Advances in Mathematics Education

Roza Leikin
Bharath Sriraman *Editors*

Creativity and Giftedness

Interdisciplinary perspectives from mathematics and beyond

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Chapter 1
Introduction to Interdisciplinary Perspectives to Creativity and Giftedness

Roza Leikin and Bharath Sriraman

Invention, innovation, originality, insight, illumination and imagination are core elements of the individual and societal progress along human history from ancient times till the modern society. While these phenomena are often considered as indicators of creativity and talent in science, technology, business, arts, and music; they are also basic mechanisms of learning. Till the past decade mathematical creativity and giftedness were overlooked in the educational research. Luckily lately more attention is paid to their nature and nature. For example, in 2010 International Group of Mathematical Creativity and Giftedness (igmcg.org) was established following five international conferences of the community of research mathematicians, mathematics educators and educational researchers. During the last decade several books and edited volumes were devoted to the constructs of mathematical creativity and mathematical talent, their identification and development (see commentary for references). Still there are many open questions remain and researches debate the question of inborn character of creative talents vs. possibility of developing creativity and ability in all students. The current volume presents international panorama of the research of creativity and giftedness, reflects the state of the art in the field and provides a broad range of views on the phenomena of creativity and giftedness with special attention to creativity and giftedness in mathematical.

Part I of the volume focuses on different aspect of creativity in mathematics and beyond. A group of studies presents possible ways of defining and evaluation mathematical creativity applied in empirical studies conducted in primary school (Pitta-Pantazi), in secondary school (Lev and Leikin), in undergraduate mathematics (Savic et al.), and in courses for mathematics teachers (Palsdottir and Sriraman;

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Voica and Singer). Some researchers describe types of mathematical tasks appropriate for the evaluation and development of mathematical creativity. Palsdottir and Sriraman argue that mathematical modeling may be viewed as a creative mathematical activity, while Voica and Singer analyze problem-posing and constructive activities as facilitators of the development of creativity in mathematics. Palsdottir and Sriraman examine the views of a group of Icelandic high school teachers about modeling activities, and characterize ways in which they implement them in the classroom. Voica and Singer analyze participants' creativity through focus on students' cognitive variety and novelty and demonstrate that creative interactions of the participants increase their problem-solving and problem-posing expertise. Pitta-Pantazi and Lev and Leikin examined relationship between creativity and giftedness. Lev and Leikin introduce a model for the evaluation of mathematical creativity using multiple solution tasks and Savic et al. introduces an assessment tool for evaluation of mathematical creativity that can be implemented in an introductory proof course.

Several chapters in the book present theoretical perspectives on mathematical creativity, on general creativity and the relationship between them. Karwowski and Dziedziewicz present a typological model of creativity made up of creative abilities, openness to experiences, and independence and suggest its consequences for early mathematics education. The authors pay special attention to the role of visual and creative imagination and on new ways of enhancing mathematical creativity using heuristic rhymes. Tan and Sriraman highlight the role of convergence in developing creativity and mathematical capacity, distinguish between convergence *in* divergence *for* emergence as three creativity mechanisms and argue that continuity, interaction and complementarity are three principles of experience that lead to the development of creativity. Hersh and John-Steiner address some psychological sources that motivate creative mathematicians, analyze their cognitive and mathematical strategies that lead to mathematical insight, and provide examples of creative breakthroughs in the teaching of mathematics. The authors argue that the pursuit of novelty, unrestricted by any other prescribed goal or objective, radically speeds up evolutionary adaptation. Mann and Chamberlin stress importance of affect in the production of creative outcomes in mathematical problem solving. In their view anxiety, aspiration(s), attitude, interest, and locus of control, self-efficacy, self-esteem, and value are major factors that affect creative problem solving. Iconoclasm is discussed by the authors as instrumental construct to the production of creative outcomes. In the chapter by Haught and Stokes creativity follows competency and the product called creative must be both novel and appropriate to its domain. they argue that paired constraints can make very young children competent in mathematics and college students more creative in composition. Beghetto and Schreiber ask "What propels creativity in learning?" They discuss abductive reasoning as a special form of creative reasoning that is triggered by states of genuine doubt that represent opportunities for creative learning.

Part II of the book devoted to research on mathematical giftedness and the education of mathematically gifted students. Clearly when discussing giftedness the authors also touch upon creativity while using different research paradigms and research methodologies. Leikin, Leikin & Waissman and Cropley, Westwell & Gabriel provide Neuro-scientific analysis of mathematical creativity and giftedness. Leikin et al. present an empirical study that uses event related potentials methodology to analyze brain activity related to solving mathematical problems by students of different levels of mathematical abilities. To analyze relationships between mathematical creativity and giftedness they employ distinctions between insigh-based (i.e. creative) and learning-based (routine) problem solving. Cropley et al. provide meta-analysis of studies on psychological and neuro-scientific perspectives on mathematical creativity and giftedness. They discuss how these approaches can inform our understanding of creativity as a component of giftedness in general and how giftedness manifests in mathematics in the creative-productive sense. As mentioned above Lev and Leikin and Pitta-Pantazi describe empirical studies that analyze relationship between mathematical creativity and giftedness. Pitta-Pantazi summarizes series of studies regarding identification of mathematically gifted students and the relation between mathematical creativity, intelligence and cognitive styles. Chapters by Leikin et al. and Lev and Leikin introduce distinctions between high achievements in mathematics, general giftedness and superior performance in mathematics. They stress that excellence in school mathematics and general giftedness are interrelated but different in nature personal characteristics related to mathematical giftedness.

As described above, while Palsdottir and Sriraman and Voica and Singer suggest approaches to teaching mathematics that develops creativity in all students, Tan and Sriraman and Hersh and John-Steiner provide theoretical perspectives on mathematics teaching that leads to creative production in mathematics and beyond. The last but not least important, Karp provides theoretical analysis of mathematically gifted education from political perspective. He stresses that "practice of recognizing certain children as more gifted than others and selecting them accordingly becomes inevitably a focus of public attention, frequently giving rise to disagreements, finding itself at the heart of political discussions, sometimes instigating such discussions, and sometimes reflecting already existing conflicts" (p. 239).

The analysis performed by Karp reflects hidden (political and educational) debate between the different authors that contributed their chapters to this volume. As one can see, some authors believe creativity is a characteristic of gifted individuals while others think it can be developed in all students; some believe that creativity is an outcome of the learning process whereas other believe creativity leads to development of mathematical proficiency. We trust that the readers will enjoy and be intrigued when reading this book. We hope that readers will hear authors' voices, will understand their positions and will be encouraged to perform further research that will shed more light on the nature and nurture of giftedness and creativity, the relationship between them, the approaches to education of gifted as well as teaching with and for creativity.

Part I
Perspectives on Creativity

Chapter 2
Creativity, Imagination, and Early Mathematics Education

Maciej Karwowski, Dorota M. Jankowska, and Witold Szwajkowski

Abstract In this chapter, we draw heavily on a new typological model of creativity and show its consequences for early maths education. According to this model, creativity is made up of three interrelated components: creative abilities (mainly creative imagination and divergent thinking), openness to experiences, and independence. This model is our starting point for the description of the importance and organization of the Mathematical Creative Problem Solving Model. We describe the assumptions, aims, and elements of this model, as well as demonstrate the practical and methodological aspects of supporting the development of mathematics. We also focus on the role played by visual and creative imagination and on new ways of enhancing mathematical creativity using heuristic rhymes.

Keywords Typological model of creativity • Creativity • Visual and creative imagination • Mathematical creative problem solving • Mathematical heuristic rhymes

2.1 Introduction

In knowledge-based society, creativity is perceived as a source of innovation and progress (Sawyer 2006). Concurrently, innovativeness is frequently equaled with mathematical thinking when it comes to engineering and invention (Wang and Shang 2014), but also with regard to teaching mathematics (see inventive mathematical thinking; Harskamp 2014, p. 371). The psychology of creativity proposes a search for connections not just between innovation and creativity but also between

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those two and imagination – the Imagination–Creativity–Innovation (ICI) model (see Beghetto 2014). As creativity and imagination are interdisciplinary constructs (Gillson and Shaley 2004; Glăveanu 2010), they have many connotations that are sometimes contradictory (Kaufman 2009). This is why we start this chapter by defining them with reference to the typological model of creativity (Karwowski 2010; Karwowski and Lebuda 2013) and the conjunctional model of creative imagination (Dziedziewicz and Karwowski 2015). Those definitional solutions will be the basis for the analysis of relations between the constructs we are interested in this chapter and will serve as reference points in the description of the role of heuristic rhymes in Mathematics Creative Problem Solving.

2.2 The Typological Model of Creativity

The history of psychological research on creativity is usually divided into two periods (see Sawyer 2006): before and after 1950, which is when Joy Paul Guilford delivered his breakthrough address during the Convention of the American Psychological Association (Guilford 1950; Kaufman 2009). As is widely known, Guilford perceived divergent thinking as the intellectual operation responsible for creative thinking, with several important characteristics, namely: (1) fluency, understood as the ability to come up with many ideas; (2) flexibility, or the ability to create solutions that are qualitatively diverse; (3) originality, responsible for producing rare and untypical ideas; and (4) elaboration – the ability to develop ideas (Guilford 1967). The number of empirical studies grew after Guilford's address, and the understanding of creativity as an egalitarian characteristic also became widespread. For example, humanist psychologists (Fromm 1959; Maslow 1959; Rogers 1970) considered it not only as a characteristic of eminent creators but also – to a greater or lesser extent – as a trait commonly found in the entire population.

Attempts undertaken by researchers and theoreticians to define creativity most frequently came down to two characteristics of its product – newness, associated with originality (Cropley 2001; Boden 2004), and value (utility) (Cropley 1999; Runco 2009). Thus, creativity is defined as activity that leads to the emergence of new (original) and useful products (Amabile 1983). With time, creativity began to be identified with a compound of personal traits. Aside from divergent thinking, the mechanisms considered by researchers to be key for creating include creative imagination (Khatena 1975; LeBoutillier and Marks 2003) as well as personality characteristics: primarily openness to experiences (Dollinger and Clancy 1993; Feist 1998; Perrine and Broderson 2005) and independence (Batey and Furnham 2006; Eysenck 1994; Nickerson 1999; Stravridou and Furnham 1996). Numerous studies of this kind made it possible to more thoroughly determine the conceptual range of creativity, but they also resulted in the emergence of a *sui generis* "hybrid of creativity" – a system of cooperating elements (traits) related to creative behaviors, which reveals the complexity and multilayer character of this phenomenon. The proposed typological model of creativity (Karwowski 2010; Karwowski and Lebuda 2013) is an attempt to systematize the relations between and among these traits. According to this model, the following hypothetical dimensions determine creativity: (1) creative abilities (cognitive dispositions that determine the effectiveness of generating, developing, and implementing solutions characterized, among other things, by a high level of originality and value, divergent thinking, and imaginative abilities); (2) openness (appreciation of intellect, willingness to meet new people and cultures, as well as learning); and (3) independence (a personality dimension marked by non-conformism and low agreeableness as well as readiness to oppose the situationally evoked influence of the group and external factors). The model implies the special importance of four creativity types, labeled and defined as follows: complex creativity (a combination of creative abilities, openness, and independence), subordinate creativity (a combination of creative abilities and openness with low independence), rebellious creativity (a combination of creative abilities and independence with low openness), and self-actualizing creativity (a combination of openness and independence with low creative abilities). Initial empirical analyses (Karwowski 2010) indicate their specific determinants (different parental attitudes as well as social and economic status), school functioning patterns (grades and satisfaction with learning), creativity styles, creative self-efficacy beliefs, and perceptions of the climate for creativity.

2.3 The Conjunctional Model of Creative Imagination

Long before the Guilford address, Francis Galton conducted the first documented study into imagination and analyzed individual differences in the clarity of representations produced by scientists (Galton 1880; Holt 1964). Almost concurrently, Théodule A. Ribot (1906) coined the concept of creative imagination. Soon after, Lev S. Vygotsky (1930/2004, 1931/1998) proposed the combinatorial (creative) imagination theory. The 1960s saw the emergence of further holistic conceptualizations of creative imagination (e.g., Rozet 1977/1982; Ward 1994). As research and theories developed, similarly as in the case of creativity, attention was drawn to the complexity of creative imagination. Its constitutive factors (properties) were indicated: the vividness (clarity) of images ("The weirdness of visions lies in their sudden appearance in their vividness while present, and in their sudden departure" Galton 1883, p. 121), the ability to manipulate the resulting images ("People can assign novel interpretations to ambiguous images which have been constructed out of parts or mentally transformed" Finke et al. 1989, p. 51), as well as the originality (newness) and value of those images ("Activity that results not in the reproduction of previously experienced impressions or actions but in the creation of new images or actions is an example of […] creative or combinatorial behavior" Vygotsky 1930/2004, p. 9). These dimensions contributed to the development of the conjunctional model of creative imagination (Dziedziewicz and Karwowski 2015), whereby creative imagination was defined as the ability to create and transform mental representations based on the material of past observations, but significantly transcending them.

In this model, the hypothetical dimensions of creative imagination are: vividness – the ability to create expressive and highly complex images, originality – ability to create unique images, and transformative ability – the ability to transform images. The model is conjunctional – that is, the combination of its three dimensions allows the typological analysis focusing on the four basic types of imaginative creative abilities: (1) creative imaging ability (high vividness of imagery, high originality, and high transformative ability), (2) pro-creative imaging ability (high originality and high transformative ability), (3) passive imaging ability (high vividness of imagery and high originality), and (4) vivid imaginative abilities (high vividness and high transformative ability).

2.4 Creativity and Imagination

Implicit and lay theories of creativity define divergent thinking and imaginativeness as traits of creative individuals (e.g., Montgomery et al. 1993). The 1960s mark the point when first correlational studies appeared. They measured the strength and direction of the relation between imaginativeness (visual and creative) and creativity (Schmeidler 1965). Researchers mainly focused on the relation between imaginativeness and creative abilities primarily via divergent thinking (e.g., Gonzales et al. 1997). Much less frequently did they analyze the relation of imagination with personality factors, such as openness and independence (Khatena 1975; Schmeidler 1965, among others). The results of these analyses reveal the existence of a relation between creative imagination and creativity, yet the strength of this relation depends on the examined domain. The combination of imaginativeness with elements of creative attitude (openness, independence) is evidently weaker than its relation with divergent thinking, especially in the domains of vividness and originality (see Dziedziewicz et al. 2013; Schmeidler 1965). On the one hand, this confirms the legitimacy of including creative imagination in the creative abilities factor in the typological model of creativity. On the other hand, though, this relation is so weak ($r = .2$–$.4$) that it is justifiable to consider these traits separately as relatively independent facets of creativity.

Further in this chapter, imaginative abilities (creating original images and transforming them) will be analyzed in conjunction with divergent thinking as creative abilities. This will render it possible to conduct detailed and systematized analysis of the role of creativity in Mathematics Creative Problem Solving in the domain of cognitive (creative imagination and divergent thinking) and personality (openness, independence) components.

2.5 Creativity in Mathematics or Mathematical Creativity?

The important question discussed among creativity scholars (e.g., Baer 1998; Chen et al. 2006; Kaufman and Baer 2005; Plucker 1998) is whether a general *c* factor – analogous to the *g* factor (Jensen 1998) associated with creativity in multiple and diverse domains, including mathematics (see Kaufman 2009, p. 57) – does indeed exist. Creativity is associated with a particular domain in the situation when researchers focus on a creative product (Plucker 2004). Consolidating the domain-specific and domain-general perspectives, Kaufman and Baer (2005) proposed the Amusement Park Theoretical Model (APT), which was meant to be the "Aristotelean golden mean."

The APT model inspired us to create profiles of mathematical creativity on the basis of the typological model of creativity. After Kaufman and Baer, we defined the general thematic framework as the "problem-solving domain," whereby mathematics became the chosen field and solving word problems became the microdomain. We claim that solving problems reflects the nature of mathematical thinking (see Silver 1994). Moreover, word problems are used in mathematics education – which is why we decided to refer to them as well. Furthermore, the analysis of their role in mathematics education frequently emphasizes the creative use and performance of particular mathematical operations.

In the early stages of mathematical education, word problems are commonly of a practical character. Generally, they are simple stories referring to childhood experiences that end with a question one needs to find the only correct answer to (closed questions), after analyzing the information that a given story contains, the data, the unknown, and the relations between them. The stories resemble brain-teasers, which are known to have a solution, and the only task at hand is to find it. If we assumed that creativity is about producing original (new) and useful solutions, it would be difficult to speak about creative solutions to word problems because they are known to mathematicians and even more so to their authors. Hence, it is the way of working towards the solution – that is, defining the problematic situation presented, competences associated with hypothesizing, and planning ways to test the hypotheses – that will provide evidence of the creativity of children solving this type of exercises. On each of the listed stages of mathematical creative problem solving, creative abilities as well as personality factors (openness and independence) will play a significant role (Table. 2.1).

Importantly, the way children solve word problems is significantly influenced by the way they formulate a particular mathematical problem. In the early stages of education, word problems are frequently built using simple and "round" numbers. The predictable form of such problems raises the (fully legitimate!) temptation to guess the result.

Example

Dorothy and Alex have 12 chocolate bars in total. Dorothy has two more bars than Alex. How many bars does Alex have?

Table 2.1 Examples of manifestations of mathematical creativity in the process of solving word problems

Mathematical creative problem solving	Mathematical creativity		
	Creative abilities	Openness	Independence
Understanding mathematical problems	The ability to define the mathematical problem illustrated in the task from multiple perspectives	Tolerance to information that is incomplete, poorly defined, or polysemous	Constructing one's own internal language, where mathematical concepts indispensable for solving the problem are set out and explained
	The ability to clearly visualize the situation presented in the task as well as vividly capture dependency relationships across data	Recognition of the potential value resulting from becoming acquainted with ways other than one's own of perceiving and describing the mathematical problem illustrated in the task at hand	Separating the meanings of mathematical concepts from the meanings of everyday language
Generating possible solutions	The ability to formulate multiple and frequently atypical hypotheses referring to the possible solutions to the mathematical problem illustrated in the task at hand	Cognitive curiosity that results in readiness to become acquainted with possible ways of solving the problem	Courage in questioning commonly accepted rules and principles in order to find new and/or atypical ways of solving the mathematical problem
	The ability to create original images that render it possible to break away from typical solutions to the mathematical problem and use analogies in order to find new ones	Ease in analyzing new information and ways of solving the problematic situation presented in the task at hand	Autonomy and perseverance in searching for possible solutions to the problematic situation
Planning for action	Flexibility in applying various strategies of solving the problem	Openness to the verification of all possible solutions to the problem	Strong belief in the success of the undertaken activities aimed at solving the problem
	The ability to transform images of possible solutions to the problematic situation illustrated in the task at hand	The acceptance of variability in applying the various problem-solving strategies	The ability to critically assess attempts – one's own and other people's –to solve the problem

Instead of counting, many pupils confronted with the above problem will respond that Alex has 5 bars and Dorothy has 7, without even being aware that they have guessed the result by making an intuitive attempt to come up with a single number, because the problem is structured in such a way that the number of potential solutions is significantly reduced. In this situation, it is necessary to reflect on whether guessing at the answer can be considered as a manifestation of creative ability. Another question to consider is this: what is the value (usefulness) – even the subjective value, for the pupils themselves – of solving the problem with the trial-and-error method, which disregards the way towards the solution and, instead, focuses solely on the solution itself? When solving word problems with this method, students frequently do not consider the relationships between the given and the unknown. They attempt to quickly reach the goal (the solution) and usually act thoughtlessly. The heuristic solution pattern anticipates the understanding of the essence of the mathematical problem illustrated by the task. It also anticipates finding the way to use the solution again in an analogous problematic situation.

Let us consider what would happen if the problem was formulated as follows:

Example
Dorothy and Alex have 5 and 2/5 of a chocolate bar in total. Dorothy has 1 and 1/3 of a bar more than Alex. How many bars does Alex have?

As it appears, in this case the method of guessing fails entirely even though we are dealing with exactly the same mathematical problem – the only difference is that numbers are no longer easy to calculate mentally. As a result of adopting the tactics of guessing at solutions to simple problems, those students who have learned to thoughtlessly follow this approach, regarding it as verified and effective, fail to understand the sense and purpose of general methods of solving a particular type of problems. Still less motivated are they to develop such methods.

What can be done to encourage children to solve problems in accordance with the heuristic scheme, which aims at the creative pursuit and discovery of ways to solve problems? We propose a "tablet of changes" – an instructional method whose aim is to create an educational situation conducive to making sense of certain mathematical concepts and operations independently. A simple task that should not pose a problem for any child at a particular stage of development is the starting point in this table. The purpose of solving this task is to strengthen self-efficacy and thereby to encourage children to attempt to solve further, more and more difficult problems. However, the most important aim is to devise a practical illustration of the relations between various branches and aspects of mathematics that will render it possible to consolidate the previously learned concepts and computational techniques. Realizing what the solution to the first problem of each row is makes it possible to apply the same method of solving the mathematical problem in the case of the remaining problems (Table 2.2).

Table 2.2 Tablet of changes – dividing fractions

Six chocolate bars were divided equally among three people. How much chocolate did each person receive?	Six chocolate bars were divided equally among four people. How much chocolate did each person receive?	Two and a half chocolate bars were divided equally among five people. How much chocolate did each person receive?	Two and a half chocolate bars were divided equally among seven people. How much chocolate did each person receive?
Solution:	Solution:	Solution:	Solution:
A group of four people have eight chocolate bars. How much chocolate will each person get?	A group of six people have eight chocolate bars. How much chocolate will each person get?	A group of three people have one and a half chocolate bar. How much chocolate will each person get?	A group of three people have two and a half chocolate bars. How much chocolate will each person get?
Solution:	Solution:	Solution:	Solution:
Fifteen chocolate bars were divided into three equal portions. How much chocolate is there in each portion?	Fifteen chocolate bars were divided into four equal portions. How much chocolate is there in each portion?	One and two thirds of a chocolate bar was divided into five equal portions. How much chocolate is there in each portion?	One third of a chocolate bar was divided equally into three fourths of a portion. How much chocolate is there in each portion?
Solution:	Solution:	Solution:	Solution:
A single portion is made up of two chocolate bars. How many such portions can be made from five chocolate bars?	A single portion is made up of two chocolate bars. How many such portions can be made from five chocolate bars?	A single portion is made up of one and one third of a chocolate bar. How many such portions can be made from five chocolate bars?	A single portion is made up of one and a half chocolate bars. How many such portions can be made from three fifths of a chocolate bar?
Solution:	Solution:	Solution:	Solution:

The problems in the tablet of changes are placed in four columns and four rows. Four is a number that everyone recognizes without making calculations. Therefore, sixteen problems in a 4×4 columnar format do not make an impression of being a large number that is hard to grasp, but that number is sufficient to conduct competence profile assessment of children within the range of problems they face solving. It makes it possible to assess the stage at which difficulties may begin to occur with regard to problem interpretation, the way of coding the solution, making calculations, or possibly even a combination of various types of complexity.

Moving along the tablet of changes to the right, along the rows, we encounter problems characterized by the same extent of conceptual difficulty but more and more complex when it comes to calculations. Moving downwards along the columns, we encounter problems with a similar degree of calculative complexity but more and more difficult when it comes to the concepts whose understanding they require. Problems in rows are usually characterized by similar wording and refer to the same objects in order not to distract children towards insignificant aspects but to keep them focused on each described mathematical problem, on the presented data, on the question posed, and on response interpretation. Such a form also renders it possible to explain to children that the purpose is to solve sixteen different problems because there are different versions of the same problem or problems. This may prevent premature discouragement from making an effort.

Tables of examples may be used at various stages of education because it is not necessary for children to solve all the problems in a given table right away. The exercise may be limited to one or two rows/columns, depending on students' skills, and then resumed after some time. Thanks to the possibility of using the same tablet of changes in a group of children with diverse levels of mathematical competence, this method enables the individualization of math classes.

2.6 Mathematical Heuristic Rhymes

Rhythm accompanies people throughout their lives. It is a constant and obvious element of nature, revealing itself in the cyclical character of astronomical phenomena, for example in the circadian rhythm that results from the Earth revolving around its own axis and determines the timing of human activity and rest. It is therefore not only a natural component of the course of human life but also an important way of perceiving the world. Colloqiual language uses the evocative concept of "being thrown off balance," which refers to undesirable disturbance or destruction of the rhythmic pattern of an activity. After all, rhythm gives a sense of order, predictability, and security. Already in prenatal life we feel and remember the rhythm of our mothers' hearts and that is why newborns calm down when they are placed on their mothers' chest. Rhythm is also present in many basic forms of human activity. We breathe, walk, and run rhythmically. The language we use also has a particular rhythm and melody (Patel and Daniele 2003). It is hard to imagine the effective performance of these activities without proper rhythm. However, we rarely realize the omnipresence of rhythm and most frequently associate it with dance, music, and poetry. Similarly to mathematics, it is not associated with creativity (Kaufman and Baer 2004), and conversely: mathematics is rarely associated with rhythm. Yet, this domain is replete with rhythms, for instance decimal rhythm in the positional system, the alteration of even and odd numbers, or number multiples. This is why the reference to students' natural sense of rhythm and the use of rhythmical rhymes in early mathematics education has multiple positive functions. Counting itself stems from rhythmical indication of objects, so it is hard to find better justification for combining structures that operate on the principle of rhythm, such as rhymes, with learning mathematics. The idea of combining a rhyme with mathematical concepts appeared as early as the nineteenth century in the stories of the famed Mother Goose (Bellos 2010):

As I was going to St Ives,
I met a man with seven wives,
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits.
Kits, cats, sacks, wives,
How many were going to St Ives?

Rhymes perform many functions in early mathematics education. In a natural way, they draw attention to the content about to be revealed and provoke children to anticipate or at least expect such words in consecutive lines that will rhythmically fit into the pattern and rhyme with those the children have already heard. This is why, frequently, there is no need to read the final word of a rhyme to children: they are able to deduce it from the previously heard content and finish saying it, especially that rhyme is an additional indicator. This develops in children the sense of rhythm and order that will be important in shaping their mathematical abilities in the future. It also fuels the sense of satisfaction, positively influences self-assessment and intrinsic motivation, and thus activates further active listening. Using rhymes in teaching mathematics also helps practice memorization and encourages imaginative creation of images, which makes it easier, for example, to assimilate abstract mathematical concepts. Moreover, when revealing a mathematical problem in many situational contexts, rhymes enhance its comprehension and teach thinking flexibility.

In early mathematical education, rhythm is helpful not just in combination with proficiency in counting. The ability to identify rhythm, associated with the way of measuring time and calendric calculations, is also important. Equally important is noticing geometric regularity on all dimensions: linear (e.g., the repetition of a sequence of items positioned in a series), surface (e.g., system design on a ball), or spatial (e.g., the regularity of architectural elements). The development of students' active sense of rhythm is one of the most important tasks for early mathematics education. The fact that rhythm is omnipresent in children's lives does not mean that they are able to give proper rhythm to the activities they perform. The ability to sing rhythmically and make appropriate use of pause, whose length results directly from the song's rhythm, may serve as an example. Many individuals encounter a problem with deciding when to begin singing the next phrase because they are unable to reproduce its rhythm by themselves when it is not chimed or accentuated for them. Reading or repeating good rhymes (including mathematical ones) promotes the development of an active sense of rhythm in children, even though this task is not always easy. Unfortunately, it happens that the authors of rhymes do not make this task any simpler because they frequently fail to observe the elementary principles of balanced and predictable distribution of accents or proper number of syllables in each line. This is why it seems crucial to select correct educational rhymes that should, among other things, (1) be made up of lines with the same number of syllables or repetitions in accordance with an easily comprehensible key, (2) be readable in the sense that accentuated syllables should make up a recognizable rhythmical pattern (somewhat resembling a melody), (3) include an intelligible idea, story, anecdote, or humor that will be clear to children, use word play, or include a surprising punch-line, as well as (4) be concise, but interesting or amusing to children even when the rhymes are read repeatedly (Szwajkowski 2011).

