}}.$$

Thus, taking $\omega = \mu_1$ and $\delta = \sqrt{\frac{2}{\pi}}$ in the above Lemma, and simplifying, we have

$$\delta_1 = 2\sigma\sqrt{\frac{2}{\pi}}\left(-1 + e^{-\pi^{-1}} + erf(\pi^{-1/2})\right),$$

which completes the proof of Theorem 2.1. □

Theorem 2.2: For $X|\mu, \sigma \sim HN(\mu, \sigma)$, the mean deviation, δ_2, about the median, m, is given by

$$\delta_2 = E|X - m| = \int_0^\infty |x - m| f(x)dx$$
$$= \sigma\sqrt{\frac{2}{\pi}}\left(k\sqrt{\pi} - 1 + 2e^{-k^2} + 2k\sqrt{\pi}erf(k)\right), \quad (2.32)$$

where $k = erf^{-1}(0.50)$.

Proof: We have

$$\delta_2 = \int_0^\infty |x - m| f(x)dx$$

As derived in Sect. 2.2.3.7 above, the median of $X|\mu, \sigma \sim HN(\mu, \sigma)$ is given by

$$m = \text{Median}(X) = \mu + \sqrt{2}erf^{-1}(0.50)\sigma = \mu + \sigma\sqrt{2}k,$$

where $k = erf^{-1}(0.50)$.
Taking $\omega = m$, we have

$$\delta = \frac{\omega - \mu}{\sigma} = \frac{m - \mu}{\sigma} = \frac{\mu + \sigma\sqrt{2}k - \mu}{\sigma} = \sqrt{2}k$$

Thus, taking $\omega = m$ and $\delta = \sqrt{2}k$ in the above Lemma, and simplifying, we have

$$\delta_2 = \sigma\sqrt{\frac{2}{\pi}}\left(k\sqrt{\pi} - 1 + 2e^{-k^2} + 2k\sqrt{\pi}erf(k)\right),$$

where $k = erf^{-1}(0.50)$. This completes the proof of Theorem 2.2. □

2.2.3.10 Renyi and Shannon Entropies, and Song's Measure of the Shape of the Half-Normal Distribution

These are derived as given below.

(i) **Renyi Entropy:** Following Renyi (1961), the entropy of the half- normal random variable X having the pdf (2.13) is given by

$$\begin{aligned}\Im_R(\gamma) &= \frac{1}{1-\gamma}\ln\int_0^{\infty}[f_X(X)]^{\gamma}\,dx,\\ &= \frac{\ln(\gamma)}{2(\gamma-1)} - \ln\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\right],\quad \sigma>0,\ \gamma>0,\ \gamma\neq 1.\end{aligned} \tag{2.33}$$

(ii) **Shannon Entropy:** Following Shannon (1948), the entropy of the half-normal random variable X having the pdf (2.13) is given by

$$H_X[f_X(X)] = E[-\ln(f_X(X)] = -\int_0^{\infty} f_X(x)\,\ln\,[f_X(x)]\,dx,$$

which is the particular case of Renyi entropy (2.31) for $\gamma \to 1$. Thus, in the limit when $\gamma \to 1$ and using L'Hospital's rule, Shannon entropy is easily obtained from the Eq. (2.33) as follows:

$$H_X[f_X(X)] = E[-\ln(f_X(X)] == \frac{1}{2} - \ln\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\right],\quad \sigma>0.$$

(iii) **Song's Measure of the Shape of a Distribution:** Following Song (2001), the gradient of the Renyi entropy is given by

$$\Im_R'(\gamma) = \frac{d}{d\gamma}\left[\Im_R(\gamma)\right] = \frac{1}{2}\left\{\frac{1}{\gamma(\gamma-1)} - \frac{\ln(\gamma)}{(\gamma-1)^2}\right\} \tag{2.34}$$

which is related to the log likelihood by

$$\Im_R'(1) = -\frac{1}{2}Var\,[\ln f(X)]\,.$$

Thus, in the limit when $\gamma \to 1$ and using L'Hospital's rule, Song's measure of the shape of the distribution of the half-normal random variable X having the pdf (2.13) is readily obtained from the Eq. (2.33) as follows:

$$\Im'_R(1) = -\frac{1}{8}\ (< 0),$$

the negative value of Song's measure indicating herein a "flat" or "platykurtic" distribution, which can be used in comparing the shapes of various densities and measuring heaviness of tails, similar to the measure of kurtosis.

2.2.3.11 Percentiles of the Half-Normal Distribution

This section computes the percentiles of the half-normal distribution, by using Maple 10. For any $p (0 < p < 1)$, the $(100p)th$ percentile (also called the quantile of order p) of the half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$, with the pdf $f_X(x)$ given by (2.13), is a number z_p such that the area under $f_X(x)$ to the left of z_p is p. That is, z_p is any root of the equation

$$F(z_p) = \int_{-\infty}^{z_p} f_X(t)dt = p. \tag{2.35}$$

Thus, from the Eq. (2.35), using the Maple program, the percentiles z_p of the half-normal distribution, $X|\mu, \sigma \sim HN(\mu, \sigma)$ can easily been obtained.

2.2.4 Folded Normal Distribution

An important class of probability distributions, known as the folded distributions, arises in many practical problems when only the magnitudes of deviations are recorded, and the signs of the deviations are ignored. The folded normal distribution is one such probability distribution which belongs to this class. It is related to the normal distribution in the sense that if Y is a normally distributed random variable with mean μ (location) and variance σ^2 (scale), that is, if $Y \sim N(\mu, \sigma^2)$, then the random variable $X = |Y|$ is said to have a folded normal distribution. The distribution is called folded because the probability mass (that is, area) to the left of the point $x = 0$ is folded over by taking the absolute value. As pointed out above, such a case may be encountered if only the magnitude of some random variable is recorded, without taking into consideration its sign (that is, its direction). Further, this distribution is used when the measurement system produces only positive measurements, from a normally distributed process. To fit a folded normal distribution, only the average and specified sigma (process, sample, or population) are needed. Many researchers have studied the statistical methods dealing with the properties and applications of the folded normal distribution, among them Daniel (1959), Leon et al. (1961), Elandt (1961), Nelson (1980), Patel and Read (1982), Sinha (1983), Johnson et al. (1994), Laughlin (http://www.causascientia.org/math_stat/Dists/Compendium.pdf,2001), and Kim (2006) are notable.

Definition: Let $Y \sim N\left(\mu, \sigma^2\right)$ be a normally distributed random variable with the mean μ and the variance σ^2. Let $X = |Y|$. Then X has a folded normal distribution with the pdf $f_X(x)$ and cdf $F_X(x) = P(X \le x)$, respectively, given as follows.

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma}\left[e^{-\frac{(x-\mu)^2}{2\sigma^2}} + e^{-\frac{(-x-\mu)^2}{2\sigma^2}}\right], & x \ge 0 \\ 0, & x < 0 \end{cases} \tag{2.36}$$

Note that the μ and σ^2 are location and scale parameters for the parent normal distribution. However, they are the shape parameters for the folded normal distribution. Further, equivalently, if $x \ge 0$, using a hyperbolic cosine function, the pdf $f_X(x)$ of a folded normal distribution can be expressed as

$$f_X(x) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\cosh\left(\frac{\mu x}{\sigma^2}\right) e^{-\frac{(x^2+\mu^2)}{2\sigma^2}}, \ x \ge 0.$$

and the cdf $\mathrm{F_X}(x)$ as

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^x \left(e^{-\frac{(y-\mu)^2}{2\sigma^2}} + e^{-\frac{(-y-\mu)^2}{2\sigma^2}}\right) dy,$$

$$x \ge 0, \ |\mu| < \infty, \ \sigma > 0. \tag{2.37}$$

Taking $z = \frac{y-\mu}{\sigma}$ in (2.37), the cdf $F_X(x)$ of a folded normal distribution can also be expressed as

$$F_X(x) = \frac{1}{\sqrt{2\pi}}\int_{-\mu/\sigma}^{(x-\mu)/\sigma} \left(e^{-\frac{1}{2}z^2} + e^{-\frac{1}{2}\left(z+\frac{2\mu}{\sigma}\right)^2}\right) dz,$$

$$z \ge 0, \ |\mu| < \infty, \ \sigma > 0, \tag{2.38}$$

where μ and σ^2 are the mean and the variance of the parent normal distribution. To describe the shapes of the folded normal distribution, the plots of the pdf (2.36) for different values of the parameters μ and σ are provided in Fig. 2.6 by using Maple 10. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

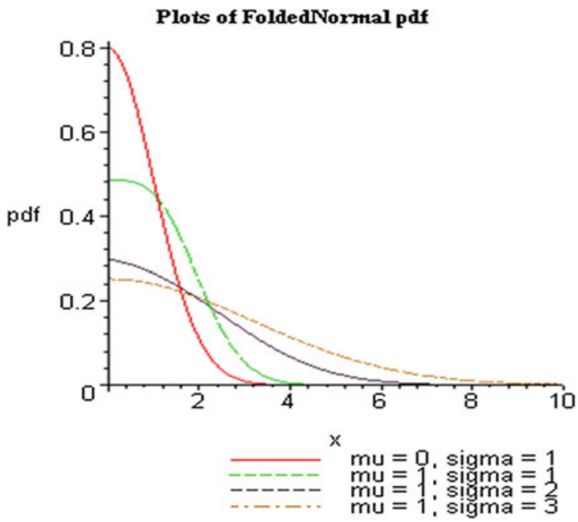


Fig. 2.6 Plots of the folded normal pdf

2.2.4.1 Some Properties of the Folded Normal Distribution

This section discusses some important properties of the folded normal distribution, $X \sim FN\left(\mu,\ \sigma^2\right)$.

2.2.4.2 Special Cases

The folded normal distribution is related to the following distributions (see, for example, Patel and Read 1982, and Johnson et al. 1994, among others).

(i) If $X \sim FN\left(\mu,\ \sigma^2\right)$, then (X/σ) has a non-central chi distribution with one degree of freedom and non-centrality parameter $\frac{\mu^2}{\sigma^2}$.
(ii) On the other hand, if a random variable U has a non-central chi distribution with one degree of freedom and non-centrality parameter $\frac{\mu^2}{\sigma^2}$, then the distribution of the random variable $\sigma\sqrt{U}$ is given by the pdf $f_X(x)$ in (2.36).
(iii) If $\mu = 0$, the folded normal distribution becomes a half-normal distribution with the pdf $f_X(x)$ as given in (2.15).

2.2.4.3 Characteristic Property

If $Z \sim N(\mu,\ \sigma)$, then $|Z| \sim FN\ (\mu,\ \sigma)$.

2.2.4.4 Mode

It is easy to see that the mode or modal value of x for which the folded normal probability density function $f_X(x)$ defined by (2.36) is maximum, is given by $x = \mu$, and the maximum value of the folded normal probability density function (2.35) is given by

$$f_X(\mu) = \frac{1}{\left(\sqrt{2\pi}\right)\sigma}\left[1 + e^{-\frac{2\mu^2}{\sigma^2}}\right]. \tag{2.39}$$

Clearly, the folded normal probability density function (2.36) is unimodal.

2.2.4.5 Moments

(i) rth **Moment of the Folded Normal Random Variable:** For some integer $r > 0$, a general formula for the rth moment, $\mu'_{f(r)}$, of the folded normal random variable $X \sim FN\left(\mu, \sigma^2\right)$ having the pdf (2.36) has been derived by Elandt (1961), which is presented here. Let $\theta = \frac{\mu}{\sigma}$. Define $\mathrm{I}_r(a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty y^r e^{-\frac{1}{2}y^2}dy, \quad r = 1, 2, \ldots,$ which is known as the "incomplete normal moment." In particular,

$$\mathrm{I}_0(a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-\frac{1}{2}y^2}dy = 1 - \Phi(a), \tag{2.40}$$

where $\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^a e^{-\frac{1}{2}y^2}dy$ is the CDF of the unit normal $N(0, 1)$. Clearly, for $r > 0$, $\mathrm{I}_r(a) = \left(\frac{1}{\sqrt{2\pi}}\right)a^{r-1}e^{-\frac{1}{2}a^2} + (r-1)\,\mathrm{I}_{r-2}(a)$. Thus, in view of these results, the rth moment, $\mu'_{f(r)}$, of the folded normal random variable X is easily expressed in terms of the I_r function as follows.

$$\begin{aligned}\mu'_{f(r)} &= E\left(X^r\right) = \int_0^\infty x f_X(x)\,dx \\ &= \left(\sigma^r\right)\sum_{j=0}^{r}\binom{r}{j}\theta^{r-j}\left[\mathrm{I}_j(-\theta) + (-1)^{r-j}\,\mathrm{I}_j(\theta)\right]. \end{aligned} \tag{2.41}$$

From the above Eq. (2.41) and noting, from the definition of the I_r function, that $\mathrm{I}_2(-\theta) - \mathrm{I}_2(\theta) = -\left[\left(\frac{2}{\sqrt{2\pi}}\right)\theta e^{-\frac{1}{2}\theta^2} + \{1 - 2\mathrm{I}_0(-\theta)\}\right]$, the first four moments of the folded normal random distribution are easily obtained as follows.

$$\mu'_{f(1)} = E[X] = \mu_f = \left(\frac{2}{\sqrt{2\pi}}\right)\sigma e^{-\frac{1}{2}\theta^2} - \mu[1 - 2\mathrm{I}_0(-\theta)]$$
$$= \left(\frac{2}{\sqrt{2\pi}}\right)\sigma e^{-\frac{1}{2}\theta^2} - \mu[1 - 2\Phi(\theta)],$$
$$\mu'_{f(2)} = E\left[X^2\right] = \sigma_f^2 = \mu^2 + \sigma^2,$$
$$\mu'_{f(3)} = E\left[X^3\right] = \left(\mu^2 + 2\sigma^2\right)\mu_f - \mu\sigma^2[1 - 2\Phi(\theta)],$$

and

$$\mu'_{f(4)} = E\left[X^4\right] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4. \tag{2.42}$$

(ii) **Central Moments of the Folded Normal Random Variable:** For some integer $n > 0$, the nth central moment (about the mean $\mu'_{f(1)} = E(X)$) of the folded normal random variable X having the pdf (2.36) can be easily obtained using the formula

$$\mu_{f(n)} = E\left[\left(X - \mu'_{f(1)}\right)^n\right] = \sum_{r=0}^{n}\binom{n}{r}\left(-\mu'_{f(1)}\right)^{n-r}E\left(X^r\right), \tag{2.43}$$

where $E(X^r) = \mu'_{f(r)}$ denotes the rth moment, given by the Eq. (2.41), of the folded normal random variable X. Thus, from the above Eq. (2.43), the first four central moments of the folded normal random variable X are easily obtained as follows.

$$\mu_{f(1)} = 0,$$
$$\mu_{f(2)} = \mu^2 + \sigma^2 - \mu_f^2,$$
$$\mu_{f(3)} = \beta_3 = 2\left[\mu_f^3 - \mu^2\mu_f - \left(\frac{\sigma^3}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}\theta^2}\right],$$

and

$$\mu_{f(4)} = \beta_4 = \left(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4\right) + \left(\frac{8\sigma^3}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}\theta^2}\mu_f + 2\left(\mu^2 - 3\sigma^2\right)\mu_f^2 - 3\mu_f^4. \tag{2.44}$$

2.2.4.6 Mean, Variance, and Coefficients of Skewness and Kurtosis of the Folded Normal Random Variable

These are easily obtained as follows:

(i) **Mean:** $E\ (X) = \alpha_1 = \mu_f = \left(\frac{2}{\sqrt{2\pi}}\right)\sigma e^{-\frac{1}{2}\theta^2} - \mu\left[1 - 2\Phi\left(\theta\right)\right]$,

(ii) **Variance:** $Var\ (X) = \beta_2 = \mu_{f(2)} = \mu^2 + \sigma^2 - \mu_f^2, \quad \sigma > 0$,

(iii) **Coefficient of Skewness:** $\gamma_1\ (X) = \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}}$,

(iv) **Coefficient of Kurtosis:** $\gamma_2\ (X) = \frac{\mu_{f(4)}}{[\mu_{f(2)}]^2}$,

where the symbols have their usual meanings as described above.

2.2.4.7 Percentiles of the Folded Normal Distribution

This section computes the percentiles of the folded normal distribution, by using Maple 10. For any $p(0 < p < 1)$, the $(100p)th$ percentile (also called the quantile of order p) of the folded normal distribution, $X \sim FN\left(\mu,\ \sigma^2\right)$, with the pdf $f_X(x)$ given by (2.36), is a number z_p such that the area under $f_X(x)$ to the left of z_p is p. That is, z_p is any root of the equation

$$F(z_p) = \int_{-\infty}^{z_p} f_X(t)dt = p. \tag{2.45}$$

Thus, from the Eq. (2.45), using the Maple program, the percentiles z_p of the folded normal distribution can be computed for some selected values of the parameters.

Note: For the tables of the folded normal cdf $F_X(x) = P(X \le x)$ for different values of the parameters, for example, $\frac{\mu_f}{\sigma_f}$ = 1.3236, 1.4(0.1)3, and x = 0.1(0.1)7, the interested readers are referred to Leon et al. (1961).

Note: As noted by Elandt (1961), the family of the folded normal distributions, $N_f\left(\mu_f\ ,\ \sigma_f\right)$, is included between the half-normal, for which $\frac{\mu_f}{\sigma_f}$ = 1.3237, and the normal, for which $\frac{\mu_f}{\sigma_f}$ is infinite. Approximate normality is attained if , for which $\frac{\mu_f}{\sigma_f} > 3$.

2.2.5 Truncated Distributions

Following Rohatgi and Saleh (2001), and Lawless (2004), we first present an overview of the truncated distributions.

2.2.5.1 Overview of Truncated Distributions

Suppose we have a probability distribution defined for a continuous random variable X. If some set of values in the range of X are excluded, then the probability distribution for the random variable X is said to be truncated. We defined the truncated distributions as follows.

Definition: Let X be a continuous random variable on a probability space (Ω, S, P), and let $T \in$ B such that $0 < P\{X \in T\} < 1$, where B is a σ- field on the set of real numbers $\Re$. Then the conditional distribution $P\{X \leq x \mid X \in T\}$, defined for any real x, is called the truncated distribution of X. Let $f_X(x)$ and $F_X(x)$ denote the probability density function (pdf) and the cumulative distribution function (cdf), respectively, of the parent random variable X. If the random variable with the truncated distribution function $P\{X \leq x \mid X \in T\}$ be denoted by Y, then Y has support T. Then the cumulative distribution function (cdf), say, $G(y)$, and the probability density function (pdf), say, $g(y)$, for the random variable Y are, respectively, given by

$$G_Y(y) = P\{Y \leq y \mid Y \in T\} = \frac{P\{Y \leq y,\ Y \in T\}}{P\{Y \in T\}} = \frac{\int\limits_{(-\infty,\ y] \cap T} f_X(u)du}{\int\limits_T f_X(u)du}, \tag{2.46}$$

and

$$g_Y(y) = \begin{cases} \frac{f_X(y)}{\int\limits_T f_X(u)du}, & y \in T \\ 0, & y \notin T. \end{cases} \tag{2.47}$$

Clearly $g_Y(y)$ in (2.47) defines a pdf with support T, since $\int_T g_Y(y)dy = \frac{\int\limits_T f_X(y)dy}{\int\limits_T f_X(u)du} = 1$. Note that here T is not necessarily a bounded set of real numbers. In particular, if the values of Y below a specified value a are excluded from the distribution, then the remaining values of Y in the population have a distribution with the pdf given by $g_L(y;\ a) = \frac{f_X(y)}{1 - F_X(a)},\ a \leq y < \infty$, and the distribution is said to be left truncated at a. Conversely, if the values of Y above a specified value a are excluded from the distribution, then the remaining values of Y in the population have a distribution with the pdf given by $g_R(y;\ a) = \frac{f_X(y)}{F_X(a)},\ 0 \leq y \leq a$, and the distribution is said to be right truncated at a. Further, if Y has a support $T = [a_1,\ a_2]$, where $-\infty < a_1 < a_2 < \infty$, then the conditional distribution of Y, given that $a_1 \leq y \leq a_2$, is called a doubly truncated distribution with the cdf, say, $G(y)$, and the pdf, say, $g(y)$, respectively, given by

$$G_Y(y) = \frac{F_X\{\max(\min(y,\ a_2),\ a_1)\} - F_X(a_1)}{F_X(a_2) - F_X(a_1)}, \tag{2.48}$$

and

$$g_Y(y) = \begin{cases} \frac{f_X(y)}{F_X(a_2) - F_X(a_1)}, & y \in [a_1,\ a_2] \\ 0, & y \notin [a_1,\ a_2]. \end{cases} \tag{2.49}$$

The truncated distribution for a continuous random variable is one of the important research topics both from the theoretical and applications point of view. It arises in many probabilistic modeling problems of biology, crystallography, economics, engineering, forecasting, genetics, hydrology, insurance, lifetime data analysis, management, medicine, order statistics, physics, production research, psychology, reliability, quality engineering, survival analysis, etc, when sampling is carried out from an incomplete population data. For details on the properties and estimation of parameters of truncated distributions, and their applications to the statistical analysis of truncated data, see, for example, Hald (1952), Chapman (1956), Hausman and Wise (1977), Thomopoulos (1980), Patel and Read (1982), Levy (1982), Sugiura and Gomi (1985), Schneider (1986), Kimber and Jeynes (1987), Kececioglu (1991), Cohen (1991), Andersen et al. (1993), Johnson et al. (1994), Klugman et al. (1998), Rohatgi and Saleh (2001), Balakrishnan and Nevzorov (2003), David and Nagaraja (2003), Lawless (2003), Jawitz (2004), Greene (2005), Nadarajah and Kotz (2006a), Maksay and Stoica (2006) and Nadarajah and Kotz (2007) and references therein.

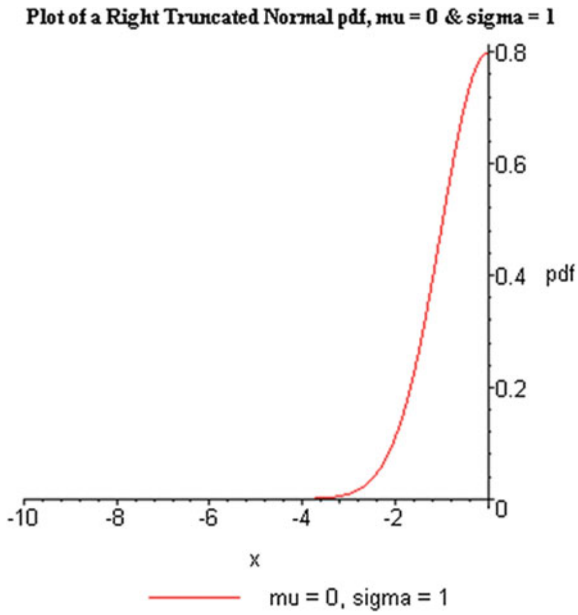
The truncated distributions of a normally distributed random variable, their properties and applications have been extensively studied by many researchers, among them Bliss (1935 for the probit model which is used to model the choice probability of a binary outcome), Hald (1952), Tobin (1958) for the probit model which is used to model censored data), Shah and Jaiswal (1966), Hausman and Wise (1977), Thomopoulos (1980), Patel and Read (1982), Levy (1982), Sugiura and Gomi (1985), Schneider (1986), Kimber and Jeynes (1987), Cohen (1959, 1991), Johnson et al. (1994), Barr and Sherrill (1999), Johnson (2001), David and Nagaraja (2003), Jawitz (2004), Nadarajah and Kotz (2007), and Olive (2007), are notable. In what follows, we present the pdf, moment generating function (mgf), mean, variance and other properties of the truncated normal distribution most of which is discussed in Patel and Read (1982), Johnson et al. (1994), Rohatgi and Saleh (2001), and Olive (2007).

Definition: Let $X \sim N\left(\mu,\ \sigma^2\right)$ be a normally distributed random variable with the mean μ and the variance σ^2. Let us consider a random variable Y which represents the truncated distribution of X over a support T = [a, b], where $-\infty < a < b < \infty$. Then the conditional distribution of Y, given that $a \leq y \leq b$, is called a doubly truncated normal distribution with the pdf, say, $g_Y(y)$, given by

$$g_Y(y) = \begin{cases} \dfrac{\frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}, & y \in [a,\ b] \\ 0, & y \notin [a,\ b] \end{cases}, \tag{2.50}$$

where $\phi(.)$ and $\Phi(.)$ are the pdf and cdf of the standard normal distribution, respectively. If $a = -\infty$, then the we have a (singly) truncated normal distribution from above, (that is, right truncated). On the other hand, if $b = \infty$, then the we have a (singly) truncated normal distribution from below, (that is, left truncated). The following are some examples of the truncated normal distributions.

Fig. 2.7 Example of a right truncated normal distribution



(i) **Example of a Left Truncated Normal Distribution:** Taking $a = 0$, $b = \infty$, and $\mu = 0$, the pdf $g_Y(y)$ in (2.50) reduces to that of the half normal distribution in (2.15), which is an example of the left truncated normal distribution.

(ii) **Example of a Right Truncated Normal Distribution:** Taking $a = -\infty, b = 0$, $\mu = 0$, and $\sigma = 1$ in (2.50), the pdf $g_Y(y)$ of the right truncated normal distribution is given by

$$g_Y(y) = \begin{cases} 2\phi(y), & -\infty < y \leq 0 \\ 0, & y > 0 \end{cases}, \quad (2.51)$$

where $\phi(.)$ is the pdf of the standard normal distribution. The shape of right truncated normal pdf $g_Y(y)$ in (2.51) is illustrated in the following Fig. (2.7).

2.2.5.2 MGF, Mean, and Variance of the Truncated Normal Distribution

These are given below.

(A) Moment Generating Function: The mgf of the doubly truncated normal distribution with the pdf $g_Y(y)$ in (2.50) is easily obtained as

$$M(t) = E\left(e^{tY}|Y \in [a, b]\right)$$
$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \left\{ \frac{\left[\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)\right]}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \right\} \quad (2.52)$$

(B) Mean, Second Moment and Variance: Using the expression for the mgf (2.52), these are easily given by

(i)
$$Mean = E(Y|Y \in [a, b]) = M'(t)\big|_{t=0}$$
$$= \mu + \sigma \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \quad (2.53)$$

Particular Cases:

(I) If $b \to \infty$ in (2.52), then we have

$$E(Y|Y > a) = \mu + \sigma h,$$

where $h = \frac{\phi\left(\frac{a-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)}$ is called the Hazard Function (or the Hazard Rate, or the Inverse Mill's Ratio) of the normal distribution.

(II) If $a \to -\infty$ in (2.53), then we have

$$E(Y|Y < b) = \mu - \sigma \left[\frac{\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \right]. \quad (2.54)$$

(III) If $b \to \infty$ in (2.54), then Y is not truncated and we have

$$E(Y) = \mu \quad .$$
$$V(Y) = \sigma^2[1 + \alpha\phi]$$

(ii)
$$Second\ Moment = E\left(Y^2|Y \in [a, b]\right) = M''(t)\big|_{t=0}$$
$$= 2\mu\{E(Y|Y \in [a, b])\} - \mu^2$$
$$= \mu^2 + 2\mu\sigma \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right]$$

$$+\sigma^2\left[1+\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \tag{2.55}$$

and

(iii) $$\begin{aligned} Variance = Var(Y|Y \in [a, b]) &= \left\{E\left(Y^2|Y \in [a, b]\right)\right\} \\ &\quad - \{E(Y|Y \in [a, b])\}^2 \\ &= \sigma^2\left\{1+\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right. \\ &\quad \left. -\left[\frac{\phi\left(\frac{b-\mu}{\sigma}\right)-\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\right\} \end{aligned} \tag{2.56}$$

Some Further Remarks on the Truncated Normal Distribution:

(i) Let $Y \sim TN\left(\mu, \sigma^2, a = \mu - k\sigma, b = \mu + k\sigma\right)$, for some real k, be the truncated version of a normal distribution with mean μ and variance σ^2. Then, from (2.53) and (2.56), it easily follows that $E(Y) = \mu$ and $Var(Y) = \sigma^2\left\{1 - \frac{2k\phi(k)}{2k\Phi(k)-1}\right\}$, (see, for example, Olive, 2007).

(ii) The interested readers are also referred to Shah and Jaiswal (1966) for some nice discussion on the pdf $g_Y(y)$ of the truncated normal distribution and its moments, when the origin is shifted at a.

(iii) A table of the mean μ_t, standard deviation σ_t, and the ratio (mean deviation/σ_t) for selected values of $\Phi\left(\frac{a-\mu}{\sigma}\right)$ and $1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$ have been provided in Johnson and Kotz (1994).

2.2.6 Inverse Normal (Gaussian) Distribution (IGD)

The inverse Gaussian distribution (IGD) represents a class of distribution. The distribution was initially considered by Schrondinger (1915) and further studied by many authors, among them Tweedie (1957a, b) and Chhikara and Folks (1974) are notable. Several advantages and applications in different fields of IGD are given by Tweedie (1957), Johnson and Kotz (1994), Chhikara and Folks (1974, 1976,1977), and Folks and Chhikara (1978), among others. For the generalized inverse Gaussian distribution (GIG) and its statistical properties, the interested readers are referred to Good (1953), Sichel (1974, 1975), Barndorff-Nielsen (1977, 1978), Jorgensen

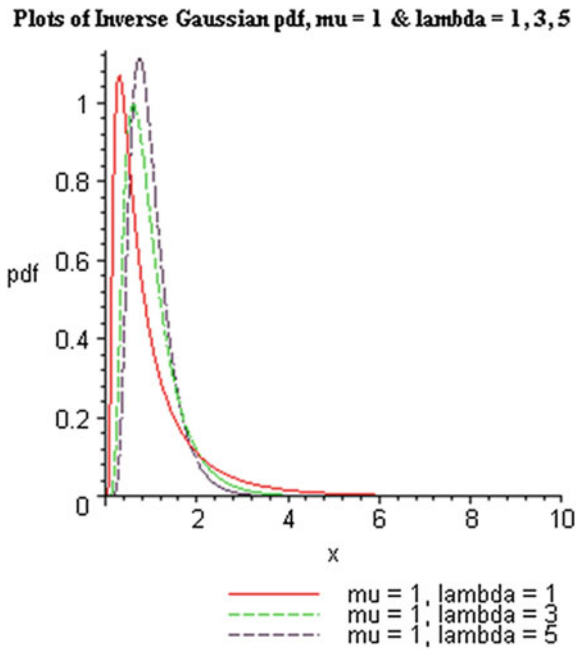


Fig. 2.8 Plots of the inverse Gaussian pdf

(1982), and Johnson and Kotz (1994), and references therein. In what follows, we present briefly the pdf, cdf, mean, variance and other properties of the inverse Gaussian distribution (IGD).

Definition: The pdf of the Inverse Gaussian distribution (IGD) with parameters μ and λ is given by

$$f(x,\ \mu,\ \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x - \mu)^2\right\} \quad x > 0, \mu > 0, \lambda > 0 \tag{2.57}$$

where μ is location parameter and λ is a shape parameter. The mean and variance of this distribution are μ and μ^3/λ respectively. To describe the shapes of the inverse Gaussian distribution, the plots of the pdf (2.57), for $\mu = 1$ and $\lambda = 1, 3, 5$ are provided in Fig. 2.8 by using Maple 10. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

Properties of IGD:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the inverse Gaussian distribution (1.1). The maximum likelihood estimators (MLE's) for μ and λ are respectively given by

$$\hat{\mu} = \bar{x} = \sum_{i=1}^{n} x_i/n,\ \tilde{\lambda} = \frac{n}{V},\ \text{where } V = \sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right).$$

It is well known that

(i) the sample mean $\bar{x}$ is unbiased estimate of μ where as $\tilde{\lambda}$ is a biased estimate of λ.
(ii) $\bar{x}$ follows IGD with parameters μ and $n\,\lambda$, whereas $\lambda\,V$ is distributed as chi –square distribution with (n-1) degrees of freedom
(iii) $\bar{x}$ and V are stochastically independent and jointly sufficient for (μ,λ) if both are unknown.
(iv) the uniformly minimum variance unbiased estimator (UMVUE) of λ is $\hat{\lambda} = (n-3)/V$ and $Var\left(\hat{\lambda}\right) = 2\,\lambda^2/(n-5) = MSE\left(\hat{\lambda}\right)$.

2.2.7 Skew Normal Distributions

This section discusses the univariate skew normal distribution (SND) and some of its characteristics. The skew normal distribution represents a parametric class of probability distributions, reflecting varying degrees of skewness, which includes the standard normal distribution as a special case. The skewness parameter involved in this class of distributions makes it possible for probabilistic modeling of the data obtained from skewed population. The skew normal distributions are also useful in the study of the robustness and as priors in Bayesian analysis of the data. It appears from the statistical literatures that the skew normal class of densities and its applications first appeared indirectly and independently in the work of Birnbaum (1950), Roberts (1966), O'Hagan and Leonard (1976), and Aigner et al. (1977). The term skew normal distribution (SND) was introduced by Azzalini (1985, 1986), which give a systematic treatment of this distribution, developed independently from earlier work. For further studies, developments, and applications, see, for example, Henze (1986), Mukhopadhyay and Vidakovic (1995), Chiogna (1998), Pewsey (2000), Azzalini (2001), Gupta et al. (2002), Monti (2003), Nadarajah and Kotz (2003), Arnold and Lin (2004), Dalla Valle (2004), Genton (2004), Arellano-Valle et al. (2004), Buccianti (2005), Azzalini (2005, 2006), Arellano-Valle and Azzalini (2006), Bagui and Bagui (2006), Nadarajah and Kotz (2006), Shkedy et al. (2006), Pewsey (2006), Fernandes et al. (2007), Mateu-Figueras et al. (2007), Chakraborty and Hazarika (2011), Eling (2011), Azzalini and Regoli (2012), among others. For generalized skew normal distribution, the interested readers are referred to Gupta and Gupta (2004), Jamalizadeh, et al. (2008), and Kazemi et al. (2011), among others. Multivariate versions of SND have also been proposed, among them Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Arellano-Valle et al. (2002), Gupta and Chen (2004), and Vernic (2006) are notable. Following Azzalini (1985, 1986, 2006), the definition and some properties, including some graphs, of the univariate skew normal distribution (SND) are presented below.

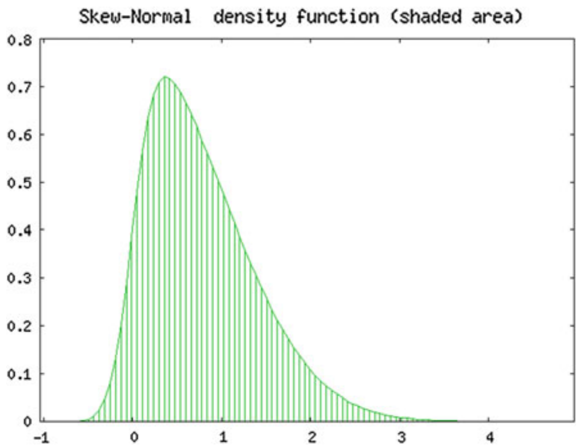


Fig. 2.9 Plot of the skew normal pdf: ($\mu = 0$, $\sigma = 1$, $\lambda = 5$)

Definition: For some real-valued parameter λ, a continuous random variable X_λ is said to have a skew normal distribution, denoted by $X_\lambda \sim SN\ (\lambda)$, if its probability density function is given by

$$f_X(x;\ \lambda) = 2\,\phi(x)\,\Phi(\lambda x),\ -\infty < x < \infty, , \tag{2.58}$$

where $\phi(x) = \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}x^2}$ and $\Phi(\lambda x) = \int\limits_{-\infty}^{\lambda x} \phi(t)\,dt$ denote the probability density function and cumulative distribution function of the standard normal distribution respectively.

2.2.7.1 Shapes of the Skew Normal Distribution

The shape of the skew normal probability density function given by (2.58) depends on the values of the parameter λ. For some values of the parameters (μ, σ, λ), the shapes of the pdf (2.58) are provided in Figs. 2.9, 2.10 and 2.11. The effects of the parameter can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

Remarks: The continuous random variable X_λ is said to have a skew normal distribution, denoted by $X_\lambda \sim SN\ (\lambda)$, because the family of distributions represented by it includes the standard $N\ (0,\ 1)$ distribution as a special case, but in general its members have a skewed density. This is also evident from the fact that $X_\lambda^2 \sim \chi^2$ for all values of the parameter λ. Also, it can be easily seen that the skew normal density function $f_X(x;\ \lambda)$ has the following characteristics:

1. when $\lambda = 0$, we obtain the standard normal density function $f_X(x; 0)$ with zero skewness;
2. as $|\lambda|$ increases, the skewness of the skew normal distribution also increases;

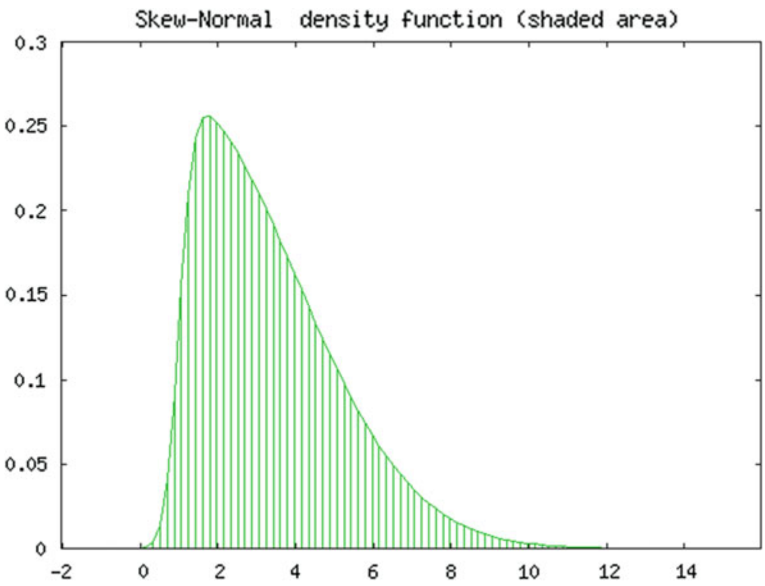


Fig. 2.10 Plot of the skew normal pdf: $(\mu = 1,\ \sigma = 3,\ \lambda = 10)$

3. when $|\lambda| \to \infty$, the skew normal density function $f_X(x;\ \lambda)$ converges to the half-normal (or folded normal) density function;
4. if the sign of λ changes, the skew normal density function $f_X(x;\ \lambda)$ is reflected on the opposite side of the vertical axis.

