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Humberto Stein Shiromoto

Design and Analysis of Control Systems

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To Carolina Pereira Gomes, my parents, and friends

Preface

> Bold ideas, unjustified anticipations, and speculative thought, are our only means for interpreting nature: our only organon, our only instrument, for grasping her. And we must hazard them to win our prize. Those among us who are unwilling to expose their ideas to the hazard of refutation do not take part in the scientific game.
>
> K. Popper, The Logic of Scientific Discover.

Over the years, technology and society have become more dependent on autonomous systems. In this book, two methodologies for the analysis and design of autonomous systems are considered: the interaction of two subsystems and controller design, i.e., the algorithm that will govern the behavior of these systems. In general, techniques for analysis and design of systems depend on the structure of the equation that describes the system dynamics which, in many cases, is nonlinear. Moreover, the presence of unknown terms such as disturbances or modeling errors may lead to a system lacking the appropriate structures.

The purpose of this book is to provide a methodology to merge different known techniques for the design and analysis of systems lacking those structures. This monograph has been written for readers varying from undergraduate students to Ph.D. candidates/researchers in the fields of sciences and engineering. Since another objective of this book is to be accessible to the largest variety of readers, I provide only sketches of proofs, making the idea behind the proofs accessible for the unfamiliar reader. In Chap. 2, the design and blending of two nonlinear controllers is presented. In Chap. 3, the analysis of two interconnected systems is considered. The Appendix A recalls fundamental results and concepts employed along the other chapters.

This monograph has mostly been adapted from my Ph.D. dissertation [1] and based on a few subsequent articles. The English translation of the title my Ph.D. manuscript is Stabilization under local and global constraints. This thesis was supervised by Christophe Prieur (senior researcher at the CNRS/GIPSA-lab, Grenoble) and co-supervised by Vincent Andrieu (junior researcher at the

CNRS/LAGEP, Lyon). I thank them for advising me in this project which made me one of the recipients of the GdR MACS best theses award. This is a French national prize awarded to up to four Ph.D. candidates every 2 years. I also thank the examiners and opponents: Antoine Chaillet, Andrew Teel, Andrea Bacciotti, Laurent Praly, Sophie Tarbouriech, and Luca Zaccarian. They enriched not only my thesis but also my work as a researcher with their comments.

Sydney, Australia Humberto Stein Shiromoto
October 2016

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1. Stein Shiromoto H (2014) Stabilisation sous contraintes locales et globales. PhD thesis, Université Grenoble Alpes

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Acronyms and Notations

$\mathscr{C}^k$	Denotes the class of k-times continuously differentiable functions;
$\mathscr{K}$	Denotes the class of continuous strictly increasing functions;
$\mathscr{K}_\infty$	Denotes class of functions belonging to $\mathscr{K}$ and that are unbounded;
$\mathscr{P}$	Denotes the class of positive definite functions;
$\mathscr{L}^\infty_{\text{loc}}$	Denotes the class of locally essentially bounded functions;
$\mathbb{R}$	Denotes the set of real values;
$\mathbb{R}_{>0}$	Denotes the set of strictly positive real values;
$\mathbb{R}_{\geq 0}$	Denotes the set of positive real values;
$\mathbf{B}_{<r}(\mathbf{K})$	Denotes the open ball centered at the set $\mathbf{K}$ and with radius $r > 0$;
$\mathbf{B}_{\leq r}(\mathbf{K})$	Denotes the closed ball centered at the set $\mathbf{K}$ and with radius $r > 0$;
$\mathbf{S}_{\neq 0}$	Denotes the set $\mathbf{S}\backslash\{0\}$;
$\mathtt{cl}\{\mathbf{S}\}$	Denotes the closure of the set $\mathbf{S}$;
$\text{co}\{\mathbf{S}\}$	Denotes the convex closure of the set $\mathbf{S}$;
$\mathscr{K}\mathscr{L}$	Denotes the class of functions $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for a fixed $t \geq 0$, the function $s \mapsto \beta(s,t)$ is of class $\mathscr{K}$ and, for a fixed $s \geq 0$, the function $t \mapsto \beta(s,t) \in \mathbb{R}_{\geq 0}$ is strictly decreasing and satisfies $\beta(s,t) \to 0$, as t tends to infinity;
$\Omega(V)_{\diamond c}$	Denotes the set $\{x \in \mathbb{R}^n : V(x) \diamond c\}$, where $\diamond$ is a binary comparison operator (i.e., $\diamond \in \{\geq, <, \neq, \text{etc}\}$;
$\mathtt{supp}(\mathtt{f})$	Denotes the set $\{x \in \mathbb{R}^n : f(x) \neq 0\}$;
$D^+_v V$	is the Dini derivative of a function $V : \mathbb{R}^n \to \mathbb{R}$ in along the vector $v \in \mathbb{R}^n$;
$L_h V$	Lie derivative of V in along the vector field h
$M \succ 0$	Stands for the matrix $M \in \mathbb{R}^{n\times n}$ being positive definite;
$M \succcurlyeq 0$	Stands for the matrix $M \in \mathbb{R}^{n\times n}$ being positive semidefinite;
$x \cdot y$	Denotes the inner product between the vectors x and $y \in \mathbb{R}^n$;
UGAS	Uniformly Globally Asymptotically Stable

Chapter 1
Introduction

Dynamical systems have an important role in sciences and technologies, since it is employed as tool for modeling physical phenomena. Mathematical control theory provides a framework to deal with dynamical systems by analyzing their behavior which can be influenced to satisfy desired constraints. In this book, the considered dynamical systems are described by ordinary differential equation.

The stability of dynamical systems is one of the main problems in control theory. In this work, the stability notion introduced by Lyapunov is considered. This concept was introduced in the seminal work entitled "General Problem of the Stability of Motion" in 1892. The interested reader may consult the English translation of the integral text in the special issue [8]. A historical development of stability theory is summarized by Parks in [9].

Roughly speaking, a point $\bar{x}$ is said to be Lyapunov stable for a dynamical system if its solutions starting from a initial condition $x(0)$ near $\bar{x}$ remains close to this point. Additionally, if these solutions eventually converge to $\bar{x}$, this point is said to be asymptotically stable. These concepts are precisely stated in Definition A.30, below. Figure 1.1 illustrates these notions.

One of the main advantages of Lyapunov's stability theory is that the properties of the solutions to differential equations can be obtained without explicitly compute these solutions. In this work, this advantage is exploited in two cases: the design of feedback laws and the analysis of the interaction of two interconnected dynamical systems. In the first case, conditions to design a feedback law to stabilize a class of dynamical systems are provided. In the latter case, a criterion to deduce the stability of interconnected systems is formulated.

In both cases, the main ingredient employed is to take into account the properties of the dynamical systems near and far from a prescribed set. For the feedback design case, two controllers are combined, according to appropriate regions, to stabilize a

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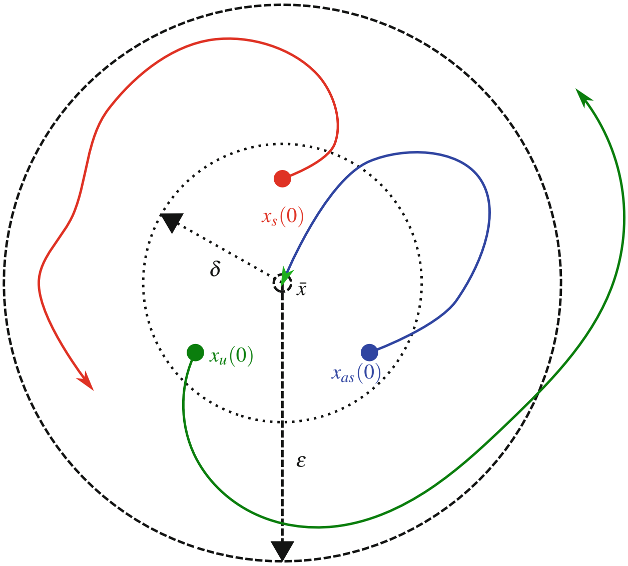

Fig. 1.1 Illustration of the concept of Lyapunov's stability. The point $\bar{x}$ is a equilibrium. The (*blue*) solution issuing from $x_{as}(0)$ remains close and eventually converge to $\bar{x}$. The (*red*) solution issuing from $x_s(0)$ remains close to $\bar{x}$. The (*green*) solution issuing from $x_u(0)$ does not remain closed to $\bar{x}$

class of dynamical systems whereas, for the interconnection analysis, an input-to-state stability condition is blended with a criterion to ensure that sets undesired of trajectories have Lebesgue measure zero.

Other works that blend local and global behaviors of dynamical systems are well known in the literature. See [2] for the design of globally inverse optimal feedback law satisfying prescribed local optimality constraints, see [1] for a combination of control-Lyapunov function and see [3] for a combination of input-to-state and integral input-to-state stability notions.

1.1 Outline and Contributions

This book is organized as follows.

In Chap. 2, the problem of designing a feedback law for a class of nonlinear systems is considered. The lack of structure of the equation describing this class of systems rends the synthesis method known as *backstepping* not suitable to obtain a single controller rendering the closed-loop system is stable.

The term backstepping has been introduced by Kokotović [5]. It is a recursive technique employed to design controllers for cascades of nonlinear systems [6, 7]. In comparison to other synthesis methods, backstepping does not require the system to be feedback linearizable and it does not assume that the first-order approximation (linearization) is controllable [4].

If the terms impeaching the application of the backstepping technique are uniformly bounded, then the backstepping can be used to drive the solutions to the system inside a compact set around the origin. The next step is to make these solutions converge to the origin.

In a compact domain of the state space that includes the origin, the nonlinear equation can be bounded by a linear differential inclusion. This domain is defined to include the compact set containing the global attractor. By employing semidefinite programming techniques, a feedback law rendering the origin locally asymptotically stable and including the global attractor in the basin of attraction is computed.

The final step is to blend both controllers. To that end, a discrete variable is introduced making the closed-loop system hybrid, i.e., with continuous and discrete dynamics. This technique allows to render the origin globally asymptotically stable for the closed-loop system.

Chapter 3 is dedicated to the analysis of the feedback interconnection of a class of nonlinear systems. More precisely, the interconnection of two input-to-state stable systems is considered.

One of the main results to ensure that origin is globally asymptotically stable for the interconnected system is the small-gain theorem. Roughly speaking, if the function resulting from composition of the systems gains is smaller than the identity function, for all positive values of the argument, then the interconnected system is asymptotically stable. The small-gain theorem has been employed in many control problems such as adaptive controllers and dynamic uncertainties (see [4] and references therein).

The question that is studied in Chap. 3 can be formulated as follows: "How the small-gain theorem can still be employed, when there exists a finite interval where it does not hold?"

To answer this question, the small-gain theorem is employed on the intervals of the real line (corresponding to regions of the state space) where it holds. In the gap region, a condition regarding the positivity of the divergence of the vector field of the interconnected system is assumed. In addition to that, the intervals where each condition hold are assumed to have nonempty intersection.

The asymptotic stability of the equilibrium of the origin is obtained by combining the two mentioned conditions. More specifically, the set of trajectories of solutions that do not converge to the origin have Lebesgue measure zero.

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Chapter 2
Blending Two Feedback Laws

2.1 Introduction

Over the years, research in control of nonlinear dynamical systems has led to many different tools for the design of feedback laws rendering the equilibrium (globally) asymptotically stable. Most of these techniques require particular structures on vector field that describes the systems dynamics. According to each case, the designer may use different approaches such as high-gain [10], backstepping [5], or forwarding [20]. However, when the vector field does not have appropriate properties (due to the presence of unstructured dynamics), some of these methods cannot be applied.

The focus of this chapter is on the class of systems with obstructions for the synthesis of controllers employing the backstepping technique. The approaches proposed in [25–27] may solve the problem by blending a backstepping feedback law that renders a suitable compact set practically globally stable with a controller that locally stabilizes the equilibrium of the origin.

By solving a semidefinite program under constraints on the basin of attraction, the feedback law that locally stabilizes the equilibrium of the origin is obtained. The main result can be seen as a design of hybrid feedback laws for systems which a priori do not have a feedback law that globally stabilizes the origin.

Methodologies to synthesize hybrid controllers are well known (see [22] and references therein, for instance). These approaches have been also applied for systems that do not satisfy the Brockett's condition [6, 12]. Hybrid feedback laws have the advantage of rendering the equilibrium of the closed-loop system robustly asymptotically stable with respect to measurement noise and actuator errors [9, 23].

Other techniques to blend feedback laws do exist in the literature. See, for instance, [1, 21] for the design of continuous controllers by blending two feedback laws. In contrast to those works, no a priori knowledge of continuous globally stabilizing controllers is assumed. Note that the problem considered in this chapter differs from the one considered in [19], where a synergistic Lyapunov function and a feedback law are designed using backstepping.

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2.1.1 Motivation and Problem Formulation

Consider the class of systems described by the equation

$$\begin{cases} \dot{x}_1(t) = f_1\left(x_1(t), x_2(t)\right) + h_1\left(x_1(t), x_2(t), u(t)\right) \\ \dot{x}_2(t) = f_2\left(x_1(t), x_2(t)\right) u(t) + h_2\left(x_1(t), x_2(t), u(t)\right), \end{cases} \tag{$\Sigma_h(u)$}$$

where, for each positive value of the *time* t, the *system states* $x_1(t)$ and $x_2(t)$, and the *input variable* $u(t)$ evolve in the Euclidean spaces $\mathbb{R}^{n-1}$, $\mathbb{R}$ and $\mathbb{R}$, respectively. The vector fields describing system $(\Sigma_h(u))$ are

$$\begin{array}{ll} f_1 \in \mathscr{C}^1\left(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}^{n-1}\right), & h_1 \in \mathscr{C}^1\left(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^{n-1}\right), \\ f_2 \in \mathscr{C}^1\left(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}\right), & h_1 \in \mathscr{C}^1\left(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right). \end{array}$$

For each index $i = 1, 2$, the functions f_i (resp. h_i) yield zero whenever $(x_1, x_2) = (0, 0)$ (resp. $(x_1, x_2, u) = (0, 0, 0)$). Moreover, $f_2(x_1, x_2) = 0$ if and only if $(x_1, x_2) = (0, 0)$. From now on, the dependence on the time variable t will be omitted. A function $v \in \mathscr{L}^{\infty}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ is said to be an *input or control for system* $(\Sigma_h(v))$. The notation $(\Sigma_h(v))$ stands for system $(\Sigma_h(u))$ under the input v.

For the time being, assume that $h_1 \equiv 0$ and $h_2 \equiv 0$. System $(\Sigma_h(u))$ is described by

$$\begin{cases} \dot{x}_1 = f_1\left(x_1, x_2\right) \\ \dot{x}_2 = f_2\left(x_1, x_2\right) u. \end{cases} \tag{$\Sigma(u)$}$$

The first assumption concerns the global stabilization of the equilibrium of the origin for system

$$\dot{x}_1 = f_1(x_1, x_2), \tag{$\Sigma_1(x_2)$}$$

when x_2 is considered as the input variable.

Assumption 2.1 There exist a feedback law $\psi_1 \in \mathscr{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$ for system $(\Sigma_1(x_2))$, a proper function $V_1 \in \left(\mathscr{C}^1 \cap \mathscr{P}\right)\left(\mathbb{R}^{n-1}, \mathbb{R}_{\geq 0}\right)$, and a locally Lipschitz function $\alpha_1 \in \mathscr{K}_\infty$ such that the inequality

$$L_{f_1} V_1\left(x_1, \psi_1\right) \leq -\alpha_1\left(V_1(x_1)\right) \tag{2.1}$$

holds, for every $x_1 \in \mathbb{R}^{n-1}$.

In other words, Assumption 2.1 states that the equilibrium of the origin is globally asymptotically stable when system $(\Sigma_1(x_2))$ is in closed loop with the feedback law ψ_1. This implies that the *backstepping* design procedure can be employed to design a feedback law $\varphi_b \in \mathscr{C}(\mathbb{R}^n, \mathbb{R})$ rendering the origin globally asymptotically stable equilibrium for system $(\Sigma(\varphi_b))$ (see Sect. 2.8 for more details).

When employing the backstepping technique for system $(\Sigma_h(u))$, the dependences of the functions h_1 and h_2 on the input variable u lead to an equation depending implicitly on u as follows. Consider the function defined as

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R}_{\geq 0} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{1}{2}(x_2 - \psi_1(x_1))^2 . \end{aligned} \tag{2.2}$$

The Lie derivative of V along the vector field f_h yields the inequality

$$\begin{aligned} L_{f_h} V(x_1, x_2, u) \leq & -\alpha_1 (V_1(x_1)) + L_{f_{h_1}} V_1(x_1, \psi_1, u) + (x_2 - \psi_1(x_1)) \\ & \cdot \Bigg(f_2(x_1, x_2)u + h_2(x_1, x_2, u) - L_{r_1}\psi_1(x_1, x_2, u) \\ & + \frac{\partial \psi_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}} \left(x_1, \eta_{x_1,x_2}(s), u\right) ds \Bigg) \end{aligned} \tag{2.3}$$

which holds, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, where the vector field r_1 is defined as $r_1 = f_1 + h_1$.

To solve inequality (2.3) on the variable u may not be an easy task. Moreover, the fact that the function f_2 yields zero at zero implies that the backstepping could not be employed. These facts motivate the introduction of an additional hypothesis on the terms h_1 and h_2 that is shown in the next section.

2.2 Global Practical Stabilizability

The second assumption provides bounds for the functions h_1 and h_2 that impeach the direct application of the backstepping method.

Assumption 2.2 There exist a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and two positive constant values $\varepsilon \in (0, 1]$ and M satisfying the following inequalities

1. (Bounds on h_1)

$$\begin{aligned} |h_1(x_1, x_2, u)| &\leq \Psi(x_1, x_2), \\ \left|\frac{\partial h_1}{\partial x_2}(x_1, x_2, u)\right| &\leq \Psi(x_1, x_2), \\ L_{h_1} V_1 (x_1, \psi_1) &\leq (1 - \varepsilon)\alpha_1 (V_1(x_1)) + \varepsilon\alpha (M) \ ; \end{aligned} \tag{2.4}$$

2. (Bound on h_2)

$$|h_2(x_1, x_2, u)| \leq \Psi(x_1, x_2),$$

for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$.

The role of the constants ε and M is explained as follows. Under Assumption 2.1, the Lie derivative of the function V_1 along the vector field r_1 yields

$$L_{r_1} V_1(x_1, x_2, u) = L_{f_1} V_1(x_1, x_2) + L_{h_1} V_1(x_1, x_2, u), \tag{2.5}$$

for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$. Consider the feedback law ψ_1 and let $x_2 = \psi_1(x_1)$. Under Assumption 2.2, Eq. (2.5) yields the inequality

$$L_{r_1} V_1(x_1, \psi_1, u) \le \varepsilon \left(\alpha_1(M) - \alpha_1(V_1(x_1))\right), \tag{2.6}$$

for every $x_1 \in \mathbb{R}^{n-1}$.

Inequality (2.6) implies that

- The constant M is an estimation of how far a solution to the system

$$\dot{x}_1 = f_1(x_1, \psi_1(x_1)) + h_1(x_1, \psi_1(x_1), u) \tag{2.7}$$

  will remain from the origin. Note that if $M = 0$, then V_1 strictly decreases along the solutions to system (2.7). Consequently, the equilibrium of the origin is globally asymptotically stable for system (2.7), for every input u. Thus, $M = 0$ could be a strong assumption;
- The constant ε is an estimation of how fast a solution to system (2.7) will approach the set $\Omega_{\le M_1}(V_1)$ which is defined by the function V_1. Thus, ε must be different from zero. Note that, when $\varepsilon = 1$, the presence of the function h_1 has no role in the speed of convergence of the solutions which may be a strong assumption as well.

The discussion regarding the role of the constants M and ε motivates the use of global practical asymptotic stabilizability of sets. This notion is recalled from [13, p. 126] and it is adapted to the context of this chapter.

Definition 2.3 A compact set $\mathbf{S} \subset \mathbb{R}^n$ containing the origin is said to be *globally practically asymptotically stabilizable for system* $(\Sigma_h(u))$ if, for every $a > 0$, there exists a feedback law $\varphi_a : \mathbb{R}^n \to \mathbb{R}$ such that the set

$$\mathbf{B}_{\le a} = \left\{x \in \mathbb{R}^n : |x|_{\mathbf{S}} \le a\right\}$$

contains a compact set that is globally asymptotically stable for system $(\Sigma_h(\varphi_a))$.

Consider the set $\mathbf{A}$ defined below.

$$\mathbf{A} = \left\{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_1(x_1) \le M, x_2 = \psi_1(x_1)\right\}. \tag{2.8}$$

Since the function V_1 is proper and ψ_1 is of class $\mathscr{C}^1$, the set $\mathbf{A}$ is compact.

The first result of this chapter is recalled from [25, 27]. It states that, under Assumption 2.2, there exists a feedback law φ_b for system $(\Sigma_h(u))$ that is continuous and rends the set $\mathbf{A}$ globally practically asymptotically stable for the closed-loop system $(\Sigma_h(\varphi_b))$.

Proposition 2.4 *Under Assumptions 2.1 and 2.2, the set* $\mathbf{A}$ *is globally practically asymptotically stabilizable for system* $(\Sigma_h(u))$.

A sketch of the proof of Proposition 2.4 is provided in Sect. 2.7.1. The main ingredients are shown as follows. Under Assumption 2.2, the functions h_1 and h_2 are bounded. This implies that the backstepping method can be employed to design a feedback law φ_b that rends the set $\mathbf{A}$ globally practically asymptotically stable for system $(\Sigma_h(\varphi_b))$.

Corollary 2.5 *Let Assumption 2.2 be satisfied with $M = 0$, then the equilibrium of the origin is globally practically asymptotically stabilizable for system $(\Sigma_h(u))$.*

Now that the set $\mathbf{A}$ is globally practically asymptotically stabilizable, it remains to show how solutions to system $(\Sigma_h(u))$ will converge towards the origin which is contained within the set $\mathbf{A}$. To do so, the existence a feedback law that locally stabilizes the origin is assumed. In a particular case, the local stabilizer can be designed as shown in Sect. 2.5.

The switching law between the two feedback laws leads to a hybrid system, i.e., a system with continuous and discrete dynamics. The framework to deal with these systems is recalled in the next section.

2.3 Hybrid Systems

The following concept of hybrid feedback law was presented in [23], and it is recalled below.

Definition 2.6 (*Hybrid feedback law*) A hybrid feedback law, denoted as $\mathscr{H}$, consists of a finite set $\mathbf{Q} \subset \mathbb{N}$,

- Two collections of closed sets $\mathbf{C}_q, \mathbf{D}_q \subset \mathbb{R}^n$ indexed by $q \in \mathbf{Q}$ and satisfying
$$\mathbf{C}_q \cup \mathbf{D}_q = \mathbb{R}^n \quad \text{and} \quad \bigcup_{q\in\mathbf{Q}} \mathbf{C}_q = \mathbb{R}^n$$
for every $q \in \mathbf{Q}$;
- A collection of continuous functions $\varphi_q : \mathbb{R}^n \to \mathbb{R}$ indexed by $q \in \mathbf{Q}$;
- A collection of outer semicontinuous set-valued maps $g_q : \mathbb{R}^n \rightrightarrows \mathbb{R}$ indexed by $q \in \mathbf{Q}$.

System $(\Sigma_h(u))$ in closed loop with $\mathscr{H}$ leads to a system with continuous and discrete dynamics described as follows.

$$\begin{cases} \dot{x} = f_h(x, \varphi_q(x)), & x \in \mathbf{C}_q, \\ q^+ \in g_q(x), & x \in \mathbf{D}_q, \end{cases} \qquad (\Sigma_h(\mathscr{H}))$$

where $x = (x_1, x_2)$, and

$$f_h(x, u) = \begin{pmatrix} f_1(x) \;\; + h_1(x, u) \\ f_2(x)u + h_2(x, u) \end{pmatrix}.$$

The state space of system $(\Sigma_h(\mathscr{H}))$ is the set $\mathbb{R}^n \times \mathbf{Q}$. The map f_h is said to be *flow map* and the maps g_q are said to be *jump maps*.

System $(\Sigma_h(\mathscr{H}))$ is analyzed in the framework provided in [7, 23] which is recalled here. The first definition concerns the concept of a hybrid time domain.

Definition 2.7 (*Hybrid time domain*) A set $\mathbf{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is said to be a *compact hybrid time domain* if

$$\mathbf{T} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j),$$

for some finite sequences of the times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. It is said to be a *hybrid time domain* if, for every $(T, J) \in \mathbf{T}$, the intersection $\mathbf{T} \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

Now that the notion of hybrid time domain has been defined, the concept of a solution to $(\Sigma_h(\mathscr{H}))$ can be recalled from [23].

Definition 2.8 (*Solution to a hybrid system*) Let $\mathbf{T}$ be a hybrid time domain and consider the functions $X : \mathbf{T} \to \mathbb{R}^n$ and $Q : \mathbf{T} \to \mathbf{Q}$. The pair (X, Q) is said to be a *solution to* $(\Sigma_h(\mathscr{H}))$ if

- For a fixed j, the function $t \mapsto X(t, j)$ is locally absolutely continuous and $(t, j) \in \mathbf{T}$;
- For a fixed j, the function $t \mapsto Q(t, j)$ is constant and $(t, j) \in \mathbf{T}$.

The hybrid time domain where (X, Q) is defined is denoted as $\mathtt{dom}(X, Q)$. Moreover,

S_1. $X(0, 0) \in \mathbf{C}_{Q(0,0)} \cup \mathbf{D}_{Q(0,0)}$;
S_2. For every $j \in \mathbb{N}$ such that $\mathbf{I}^j := \{t \in \mathbb{R} : (t, j) \in \mathtt{dom}(X, Q)\}$, and for almost every $t \in \mathbf{I}^j$,

$$\dot{X}(t, j) = f_h\left(X(t, j), \varphi_{Q(t,j)}(X(t, j))\right), \quad X(t, j) \in \mathbf{C}_{Q(t,j)}\ ;$$

S_3. For every $(t, j) \in \mathtt{dom}(X, Q)$ such that $(t, j + 1) \in \mathtt{dom}(X, Q)$,

$$Q(t, j + 1) \in g_{Q(t,j)}(X(t, j)), \quad X(t, j) \in \mathbf{D}_{Q(t,j)}.$$

A solution (X, Q) is said to be

- *Nontrivial* if $\mathtt{dom}(X, Q)$ contains at least two points;
- *Complete* if $\sup \mathtt{dom}(X, Q) = \infty$;

- *Maximal* if there exists no other solution $\left(\overline{X}, \overline{Q}\right)$ to $(\Sigma_h(\mathscr{H}))$ such that $\mathtt{dom}(X, Q) \subsetneq \mathtt{dom}\left(\overline{X}, \overline{Q}\right)$, and for every $(t, j) \in \mathtt{dom}(X, Q)$, $(X(t, j), Q(t, j)) = \left(\overline{X}(t, j), \overline{Q}(t, j)\right)$.

Let (X, Q) be a solution to system $(\Sigma_h(\mathscr{H}))$. Whenever $X \in \mathbf{C}_Q$, the evolution of (X, Q) is given by the differential equation and the solution is said to *flow*. On the other hand, whenever $X \in \mathbf{D}_Q$, the evolution of (X, Q) is given by the difference equation and the solution is said to *jump*.

The following definition concerns the regularity that the functions f and g satisfy to ensure that solutions to system $(\Sigma_h(\mathscr{H}))$ exist. These conditions are based on [7, Assumption 6.10] and [23].

Definition 2.9 (*Basic Assumptions*) System $(\Sigma_h(\mathscr{H}))$ is said to satisfy the *basic assumptions for the existence of solutions* if, for every index $q \in \mathbf{Q}$,

1. The sets $\mathbf{C}_q$ and $\mathbf{D}_q$ are closed subsets of $\mathbb{R}^n$;
2. The map $f_h(\cdot, \varphi_q(\cdot)) : \mathbf{C}_q \to \mathbb{R}^n$ is continuous;
3. The set-valued map $g_q : \mathbf{D}_q \rightrightarrows \mathbf{Q}$ is outer semicontinuous, locally bounded, and for every $x \in \mathbf{D}_q$, the set $g_q(x)$ is nonempty.