In order to stimulate students' mathematical creativity, it is a good idea to combine rhymes with solving diverse word problems. A good rhyme written into the content of a word problem may make it easier for children to remember and elicit information that is crucial for solving that problem. Additionally, thanks to the attractive form, it may also encourage students to face the presented mathematical problem. If, additionally, the problem will be constructed in a way that renders multiple solutions possible, the problem itself may become even more contextually interesting and motivating.

Below is an example of a mathematical heuristic rhyme (The rhyme is followed by a picture of hamster and cookies in four colors: 2 green - apple, 7 orange – orange, 4 red – cherry, 6 blue – "berry"):

HAMSTER COOKIES

Hamster George who was so cute
Got some cookies made of fruit:
He bought apple, orange, cherry,
And one more that ends with 'berry.'
Never ate so much before -
He had just eleven more!
Just two flavors he left behind
How many ate he and of what kind?

The heuristic scheme of this problem fosters active learning. The rhyme makes it possible for children to independently discover data. It may also interestingly illustrate a problematic situation when mathematical data needed to solve the problem are provided after or during the reading of its content. Finally, children actively and creatively seek possible problem solutions.

Let us analyze a model way of seeking possible solutions. It is not a simple one, as when the problem is approached on a heuristic level. A number of solutions exist and they can be reached through a number of stages of elimination and inference.

Data: The hamster bought 2 apple cookies, 7 orange ones, 4 cherry ones, and 6 berry ones

Looking for potential solutions: The hamster bought a total of 19 cookies (2 + 7 + 4 + 6 = 19). Since he had 11 left, he must have eaten 8 (19–11 = 8). He could not have any apple cookies left because they do not add up to 11 when taken together with any of the remaining flavors. Consequently, he surely ate 2 apple cookies. He could not have any cherry or berry cookies left because there are not enough of them in total (4 + 6 = 10). He may have had orange and cherry cookies left (7 + 4 = 11), which would mean that he ate 2 apple cookies, 4 cherry ones, and any two of all orange and berry ones, which leaves 3 more possibilities.

The hamster may have had orange and cherry cookies left (7 + 4 = 11), which would mean that he ate 2 apple and 6 berry cookies. It is also possible that orange and cherry cookies were left, which would mean that the hamster ate 2 apple cookies, 4 cherry ones, and any two of orange and berry ones, which leaves 3 more possibilities.

Observing children when they are solving such a problem opens many possibilities for analyzing various aspects of mathematical creativity, such as flexibility in applying various strategies of dealing with an open mathematical problem (e.g., searching for a possible solution using a functional method – that is, by manipulating objects – or searching for solutions with the use of a drawing or symbolic calculative method) or the ability to formulate original hypotheses that refer to the probability of one of those solutions to occur (e.g., response to the question of why the hamster ate cookies of only two flavors, namely apple and orange, when he had the opportunity to try all four?). Based on a class conducted using a mathematical heuristic rhyme, the teacher can indicate the strengths and weaknesses of creatively solving open mathematical problems; such a class also allows the teacher to infer her or his students' mathematical creativity profile. Tasks of this type also foster students' integration with their teacher because they create many opportunities to follow students' reasoning as well as support this process by means of asking additional questions and providing guidelines.

The result of observing a group of 9- and 10-year-old children solving this problem indicates that solving the first part of the problematic situation, namely answering the question of how many cookies the hamster ate, is quite simple. The possibility of providing the answer to this question relatively quickly encourages children to perform further inquiries into the problem and to seek to answer the second part of the mathematical problem that refers to the kind of cookies the hamster ate, namely to the stage of analyzing various solutions and drawing conclusions. The conclusion that the hamster could not have any apple cookies left because they do not add up to 11 with any of the remaining flavors, so he surely ate 2 apple cookies, is not so certain, but the children reached this conclusion by modelling various situations with the aid of disks and a specially prepared board that featured the hamster while performing simple calculations. These calculations did not constitute an end in itself, but were an activity that supported making conclusions.

2.7 Mathematical Creative Problem Solving

At the end of this chapter, let us present a mathematics class interaction using a tablet of changes and a mathematical rhyme. We have prepared this interaction with early mathematical education in mind, and this is why we used the so-called balance beam – an original teaching aid that makes it easier for children to move from the level of concrete things (counting particular objects) to the symbolic level (numerical record of calculations).

A balance beam is a small device that resembles scales with weights in form of colorful disks. The disks are identical in dimensions and weight. They have holes in the middle thanks to which it is possible to easily place them on the scales' pins in defined positions. As Fig. 2.1 shows, the distance between the pins and the center of the scales is a multiple of the disk's diameter.

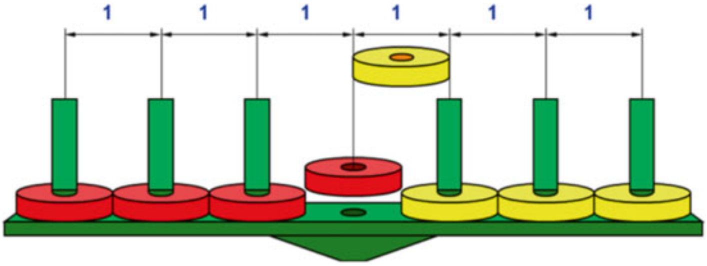

Fig. 2.1 Balance beam

"This is to believe in :
scales are really, truly even
when the sides on both those ends
weigh the same and when none bends.

Disks give weight,
but which, my mate,
weigh much heavier: in the middle,
or away? Now, that's a riddle!

How to measure weight transitions
in these very cool positions?"

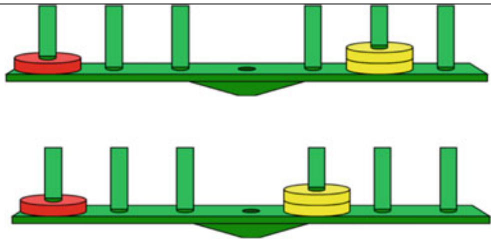

Fig. 2.2 Mathematical rhyme about the balance beam

Sample interactions with the use of the balance beam are carried out according to consecutive steps of mathematical creative problem solving: (1) understanding mathematical problem; (2) generating possible solutions; (3) planning for action. In the beginning, thanks to a short rhyme, children become acquainted with the general principles of balance beam's operation and in this way they familiarize themselves with the mathematical problem that relates to the equilibrium condition (Fig. 2.2).

Our experiences with the use of the balance beam show that children do not experience problems in applying the number of disks. They also notice that the farther away the disk is from the middle, the more weight it applies to a particular side of the scales.

While practicing with the balance beam, children have an opportunity to experiment and test their hypotheses in practice. Solving simple problems provided by the teacher (see Fig. 2.3), they have a chance to independently comprehend the equilibrium condition and verify its correctness by arranging the disks in a way that renders it possible to make a calculation.

Finding a solution to a mathematical problem, such as the equilibrium condition on the balance beam, is an interesting challenge for children. It is an excellent exercise in mathematical creativity, because solving it requires generating a new amount that is a product of the number of disks and their distance from the middle of the scales.

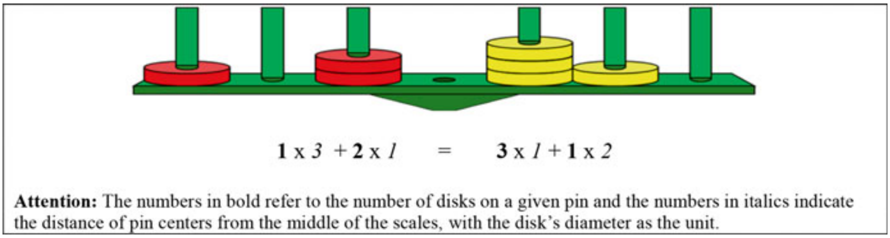

Fig. 2.3 Equilibrium condition as exemplified by the balance beam

2.8 Conclusion

Creativity is usually analyzed within the domain of art and is not often considered to be an important part of mathematical thinking or math education (Sriraman 2004, 2005). Math, by contrast, is highly algorithmic and is often perceived by teachers and students as not allowing much space for heuristics or creative thinking. In this chapter, we intended to show the possibilities of using creative thinking while solving mathematical problems and the possibilities of using creative tasks to enhance children's mathematical abilities.

The methods described in this chapter were deduced from the theoretical models of creativity and imagination we started with. We have especially focused on using heuristic rhymes and the process of mathematical problem solving as illustrations of simultaneous engagement of creative abilities, openness, independence, and different aspects of creative imagination, especially vividness, originality, and transformativeness. Both heuristic rhymes and creative problem solving of mathematical problems are being intensively introduced within classes taught by teachers we cooperate with; the effects are promising. We do hope that the methods described above as well as similar attempts at developing children's creativity in the domain of mathematics will enhance both their creative abilities and imagination as well as improve their school achievement. Future studies will assess the effectiveness of these methods.

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Chapter 3
Formative Assessment of Creativity in Undergraduate Mathematics: Using a Creativity-in-Progress Rubric (CPR) on Proving

Milos Savic, Gulden Karakok, Gail Tang, Houssein El Turkey, and Emilie Naccarato

Abstract Creativity is one of the most important aspects of mathematicians' work (Sriraman 2004), whether it is an enlightenment that is somewhat unexpected or a product that is aesthetically pleasing (Borwein, Liljedahl & Zhai 2014). There are studies in the primary and secondary levels on mathematical creativity of students (e.g., Leikin 2009; Silver 1997), and recent efforts have included mathematical creativity in K-12 education standards (e.g., Askew 2013). However, there is little research in undergraduate mathematics education on creativity. The project described in this chapter introduces an assessment framework for mathematical creativity in undergraduate mathematics teaching and learning. One outcome of this project is a formative assessment tool, the Creativity-in-Progress Rubric (CPR) on proving, that can be implemented in an introductory proof course. Using multiple methodological tools on a case study, we demonstrate how implementing the CPR on proving can help researchers and educators to observe and assess a student's development of mathematical creativity in proving. We claim if mathematicians who regularly engage in proving value creativity, then there should be some explicit discussion of mathematical creativity in proving early in a young mathematician's career. In this chapter, we also outline suggestions on how to introduce mathematical creativity in the undergraduate classroom.

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3.1 Introduction

Though creativity is an important aspect of professional mathematicians' work (Borwein et al. 2014), it is a complicated subject for mathematics educators to research, given that there are over 100 different definitions of creativity (Mann 2006). A considerable amount of literature concentrates on mathematical creativity at the primary and secondary levels (e.g., Silver 1997; Lev-Zamir and Leikin 2011). However, our examination of research at the tertiary, or post-secondary, level revealed little discussion of how students are creative or how creativity can be fostered in undergraduate courses, particularly in proving or proof-based courses. Given that the students in tertiary courses are the next generation of mathematicians, engineers, or math educators, developing their mathematical creativity is crucial. Mann (2006) stated that avoiding the acknowledgment of creativity could "drive the creatively talented underground or, worse yet, cause them to give up the study of mathematics altogether" (p. 239). Although Mann referenced secondary students, this recognition is important for young mathematicians at all levels, including post-secondary students.

Aiming to spark discussions about creativity at the tertiary levels, our focus in valuing creativity during the proving process yielded a Creativity-in-Progress Rubric on Proving (see Table 3.1). The rubric is intended to be used as a formative assessment, so that students can improve their metacognition in proving, and eventually, their final proofs. We begin this discussion by introducing the pertinent literature and theoretical framework involved. The rubric is introduced with explanations of its two main categories: *Making Connections* and *Taking Risks*. Two case studies are presented to further illustrate these categories. Finally, we discuss observations of the case studies, implications of utilizing the rubric as a formative assessment, suggestions of additional ways of implementing it in a proof-based tertiary course, and provide future research with the rubric.

3.2 Background

3.2.1 Perspectives of Creativity

Researching an individual's creativity and the ways in which to enhance it has been an endeavor yielding many definitions and approaches. For example, Kozbelt et al. (2010) provided a summary of contemporary theories through a meta-analysis and outlined ten major perspectives of creativity: Developmental, Psychometric, Economics, Stage and Componential Process, Cognitive, Problem Solving and

Table 3.1 Creativity-in-Progress Rubric (CPR) on proving

MAKING CONNECTIONS:	**Beginning**	**Developing**	**Advancing**
Between Definitions/Theorems	Recognizes some relevant definitions/theorems from the course or textbook with no attempts to connect them in their proving	Recognizes some relevant definitions/theorems from the course and attempts to connect them in their proving	Implements relevant definitions/theorems from the course and/or other resources outside the course in their proving
Between Representations[1]	Provides a representation with no attempts to connect it to another representation	Provides multiple representations and recognizes connections between representations	Provides multiple representations and uses connections between different representations
Between Examples	Generates one or two specific examples with no attempt to connect them	Generates one or two specific examples and recognizes a connection between them	Generates several specific examples and uses the key idea synthesized from their generation
TAKING RISKS:	**Beginning**	**Developing**	**Advancing**
Tools and Tricks	Uses a tool or trick that is algorithmic or conventional for the course or the student	Uses a tool or trick that is model-based or partly unconventional for the course or the student	Creates a tool or trick that is unconventional for the course or the student
Flexibility	Attempts one proof technique	Acknowledges the possibility of different proving approaches, but attempts no further examination	Acts on different proving approaches
Perseverance	Begins to engage with proving	Continues to engage with surface level features but not with the key ideas	Continues to engage with the key ideas
Posing Questions	Recognizes a question should be asked, but does not formulate a question	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof
Evaluation of the Proof Attempt	Checks work locally	Recognizes a successful or unsuccessful proving attempt	Recognizes the key idea that makes the proving attempt successful or unsuccessful

[1] We define a *mathematical representation* similar to NCTM's (2000) definition. It includes written work in the form of diagrams, graphical displays, and symbolic expressions. We also include linguistic expressions as a form of lexical or oral representation. For example, a student can use the lexical or oral representation, "the intersection of sets A and B"; a Venn Diagram to depict his/her mathematical thinking; a symbolic representation $A \cap B$; or set notation $\{x | x \in A \text{ and } x \in B\}$ (which is also a symbolic representation). Note the last two representations are in the same category, e.g. symbolic, but they are still considered two different representations.

Expertise-Based, Problem Finding, Evolutionary, Typological, and Systems. All of these theoretical approaches focus mainly on *domain-general* creativity, or "the creative ability to generate divergent or original ideas in a wide variety of domains" (Hong and Milgram 2010, p. 272). However, Baer (1998) cautioned against studying only general creativity practices; instead he advocated for research in *domain-specific* creativity. Plucker and Zabelina (2009) built on this idea: "some even argue that creativity is not only domain-specific, but that it is necessary to define specific ability differences within domains and even on specific tasks" (p. 6).

In both domain-specific and domain-general studies, creativity researchers either focus on the *end product* as original and useful (Runco and Jaeger 2012), or on a *process* that involves convergent or divergent thinking (Guilford 1967). Even though researching the creative process creates difficult hypotheses for testing (Torrance 1966), researching the creative product may not provide full understanding of the development of creativity, or may not reflect the creativity used to reach that product. For example, Pelczer and Rodriguez (2011) pointed out that "it is important that when judging the creativity of a student we pay attention also to the process by which [s/]he arrived to the results and not only to the final problem" (p. 394).

Viewing personal creativity as a product or a process brings up the following question: *For whom* is the product or process creative? This issue has been acknowledged through the discussion of *relative* and *absolute* creativity. Relative creativity is described as "the discoveries by a specific person within a specific reference group, to human imagination that creates something new" (Vygotsky 1982, 1984; as cited by Leikin 2009, p. 131). That is, a person may create something that is new to him/her or to his/her peers in a given subculture, but it may not be new to the community of more knowledgeable others. Absolute creativity, on the other hand, considers discoveries at a global level, such as the proof of Fermat's Last Theorem by Andrew Wiles (1995). In the education field, Liljedahl (2013) stated, "students have moments of creativity that may, or may not, result in the creation of a product that may, or may not, be either useful or novel" (p. 256). Thus, it is reasonable that the "relative creativity" perspective has been implemented frequently in previous educational creativity research (e.g., Liljedahl and Sriraman 2006; Leikin 2009).

3.2.2 What Is Mathematical Creativity?

Early researchers aimed to define mathematical creativity by focusing on experts. For example, Hadamard (1945) explored mathematical creativity of prominent mathematicians across the world through the use of surveys by mail. He theorized that the four stages the psychologist Wallas (1926) conjectured were applicable in describing the work of a mathematician. The four stages are preparation (thoroughly understanding the problem), incubation (when the mind solves a problem subconsciously and automatically), illumination (internally generating an idea after the incubation process), and verification (determining if that idea is correct).

However, Guilford (1950) found Hadamard's stages "superficial from the psychological point of view" (p. 451). He was concerned that these stages were not providing sufficient detail about the mental processes that occur. Guilford, then, created a list of testable factors that were later refined by other researchers: *fluency*, *flexibility*, *originality*, and *elaboration. Fluency* refers to the "number of ideas generated in response to a prompt" (Silver 1997, p. 76). *Flexibility* is the ability to shift approaches when the current approach is unproductive for generating a response to a prompt (Silver 1997). *Originality* (or novelty) is described as the ability to create a unique production or an unusual thought (Torrance 1966). *Elaboration* refers to the ability to produce a detailed plan and generalize ideas (Torrance, ibid). These factors of creativity have been used at the primary and secondary stages of schooling to determine students' levels of creativity (e.g., Balka 1974; Leikin 2009).

3.2.3 Mathematical Creativity at the Tertiary Level

There are on-going efforts to introduce challenging tasks in tertiary mathematics courses (e.g., through implementation of new pedagogical strategies such as inquiry-based learning (Smith 2006) or realistic mathematics education (Gravemeijer and Doorman 1999)). Such tasks are useful to elicit students' mathematical creativity

(Leikin 2014). Zazkis and Holton (2009) suggested mathematical problems that could challenge students at this level to possibly promote creative processes, and provided an outline of the importance of mathematical creativity. However, we know little about how to explicitly value tertiary students' creativity when such mathematical problems are implemented.

An essential aspect in tertiary mathematics is to ask students to communicate their reasoning through written proofs. Alcock and Weber (2008) stated that "a central unresolved issue in mathematics education is that of how to help students develop their conceptions of proof and ability to write proofs" (p. 101). While there is research about proving from many different perspectives (e.g., Selden and Selden 1995; Harel and Sowder 1998; Weber 2001), few have investigated mathematical creativity in proving (e.g., Leikin 2014).

To address this need, our research group developed a formative assessment tool that could be used to promote each student's development of mathematical creativity on a given mathematical task in a proof-based course and to examine this development over the duration of the course. The Creativity-in-Progress Rubric (*CPR*) on Proving can inform both students and teachers about the progression that a student is making in developing his/her own mathematical creativity.

3.3 Creativity-in-Progress Rubric on Proving

3.3.1 Development of CPR on Proving

Development of the CPR on Proving was motivated by the aforementioned studies, as well as our investigation of mathematicians' perspectives on students' mathematical creativity in tertiary level courses (Karakok et al. 2016). We interviewed six active research mathematicians (with pseudonyms Drs. A-F), who teach undergraduate and graduate level mathematics courses, and asked them about the role of mathematical creativity in proving, in teaching mathematics, and in students' learning. The mathematicians in our study also examined three student-created proofs of a theorem in number theory (Birky et al. 2011) using a domain-general creative thinking rubric (Rhodes 2010) created by the American Association of Colleges and Universities (AAC&U). This domain-general rubric was created to record growth and value creativity in a broad range of interdisciplinary student work samples. We utilized the mathematicians' ideas to modify the AAC&U rubric to make it domain-specific to mathematics. Our modification was also influenced by Leikin's (2009) rubric on mathematical creativity in problem solving, since the proving process is considered a subset of the problem-solving process (Furinghetti and Morselli 2009). We leveraged these ideas to develop a rubric (Savic et al. 2015) which we then refined using students' interview data (Tang et al. 2015).

Our development and refinement of the CPR on Proving were grounded in two of the ten aforementioned perspectives of creativity: *Developmental* and *Problem Solving and Expertise-Based* (Kozbelt et al. 2010). The primary assertion of the creativity theories in the *Developmental* perspective is that creativity develops over time, and the main focus of investigation is a person's developing process of creativity. This perspective also emphasizes the role of the environment surrounding a student, in which interactive elements occur to enhance a student's creativity. The second perspective that helped shape our project is *Problem Solving and Expertise-Based*, which emphasizes the role of an individual's problem-solving process and also argues that "creative thought ultimately stems from mundane cognitive processes" (Kozbelt et al. 2010, p. 33). This particular idea highlights that during problem solving or proving, implementing seemingly "mundane" tasks (such as finding relevant examples or representing the same concept in multiple ways) help the development of creativity by laying the foundations for creativity in novel situations. For example, Kozbelt et al. (2010) noted that "archival study of individual creative episodes of eminent scientists has generated a number of computational models of the creative process" (p. 33). These computational models included key components such as problem-solving processes, heuristics (ways that experts solve problems), and tasks. Furthermore, this perspective underscores the use of open-ended problems to challenge students' thinking processes, providing opportunities for students to use experts' ways of solving problems to be creative in such novel situations.

Overall, the CPR on Proving was developed from a relative, domain-specific approach, focusing on an individual student's progress on tasks and the development of his/her creativity over time. The next section provides a brief description of each of the categories developed.

3.3.2 Categories and Levels of the CPR on Proving

The CPR on Proving has two categories: *Making Connections* and *Taking Risks,* which are divided into subcategories that are reflective of the different aspects of creativity found in prior research. For each subcategory, the rubric provides three general levels: *Beginning*, *Developing*, and *Advancing*, each of which serves as a marker along the continuum of a student's progress in that subcategory. This continuum among levels of the rubric communicates the possible states of growth. Our research group acknowledges that some students may exhibit qualities that place them further along in one level, or in between two levels. Hence, the continuum of levels for each subcategory allows for a better approximation of placing proving attempts on the rubric based on the work provided. The user of the rubric can indicate the corresponding level by tracing the arrow using a highlighter or a marker (for example, see Fig. 3.2 in Sect. 3.4).

The descriptions of the subcategories in Table 3.1 are derived from either the research literature on creativity, quotes from our study of creativity with mathematicians (Karakok et al. 2015), or both. The description of the rubric is followed by two case studies in Sect. 3.4, which will provide further examples for each subcategory.

3.3.2.1 Making Connections

During the proving process, a student should be encouraged to make connections from previously learned material and apply these connections to new tasks. This category originated from the AAC&U Creative Thinking Rubric (Rhodes 2010) category *Connecting, Synthesizing, Transforming*, where a milestone level is achieved by a student who "connects ideas or solutions into novel ways" (p. 2). In our prior studies (Karakok et al. 2015; Tang et al. 2015), mathematicians commented that connecting ideas from other areas of mathematics was a crucial process in their work. For example, Dr. C. said,

> [F]inally I found some nice books in an area totally unrelated to mine, in matrix theory, and at some point I realized that I could apply this [aspect of Matrix Theory] that no one ever thought of applying to differential equations before and solved my problem … [I]n the process of applying it, I think I created … some new connections.

We define the category, **Making Connections**, as *the ability to connect the proving task with definitions, theorems, multiple representations, and examples from the current course that a student is in, and possible prior experiences from previous courses*. In this category, we consider making connections: between definitions/theorems, between representations, and between examples. Each of these subcategories is described below.

Between Definitions/Theorems To enhance connection-making abilities, students should make use of definitions/theorems previously discussed in the course and perhaps, from other courses. Dr. A, one of the mathematicians in our previous study (Karakok et al. 2015) stated, "Somehow your mind has to spread out a little bit to see…connections to other theorems you could use…That's creativity also." The way in which students use previous definitions/theorems in their proving processes defines the Beginning, Developing, or Advancing levels.

At the *beginning* level, a student recognizes some relevant (or irrelevant) definitions/theorems from the course or textbook (or resources related to the course) with no evidence of explicit attempts to connect those definitions/theorems to the task during the current proving process. For example, a student might list definitions related to a concept (e.g., function, onto, 1–1) that s/he has read in the task without providing any evidence of connecting these definitions to the proof in his/her attempts. At the *developing* level, a student recognizes some relevant definitions/theorems from the course or textbooks (or resources related to the course) with evidence of an attempt to connect these to the task during the proving process. At the *advancing* level, a student implements definitions/theorems from the course and/or other resources outside the course in his/her proving. While discussing creativity in the post-secondary classroom, Dr. E stated, "I think when students realize that they can solve these problems with things that are not just in this section. It can be from some other part of the course. Be somewhat creative." So, at the advancing level, a student not only recognizes relevant definitions and theorems, but also explicitly illustrates using them.

Between Representations Creating or using multiple representations can be important for solving or understanding problems. The National Council of Teachers of Mathematics (2000) referred to a representation as one way a student might depict his/her mathematical thinking. Other researchers have emphasized making connections among and between representations of a concept, such as representing a function as a table, graph, verbally and symbolically (Carlson et al. 2010). The connections students make between representations is also important for their development of mathematical creativity. For instance, Dr. F said, "[Creativity] is primarily to look at things differently. For example, notice that some equations result in some geometry, with that some geometry connects to some algebra." In this subcategory, students may not always make broad connections across the two fields of geometry and algebra, but perhaps attempt to utilize representations within each field.

Representations include written work in the form of diagrams, graphical displays, and symbolic expressions. We also include linguistic expressions, either oral or written, as a form of a lexical representation. For example, a student can write, "the intersection of sets A and B," or orally state "A intersect B is the set of common elements between A and B." A Venn Diagram, the symbolic representation $A \cap B$, the set notation $\{x | x \in A \text{ and } x \in B\}$ (which is also a symbolic representation), are other possible representations a student can use to depict his/her mathematical thinking about the concept of intersection. Note the last two representations are in the same category (i.e., symbolic), but they are still considered to be two different representations.

At the *beginning* level, a student is able to provide a representation with no evidence of attempting to connect it to another representation. At the *developing* level, a student should recognize connections between some representations and attempt to connect them to the proving task on hand. Students on this level may not recognize all the related representations of a mathematical object, but at least demonstrate connections to more than one representation. At the *advancing* level, a student should utilize and implement different representations in his/her proving process, hence making explicit connections between the different possible representations of a mathematical concept and applying these connections to their proof attempt.

Between Examples This subcategory refers to students' scratch work or "play time" where they experiment with different ideas to attempt the task. They can do this by creating examples, comparing and contrasting examples, or by providing counterexamples that are sufficient to disprove a claim. Students usually practice with examples as a method to understand the definition of a concept or to validate the verity of a mathematical statement. Doing so could help students to develop their creativity. Dr. A, for example, stated:

> Thinking to yourself: 'Can I make up a little example that'll help me get a sense of this?' 'Is there something I can try to do that will help me get my hands on this new concept?' And, again, I think that is a creative process.

However, students need to further their example generation to see possible connections to a pattern, which is somewhat of a difficulty for students (Dahlberg and Housman 1997). Merely asking students to create examples does not necessarily lead to a proof production. As Iannone et al. (2011) state, "[I]f example generation is to be a useful pedagogical strategy, more nuance is needed in its implementation" (p. 11). Thus, in this subcategory we aim to push students to make connections between examples to generalize to a key idea, or pattern.

At the *beginning* level, a student generates one or two specific examples with no attempt to connect them. However, at the *developing* level, a student recognizes a connection between the generated examples. At the *advancing* level, a student utilizes the key idea synthesized from generating examples. One way to see this is when students recognize patterns from examples and symbolize these patterns formally to assist in the proving process.

3.3.2.2 Taking Risks

During the proving process, a student should be encouraged to explore concepts, create new ideas, and evaluate those attempts in order to ultimately create a valid proof. Those explorations require a student to take risks during the proving process. The category Taking Risks originated from the AAC&U Creative Thinking Rubric (Rhodes 2010), where the highest level is achieved by a student who "[a]ctively seeks out and follows through on untested and potentially risky directions or approaches to the assignment in the final product" (p. 2). The category and the forthcoming subcategories were also influenced by our interviews with mathematicians about the proving process (Karakok et al. 2015; Tang et al. 2015). For example, Dr. B, stated:

> [O]ccasionally when you are trying to prove something, you know where you want to go, so it's just a matter of trying several different things, and seeing what fits in order to get you there. But other times, you don't know where you are going. Proving means you're saying, "There is this problem, and I'm going to try this approach and this approach. I don't even know what the next step should be." So I think the creativity part of it affects the proof differently.