2.2.7.2 Some Properties of Skew Normal Distribution

This section discusses some important properties of the skew normal distribution, $X_\lambda \sim SN(\lambda)$.
Properties of $SN(\lambda)$:

(a) $SN(0) = N(0,\ 1)$.
(b) If $X_\lambda \sim SN(\lambda)$, then $-X_\lambda \sim SN(-\lambda)$.
(c) If $\lambda \to \pm\infty$, and $Z \sim N(0,\ 1)$, then $SN(\lambda) \to \pm|Z| \sim HN(0,\ 1)$, that is, $SN(\lambda)$ tends to the half-normal distribution.
(d) If $X_\lambda \sim SN(\lambda)$, then $X_\lambda^2 \sim \chi^2$.
(e) The MGF of X_λ is given by $M_\lambda(t) = 2e^{\frac{t^2}{2}}\Phi(\delta t),\ t \in \Re$, where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.
(f) It is easy to see that $E(X_\lambda) = \delta\left(\sqrt{\frac{2}{\pi}}\right)$, and $Var(X_\lambda) = \frac{\pi - 2\delta^2}{\pi}$.
(g) The characteristic function of X_λ is given by $\psi_\lambda(t) = e^{\frac{-t^2}{2}}[1 + ih(\delta t)],\ t \in \Re$, where $h(x) = \left(\sqrt{\frac{2}{\pi}}\right)\int\limits_0^x e^{\frac{y^2}{2}}dy$ and $h(-x) = -h(x)$ for $x \geq 0$.

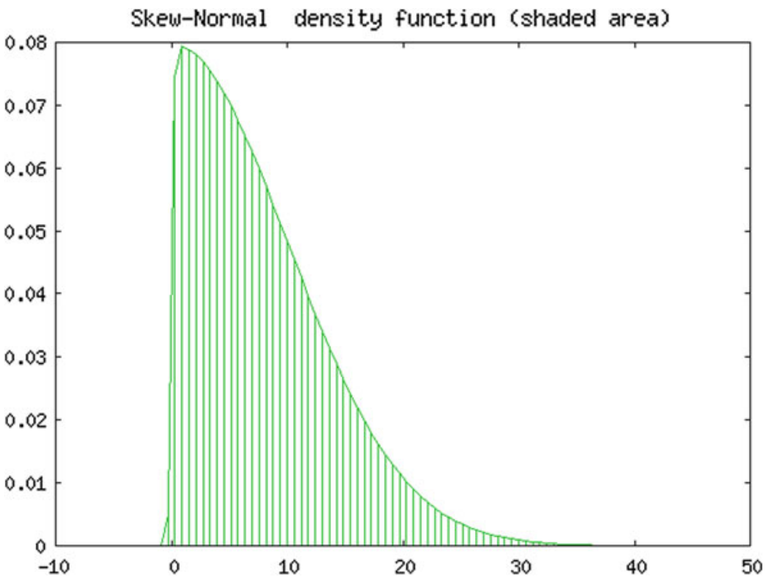


Fig. 2.11 Plots of the skew normal pdf: ($\mu = 0,\ \sigma = 10,\ \lambda = 50$)

(h) By introducing the following linear transformation $Y = \mu + \sigma X$, that is, $X = \frac{Y-\mu}{\sigma}$, where $\mu \geq 0,\ \sigma > 0$, we obtain a skew-normal distribution with parameters $(\mu,\ \sigma,\ \lambda)$, denoted by $Y \sim SN\left(\mu,\ \sigma^2,\ \lambda\right)$, if its probability density function is given by

$$f_Y(y;\ \mu,\ \sigma,\ \lambda) = 2\phi\left(\frac{y-\mu}{\sigma}\right)\Phi\left(\frac{\lambda(y-\mu)}{\sigma}\right),\ -\infty < y < \infty, \tag{2.59}$$

where $\phi(y)$ and $\Phi(\lambda y)$ denote the probability density function and cumulative distribution function of the normal distribution respectively, and $\mu \geq 0,\ \sigma > 0$ and $-\infty < \lambda < \infty$ are referred as the location, the scale and the shape parameters respectively. Some characteristic values of the random variable Y are as follows:

I. Mean: $E\left(Y\right) = \mu + \left(\sigma\delta\sqrt{\frac{2}{\pi}}\right)$

II. Variance: $Var\left(Y\right) = \frac{\sigma^2\left(\pi - 2\delta^2\right)}{\pi}$

III. Skewness: $\gamma_1 = \left(\frac{4 - \pi}{2}\right)\frac{\left[E\left(X_\lambda\right)\right]^3}{\left[Var\left(X_\lambda\right)\right]^{\frac{3}{2}}}$

IV. Kurtosis: $\gamma_2 = 2\left(\pi - 3\right)\frac{\left[E\left(X_\lambda\right)\right]^4}{\left[Var\left(X_\lambda\right)\right]^2}$

2.2.7.3 Some Characteristics Properties of Skew Normal Distribution

Following Gupta et al. (2004), some characterizations of the skew normal distribution (SND) are stated below.

(i) Let X_1 and X_2 be *i.i.d.* F, an unspecified distribution which admits moments of all order. Then $X_1^2 \sim \chi_1^2$, $X_2^2 \sim \chi_1^2$, and $\frac{1}{2}\left(X_1 + X_2\right)^2 \sim H_0\left(\lambda\right)$ if and only if $F = SN\left(\lambda\right)$ or $F = SN\left(-\lambda\right)$ where $H_0\left(\lambda\right)$ is the distribution of $\frac{1}{2}\left(X + Y\right)^2$ when X and Y are *i.i.d.* $SN\left(\lambda\right)$.

(ii) Let $H_0\left(\lambda\right)$ be the distribution of $\left(Y + a\right)^2$ where $Y \sim SN\left(\lambda\right)$ and $a \neq 0$ is a given constant. Let X be a random variable with a distribution that admits moments of all order. Then $X^2 \sim \chi_1^2$, $\left(X + a\right)^2 \sim H_0\left(\lambda\right)$ if and only if $X \sim SN\left(\lambda\right)$ for some λ.

For detailed derivations of the above and more results on other characterizations of the skew normal distribution (SND), see Gupta et al. (2004) and references therein. The interested readers are also referred to Arnold and Lin (2004), where the authors have shown that the skew-normal distributions and their limits are exactly the distributions of order statistics of bivariate normally distributed variables. Further, using generalized skew-normal distributions, the authors have characterized the distributions of random variables whose squares obey the chi-square distribution with one degree of freedom.

2.3 Goodness-of-Fit Test (Test For Normality)

The goodness of fit (or GOF) tests are applied to test the suitability of a random sample with a theoretical probability distribution function. In other words, in the GOF test analysis, we test the hypothesis if the random sample drawn from a population follows a specific discrete or continuous distribution. The general approach for this is to first determine a test statistic which is defined as a function of the data measuring the distance between the hypothesis and the data. Then, assuming the hypothesis is true, a probability value of obtaining data which have a larger value of the test statistic than the value observed, is determined, which is known as the p-value. Smaller p-values (for example, less than 0.01) indicate a poor fit of the distribution. Higher values of p (close to one) correspond to a good fit of the distribution. We consider the following parametric and non-parametric goodness-of-fit tests

2.3.1 χ^2(Chi-Squared) Test

The χ^2 test, due to Karl Pearson, may be applied to test the fit of any specified continuous distribution to the given randomly selected continuous data. In χ^2 analysis, the data is first grouped into, say, k number of classes of equal probability. Each class should contain at least 5 or more data points. The χ^2 test statistic is given by

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \tag{2.60}$$

where O_i is the observed frequency in class $i, i = 1, \ldots, k$ and E_i is the expected frequency in class i, if the specified distribution were the correct one, and is given by

$$E_i = F(x_i) - F(x_{i-1}),$$

where $F(x)$ is the cumulative distribution function (CDF) of the probability distribution being tested, and x_i , x_{i-1} are the limits for the class i. The null and alternative hypotheses being tested are, respectively, given by:

H_0: The data follow the specified continuous distribution;
H_1: The data do not follow the specified continuous distribution.

The null hypothesis (H_0) is rejected at the chosen significance level, say, α, if the test statistic is greater than the critical value denoted by $\chi^2{}_{1-\alpha,\, k-1}$, with $k - 1$ degrees of freedom (df) and a significance level of α. If r parameters are estimated from the data, df are $k - r - 1$.

2.3.2 Kolmogorov-Smirnov (K-S) Test

This test may also be applied to test the goodness of fit between a hypothesized cumulative distribution function (CDF) $F(x)$ and an empirical CDF $F_n(x)$. Let $y_1 < y_2 < \ldots < y_n$ be the observed values of the order statistics of a random sample $x_1, x_2, \ldots, x_n$ of size n. When no two observations are equal, the empirical CDF $F_n(x)$ is given by, see Hogg and Tanis (2006),

$$F_n(x) = \begin{cases} 0, & x < y_1, \\ \frac{i}{n}, & y_i \le x < y_{i+1}, \quad i = 1, 2, \ldots, n-1, \\ 1, & y_n \le x. \end{cases} \tag{2.61}$$

Clearly,

$$F_n(x) = \frac{1}{n}[\text{Number of Observations} \le x].$$

Following Blischke and Murthy (2000), the Kolmogorov-Smirnov test statistic, D_n, is defined as the maximum distance between the hypothesized CDF $F(x)$ and the empirical CDF $F_n(x)$, and is given by

$$D_n = \max\left\{D_n^+, D_n^-\right\},$$

where

$$D_n^+ = \max_{i=1,2,\ldots,n}\left[\frac{i}{n} - F_n(y_i)\right]$$

and

$$D_n^- = \max_{i=1,2,\ldots,n}\left[F_n(y_i) - \frac{i-1}{n}\right].$$

For calculations of fractiles (percentiles) of the distribution of D_n, the interested readers are referred to Massey (1951). In Stephens (1974), one can find a close approximation of the fractiles of the distribution of D_n, based on a constant denoted by d_α which is a function of n only. The values of d_α can also be found in Table 11.2 on p. 400 of Blischke and Murthy (2000) for $\alpha = 0.15,\ 0.10,\ 0.05,$ and 0.01. The critical value of D_n is calculated by the formula $d_\alpha / \left(\sqrt{n} + \frac{0.11}{\sqrt{n}} + 0.12\right)$. The null and alternative hypotheses being tested are, respectively, given by:

H_0: The data follow the specified continuous distribution;
H_1: The data do not follow the specified continuous distribution.

The null hypothesis (H_0) is rejected at the chosen significance level, say, α, if the Kolmogorov-Smirnov test statistic, D_n, is greater than the critical value calculated by the above formula.

2.3.3 Anderson-Darling (A-D) Test

The **Anderson-Darling test** is also based on the difference between the hypothesized CDF $F(x)$ and the empirical CDF $F_n(x)$. Let $y_1 < y_2 < \ldots < y_n$ be the observed values of the order statistics of a random sample $x_1, x_2, \ldots, x_n$ of size n. The A-D test statistic (A^2) is given by

$$A^2 = A_n^2 = \frac{-1}{n} \sum_{i=1}^{n} (2i - 1) \left\{ \ln F_n(y_i) + \ln\left[1 - F_n(y_{n-i+1})\right]\right\} - n.$$

Fractiles of the distribution of A_n^2 for $\alpha = 0.15,\ 0.10,\ 0.05,$ and 0.01, denoted by a_α, are given in Table 11.2 on p. 400 of Blischke and Murthy (2000). The null and alternative hypotheses being tested are, respectively, given by:

H_0 : The data follow the specified continuous distribution
H_1: The data do not follow the specified continuous distribution.

The null hypothesis (H_0) is rejected if the A-D test statistic, A_n^2, is greater than the above tabulated constant a_α (also known as the critical value for A-D test analysis) at one of the chosen significance levels, $\alpha = 0.15,\ 0.10,\ 0.05,$ and 0.01. As pointed out in Blischke and Murthy (2000), "the critical value a_α does not depend on n, and has been found to be a very good approximation in samples as small as $n = 3$".

2.3.4 The Shapiro-Wilk Test for Normality

The **Shapiro-Wilk test** (also known as the W test) may be applied to test the goodness of fit between a hypothesized cumulative distribution function (CDF)$F(x)$ and an empirical CDF $F_n(x)$.

Let $y_1 < y_2 < \ldots < y_n$ be the observed values of the order statistics of a random sample $x_1, x_2, \ldots, x_n$ of size n with some unknown distribution function $F(x)$. Following Conover (1999), the Shapiro-Wilk test statistic, W, is defined as

$$W = \frac{\sum_{i=1}^{k} a_i (y_{n-i+1} - y_i)^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2},$$

where $\overline{x}$ denotes the sample mean, and, for the observed sample size $n \le 50$, the coefficients a_i, $i = 1, \ldots, k$, where k is approximately $\frac{n}{2}$, are available in Table A16, pp. 550–552, of Conover (1999). For the observed sample size $n > 50$, the interested readers are referred to D'Agostino (1971) and Shapiro and Francia (1972) (Fig. 2.12).

For the Shapiro-Wilk test, the null and alternative hypotheses being are, respectively, given by:

H_0: $F(x)$ is a normal distribution with unspecified mean and variance
H_1: $F(x)$ is non-normal.

The null hypothesis (H_0) is rejected at one of the chosen significance levels α if the Shapiro-Wilk test statistic, W, is less than the α quantile as given by Table A 17,

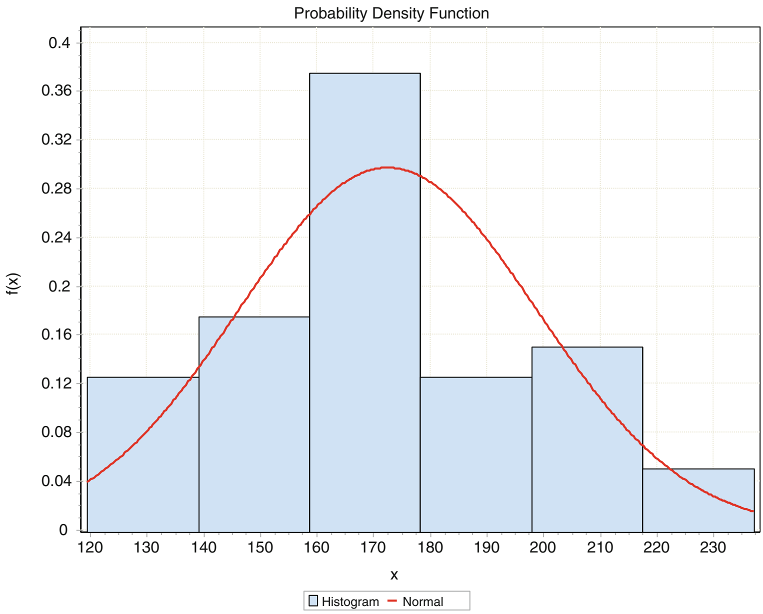


Fig. 2.12 Frequency histogram of the weights of 40 adult men

pp. 552–553, of Conover (1999). The p- value for the Shapiro-Wilk test may be calculated by following the procedure on p. 451 of Conover (1999).

Note: The Shapiro-Wilk test statistic may be calculated using the computer softwares such as R, Maple, Minitab, SAS, and StatXact, among others.

2.3.5 Applications

In order to examine the applications of the above tests of normality, we consider the following example of weights of a random sample of 40 adult men (Source: *Biostatistics for the Biological and Health Sciences*, Mario F Triola, Publisher: Pearson, 2005).

Example: We consider the weights of a random sample of 40 adult men as given below:

{169.1, 144.2, 179.3, 175.8, 152.6, 166.8, 135.0, 201.5, 175.2, 139.0, 156.3, 186.6, 191.1, 151.3, 209.4, 237.1, 176.7, 220.6, 166.1, 137.4, 164.2, 162.4, 151.8, 144.1, 204.6, 193.8, 172.9, 161.9, 174.8, 169.8, 213.3, 198.0, 173.3, 214.5, 137.1, 119.5, 189.1, 164.7, 170.1, 151.0}.

Table 2.2 Descriptive statistics

Statistic	**Value**	**Percentile**	**Value**
Sample size	40	Min	119.5
Range	117.6	5 %	135.11
Mean	172.55	10 %	137.56
Variance	693.12	25 % (Q1)	152.0
Standard deviation	26.327	50 % (Median)	169.95
Coefficient of variation	0.15258	75 % (Q3)	190.6
Standard error	4.1627	90 %	212.91
Skewness	0.37037	95 %	220.29
Excess Kurtosis	−0.16642	Max	237.1

Table 2.3 Normality for the Weights of 40 Adult Men

Test statistics	Value of the test statistics	P-value	Decision at 5 % level of significance
K-S test	0.112	0.652	Do not reject H_0
A-D test	0.306	0.552	Do not reject H_0
Chi-Squared test	2.712	0.844	Do not reject H_0
Shapiro-Wilk test	0.967	0.379	Do not reject H_0

Using the software EasyFit, the descriptive statistics are computed in the Table 2.2 below. The frequency histogram of the weights of 40 adult men is drawn in Fig. 2.12.

The goodness of fit (or GOF) tests, as discussed above, are applied to test the compatibility of our example of weights of the random sample of 40 adult men with our hypothesized theoretical probability distribution, that is, normal distribution, using various software such as EasyFit, Maple, and Minitab. The results are summarized in the Table 2.2 below. The chosen significance level is $\alpha = 0.05$. The null and alternative hypotheses being tested are, respectively, given by:

H_0: The data follow the normal distribution;
H_1: The data do not follow the normal distribution.

It is obvious from Table 2.3 is that the normal distribution seems to be an appropriate model for the weights of 40 adult men considered here. Since the sample size is large enough, all tests are valid for this example. In this section, we have discussed various tests of normality to test the suitability of a random sample with a theoretical probability distribution function. In particular, we have applied to test the applicability of normal distribution to a random sample of the weights of 40 adult men. It is hoped that this study may be helpful to apply these goodness of fit (or GOF) tests to other examples also.

2.4 Summary

The different forms of normal distributions and their various properties are discussed in this chapter. The entropy of a random variable having the normal distribution has been given. The expressions for the characteristic function of a normal distribution are provided. Some goodness of fit tests for testing the normality along with applications is given. By using Maple 10, various graphs have been plotted. As a motivation, different forms of normal distributions (folded and half normal etc.) and their properties have also been provided.

Chapter 3
Student's *t* Distribution

3.1 Student's *t* Distribution

The Student's t distribution or t distribution defines a family of continuous probability distributions. It has a wide range of applications in probability, statistics, and other fields of sciences. It was first developed by Willieam S. Gosset (1908) in his work on "the probable error of a mean," published by him under the *nom de plume* of *Student*. Further developments continued with the contributions of Fisher (1925) and others later. For detailed discussions on the development of the t distribution and its usages, see, example, Pearson (1967, 1970), Eisenhart (1979), Box (1981), Patel and Read (1982), Johnson et al. (1995), Wiper et al. (2005), Finner (2008), and Zabell (2008), and references therein. The graph of the probability density function of the Student's t distribution is a symmetric and bell-shaped curve, differing for different sample sizes. The Student's t distribution has mean $= 0$ and standard deviation is greater 1 for degrees of freedom greater than 2 and it does not exists for 1 and 2 degrees of freedom. As the sample size $n \rightarrow \infty$, the Student's t distribution approaches the standard normal distribution. If we compare the z-table and t table, we can see that the percentile points for of normal distribution and Student $-t$ distribution for large degrees of freedom are approximately equal.

The 95th and 99th percentile points of Standard Normal with pdf $f_Z(z)$ and Student$-t$ distribution with n ($\geq$1) degrees of freedom with pdf $f_{t_n}(x)$, where

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty \quad \text{and}$$

$$f_{t_n}(x) = \frac{1}{\sqrt{n} B\left(\frac{n}{2}, \frac{1}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}}, -\infty < x < \infty.,$$

are given below.

The classical theory of statistical inference is mainly based on the assumption that errors are normally and independently distributed. Recently many researchers

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Percentile points	N(0,1)	t_{40}	t_{50}	t_{60}	t_{80}	t_{100}
95	1.645	1.684	1.676	1.671	1.664	1.660
99	2.33	2.423	2.403	2.390	2.374	2.364

have investigated that how inferences are affected if the population model departs from normality. In reality, many economic, finance and business data exhibit fat-tailed distributions. The suitability of independent *t*-distributions for stock return data was performed by Blattberg and Gonedes (1974). Zellner (1976) analyzed the stock prices data by a simple regression model under the assumption that errors have a multivariate *t*-distribution. However, errors in this model are uncorrelated but not independent. In a later date, Prucha and Kelejian (1984) discussed the inadequacy of normal distribution and suggested a correlated *t*-model for many real world problems as a better alternative of normal distribution. Kelejian and Prucha (1985) proved that the uncorrelated *t*-distributions are better to capture heavy-tailed behavior than independent *t*-distributions. For detailed on the multivariate *t* distribution and its applications in linear regression model, we refer our interested readers to Kelker (1970), Canmbanis et al. (1981), Fang and Anderson (1990), Kibria (1996), Kibria and Haq (1998, 1999), Kibria and Saleh (2003), Kotz and Nadarajah (2004), Joarder (1998), Joarder and Ali (1997), Joarder and Sing (1997), and the references therein.

In this chapter, we present some basic properties of student's *t* distribution, (for details, see, for example, Whittaker and Robinson (1967), Feller (1968, 1971), Patel et al. (1976), Patel and Read (1982), Johnson et al. (1994), Joarder and Ali (1997), Joarder and Singh (1997), Joarder (1998), Evans et al. (2000), Balakrishnan and Nevzorov (2003), and Kapadia et al. (2005), among others).

Definition 3.1.1 (General Form of Student's *t* Distribution): A continuous random variable X with location parameter μ, scale parameter $\sigma > 0$, and degrees of freedom $v > 0$ is said to have the general form of the Student's t distribution if its pdf $g_X(x)$ is given by, (for details, see Blattberg and Gonedes (1974)):

$$g_X(x) = \frac{1}{\sigma\sqrt{v}B\left(\frac{v}{2},\ \frac{1}{2}\right)}\left[1 + \frac{1}{v}\left(\frac{x - \mu}{\sigma}\right)^2\right]^{\frac{-(1+v)}{2}},$$
$$-\infty < x < \infty,\ v > 0,\ \sigma > 0, \tag{3.1}$$

where $B(.,.)$ denotes beta function.

The general Student's t distribution has the following properties, (for details, see Moix (2001)):

(i) E(X) = μ for $v > 1$ and E(X) does not exist for $v = 1$;
(ii) var(X) = $= \frac{v\sigma^2}{v-2}$ for $v > 2$ and var (X) does not exist for $v \leq 2$;
(iii) In general, all moments of order $r < v$ are finite;
(iv) When $v = 1$, the general Student's t distribution reduces to the Cauchy distribution;

(iv) When $v \to \infty$, the general Student's t distribution converges to the normal distribution;
(v) For $\mu = 0$ and $\sigma = 1$, the general Student's t distribution reduces to the standard Student's t distribution;
(vi) The probability density function of the standardized Student's t random variable exhibits fatter tails than the probability density function of the standardized normal random variable;

For a comparison between the stable and the Student's t distributions, see, for example, Embrechts et al. (1997), and Moix (2001), among others.

Definition 3.1.2 (Student's t as a Mixture of Normal and Inverted Gamma Distributions): It is well-known that the pdf of the Student's t distribution can be expressed as

$$g_X(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\omega^2\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\omega\sigma}\right)^2} h(\omega)\, d\omega, \quad -\infty < x < \infty, \omega > 0, \sigma > 0, \tag{3.2}$$

which is the mixture of the normal distribution $N\left(\mu, \omega^2\sigma^2\right)$ and the inverted gamma distribution with v degrees of freedom and pdf given by

$$h_\Omega(\omega) = \frac{2\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \omega^{-(v+1)} e^{-\frac{1}{2}\left(\frac{v}{\omega^2}\right)}, \quad \omega > 0, v > 0.$$

Definition 3.1.3 (Student's t as a Scale Mixture of Normal Distributions): Let T_v be a Student's t random variable with v degrees of freedom and pdf $f_{\mathrm{T}_v}(t)$. The pdf of the Student's t distribution can be expressed as a scale mixture of normal distributions given by

$$f_{\mathrm{T}_v}(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2 x}{2v}} \frac{\sqrt{x}}{\sqrt{v}} \frac{1}{\Gamma\left(\frac{v}{2}\right) 2^{\frac{v}{2}}} x^{\frac{v}{2}-1} e^{-\frac{x}{2}} dx, \quad v > 0, \tag{3.3}$$

(for details, see Casella and Berger (2002)).

Definition 3.1.4 (Student's t as a Predictive Distribution, a Bayesian Approach): Let $X \sim N\left(0, \frac{1}{\theta}\right)$ be a normal random variable with pdf $f_X(x|\theta)$ given by

$$f_X(x|\theta) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{\theta x^2}{2}}, \quad -\infty < x < \infty,$$

where $\theta = \frac{1}{\sigma^2}$, the inverse of the variance, is called the precision of X. Suppose that θ has the gamma distribution with pdf $h(\theta)$ given by

$$h(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{\alpha-1} e^{-\frac{\theta}{\beta}}, \quad 0 < \theta < \infty.$$

Then the predictive pdf is given by

$$\begin{aligned} k_1(x) &= \int_0^{\infty} \frac{\theta^{\alpha+\frac{1}{2}-1}}{\Gamma(\alpha)\beta^{\alpha}\sqrt{2\pi}} \, e^{-\left(\frac{x^2}{2}+\frac{1}{\beta}\right)\theta} d\theta \\ &= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)\ \beta^{\alpha}\sqrt{2\pi}} \left(\frac{1}{\beta}+\frac{x^2}{2}\right)^{-\left(\alpha+\frac{1}{2}\right)}, \quad -\infty < x < \infty, \end{aligned} \tag{3.4}$$

which, for $\alpha = \frac{r}{2}$ and $\beta = \frac{2}{r}$, reduces to a Student's t pdf with r degrees of freedom given by

$$k_1(x) \propto \left(1+\frac{x^2}{r}\right)^{-\frac{r+1}{2}}, \quad -\infty < x < \infty,$$

(for details, see Hogg et al. (2005), and Hogg and Tanis (2006), among others).

Definition 3.1.5 (Standard Student's t Distribution): A continuous random variable X is said to have the standard Student's t distribution with v degrees of freedom if, for some integer $v > 0$, its pdf $g_X(x)$ and cdf $G_X(x) = P(X \le x)$ are, respectively, given by

$$g_x(x) = \frac{1}{\sqrt{v}B\left(\frac{v}{2},\ \frac{1}{2}\right)} \left(1+\frac{x^2}{v}\right)^{-(1+v)/2}, \ -\infty < x < \infty, \quad v > 0 \quad \ldots \quad (3.5)$$

and

$$G_X(x) = \begin{cases} 1 - \frac{1}{2} I_t\left(\frac{v}{2}, \frac{1}{2}\right), & if\ x > 0, \\ \frac{1}{2} I_t\left(\frac{v}{2},\ \frac{1}{2}\right), & otherwise, \end{cases} \quad \ldots \quad (3.6)$$

with $t = \left(1+\frac{x^2}{v}\right)^{-1}$, where $B(.,.)$ and $I_t(.,.)$ denote beta and incomplete beta functions, respectively. For special cases, we have

for $v = 1$, $g_1(x) = \dfrac{1}{\pi(1+x^2)}$, $G_1(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\arctan(x)$,

for $v = 2$, $g_2(x) = \dfrac{1}{(2+x^2)^{\frac{3}{2}}}$, $G_2(x) = \dfrac{1}{2} + \dfrac{x}{2\sqrt{2+x^2}}$.

for $v = 3$, $g_3(x) = \frac{6\sqrt{3}}{\pi(3+x^2)^2}$ and as $v \to \infty$, $G_v(x)$ converges in distribution to the cdf of N(0,1). Also,

$$G_1(x) = \frac{1}{2} + \frac{1}{\pi}\text{arc tan } (x)$$

and letting $\theta = \text{arc tan } (x/v)$, we have

$$G_{v(x)} = \frac{1}{2} + \frac{1}{\pi}\left[\theta + \left\{\cos\theta + \frac{2}{3}\cos^2\theta + \frac{2(4)\ldots(v-3)}{3(5)\ldots(v-2)}\cos^{v-2}\theta\right\}\sin\theta\right],$$

for $v = 2n + 1, n = 1, 2, \ldots$ and

$$G_{v(x)} = \frac{1}{2} + \frac{1}{2}\left\{1 + \frac{1}{2}\cos^2\theta + \frac{1(3)}{2(4)}\cos^4\theta + \cdots + \frac{1(3)\ldots(v-3)}{3(5)\ldots(v-2)}\cos^{v-2}\theta\right\}\sin\theta,$$

for $v = 2n, n = 1, 2, \ldots$

Another simple form of cdf $G_X(x)$ of the Student's t distribution with v degrees of freedom, which appears in the literature, is given by

$$G_X(x) = \frac{1}{2} + \frac{x\,\Gamma\left(\frac{v+1}{2}\right)\,{}_2F_1\left(\frac{1}{2},\ \frac{v+1}{2};\ \frac{3}{2};-\frac{x^2}{v}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}, \tag{3.7}$$

where ${}_2F_1$ (.) denote the generalized hypergeometric function of order (2.1), (see, for example, Wikipedia (2007), among others). To describe the shapes of the Student's t distribution, the plots of the pdf (3.5) and cdf (3.6), for different values of degrees of freedom, v, are provided in Figs. 3.1 and 3.2, respectively, by using Maple 10. The effects of the parameter, v, can easily be seen from these graphs. It is also clear that the graph of the pdf $g_X(x)$ of a Student's t distribution is symmetric about mean.

3.2 Some Properties of the Student's t Distribution

This section discusses the mode, moments, mean, variance, coefficients of skewness and kurtosis, and entropy of the Student's t distribution. For detailed derivations of these, see, for example, Lukacs (1972), Dudewicz and Mishra (1988), Johnson et al. (1995), Rohatgi and Saleh (2001), Balakrishnan and Nevzorov (2003), and Kapadia et al. (2005), among others.

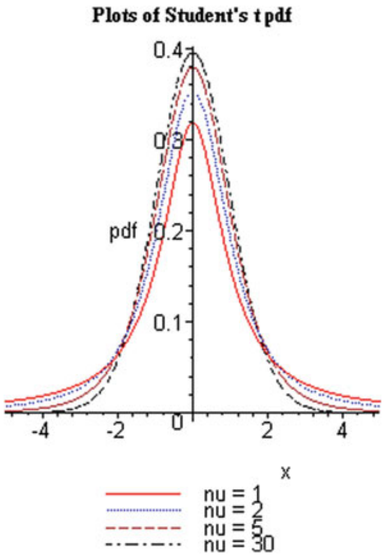

Fig. 3.1 Plots of the Student's *t* pdf, for different values of v

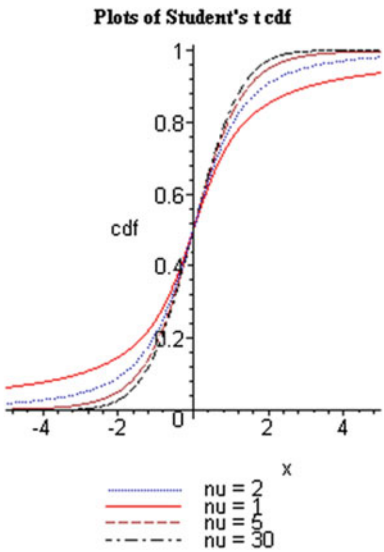

Fig. 3.2 Plots of the Student's *t* cdf, for different values of v

3.2.1 Mode

The Mode or modal value is that value of x for which the probability density function $g_x(x)$ defined by (3.5) is maximum. Now, differentiating Eq. (3.6), we have

$$g'_X(x) = -\frac{(v+1)\,x\left(1+\frac{x^2}{v}\right)^{-\frac{(v+3)}{2}}}{v\sqrt{v}B\left(\frac{v}{2},\frac{1}{2}\right)} \tag{3.8}$$

which, when equated to 0, gives the mode to be $x = 0$. It can be easily seen that $g''_{\mathrm{x}}(0) < 0$. Thus, the maximum value of the probability density function $g_X(x)$ is easily obtained from (3.5) as $g_{\mathrm{x}}(0) = \frac{1}{\sqrt{v}B\left(\frac{v}{2},\frac{1}{2}\right)}$. Since the equation $g'_v(x)$ has a unique root at x = 0 $g''_v(x)$ <0, the Student $-t$ distribution is unimodal.

3.2.2 Moments

For some degrees of freedom $v > 0$ and some integer $r > 0$, the rth moment about the mean of a random variable X having the pdf (3.5) is given by

$$E\left(X^r\right) = \begin{cases} \dfrac{v^{r/2}\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v-r}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{v}{2}\right)} & , \ when\ r\ is\ even,\ r < v; \\ 0, \quad when\ r\ is\ odd \end{cases} \tag{3.9}$$

Using the properties of gamma function, Eq. (3.9) can be written as

$$E(X^r) = v^{r/2}\prod_{j=1}^{r/2}\frac{2i-1}{v-2i}, \text{ for even r and } 0 < \text{ r} < \ v \text{ and } v > 2.$$

3.2.3 Mean, Variance, and Coefficients of Skewness and Kurtosis

From the expression (3.9), the mean, variance, coefficients of skewness and kurtosis of a Student's t random variable X having the pdf (3.5) are easily obtained as follows:

(i) **Mean:** $\alpha_1 = E\ (X) = 0, \quad v > 0;$
(ii) **Variance:** $Var\,(X) = \beta_2 = \frac{v}{v-2}, \quad v > 2;$
(iii) **Coefficient of Skewness:** $\gamma_1\ (X) = \frac{\beta_3}{\beta_2^{3/2}} = 0;$
(iv) **Coefficient of Kurtosis:** $\gamma_2\ (X) = \frac{\beta_4}{\beta_2^2} = \frac{3\,(v-2)}{v-4}, \quad v > 4.$

Since the coefficient of kurtosis, $\gamma_2\ (X) > 4$ for $v > 4$, it follows that the Student's t distributions are leptokurtic distributions for $v > 4$.

3.2.4 Mean Deviation and Coefficient of Variation of a Student's t Random Variable

(i) **Mean Deviation:** $\left(\frac{2\,\sigma^2}{\pi}\right)^{\frac{1}{2}}$
(ii) **Coefficient of Variation:** Undefined

3.2.5 Moment Generating Function

Does not exist (for details, see, for example, Mood et al. (1974), among others).

3.2.6 Entropy

For some degrees of freedom $v > 0$, entropy of a random variable X having the pdf (3.5) is easily given by

$$H_X\left[g_x\left(x\right)\right] = E[-\ln(g_x\left(X\right)] = -\int_{-\infty}^{\infty} g_x\left(x\right)\ln\left[g_x(x)\right]dx$$
$$= \left(\frac{v+1}{2}\right)\left[\psi\left(\frac{v+1}{2}\right) - \psi\left(\frac{v}{2}\right)\right] + \ln\left(\sqrt{v}B\left(\frac{v}{2}, \frac{1}{2}\right)\right), \quad (3.10)$$

where ψ (.) and B (.) denote digamma and beta functions, respectively, (see, for example, Lazo and Rathie (1978), and Kapur (1993), among others). The possible shape of the entropy for different values of the parameter v is provided in Fig. 3.3, by using Maple 10. The effects of the parameter v on entropy can easily be seen from the graph. Clearly, the entropy H_X (.) of Student's t distribution is monotonic decreasing in v. Moreover, as $v \to \infty$, the entropy H_X (.) of Student's t distribution tends to $\frac{1}{2} + \frac{1}{2}\ln\left(2\pi\right)$, which is the entropy of standard normal distribution.