Items 1 and 2 have been added here for the sake of completeness of this chapter. Note that these conditions hold for system $(\Sigma_h(\mathscr{H}))$, due to Definition 2.6.

Now that the concept of a hybrid solution and the conditions for its existence have been stated, the notion of asymptotic stability can be recalled from [7, 23].

Definition 2.10 (*Uniform Local Asymptotic Stability*) Let $\mathbf{A} \subset \mathbb{R}^n$ be a compact set. The set $\mathbf{A}$ is said to be

- *Uniformly stable for system* $(\Sigma_h(\mathscr{H}))$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every solution (X, Q) to $(\Sigma_h(\mathscr{H}))$ satisfying $|X(0, 0)|_{\mathbf{A}} \leq \delta$, the inequality $|X(t, j)|_{\mathbf{A}} \leq \varepsilon$ holds for every $(t, j) \in \mathtt{dom}(X, Q)$;
- *Attractive for system* $(\Sigma_h(\mathscr{H}))$ if there exists $\delta_0 > 0$ such that every solution (X, Q) to $(\Sigma_h(\mathscr{H}))$ satisfying $|X(0, 0)|_{\mathbf{A}} \leq \delta_0$ is complete and satisfies the limit

$$\lim_{t+j\to\infty} |X(t, j)|_{\mathbf{A}} = 0 \; ; \tag{2.9}$$

- *Asymptotically stable for system* $(\Sigma_h(\mathscr{H}))$ if it is stable and attractive.

The *uniform basin of attraction* of the set $\mathbf{A}$ is the set of all $X(0, 0) \in \mathbb{R}^n$ such that, for every $Q(0, 0) \in \mathbf{Q}$, there exists a solution (X, Q) to $(\Sigma_h(\mathscr{H}))$ that is complete and satisfies the limit (2.9).

Note that the uniformity of the stability of $\mathbf{A}$ is with respect to the indexes $q \in \mathbf{Q}$.

2.4 Blending Feedback Laws

Assumption 2.11 (*Locally stabilizing feedback law*) There exists a hybrid feedback law $\mathscr{H}_\ell$ for system $(\Sigma_h(u))$ which is defined as follows. Given a finite set $\mathbf{L} \subset \mathbb{N}$, for every index $l \in \mathbf{L}$, there exist

1. A collection of closed sets $\mathbf{C}_l, \mathbf{D}_l \subset \mathbb{R}^n$ satisfying

$$\mathbf{C}_l \cup \mathbf{D}_l = \mathbb{R}^n \quad \text{and} \bigcup_{l \in \mathbf{L}} \mathbf{C}_l = \mathbb{R}^n \ ;$$

2. A collection of continuous feedback laws $\varphi_l : \mathbb{R}^n \to \mathbb{R}$;
3. A collection of outer semicontinuous set-valued maps $g_l : \mathbb{R}^n \rightrightarrows \mathbf{L}$;
4. A collection of functions V_l for system $(\Sigma_h(\mathscr{H}_\ell))$ for which there exist functions $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}_\infty$ and a constant value $c_\ell > 0$ satisfying the following inequalities

$$\forall x \in \mathbb{R}, \ \ \underline{\alpha}(|x|) \leq V_q(x) \leq \overline{\alpha}(|x|) \ ; \qquad (2.10a)$$

$$\forall x \in (\Omega_{\leq c_\ell}(V_l)) \cap \mathbf{C}_l) \setminus \{0\}, \ \ L_{f_h} V_l(x, \varphi_q(x)) < 0 \ ; \qquad (2.10b)$$

$$\forall x \in (\Omega_{\leq c_\ell}(V_l) \cap \mathbf{D}_l) \setminus \{0\}, g \in g_l(x), \ \ V_g(x) - V_l(x) < 0. \qquad (2.10c)$$

Items 1–3 of Assumption 2.11 concern the definition of a hybrid feedback law for system $(\Sigma_h(u))$, according to Definition 2.6. Item 4 states that the functions V_l are positive definite and proper and that each one strictly decreases along the solutions to the closed-loop system $(\Sigma_h(\mathscr{H}_\ell))$. The functions satisfying the set of inequalities (2.10) have a special denomination as follows.

Definition 2.12 Each function V_l satisfying the set of inequalities described in Eq. (2.10) is said to be a *hybrid Lyapunov function for system* $(\Sigma_h(\mathscr{H}_\ell))$.

Assumption 2.11 implies that the set $\{0\} \times \mathbf{L}$ is locally asymptotically stable for $(\Sigma(\mathscr{H}_\ell))$. To see this claim, note that whenever a solution starts in a neighborhood of the origin, Eq. (2.10b) ensures that, for every $l \in \mathbf{L}$, hybrid the Lyapunov function V_l is strictly decreasing during a flow. Eq. (2.10c) ensures that, for a solution starting in a neighborhood of the origin, during a transition from a feedback law φ_l to a feedback law determined by $g \in g_l$, the value $V_l(x)$ strictly decreases to $V_g(x)$.

Recall the set $\mathbf{A}$ defined in Eq. (2.8). The next assumption concerns the inclusion of $\mathbf{A}$ in the basin of attraction of the origin for system $(\Sigma_h(\mathscr{H}_\ell))$.

Assumption 2.13 For every $l \in \mathbf{L}$, the function V_l and the constant c_l of Assumption 2.11 satisfy the inequality

$$\max_{x \in \mathbf{A}} < c_\ell .$$

Recall that, under Assumptions 2.1 and 2.2, the set $\mathbf{A}$ is globally practically asymptotically stabilizable for system $(\Sigma_h(u))$. It remains to show that a hybrid feedback law can steer the solutions to system $(\Sigma_h(u))$ towards the origin. This result is recalled from [25, 27].

Theorem 2.14 *Under Assumptions 2.1, 2.2, 2.11, and 2.13, there exist*

- *A continuous feedback law* $\varphi_b : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$*;*
- *A constant value* b_ℓ *satisfying the inequality* $0 < b_\ell < c_\ell$*;*
- *A hybrid feedback law for system* $(\Sigma_h(u))$ *defined by the discrete set* $\mathbf{Q} = \{1, 2\} \times \mathbf{L}$ *such that, for every* $l \in \mathbf{L}$*,*
  - *The closed subsets of* $\mathbb{R}^n$ *are defined by*

$$\begin{aligned} \mathbf{C}_{1,l} &= \Omega_{\leq c_\ell}(V_l) \cap \mathbf{C}_l, & \mathbf{D}_{1,l} &= \big(\Omega_{\leq c_\ell}(V_l) \cap \mathbf{D}_l\big) \cup \Omega_{\geq c_l}(V_l) \\ \mathbf{C}_{2,l} &= \Omega_{\geq b_\ell}(V_l), & \mathbf{D}_{2,l} &= \Omega_{\leq b_\ell}(V_l); \end{aligned} \tag{2.11a}$$

  - *The continuous feedback laws* $\varphi_{q,l} : \mathbf{C}_{q,l} \to \mathbb{R}$ *are defined by*

$$\varphi_{q,l}(\cdot) = \begin{cases} \varphi_l(\cdot), & if\, q{=}1, \\ \varphi_g(\cdot), & if\, q{=}2\ ; \end{cases} \tag{2.11b}$$

  - *The outer semicontinuous set-valued maps* $g_{q,l} : \mathbf{D}_{q,l} \rightrightarrows \mathbf{Q}$ *are defined by*

$$\begin{aligned} g_{2,l} : \mathbf{D}_{2,l} &\rightrightarrows \mathbf{Q} \\ x &\mapsto \{(1, l)\} \end{aligned} \tag{2.11c}$$

  and

$$\begin{aligned} g_{1,l} : \mathbf{D}_{1,l} &\rightrightarrows \mathbf{Q} \\ x &\mapsto \begin{cases} \{(1, g_l(x))\}, & x \in \Omega_{<c_\ell}(V_l) \cap \mathbf{D}_l, \\ \{(2, l)\}, & x \in \Omega_{>c_\ell}(V_l), \\ \{(1, g_l(x)), (2, l)\}, & x \in \Omega_{=c_\ell}(V_l) \cap \mathbf{D}_l, \end{cases} \end{aligned} \tag{2.11d}$$

rendering the set $\{0\} \times \mathbf{L}$ *globally asymptotically stable for* $(\Sigma_h(\mathscr{H}))$*.*

Figure 2.1 illustrates the sets defined in Eq. (2.11a), when system $(\Sigma_h(u))$ is bidimensional. A sketch of the proof of Theorem 2.14 is provided in Sect. 2.7.2.

Note that Theorem 2.14 can also be employed to design controllers for systems that do not satisfy Brockett's necessary condition for the existence of a continuous stabilizing feedback law (see, e.g., [8, Example 38] and [11]).

Remark 2.15 A concept of robust stability of system $(\Sigma_h(\mathscr{H}))$ is recalled from [7, Definition 6.27] and [23, Definition 4.1] as follows. A compact invariant set $\mathbf{S} \subset \mathbb{R}^n$ is said to be *robustly asymptotically stable for system* $(\Sigma_h(\mathscr{H}))$ *with respect to measurement noise* if $\mathbf{S}$ is asymptotically stable for $(\Sigma_h(\mathscr{H}))$ and there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\mathbf{S}$ is also asymptotically stable for system

$$\begin{cases} \dot{x} \in f_q^\rho(x), & x \in \mathbf{C}_q^\rho, \\ q^+ \in g_q^\rho(x), & x \in \mathbf{D}_q^\rho, \end{cases} \tag{$\Sigma_h^\rho(\mathscr{H})$}$$

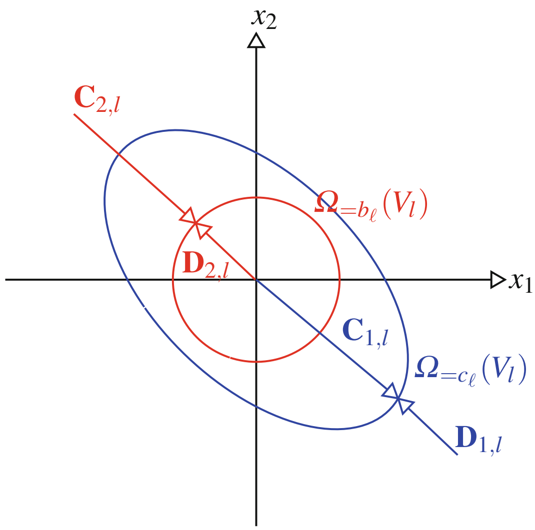

Fig. 2.1 Illustration of the proposed approach, when system $(\Sigma_h(u))$ is bidimensional

where

$$
\begin{aligned}
f_q^\rho(x) &= \operatorname{co}\left\{ f_h\Big(x, \varphi_q(\mathbf{B}_{\leq\rho(x)}(x) \cap \mathbf{C}_q)\Big)\right\}, \\
g_q^\rho(x) &= g_q\Big(\mathbf{B}_{\leq\rho(x)}(x) \cap \mathbf{D}_q\Big), \\
\mathbf{C}_q^\rho &= \left\{x \in \mathbb{R}^n : \mathbf{B}_{\leq\rho(x)}(x) \cap \mathbf{C}_q \neq \emptyset\right\}, \\
\mathbf{D}_q^\rho &= \left\{x \in \mathbb{R}^n : \mathbf{B}_{\leq\rho(x)}(x) \cap \mathbf{D}_q \neq \emptyset\right\}.
\end{aligned}
$$

From the definition of a hybrid feedback law, the set $\{0\} \times \mathbf{L}$ is robustly asymptotically stable for system $(\Sigma_h(\mathscr{H}))$ (c.f. [23, Theorem 4.3]).

Note that the definition of the sets $\mathbf{C}_q$ and $\mathbf{D}_q$ imposes a limitation on the magnitude of the perturbation ρ. More precisely, if ρ is such that $\mathbf{D}_{1,l}^\rho \cap \mathbf{D}_{2,l}^\rho \neq \emptyset$, then the hysteresis region is empty and solutions to system $(\Sigma_h(\mathscr{H}))$ may chatter between the feedback laws φ_b and φ_l. In this case, the x-component of the solution would remain in the region $\mathbf{D}_{1,l}^\rho \cap \mathbf{D}_{2,l}^\rho$ and never converge to zero.

So far, the global asymptotic stability of the set $\{0\} \times \mathbf{L}$ has been obtained by assuming the existence of a local stabilizer and combining it with a controller that renders a set globally practically asymptotically stable. In the next section, an approach is proposed to design the local stabilizer satisfying constraints on the basin of attraction.

2.5 Semiglobal Stabilization

In this section, another notion of stabilization of the set $\{0\} \times \mathbf{L}$ is considered. This concept is recalled from [4] and [13, p. 126] and is employed to design a feedback law satisfying Assumptions 2.11 and 2.13.

Definition 2.16 (*Semiglobal Asymptotic Stabilizability*) The origin is said to be *semiglobally asymptotically stabilizable for system* $(\Sigma_h(u))$ if, for every compact set $\mathbf{K} \subset \mathbb{R}^n$ containing the origin, there exists a feedback law $\varphi_\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{K}$ is contained in the basin of attraction of $(\Sigma_h(\varphi_\ell))$.

Using the approach proposed in [2], the nonlinear dynamics of $(\Sigma_h(u))$ is formulated in terms of a linear differential inclusion (LDI). A brief explanation of, this method can be given as follows. Let $\mathbf{L}$ be a singleton, define a neighborhood $\mathbf{N}_{\leq r}$ of the origin strictly containing the set $\mathbf{A}$. In this neighborhood, the search of a continuous stabilizing feedback law $\varphi_l : \mathbb{R}^n \to \mathbb{R}$ for system $(\Sigma_h(u))$ is formulated in terms of semidefinite programming with the constraint that the basin of attraction of the closed-loop system $(\Sigma_h(\varphi_l))$ must contain the set $\mathbf{A}$.

Under Assumption 2.1 the set $\mathbf{A}$ is compact. Thus, there exist a finite set of indexes $\mathbf{P} \subset \mathbb{N}$ and a set of vectors $\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}$ such that

$$\mathbf{A} \subset \operatorname{co}\left\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\right\}. \tag{2.12}$$

Let $r_u > 0$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ be such that

$$\operatorname{co}\left\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\right\} \subset \mathbf{N}_{\leq r} = \left\{x \in \mathbb{R}^n : |x_i| \leq r_i, i = 1, 2, \ldots, n\right\}.$$

Define the matrices $F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times m}$ as

$$F = \frac{\partial f_h}{\partial x}(0) \text{ and } G = \frac{\partial f_h}{\partial u}(0). \tag{2.13a}$$

These matrices correspond to the first approximation of $(\Sigma_h(u))$ at the origin, i.e.,

$$\dot{x} = Fx + Gu. \tag{2.13b}$$

Define the function

$$\widetilde{f}_h(x, u) = f_h(x, u) - Fx - Gu. \tag{2.14}$$

Note that $\widetilde{f}_h$ is of class $\mathscr{C}^1$.

Consider the set of indexes $\mathbf{M} = \left\{m \in \mathbb{N} : 1 \leq m \leq 2^{n\times n}\right\}$. For each index $m \in \mathbf{M}$, let $C_m \in \mathbb{R}^{n\times n}$ be a matrix with components $c_{ij}^{\pm}$, where $i, j = 1, 2, \ldots, n$, are given by either

$$c_{ij}^{+} = \max\left\{\frac{\partial \widetilde{f}_{h,i}}{\partial x_j}(x, u) : x \in \mathbf{N}_{\leq r}, |u| \leq r\right\} \tag{2.15a}$$

or

$$c_{ij}^- = \min\left\{\frac{\partial \widetilde{f}_{h,i}}{\partial x_j}(x,u) : x \in \mathbf{N}_{\leq r}, |u| \leq r\right\}. \tag{2.15b}$$

Analogously to the previous paragraph, consider the set of indexes $\mathbf{V} = \{v \in \mathbb{N} : 1 \leq v \leq 2^n\}$. For each index $v \in \mathbf{V}$, let $D_v \in \mathbb{R}^{n\times 1}$ be a vector with components $d_i^{\pm}$, where $i = 1, 2, \ldots, n$, are given by either

$$d_i^+ = \max\left\{\frac{\partial \widetilde{f}_{h,i}}{\partial u}(x,u) : x \in \mathbf{N}_{\leq r}, |u| \leq r\right\} \tag{2.16a}$$

or

$$d_i^- = \min\left\{\frac{\partial \widetilde{f}_{h,i}}{\partial u}(x,u) : x \in \mathbf{N}_{\leq r}, |u| \leq r\right\}. \tag{2.16b}$$

Remark 2.17 To see that the maximum number of matrices C_m and vectors D_v are $2^{n\times n}$ and 2^n, respectively, consider a differentiable function $g : \mathbb{R}^p \to \mathbb{R}^q$. The (i, j)-th element of the Jacobian matrix J of g is given by

$$J_{ij} = \frac{\partial g_i}{\partial x_j},$$

where $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.

Consider a matrix whose elements are given by either J_{ij}^- or J_{ij}^+. In particular, consider the matrix with only one element J_{ij}^-. When $i = j = 1$, this matrix is represented by

$$\begin{bmatrix} - & + & \cdots & + \\ + & + & \cdots & + \\ \vdots & \vdots & \ddots & \vdots \\ + & + & \cdots & + \end{bmatrix}.$$

The number of matrices with only one element J_{ij}^- is given by the combination of the symbol "$-$" among all of the $(p \times q - 1)$ symbols "$+$". Thus,

$$\binom{p\times q}{1} = \frac{(p\times q)!}{1!(p\times q-1)!} = (p\times q)!$$

which is the permutation of "$-$" among all "$+$".

Hence, the total number of matrices is given by the sum of all the above combinations. More precisely,

$$\sum_{k=1}^{p\times q}\binom{p\times q}{k} = \sum_{k=1}^{p\times q}\frac{(p\times q)!}{k!(p\times q-k)!} = 2^{p\times q},$$

Therefore, the number of matrices grows exponentially with the dimension of the domain and image sets of g.

Note also that, depending on the structure of the function g, the number of matrices may be smaller than $p \times q$. Example 2.18 illustrates this case.

Example 2.18 Consider the nonlinear function $f_{h,1} \in \mathscr{C}^1(\mathbb{R}^3, \mathbb{R})$, and the linear function $f_{h,2} \in \mathscr{C}^1(\mathbb{R}^3, \mathbb{R})$. Recall the definition of the function $\widetilde{f}_h$ from (2.14),

$$\widetilde{f}_h(x, u) = f_h(x, u) - Fx - Gu,$$

where $x := (x_1, x_2)$. Since $f_{h,2}$ is linear, the function (2.14) is given, for every $(x_1, x_2, u) \in \mathbb{R}^3$, by

$$\widetilde{f}_h(x_1, x_2, u) = \left(\widetilde{f}_{h,1}(x_1, u), 0\right)^\top.$$

Following the previous definitions of the derivatives, the 16 matrices C_m are

$$\begin{aligned}
&\begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^- & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^- & c_{22}^- \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^- & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^- & c_{22}^- \end{bmatrix}.
\end{aligned} \tag{2.17}$$

Because of the structure of $\widetilde{f}_{h,1}$ and $\widetilde{f}_{h,2}$, $c_{12}^- = c_{12}^+ = 0$, and for every $i = 1, 2$, $c_{2,i}^+ = c_{2,i}^- = 0$. Thus, the matrices (2.17) are reduced to the first pair of the first line.

For the derivatives with respect to the input u, the 4 vectors D_v are

$$\begin{bmatrix} d_1^+ \\ d_2^+ \end{bmatrix}, \quad \begin{bmatrix} d_1^- \\ d_2^+ \end{bmatrix}, \quad \begin{bmatrix} d_1^+ \\ d_2^- \end{bmatrix}, \quad \begin{bmatrix} d_1^- \\ d_2^- \end{bmatrix},$$

Since $d_2^- = d_2^+ = 0$, these vectors are also reduced to the first pair.

From Eq. (2.14), for every $x \in \mathbf{N}_{\leq r}$ and for every $|u| \leq r_u$, the value $f_h(x, u)$ is contained in the convex set formed by the first-order approximation (2.13) and the matrices C_m and vectors D_v, where $m \in \mathbf{M}$ and $v \in \mathbf{V}$. More precisely, for every $x \in \mathbf{N}_{\leq r}$ and for every $|u| \leq r_u$, system $(\Sigma_h(u))$ satisfies the following linear differential inclusion

$$\dot{x} \in \mathrm{co}\left\{(F + C_m)x + (G + D_v)u\right\},$$

where $m \in \mathbf{M}$ and $v \in \mathbf{V}$. For a proof of this claim, the interested reader may address [24, Sect. 2.5.4].

Let the set of vectors $\{e_i\}_{i\in\mathbf{I}}$, where $\mathbf{I} = \{i \in \mathbb{N} : 1 \le i \le n\}$, be the canonical basis in $\mathbb{R}^n$. In other words, the vector e_i has all components equal to zero except the i-th which is equals to 1. Let $M \in \mathbb{R}^{n\times n}$ be a matrix. The notation $M \succ 0$ (resp. $M \succeq 0$) stands for M being positive (semi)definite.

The next result is recalled from [27]. It concerns the design a local stabilizing feedback φ_1 for system (2.14) satisfying the constraint that the set $\mathbf{A}$ is included in the basin of attraction of the origin of $(\Sigma_h(\varphi_1))$.

Proposition 2.19 *Assume that there exist for a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$ and a matrix $H \in \mathbb{R}^{n\times 1}$ satisfying, for every indexes $m \in \mathbf{M}$ and $v \in \mathbf{V}$, the following set of linear matrix inequalities*

$$W(F + C_m)^T + H(G + D_v)^T + (F + C_m)W + (G + D_v)H^T \prec 0, \qquad (2.18a)$$

$$\forall i \in \mathbf{I}, \quad \begin{bmatrix} r_i^2 W & We_i \\ * & 1 \end{bmatrix} \succeq 0, \qquad (2.18b)$$

$$\forall p \in \mathbf{P}, \quad \begin{bmatrix} 1 & x_p^T \\ * & W \end{bmatrix} \succeq 0, \qquad (2.18c)$$

and

$$\begin{bmatrix} r_u^2 W & H \\ * & 1 \end{bmatrix} \succeq 0. \qquad (2.18d)$$

Then, by letting $\mathbf{L} = \{1\}$, $V_1(x) = x^T Px$, where $P = W^{-1}$, $c_\ell = 1$, $\mathbf{C}_1 = \mathbb{R}^n$, $\mathbf{D}_1 = \Omega_{\ge 1}(V_1)$, $g_1(x) \equiv \{1\}$ and $\varphi_1(x) = Kx$, where $K = H^T P$, Assumptions 2.11 and 2.13 hold.

A sketch of the proof is given as follows. Eq. (2.18a) implies that the function V_1 is a Lyapunov function in the small (cf. Definition A.34) for system $(\Sigma_h(\varphi_1))$.

From Schur's complement,[1] the linear matrix inequalities (2.18b)–(2.18d) are, respectively, equivalent to the following set of inequalities in the variables W and H

$$We_i e_i^T W^T \preceq r_i^2 W, \qquad (2.18b.bis)$$

$$x_p W^{-1} x_p^T \le 1, \qquad (2.18c.bis)$$

$$HH^T \preceq Wr_u^2. \qquad (2.18s.bis)$$

Equation (2.18b.bis) implies the inclusion $\Omega_{\le 1}(V_1) \subset \mathbf{N}_{\le r}$ holds. Equation (2.18c.bis) implies that $\text{co}\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\} \subset \Omega_{\le 1}$. Finally, Eq. (2.18d.bis) implies $|u| \le r_u$.

Note that, from Proposition 2.19, the origin is semiglobally asymptotically stabilizable for system $(\Sigma_h(u))$. Global asymptotic stability of $\{0\} \times \{(1, 1)\}$ is achieved by blending the practical asymptotic stability of $\mathbf{A}$ with the semiglobal asymptotic

[1] See Theorem 2.25.

stability of the origin. This is formalized in the next result which is a consequence of Theorem 2.14 and is recalled from [24].

Corollary 2.20 *Under Assumptions 2.1 and 2.2 and the hypotheses of Proposition 2.19, by defining the hybrid feedback law* $\mathscr{H}$ *as in (2.11), the set* $\{0\} \times \{(1, 1)\}$ *is globally asymptotically stable for* $(\Sigma_h(\mathscr{H}))$.

2.6 Illustration

Consider the system presented in [27] and given by

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + 0.1[x_1^2 + (1 + x_1)\sin(u)], \\ \dot{x}_2 = u. \end{cases} \tag{2.19}$$

Define the functions

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + x_2 + 0.1x_1^2, & h_1(x_1, x_2, u) &= 0.1(1 + x_1)\sin(u), \\ f_2(x_1, x_2) &\equiv 1, & h_2(x_1, x_2, u) &\equiv 0. \end{aligned}$$

Due the presence of the term $0.1(1 + x_1)\sin(u)$, the use of the backstepping technique requires to solve an implicit equation in the variable u (see more details in Remark 2.22). This motivates the approach proposed in this chapter.

Checking Assumptions

First, Assumptions 2.1 and 2.2 are verified. Then, Proposition 2.19 is employed to design a controller such that Assumptions 2.11 and 2.13 are satisfied. Next, Proposition 2.4 and Corollary 2.20 are employed to design a hybrid feedback law $\mathscr{H}$ rendering the set $\{0\} \times \{(1, 1)\}$ globally asymptotically stable for system (2.19) in closed loop.
Assumption 2.1. Consider the function defined, for every $x_1 \in \mathbb{R}$, as $V_1(x_1) = x_1^2/2$. The Lie derivative of V_1 along the vector field f_1 yields, for every $x_1 \in \mathbb{R}$,

$$L_{f_1} V_1(x_1, x_2) = x_1^2 + x_1 x_2 + 0.1x_1^3.$$

Consider the feedback law ψ_1 for subsystem x_1 of (2.19) defined, for every $x_1 \in \mathbb{R}$, as $\psi_1(x_1) = -(1 + K_1)x_1 - 0.1x_1^2$, where $K_1 > 0$ is a constant value. Letting $x_2 = \psi_1(x_1)$, the Lie derivative $L_{f_1} V_1$ yields the equation

$$L_{f_1} V_1(x_1, \psi_1(x_1)) = -K_1 x_1^2 = -\alpha(V_1(x_1)),$$

where, for every $s \in \mathbb{R}_{\geq 0}$, $\alpha(s) := 2K_1 s$. Thus, Assumption 2.1 holds.

Assumption 2.21 The following remark explains the need of Assumption 2.2 for the example in consideration.

Remark 2.22 Since Assumption 2.1 holds, the backstepping technique could be employed to design a stabilizing feedback law for system (2.19), if the term h_1 was identically zero (see more details on the backstepping procedure in Sect. 2.8).

Based on the backstepping procedure, consider the function V defined in Eq. (2.2) and with $n = 2$. The Lie derivative of V along the vector field $(f_1 + h_1,\ f_2)^\top$ satisfies the inequality

$$\begin{aligned} L_{f_h} V(x_1, x_2, u) \leq &- K_1 x_1^2 + x_1 0.1(1 + x_1) \cdot \sin(u) + (x_2 - \psi_1(x_1)) \\ &\cdot \left(u + \frac{x_1}{2} + (1 + K_1 + 0.2 K_1 x_1) \right. \\ &\left. \cdot (x_1 + x_2 + 0.1[x_1^2 + (1 + x_1) \cdot \sin(u)]) \right), \end{aligned} \tag{2.20}$$

for every $(x_1, x_2, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

To obtain a term proportional to the term $(x_2 - \psi_1(x_1))^2$ on the right-hand side of the inequality (2.20), it would be necessary to solve an implicit equation in the variable u defined as $E(x_1, x_2, u) \leq -K_1 x_1^2 - L(x_2 - \psi_1(x_1))^2$, where E is the right-hand side of (2.20), and $L > 0$ is a constant value. This procedure seems to be difficult (if not impossible), and motivates the design a hybrid feedback by applying Theorem 2.14.