Therefore, we define the category **Taking Risks** as the *ability to actively attempt a proof, perhaps using multiple proof approaches and/or techniques, posing questions about reasoning within the attempts, and evaluating those attempts*. The five subcategories, *Tools and Tricks, Flexibility, Perseverance, Posing Questions,* and *Evaluation of the Proof Attempt,* are described below.

Tools and Tricks We found through interviewing mathematicians that creativity also can involve creating tools or tricks in the proving process. Dr. E stated, "You can be very creative about the way in which you approach the question, either with new tools or with a really good idea for a partial result." Using these tools or tricks can be original to the student or the course, thus leading to *relative* creativity in their proving. A common example of a tool or trick that is original is involved in the proof of the theorem, "There are infinitely many prime numbers." One must assume a finite amount of prime numbers, $p_1, \ldots, p_n$, and create a new number $(p_1 \cdot \ldots \cdot p_n + 1)$ that is larger than the largest prime p_n which one then shows is still prime. The usual question asked by students when presented with this proof is, "Where did this come from?" This new number is an example of an unexpected object (tool) created to assist with creation of the proof.

The creation of an entirely new tool or trick is creative, and we believe it is evidence of a risk taken; however, the tool or trick need not be original to be considered in line with creative thought. Adapting a previous tool or trick to new contexts is also considered unconventional. At the *beginning* level, a student uses a tool or trick that is algorithmic or conventional. Conventional solutions are "generally recommended by the curriculum, displayed in textbooks, and usually taught by the teachers" (Leikin 2009, p. 133). For example, if an instructor presented the trick that you should "add zero" while completing a square, a student at the beginning level would employ the same trick in a proof that required completing the square. At the *developing* level, a student uses a tool or trick that is model-based or partly unconventional. If the student used that trick in a new context or in a proof that did not require completing the square in the same course, the student would be considered developing. For example if the student had to prove $4 \mid \left(5^n - 1\right)$ for every natural number n, and in the inductive case wrote $5^{k+1} - 1 = \left(5^k - 1 + 1\right)5 - 1$, then the student would be considered at the developing level. Finally, at the *advancing* level, a student creates a tool or trick that is unconventional for the course or the student. If a student thought of "adding zero" without any prompting or previous knowledge in the course, this would be considered advancing.

Flexibility In the category Making Connections, we discussed recognizing the need to use a proof technique used on previous proofs on a new proof. Flexibility is the ability to shift approaches in proving a theorem or claim. This idea was adapted from Silver's (1997) definition of flexibility for problem solving. For example, a student might begin a proof using a direct proof, but then shift to a proof by contradiction if the student did not find the first technique helpful. Dr. D found this ability helpful during her proof attempts, "If it doesn't work you say 'let me try something different and use some information I gathered to [come] up [with] something that might be more useful.'"

In this subcategory continuum, at the *beginning* level, a student attempts one proof technique in his/her proof. At the *developing* level, a student acknowledges the possibility of using different proof techniques, but does not act on it. Finally, at the *advancing* level, a student acts on different proving approaches. A student at the advancing level would act on multiple proof techniques, perhaps because the student did not find the initial proof technique(s) helpful, or s/he wanted to attempt a more efficient proof.

Perseverance Perseverance is a quality that many mathematicians possess either consciously or sub-consciously, and that many instructors want their students to exhibit in their courses. In his seminal work on problem solving, Schoenfeld (1992) pointed out that students "give up working on a problem after a few minutes of unsuccessful attempts even though they might have solved it had they persevered" (p. 359). Dr. B echoes the importance of perseverance in his statement:

> [T]he creativity part is, 'ok I know I'm going to get this far, and I want to get here, kind of four steps down, but somehow able to pinpoint the key idea,' so that's part of the creative process… sometimes it'll just be a matter of trying various [ideas] to get that [next] step.

We identify perseverance in a proving process of a student when he/she is continuing to engage in the proving process no matter the hardship. Time is not an aspect of our definition of perseverance; rather the engagement with key aspects and the challenges of the proof is the gauge of perseverance. According to Thom and Pirie (2002), perseverance is a "sense (i.e. intuitive and experiential) in knowing when to continue with, and not to give up too soon on a chosen strategy or action" (p. 2). Therefore, to develop creativity in proving, perseverance is needed.

At the *beginning* level, a student demonstrates perseverance by engaging with the proving process minimally. For example, a student would not finish his/her first proving attempt, and would not try any other proof attempt. At the *developing* level, a student would continue to engage with surface level features of the proving process, but there is no evidence of engagement with the key ideas of the proof. Finally, at the *advancing* level, a student perseveres by engaging with key ideas of the proof. S/he may or may not have a final valid proof, but is engaging with the key ideas or reasoning for the proof.

Posing Questions In the proving process, there are certain times when a question can lead to a creative thought. Dr. B acknowledged that, while researching, he asks himself, "What do I need to do in order to make that step so the rest of it is downhill?" Pelczer and Rodriguez (2011), citing Jensen (1973), stated that if students want to be creative, they "should be able to pose mathematical questions that allow exploration of the original problem" (p. 384). Posing questions can occur throughout the proving process, but there are different qualities of questions that students can pose.

At the *beginning* level, a student will recognize that a question should be asked (perhaps with a question mark next to his/her proof), but will not formulate a full question. At the *developing* level, a student will pose a clarifying question about a statement of a definition or theorem, for example to clarify terms used within a theorem statement. Finally, at the *advancing* level, a student will pose a clarifying question about the reasoning in the proof.

Evaluation of the Proof Attempt We define a successful proof as a correct proof which "establishes the truth of a theorem" (Selden and Selden 2003, p. 5). A *successful* proof is neither a necessary nor sufficient condition for a *creative* proof attempt. That being said, understanding the *key ideas* that make a proof attempt successful or unsuccessful can provide insight for future proof attempts. Key ideas are defined by Raman (2003) as, "a heuristic idea which one can map to a formal proof with appropriate sense of rigor. It … gives a sense of *understanding* and *conviction*. Key ideas show *why* a particular claim is true" (p. 323). For example, Dr. D stated that she re-evaluated a result to find a visual application and ended up "go[ing] back to the drawing board because the stuff that we thought we proved was wrong. My thinking about [a] different way to visualize it and seeing something completely unexpected got us there."

At the *beginning* level, a student is one who only checks work locally, that is, for small errors or typos. At the *developing* level, a student recognizes a successful or unsuccessful proof attempt without identifying the key idea that makes the attempt successful or unsuccessful. A student may look at his/her proof, realize that it is incorrect, but not realize exactly why the proof is incorrect. Finally, an ability to recognize the key idea in a proof attempt, successful or unsuccessful, describes an *advancing*-level in this subcategory.

In the next section, we provide two students' proving attempts on two different tasks to illustrate student work that falls into various levels of these subcategories.

3.4 Case Studies

The data presented in this section was collected in Spring 2014 at a large research university in the United States. In an inquiry-based, introduction-to-proof course, 24 students were given LiveScribe pens, a data collection tool capable of capturing audio and written work in real time (For details on LiveScribe pens, see Savic 2015). Use of this technology was an intentional attempt to capture the processes of students' proof development, including scratch work and verbal expressions. All students were required to do and turn in their homework using the pen and special paper; all homework was downloaded to the professor's computer for both grading and analysis. The LiveScribe pen data was examined for the eight students that participated in an exit interview to relate exit interview data with performance throughout the semester. Our research group narrowed the data analysis to five proving tasks that we jointly agreed could elicit creativity in the proving process. Those tasks were coded using the CPR on Proving. Here we report on two of those five tasks.

Theorem 29 is stated: "If 3 divides the sum of the digits of n, then 3 divides n". This theorem was the third theorem in the number theory section of the course, located after the definition of even and odd numbers, divisibility, (Definition S: $a \mid b \Leftrightarrow b = na \, for \, some \, n \in \mathbb{Z}$) and the following theorems (27 and 28): "If m and n are even numbers, prove that $m+n$ and $m \cdot n$ are even numbers" and "If $a \mid b$ and $a \mid c$, then $a \mid (br+cs)$ for any $r,s \in \mathbb{Z}$." respectively. We will refer to Theorem 29 as the "digit" theorem. The other task is the second theorem on the second test: "If 3 divides a natural number n, then n is a trapezoidal number." A trapezoidal number, defined on the test, is a natural number that can be expressed as the sum of two or more consecutive natural numbers. We will refer to Theorem 2 of Test 2 as the "trapezoid" theorem.

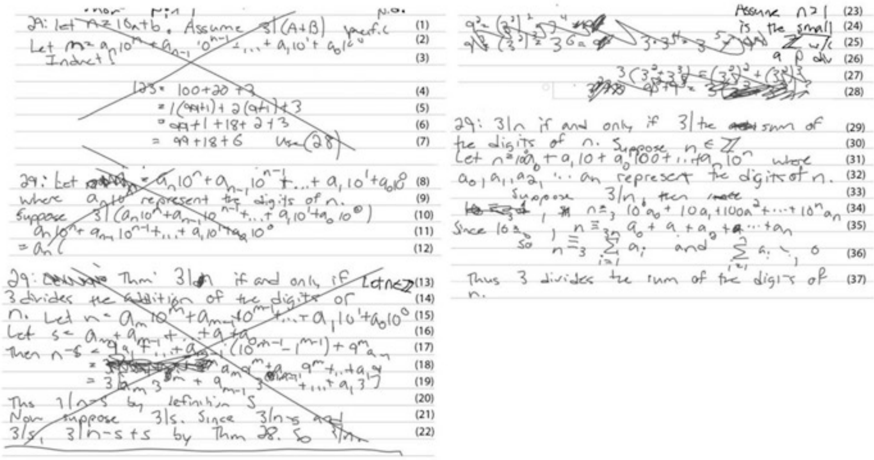

Fig. 3.1 Greg's proving process for the digit theorem

To demonstrate the aspects of mathematical creativity in proving that the CPR reveals, we focus on two students in the course, Greg and Marty. At the end of Spring of 2014, Greg was a mathematics major with enough credits to be a fourth-year student; Marty was a fourth year dual-major in Economics and History. When coding students' work for one proof task, we used a holistic approach to focus on the students' entire collection of attempts rather than each individual attempt. For instance, we coded Greg's four attempts of the digit theorem (see Fig. 3.1) rather than each individual attempt. We present both students' proving attempts for both theorems.

3.4.1 *The Digit Theorem*

All of the proving attempts by Greg for the digit theorem are located in Fig. 3.1. The LiveScribe Pen time-stamped each recording so we were able to see that the student attempted this proof at least four times (Fig. 3.1 Lines 1–7, Lines 8–12, Lines 13–28, and Lines 29–37) over two days. Figure 3.2 provides the aggregation of our coding of all of these attempts. We provide an explanation for the coding of each subcategory (written in italics) below.

Between Definitions/Theorems We observed that Greg used Definition S (Line 20) and Theorem 28 (Lines 7 and 22) and tried to implement both into his proving attempts. Therefore, since he implemented definitions and theorems in his proving attempts, we coded his work as "advancing". This is indicated with the dark arrow in Fig. 3.2.

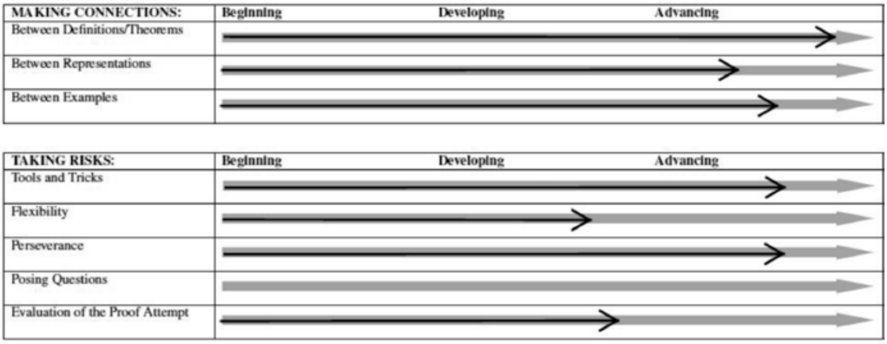

Fig. 3.2 Levels of Greg's work on the digit theorem

Between Representations Greg also had two symbolic representations for the natural number $n : a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0 10^0 \equiv_3 \sum_{i=1}^{n} a_i$ (Lines 34 and 36). We observed that he used connections between these two representations in his proving process, but it is unclear how the second representation was helpful to his attempt. Thus, his work is categorized in the "advancing" level, yet not as complete as in *Between Definitions/Theorems*.

Between Examples Greg explored an example, $n = 123$ (Lines 4–7), and when he tried to factor a 3 from 9^n (Line 19), he generated examples (Lines 23–28). Since he used the key idea generated from "123" (Lines 17–19), he demonstrated an "advancing" level in our coding.

Tools and Tricks Greg used a notation "$\equiv_3$", which means "equivalent modulo 3," that was unconventional for the course because it had not been previously discussed. He also used the trick of rewriting 10 as $9+1$ and 100 as $99+1$ (Lines 4 and 5). Therefore, we coded Greg's work as "advancing."

Flexibility On Line 3, he indicated that the proof might be approached using induction, but largely used direct proof, hence he acknowledged the possibility of a different approach with no further examination. For this reason we coded his work in the "developing" level.

Perseverance Greg engaged with key ideas of the proof (Lines 17–22), and also created many proving attempts, so he was coded as "advancing" in this subcategory.

Posing Questions There were no questions posed either in his written or oral recorded work. Due to this lack of evidence, no levels were assigned to this subcategory of his work.

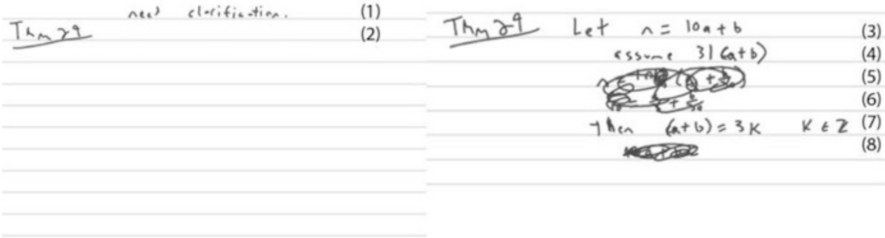

Fig. 3.3 Marty's proving process for the digit theorem

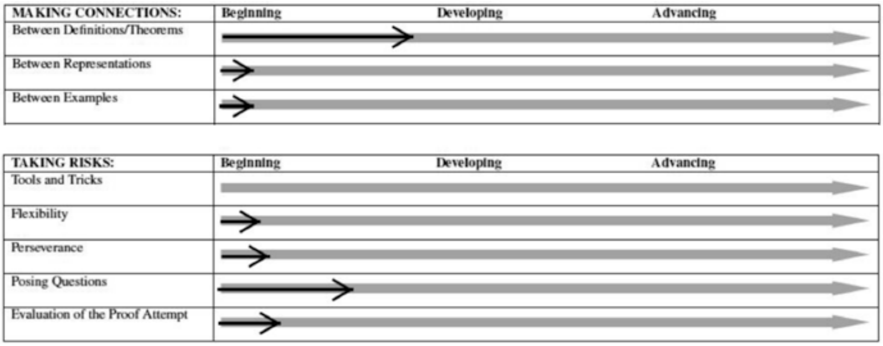

Fig. 3.4 Levels of Marty's work on the digit theorem

Evaluation of the Proof Attempt Though he recognized unsuccessful attempts by crossing them out, there is no explicit acknowledgment of the key idea(s) that made the attempt unsuccessful, thus his work was coded as "developing." However, the multiple evaluations of his proving contributed to the arrow being closer to advancing.

For the digit theorem, all of Marty's attempts are demonstrated in Fig. 3.3. Marty attempted the proof first (Fig. 3.3 Lines 1–2), and attempted the proof again two days later (Fig. 3.3 Lines 3–8). In Fig. 3.4, we present the summary of our coding of Marty on the digit theorem.

As it can be observed in Fig. 3.3, Marty's demonstrated proof process was brief. Thus, we share descriptions for some of the subcategories. Marty recognized but did not implement the definition of divisibility by stating that $(a+b)=3k$ (Line 7). Also, he wrote "need clarification," (Line 1) which was recognition to pose a question without fully formulating the question. Those two actions provided evidence for the positioning of the arrows in between the "beginning" and "developing" levels for the subcategories of *Between Definitions/Theorems* and *Posing Questions*. For other subcategories, we only observed "beginning" level actions, with the exception of the *Tools and Tricks* subcategory. Marty's work did not provide evidence for this subcategory, so no level is indicated.

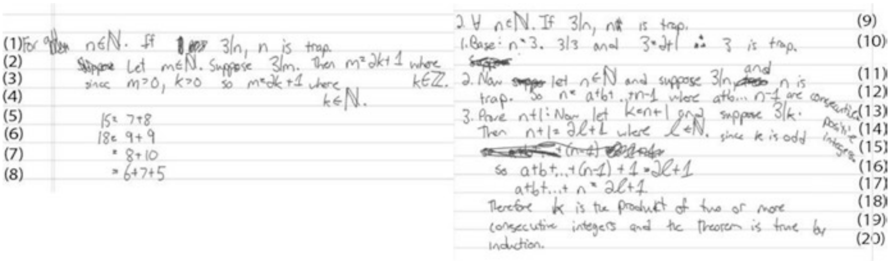

Fig. 3.5 Proving attempts of Greg on the trapezoid theorem

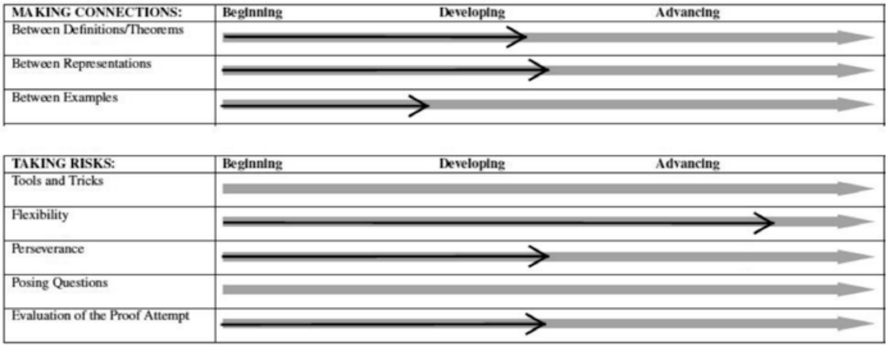

Fig. 3.6 Levels of Greg's work on the trapezoid theorem

3.4.2 The Trapezoid Theorem

The proving attempts of the trapezoid theorem from Greg are presented in Fig. 3.5, while the levels coded for Greg are in Fig. 3.6.

In the Making Connections category, Greg implemented definitions for trapezoidal (Line 12) and odd numbers (Line 14) in his proof, and attempted to use both definitions in his proving ("developing" in *Between Definitions/Theorems*). He used a representation of a trapezoidal number (Line 12) and attempted to connect it to the definition of a trapezoidal number ("developing" in *Between Representations*). Finally, he generated some examples (Lines 5–8), but there was no indication that he recognized a connection between his examples so as to generate a key idea to use in the proof attempt ("beginning" in *Between Examples*).

In the Taking Risks category, Greg did not use a trick or tool or pose a question (both left blank in the rubric). He acted on two different proving approaches, namely direct proof and induction ("advancing" in *Flexibility*). Greg continued to engage with the surface level, but not the key ideas, of the proof ("developing" in *Perseverance*). Finally, he evaluated his first proof attempt but did not recognize the key idea that made it unsuccessful ("developing" in *Evaluation of the Proof Attempt*).

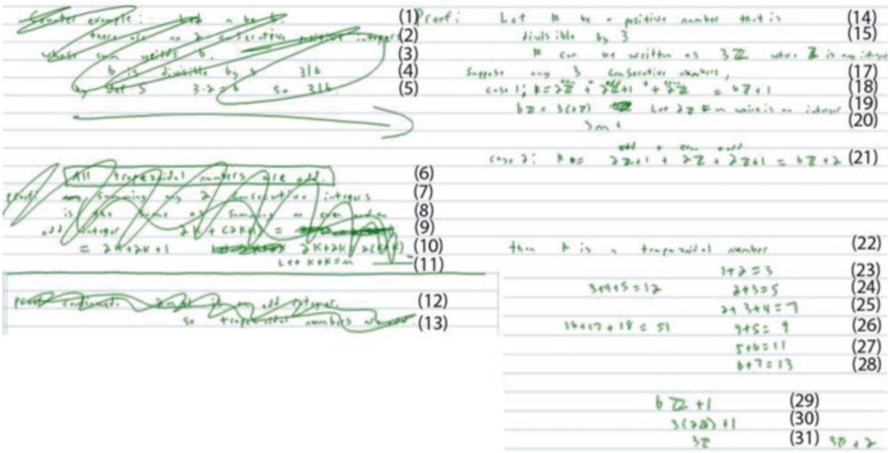

Fig. 3.7 Proving attempts of Marty on the trapezoid theorem

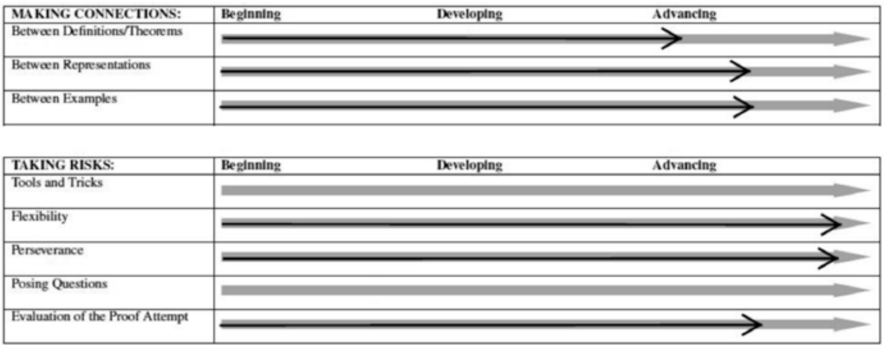

Fig. 3.8 Levels of Marty's work on the trapezoid theorem

Marty's proving attempts for the trapezoid theorem are located in Fig. 3.7. The levels coded for Marty are in Fig. 3.8.

Note that Marty produced examples in his proving (Lines 23–31), and used those examples to find the key idea (lines 18 and 21). We did not see any evidence of the subcategories *Tools and Tricks* and *Posing Questions*. Also, Marty's work was placed in "advancing" levels for many of the other subcategories, despite the production of an incorrect proof.

Overall, we noticed that Marty mostly demonstrated "beginning" level actions on both categories (Making Connections and Taking Risks) for the digit theorem (see Fig. 3.4) with the exception of no level for the subcategory *Tools and Tricks* of the Taking Risks category. As shown in Fig. 3.8, he had mostly "advancing" level actions on both categories (Making Connections and Taking Risks) for his work on the trapezoid theorem, with the exception of no levels for subcategories, Tools and

Tricks and Posing Questions. Greg, on the other hand, had "advancing" level actions for the Making Connection subcategories and mostly "developing" for the Taking Risks for the digit theorem (see Fig. 3.2), with the exception of no level for subcategory Posing Questions. For the trapezoid theorem, Greg's work had varying levels of actions for subcategories: mostly "developing" for Making Connections, and varying levels between mid- "developing" to high- "advancing" for Taking Risks (see Fig. 3.6).

3.5 Discussion

In this section, we delineate some observations and provide hypotheses that were generated from the case studies in the previous section. Specifically, we highlight the following: (i) the similarity between the coded levels of some categories between the two students; (ii) the connection between students' work and their perspectives on creativity; (iii) the fact that correctness was not taken into consideration while coding; and (iv) the association between some of the subcategories in the rubric. In addition, we suggest some teaching practices that complement the CPR on Proving.

3.5.1 Remarks on the Coding of Students' Work

The coding of Greg and Marty's work illustrates the evaluation of the whole proving process rather than the final product. This analysis highlighted how two students' different proving processes on the same task could be coded at the same level for a subcategory. For example, both students' differing processes were coded "advancing" for the subcategory *Flexibility* on the trapezoid theorem, but there was variation in the number of proof attempts and approaches between Greg and Marty. We developed our rubric intentionally to capture such instances to stress the valuing of assessing individual student's work.

Our second observation is that both students' coded work had some alignment with their perceived notion of creativity as they shared it in the exit interview conducted at the end of the semester. For example, Greg stated that mathematical creativity is "coming up with the little tricks that make each proof flow." He utilized modular arithmetic (Lines 34 and 36 in Fig. 3.1), which was unconventional for the course, and thus we considered it a "trick." Marty, on the other hand, defined mathematical creativity as "start[ing] from a different place or us[ing] a different method. You know, induction, contradiction, all those sorts of things." This description was observed in his proving attempts of the trapezoid theorem, since he first attempted to disprove this task using a counterexample, then shifted his approach to two different direct proofs (Fig. 3.7), thereby demonstrating an "advancing" level of Flexibility. This possible link between students' perceived notion of creativity and their proof attempts requires further investigation. However, we hypothesize that students' use of the CPR on their proof attempts could potentially help them to reach their creative potentials by illuminating other aspects of the proving process that students may not have focused on.

In our data, incorrect final proofs did not associate with less potential for mathematical creativity. Marty's work for the trapezoid theorem is an incorrect proof, however many of his actions in his proving attempts were coded in the Taking Risks category as "advancing." The CPR on Proving was designed to purposefully remove judgment of validity in order to encourage students to take risks and to be engaged in the proving process. Also, it is possible that some students can be relatively creative while being incorrect, which is a sentiment shared by one of our interviewed mathematicians, Dr. F:

> I will risk it and say that [a proof] doesn't have to be correct to be creative. But at least it [the proof] should be fixable. It can happen that you have an original idea and you mess up details, which is not surprising because if it is an original idea then it means that you haven't practiced that, [so] you would make mistakes.

Some subcategories can help develop or enhance other subcategories in a student's proving process. Greg demonstrated "advancing" level actions in the *Tools and Tricks* and *Perseverance* subcategories of Taking Risks for the digit theorem (Fig. 3.2). In the coding analysis, we noticed that his evaluation actions for each of his individual attempts combined with his perseverance contributed to his overall "developing" level of *Flexibility* on this task. Similarly, our analysis of Marty's different attempts indicated that his process of evaluation of each attempt and his overall perseverance allowed him to demonstrate advancing level actions of flexibility. Noticing such overlaps with the subcategory of *Perseverance,* and subcategories *Flexibility* and *Evaluation*, we revised the category of Taking Risks and eliminated *Perseverance* subcategory (for details see Karakok et al. 2016). It is also possible to think there are overlaps between the categories of Making Connections and Taking Risks. For instance, Greg took a risk to employ a trick from a previous course to prove the digit theorem, thereby making a connection. However, these subcategories include unique elements and actions which help distinguish them from each other, such as utilization of a "trick" in Greg's example. The user can leverage those unique elements to improve or enhance his/her proving process.

We believe that it is possible for a student to engage in proving attempts that do not necessarily exhibit all advancing-level actions but still demonstrate relative creativity. Reflection on each subcategory of the CPR on Proving can help develop a student's creative potential for proving. For instance, a student might approach a proof with one technique (a "beginning" level in *Flexibility*), without creating any Tools and Tricks, but persevere in the process at the advanced level by creating many examples and generalizing them to gain an understanding of the key idea of the proof (advancing level in *Between Examples*). Another example comes from Marty, his work did not have any evidence of actions to be coded in the *Tools and Tricks* subcategory, but showed the potential of creativity in his proving.

3.5.2 Teaching Implications

The CPR on Proving was created for instructors' or students' use. For each subcategory, instructors can evaluate their students' proving attempts and decide which part of the continuum the work could be placed. Once trained on how to use the rubric, students can also do this for their own and also for their peers' work. Our intention is that if an instructor or a student evaluates proving attempts, s/he can hopefully see what needs to be improved or worked on during future tasks (attempts).

We have found that the use of the CPR on Proving may not elicit all of the subcategories for certain tasks. For example, the rubric may not be very useful for either a student or instructor in examining work on a routine task or typical exercise (as opposed to Schoenfeld's (1982) definition of a problem). However, a task that can be proved using several different proof techniques can challenge students to be creative in their explorations during their proof attempts. Both theorems used in the case studies are examples of such tasks (e.g., Zazkis and Holton 2009; Leikin 2014). The use of CPR on Proving could also help in the scenario of "getting stuck" or encountering a proving impasse (Savic 2015), since most students "[produce] either no solution, incomplete solutions, or the 'standard' ones" (Zazkis and Holton 2009, p. 349).