3.2.7 Characteristic Function

The characteristic function of the student's t distribution is a research topic of considerable importance and interest in statistics both from the theoretical and applications point of view. It has been studied by many authors, (among them, Ifram (1970), Mitra

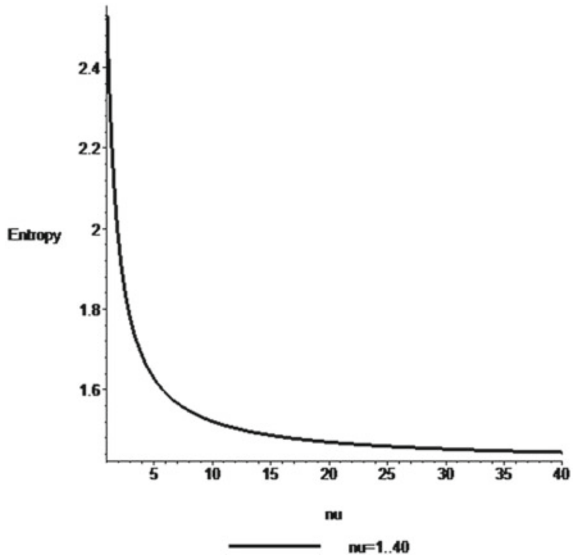

Fig. 3.3 Plot of the entropy for $v = 1..40$

(1978), Pastena (1991), Hurst (1995), Dreier and Kotz (2002) for the characteristic function of the univariate t distributions, and Joarder and Ali (1996) for the characteristic function of the multivariate t distributions, are notable). The purpose of this section is to present briefly some of the expressions for the characteristic function of the Student's t distribution with degrees of freedom v as developed by different authors which are provided below (for details, see Johnson et al. (1995), and Dreier and Kotz (2002), and references therein).

(i) Ifram (1970) derives the characteristic function of the Student's t distribution with degrees of freedom v as given by

$$\psi_X(t) = E\left(e^{itX}\right) = \frac{1}{B\left(\frac{1}{2}, \frac{v}{2}\right)} \int_{-\infty}^{\infty} e^{it(\sqrt{v})x} \left(1 + x^2\right)^{-\left(\frac{1}{2}+\frac{v}{2}\right)} dx, \tag{3.11}$$

and discusses both the cases for odd and even degrees of freedom.

(ii) The following expressions for the characteristic function of the Student's t distribution are obtained by Mitra (1978):

$$\psi_X(t) = e^{-|\sqrt{v}t|} \sum_{j=0}^{m-1} c_{j,(m-1)} \left|\sqrt{v}t\right|^j,$$

where $m = \frac{v+1}{2}$ and the $c_{j,m}$'s satisfy the following recurrence relations

$$
\begin{aligned}
c_{0,m} &= 1, \\
c_{1,m} &= 1, \\
c_{(m-1),m} &= \frac{1}{1.3\ldots(2m-5)(2m-3)}, \\
c_{j,m} &= \frac{c_{(j-1),(m-1)} + (2m-3-j)c_{j,(m-1)}}{(2m-3)}, \quad 1 \le j \le m-1.
\end{aligned}
$$

(iii) Further development on the characteristic function of the Student's t distribution continued with the work of Pastena (1991) who provides comments and corrections to Ifram's results.

(iv) Using the characteristic function of the symmetric generalized hyperbolic distribution, Hurst (1995) derives the expression for the characteristic function of the Student's t distribution in terms of Bessel functions.

(v) Dreier and Kotz (2002) derived the characteristic function of the Student's t distribution with degrees of freedom v as given by

$$
\psi_X(t) = E\left(e^{itX}\right) = \frac{2^v v^{\frac{v}{2}}}{\Gamma(v)} \int_0^\infty e^{-(\sqrt{v})(2x+|t|)} \left[x\left(x+|t|\right)\right]^{\left(\frac{v}{2}-\frac{1}{2}\right)} dx, \quad t \in \Re. \tag{3.12}
$$

3.3 Percentiles

This section computes the percentiles of the Student's t distribution, by using Maple 10. For any $p(0 < p < 1)$, the $(100p)$th percentile (also called the quantile of order p) of the Student's t distribution with the pdf $g_X(x)$ is a number t_p such that the area under $g_X(x)$ to the left of t_p is p. That is, t_p is any root of the equation

$$
G(t_p) = \int_{-\infty}^{t_p} g_X(u)du = p. \tag{3.13}
$$

Using the following Maple program, the percentiles t_p of the Student's t distribution are computed for some selected values of p for the given values of v, which are provided in Table 3.1.

3.4 Different Forms of t Distribution

This section presents different forms of t distribution and some of their important properties, (for details, see, for example, Whittaker and Robinson (1967), Feller (1968, 1971), Patel et al. (1976), Patel and Read (1982), Johnson et al. (1994),

Table 3.1 Percentiles of Student's *t* distribution

v	75 %	80 %	85 %	90 %	95 %	99 %
1	1.00000	1.37638	1.96261	3.07768	6.31375	31.82051
5	0.72668	0.91954	1.15576	1.47588	2.01504	3.36493
15	0.69119	0.86624	1.07353	1.34060	1.75305	2.60248
30	0.68275	0.86624	1.05466	1.31041	1.69726	2.45726

Evans et al. (2000), Balakrishnan and Nevzorov (2003), and Kapadia et al. (2005), among others).

3.4.1 Half t Distribution

A random variable X said to have a half-*t* distribution with parameters ξ, τ, and λ if its pdf can be written as

$$f(x|\xi,\tau,\lambda) = 2\frac{\Gamma(\lambda+1/2)\sqrt{\tau}}{\Gamma(\lambda/2)\sqrt{\lambda\pi}}\left[1+\frac{1}{\lambda}(\sqrt{\tau}(x-\xi)^2\right]^{-(\lambda+1/2)}$$
$$\text{for x} > \xi, -\infty < \xi < \infty, \tau > 0, \lambda > 0 \quad (3.14)$$

Note that, as $\lambda \to \infty$ in Eq. (3.14), the half-*t* distribution approaches the half-normal distribution, which follows from the definition of the exponential function, $\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$. Also, note that as $\lambda \to 0$ in Eq. (3.14), the right-hand tail of the half-t distribution becomes increasingly heavier relative to that of the limiting half-normal distribution, obtained as $\lambda \to \infty$.

3.4.2 Skew t Distribution

A random variable *X* is said to have the skew-*t* distribution if its pdf is $f(x) = 2g(x)G(\lambda x)$, where g(x) and G(x), respectively, denote the pdf and the cdf of the Student's *t* distribution with degrees of freedom *v*. For different degrees of freedoms the skew t density are given in Fig. 3.4.

3.5 Summary

In this chapter, we first present the motivation and importance of studying the Student's *t* distribution. Then some basic ideas, definitions and properties of the

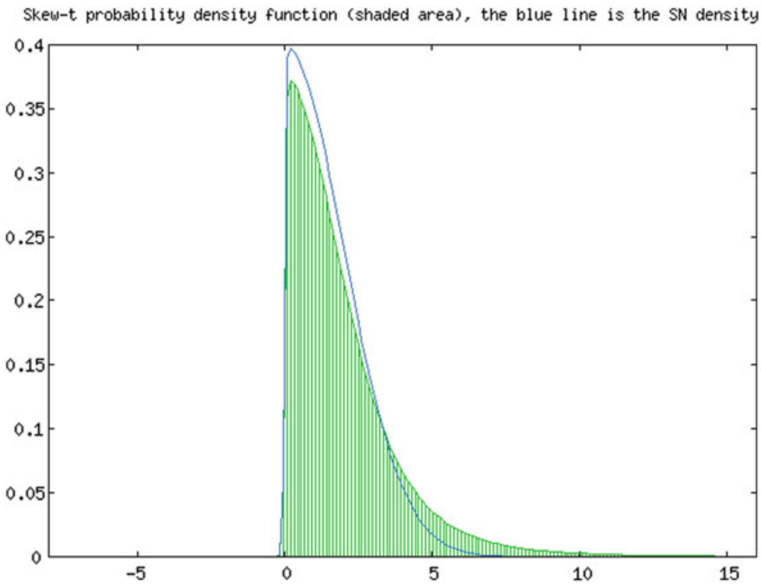

Fig. 3.4 Parameters of the Skew-t density: location = 0; scale = 2; α = 50; ν = 4 and 40

Student's *t* distributions have been reviewed. The entropy of a random variable having the Student's *t* distributions has been given. The expressions for different forms of the characteristic function of the Student's *t* distribution are provided. As a motivation, different forms of the Student's distribution, such as Half-t and Skew-t distributions, which are areas of current research, have been provided.

Chapter 4
Sum, Product and Ratio for the Normal Random Variables

4.1 Introduction

The distributions of the sum, product, and ratio of two independent random variables arise in many fields of research, for example, automation, biology, computer science, control theory, economics, engineering, fuzzy systems, genetics, hydrology, medicine, neuroscience, number theory, statistics, physics, psychology, reliability, risk management, etc. (for details, see Grubel (1968), Rokeach and Kliejunas (1972), Springer (1979), Kordonski and Gertsbakh (1995), Ladekarl et al. (1997), Amari and Misra (1997), Sornette (1998), Cigizoglu and Bayazit (2000), Brody et al. (2002), Galambos and Simonelli (2005), among others). The distributions of the sum $X+Y$, product XY, and ratio X/Y, when X and Y are independent random variables and belong to the same family, have been extensively studied by many researchers, among them, the following are notable:

(a) Ali (1982), Farebrother (1984), Moschopoulos (1985), Provost (1989a), Pham-Gia and Turkkan (1994), Kamgar-Parsi et al. (1995), Hitezenko (1998), Hu and Lin (2001), Witkovsky (2001), and Nadarajah (2006a) for the sum $X+Y$.
(b) Sakamoto (1943), Harter (1951) and Wallgren (1980), Springer and Thompson (1970), Stuart (1962) and Podolski (1972), Steece (1976), Bhargava and Khatri (1981), Abu-Salih (1983), Tang and Gupta (1984), Malik and Trudel (1986), Rathie and Rohrer (1987), Nadarajah (2005a, b), Nadarajah and Gupta (2005), Nadarajah and Kotz (2006a) for the product XY.
(c) Marsaglia (1965), and Korhonen and Narula (1989), Press (1969), Basu and Lochner (1971), Shcolnick (1985), Hawkins and Han (1986), Provost (1989b), Pham-Gia (2000), Nadarajah (2005c, 2006b), Nadarajah and Gupta (2005, 2006), and Nadarajah, and Kotz (2006b) for the ratio X/Y.

The algorithms for computing the probability density function of the sum and product of two independent random variables, along with an implementation of the algorithm in a computer algebra system, have also been developed by many authors, among them, Agrawal and Elmaghraby (2001), and Glen et al. (2004) are notable.

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Atlantis Studies in Probability and Statistics 4, DOI: 10.2991/978-94-6239-061-4_4,

This chapter presents the distributions of the sum $X + Y$, product XY, and ratio X/Y when X and Y are independent random variables and have the same normal distributions. For the sake of completeness, the definitions of sum, product, and ratio of two independent random variables are given below, (for details, see Lukacs (1972), Dudewicz and Mishra (1988), Rohatgi and Saleh (2001), Kapadia et al. (2005), and Larson and Marx (2006), among others).

Let X and Y be any two independent, absolutely continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Note that the sum $X + Y$, product XY, and ratio X/Y of X and Y are also random variables, (for details, see Lukacs (1972), among others).

4.1.1 Definition (Distribution of a Sum)

Let $Z = X + Y$ for $-\infty < X, Y < +\infty$. Then

(i) $$F_Z(z) = P(Z \leq z) = P(X + Y \leq z) = \int_{-\infty}^{\infty} f_X(x) F_Y(z - x)\, dx \tag{4.1}$$

(ii) $$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx \tag{4.2}$$

(iii) **Convolution:** The probability density function $f_Z(z)$ in (ii) above is also called the convolution of the pdfs $f_X(x)$ and $f_Y(y)$, which is expressed as $\{f_Z(z)\} = \{f_X(z)\} * \{f_Y(z)\}$.

4.1.2 Definition (Distribution of a Product)

Let $W = XY$ for $-\infty < X, Y < +\infty$. Then

(i) $$F_W(w) = P(W \leq w) = P(XY \leq w) = P\left(X \leq \frac{w}{Y}\right)$$
$$= \begin{cases} \int_0^{\infty} F_X\left(\frac{w}{y}\right) f_Y(y)\, dy, & Y > 0 \\ F_Y(0) - \int_{-\infty}^{0} F_X\left(\frac{w}{y}\right) f_Y(y)\, dy, & Y < 0 \end{cases} \tag{4.3}$$

(ii) $$f_W(\mathrm{w}) = \begin{cases} \int\limits_0^\infty \frac{1}{y} f_X\left(\frac{w}{y}\right) f_Y(y)\,dy, & Y > 0,\ -\infty < w < \infty \\ \int\limits_{-\infty}^\infty \left|\frac{1}{y}\right| f_X\left(\frac{w}{y}\right) f_Y(y)\,dy, & Y < 0,\ -\infty < w < \infty \end{cases} \tag{4.4}$$

4.1.3 Definition (Distribution of a Ratio)

Let $U = \frac{X}{Y}$ for $-\infty < X, Y < +\infty$. Then

(i)

$$F_U(u) = P(U \le u) = P\left(\frac{X}{Y} \le u\right) = P(X \le uY)$$

$$= \begin{cases} \int\limits_0^\infty F_X(uy) f_Y(y)\,dy, & Y > 0 \\ F_Y(0) - \int\limits_{-\infty}^0 F_X(uy) f_Y(y)\,dy, & Y < 0 \end{cases} \tag{4.5}$$

(ii)

$$f_U(u) = \begin{cases} \int\limits_0^\infty y f_X(uy) f_Y(y)\,dy, & Y > 0,\ -\infty < u < \infty \\ \int\limits_{-\infty}^\infty |y|\, f_X(uy) f_Y(y)\,dy, & Y < 0,\ -\infty < u < \infty \end{cases} \tag{4.6}$$

4.1.4 Mean and Variance of the Sum and Product of Independent Random Variables

Let X and Y be any two independent, absolutely continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Suppose that the first and second moments of both X and Y exist. Then

(i) $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
(ii) $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$
(iii) $Var(X + Y) = Var(X) + Var(Y)$.

Note that the above results can be extended to any finite sums or products of independently distributed random variables, (for details, see Lukacs (1972), and Dudewicz and Mishra (1988), among others).

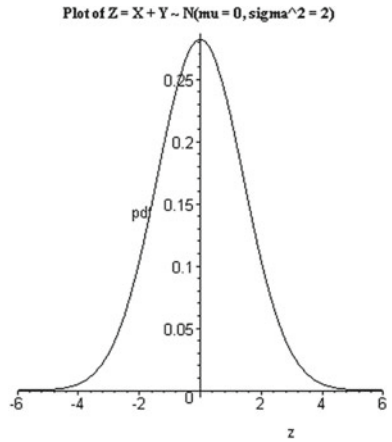


Fig. 4.1 Plot of the $Z = X + Y \sim N(\mu = 0, \sigma^2 = 2)$

4.2 Distribution of the Sum of Independent Normal Random Variables

This section presents the distributions of the sum of independent normal random variables as described below.

(i) Let X and Y be two independent $N(0, 1)$ random variables. Then $Z = X + Y \sim N(0, 2)$ with pdf given by

$f_Z(z) = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}, -\infty < z < \infty,$

(for details, see Dudewicz and Mishra (1988), and Balakrishnan and Nevzorov (2003), among others). A Maple plot of this pdf is given in Fig. 4.1.

(ii) Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_y^2)$ be two independent random variables. Then $Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ with pdf given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi\left(\sigma_X^2 + \sigma_Y^2\right)}} e^{-(z - \mu_X - \mu_Y)^2/2\left(\sigma_X^2 + \sigma_Y^2\right)}, \quad -\infty < z < \infty, \tag{4.7}$$

(for details, see Lukacs (1972), and Balakrishnan and Nevzorov (2003), among others).

(iii) Let $X_1, X_2, ..., X_n$ be a set of independently distributed $N(\mu_i, \sigma_i^2)$ $(i = 1, 2, ..., n)$ random variables. Let $Z = \sum_{i=1}^{n} c_i X_i$ where c_i are some constants. Then $Z \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$, (for details, see Lukacs (1972), Dudewicz and Mishra (1988), and Kapadia et al. (2005), among others).

Note that the mean and variance of the sum of independent normal random variables can be easily derived by following the above Sect. 4.1.4.

4.3 Distribution of the Product of Independent Normal Random Variables

This section presents the distributions of the product of independent normal random variables as described below, (for details, see Epstein (1948), Zolotarev (1957), Kotlarski (1960), Donahue (1964), Springer and Thompson (1966, 1970), Lomnicki (1967), and Glen et al. (2002), among others).

(i) Let $X_1, X_2, ..., X_n$ be a set of independently distributed $N\left(0, \sigma_i^2\right)$ $(i = 1, 2, ..., n)$ random variables. Let $W = \prod_{i=1}^{n} X_i$. Then the random variable W follows a distribution with pdf given, in terms of a Meijer G-function, by

$$g_W(w) \;=\; H \;\; G_{0\;n}^{n\;0}\left(w^2 \prod_{i=1}^{n} \frac{1}{2\;\sigma_i} \;\mid 0 \right) \tag{4.8}$$

where H is a normalizing constant given by

$$H \;=\; \left[(2\pi)^{\frac{n}{2}} \prod_{i=1}^{n} \sigma_i \right]^{-1},$$

(for details, see Springer and Thompson (1966, 1970)), where $G_{0n}^{n0}\left(w^2 \prod_{i=1}^{n} \frac{1}{2\sigma_i} \mid 0 \right)$ denotes the Meijer G-Function. It is defined as follows

$$G_{p,q}^{m,n}\left(x|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p} \right) = \frac{1}{2\pi i} \int_L \frac{x^{-t}\Gamma(b_1+t)\cdots\Gamma(b_m+t)\Gamma(1-a_1-t)\cdots\Gamma(1-a_n-t)}{\Gamma(a_{n+1}+t)\cdots\Gamma(a_p+t)\Gamma(1-b_{m+1}-t)\cdots\Gamma(1-b_q-t)} dt,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial and L denotes an integration path (for details on Meijer G-Function, see, Gradshteyn and Ryzhik (2000), Sect. 9.3, p. 1068).

(ii) Let X and Y be independently distributed as N (0, 1), and Z = XY. Then the characteristic function of the product of two independent normal random variables is given by:

$$E(e^{itXY}) = E_Y(E(^{itXY}|Y)$$

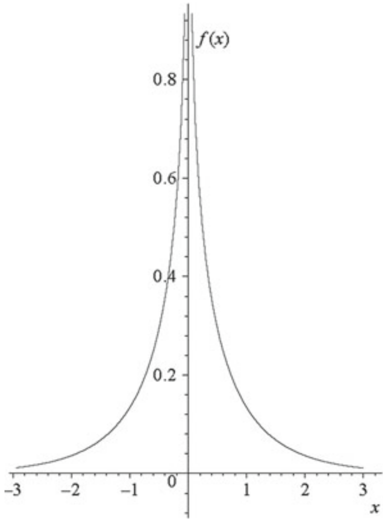


Fig. 4.2 Plot of the PDF of $W = XY$, when $X \sim N(0, 1)$ and $Y \sim N(0, 1)$

That is,

$$E(^{itXY}|Y) = e^{-\frac{1}{2}t^2Y^2}$$

Thus we have,

$$\begin{aligned}E_Y(E(^{itXY}|Y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2y^2} e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{\sqrt{1+t^2}}\end{aligned}$$

Inverting this characteristic function we get the pdf of Z as

$$f_Z(z) = \frac{1}{\pi}K_0(z),$$

where $K_0(z)$ is the Bessel function of the second kind.

(iii) Let $X \sim N\left(0, \sigma_X^2\right)$ and $Y \sim N\left(0, \sigma_y^2\right)$ be two independent random variables. Let $W = XY$. Then the random variable $W = XY$ follows a distribution with pdf given by

$$f_W(w) = \frac{1}{\pi\sigma_X\sigma_Y}\mathrm{K}_0\left(\frac{w}{\sigma_X\sigma_Y}\right), \tag{4.9}$$

where K_0 (.) denotes modified Bessel function of the second kind, (for details, see Lomnicki (1967), and Glen et al. (2002), among others). A Maple plot of the pdf of the random variable $W = XY$, when $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, is given in Glen et al. (2002), which, for the sake of completeness, is reproduced and presented in Fig. 4.2.

(iv) **Alternative Derivation of the PDF (4.9) of the Product of Two Independently Distributed Normal Random Variables:**
Suppose X_1 and X_2 are two independently distributed $N(0, \sigma_i^2)$, $i = 1.2$, random variables, and $Y = X_1X_2$, then the pdf $f_Y(y)$ of Y is given by

$$f_Y(y) = \frac{1}{\pi\sigma_1\sigma_2} K_0\left(\frac{y}{\sigma_1\sigma_2}\right),$$

where $K_0(x)$ is the modified Bessel function of the second kind (see Abramowitz and Stegun (1970, p. 376).

Proof. Let $\phi_Y(t)$ be the characteristic of Y, then

$$\begin{aligned}
\phi_Y(t) &= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{=\frac{x^2}{2\sigma_1^2}} \left(\int_{-\infty}^{\infty} e^{itxy} \frac{1}{\sigma_2\sqrt{2\pi}} e^{=\frac{y^2}{2\sigma_2^2}} dy \right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} e^{-\frac{t^2x^2\sigma_2^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}(1+t^2\sigma_1^2\sigma_2^2)} dx
\end{aligned}$$

Let $x\sqrt{(1 + t^2\sigma_1^2\sigma_2^2)} = u$, then

$$\begin{aligned}
\phi_Y(t) &= \frac{1}{\sqrt{1 + t^2\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma_2^2}} du \\
&= \frac{1}{\sqrt{1 + t^2\sigma_1^2\sigma_2^2}}
\end{aligned}$$

Using the inverse of the characteristic function, the pdf $f_Y(y)$ of Y is given by

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \frac{1}{\sqrt{1 + t^2\sigma_1^2\sigma_2^2}} dt \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{\cos ty}{\sqrt{1 + t^2\sigma_1^2\sigma_2^2}} dt
\end{aligned}$$

Using the transform $t\sigma_1\sigma_2 = w$, we obtain

$$f_Y(y) = \frac{1}{\pi\sigma_1\sigma_2}\int_0^{\infty}\frac{\cos(\frac{y}{\sigma_1\sigma_2}w)}{\sqrt{1+w^2}}dw$$
$$= \frac{1}{\pi\sigma_1\sigma_2}K_0(\frac{y}{\sigma_1\sigma_2}),$$

where $K_0(.)$ is the Bessel function of the second kind. This completes the proof.

(v) **Distribution of the Product of Two Independently Distributed Standard Normal Random Variables:** As a special case of the Eq. (4.9) in (iii) above, for the sake of completion, using the definition of the characteristic function of a random variable, we derive below independently the distribution of the product of two independently distributed standard normal random variables.

Let X and Y be independently distributed random variables as N (0, 1), and Z = XY. Then the characteristic function of the product of two independent normal random variables is given by:

$$E(e^{itXY}) = E_Y(E(^{itXY}|Y).$$

That is,

$$E(^{itXY}|Y) \ = \ e^{-\frac{1}{2}t^2Y^2}.$$

Thus, we have

$$E_Y(E(^{itXY}|Y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2y^2}e^{-\frac{1}{2}y^2}dy$$
$$= \frac{1}{\sqrt{1+t^2}}$$

Inverting this characteristic function, we get the pdf of Z as given below

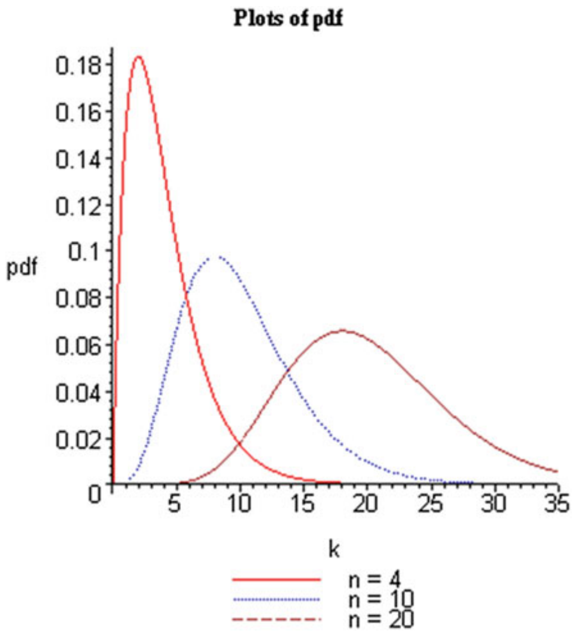
$$f_Z(z) = \frac{1}{\pi}K_0(z), \tag{4.10}$$

where $K_0(z)$ is the Bessel function of the second kind. Obviously, Eq. (4.10) is a special case of Eq. (4.9), when $\sigma_X = 1$ and $\sigma_Y = 1$.

(vi) Let $X_1, X_2, ..., X_n$ be a set of independently distributed N (0, 1) ($i = 1, 2, ..., n$) random variables. Let $K = \sum_{i=1}^{n} X_i^2$. Then the random variable $K = \sum_{i=1}^{n} X_i^2$ follows a chi-square distribution with n degrees of freedom whose pdf is given by

$$\begin{cases} p_K(k) \ = \ \frac{1}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\, w^{\frac{n}{2}\,-\,1}\, e^{\frac{-k}{2}} & if\ k > 0, \\ \quad\quad\quad = \ 0 & if\ w < 0. \end{cases}, \tag{4.11}$$

Fig. 4.3 Plots of $K = \sum_{i=1}^{n} X_i^2$, where each $X_i \sim N(0, 1)$



with the Mean $(K) = n$ and Variance $(K) = 2n$. Note that the random variable $K = \sum_{i=1}^{n} X_i^2$ also follows a *gamma* $\left(\frac{1}{2}, \frac{n}{2}\right)$ distribution, (for details, see Lukacs (1972), and Kapadia et al. (2005), among others). A Maple plot of the pdf (4.11) in (vi) is given in Fig. 4.3.

4.4 Distribution of the Ratio of Independent Normal Random Variables

In many statistical analysis problems, the ratio of two normally distributed random variables plays a very important role. The distributions of the ratio *X/Y*, when *X* and *Y* are normally random variables, have been extensively studied by many researchers, notable among them are Geary (1930), Fieller (1932), Curtiss (1941), Kendall (1952), Marsaglia (1965), Hinkley (1969), Hayya et al. (1975), Springer (1979), Cedilnik (2004), and Pham-Gia et al. (2006). This section presents the distributions of the ratio of independent normal random variables as described below.

(i) Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ be two independent random variables. Let $U = X/Y$. Then the random variable $U = X/Y$ follows a standard Cauchy distribution with pdf and cdf respectively given by

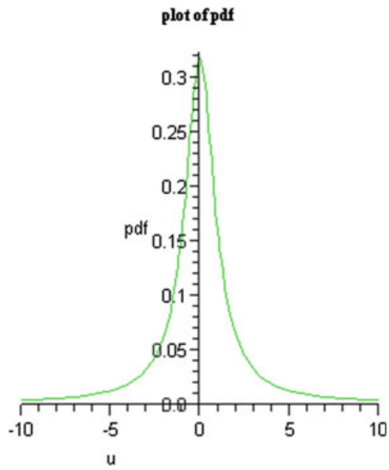


Fig. 4.4 Plot of $U = X/Y$, where $X \sim N(0, 1)$ and $Y \sim N(0, 1)$

$$f_U(u) = \frac{1}{\pi\left(1 + u^2\right)}, \quad -\infty < u < \infty, \text{ and}$$

$$F_U(u) = \frac{1}{2} + \frac{1}{\pi}\arctan(u), \quad -\infty < u < \infty, \tag{4.12}$$

(for details, see Feller (1971), Casella and Berger (2002), and Severini (2005), among others). A Maple plot of the pdf (4.12) in (i) is given in Fig. 4.4.

(ii) Let $X \sim N\left(0, \sigma_X^2\right)$ and $Y \sim N\left(0, \sigma_y^2\right)$ be two independent random variables. Let $U = X/Y$. Then the random variable $U = X/Y$ follows a generalized form of Cauchy distribution with pdf given by

$$f_U(u) = \frac{\sigma_X \sigma_Y}{\pi\left(\sigma_X^2 + \sigma_Y^2 u^2\right)}, \quad -\infty < u < \infty \tag{4.13}$$

(for details, see Pham-Gia et al. (2006), among others).

(iii) Let $X \sim N\left(0, \sigma_X^2\right)$ and $Y \sim N\left(0, \sigma_y^2\right)$ be two independent random variables. Let $U = X/Y$. If $\sigma_X \neq 0$, $\sigma_Y \neq 0$ or $\rho \neq 0$, then the random variable $U = X/Y$ follows a more general Cauchy distribution with pdf given by

$$f_U(u) = \frac{1}{\pi}\left[\frac{\beta}{(u - \alpha)^2 + \beta^2}\right], \quad -\infty < u < \infty, \tag{4.14}$$

where ρ is the coefficient of correlation between X and Y, and $\alpha = \rho\frac{\sigma_X}{\sigma_Y}$ and $\beta = \frac{\sigma_X}{\sigma_Y}\sqrt{1 - \rho^2}$. Note that for $\rho = 0$, this pdf reduces to Cauchy distribution $C\left(0, \frac{\sigma_X}{\sigma_Y}\right)$ as given in Eq. (4.13) in (ii) above. Also for $\rho = 0, \sigma_X = \sigma_Y = 1$, we have a standard Cauchy distribution $C(0, 1)$ as given in Eq. (4.12) in (i) above.

(iv) Let $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_y^2\right)$ be two independent random variables, (that is, $cor\,(X, Y) = 0$). Let $U = X/Y$. Then, following Hinkley (1969), the pdf of the random variable $U = X/Y$ is given by

$$f_U(u) = \frac{b\,(u) \cdot c\,(u)}{a^3\,(u)} \, \frac{1}{\sqrt{2\pi}\;\sigma_X \sigma_Y} \left[2\;\Phi\left(\frac{b\,(u)}{a\,(u)}\right) - 1\right] + \frac{1}{a^2\,(u)\,.\,\pi \sigma_X \sigma_Y} \left[e^{-\frac{1}{2}\left\{\left(\frac{\mu_X}{\sigma_X}\right)^2 + \left(\frac{\mu_Y}{\sigma_Y}\right)^2\right\}} \right] \tag{4.15}$$

where

$$a(u) = \sqrt{\frac{1}{\sigma_x^2}u^2 + \frac{1}{\sigma_Y^2}}, \quad b(u) = \frac{\mu_X}{\sigma_x^2}u^2 + \frac{\mu_Y}{\sigma_Y^2},$$

$$c(u) = e^{\frac{1}{2}\frac{b^2(u)}{a^2(u)} - \frac{1}{2}\left\{\left(\frac{\mu_X}{\sigma_X}\right)^2 + \left(\frac{\mu_Y}{\sigma_Y}\right)^2\right\}},$$

and

$$\Phi\,(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}v^2} dv,$$

denote the standard normal cumulative distribution function. It is easy to see that the above pdf reduces to a standard Cauchy distribution $C\,(0, 1)$ if
$\mu_X = \mu_Y = 0$, and $\sigma_X = \sigma_Y = 1$, that is, $b\,(u) = 0$.

(v) Recently, in a very detailed paper, Pham-Gia et al. (2006) have considered the density of the ratio X/Y of two normal random variables X and Y and applications, and have obtained some closed form expressions of the pdf of *X/Y* in terms of Hermite and Kummer's confluent hypergeometric functions, by considering all cases, that is, when X and Y are standardized and nonstandardized, independent or correlated, normal random variables. For the sake of brevity, only the case when X and Y are independent normal random variables is stated below. For other cases and applications, please visit Pham-Gia et al. (2006). Let $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_y^2\right)$ be independent normal random variables. Let $U = \frac{X}{Y}$. Then, following Pham-Gia et al. (2006), the random variable $U = \frac{X}{Y}$ has a distribution with pdf given by

$$f_U(u) = \frac{K_1 . \left[{}_1F_1\left(1;\ \frac{1}{2};\ \theta_1\,(u)\right)\right]}{\sigma_X^2 + \sigma_Y^2 u^2}, \quad -\infty < u < \infty \tag{4.16}$$

where ${}_1F_1\,(.)$ denotes Kummer's confluent hypergeometric function, and

$$\theta_1\,(u) \;=\; \frac{1}{2\sigma_X^2\,\sigma_Y^2}.\left[\frac{\left(\mu_Y\sigma_X^2 \;+\; u\mu_X\sigma_Y^2\right)^2}{\sigma_X^2 \;+\; \sigma_Y^2u^2}\right] \;\geq\; 0,$$

and $K_1 \;=\; \frac{\sigma_X\sigma_Y}{\pi}\, e^{-\frac{1}{2}\left\{\left(\frac{\mu_X}{\sigma_X}\right)^2 + \left(\frac{\mu_Y}{\sigma_Y}\right)^2\right\}}$.
By using the following relation

$$H_{-2}\,(s) \;+\; H_{-2}\,(-s) \;=\; {}_1F_1\left(1;\;\frac{1}{2};\;s^2\right), \quad \forall s,$$

where $H_\nu\,(s)$ denotes Hermite function (for details, see Lebedev (1972, pp. 283–299, among others), it is easy to see that the random variable $U = \frac{X}{Y}$ follows a distribution with pdf given, in terms of Hermite function, as

$$f_U(u) = \frac{K_1.\left\{H_{-2}\left(\sqrt{\theta_1\,(u)}\right)\right\}}{\sigma_X^2 \;+\; \sigma_Y^2u^2}, \quad 0 \;\leq\; u \;<\; \infty, \tag{4.17}$$

where

$$\theta_1\,(u) \;=\; \frac{1}{2\sigma_X^2\,\sigma_Y^2}.\left[\frac{\left(\mu_Y\sigma_X^2 \;+\; u\mu_X\sigma_Y^2\right)^2}{\sigma_X^2 \;+\; \sigma_Y^2u^2}\right] \;\geq\; 0,$$

$$K_1 \;=\; \left\{\frac{\sigma_X\sigma_Y}{\prod_{i\,=\,1}^{2}\left[1 \;-\; \Phi\left(\frac{-\mu_i}{\sigma_i}\right)\right]}\right\}\, e^{-\frac{1}{2}\left\{\left(\frac{\mu_X}{\sigma_X}\right)^2 + \left(\frac{\mu_Y}{\sigma_Y}\right)^2\right\}},$$

and

$$\Phi\,(t) \;=\; \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}}\, e^{-\,\frac{1}{2}\,v^2} dv$$

denote the standard normal cumulative distribution function.

The above results of Pham-Gia et al. are valid $\forall\sigma_X, \sigma_Y \;>\; 0$, and $\forall\mu_X, \mu_Y \;\in\; R(\textit{set of real numbers})$. Since ${}_1F_1\left(1;\,\frac{1}{2};\,0\right) \;=\; 1$, it is easy to see that, when $\mu_X \;=\; \mu_Y \;=\; 0$, the random variable $U \;=\; X/Y$ follows a generalized form of Cauchy distribution with pdf given by

$$f_U(u) = \frac{\sigma_X\sigma_Y}{\pi\;\left(\sigma_X^2 + \sigma_Y^2u^2\right)}, \quad -\infty \;<\; u \;<\; \infty, \tag{4.18}$$

which reduces to the pdf of a standard Cauchy distribution $C\,(0,\,1)$ when $\sigma_X = \sigma_Y$.

(vi) The following results about the ratio X/Y of two normal random variables X and Y found in the literature are also worth noting.

A. Kendall (1952), Kapadia et al. (2005, pp. 210–211): Let $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_y^2\right)$ be independent normal random variables. Assume that μ_Y is so large compared to σ_y that the range of Y is effectively positive. Then, as obtained by Kendall (1952), the random variable $U = X/Y$ has a distribution with pdf given by

$$f_U(u) = \frac{1}{\sqrt{2\pi}} \left[\frac{\left(\mu_Y \sigma_X^2 + \mu_X u \sigma_Y^2\right)}{\left(\sigma_X^2 + u^2 \sigma_Y^2\right)^{\frac{3}{2}}} \right] e^{\left[-\frac{1}{2} \frac{(u\mu_Y - \mu_X)^2}{(\sigma_X^2 + \sigma_Y^2 u^2)} \right]},$$
$$-\infty < u < \infty \tag{4.19}$$

B. Kamerud (1978) has obtained a particular Cauchy-like distribution by considering the ratio X/Y of two non-centered independent normal random variables X and Y.

C. Marsaglia (1965) has investigated the ratio X/Y of two arbitrary normal random variables X and Y, and has also obtained a Cauchy-like distribution.

D. Mood et al. (1974, p. 246), Patel et al. (1976, p. 209): Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ be independent standard normal random variables. Let $U = (X/Y)^2$. Then the random variable $U = (X/Y)^2$ has an F distribution with pdf $f_U(u, 1, 1)$.

4.5 Distributions of the Sum, Product and Ratio of Dependent Normal Variables

Since the distribution of the sums, differences, products and ratios (quotients) of random variables arise in many fields of research such as automation, biology, computer science, control theory, economics, engineering, fuzzy systems, genetics, hydrology, life testing, medicine, neuroscience, number theory, statistics, physics, psychology, queuing processes, reliability and risk management, among others, the derivations of these distributions for dependent (correlated) random variables have also received the attention of many authors and researchers. For detailed discussions on the sum, difference, product and ratio of dependent (correlated) random variables, the interested readers are referred to Springer (1979), and references therein.