To see that items 1 and 2 of Assumption 2.2 hold, note that the equations

$$\begin{aligned} |h_1(x_1, x_2, u)| &= |0.1(1 + x_1) \sin(u)| \leq 0.1(1 + |x_1|), \\ |h_2(x_1, x_2, u)| &\equiv 0, \\ \left| \frac{\partial h_1}{\partial x_2}(x_1, x_2, u) \right| &\equiv 0, \\ L_{h_1} V_1(x_1, \psi_1(x_1), u) &\leq |x_1| 0.1 + x_1^2 0.1 \leq \frac{x_1^2}{2} 1.2 + \frac{0.1^2}{2}, \end{aligned}$$

hold, for every $(x_1, x_2, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

It remains to define the function Ψ and constant values ε and M. Let, for every $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, the function $\Psi(x_1, x_2) = 0.1(1 + |x_1|)$. For the constants, letting them satisfy the inequalities $\varepsilon \leq 1 - 1.2/(2K_1)$ and[2] $M \geq 0.1/(4K_1\varepsilon)$, items 1 and 2 of Assumption 2.2 are satisfied.

Using Proposition 2.19 to verify Assumptions 2.11 and 2.13. In the following, the process described in Sect. 2.5 is illustrated.

From the definitions of the storage function V_1 and the feedback law ψ_1, the set $\mathbf{A}$ is given by

$$\mathbf{A} = \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1| \leq \sqrt{2M}, x_2 = -(1 + K_1)x_1 - 0.1 x_1^2 \right\}.$$

[2]The conditions $\varepsilon \leq 1 - 1.2/(2K_1)$ and $\varepsilon > 0$ imply the lower bound $K_1 > 0.6$.

Since $\psi_1 \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$, from the mean value theorem, for every $x_1 \in \left[-\sqrt{2M}, \sqrt{2M}\right] \setminus \{0\}$, there exits $c \in \left[-\sqrt{2M}, \sqrt{2M}\right]$ such that

$$\psi_1(x_1) = \frac{\partial \psi_1}{\partial x_1}(c) \cdot x_1.$$

Define the constant values

$$a^+ = \max_{|x_1| \le \sqrt{2M}} \frac{\partial \psi_1}{\partial x_1}(x_1) = \max_{|x_1| \le \sqrt{2M}} -(1 + K_1) - 0.2x_1 = -(1 + K_1) + 0.2\sqrt{2M}$$
$$a^- = \min_{|x_1| \le \sqrt{2M}} \frac{\partial \psi_1}{\partial x_1}(x_1) = \min_{|x_1| \le \sqrt{2M}} -(1 + K_1) - 0.2x_1 = -(1 + K_1) - 0.2\sqrt{2M}$$

and the set $\mathbf{P} = \{1, 2, 3, 4\}$ of indexes. From these definitions, for every $|x_1| \le \sqrt{2M}$, the inequality

$$a^- \cdot x_1 \le \psi_1(x_1) \le a^+ \cdot x_1$$

holds. This implies that, for every $(x_1, x_2) \in \mathbf{A}$, the inequality $a^- \cdot x_1 \le x_2 \le a^+ \cdot x_1$ holds. Consequently,

$$\mathbf{A} \subseteq \mathrm{co}\Bigg\{ \left(\left\{\sqrt{2M}\right\} \times \left\{x_2^{+,<0}, x_2^{-,<0}\right\}\right) \cup \left(\left\{-\sqrt{2M}\right\} \times \left\{x_2^{+,>0}, x_2^{-,>0}\right\}\right) \Bigg\}, \tag{2.21}$$

where

$$\begin{aligned} x_2^{+,>0} &= -a^+\sqrt{2M},\ x_2^{+,<0} = a^+\sqrt{2M}, \\ x_2^{-,>0} &= -a^-\sqrt{2M},\ x_2^{-,<0} = a^-\sqrt{2M}. \end{aligned} \tag{2.22}$$

Figure 2.2 illustrates the inclusion (2.21).

A condition for feasibility of the linear matrix inequalities (2.18) of Proposition 2.19 is the inclusion $\mathbf{A} \subset \mathbf{N}_{\le r}$. This inclusion holds, whenever the inequalities $\sqrt{2M} < r_1$ and $|a^{\pm}\sqrt{2M}| < r_2$ are satisfied. These inequalities imply that K_1 is bounded as follows.

$$\frac{0.1}{2r_1^2} + 0.6 < K_1 < \frac{r_2}{r_1} - 0.2r_1 - 1. \tag{2.23}$$

Note that the bounds on K_1 described in Eq. (2.23) impose a limitation on the speed of response.

Employing the technique presented in Sect. 2.5, let $\theta = 0.1$, $r = [1, 2]$, $\mathbf{N}_{\le r} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1| \le 1, |x_2| \le 2\}$, and $|u| \le 2\pi$. For these choices, Eq. (2.23) holds with $K_1 = 1.45$. To satisfy Assumption 2.2, let $M = 0.03$ and $\varepsilon = 0.6$.

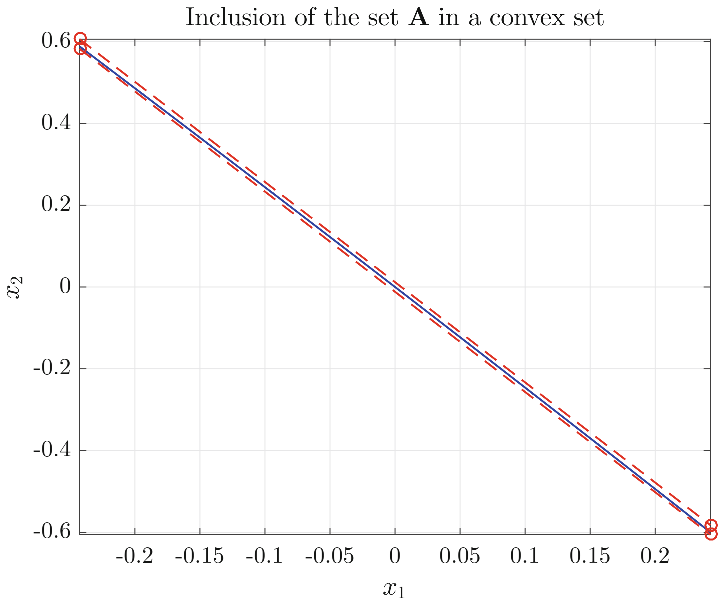

Fig. 2.2 The sets **A** (*solid blue line*) and the convex set defined in (2.21) (*dashed red line*) are presented. The *circles* are the vertexes of the convex set

The matrices F and G defined in (2.13a) are given by

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}.$$

Function $\tilde{f}_h$ is given by

$$\tilde{f}_h(x, u) = \begin{bmatrix} 0.1\sin(u)(x_1 + 1) - 0.1u + 0.1x_1^2 \\ 0 \end{bmatrix},$$

its derivatives with respect to the state and input variable are, respectively, given by

$$\frac{\partial \tilde{f}_h}{\partial x}(x_1, x_2) = \begin{bmatrix} 0.2x_1 + 0.1\sin(u) & 0 \\ 0 & 0 \end{bmatrix}$$
$$\frac{\partial \tilde{f}_h}{\partial u}(x_1, x_2) = \begin{bmatrix} 0.1\cos(u)(x_1 + 1) - 0.1 \\ 0 \end{bmatrix}.$$

The matrices $C_m \in \mathbb{R}^{2\times 2}$, indexed by $m \in \{m \in \mathbb{N} : 1 \leq m \leq 2\}$, and vectors $D_v \in \mathbb{R}^2$, index by $\{v \in \mathbb{R} : 1 \leq v \leq 2\}$, have components defined by the set of Eqs. (2.15) and (2.16), respectively. The matrices and vectors that are not identically zero are given by

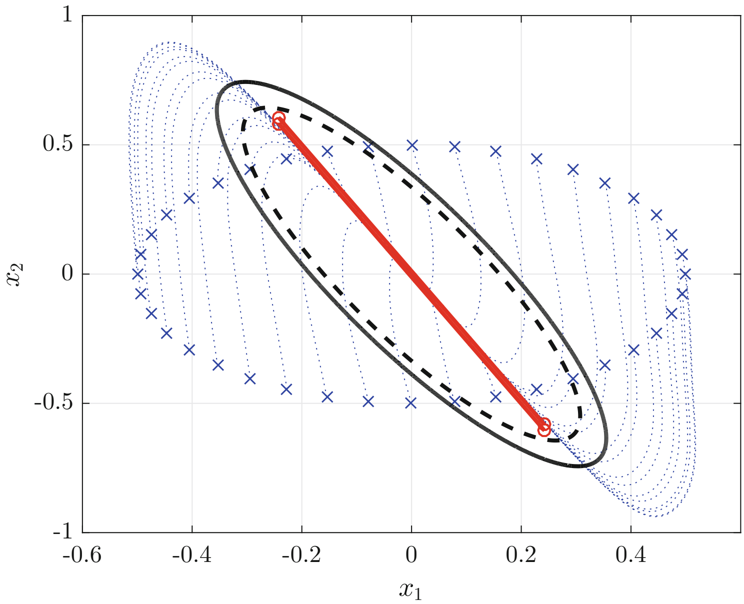

Fig. 2.3 The sets $\Omega_{=1}(x^T P x)$ (in *black* with *solid line*), $\Omega_{=0.75}(x^T P x)$ (in *black* with *dashed line*), and the set $\mathrm{co}\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}$ (in *red*) at the center. Initial conditions (*blue crosses*) are points given in a ball of radius 0.5 and centered at the origin

$$C_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \; \text{and } D_2 = \begin{bmatrix} -0.3 \\ 0 \end{bmatrix}.$$

Using the optimization parser Yalmip [18] and solver Mosek to solve the linear matrix inequalities (2.18), the following matrices P and K have been obtained

$$P = \begin{bmatrix} 29.1836 & 11.8910 \\ 11.8910 & 6.6548 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -12.7326 & -6.3529 \end{bmatrix}.$$

From Proposition 2.19, Assumptions 2.11 and 2.13 hold with $c_\ell = 1$.

Figure 2.3 shows some solutions of system (2.19) in closed loop with the feedback law φ_ℓ, the inclusions $\mathbf{A} \subset \Omega_{\leq 1}(V_1)$ and $\Omega_{\leq 1}(V_1) \subset \mathbf{N}_{\leq r}$.

Main Result

Since Assumptions 2.1 and 2.2 hold, from the proof of Proposition 2.4 the feedback law defined, for every $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, as

$$\begin{aligned} \varphi_g(x_1, x_2) = & -(1 + K_1 + 2\theta x_1)(x_1 + \theta x_1^2 + x_2) - \frac{x_1}{2K_V} \\ & - \frac{x_1 - \psi_1(x_1)}{K_V} \left[c + \frac{c}{4} \Delta(x_1, x_2)^2 \right], \end{aligned}$$

where

$$\Delta(x_1, x_2) = |x_1|\theta(1 + |x_1|) + K_V\theta(1 + |x_1|)(1 + |1 + K_1 + 2\theta x_1|),$$

and with parameters $\theta = 0.1$ and $c = K_v = 350$ renders the set $\mathbf{A}$ globally practically asymptotically stable for system (2.19) in closed loop with φ_g.

To employ the hybrid feedback law $\mathscr{H}$ provided by Theorem 2.14, let $b_\ell = 0.75$, $\mathbf{Q} = \{1, 2\} \times \{1\}$. From Eq. (2.11a), the subsets of $\mathbb{R}^2$ are given by

$$\begin{aligned} \mathbf{C}_{1,1} &= \Omega_{\leq 1}(x^T P x), \quad &\mathbf{D}_{1,1} &= \Omega_{\geq 1}(x^T P x), \\ \mathbf{C}_{2,1} &= \Omega_{\geq 0.75}(x^T P x), \quad &\mathbf{D}_{2,1} &= \Omega_{\leq 0.75}(x^T P x). \end{aligned}$$

From (2.11b), the function $\varphi_{q,1}$ is given by

$$\varphi_{q,1}(\cdot) = \begin{cases} \varphi_1(\cdot), \text{ if } q = 1, \\ \varphi_g(\cdot), \text{ if } q = 2, \end{cases}$$

while, from (2.11c) and (2.11d), the function $g_{q,1}$ given by

$$\mathbf{D}_{2,1} \ni x \mapsto g_{2,1}(x) = \{(1, 1)\}$$

and

$$\begin{aligned} g_{1,1} : \mathbf{D}_{1,1} &\rightrightarrows \mathbf{Q} \\ x &\mapsto \begin{cases} \{(1, 1)\}, \; x \in \Omega_{<1}(x^T P x) \cap \mathbf{D}_1, \\ \{(2, 1)\}, \; x \in \Omega_{>1}(x^T P x), \\ \{(1, 1), (2, 1)\}, \; x \in \Omega_{=1}(V_l) \cap \mathbf{D}_1. \end{cases} \end{aligned}$$

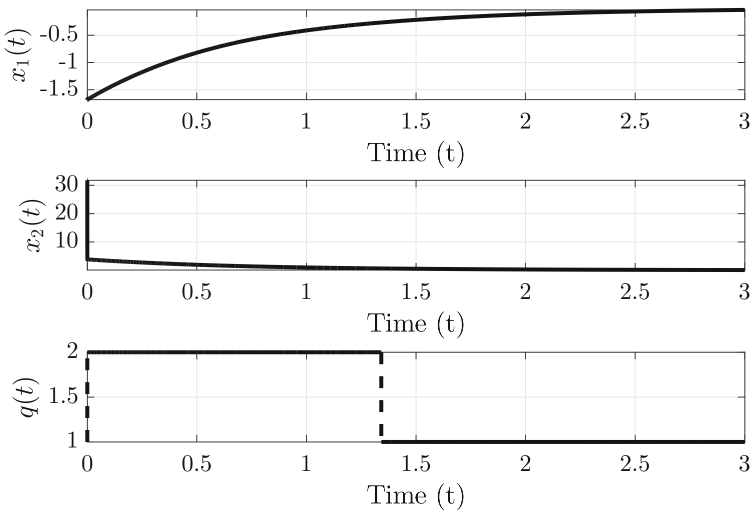

Fig. 2.4 Continuous-time evolution of a solution of (2.19) in closed loop with $\mathscr{H}$ starting from $(-1.6814, 31.7552, 1)$

Moreover, from Corollary 2.20, the set $\{0\} \times \{(1, 1)\}$ is globally asymptotically stable for system (2.19) in closed loop with $\mathscr{H}$.

A simulation of (2.19) in closed loop with $\mathscr{H}$ with initial condition $(x_1, x_2, q) = (-1.6814, 31.7552, 1)$ is presented in Fig. 2.4. The continuous-time evolution of the components[3] x_1, x_2 and q of the solution is shown. First, system (2.19) is in closed loop with φ_g (for $t \in [0, 1.4]$), and after (2.19) is in closed loop with φ_1, and the solution converges to the origin.

2.7 Sketches of the Proofs of Chap. 2

2.7.1 Sketch of the Proof of Proposition 2.4

Under Assumptions 2.1 and 2.2, recall the proper function $V_1 \in (\mathscr{C}^1 \cap \mathscr{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geq 0})$, the feedback law $\psi_1 \in \mathscr{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$ for system $(\Sigma_1\ (x_2))$ and the set $\mathbf{A}$ defined in Eq. (2.8). To sketch a proof of Proposition 2.4, the following lemma is recalled from [27].

Lemma 2.23 *There exist constant values $a' > 0$ and $K_V > 0$, and a function given by*

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{K_V}{2}(x_2 - \psi_1(x_1))^2 \end{aligned} \tag{2.24}$$

such that the set $\Omega_{\leq a'}(V)$ satisfies the inclusion

$$\Omega_{\leq a'}(V) \subset \mathbf{B}_{\leq a}(\mathbf{A}). \tag{2.25}$$

In other words, Lemma 2.23 shows that it is possible to choose the gain K_V such that there exist sublevel sets of the function given by (2.24) that are contained in $\mathbf{B}_{\leq a}(\mathbf{A})$.

Remark 2.24 What remains is to estimate the values K_V and a'. By letting the constant values $a' \leq M+a$ and $K_v = {}^{(M+a)}/_{(2a^2)}$ the inclusion $\Omega_{\leq a'}(V) \subset \mathbf{B}_{\leq a}(\mathbf{A})$ is satisfied. To see this claim, pick any $(x_1^*, x_2^*) \in \Omega_{\leq a'}(V)$ such that[4]

$$\begin{cases} V_1(x_1^*) \leq \dfrac{a'}{2} \\ |x_2^* - \psi_1(x_1^*)| \leq a. \end{cases}$$

---

[3]Regarding q, here it is shown only its first component, because the second one does not change.

[4]Note that with these constraints, $V(x_1^*, x_2^*) \leq a'$.

This implies that

$$V(x_1^*, x_2^*) = V_1(x_1^*) + \frac{K_V}{2}(x_2^* - \psi_1(x_1^*))^2 \leq \frac{M+a}{2} + \frac{M+a}{4} = 3\frac{M+a}{4}$$

and the inclusion (2.25) holds.

Note that K_V makes V dependent on the parameter a. In other words, Eq. (2.24) is also parametrized by a. A theorem to deduce uniform global practical stability is provided in [3, Theorem 7.5] (see Theorem 2.26), where an additional condition regarding the behavior Lyapunov function with respect to the parameter a is introduced.

A sketch of the proof of Proposition 2.4 can now be provided. The objective of this sketch is to show that there exist a continuous feedback law $\varphi_g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ and a constant value $a' > 0$ such that the set $\Omega_{a'}(V) \subset \mathbf{B}_{\leq a}(\mathbf{A})$ is globally asymptotically stable for the closed-loop system $\Sigma_h(\varphi_g)$.

Define the function

$$\begin{aligned} r_1 : \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} &\to \mathbb{R}^{n-1} \\ (x_1, x_2, u) &\mapsto f_1(x_1, x_2) + h_1(x_1, x_2, u). \end{aligned}$$

The Lie derivative of V_1 along r_1 yields the inequality

$$\begin{aligned} L_{r_1} V_1(x_1, x_2, u) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] &+ L_{r_1} V_1(x_1, x_2, u) \\ &- L_{r_1} V_1(x_1, \psi_1(x_1), u) \end{aligned} \tag{2.26}$$

which holds for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, due to Assumption 2.1 and item 1 of[5] Assumption 2.2.

Fix $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and let the function

$$\begin{aligned} \eta_{x_1,x_2} : [0, 1] &\to \mathbb{R} \\ s &\mapsto s x_2 + (1 - s)\psi_1(x_1). \end{aligned}$$

Since $r_1 \in \mathscr{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^{n-1})$ and $\eta_{x_1,x_2} \in \mathscr{C}^1([0, 1], \mathbb{R})$, the equation

$$r_1(x_1, x_2, u) - r_1(x_1, \psi_1(x_1), u) = (x_2 - \psi_1(x_1)) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), u)\, ds$$

holds.

[5] Recall that the inequality $L_{h_1} V_1(x_1, \psi_1(x_1), u) \leq (1 - \varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M)$ is assumed.

Hence, from Eq. (2.26), the inequality

$$L_{r_1} V_1(x_1, x_2, u) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \frac{\partial V_1}{\partial x_1}(x_1) \cdot (x_2 - \psi_1(x_1)) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), u)\, ds \tag{2.27}$$

holds, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$.

Consider the function $\tilde{\psi} : \left(\left(\mathbb{R}^{n-1} \times \mathbb{R}\right) \setminus \{(0,0)\}\right) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{\psi}(x_1, x_2, \bar{u}) = \frac{1}{f_2(x_1, x_2)} \left(\frac{\bar{u}}{K_V} + L_{f_1} \psi_1(x_1, x_2) \right) \frac{1}{K_V} \frac{\partial V_1}{\partial x_1}(x_1) \left(\cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \right), \tag{2.28}$$

where K_V is given by Lemma 2.23.

Denote $\tilde{\psi}(x_1, x_2, \bar{u})$ as $\tilde{\psi}(\bar{u})$ and let $u = \tilde{\psi}(\bar{u})$. From inequality (2.27), the Lie derivative of function V (defined by Eq. (2.24)) along the vector field f_h yields the inequality

$$\begin{aligned} L_{f_h} V(x_1, x_2, \tilde{\psi}(\bar{u})) &\leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ &+ \frac{\partial V_1}{\partial x_1}(x_1)(x_2 - \psi_1(x_1)) \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &+ K_V(x_2 - \psi_1(x_1)) \\ &\cdot \left[f_2(x_1, x_2)\tilde{\psi}(\bar{u}) + h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - L_{r_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u})) \right]. \end{aligned} \tag{2.29}$$

Now, replacing the function (2.28) in inequality (2.29) yields

$$\begin{aligned} L_{f_h} V(x_1, x_2, \tilde{\psi}(\bar{u})) &\leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ &+ (x_2 - \psi_1(x_1)) \Bigg[\bar{u} + \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial h_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &+ K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - K_V L_{h_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u})) \Bigg]. \end{aligned} \tag{2.30}$$

Let

$$\Upsilon(x_1, x_2, \tilde{\psi}(\bar{u})) = \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial h_1}{\partial x_2}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds$$
$$+ K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - K_V L_{h_1} \psi_1(x_1, x_2, \tilde{\psi}(\bar{u})). \tag{2.31}$$

Replacing (2.31) into (2.30), the Lie derivative $L_{f_h} V$ yields the inequality

$$L_{f_h} V(x_1, x_2, \tilde{\psi}(\bar{u})) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))]$$
$$+ (x_2 - \psi_1(x_1))[\bar{u} + \Upsilon(x_1, x_2, \tilde{\psi}(\bar{u}))]. \tag{2.32}$$

Define, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the function

$$\Delta(x_1, x_2) = \left|\frac{\partial V_1}{\partial x_1}(x_1)\right| \int_0^1 \Psi(x_1, \eta_{x_1,x_2}(s))\, ds + K_V \Psi(x_1, x_2) \left(1 + \left|\frac{\partial \psi_1}{\partial x_1}(x_1)\right|\right).$$

From Assumption 2.2, the inequality

$$|\Upsilon(x_1, x_2, \tilde{\psi}(\bar{u}))| \leq \Delta(x_1, x_2),$$

holds, for every $(x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$.

From the Cauchy–Schwartz inequality, and for any constant value $c > 0$, the inequality

$$(x_2 - \psi_1(x_1))\Upsilon(x_1, x_2, \bar{u}) \leq \frac{1}{c} + \frac{c}{4}(x_2 - \psi_1(x_1))^2 \Delta(x_1, x_2)^2. \tag{2.33}$$

holds, for every $(x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$.

Let, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\bar{u} = -(x_2 - \psi_1(x_1)) \left[c + \frac{c}{4} \Delta(x_1, x_2)^2\right]. \tag{2.34}$$

Inequality (2.32) together with (2.33) and (2.34) yields, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and for any $c \geq 1$, the inequality

$$L_{f_h} V(x_1, x_2, \tilde{\psi}) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \frac{1}{c} - c(x_2 - \psi_1(x_1))^2, \tag{2.35}$$

where, $\tilde{\psi}(x_1, x_2)$ is denoted by $\tilde{\psi}$, to simplify the presentation.

Since V_1 is a proper function, the set

$$\mathbf{A}_{\geq 0} := \left\{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}^n : \varepsilon\alpha(V_1(x_1)) + c(x_2 - \psi_1(x_1))^2 \leq \varepsilon\alpha(M) + \frac{1}{c}\right\}$$

is compact. Note that $\mathbf{A}_{\geq 0}$ is the set of values of (x_1, x_2) for which the inequality $L_{f_h} V(x_1, x_2, \tilde{\psi}) \geq 0$ holds.

Define the value

$$\zeta = \max\{V(x_1, x_2) : (x_1, x_2) \in \mathbf{A}_{\geq 0}\},$$

for every $c > 1$. Note that, for every $(x_1, x_2) \in \Omega_{>\zeta}(V)$, the inequality $L_{f_h} V(x_1, x_2, \tilde{\psi}) < 0$ holds. In other words, the set $\Omega_{\leq\zeta}(V)$ is globally asymptotically stable for the closed-loop system $\Sigma_h(\tilde{\psi})$.

Let $K_\alpha > 0$ be the Lipschitz constant of α in the compact set $[0, \zeta]$. With this constant, the inequality

$$|\alpha(V_1(x_1)) - \alpha(V(x_1, x_2))| \leq \frac{K_V K_\alpha}{2}(x_2 - \psi_1(x_1))^2$$

holds, for every $(x_1, x_2) \in \Omega_{\leq\zeta}(V)$.

From Eq. (2.35), for every $c > 1$, and for every $(x_1, x_2) \in \Omega_{\leq\zeta}(V)$, the inequality

$$L_{f_h} V(x_1, x_2, \tilde{\psi}) \leq \varepsilon[\alpha(M) - \alpha(V(x_1, x_2))] + \frac{1}{c} - \left(c - \varepsilon\frac{K_V K_\alpha}{2}\right)(x_2 - \psi_1(x_1))^2$$

holds.

Consider the constant value a' given by Lemma 2.23 and let the value

$$c_g = \max\left\{\frac{1}{\varepsilon[\alpha(a') - \alpha(M)]}, \varepsilon\frac{K_V K_\alpha}{2}, 1\right\}.$$

For every $c > c_g$, and for every $(x_1, x_2) \in \Omega_{\leq\zeta}(V)$, the inequality

$$L_{f_h} V(x_1, x_2, \tilde{\psi}) \leq \varepsilon\left[\alpha(a') - \alpha(V(x_1, x_2))\right]$$

holds. Thus, for every $c > c_g$, and for every $(x_1, x_2) \in \Omega_{>a'}(V)$,[6] the inequality

$$L_{f_h} V(x_1, x_2, \tilde{\psi}) < 0$$

holds. Hence, the set $\Omega_{\leq a'}(V)$ is an attractor for $\Sigma_h(\tilde{\psi})$.

[6]Let $|x|_{a'} := \mathtt{dist}(x, \Omega_{\leq a'}(V))$. Since $\alpha \in \mathscr{K}_\infty$, define the function $\alpha_{a'}(|x|) := \alpha(|(x_1, x_2)|_{a'}) = -\alpha(a') + \alpha(V(x_1, x_2))$ which is of class $\mathscr{K}_\infty$. Thus, for every $(x_1, x_2) \in \Omega_{\geq a'}(V)$, $L_{f_h} V(x_1, x_2, \tilde{\psi}) \leq -\alpha_{a'}(|x|_{a'})$ and Eq. (2.39b) is satisfied.

From Lemma 2.23 the inclusion $\Omega_{\leq a'}(V) \subset \mathbf{B}_{\leq a}(\mathbf{A})$ holds, this implies that solutions to the closed-loop system $\Sigma_h(\tilde{\psi})$ with initial conditions belonging to $\mathbf{B}_{>a}(\mathbf{A})$ will converge to a set contained in $\mathbf{B}_{\leq a}(\mathbf{A})$. Thus, $\mathbf{A}$ is practically asymptotically stable for $\Sigma_h(\tilde{\psi})$.

From (2.28) and (2.34), consider the function

$$\varphi_g(x_1, x_2) = \frac{1}{K_V f_2(x_1, x_2)} \Bigg[K_V L_{f_1} \psi_1(x_1, x_2) - \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial x_2}(x_1, \eta_{x_1,x_2}(s))\, ds \\ -(x_2 - \psi_1(x_1)) \cdot \left(c + \frac{c}{4} \Delta^2(x_1, x_2) \right) \Bigg]$$

which is defined, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \setminus \{(0, 0)\}$ and $c > c_g$, the feedback law $\tilde{\psi}(\cdot) = \varphi_g(\cdot)$ rends the set $\Omega_{\leq a'}(V)$ globally asymptotically stable for the closed-loop system $\Sigma_h(\varphi_g)$. This concludes the sketch of the proof of Proposition 2.4.