Mathematicians such as Pólya (1954) have noted the importance of guessing theorems as a creative endeavor. Therefore, we suggest engaging students in the process of creating conjectures, posing problems, or solving open-ended tasks (Silver 1997; Brown and Walter 2005) with the CPR on Proving. Asking students to conjecture can challenge them to create examples in order to generalize, can encourage them to ask questions about their conjectures, or to evaluate the key ideas that make their conjectures true or false. Use of the CPR on Proving while students are engaged in these tasks may both improve their proving process and increase their potential for mathematical creativity.

The CPR on Proving provides an opportunity for teachers to be informed on students' proving processes. However, it is crucial that the instructor does not use the rubric to label or categorize students' level of creativity in broader generality. This might cause students to feel less creative, and perhaps consequently do poorly in a proof course. We recommend an open discussion on the rubric between the instructor and the students early on to unpack the meaning of each category, subcategory and underlying levels, and demonstrate the usage of the rubric. Some suggestions might include an instructor using the CPR on Proving after a student demonstrates his/her proof attempt in class, or having students discuss their own creative proof attempts using laminate copies of the rubric (for re-use) with dry-erase markers. Also, the CPR does not have to be utilized in its entirety. It might be appropriate for an instructor or a student to focus on one subcategory in order to highlight certain areas of improvement. For example, it might be useful for the students to examine an unsuccessful or successful proof and ask them to reflect on the key ideas that made it so. This would help them in the *Evaluation of the Proof Attempt*.

Lastly, we suggest creating an environment in the classroom where creativity can be nurtured. This includes allowing students to make mistakes, perhaps even valuing and discussing them in class. In order to successfully achieve this, students' grades may not be penalized as frequently for making mistakes (see Burger and Starbird (2012) for similar suggestions). Instructors could be explicit about the tasks for which they expect students to provide a correct proof and those for which they expect students to explore freely without judging correctness. Incorrectness, as we have seen in both Greg and Marty's work, and as Dr. F stated above, can be a catalyst for creativity.

3.6 Conclusion

The development, and subsequent implementation, of our rubric is meant to start a discussion to encourage and promote mathematical creativity in the tertiary level. We do not claim that this rubric encompasses all the behaviors that can lead to mathematical creativity. Furthermore, we recognize that there may be other subcategories, so it is not our intention to limit the discussion of the process of creativity with these subcategories. The intention of the CPR on Proving is to make the behaviors that mathematicians themselves exhibit during their pursuit of new ideas explicit to the students.

As a research group, our overall goal for the Creativity-in-Progress Rubric (CPR) on Proving is to help tertiary students reflect on their proving processes, and hopefully, alleviate potential proving difficulties that are common to most students at the post-secondary level. Guilford (1975) highlighted the importance of reflection, stating that "the student [should] be taught about the nature of his[/her] own intellectual resources, so that [s/]he may gain more control over them" (p. 120). The function of the rubric, as we see it, is not to determine students' exact levels of creativity, nor to state that one student is or is not creative. Instead, our rubric is about fostering growth. It is about encouraging students to engage in behaviors that mathematicians claimed may lead to mathematical creativity. We believe crucial actions that can lead to the potential for mathematical creativity in proving are embedded in the advancing levels of the rubric. Regardless of the school level of the student:

> It must not be forgotten that the basic law of children's creativity is that its value lies not in its results, not in the product of creation, but in the process itself. It is not important what children create, but that they do create, that they exercise and implement their creative imagination. (Vygotsky 2004, p. 72)

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Chapter 4
Teacher's Views on Modeling as a Creative Mathematical Activity

Gudbjorg Palsdottir and Bharath Sriraman

Abstract In this chapter we examine mathematical modeling activities presented in high school textbooks in Iceland with respect to how teachers utilize such activities. We first argue on the basis of the existing literature that mathematical modeling may be viewed as a creative mathematical activity. We ask whether the institutionalization of mathematical modeling through school textbooks in the form of activities convey the creative aspect of modeling. To answer the question, we examine the views of a group of Icelandic high school teachers about modeling activities, and ways in which they implement them in the classroom. Preliminary results indicate that teacher's use a dialogic and practical approach to modeling activities as opposed to a strictly mathematical approach. We discuss their views within the Icelandic context.

Keywords Mathematical modeling • Mathematical creativity • Modeling tasks • Math textbooks • Inservice teachers • Iceland

4.1 Introduction

The experiential world of the twenty-first century student and teacher is characterized by complex systems such as the Internet, multi-medias, sophisticated computing tools, global markets, virtual realities, access to online educational environments, etc., and emerging fields such as bio-informatics and mathematical genetics, cryptography, mathematical biology, etc., which call for different mathematical skills such as the ability to model complex systems and problem solving. Mathematical modeling can be considered as a creative mathematical creativity for many reasons. Modeling in its true sense requires understanding a real situation, be it a natural

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phenomenon like the flow of water or social phenomenon such as the spread of information via Twitter, in order to create a "model" that both validates empirical data and is useful for predictive purposes. In one sense modeling can be considered to be tautological since modeling means to create a model, and a model is the result of modeling. There are other definitional issues that arise. For instance a model is the realization or interpretation of an axiomatic system in pure mathematics. In other words given a set of axioms, a model allows one to understand and properly understand otherwise abstract properties. In mathematics education modeling is often understood as an aspect of problem solving (English and Sriraman 2009) whereas others view problem solving as an aspect of modeling (Lesh and Doerr 2003). In other words modeling can be categorically viewed as an umbrella terms that practically includes everything- ranging from the realization of an axiomatic system, to problem solving, to something "applied" or having a predictive purpose to understand real world phenomena. Given this confusion that abounds on what modeling is, it is natural that teachers often have to rely on textbooks that prescribe modeling in the form of diagrams and associated activities. For the purpose of this chapter modeling is viewed as a creative mathematical activity because it is often open-ended, ambiguous (requiring the modeler to make assumptions), time consuming, and interpretive (Sriraman 2005). Numerous curricular documents in the U.S., U.K., Australia extol the virtues of modeling as a means to prepare students to think flexibly and creatively when faced with real world problems (Lesh and Doerr 2003; English 2006). Similarly in Iceland, it is highlighted in the national curriculum guide that in our society many activities are based on the utilization of mathematics and therefore the students both need mathematical literacy and to be able to use mathematics as a living tool in understanding mathematical problems and seeking solutions by applying creative thinking, reflection, arguments and the presentation of mathematical models. In the competence of being able to ask and answer in and with mathematics it is expected of tenth grade students to:

- express themselves about mathematical topics and reality by using the language of mathematics, verbally explain to others their thoughts about it, seek solutions and present mathematical problems invarious mathematical forms by applying creative thinking, reflection and reasoning, present, analyse, interpret and evaluate mathematical models,
- find, propose and define mathematical problems, both related to everyday life and mathematical issues, evaluate solutions, for example with the aim of forming generalisations from them,
- set up, interpret and critically scrutinise a mathematical model of real conditions; this may include calculation, drawings, graphs, equations and functions, (Ministry of Education, Science and Culture 2013, p. 218)

This century the focus in the Icelandic national curricula has much been on mathematical thinking and mathematical processes. The curriculum is divided into processes and content. The new one has the competence criteria for mathematics arranged in seven categories: (1) Being able to ask and answer in and with mathematics, (2) being able to use the language and tools of mathematics, (3) work methods and application of mathematics, (4) numbers and calculation, (5) algebra, (6) geometry and measurement, (7) statistics and probability. Modeling is not a concept in focus but the ideas modeling is based on are clearly present as can been seen in the writings about the main objectives of mathematics:

- *Mathematics helps us to describe circumstances precisely and to explain causation within them, interpret data, and to predict and influence development* (Ministry of Education, Science and Culture 2013, p. 216)

In the general literature on creativity numerous definitions can be found. Craft (2002) uses the term life wide creativity to describe the numerous contexts of day-to-day life in which the phenomenon of creativity (c) manifests. Other researchers have described creativity as a natural survival or adaptive response of humans in an ever-changing environment. Richards (1993) uses the term everyday creativity" ("little c") to describe such activities as improvising on a recipe. It is generally accepted that works of extraordinary creativity can be judged only by experts within a specific domain of knowledge. For instance, Andrew Wiles' proof of Fermat's Last Theorem could only be judged by a handful of mathematicians within a very specific subdomain of number theory. Taking the view of Craft and Richardson we view creativity as "little c" or "ordinary/every day creativity" as being relevant for the school setting.

4.2 Modeling as a Creative Mathematical Activity

In general, there is a paucity of literature in mathematics education which views modeling from a creative viewpoint. However there is some literature which views modeling as a creative aspect of problem solving. English and Sriraman (2009) suggested four salient points for the incorporation of modeling in the curriculum, namely

1. It provides a new way of looking at quantities in realistic settings requiring accumulations, probabilities, frequencies, ranks, and vectors.
2. Modeling goes beyond traditional word problems in the sense that it involves a cyclic process of interpreting the problem information, selecting relevant quantities, identifying operations that may lead to new quantities, and creating meaningful representations (Lesh and Doerr 2003).
3. Modeling is ubiquitous with other disciplines such as economics, information systems, social and environmental science which incorporate powerful mathematical models for dealing with a range of complex problems
4. Modeling problems facilitate small group work and present teachers with an opportunity to let students pose questions, resolve issues and conflicts that arise as students develop models.

Taken as a whole, mathematical modeling incorporates fluency, flexibility, elaboration and originality which are the basic constructs of creativity. These constructs are also applicable to teacher's conceptions of creativity. Lev-Zamir and Leikin (2013) analyzed teacher's declarative conceptions of mathematical creativity in teaching and found a mismatch between declarations and actions. These researchers

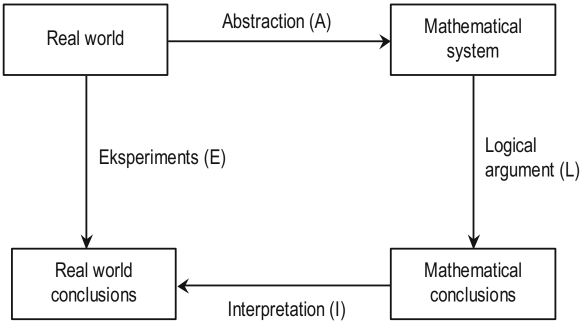

Fig. 4.1 Modeling cycle presented in Icelandic high school textbooks

suggested that the mismatch could be explained in terms of deep beliefs versus surface beliefs. In other words while a teacher may espouse a flexible approach to teaching and learning, their practice may reveal otherwise.

Another aspect of viewing modeling as a creative activity is the possibility of allowing teachers to engage in problem posing. Following the suggestion of Stoyanova and Ellerton (1996) problem-posing situations can be free; semi-structured or structured. When presented with a modeling cycle from a textbook (Fig. 4.1), mathematics education researchers have the possibility of letting teachers engage in problem posing- i.e., to create problems that would meet the criteria of the modeling cycle depicted. This makes it a semi-structured situation in that not every problem can be subsumed under the modeling cycle. This was the approach of this particular research with high school teachers- to make them verbalize viable and non-viable situations that could be considered as "creative" modeling activities.

As mentioned in the introduction there are different approaches to modeling found in textbooks- this can vary from contrived situations with data presented in order for students to procedurally use functions to somehow model the date by trying different regression equations, or it can be completely open ended tasks requiring teachers to guide students into productive ways of thinking about the situation. Given this disparity in the ways modeling is incorporated in the classroom, we conducted a study with in-service teachers in Iceland in order to answer the following question:

1. What are teachers' views of modeling as a creative activity in their classroom?

4.3 Methodology

In order to answer the question above high school teachers in Iceland were invited to participate in a series of workshops on modeling organized by the authors. Four newly graduated teachers and one experienced teacher agreed to participate in these workshops on a voluntary basis. The authors wanted to get some knowledge of the teachers' ideas on modeling when discussing them and when using them in planning teaching. The initial goal of the workshops was for teachers to collaborate and make a teaching plan on modeling activities for their students. There were three workshops for 2 h each. In the first workshop, the focus was on the concept of modeling, namely how were teachers working with modeling with their students, and what they thought was the meaning of modeling? In the next workshop the focus was on how the teachers could collaborate on creating a project on mathematical modeling? The last workshop was concentrated on planning and making a concrete modeling project for incorporation in the classroom. The discussions were informal and led by the first author with the purpose of getting a picture of the teachers' understanding of modeling. All the teachers participated actively and they created a project on modeling they were going to use in their teaching. We present important elements in the discussion that emerged from the workshops. By centering the discussion around what teachers considered to be modeling and modeling as a creative activity we were unconsciously using the Freirean notion that "through dialogue, the teacher-of-the-students and the students-of-the- teacher cease to exist and a new term emerges: teacher-student with student-teachers. The teacher is no longer merely the one-who-teaches, but one who is himself taught in dialogue with the students, who in turn while being taught also teach. They become jointly responsible for a process in which all grow (Freire 2005, p. 264).

4.4 Results

Based on the guidelines of the national curriculum, the role of the teacher is to create a meaningful learning environment where the students are researching problems they find interesting. In the curriculum materials that the teachers in this study were using, there is a focus on connecting mathematics to the society and the students' interest. In the tasks, emphasis is put on giving the students opportunity to develop the use of mathematical processes. In the material for the tenth grade there is a chapter on models. There the students are presented various models and the concept to mathematize as seen in the modeling cycle in Fig. 4.1. Building on that, students are supposed to create their own models for various topics from daily life, as sharing, estimating pollution, designing and predicting the growth of the population (Palsdottir and Gunnarsdottir 2007, pp. 42–47). This is the context in which the teachers were working in their classroom and hence important for the reader to know. We now present some of the teacher's views on modeling as a creative activity. We describe in thick detail what transpired in each workshop and distill important elements from the dialogue that occurred between the participants and the authors.

In the first workshop both authors were present and we discussed the objective of our study and why the teachers were invited and what we had planned for them. All the teachers were using the same curriculum materials and we asked them how they were using the chapter in the textbook on modeling. The intention was to make it easier for the teachers to give examples of their teaching on the topic and so the other teachers could refer to their experience about the use of same modeling tasks from the textbook. We were surprised to find that none of the teachers had been using this chapter but they all knew about the modeling cycle in the chapter. They had been using the picture of the cycle in their teaching when they were working with open themes about mathematics in real world. The modeling cycle was seen as a lens through which students could look at the world. They said that they were not using the chapter from the curriculum material because t**hey didn't find the content very important** and that it was more interesting to do some of the content in a creative collaboration with the students and in open group work projects with the students. When asked, the teachers also admitted that they had not seen modeling as a vital part of the curriculum. Although they knew it was clearly stated in the national curriculum guide, they were aware that students were not tested on it in the national tests. The teachers were not explicitly using the concept of model or modelling but they were aware of the notion of mathematization and expressing this by using the terms -generalizing and looking at a structure. They expressed the view that when working with models it is best for them to make their own tasks that are close to the students' daily lives. They like the picture in the textbook of mathematizing but they did not connect the cycle with the tasks from daily life. One of the teachers' said "It is hard to work with tasks that are created by somebody else". In this case "tasks" referred to themes from the real world that connected to their students' lives and not tasks found in the book.

They wanted to give the students some opportunities to be creative and say that it is important for the students' motivation in their mathematical learning. For the teachers models were- "reiknilíkön" – models to calculate things. The teachers gave examples of projects that are mostly about money and not about things from nature and the development of the society. One example was given of an environmental issue involving the use of disposable diapers in society. They expressed the view that modeling was about seeing structure in problems from daily life and that the students should find or be given an open task or a concept to dig into. Again the examples that were given related to "costs of living, prices of goods etc" because their opinion was that students did not have a realistic idea about the cost of daily living.

The second author introduced a task based the problem of interpreting a will in which certain proportional relationships had to be met. The purpose of using a concrete task was to see if the teachers would consider it to meet the criteria of a creative modeling activity. The teachers were curious about the problem and they found the task interesting and speculated about the mathematics in their discussion. They were thinking about how to interpret the conditions and how it could be calculated. They were addressing some of the modeling aspects and interested to hear and discuss others ways and solutions. When probed on whether this could be considered a modeling activity, there was no consensus on the criteria that needed to be applied to make the determination.

At the second workshop the teachers exchanged ideas on projects. For them planning the starting point when working on modeling was to decide what could be interesting for the students to research. We started by looking through the chapter on models in the textbook. For them many of the tasks were exotic and strange. They also felt that it would be much better and more attractive to make their own projects that were open. The ideas that appeared were on import of goods to Iceland, the cost of starting a new school, the cost and planning of Christmas and environmental things. For the teachers the solutions to these scenarios were students making models as formulas for growth. Christmas was coming so the teachers decided to work with the Christmas as a theme at the next workshop. They felt it important to make their own project and wanted to be creative, searching for work that would be challenging for them.

At the third workshop the teachers met in an excited state since they had all been thinking about the Christmas theme and how it could fit in their time schedule. They had not discussed with their students but reported that now the students were starting to think and discuss about Christmas traditions so this theme would interest their students. They all were looking forward to planning and hear what the other teachers' ideas about this theme. They started with the idea to find out about how much families were spending on Christmas. They discussed ideas about creating realistic scenarios of families or friends by making different groups of students representing different types of families or friends (with variations in the number of members and age). There was more discussion about the social aspects of this task and very little focus on the mathematical content of such a task. They soon found out that the theme needed to be narrowed down and in the end they made a task where a group of three to five students should make a plan for Christmas dinner, the three course menu and find out about the cost of buying the ingredients/raw materials for this group. The task consisted of the group having to decide what to buy and find out about the prices of things. They were later supposed to use the information to make a formula for the cost of the whole menu and each course so they could determine the cost for different group sizes. The general idea was to involve the entire class to discuss the information gathered by different groups to get a realistic idea about costs of such a dinner and to be more conscious about spending. Again, in this discussion there was very little thinking about modeling as the mathematical theme but more a discussion of practical things, the process of planning- i.e., the use of webpages, spreadsheets and arithmetic to find the relevant information. When reminded of modeling by the first author, they expressed that the information could be used to generalize and give examples of models that could be used to find out about the cost of a Christmas dinner for different group sizes and different menus. There was also a discussion about socio-economic status and the types of menus served. An ethical dimension came up with respect to displaying expensive menus to students from different socio-economic circumstances. In general, the teachers were conscious that any discussion about realistic problems brought an element of ethics which is an important consideration for any classroom with students from different socio-economic backgrounds and different traditions.

4.5 Discussion

Based on these workshops with the high school teachers we realized that there is no clear consensus on what is considered to be a creative modeling activity. By allowing the teachers to engage in a dialogue with each other and with us, it became clear that the modeling cycle represented in the textbooks did not necessarily play a big role in shaping teacher's views but it was used in a more practical sense. As found in the work of Lev-Zamir and Leikin (2013) teacher's declarative statements about creativity did not always coincide with the types of problem situations that were discussed. Tasks that revolved around calculating costs can be viewed as a procedural activity. Most of the teachers viewed models as something used in applied mathematics. However the dialogism between teachers and the researchers revealed that there was consensus in the need to give their students space to be creative which could in turn affect their interest. The teachers' goal was to inspire their students and help them to collaborate and connect their mathematical learning with their daily life. To this end models were viewed as one way in which this could be achieved. Even though teachers voiced the necessity to make appropriate tasks to motivate deeper mathematical thinking, as we have presented in these findings the tasks were more practical and not mathematically challenging. These views coincide with the recommendations on the four salient points for the incorporation of modeling in the curriculum (English and Sriraman 2009).

The teachers were aware of their ideas behind modeling without using the term "modeling" and saw that as a way for children to be creative. Modeling was something that called for them to come up with their own applied problems and not use problems from a book. For the teachers it was obvious or a fact that children should mathematize from problems that the teacher's created based on the contextual environment. This was the reason why they were not keen on using problems from the book. The modeling cycle served as a framework within which they were free to create their own problems. Teachers also believed for students to be creative there was a need for group work and free space. The word "creative" did not occur in the teacher's vocabulary and discourse when referring to modeling but very often words that were synonymous to it were mentioned. In other words creativity and mathematics were seen as being disjoint until the authors started using the word in reference to teachers creating their own problems and conferring ownership of the problems to the teachers. In the teacher's discussion on what constitutes a creative task there was less emphasis on modeling but more on group tasks about daily life with a lot of freedom for students to decide and be creative but maybe not so much on the mathematics, more on the ways to work with the tasks and how to present the results.

The teachers were interested in improving their teaching and viewed these workshops as an opportunity to get new ideas to better their teaching of the content. They also stated that it was important to have students that like to do mathematics and that according to their experience working with problems from daily life was interesting to achieve this goal. One teacher suggested "the starting point in math teaching could be daily life, and everything could be built up from that". This view is quite consistent with realistic mathematics education (Freudenthal 1972) and being conscious of the mathematical content that needed to be addressed. Another teacher said "it matters what kind of a message teachers give their students" and the necessity to use sufficient time on tasks to allow students to elicit their thinking and discuss their ideas with each other. The Freirean notion of dialogism was very strong in terms of both the teacher and the students learning from each other in a modeling activity. The findings in terms of creativity research, the focus on daily life and problems associated with costs and living suggest that teachers emphasized "little c" creativity (Richards 1993) to their students. The teachers were concerned about their students, were able to involve their students in meaningful discussions. Modeling for mathematics education researchers revolves around big data sets about larger societal concerns (whale population, deforestation, etc.) but for these teachers it was more about understanding the world around the students focused on problems from daily life. This finding suggests that the modeling problems in the textbook are not necessarily relevant to teachers given their contexts. One criticism of a modeling-based curricula is the emphasis on predicting the future through regression models but as we have seen in this study, a teacher is more concerned about the here and now relevant to 13–15 year olds.

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Chapter 5
The Prominence of Affect in Creativity: Expanding the Conception of Creativity in Mathematical Problem Solving

Eric L. Mann, Scott A. Chamberlin, and Amy K. Graefe

Abstract Constructs such as fluency, flexibility, originality, and elaboration have been accepted as integral components of creativity. In this chapter, the authors discuss affect (Leder GC, Pehkonen E, Törner G (eds), Beliefs: a hidden variable in mathematics education? Kluwer Academic Publishers, Dordrecht, 2002; McLeod DB, J Res Math Educ 25:637–647, 1994; McLeod DB, Adams VM, Affect and mathematical problem solving: a new perspective. Springer, New York, 1989) as it relates to the production of creative outcomes in mathematical problem solving episodes. The saliency of affect in creativity cannot be underestimated, as problem solvers require an appropriate state of mind in order to be maximally productive in creative endeavors. Attention is invested in commonly accepted sub-constructs of affect such as anxiety, aspiration(s), attitude, interest, and locus of control, self-efficacy, self-esteem, and value (Anderson LW, Bourke SF, Assessing affective characteristics in the schools. Lawrence Erlbaum Associates, Mahwah, 2000). A new sub-construct of creativity that is germane and instrumental to the production of creative outcomes is called iconoclasm and it is discussed in the context of mathematical problem solving episodes.

Keywords Affect • Creativity • Iconoclasm • Mathematics • Mathematical problem solving

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5.1 Introduction

Mathematics is a human endeavor. Yet, it is often portrayed in the K-12 classroom as a tool to communicate information or help answer questions encountered in daily life, overshadowing the creativity that spawned the rules and algorithms children work so diligently to master. Studies of the creative works of eminent mathematicians often mention curiosity and a willingness to embrace challenge as necessary attributes of creativity. Movshovitz-Hadar and Kleiner (2009) stress the importance of courage as well - both the social courage needed to take a risk and the intellectual courage to follow a path not knowing if the end result will bring success or failure. Several examples offered by Movshovitz-Hadar and Kleiner include the work of Janos Bolyai and Nicolai Loabachevsky. Both had the intellectual courage to ask, "What if parallel lines do meet?" and the social courage to share their work with the world. Both were discouraged from pursing this line of study. Bolyai was discouraged by both his father and Gauss who claimed to have made the same discovery earlier but did not seek to publish for fear of controversy. Loabachevsky was named the "madman of Kazan" when his manuscript was rejected by the St. Petersburg Academy of Sciences. Yet their courage to persist provided the geometric basis for the understanding of physical time and space (Cannon et al. 1997). For a more in-depth discussion of courage and mathematical creativity and several more examples of mathematical creations initially viewed as heretical see Movshovitz-Hadar and Kleiner.

A seminal work in mathematical creativity is Hadamard's (1945) essay, *The Psychology of Intervention in the Mathematical Field*, in which he summarizes and extends the work of others seeking to understand the process of mathematical thought. In that work, Hadamard discusses various types of mathematical minds and the products that they may create. He classified these mathematical minds as either logical (those that follow predetermined conventions, routines, or procedures) or intuitive (those that are often guided by common sense). One of the challenges for intuitive problem solvers, according to Hadamard, is that they need to have the courage to share their answers with peers and the mathematical community. As an example, Hadamard describes a situation in which a student, "guided by common sense, knew the right answer to my question, but did not feel he was allowed to give it and did not realize that …[it] could be easily translated into a rigorous and correct proof" (p. 105–106). He also shares a note found in Riemann's papers that read, "These properties of ς(s)…are deduced from an expression of it which, however, [sic] I did not succeed in simplifying enough to publish it" (p. 118). In the first example the student's lack of courage resulted in a lost opportunity to develop his mathematical talent and understanding. In the second example, Riemann's courage in sharing his work, even though he had not arrived at a publishable-ready expression, brought new insight to the study of prime numbers. For more on the search for a proof for the properties of the Riemann Zeta function, see the Clay Mathematics Institute's Millennium Problems.[1]

The difference between Hadamard's student and Riemann or Bolyai and Gauss was the courage to take a risk, to ask a question, to act on intuition, and to share ideas. Without that courage, potentially creative mathematical ideas remain unknown and unexplored; students are left to follow systematic solution paths without exploring the synthesis of thought that is necessary to develop mathematical understanding (Hadamard 1945). Hadamard refers to intuitive problem solvers working with almost reckless abandon in seeking solutions, and he borrows Poincaré's terminology when he refers to them as "bold cavalrymen of the advance guard seeking conquests" (p. 106).

Conquests are rarely achieved by simply doing what has always been done. Rather, mathematical conquests are gained by some level of risk taking (reckless abandon) and the courage to pursue a line of thought or an approach that often challenges iconic (conventional or traditional) practices and beliefs, hence an iconoclastic view of mathematical creativity. In this chapter the authors propose iconoclasm as a necessary fifth component in developing an understanding of mathematical creativity.

5.2 Creativity and Mathematics

Analysis of the literature affirms the fact that creativity is multifaceted in the field of mathematics education (Mann 2006, 2009; Sriraman 2006). Nevertheless, four components that recur in nearly all creativity literature are fluency, flexibility, originality, and elaboration. In this section, these four components are discussed.

Fluency, or the number of relevant responses that can be created by any one individual, is an indicator of creativity. Often compared to brainstorming, fluent thinkers are able to generate many ideas, possibilities, and potential approaches to finding solutions to a problem. Generation of ideas is the focus here, though once completed, creativity assessment evaluations do consider the relevancy of the responses. For example, in scoring a stimulus response on the Torrance® Tests of Creative Thinking figural forms (Torrance et al. 2008), the evaluator is instructed to score responses based on the relevancy and meaningfulness of the response.

Flexibility in thinking (Krutetski 1976; Torrance 1966) is considered one's ability to think about a problem solving task from more than one perspective and/or to reverse mental processes. It is not uncommon for problem solvers working on a task to be constrained by a preconceived solution path. This is especially true when the predominant "view of school mathematics is one of rules and procedures, memorization and practice, and exactness in procedures and answers" (Linquist 1997, p. xiv).

[1] http://www.claymath.org/millenium-problems/riemann-hypothesis

This is a limited view of mathematics in which mathematical creativity does not have the freedom to develop.