In what follows, we provide the distributions of the sums, differences, products and ratios of dependent (correlated) normal variables.

4.5.1 Some Basic Definitions

For the sake of completeness, some basic definitions are given below.
Let X and Y be any two absolutely continuous random variables with p.d.f.'s $f_X(x)$ and $f_Y(y)$ respectively. Let $f_{X,Y}(x, y)$ be the joint p.d.f. of X and Y.

Definition 4.5.1. The random variables X and Y are said to **dependent** if and only if

$$f_{X,Y}(x, y) \neq f_X(x) . f_Y(y), \quad \forall (x, y) \in \Re^2, \; \Re = \{\text{all real numbers}\};$$

otherwise X and Y are said to be **independent**.

Definition 4.5.2. The correlation coefficient between the random variables X and Y, denoted by r_{XY}, is defined as

$$r_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y},$$

where σ_{XY} denotes the covariance of X and Y, σ_X is the standard deviation of X, and σ_Y is the standard deviation of Y. The random variables X and Y are said to correlated if the correlation coefficient between them, that is, $r_{XY} \neq 0$; otherwise X and Y are said to be **uncorrelated**.

Remark 4.5.1. It can easily be seen that two independent random variables are uncorrelated. But the converse is not true, that is, two uncorrelated random variables may not be independent. For example, if a random variable X has a standard normal distribution and $Y = X^2$, then it is easy to see that X and Y are uncorrelated but not independent. For details on correlation and dependence (independence), the interested readers are referred to Lukacs (1972), Tsokos (1972), Springer (1979), Dudewicz and Mishra (1988), Rohatgi and Saleh (2001), Mari and Kotz (2001), and Gupta and Kapoor (2002), among others.

Definition 4.5.3. Let

$$Z = X + Y, U = X - Y, V = XY, \text{and } W = X/Y$$

denote the sum, difference, product, and ratio of the random variables X and Y respectively. Then, following Theorem 3, p. 139, of Rohatgi and Saleh (2001), the p.d.f.'s of Z, U, V, and W are, respectively, given by

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z - x) dx,$$

$$f_U(u) = \int_{-\infty}^{+\infty} f(u + y, y) dy,$$

$$f_V(\nu) = \int_{-\infty}^{\infty} f\left(x, \frac{\nu}{x}\right) \frac{1}{|x|} dx,$$

and

$$f_W(w) = \int_{-\infty}^{\infty} f(xw, x) |x| dx.$$

4.5.2 *Distributions of the Sums and Differences of Dependent (Correlated) Normal Random Variables*

Here, the distributions of the sums and differences of dependent (correlated) normal random variables are briefly provided. For details, one is referred to Springer (1979, pp. 67–75), among others.

Let X_1 and X_2 denote the two dependent (correlated) normal random variables (r.v.'s) with zero mean, correlation coefficient ρ, and variances σ_1^2 and σ_2^2, respectively. Let $Z_S^* = X_1 + X_2$ denote the sum of X_1 and X_2, with the p.d.f. $g(z^*)$. Then, following Springer (1979), the joint p.d.f. of X_1 and X_2 is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1 - \rho^2)^{1/2}} \exp\left[-\frac{1}{2(1 - \rho^2)}\left(\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)\right],$$
$$|\rho| < 1, \quad -\infty < x_i < \infty, \quad \sigma_i > 0, \quad i = 1, 2, \tag{4.20}$$

whereas the p.d.f of the sum $Z_S^* = X_1 + X_2$ is given by

$$g(z^*) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}} \exp\left[-\frac{z^2}{2(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)}\right],$$
$$|\rho| < 1, \quad -\infty < z < \infty. \tag{4.21}$$

Obviously $Z_S^* \sim N\left(0, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$. It follows that if $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim N\left(\mu_2, \sigma_2^2\right)$, and $Z_S = X_1 + X_2$, then $Z_S \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$.

Remark 4.5.2. Similar to the above, it can easily be shown that if $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim N\left(\mu_2, \sigma_2^2\right)$, and $Z_D = X_1 - X_2$, then $Z_D \sim N\left(\mu_1 - \mu_2, \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$.

Remark 4.5.3. It is interesting to note that the following transformation

$$y_1 = \frac{1}{\left(1 - \rho^2\right)^{1/2}} \left(\frac{x_1}{\sigma_1} - \frac{\rho x_2}{\sigma_2}\right), \text{ and } y_2 = \frac{x_2}{\sigma_2},$$

to the above joint p.d.f. of the two dependent (correlated) normal random variables X_1 and X_2 with zero mean, correlation coefficient ρ, and variances σ_1^2 and σ_2^2, respectively, gives

$$g(y_1, y_2) = \left(\frac{1}{\sqrt{2\pi}} e^{-y_1^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-y_2^2/2}\right),$$

that is, the transformed random variables Y_1 and Y_2 are independent and standard normally distributed, but the original random variables X_1 and X_2 are dependent (correlated) normal random variables. For details, see Springer (1979).

Remark 4.5.4. For the p.d.f. of the sum of n dependent (correlated) normal r.v.'s, which can be obtained in the same manner, see Springer (1979, Eq. 3.4.20, p. 72).

Remark 4.5.5. However, in general, normality of marginal random variables X_1 and X_2 does not imply normality of their joint distribution and thus does not imply normality of their sum. For example, it has been observed by Holton (2003) and Novosyolov (2006) that the sum of dependent normal variables may be not normal.

4.5.3 Distributions of the Products and Ratios of Dependent (Correlated) Normal

Random Variables: In what follows, the distributions of the product and ratio of dependent (correlated) normal random variables are briefly provided. For details, one is referred to Springer (1979, p. 151), and references therein.

Let $X_1 \sim N(0, 1)$ and $X_2 \sim N(0, 1)$ denote the two dependent (correlated) normal random variables (r.v.'s) with zero mean, variances 1, and correlation coefficient ρ. Let $Y = X_1 X_2$ denote the product of X_1 and X_2, with the p.d.f. $h(y)$. Also, let $W = \frac{X_1}{X_2}$ denote the quotient of X_1 and X_2, with the p.d.f. $g(w)$. Then, following Springer (1979), the p.d.f. of the product $Y = X_1 X_2$ is given by

$$h(y) = \frac{1}{\pi\left(1 - \rho^2\right)^{1/2}} \exp\left[\frac{\rho\, y}{1 - \rho^2}\right] K_0\left[\frac{y}{1 - \rho^2}\right], \quad |\rho| < 1,$$
$$-\infty < y < \infty, \tag{4.22}$$

where $K_0(x)$ is the modified Bessel function of the second kind of order zero. The p.d.f of the quotient $W = \frac{X_1}{X_2}$ is given by

$$g(w) = \frac{\left(1 - \rho^2\right)^{1/2}}{\pi} \frac{1}{w^2 - 2\rho w + 1}, \quad |\rho| < 1, \quad -\infty < w < \infty.$$

Remark 4.5.6. For the p.d.f. of the quotient $W = X_1/X_2$, where $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim N\left(\mu_2, \sigma_2^2\right)$ denote the two dependent (correlated) normal random variables (r.v.'s), with means μ_i, variances $\sigma_i^2 \quad (i = 1, 2)$, and correlation coefficient ρ, see Hinkley (1969).

4.6 Summary

First, this chapter presents some basic definitions and ideas on the sum, product, and ratio of two independent random variables. The distributions of the sum $X + Y$, product XY, and ratio, X/Y, when X and Y are independent random variables and have the normal distributions, have been reviewed in details. A short discussion, when X and Y are dependent or correlated, is also provided. The expressions for the pdfs as proposed by different authors are presented. By using Maple programs, various graphs have been plotted.

Chapter 5
Sum, Product and Ratio for the Student's *t* Random Variables

This chapter presents the distributions of the sum $X + Y$, product XY, and ratio X/Y when X and Y are independent random variables and have the Student's t distributions with appropriate degrees of freedoms.

5.1 Distribution of the Sum of Independent Student's *t* Random Variables

The distributions of the sum of independent Student's t random variables play an important role in many areas of research. The general theories of the distributions of linear combinations of independent Student's t random variables began with the work of Behrens (1929) and later by Fisher (1935). Further developments continued with the contributions of Bose and Roy (1938), Sukhatme (1938), Fisher (1939, 1941), Chapman (1950), Fisher and Healy (1956), Fisher and Yates (1957), James (1959), Ruben (1960), Patil (1965), Scheffe (1970), Ghosh (1975), Walker and Saw (1978), and recently by Fotowicz (2006) and Nadarajah (2006a). The distributions of the weighted sum of independent Student's t random variables in some cases are described below.

(i) Distribution of the weighted sum of two Student's t random variables: Fisher (1935), in his studies of the fiducial argument in statistical inference, developed a statistic, called the Behrens-Fisher statistic, in the form of a weighted sum $Z = \alpha X_m + \beta Y_n$ for the given Student's t random variables X_m and Y_n of degrees of freedom m and n, respectively. By approximating the integral of the fiducial density of Z through a Riemann sum, Sukhatme (1938) computed the percentage points of the distribution of Z. On the other hand, Chapman (1950) considered the case when $\alpha = \beta = 1$ and $m = n = \nu$ (say), and derived an expression for the density of Z for odd values of ν and computed the values of the cdf of Z through numerical integration.

M. Ahsanullah et al., *Normal and Student's t Distributions and Their Applications*, Atlantis Studies in Probability and Statistics 4, DOI: 10.2991/978-94-6239-061-4_5,

(ii) Distribution of the difference (or sum) of two Student's t random variables: Let X_m and Y_n be two independent Student's t random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Then, for $m = n > 0$, Ghosh (1975) derived the pdf of the random variable $Z = X_m - Y_n$ given by

$$f_z(z) = \int_{-\infty}^{+\infty} f_Y(y) f_Y(z+y)\, dy = \frac{\Gamma\left(\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2^n\sqrt{n}\Gamma^2\left(\frac{1}{2}n\right)\Gamma\left(\frac{1}{2}n+1\right)}\left(\frac{4n}{4n+z^2}\right)^{n+1} \times \\ \times {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; \frac{1}{2}n+1; \frac{z^2}{4n+z^2}\right), \quad -\infty < z < \infty \tag{5.1}$$

where ${}_2F_1(.)$ denotes Gauss hypergeometric function. Further, when $m \neq n$, Ghosh (1975) also derived an approximation of the distribution of $Z = X_m - Y_n$ by using a series expansion. The results of Ghosh (1975) are applicable to the sum $Z = X_m + Y_n$ as well, since $X_m - Y_n$ and $X_m + Y_n$ are identically distributed.

(iii) **Distribution of the linear combinations of Student's t random variables:** Walker and Saw (1978) have considered the distribution of a linear combination of independent Student's t random variables $X_1, X_2, \ldots, X_n$, with degrees of freedom $v_1, v_2, \ldots, v_n$, respectively, given by

$$Z^* = c_1X_1 + c_2X_2 + \cdots + c_nX_n.$$

where $c_1, c_2, \ldots, c_n$ denote the real arbitrary coefficients. For odd degrees of freedom, they derived the cdf of Z^* as a mixture of Student's t distributions, which enables one to calculate the percentage points using only tables of the t distribution. If $\nu_i = 1$ for $i = 1, 2, \ldots, n$, the random variable Z^* will have the Cauchy distribution. Further, Z^* will have the normal distribution if $\nu_i \to \infty \quad \forall i$. The distribution of Z^* does not have a closed form when $1 < \nu_i < \infty$, for some i, (for details, see Walker and Saw (1978), among others).

(iv) Special cases:
(a) Let X and Y be two independent standard Student's t random variables each with 1 degree of freedom, and pdfs $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Then, the pdf of the random variable Z given by

$$f_z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx = \frac{2}{\pi(4+z^2)}, \quad -\infty < z < \infty, \tag{5.2}$$

which is also the pdf of the sum of two standard Cauchy random variables $X \sim C(0, 1)$ and $Y \sim C(0, 1)$, (for details, see Kapadia et al. (2005), among others). The above expression is also easily obtained by substituting $n = 1$ in the Eq. (5.1) above, and noting that ${}_2F_1(\alpha, \beta; \beta; t) = (1-t)^{-\alpha}$, (for details, see Abramowitz and Stegun (1970), Lebedev (1972), Prudnikov et al. (1986), and Gradshteyn and Ryzhik (2000), among others). A Maple plot of the above pdf is given in Fig. 5.1.

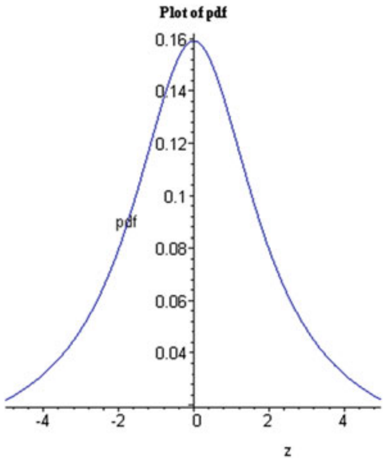


Fig. 5.1 Plot of $Z = X + Y$, where X and Y are standard Student's t random variables having 1 degree of freedom each

(b) The general form of the pdf of the sum of n independent Student's t random variables $X_1,\ X_2, \ldots, X_n$, each having 1 degree of freedom, is given by

$$f_z(z) = \frac{n}{\pi(n^2 + z^2)}, \quad -\infty < z < \infty, \tag{5.3}$$

which is also the pdf of the sum of n standard Cauchy random variables, (for details, see Kapadia et al. (2005), among others).

(c) Let X and Y be two independent standard Student's t random variables with 2 degrees of freedom, and pdfs $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Then, by substituting $n = 2$ in the expression for pdf in Eq. (5.1), and using the equation (7.3.2.106/p 474) from Prudnikov et al. volume 3 (1986), the pdf of the random variable Z is obtained as follows

$$f_z(z) = 2\left[\left(8 + z^2\right) \mathrm{K}\left(\frac{z}{\sqrt{8 + z^2}}\right) + \left(8 - z^2\right) D\left(\frac{z}{\sqrt{8 + z^2}}\right)\right], -\infty < z < \infty, \tag{5.4}$$

where $D(.)$ and $\mathrm{K}(.)$ denote the complete elliptic integrals.

(d) Let X and Y be two independent standard Student's t random variables with 3 degrees of freedom, and pdfs $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Then, by substituting $n = 3$ in the expression for pdf in (i) above as derived by Ghosh (1975), and using the equation (7.3.2.121/p 475) from Prudnikov et al. volume 3 (1986), the pdf of the random variable Z is obtained as follows

$$f_z(z) = \frac{12\sqrt{3}\left(60 + z^2\right)}{\pi\left(12 + z^2\right)^3}, \quad -\infty < z < \infty \tag{5.5}$$

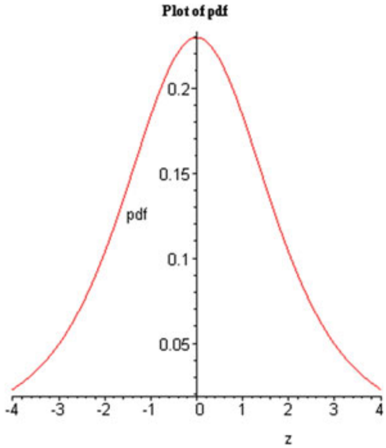


Fig. 5.2 Plot of $Z = X + Y$, where X and Y are standard Student's t random variables having 3 degrees of freedom each

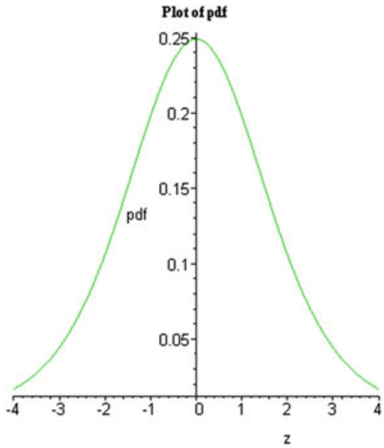


Fig. 5.3 Plot of $Z = X + Y$, where X and Y are standard Student's t random variables with 5 degrees of freedom

A Maple plot of the above pdf is given in Fig. 5.2, which certainly represent a symmetric distribution.

(e) Let X and Y be two independent standard Student's t random variables with 5 degrees of freedom, and pdfs $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Then, the pdf of the random variable Z is given by

$$f_z(z) = \frac{400\sqrt{5}\left(8400 + 120z^2 + z^4\right)}{3\pi\left(20 + z^2\right)^5}, \quad -\infty < z < \infty,$$

(for details, see Casellla and Berger (2002)). A Maple plot of the above pdf is given in Fig. 5.3, which is a form of symmetric distribution.

(v) Computation of the Coverage Interval:
Recently, the computation of the coverage interval for the convolution of two independent Student's t random variables has been investigated by Fotowicz (2006), and Nadarajah (2006a), which is defined as follows.

Let X_m and Y_n be two independent Student's t random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Let Z denote the convolution $Z = X_m + Y_n$. Obviously, Z is a symmetric random variable. Let the pdf and cdf of Z be denoted by $g(\cdot)$ and $G(\cdot)$ respectively. Then, the coverage interval of $Z = X_m + Y_n$ corresponding to coverage probability $p = 1 - \alpha$ is defined by

$$[z_{low}, z_{high}],$$

where $z_{low} = G^{-1}\left(\frac{\alpha}{2}\right)$ and $z_{high} = G^{-1}\left(1 - \frac{\alpha}{2}\right)$. For details of the methods for the computation of the coverage interval for the convolution of two independent Student's t random variables, see Fotowicz (2006), and Nadarajah (2006a).

5.2 Distribution of the Product of Independent Student's t Random Variables

This section discusses the distribution of the product of independent Student's t random variables. The distribution of the product independent random variables is one of the important research topics both from theoretical and applications point of view. It arises in many applied problems of biology, economics, engineering, genetics, hydrology, medicine, number theory, order statistics, physics, psychology, etc, (see, for example, Cigizoglu and Bayazit (2000), Frisch and Sornette (1997), Galambos and Simonelli (2005), Grubel (1968), Ladekarl et al. (1997), Rathie and Rohrer (1987), Rokeach and Kliejunas (1972), Springer (1979), and Sornette (1998, 2004), among others). The distributions of the product $Z = XY$, when X and Y are independent random variables and belong to the same family, have been studied by many authors, (see, for example, Sakamoto (1943) for the uniform family, Springer and Thompson (1970) for the normal family, Stuart (1962) and Podolski (1972) for the gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for the beta family, AbuSalih (1983) for the power function family, and Malik and Trudel (1986) for the exponential family, among others). The distribution of the product of two correlated t variates has been considered by Wallgren (1980). Recently, Nadarajah and Dey (2006) have studied the distribution of the product $Z = XY$ of two independent Student's t random variables X and Y. It appears from the literature that not much study has been done for the distributions of the product of independent Student's t random variables and applications, which need further investigation. Following Nadarajah and Dey (2006), the distributions of the product of independent Student's t random variables are summarized below. For details, see Nadarajah and Dey (2006).

(i) Distribution of the product of two independent Student's t random variables:
(a) Let X_m and Y_n be two independent Student's t random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Then, as derived by Nadarajah and Dey (2006), the cdf of the random variable $Z = X_m Y_n$, for m odd, is given by

$$F(z) = \left(\frac{4}{\pi\sqrt{n}B\left(\frac{n}{2},\frac{1}{2}\right)}\right)\int_0^\infty \tan^{-1}\left(\frac{z}{\sqrt{m}y}\right)\left(1+\frac{y^2}{n}\right)^{-\frac{1}{2}-\frac{n}{2}} dy + \left(\frac{z}{\pi\sqrt{mn}B\left(\frac{n}{2},\frac{1}{2}\right)}\right)$$
$$\times \sum_{k=1}^{(m-1)/2}\left[B\left(\frac{1+n}{2},k\right)B\left(k,\frac{1}{2}\right){}_2F_1\left(k,\frac{1+n}{2};k+\frac{1+n}{2};1-\frac{z^2}{mn}\right)\right], \tag{5.6}$$

where $B(\cdot)$ and ${}_2F_1(\cdot)$ denotes the beta and Gauss hypergeometric functions respectively.

(b) Let X_m and Y_n be two independent Student's t random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Then, as derived by Nadarajah and Dey (2006), the cdf of the random variable $Z = X_m Y_n$, for m even, is given by

$$F(z) = \left(\frac{z}{\pi\sqrt{mn}B\left(\frac{n}{2},\frac{1}{2}\right)}\right)$$
$$\times \sum_{k=1}^{m/2}\left[B\left(\frac{1+n}{2},k-\frac{1}{2}\right)B\left(k-\frac{1}{2},\frac{1}{2}\right){}_2F_1\left(k-\frac{1}{2},\frac{1+n}{2};k+\frac{n}{2};1-\frac{z^2}{mn}\right)\right], \tag{5.7}$$

where $B(\cdot)$ and ${}_2F_1(\cdot)$ denotes the beta and Gauss hypergeometric functions respectively.

(c) For the cdf of the random variable $Z = X_m Y_n$ for particular cases of the degrees of freedom $m = 2, 3, 4, 5$ and $n = 1, 2, 3, 4, 5$, see Nadarajah and Dey (2006).

(d) The possible shapes of the pdf of $Z = |X_m Y_n|$ for a range of values of degrees of freedom $m > 0$ and $n > 0$ are illustrated in Fig. 5.4a, b. We observe that the shapes of the pdf of $Z = X_m Y_n$ are unimodal. The effects of the parameters can easily be seen from these graphs.

(ii) Special Cases:
(a) Let X and Y be two independent Student's t random variables with 1 degree of freedom, and pdf $f_X(x)$ and pdf $f_Y(y)$, respectively. Then, the pdf of the random variable $Z = XY$ is given by

$$g_z(z) = \frac{2\,\text{In}|z|}{\pi^2(z^2-1)} - \infty < z < \infty,$$

(for details, see Rider (1965), Springer and Thompson (1966), and Springer (1979)). A Maple plot of the above pdf is given in Fig. 5.5.

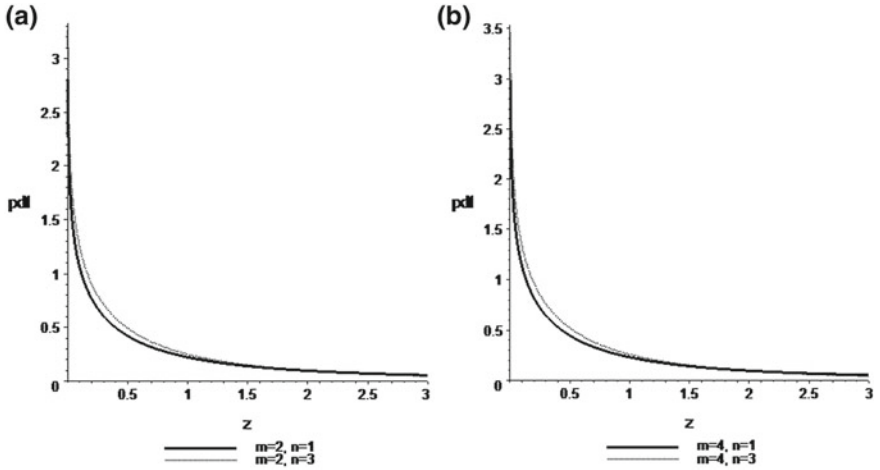
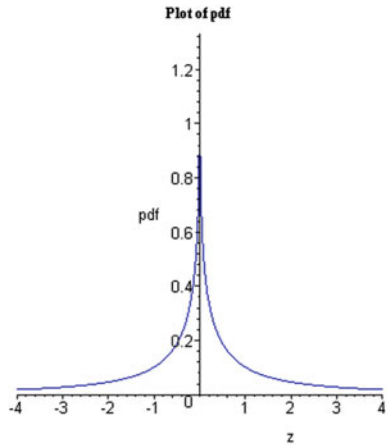


Fig. 5.4 Plots of the pdf of $Z = |X_m Y_n|$ for (**a**) $m = 2$ and $n = 1, 3$ and (**b**) $m = 4$ and $n = 1, 3$

Fig. 5.5 Plot of the pdf of $Z = XY$, where and are standard Student's *t* random variables with 1 degree of freedom



(b) Let X be a Student's t random variables with pdf $f_X(x)$ and degrees of freedom $\nu > 0$. Then, the pdf of the random variable $Z = X^2$ is given by

$$g_Z(z) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{z}{2}\right)^{-\left(\frac{1+\nu}{2}\right)} z^{-\frac{1}{2}}, z > 0, \nu > 0, \tag{5.8}$$

which defines a $F_{1,\nu}$ distribution (see, for example, Lukacs (1972) and Patel and Read (1982), among others).

(c) Let X be a Student's t random variables with pdf $f_X(x)$ and 1 degree of freedom. Then, the pdf of the random variable $Z = X^2$ is given by

$$g_z(z) = \frac{1}{\pi}\,(1+z)^{-1}\,z^{-\frac{1}{2}},\, z > 0$$

which defines a $F_{1,1}$ distribution (see, for example, Lukacs (1972), among others).

5.3 Distribution of the Ratio of Independent Student's *t* Random Variables

This section presents the distributions of the ratio of independent Student's random variables, which play an important role in many areas of research, for example, in economics, genetics, meteorology, nuclear physics, statistics, etc. The distributions of the ratio $Z = X/Y$, when X and Y are independent random variables and belong to the same family, have been studied by many researchers, (see, for example, Marsaglia (1965) and Korhonen and Narula (1989) for the normal family, Basu and Lochner (1971) for the Weibull family, Shcolnick (1985) for the stable family, Hawkins and Han (1986) for the non-central chi-squared family, Provost (1989b) for the gamma family, and Pham-Gia (2000) for the beta family, among others). Recently, Nadarajah and Dey (2006) have studied the distributions of the ratio $Z = X/Y$ of two independent Student's t random variables X and Y. It appears from the literature that not much attention has been paid to the study of the distributions of the ratio of independent Student's t random variables and applications, except Nadarajah and Dey (2006), which need further investigation. For the sake of completeness of this project, following Nadarajah and Dey (2006), the distributions of the ratio of independent Student's t random variables are summarized below. For details, see Nadarajah and Dey (2006).

(i) Distribution of the ratio of two independent Student's *t* random variables:
(a) Let X_m and Y_n be two independent Student's t random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Then, as derived by Nadarajah and Dey (2006), the cdf of the random variable $Z = X_m/Y_n$, for m odd, is given by

$$F(z) = \left(\frac{4}{\pi\sqrt{n}B\left(\frac{n}{2},\frac{1}{2}\right)}\right)\int_0^\infty \tan^{-1}\left(\frac{zy}{\sqrt{m}}\right)\left(1+\frac{y^2}{n}\right)^{-\frac{1}{2}-\frac{n}{2}} dy + \left(\frac{2\sqrt{n}z}{\pi\sqrt{m}B\left(\frac{n}{2},\frac{1}{2}\right)}\right)$$
$$\times \sum_{k=1}^{(m-1)/2}\left[\frac{m^k B(k,\frac{1}{2})}{z^{2k}n^k(2k+n-1)}\,{}_2F_1\left(k, k+\frac{n-1}{2}; k+\frac{1+n}{2}; 1-\frac{m}{nz^2}\right)\right], \tag{5.9}$$

where $B(\cdot)$ and ${}_2F_1(\cdot)$ denotes the beta and Gauss hypergeometric functions respectively.

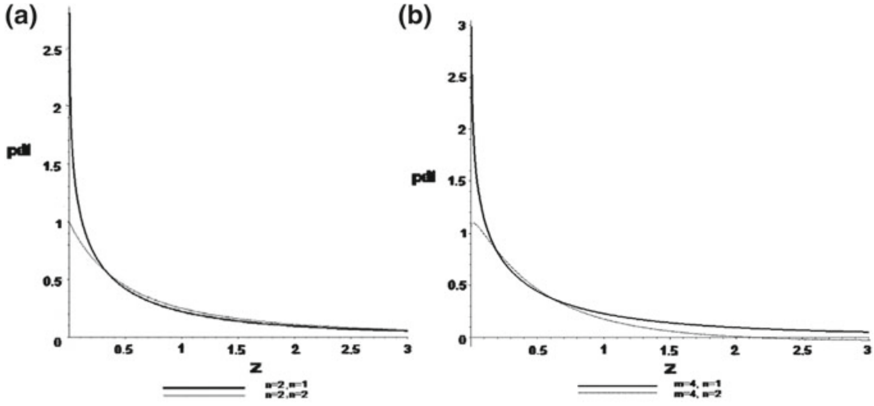


Fig. 5.6 Plots of the pdf of $Z = \left|\frac{X_m}{Y_n}\right|$ for (**a**) $m = 2$ and $n = 1, 2$ and (**b**) $m = 4$, and $n = 1, 2$

(b) Let X_m and Y_n be two independent Student's *t* random variables with pdfs $f_X(x)$ and $f_Y(y)$, and degrees of freedom $m > 0$ and $n > 0$, respectively. Then, as derived by Nadarajah and Dey (2006), the cdf of the random variable $Z = X_m/Y_n$, for m even, is given by

$$F(z) = \left(\frac{2\sqrt{n}z}{\pi\sqrt{m}B\left(\frac{n}{2},\frac{1}{2}\right)}\right) \times \sum_{k=1}^{(m/2)}\left[\frac{m^{k-\frac{1}{2}}B\left(k-\frac{1}{2},\frac{1}{2}\right)}{z^{2k-1}n^{k-\frac{1}{2}}(2k-2+n)} {}_2F_1\left(k-\frac{1}{2}, k+\frac{n}{2}-1; k+\frac{n}{2}; 1-\frac{m}{nz^2}\right)\right], \tag{5.10}$$

where $B(\cdot)$ and ${}_2F_1(\cdot)$ denotes the beta and Gauss hypergeometric functions respectively.

(c) For the cdf of the random variable $Z = X_m/Y_n$ for particular cases of the degrees of freedom $m = 2, 3, 4, 5$ and $n = 1, 2, 3, 4, 5$ see Nadarajah and Dey (2006).

(d) The possible shapes of the pdf of $Z = \left|\frac{X_m}{Y_n}\right|$ for a range of values of degrees of freedom and are illustrated in Fig. 5.6a, b. We observe that the shapes of $Z = \left|\frac{X_m}{Y_n}\right|$ are unimodal. The effects of the parameter can easily be seen from these graphs.

(ii) Special Case: Let *X* and *Y* be two independent Student's *t* random variables with 1 degree of freedom, and pdfs $f_X(x)$ and $f_Y(y)$, respectively. Then, the pdf of the random variable $Z = |X/Y|$ is given by

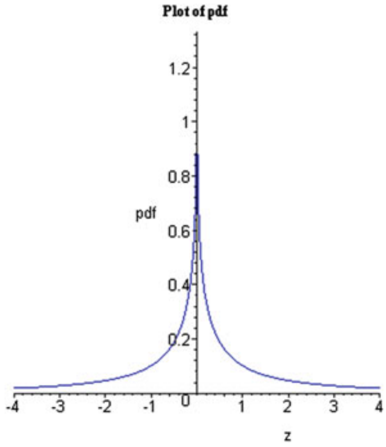


Fig. 5.7 Plot of the pdf of $Z = X/Y$, where X and Y are standard Student's t random variables with 1 degree of freedom

$$g_z(z) = \frac{2\ln|z|}{\pi^2(z^2-1)} - \infty < z < \infty,$$

which is identical to the pdf of the product random variable $Z = |XY|$ of two independent Student's t random variables X and Y as discussed above, (for details, see Rider (1965), Springer and Thompson (1966), and Springer (1979)). A Maple plot of the above pdf is given in Fig. 5.7.

Since the Cauchy distribution with median zero (standard) the pdf of X and 1/X are identical, the pdfs of product and the ratio of two standard Cauchy distribution are identical.

5.4 Distributions of the Sums, Differences, Products and Ratios of Dependent (Correlated) Student *t* Random Variables

It appears from literature that not much attention has been paid to the distributions of the sums, differences, products and ratios of dependent (correlated) Student's t random variables, and therefore needs further research investigation.

5.5 Summary

This chapter has reviewed the distributions of the sum $X + Y$, product XY, and ratio X/Y, when X and Y are independent random variables and have the Student's t distributions. The expressions for the pdfs as proposed by different authors are presented. Some special cases of the sum, product and ratio distributions are given. By using Maple programs, various graphs have been plotted.

Chapter 6
Sum, Product and Ratio for the Normal and Student's *t* Random Variables

The distributions of the sum $X + Y$, product XY, and ratio X/Y, when X and Y are independent random variables and belong to different families, are of considerable importance and current interest. These have been recently studied by many researchers, (among them, Nadarajah (2005b, c, d) for the linear combination, product and ratio of normal and logistic random variables, Nadarajah and Kotz (2005c) for the linear combination of exponential and gamma random variables, Nadarajah and Kotz (2006d) for the linear combination of logistic and Gumbel random variables, Nadarajah and Kibria (2006) for the linear combination of exponential and Rayleigh random variables, Nadarajah and Ali (2004) for the distributions of the product XY when X and Y are independent Laplace and Bessel random variables respectively, Ali and Nadarajah (2004) for the product and the ratio of t and logistic random variables, Ali and Nadarajah (2005) for the product and ratio of t and Laplace random variables, Nadarajah and Kotz (2005b) for the ratio of Pearson type VII and Bessel random variables, Nadarajah (2005c) for the product and ratio of Laplace and Bessel random variables, Nadarajah and Ali (2005) for the distributions of XY and X/Y, when X and Y are independent Student's and Laplace random variables respectively, Nadarajah and Kotz (2005a) for the product and ratio of Pearson type VII and Laplace random variables, Nadarajah and Kotz (2006a) for the product and ratio of gamma and Weibull random variables, Shakil, Kibria and Singh (2006) for the ratio of Maxwell and Rice random variables, Shakil and Kibria (2007) for the ratio of Gamma and Rayleigh random variables, Shakil, Kibria and Chang (2007) for the product and ratio of Maxwell and Rayleigh random variables, and Shakil and Kibria (2007) for the product of Maxwell and Rice random variables, are notable). This chapter studies the distributions of the sum $X + Y$, product XY, and ratio X/Y, when X and Y are independent normal and Student's t random variables respectively.

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6.1 Distribution of the Sum of Normal and Student's *t* Random Variables

This section discusses the distributions of the sum of the independent normal and Student's *t* random variables. The distributions of the sum of the normal and Student's *t* random variables were first studied by Kendall (1938) who formulated the pdf of the random variable $Z = X + Y$, for 1 degree of freedom. Recently, Nason (2006) has studied the distributions of the sum $X + Y$, when X and Y are independent normal and sphered Student's *t* random variables respectively. It appears from the literature that not much attention has been paid to the distributions of the sum of the normal and Student's *t* random variables. This chapter introduces and develops some new results on the pdfs for the sum of the normal and Student's *t* random variables, which have been independently derived here.

6.1.1 Kendall's Pdf for the Sum X + Y

Let $X \sim N(\mu, \sigma^2)$ be a normal random variable with pdf $f_X(x)$. Let Y be a Student's *t* random variable with degrees of freedom $\nu = 1$, and pdf $f_Y(y)$. Let $Z = X + Y$ Then, the pdf of the random variable Z, when $\nu = 1$, as derived by Kendall (1938), is given by

$$f_z(z) = \frac{\sqrt{2}}{\pi\,\sigma}\,\mathrm{Re}\left\{e^{z^2}\,erfc(z)\right\}, \tag{6.1}$$

where $z = d - \frac{ip}{2}, d = \frac{1}{\sqrt{2}\sigma}, p = \frac{\sqrt{2}\mu}{\sigma}$, *erfc* (.), denotes complementary error function, and means 'take the real part'.

6.1.2 Nason's Pdfs for the Sum X + Y, Based on Sphered Student's t Density

Recently, Nason (2006) has studied the distributions of the sum $X + Y$, when X and Y are independent normal and sphered student's *t* random variables respectively. The pdf of a sphered student's *t* random variable is defined as.

Definition (Sphered Student's *t* density): The sphered Student's *t*-density on $\nu \geq 3$ degrees of freedom is defined by $t_\nu : \Re \rightarrow (0, \infty)$ such that

$$t_\nu(x) = \frac{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}\left[1+\frac{x^2}{\nu-2}\right]^{\frac{-(\nu+1)}{2}}, \quad -\infty < x < \infty, \nu > 0 \tag{6.2}$$

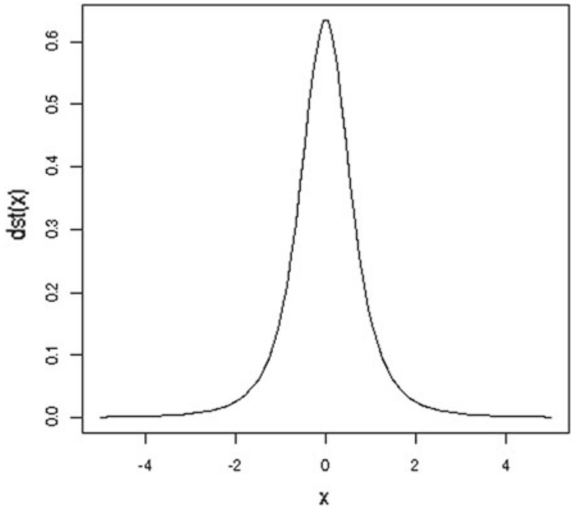

Fig. 6.1 Plot of the pdf of sphered student's *t* for degrees of freedom $\nu = 3$

Note that the sphered Student's *t* density is the standard Student's *t* distribution rescaled to have unit variance. For a multivariate version of the sphered Student's *t* distribution, see, for example, Krzanowski and Marriott (1994), and Nason (2001), among others. The possible shape of the above sphered Student's *t* density for degrees of freedom $\nu = 3$ is illustrated in Fig. 6.1, which is reproduced from Nason (2006).