2.7.2 *Sketch of the Proof of Theorem 2.14*

Let the constant values a, b_ℓ, and c_ℓ be such that the inequalities $0 < b_\ell < c_\ell$ and

$$\max\{V_l(x) : x \in \mathbf{B}_{\leq a}(\mathbf{A})\} < b_\ell, \tag{2.36}$$

hold, for every $l \in \mathbf{L}$.

Under Assumptions 2.1 and 2.2, Proposition 2.4 provides the continuous feedback law $\varphi_g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ which is used to design a hybrid feedback law $\mathscr{H}$ building an hysteresis of local and nonlocal feedback laws on appropriate domains.

Define the set of indexes $\mathbf{Q} = \{1, 2\} \times \mathbf{L}$. Consider the subsets (2.11a) and the maps defined in Eqs. (2.11b)–(2.11d). Recall that the state of the closed-loop system $(\Sigma_h(\mathscr{H}))$ is $(x, q) \in \mathbb{R}^n \times \mathbf{Q}$. The remainder of this sketch consists of the analysis of solutions to $(\Sigma_h(\mathscr{H}))$, according to the region where the initial condition belongs to.

Case 1. Assume that $q = \{(2, l)\}$.

1.i. If $x \in \mathbf{C}_{2,l}$, then Eq. (2.11b) implies that $\varphi_{2,l}(x) = \varphi_g(x)$. From Proposition 2.4, the set $\mathbf{A}$ is globally practically asymptotically stable for the closed-loop system $\Sigma_h(\varphi_g)$. Moreover, Eqs. (2.11a) and (2.36) imply that the inclusion $\mathbf{A} \subset \mathbf{D}_{2,l}$ holds. Together with the fact that solutions of $(\Sigma_h(\mathscr{H}))$ will not jump until the x-component be in the set $\mathbf{D}_{2,l}$, they will converge to $\mathbf{D}_{2,l}$;

1.ii. If $x \in \mathbf{D}_{2,l}$, then from (2.11c), $g_{2,l}(x) = \{(1, l)\}$ and, after the jump, the local hybrid feedback law is selected. Since the value of x does not change during a jump, $x \in \mathbf{D}_{2,l}$ after a jump. Moreover, Eqs. (2.11a) and (2.36) imply that the inclusion $\mathbf{D}_{2,l} \subset \Omega_{<c_\ell}(V_l)$ holds. From the local asymptotic stability of the set $\{0\} \times \mathbf{L}$, solutions to the closed-loop system $(\Sigma_h(\mathscr{H}_\ell))$ starting in $\mathbf{D}_{2,l}$ will converge to $\{0\} \times \mathbf{L}$;

To sum up Case 1, whenever $(X(0,0), Q(0,0)) \in \mathbb{R}^n \times \{(2,l)\}$, the issuing solutions of $(\Sigma_h(\mathscr{H}))$ converge to $\{0\} \times \mathbf{L}$.

Case 2. Assume that $q = \{(1,l)\}$.

2.i. If $x \in \mathbf{C}_{1,l}$, then from Eq. (2.11b), $\varphi_{1,l}(x) = \varphi_l(x)$, and the local hybrid feedback law is selected. From the local asymptotic stability of $\{0\} \times \mathbf{L}$, solutions to $\Sigma_h(\mathscr{H}_\ell)$ starting in $\mathbf{C}_{1,l}$ will converge to $\{0\} \times \mathbf{L}$;

2.ii. If $x \in \mathbf{D}_{1,l}$. Then from the definition of sets (2.11a) and the jump map (2.11d), either

- 2ii.a. $q^+ = \{(2,l)\}$ and, after the jump, φ_g is selected. Since before this jump $x \in \Omega_{\geq c_\ell}(V_l)$, and $\Omega_{\geq c_\ell}(V_l) \subset \mathbf{C}_{2,l}$, and the x-component remains constant after the jump; from Case 1.i., solutions of $(\Sigma_h(\mathscr{H}))$ converge to $\mathbf{D}_{2,l}$;
- 2ii.b. or $q^+ = \{(1, g_l(x))\}$ and, after the jump, a local feedback law is selected. Since before this jump, $x \in \Omega_{\leq c_\ell}(V_l) \cap \mathbf{D}_l$ and the x-component of the solutions remains constant just after the jump, from the local asymptotic stability of $\{0\} \times \mathbf{L}$, solutions of $\Sigma_h(\mathscr{H}_\ell)$ starting in $\Omega_{\leq c_\ell}(V_l) \cap \mathbf{D}_l$ will converge to $\{0\} \times \mathbf{L}$;

To sum up Case 2, whenever $(X(0,0), Q(0,0)) \in \mathbb{R}^n \times \{(1,l)\}$, the solutions of $(\Sigma_h(\mathscr{H}))$ converge to $\{0\} \times \mathbf{L}$.

Therefore, the set $\{0\} \times \mathbf{L}$ is locally stable and globally attractive for system $(\Sigma_h(\mathscr{H}))$. Hence, it is globally asymptotically stable for this system. This concludes the sketch of the proof of Theorem 2.14.

2.8 Summary

In this chapter, a method to design hybrid feedback laws has been presented. The hybrid controller is employed to combine a nonlinear feedback law, obtained by employing the backstepping method that practically stabilizes a compact set with a feedback law that renders the equilibrium of the origin locally asymptotically stable. The approach employed to design both controllers is constructive. In particular, the second controller is obtained by solving a semidefinite program under constraints in the basin of attraction.

Appendix of Chap. 2

The Backstepping Procedure

The backstepping is a well-known method to design a feedback law rendering a class of cascaded systems asymptotically stable. This procedure is recalled here. For more details, the interested reader may address [13, 15–17] and references therein.

Consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2)u \end{cases} \tag{2.37}$$

where the functions $f_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}_{\neq 0}$ are of class $\mathscr{C}^1$.

Assume that, there exists a feedback law $\phi_1 \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ with $\phi_1(0) = 0$ for the x_1-subsystem rendering the equilibrium of the origin globally asymptotically stable for

$$\dot{x}_1 = f_1(x_1, \phi_1(x_1)).$$

From the converse Lyapunov theorem,[7] there exist a proper function $V_1 \in (\mathscr{C}^1 \cap \mathbf{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geq 0})$ and a function $\alpha_1 \in \mathscr{K}_\infty$ such that the Lie derivative of V_1 along the vector field f_1 yields the inequality $L_{f_1} V_1(x_1, \phi_1(x_1)) \leq -\alpha_1(|x_1|)$, for every $x_1 \in \mathbb{R}^{n-1}$.

Fix the pair $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and consider the function

$$\begin{aligned} \eta_{x_1,x_2} : [0, 1] &\to \mathbb{R} \\ s &\mapsto s x_2 + (1 - s)\phi_1(x_1). \end{aligned}$$

Since for every pair $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the function f_2 is nonzero, by letting $u = v / f_2(x_1, x_2)$, system (2.37) can be rewritten as

$$\begin{cases} \dot{x}_1 = f_1(x_1, \phi_1(x_1)) + (x_2 - \phi_1(x_1)) \displaystyle\int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \\ \dot{x}_2 = v, \end{cases}$$

where $v \in \mathbb{R}$.

Consider the change of variables $e := x_2 - \phi_1(x_1)$. The time derivative of e yields the differential equation $\dot{e} = \dot{x}_2 - L_{f_1}\phi_1(x_1, x_2)$. System (2.37) rewritten in the new variable e is given by

$$\begin{cases} \dot{x}_1 = f_1(x_1, \phi_1(x_1)) + e \displaystyle\int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \\ \dot{e} = w, \end{cases} \tag{2.38}$$

where $w = v - L_{f_1}\phi_1(x_1, x_2)$. System (2.38) is denoted by $\dot{x} = f(x)$.

Consider the candidate Lyapunov function for system (2.38) given by

$$V(x_1, e) = V_1(x_1) + \frac{e^2}{2}.$$

[7] See Theorem A.36.

Its Lie derivative along the vector field f yields the inequality

$$L_f V(x_1, e) \leq \alpha_1(|x_1|) + e\left[\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds + w\right].$$

Consider the feedback law defined by the equation

$$\phi(x_1, e) = -\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - Ke,$$

where $K > 0$ is a constant value. Letting $w = \phi(x_1, e)$, the Lie derivative $L_f V$ yields the inequality

$$L_f V(x_1, e) \leq -\alpha_1(|x_1|) - Ke^2$$

which holds for every $(x_1, e) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Thus, the equilibrium of the origin is globally asymptotically stable for system (2.38) in closed loop with ϕ. Hence, it is also asymptotically stable for (2.37).

Since $\phi(x_1, e) = w = v - L_{f_1}\phi_1(x_1, x_2)$, it follows that

$$v = -\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - (x_2 - \phi_1(x_1)) + L_{f_1}\phi_1(x_1, x_2).$$

Therefore, the feedback law defined, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by[8]

$$\varphi_b(x_1, x_2) = \frac{1}{f_2(x_1, x_2)}\Bigg[-\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - (x_2 - \phi_1(x_1)) + L_{f_1}\phi_1(x_1, x_2)\Bigg].$$

rends the equilibrium of the origin globally asymptotically stable for (2.37) in closed loop.

The Schur's Complement

The Schur's complement is employed in this chapter to design a linear feedback law rendering the origin locally asymptotically stable. For further reading on the Schur's

[8]Note that, if $f_2(0, 0) = 0$ the feedback law φ_b would be discontinuous at the origin.

complement, the interested reader is invited to address [14, 28]. Here, some of the basic concepts are recalled.

Consider the matrices $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{q\times q}$ and the block matrix $M \in \mathbb{R}^{(p+q)\times(p+q)}$ given by

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and assume that $\det(A) \neq 0$. Consider a vector $z = (x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. The linear system $Mz^T = 0$, i.e., the system

$$\begin{cases} Ax + By = 0, \\ Cx + Dy = 0. \end{cases}$$

Multiplying the first equation by $-CA^{-1}$, on the left, and adding it to the second one, the x-component of the vector is eliminated, and the linear system is given by

$$(D - CA^{-1}B)y = 0.$$

The matrix $S = D - CA^{-1}B$ is called *Schur complement* of A in M (cf. [14]).

The next result is recalled from [28] and adapted to the context of this book.

Theorem 2.25 *Let $M \in \mathbb{R}^{(p+q)\times(p+q)}$ be a symmetric matrix given by*

$$M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix},$$

where $A \in \mathbb{R}^{p\times p}$ and $\det(A) \neq 0$. Then, $M \succ 0$ if and only if $A \succ 0$ and $(D - B^T A^{-1} B) \succ 0$.

From [14, Proposition 1], under the hypothesis of Theorem 2.25, $M \succ 0$ if and only if $D \succ 0$ and $A - BD^{-1}B^T \succ 0$.

A Remark on the Lyapunov Sufficient Conditions for Practical Stability

Recall the concept of global practical asymptotic stabilizability stated in Definition 2.16.

Under Assumptions 2.1 and 2.2, for every value $a > 0$, there exists (cf. Proposition 2.4, above.) a feedback law $\varphi_g : \mathbb{R}^n \to \mathbb{R}$ such the set $\mathbf{B}_{\leq a}(\mathbf{A})$ contains a compact invariant set that is globally asymptotically stable for the closed-loop system $(\Sigma_h(\varphi_g))$, where $\mathbf{A}$ is the set given by

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_1(x_1) \leq M, x_2 = \psi_1(x_1)\}. \tag{2.8}$$

Because of the value K_V, the feedback law φ_g is parametrized by a (cf. the proof of Proposition 2.4). Consequently, the closed-loop system $(\Sigma_h(\varphi_g))$ and the candidate Lyapunov function

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{K_V}{2}(x_2 - \psi_1(x_1))^2 \end{aligned} \tag{2.24}$$

are also parametrized by a.

From now on, the dependence of aforementioned functions on the parameter a is highlighted by adding it as subscript, for instance, $(\Sigma_h(\varphi_{g,a}))$.

The following theorem is presented in [3, Theorem 7.5] and it has been reformulated to the context of this chapter. It provides a sufficient condition for the stability of a compact invariant set when the candidate Lyapunov function and the feedback law depend on the parameter.

Theorem 2.26 (Lyapunov sufficient conditions for global practical asymptotic stability) *Let $\boldsymbol{A} \subset \mathbb{R}^n$ be a compact set. Suppose that, given any $a > 0$, there exist a continuous differentiable Lyapunov function $V_a : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and functions $\underline{\alpha}_a$, $\overline{\alpha}_a$ and α_a of class $\mathcal{K}_\infty$ such that, for every $x \in \boldsymbol{B}_{\geq a}(\boldsymbol{A})$, the following inequalities*

$$\underline{\alpha}_a(|x|_{\boldsymbol{A}}) \leq V_a(x) \leq \overline{\alpha}_a(|x|_{\boldsymbol{A}}), \tag{2.39a}$$

$$L_{f_h} V_a(x, \varphi_{g,a}) \leq -\alpha_a(|x|_{\boldsymbol{A}}), \tag{2.39b}$$

$$\lim_{a \to 0} \underline{\alpha}_a^{-1} \circ \overline{\alpha}_a(a) = 0 \tag{2.39c}$$

hold. Then, the set $\boldsymbol{A}$ is globally practically asymptotically stable for the closed-loop system $\Sigma_h(\varphi_g)$.

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Chapter 3
Analysis Under Nested Criteria

The small-gain theorem provides a sufficient condition for the stability of two feedback interconnected dynamical systems for which input-to-state (or input–output) gains can be defined. Roughly speaking, to apply this theorem, the resulting gains' composition is required to be continuous, increasing, and strictly smaller than the identity function.

In this chapter, an alternative sufficient condition is presented for the case in which such criterion fails due to either lack of continuity or when gains composition results in a function that is larger than the identity function, on a finite interval of the real line. A notion of asymptotic stability of the origin is ensured by merging a region-dependent small gain with an additional condition on the divergent of the vector field of the interconnected system. This latter condition implies that trajectories of solutions that do not converge to the origin have zero Lebesgue measure. An example illustrates the approach.

3.1 Introduction

The use of nonlinear input–output gains for stability analysis was introduced in [35] by considering a system as an input–output operator. A result that ensures stability of interconnected systems is provided by the so-called small-gain theorem.

Sontag introduced a new concept of system gain relating the input to system states [27]. This stability notion, called input-to-state stability, links Zames' and Lyapunov's approaches [28]. Characterizations in terms of dissipation inequalities and Lyapunov functions are given in [30].

In [18], the small-gain theorem has been reformulated for systems that are input-to-state stable. This theorem has also been formulated in terms of Lyapunov functions in [17].

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Besides stability analysis, the importance of the small-gain theorem is also due to its use for the design of feedback laws satisfying robustness constraints. The interested reader is invited to see [12, 26] and references therein. Other versions of the small-gain theorem do exist in the literature, see, for instance, [4, 5] for not necessarily input-to-state stable systems, [15, 16] for the context of integral input-to-state gains, and [8] for the large-scale interconnection of input-to-state stable systems.

One of the requirements to employ the small-gain theorem for input-to-state stable system is that the composition of the system gains must be smaller than the identity function for all positive values. Such a condition, called small-gain condition, restricts the application of the small-gain theorem to a composition of well chosen gains.

The approaches proposed in [33, 34] provide an alternative criterion for the stabilization of interconnected systems, when a single small-gain condition does not hold for every positive values. It consists of showing that two small-gain conditions hold on two intervals of the set of positive real numbers, these intervals correspond to regions of the state space. More specifically, these two regions are: a compact set containing the origin and the infinity. Additionally, in the gap between these regions, a condition on the divergence of the vector field is provided to ensure that the set of trajectories of solutions not converging to the compact set has Lebesgue measure zero. Then, the combination of these conditions implies that equilibrium of the origin is almost globally asymptotically stable (this notion is precisely defined below) for the interconnected system. For planar systems, an extension of the Bendixson's criterion to regions which are not simply connected is provided.

This approach may be seen as a blend of two small-gain conditions with a criterion on the derivative of the vector field. The use of a unifying approach for local and nonlocal properties is well known in the literature, see [2] in the context of control Lyapunov functions, and [6] for blending iISS and ISS properties.

3.2 Background, Motivation, and Problem Statement

3.2.1 Background

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a function of class $\mathscr{C}^1$ and consider the system

$$\dot{x}(t) = f(x(t), u(t)), \tag{3.1}$$

where for every positive time t, the *system state* $x(t)$ and the *input variable* evolve in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, for some positive integers n and m. From now on, arguments t will be omitted.

A function $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ is said to be an *input or control for system* (3.1). A solution to system (3.1) with initial condition $x \in \mathbb{R}^n$, under the input u, and computed at time t is denoted as $X(t, x, u)$.

The following stability notion is recalled from [29, p. 8].

Definition 3.1 (*Input-to-State Stability*) The equilibrium of the origin is said to be *input-to-state stable for* (3.1) if there exist functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the inequality

$$|X(t, x, u)| \leq \beta(|x|, t) + \gamma\left(\left|u_{[0,t]}\right|_\infty\right), \tag{3.2}$$

holds, for every initial condition $x \in \mathbb{R}^n$, for every input u for system (3.1), and for every time $t \geq 0$. The function $\gamma\left(\left|u_{[0,t]}\right|_\infty\right)$ is said to be an *ISS gain for* (3.1).

As remarked in [29, p. 9], since $\gamma \in \mathcal{K}_\infty$, the value $\sup\{\gamma(|u(s)|) : s \in [0, t]\}$ is equivalent to $\gamma(\sup\{|u(s)| : s \in [0, t]\}) = \gamma(|u_{[0,t]}|_\infty)$.

From now on, to state that *system* (3.1) *is ISS* is equivalent to state that the equilibrium of the origin is input-to-state stable for system (3.1).

Inequality (3.2) implies that solutions to ISS systems converge to a ball centered at the origin with radius given by $\gamma(|u|_\infty)$ (see (3.2)).

The next definition is based on [8, 19, 29].

Definition 3.2 Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function. It is said to be an *ISS-Lyapunov function for system* (3.1) if

- There exist functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that the inequality

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{3.3}$$

  holds, for every $x \in \mathbb{R}^n$.
- There exist a function $\alpha_x \in \mathcal{K}$ that is said to be an *ISS-Lyapunov gain for system* (3.1), and a proper function $\lambda_x \in (\mathcal{C} \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ such that the condition

$$V(x) \geq \alpha_x(|u|) \tag{3.4a}$$

  implies that the inequality

$$D_f^+ V(x, u) \leq -\lambda_x(x) \tag{3.4b}$$

  holds.

As remarked in [8], the proof that a locally Lipschitz ISS-Lyapunov function for system (3.1) implies that system (3.1) is ISS goes along the lines presented in [30].

3.2.2 Motivation

Consider the system

$$\dot{z} = g(v, z), \tag{3.5}$$

where $g \in \mathscr{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$. A solution to (3.5) with initial condition z, under the input v and computed at time t is denoted as $Z(t, z, v)$.

From now on, assume that there exists an ISS-Lyapunov function $W : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ for system (3.5). Consequently, there exist functions $\underline{\alpha}_z, \overline{\alpha}_z \in \mathscr{K}_\infty$ such that the inequality

$$\underline{\alpha}_z(|z|) \leq W(z) \leq \overline{\alpha}_z(|z|) \tag{3.6}$$

holds, for every $z \in \mathbb{R}^m$. Also, there exist functions $\alpha_z \in \mathscr{K}$ and $\lambda_z \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ such that the condition

$$W(z) \geq \alpha_z(|v|) \tag{3.7a}$$

implies

$$D_g^+ W(v, z) \leq -\lambda_z(z). \tag{3.7b}$$

Interconnecting systems (3.1) and (3.5) yields the system

$$\begin{cases} \dot{x} = f(x, z) \\ \dot{z} = g(x, z). \end{cases} \tag{3.8}$$

Using vectorial notation, system (3.8) is denoted as $\dot{y} = h(y)$. A solution to (3.8) starting at $y \in \mathbb{R}^{n+m}$ and evaluated at time t is denoted as $Y(t, y)$. The set of inequalities (3.4) and (3.7) are rewritten, respectively, as

$$V(x) \geq \gamma(W(z)) \Rightarrow D_f^+ V(x, z) \leq -\lambda_x(x) \tag{3.9a}$$

and

$$W(z) \geq \delta(V(x)) \Rightarrow D_g^+ W(x, z) \leq -\lambda_z(z). \tag{3.9b}$$

The ISS-Lyapunov gains are defined as the functions $\gamma(\cdot) = \alpha_x \circ \overline{\alpha}_z^{-1}(\cdot) \in \mathscr{K}$ and $\delta(\cdot) = \alpha_z \circ \overline{\alpha}_x^{-1}(\cdot) \in \mathscr{K}$.

A sufficient condition to ensure that the equilibrium of the origin is globally asymptotically stable for system (3.8) is provided by the small-gain theorem. This result is roughly stated here, the interested reader may consult [17, Theorem 3.1] for further details.

If the ISS-Lyapunov gains γ and δ are such that the *small-gain condition*

$$\gamma \circ \delta(s) < s \tag{SGC}$$

holds, for every $s > 0$, then the equilibrium of the origin is globally asymptotically stable for system (3.8).

In this chapter, systems for which the composition of the ISS-Lyapunov gains does not satisfy the small-gain condition (SGC) on an interval of $\mathbb{R}_{>0}$ are considered. Moreover, if

- There exist ISS-Lyapunov gains γ_ℓ and γ_g, for the x-subsystem of (3.8);
- There exist ISS-Lyapunov gains δ_ℓ and δ_g, for the z-subsystem of (3.8);
- The compositions $\gamma_\ell \circ \delta_\ell$ and $\gamma_g \circ \delta_g$ satisfy the small-gain condition, not for all values of the arguments, but for two different intervals of $\mathbb{R}_{>0}$;
- In the region where the small-gain condition does not hold, a criterion on the derivative of the vector field is satisfied.

Then, for almost every initial condition, solutions to (3.8) converge to the origin. Moreover, if $n = m = 1$, the previous conclusion holds true for every initial condition. These results are formally stated in Theorems 3.11 and 3.12.

3.3 Standing Assumptions

The first assumption, recalled from [34], concerns the region where the set of implications (3.9) holds.

Assumption 3.3 There exist constant values

$$0 \le \underline{M} < \overline{M} \le \infty \quad \text{and} \quad 0 \le \underline{N} < \overline{N} \le \infty,$$

functions γ and δ of class $\mathcal{K}$ such that the limit

$$b = \limsup_{s\to\infty} \gamma(s) > \begin{cases} \overline{M}, & \text{if } \overline{M} < \infty \\ \underline{M}, & \text{if } \overline{M} = \infty \end{cases} \tag{3.10}$$

holds. If $\min\{\overline{M}, \overline{N}\} < \infty$, assume also that the inequality

$$\max\{\gamma^{-1}(\underline{M}), \underline{N}\} < \min\{\delta(\overline{M}), \overline{N}\} \tag{3.11}$$

holds. Define the set

$$\begin{aligned} \mathbf{S} = &\left\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : \underline{M} \le V(x) \le \overline{M}, W(z) \le \overline{N}\right\} \\ &\cup \left\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : V(x) \le \overline{M}, \underline{N} \le W(z) \le \overline{N}\right\}. \end{aligned} \tag{3.12}$$

The ISS-Lyapunov functions V and W satisfy, for every $(x, z) \in \mathbf{S}$, the set of implications (3.9).

Assumption 3.3 provides conditions for the ISS-Lyapunov inequalities (3.9a) and (3.9b) to hold on the set $\mathbf{S} \subset \mathbb{R}^{n+m}$, which is defined as the union of sublevel sets of the ISS-Lyapunov functions V and W.

Equation (3.10) states that, for the ISS-Lyapunov gain γ, its maximum value is reached outside the set $(\underline{M}, \overline{M})$ (when $\mathbf{S}$ is bounded) or inside it (when $\mathbf{S}$ being unbounded).

The role of Eq. (3.11) becomes clear in Proposition 3.6. This equation is employed for the estimation of the basin of attraction of suitable attractors.

The next assumption is also recalled from [34].

Assumption 3.4 The inequality

$$\gamma \circ \delta(s) < s, \quad \forall s \in \begin{cases} \left[\underline{M}, \overline{M}\right] \setminus \{0\}, & \text{if } \overline{M} < \infty \\ [\underline{M}, \overline{M}) \setminus \{0\}, & \text{if } \overline{M} = \infty \end{cases} \tag{3.13}$$

holds.

Assumption 3.4 implies that the small-gain condition holds on an interval of $\mathbb{R}_{\geq 0}$ corresponding to the set **S**.

Remark 3.5 Equation (3.13) is equivalent to

$$\begin{array}{l} \text{if } \overline{M} < \infty,\ s \in [\gamma^{-1}(\underline{M}), \gamma^{-1}(\overline{M})] \setminus \{0\},\ \delta \circ \gamma(s) < s, \\ \text{if } \overline{M} = \infty,\ s \in [\gamma^{-1}(\underline{M}), \gamma^{-1}(\overline{M})) \setminus \{0\},\ \delta \circ \gamma(s) < s. \end{array} \tag{3.14}$$

3.4 Results

The first result is recalled from [34].

Proposition 3.6 *Under Assumptions 3.3 and 3.4, define the values*

$$\widetilde{M} = \max\{\gamma^{-1}(\underline{M}), \underline{N}\} \quad \text{and} \quad \widehat{M} = \min\{\delta(\overline{M}), \overline{N}\}.$$

There exists a proper function $U \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ that is locally Lipschitz on $\mathbb{R}^{n+m} \setminus \{0\}$ and such that the limit

$$\lim_{t \to \infty} U(Y(t, y)) \leq \widetilde{M}$$

holds, for every $y \in \Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$. Moreover, if the ISS-Lyapunov gains γ and δ are differentiable, then a suitable U is given by

$$U(x, z) = \max\left\{\sigma(V(x)), W(z)\right\}, \tag{3.15}$$

where $\sigma(\cdot) = (\delta(\cdot) + \gamma^{-1}(\cdot))/2$.

A sketch of the proof of Proposition 3.6 is provided in Sect. 3.6.1. The main lines of reasoning are presented here as follows. Assumptions 3.3 and 3.4 provide the function $U : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ whose Dini's derivative (see Definition A.39 below) is negative-definite in **S**. From inequality (3.11), the set $\Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$ is nonempty and included in **S**. Thus, solutions to system (3.8) starting in $\Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$ converge to the set $\Omega_{\leq \widetilde{M}}(U)$.

Remark 3.7 Since the function U given by Proposition 3.6 is not locally Lipschitz at the origin, the construction of the function $\sigma \in (\mathcal{K}_\infty \cap \mathcal{C}^1)$ is slightly different. The interested reader may consult the proofs of [25, Lemma 4.1 and Theorem 4.2].

The next two corollaries of Proposition 3.6 are recalled from [34] (see also [7, 8])

Corollary 3.8 (Local stabilization) *Consider Assumptions 3.3 and 3.4 with the constant values* $\underline{M} = \underline{N} = 0$, $M_\ell := \overline{M} < \infty$ *and* $N_\ell := \overline{N} < \infty$. *The set* $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ *is included in the basin of attraction of the origin of (3.8), where* U_ℓ *and* $\widehat{M}_\ell$ *are given by Proposition 3.6.*

Corollary 3.9 (Global attractivity) *Consider Assumptions 3.3 and 3.4 with the constant values* $M_g := \underline{M} > 0$ *and* $N_g := \underline{N} > 0$, *and* $\overline{M} = \overline{N} = \infty$. *The set* $\Omega_{\leq \widetilde{M}_g}(U_g)$ *is globally attractive for (3.8), where* U_g *and* $\widetilde{M}_g$ *are given by Proposition 3.6.*

From Corollary 3.8 (resp. Corollary 3.9), solutions to (3.8) starting in $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ (resp. $\Omega_{> \widetilde{M}_g}(U_g)$) converge to the origin (resp. to the set $\Omega_{\leq \widetilde{M}_g}(U_g)$).