Ervynck (1991) outlined the development of mathematical creativity as a series of stages with algorithmic activity as precursors to creative activities. When producing creative solutions, it is necessary to be able to draw on the foundational knowledge of the technical and computational aspects of mathematics. However, when the emphasis does not transition to the next stage of development, the student is stuck in the view of mathematics as a world of right and wrong answers (Ginsburg 1996). When this happens, flexibility is generally precluded and individuals are locked into searching for the "right" solution path rather than looking for multiple paths to a solution. With sufficient mathematical knowledge and experience, flexible thinkers can evaluate the results for appropriateness *and* elegance. Developing adaptive expertise (National Research Council 2000) is important for successful learning: "Adaptive experts are able to approach new situations flexibly … they don't simply attempt to do the same things more efficiently; they attempt to do things better" (p. 48). Developing the ability to be flexible in one's approach to problem solving is essential for creative development. As an example, if a mathematical problem was provided in which most problem solvers used number sense to solve the problem, a flexible thinker may revise an initial solution to find a more efficient approach or look for connections to other mathematical domains such as statistics and probability or algebra. A highly flexible individual may chart new waters simply by having the courage asking 'what if' in looking for a better approach. Questioning the validity of Euclid's fifth postulate, the parallel postulate, was a significant 'what if' that challenged centuries of mathematical study and created new discoveries in mathematics with many applications in a universe much bigger than the ancient Greeks ever imagined. Flexible thinkers lend themselves to highly creative solutions due to their ability to think 'in addition' to the manner in which others might typically think.

Often the concepts of flexibility and fluency are confused. While fluency is considered the number of responses generated, flexibility is focused on the variety of approaches that an individual is able to use in solving a problem. From a research/assessment perspective, blurred lines between constructs can be problematic; in practice, the two constructs go hand-in-hand. As an example, the National Council of Teachers of Mathematics (2014) position statement on procedural fluency invokes a focused perspective on flexibility, calling it,

> …the ability to apply procedures accurately, efficiently and flexibly, to transfer procedures to different problems and contexts; to build or modify procedures from other procedures … building on familiar procedures as they create their own informal strategies and procedures.

Originality, the ability to create novel responses, was initially considered the only measure of creativity (Chassell 1916) and is likely the most regularly used synonym by those not familiar with creativity research. It may be common for teachers to only see highly creative products in mathematics classrooms as ones that are original or novel. The ability to create novel products (e.g., physical models, mathematical models, or on-paper prototypes) serves as one piece of evidence that creative potential exists. Along with the aforementioned manifestations of originality, it is important to note that mathematical processes, procedures, and algorithms also can be highly original. A view of originality as something new to the world on par with the works of Euler, Gauss, Cantor, and da Vinci, among many others, is a narrow view of the construct. Attaining this level of creative recognition, legendary Big-C status, (Csikszentmihalyi 1999) is a complex task often not achieved in a creator's lifetime. Kaufman and Beghetto (2009) offered a broader view in their Four C Model of Creativity that presents different dimensions of creativity and originality. In this model, the developmental progression of creativity is recognized as inherent in the learning process:

- Mini-c: Novel and personally meaningful interpretation of experiences, actions, and events.
- Little-c: Everyday expressions of novel and task appropriate behaviors, ideas, or products
- Pro-c: Expert expressions of novel and meaningful behaviors, ideas, or products
- Big-C: Legendary novel and meaningful accomplishments, which often redirect an entire field of study or domain

Assessments attempting to identify creative potential in individuals are focused on the mini-c and little-c levels of creativity. Society, however, generally assesses Pro-c and Big-C. As a short side conversation, some debate exists about whether a problem's solution needs to be useful and utilitarian to be considered creative. On one hand, Amabile (1996) asserts that a solution needs to be appropriate to the task, and Torrance (1966, 2008) asserts that solutions need to be interpretable, meaningful, and have relevant ideas. Sriraman (2006), however, argues that problem solving solutions can be highly theoretical and not have any immediate or direct applications. It may be the case that the application of a highly theoretical solution will realize its significance long after created (e.g., many decades), thus substantiating Sriraman's point.

For several years, creativity in mathematics was comprised of only fluency, flexibility, and originality (Haylock 1997; Kim et al. 2003; Tuli 1980). More recently, the notion of elaboration (Imai 2000) was connected to creativity in mathematics. Elaboration pertains to the ability of an individual to provide depth beyond what most problem solvers can provide in an explanation. Individuals with a high degree of elaborative skill may identify and be capable of expounding on intricacies of a solution that many peers may not recognize.

5.3 Affect and Creativity

Positive affect (feelings, emotions, dispositions and beliefs) have been associated with the creative process (Eubanks et al. 2010; Leu and Chiu 2015). Much of the research in this area is focused on developing a work place environment to encourage creativity (Amabile et al. 2005; Bledow et al. 2013) and in the field of social psychology (Baas et al. 2008; Jauk et al. 2014; Nijstade et al. 2010). A common premise in these studies is that affective states play a significant role in stimulating creative thinking and is a factor that can be influenced.

Amabile et al. (2005) studied the relationship between an individual's affective state and their daily creative activities in the work place. Their findings suggest that affect plays a more prominent role in organizational theories of creativity than previously thought and that these findings might extend beyond the study of creativity in the work place to a broader concept of the nature of creativity.

In their Dual Pathway to Creativity Model, Nijstad et al. (2010) theorized that creativity could be achieved via a flexibility pathway or a persistence pathway. The flexibility pathway acknowledges prior work in assessing flexibility, fluency, and originality with respect to creativity. This pathway is associated with breaking away from habitual thinking and the ability to switch flexibly between multiple approaches to a task. The persistence pathway acknowledges that creative ideas may also emerge as a result of hard work and systematic, in-depth explorations of a few perspectives. While one might infer that these two pathways are somewhat diametrically opposed, Nijstad, De Dreu, Reitzschel and Baas suggest that the creative individual may switch between pathways over the course of solving a problem. Applications of this model in seeking to improve creativity in the work place are readily apparent as are connections to several of the Mathematical Practices in the Common-Core State Standards – Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010).

While most of the work in this area is recent, Fiest's 1998 meta-analysis of personality and creativity concluded, "that in general, a "creative personality" does exist and personality dispositions do regularly and predictably relate to creative achievement" (as cited in Runco 2014, p. 267). Seminal work done by Donald MacKinnon at the Institute for Personality Assessment and Research (IPAR) still holds true today (Runco 2014). MacKinnon identified lability as a measure of creativity. His colleague, Harrison Gough noted that, "though there is a facet of high ego strength in this scale [lability], an adventurous delighting in the new and different and a sensitivity to all that is unusual and challenging, the main emphasis seems to be on an inner restlessness and inability to tolerate consistency and routine" (as citied in Runco 2014, p. 269).

The inability to tolerate consistency and routine is mirrored in Goldin's (2009) issues of integrity and intimacy. When mathematics is taught as mainly a series of rules and procedures, serious issues of integrity may arise for the child. She writes, "at some level, I would conjecture, the child *knows* that something is missing… children who have greater mathematical ability and potential for developing inventiveness are likely to have the more serious integrity issues around conceptual understanding" (p. 190). Eventually these integrity issues result in one of three outcomes for the child: (1) take a risk and pose a question, (2) accept the lack of meaning, or (3) assume that I am not good with math. Mathematical problem solving is an intimate task, one in which the individual invests a significant amount of attention and energy in seeking a solution. It takes social courage to present a new approach or idea publicly, especially when it may challenge the accepted truth as conveyed by the teacher or textbook.

Goldin's (2009) concluding thoughts connect to MacKinnon's construct of lability (see Runco 2014), to Nijstad et al. (2010) Dual Pathway to Creativity Model, and to Movshovitz-Hadar and Kleiner (2009) discussion of courage. She writes:

> Thoughtful attention [to] the affective domain can result, over many years, in a kind of *strength of* purpose in the pursuit of mathematical understanding. Then the growing child builds affective structures that literally last a lifetime, enabling continuing curiosity and mathematical persistence and perseverance, representing essential information, evoking powerful problem-solving heuristics and learning strategies, stimulating inventiveness and following out the resulting ideas, and promising the continuing thrill and long-term satisfaction associated with the achievement of new mathematical insights (p. 193).

5.4 Mathematical Problem Solving

In his work, *Mathematics as a Creative Art*, Halmos (1983) wrote:

> Mathematics – this may surprise you or shock you some – is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof. The conviction is not likely to come early – it usually comes after many attempts, many failures, many discouragements, and many false starts (p 256–257).

Mathematical problem solving (MPS) differs from the mathematical exercises on which K-16 students spend time working to develop mathematical technical skills, exercises often comprised of repetitive tasks in which students are asked to find answers to a series of short, similar exercises. MPS tasks used to develop creativity are not dissimilar from the work of mathematicians as described by Halmos. Indeed they "are ones for which students have no memorized rules, nor for which they perceive there is one right solution method. Rather, the tasks are viewed as opportunities to explore mathematics and come up with reasonable methods for solution" (Hiebert et al. 2000, p. 8).

As with the construct of mathematical creativity, MPS continues to be studied and the basic concept expanded. MPS simultaneously enjoys and suffers from multiple operational definitions. Chamberlin (2008) in a survey of 20 mathematics education experts from North America, Europe, and Israel, refined the conception of MPS in a Delphi Study, ultimately defining it as comprised of both processes and characteristics. The most commonly agreed upon processes were: (1) engaging in cognition, (2) seeking a solution, (3) communicating ideas, (4) engaging in iterative cycles, (5) defining mathematical goals, and (6) mathematizing situations to solve problems. Regarding characteristics, the best problem solving tasks can: (1) be solved with more than one tool and more than one approach, (2) be used to assess level of understanding, and (3) require the implementation of multiple algorithms for a successful solution.

5.5 Iconoclasm

“Dare to be a radical, but don’t be a damn fool,” (Baron as cited in Runco 2014, p. 275). In his writing about the relationships between creativity, personality and motivation, Runco shares with his reader descriptors of the creative individual from a variety of scholars such as independent, non-conformist, rebellious, unconventional, norm-doubting, and contrarian. Also in 2014, the authors[2] proposed adding iconoclasm to flexibility, fluency, originality, and elaboration as components to measure creative potential. Iconoclasm differs from the four previous components in that they are manners through which creative products are manifested during mathematical problem solving episodes, while iconoclasm is an affective state that must be met for creativity to emerge.

Iconoclasm pertains to affect (feelings, emotions, dispositions, and beliefs) more than it does to cognition, though the two are intricately intertwined. In essence, if one perceives either restrictive policies within an educational environment (such as strict adherence to textbook solution methods) or the classroom teacher (with all the “right” answers) as the ultimate authority, then the individual’s creative abilities will likely be curtailed. As iconoclasm pertains to the world of psychology of mathematics, MPS, and creativity, it may be considered unthinkable to challenge conventions in MPS as perhaps less than efficient. As an example, a teacher of young elementary students may be stuck with a partial products method to multiply several digit numerals because the textbook expects students to learn multiplication in that manner. However, a promising young mathematician may find a more efficient and insightful manner to conduct multiplication and share it with the class. When this is done, the act of iconoclasm has occurred.

Another way in which iconoclasm varies from the previous four subcomponents of creativity is that it precedes creative product output rather than being manifested in the products. That is to say that without iconoclasm, the remaining four sub constructs would never be considered germane to the study of mathematical creativity because, although there would continue to be mathematical products, few would be creative.

Mathematical problem solvers that have substantial levels of iconoclasm are theoretically more likely to recognize novel solutions, possibly because they may not experience (or have learned to embrace) high levels of anxiety, common to

[2] Chamberlin and Mann 2014.

many others, when solving a problem. Additionally, such problem solvers may choose not to employ commonly accepted algorithms simply to deviate from peer problem solvers, and they may do it with relatively low anxiety (a higher level of intellectual courage). Moreover, they may be inclined to share their solution(s) with peers with very little fear of how they will be perceived (high level of social courage).

Iconoclasm has its roots in religion and technically translates to the breaking of icons. The interpretation of iconoclasm from centuries ago pertains to tearing down or attacking cherished idols, a rebellious act viewed as dissension or heresy. More recently, the term iconoclast has surfaced and pertains to the person that precipitates or originates an act of iconoclasm. The two applications of iconoclasm, in religion and in the mathematical problem solving sense, converge in that in each instance, the commonality is courage to deviate from what is expected, while the default is to accept commonly held beliefs, such as algorithms, without question. In the religious context, courage is required to challenge cherished idols and is often undertaken at great risk to personal safety. Though situations in mathematical settings (e.g., a mathematics classroom) may not be life threatening, a problem solver also needs courage to deviate from the norm in order to identify *potentially* creative solutions and then to share such solutions with peers and teachers. The term *potentially* is important here – not all creative attempts at solving a problem will be successful – but having the courage to share an approach offers opportunities for collaboration and discourse that may eventually result in a successful solution (Halmos's vague guesses, broad generalizations, unwarranted conclusions).

For example, if two students are solving the same problem and the student with the higher degree of iconoclasm is confident in sharing a highly creative problem solving solution (e.g., one that is particularly original/novel with respect to the "taught" methods or other approaches in the classroom), then the teacher's willingness to listen to the solution and provide constructive feedback rewards the attempt and encourages further exploration. On the contrary, the student with a low degree of iconoclasm may be fearful that the teacher will not be receptive to an alternate solution, so the student may have little impetus to follow through or invest the energy in producing and/or sharing a highly creative response. If unsuccessful, it is not unreasonable to assume this student labels either the problem as too hard or his abilities as inadequate – both of which are unfortunate and avoidable outcomes. While the second student may benefit from observing the teacher's interactions with the first student, encouraging a more iconoclastic approach to future mathematical problem solving activities, it is all too common to see students disengage with an "I'm just not good at math" attitude when struggling with a problem. The teacher is thus challenged to use all the pedagogical skills at his or her disposal to encourage students to "solve problems in novel ways and post new mathematical questions of interest to investigate" (Johnsen and Sheffield 2012, p. 16).

In short, the teacher or learning facilitator needs to create a favorable environment (Amabile et al. 2005; Bledow et al. 2013; Goldin 2009; Hiebert et al. 2000; Merkel et al. 1996) for individuals to manufacture creative products. In this climate, all students benefit. However, in situations in which the teacher does *not* create a climate conducive to the emergence of creative products, it is likely that only students with a high degree of iconoclasm will be inclined to develop creative products, despite the fact that students with low levels of iconoclasm may have similar levels of creative potential. This is because individuals that are not fearful of higher authorities or peers likely have lower levels of anxiety, higher self-efficacy and self-esteem, and a better attitude about mathematical problem solving.

Fluency, flexibility, originality, and elaboration are manners in which creative products are measured and have been investigated to a large degree. Individuals that study creativity in mathematics value creative output by asking whether such products were highly novel (original), added to the number of solutions (fluency), exhibited high degrees of flexibility in thinking, or added to the depth of explanations (elaboration). Runco and Albert's (1986) Threshold Theory of Creativity found a relationship between creativity and intelligence up to a point (~120 IQ). Feldhusen and Westby (2003) determined that an individual's knowledge base is the fundamental source of his or her creative thought. Bern (2008), writes about his efforts to understand the neurological base of creativity in his non-technical book, *Iconoclast: A Neuroscientist Reveals How To Think Differently*. Bern's book reviews the lives of several well-known iconoclasts and introduces the reader to the field of neuroeconomics, a multi-disciplinary discipline less than two decades old, that seeks to understand human choice and decision making (for a brief history of neuroeconomics see Glimcher et al. 2009). While Bern does mention creativity briefly, there is no connection made to creativity research literature. That said Sternberg (2009) finds merit in Bern's work, especially the recognition of iconoclasticty as a quality that can be developed, a necessary condition to the study of means to develop creativity by encouraging iconoclasm. Combined, these various approaches to understanding the nature of creativity suggest that a combination of some degree of intelligence, knowledge, and iconoclasm are necessary for creative products to emerge from problem solvers. While there is a body of research to support the first two conclusions, research is needed to develop the means to assess an individual's level of iconoclasm in mathematical problem solving situations and to explore the relationships with other contributing factors.

5.5.1 Examples of Iconoclasm in Mathematics

It appears as though iconoclasm has always been a trait of creative mathematics because each novel revolutionary mathematical idea is met with skepticism, and it often requires significant time for the field to accept the new paradigm (Movshovitz-Hadar and and Kleiner 2009). Negative numbers, for example, were initially thought to be a pointless idea and therefore not relevant to mathematics. The Chinese were originally credited with conceiving the idea of negative numbers, though the Greeks (Diophantus, specifically) used them with some degree of regularity to explore concepts in what is now known as algebra some 500 years later (Rogers 2014). In 620 CE, the Indians saw use for them in the context of fortunes and debts. Despite Greek use of negative numbers, Europeans were not wholly accepting of them until around 1400 AD (Rogers). As recent as the 1800s, only 200 years ago, some such as Carnot (in 1803) and Busset (in 1843) did not accept negative numbers as a workable concept in mathematics (Boyé no date). In fact, Busset saw negative numbers as the reason that mathematics was difficult to teach. Busset went so far as to mention that mental aberrations, such as the concept of negative numbers, might prevent gifted minds from studying mathematics (Boyé). It is thus readily apparent that a significant portion of mathematics, commonly accepted by today's mathematicians and introduced to students in grade 6 (Common Core Standards Writing Team 2012), was, at one time, a subject of much debate.

A characteristic that mathematically creative individuals may possess is iconoclasm. This is because they may feel more comfortable or less anxious posing solution paths to problem solving tasks that are not commonly accepted solutions than peers without similar levels of courage. Few have discussed the reason for the emergence of creative output. In other words, why does a mathematical problem solver pursue highly creative solutions? This is the point at which iconoclasm, as a necessary aspect of creativity, begins to take meaning. Within the construct of iconoclasm, individuals seek creative and innovative solutions to mathematical problems because they feel that the currently agreed upon solutions are not adequate or they are convinced that their solution can add to the body of knowledge in a particular mathematical content area. In some cases, iconoclastic problem solvers (i.e., those with high degrees of iconoclasm) may seek innovative solutions because they want to stand out among colleagues. Few educational psychologists have contemplated the reason why creative products emerge from individuals. In considering one's motive to create a new solution, the psychological constructs of affect in MPS (McLeod and Adams 1989), such as feelings, emotions, and dispositions, are considered relative to why individuals are creative. With the commonly used constructs of creativity (e.g., fluency, flexibility, originality, and elaboration), the discussion of why creativity emerges is rarely considered. With motivation being the sum-total of affect (Anderson and Bourke 2000), the theory of iconoclasm suddenly warrants serious consideration as a component of creativity because it explains why problem solvers create solutions to tasks. More specifically, problem solvers with high degrees of iconoclasm may be motivated to be different and/or more efficient, explicit, or novel than peers.

Giftedness, like mathematical problem solving, has many conceptions. A commonly accepted conception of giftedness is Renzulli's Three-Ring Conception of Giftedness (1978) in which he defined it as being comprised of above average ability, task commitment, and creativity. Regarding task commitment, Renzulli (1998) uses the terms perseverance, determination, dedication, high levels of interest, enthusiasm, and fascination, all of which are components of motivation. Subsequently, Renzulli added co-cognitive traits through his Operation Houndstooth (Renzulli 2002) research. Each of the six areas outlined by Renzulli in Operation Houndstooth (i.e., optimism, courage, romance with a discipline, sensitivity to human concerns, vision, and physical/mental energy) has a strong connection to the affective domain and adds fuel to the discussion of why problem solvers seek solutions.

Baer and Kaufman's Amusement Park Theory of Creativity (2005) seconds this notion with initial requirements for the emergence of creativity, or as they call it, the 'ticket into the park,' which includes intelligence, motivation, and the environment.

5.5.2 Relationship of Iconoclasm to Fluency, Flexibility, Originality, and Elaboration

Given an established relationship between iconoclasm and motivation, one may wonder about the interrelationship between iconoclasm and the remaining four components of creativity (i.e., fluency, flexibility, originality, and elaboration). The theory of iconoclasm as a fifth construct of creativity in mathematics helps researchers understand the connection of the remaining four. This is because it explains the motive for creativity as Forgeard and Mecklenberg (2013) attempted. Critical to their work was the component of intrinsic motivation in relation to the generation of creative products. Further, they utilized the concept of pro-social motivation (Grant and and Berg 2010), which Forgeard and Mecklenberg distinguish as "one's desire to contribute to other peoples' lives" (pp. 255, 257). When iconoclasm is viewed relative to motivational factors, for example pro-social motivation, its relationship to the four subcomponents of creativity is revealed; the iconoclast has an intrinsic need to find a better solution. Prior to realizing that iconoclasm may have this degree of interaction with the model of creativity, the four accepted sub-constructs were largely disparate facets (with the exception of flexibility and fluency in thinking).

While examples abound of iterative improvements in things that were once "good enough" in the our daily life (e.g. each new generation of cell phones is more efficient, multidimensional, smaller, and faster), often in a classroom setting finding the anticipated answer by a prescribed path is highly valued. The motive for individuals seeking more than one solution, or fluency, to a problem may be to identify the most efficient solution, thus placing themselves in a position to capitalize on the most sophisticated and quickest method in future situations. In many cases, the impetus for fluency, or a greater number of options in solution paths, pertains to problem solvers wanting more options than they currently have. To accomplish this objective, problem solvers may need to be iconoclastic and challenge the system (e.g., teacher or textbook) in order to identify a greater number of solution paths. In many cultures, be it a country, classroom, or work environment, it may be unacceptable to challenge a system that is already producing *adequate* results. Iconoclastic problem solvers, however, may feel that "good enough" just is *not* good enough and thus seek a more efficient solution path than what their environment accepts. Fluency and flexibility are intricately intertwined in the respect that individuals may seek novel perspectives because they feel the process may prove fruitful in a particularly original solution path. The solution path may have positive social outcomes thus reinforcing the decision to challenge the system.

The use of multiple representations may help develop an increased number of solution paths. As an example, van Dyke and Craine (1997) suggest that algebra has at least four representations (i.e., verbal statement or text, equation, table of values, and graph) that can be utilized to solve a problem. Elaboration during problem solving may provide opportunities to create new solutions because, in explaining solutions, flaws may be uncovered. Ideally, through this process, novel solution paths are precipitated. After all, if the previous solutions were acceptable and worked well, the impetus to explore additional representations may only emerge if there are reasons to examine the processes used in depth.

In identifying such inadequacies in commonly accepted solutions or algorithms, creative problem solvers are intrinsically driven, or have a motivation, (Forgeard and and Mecklenburg 2013) to identify a more creative, sophisticated, or innovative solution because the need exists, if only from a pro-social perspective. They may also develop the solution out of pure enjoyment or aesthetic appreciation of mathematics as Krutetskii (1976) suggests. Csikszetentmihalyi (2014) writes, "the creative person cannot be entirely invested in the commonly accepted conceptual configurations of his or her domain…a creative person should be dissatisfied with the state of knowledge and be motivated to search for alternatives" (p. 164). This dissatisfaction may manifest itself in a young child inventing her own multiplication and division algorithms (Ambrose and et al. 2003) or in a seasoned mathematician questioning a long accepted "truth" in mathematics and disputing it.[3] Alternatively, highly creative individuals in mathematics may opt not to seek alternate solutions in MPS situations if they perceive the problem as uninteresting, lacking challenge, and thus unworthy of devoting additional creative and cognitive resources or if they are sufficiently pleased with the solution provided. In this respect, the iconoclastic nature of a creative problem solver must be awakened to realize a need for a *better* solution.

5.6 Areas for Future Research

One of the aspects of educational psychology that makes it a well-respected discipline is the reluctance of the field to accept newly proposed theories without empirical evidence to substantiate them. Consequently, researchers interested in iconoclasm in mathematical problem solving have several options. Foremost among them is the development of an instrument to investigate whether iconoclasm, in the form of challenging commonly accepted algorithms, is something that mathematics problem solvers will embrace when faced with a relatively inefficient solution. The work of Leu and Chiu (2015) and Tjoe (2015) are moving the field in this direction. Chamberlin (2010) has developed the Chamberlin Affective Instrument for

[3] Euler's sum of power conjecture stood for almost 200 years before a short paper (two sentences) was published in the Bulletin of the American Mathematical Society disproved the theory (Lander and and Parkin 1966).

Mathematical Problem Solving (Chamberlin and Powers 2013), and the authors are now working to construct and validate an integrated assessment tool. Paper and pencil assessments often lack the "intimacy" needed to understand the thought process and affective engagement involved in creative problem solving activities. Krutetskii's (1976) approach of having mathematical problem solvers create solutions, look at their solutions for evidence of creativity, and then interview such individuals in an attempt to understand their thought process is difficult to use with large sample sizes, yet offers the opportunity to explore deeply the level of iconoclasm in individual responses. In the end, the construct of iconoclasm needs to be empirically tested, and while several prospective approaches appear to exist for such an investigation, multiple studies are needed.

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Chapter 6
When Mathematics Meets Real Objects: How Does Creativity Interact with Expertise in Problem Solving and Posing?

Florence Mihaela Singer and Cristian Voica

Abstract The paper analyzes the results of activities undertaken by Mathematics students enrolled in a pre-service teacher-training program. Students were given the task to describe the way of building a figure from which one could get a box, to identify the geometric properties that allow producing the box, and to propose new models from which new boxes can be obtained. For creativity analysis, a cognitive flexibility framework has been used, within which students' cognitive variety, cognitive novelty, and their capacity to make changes in cognitive framing are analyzed. The analysis of some specific cases led to the conclusion that creativity manifestation is conditioned by a certain level of expertise. In the process of building a solution for a nonstandard problem, expertise and creativity support and mutually develop each other, enabling bridges to the unknown. This interaction leads also to an increase in expertise. Moreover, in order to get individual relevant data, the identification of creativity should take place based on tasks situated in the proximal range of the person's expertise but exceeding his/her actual level of expertise at a time.

Keywords Mathematical creativity • Modelling • Cognitive flexibility • Expertise

6.1 Introduction

What is the relationship between expertise and creativity? This is a question that has generated lots of controversy in literature over time. Some authors (eg Diezmann and Watters 2000) argue that expertise is a precondition for creativity. Other authors (eg Craft 2005), accepting the existence of "small *c* creativity", say the contrary, arguing that because creativity can occur in any person, we must accept a spectrum of knowledge – therefore of expertise, in connection with creativity (Craft 2005).

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We started our project with the intention to answer the question: *How does students' mathematical creativity manifest in a context in which technology and modelling interact with theoretical mathematics?* Our study was initiated by the fact that we noticed very different behaviors in terms of creativity when placing a group of students - prospective mathematics teachers in a context of problem solving and problem posing that involves modelling. Thus, while progressing with our analysis, another question became dominant, which actually includes the previous one: *How does creativity interact with expertise in problem solving and posing?* As a result, we intend to study the link between creativity and expertise in a complex situation, which occurs in a context combining problem solving, problem posing, and modelling.

6.2 Framework

6.2.1 Problem Solving and Problem Posing

In his well-known book *How to Solve It*, Pólya (1945) identified four steps in solving a problem: (i) understanding the problem; (ii) developing a plan; (iii) carrying out the plan; and (iv) looking back. Subsequently, lots of frameworks have been developed for studying the problem solving process (eg Schoenfeld 1992).

We have found a variety of approaches for studying problem posing in the literature, as well (eg Brown and Walter 2005; Jay and Perkins 1997; Singer et al. 2015). In this paper, we accept Silver's position, stating that problem posing refers to the generation of (completely) new problems, and also to the re-formulation/modification of given problems (Silver 1994). We specifically address here the context of problem modification.

A conceptual cognitive framework for problem solving, with various applications in problem posing was developed by Singer and Voica (2013). This framework highlights four operational categories: decoding, representing, processing, and implementing (Singer and Voica 2013).

6.2.2 Mathematics Modelling

Mathematical modelling can be seen as a process of translating between the real world and mathematics in both directions (Borromeo Ferri 2006). In recent years, the following description of an ideal modelling process (according to Blum and Leiss 2007) is frequently discussed: starting from *a real world situation*, this is simplified and/or structured: one thus arrives to *a real model of the situation*. This is transposed in a mathematical language, thus generating *a mathematical model*. The processing of the mathematical model leads to some results, which are then interpreted and validated into the real situation.

In the present study, students had to describe mathematically a real object – therefore to build a mathematical model of the real object, and then to extend the model so that to design new more complex objects. Using Kaiser and Sriraman's (2006) terminology, this type of task is framed into realistic or applied modelling (solving real world problems, understanding of the real world, promotion of modelling competencies).