Nason (2006) has studied the distributions of the sum of normal and sphered student's *t* random variables. Let $X \sim N(\mu, \sigma^2)$ be a normal random variable with pdf $\varphi_{\mu,\ \sigma}(\mathrm{x})$. Let T_ν be a random variable distributed according to sphered Student's *t*-distribution on degrees of freedom, with pdf $h_\nu(t)$. Let $\mathrm{Y} = \mathrm{X} + \mathrm{T}_\nu$. Then, the pdf of the random variable Y can be represented as the convolution of the density functions of X and T_ν , as follows:

$$f_Y(y) = \int_{-\infty}^{+\infty} \varphi_{\mu,\sigma}(y - t)h_\nu(t)dt = \int_{-\infty}^{+\infty} \varphi_{\mu+y,\sigma}(t)h_\nu(t)dt = \left\langle \varphi_{\mu+y,\,\sigma},\ h_\nu \right\rangle, \qquad (6.3)$$

where $\langle , \rangle$ is the usual inner product $\langle f, g\rangle = \int_{-\infty}^{+\infty} f(u)g(u)du$. Then, Nason has derived some nice results for the inner products of $h_\nu(t)$ with $\varphi_{\mu,\sigma}(x)$ for both cases when $\nu = 3$ and $\nu > 3$. For details of these results, see Nason (2006).

As pointed out by Nason (2006), some of the applications and importance of these results are in Bayesian wavelet shrinkage, Bayesian posterior density derivations, calculations in the theoretical analysis of projection indices, and computation of certain moments.

6.1.3 Some New Results on the Pdfs for the Sum X + Y

In what follows, we introduce and develop some new results on the pdfs for the sum of the normal and Student's t random variables. Let $X \sim N\left(0, \sigma^2\right)$ be a normal random variable with pdf $f_X(x)$. Let Y be a Student's t random variable with ν degrees of freedom, and pdf $f_Y(y)$. Let $Z = X + Y$. Then, the pdf of the random variable Z is given by

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_X(z-t) f_Y(t)\, dt \\
&= \frac{1}{\sqrt{2\pi\nu}\,\sigma B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2\sigma^2}} \left(1 + \frac{t^2}{\nu}\right)^{\frac{-(1+\nu)}{2}} dt \\
&= \frac{2\, e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\nu}\,\sigma B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_{0}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \cosh\left(\frac{zt}{\sigma^2}\right) \left(1 + \frac{t^2}{\nu}\right)^{\frac{-(1+\nu)}{2}} dt \\
&= \frac{2\, \nu^{\frac{\nu}{2}} e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\nu}\,\sigma B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z^2}{2\sigma^4}\right)^n \int_{0}^{\infty} t^{2n} e^{-\frac{t^2}{2\sigma^2}} \left(t^2 + \nu\right)^{\frac{-(1+\nu)}{2}} dt \qquad (6.4)
\end{aligned}
$$

Substituting $t^2 = u$ in the above integral, and using the Eq. (2.3.6.9), We obtain,

$$
\int_{0}^{\infty} w^{\alpha-1} e^{-pw} (w + \kappa)^{-\rho}\, dw = \Gamma(\alpha) \kappa^{\alpha-\rho} \psi(\alpha,\ \alpha + 1 - \rho; p\kappa)
$$

from Prudnikov et al. (1986, volume 1), the above expression for the pdf reduces to:

$$
f_Z(z) = \frac{\nu^{\frac{\nu}{2}}\, e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z^2}{2\sigma^4}\right)^n \Gamma\left(n + \frac{1}{2}\right) \nu^{n-\frac{\nu}{2}} \psi\left(n + \frac{1}{2}, n + 1 - \frac{\nu}{2}; \frac{\nu}{2\sigma^2}\right),
$$

where $|z| < \infty$, $\nu > 0$, $\sigma > 0$ and $\psi(.)$ denotes Kummer's hypergeometric function. In view of the fact that $|z| < \infty$, ignoring all the terms after the first term in the above series, an approximate pdf of the random variable $Z = X + Y$ is easily obtained as

$$
f_Z(z) = \frac{1}{\sqrt{2}\,\sigma B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \psi\left(\frac{1}{2}, 1 - \frac{\nu}{2}; \frac{\nu}{2\sigma^2}\right) e^{-\frac{z^2}{2\sigma^2}} \qquad (6.5)
$$

Substituting $\nu = 1$, and using the Eq. (13.6.39) from Abramowitz and Stegun (1972), the above expression for the pdf easily reduces to:

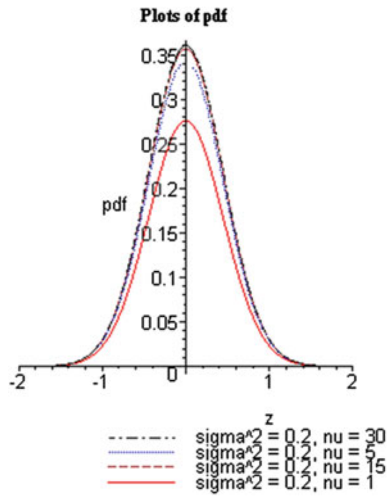

Fig. 6.2 Plots of the pdf of $Z = X + Y$

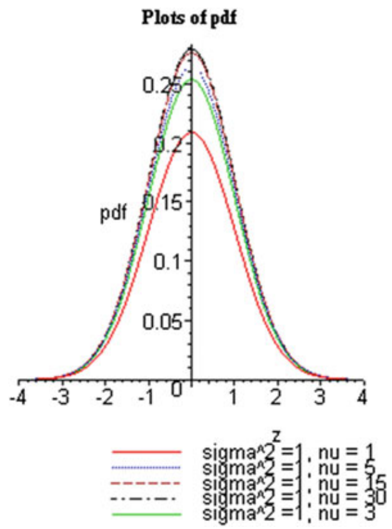

Fig. 6.3 Plots of the pdf of $Z = X + Y$

$$f_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{1}{2\sigma^2}} \, erfc\left(\frac{1}{\sqrt{2}\,\sigma}\right) e^{-\frac{z^2}{2\sigma^2}},$$

where *erfc* (.) denotes complementary error function. Using Maple, the possible shapes of the above approximate pdf of $Z = X + Y$ for a range of values of $\sigma > 0$ and degrees of freedom $\nu > 0$ are illustrated in Figs. 6.2, 6.3, and 6.4.

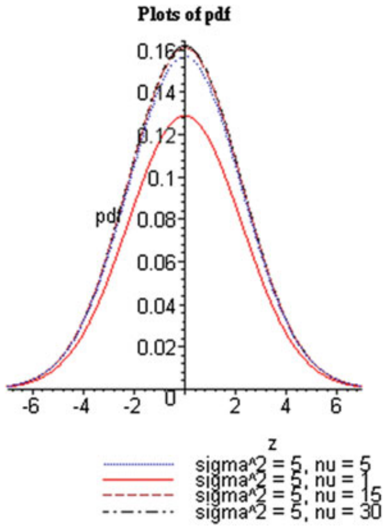

Fig. 6.4 Plots of the pdf of $Z = X + Y$

6.2 Distribution of the Product of Normal and Student's *t* Random Variables

The distributions of the product of independent normal and Student's t random variables arise in many areas of engineering, medicine, science, and statistics. These distributions play an important role in many areas of statistics, for example, Bayesian analysis, projection pursuit, and wavelet shrinkage, to mention a few. In case of Bayesian wavelet shrinkage, it has been shown by Johnstone and Silverman (2004, 2005) that excellent performance is obtained by using heavy-tailed distributions as part of a wavelet coefficient mixture prior instead of the standard normal. A quantity of interest is the product of the heavy-tailed distribution with the standard normal. It is possible that Student's t distribution might also be an interesting distribution to use in this context. This section presents the distributions of the product of independent normal and Student's t random variables, independently derived by the author. These results are believed to be new.

6.2.1 Derivation of the Pdf for the Product of Normal and Student's t Random Variables

Let $X \sim N\left(0, \sigma^2\right)$ be a normal random variable with pdf $f_X(x)$. Let Y be a Student's t random variable with ν degrees of freedom, and pdf $f_Y(y)$. Let $Z = XY$. Then, the pdf of the random variable Z is given by

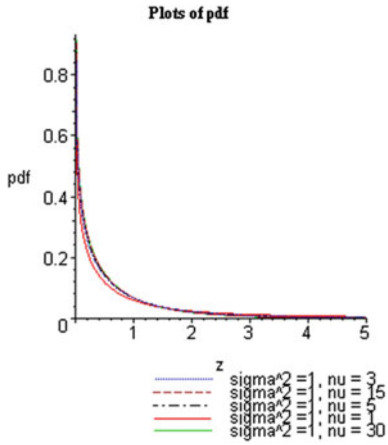

Fig. 6.5 Plots of the pdf of $Z = |XY|$

$$f_z(z) = \sqrt{\frac{2}{\pi \nu}} \frac{1}{B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{1}{y} e^{-\frac{z^2 y^2}{2\sigma^2}} \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} dy \tag{6.6}$$

Substituting $\frac{1}{y^2} = t$, and using following result

$$\int_0^\infty w^{\alpha-1} e^{-pw} (w + \kappa)^{-\rho} \, dw = \Gamma(\alpha) \kappa^{\alpha-\rho} \psi(\alpha, \alpha + 1 - \rho; p\kappa),$$

in the Eq. (6.6), (see the Eq. (2.3.6.9), p. 324, Prudnikov et al. (1986, volume 1)), the pdf of the random variable $Z = XY$ reduces to

$$f_z(z) = \sqrt{\frac{1}{2\nu}} \frac{\Gamma\left(\frac{\nu}{2}, \frac{1}{2}\right)}{B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \psi\left(\frac{\nu}{2} + \frac{1}{2}, 1, \frac{z^2}{2\sigma^2 \nu}\right), \ \text{-}\infty < z < \infty, \ \nu > 0.\sigma > 0. \tag{6.7}$$

where $\psi(.)$ denotes Kummer's hypergeometric function. Substituting $\nu = 1, \sigma = 1$, and using the Eq. (13.6.39), p. 510, from Abramowitz and Stegun (1972), the above expression for the pdf easily reduces to:

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} erfc\left(\frac{z}{\sqrt{2}}\right), \tag{6.8}$$

where *erfc* (.) denotes complementary error function. Using Maple, the possible shapes of the pdf of the random variable $Z = |XY|$ for a range of values $\sigma > 0$ of and degrees of freedom $\nu > 0$ are illustrated in Fig. 6.5.

6.3 Distribution of the Ratio of Normal and Student's *t* Random Variables

The distributions of the ratio of independent normal and Student's *t* random variables are of interest in many areas of engineering, medicine science, and statistics. For example, in the analysis of Bayesian wavelet shrinkage, Johnstone and Silverman (2004, 2005) show that excellent performance is obtained by using heavy-tailed distributions as part of a wavelet coefficient mixture prior instead of the standard normal. A quantity of interest is the ratio of the heavy-tailed distribution with the standard normal. It is possible that Student's *t* distribution might also be an interesting distribution to use in this context. This section presents the distributions of the ratio of independent normal and Student's *t* random variables, independently derived by the author. These results are believed to be new.

6.3.1 Derivation of the Pdf for the Ratio of Normal and Student's t Random Variables

Let $X \sim N\left(0, \sigma^2\right)$ be a normal random variable with pdf $f_X(x)$. Let Y be a Student's t random variable with ν degrees of freedom, and pdf $f_Y(y)$. Let $Z = X/Y$. Then, the pdf of the random variable Z is given by

$$
\begin{aligned}
f_z(z) &= \int_{-\infty}^{\infty} |y| f_X(zy) f_Y(y) dy \\
&= \frac{\sqrt{2}}{\sqrt{\pi\nu}\sigma \, \mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^{\infty} y e^{-\frac{z^2 y^2}{2\sigma^2}} \left(1 + \frac{y^2}{\nu}\right)^{-\frac{1+\nu}{2}} dy \qquad (6.9)
\end{aligned}
$$

Substituting $y^2 = t$, and using following result

$$
\int_0^{\infty} w^{\alpha-1} e^{-pw} (w+\kappa)^{-\rho} dw = \Gamma(\alpha)\kappa^{\alpha-\rho} \psi(\alpha, \alpha+1-\rho; p\kappa),
$$

in the Eq. (6.9), (see the Eq. (2.3.6.9), p. 324, Prudnikov et al. (1986, volume 1)), the pdf of the random variable $Z = X/Y$ reduces to

$$
f_z(z) = \frac{\sqrt{\nu}}{\sqrt{2\pi}\sigma \, B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \psi\left(1, \frac{3-\nu}{2}; \frac{\nu z^2}{2\sigma^2}\right), -\infty < z < \infty, \; \nu > 0, \; \sigma > 0. \qquad (6.10)
$$

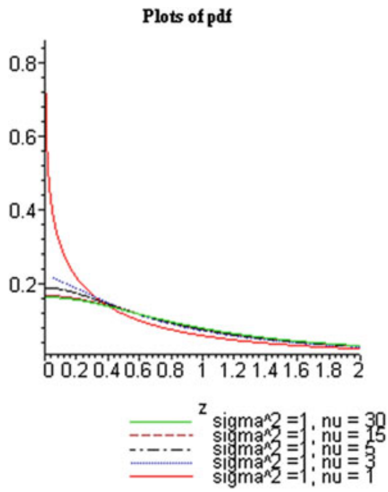

Fig. 6.6 Plots of the Pdf of $Z = |X/Y|$

where ψ (.) denotes Kummer's hypergeometric function. Substituting $\nu = 1, \sigma = 1$, and using the Eq. (13.6.39), p. 510, from Abramowitz and Stegun (1972), the above expression for the pdf easily reduces to:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} erfc\left(\frac{z}{\sqrt{2}}\right),$$

where $erfc$ (.) denotes complementary function. Note that, for $\nu = 1, \sigma = 1$, the expressions for the pdfs of $Z = XY$ and $Z = X/Y$ are identical. Using Maple 10, the possible shapes of the pdf of the random variable $Z = X/Y$ for a range of values of $\sigma > 0$ and degrees of freedom $\nu > 0$ are illustrated in Fig. 6.6.

6.4 Summary

This chapter has studied the distributions of the sum $X+Y$, product XY, and ratio X/Y, when X and Y are independent normal and Student's t random variables respectively. The distributions of the sum, product, and ratio of the independent normal and Student's t random variables as proposed by different researchers have been reviewed. Some new results have been included. To describe the possible shapes of these pdfs, several graphs have been plotted using Maple programs.

Chapter 7
Product of the Normal and Student's *t* Densities

7.1 Introduction

The distributions of the product of two random variables have a great importance in many areas of research both from the theoretical and applications point of view. Srivastava and Nadarajah (2006) have studied some families of Bessel distributions and their applications by taking products of a Bessel function pdf of the first kind and a Bessel function pdf of the second kind. It appears from the literature that not much attention has been paid to this kind of study. The normal and Student's t distributions arise in many fields and have been extensively studied by many researchers in different times. This chapter introduces and develops a new symmetric type distribution with its probability density function (pdf) taken to be of the form $p_{\mathrm{X}}(x) = C.f_{\mathrm{X}}(x).g_{\mathrm{X}}(x)$, where C is the normalizing constant, and $f_{\mathrm{X}}(x)$ and $g_{\mathrm{X}}(x)$ denote the pdfs of normal and Student's distributions, respectively. More on this topic the readers are referred to Frisch and Sornette (1997), Sornette (1998, 2004), Galambos and Simonelli (2005), Shakil and Kibria (2007), and recently Shakil and Kibria (2009), among others.

7.1.1 Some Useful Lemmas

The following Lemmas will be used to complete the derivations.

Lemma 7.1.1 (Prudnikov et al. 1986, volume 1, Equation 2.3.6.9, page 324). For $|\arg z| < \pi$, $\rho > 0$, $z > 0$, $\mathrm{Re}\,(\alpha) > 0$, and $p > 0$,

$$\int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} e^{-px} dx = \Gamma(\alpha)\, z^{\alpha-\rho}\, \psi\,(\alpha,\ \alpha+1-\ \rho;\, pz)$$

M. Ahsanullah et al., *Normal and Student's t Distributions and Their Applications*, Atlantis Studies in Probability and Statistics 4, DOI: 10.2991/978-94-6239-061-4_7,

where $\psi(\cdot)$ denotes Kummer's function.

Proof: The proof of Lemma 7.1.1 easily follows by replacing z by $z.\,p$, and substituting $\gamma = \alpha + 1 - \rho$ and $t = \frac{x}{z}$ in the Eq. (1.1) of Chap. 1.

Lemma 7.1.2 (Prudnikov et al. 1986, volume 1, equation 2.3.8.1, page 328). For $|\arg\left(1 + \frac{a}{z}\right)| < \pi$, $a > 0$, $\rho > 0$, $z > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $p > 0$,

$$\int_0^a \frac{x^{\alpha-1}(a-x)^{\beta-1}}{(x+z)^{\rho}} e^{-px} dx = B(\alpha, \beta)\, z^{-\rho} a^{\alpha+\beta-1} \Phi_1\left(\alpha, \rho; \alpha+\beta; \frac{-a}{z}, ap\right),$$

where $B(\cdot)$ and $\Phi_1(\cdot)$ denote the beta and generalized hypergeometric functions respectively.

Proof: The proof of Lemma 7.1.2 easily follows by using the substitution $x = a \cdot u$, and applying the Eq. (1.2) of Chap. 1 and the definition of beta function.

7.2 Product of the Densities of Normal and Student's *t* Random Variables

This section develops a new symmetric Student's t-type distribution with its pdf taken to be the product of the densities of normal and Student's t random variables.

7.2.1 Expressions for the Normalizing Constant and PDF

Consider the product function $p_{\mathrm{X}}(x) = C \cdot f_{\mathrm{X}}(x) \cdot g_{\mathrm{X}}(x)$, $-\infty < x < \infty$, $C > 0$, where C denotes a normalizing constant, and $f_{\mathrm{X}}(x)$ and $g_{\mathrm{X}}(x)$ denote the pdf of the standard normal and Student's t distributions respectively. The pdf $f_{\mathrm{X}}(x)$ of $X \sim N(0, 1)$ is given by

$$f_{\mathrm{X}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty, \tag{7.1}$$

whereas the pdf $g_X(x)$ of a Student's t distribution with v degrees of freedom (for some integer $v > 0$), in terms of beta function, is, given by

$$g_{\mathrm{X}}(x) = \frac{1}{\sqrt{v} B\left(\frac{v}{2}, \frac{1}{2}\right)} \left(1 + \frac{x^2}{v}\right)^{-(1+v)/2}, -\infty < x < \infty, v > 0 \tag{7.2}$$

Theorem 7.2.1 The product function $p_X(x) = C.f_X(x).g_X(x)$, $-\infty < x < \infty$, $C > 0$, defines a pdf if the normalizing constant is given by

$$C = \frac{\sqrt{2}B\left(\frac{v}{2}, \frac{1}{2}\right)}{\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}, \qquad v > 0, \tag{7.3}$$

where $B(\cdot)$ and $\psi(\cdot)$ denote beta and Kummer's functions respectively.

Proof: Clearly, $p_X(x) \geq 0, \forall x \in (-\infty, +\infty)$ and $C > 0$. Hence for $p_X(x)$ to be a pdf, we must have $\int_{-\infty}^{+\infty} p_X(x)dx = 1$, where $p_X(x) = C.f_X(x).g_X(x)$. Clearly, in view of the even properties of the pdfs of normal and Student's t distributions, we have $p_X(-x) = p_X(x)$ that is, $p_X(x)$ is an even function. Thus, we have

$$\int_{-\infty}^{+\infty} p_X(x)dx = 2\int_0^{\infty} p_X(x)dx = 2\int_0^{\infty} C.f_X(x).g_X(x)dx = 1,$$

which, on substituting the expressions (7.1) and (7.2) for the pdfs of standard normal and Student's t distributions respectively, gives

$$2\int_0^{\infty} C.\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.\frac{1}{\sqrt{v}B\left(\frac{v}{2}, \frac{1}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-(1+v)/2} dx = 1,$$

that is,

$$\int_0^{\infty} e^{-x^2/2}.\left(1+\frac{x^2}{v}\right)^{-(1+v)/2} dx = \frac{\sqrt{2\pi v}B\left(\frac{v}{2}, \frac{1}{2}\right)}{2C}. \tag{7.4}$$

Now, by substituting $\frac{x^2}{2} = t$ and using Lemma 7.1.1 in the above integral (7.4), the proof of Theorem 7.2.1 easily follows.

Theorem 7.2.2 If $f_X(x)$ and $g_X(x)$ are the pdfs of standard normal and Student's t distributions, as defined in (7.1) and (7.2), respectively, for some continuous random variable X, and C denotes the normalizing constant given by (7.3), the product function given by

$$\begin{aligned} p_X(x) &= C.f_X(x) \cdot g_X(x) \\ &= \frac{e^{-x^2/2}.\left(1+\frac{x^2}{v}\right)^{\frac{-(1+v)}{2}}}{(\sqrt{\pi v})\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}, \quad -\infty < x < \infty, v > 0, \end{aligned} \tag{7.5}$$

defines a pdf of the random variable X, where $\psi(\cdot)$ denotes Kummer's function. It appears that $p_X(x)$ is symmetric and t-type distribution with v degrees of freedom.

Proof: The proof easily follows from Theorem 7.2.1.
Special Case: For $v = 1$, the pdf in (7.5) reduces to the product of densities of normal and Cauchy, with the normalizing constant given by

$$C = \frac{(\sqrt{2})\pi}{\psi\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}\right)}.$$

7.2.2 Derivation of the CDF

This section derives the associated cdf of the random variable X, when the normalizing constant $C(> 0)$ satisfies the requirements for the product function $p_X(x)$ to be a density function, as shown in Sect. 7.2.1.

Theorem 7.2.3 Let $f_X(x)$ and $g_X(x)$ be the pdfs of standard normal and Student's t distributions defined by (7.1) and (7.2).Then, the cdf of the random variable X is given by

$$F_X(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi v}\ \psi\left(\frac{1}{2},\ 1 - \frac{v}{2}; \frac{v}{2}\right)} \left[x\,\Phi_1\left(\frac{1}{2}, \frac{1}{2} + \frac{v}{2}; \frac{3}{2}; \frac{-x^2}{v}, \frac{x^2}{2}\right)\right], \quad (7.6)$$

where $|\arg\left(1 + \frac{x^2}{v}\right)| < \pi, v > 0$, and $\psi(\cdot)$ and $\Phi_1(\cdot)$ denote Kummer's and generalized hypergeometric functions respectively.

Proof: Using the expression for the pdf (7.5) as derived in Theorem 7.2.2 above, we have

$$\begin{aligned} F_X(x) &= \Pr(X \le x) \\ &= \frac{1}{\sqrt{\pi v}\psi\left(1/2, 1 - \frac{v}{2}; \frac{v}{2}\right)} \int_{-\infty}^{x} \left\{\left(1 + \frac{t^2}{v}\right)^{-(1+v)/2} e^{-t^2/2}\right\} dt \\ &= 1 - \frac{1}{\sqrt{\pi v}\psi\left(1/2, 1 - \frac{v}{2}; \frac{v}{2}\right)} \left[\int_{0}^{\infty} \left\{\left(1 + \frac{t^2}{v}\right)^{-(1+v)/2} e^{-t^2/2}\right\} dt \right. \\ &\quad \left. - \int_{0}^{x} \left\{\left(1 + \frac{t^2}{v}\right)^{-(1+v)/2} e^{-t^2/2}\right\} dt\right] \end{aligned} \quad (7.7)$$

Now, by substituting $\frac{t^2}{2} = u$ and using Lemma 7.1.2 in the above integral (7.7), the proof of Theorem 7.2.3 easily follows.

7.3 Some Properties of the Symmetric Distribution

This section discusses some characteristics of the proposed new symmetric distribution.

7.3.1 Mode

Mode is the value of x for which the product probability density function $p_{\mathrm{x}}(x)$ defined by (7.5) is maximum. Now, differentiating equation (2.5), we have

$$p'_{\mathrm{x}}(x) = -\frac{xe^{-x^2/2}\left(1+\frac{x^2}{v}\right)^{-(1+v)/2}}{\sqrt{\pi v}\psi\left(1/2, 1-\frac{v}{2}; \frac{v}{2}\right)}\left[1+\left(\frac{1+v}{v}\right)\left(1+\frac{x^2}{v}\right)^{-1}\right],$$

which, when equated to 0, gives the mode to be $x = 0$. It can be easily seen that $p''(x) < 0$. Thus, the maximum value of the product probability density function $p_{\mathrm{x}}(x)$ is easily obtained from (7.5) as $p_{\mathrm{x}}(0) = \frac{1}{\sqrt{\pi v}\psi(1/2, 1-\frac{v}{2}; \frac{v}{2})}$. Clearly, the product probability density function (7.5) is unimodal.

7.3.2 Moments

Theorem 7.3.4 For some degrees of freedom $v > 0$ and some integer $k > 0$, kth moment of a random variable x having the pdf (7.5) is given by

$$E\left(X^k\right) = \begin{cases} \frac{\Gamma\left(\frac{k+1}{2}\right)v^{\frac{k}{2}}\ \psi\left(\frac{k+1}{2},\ \frac{k-v}{2},\ +1; \frac{v}{2}\right)}{\sqrt{\pi}\ \psi\left(\frac{1}{2},\ 1-\frac{v}{2}; \frac{v}{2}\right)}, when\, k\, is\, even; \\ 0, \quad when\, k\, is\, odd; \end{cases} \tag{7.8}$$

where $k > 0$ is integer, and $\psi(\cdot)$ denotes Kummer's function.

Proof: Using the expression for the pdf (7.5), we have

$$E\left(X^k\right) = \frac{1}{\sqrt{\pi v}\ \psi\left(\frac{1}{2},\ 1-\frac{v}{2}; \frac{v}{2}\right)}\int_{-\infty}^{+\infty}\left\{x^k\left(1+\frac{x^2}{v}\right)^{-\frac{(1+v)}{2}} e^{\frac{-x^2}{2}}\right\}dx \tag{7.9}$$

Case I: When k is even.
Let $k = 2n$, where $n > 0$ is an integer. Then, since, clearly, the integrand in (7.9) is an even function, we have

$$E\left(X^k\right) = \frac{2v^{\frac{v}{2}}}{\sqrt{\pi}\ \psi\left(\frac{1}{2},\ 1-\frac{v}{2};\ \frac{v}{2}\right)} \int_0^{+\infty} \left\{ x^{2n} \left(x^2+v\right)^{-\frac{(1+v)}{2}} e^{\frac{-x^2}{2}} \right\} dx \qquad (7.10)$$

Now, by substituting $\frac{x^2}{2} = u$ and using Lemma 7.1.1 in the above integral (7.10), the proof of Theorem 7.3.4 (when k is an even integer) easily follows.

Case II: When k is odd.
Let $k = 2n - 1$, where $n > 0$ is an integer. Then, since, clearly, the integrand in (7.10) is an odd function, the proof of Theorem 7.3.4 (when k is an odd integer) easily follows.

Special Cases: By taking $k = 2$ and $k = 4$ respectively in (7.10), for some degrees of freedom $v > 0$, the second and fourth moments are easily obtained as follows

$$E\left(X^2\right) = \frac{1}{2}\frac{v\psi\left(\frac{3}{2}, 2-\frac{v}{2}; \frac{v}{2}\right)}{\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}, \qquad (7.11)$$

and

$$E\left(X^4\right) = \frac{3}{4}\frac{v^2\psi\left(\frac{5}{2}, 3-\frac{v}{2}; \frac{v}{2}\right)}{\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}, \qquad (7.12)$$

where $\psi(\cdot)$ denotes Kummer's function.

7.3.3 *Mean, Variance, and Coefficients of Skewness, and Kurtosis*

From (7.12), the mean, variance, and coefficients of skewness and kurtosis of probability density function (7.5) are easily obtained as follows:

(i) Mean: $\mu = E(x) = 0$;

(ii) Variance: $Var\,(X) = \sigma^2 = \frac{v\psi\left(\frac{3}{2},2-\frac{v}{2};\frac{v}{2}\right)}{2\psi\left(\frac{1}{2},1-\frac{v}{2};\frac{v}{2}\right)}, \quad v > 0$

(iii) Coefficient of Skewness: $\gamma_1 = \frac{\beta_3}{\beta_2^{3/2}} = 0$

(iv) Coefficient of Kurtosis: $\gamma_2 = \frac{\beta_4}{\beta_2^2} = \frac{3\psi\left(\frac{5}{2},3-\frac{v}{2};\frac{v}{2}\right)\psi\left(\frac{1}{2},1-\frac{v}{2};\frac{v}{2}\right)}{\left[\psi\left(\frac{3}{2},2-\frac{v}{2};\frac{v}{2}\right)\right]^2}, \quad v > 0$

7.4 Entropy

The Shannon (1948) entropy of an absolutely continuous random variable x having the probability density function $\phi_{\mathrm{x}}(x)$ is defined as

$$H[X] = \mathrm{E}[-\ln\,(\phi_{\mathrm{x}}(x)] = -\int_{S} \phi_{\mathrm{x}}(x)\,\ln\,[\phi_{\mathrm{x}}(x)]dx \tag{7.13}$$

where $S = \{x : \phi_{\mathrm{x}}(x) > 0\}$.

Theorem 7.4.1 For $v > 0$, the entropy of a random variable x having the probability density function $p_x(x)$ in (7.5) , is given by

$$H\,[X] = \ln\left\{\sqrt{\pi v}\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)\right\} + \frac{1}{4}\frac{v\psi\left(\frac{3}{2}, 2-\frac{v}{2}; \frac{v}{2}\right)}{\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}$$
$$+\left[\frac{(1+v)}{2\sqrt{\pi}}\right]\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\Gamma\left(j+\frac{1}{2}\right)}{j}\frac{\psi\left(j+\frac{1}{2}, j-\frac{v}{2}+1; \frac{v}{2}\right)}{\psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)},$$

where $\psi(\cdot)$ denotes Kummer's function.

Proof: From (7.13), we have

$$H\,[X] = \mathrm{E}\,[-\ln\,(p_{\mathrm{x}}\,(x))] = -\int_{-\infty}^{+\infty} p_{\mathrm{x}}\,(x)\;\ln\;[p_{\mathrm{x}}\,(x)]\,dx$$

$$= -\int_{-\infty}^{+\infty}\frac{\left(1+\frac{x^2}{v}\right)^{-\frac{(1+v)}{2}} e^{\frac{-x^2}{2}}}{\sqrt{\pi\ v}\ \psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}\mathrm{x}\ \ln\left\{\frac{\left(1+\frac{x^2}{v}\right)^{-\frac{(1+v)}{2}} e^{\frac{-x^2}{2}}}{\sqrt{\pi\ v}\ \psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)}\right\} dx \tag{7.14}$$

In view of the definitions of the pdf (7.5) (Theorem 7.2.2) and moments, the above expression (7.14) of entropy easily reduces as

$$H\,[X] = \ln\left\{\sqrt{\pi v}\ \psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)\right\} + \frac{1}{2}E\left(X^2\right)$$
$$+\left(\frac{1+v}{2}\right)E\left[\ln\left(1+\frac{X^2}{v}\right)\right]$$
$$= \ln\left\{\sqrt{\pi v}\ \psi\left(\frac{1}{2}, 1-\frac{v}{2}; \frac{v}{2}\right)\right\} + \frac{1}{2}E\left(X^2\right)$$

Table 7.1 Percentiles of the new symmetric distribution

v	50 %	60 %	70 %	75 %	80 %	85 %	90 %	95 %	99 %
3	≈ 0	0.1750	0.3640	0.4692	0.5881	0.7287	0.9095	1.1865	1.7385
4	≈ 0	0.1761	0.3656	0.4714	0.5902	0.7304	0.9098	1.1831	1.7230
5	≈ 0	0.1767	0.3667	0.4727	0.5914	0.7313	0.9097	1.1804	1.7118
10	≈ 0	0.1780	0.3689	0.4750	0.5940	0.7325	0.9090	1.1734	1.6840
15	≈ 0	0.1784	0.3696	0.4757	0.5942	0.7327	0.9081	1.1704	1.6721
20	≈ 0	0.1786	0.3699	0.4760	0.5944	0.7328	0.9077	1.1687	1.6659
25	≈ 0	0.1787	0.3701	0.4762	0.5946	0.7329	0.9074	1.1677	1.6620
30	≈ 0	0.1789	0.3702	0.4763	0.5947	0.7329	0.9073	1.1670	1.6593
35	≈ 0	0.1788	0.3703	0.4764	0.5947	0.7329	0.9071	1.1664	1.6573
40	≈ 0	0.1788	0.3703	0.4765	0.5948	0.7329	0.9070	1.1660	1.6560
50	≈ 0	0.1789	0.3704	0.4766	0.5949	0.7329	0.9069	1.1655	1.6537
60	≈ 0	0.1790	0.3705	0.4767	0.5914	0.7329	0.9068	1.1651	1.6524
70	≈ 0	0.1792	0.3708	0.4770	0.5953	0.7333	0.9073	1.1660	1.6560
75	≈ 0	0.1790	0.3706	0.4767	0.5950	0.7329	0.9067	1.1648	1.6513

$$+\left(\frac{1+v}{2}\right)\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{jv^{j}}E\left(X^{2j}\right) \tag{7.15}$$

By using the moment expressions (7.10) and (7.11) in (7.15), the proof of Theorem 7.4.1 easily follows.

7.5 Percentage Points

This section computes the percentage points of the new symmetric distribution. For any α, where $0 < \alpha < 1$, the (100α)th percentile or the quantile of order α of the new symmetric distribution with the pdf $p_{\mathrm{X}}(x)$ is a number t_α such that the area under $p_{\mathrm{X}}(x)$ to the left of t_α is α. That is, t_α is any root of the equation

$$F\left(t_\eta\right) = \int_{-\infty}^{t_a} p_X\left(u\right) du = \alpha\ .$$

Using Maple, the percentiles t_α of the new symmetric distribution are computed for some selected values of α for the given values of degrees of freedom v, which are provided in Table 7.1.

7.6 Summary

This chapter has derived a new symmetric type distribution and its properties, with its probability density function (pdf) taken to be the product of a normal pdf $f_X(x)$and a Student's t distribution pdf $g_X(x)$ for some continuous random variable X. The expressions for the associated pdf, cdf, *kth* moment, mean, variance, skewness, kurtosis, and entropy have been derived in terms of Kummer's functions. It is shown that the pdf of the proposed distribution is unimodal. It is noted that for $v = 1$, we obtain a new symmetric distribution which is the product of the standard normal and Cauchy distributions, and for large v we will obtain a distribution which is product of two standard normal densities. The percentage points have also been provided. We hope the findings of this chapter paper will be useful for the practitioners in various fields of sciences.

Chapter 8
Characterizations of Normal Distribution

8.1 Introduction

Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various methods. In recent years, many authors have studied characterizations of various distributions. This chapter discusses characterizations of normal distribution.

8.2 Characterizations of Normal Distribution

In this section, we will consider several characterizations of normal distribution. Let Z be a standard normal distribution (N(0, 1)) with pdf f(z), then

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\right)z^2}, -\infty < z < \infty \tag{8.1}$$

Suppose the random variable X has a normal distribution with mean μ and standard deviation σ (N(μ, σ)) with the pdf f(x) as

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{8.2}$$
$$-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0.$$

The characteristic functions $\phi(t)$ and $\phi_1(t)$ of Z and X are respectively

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$$\phi(t) = e^{-\frac{1}{2}t^2}. \tag{8.3}$$

and

$$\phi_1(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}, \quad i = \sqrt{-1} \quad \text{for all t}. \tag{8.4}$$

Here we present some basic properties of the characteristic function $\phi(t)$ which we need for our characterization theorems.

1. $\phi(t)$ always exists.
2. $\phi(t)$ is uniformly continuous on entire space.
3. $\phi(t)$ is non vanishing in a region around zero, $\phi(0) = 1$.
4. $\phi(t)$ is bounded. $|\phi(t)| \le 1$.
5. $\phi(t)$ is Hermitian, $\phi(-t) = \overline{\phi}(t)$, where $\overline{\phi}(t)$ is the complex conjugate of $\phi(t)$.
6. $\phi(t)$ for a symmetric (around zero) random variable is real and even function.
7. There is a one to one correspondence between $\phi(t)$ and the cdf

Polya (1923) gave the following characterization theorem.

Theorem 8.2.1. Suppose X_1 and X_2 are independent and identically distributed random variables. Then X_1 and $\frac{(X_1+X_2)}{\sqrt{2}}$ are identically distributed if and only if X_1 and X_2 are normally distributed.

Proof. It is easy to show that $E(X_1) = E(X_2) = 0$. Let $\phi(t)$ and $\phi_1(t)$ be the characteristic functions of X_1 and $\frac{(X_1+X_2)}{\sqrt{2}}$ respectively.