Note that if the inclusion $\Omega_{\leq \widetilde{M}_g}(U_g) \subset \Omega_{\leq \widehat{M}_\ell}(U_\ell)$ holds, then global asymptotic stability of system (3.8) is achieved by combining the local stability of origin together with the global attraction of the set $\Omega_{\leq \widetilde{M}_g}(U_g)$. When that inclusion does not hold, solutions to system (3.8) starting in the set $\Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)$ may converge to a ω-limit set instead of $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ (see [14] Birkhoff's theorem). The following theorems provide sufficient conditions for sets of trajectories of solutions that converge to these ω-limit sets to have Lebesgue measure zero. Before presenting these results, the following stability notion is recalled from [3].

Definition 3.10 Let $\mathbf{A} \subset \mathbb{R}^{n+m}$ be a compact set. It is said to be *almost globally asymptotically stable for system* (3.8) if it is locally stable in the Lyapunov sense, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \quad : \quad |y|_{\mathbf{A}} \leq \delta \Rightarrow |Y(t, y)| \leq \varepsilon, \quad \forall t \geq 0,$$

and attractive for almost every initial condition. More precisely, there exists a set $\aleph \subset \mathbb{R}^{n+m}$ with Lebesgue measure $\mu(\aleph) = 0$ such that the limit

$$\lim_{t \to \infty} |Y(t, y)|_{\mathbf{A}} = 0$$

holds, for every $y \in \mathbb{R}^{n+m} \setminus \aleph$.

Now, the first result is to be recalled from [34].

Theorem 3.11 *Under Assumptions 3.3 and 3.4, assume that the constant values of Corollaries 3.8 and 3.9 are such that*

$$M_\ell < M_g \quad or \quad N_\ell < N_g.$$

Let

$$\mathbf{R} = \texttt{cl}\left\{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)\right\}.$$

If there exists a function $\rho \in \mathscr{C}^1(\mathbb{R}^{n+m} \setminus \{0\}, \mathbb{R}_{\ge 0})$ *with* $\texttt{supp}(\rho) \supseteq \mathbf{R}$ *such that the inequality*

$$\texttt{div}\,(h\rho)(y) > 0$$

holds, for every $y \in \mathbf{R}$*, then the equilibrium of the origin is almost globally asymptotically stable for system* (3.8)*.*

A sketch of the proof of Theorem 3.11 is provided in Sect. 3.6.2 and is based on [3, 22]. The main reasoning can be stated as follows. From Liouville's theorem (see Lemma 3.29), the Lebesgue measure of the sets of trajectories of solutions to (3.8) is related to the divergence of vector field of system (3.8). Under the hypothesis of Theorem 3.11, the Lebesgue measure of sets of trajectories of solutions that does not converge to $\Omega_{\le \widehat{M}_\ell}(U_\ell)$ is zero. Thus, the conclusion of Theorem 3.11 follows.

The next result is recalled from [34].

Theorem 3.12 *Let* $n = m = 1$*. Under Assumptions 3.3 and 3.4, assume that the constant values of Corollaries 3.8 and 3.9 are such that*

$$M_\ell < M_g \quad or \quad N_\ell < N_g.$$

Let

$$\mathbf{R} = \texttt{cl}\left\{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)\right\}.$$

If

$$\texttt{div}\,h(y) \ne 0 \quad and \quad h(y) \ne 0 \tag{3.16}$$

holds, for every $y \in \mathbf{R}$*, then the equilibrium of the origin is globally asymptotically stable for system* (3.8)*.*

The main ingredient of the sketch of the proof of Theorem 3.12 is to show the absence of ω-limit sets in $\mathbf{R}$. This is achieved by exhibiting a contradiction between

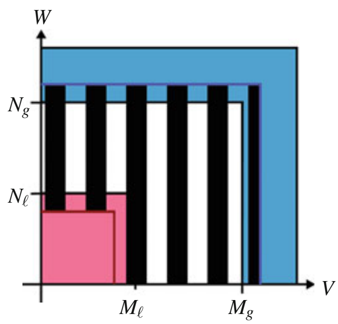

Fig. 3.1 Illustration of sets $\Omega_{\le M_\ell}(V) \times \Omega_{\le N_\ell}(W)$ (*red region*), $\Omega_{=\widetilde{M}_\ell}(U_\ell)$ (*dark red line*), $\Omega_{\ge \underline{M}}(V) \times \Omega_{\ge \underline{N}}(W)$ (*blue region*), $\Omega_{=\widetilde{M}_g}(U_g)$ (*blue line*), and $\mathbf{R} = \texttt{cl}\{\Omega_{\le \widehat{M}_g}(U_g) \setminus \Omega_{\le \widetilde{M}_\ell}(U_\ell)\}$ (*pattern filled*)

its existence and the assumption that, for every $y \in \mathbf{R}$, $\mathtt{div}\ h(y) \neq 0$ and $h(y) \neq 0$. A sketch of the proof of Theorem 3.12 is provided in Sect. 3.6.3.

Figure 3.1 illustrates the region $\mathbf{R}$ obtained from the hypothesis of Corollaries 3.8 and 3.9, when $M_l < M_g$ and $N_l < N_g$.

3.5 Illustration

In this section, an example where the small-gain condition cannot be applied is given.[1] Corollaries 3.8 and 3.9, and Theorem 3.12 are illustrated.

Consider the system

$$\begin{cases} \dot{x} = f(x,z) = -\rho_x(x) + z, \\ \dot{z} = g(x,z) = -z + \rho_z(x), \end{cases} \tag{3.17}$$

where for every $x \in \mathbb{R}$, the functions ρ_x and ρ_z are defined by

$$\rho_x(x) = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x \quad \text{and} \quad \rho_z(x) = 0.8\rho_x(x).$$

Let the functions

$$\mathbb{R} \ni x \mapsto V(x) = |x| \in \mathbb{R}_{\geq 0},$$
$$\mathbb{R} \ni z \mapsto W(z) = |z| \in \mathbb{R}_{\geq 0}.$$

The Dini derivative (cf. Definition A.39) of V along the vector field f yields the inequality

$$D_f^+ V(x,z) \leq -\rho_x(V(x)) + W(z),$$

which holds, for every $(x,z) \in \mathbb{R} \times \mathbb{R}$. Define the function $\lambda_x(x) = \varepsilon_x \rho_x(V(x))$, for every $x \in \mathbb{R}$, where $\varepsilon_x \in (0,1)$ is a constant value. The condition

$$\rho_x(V(x)) \geq \frac{W(z)}{1-\varepsilon_x} \tag{3.18a}$$

implies that the inequality

$$D_f^+ V(x,z) \leq -\lambda_x(x) \tag{3.18b}$$

holds.

[1] To conclude about the asymptotic stability of this example, one may infer from the LaSalle invariance principle together with the Lyapunov function $V + W$. Other techniques also apply, see [4], for example.

Since the function ρ_x is strictly decreasing on the interval $(1, 2)$, it is not possible to use its inverse as an ISS-Lyapunov gain. From now on, let the constant value $\varepsilon_x = 0.05$. Consider the piecewise continuous and positive-definite function[2]

$$\Gamma(s) = \begin{cases} \rho_x^{-1}\left(\frac{s}{0.95}\right), \ s \in [0, 0.6), \\ \\ \rho_{x,+}^{-1}\left(\frac{s}{0.95}\right), \ s \in [0.6, \infty), \end{cases} \tag{3.19}$$

where $\rho_{x,+} : [2, \infty) \to [\rho_x(2), \infty)$ is given by $\rho_{x,+}(\cdot) = \rho_x(\cdot)$.

Proposition 3.13 *The positive-definite function Γ can be viewed as a nonsmooth ISS-Lyapunov gain of the x-subsystem of (3.17), because the condition*

$$V(x) \geq \Gamma(W(z)) \tag{3.20a}$$

implies that the inequality

$$D_f^+ V(x, z) \leq -\lambda_x(x) \tag{3.20b}$$

holds. Moreover, the function Γ is "tight". More precisely, if there exist a function $\Gamma^* : \mathbb{R} \to \mathbb{R}$ and a positive value s^* such that $\Gamma^*(s^*) < \Gamma(s^*)$, then there exists $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfying the inequalities $V(x^*) \geq \Gamma^*(W(z^*))$ and $D_f^+ V(x^*, z^*) > 0$.

A sketch of the proof of Proposition 3.13 is provided in Sect. 3.6.4. Note that any function $\gamma \in \mathscr{K}$ satisfying the inequality $\Gamma(\cdot) \leq \gamma(\cdot)$ is an ISS-Lyapunov gain for the x-subsystem of (3.17).

The above reasoning can be applied to the z-subsystem. The condition

$$W(z) \geq \Delta(V(x)) \tag{3.21a}$$

implies that the inequality

$$D_g^+ W(x, z) \leq -\lambda_z(z) \tag{3.21b}$$

holds, where Δ is the tight ISS-Lyapunov gain of the z-subsystem of (3.17) defined by

$$\Delta(s) = \begin{cases} \underline{\rho}(s), \ s \in [0, 1), \\ \underline{\rho}(1), \ s \in [1, 2.5), \\ \underline{\rho}(s), \ s \in [2.5, \infty), \end{cases}$$

where the function $\underline{\rho}$ is defined as $\underline{\rho}(s) = \rho_z(s)/0.95$, for every $s \in \mathbb{R}_{>0}$. Note also that any function $\delta \in \mathscr{K}$ satisfying the inequality $\Delta(\cdot) \leq \delta(\cdot)$ is an ISS-Lyapunov gain for the z-subsystem of (3.17).

[2] Note that $0.95\rho_x(2) = 0.6$.

Now that the tight ISS-Lyapunov gains have been obtained, their composition shows that the small-gain condition (SGC) does not hold. This claim is formalized in the following proposition:

Proposition 3.14 *The inequality* $s \leq \Gamma \circ \Delta(s)$ *holds, for every* $s \in (0.65, 2.5)$.

A sketch of the proof of Proposition 3.14 is provided in Sect. 3.6.5.

Note that the results from [5, 15, 16] cannot be applied here, since they require a composition of gains to be smaller than its argument. Although the results presented in [4] could be applied to this example, the main advantage of this approach over that method is that a Lyapunov function can be obtained in the regions where the small-gain condition holds.

Illustration of Corollary 3.8. Consider a function $\gamma_\ell \in \mathcal{K}$ such that, for every $s \in [0, 0.6]$, $\gamma_\ell(s) = \Gamma(s)$. Let the function $\delta_\ell \in \mathcal{K}$ be such that

$$\delta_\ell(s) \begin{cases} = \Delta(s), & s \in [0, 1), \\ \geq \Delta(s), & s \in [1, 2.5]. \end{cases}$$

Verifying Assumption 3.3. Pick[3] $\overline{M} = M_\ell = \overline{N} = N_\ell = 0.3$, and $\underline{M} = \underline{N} = 0$. Note that $\max\{\gamma_\ell^{-1}(\underline{M}), \underline{N}\} = 0$ and $\min\{\delta_\ell(M_\ell), N_\ell\} = 0.3$. Moreover, for every $(x, z) \in (\Omega_{\leq M_\ell}(V) \times \Omega_{\leq N_\ell}(W))$, the implications

$$\begin{aligned} V(x) \geq \gamma_\ell(W(z)) &\Rightarrow D_f^+ V(x, z) \leq -\lambda_x(x), \\ W(z) \geq \delta_\ell(V(x)) &\Rightarrow D_g^+ W(x, z) \leq -\lambda_z(z) \end{aligned}$$

hold.

Verifying Assumption 3.4. From Remark 3.5 the inequality

$$\delta_\ell \circ \gamma_\ell(s) = \frac{\rho_z \circ \rho_x^{-1}(s/0.95)}{0.95} = \frac{0.8}{0.95} \rho_x \circ \rho_x^{-1}\left(\frac{s}{0.95}\right) < s$$

holds, for every $s \in (0, \gamma_\ell^{-1}(M_\ell)] = (0, 0.6]$.

From Corollary 3.8, the set $\Omega_{\leq 0.3}(U_\ell)$ is included in the basin of attraction of the origin. Moreover,

$$U_\ell(x, z) = \max\left\{\frac{\delta_\ell(V(x)) + \gamma_\ell^{-1}(V(x))}{2}, W(z)\right\}.$$

Illustration of Corollary 3.9. Consider a function $\gamma_g \in \mathcal{K}$ such that, for every $s \in [0.7, \infty)$, $\gamma_g(s) = \Gamma(s)$. Let also the function $\delta_g \in \mathcal{K}$ be such that, for every $s \in [2, \infty)$, $\delta_g(s) = \underline{\rho}(s)$.

[3] Note that $b = \infty$.

Verifying Assumption 3.3. Pick $\underline{M} = M_g = 4$ and $\underline{N} = N_g = 1$, and $\overline{M} = \overline{N} = \infty$. Note that, $\max\{\gamma_g^{-1}(\underline{M}), \underline{N}\} = 4$ and $\min\{\delta_g(\overline{M}), \overline{N}\} = \infty$. Moreover, the implications

$$\begin{aligned} V(x) \geq \gamma_g(W(z)) &\Rightarrow\ D_f^+ V(x,z) \leq -\lambda_x(x), \\ W(z) \geq \delta_g(V(x)) &\Rightarrow D_g^+ W(x,z) \leq -\lambda_z(z) \end{aligned}$$

hold, for every $(x, z) \in (\Omega_{\geq M_g}(V) \times \Omega_{\geq N_g}(W))$.

Verifying Assumption 3.4. From Remark 3.5 the inequality

$$\delta_g \circ \gamma_g(s) = \frac{0.8}{0.95} \rho_x \circ \rho_x^{-1} \left(\frac{s}{0.95} \right) < s$$

holds, for every $s \in [\gamma_g^{-1}(M_g), \infty) = [4.5, \infty)$.

From Corollary 3.9 the set $\Omega_{\leq 4}(U_g)$ is globally attractive for (3.17). Moreover,

$$U_g(x,z) = \max \left\{ \frac{\delta_g(V(x)) + \gamma_g^{-1}(V(x))}{2}, W(z) \right\}.$$

Illustration of Theorem 3.12. Note that $M_\ell = 0.3 < 4 = M_g$ and $N_\ell = 0.3 < 1 = N_g$. Since the vector field of system (3.17) has a continuous derivative, and the only equilibrium point is the origin, and the equation

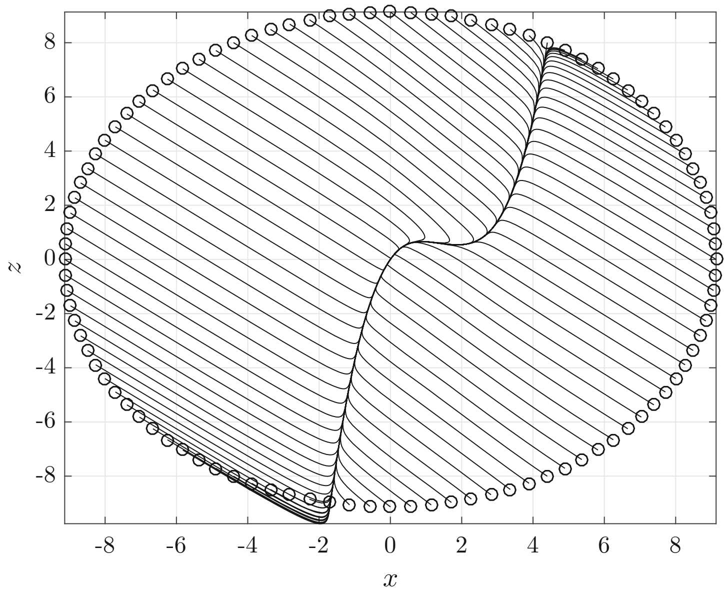

Fig. 3.2 Simulation of (3.17)

$$\frac{\partial f}{\partial x}(x,z)+\frac{\partial g}{\partial z}(x,z)=-x^2+3x-3=0$$

has no zeros on the set

$$\mathbf{R}=\mathtt{cl}\left\{\Omega_{\leq 4}(U_g)\setminus\Omega_{\leq 0.3}(U_\ell)\right\},$$

from Theorem 3.12, the equilibrium of the origin is globally asymptotically stable for system (3.17).

Figure 3.2 shows a simulation of (3.17) for some initial conditions.

3.6 Sketches of the Proofs of Chap. 3

3.6.1 Sketch of the Proof of Proposition 3.6

Before sketching a proof for Proposition 3.6, the following lemma is recalled from [33].

Lemma 3.15 *Under Assumptions 3.3 and 3.4, there exists a function $\tilde{\gamma}\in\mathscr{K}_\infty$ such that the inequality*

$$\delta(s)<\widetilde{\gamma}(s), \tag{3.22}$$

holds, for every $s\geq 0$. Moreover,

$$\begin{array}{l} \textit{if}\ \ \overline{M}<\infty,\ \textit{then}\ \forall s\in[\underline{M},\overline{M}]\setminus\{0\},\ \widetilde{\gamma}(s)<\gamma^{-1}(s),\\ \textit{if}\ \ \overline{M}=\infty,\ \textit{then}\ \ \forall s\in[\underline{M},b)\setminus\{0\},\ \ \widetilde{\gamma}(s)<\gamma^{-1}(s). \end{array} \tag{3.23}$$

The proof of Lemma 3.15 is based on [17] and is provided in [32].

The ideas to prove Proposition 3.6 are adapted from [17, proof of Theorem 3.1]. The sketch is divided into two parts. First, the Dini derivative of a proper function $U\in(\mathscr{C}\cap\mathscr{P})(\mathbb{R}^{n+m},\mathbb{R}_{\geq 0})$ is shown to be strictly negative in the set **S** defined in (3.12). In the second part, solutions to (3.8) starting in $\Omega_{\leq\widehat{M}}(U)\setminus\Omega_{\leq\widetilde{M}}(U)$ are shown to converge to $\Omega_{\leq\widetilde{M}}(U)$.

First Part. Under Assumptions 3.3 and 3.4, consider the function $\widetilde{\gamma}\in\mathscr{K}_\infty$ obtained in Lemma 3.15. Since δ is of class $\mathscr{K}$ and the function $\widetilde{\gamma}$ satisfies (3.23), there exists ([17, Lemma A.1]) a function $\sigma\in\mathscr{K}_\infty\cap\mathscr{C}^1$ whose derivative is strictly positive and satisfies the inequality

$$\delta(s)<\sigma(s)<\widetilde{\gamma}(s) \tag{3.24}$$

for every $s>0$.

Let

$$\begin{aligned} U : \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R}_{\geq 0} \\ (x, z) &\mapsto \max\{\sigma(V(x)), W(z)\}. \end{aligned}$$

Note that $U \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ is a proper function. For any $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, one of the following cases is possible:

Case 1. $\sigma(V(x)) < W(z)$;
Case 2. $W(z) < \sigma(V(x))$ or;
Case 3. $W(z) = \sigma(V(x))$.

The proof follows by showing that the Dini derivative of U is strictly negative. For each case, assume that

$$(x, z) \in \mathbf{S}_{\neq 0} := \mathbf{S} \setminus \{(0, 0)\},$$

where $\mathbf{S}$ is defined in (3.12).
Case 1. Assume that

$$\sigma(V(x)) < W(z).$$

This implies that

$$U(x, z) = W(z) \quad \text{and} \quad D^+_{f,g} U(x, z) = D^+_g W(x, z).$$

From (3.24), the inequality

$$\delta(V(x)) < \sigma(V(x)) < W(z)$$

holds. From implication (3.9b), the inequality $D^+_g W(x, z) \leq -\lambda_z(z)$ holds. Thus,

$$W(z) > \sigma(V(x)) \Rightarrow D^+_{f,g} U(x, z) \leq -\lambda_z(z).$$

Case 2. Assume that the inequality

$$W(z) < \sigma(V(x))$$

holds. This implies that

$$U(x, z) = \sigma(V(x)) \quad \text{and} \quad D^+_{f,g} U(x, z) = \sigma'(V(x)) D^+_f V(x, z).$$

Since $(x, z) \in \mathbf{S}_{\neq 0}$, Eq. (3.24) implies that the inequality

$$W(z) < \sigma(V(x)) < \widetilde{\gamma}(V(x))$$

holds.

If $\overline{M} < \infty$, then from (3.23), the inequality

$$W(z) < \sigma(V(x)) < \widetilde{\gamma}(V(x)) < \gamma^{-1}(V(x)) \tag{3.25}$$

also holds. Together with implication (3.9a), the inequality $D_f^+ V(x,z) \leq -\lambda_x(x)$ holds.

If $\overline{M} = \infty$, then two regions of x must be analyzed: $b < V(x)$ and $\underline{M} \leq V(x) \leq b$.

Case 2.a. In the region where $b < V(x)$, Eqs. (3.9a) and (3.10) yield the implication

$$V(x) > b > \gamma(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x).$$

Case 2.b. In the region where $\underline{M} \leq V(x) \leq b$ the implication

$$V(x) > \gamma(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x)$$

holds, due to Eqs. (3.23), (3.25), and (3.9a).

Summing up Case 2,

$$W(z) < \sigma(V(x)) \Rightarrow D_{f,g}^+ U(x,z) \leq -\sigma'(V(x))\lambda_x(x).$$

Case 3. Assume that

$$W(z) = \sigma(V(x)).$$

Note that,

$$\begin{aligned} D_{f,g}^+ U(x,z) &= \limsup_{t \searrow 0} \frac{1}{t} \left[\max\left\{\sigma\left(V(X(t,x,z))\right), W(Z(t,z,x))\right\} - U(x,z)\right] \\ &= \limsup_{t \searrow 0} \max\left\{\frac{\sigma(V(X(t,x,z))) - \sigma(V(x))}{t}, \frac{W(Z(t,z,x)) - W(z)}{t}\right\} \\ &= \max\left\{\sigma'(V(x)) D_f^+ V(x,z), D_g^+ W(x,z)\right\}. \end{aligned}$$

The analysis of $D_{f,g}^+ U$ is divided into two subcases. In the first, the function $D_g^+ W$ is analyzed while in the last the function $D_f^+ V$ is analyzed.

Case 3.a. The analysis of $D_g^+ W$. From (3.24), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $\delta(V(x)) < \sigma(V(x)) = W(z)$ holds. Following the reasoning employed in Case 1, $D_g^+ W(x,z) \leq -\lambda_z(z)$.

Case 3.b. The analysis of $D_f^+ V$. From (3.24), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $W(z) = \sigma(V(x)) < \widetilde{\gamma}(V(x))$ holds. Following the same reasoning as in Case 2, $D_f^+ V(x,z) \leq -\lambda_x(x)$.

Summing up Case 3, the condition

$$0 \neq W(z) = \sigma(V(x))$$

implies that the inequality

$$D^{+}_{f,g}U(x,z) \leq -\min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}$$

holds.

From [34, Claim 1], there exists a value $c > 0$ such that the inclusion $\Omega_{\leq c}(U) \subset \left(\Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)\right)$ holds. Moreover, the constant values $\widetilde{M}$ and $\widehat{M}$ are such that

$$\left(\Omega_{\leq \underline{M}}(V) \times \Omega_{\leq \underline{N}}(W)\right) \subset \Omega_{\leq \widetilde{M}}(U) \subset \Omega_{\leq \widehat{M}}(U) \subset \left(\Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)\right). \tag{3.26}$$

The chain of inclusions (3.26) implies that the inequality

$$\widetilde{M} \leq U(x,z) \leq \widehat{M}$$

holds. Consequently, also the inequality

$$D^{+}_{f,g}U(x,z) \leq -E(x,z)$$

holds, where the function E is defined by

$$\begin{aligned} E : \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R}_{\geq 0} \\ (x,z) &\mapsto \min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}. \end{aligned}$$

Since $E \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}_{\geq 0})$, from Lemma A.33 (see also [27, p. 13]) there exists a function $\alpha \in \mathscr{K}_\infty$ such that, for every $(x,z) \in \mathbb{R}^{n+m}$, the inequality $\alpha(|(x,z)|) \leq E(x,z)$ holds. Moreover, since the set $\Omega_{<\widetilde{M}}(U)$ contains the origin

$$\alpha\left(|(x,z)|_{\widetilde{M}}\right) \leq \alpha(|(x,z)|) \leq E(x,z)$$

holds. Thus, the condition

$$\widetilde{M} \leq U(x,z) \leq \widehat{M}$$

implies that the inequality

$$D^{+}_{f,g}U(x,z) \leq -\alpha\left(|(x,z)|_{\widetilde{M}}\right)$$

holds.

Second part. The locally Lipschitz continuity of the function U on $(\mathbb{R}^n \times \mathbb{R}^m) \setminus \{(0,0)\}$ follows from the Lipschitz continuity of W and $\sigma \circ V$ on that region. As explained in [19, Remark III.1], the function U can be modified to be Lipschitz also at the origin.

From Theorem A.42, for every $y \in \mathbb{R}^{n+m}$, and for every $t \in \mathbb{R}_{\geq 0}$, along solutions to system (3.8), the equation

$$D^+U(Y(t,y)) = D_h^+U(Y(t,y))$$

holds. Thus, solutions to (3.8) starting in $\Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$ converge to $\Omega_{\leq \widetilde{M}}(U)$.

To see that U can be given as in (3.15), note that U relies on the computation of σ. Define, for every $s \in \mathbb{R}_{\geq 0}$, the function $\sigma(s) = (\delta(s) + \gamma^{-1}(s))/2$. The derivative of this function yields for every $s \in \mathbb{R}_{>0}$,

$$2\frac{d\sigma}{ds}(s) = \frac{d\delta}{ds}(s) + \frac{1}{\frac{d\gamma}{ds}\left(\gamma^{-1}(s)\right)}.$$

Note that this derivative is positive, because $d\delta(s)/ds > 0$ and $d\gamma\left(\gamma^{-1}(s)\right)/ds > 0$. Moreover, the function σ satisfies inequality (3.24). This concludes sketch of the proof of Proposition 3.6.

3.6.2 Sketch of the Proof of Theorem 3.11

Before sketching a proof of Theorem 3.11 the following lemma, based on [3, 22], is recalled from [34].

Lemma 3.16 *Under the hypotheses of Theorem 3.11, if there exists a function* $\rho \in \mathscr{C}^1(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ *with* $\mathtt{supp}(\rho) \supseteq \mathbf{R}$ *such that the inequality*

$$\mathtt{div}\,(h\rho)(y) > 0 \tag{3.27}$$

holds, for every $y \in \mathbf{R}$*, then for almost every initial condition in* $\mathbf{R}$*, the solutions to system* (3.8) *converge to* $\Omega_{\leq \widehat{M}_\ell}(V)$.

A sketch of the proof of Lemma 3.16 is provided after the sketch of the proof of Theorem 3.11.

This sketch is divided into four parts. First, every solution starting in $\Omega_{>\widetilde{M}_g}(U_g)$ is shown to converge to $\Omega_{\leq \widetilde{M}_g}(U_g)$. The second part shows that every solution to system (3.8) starting in $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ converges to the origin. In the third part, the set of trajectories of solutions to (3.8) starting in the set $\mathbf{R}$ that do not converge to the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is shown to have Lebesgue measure zero. The fourth part concludes the almost global asymptotic stability of the origin.

First Part. From Corollary 3.9, the set $\Omega_{\leq \widetilde{M}_g}(U_g)$ is globally attractive for (3.8) with the constant values $\widetilde{M}_g = \max\{\gamma_g^{-1}(M_g), N_g\}$, M_g, and N_g, where the ISS-Lyapunov function γ_g is given by Assumption 3.3.

Second Part. From Corollary 3.8, the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is contained in the basin of attraction of the origin with the constant values $\widehat{M}_\ell = \min\{\delta_\ell(M_\ell), N_\ell\}$, M_ℓ, and N_ℓ, and ISS-Lyapunov function γ_ℓ is given by Assumption 3.3.