6.2.3 Creativity

Creativity had long been viewed as a domain-general phenomenon. However, recently, new evidence show that creativity is not only domain-specific, but it even seems to be task specific within content areas (eg Baer 2012).

There is no consensus concerning the definition of creativity and its framework of study; there is no consensus in studying mathematical creativity either. There is however certain consensus regarding the difference between (advanced) research mathematicians creativity –considered as "extraordinary" or "absolute" creativity, and creativity in school mathematics – part of "everyday" or "relative" creativity (eg Craft 2003; Lev and Leikin 2013; Sriraman 2005). In addition, "big "C" creativity" and "small "c" creativity" are largely discuss (eg Bateson 1999; Gardner 2008).

Usually, creativity is studied starting from Torrance's tests, which is based on four related components: fluency, flexibility, novelty, and elaboration. Starting from here, various frameworks for studying creativity have been generated, usually adapted to specific types of tasks.

For problem solving context, Leikin (2013) uses multiple-solution tasks as a lens to observe creativity. The interplay between individual and expert solution space is an expression of creativity and the dimensions of her model are originality, fluency and flexibility, which are aggregated into creativity score by a research-based and, subsequently refined, scoring technique.

The construct of *spaces of discovered* properties is at the core of a new framework (Leikin and Elgrabli 2015), advanced to explore the complex relationship between creativity and knowledge in the context of an investigation task set in a dynamic geometry environment. The discovered properties were assessed from the point of view of their novelty, complexity of auxiliary constructions, and the complexity of their proofs.

For problem posing context, Kontorovich and Koichu suggested a framework based on four "facets": resources, heuristics, aptness, and social context in which problem posing occurs (Kontorovich and Koichu 2009). A more recent refinement of this framework has integrated task organization, knowledge base, problem posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness as parameters in analyzing creativity in problem posing situation (Kontorovich et al. 2012).

A different approach to creativity, one based on organizational theory, has been taken by Voica and Singer (2011, 2013). Their framework relies on the concept of cognitive flexibility. Cognitive flexibility is described by: cognitive variety, cognitive novelty, and changes in cognitive framing. Cognitive variety manifests in the formulation of different new problems/properties from an input stimulus; cognitive novelty captures the innovative aspect in the posed problem – its distance from the starting element; while changes in the participant's mental frame refer to shifts in the "on-focus" elements during the problem posing. Thus, cognitive flexibility arises as a complex, non-linear interplay between these dimensions. Consequently, the construct of cognitive flexibility opens up the possibility to capture different ways of being creative, namely through the differing loads on the three dimensions.

In the present study, we use the cognitive-flexibility framework in analyzing data. We consider that this framework better corresponds to our case, in which communication tasks related to problem solving, problem posing and modeling of problem situations occur. By using this framework, we can capture, beyond mathematical creativity, implications related to communication and social interactions reflected in problem posers' cognitive approach.

6.2.4 Experts Versus Novices

Expertise implies the existence and use of two types of knowledge: explicit knowledge of facts, principia, formulae pertaining to the domain, and implicit knowledge of how to operate with them (Sternberg 1998).

Glaser (1999) argued that, because self-monitoring – the ability to observe and, if necessary, reshape one's performance – is a hallmark of expertise, this skill should be emphasized in instruction. How to arrive at doing these in the real classroom? Although a very tempting concept from the point of view of artificial intelligence, the idea of expertise was not very much explored in psychology in relation to education. The criteria developed by Glaser (1988) for comparing experts and novices are still valid. Glaser characterizes expertise through six features ("generalizations" in Glaser's terminology: knowledge organization and structure, depth of problem representation, theory and schema change, proceduralized and goal oriented knowledge, automaticity, and metacognitive self-regulatory skills; because we use these features further, we detail them below.

- In terms of knowledge structure and organization, the expert has structured information items that are integrated into previous knowledge organizations so that they are rapidly selected from memory in large units, while novices possess punctual knowledge, consisting of isolated elements that display a superficial understanding of domain-specific key concepts and terms. (A)
- Regarding the complexity of problem-solving representation, the novice solves a task starting from its surface features, while the expert makes interferences and identifies principles underlying the surface structures. (B)
- In changing thinking schemes, the expert amends his/her own knowledge theories, and develops schemes that facilitate more advanced thinking, while novice manifests rigidity in changing a thinking scheme. (C)

- In terms of goal-oriented procedural knowledge, the expert displays functional knowledge, while novice possesses information without clearly understanding the applicability conditions. (D)
- In terms of automation that reduces the concentration of attention, an expert can focus attention while alternates between basic capacity and higher levels of strategy and understanding, using automate thinking to achieve good performance, while novices have difficulty in sharing attention, they frequently get lost in details and are unable to concentrate on essential facts. (E)
- Regarding metacognitive capacities of self-regulation, the expert check rapidly and intuitively the solution to a problem, proves accuracy in judging its difficulty, in assessing own knowledge and understanding, can ask questions, predict the outcome of the work, and use time effectively, while the novice tackles a linear approach, without looking ahead, and without controlling timing and work outcomes. (F)

6.2.5 Expertise and Creativity

During this study, we have started by exploring students' creativity and we came up by analyzing the students' level of expertise. We therefore ask ourselves: what is the relationship between these concepts? There are conflicting views about it, depending on how creativity is defined, but also depending on the domain being surveyed. We will further refer to creativity in school mathematics. For Diezmann and Watters, for example, for a student to be creative, he/she needs some intellectual autonomy and expertise (Diezmann and Watters 2000). Expertise is therefore seen as a necessary precondition for the manifestation of creativity. In his studies, Baer nuanced this relationship: he admits as obvious that some degree of expertise is important for the expression of creativity, but the question is what kind of expertise is required in a particular domain (Baer 1998, 2010).

On the other hand, Craft (2005) admits that every student is capable of creative manifestations; the consequence would be that expertise is not absolutely necessary for the manifestation of creativity or, at least, that we should accept a spectrum of knowledge at different levels.

6.3 Method

6.3.1 Sample and Task

The data comes from students in mathematics – prospective teachers who have received the same task during a Mathematics Education course. The task (listed in Annex) had two parts. In the first part, students approached a task of communication ("telephoned" description of a geometric configuration – Fig. 6.1.) consisting of producing a list of instructions based on which an interlocutor who did not have access to seeing the configuration have to reproduce it. After finishing this activity,

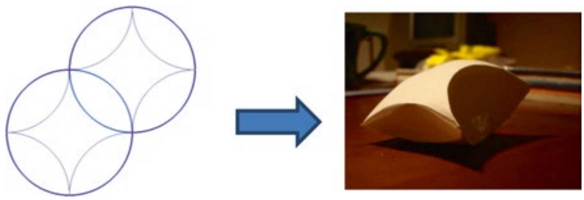


Fig. 6.1 The images initially showed to students (the box was presented as physical object, in the classroom)

students had to interact with the person "at the other end (of the phone)", and to improve the instruction list taking into account the received feedback and, eventually, to validate the new list of instructions with another partner (the validation consisted of that the partner was able to make an object that meets certain geometric properties).

In the second part of the task, students explored geometric properties of the given configuration, and tried to develop generalizations.

For the first part, students could work in groups of two, while for the second they had to work individually. There were students who preferred to work alone for the entire task. To solve the task, the students had a period of three weeks. In total, 26 students responded to this task: they constitute our sample for research.

6.3.2 A Modeling Context

The task proposed to students involves a modelling process. This is because, in a first phase of the task, properties related to the technological process for obtaining the box are to be interpreted in mathematical terms; thus one builds the mathematical model of the real object. In the second phase, this model was faced up with the possibility of extension, which allows obtaining new objects of the same category. The validation of the new mathematical proposals was made by obtaining geometric configurations and the actual construction of new boxes.

In achieving the mathematical model, students were exposed to a context of communication and social interaction, which led to the description of the model in an implementable technological manner (the students listed the steps of a technological process). This is another argument for interpreting the task as being a modelling one.

6.4 Results

6.4.1 What Elements of the Geometrical Configuration Were Relevant for Students?

Students' instruction lists and their recommendations for constructions show that they focused on the decomposition of the given figure in certain components. We briefly present the elements that students highlight in formulating instructions

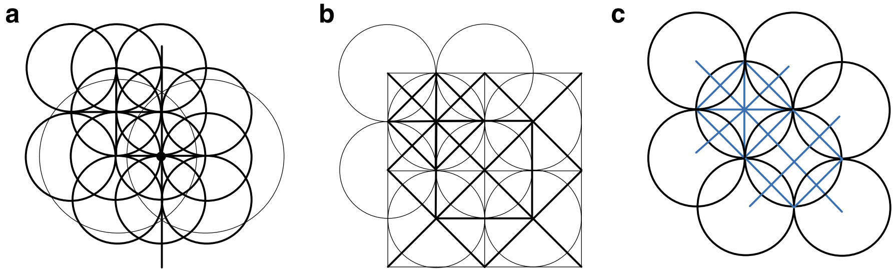


Fig. 6.2 Configurations that students perceived within the initial figure: (**a**) network circles; (**b**) squares and inscribed circles; (**c**) squares and circumscribed circles

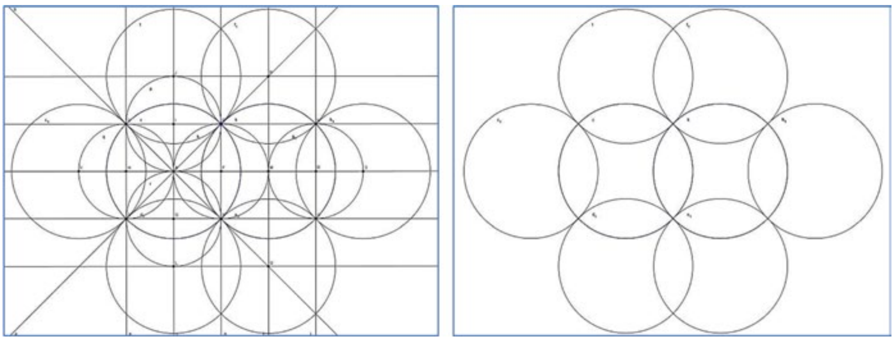


Fig. 6.3 Drawings made by Miron and Ana using GeoGebra

for (telephoned) reproduction of the given figure and for obtaining the box. (A broader discussion on the results presented in this paragraph is found in Pelczer et al. 2015). We have seen a variety of starting approaches. Below, there are a few selected. In the issued instruction lists, students frequently showed some networks/ tessellations of the plan, which guided the construction achievement. Most often, there is about a network of circles or plane coverage with squares and circles inscribed or circumscribed to them. Figure 6.2 shows three configurations that students perceived within the initial figure, namely: A) network of circles; B) squares and inscribed circles; C) squares and circumscribed circles.

A particular situation occured in the response given by one of the teams who used GeoGebra (although this software is not recommended by the curriculum). The team Miron & Ana included Fig. 6.3 in their solving. Here, the first figure shows a plan coverage with squares and circles inscribed and circumscribed to them, while the second figure (“clean and ready to cut”) only highlights a pattern of circles.

6.4.2 What Geometric Properties Do Students Identify?

For identifying geometrical properties it is not enough for the students to observe the initial configuration because the task statement does not contain data about the figure; they need to translate facts related to the technological process into a mathematical language. Therefore, in describing the configuration, students had to identify dominant perceptual elements of the mathematical model.

For example, students have noticed that to obtain the box, some parts of the figure should coincide when overlapped. "Perfect" overlap was expressed in some cases through congruence. There are also cases where students retain from overlapping parts of the figure only the equality of their areas: the mathematical property identified in this case is "weak" because it does not translate, in mathematical terms, the complexities of the real object. In other words, in this case the properties suggested by students do not allow a unique characterization of the given configuration, but have degrees of freedom that lead to a broader class of configurations. Consequently, two categories of properties that students remark within the given configuration occur: strong properties and weak properties.

More precisely

Strong property: is part of a mathematical model that uniquely characterizes the initial figure from which the box is obtained. In other words, it is a property belonging to a minimal set of necessary and sufficient conditions that ensure identical reproduction of the object.

Weak property: expresses mathematical features necessary but not sufficient, of the given figure. In other words, it provides a class of possible configurations of the given basic elements in which the initial configuration is found, but one can find there other configurations as well.

Table 6.1 shows the geometrical properties identified by the students from our sample through the model specifications that allow building the box. For the clarity of presentation, we organized the students identified properties into 5 categories of content. We have also selected some significant comments of students for the characterization of the respective property. They reveal types of constraints identified in the mathematical model, which conditioned the making of the box.

Most students identify, in the given configuration, equal circles and regular polygons. Out of these, some remain in the straightedge-and-compass constructability, ie they focus on polygons that can be built in this way. Table 6.1 shows separately content categories inscribed/circumscribed and regular polygons. Although this seems to be a redundancy, because any regular polygon is an inscriptible one, we distinguished among these categories because while some students consistently use circles to build polygons, others operate with regular polygons without needing the support of a circle.

Table 6.1 Dominant features of the mathematical model used for building the box

Dominants identified by students	Weak properties	Strong properties
Highlighting congruence	The lateral faces ("lenses") have equal surfaces.	Lateral faces of the box are congruent figures.
	"*When pasting the figure it should perfectly overlap*" – Paul	"*The figures formed by the intersections of circles are all congruent each other – so we can put perfectly on each other and form the sides of the box*" – Andreea
	The number of "convex lens" is even.	Interior arches are congruent with each other and are congruent with large arcs on the circles.
	"*Because they overlap two by two*" – Paul	"*Interior arches are equal to itself – otherwise, bonding would not be possible*" – Catalin
Emphasis on geometric transformations	The plane figure has "stability" in rotations towards the centers of the circles – Cristina	Figure axis of symmetry is the common chord "*If you fold on the dotted line, figures overlap*" – Anca
	The figure has as a symmetry line the centers line. – Dana	"*It helps to assemble the box*" – Rodica
	The faces of the box have symmetry axes. – Madalina	The second circle is a translation of the first circle – Adriana
Emphasizing tessellations	The squares used have sides equal to the diameter of the initial circle – ie one can use the circle inscribed in the square.	
Highlighting inscribed/ circumscribed polygons	Some polygons are cyclic.	The property of inscriptibility essentially intervenes in the square.
	"*In a circle a polygon can be inscribed*" (Georgeta)	"*The Square fits" perfectly "in a circle*" – Andreea
Emphasis on regular polygons	Square	Regular polygons can be constructed with compass and straightedge.
	"*We can see equidistant points corresponding to a square*" – Catalin	"*In fact, the essential property in the construction of this figure is breaking the circle into four equal arcs, namely the opportunity to build the angle* $\frac{\pi}{2}$. *Reformulated, it is constructible regular polygons*" – Miron. (Miron states this without being asked a generalization at this stage.)
	"*Square is a regular polygon*" – Gabriela	

6.4.3 What Changes Do Students Propose to the Initial Figure to Get Other Boxes?

The second task proposed to students required from them to alter the initial figure to get two new boxes of different shapes. To meet this requirement, students had to consider the mathematical model (reached by identifying the properties of the original figure), to extend/modify the model, and to validate the new model by effectively obtaining new boxes. Table 6.2 shows the dominants of the mathematical models used by the students in our sample to develop other types of boxes, different from the original. The dominants are given in terms of geometric properties that students perceived as essential in guiding the transfer from the initial object to new constructions.

Table 6.2 Dominants of the mathematical model used by students in the generation of new boxes

Dominant used by students	Solution – the modified box	Nr of stud arriving to the solution	Comment
Focus on the net of a solid	icosahedron	**1**	Concentration on the final product, they just keep the idea of container.
	dodecahedron	**1**	
	parallelepiped	**1**	
	(regular) octahedron	**2**	
	right-regular pyramid with congruent edges	**1**	
	cylinder	**1**	
Plan coverage with regular polygons	triangular box	**1**	Students use tessellations with squares or equilateral triangles.
	"spectacle case" box	**3**	
	hexagonal box	**1**	
	"heart-shaped box"	**1**	
Focus on inscriptibile/ circumscriptibile polygons	regular octagon	**1**	Metric aspects are ignored; for the first three cases, the common chord is a diagonal in polygons, not a side.
	regular dodecagon	**1**	
	regular 16-gon	**1**	
	equilateral triangle	**1**	
	regular hexagon	**1**	
Emphasis on the use of a regular polygon	regular pentagon	**2**	To achieve the figure, students use practical tools (ruler to scale, protractor, square ruler) or technology (GeoGebra)
	regular hexagon	**2**	
	regular octagon	**1**	
Focus on constructability with compass and straightedge	regular pentagon *and* hexagon	**1**	Students presented effective (ideal) constructions, using onlycompass and straightedge
	regular hexagon *and* octagon	**8**	

We have identified three types of approaches used by students in modifying the initial given figure.

(a) **A Theoretical Approach**
This approach is characterized by "perfect" figures: the students that adopted this approach propose changes related to the idea of regular polygon that can be constructed with compass and straightedge. Typically, students who have this approach minimally change the initial context, they just change the number of sides. In general, these students did not pay attention to the practical purpose of the task, focusing on the rigor of the mathematical constructions.

(b) **A Technological Approach**
Students who adopt this approach are not interested in the rigorous construction of the figure because they have alternative instruments (ruler, protractor, or square ruler; graphic computer programs), and the focus is on obtaining the final product. For these students, the practical verification (even if there are flaws in combining the elements to obtain the product) replaces proof and argumentation.

(c) **Focus on plane figures**, **with no analogical 3D transfer**
Some students retain from the task only that we want to form "a container". These students went back to their basic knowledge (such as the classical net pattern of a cylinder or octahedron), actually neglecting the task constraints.

6.5 Discussion

6.5.1 Some General Comments

We will comment on the geometric properties identified by students (Table 6.1) and on their perceptual clues in generating new boxes (Table 6.2) from the view of modeling. We note that geometric transformations have not been used to generate new configurations: they just remained at the level of the language used by students to describe the mathematical model. The properties that highlight the congruence of elements of the original figure was obtained by the mathematical translation of a technological process (the effective realization of the box), while the students who relate to an unfolded net of a solid as a way of generating new "products" seem to retain only this aspect – ie that the connection plane-space goes through unfolding and make a transfer conditioned by this stereotype.

Most of the students' proposed changes (18 new proposals) are based on regular polygons, constructible with compass and straightedge. In fact, starting from square (seen as a regular constructible polygon), students undergo a process of generalization and propose in 8 out of the 9 cases, boxes that use regular hexagons and regular octagons. For these students, we found a certain automatism: they use an algorithm corresponding to a general property (constructability of polygons with compass and straightedge).

Plane tessellation with regular polygons as a dominant feature of the mathematical model represents a creative potential, yet untrained in the Romanian school. Some students retain that, following the instructions indicated by them, a square coverage of the plane appears as background. Subsequently, they do coverage plane with squares or equilateral triangles and, starting from this background, they propose new geometric configurations that can lead to obtain boxes.

The weak properties identified by students appear in an incomplete mathematical modelling. However, they allow more degrees of freedom, because they can lead to a wider class of geometrical configurations: therefore, they have the potential to facilitate a more creative approach. Strong properties usually lead to a mathematical model very well-articulated. The existence of this model seems to be sufficiently rigid to direct the solution and push students towards a theoretical approach. We note that, the more theoretically advanced is the mathematical model (as in the case of the theorem of characterization of regular constructible polygons), the stronger it controls generalizations. As a result, although in this case many potential solutions appear, they follow the same pattern, they are in the same equivalence class, so once the student has demonstrated mastery of this instrument, his/her results cannot be recorded as cognitive variety.

6.5.2 A Few Case Studies

The sample of students used for this study is relatively small. Therefore, a quantitative analysis would not be relevant. On the other hand, we try to understand the relationship between creativity and expertise. Both features can be better captured by analyzing individual student responses. Therefore, we further include case studies in which students discuss how they have responded to the task, from two perspectives: proven expertise in the formulation of solutions, and their degree of creativity. We will try every time to identify, in student's cognitive behavior, evidence for the criteria that distinguish between novice and expert, detailed in Sect. 6.2.4, and the cognitive flexibility components, within the framework used to identify creativity. In some cases it was possible to make, for a student, clear distinctions novice – expert or creative – uncreative. There are also situations where, based on available data, we could not make such distinctions.

6.5.2.1 Case 1 (Emilian)

Emilian identifies the following geometric properties of the given figure: the points on the two circles are equidistant, forming two squares; the interior arches are congruent with the "proper" arches; inner arcs do not intersect (except their ends). These conditions define the initial geometric configuration, which shows that Emilian is able to infer the necessary and sufficient conditions underpinning this configuration.

In modifying the configuration, Emilian first tries to customize to the triangular case – this shows specific behavior in problem solving. He realizes that, in this case, one of the identified conditions regarding the arcs (condition which is automatically checked in the given configuration!) cannot be met. He has a moment of doubt (he writes: "I think the box cannot be made, at least not in this way"), then he returns and delete/cut out some of these comments. He has a few paragraphs originally written, on which he returned and cut out. This behavior, on the type "step back" of observing own solving process, is also obvious in the analysis of the list of instructions. Initially, it contained 13 items; subsequently, based on observations made on the person who followed these instructions, the list was reformulated. Even if the task did not require (explicitly) this experiment being rebuilt (ie the new list of instructions to be proposed to another person), based on the new observations made, the list of instructions was changed again. This behavior shows, from a cognitive view, that Emilian has the ability to change his thinking schemes. The changes proposed by Emilian – boxes using regular hexagon and regular octagon denote abstract mathematical thinking. Even if Emilian do not explain why "skip" over the case of polygons with 5 or 7 sides, the fact that he began his analysis with the case of triangle proves understanding of the restrictions imposed by the constructability with compass and straightedge. The avoidance of certain numbers shows that Emilian possesses structured information (results about constructability with compass and straightedge), which he activates in this case. All these bring evidence for the existence of a certain way of structured organization of knowledge. Emilian obtains the figures through constructions made using only compass and straightedge, and claims that in the hexagonal box type, he checked his conjecture by building the box; thus proving purposely oriented procedural knowledge.

Once the checking made for one of the boxes, Emilian seems convinced that the other box fulfills the requirements without any supplementary checking. He thus expands the observed properties to the octagonal box, proving metacognitive capacities of self-regulation.

Previous comments show that Emilian proves theoretical expertise: he shows abstract thinking, he explicitly identify necessary and sufficient geometrical conditions allowing the construction of the object. In other words, his expertise compels him to assume a rigorous mathematical modeling of the object. To what extent does he show creativity in solving the task? The fundamental element to which Emilian refers is a regular polygon constructible with compass and straightedge. Once this frame built (mentally) – shaped by defining geometric properties (ie necessary and sufficient), he manages to identify and further modify essential elements (in this case – the number of sides of the polygon) and generate new valid configurations; this is about the capacity of changing an initial mental frame within the persistence of his assumed mathematical model. Emilian includes proof of the impossibility of building a triangular box in his response, by the same process. Subsequently, he generates boxes of different number of sides (6 and 8). This approach, of inductive type (starts with the minimum possible number of sides continues by varying the number) suggests that Emilian knows that the generalization process can be continued. We interpret this behavior as specific to cognitive variety. This shows that Emilian approaches creatively the given task.

6.5.2.2 Case 2 (Andreea)

The solving proposed by Andreea focuses on the technological process. Her instructions state, first, the materials necessary to achieve the box construction and are very detailed ("make notches using the cutter – but do not cut (!) the arches inside, so we can easily bend them"). Unlike Emilian, her constructions use ruler to scale and square ruler. This shows that Andreea is not interested in the mathematical "theoretical/abstract" aspect of the task, but of the pragmatic ones. She intuitively identifies the geometric properties that allow obtaining the box, which she formulate using a common language (such as "so we can put perfectly one on each other and form the lateral sides of the box", "the square perfectly falls in a circle", etc.). Andreea proves a type of goal oriented procedural knowledge.

Andreea identifies a defining property of the initial configuration – namely, that the marked points on the two circles determine a square in each. She claims that the square "fits perfectly in a circle" (meaning that it is a cyclic polygon), and that this applies to any regular polygon; as a result, we can use any regular polygon instead of the square, the only changes being that the number of lateral sides of the box increases and the box shape changes. In other words, Andreea identifies principles underlying the original structure – ie the property that the used polygons should be regular.

Andreea proves effectiveness in solving the task. She does not question constructability – as Emilian, but construction: for this, she neglects the details of the figure, focusing on the property she found as dominant, and generate (for example) a non-rigorous drawing, yet very clear in respect to information transmitted (Fig. 6.4). This shows that Andreea can develop her thinking schemes by synthesizing information.

As evidence of her technological orientation skills, Andreea uses GeoGebra to get the figures she suggested. The existence of this universal tool – GeoGebra ensures Andreea that the construction can be made for an arbitrary number of sides of regular polygons. Once generated the plane configuration, Andreea seems convinced that the effective realization of the box doesn't bring any difficulty – it is made similarly with the original case. This shows metacognitive capacities of self-regulation – it is no needed to recheck something that works analogically!

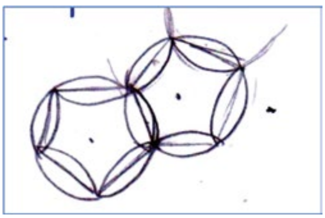


Fig. 6.4 Representation made by Andreea to explain how to obtain a pentagonal box

Is Andreea creative? We will further analyze this issue, to show that the answer is affirmative. Andreea succeeds to understand the properties of the original figure, even if sometimes her language is too approximate from a strictly mathematical view. For example, she notices that the arches that appear in the initial figure are equal, and the common side of the squares determines “congruent arcs on both sides” – ie, she notices the symmetry of the figure. Starting from the fact that the square is a regular polygon, Andreea says that we can use for the requested construction any regular polygon, but it is difficult to identify the centers of the circle describing the inner arcs. This looks as she evolves within a well-defined frame, but she does not pay attention to metric details because she can use a tool (GeoGebra) for every conceivable situation. Once the frame built, the variations she proposes (changing the number of sides of polygons consistently) show capacity of frame change.

In her response, Andreea includes only one box – namely, a pentagonal box. However, we concluded that she displays, in fact, cognitive variety: once a property with potential for generalization (ie the regularity of the polygons) determined, Andreea knows that she can unrestrictedly change the number of points of division – ie she can get many (new) models of boxes! She is not “restricted” in the construction of these new boxes, as the used instrument (GeoGebra) allows unrestricted freedom to vary a parameter of the geometric configuration (ie number of sides).

6.5.2.3 Case 3 (Paul)

Paul is prolific in identifying geometrical properties of the given configuration. He sets out 10 geometric properties, some of which are “dependent” (can be deduced from the properties listed above) – and, consequently, could be missing. We interpret his desire to formulate more geometric properties than necessary as an argument for the fact that Paul can change his thinking schemes and to focus in turn on some other aspects of the given geometric configuration.

Paul expresses the properties of the given configuration in two language registers. On the one hand, Paul connects the geometric context of the initial figure with a strong mathematical result, such as the theorem of characterization of regular polygons constructible with straightedge and compass (ie a regular polygon with n sides is built if and only if $n = 2^k \Pi p$, where the product contains only prime distinct Fermat numbers). The correlation of these properties with the context (in which the constructability with compass and straightedge was not explicitly stated) suggests that Paul has a knowledge organization of expert type, because he can quickly select, from memory, that specific information which is necessary and useful in solving the current task.

On the other hand, Paul seems that he does not only want to identify and convey properties, but he also wants to explain them suggestively. In this respect, he uses intuitive descriptions or names, such as “biconvex lens”, “the box resembles to a cuboid covered with two blankets bond in the corners”. For Paul, the link between

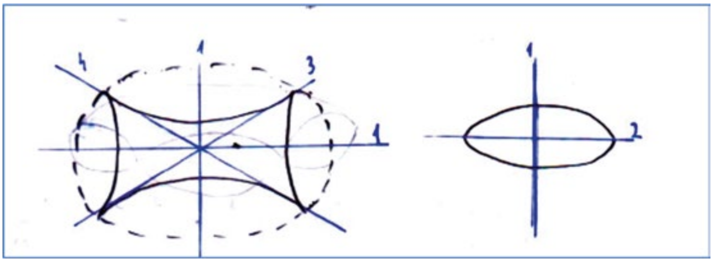


Fig. 6.5 Paul's drawing, which suggests that he may use an elliptical figure for getting a box

theory and practice is much stronger than for Emilian or Andreea. This is reflected in the plasticity of language and in the fact that, unlike other colleagues, he moves from the general result (ie the theorem of constructability of a regular polygon) to the concrete situation in which the theorem is applied. Paul's expertise doubles and supports his creativity. He is prolific in identifying properties of the given figure, which indicates cognitive variety. Moreover, although he does not explicitly state, he seems convinced that one can build a box by distorting the initial figure such as circles become ellipses (see Fig. 6.5). If this was indeed his intention, Paul shows reframing, therefore a high level of creativity.