If X_1 and X_2 are N(0, σ^2), then

$$\phi_1(t) = [\phi(t/\sqrt{2})]^2 = \left(e^{-(1/2)\sigma^2(t/\sqrt{2})^2}\right) = e^{-(1/2)\sigma^2 t^2} \tag{8.5}$$

Thus $\frac{(X_1+X_2)}{\sqrt{2}}$ is distributed as N(0, σ^2).
Suppose X_1 and $\frac{(X_1+X_2)}{\sqrt{2}}$ are identically distributed.
Then

$$\phi_1(t) = [\phi(t/\sqrt{2})]^2$$

i.e.

$$\phi_1(\sqrt{2}t) = [\phi(t)]^2 \quad \text{for all t.}$$

Therefore recurrently

$$\phi_1\left(t2^{\frac{k}{2}}\right) = (\phi(t))^{2^k} \quad \text{for all t.} \tag{8.6}$$

Let us take a t_0 such that $\phi(t_0) \neq 0$, such a t_0 can be found since $\phi(t)$ is continuous and $\phi(0) = 1$. Let $\sigma^2 > 0$ be such that $\phi(t_0) = e^{-\sigma^2}$, then we have

$$\phi_1(t_0 2^{-\frac{k}{2}}) = e^{-\sigma^2 2^{-k}} \quad \text{for } k = 0, 1, 2, \ldots\ldots \tag{8.7}$$

Thus $\phi_1(t) = e^{-t^2\sigma^2}$ for all t and the theorem is proved.
The following theorem is due to Cramer (1936).

Theorem 8.2.2. Let X_1 and X_2 be two independent but not necessarily identically distributed random variables and $Z = X_1 + X_2$. If Z is normally distributed, then X_1 and X_2 are normally distributed.

To prove the theorem, we need the following two Lemmas.

Lemma 8.2.1. Hadamard's factorial Theorem.
If g(t) is a integral function of finite order ρ which has zeros $\beta_1, \beta_2, \ldots$and does not vanish at the origin, then g(t) can be written as

$$g(t) = m(t)e^{n(t)},$$

where m(t) is the canonical product formed with the zeros of $\beta_1, \beta_2, \ldots$ and n(t) is a polynomial of degree not exceeding ρ.

Lemma 8.2.2. If $e^{n(t)}$, where n(t) is a polynomial, is a characteristic function, then the degree of n(t) cannot exceed 2.

Proof of Theorem 8.2.2. The necessary condition is easy to prove. We will prove here the sufficiency. We will prove the Theorem under the assumption that Z has mean zero and standard deviation $= \sigma$. The characteristic function ϕ(t) of Z can be written as $\phi(t) = \phi_1(t)\phi_2(t)$, where $\phi_1(t)$ and $\phi_2(t)$ are the characteristic functions of X_1 and X_2 respectively. Now $\phi_1(t) = e^{-\frac{1}{2}\sigma^2 t^2}$ is an entire function without zero. Thus $\phi_1(t) = e^{p(t)}$, where p(t) is a polynomial of degree not exceeding 2. Hence we can write

$$\phi_1(t) = e^{-a_0 + a_1 t + a_2 t^2} \text{ for some real } a_0,\ a_1 \text{ and } a_2.$$

For any characteristic function $\phi(t)$, $|\phi(t)\} \leq 1$, hence a_2 must be negative. Assuming mean of $X_{1-} = \mu$ and standard deviation $X_1 = \sigma$. We obtain

$$\phi_1(t) = e^{i\mu t - (\frac{1}{2})\sigma^2 t^2} \text{ for all t.}$$

Thus X_1 is normally distributed. Similarly it can be proved that X_2 is also normally distributed.

Remark 8.2.1. If Z is normally distributed, then we can write $Z = X_1 + X_2 + \cdots + X_n$, where $X_i' s$, $i = 1, 2, \ldots, n$ are independent and normally distributed.

Remark 8.2.2. If $X_1, X_2, \ldots, X_n$ are n independent and identically distributed random variable with mean zero and variance 1, then by central limit theorem

$$S_n = \frac{X_1}{\sqrt{n}} + \frac{X_2}{\sqrt{n}} + \cdots + \frac{X_n}{\sqrt{n}} \to N(0, 1).$$

But by Cramer's theorem if S_n is N(0, 1), then each $\frac{X_i}{\sqrt{n}}, i = 1, 2, \ldots n$ must be normal.

Gani and Shanbag (1975) proved that if $Z = X_1 + X_2$ is a sum of two independent random variables is normal, then Z can be decomposed as such that the conditional distribution of one given the other is normal.

Basu and Ahsanullah (1983) gave a generalization of Cramer's decomposition theorem in the case of sum of dependent random variables based on Gani and Shanbag's decomposition in the following theorem.

Theorem 8.2.3. Let Z be a normal random variable with zero mean and variance σ^2 which has Gani and Shanbag decomposition as sum of two random variables $(X_1 + X_2)$ with additional property that X_1 and X_2 are identically distributed with finite strictly positive second moment and correlation coefficient $\rho, 0 <| \rho |< 1$ Then X_1 and X_2 both follow the normal distribution.

Proof. See Basu and Ahsanullah (1983).

The following characterization theorem is due to Darmois (1951) and Basu (1951).

Theorem 8.2.4. Suppose $X_1, X_2, \ldots, X_n$ be a set of independent but need not be identically distributed random variables and let

$$L_1 = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \tag{8.8}$$

and

$$L_2 = b_1 X_1 + b_2 X_2 + \cdots + b_n X_n \tag{8.9}$$

where a's and b's are constant. If L_1 and L_2 are independently distributed, then X_i for which $a_i b_i \neq 0$ is normally distributed.

For an interesting proof of the theorem see Linnik (1964 p. 97).

Kagan et al. (1965) showed that if $n(\geq 3)$ independent and identically distributed random variable with $E(X_i) = 0$ and $E(\bar{X} \mid X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X}_n) = 0$, where $n\bar{X} = \sum_{i=1}^{n} X_i$, then$X_i's(i = 1, 2, \ldots, n)$ are normally distributed.

Rao (1967) proved that for $n(n \geq 3)$ i.i.d.rv's $X_1, \ldots, X_n$ if $E(X_i) = 0$ and $E(X_i^2) < \infty$, $i = 1, 2, \ldots, n$. Then if $E(\bar{X} \mid X_i - \bar{X}) = 0$ for a fixed i, then X's are normal.

The following example shows that the result need not be true for $n = 2$.

Example 8.2.1. Let the random variable X_1 and X_2 have the following joint pdf,

$$f(x_1, x_2) = \frac{1}{4}, -1 \leq x_1, x_2 \leq 1, \tag{8.10}$$
$$= 0, \text{ otherwise.}$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, the conditional pdf of $Y_1 \mid Y_2 = Y_2$ is given by

$$f(y_1 \mid Y_2 = y_2) = \frac{1}{2(2 - y_2)}, \quad 0 < y_2 < 2, \ -2 < y_1 < 2 - y_2$$

Hence the result.

Kagan and Zinger (1971) proved the normality of the X's under the following conditions.

$$E(\mid X_i \mid^2) < \infty, \mathrm{i} = 1, 2, \ldots, n$$
$$E(L_1^{k-1} \mid L_2) = 0, k = 1, 2, \ldots, n$$

where $L_1 = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$ and $L_2 = b_1 X_1 + b_2 X_2 + \cdots + b_n X_n$.

Theorem 8.2.5. Suppose X_1 and X_2 independent and identically distributed random variables with mean zero and variance 1. If $X_1 + X_2$ and $X_1 - X_2$ are independent, then the common distribution of X_1 and X_2 is normal.

Proof. Let $\phi(t)$ be the common characteristic function of X_1 and X_2. Since $X_1 + X_2$ and $X_1 - X_2$ are independent, we must have

$$\phi(2t) = (\phi(t))^3 \phi(-t).$$

The function $\phi(t)$ never vanishes. Writing $\rho(t) = \frac{\phi(t)}{\phi(-t)}$, we have $\rho(2t) = (\rho(t))^2$. By induction, we obtain

$$\rho(t) = \left(\rho(\frac{t}{2^k})\right)^k - \left(1 + o(\frac{t}{2^k})\right)^{2^k} \rightarrow 1.$$

Thus $\rho(t) - 1$ and $\phi(t) = \phi(-t)$ for all t.

We have $\phi(t) = [\phi(t/2^k)]^{2^k} = e^{-(1/2)t^2}$
The theorem is proved.

The following lemma is due to Roberts (1971).

Lemma 8.2.3. If is a characteristic function of the random variable Z. Suppose Z^2 is distributed as chi-square with one degree of freedom, then

$$\phi(t) + \phi(-t) = 2e^{-(\frac{1}{2})t^2} \tag{8.11}$$

Proof. Let g(.) be the probability density function of Z^2 and h(.) be the pdf of Z. Then

$$g(u) = \frac{u^{-(\frac{1}{2})}}{\sqrt{2\pi}} e^{-(\frac{1}{2})u} = \frac{1}{2\sqrt{u}} h(u) + \frac{1}{2\sqrt{u}} h(-u).$$

Thus

$$h(u) + h(-u) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}u}$$

Now

$$\begin{aligned}
\phi(t) + \phi(-t) + & \int_{-\infty}^{\infty} e^{itx} h(x)dx + \int_{-\infty}^{\infty} e^{-itx} h(x)dx \\
&= \int_{-\infty}^{\infty} e^{itx} h(x)dx + \int_{-\infty}^{\infty} e^{itx} h(-x)dx - \int_{-\infty}^{\infty} e^{itx} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2} dx \\
&= 2e^{-\frac{1}{2}t^2} \qquad (8.12)
\end{aligned}$$

Ahsanullah (1989) gave the following characterization theorem using the above lemma.

Theorem 8.2.6. Suppose $X_1, X_2, \ldots X_n (n \geq 2)$ are n independent and identically distributed random variables. Suppose $L_1 = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \ldots, a_n$ are constants, not all them are zero and X's are symmetric around zero. Then if L_1^2 is distributed as a chi-square with one degree of freedom, then X's are normal.

Let $\emptyset_1^*(t)$ be the characteristic function of L_1, then by Lemma 8.1,

$$\begin{aligned}
2e^{-\frac{1}{2}t^2} &= \emptyset_1^*(t) + \emptyset_1^*(-t) \\
&= \prod_{j=1}^{n} \phi(a_j t) + \prod_{j=1}^{n} \phi(-a_j t)
\end{aligned}$$

where $\emptyset(t)$ is the characteristic function of $X_i's, i = 1, \ldots, n$

$= 2 \prod_{j=1}^{n} \phi(a_j t)$ by the symmetry of the X's.

It is known (Linnik and Zinger (1955) that if $\emptyset_1(t), \emptyset_2(t) \ldots, \emptyset_n(t)$ are characteristic functions and $a_1, a_2, \ldots a_n$ are positive constants, then if

$$(\phi_1(t))^{a_1} (\phi_2(t))^{a_2} \ldots\ldots (\phi_n(t))^{a_n} = e^{-[i\mu t - \frac{\sigma^2 t^2}{2}] \sum_{i=1}^{n} a_n},$$

$-\infty < \mu <$ œ, $\sigma > 0$ for $|t| < \delta, \delta > 0$. holds if then $\emptyset_1(t), \ldots \emptyset_n(t)$ are the characteristic function of normal distribution. Then it follows that $\emptyset(t)$ is the characteristic function of a normal distribution.

Remark 8.2.3. Taking $a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{n}}$, it follows from the above theorem that $n\bar{X}^2$ is distributed as chi-square with one degree of freedom and X's are symmetric around zero imply the normality of the X's.

The following Theorem (Ahsanullah 1990) gives a characterization of normal distribution using chi-square distribution.

Theorem 8.2.7. Let X_1 and X_2 be two independent and identically distributed random variables. Suppose $L_1 = aX_1 + \sqrt{1-a^2}X_2,\ 0 < |a| < 1$, and assume L_1^2 and X_1^2 are each distributed as chi-square with one degree of freedom, this X_1 and X_2 are normally distributed.

Proof. Let $\phi(t)$ be the characteristic function of X_1, then $2e^{-\frac{1}{2}t^2} = \emptyset(at)\phi(\sqrt{1-a^2}t) + \phi(-at) + \phi(-\sqrt{1-a^2}t)$ and $2e^{-\frac{1}{2}t^2} = \emptyset(t) + \emptyset(-t)$

Then $\emptyset(at)\emptyset\sqrt{1-a^2}t) = e^{-\frac{1}{2}t^2}$

It is known (Linnik and Zinger (1955) that if $\emptyset_1(t), \emptyset_2(t) \cdots, \emptyset_n(t)$ are characteristic functions and $a_1, a_2, \ldots a_n$ are positive constants, then if

$$(\phi_1(t))^{a_1}(\phi_2(t))^{a_2}\ldots(\phi_n(t))^{a_n} = e^{-\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)\sum_{k=1}^{n} a_k} \tag{8.13}$$

where $-\infty < \mu < \infty, \sigma < 0$ and $i = \sqrt{-1}$, holds for all $|t| < \delta$, for all $\delta > 0$, then $\phi_1(t), \ldots, \phi_n(t)$ are characteristic functions of normal distributions. Thus $\phi(t)$ is the characteristic function of a normal distribution and X's are normally distributed.

Ahsanullah and Hamedani (1988) proved the following theorem.

Theorem 8.2.8. If X_1 and X_1 be independent and identically distributed symmetric (about zero) random variables with pdf $f(\cdot)$ and let $Z = \min(X_1, X_2)$. If Z^2 is distributed as chi-square with one degree of freedom. then X_1 and X_2 are normally distributed.

Proof. Let $\emptyset(t)$ be the characteristic function of Z, then $\emptyset(t) = 2\int_{-\infty}^{\infty} e^{itx}\bar{F}(x) f(x)dx$, where

$$\bar{F}(x) = 1 - F(x) \text{ and } F(x) = \int_{-\infty}^{x} f(u)du.$$

Hence

$$\begin{aligned}\emptyset(t) + \emptyset(-t) &= 4\int_{-\infty}^{\infty} \cos(tx)\bar{F}(x)f(x)dx \\ &= 4\int_{0}^{\infty} \cos(tx)(1 - F(x))f(x)dx \\ &\quad + 4\int_{0}^{\infty} \cos(tx)F(x)f(x)dx \\ &= 4\int_{0}^{\infty} \cos(tx)f(x)dx \end{aligned} \tag{8.14}$$

$= 2\int_{-\infty}^{\infty} \cos(tx) f(x)dx$ (by the symmetric property of $f(x)$)
$= 2\phi_X(t)$, where $\phi_X(t)$ is the characteristic function of X.
By Lemma 2.3, $\emptyset(t) + \emptyset(-t) = 2e^{-\frac{t^2}{2}}$, and hence $\phi_X(t) = e^{-\frac{t^2}{2}}$.
Thus the common distribution of X_1 and X_2 is normal.

Remark 8.2.4. It is easy to see that we can replace in Z the min by the max.

The following two theorems were proved by Ahsanullah (1989).

Theorem 8.2.9. Let X_1 and X_2 be independent and identically distributed random variables and

Suppose

$$Z_1 = a_1 X_1 + a_2 X_2$$

$$Z_2 = b_1 X_1 + b_2 X_2$$

Such that

$$-1 < a_1, a_2 < 1, -1 < b_1, b_2 < 1, 1 = a_1^2 + a_2^2 = b_1^2 + b_2^2$$

and $a_1 b_2 + a_2 b_1 = 0$.

If z_1^2 and z_2^2 are each distributed as chi-square with one degree of freedom, then X_1 and X_2 are normally distributed.

Proof. Let $\emptyset_3$(t) and $\emptyset_4$(t) be the characteristic functions of Z_1 and Z_2 respectively. Then we have

$$\phi_1(t) + \phi_1(-t) = 2e^{-\frac{1}{2}t^2} = \phi_2(t) + \phi_2(-t)$$

Now if $\emptyset(t)$ is the characteristic function of X, then

$$\emptyset_3(t) = \emptyset(a_1 t)\emptyset(a_2 t), \text{ while } \emptyset_4(t) = \emptyset(b_1 t)\emptyset(b_2 t). \tag{8.15}$$

Substituting $b_1 = -a_1 b_2 \mid a_2$ and using the relation $a_1^2 + a_2^2 = 1$, we must have $a_2^2 = b_2^2$, Taking $a_2 = b_2$ or $a_2 = -b_2$ and writing $\emptyset_3(t)$, $\emptyset_4(t)$ in terms of $\emptyset(t)$, we get on simplification

$$\begin{aligned}
&\emptyset(a_1 t)\emptyset(a_2 t) + \emptyset(-a_1 t)\emptyset(-a_2 t) \\
&\quad = \emptyset(-a_1 t)\emptyset(-a_2 t) + \emptyset(a_1 t)\emptyset(-a_2 t) \\
&\quad = 2e^{-(\frac{1}{2})t^2}, \text{ for all } t, -\infty < t < \infty.
\end{aligned} \tag{8.16}$$

From (8.16) we obtain directly

$$(\emptyset(a_1 t) + \emptyset(-a_1 t))(\emptyset(a_2 t) + \emptyset(-a_2 t)) = 4e^{-\frac{t}{2}^2} \tag{8.17}$$

$$(\emptyset(a_1t) - \emptyset(-a_1t))(\emptyset(a_2t) - \emptyset(-a_2t)) = 0 \tag{8.18}$$

for all t, $-\infty < t < \infty$. From (8.18) it follows that we must have $\emptyset(t) = \emptyset(-t)$ for all t. Thus

$$\emptyset(a_it)\emptyset(a_2t) = e^{-\frac{t}{2}^2}. \tag{8.19}$$

Hence by Cramer's theorem X_1 and X_2 are normally distributed.

Theorem 8.2.10. Let X_1 and X_2 be independent and identically distributed random variables and suppose $U = aX_1 + bX_2$ such that $0 < a, b < 1$ and $a^2 + b^2 = 1$. If U^2 and X_1^2 are each distributed as chi-square with one degree of freedom, then X_1 and X_2 are both distributed as normal.

Proof. Let $\phi_1(t)$ and $\phi(t)$ be the characteristic function of U and X_1 respectively. Then by lemma 8.1, we have

$$\begin{aligned} 2e^{-\frac{1}{2}t^2} &= \phi_1(t) + \phi_1(-t) \\ &= \phi(at)\phi(bt) + \phi(-at)\phi(-bt) \\ &= \phi(t) + \phi(-t), \end{aligned} \tag{8.20}$$

for all t.

From (8.20), we have

$$\phi(at) + \phi(-at) = 2e^{-(1/2)a^2t^2},$$
$$\phi(bt) + \phi(-bt) = 2e^{-(1/2)b^2t^2}$$

and hence

$$(\phi(at) + \phi(-at))(\phi(bt) + \phi(-bt)) = 4e^{-(1/2)t^2} \tag{8.21}$$

We have also

$$(\phi(at) - \phi(-at))(\phi(bt) - \phi(-bt)) = 0 \tag{8.22}$$

Since (8.22) is true for all t, $-\infty < t < \infty$, we must have

$\phi(t) = \phi(-t)$ for all t, $-\infty < t < \infty$.

Hence we obtain

$$\phi(at)\phi(bt) = e^{-(1.2)t^2}, \quad \text{for all t, } -\infty < \text{t} < \infty. \tag{8.23}$$

By Cramer's theorem it follows that X_1 and X_2 are normally distributed.

The following theorem is due to Ahsanullah et al. (1991).

Theorem 8.2.11. Suppose $X_1, X_2, \ldots, X_n$ be n independent and identically distributed random variables for some fixed n, $(n \geq 2)$. Let $T_1 = \sum_{i=1}^n X_i^2$ and $T_2 = n\bar{X}^2$.

If T_1 and T_2 are distributed as chi-squares with n and 1 degrees of freedom, then X_1's, $i = 1, 2, \ldots$ are $N(0, 1)$.

Proof. Let $\varphi_2(t)$ be the characteristic function of $\sqrt{n}\bar{X}$, then by Lemma 8.3.

$$\varphi_2(t)\,\varphi_2(-t) = 2e^{-(1/2)t^2}.$$

Now $\varphi_2(t) = \left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n$, where $\varphi(t)$ is the characteristic function of t. Thus we have

$$2e^{-(1/2)t^2} = \varphi_2(t) + \varphi_2(-t) = \left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n + \left(\varphi\left(\frac{-t}{\sqrt{n}}\right)\right)^n$$

$$\text{i.e., } 2e^{-\left(\frac{1}{2}\right)nt^2} = \varphi_2\left(\sqrt{nt}\right) + \varphi_2\left(\sqrt{-nt}\right) = (\varphi(t))^n + (\varphi(-t))^n. \tag{8.24}$$

Since $X_i^2(i = 1, 2, \ldots, n)$ is distributed as a chi-square with one degree of freedom, we have

$$2e^{-\left(\frac{1}{2}\right)t^2} = \varphi(t) + \varphi(-t)$$

and hence

$$2e^{-(1/2)nt^2} = \varphi\left(\sqrt{nt}\right) + \varphi\left(-\sqrt{nt}\right). \tag{8.25}$$

Now $e^{-(1/2)nt^2} = \left(\frac{\varphi(t)+\varphi(t)}{2}\right)^n$, and therefore

$$2 = \frac{\varphi\left(\sqrt{nt}\right) + \varphi\left(-\sqrt{nt}\right)}{\left(\frac{\varphi(t)+\varphi(-t)}{2}\right)^n} = \left(\frac{2\varphi(t)}{\varphi(t) + \varphi(-T)}\right)^n + \left(\frac{2\varphi(-t)}{\varphi(t) + \varphi(-t)}\right)^n \quad \text{for all } t. \tag{8.26}$$

Thus we have

$$\frac{\varphi(\sqrt{nt}) + \varphi(-\sqrt{nt})}{2} = \left(\frac{\varphi(t) + \varphi(-t)}{2}\right)^n = \frac{(\varphi(t))^n + (\varphi(-t))^n}{2} \text{ for } n \geq 2.$$

Since $\left(\frac{\varphi(t)+\varphi(-t)}{2}\right)^n = \frac{(\varphi(t))^n+(\varphi(-t))^n}{2}$ for some fixed $n \geq 2$ and all t and further $\varphi(t) \neq 0$ and $\varphi(t) \neq 1$, hence $\varphi(t) = \varphi(-t)$ and x's are symmetric around zero. Thus

$$\varphi(t) = \left(\varphi\left(\sqrt{nt}\right)\right)^{1/n}, \text{ for all } t. \tag{8.27}$$

$$= (\varphi(nt))^{1/n^2} = \left(\varphi\left(n^k t\right)\right)^{1/n^{2k}}$$
$$= \left(\varphi\left(t\sqrt{N}\right)\right)^{1/N}, \; N = (n)^{2k}, \; k \geq 1. \tag{8.28}$$

Since the X's have zero mean and unit variance, writing $\varphi(t) = 1 - \frac{t^2}{2} + \eta(t)$, where $\eta(t)/t^2 \to 0$ as $t \to 0$, we have

$$\varphi(t) = \lim_{N\to\infty}\left(1 - \frac{t^2}{2N} + \eta\left(\frac{t}{\sqrt{N}}\right)\right)^N = e^{-\frac{1}{2}t^2}$$

Thus X's are normal. The proof is complete.

The result of the following theorem has lots of application in statistical inferences.

Theorem 8.2.12. Let $X_1, X_2, \ldots, X_n$ be a simple random sample from a normal population with pdf *f(x)*. Then the sample mean $\overline{X}\left(= \frac{1}{n}\sum_{k=1}^{n} X_k\right)$ and sample variance $S^2\left(= \frac{1}{n}\sum_{k=1}^{n}\left(X_k - \overline{X}\right)^2\right)$ are independent if and only if the distribution of the X's is normal.

Proof. The proof of necessity.

Suppose X_i, i = 1, 2, ..., n, is distributed as normal with mean = μ and variance = σ^2. The joint pdf of X_i, i = 1, 2, , ..., n is

$$f_X(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2}\sum_{I=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

Let us make the following transformation

$$Y_1 = \overline{X}$$
$$Y_2 = X_2 - \overline{X}$$
$$Y_3 = X_3 - \overline{X}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$Y_{\mathrm{n}} = X_{\mathrm{n}} - \overline{X}$$

The jacobian of the transformation is n. We have

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma^2}\right)^2 = \frac{1}{\sigma^2}[\left(X_1-\overline{X}\right)^2+\sum_{i=2}^{n}\left(X_i-\overline{X}\right)^2+n\left(\overline{X}-\mu\right)^2]$$
$$=\frac{1}{\sigma^2}[(\sum_{i=2}^{n}\left(X_i-\overline{X}\right)^2+\sum_{i=2}^{n}\left(X_i-\overline{X}\right)^2+n\left(\overline{X}-\mu\right)^2]$$
$$=\frac{1}{\sigma^2}\left[\left(\sum_{i=2}^{n}Y_i\right)^2+\sum_{i=2}^{n}Y_i^2+nY_1^2\right]$$

The joint pdf of $Y_1 \ldots, \mathrm{Y}_n$ is

$$f_Y\left(y_1,\ y_2,\ldots,y_n\right)=\frac{n}{(2\pi)^{n/2}\,\sigma^n}e^{-\frac{1}{2\sigma^2}[\sum_{i=2}^{n}y_i^2+(\sum_{i=2}^{n}y_i)^2+nY_1^2]},$$
$$-\infty<y_i<\infty,\ \mathrm{i}=1,2,\ldots,\mathrm{n}$$

Thus $Y_1(=\overline{X})$ is independent of $\mathrm{Y}_2, \ldots, \mathrm{Y}_n$.

Now
$$nS^2=\sum_{k=1}^{n}(X_k-\overline{X})^2=\sum_{k=2}^{n}Y_k^2+(\sum_{k=2}^{n}Y)^2.$$

Thus $\overline{X}$ and S^2 are independent.

The proof of the sufficiency.

Let the characteristic function, $\phi(t)$, of the distribution is given by $\phi(t)=\int e^{itx} f(x)dx$.

The joint characteristic function of the statistics $\overline{X}$ and S^2 is given by

$$\phi(t_1,t_2)=\int\cdots\int, e^{it_1\overline{x}+it_2s^2}\ f(x_1)\ldots f(x_n)dx_1\ldots dx_n$$

The characteristic function of the mean $\overline{X}$ is

$$\phi_1(t_1)=\phi(t_1,0)=\int\cdots\int, e^{it_1\overline{x}}\ f(x_1)\ldots f(x_n)dx_1\ldots dx_n$$

and the characteristic function of the variance S^2 is given by

$$\phi_2(t_2)=\phi(0,t_2)=\int\cdots\int, e^{it_2s^2}\ f(x_1)\ldots f(x_n).$$

The independence of the distribution of $\overline{X}$ and S^2 means in terms of the characteristic function
$$\phi(t_1,t_2)=\phi_1(t_1)\phi_2(t_2).$$

Using the fact $\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$, we can write

$$\phi_1(t_1) = \prod_{k=1}^{n} \int e^{it_1 x_k/n} f(x_k) dx_k = (\phi(t_1/n))^n$$

We have

$$\begin{aligned}
&\frac{d}{dt_2}\phi(t_1, t_2) \mid_{t_2=0} \\
&= \int \cdots \int is^2 e^{it_1 x_1/n} \cdots e^{it_1 x_n/n} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n \\
&= i(\phi(t_1/n))^{n-1} E(s^2) \\
&= \frac{n-1}{n} i\sigma^2 (\phi(t_1/n))^{n-1},
\end{aligned}$$

where σ^2 is the variance of the X's.

Using the relation $\frac{d}{dt_2}\phi\left(t_1, t_2\right)\Big|_{t_2=0} = \phi_1\left(t_1\right) \frac{d}{dt_2}\phi_2\left(t_2\right)\Big|_{t_2=0,}$ we obtain

$$\begin{aligned}
&\int \cdots \int is^2 e^{it\bar{x}} f\left(x_1\right) \ldots f\left(x_n\right) dx_1 \ldots dx_n \\
&= \tfrac{n-1}{n} i\sigma^2 \int \cdots \int e^{it\bar{x}} f\left(x_1\right) \ldots f\left(x_n\right) dx_1 \ldots dx_n
\end{aligned}$$

Now $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_1 - \bar{x})^2 = \frac{1}{n}[\frac{n-1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\sum_{i,j=1, i\neq j}^{n} x_i x_j.$

We have

$$\frac{d}{dt}\phi(t) = \phi'\left(t\right) = i\int xe^{itx} f(x)\, dx,$$

and

$$\frac{d^2}{dt^2}\phi\left(t\right) = \phi''\left(t\right) = -\int x^2 e^{itx} f(x)\, dx,$$

Thus we can write the following differential equation

$$\phi''\left(t\right)\left(\phi(t)\right)^{n-1} - \left(\phi'\left(t\right)\right)^2 \left(\phi\left(t\right)\right)^{n-2} = \sigma^2\phi(t)^n$$

Since $\phi(0) = 1$ and ϕ(t) is continuous, there exist a neighborhood around zero where ϕ(t) is not zero. Now restricting t in that region, we can write

$$\phi''(t) - (\phi'(t))^2(\phi(t))^{-1} = \sigma^2\phi(t)$$

i.e

$$\frac{\phi''(t)}{\phi(t)} - \frac{(\phi'(t))^2}{(\phi(t))^2} = -\sigma^2$$

We can write the above equation as

$$\frac{d^2}{dt^2}\ln\phi(t) = -\sigma^2.$$

The solution of the above equation is $\phi(t) = e^{at+bt^2}$, where a and b are constants.
Using the condition $E(X) = \mu$ and Var(X) $= \sigma^2$, we have

$$\phi(t) = e^{i\mu t-\frac{1}{2}\sigma^2 t^2}.$$

Thus X is normally distributed.

We have presented here the proof of the sufficiency given by Lukacs (1942). Geary (1934) was the first to prove the sufficiency of the theorem.

We know that if X_1 and X_2 are distributed as N(0, 1), then the ratio X_1/X_2 is distributed as Cauchy with median zero (C(0)). The following example shows that the converse is not true.

Example 8.2.2. Consider the following density function.

$$f(\mathrm{x}) = \frac{2^{1/2}}{\pi(1+x^4)}, -\infty < x < \infty.$$

Then X_1/X_2 is C(0).

It is natural to ask that if X_1 and X_2 are *i.i.d* and X_1/X_2 is distributed as C(0), what additional condition will guarantee the normality of X_1 and X_2. The following theorem (Ahsanullah and Hamedani (1988) gives an answer to the question.

Theorem 8.2.13. Let X and Y be independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue measure) distribution function and let Z = min (X,Y). If $Z^2$and X/Y are distributed as chi-square with one degree of freedom and C(0) respectively, then X and Y are distributed as standard normal.

Proof. Let f(x) be the pdf of X. Since X/Y is distributed as C(0), we have

$$\int_{-\infty}^{\infty} f(uv)f(v)|v|dv = \frac{1}{\pi(1+u^2)}, -\infty < u < \infty.$$

Or

$$\int_0^{\infty} [f(uv)f(v) + f(-uv)f(-v)]vdu = \frac{1}{\pi(1+u^2)}, -\infty < u < \infty..$$

Now letting $u \to 1$ and u $\to -1$, we obtain respectively

$$\int_0^{\infty} [(f(v))^2 + (f(-v))^2]vdu = \frac{1}{2\pi} \tag{8.29}$$

and

$$\int_0^{\infty} 2f(v)f(-v)vdu = \frac{1}{2\pi} \tag{8.30}$$

From (8.29) and (8.30) it follows that

$$\int_0^{\infty} [(f(v) - f(-v))^2]vdu = 0 \tag{8.31}$$

Hence f(v) = f(−v) for almost all v, v< ∞. Thus f(x) is symmetric and hence the conclusion follows from Theorem 8.2.8.

8.3 Summary

A probability distribution can be characterized through various methods. The purpose of this chapter was to discuss various characterizations of normal distribution. It is hoped that the findings of the chapter will be useful for researchers in different fields of applied sciences.

Chapter 9
Characterizations of Student's *t* Distribution

9.1 Introduction

As pointed out in Chap. 8, characterization of a probability distribution plays an important role in probability and statistics. In this chapter, we will present some characterizations of Student's t distribution.

9.2 Characterizations of Student's *t* Distribution

Student's *t* distributions have been widely used in both the theoretical and applied work in statistics. In this section, we will consider some characterizations of Student's t distribution.

The probability density function of the t-distribution with v degrees of freedom is given by

$$f_v(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-(v+1)/2}, -\infty < x < \infty. \tag{9.1}$$

We will denote Student's t distribution with v degrees of freedom as t_v distribution. In this chapter We will present some characterizations of Student's t-distribution.

For $v = 1$, the pdf of t_1 is given by

$$f_1(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty.$$

This is a the standard Cauchy (C(0)) distribution. The following are some of the characteristic properties of standard Cauchy distribution.
If X is distributed as C(0), then

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(i) 1/X is also distributed as C(0).
(ii) $2X/(1-X^2)$ is also distributed as C(0),

However identical distribution of X and 1/X does not characterize the Cauchy distribution (C(0)). The identical distribution of X and $2X/(1-X^2)$ Characterizes the distribution C(0). If X_1 and X_2 are two independent and identically distributed continuous random variables, then any one of following two conditions (see Arnold, (1979)) characterize the standard Cauchy distribution.

(i) X_1 and $(X_1+X_2)/(1-X_1X_2)$ are independent,
(ii) X_1 and $(X_1+X_2)/(1-X_1X_2)$ are identically distributed,

The pdf of t_2 distribution is as given below:

$$f_2(x) = \frac{1}{(2+x^2)^{\frac{3}{2}}}, \tag{9.2}$$

and the corresponding cdf is

$$F_2(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{2+x^2}}\right) \tag{9.3}$$

Several characterizations of t_2 distribution based on regression properties of order statistics were obtained in Akundov et al. (2004); Balakrishnan and Akundov (2003); Nevzorov et al. (2003).

It is interesting to note that t_2 distribution is a member of a general family of distribution satisfying the relation

$$(F(x))(1-F(x))^{\alpha} = cf(x), \tag{9.4}$$

where c is a constant. The logistic distribution satisfies the relation with $\alpha = 1$, the uniform distribution satisfies the relation with $\alpha = 0$ and the squared sign distribution with cdf $F(x) = \sin^2(x), 0 \le x \le \pi/2$ satisfies the relation with $\alpha = 1/2$. Here we will present a characterization of t_2 distribution as given by Nevzorov et al. (2003) in Theorem 9.1 satisfying the relation (9.4) with $\alpha = 3/2$.

Theorem 9.2.1. Suppose that X_1, X_2 and X_3 are independent and identically distributed with cdf F(x) and pdf f(x). Let $X_{1,2} \le X_{2,3} \le X_{3,3}$ be the corresponding order statistics. Further let $W_3 = (X_{1,3}+X_{3,3})/2$ and $M_3 = X_{2,3}$. Then $E(W_3|M_3 = x) = x$, where $\gamma < x < \delta$, $\gamma = \inf\{x|F(x) > 0\}$, $\delta = \sup\{x\}F(x) < 1\}$ if and only if

$$F(x) + F_2\left[\frac{x-\mu}{\sigma}\right], -\infty < x < \infty, \sigma < 0,$$

where $F_2(.)$ is the cdf as given in (9.3).