Third Part. It remains to show that the inclusion $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leq \widetilde{M}_g}(U_g)$ holds. From the proof of [34, Claim 1], the implications

$$U_\ell(x,z) \le \widehat{M}_\ell \Rightarrow \max\{V(x), W(z)\} \le \min\{M_\ell, N_\ell\},$$
$$U_g(x,z) \ge \widetilde{M}_g \Rightarrow \min\{V(x), W(z)\} \ge \max\{M_g, N_g\}$$

hold. Moreover, inequality $\min\{M_\ell, N_\ell\} < \max\{M_g, N_g\}$ implies that inclusion $\Omega_{\le \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\le \widetilde{M}_g}(U_g)$ holds. Consequently, the set

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)\right\}$$

is nonempty.

Because of the existence of the function $\rho \in \mathscr{C}^1(\mathbb{R}^{n+m}, \mathbb{R}_{\ge 0})$ with $\mathtt{supp}(\rho) \supseteq \mathbf{R}$ and such that the inequality $\mathtt{div}\,(h\rho)(y) > 0$ holds, for every $y \in \mathbf{R}$, the set $\mathbf{Z}$ of initial conditions in $\mathbf{R}$ from which solutions do not converge to $\Omega_{\le \widehat{M}_\ell}(U_\ell)$ and the set of trajectories of solutions to system (3.8) have Lebesgue measure zero, due to Lemma 3.16.

Fourth Part. Trajectories of solutions to (3.8) that converge to the set $\mathbf{Z} \subset \mathbf{R}$ (cf. proof of Lemma 3.16) are shown to have also measure zero. For every t, let

$$Y(t, \mathbf{Z}) = \{Y(t, z) : t \in \mathtt{dom}(z), z \in \mathbf{Z}\},$$

where $\mathtt{dom}(z)$ is the maximum time interval where $Y(t, z)$ exists. Since $\mathbf{Z}$ is positively invariant, for every $t_1, t_2 \in \mathtt{dom}(z)$,

$$t_1 < t_2 \le 0 \Rightarrow Y(t_2, \mathbf{Z}) \subset Y(t_1, \mathbf{Z}).$$

This inclusion implies that[4]

$$\mathbf{Y} := \bigcup_{t \le 0} \{Y(t, \mathbf{Z})\} = \bigcup_{l \in \mathbb{Z}_{<0}} \{Y(t, \mathbf{Z}) : t \le l\}.$$

Hence, the set $\mathbf{Y}$ is a countable union of images of $\mathbf{Z}$ by the flow. Since $\mathbf{Z}$ is measurable (cf. proof of Lemma 3.16) and, for every $t \in \mathtt{dom}(y)$, the map $\mathbf{Z} \ni y \mapsto Y(t, y)$ is a diffeomorphism,[5] $\mathbf{Y}$ is also measurable. Consequently, the equation

$$\int_{Y(t,\mathbf{Z})} dz \le \int_{\mathbf{Z}} |\,\mathtt{grad}\; Y(t, y)|\, dy = 0$$

holds, for every value of the time $t \in \mathtt{dom}(\mathbf{Z})$, because $\mathbf{Z}$ has Lebesgue measure zero. This implies that, for every $t \in \mathtt{dom}(\mathbf{Z})$, the set $Y(t, \mathbf{Z})$ has Lebesgue measure zero. Since $\mathbf{Y}$ is the countable union of sets of measure zero, it has also measure zero.

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[4] Note also that, when $Y(t, \mathbf{Z})$ does not exist, $Y(t, \mathbf{Z}) = \emptyset$.

[5] Because (3.8) is of class $\mathscr{C}^1$ and solutions are unique (see also [13, Corollary 3.1]).

From the above discussion, the equilibrium of the origin is locally stable and almost globally attractive for (3.8). Thus, it is almost globally asymptotically stable for (3.8). This concludes the sketch of the proof of Theorem 3.11.

Sketch of the proof of Lemma 3.16. The objective of this sketch is to show the set of trajectories of solutions to (3.8) that do not converge to the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ has Lebesgue measure zero. To do so, the same reasoning employed in [22, Theorem 1] and [3, Theorem 3] is used. However, here a less conservative condition is required, since a set that positively invariant is considered, the divergence is required to be positive only in a compact set.

Let $\mathbf{Z} \subset \mathbb{R}^{n+m}$ be given by

$$\mathbf{Z} = \bigcap_{l=1}^{\infty} \left\{ y \in \Omega_{\leq \widetilde{M}_g}(U_g) : U_\ell(Y(t, y)) > \widehat{M}_\ell, t > l \right\},$$

Since $\mathbf{Z}$ is a countable intersection of open sets, it is measurable (cf. Propositions A.8 and A.9).

Note that $\mathbf{Z}$ is the set of all initial conditions belonging to $\Omega_{\leq \widetilde{M}_g}(U_g)$ from which issuing solutions do not converge to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$. Since $\Omega_{\leq \widetilde{M}_g}(U_g)$ is positively invariant (cf. Corollary 3.9), the set $\mathbf{Z}$ is also positively invariant. Thus, given a fixed time instant $\tau > 0$, for every time $t \geq \tau$, the inclusion $Y(t, \mathbf{Z}) \subset Y(\tau, \mathbf{Z})$ holds. Hence, the inequality

$$\int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy \leq 0 \tag{3.28}$$

holds, for every $t \geq 0$.

From Lemma 3.29, for every $t \geq 0$,

$$\int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}\,(h\rho)(y)\, dyds = \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy.$$

Since the inequality $\mathtt{div}\,(h\rho)(y) > 0$ holds, for every $y \in \mathbf{R}$, and $\mathbf{Z}$ belongs to the set $\mathbf{R}$, the inequality

$$\begin{aligned} t \int_{Y(t,\mathbf{Z})} \mathtt{div}\,(h\rho)(y)\, dy &\leq \int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}\,(h\rho)(y)\, dyds \\ &\leq \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy \end{aligned}$$

holds, for every time $t \geq 0$.

Inequality (3.28) implies that the inequality

$$\int_{Y(t,\mathbf{Z})} \mathtt{div}\,(h\rho)(y)\,dy \leq 0$$

holds, for every positive value of the time t. Consequently, inequality (3.27) implies that the equation

$$\int_{Y(t,\mathbf{Z})} \mathtt{div}\,(h\rho)(y)\,dy = 0$$

holds, for every positive value of the time t. Thus, the set $Y(t, \mathbf{Z})$ has Lebesgue measure zero, for every $t \geq 0$. From the continuity of Y, the set $\mathbf{Z}$ has also Lebesgue measure zero. Consequently, the inequality

$$\limsup_{t\to\infty} U_\ell(Y(t, y)) \leq \widehat{M}_\ell$$

holds, for almost every $y \in \mathbf{R}$. This concludes the sketch of the proof of Lemma 3.16.

3.6.3 Sketch of the Proof of Theorem 3.12

Before showing a sketch of the proof of Theorem 3.12, the following lemma is recalled from [34].

Lemma 3.17 (Extended Bendixson's criterion) *Let $n = m = 1$, under the hypotheses of Theorem 3.12 if the conditions*

$$\mathtt{div}\, h(y) \neq 0 \quad \textit{and} \quad h(y) \neq 0, \tag{3.29}$$

hold, for every $y \in \mathbf{R}$, then all solutions to (3.8) issuing from $\mathbf{R}$ converge to the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$.

A sketch of the proof of Lemma 3.17 is provided after the sketch of the proof of Theorem 3.12.

This sketch follows the same reasoning as the sketch of the proof of Theorem 3.11. The difference consists of the third and fourth parts. More specifically, condition (3.29) is assumed instead of the existence of the function ρ. Moreover, Lemma 3.17 implies that there exist no ω-limit sets in $\mathbf{R}$. In the fourth part, no considerations concerning the measure of the sets of initial conditions are needed.

Similarly to the discussion of the sketch of the proof of Theorem 3.11, the equilibrium of the origin is locally stable and globally attractive for (3.8). Thus, it is globally asymptotically stable for (3.8). This concludes the sketch of the proof of Theorem 3.12.

Before sketching a proof for Lemma 3.17, some concepts regarding the asymptotic behavior of solutions are recalled. For planar systems, a closed curve $C \subset \mathbb{R}^2$ is called *closed orbit* if C is not an equilibrium point and there exists a time $T < \infty$ such that, for each $(x, z) \in C$, the equality $(X(kT, x, z), Z(kT, x, z)) = (x, z)$ holds, for every integer k (cf. [26, Definition 2.6]).

Sketch of the Proof of Lemma 3.17. Consider the proper function $U_\ell \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ (resp. $U_g \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$) and the constant values given by Corollary 3.8 (resp. Corollary 3.9). From the proof of [34, Claim 1], the implications

$$\begin{aligned} U_\ell(x, z) \leq \widehat{M}_\ell &\Rightarrow \max\{V(x), W(z)\} \leq \min\{M_\ell, N_\ell\},\\ U_g(x, z) \geq \widetilde{M}_g &\Rightarrow \min\{V(x), W(z)\} \geq \max\{M_g, N_g\} \end{aligned}$$

hold. The inequality $\min\{M_\ell, N_\ell\} < \max\{M_g, N_g\}$ implies the inclusion $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leq \widetilde{M}_g}(U_g)$. Consequently, the set

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)\right\}$$

is nonempty.

From Lemma 3.19, there exists a proper function $U_\infty \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ that is locally Lipschitz on $\mathbb{R}^{n+m} \setminus \{0\}$ (resp. a function $\tilde{h} \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R}^2)$) with $\mathtt{supp}(U_\infty)$ (resp. $\mathtt{supp}(\tilde{h})$) satisfying the inclusion $\mathtt{supp}(U_\infty) \supset \mathbf{R}$ (resp. $\mathtt{supp}(\tilde{h}) \supset \mathbf{R}$). Moreover, for every $y \in \mathbf{R}$, $U_\infty(y) = U_g(y)$ (resp. $\tilde{h}(y) = h(y)$).

From Theorem 3.25,

- The set $\Omega_{=M_g}(U_\infty)$ has finite perimeter;
- The function U_∞ is almost everywhere (in the Hausdorff measure sense) differentiable in $\Omega_{=M_g}(U_\infty)$;
- Let $\mathbf{S}_\infty \subset \Omega_{=M_g}(U_\infty)$ be set of points, where U_g is not differentiable. There exists a Lipschitz parametrization $p_\infty : [a_\infty, b_\infty] \subset \mathbb{R} \to \Omega_{=M_g}(U_\infty)$ that is injective and satisfies, for almost (in the Lebesgue measure sense) every $s \in [a_\infty, b_\infty]$, $p_\infty(s) \notin \mathbf{S}_\infty$. Moreover, in the points where p_∞ is differentiable, $\frac{dp_\infty(s)}{ds}$ is perpendicular to $\frac{\partial U_\infty}{\partial p_\infty}(p_\infty(s))$.

From Theorem 3.27,

$$\iint\limits_{\Omega_{\leq M_g}(U_\infty)} \mathtt{div}\ \tilde{h}(y)\, dy = \oint\limits_{\Omega_{=M_g}(U_\infty)} \tilde{h}(y) \cdot n_\infty(y)\, dy, \tag{3.30}$$

where n_∞ is the outward normal of $\Omega_{\leq M_g}(U_\infty)$ defined, for every $y \in \Omega_{=M_g}(U_\infty)$,[6] by

$$n_\infty(y) = \begin{cases} \dfrac{\frac{\partial U_\infty}{\partial y}(y)}{\left|\frac{\partial U_\infty}{\partial y}(y)\right|}, \text{ if } \frac{\partial U_\infty}{\partial y}(y) \text{ exists,}\\ \qquad 0, \text{ if otherwise.} \end{cases}$$

[6]Note that, from the previous paragraph, for almost every $y \in \Omega_{=M_g}(U_\infty)$, $\frac{\partial U_g}{\partial y}(y)$ exists.

From the sketch of the proof of Proposition 3.6 and from Corollary 3.9, there exists a function $E_g \in (\mathscr{C} \cap \mathscr{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ such that the inequality

$$D^+_{\tilde{h}} U_\infty(y) \leq -E_g(y) < 0$$

holds, for every $y \in \Omega_{=M_g}(U_\infty)$.

From the above discussions, the existence of the parametrization p_∞ and Remark A.40, the inequality

$$\frac{D^+_{\tilde{h}} U_\infty(p_\infty(s))}{\left|\frac{\partial U_\infty}{\partial y}(y)(p_\infty(s))\right|} = \tilde{h}(p_\infty(s)) \cdot n_\infty(p_\infty(s)) < 0$$

holds, for almost every $s \in [a_\infty, b_\infty]$.

Applying the generalized divergence theorem to the level curve $\Omega_{=M_g}(U_\infty)$, the inequality

$$\iint\limits_{\Omega_{\leq M_g}(U_g)} \mathtt{div}\ h(y)\,dy = \int\limits_{[a_\infty, b_\infty]} h(p_\infty(s)) \cdot n_\infty(p_\infty(s))\,ds < 0, \tag{3.31}$$

is obtained from Eq. (3.30), because $U_\infty(y) = U_g(y)$ and $\tilde{h}(y) = h(y)$, for every $y \in \mathbf{K}$.

Analogously to the reasoning that lead to inequality (3.31) and by letting $p_\ell : [a_\ell, b_\ell] \to \Omega_{=M_\ell}(U_\ell)$ be a parametrization of $\Omega_{=M_\ell}(U_\ell)$ with outward unit normal n_ℓ, the inequality

$$\iint\limits_{\Omega_{\leq \widehat{M}_\ell}(U_\ell)} \mathtt{div}\ h(y)\,dy = \int\limits_{[a_\ell, b_\ell]} h(p_\ell(s)) \cdot n_\ell(p_\ell(s))\,ds < 0 \tag{3.32}$$

holds.

Suppose, by absurd, that there exists a closed orbit[7] $C \in \mathbb{R}^2$, parametrized by $p : [a, b] \to C$ and with outward unit normal n, and contained in $\mathbf{R}$. From the generalized divergence theorem,

$$\iint\limits_{\mathbf{D}_C} \mathtt{div}\ h(y)\,dy = \int\limits_{[a,b]} h(p(s)) \cdot n(p(s))\,ds = 0, \tag{3.33}$$

where $\mathbf{D}_C$ is the simply connected region bounded by C.

[7] From the uniqueness of solutions with respect to initial conditions, the closed orbit C is a simple closed curve.

Note that

$$\iint\limits_{\Omega_{\leq M_g}(U_g)\backslash \mathbf{D}_C} \mathtt{div}\ h(y)\,dy = \iint\limits_{\Omega_{\leq M_g}(U_g)} \mathtt{div}\ h(y)\,dy - \iint\limits_{\mathbf{D}_C} \mathtt{div}\ h(y)\,dy$$
$$= \iint\limits_{\Omega_{\leq M_g}(U_g)} \mathtt{div}\ h(y)\,dy,$$

where the last equality is due to (3.33). From inequality (3.31), the following inequality

$$\iint\limits_{\Omega_{\leq M_g}(U_g)\backslash \mathbf{D}_C} \mathtt{div}\ h(y)\,dy < 0 \tag{3.34}$$

holds. On the other hand,

$$\iint\limits_{\mathbf{D}_C\backslash \Omega_{\leq \widehat{M}_\ell}(U_\ell)} \mathtt{div}\ h(y)\,dy = \iint\limits_{\mathbf{D}_C} \mathtt{div}\ h(y)\,dy - \iint\limits_{\Omega_{\leq \widehat{M}_\ell}(U_\ell)} \mathtt{div}\ h(y)\,dy$$
$$= - \iint\limits_{\Omega_{\leq \widehat{M}_\ell}(U_\ell)} \mathtt{div}\ h(y)\,dy,$$

where the last equality is also due to (3.33). From inequality (3.32), the inequality

$$\iint\limits_{\mathbf{D}_C\backslash \Omega_{\leq \widehat{M}_\ell}(U_\ell)} \mathtt{div}\ h(y)\,dy > 0 \tag{3.35}$$

holds.

From inequalities (3.34) and (3.35) and the fact that C is arbitrary, the function $\mathtt{div}\ h$ changes sign in $\mathbf{R}$. From the continuity of $\mathtt{div}\ h$, there exists $\bar{y} \in \mathbf{R}$ such that $\mathtt{div}\ h(\bar{y}) = 0$. This is a contradiction with the condition (3.29). Thus, there exist no closed orbits C contained in $\mathbf{R}$.

From the Poincaré–Bendixson Theorem (cf. [26, Theorem 2.15]), the ω-limit set of a solution starting in $\mathbf{R}$ is a closed orbit or equilibria. Since equilibria do not exist in $\mathbf{R}$ by assumption, there exist no ω-limit sets in $\mathbf{R}$, every solution starting in $\mathbf{R}$ will converge to the set $\Omega_{\leq M_\ell}(U_\ell)$. This concludes the sketch of the proof of Lemma 3.17.

Corollary 3.18 (Bendixson criterion) *Let $n = m = 1$, under the hypotheses of Theorem 3.12 if $\mathbf{R}$ is a simply connected region such that the conditions*

$$\mathtt{div}\ h(y) \neq 0 \quad \textit{and} \quad h(y) \neq 0 \tag{3.29}$$

hold, for every $y \in \mathbf{R}$, then all solutions to (3.8) issuing from the set $\mathbf{R}$ converge to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$.

3.6.4 Sketch of the Proof of Proposition 3.13

Assume for a fixed $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, the inequality $V(x) \geq \Gamma(W(z))$ holds. Suppose also that $z \in \Omega_{<0.6}(W)$. From (3.19),

$$\Gamma(W(z)) = \rho_x^{-1}\left(\frac{W(z)}{0.95}\right).$$

Consequently, the inequality

$$V(x) \geq \rho_x^{-1}\left(\frac{W(z)}{0.95}\right)$$

holds. Because ρ_x^{-1} is strictly increasing in $[0, 0.6)$, it is invertible and the condition

$$\rho_x(V(x)) \geq \frac{W(z)}{0.95}$$

holds. This implies that the inequality $D_f^+ V(x, z) \leq -\lambda_x(x)$ holds, due to (3.18). The above reasoning, applied for $z \in \Omega_{\geq 0.6}(W)$, yields an analogous conclusion.

Now it remains to show that Γ is tight. From the surjectivity and continuity of W, there exists $z^* \in \mathbb{R}_{>0}$ such that $s^* = W(z^*)$. Thus, the inequality $\Gamma^*(W(z^*)) < \Gamma(W(z^*))$ holds.

Assume that $z^* \in (\mathbb{R}_{>0} \cap \Omega_{<0.6}(W))$. From (3.19),

$$\Gamma(W(z^*)) = \rho_x^{-1}\left(\frac{W(z^*)}{0.95}\right).$$

Since ρ_x^{-1} is strictly increasing in the interval $[0, 0.6)$, it is invertible and the inequality $\rho_x(\Gamma^*(W(z^*)))0.95 < W(z^*)$ holds. From the surjectivity and continuity of V, there exists $x^* \in \mathbb{R}_{>0}$ such that $\rho_x(\Gamma^*(W(z^*)))0.95 \leq \rho_x(V(x^*)) < W(z^*)$. Since $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, the Dini derivative of V yields the inequality

$$D_f^+ V(x^*, z^*) = -\rho_x(V(x^*)) + W(z^*) > 0.$$

The case in which $z^* \in \Omega_{\geq 0.6}(W)$ is parallel. Thus, there exists $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that $D_f^+ V(x^*, z^*) > 0$. This concludes the sketch of the proof of Proposition 3.13.

3.6.5 Sketch of the Proof of Proposition 3.14

In the interval $(0.65, 2.5)$ two regions are analyzed: $(0.65, 1)$ and $[1, 2.5)$.

Suppose that $s \in (0.65, 1)$. In this region, Δ is strictly increasing and satisfies the inequality $0.6 < \Delta(s) < 0.7$. From (3.19), the function Γ is strictly increasing in the interval $(0.6, 0.7)$. Moreover, the function ρ_x^{-1} satisfies the inequality, $2.2 < \rho_x^{-1}(s) < 2.3$, for every $s \in (0.65, 0.7)$. Hence, for every $s \in (0.65, 1)$, the inequality $s \leq \Gamma \circ \Delta(s)$ holds. The analysis for the interval $[1, 2.5)$ is analogous. This concludes the sketch of the proof of Proposition 3.14.

3.7 Conclusion

A condition to verify the stability of interconnected ISS systems that do not satisfy the small-gain condition is proposed. The approach consists in verifying if the small-gain conditions hold in two different intervals corresponding to different regions of the state space: close to the origin and at infinity. In the gap between both regions, these small-gain conditions are blended with a divergence criterion for the absence of ω-limit sets. For planar system, an extension of Bendixson criterion is proposed. For higher dimensional systems, the set of trajectories of solutions that does not converge to the origin has Lebesgue measure zero.

Appendix of Chap. 3

Technical Lemma

Lemma 3.19 *Let $k \geq 0$ and $p > 0$ be constant integers. Given a function $h \in \mathscr{C}^k(\mathbb{R}^n, \mathbb{R}^p)$, and compact set $\mathbf{K} \subset \mathbb{R}^n$ such that, for every $y \in \mathbf{K}$, $h(y) \neq 0$. Then, there exists $\tilde{h} \in \mathscr{C}^k(\mathbb{R}^n, \mathbb{R}^p)$ such that $\mathtt{supp}(\tilde{h}) \supset \mathbf{K}$ and, for every $y \in \mathbf{K}$, $\tilde{h}(y) = h(y)$.*

The proof of Lemma 3.19 is based on [24, p. 370] and can be found in [32].

The Divergence Theorem for Level Sets of a Lyapunov Function

The following definition is recalled from [23]:

Definition 3.20 (*Gamma function*) The function

$$\begin{aligned} \Gamma : \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ s &\mapsto \int_0^\infty t^{s-1} e^{-t}\, dt \end{aligned}$$

is called *gamma function*.

The next definition is recalled from [21].

Definition 3.21 (*Hausdorff measure*) Let $\mathbf{E} \subset \mathbb{R}^n$, the *diameter* of the set $\mathbf{E}$ is the function

$$\begin{aligned} \mathtt{diam} : \mathbf{E} \times \mathbf{E} &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \sup\{|x - y|\}. \end{aligned}$$

Let $0 \leq n < \infty$ and define, for $0 < \delta \leq \infty$, the value

$$\mathscr{H}_\delta^n(\mathbf{E}) = \inf \left\{ \sum_{j \in \mathbb{N}} \mathtt{diam}(\mathbf{E}_j)^n : \mathbf{E} \subset \bigcup_{j \in \mathbb{N}} \mathbf{E}_j, \mathtt{diam}(\mathbf{E}_j) < \delta, \mathbf{E}_j \subset \mathbb{R}^n \right\}.$$

The n-dimensional *unnormalized Hausdorff measure of* $\mathbf{E}$ is the limit

$$\widetilde{\mathscr{H}^n}(\mathbf{E}) = \lim_{\delta \to 0} \mathscr{H}_\delta^n(\mathbf{E}) = \sup_{\delta > 0} \mathscr{H}_\delta^n(\mathbf{E}).$$

The n-dimensional *Hausdorff measure of* $\mathbf{E}$ is given by

$$\mathscr{H}^n(\mathbf{E}) = \frac{\alpha(s)}{2^n} \widetilde{\mathscr{H}^n}(\mathbf{E}),$$

where

$$\alpha(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

The relation between Hausdorff and Lebesgue measures is explained in the following remark which is based on [10, Sect. 2.2] and [20]:

Remark 3.22 Note that the n-dimensional Lebesgue measure of a set $\mathbf{E} \subset \mathbb{R}^n$ is the n-fold product of unidimensional Lebesgue measures (cf. Definitions A.5 and A.7) while the Hausdorff measure is computed in terms of arbitrarily coverings of $\mathbf{E}$ with small diameters. Moreover, the Lebesgue measure in $\mathbb{R}^n$ is equivalent to the n-dimensional Hausdorff measure, i.e., $\mu = \mathscr{H}^n$. Also, if $\mathscr{H}^n(\mathbf{E}) < \infty$, then $\mathscr{H}^{n-1}(\mathbf{E}) = \infty$ and $\mathscr{H}^{n+1}(\mathbf{E}) = 0$.

The next concept, recalled from [21, p. 50], is a measure-theoretical notion of boundaries of a set.

Definition 3.23 (*Essential boundaries*) For a set $\mathbf{E} \subset \mathbb{R}^n$,

- The *essential exterior* is the set

$$\mathtt{ext}_*(\mathbf{E}) = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mu\left(\mathbf{E} \cap \mathbf{B}_{\leq r}(x)\right)}{\mu\left(\mathbf{B}_{\leq r}(x)\right)} = 0 \right\};$$

- The *essential interior* is the set $\mathtt{int}_*(\mathbf{E}) = \mathtt{ext}_*(\mathbb{R}^n \setminus \mathbf{E})$;

- The *essential closure* is the set $\texttt{cl}_*\{\mathbf{E}\} = \mathbb{R}^n \setminus \texttt{ext}_*(\mathbf{E})$;
- The *essential boundary* is the set $\partial_*\mathbf{E} = \texttt{cl}_*\{\mathbf{E}\} \setminus \texttt{int}_*(\mathbf{E})$.

The following properties hold[8]:

$$\texttt{int}_*(\mathbf{E}) \subset \texttt{cl}_*\{\mathbf{E}\}, \texttt{int}_*(\mathbb{R}^n \setminus \mathbf{E}) = \texttt{ext}_*(\mathbf{E}),$$
$$\partial_*\mathbf{E} = \texttt{cl}_*(\mathbf{E}) \cap \texttt{cl}_*\{\mathbb{R}^n \setminus \mathbf{E}\} = \partial_*(\mathbb{R}^n \setminus \mathbf{E}) = \mathbb{R}^n \setminus (\texttt{int}_*(\mathbf{E}) \cup \texttt{ext}_*(\mathbf{E})).$$

Definition 3.23 is related to the usual topological concepts as follows:

$$\texttt{int}(\mathbf{E}) \subset \texttt{int}_*(\mathbf{E}), \quad \texttt{cl}_*\{\mathbf{E}\} \subset \texttt{cl}\{\mathbf{E}\}, \quad \partial_*\mathbf{E} \subset \partial\mathbf{E}.$$

Moreover,

$$\partial_*\mathbf{E} = \partial\mathbf{E} \Leftrightarrow \texttt{int}(\mathbf{E}) = \texttt{int}_*(\mathbf{E}) \quad \text{and} \quad \texttt{cl}_*\{\mathbf{E}\} = \texttt{cl}\{\mathbf{E}\},$$

and the inclusion

$$\texttt{int}_*(\mathbf{E}) \subset \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mu\left(\mathbf{E} \cap \mathbf{B}_{\leq r}(x)\right)}{\mu\left(\mathbf{B}_{\leq r}(x)\right)} = 1 \right\}$$

becomes an inequality, when $\mathbf{E}$ is measurable.

The following measure-theoretical notion of perimeter of a set is recalled from [21, Definition 4.5.1]:

Definition 3.24 (*Perimeter of a set*) The *perimeter* of a set $\mathbf{E} \subset \mathbb{R}^n$ is the measure

$$P(\mathbf{E}) = \mathscr{H}^{n-1}(\partial_*\mathbf{E}).$$

The perimeter is *finite* if $\mu(\mathbf{E}) + P(\mathbf{E}) < \infty$.

The notion of a perimeter of a set is an important concept for the next theorem, adapted from [1, Theorem 2.5].

Theorem 3.25 *Let $k \geq 0$ be a constant integer, and consider the Lipschitz map $V \in \mathscr{C}^k(\mathbb{R}^2, \mathbb{R}_{\geq 0})$ with $\texttt{supp}(V)$ compact. The following statements hold, for almost every $c \in \mathbb{R}_{\geq 0}$:*

1. *$\Omega_{=c}(V)$ is 1-rectifiable[9] and $\mathscr{H}^1(\Omega_{=c}(V)) < \infty$;*
2. *For $\mathscr{H}^1$-almost every $x \in \Omega_{=c}(V)$, the map V is differentiable at x;*
3. *Every connected component C of $\Omega_{=c}(V)$ is either a point or a closed simple curve with a Lipschitz parametrization $p : [a, b] \to C$ which is injective and satisfies, for almost every $t \in [a, b]$,*

---

[8]Pfeffer [21, p. 49].