6.5.2.4 Case 4 (Dana)

Dana has generated a list of instructions containing eight items. Her instruction list starts from two secant circles and from the symmetrical points of the centers of the circles to the intersection points. Subsequently, she builds arcs of circles with centers in these points. Dana's instructions and comments do not specify whether the initial circles are equal, or if quadrilaterals obtained are squares. Dana implicitly assumes, however, that these conditions are met. In fact, if we follow her instruction list (with the supplementary hypothesis of congruence of the initial circles), we get a box in which the base is a rectangle (see Fig. 6.6) Dana is however not aware of this fact that could lead her to an immediate generalization; she is focused only on the figure and she believes that in this way, she gets squares, regardless of the distance between the centers of the two circles.

For the initial figure, Dana notes that "the intersection of the two squares is another square having as side the radius of the two circles". She breaks down the initial figure into "small" squares (as in Fig. 6.7), and then she generates new figures, made of triangles, which keep the "zigzag" pattern.

It seems that Dana retains only surface features of the task (ie a specific pattern of squares that cover, in her perception, the initial figure) and uses this pattern for another geometric figure – ie equilateral triangle. Not coincidentally, the figures

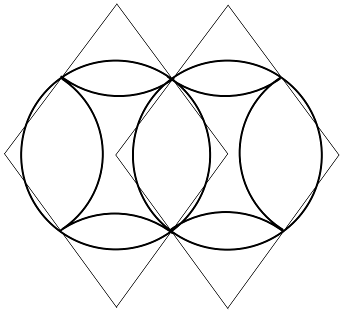


Fig. 6.6 The figure generated following Dana's instructions

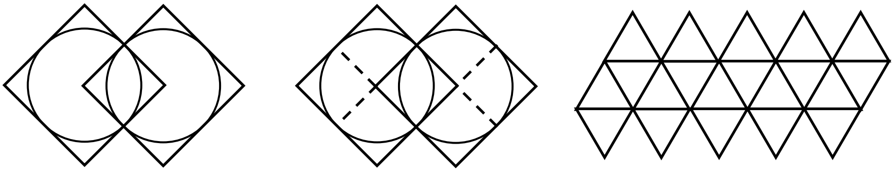


Fig. 6.7 Dana's patterns identified in the initial figure and applied to the figure she generated

generated by Dana (as alternatives to the given figure) no longer contain circles or circle arcs: Dana identifies only superficially the geometric figure baseline (the 2 circles have equal radii; we built two symmetrical squares; their intersection is also a square; the figure has two symmetry axes), and none of them is about the built arches. All this shows that Dana is rather novice in exploring the task.

Dana retains only one aspect – namely, that in the end, we obtain a container. The background she identified, consisting of matching squares arranged diagonally, suggests the use of figures previously known, representing the unfolded net of some regular polyhedron (octahedron and icosahedron).

We can say that Dana denotes cognitive novelty because her chosen changes are significantly far from the initial context. However, she thus slides out of the problem frame due to insufficient understanding of the geometric properties of the given figure (her generated construction leads to circumscribed rhombuses and inscribed rectangles, missing the condition of equal circles). At a careful analysis, we note that, in fact, she exploits a simple regular easily identifiable pattern. This is a relevant case for the situation that creativity does not advance too much because expertise is missing (in the Glaser's sense). Apparently, this is in contradiction with the fact that Dana is a student with high academic results. Perhaps her learning is often a surface one, based on memorization and not on depth analysis of mathematical contexts – but we do not have more data to advance this hypothesis.

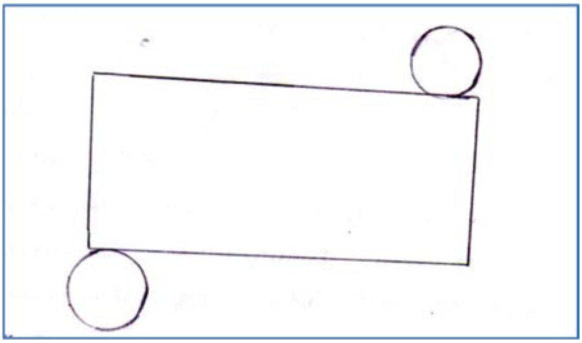


Fig. 6.8 The change proposed by Georgeta

6.5.2.5 Case 5 (Georgeta)

The structure generated by Georgeta differs from those of all his colleagues. All the other students in our sample have generated a list of instructions specifying (and numbering) the steps. Georgeta has designed its instructions as a descriptive prose. Many of her instructions are non-essential and unclear. For example, there are indications of the colors to be used for certain details of figure and comments like "common part of the two circles must be quite large, but smaller than the radius". By this, she proves superficial understanding of key concepts and terms. Georgeta believes that the defining geometrical property of the given configuration is that "in a circle can be inscribed any geometric figure, more exactly, polygons". These statements have shown us that she is novice. As a change from the original, Georgeta proposes the drawing of Fig. 6.8, in which an unfolded cylinder appears. She insists that it causes a box, while it has no other geometric properties compared to the initial context.

With the proposed change, Georgeta depart significantly from the given pattern. Is this evidence of cognitive novelty? We incline to think it is not.

6.5.2.6 Case 6 (Cristina)

Cristina's instruction list starts from the description of three special "preliminary" constructions with compass and straightedge: the midpoint of a segment, the perpendicular from a point on a line, the circle inscribed in a square. Her instructions contain 11 items: most of them are synthetically formulated. Cristina gives in her instruction list "milestones" – brief indications to verify the construction accuracy. This ability to synthesize the information transmitted, but also to keep a protective attitude towards the reader, proves the goal oriented procedural knowledge – which is guiding the solver.

Cristina equally proves synthetic when she identifies geometrical properties of the given configuration: they refer to invariance through symmetries and rotations. These properties are seen in relation to the final object (the box); for example, the symmetry to the common chord is the condition that "causes the box to have

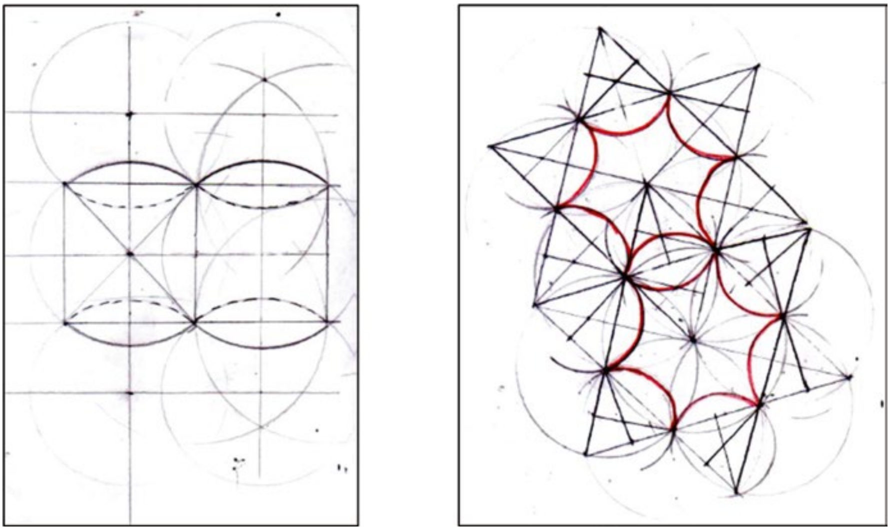


Fig. 6.9 Cristina's drawings for getting new boxes

walls – when bending the box, the walls have to overlap". The element through which Cristina seems to modify the initial configuration is the coverage of the plane with figures of the same shape. For the given figure, the background she perceives is a tessellation with congruent squares. Cristina keeps this tessellation as a way of generating a new box (Fig. 6.9a) or use a tessellation with equilateral triangles (Fig. 6.9b).

Cristina keeps a method similar to that of her instruction list for drawing the inside arcs. More precisely, these are arcs of the circles circumscribed to squares or equilateral triangles from the tessellation. Cristina works with a weaker condition: in the second proposal, the arcs are no longer symmetrical towards the common chord, and this is why the sides of the box do not perfectly match. In principle, this weakening of a condition could allow a bigger number of possible solutions (at the expense of object's "perfection"). Could this be an evidence for creativity?

The weakening of conditions is actually a gap in her response, to the extent that she is not aware of the consequences: she actually did not realize the implications, even if she made the box and so checked that it can be built. Specifically, Cristina is unaware that in the new construction, the sides of the box do no longer "perfectly" overlap, as happens in the initial model.

6.5.2.7 Case 7 (Adelina)

Adelina preferred to solve alone the whole task (not in a team of two, as most of her colleagues did). Her list of instructions contains 10 items; the language used is not mathematically rigorous, but instructions can be easily followed. Her instructions are focused on obtaining the figure, not on getting the box: once the figure drawn,

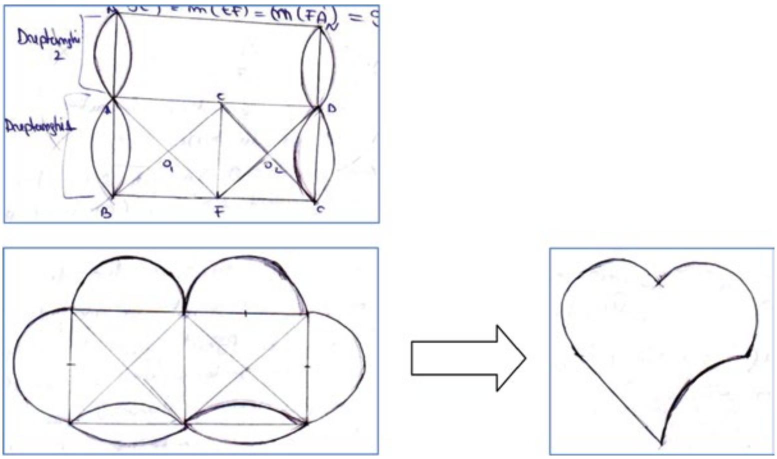


Fig. 6.10 Adelina's proposals for new boxes

Adelina believes that her mission was accomplished. Adelina identifies only two geometric properties of the initial configuration, namely: 1. quadrilateral determined by the points of intersection of the two circles of the figure and the centers of the circles is a square; 2. the circles have been divided into four equal parts.

Even if the mathematical model described by Adelina is incomplete (it does not say anything about the inner circle arcs), her instruction list shows that she internalized the context and can give directions to complete its reproduction. This shows us that Adelina displays functional knowledge.

To change the initial figure, Adelina proposes the models shown in Fig. 6.10.

Adelina achieved a first product resulting from a correct mathematical modeling, whose shape is found even among usual items around us (a spectacle case), although she does not mentioned this as such. The utility of the obtained product indicates transfer capacities (Gardner 1993). The second product obtained – also by a correct mathematical modeling, has, in addition, aesthetic value. The fact that these new objects have practical and aesthetic values is another argument for her procedural functional knowledge. Compared to its peers, Adelina proposes very different models. So we can say that she denotes cognitive novelty. For her both new models, she keeps the same background (easily to identify congruent squares) and the same way of building arches (parts of the circumscribed circles to such squares). Adelina evolves within a well-defined framework and manages to make substantial changes to it, while keeping it consistent.

6.5.2.8 Case 8 (Anca)

Anca has generated a list of six initial instructions. In her instructions, she implicitly assumes that the person to follow the list knows some mathematical concepts, at least at a basic level (eg perpendicular lines, reflective symmetry of a point, square circumscribed to a circle, etc.). At the end of this list, Anca includes a commentary

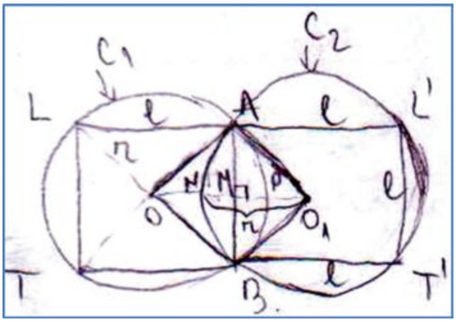


Fig. 6.11 Notations made by Anca for the metric description of the initial configuration

under the title "philosophy of the instructions," in which she claims the construction accuracy. She also includes extensive comments on the difficulties faced by people to whom he proposed making the box: some of these difficulties arise from misunderstandings on mathematical concepts. The fact that Anca redesigned not just the lists of instructions, but the entire solution to the task as a whole (she asked to resubmit a new version of solving the whole task, because she believed that she can better explain how to solve it) shows, on the one hand, her capacity of changing thinking schemes, and, on the other hand, she proves metacognitive capacities of self-regulation. Anca prefers to describe metrically the geometric properties of the given configuration: she expresses the lengths of the various segments as function of the radius of the initial circles (Fig. 6.11).

Typically, the quantitative metric approach of a configuration is a barrier to generalization/transfer because quantitative information limits the chance of identified generic properties. Anca proposes three modifications to the initial configuration: she replaces squares with regular octagons, with regular dodecagon, respectively with regular 16-gons. For the new situations, she explains how regular polygons can be built with straightedge and compass (mainly building bisectors of angles, but she does not perform the constructions, including only schematic representations of them). Anca possesses goal oriented procedural knowledge.

We note that Anca manages to overcome the "barrier" of metric results and identifies a property with potential for generalization – ie "square is a regular polygon." Perhaps, she sees regular polygons in quantitative context (lengths of sides and measures of angles), not in a qualitative one (invariance over symmetries and rotations). The focus on a particular property of the initial configuration, which allows generalization shows that Anca may overcome interferences and identify principles underlying the surface structures. We may ask where her "jump over hexagon" comes from – ie why Anca, unlike the majority of students who have generalized based on the idea of a regular polygon did not consider the case of hexagon. A possible answer is suggested by the way she imagine the new boxes (Fig. 6.12). Anca keeps as invariant the configuration of two equal circles that intersect over arches of 90°. She then divides each of these circles in a same number of congruent arcs, such as the intersection points of circles to be the dividing points. Therefore, her self-imposed restriction (the relative position of the two circles) requires dividing the number of points to be multiple of 4.

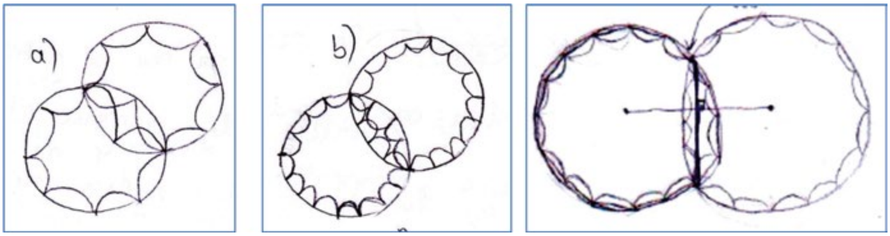
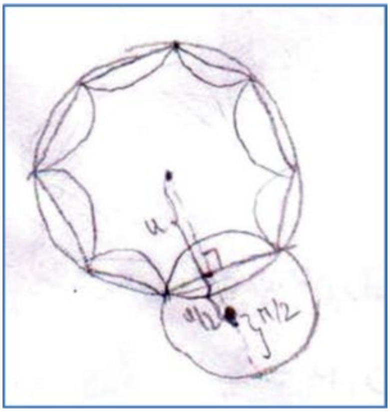


Fig. 6.12 Anca's imagined configurations for her new boxes

Fig. 6.13 The construction pattern of the inner circle arcs indicated by Anca



Anca is thoughtless in tracing the arcs of the circle. She proposes a construction described in metric terms that he believes is generally applicable (Fig. 6.13), but which cannot be applied in all the described cases. Because of this, the "lenses" Anca obtained shows no symmetry and the boxes imagined do not close "perfectly" (as in Cristina's case).

Anca evolves in a well-defined cognitive frame and makes changes in this frame, varying the number of sides of polygons. She also proves cognitive variety – by her new generated models. Anca identifies a general process of obtaining new configurations – namely, for a given configuration, doubling the number of points of division by building bisectors of angles. In this way, the idea that implicitly appears is that the number of sides may vary indefinitely – which is another argument for cognitive variety.

6.5.2.9 Case 9 (Miron)

Miron's instruction list begins with mentioning a list of the necessary materials and continues in some detail (eg: he mentions the fact that two distinct points determine a line, and lists basic compass and straightedge constructions, such as drawing a segment determined by two points). The proper list of instructions contains 14 items. The instructions contain milestones – indications on how the solver can verify his/her construction. Miron proves very synthetic in identifying the essential elements of the given figure:

In fact, the essential property in the construction of this figure is the possibility of breaking the circle into four equal arcs, namely the opportunity to build the angle π/2. Reformulated, it is about constructible regular polygons.

He thus proves that he can mobilize thinking schemes, easily moving from the original context to a generalized representation of it. He recalls the theorem about regular polygons constructible with straightedge and compass ("A regular polygon with n sides is constructive $\Leftrightarrow n = 2^k\, p_1 p_2 \ldots p_r$, where p_i are distinct prime Fermat numbers- ie $2^{2^m} + 1$"), proving that he can rapidly select from memory items of structured information when needed. Miron notes that the square obviously satisfies the theorem conditions, but the instruction list for the initial figure are specific to this case and are not useful in the generalizations that follow. Miron's proposed new cases are those of a regular hexagon and regular octagon. He presents the constructions steps in a highly synthetic and generalized formulation:

- *We choose n equally-spaced points on the circle (how exactly to do that depends on n, but it is always possible).*
- *For any two consecutive, we build another circle that contains them and has the same radius as the initial (a compass and straightedge elementary construction in at most four steps).*
- *We now have a "star" with n corners inside the initial circle. We choose any of the other circles and repeat the procedure (of the construction of another "star" inside it).*
- *We reached the desired figure that can be cut.*

When putting the construction into act, he uses GeoGebra to make the "classical" compass and straightedge construction (to specify the division of a circle into n equal arcs). The technology in this case is just a good instrument (it has accuracy and shortens time) that replaces physical objects such as paper, straightedge and compass, keeping all valences of the ideal construction.

He alternates schemes and procedures which he combines in a manner that focuses on optimization and getting results simultaneously. Miron proves metacognitive capacity of self-regulation, high transfer capacity and, in general, the type of expertise of a mathematician.

Comparing to how another student (Andreea) used GeoGebra, we can see that Miron – with mathematics expertise, used the software as only a support to enhance and concentrate the force of the theory, while Andreea – with a rather pragmatic expertise uses the facilities of the software in actual construction without questioning the geometric accuracy. Figure 6.14 shows the images used by Miron to construct the regular octagon-based box.

He does no need to identify the initial figure geometric properties that allow the construction of the box (of the type: symmetry, congruency) because he internalized a general scheme available for construction. This scheme – the constructability theorem – offers the individual cases to perform the initial construction and the pattern that allows generalization. In these circumstances, we can ask how creative is a solution induced by the in-depth knowledge of a strong theorem. Perhaps the given context is not enough challenging for him to provoke creativity.

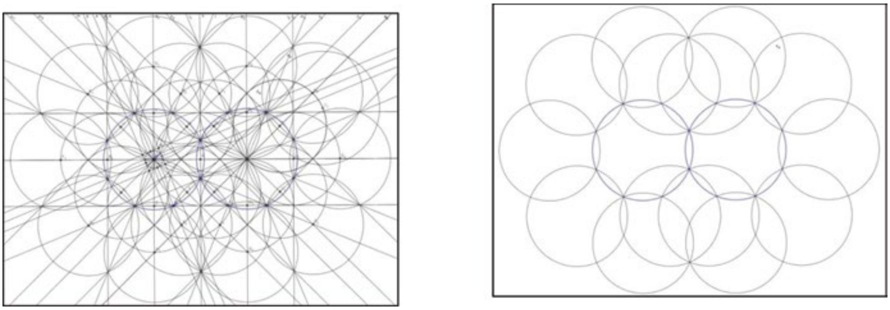


Fig. 6.14 Miron's images obtained using GeoGebra

6.5.3 Comparative Remarks

We will look in more detail into the cases presented above.

Emilian, Andrei and Paul proved validated task-related expertise because their cognitive behavior allowed checking the assumed criteria that confront expert-novice abilities. However, their type of expertise is manifested in different ways: Emilian proves metacognitive capacity for self-regulation and general use of problem-solving tactics; Andreea is a practitioner expert type focused on a technological approach, showing high procedural and goal-oriented knowledge; Paul proved stronger transfer skill for making connections between theory and practice than Emilian and Andreea, and has developed a meta-cognitive capacity to explain the identified properties using suggestive expressions.

We note that in all these three cases, students manifested a creative behavior:

Emilian firstly investigates the case of a box with a triangular base, he identifies arguments by showing the impossibility of construction, and then he generalizes. Andreea includes a single new model box (pentagonal). She however indicates a construction with potential of generalization, performed with a "universal" instrument – GeoGebra: she is confident that this process will work for any number of sides, and therefore she does not need to include other cases. Paul is prolific in identifying properties, showing cognitive variety. Through drawings, he suggests a substantial change frame, because he finally replaces circles with ellipses. Paul recalls a general result, regarding the polygon constructability with straightedge and compass; once identified the theoretical background, he particularizes the theorem and provides two new constructions.

Dana and Georgeta behave as novices. The properties they identified with respect to the initial configuration are weak properties. Dana identifies a pattern and appears to extend this pattern to generate new boxes. Georgeta relates to unfolding a solid and proposes as a new model an unfolded net of a cylinder. At a first view, Dana and Georgeta seem more creative because their proposals are significantly far from the initial model. They yet focus on superficial aspects, such as a simple pattern of squares distribution and/or the idea of container, and this lack of consistency shows that, in fact, they do not behave mathematically creative.

It would be expected that weak properties, allowing more degrees of freedom, have the potential to facilitate a more creative approach. However, we found that they actually lead to an insufficiently consolidated frame (probably caused by an insufficient level of expertise), instead of leading to spectacular generalizations.

The above comments suggest the conclusion that expertise seems to be a precondition for creativity. We will show that this statement should be at least nuanced.

The data available for Cristina, Adelina and Anca did not allowed us to consider them experts in every sense of the word. Rather, they have a moderate level of expertise, having characteristics of expert behavior, but also novice features. For example, Cristina is not aware of the consequences of weakening some requirements; Adelina shows a superficial understanding of some of the concepts used; and Anca based her constructions on metric inputs, but which may not apply in certain situations. We classified the proposals of the three students as creative. Cristina proposes two new boxes, totally different proving cognitive variety. Adelina's proposals have functional and aesthetic valences, and are very different from all the other proposals. Anca suggests a construction of a generalized manner that allows many more new products, thus showing cognitive variety.

Therefore Cristina, Anca and Adelina behave creatively. It seems that a rather moderate level of expertise allows expression of their creativity.

To verify this hypothesis, we consider the case of Miron. Obviously, Miron is the expert par excellence. He summarizes, in his solving, the problem nature, he quickly selects items he needs from memory, and "closes" the problem by applying a general result that solves a whole class of problems of the same type. Moreover, he "hijacks" a tool like GeoGebra using it for a compass and straightedge construction, and including it in his theoretical approach.

In his case, his high level of expertise as related to the task practically cancels the problem. In this case, it becomes legitimate to ask if does make sense to put the question of a creative answer in Miron's case Why did this question arise? Because Miron, by mastering powerful mathematical tools, manages to reduce a problem that for others is complex to schemes automatically activated. For this reason, because the solution is based on results already known to him, his creative contribution is at most in appropriately correlating concepts and procedures, ie in small changes well controlled within a cognitive frame clearly emphasized from the beginning. Meanwhile, cognitive novelty, and cognitive variety are practically undetectable. As a result, we believe that, in the Miron's case, we cannot detect creativity on this task. We make the assumption that facing more complex tasks that would require a higher level of expertise, Miron could be highly creative. This hypothesis was confirmed by the information later obtained about him, beyond this task. We learned that Miron is already included in a mathematics research program and that he has already published original results.

Therefore, the determination of creativity should happen at a level that exceeds the person's expertise at that moment. It appears as a corollary that *creativity is not an absolute parameter*. The manifestation of creativity depends on the context, as confirmed by other studies.

6.6 Conclusions

In this paper, we have studied how students' creativity manifests in a complex context that involves modeling, problem solving and problem posing. A first conclusion refers to how students use the defining elements perceived in an initial given figure. We have seen that these elements are further used for generalization and transfer. So, the way in which students perceive the initial figure is fundamental for solving the task and for posing new coherent modifications.

A second conclusion refers to the relationship creativity – expertise. The students that seemed more creative at a first sight, proving that they are novices in the domain, produced either non-functional or inappropriate objects. Conversely, students who showed a high degree of expertise utilized strong mathematical results (such as constructability with compass and straightedge) and made incremental changes by varying a simple parameter (in our example, the number of sides of regular polygons).

The analysis of some specific cases led to the conclusion that creativity manifestation is conditioned by a certain level of expertise. In the process of building a solution for a nonstandard problem, expertise and creativity support each other and enable bridges to the unknown, mutually developing each other. This interaction leads also to an increase in expertise.

We have seen that, because contextualization, it is practically not possible to find tasks that would allow discerning creativity for a broad range of skills. If the task is at a cognitive level accessible to a majority, a person with high level of expertise will make appeal to tools that automatize the response; if the task is challenging for a person with a high level of expertise, then it is not cognitively accessible to a larger sample, in order to make comparisons.

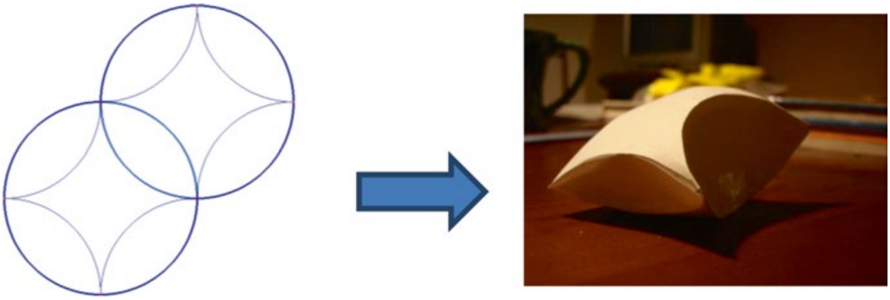
We unravel from here that a possible method of training excelling students is through practicing tasks appropriate to their level of mathematical abilities, but containing nonstandard challenging components, in order to train metacognitive self-regulation capabilities through creative leaps.

Therefore, to create the context in which a student can advance, it is necessary to determine the type of task for which he/she manifests expertise and to integrate this task in a challenging context. Our study shows that this approach seems to work for advanced students. Further research will focus on a methodology to check if it may work for students of any level.

Annex

The Given Task

From the figure below one can get a “fantasy box” [a.n. the box was presented “physically” by the teacher].



I. *The first two questions constitute a group task (2 people). For this part, the group members will receive the same score.*

1. Write specific instructions for constructing this figure. The instructions will contain only words, no drawings, diagrams or pictures.
2. Give these instructions to another person who does not know what you want to achieve. Ask that person to follow instructions. Do not interact with that person, do not give indications, or help. Note (or record) what happens. If the person has difficulty in representing the figure, or something unforeseen happens, it’s OK: this only shows that your instructions are not enough precise and should be reviewed. You will not be penalized if the first set of instructions is not quite accurate.

   (a) Write a report as detailed as possible (but no longer than 3 pages!) about what happened;

   (b) Write a revised instruction list and possibly repeat the experiment with another person.

II. *Answer the following 3 questions individually.*

3. What geometric properties are used in the construction of this box? Explain your answer.
4. The fantasy-box has a “squared” shape ☺. How could you modify the original drawing to get boxes of other shapes? Build two new figures and make sure you can get boxes starting from the figures you indicated.
5. Do the proposed figures above use other geometric properties than the ones of the original box? Explain your answer, and if it is yes, please specify which are these properties.