Proof.

$$E(W_3|M_3 = x) = x$$

$$\text{i.e. } E\left(\frac{X_{1,3} + X_{2,3}}{2} + \frac{X_{2,3}}{2}|X_{2,3} = x\right) = \frac{3x}{2}$$

and

$$E(\overline{X}|X_{2,3} = x) = x \quad \text{or} \quad E(X_1|X_{2,3} = x) = x$$

For t_2 distribution,

$$E(X_1|X_{2,3}) = \frac{1}{3}\left\{x + \frac{1}{F_2(x)}\int_{-\infty}^{0} uf(u)du + \frac{1}{1 - F_2(x)}\int_{0}^{\infty} uf(u)du\right\},$$

where $F_2(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{2+x^2}}\right)$
We have

$$\frac{1}{F_2(x)}\int_{-\infty}^{x} uf_2(u)du = x - \frac{1}{F_2(x)}\int_{x}^{\infty} F_2(u)du,$$

$$\frac{1}{1 - F_2(x)}\int_{-\infty}^{x} uf_2(u)du = x + \frac{1}{1 - F_2(x)}\int_{x}^{\infty} F_2(u)du,$$

Thus

$$E(X_1|X_{2,3}) = x - \frac{1}{3}\left\{\frac{1}{F_2(x)}\int_{-\infty}^{0} F_2(u)du - \frac{1}{1 - F_2(x)}\int_{0}^{\infty}(1 - F_2(u))du\right\}.$$

But

$$-\frac{1}{F_2(x)}\int_{-\infty}^{x} F_2(u)du + \frac{1}{1 - F_2(x)}\int_{x}^{\infty}(1 - F_2(u))dy = 0$$

Hence for t_2 distribution

$$E(W_3|M_3 = x) = x.$$

We now prove the sufficiency condition.

$$E(W_3|M_3 = x) = x \text{ implies}$$

$$E(X_1|X_{2,3} = x) = x,$$

$$E(W_3|M_3 = x) = x \text{ implies}$$

$$\frac{1}{F(x)}\int_{-\infty}^{x} F(u)du - \frac{1}{1 - F(x)}\int_{x}^{\infty}(1 - F(u))du = 0 \qquad (9.5)$$

The Eq. (9.5) is equivalent to

$$(1-F(x))\int_{-\infty}^{x}F(u)du - F(x)\int_{x}^{\infty}(1-F(u))du = 0$$

or

$$\frac{d}{dx}\left[\int_{x}^{\infty}(1-F(u))du\int_{-\infty}^{x}F(u)du\right] = 0.$$

Thus we have

$$\int_{-\infty}^{x}F(u)du\int_{x}^{\infty}(1-F(u))du = c \tag{9.6}$$

where c is a constant. We can write the Eq. (9.6) as

$$\int_{x}^{\infty}(1-F(u))du = \frac{c}{\int_{-\infty}^{x}c, F(u)du}$$

Differentiating the above equation with respect to x, we obtain

$$1-F(u) = \frac{cF(x)}{\left(\int_{-\infty}^{x}F(u)du\right)^{2}},$$

which is equivalent to

$$\int_{-\infty}^{x}F(u)du = \left(\frac{cF(x)}{(1-F(x))}\right)^{1/2.}$$

Differentiating the above equation with respect to x, we obtain

$$\{F(x)(1-F(x))\}^{3/2} = cf(x), c > 0. \tag{9.7}$$

Let G(x) be the inverse function of F(x) such that F(G(x)) = x, 0 < x<1, we obtain from (9.7)

$$(u(1-u))^{3/2} = cf(G(x)), 0 > x < 1. \tag{9.8}$$

Since $\frac{d}{dx}G(x) = \frac{1}{f(G(x))}$, the general solution of (9.8) is

$$G(x) = d + \int_{1/2}^{x}\frac{du}{(u(1-u))^{3/2}}, \text{c} > 0 \text{ and, } -\infty < \text{d} < \infty.$$

We have $G(0) = \gamma = -\infty$ and $G(1) = \delta = \infty$, hence

$$G(x) = \frac{2x-1}{2\sqrt{x(1-x)}}, 0 < x < 1 \tag{9.9}$$

The inverse of G(x) is

$$F(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right), -\infty < x < \infty. \tag{9.10}$$

So we have

$$\begin{aligned} F(x) &= F_0\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{2}\left\{1 + \frac{x-\mu}{\sqrt{\{\sigma^2 + (x-\mu)^2\}}}\right\}, -\infty < x < \infty, \quad -\infty < \mu < \infty, \sigma > 0. \end{aligned}$$

We now consider t_3 distribution.
The pdf of t_3 distribution is given by

$$f_3(x) = \frac{2}{\pi\sqrt{3}\left(1 + \frac{x^2}{3}\right)^2}, -\infty < x < \infty. \tag{9.11}$$

The following characterization Theorem (Theorem 9.2) is due to Akundov and Nevzorov (2010).

Theorem 9.2.2. Suppose that X_1, X_2 and X_3 are independent and identically distributed with cdf F(x) and pdf f(x). We assume that $E(X_1^2) < \infty$. We assume without any loss of generality E(X) = 0 and $E(X_1^2) = 1$. Let $X_{1,2} \le X_{2,3} \le X_{3,3}$ be the corresponding order statistics. Then the following conditions are equivalent.

$$\begin{aligned} &(a)\ E((X_{2,3} - X_{1,3})^2 | X_{2,3} = x) \\ &\qquad = E((X_{3,3} - X_{2,3})^2 | X_{2,3} = x) \quad a.s. \\ &(b)\ F(x) = F_3\left(\frac{x-\mu}{\varsigma}\right), -\infty < x < \infty, \sigma > 0, \end{aligned} \tag{9.12}$$

where $F_3(.)$ is the t_3 distribution with the pdf given in (9.11).

Proof. We can rewrite (9.12) as

$$\begin{aligned} &2xE(X_{3,3}|X_{2,3} = x) - 2xE(X_{1,3,}|X_{2,3} = x) \\ &\qquad = E(X_{3,3}^2|X_{2,3} = x) - E(X_{1,3}^2|X_{2,3} = x) \end{aligned} \tag{9.13}$$

We know that

$$E(X_{1:3}|X_{2:3} = x) = \frac{1}{F(x)}\int_{-\infty}^{x} t\,dF(t).$$

and

$$E(X_{3:3}|X_{2:3} = \mathrm{x}) = \frac{1}{1-F(x)}\int_x^{\infty} t dF(t).$$

Using these results we obtain from (9.13) that

$$\frac{2x}{1-F(x)}\int_x^{\infty} t dF(t) - \frac{2x}{F(x)}\int_{\infty}^{x} t dF(t)$$
$$= \frac{1}{1-F(x)}\int_x^{\infty} t^2 dF(t) - \frac{1}{F(x)}\int_{\infty}^{x} t^2 dF(t)$$

If we define

$$I(x) = \int_{-\infty}^{x} t dF(t) \text{ and } R(x) = \frac{1}{1-F(x)}\int_{-\infty}^{x} t^2 dF(t).$$

Then we have

$$\int_x^{\infty} t dF(t) = -I(x) \text{ and } \int_x^{\infty} t^2 dF(t) = 1 - R(x)$$

It follows immediately that

$$\int_t^{\infty} t dF(t) = I(x) \text{ and } \int_t^{\infty} t^2 dF(t) = 1 - R(x) \tag{9.14}$$

since E(X) = 0 and $E(\mathrm{X}^2) = 1$. Now we can write

$$2xI(x)\left(\frac{1}{1-F(x)} + \frac{1}{F(x)}\right)$$
$$= \frac{1}{1-F(x)} - R(x)\left(\frac{1}{1-F(x)} + \frac{1}{F(x)}\right)$$

or

$$R(\mathrm{x}) = \mathrm{F(x)} + 2xl(x). \tag{9.15}$$

Differentiating (9.15) with respect to x, we obtain

$$x^2 f(x) = f(x) + 2I(x) + 2x^2 f(x)$$

or

$$-2l(x) = f(x) + x^2 f(x). \tag{9.16}$$

Since the left-hand side of (9.16) is differentiable it follows that f(x) is also differentiable and then differentiating (9.16) with respect to x, we obtain

$$-2xf(x) = f'(x) + 2xf(x) + x^2 f(x)$$

Or

$$\frac{f'(x)}{f(x)} = \frac{-4x}{1+x^2} \tag{9.17}$$

Upon solving the differential equation in (9.17), we arrive at

$$f(x) = \frac{2}{\pi\left(1+x^2\right)^2} \tag{9.18}$$

Note that if X has the probability density function given by (9.18), then the random variable
Y = $\sqrt{3}$ X has the Student's t_3 distribution as given by

$$f_3(x) = \frac{1}{\pi\sqrt{3}}\left(1+\frac{x^2}{3}\right)^{-2}$$

Since we restricted ourselves so far to E(x) = 0 and E(x^2) = 1, it is clear that considering now arbitrarily expected value (μ) and variance (σ^2) we arrive on the result of theorem that (a) $\to$ (b)

Next, looking at the steps above, it can be readily checked that (b) $\to$ (a). The proof of the theorem is complete.

Remark 9.2.1. The characterization result established in this theorem can equivalently be stated in terms of variances of the left-truncated and right-truncated random variables, i.e. the theorem holds if we replace its condition (a) by the condition

$$\begin{aligned}&\text{Vac}(X_{1:3}X|_{2:3} = \text{x}) - \text{Var}(X_{3:3}|X_{2:3} = x)\\&\qquad = E(X_{3:3}X_{1:3}|X_{2:3} = \text{x})[E(X_{3:3} + X_{1:3}|X_{2:3} = \text{x}) - 2x].\end{aligned}$$

Let Q(x) be the quantile function of a random variable X with cdf F(x), i.e. F(Q(x) = x, $0 < x < 1$. Akundov et al. (2004) proved that for $0 < x < 1$, the relation

$$E(\lambda X_{2,3} + (1-\lambda)X_{3,3}|X_{2,3} = x) = x. \tag{9.19}$$

Characterizes a family of distributions with quantile function

$$Q_\lambda(x) = \frac{c(x-\lambda)}{\lambda(1-\lambda)(1-x)^\lambda x^{1-\lambda}} + d, 0 < x < 1, \tag{9.20}$$

where $0 < \text{c} < \infty, -\infty < \text{d} < \infty$. Let us call this family of distribution as Q family.

Theorem 9.2.3. (Q family) Assume that E|X| < ∞ and n ≥ 3 is a positive integer. The random variable X belongs to the Q family if for some k, 2 ≤ k ≤ n − 1 and λ, 0 < λ < 1,

$$\lambda E\left[\frac{1}{k-1}\sum_{i=1}^{k-1}(X_{k,n}-X_{i,n}|X_{k,n}=x\right]$$
$$=(1-\lambda)E\left[\frac{1}{n-k}\sum_{j=k+1}^{n}(X_{j,n}-X_{k,n}|X_{k,n}=x\right] \tag{9.21}$$

Proof. The Eq. (9.21) can be written as

$$\lambda E\left[\frac{1}{k-1}\sum_{i=1}^{k-1}X_{i,n}|X_{k,n}=x\right]$$
$$+(1-\lambda)E\left[\frac{1}{n-k}\sum_{j=k+1}^{n}(X_{j,n}|X_{k,n}=x\right]=x \tag{9.22}$$

Clearly for n = 3 and k = 2 Eq. (9.22) reduces to (9.20).
Q family with different values of λ approximates well member of common distribution including Tukey's Lambda, Cauchy and Gumbel (for maximum). t_2 distribution belongs to the Q family having quantile function (9.20) with

$$Q_{1/2}(x)=\frac{2^{1/2}(x-1/2)}{(x(1-x))^{1/2}},\,0<x<1.$$

A generalization of Theorem 9.1 can easily be established by considering 2n+1 samples and using the condition $E(\overline{X}/M_{2n+1}=x)=x$, where $\overline{X}=\frac{1}{2n+1}\sum_{j=1}^{2n+1}X_j$ and $M_{2n+1}=[X_{1,2n+1}+X_{2n+1,2n+1}]/2$.
The following Theorem 9.2.4 (See Yanev and Ahsanullah (2012)) is a generalization of Theorem 9.2.3.

Theorem 9.2.4. Assume that the random variable X has cdf F(x) with $E(X^2)<\infty$. Let $v \geq 3$ and n ≥ 3 positive integers. Then

$$F(x)=Fv\left(\frac{x-\mu}{\sigma}\right).\text{ for }-\infty<\mu<\infty,\sigma>0. \tag{9.23}$$

where $F_v(.)$ is the t distribution with v degrees of freedom, if and only if

$$E\left[\frac{1}{k-1}\sum_{i=1}^{k}\left(\frac{v-1}{2}X_{k,n}-\left(v-2\right)X_{i,n}\right)\right)^2|X_{k,n}=x\right]$$
$$=E\left[\frac{1}{n-k}\sum_{j=1}^{k}\left(\left(v-2\right)X_{j,n}\right)-\frac{v-1}{2}X_{k,n}\right)^2|X_{k,n}=x\right] \tag{9.24}$$

To prove the theorem, we need the following two lemmas.

Lemma 9.2.1. The cdf F(x) of a random variable X with quantile function (9.20) is the only continuous cdf solution of the equation

$$(F(x))^{\lambda-1}(1-F(x))^{1+\lambda}=cF'(x), c>0. \tag{9.25}$$

Lemma 9.2.2. Let $v \geq 1, n \geq 2$, integers. Then

$$\frac{1}{\mathrm{k}-1}\sum_{\mathrm{i}=1}^{\mathrm{k}-1}\mathrm{E}([\mathrm{X}_{\mathrm{i,n}}^{\mathrm{r}}|\mathrm{X}_{\mathrm{k,n}}=\mathrm{x}]=\frac{1}{\mathrm{F(x)}}\int_{-\infty}^{x}t^r dF(t), \quad 2\leq \mathrm{k}\leq \mathrm{n}, \tag{9.26}$$

and

$$\frac{1}{n-k}\sum_{j=1}^{k-1}E([X_{j,n}^r|X_{k,n}=x)=\frac{1}{1-F(x)}\int_{x}^{\infty}t^r dF(t), 1\leq k\leq n-1. \tag{9.27}$$

Proof. Using the formulas of the conditional density of $X_{j,n}$ given $X_{k,n}=x(j<k)$ (see Ahsanullah and Hamedani (2010), p.13, Ahsanullah and Nevzorov (2001), p.3 and Ahsanullah et al. (2013)), we obtain for r $>$ 1

$$\frac{1}{k-1}\sum_{j=1}^{k-1}E(X_{j,n}^r|X_{k,n}=x$$
$$=\frac{1}{k-1}\frac{k-1}{(F(x))^{k-1}}\sum_{j=1}^{k-1}\binom{k-2}{j-1}\int_{-\infty}^{x}(F(t))^{j-1}(F(x)-F(t))^{k-1-j}t^r dF(t)$$
$$=\frac{1}{[F(s)]^{k-1}}\sum_{i=0}^{k-2}\binom{k-2}{i}\int_{-\infty}^{x}[F(t)]^i[F(x)-F(t)]^{k-2-i}t^r dF(t)$$
$$=\frac{1}{F(x)}\int_{-\infty}^{x}t^r dF(t)$$

This verifies the relation (9.26). The relation (9.27) can be proved similarly.

Proof of Theorem 9.2.4. First we prove (9.21) implies (9.20). Applying Lemma (9.2) for the left hand side of Eq. (9.22), we obtain

$$\lambda \sum_{j=1}^{k-1} \mathrm{E}(\mathrm{X_{j,n}}|\mathrm{X_{k,n}} = \mathrm{x}) + \frac{1-\lambda}{\mathrm{n}-\mathrm{k}} \sum_{j=k+1}^{n} \mathrm{E}([\mathrm{X_{j,n}}|\mathrm{X_{k,n}} = \mathrm{x}].$$

$$= \frac{\lambda}{F(x)} \int_{-\infty}^{x} tF(t) + \frac{1-\lambda}{1-F(x)} \int_{x}^{\infty} tF(t) \tag{9.28}$$

Since that $\mathrm{E}(|\mathrm{X}|) < \infty$, we have

$$\lim_{x \to -\infty} xF(x) = 0 \text{ and } \lim_{x \to \infty} x[1-F(x)] = 0. \tag{9.29}$$

Therefore integrating by parts, we obtain from (9.28)

$$= \frac{\lambda}{F(x)} \int_{-\infty}^{x} tF(t) + \frac{1-\lambda}{1-F(x)} \int_{x}^{\infty} tF(t)$$

$$= x - \frac{\lambda}{F(x)} \int_{-\infty}^{x} F(t)dt + \frac{1-\lambda}{1-F(x)} \int_{x}^{\infty} (-F(t))dt \tag{9.30}$$

Thus, from Eqs. (9.28) and (9.30), it follows from Eq. (9.21) is equivalent to

$$\lambda(1-F(x)) \int_{-\infty}^{x} F(t)dt = (1-\lambda)F(x) \int_{x}^{\infty} (1-F(t))dt$$

The above equation can be written as

$$-\frac{\lambda}{1-\lambda} \int_{-\infty}^{x} F(t)dt \frac{d}{dx} \int_{x}^{\infty} (1-F(t))dt = \int_{x}^{\infty} (1-F(t))dt \frac{d}{dx} \int_{-\infty}^{x} F(t)dt$$

which leads to

$$\int_{-\infty}^{x} F(t)dt = c(\int_{x}^{\infty} (1-F(t))dt)^{-\lambda/(1-\lambda)}, \mathrm{x} > 0$$

Differentiating both sides of the above equation with respect to x, we obtain

$$\int_{x}^{\infty} (1-F(t))dt) = c_1 \left(\frac{1}{F(x)} - 1 \right)^{1-\lambda}, \mathrm{x} > 0$$

Differentiating one more time, we have

$$(F(x))^{2-\lambda}(1-F(x))^{1+\lambda} = c_2 F'(x), c_2 > 0, \tag{9.31}$$

which is Eq. (9.25). Referring to Lemma 9.1, we see that (9.21) implies (9.20).

To complete the proof of the theorem it remains to verify that F(x) with quantile function (9.20) satisfies Eq. (9.21). Differentiating Eq. (9.20) with respect to x, we obtain

$$Q'_\lambda(x) = c(1-x)^{-(1+\lambda)}x^{-(2+\lambda)}, c > 0,$$

On the other hand, since F(Q(x)) = x, we have $Q'(x) = F'(Q(x))^{-1}$.
Note that the left hand side is differentiable so is the right hand side. Therefore

$$(1-x)^{1-\lambda}x^{2-\lambda} = cF'(Q_\lambda(x))$$

which is equivalent to Eq. (9.30) and then to Eq. (9.21).

Proof of Theorem 9.2.4. The Eq. (9.21) can be written as

$$(v-1)x\left\{\frac{1}{n-k}\sum_{j=k+1}^{n}E(X_{j,n}|X_{k,n}-x)-\frac{1}{k-1}\sum_{j=1}^{k-1}E(X_{j.n}|X_{k,n}=x)\right\}$$
$$=(v-2)\left\{\frac{1}{n-k}\sum_{j=k+1}^{n}E(X_{j,n}^2|X_{k.n}-x)-\sum_{j=n}^{k-1}E(X_{j,n}^2|X_{k,n}=x)\right\}$$

Referring to Lemma 9.2, with r = 1 and r = 2, we see that this equivalent to

$$(v-1)x\left\{\frac{1}{F(x)}\int_x^\infty tdF(t)-\frac{1}{F(x)}\int_{-\infty}^x tdF(t)\right\}$$
$$=(v-2)\left\{\frac{1}{1-F(x)}\int_x^\infty t^2dF(t)-\frac{1}{F(x)}\int_{-\infty}^x t^2dF(t)\right\} \tag{9.32}$$

If E(X) = 0 and $E(X^2) = 1$. Then

$$\int_x^\infty dF(t) = -\int_{-x}^\infty tdF(t) \text{ and } \int_x^\infty t^2dF(t) = 1-\int_{-\infty}^x t^2dF(t)$$

and the Eq. (9.32) is equivalent to

$$-(v-1)x\left\{\frac{1}{1-F(x)}+\frac{1}{F(x)}\right\}\int_{-\infty}^x tdF(t)$$
$$=\frac{v-2}{1-F(x)}-(v-2)\left\{\frac{1}{1-F(x)}+\frac{1}{F(x)}\right\}\int_{-\infty}^x t^2dF(t)$$

Multiplying both sides of the above equation by F(x)(1−F(x)), we obtain

$$-(v-1)x\int_{-\infty}^x tdF(t) = (v-2)[F(x)-1]\int_{-\infty}^x t^2dF(t)] \tag{9.33}$$

Differentiating both sides of the above equation, we have

$$-(v-1)\int_{-\infty}^{x} t dF(t) = f(x)(x^2+v-2)$$

Since the left hand side of the above equation is differentiable, we have $f'(x)$ exists. Differentiating with respect to x, we obtain

$$\frac{f'(x)}{f(x)} = -\frac{v+1}{v-2}\frac{x}{1+x^2/(v-2)}$$

Integrating both sides of the equation and making use of the fact that f(x) is a pdf, we have

$$f(x) = c\left(1+\frac{x^2}{v-2}\right)^{-(v-1)/2}, \text{ where c} = \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \tag{9.34}$$

It is not difficult to see that if Z has the pdf as given in (9.34), then

$$X = Z\sqrt{\frac{v}{v-2}}$$

follows t_v distribution with pdf as given in (9.1).
The following theorem (Theorem 9.2.5) gives a characterization of folded t_3 distribution based on truncated first moment.

Theorem 9.2.5. Suppose that the random variable X has an absolutely continuous (with respect to Lebesgue measure) cdf F(x), with $f(x) = \frac{2}{\pi\sqrt{3}}\left(1+\frac{x^2}{3}\right)^{-2}$ (folded t_3 distribution) if and only if $E(X|X \le x) = g(x)\tau(x)$, where $g(x) = \frac{x^2(x^2+3)}{6}$ and $\tau(x) = \frac{f(x)}{F(x)}$ is the reversed hazard rate.

Proof. Suppose that $f(x) = \frac{2}{\pi\sqrt{3}}\left(1+\frac{x^2}{3}\right)^{-2}$, x $\ge$ 0, then

$$\text{g(x)} = \frac{\int_0^x x\left(1+\frac{t^2}{3}\right)dt}{\left(1+\frac{x^2}{3}\right)^{-2}} = \frac{1}{6}x^2(x^2+3)$$

Suppose g(x) $= \frac{1}{6}x^2(x^2+3)$,
Then $E(X|X \le x) = g(x)\tau(x)$ implies that

$$\int_0^x tf(t)dt = g(x)f(x) \tag{9.35}$$

Differentiating (9.35) and using g(x) $= \frac{1}{6}x^2(x^2+3)$, we obtain

$$x\mathrm{f}(\mathrm{x}) = \frac{1}{6}x^2(x^2+3)\mathrm{f}'(\mathrm{x}) + \frac{1}{6}x(4x^2+6)f(x) \tag{9.36}$$

On simplification we obtain from (9.36)

$$\frac{f'(x)}{f(x)} = -\frac{4x}{x^2+3} \tag{9.37}$$

On integrating (9.37) and using the fact f(x) is a pdf, we have

$$\mathrm{f}(\mathrm{x}) = \frac{2}{\pi\sqrt{3}}(1+\frac{x^2}{3})^{-2}, \mathrm{x} \geq 0,$$

9.3 Summary

In this chapter, some characterizations of Student's t distribution have been discussed. It is hoped that the findings of this chapter will be useful for the practitioners in various fields of studies and further enhancement of research in the field of distribution theory and its applications.

Chapter 10
Concluding Remarks and Some Future Research

The normal and Student's t distributions are two of the most important distributions in statistics. This book has reviewed the normal and Student's t distributions, and their applications. The sum, product and ratio for the normal distributions, and the sum, product and ratio for the Student's t distributions have been discussed extensively. Their properties and possible applications are discussed. Some special cases for each of the chapters are given. The distributions of the sum, product, and ratio of independent random variables belonging to different families are also of considerable importance and one of the current areas of research interest. This book introduces and develops some new results on the distributions of the sum of the normal and Student's t random variables. Some properties are discussed. Further, a new symmetric distribution has been derived by taking the product of the probability density functions of the normal and Student's t distributions. It is observed that the new distribution is symmetric and carries most of the properties of symmetric distributions. Some characteristics of the new distributions are presented. The entropy expression has been given. The percentage points have also been provided. It is shown that the pdf of the proposed distribution is unimodal. It is noted that for the degrees of freedom $v = 1$, we obtain a new symmetric distribution which is the product of the standard normal and Cauchy distributions and for large v we will obtain a distribution which is the product of two standard normal densities. The percentage points of the new distributions have also been provided. The characterizations of normal and Student's t are given. We hope the findings of the book will be useful for the practitioners in various fields of sciences. Finally, the given references of the book will be a valuable asset for those researchers who want to do research in these areas.

The purpose of this book was to provide the distribution of the sums, differences, products and ratios of independent (uncorrelated) normal and student t random variables. Most of the recent works are reviewed. It appears that not much attention has been paid to the distribution of the sums, differences, products and ratios of dependent (correlated) student t random variables, and therefore needs further research investigation. It is also evident that very little attention has been paid to the estimates of parameters, inferential and prediction properties based the distribution of the sums, differences, products and ratios of dependent (correlated) normal and

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student's t random variables. Also, one can investigate the distribution of order statistics and record values based on the distribution of the sums, differences, products and ratios of dependent (correlated) normal and student's t random variables. The inferential properties and prediction of future order statistics and record values based on existing ones from the distribution of order statistics and record values based on the distribution of the sums, differences, products and ratios of dependent (correlated) normal and student t random variables are also open problems. We hope that the materials of this book will be useful for the practitioners in various fields of studies and further enhancement of research on the distribution of the sums, differences, products and ratios (quotients) of random variables and their applications.

Finally, as stated above, this book primarily provided the theoretical contributions of normal and student's t distributions and their possible applications. However, it appears that not much attention has been paid to the estimates of parameters and inferential properties based on the distribution of the sums, differences, products and ratios of dependent (correlated) normal and student's t random variables. Therefore, we hope that, using a real world data, one can pursue further research based on the results provided in this book, specially, parameter estimates, inferences about the parameters, goodness-of-fit, and prediction of the future observations for these distributions are open problems.

References

Abramowitz, M., & Stegun, I. A. (1970). *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*. New York: Dover.

Abu-Salih, M. S. (1983). Distributions of the product and the quotient of power-function random variables. *Arab Journal of Mathematics*, *4*, 77–90.

Agrawal, M. K., & Elmaghraby, S. K. (2001). On computing the distribution function of the sum of independent random variables. *Computers and Operations Research*, *28*, 473–483.

Ahsanullah, M. (1987). A note on the characterization of the normal distribution. *Biometrical Journal*, *29*(7), 885–888.

Ahsanullah, M. (1989a). Characterizations of the normal distribution by properties of linear functions and chi-squared. *Pakistan Journal of Statistics*, *5*(3), 13, 267–276.

Ahsanullah, M. (1989b). On characterizations of the normal law. *Bulletin of the Institute of Mathematics Academia Sinica*, *17*(2), 229–232.

Ahsanullah, M. (1990). Some characterizations properties of normal distribution. *Computational Statistics and Data Analysis*, *10*, 117–120.

Ahsanullah, M. (2011). *Handbook of univariate statistical distributions*. MI, USA: Trafford Publishers.

Ahsanullah, M., Bansal, N., & Hamedani, G. G. (1991). On characterization of normal distribution. *Calcutta Statistical Assciation Bulletin*, *41*(161–164), 157–162.

Ahsanullah, M., & Hamedani, G. G. (1988). Some characterizations of normal distribution. *Calcutta Statistical Association Bulletin*, *37*, 145–146.

Ahsanullah, M., & Hamedani, G. G. (2010). *Exponential distribution*. New York: Nova Science Publishers.

Ahsanulllah, M., & Kirmani, S. N. U. A. (2008). *Topics in extreme value distributions*. New York, USA: Nova Science Publishers.

Ahsanullah, M., & Nevzorov, M. (2001). *Ordered random variables*. New York: Nova Science Publishers.

Ahsanullah, M., & Nevzorov, V. B. (2005). *Order statistics :Examples and exercises*. New York: Nova Publishers.

Ahsanullah, M., Nevzorov, V. B., & Shakil, M. (2013). *An introduction to order statistics*. Paris, France: Atlantis Press.

Aigner, D. J., Lovell, C. A. K., & Schmidt, P. (1977). Formulation and estimation of stochastic frontier production function models. *Journal of Econometrics*, *6*, 21–37.

Airy, G. B. (1861). *On the algebraical and numerical theory of errors of observations and the combination of observations*. London: Macmillan.

M. Ahsanullah et al., *Normal and Student's t Distributions and Their Applications*, Atlantis Studies in Probability and Statistics 4, DOI: 10.2991/978-94-6239-061-4,

Akundov, I. S., Balakrishnan, N., & Nevzorov, V. B. (2004). New characterizations by properties of midrange and related statistics. *Communications in Statistics-Theory and Methods*, *33*(12), 3133–3143.

Akundov, I. S., & Nevzorov, V. B. (2010). A simple characterization of Student t_2 distribution. *Statistics and Probability Letters*, *80*(5–6), 293–295.

Ali, M. M. (1982). Distribution of linear combinations of exponential variates. *Communications in Statistics-Theory and Methods*, *11*, 1453–1463.

Ali, M. M., & Nadarajah, S. (2004). On the product and the ratio of *t* and logistic random variables. *Calcutta Statistical Association Bulletin*, *55*, 1–14.

Ali, M. M., & Nadarajah, S. (2005). On the product and ratio of *t* and Laplace random variables. *Pakistan Journal of Statistics*, *21*(1), 1–14.

Altman, D. G. (1993). Construction of age-related reference centiles using absolute residuals. *Statistics in Medicine*, *12*(10), 917–924.

Amari, S. V., & Misra, R. B. (1997). Closed-form expressions for distribution of sum of exponential random variables. *IEEE Transactions on Reliability*, *46*, 519–522.

Amoroso, L. (1925). Richerche intorno alla curve die redditi. *Annali di Matematica Pura ed Applicata*, *21*, 123–159.

Andersen, P. K., Borgan, O., Gill, R. D., & Keiding, N. (1993). *Statistical models based on counting processes*. New York: Springer.

Arellano-Valle, R. B., & Azzalini, A. (2006). On the unification of families of skew-normal distributions. *Scandinavian Journal of Statistics*, *33*, 561–574.

Arellano-Valle, R. B., Del Pino, G., & San Martin, E. (2002). Definition and probabilistic properties of skew-distributions. *Statistics and Probability Letters*, *58*(2), 111–121.

Arellano-Valle, R. B., Gómez, H. W., & Quintana, F. A. (2004). A new class of skew-normal distributions. *Communications in Statistics-Theory and Methods*, *33*(7), 1465–1480.

Arnold, B. C. (1979). Some characterizations of the Cauchy distribution. *Australian Journal of Statistics*, *21*, 166–169.

Arnold, B. C., & Lin, G. D. (2004). Characterizations of the skew-normal and generalized chi distributions. *Sankhyā: The Indian Journal of Statistics*, *66*(4), 593–606.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, *12*, 171–178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. *Statistica*, *46*, 199–208.

Azzalini, A. (2001). A note on regions of given probability of the skew-normal distribution. *Metron*, *LIX*, 27–34.

Azzalini, A. (2005). The skew-normal distribution and related multivariate families. *Scandinavian Journal of Statistics*, *32*, 159–188.

Azzalini, A. (2006). Some recent developments in the theory of distributions and their applications. Atti della XLIII Riunione della SocietÃ Italiana di Statistica, volume Riunioni plenarie e specializzate, pp. 51–64.

Azzalini, A., & Capitanio, A. (1999). Statistical applications of the multivariate skew-normal distribution. *Journal of Royal Statistical Society B*, *61*, 579–602.

Azzalini, A., & Dalla_Valle, A. (1996). The multivariate skew-normal distribution. *Biometrika*, *83*, 715–726.

Azzalini, A., & Regoli, G. (2012). Some properties of skew-symmetric distributions. *Annals of the Institute of Statistical Mathematics*, *64*, 857–879.

Babbitt, G. A., Kiltie, R., & Bolker, B. (2006). Are fluctuating asymmetry studies adequately sampled? Implications of a new model for size distribution. *The American Naturalist*, *167*(2), 230–245.

Bagui, S., & Bagui, S. (2006). Computing percentiles of Skew normal distributions. *Journal of Modern Applied Statistical Methods*, *5*(2), 283–595.

Balakrishnan, N., & Nevzorov, V. B. (2003). *A primer on statistical distributions*. New Jersey: Wiley.

Barndorff-Nielsen, O. (1977). Exponentially decreasing log-size distributions. *Proceedings of the Royal Society of London Series A*, *353*, 401–419.

Barndorff-Nielsen, O. (1978). Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of Statistics*, *5*, 151–159.

Barranco-Chamorro, I., Moreno-Rebollo, J. L., Pascual-Acosta, A., & Enguix-Gonzalez, A. (2007). An overview of asymptotic properties of estimators in truncated distributions. *Communications in Statistics-Theory and Methods*, *36*(13), 2351–2366.

Basu, D. (1951). On the independence of linear functions of independent chance variables. *Bulletin of the International Statistics Institute*, *39*, 83–96.

Basu, A. K., & Ahsanullah, M. (1983). On decomposition of normal and Poisson law. *Calcutta Statistical Association Bulletin*, *32*, 103–109.

Basu, A. P., & Lochner, R. H. (1971). On the distribution of the having generalized life distributions. *Technometrics*, *13*, 281–287.

Behrerens, W. V. (1929). Ein Beitrag zur Fehlerberechnung bei wenigen Beobachtungen. *Landw Jb*, *68*, 807–837.

Bessel, F. W. (1818). Astronomiae pro anno MDCCLV deducta ex observationibus viri incomporabilis James Bradley specula Grenovicensi per annos 1750–1762.

Bessel, F. W. (1838). Untersuchungen uber der Wahrscheinlichkeit der Beobachtungsfehler. *Astronomische Nachrichten*, *15*, 368–404.

Bhargava, R. P., & Khatri, C. G. (1981). The distribution of product of independent beta random variables with application to multivariate analysis. *Annals of the Institute of Statistical Mathematics*, *33*, 287–296.

Birnbaum, Z. W. (1950). Effect of linear tuncation on a multinormal population. *The Annals of Mathematical Statistics*, *21*, 272–279.

Bland, J. M. (2005). The Half-Normal distribution method for measurement error: two case studies. Unpublished talk available on http://www-users.york.ac.uk/~mb55/talks/halfnor.pdf

Bland, J. M., & Altman, D. G. (1999). Measuring agreement in method comparison studies. *Statistical Methods in Medical Research*, *8*, 135–160.

Blatberg, R. C., & Gonedes, N. J. (1974). A comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business*, *47*, 224–280.

Blischke, W. R., & Murthy, D. N. P. (2000). *Reliability, modeling, prediction, and optimization*. New York: Wiley.

Bose, R. C., & Roy, S. N. (1938). The distribution of the D^2 studentized-statistic. *Sankhya*, *4*, 19–38.

Box, J. F. (1981). Gosset, Fisher and *t* distribution. *The American Statistician*, *35*, 61–67.

Bravais, A. (1846). *Analyse mathematique sur les probabilities des erreurs de situation d'un point, Memores des Savans Etrangers* (Vol. 9, pp. 255–332). Paris: Academie (Royale) des Sciences del'Institut de France.

Brody, J. P., Williams, B. A., Wold, B. J., & Quake, S. R. (2002). Significance and statistical errors in the analysis of DNA microarray data. *Proceedings of the National Academy of Sciences USA*, *99*(20), 12975–12978.

Buccianti, A. (2005). Meaning of the parameter of skew-normal and log-skew normal distributions in fluid geochemistry. *CODAWORK'05*, *Girona*, *Spain* October 19–21, 2005.

Buckland, S. T., Anderson, D. R., Burnham, K. P., & Laake, J. L. (1993). *Distance sampling: Estimating abundance of biological populations*. London: Chapman and Hall.

Cambanis, S., Huang, S., & Simons, G. (1981). On the theory of elliptically contoured distributions. *Journal of Multivariate Analysis*, *11*, 368–385.

Casella, G., & Berger, R. L. (2002). *Statistical inference*. California: Thomson Learning.

Cedilnik, A., Kosmelj, K., & Blejee, A. (2004). The distribution of the ratio of jointly normal variables. *Metodoloskizvezki*, *1*(1), 99–108.

Chakraborty, S., & Hazarika, P. J. (2011). A survey on the theoretical developments in univariate skew normal distributions. *Assam Statistical Review*, *25*(1), 41–63.

Chapman, D. G. (1950). Some Two Sample Tests. *Annals of Mathematical Statistics*, *21*, 601–606.

Chapman, D. G. (1956). Estimating the parameters of a truncated Gamma distribution. *The Annals of Mathematical Statistics*, *27*(2), 498–506.
Charlier, C. V. L. (1905). Uber die Darstellung willkurlicher Funktionen. *Arkiv for Matematik, Astronomi och Fysik*, *2*(20), 1–35.
Chen, Y. Y., & Wang, H. J. (2004). A method of moments estimator for a stochastic frontier model with errors in variables. *Economics Letters*, *85*(2), 221–228.
Chiogna, M. (1998). Some results on the scalar skew-normal distribution. *Journal of the Italian Statistical Society*, *7*(1), 1–13.
Chou, C. Y., & Liu, H. R. (1998). Properties of the half-normal distribution and its application to quality control. *Journal of Industrial Technology*, *14*(3), 4–7.
Cigizoglu, H. K., & Bayazit, M. (2000). A generalized seasonal model for flow duration curve. *Hydrological Processes*, *14*, 1053–1067.
Coffey, C. S., Kairalla, J. A., & Muller, K. E. (2007). Practical methods for bounding type I error rate with an internal pilot design. *Communications in Statistics-Theory and Methods*, *36*(11), 2143–2157.
Conover, W. J. (1999). *Practical nonparametric statistics*. New York: Wiley.
Cohen, A. C. (1959). Simplified estimators for the normal distribution when samples are singly censored or truncated. *Technometrics*, *1*, 217–237.
Cohen, A. (1991). *Truncated and censored samples*. New York: Marcel Dekker.
Cooray, K., & Ananda, M. M. (2008). A generalization of the half-normal distribution with applications to lifetime data. *Communications in Statistics-Theory and Methods*, *37*(9), 1323–1337.
Copson, E. T. (1957). *An introduction to the theory of functions of a complex variable*. Oxford, U.K.: Oxford University Press.
Cramer, H. (1936). Uber cine eigensehaft der mornalen verteilungs function. *Mathematische Zeitschriff Chribt*, *11*, 405–411.
Curtiss, J. H. (1941). On the distribution of the quotient of two chance variables. *The Annals of Mathematical Statistics*, *12*(4), 409–421.
D'Agostino, R. B. (1971). An omnibus test of normality for moderate and large size samples. *Biometrika*, *58*, 341–348.
Dalla Valle, A. (2004). "The skew-normal distribution". In M. G. Genton (Ed.), *Skew-elliptical distributions and their applications: A journey beyond normality* (pp. 3–24). Boca Raton, FL: Chapman & Hall.
Daniel, C. (1959). Use of half-normal plots in interpreting factorial two-level experiments. *Technometrics*, *1*(4), 311–341.
Darmois D. (1951). Sur diverse propriete caracteristique de la loi de probabilite de Laplace- Gauss. *Bulletin of the Institute of Economics and Statistics Institute*, *13*, 79–82.
David, H. A., & Nagaraja, H. N. (2003). *Order statistics* (3rd ed.). Hoboken: Wiley.
Dobzhansky, T., & Wright, S. (1947). Genetics of natural populations. XV. Rate of diffusion of a mutant gene through a population of *Drosophila pseudoobscura*. *Genetics*, *32*(3), 303.
Donahue, J. D. (1964). *Products and quotients of random variables and their applications*. Denver, Colorado: The Martin Company.
Dreiera, I., & Kotz S. (2002). A note on the characteristic function of the *t*-distribution. *Statistics and Probability Letters*, *57*, 221–224.
Dudewicz, E. J., & Mishra, S. N. (1988). *Modern mathematical statistics*. New York: Wiley.
Edgeworth, F. Y. (1892). Correlated averages. *Philosophical Magazine Series*, *5*(34), 190–204.
Edgeworth, F. Y. (1883). The law of error. *Philosophical Magazine*, *16*, 300–309.
Edgeworth, F. Y. (1905). The law of error. *Proceedings of the Cambridge Philosophical Society*, *20*, 36–65.
Eisenhart, C. (1979). On the transition from "Student's" z to "Student's" t. *The American Statistician*, *33*, 6–11.
Elandt, R. C. (1961). The folded normal distribution: Two methods of estimating parameters from moments. *Technometrics*, *3*(4), 551–562.