[9]In other words, the set $\Omega_{=c}(V)$ can be $\mathscr{H}^1$-almost everywhere covered by countably many 1-dimensional curves of class $\mathscr{C}^1$.

$$\frac{dp}{dt}(t) = \tau(p(t)),$$

where, for every $x \in C$, $\tau(x)$ *is the vector tangent to* C.

From item 2 and since the level set $\Omega_{=c}(V)$ is either a point or a simple closed curve of $\mathbb{R}^2$, $\partial\Omega_{=c}(V) = \mathtt{cl}\{\Omega_{=c}(V)\} = \Omega_{=c}(V)$. Moreover, $\partial_*\Omega_{=c}(V) \subset \mathtt{cl}\{\Omega_{=c}(V)\}$. From item 1, the sublevel set $\Omega_{\leq c}(V)$ has finite perimeter. Thus the inequality

$$\int_{\Omega_{=c}(V)} d\mathscr{H}^1 < \infty$$

holds. Note that, from Remark 3.22, this integral is defined in the Lebesgue sense in $\mathbb{R}^1$.

The next definition, based on [20, Definition 1.6] and [21, pp. 127–128], recalls the concept of vector being an outward normal to a set.

Definition 3.26 (*Outward normal*) For every $x \in \partial_*\mathbf{E}$, denote by $n_{\mathbf{E}}(x)$ the unit vector of $\mathbb{R}^n$ such that

$$\mathbf{H}_{\pm}(\mathbf{E}, x) = \{y \in \mathbb{R}^n : \pm n_{\mathbf{E}}(x) \cdot (y - x) \geq 0\}.$$

The function $n_{\mathbf{E}}$ is called *outward unit normal of* $\mathbf{E} \subset \mathbb{R}^n$ if, for every $x \in \partial_*\mathbf{E}$,

$$\begin{aligned} \lim_{r\to 0} \frac{\mu\left(\mathbf{B}_{\leq r}(x) \cap \mathbf{H}_{+}(\mathbf{E}, x) \cap \mathbf{E}\right)}{\mu\left(\mathbf{B}_{\leq r}(x)\right)} &= 0, \\ \lim_{r\to 0} \frac{\mu\left(\mathbf{B}_{\leq r}(x) \cap (\mathbf{H}_{-}(\mathbf{E}, x) \setminus \mathbf{E})\right)}{\mu\left(\mathbf{B}_{\leq r}(x)\right)} &= 0 \end{aligned} \tag{3.36}$$

hold.

From item 2 of Theorem 3.25, for $\mathscr{H}^1$-almost every $x \in \Omega_{=c}(V)$, $\mathtt{grad}\ V(x)$ exists. Thus, the vector field

$$\begin{aligned} n : \Omega_{=c}(V) &\to \mathbb{R}^2 \\ x &\mapsto \begin{cases} \dfrac{\mathtt{grad}\ V(x)}{|\mathtt{grad}\ V(x)|}, & \text{if } \mathtt{grad}\ V(x) \text{ exists,} \\ 0, & \text{if otherwise} \end{cases} \end{aligned} \tag{3.37}$$

is $\mathscr{H}^1$-almost everywhere an outward normal to the set $\Omega_{\leq c}(V)$. Since the outward normal to sets of finite perimeter is unique (cf. [11, Theorem 3.4]), the vector n satisfies the limits (3.36).

For a further reading on the sets of finite perimeters and on the construction of outward normals for them, the interested reader is invited to see [21, Chaps. 5 and 6].

The next result shows the relationship between the line integral along a closed curve and the integral on the domain bounded by this curve.

Theorem 3.27 (Generalized divergence theorem)[10] *Under the assumptions of Theorem 3.25. Let $k \geq 0$ be a constant integer, and consider the map $f \in \mathscr{C}^k(\mathbb{R}^2, \mathbb{R}^2)$. Then, the formula*

$$\iint_{\Omega_{\leq c}(V)} \texttt{div}\ f(x)\, dx = \oint_{[a,b]} f(p(s)) \cdot n(p(s))\, ds$$

holds, where the integral of the left-hand side (resp. right-hand side) is taken in the Lebesgue (resp. 1-dimensional Hasudorff) measure on $\mathbb{R}^2$ (resp. $\mathbb{R}$), and $p : [a, b] \to \Omega_{=c}(V)$ is a parametrization of $\Omega_{=c}(V)$.

Before showing a sketch of the proof of Theorem 3.27, the following lemma, based on [31, p. 106], is needed. For a detailed proof in $\mathbb{R}^n$, the interested reader may consult [21, Chaps. 1–6].

Lemma 3.28 (Green's Theorem) *Let $C \subset \mathbb{R}^2$ be a positively oriented, piecewise smooth, simple closed curve with finite length, let $\mathbf{D}_C$ be the region bounded by C, and let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$. If $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are defined on an open region containing $\mathbf{D}_C$, and f is differentiable in such a region, then*

$$\oint_C (f_1(x_1, x_2)\, dx_1 + f_2(x_1, x_2)\, dx_2) = \iint_{\mathbf{D}_C} \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2, \qquad (3.38)$$

where the path of integration along C is counterclockwise.

Sketch of the *(of Lemma 3.28)*

This proof is based on [31, p. 108]. Since C is a simple closed curve in the plane, the region $\mathbf{D}_C$ is bounded. The projection of the curve in the x-axis (resp. y-axis) yields an interval $[a, b]$ (resp. $[e, f]$). Consider the points of $A, B \in C$ (resp. $E, F \in C$) corresponding to the points a and b (resp. e and f) on the x-axis, the curve C can be seen as the union of the curves AEB and AFB. Figure 3.3 illustrates the curve C, and the intervals $[a, b]$, and $[e, f]$.

Let the equation of the curve containing the points AEB (resp. AFB) be given by a piecewise continuous function $\eta_1 : [a, b] \to \mathbb{R}^2$ (resp. $\eta_2 : [a, b] \to \mathbb{R}^2$).

Integrating the partial derivative of f_1 with respect to x_2 in $\mathbf{D}_C$ yields

$$\iint_{\mathbf{D}_C} \frac{\partial f_1}{\partial x_2}(x_1, x_2)\, dx_1 dx_2 = \int_a^b \int_{\eta_1(x_1)}^{\eta_2(x_1)} \frac{\partial f_1}{\partial x_2}(x_1, x_2)\, dx_2 dx_1,$$

[10] Adapted from [20, Theorem 1.7] or [21, Theorem 6.5.4]. In the latter, the set where the integral is computed is assumed to have *Bounded Variation*, in [21, Theorem 6.5.5] it is shown that a set has bounded variation if and only if it has finite perimeter.

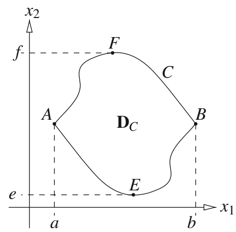

Fig. 3.3 Illustration of the curve C

where the equality is due to Fubini's theorem (cf. [9, Theorem 14.1]). Moreover, since C has finite length, the equality

$$\iint\limits_{\mathbf{D}_C} \frac{\partial f_1}{\partial x_2}(x_1, x_2)\, dx_1 dx_2 = \int\limits_a^b (f_1(x_1, \eta_2(x_1)) - f_1(x_1, \eta_1(x_1)))\, dx_1$$

$$= -\int\limits_a^b f_1(x_1, \eta_1(x_1))\, dx_1 - \int\limits_b^a f_1(x_1, \eta_2(x_1))\, dx_1$$

$$= -\oint\limits_C f_1(x_1, x_2)\, dx_1.$$

holds.

Analogously, integrating the partial derivative of f_2 with respect to x_1 in $\mathbf{D}_C$ yields

$$\iint\limits_{\mathbf{D}_C} \frac{\partial f_2}{\partial x_1}(x_1, x_2)\, dx_1 dx_2 = \oint\limits_C f_2(x_1, x_2)\, dx_2.$$

From where the conclusion follows.

Now, it is possible to present an idea of the proof of Theorem 3.27.

From Theorem 3.25,

- The curve $\Omega_{=c}(V)$ is piecewise $\mathscr{C}^1$, because it is rectifiable. Moreover, it is also simple and closed;
- Since V is $\mathscr{H}^1$-a.e. differentiable in $\Omega_{=c}(V)$, the outward normal vector defined by (3.37) is $\mathscr{H}^1$-a.e. non-nil;
- The curve $\Omega_{=c}(V)$ has finite length, because $\mathscr{H}^1(\Omega_{=c}(V)) < \infty$;
- There exists a injective and Lipschitz continuous parametrization $p : [a, b] \to \Omega_{=c}(V)$ that is a.e. differentiable.

Consider the vector field $\tilde{f} = (-f_2, f_1)$ that is perpendicular to $f = (f_1, f_2)$. Since $f = (f_1, f_2) \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, $\tilde{f} \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. Together with the above, from[11] Lemma 3.28,

[11] More specifically from (3.38).

$$\oint_{\Omega_{=c}(V)} (-f_2(x_1, x_2)\, dx_1 + f_1(x_1, x_2)\, dx_2) = \oint_{\Omega_{=c}(V)} (-f_2(x_1, x_2), f_1(x_1, x_2)) \cdot (dx_1, dx_2)$$

Consider a point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in C$ for which there exists $\bar{s} \in [a, b]$ such that $p(\bar{s}) = (\bar{x}_1, \bar{x}_2)$ and $p'(\bar{s})$ is defined. The unit tangent vector to C at $\bar{x}$ is given by $T(\bar{s}) = {}^{p'(\bar{s})}/_{|p'(\bar{s})|} = (\tau(\bar{s}), \sigma(\bar{s}))$. The unit normal vector at $\bar{x}$ is given by $N(\bar{s}) = n(p(s)) = (\sigma(\bar{s}), -\tau(\bar{s}))$. For almost every $s \in [a, b]$,

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = T(s)\, ds = \begin{pmatrix} \tau(s) \\ \sigma(s) \end{pmatrix} ds.$$

Thus,

$$\begin{aligned} \oint_{\Omega_{=c}(V)} (-f_2(x_1, x_2)\, dx_1 + f_1(x_1, x_2)\, dx_2) &= \oint_{[a,b]} (-f_2(p(s)), f_1(p(s))) \cdot (\tau(s), \sigma(s))\, ds \\ &= \oint_{[a,b]} (f_1(p(s)), f_2(p(s))) \cdot (\sigma(s), -\tau(s))\, ds \\ &= \oint_{[a,b]} (f_1(p(s)), f_2(p(s))) \cdot n(p(s))\, ds \end{aligned}$$

From (3.38),

$$\iint_{\Omega_{\leq c}(V)} \left(\frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2) \right) dx_1 dx_2 = \oint_{[a,b]} (f_1(p(s)), f_2(p(s))) \cdot n(p(s))\, ds.$$

This concludes the sketch of the proof of Theorem 3.27.

Integration Along Solutions of an ODE

Before recalling the main result, the following lemma which is recalled from [22, Lemma A.1] is needed.

Lemma 3.29 (Liouville's Theorem) *Let $k \geq 1$ and $p \geq 1$ be constant integers, the function $\rho \in (\mathscr{C}^k \cap \mathscr{L}^p)(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$. Let also $Y(t, y)$ be a solution of (3.8) starting in $y \in \mathbb{R}^{n+m}$ and computed at time $t \in \mathbb{R}_{\geq 0}$. For a measurable set $\mathbf{Z}$, let $Y(\cdot, \mathbf{Z}) = \{Y(\cdot, z) : z \in \mathbf{Z}\}$. Then,*

$$\int\limits_{Y(t,\mathbf{Z})} \rho(y)\,dy - \int\limits_{\mathbf{Z}} \rho(y)\,dy = \int\limits_0^t \int\limits_{Y(\tau,\mathbf{Z})} \mathtt{div}\,(\rho h)(y)\,dyd\tau.$$

The main result in this section is recalled from [22, Theorem 1].

Theorem 3.30 (Almost attractivity) *Let $k \geq 1$ and $p \geq 1$ be constant integers. Suppose that there exists $\rho \in (\mathscr{C}^k \cap \mathscr{L}^p)(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ such that,*

$$\int_{\mathbf{B}_{\geq 1}(0)} \frac{(h\rho)(y)}{|y|}\,dy < \infty$$

and

$$y \in \mathbb{R}^{n+m}, \quad \mathtt{div}\,(h\rho)(y) > 0.$$

Then, for almost every initial condition $y \in \mathbb{R}^{n+m}$,

$$\limsup_{t\to\infty} |Y(t, y)| = 0.$$

Moreover, if the origin is stable, then the conclusion remains valid when ρ takes negative values.

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Appendix A
Appendix

A.1 Notation

The set of real values and (resp. strictly) positive are denoted, respectively, by $\mathbb{R}$ and (resp. $\mathbb{R}_{>0}$) $\mathbb{R}_{\geq 0}$.

Throughout this book, the Euclidean space $\mathbb{R}^n$, for some integer $n > 0$, is considered endowed with the inner product denoted as $x \cdot y$, for any two vectors $x, y \in \mathbb{R}^n$. The *Euclidean norm* of the vector x is denoted as $|x|$. Let $\mathbf{K} \subset \mathbb{R}^n$ be a compact set and $x \in \mathbb{R}^n \setminus \mathbf{K}$, the notation $|x|_{\mathbf{K}}$ stands for *point-to-set distance* $\inf_{y \in \mathbf{K}} |y - x|$. The *open* (resp. *closed*) *ball centered* at the compact set $\mathbf{K} \subset \mathbb{R}^n$ *with radius* $r > 0$ is defined as $\mathbf{B}_{<r}(\mathbf{K}) = \{x \in \mathbb{R}^n : |x|_{\mathbf{K}} < r\}$ (resp. $\mathbf{B}_{\leq r}(\mathbf{K}) = \{x \in \mathbb{R}^n : |x|_{\mathbf{K}} \leq r\}$).

Let $\mathbf{S} \subset \mathbb{R}^n$ be a set containing the origin, the notation $\mathbf{S}_{\neq 0}$ stands for the set $\mathbf{S} \setminus \{0\}$. The *closure* of $\mathbf{S}$ is denoted as $\mathtt{cl}\{\mathbf{S}\}$. The notation $\mathtt{co}\{\mathbf{S}\}$ stands for the *convex closure* of the set $\mathbf{S}$.

The *support* of a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is denoted as $\mathtt{supp}(f)$ and is the set $\{x \in \mathbb{R}^n : f(x) \neq 0\}$.

A.2 Lebesgue Measure and Integral

Some concepts of Lebesgue measure an integral, used mainly in Chap. 3, are recalled here.

A.2.1 The Lebesgue Measure

Definition A.1 [21] A collection $\mathfrak{S}$ of subsets of a set $\mathbb{R}^n$ is said to be a *σ-algebra of sets* if

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1. $\emptyset, \mathbb{R}^n \in \mathfrak{S}(\mathbb{R}^n)$;
2. if $\mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$, then $\mathbb{R}^n \setminus \mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$;
3. if $\{\mathbf{A}_i : i \in \mathbb{N}\} \subset \mathfrak{S}(\mathbb{R}^n)$, then

$$\bigcup_{i\in\mathbb{N}} \mathbf{A}_i \in \mathfrak{S}(\mathbb{R}^n) \quad \text{and} \quad \bigcap_{i\in\mathbb{N}} \mathbf{A}_i \in \mathfrak{S}(\mathbb{R}^n).$$

The pair $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ is said to be a *measurable space*, and the sets $\mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$ are said to be *measurable sets*.

Definition A.2 [12] Let $\mathfrak{T}(\mathbb{R}^n)$ be collection of all open sets of $\mathbb{R}^n$. The *Borel σ-algebra* $\mathfrak{B}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the σ-algebra generated by the open sets, $\mathfrak{B}(\mathbb{R}^n) = \mathfrak{S}(\mathfrak{T}(\mathbb{R}^n))$. A set that belongs to $\mathfrak{B}(\mathbb{R}^n)$ is said to be a *Borel* set.

Definition A.3 (*Based on* [14]) Consider a measurable space $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$, a mapping $\mathtt{mes} : \mathfrak{S}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a *measure* if

1. $\mathtt{mes}(\emptyset) = 0$;
2. $\mathtt{mes}$ is countably additive, i.e.,

$$\mathtt{mes}\left(\bigcup_{i\in\mathbb{N}} \mathbf{A}_i\right) = \sum_{i\in\mathbb{N}} \mathtt{mes}(\mathbf{A}_i),$$

for every sequence of pairwise disjoint sets $\{\mathbf{A}_i\}_{i\in\mathbb{N}} \subset \mathfrak{S}(\mathbb{R}^n)$.

The triple $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n), \mathtt{mes})$ is said to be a *measure space*.

Definition A.4 (*Based on* [12, 14]) The *length* of an interval $[a, b]$ is defined by $\ell([a, b]) = b - a$. A *rectangle* $\mathbf{R} \subset \mathbb{R}^n$ is a set of the form

$$\mathbf{R} = \times_{i=1}^{n} [a_i, b_i].$$

A rectangle is said to be *open* if $\mathbf{R} = \mathtt{int}(\mathbf{R})$. Two rectangles $\mathbf{R}_1$ and $\mathbf{R}_2$ are said to be *almost disjoint* if $\mathtt{int}(\mathbf{R}_1) \cap \mathtt{int}(\mathbf{R}_2) = \emptyset$. The set of all n-dimensional rectangles of $\mathbb{R}^n$ is denoted by $\mathfrak{R}$. The *volume* of a rectangle $\mathbf{R}$ is defined by

$$\mathtt{vol}(\mathbf{R}) = \prod_{i=1}^{n} \ell([a_i, b_i])$$

with the convention that $0 \cdot \infty = 0$.

Definition A.5 (*Based on* [12, 14]) Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ be a measurable space the *Lebesgue outer measure* of $\mathbf{E} \subset \mathbb{R}^n$ is defined by

$$\mu^*(\mathbf{E}) = \inf\left\{\sum_{i\in\mathbb{N}} \mathtt{vol}(\mathbf{R}_i) : \mathbf{E} \subset \bigcup_{i\in\mathbb{N}} \mathbf{R}_i, \mathbf{R}_i \in \mathfrak{R}\right\},$$

where the infimum is taken over all countable collection whose union contains $\mathbf{E}$. The mapping

$$\begin{aligned} \mu^* : 2^{\mathbb{R}^n} &\to \mathbb{R}_{\geq 0} \\ \mathbf{E} &\mapsto \mu^*(\mathbf{E}), \end{aligned}$$

where $2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$, is called *outer Lebesgue measure*.

Proposition A.6 [12] *The Lebesgue outer measure has the following properties*

P_1 $\mu^*(\emptyset) = 0$;
P_2 *if* $\mathbf{E} \subset \mathfrak{S}(\mathbb{R}^n)$, *then* $\mu^*(\mathbf{E}) \leq \mu^*(\mathfrak{S}(\mathbb{R}^n))$;
P_3 *if* $\{\mathbf{E}_i \subset \mathbb{R}^n : i \in \mathbb{N}\}$ *is a countable collection of subsets of* $\mathbb{R}^n$, *then*

$$\mu^*\left(\bigcup_{i=1}^{n} \mathbf{E}_i\right) \leq \sum_{i=1}^{m} \mu^*(\mathbf{E}_i).$$

Definition A.7 [12] A set $\mathbf{E} \subset \mathbb{R}^n$ is said to be *Lebesgue measurable* if, for every $\mathbf{A} \subset \mathbb{R}^n$,

$$\mu^*(\mathbf{A}) = \mu^*(\mathbf{A} \cap \mathbf{E}) + \mu^*(\mathbf{A} \setminus \mathbf{E}). \tag{A.1}$$

Let $\mathfrak{L}(\mathbb{R}^n)$ denote the σ-algebra of Lebesgue measurable sets, the restriction of the Lebesgue outer measure μ^* to the Lebesgue measurable sets, $\mu = \mu^*|_{\mathfrak{L}(\mathbb{R}^n)}$, $\mu : \mathfrak{L}(\mathbb{R}^n) \to [0, \infty]$, is called *Lebesgue measure*.

Proposition A.8 [12] *Every rectangle is Lebesgue measurable.*

Proposition A.9 [12] *Every open set is a countable union of almost disjoint rectangles.*

A consequence of Propositions A.8 and A.9 is that every open set is Lebesgue measurable.

Definition A.10 [12] Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ and $(\mathbb{R}^m, \mathfrak{S}(\mathbb{R}^m))$ be measurable spaces. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *measurable* if $f^{-1}(\mathbf{B}) \in \mathfrak{S}(\mathbb{R}^n)$, for every $\mathbf{B} \in \mathfrak{S}(\mathbb{R}^m)$.

A.2.2 The Lebesgue Integral

Definition A.11 [12] A *characteristic function of a subset* $\mathbf{E} \subset \mathbb{R}^n$ is defined as

$$\begin{aligned} \chi_{\mathbf{E}} : \mathbb{R}^n &\to \mathbb{R} \\ y &\mapsto \begin{cases} 1, \text{ if } y \in \mathbf{E}, \\ 0, \text{ if } y \notin \mathbf{E}. \end{cases} \end{aligned}$$

A *simple function* $\phi : \mathbb{R}^n \to \mathbb{R}$ on a measurable space $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ has the form

$$\phi(y) = \sum_{i=1}^{N} c_i \chi_{\mathbf{E}_i}(y),$$

where, for every $i = 1, \ldots, N$, $c_i \in \mathbb{R}$, and $\mathbf{E}_i \in \mathfrak{S}(\mathbb{R}^n)$. It is said to be *positive simple function* if, in addition, for every $i = 1, \ldots, N$, $c_i \in \mathbb{R}_{\geq 0}$.

Definition A.12 [12] Let $(\mathbb{R}^n, \mathfrak{S}, \mu)$ be a measure space and $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a positive simple function the *integral of* ϕ *with respect to* μ is defined as

$$\int \phi \, d\mu = \sum_{i=1}^{N} c_i \mu(\mathbf{E}_i)$$

with the convention that if $c_i = 0$ and $\mu(\mathbf{E}_i) = \infty$, then $0 \cdot \infty = 0$.

Definition A.13 (*Based on* [12]) Let $(\mathbb{R}^n, \mathfrak{S}, \mu)$ be a measure space, and $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a be measurable function, then

$$\int h \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq h, \phi \text{ simple} \right\}.$$

The function h is said to be *integrable* if

$$\int h \, d\mu < \infty.$$

Remark A.14 [12] Definitions A.12 and A.13 can also be applied to vector fields. In this case, the integral of a vector field simple function ϕ is defined exactly as in Definition A.12. A vector field $f : \mathbb{R}^n \to \mathbb{R}^m$ is called integrable if there exists a sequence of integrable simple functions $\{\phi_i\}_{i \in \mathbb{N}}$, where $\phi_i : \mathbb{R}^n \to \mathbb{R}^m$, such that the convergence $\phi_i \to f$ pointwisely[1] with respect to the norm on $\mathbb{R}^m$, and

$$\int ||f - \phi_n|| \, d\mu \to 0, \quad \text{as } n \to \infty.$$

Definition A.15 (*Based on* [12, 26]) Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n), \mu)$, and $p \in [1, \infty)$. The Lebesgue measurable function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally* p*-integrable* (or p*-summable*) on $\mathbb{R}^n$ if, for every compact set $\mathbf{K} \subset \mathbb{R}^n$, and for $p \in [1, \infty)$, the inequality

$$\int_{\mathbf{K}} |h(y)|^p \, d\mu < \infty$$

[1]Let $\mathbf{S} \subset \mathbb{R}^n$, and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real valued functions defined on $\mathbf{S}$. The sequence $\{f_n\}_{n \in \mathbb{N}}$ is said to *converge pointwisely to* f if there exists $\bar{x} \in \mathbf{S}$ such that, for every $\varepsilon > 0$, there exists $N(\bar{x}, \varepsilon) > 0$ satisfying, for every $n \geq N(\bar{x}, \varepsilon)$, the inequality $|f_n(\bar{x}) - f(\bar{x})| < \varepsilon$.

holds. The class of locally p-integrable functions h on $\mathbb{R}^n$ is denoted by $\mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$. For $p = \infty$, the inequality

$$\operatorname{ess\,sup}_{y \in \mathbf{K}} |h(y)| < \infty$$

holds, where

$$\operatorname{ess\,sup}_{y \in \mathbf{K}} h(y) = \inf\{a \in \mathbb{R} : \mu(y \in \mathbf{K} : h(y) > a) = 0\}.$$

The class of locally ∞-integrable functions h on $\mathbb{R}^n$ is called *locally essentially bounded*, and is denoted by $\mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$. The *essential supremum norm of h in* $\mathbb{R}^n$ is the positive value $|h|_\infty := \operatorname{ess\,sup}\{|h(y)| : y \in \mathbb{R}^n\}$.

A.3 Continuity of Functions

Definition A.16 (*Uniform continuity*) Let $\mathbf{Y} \subset \mathbb{R}^n$ be an open set. A function $h : \mathbf{Y} \to \mathbb{R}^n$ is said to be *continuous* if,

$$\forall \bar{y} \in \mathbf{Y}, \forall \varepsilon > 0, \exists \delta(\bar{y}, \varepsilon) > 0 : \forall y \in \mathbf{Y}, |y - \bar{y}| < \delta(\bar{y}, \varepsilon) \Rightarrow |h(y) - h(\bar{y})| < \varepsilon.$$

The class of k-times continuously differentiable functions $h : \mathbf{Y} \to \mathbb{R}^n$ is denoted by $\mathscr{C}^k(\mathbf{Y}, \mathbb{R}^m)$. The function h is said to be *uniformly* continuous if,

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \forall y_1 \in \mathbf{Y}, \forall y_2 \in \mathbf{Y}, |y_2 - y_2| < \delta(\varepsilon) \Rightarrow |h(y_1) - h(y_2)| < \varepsilon.$$

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and c be a constant value, the notation $\Omega(V)_{\diamond c}$ stands for the the set $\{x \in \mathbb{R}^n : V(x) \diamond c\}$, where $\diamond$ is a binary comparison operator, i.e., $\diamond \in \{\geq, <, \neq, \text{etc}\}$.

Definition A.17 A continuous function $h : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *locally Lipschitz* if, for every compact set $\mathbf{K} \subset \mathbb{R}^n$, there exists a constant value $M > 0$, called *Lipschitz constant* such that, for every $y_1, y_2 \in \mathbf{K}$, the inequality

$$|h(y_1) - h(y_2)| \leq M|y_1 - y_2|$$

holds.

Definition A.18 (*Absolute continuity*) Let $[a, b] \subset \mathbb{R}$ be a compact set. A function $h : [a, b] \to \mathbb{R}^n$ is said to be *absolutely* continuous if there exists a function $g \in \mathscr{L}^1([a, b], \mathbb{R}^n)$ such that, for every $t \in [a, b]$,

$$h(t) = h(a) + \int_a^t g(s)\, ds.$$

Definition A.19 [18] A function $h : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *locally bounded* if, for all $x \in \mathbb{R}^n$, there exists an open set $\mathbf{O} \subset \mathbb{R}^n$ such that h is bounded on $\mathbf{O}$.

Theorem A.20 [18] *If the function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous, then it is locally bounded.*

A.3.1 Set-Valued Maps

The following definition is recalled from [8].

Definition A.21 (*Outer semicontinuity*) A set-valued map $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be *outer semicontinuous at* $x \in \mathbb{R}^n$ if, for every sequence of points $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$ convergent to $x \in \mathbb{R}^n$, and any convergent sequence of points $\{y_i\}_{i\in\mathbb{N}} \subset M(\{x_i\}_{i\in\mathbb{N}})$, one has $y \in M(x)$, where $y_i \to y$. The map is *outer semicontinuous* if it is outer semicontinuous for every $x \in \mathbb{R}^n$. Given $\mathbf{S} \subset \mathbb{R}^n$, $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous relative* to $\mathbf{S}$ if the set-valued map from $\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by $M(x)$, for $x \in \mathbf{S}$, and $\emptyset$, for $x \notin \mathbf{S}$, is *outer semicontinuous* at each $x \in \mathbf{S}$.