Eling, M. (2011). Fitting insurance claims to skewed distributions: Are the Skew-normal and Skew-student good models? Working papers on risk management and insurance, No. 98-November, 2011, Institute of Insurance Economics, University of St. Gallen, St. Gallen, Switzerland.

Embrechts, P., Klüppelberg, C., & Mikosch, T. (1997). *Modelling extremal events: For insurance and finance*. Berlin: Springer.

Epstein, B. (1948). Some applications of the Mellin transform in statistics. *Annals of Mathematics Statistics*, *19*, 370–379.

Evans, M., Hastings, N., & Peacock, B. (2000). *Statistical distributions* (3rd ed.). New York: Wiley.

Fang, K. T., & Anderson, T. W. (1990). *Statistical inference in elliptically contoured and related distributions*. New York: Allerton Press.

Farebrother, R. W. (1984). Algorithm AS 204: The distribution of a positive linear combination of chi-squared random variables. *Applied Statistics*, *33*, 332–339.

Feller, W. (1968). *An introduction to probability theory and applications* (Vol. 1). New York: Wiley.

Feller, W. (1971). *An introduction to probability theory and applications* (Vol. 2). New York: Wiley.

Fernandes, E., Pacheco, A., & Penha-Gonçalves, C. (2007). Mapping of quantitative trait loci using the skew-normal distribution. *Journal of Zhejiang University Science B*, *8*(11), 792–801.

Fieller, E. C. (1932). The distribution of the index in a normal bivariate population. *Biometrika*, *24*(3/4), 428–440.

Fisher, R. A. (1925). Applications of student's t distribution. *Metron*, *5*, 90–104.

Fisher, R. A. (1930). The moments of the distribution for normal samples of measures of departure from normality. *Proceedings of the Royal Society of London*, *A130*, 16–28.

Fisher, R. A. (1931). The truncated normal distribution. British Association for the Advancement of Science. *Mathematical Tables*, *5*, xxxiii–xxxiv.

Fisher, R. A. (1935). The fiducial argument in statistical inference. *Annals of Eugenics*, *6*, 391–398.

Fisher, R. A. (1939). The comparison of samples with possibly unequal variances. *Annals of Eugenics*, *9*, 174–180.

Fisher, R. A. (1941). The asymptotic approach to Behrens' integral, with further tables for the d test of significance. *Annals of Eugenics*, *11*, 141–173.

Fisher, R. A., & Healy, M. J. R. (1956). New tables of Behrens' test of significance. *Journal of the Royal Statistical Society Series B*, *18*(2), 212–216.

Fisher, R. A., & Yates, F. (1957). *Statistical tables for biological agricultural and medical research*. Edinburgh: Oliver and Boyd.

Folks, J. L., & Chhikara, R. S. (1978). The inverse Gaussian distribution and its statistical application-a review. *Journal of the Royal Statistical Society Series B (Methodological)*, *40*, 263–289.

Fotowicz, P. (2006). An analytical method for calculating a coverage interval. *Metrologia*, *43*, 42–45.

Frisch, U., & Sornette, D. (1997). Extreme deviations and applications. *Journal of Physics I France*, *7*, 1155–1171.

Galambos, J., & Simonelli, I. (2005). *Products of random variables—applications to problems of physics and to arithmetical functions*. Boca Raton, Atlanta: CRC Press.

Galton, F. (1875). Statistics by inter comparison, with remarks on the law of frequency of error. *Philosophical Magazine*, *49*, 33–46.

Galton, F. (1889). *Natural inheritance*. London: Macmillan.

Gauss, C. F. (1809). Theoria Motus Corporum Coelestium, Lib. 2, Sec. III, 205–224. Hamburg.: Perthes u. Besser.

Geary, R. C. (1930). The frequency distribution of the quotient of two normal variates. *Journal of the Royal Statistical Society*, *93*(3), 442–446.

Geary, R. C. (1934). Distribution of student's ratio for non normal distribution. *Journal of Royal Statistitcs Society Series B*, *3*, 178–184.

Genton, M. G. (Ed.). (2004). *Skew-elliptical distributions and their applications: A journey beyond normality*. Boca Raton, FL: Chapman & Hall.

Ghosh, B. K. (1975). On the distribution of the difference of two t-Variables. *Journal of the American Statistical Association*, *70*(350), 463–467.

Glen, A. G., Leemis, L. M., & Drew, J. H. (2004). Computing the distribution of the product of two continuous random variables. *Computational Statistics and Data Analysis*, *44*(3), 451–464.

Goldar, B., & Misra, S. (2001). Valuation of environmental goods: Correcting for bias in contingent valuation studies based on willingness-to-accept. *American Journal of Agricultural Economics*, *83*(1), 150–156.

Good, I. J. (1953). The population frequencies of species and the estimation of population parameters. *Biometrika*, *40*, 237–260.

Gosset, W. S. ("Student") (1908). The probable error of a mean. *Biometrika*, *6*, 1–25.

Gradshteyn, I. S., & Ryzhik, I. M. (2000). *Table of integrals, series, and products* (6th ed.). San Diego: Academic Press.

Greene, W. H. (2005). Censored data and truncated distributions. NYU Working Paper No. 2451/26101, Available at SSRN (Social Science Research Network) 825845.

Grubel, H. G. (1968). Internationally diversified portfolios: Welfare gains capital flows. *American Economic Review*, *58*, 1299–1314.

Gupta, R. C., & Gupta, R. D. (2004). Generalized skew normal model. *Test*, *13*(2), 501–524.

Gupta, A. K., Chang, F. C., & Huang, W. J. (2002). Some skew-symmetric models. *Random Operators Stochastic Equations*, *10*, 133–140.

Gupta, A. K., & Chen, J. T. (2004). A class of multivariate skew-normal models. *Annals of the Institute of Statistical Mathematics*, *56*(2), 305–315.

Gupta, S. C., & Kapoor, V. K. (2002). *Fundamentals of mathematical statistics*. New Delhi: Sultan Chand & Sons.

Gupta, A. K., Nguyen, T. T., & Sanqui, J. A. T. (2004). Characterization of the skew-normal distribution. *Annals of the Institute of Statistical Mathematics*, *56*(2), 351–360.

Haberle, J. G. (1991). Strength and failure mechanisms of unidirectional carbon fibre-reinforced plastics under axial compression, Unpublished Ph.D. thesis. London, U.K.: Imperial College.

Hald, A. (1952). *Statistical theory with engineering applications*. New York: Wiley.

Harter, H. L. (1951). On the distribution of Wald's classification statistic. *Annals of Mathematical Statistics*, *22*, 58–67.

Hausman, J. A., & Wise, D. A. (1977). Social experimentation, truncated distributions, and efficient estimation. *Econometrica: Journal of the Econometric Society*, *45*(4), 919–938.

Hawkins, D. I., & Han, C. P. (1986). Bivariate distributions noncentral chi-square random variables. *Communications in Statistics-Theory and Methods*, *15*, 261–277.

Hayya, J., Armstrong, D., & Gressis, N. (1975). A note on the ratio of two normally distributed variables. *Management Science*, *21*(11), 1338–1341.

Helmert, F. R. (1876). Die Genauigkeit der Formel von Peters zur Berechnung des wahrscheinlichen Beobachtungsfehlers director Beobachtungen tungen gleicher Genauigkeit. *Astronomische Nachrichten*, *88*, 113–120.

Henze, N. (1986). A probabilistic representation of the 'skew-normal' distribution. *Scandinavian Journal of Statistics*, *13*, 271–275.

Hinkley, D. V. (1969). On the ratio of two correlated normal random variables. *Biometrika*, *56*(3), 635–639.

Hitezenko, P. (1998). A note on a distribution of weighted sums of iid Rayleigh random variables. *Sankhya, A*, *60*, 171–175.

Hogg, R. V., McKean, J., & Craig, A. T. (2005). *Introduction to mathematical statistics*. New Jersey: Pearson Prentice Hall.

Hogg, R. V., & Tanis, E. A. (2006). *Probability and statistical inference*. New Jersey: Pearson Prentice Hall.

Holton, G. A. (2003). *Value-at-risk: Theory and practice*. San Diego, CA: Academic Press.

Hu, C.-Y., & Lin, G. D. (2001). An inequality for the weighted sums of pairwise i.i.d. generalized Rayleigh random variables. *Journal of Statistical Planning and Inference*, *92*, 1–5.

Hurst, S. (1995). The characteristic function of the Student t-distribution. Financial Mathematics Research Report 006–95. Australia: Australian National University, Canberra ACT 0200.

Ifram, A. F. (1970). On the characteristic functions of the F and t-distributions. *Sankhya*, *A32*, 350–352.

Jamalizadeh, A., Behboodian, J., & Balakrishnan, N. (2008). A two-parameter generalized skew-normal distribution. *Statistics and Probability Letters*, *78*(13), 1722–1726.

James, G. S. (1959). The Behrens-Fisher distribution and weighted means. *Journal of the Royal Statistics Society B*, *21*, 73–90.

Jawitz, J. W. (2004). Moments of truncated continuous univariate distributions. *Advances in Water Resources*, *27*, 269–281.

Joarder, A. H. (1998). Some useful Wishart expectations based on the multivariate t-model. *Statistical Papers*, *39*(2), 223–229.

Joarder, A. H., & Ali, M. M. (1996). On the characteristic function of the multivariate t distribution. *Pakistan Journal of Statistics*, *12*, 55–62.

Joarder, A. H., & Ali, M. M. (1997). Estimation of the scale matrix of a multivariate t-model under entropy loss. *Metrika*, *46*(1), 21–32.

Joarder, A. H., & Singh, S. (1997). Estimation of the trace of the scale matrix of a multivariate t-model using regression type estimator. *Statistics*, *29*, 161–168.

Johnson, A. C. (2001). *On the truncated normal distribution: characteristics of singly- and doubly-truncated populations of application in management science, PhD. Dissertation*. Illinois.: Stuart Graduate School of Business, Illinois Institute of Technology.

Johnson, N. L., Kotz, L., & Balakrishnian, N. (1994). *Continuous univariate distributions* (2nd ed., Vol. 1). New York: Wiley.

Johnson, N. L., Kotz, L., & Balakrishnan, N. (1995). *Continuous univariate distributions* (2nd ed., Vol. 2). New York: Wiley.

Johnstone, I., & Silverman, B. (2004). Needles and hay in haystacks: Empirical Bayes estimates of possibly sparse sequences. *Annals of Statistics*, *32*, 1594–1649.

Johnstone, I., & Silverman, B. (2005). Empirical Bayes selection of wavelet thresholds. *Annals of Statistics*, *33*, 1700–1752.

Jones, D. S. (1979). *Elementary information theory*. Oxford: Clarendon Press.

Jørgensen, B. (1982). *Statistical properties of the generalized inverse gaussian distribution*. New York: Springer.

Kagan, A. M., Linnik, Yu V, & Rao, C. R. (1973). *Characterization problems in mathematical statistics*. New York: Wiley.

Kagan, A. M., & Zinger, A. R. (1971). Sample mean as an alternative of location parameter: Case of non quadratic loss function. *Sankhya Series A*, *33*, 351–358.

Kamgar-Parsi, B., & Brosh, M. (1995). Distribution and moments of weighted sum of uniform random variables with applications in reducing Monte Carlo simulations. *Journal of Statistical Computation and Simulation*, *52*, 399–414.

Kamerud, D. (1978). The random variable X/Y, X, Y normal. *The American Monthly*, *85*(3), 206–207.

Kapadia, A. S., Chan, W., & Moyé, L. A. (2005). *Mathematical statistics with applications* (Vol. 176). New York.: Chapman & Hall.

Kapur, J. N. (1993). *Maximum-entropy models in science and engineering*. New Delhi: Wiley-Eastern.

Kappenman, R. F. (1971). A note on the multivariate t ratio distribution. *Annals of Mathematical Statistics*, *42*, 349–351.

Kazemi, M. R., Haghbin, H., & Behboodian, J. (2011). Another generalization of the Skew normal distribution. *World Applied Sciences Journal*, *12*(7), 1034–1039.

Keceioglu, D. (1991). *Reliability engineering handbook* (Vol. 1). Englewood Cliffs, NJ:Prentice Hall.

Kelejian, H. H., & Prucha, I. R. (1985). Independent or uncorrelated disturbances in linear regression. *Economic Letters*, *19*, 35–38.

Kelker, D. (1970). Distribution theory of spherical distributions and locations scale parameters. *Sankhya, A*, *32*, 419–430.
Kendall, D. (1938). Effect of radiation damping and doppler broadening on the atomic absorption coefficient. *Z. Astrophysics*, *16*, 308–317.
Kendall, M. G. (1952). *The advanced theory of statistics* (5th ed., Vol. 1). London: Charles Griffin and Company.
Kendall, M. G., & Stuart, A. (1958). *The advanced theory of statistics, 1*. New York: Hafner Pub. Co.
Kibria, B. M. G. (1996). On shrinkage ridge regression estimators for restricted linear models with multivariate t disturbances. *Students*, *1*(3), 177–188.
Kibria, B. M. G., & Haq, M. S. (1998). Marginal likelihood and prediction for the multivariate ARMA(1,1) linear models with multivariate t error distribution. *Journal of Statistical Research.*, *32*(1), 71–80.
Kibria, B. M. G., & Haq, M. S. (1999). The multivariate linear model with multivariate t and intra-class covariance structure. *Statistical Paper*, *40*, 263–276.
Kibria, B. M. G., & Nadarajah, S. (2007). Reliability modeling: Linear combination and ratio of exponential and Rayleigh. *IEEE Transactions on Reliability*, *56*(1), 102–105.
Kibria, B. M. G., Saleh, A. K. Md. E. (2003). Preliminary test ridge regression estimators with student's t errors and conflicting test-statistics. *Metrika*, *59*, 105–124.
Kim, H. J. (2006). On the ratio of two folded normal distributions. *Communications in Statistics-Theory and Methods*, *35*(6), 965–977.
Kimber, A. C., & Jeynes, C. (1987). An application of the truncated two-piece normal distribution to the measurement of depths of arsenic implants in silicon. *Applied statistics*, 352–357.
Klugman, S., Panjer, H., & Willmot, G. (1998). *Loss models: From data to decisions*. New York: Wiley.
Korhonen, P. J., & Narula, S. C. (1989). The probability distribution of the ratio of the absolute values of two normal variables. *Journal of Statistical Computation and Simulation*, *33*, 173–182.
Kotlarski, I. (1960). On random variables whose quotient follows the Cauchy law. *Colloquium Mathematicum*, *7*, 277–284.
Kotz, S., & Nadarajah, S. (2004). *Multivariate t Distributions and their Applications*. New York: Cambridge University Press.
Krzanowski, W. J., & Marriot, F. H. C. (1994). *Multivariate Analysis*. London: Arnold.
Ladekarl, M., Jensen, V., & Nielsen, B. (1997). Total number of cancer cell nuclei and mitoses in breast tumors estimated by the optical disector. *Analytical and Quantitative Cytology and Histology*, *19*, 329–337.
Laha, R. G. (1957). On a characterization of the normal distribution from properties of suitable linear statistics. *Ann. Math. Statistics*, *28*, 126–139.
Laplace, P. S. (1812). *Theorie Analytique des Probabilites* (Vol. 7). Paris: Oeuvres Completes.
Larson, R. J., & Marx, M. L. (2006). *An introduction to mathematical statistics and its applications*. New Jersey: Pearson Prentice Hall.
Lawless, J. F. (2003). *Statistical models and methods for lifetime data* (2nd ed.). New York: Wiley.
Lazo, A. C. G. V., & Rathie, P. N. (1978). On the entropy of continuous probability distributions. *IEEE Transactions on Information Theory*, *IT–24*, 120–122.
Lebedev, N. N. (1972). *Special functions and their applications*. New York: Dover.
Lee, R. Y., Holland, B. S., & Flueck, J. A. (1979). Distribution of a ratio of correlated gamma random variables. *SIAM Journal on Applied Mathematics*, *36*, 304–320.
Legendre, A. M. (1805). Nouvelles Methodes pour la Determination des Orbites des Cometes, Paris.
Leone, F. C., Nelson, L. S., & Nottingham, R. B. (1961). The folded normal distribution. *Technometrics*, *3*(4), 543–550.
Levy, H. (1982). Stochastic dominance rules for truncated normal distributions: A note. *The Journal of Finance*, *37*(5), 1299–1303.
Linnik, Yu V. (1964). *Decomposition of probability distributions*. London: Oliver & Boyd.

Lomnicki, Z. A. (1967). On the distribution of products of random variables. *Journal of the Royal Statistical Society. Series B*, *29*(3), 513–524.

Lukas, E. (1942). A characterization of the normal distribution. *Annals of Mathematical Statistics*, *13*, 91–93.

Lukacs, E. (1972). *Probability and mathematical statistical statistics—an introduction*. New York: Academic Press.

Lukacs, E., & Laha, R. G. (1964). *Applications of characteristic functions*. New York: Hefner Publishing Company.

Lyapunov, A. M. (1901). Nouvelle formed u theoreme sur la limite de probabilite, Memores de l'Academie Imperiale des Sciences de St. *Petersbourg*, *12*, 1–24.

Maksay, S., & Stoica, D. M. (2006). Considerations on modeling some distribution laws. *Applied Mathematics and Computation*, *175*(1), 238–246.

Malik, H. J., & Trudel, R. (1986). Probability density function of the product and quotient of two correlated exponential random variables. *Canadian Mathematical Bulletin*, *29*, 413–418.

Mari, D. D., & Kotz, S. (2001). *Correlation and dependence*. London: Imperial College Press.

Markov, A. A. (1899–1900). The law of large numbers and the method of least squares. *Izvestia Physiko-mathematicheskago Obschvestva pri Imperatorskom Kazanskom Universitet*, *8*, 110–128.

Marsaglia, G. (1965). Ratios of normal variables and ratios of sums of uniform variables. *Journal of the American Statistical Association*, *60*, 193–204.

Mateu-Figueras, G., Pedro, P., & Pewsey, A. (2007). Goodness-of-fit tests for the Skew-normal distribution when the parameters are estimated from the data. *Communications in Statistics: Theory and Methods*, *36*(9), 1735–1755.

McLaughlin, M. P. (1999). *A compendium of common probability distributions*. http://www.ub.edu/stat/docencia/Diplomatura/Compendium.pdf

Meeusen, W., & van den Broeck, J. (1977). Efficiency estimation from Cobb-Douglas production functions with composed error. *International Economic Review*, *8*, 435–444.

Mitra, S. S. (1978). *Recursive formula for the characteristic function of Student t- distribution for odd degrees of freedom, Manuscript*. University Park, PA: Pennsylvania State University.

Mood, A. M., Graybill, F. A., & Boes, D. C. (1974). *Introduction to the theory of statistics*. New York: McGraw-Hill Book Co.

Moivre, A. de (1733). Approximatio ad Summam Ferminorum Binomii in Seriem expansi, Supplementum II to Miscellanae. *Analytica*, 1–7.

Moivre, A. de (1738). *The doctrine of chances*. London: Fank Cass & Co (Reprint 1967).

Moix, P.-Y. (2001). *The measurement of market risk: Modelling of risk factors, asset pricing, and approximation of portfolio distributions*. New York: Springer.

Monti, A. C. (2003). A note on the estimation of the skew normal and the skew exponential power distributions. *Metron*, *61*(2), 205–219.

Moschopoulos, P. G. (1985). The distribution of the sum of independent gamma random variables. *Annals of the Institute of Statistical Mathematics*, *37*, 541–544.

Mukhopadhyay, N. (2006). *Introductory statistical inference*. London: Chapman & Hall.

Mukhopadhyay, S., & Vidakovic, B. (1995). Efficiency of linear Bayes rules for a normal meanskewed priors class. *The Statistician*, *44*, 389–397.

Nadarajah, S. (2005a). A generalized normal distribution. *Journal of Applied Statistics*, *32*(7), 685–694.

Nadarajah, S. (2005b). Products, and ratios for a bivariate gamma distribution. *Applied Mathematics and Computation*, *171*(1), 581–595.

Nadarajah, S. (2005c). On the product XY for some elliptically symmetric distributions. *Statistics and Probability Letters*, *72*, 67–75.

Nadarajah, S. (2005d). On the product and ratio of laplace and bessel random variables. *Journal of Applied Mathematics*, *4*, 393–402.

Nadarajah, S. (2005e). Linear combination, product and ratio of normal and logistic random variables. *Kybernetika*, *41*(6), 787–798.

Nadarajah, S. (2006a). Exact calculation of the coverage interval for the convolution of two student's t distributions. *Metrologia*, *43*, L21–L22.
Nadarajah, S. (2006b). On the ratio X/Y for some elliptically symmetric distributions. *Journal of Multivariate Analysis*, *97*(2), 342–358.
Nadarajah, S., & Ali, M. M. (2004). On the product of Laplace and Bessel random variables. *Journal of the Korean Data & Information Science Society*, *15*(4), 1011–1017.
Nadarajah, S., & Ali, M. M. (2005). On the product and ratio of t and laplace random variables. *Pakistan Journal of Statistics*, *21*, 1–1.
Nadarajah, S., & Dey, D. K. (2006). On the product and ratio of t random variables. *Applied Mathematics Letters*, *19*, 45–55.
Nadarajah, S., & Gupta, A. K. (2005). On the product and ratio of bessel random variables. *International Journal of Mathematics and Mathematical Sciences*, *18*, 2977–2989.
Nadarajah, S., & Gupta, A. K. (2006). On the ratio of logistic random variables. *Computational Statistics & Data Analysis*, *50*(5), 1206–1219.
Nadarajah, S., & Kotz, S. (2003). Skewed distributions generated by the normal kernel. *Statistics and Probability Letters*, *65*, 269–277.
Nadarajah, S., & Kotz, S. (2005a). On the product and ratio of Pearson type VII and laplace random variables. *Austrian Journal of Statistics*, *34*(11), 11–23.
Nadarajah, S., & Kotz, S. (2005b). On the ratio of Pearson type VII and Bessel random variables. *Journal of Applied Mathematics and Decision Sciences*, *9*(4), 191–199.
Nadarajah, S., & Kotz, S. (2005c). On the linear combination of exponential and gamma random variables. *Entropy*, *7*(2), 161–171.
Nadarajah, S., & Kotz, S. (2006a). Skew distributions generated from different families. *Acta Applicandae Mathematica*, *91*(1), 1–37.
Nadarajah, S., & Kotz, S. (2006b). On the product and ratio of gamma and Weibull random variables. *Econometric Theory*, *22*, 338–344.
Nadarajah, S., & Kotz, S. (2006c). The linear combination of logistic and gumbel random variables. *Fundamenta Informaticae*, *74*(2–3), 341–350.
Nadarajah, S., & Kotz, S. (2006d). On the product XY for the elliptically symmetric Pearson type VII distribution. *Mathematical Proceedings of the Royal Irish Academy*, *106A*(2), 149–162.
Nadarajah, S., & Kotz, S. (2006e). The Bessel ratio distribution. *Comptes rendus de l'Académie des Sciences, Series I*, *343*, 531–534.
Nadarajah, S., & Kotz, S. (2007). Programs in R for computing truncated t distributions. *Quality and Reliability Engineering International*, *23*(2), 273–278.
Nason, G. (2001). Robust projection indices. *Journal Royal Statistical Society B*, *63*, 551–567.
Nason, G. (2006). On the sum of t and Gaussian random variables. *Statistics & Probability Letters*, *76*, 1280–1286.
Nevzorov, V. B. (2002). On a property of student t distribution with 2 degrees of freedom. Zap Nauchn Sem. POMI294, 148–157 (Russian).
Nevzorov, V. B., Balakrishnan, N, and Ahsanullah, M. (2003). Simple Characterizations. of Students $t_2$distribution. *Statistician*, *52*(3), 395–400.
Nevzorov, L., Nevzorov, V.B. and Akundov, I. (2007). A simple characterization of Student's t_2 distribution. *Metron*, *LXV*(1), 53–57.
Novosyolov, A. (2006). The sum of dependent normal variables may be not normal. Krasnoyarsk, Russia: Institute of Computational Modelling, Siberian Branch of the Russian Academy of Sciences.
O'Hagan, A., & Leonard, T. (1976). Bayes estimation subject to uncertainty about parameter constraints. *Biometrika*, *63*, 201–203.
Pastena, D. (1991). Note on a paper by Ifram. *Sankhya*, *A39*, 396–397.
Patel, J. K., Kapadia, C. H., & Owen, D. B. (1976). *Handbook of statistical distributions*. New York: Marcel Dekker Inc.
Patel, J. K., & Read, C. B. (1982). *Handbook of the normal distribution*. New York: Marcel Dekker Inc.

Patil, V. H. (1965). Approximation to the Behrens-Fisher distributions. *Biometrika*, *52*, 267–271.
Pearson, E. S. (1967). Some reflexions on continuity in the development of mathematical statistics, 1885–1920. *Biometrika*, *54*, 341–355.
Pearson, E. S. (1970). William Sealy Gosset, 1876–1937: "Student" as a statistician. In E. S. Pearson, & M. G. Kendall, (Eds.), *Studies in the history of statistics and probability* (pp. 360–403). New York: Hafner.
Pearson, K. (1896). Mathematical contributions to the theory of evolution, III: Regression, heredity, and panmixia. *Philosophical Transactions of the Royal Society of London*, *A187*, 253–318.
Pewsey, A. (2000). Problems of inference for Azzalini's skewnormal distribution. *Journal of Applied Statistics*, *27*(7), 859–870.
Pewsey, A. (2002). Large-sample inference for the general half-normal distribution. *Communications in Statistics-Theory and Methods*, *31*, 1045–1054.
Pewsey, A. (2004). Improved likelihood based inference for the general half-normal distribution. *Communications in Statistics-Theory and Methods*, *33*, 197–204.
Pewsey, A. (2006). Modelling asymmetrically distributed circular data using the wrapped skew-normal distribution. *Environmental and Ecological Statistics*, *13*(3), 257–269.
Pham-Gia, T. (2000). Distributions of the ratios of independent beta variables and applications. *Communications in Statistics-Theory and Methods*, *29*, 2693–2715.
Pham-Gia, T. G., & Turkkan, N. (1994). Reliability of a standby system with beta-distributed component lives. *IEEE Transactions on Reliability*, *43*, 71–75.
Pham-Gia, T., Turkkan, N., & Marchand, E. (2006). Density of the ratio of two normal random variables and applications. *Communications in Statistics-Theory and Methods*, *35*(9), 1569–1591.
Podolski, H. (1972). The distribution of a product of n independent random variables with generalized gamma distribution. *Demonstratio Mathematica*, *4*, 119–123.
Polya, G. (1923). Herleitung der Gausschen fehler genertzes aux einer functional gieichung. *Math-Zeit*, *18*, 96–108.
Press, S. J. (1969). The t ratio distribution. *Journal of the American Statistical Association*, *64*, 242–252.
Provost, S. B. (1989a). On sums of independent gamma random variables. *Statistics*, *20*, 583–591.
Provost, S. B. (1989b). On the distribution of the ratio of powers of sums of gamma random variables. *Pakistan Journal Statistics*, *5*, 157–174.
Prucha, I. R., & Kelejian, H. H. (1984). The structure of simultaneous equation estimators: A generalization towards nonnormal disturbances. *Econometrica*, *52*, 721–736.
Prudnikov, A. P., Brychkov, Y. A., & Marichev, O. I. (1986). *Integrals and series (Vols. 1, 2, and 3)*. Amsterdam: Gordon and Breach Science Publishers.
Rathie, P. N., & Rohrer, H. G. (1987). The exact distribution of products of independent random variables. *Metron*, *45*, 235–245.
Rényi, A. (1961). On measures of information and entropy. *Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability*, *1960*, 547–561.
Rider, P. R. (1965). Distributions of product and quotient of Cauchy variables. *The American Mathematical Monthly*, *72*(3), 303–305.
Roberts, C. (1966). A correlation model useful in the study of twins. *Journal of the American Statistical Association*, *61*, 1184–1190.
Rohatgi, V. K., & Saleh, A. K. M. E. (2001). *An introduction to probability and statistics*. New York: Wiley.
Rokeach, M., & Kliejunas, P. (1972). Behavior as a function of attitude-toward-object and attitude-toward-situation. *Journal of Personality and Social Psychology*, *22*, 194–201.
Ruben, H. (1960). On the distribution of the weighted difference of two independent student variables. *Journal of the Royal Statistical Society, Series B*, *22*(1), 188–194.
Sakamoto, H. (1943). On the distributions of the product and the quotient of the independent and uniformly distributed random variables. *Tohoku Mathematical Journal*, *49*, 243–260.
Savchuk, V., & Tsokos, C. P. (2011). *Bayesian theory and methods with applications*. Paris, France: Atlantis Press.

Scheffe, H. (1970). Practical solutions of the Behrens-Fisher problem. *Journal of the American Statistical Association*, *65*, 1501–1508.

Schneider, H. (1986). *Truncated and censored samples from normal populations*. New York: Marcel Dekker Inc.

Schrodinger, E. (1915). Zur Theorie der Fall-und Steigversuche an Teilchen mit Brownscher Bewegung. *Physikalische Zeitschrift*, *16*, 289–295.

Severini, T. A. (2005). *Elements of distribution theory*. New York: Cambridge University Press.

Shah, S. M., & Jaiswal, M. C. (1966). Estimation of parameters of doubly truncated normal distribution from first four sample moments. *Annals of the Institute of Statistical Mathematics*, *18*(1), 107–111.

Shakil, M., Kibria, B. M. G., & Singh, J. N. (2006). Distribution of the ratio of Maxwell and Rice random variables. *International Journal of Contemporary Mathematical Sciences*, *1*(13), 623–637.

Shakil, M., & Kibria, B. M. G. (2006). Exact distribution of the ratio of gamma and rayleigh random variables. *Pakistan Journal of Statistics and Operations Research*, *2*(2), 87–98.

Shakil, M., & Kibria, B. M. G. (2007). On the product of Maxwell and Rice random variables. *Journal of Modern Applied Statistical Methods*, *6*(1), 212–218.

Shakil, M., & Kibria, B. M. G. (2009). A new student's t-type distribution and its properties. *Journal of Applied Statistical Science.*, *16*(3), 355–364.

Shakil, M., Kibria, B. M. G., & Chang, K.-C. (2008). Distributions of the product and ratio of Maxwell and Rayleigh random variables. *Statistical Papers*, *49*, 729–747.

Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, *27*, 379–432.

Shapiro, S. S., & Francia, R. S. (1972). An approximate analysis of variance test for normality. *Journal of the American Statistical Association*, *67*, 215–216.

Shcolnick, S. M. (1985). On the ratio of independent stable random variables. *Lecture Notes in Mathematics* (Vol. 1155, pp. 349–354). Berlin, Germany: Springer.

Shkedy, Z., Aerts, M., & Callaert, H. (2006). The weight of Eurocoins: Its distribution might not be as normal as you would expect. *Journal of Statistics Education*, *14*(2), 102–129.

Sichel, H. S. (1974). On a distribution representing sentence-length in written prose. *Journal of the Royal Statistical Society Series A*, *137*, 25–34.

Sichel, H. S. (1975). On a distribution law for word frequencies. *Journal of the American Statistical Association*, *70*, 542–547.

Sinha, S. K. (1983). Folded normal distribution: A Bayesian approach. *Journal of the Indian Statistical Association*, *21*(1083), 31–34.

Song, K.-S. (2001). Rényi information, loglikelihood and an intrinsic distribution measure. *Journal of Statistical Planning and Inference*, *93*(1–2), 51–69.

Sornette, D. (1998). Multiplicative processes and power laws. *Physical Review, E*, *57*, 4811–4813.

Sornette, D. (2004). *Critical phenomena in natural sciences, chaos, fractals, self-organization and disorder: Concepts and tools* (2nd ed.). Heidelberg, Germany: Springer Series in Synergetics.

Springer, M. D. (1979). *The algebra of random variables*. New York: Wiley.

Springer, M. D., & Thompson, W. E. (1966). The distribution of product of independent random variables. *SIAM Journal of Applied Mathematics*, *14*(3), 511–526.

Springer, M. D., & Thompson, W. E. (1970). The distribution of products of beta, gamma and Gaussian random variables. *SIAM Journal on Applied Mathematics*, *18*, 721–737.

Srivastava, H. M., & Nadarajah, S. (2006). Some families of Bessel distributions and their applications. *Integral Transforms and Special Functions*, *17*(1), 65–73.

Steece, B. M. (1976). On the exact distribution for the product of two independent beta-distributed random variables. *Metron*, *34*, 187–190.

Stigler, S. M. (1999). *Statistics on the table*. Cambridge, MA: Harvard University Press.

Stuart, A. (1962). Gamma-distributed products of independent random variables. *Biometrika*, *49*, 564–565.

Sugiura, N., & Gomi, A. (1985). Pearson diagrams for truncated normal and truncated Weibull distributions. *Biometrika*, *72*(1), 219–222.

Suhir, E. (1997). *Applied probability for engineers and scientists*. New York: McGraw-Hill.

Sukhatme, P. V. (1938). On Fisher and Behrens test of significance for the difference in means of two normal samples. *Sankhya*, *4*, 39–48.

Tang, J., & Gupta, A. K. (1984). On the distribution of the product of independent beta random variables. *Statistics & Probability Letters*, *2*, 165–168.

Tchebyshev, P. L. (1890). Sur deux theorems relatifs aux probabilites. *Acta Mathematica*, *14*, 305–315; Reprinted (1962) in Oeuvres, Vol. 2, Chelesa, New York.

Thomopoulos, N. T. (1980). *Applied forecasting methods*. Englewood Cliffs, New Jersey: Prentice-Hall.

Triola, M. F. (2005). *Biostatistics for the biological and health science*. New Jersey: Pearson.

Tsokos, C. P. (1972). *Probability distributions: An introduction to probability theory with applications*. California: Duxbury Press.

Tweedie, M. C. K. (1957a). Statistical properties of inverse Gaussian distribution. I. *Annals of Mathematical Statistics*, *28*, 362–377.

Tweedie, M. C. K. (1957b). Statistical properties of inverse Gaussian distribution. II. *Annals of Mathematical Statistics*, *28*, 696–705.

Vernic, R. (2006). Multivariate skew-normal distributions with applications in insurance. *Insurance: Mathematics and Economics*, *38*, 413–426.

Walker, G. A., & Saw, J. G. (1978). The distribution of linear combinations of t-variables. *Journal of the American Statistical Association*, *73*(364), 876–878.

Wallgren, C. M. (1980). The distribution of the product of two correlated variates. *Journal of the American Statistical Association*, *75*, 996–1000.

Watson, G. N. (1962). *A treatise on the theory of Bessel functions* (2nd ed.). London: Cambridge University Press.

Weisstein, E. W. (2007). "Half-normal distribution". http://mathworld.wolfram.com

Weisstein, E. W. (2007). "Inverse Erf". From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/InverseErf.html

Whittaker, E. T., & Robinson, G. (1967). *The calculus of observations: A treatise on numerical mathematics* (4th ed.). New York: Dover.

Wiper, M. P., Giron, F. J., and Pewsey, A. (2005). Bayesian inference for the half-normal and half-t distributions. *Statistics and Econometrics Series 9*, Working Papers 05–47.

Witkovsky, V. (2001). Computing the distribution of a linear combination of inverted gamma variables. *Kybernetika*, *37*, 79–90.

Yanev, G. P., & Ahsanullah, M. (2012). Characterizations of student's t-distribution via regression of order statistics. *Statistics*, *46*(4), 425–429.

Zabell, S. L. (2008). On student's t 1908 article "The probable error of a mean". *Journal of the American Statistical Association*, *103*(481), 1–7.

Zellner, A. (1976). Bayesian and non-Bayesian analysis of the regression model with multivariate student error terms. *Journal of the American Statistical Association*, *71*, 400–405.

Zolotarev, V. M. (1957). Mellin-Stieltjes transforms in probability theory. *Teoriya Veroyat. Primen*, *2*, 444–469.