Note that continuous functions are outer semicontinuous set-valued maps.

A.4 Initial-Value Problem

A.4.1 Existence and Uniqueness of Solutions

Consider the locally Lipschitz function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the initial-value problem

$$\begin{cases} \dot{y}(t) = h(y(t), u(t)), \\ y(t_0) = y_0, \end{cases} \tag{IVP}$$

where, for every value of the *time* $t \in \mathbb{R}$, the *system state* y and the *input variable* u evolve in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. *Inputs for* (IVP) are functions $v \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$.

With an abuse of notation and concept, the input variable u and inputs for system (IVP) are denoted with the same letter, unless specified otherwise.

From the fundamental theorem of calculus, for almost every $t \in \mathbb{R}$, Eq. (IVP) is equivalent to

$$y(t) = y_0 + \int_{t_0}^{t} h(y(s), u(s))\, ds.$$

Definition A.22 (*Solution*) Let $y_0 \in \mathbb{R}^n$, $\mathbf{I} \subset \mathbb{R}$ with $t_0 \in \mathbf{I}$, and a fixed input u for (IVP). A function $Y : \mathbf{I} \times \{y_0\} \times \{u\} \to \mathbb{R}^n$ is said to be a *solution to* (IVP) *with initial condition* y_0 *and input* u if

1. $Y(t_0) = y_0$;
2. The function $\mathbf{I} \ni t \mapsto Y(t, y_0, u) \in \mathbb{R}^n$ is absolutely continuous;
3. For almost every $t \in \mathbf{I}$, the equation

$$\frac{dY}{dt}(t, y, u) = h(Y(t, y, u))$$

holds.

The set of solutions to (IVP) with initial condition y_0 and input u is denoted by $\mathscr{S}_h(y_0, u)$. From now on, the interval $\mathbf{I}$ is denoted by $\mathtt{dom}(Y)$. When $u \equiv 0$, the solution is denoted by $Y(t, y_0)$ and the set of solutions by $\mathscr{S}_h(y_0)$.

Definition A.23 [10] Let Y and $\bar{Y}$ be solutions to (IVP), $\bar{Y}$ is said to be *continuation* of Y if $\mathtt{dom}(Y) \subsetneq \mathtt{dom}(\bar{Y})$, and for every $t \in \mathtt{dom}(Y)$, the equation $Y(t, y_0, u) = \bar{Y}(t, y_0, u)$ holds, and for almost every $t \in \mathtt{dom}(\bar{Y})$,

$$\frac{d\bar{Y}}{dt}(t, y_0, u) = h(\bar{Y}(t, y_0, u)).$$

A solution Y is called

- *complete* if $\mathtt{dom}(Y)$ is unbounded. If $\sup \mathtt{dom}(Y) = \infty$, then Eq. (IVP) is said to be *forward complete*;
- *maximal* if cannot be continued.

Definition A.24 Let Y be a solution to (IVP), the set

$$\bigcup_{t \in \mathtt{dom}(Y)} \{Y(t, y_0, u)\} \tag{A.2}$$

is said to be the *trajectory* of Y.

Note that the concept of trajectory of solutions employed here is different from what is stated in [10]. There, the set (A.2) belongs to $\mathbb{R}^{n+1}$, because the time is considered in the definition.

The next three theorems are recalled from [10, 11, 25] and adapted to the context of this book.

Theorem A.25 (Existence) *Let u be an input for system (IVP). If the vector field h is continuous, then for every initial condition $y_0 \in \mathbb{R}^n$, there exists at least one solution Y to (IVP) that is locally Lipschitz continuous and can be continued to a maximal interval of existence. Moreover, if Y is maximal, then $Y(t, y_0, u)$ tends to infinity as $t \to \partial\mathtt{dom}(Y)$.*

Theorem A.26 (Uniqueness) *Let u be an input for system (IVP). If the vector field h is locally Lipschitz continuous with respect to y and uniformly continuous with respect to u on compact sets, then for every initial condition $y_0 \in \mathbb{R}^n$, there exists a unique solution Y to (IVP).*

Theorem A.27 (Regularity) *Let u be an input for (IVP). If the vector field h is of class $\mathscr{C}^1$, then for every initial condition $y_0 \in \mathbb{R}^n$, the solution Y to (IVP) is of class $\mathscr{C}^1(\mathtt{dom}(Y) \times \{y_0\} \times \{u\}, \mathbb{R}^n)$ and the map $\mathbb{R}^n \ni y_0 \mapsto Y(\cdot, y_0, u) \in \mathbb{R}^n$ is a diffeomorphism of class $\mathscr{C}^1$.*

A.4.2 Stability of Solutions

Continuous feedback laws for system (ϕ-ODE) are continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$. Letting $u = \phi$, it yields

$$\begin{cases} \dfrac{dy}{dt}(t) = h(y(t), \phi(y(t))), \\ y(t_0) = y_0. \end{cases} \tag{ϕ-ODE}$$

With an abuse of notation the function $h(y(\cdot), \phi(y(\cdot)))$ is denoted as $h(y(\cdot))$. Also, assume that (ϕ-ODE) is forward complete.

The following definition is based on [13, 22].

Definition A.28 (*ω-limit and invariant sets*) The element $y_\infty \in \mathbb{R}^n$ is said to be an *ω-limit point of Y* if there exists a sequence of times $\{t_i\}_{i \in \mathbb{N}} \subset \mathtt{dom}(Y)$ with $t_i \nearrow \infty$ such that, for every solution $Y \in \mathscr{S}_h(y_0)$, the limit $Y(t_i, y_0) \to y_\infty$ holds as $i \to \infty$. The set of all ω-limit points of Y is called *ω-limit set*, and it is denoted by $\omega(y_0)$.

A set $\mathbf{M} \subset \mathbb{R}^n$ is said to be *positively invariant* with respect to (ϕ-ODE) if, for every initial condition $y_0 \in \mathbf{M}$, and for every issuing solution $Y \in \mathscr{S}_h(y_0)$, $Y(t, y_0) \in \mathbf{M}$, for every $t \geq t_0$.

Definition A.29 [22] The point $\bar{y}$ is said to be an *equilibrium point of* (ϕ-ODE) if $h(\bar{y}) = 0$.

Note that, if (ϕ-ODE) has only one equilibrium point in $\mathbf{Y}$, then there exists a coordinate change rendering the origin an equilibrium point. Thus, the properties referring to the origin as an equilibrium point are equivalent to refer to any other single equilibrium point.

The next definition is recalled from [1, 2, 22] and adapted to the context of this book.

Definition A.30 Let $\mathbf{A} \subset \mathbb{R}^n$ be a compact and positively invariant set with respect to (ϕ-ODE). This set is said to be

- *Stable for* (ϕ-ODE) if, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for every initial condition $y_0 \in \mathbb{R}^n$, and for every issuing solution $Y \in \mathscr{S}_h(y_0)$, the implication
$$|y_0|_{\mathbf{A}} \leq \delta(\varepsilon) \Rightarrow |Y(t, y_0)|_{\mathbf{A}} \leq \varepsilon$$
holds, for every $t \geq t_0$;
- *Unstable for* (ϕ-ODE) if not stable;
- *Locally attractive for* (ϕ-ODE) if, there exists $\delta_0 > 0$ such that, for every initial condition $y_0 \in \mathbb{R}^n$, and for every issuing solution $Y \in \mathscr{S}_h(y_0)$, the implication
$$|y_0|_{\mathbf{A}} \leq \delta_0 \Rightarrow \lim_{t \to \infty} |Y(t, y_0)|_{\mathbf{A}} = 0$$
holds;
- *Locally asymptotically stable for* (ϕ-ODE) if it is stable and locally attractive. The adjective locally is replaced by *globally*, when the choice of δ_0 can be taken as large as desired.

Note that stability and attractivity are different concepts. An example provided in [9, Paragraph 40] shows a system that is unstable but attractive.

A.5 Differentiation Along Solutions of an ODE

The next definition of classes of functions is recalled from [13, 22, 23].

Definition A.31 (*Comparison functions*)

- A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *locally positive definite with respect to a set* $\mathbf{S} \subset \mathbb{R}^n$ if there exists a constant value $r > 0$ such that, for every $y \in \mathbf{B}_{\leq r}(\mathbf{S}) \setminus \mathtt{cl}\{\mathbf{S}\}$, $V(y) > 0$, and $V(y) = 0$ if and only if $y \in \mathbf{S}$. The class of such functions is denoted $\mathscr{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$. It is said to be *positive definite* if r can be taken as large as desired and $\mathbf{S} = \{0\}$, in this case, the class of functions is denoted by $\mathscr{P}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$;
- A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *proper*[2] if $V(y) \to \infty$, as $|y| \to \infty$;
- A continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is said to be *strictly increasing* if, for every $s_1, s_2 \in [0, a)$ with $s_1 < s_2$, $\alpha(s_1) < \alpha(s_2)$;
- A continuous function $\alpha \in \mathscr{P}([0, a), \mathbb{R}_{\geq 0})$ is said to be *of class* $\mathscr{K}([0, a), \mathbb{R}_{\geq 0})$ if it is strictly increasing. It is denoted by $\mathscr{K}$, if a can be taken as large as desired. It is said to be of *class* $\mathscr{K}_\infty$ if it is of class $\mathscr{K}$ and unbounded;
- A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *of class* $\mathscr{KL}$ if, for a fixed $t \geq 0$, the function $s \mapsto \beta(s, t)$ is of class $\mathscr{K}$ and, for a fixed $s \geq 0$, the function $t \mapsto \beta(s, t) \in \mathbb{R}_{\geq 0}$ is non-increasing and satisfies $\beta(s, t) \to 0$, as $t \to \infty$.

[2]This property is also called radial unboundedness.

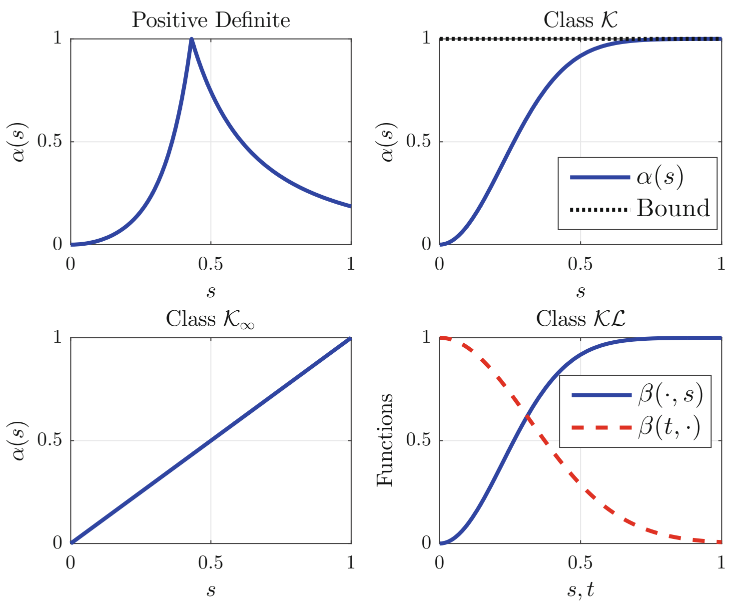

Fig. A.1 Illustration of some of the functions described in Definition A.31

Figure A.1 illustrates the functions presented in Definition A.31.

Proposition A.32 [13] *Let α_1, $\alpha_2 \in \mathcal{K}$, α_3, $\alpha_4 \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$. Then the following properties hold:*

- $\alpha_1^{-1} \in \mathcal{K}$ *is defined on* $[0, \alpha_1(a)]$*;*
- $\alpha_3^{-1} \in \mathcal{K}_\infty$*;*
- $\alpha_1 \circ \alpha_2 \in \mathcal{K}$*;*
- $\alpha_3 \circ \alpha_4 \in \mathcal{K}_\infty$*;*
- $\beta(s, t) = \alpha_1(\beta(\alpha_2(s), t)) \in \mathcal{KL}$*;*
- $\beta(s, t) \leq \alpha_1(\alpha_2(s)e^{-t}) \in \mathcal{KL}$.[3]

The following lemma is recalled from [13] and adapted to the context of this book.

Lemma A.33 *The continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is locally positive definite with respect to the origin if and only if there exist a constant value $r > 0$, and functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}([0, r), \mathbb{R}_{\geq 0})$ such that the inequalities*

$$\underline{\alpha}(|y|) \leq V(y) \leq \overline{\alpha}(|y|) \tag{A.3}$$

hold, for every $y \in \mathbf{B}_{\leq r}(0)$. Additionally, V is proper if and only if $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$.

[3] [24, Proposition 7].

Definition A.34 (*Lie derivative, Lyapunov function*) The locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to a *Lyapunov function candidate*[4] *for* (ϕ-ODE) (resp. *in the small*) if there exist $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}_\infty$ (resp. $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}([0, r), \mathbb{R}_{\geq 0})$) such that, for every $y \in \mathbb{R}^n$ (resp. for every $y \in \mathbf{B}_{\leq r}(0)$), the inequality (A.3) holds.

If V is differentiable, the function

$$\begin{aligned} L_h V : \mathbb{R}^n &\to \mathbb{R} \\ y &\mapsto \frac{\partial V}{\partial y}(y) \cdot h(y, \phi) \end{aligned}$$

is said to be the *Lie derivative of* V *in along the vector field* h. If $-L_h V$ is continuous and (resp. locally) positive definite, then V is said to be *Lyapunov function* (resp. in the small) for (ϕ-ODE).

From Lemma A.33 and Definition A.34, $-L_h V \in (\mathscr{C} \cap \mathscr{P}_{\text{loc}})(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ implies that there exists a function $\alpha \in \mathscr{K}([0, r), \mathbb{R}_{\geq 0})$ such that, for every $y \in \mathbf{B}_{\leq r}(0)$, the inequality $L_h V(y) \leq -\alpha(|y|)$ holds. Remark 4.1 from [16] shows that it is not restrictive to choose the functions $\underline{\alpha}$, $\overline{\alpha}$ satisfying the inequalities (A.3), and α as belonging to the set $(\mathscr{C}^\infty \cap \mathscr{K})([0, r), \mathbb{R}_{\geq 0})$.

Theorem A.35 [22] *If*

1. $V \in \mathscr{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ *and proper, and* $-L_h V \in \mathscr{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$*, then the origin is locally uniformly asymptotically stable for (*ϕ*-ODE);*
2. $V \in \mathscr{P}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ *and proper, and* $-L_h V \in \mathscr{P}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$*, then the origin is globally uniformly asymptotically stable for (*ϕ*-ODE).*

Theorem A.36 [1] *If the origin is globally (resp. locally) asymptotically stable for (*ϕ*-ODE), then there exists a Lyapunov function (resp. in the small) for (*ϕ*-ODE).*

Proposition A.37 [13] *Consider the scalar differential equation*

$$\begin{cases} \dot{y} = -\alpha(y) \\ y(t_0) = y_0, \end{cases}$$

where α *is a locally Lipschitz function of class* $\mathscr{K}([0, a), \mathbb{R}_{\geq 0})$*. For every* $0 \leq y_0 < a$*, this equation has a unique solution* Y *defined on* $[t_0, \infty)$*. Moreover, there exists* $\beta \in \mathscr{KL}(([0, a), \mathbb{R}_{\geq 0}) \times \mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ *such that, for every* $t \geq t_0$*,* $y(t) = \beta(y_0, t - t_0)$*.*

Proposition A.38 [13] *Assume that the origin is an equilibrium point of (*ϕ*-ODE). Then,*

- *It is uniformly stable if and only if there exist* $\alpha \in \mathscr{K}$*, and a constant value* $c > 0$ *such that for every* $|y_0| \leq c$*, the inequality*

[4] This function is also known as storage function.

$$|Y(t, y_0)| \leq \alpha(|y_0|)$$

holds, for every $t \geq 0$*;*

- *It is locally uniformly asymptotically stable if and only if there exist* $\beta \in \mathcal{KL}$*, and a constant value* $c > 0$ *such that for every* $|y_0| \leq c$*, the inequality*

$$|Y(t, y_0)| \leq \beta(|y_0|, t) \tag{A.4}$$

holds, for every $t \geq t_0$*;*

- *It is globally uniformly asymptotically stable if and only if the constant value c can be taken as large as desired, in inequality* (A.4).

In some cases, the requirement for a continuously differentiable Lyapunov function candidate can be quite restrictive. Inspired by works such as [7, 15, 19], relaxed notions of derivatives and sufficient conditions ensuring asymptotic stability are recalled.

The next definition is recalled from [5, 17, 20] and adapted to the context of this book.

Definition A.39 (*Dini Derivatives*) Consider a function $f : [a, b) \to \mathbb{R}$, if the limits at $t \in [a, b)$

$$D^+ f(t) = \limsup_{\tau \searrow 0} \frac{f(t+\tau) - f(t)}{\tau},$$
$$D_+ f(t) = \liminf_{\tau \searrow 0} \frac{f(t+\tau) - f(t)}{\tau},$$
$$D^- f(t) = \limsup_{\tau \nearrow 0} \frac{f(t+\tau) - f(t)}{\tau},$$
$$D_- f(t) = \liminf_{\tau \nearrow 0} \frac{f(t+\tau) - f(t)}{\tau},$$

exist, then they are said to be *Dini derivatives*. Let the vectors $y, v \in \mathbb{R}^n$, and the function $V : \mathbb{R}^n \to \mathbb{R}$. The limit

$$D_v^+ V(y) = \limsup_{\tau \searrow 0} \frac{V(y + \tau v) - V(y)}{\tau} \tag{A.5}$$

(if it exists) is said to be the *Dini derivative of* V *along the vector v and computed at* y. The other three Dini derivatives can be analogously defined in along v. The set

$$\mathtt{grad}_D\, V(y) = \{\xi \in \mathbb{R}^n : D_{+,v} V(y) \geq \xi \cdot v, \forall v \in \mathbb{R}^n\}$$

is said to be the *Dini subdifferential of* V and each ξ is called *Dini subgradient*.

Remark A.40 [6] Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then, for every $y \in \mathbb{R}^n$, $D_v^+ V(y)$ exists. Moreover, if V is of class $\mathscr{C}^1$, then, for every $y \in \mathbb{R}^n$,

$$D_v^+ V(y) = \tfrac{\partial V}{\partial y}(y) \cdot v.$$

Proposition A.41 [20] *Let the function $f : (a, b) \to \mathbb{R}$ be continuous. It is increasing in (a, b) if and only if, on the interval (a, b), the four Dini derivatives are larger or equal to zero.*

The following theorem is credited to [27], and it is presented and proved in [20, Theorem 4.3]. It states that the Dini derivative with respect to time of a locally Lipschitz function computed along solutions of (ϕ-ODE) is equal to its Dini derivative computed along solutions of (ϕ-ODE) in the direction of the vector field.

Theorem A.42 *Let Y be a solution to (ϕ-ODE), and $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then, for every $t \in \mathtt{dom}(Y)$, the equation*

$$D^+ V(Y(t, y)) = D_h^+ V(Y(t, y))$$

holds.

A consequence of Proposition A.41 and Theorem A.42 is that if the inequality $D_h^+ V(y) \leq 0$ holds, for every $y \in \mathbb{R}^n$, then V is non-increasing along the solutions to (ϕ-ODE). Moreover, the above statements and consequences remain true for the other three Dini derivatives (see [20]).

Remark A.43 Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be Lyapunov function candidate for system (ϕ-ODE). If there exists a function $\alpha \in \mathscr{K}_\infty$ such that, for every $y \in \mathbb{R}^n$, the inequality

$$D_f^+ V(y) \leq -\alpha(|y|)$$

holds, then V is strictly decreasing along solutions of (ϕ-ODE).

Note that, for a function $W \in (\mathscr{C}^0 \cap \mathscr{P})(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ the condition $D_h^+ W(y) < 0$, for every $y \in \mathbb{R}^n$, is not enough to ensure that $D_h^+ W$ is strictly decreasing. Since $D_h^+ W$ may not exist in a given $\bar{y}$, the condition $D_h^+ W(\bar{y}) < 0$ may imply $\sup D_h^+ W(\bar{y}) = 0$.

Definition A.44 [4, 5] The *Clarke upper and lower derivatives* of a function $V : \mathbb{R}^n \to \mathbb{R}$ are given, respectively, by the limits

$$V_v^\circ(y) = \limsup_{\substack{x \to y \\ \tau \searrow 0}} \frac{V(x + \tau v) - V(x)}{\tau},$$

$$V_{\circ,v}(y) = \liminf_{\substack{x \to y \\ \tau \searrow 0}} \frac{V(x + \tau v) - V(x)}{\tau}.$$

The set

$$\mathtt{grad}_C\, V(y) = \{\xi \in \mathbb{R}^n : V_v^\circ(y) \geq \xi \cdot v, \forall v \in \mathbb{R}^n\}$$

is said to be the *Clarke generalized gradient of* V at $y \in \mathbb{R}^n$.

Since the Clarke generalized gradient has the following property[5]

$$\mathtt{grad}_C\, V(y) = \{\xi \in \mathbb{R}^n : V_{\circ,v}(y) \le \xi \cdot v \le V_v^{\circ}(y)\},$$

the Clarke derivatives can be reconstructed as

$$\begin{aligned} V_{\circ,v}(y) &= \inf\, \{\xi \cdot v : \xi \in \mathtt{grad}_C\, V(y)\}, \\ V_v^{\circ}(y) &= \sup\, \{\xi \cdot v : \xi \in \mathtt{grad}_C\, V(y)\}. \end{aligned}$$

The next result is recalled from [5].

Proposition A.45 *The following properties hold for the Clarke generalized gradient*

- *If* $V \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}_{\ge 0})$*, then* $\mathtt{grad}_C\, V(y) = \{\mathtt{grad}\, V(y)\}$*;*
- *Let* $x \in \mathbb{R}^n$ *and* $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ *be locally Lipschitz. Let also* $\aleph$ *be any subset of measure zero in* $\mathbb{R}^n$*, and* $\aleph_V$ *be the set of points at which* V *fails to be differentiable. Then,*

$$\mathtt{grad}_C\, V(y) = \mathtt{co}\left\{\lim_{i\to\infty} \mathtt{grad}\, V(y_i) : y_i \to y,\, y_i \notin \aleph \cup \aleph_V\right\}.$$

The result that gives sufficient a condition for the monotonicity of a function $V : \mathbb{R}^n \to \mathbb{R}$ is recalled from [5].

Theorem A.46 *Let* $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ *be a Lyapunov function candidate for (*ϕ*-ODE). If, for every* $y \in \mathbb{R}^n$*, and for every* $\xi \in \mathtt{grad}_C\, V(y)$*, the inequality*

$$\max\{h(y) \cdot \xi\} \le 0$$

holds, then V *is non-increasing along solutions to (*ϕ*-ODE).*

Remark A.47 Under the conditions of Theorem A.46. If there exists $\alpha \in \mathscr{K}_\infty$ such that, for every $y \in \mathbb{R}^n$, and for every $\xi \in \mathtt{grad}_C\, V(y)$,

$$\max\{h(y) \cdot \xi\} \le -\alpha(|y|), \tag{A.6}$$

then V is strictly decreasing, along solutions of (ϕ-ODE). From [6, Proposition 5.3], Eq. (A.6) is equivalent to

$$\inf_{x \in \mathbb{R}^n} D_{+,h} V(y) \le -\alpha(|y|).$$

The conclusion follows from Remark A.43.

[5] See also [3, p. 22].

Trade-offs. Note that,[6] for every $\tau \in \mathbb{R}_{>0}$,

$$\liminf_{x \to y} \frac{V(x + \tau v) - V(x)}{\tau} \leq \frac{V(y + \tau v) - V(y)}{\tau} \leq \limsup_{x \to y} \frac{V(x + \tau v) - V(x)}{\tau}.$$

From Definitions A.39 and A.44, and the continuity of V, these inequalities imply

$$V_{\circ,v}(y) \leq D_{+,v}V(y) \leq D_v^+ V(y) \leq V_v^\circ(y).$$

Consequently,

Proposition A.48 (Based on [5, Proposition 11.49]) *If the function* $V : \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz, then for every* $y \in \mathbb{R}^n$*, the inclusion*

$$\mathtt{grad}_D\, V(y) \subset \mathtt{grad}_C\, V(y)$$

holds.

From Definition A.39, the Dini subdifferential reduces to the derivative, when it exists. This is not necessarily the case for the Clarke generalized gradient, because it is defined on the dual set. On the other hand, the Clarke derivatives can be reconstruct directly from the Clarke generalized gradient while the four Dini derivatives can not be directly reconstruct from the Dini subdifferential [5].

From Proposition A.48, requiring a property to hold with respect to the Clarke generalized gradient is more restrictive than require for it to hold with respect to Dini subdifferential.

A.5.1 The System with Inputs

Recall system (IVP) defined by

$$\begin{cases} \dot{y}(t) = h(y(t), u(t)), \\ y(t_0) = y_0. \end{cases} \tag{IVP}$$

From now on assume that (IVP) is forward complete and, for every $t \geq 0$, $u(t) \in \mathbf{K}_m$, where $\mathbf{K}_m \subset \mathbb{R}^m$ is a compact set.

The objective of this section is to recall the existing results on the stability analysis of (IVP) with inputs in a compact set. The concepts of stability and Lyapunov functions are slightly different.

[6] See also [3, p. 22].

Definition A.49 [16] A closed set $\mathbf{M} \subset \mathbb{R}^n$ is said to *invariant set with respect to* (IVP) if

$$\forall y_0 \in \mathbf{M}, \forall u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbf{K}_m), \forall t \geq 0, \quad Y(t, y_0, u) \in \mathbf{M}.$$

System (IVP) is uniformly globally asymptotically stable (UGAS) with respect to $\mathbf{M}$ if it is

- Uniformly stable: there exists a function $\delta \in \mathscr{K}_\infty$ such that, for every $\varepsilon \geq 0$, and for every $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbf{K}_m)$, and for every $t \geq 0$, the implication

$$|y_0|_{\mathbf{M}} \leq \delta(\varepsilon) \Rightarrow |Y(t, y_0, u)| \leq \varepsilon$$

  holds;
- Uniformly attractive: for any $r, \varepsilon > 0$, there exists $T > 0$ such that, for every $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbf{K}_m)$, and for every $t \geq T$, the implication

$$|y_0| \leq r \Rightarrow |Y(t, y_0, u)| \leq \varepsilon$$

  holds.

Analogously to Proposition A.38, the UGAS of (IVP) is also characterized in terms of comparison and Lyapunov functions. Namely,

Proposition A.50 [16] *System (IVP) is UGAS with respect to a closed invariant set* $\mathbf{M} \subset \mathbb{R}^n$ *if and only if there exists a function* $\beta \in \mathscr{K}\mathscr{L}$ *such that, for every* $y_0 \in \mathbb{R}^n$, *for every* $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbf{K}_m)$, *and for every* $t \in \mathbb{R}_{\geq 0}$, *the inequality*

$$|Y(t, y_0, u)|_{\mathbf{M}} \leq \beta(|y_0|, t) \tag{A.7}$$

holds.

The next definition is recalled from [16] and adapted to the context of this book.

Definition A.51 Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Lyapunov function candidate for (IVP) with respect to a nonempty, closed invariant set $\mathbf{M} \subset \mathbb{R}^n$. Let also V be smooth on $\mathbb{R}^n \setminus \mathbf{M}$. The function V is said to be a smooth *Lyapunov function for* (IVP) *with respect to* $\mathbf{M}$ if there exists a function $\alpha \in \mathscr{K}_\infty$ such that, for every $y \in \mathbb{R}^n \setminus \mathbf{M}$, and for every $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbf{K}_m)$, the inequality

$$L_h V(y, u) \leq -\alpha\left(|y|_{\mathbf{M}}\right)$$

holds.

Theorem A.52 [16] *Let* $\mathbf{M} \subset \mathbb{R}^n$ *be a nonempty compact invariant set with respect to (IVP). Then, system (IVP) is UGAS with respect to* $\mathbf{M}$ *if and only if there exists a smooth Lyapunov function* V *for (IVP) with respect to* $\mathbf{M}$.

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