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Chapter 7
Constraints, Competency and Creativity in the Classroom

Catrinel Haught-Tromp and Patricia D. Stokes

Abstract Constraints define domains, specifying goal criteria and accepted means of meeting them. Competency, the ability to problem solve in a domain, depends on mastering basic constraints. Creativity follows competency, as the product called creative must be both novel and appropriate to its domain. In this chapter, we discuss and demonstrate how different kinds of constraints affect early competency in mathematics and later creativity in composition. Applications of our constraint model to other do mains are also suggested.

Keywords Creativity • Constraints • Competency • Education

Constraints define domains, specifying goal criteria and accepted means of meeting them. Competency, the ability to problem solve in a domain, depends on mastering basic constraints. Creativity follows competency, the product called creative must be both novel and appropriate to its domain. In this chapter, we discuss and demonstrate how paired constraints made very young children very competent in math and college students more creative in composition. Applications of the constraint model to other domains are also suggested. Since all readers will not be familiar with the problem-solving literature, definitions precede applications and suggestions.

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7.1 Definitions

7.1.1 Constraint Pairs and Problem Space

The dictionary defines constraints as restrictions or confinements. In contrast, the problem-solving model presented in this paper (Reitman 1965; Simon 1973; Stokes 2006, 2014b) considers constraints as pairs. One of the pair satisfies the dictionary definition – it precludes something. The other expands the definition – it directs search for and promotes a substitute. This solution-by-substitution process takes place in what Newell and Simon (1972) called a problem space. A problem space has three parts, an initial state, a goal state, and between the two, a search space in which a solution path is constructed. Table 7.1 presents a simplified problem space for mastering single digit addition.

The initial state (4 + 5 = x) is the given problem. The goal (solve for x) has a criterion: solve with the most efficient strategy. In the search space, the preclude column orders addition strategies identified as least (guess) to most (retrieve) efficient by Siegler and Jenkins (1989). Each is paired (in the corresponding promote column) with its substitute (the next most efficient strategy). As the table shows, mastery of single digit addition is constraint based: less efficient strategies are precluded, more efficient ones are promoted.[1]

For tasks that require creativity, a similar set of steps applies. Here, the solution stems from a nondeterministic process: at each step there are multiple options, and choice to be made. In computational parlance, the same initial state can yield several different outcomes (Johnson-Laird 1988) – unlike, say, addition or multiplication, where the same input can only yield one correct outcome, via a deterministic process. The goal (generate a creative output) has a criterion: for the task at hand, find something novel and, at lower levels, useful or appropriate, and at higher ones, generative or influential.[2] In the search space, one precludes often-used, less efficient strategies with unexpected, lower-probability, hence more efficient ones.

Table 7.1 Simplified problem space for single digit addition

Parts	Description		
Initial state	4 + 5 = x		
Search space	Preclude		Promote
	Guess	→	Count all
	Count all	→	Count on
	Count on	→	Count from higher addend
	Counting	→	Retrieve from memory
Goal state	Solve for x using the most efficient strategy		

[1] This does not mean that the less efficient strategies disappear. Rather, the distribution of strategies shifts so that the more efficient ones are used more often.

[2] Generative means leading to variations, i.e., other solutions; influential means expanding a domain (Stokes 2006).

7.1.2 Constraints and Domains

Domains, well-developed areas of skill, are defined by agreed-upon/recognized goal, source, subject and task constraints (Stokes and Fisher 2005). *Goal constraints* are performance or stylistic criteria that must be met for an equation or composition to be considered correct and perhaps, at higher levels, elegant or creative.[3] S*ource constraints* provide elements to be worked with (promote) or against (preclude). *Subject constraints* specific content or motif. *Task constraints* govern materials and their application.

Competency, or the ability to problem-solve in a domain, begins with mastering the basic constraints that constitute a domain. Competency in mathematics includes counting and calculation. As a later section will delineate, competency in composition involves vocabulary and grammar. Creativity necessarily follows competency: a solution or composition must be appropriate, as well as novel, in its domain.

7.2 Applications: Constraints in Our Classrooms

Both of us applied the constraint model in the classroom. One used it to create an early math curriculum; the other, to help students become more creative in writing. For each example, we delineate the problem, the solution, the outcome, and, importantly, the next steps. The “next steps” section demonstrates how paired constraints can help refine or continue curriculum and lesson plan development.

7.2.1 Creating a Math Curriculum

The Problem The catalyst for creating the new curriculum was the place-value problem. The term place-value is self explanatory: the value of each digit in a multi-digit number is determined by its placement. For example, in a two-digit number, the digit on the left is a ten, the digit on the right is a one. The problem is that American children, who call the number 13 “thirteen,” mistake the 3 as being of greater value than the 1. Japanese, Chinese, and Korean children, who use an explicit base-10 count and call the same number “ten-three,” do not make the same mistake (Fuson 1990; Miura and Okamoto 2003). To show how “ten-three” fits in the Asian (Korean, Chinese, Japanese) counts, Table 7.2 shows the numbers and number names from 1 through 29. Notice there are only ten number names (1–10), which combine to form the higher numbers. Notice too that ten appears in every number above ten: 11 is *ten*-one; 21 is two-*ten*-one.

[3] The initial state in a problem space is a prior goal constraint. It is the preclude half of the constraint pair which promotes its substitute, the new goal constraint.

Table 7.2 Explicit base-10 count

	Ones		Tens		Twenties
		10	*Ten*	20	Two-*ten*
1	One	11	*Ten*-one	21	Two-*ten*-one
2	Two	12	*Ten*-two	22	Two-*ten*-two
3	Three	13	*Ten*-three	23	Two-*ten*-three
4	Four	14	*Ten*-four	24	Two-*ten*-four
5	Five	15	*Ten*-five	25	Two-*ten*-five
6	Six	16	*Ten*-six	26	Two-*ten*-six
7	Seven	17	*Ten*-seven	27	Two-*ten*-seven
8	Eight	18	*Ten*-eight	28	Two-*ten*-eight
9	Nine	19	*Ten*-nine	29	Two-*ten*-nine

Table 7.3 Problem space for new math curriculum

Parts	Description		
Initial state	Current curricula		
Search space	Preclude		Promote
	English language count	→	Explicit base-10 count
	Non-numeric	→	Numbers, symbols, patterns
	Multiple manipulatives	→	Single manipulative
	Split practice	→	Continuous, focused practice
Goal state	New curriculum		
	Criterion: thinking in numbers, symbols, and patterns		

In contrast to American children who think of numbers as chains of ones (21 means 21 ones), Asian children think of numbers as tens and ones (21 means 2 tens and 1 one). For children who think this way, place-value is not a problem.

The Solution The proposed solution (to the place-value problem and by extension to multi-digit addition and subtraction problems) was not simply to introduce an explicit base-10 count, but to embed it in a curriculum that taught children to think mathematically, in large meaningful patterns. Table 7.3 shows the problem space. The initial state was current curricula. The goal state, a new curriculum, had a criterion: thinking in numbers, symbols, and patterns.

The first pair (which involve source constraints) precluded the English language count and promoted an explicit base-10 count. The next three are task constraints designed to further satisfy the new goal criterion. Non-numeric meant videos with cartoon characters, work sheets with stories, and word problems that can distract children from the strictly numeric. The single manipulative was meant, like the abacus, to make base-10 numbers and patterns tangible and concrete.

Figure 7.1 shows the manipulative, called the count-and-combine chart, with the numbers 1 through 10. Notice that 10 is represented both as 10 "one" blocks and as a single "ten" block.

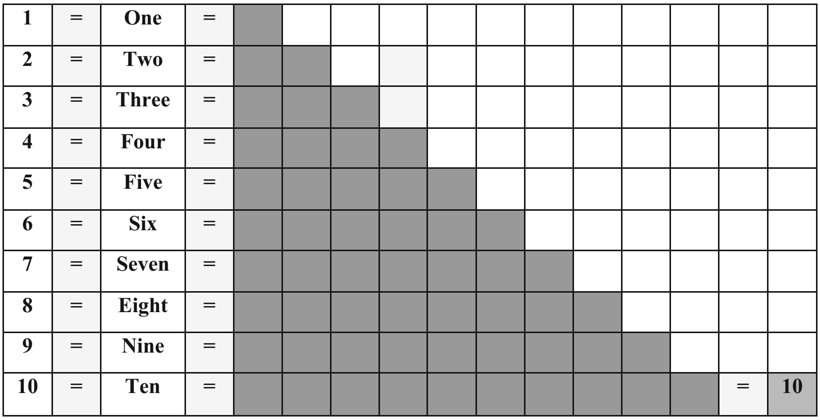
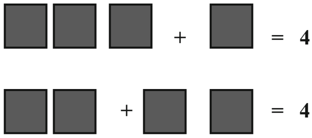


Fig. 7.1 Count-and-combine chart for numbers 1–10

Fig. 7.2 Two addition combinations for 4



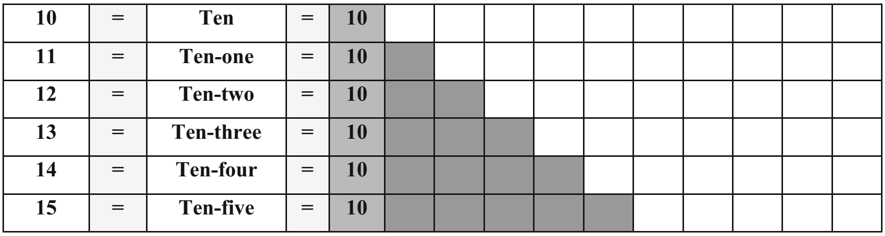


Fig. 7.3 Count-and-combine chart for numbers 10–15

The numbers, number names, symbols, and colored boxes representing ones are all moveable. Children began by reciting the rows. The top row is read "number one same as word one equals one block." They continued by recombining the blocks creating addition combinations for each number. Figure 7.2 shows two of eight possible combinations for the number 4.[4] As the numbers increased in value, so did the possible combinations.

Figure 7.3 shows a count-and-combine chart with the numbers 10–15 (ten-five). As in Fig. 7.1, ten is represented as a unit, by a single block marked "10." There are other similarities. In each chart, the block pattern mirrors the reiterations in the count: four equals 4 one blocks; ten-four equals one 1 ten block and 4 one blocks.

[4] The six combinations are: 4, 2 + 2, 3 + 1, 1 + 3, 2 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, and 1 + 1 + 1 + 1.

Later on, the children learned that 44 (four-ten-four) equals 4 ten blocks and 4 one blocks. They also learned to add tens to tens, before adding ones to ones, and to take tens away from tens, before taking ones from ones. To clarify what "take away" meant, children physically took away the same number of ten and one blocks from either side of the minus sign. The blocks left over were the remainder.

The Outcome The children were tested at the end of the school year. Did those using the new curriculum learn more math than a comparison group using the district curriculum (*New Jersey Mathematics: Scott Foresman – Addison Wesley*)? Yes. On place-value, single- and double-digit addition and subtraction, and number line estimation, children taught with the new curriculum (*Only the NUMBERS Count*©) outperformed those in the comparison group (Stokes 2013, 2014a). On number line estimation, they performed as well as Chinese students of the same ages (Siegler and Mu 2008). In sum, they became highly competent at thinking in numbers, symbols, and patterns.

The Next Step The next step was expanding the curriculum to second grade. The problem became how to teach multiplication and division. The solution was again derived from the Asian classroom. This time we *precluded* our multiplication table and *substituted*, in its place, the Chinese table. Like the count-and-combine charts, the table is chanted. The chant defines the key difference in the learning: children do not count ("two, four, six …"), they multiply ("two-twos are four, two-threes are six …"). The table itself is much simpler than ours. The simplicity suggested two new uses: to visually demonstrate the "flips"[5] (2×3 is the same as 3×2) in division and, more importantly, how division un-does multiplication. This expansion is being piloted as this chapter is being written. So far, so good.

7.2.2 *Making Composition More Creative*

The Problem When an essay or a presentation for work is due, many of us – children and adults – find ourselves stuck in old "solutions." When a friend's birthday is coming up, we struggle to create a message that says what we mean. What makes that new Word document, white sheet of paper, or blank greeting card so intimidating? There are several possibilities. One, there are too many possible solution paths: choice is stressful. Two, without constraints we all repeat what has worked best in the past: familiar solutions surface sooner than novel ones (Maltzman 1960; Runco 1986; Ward 1969), the most-traveled path prevails.[6]

Language in general and figurative language in particular (Glucksberg and Haught 2006; Haught 2013, 2014) is a prime example of creativity, which operates within a

[5] This is the term children already used to indicate that addends could be reversed, i.e., $2+3$ is the same as $3+2$.

[6] This is called operant conditioning.

Table 7.4 Problem space for creative composition

Parts	Description		
Initial state	Clichéd composition		
Search space	Preclude		Promote
	Existing, clichéd associations	→	Novel, unexpected associations
	Too large a search space	→	Narrower, more focused search space
	Superficial exploration of many alternatives	→	In-depth exploration of fewer alternatives
	'Blank page'-induced writer's block	→	Constraint-induced creative writing
Goal state	Creative composition		

given set of constraints. But, you ask, what about artistic freedom? Indeed, an unconstrained field does invite free, unencumbered exploration. By chance alone, you *might* stumble upon an unexpected, fortuitous turn of phrase. But most of the time, the most-traveled path will prevail, drawing you into prosaic, formulaic phrasing.

The problem is obvious: how can we make composing more creative?

The Solution The strategy we suggest is straightforward: seek and embrace constraints. Remember, constraints do two things. They limit search along those predictable (albeit reliable) old paths, precluding widely-used associations. They direct search along less-traveled paths, promoting in-depth exploration of unexpected, surprising associations. Tables 7.4 presents a generalized problem space for creative composition. The initial state is the clichéd, the goal is the creative. To actually reach the goal, each of the pairs must be further specified. For example, imagine the cliché to be precluded is "Once upon a time…." The opening suggests, of course, a fairy tale. Substituting "3 am, again…" suggests several quite different tales: one about insomnia, another about surveillance, a third about what?

The Outcome Constraints work especially well for professionally literary composition. Members of OULIPO (the Ouvroir de Littérature Potentielle) self-impose formal and combinatorial constraints on their writing. For example, they have successfully excluded specific letters – see Perec's (1969) 300-page novel, which excluded the letter E – , allowed a single vowel, and replaced each noun with the seventh noun after it in the dictionary.[7] Theodore Geisel, well-known as Dr. Seuss, wrote *Green Eggs and Ham* in response to a challenge: write a children's book using only 50 words. In his novels (one of which[8] won a Nobel Prize), Jose Saramago precludes quotation marks around conversations. The constraint makes the reader pay very close attention to the phrasing that identifies a speaker.

What about the non-professional writer? Do constraints work as well? It seems so. When college students were asked to generate creative sentences in response to pictures or words, their outputs were judged more creative for pictures (Haught 2015).

[7] For more examples and a history of OULIPO, see Becker (2012)

[8] *The History of the Siege of Lisbon*

Pictures proved more provocative, more suggestive than words, “worth a thousand” of them, as the saying goes. For example, pictures of a LION, a STRAWBERRY, and a HARP produced sentences like “The harp had a strawberry-colored lion carved in its post.” The words alone led to sentences like “I ate a strawberry while listening to harp music and watching a lion at the zoo.” In short, the search space was constrained by the visual representations, which in turn guided the construction of the sentences.

College students also wrote more creative rhymes for a special occasion – a birthday or anniversary – when their task was constrained by including a given noun (Haught-Tromp 2016). You should try this yourself. How would you say *I love you* in a two-line rhyme that must include the noun *vest*? How could you express *I am sorry* in a rhyme that includes the noun *shirt*? Here are what two participants wrote:

We belong together like a sweater and vest.
“I love you” and I’ll write it across my chest.

Here is a wool shirt for you to keep.
I am sorry, and so is the sheep.

Interestingly, even after removing the constraint of a given noun, the rhymes were more creative. Mere practice with a constraint seems to help, even immediately after it is explicitly removed.

In another study, preliminary results indicate that rhymes required to start with a given letter of the alphabet were more creative than those written without the constraint. The first letter appears to have acted as an anchor, precluding search for rhymes starting with any of the other 25 letters of the English alphabet, promoting more efficient search within a given, narrower field.

The Next Step We have examined externally imposed (the student examples) and internally imposed constraints (the professional writers). The pair pose two interesting questions. One, can we teach students to use their own constraints? Two, can we teach students to use the professional writers constraints?

To answer the first question, college students were instructed to write down the first concrete nouns that came to mind (the internal constraint), and then incorporate those nouns in greeting-card type rhymes (the external constraint). The internal constraint led to more creative greetings than those written by students using only the external constraint. To convey *Thank you* in a rhyme that also incorporates a self-generated word (*sunflower*, *dog)*, two participants wrote:

Thank you for making my life a beautiful <u>sunflower</u> ;
You sure do have some magical power.

Thank you <u>dog</u>;
Stay solid like a log.

The external constraint alone yielded mostly uninspired rhymes, of the sort:

Thank you for being so great,
It’s something I really appreciate.

But, interestingly, once again, when students first worked with both constraints (external and internal), and then with only the external constraint, their rhymes in the latter exercise were more creative than when they had never been introduced to an internal constraint. Students seem to have continued to seek constraints, after being initiated into the practice of using them, which they liked and which enhanced creativity.

To answer the second question, students rewrote their own short memoirs using the constraints described and used by A.S. Byatt, Italo Calvino, and Milan Kundera. The students were surprised at how easy it was to "try on" another writer's constraints, and importantly, at how much more imaginative their stories had become.

More next steps are suggested in the next section.

7.3 Suggestions: Using Constraints in Your Classroom

For tasks that are largely unrestricted, especially those that require creativity, imposing constraints can help. We include several examples.

- In literary composition, the challenge of developing an essay on a given theme can be overcome by anchoring it with a set of semantic (e.g., include a given set of words) or formal (e.g., start with a given letter of the alphabet) constraints.
- Again in composition, teach your students how to create their own constraints. They can begin by practicing (as shown in the preceding section) with combined internal (student-generated) and external (teacher-generated) constraints. They can also practice using constraint pairs to (1) identify elements in their current writing style, (2) pick specific elements to preclude, and (3) specify substitutes.
- In developing vocabulary, parsing sentences can provide a useful structure. Once the parts of the sentence are diagrammed, students can be asked to suggest multiple substitutions for nouns, verbs, adjectives. How many ways can this sentence be expanded, made more specific, more interesting?

$$\frac{\text{Sally / baked / cookies}}{\text{/ today / chocolate – chip}}$$
$$\text{/ twenty}$$

- In art, a set of small canvas boards with the same cartoon (a subject constraint) to be painted in different styles (a task constraint) or in different color combinations (also a task constraint) can be a catalyst for creativity. The search space could be narrowed further by specifying a style or palette the student does not usually employ.
- In history, a seemingly dry series of texts, dates, people, and events can come to life within the framework of a beautiful constraint: have the students immerse themselves and become active participants in critical events like the trials of Socrates or Anne Hutchinson. "Reacting to the Past," a program created by Mark Carnes at Barnard College, uses just this constraint to facilitate student engagement and improve critical thinking, problem solving and communication skills (Carnes 2014).

Table 7.5 Problem space for ______________________

Parts	Description	
Initial state		
Search space	Preclude	Promote
Goal state		

Not to be ignored are those inevitable classroom constraints (time, technology, assessment needs) that are not obviously occasions for incrementing creativity. Our suggestion is to make these constraints the "preclude" half of a constraint pair. What can be promoted as a direct result of such a constraint? Let's focus on testing, on using a required assessment tool for pedagogical purposes. Research shows that there are advantages to testing. Test-enhanced learning refers to the finding that taking a test on studied material produces both better learning and retention than re-studying the material for the same amount of time as the test (Roediger and Karpicke 2006; McDaniel et al. 2007). So, some suggestions for all testing on all topics:

- Retrieving is more effective than re-reading. Test frequently.
- Give immediate feedback, which has also shown to enhance competency (Brosvic and Epstein 2007).
- Incorporating "teaching-the-test" into regular lesson plans. This will make your teaching more variable and more effective (Stokes 2013).
- Frame questions that preclude rote memorization and promote meaningful understanding.

Whether your challenge is enhancing competency or creativity, think in terms of paired constraints. Start by filling in the blank problem space below. Identify the initial state and the goal state. Define what each constraint pair precludes and what it promotes, and remember, every solution path starts with a single substitution (Table 7.5).

7.4 Conclusions and Concluding Caveats

Our conclusions are two.

First, paired constraints are tools that can help teachers *design* curricula and lessons to help students develop competency and creativity. Second, re-iterative use of paired constraints (to *re-design* that lesson plan) can help keep your thinking and your teaching new.

Our caveats are also paired. Learning to use paired constraints is like all learning – it takes time and practice. Persistence pays off. The payoff could be finding a new path to a creative solution. Faced with a frustrating outcome, don't cut exploration short and go back to where you started. Chances are you'll only embark on another, equally frustrating search down yet another – or, sometimes, the same! – well-traveled path.

Take your time. Practice.

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Chapter 8
Convergence in Creativity Development for Mathematical Capacity

Ai-Girl Tan and Bharath Sriraman

Abstract In this chapter, we highlight the role of convergence in developing creativity and mathematical capacity. We renew our understanding of creativity from the relations of three creativity mechanisms: Convergence *in* divergence *for* emergence, and three principles of experience: Continuity, interaction and complementarity. Convergence in the context of creativity development is an incidence of learning for capacity building and knowledge construction. Examples of convergent processes in learning are: setting a plan, having a structure, and possessing coordinated capacity to complete a task. To elaborate, we refer to theories of development and creativity on how people develop their capacity in convergence (e.g., collaboration), through mathematical learning (e.g., with coherence, congruence), and for creativity (e.g., imagination). We make reference to convergent creativity of an eminent mathematician *Srinivasa Ramanujan* (1887–1920) for a reflection on developing creativity and building capacity for good life.

Keywords Convergence • Mathematics • Collaboration • Creativity

8.1 Introduction

8.1.1 Scope of the Chapter

This chapter comprises three parts. In the first part we present our assumptions, mechanisms and principles of creativity and creativity development. In the second part we review briefly contemporary views on creativity development and

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knowledge construction. We reflect on the role of convergent creativity in developing capacity through learning a subject matter. In the third part we draw preliminary conclusions that convergent creativity is essential for knowledge construction and for good life. We make reference to the legend-like encounters of a mathematician *Srinivasa Ramanujan* (1887–1920) for some elaboration on convergent creativity, knowledge construction, and mathematical capacity. In addition we refer to the works of *Vadim Krutetskii* (1917–1991) who developed a systematic means of promoting convergent thinking in highly able students. Even though our inquiry into convergent creativity in this chapter is preliminary, we consider reflecting on convergent creativity an important aspect of developing high ability in education.

8.1.2 Assumptions, Mechanisms, and Principles

Convergence in creativity is a basis of knowledge construction. It is necessary for creativity development. We propose four assumptions of convergent creativity in the context of creativity for capacity building and knowledge construction: (a) Convergence is a mechanism of creativity; complementing divergence and emergence, two other mechanisms of creativity. (b) Creativity is an incidence of learning (Guilford 1950); and learning is a continual, interactive and developmental experience (Dewey 1937/1997). (c) Our basic words are relational to the objects, people, and intelligent systems (Buber 1937). (d) The world of knowledge and meanings is constructed through self-discovery (Sundararajan and Raina 2013) and in collaboration (Zittoun et al. 2007). The first two assumptions are related to mechanisms and principles of creativity. The last two assumptions concern construction in the relational worlds of human wisdom, traditions and affordances (what the environment offers). To lead a balance life, convergent *in* creativity takes developing coordinated abilities (structure) and collaborative capacity as core activities in learning. In the world of mathematics, convergent thinking forms the basis of reasoning required to discover invariant principles or properties, as well as to formulate generalizations from seemingly different situations by focussing on structural properties during abstraction (Sriraman 2003, 2004a, b, c).

As a mechanism, convergence complements divergence and emergence, the two other mechanisms of creativity. While convergence and divergence involves a positional change of a set of elements, emergence is about a change of the very set of a system of elements (Kastenhofer 2007, p. 363). Tan (2014) uses a preposition "*in*" to represent complementarity of the mechanisms and their continuous and interactive experience. Briefly, we can consider divergence (e.g., variation and differentiation) as part of *multiplicity* and *variety* in natural and psychological life. Convergence is "getting ready" to transform *randomness* to create patterns in the spirit of upholding the good (Nishitari 1991). The transition of convergence *in* divergence can be *spontaneous* or *goal-directed* which involves *open* acceptance of realities as they are, as well as *effortful* preparation, selection, and cross-checking what is in the mind with what is acceptable. From a mathematical viewpoint, convergent thinking is a vital aspect of moving the field forward, when patterns from seemingly different sets of examples suggest abstractions that reveal structural invariances-when this occurs, seemingly random ideas cohere to form the basis of deep theorems. It is common amongst mathematicians to use the term "beauty" in an aesthetic sense when a result coheres properties from different areas (Brinkmann and Sriraman 2009), and this can be viewed in terms of convergent thinking . Although classical examples abound (e.g., Sriraman 2005), a modern example is the Modularity theorem established by Andrew Wiles, that connects the areas of elliptic curves with modular forms.

Knowing (the present, tacit) and knowledge (the past, domain-relevant) guides emergence of new experiences (Cropley 2006). Tan (2014) recommends the proposition "*for*" to represent the direction to which "convergence *in* divergence" heads, adhering to the continual, interactive, and complementary principles (Bohr 1950; Dewey 1937/1997). Creative learning can be spontaneous and goal-directed. In experiencing creative learning the principle of interaction intercepts and unites with the principle of continuity (Dewey 1937/1997). Interaction precedes and mediates development of knowledge and capacity of creativity (Ponomerav 2008a). Interception is an example of convergence in divergence for emergence of any novel experience. With reference to the principle of complementarity, perception and emotion which are qualitatively different from logical thinking enrich our creativity experiences. Generating novel ideas (Sternberg and Lubart 1999) is in coordination with recognizing the best idea (Amabile 1983), getting ready for construction of meanings. In capacity building (emergence) following the principle of interaction (Dewey 1937/1997) convergence serves as a pulling force of cross-domain influences in learning. Spontaneous learning complements goal-directed learning. Actions and activities are in coherence with the affordances, and in congruence with the minds of the actor and the audience (Glaveanu 2011). In everyday life, we undertake multiple complementary roles (e.g., the actor and the audience) and maintain congruence in these role-identities. The actor in action sets up the plan, coordinates his(her) abilities, transforms him(her)self and the environment when s/he encounters the activity with the audience to which s/he is part. S/he emerges with joy, contentment, and inspiration to bring his(her) audience to witness the beauty of his(her) creative (inter)action.

Emergence is about becoming (Rogers 1961), and bringing something into existence (Frankl 1984). In creative production, the motivated person identifies a theme of interest (*convergent thinking*). S/he proceeds to ideation (*divergent thinking*) and applying domain-relevant knowledge to select the best idea (*convergent thinking*). New ideas or variabilities are refined with reference to domain-relevant knowledge (Amabile 1983). Creative products that are acceptable emerge after a series of iterative convergent processes such as seeing limits, zeroing on the potential, and drawing on the "correct" conclusions (Cropley 2006). As stated earlier mathematics as a field provides astonishing examples of this process. Teaching experiments with high school students indicate that this is not limited to professional mathematicians given the right pedagogical conditions in the classroom (Sriraman 2005).

8.2 Creativity Development

8.2.1 Developing Capacity

In the field of psychology, after the Second World War there have been conscious efforts to remove the "*fear*" of conducting scientific investigations on human creativity (see Guilford 1950). A call to restore the "*faith*" in humanism renewed dialogues on sciences for the good. Spaces were created for interdisciplinary sharing of conceptions of creativity. During this period, some psychologists presented their views of creativity (e.g., Carl Rogers and Joy Guilford) and designed tests to identify creative talent (see Guilford 1957). Subsequent efforts were observed in constructing measurements (e.g., Torrance Tests of Creative Thinking, see Torrance 1966), and proposing models (e.g., a componential model of creativity, e.g., Amabile 1983) and theories of creativity (e.g., a three-systems theory, Csikszentmihalyi 1988). Decades later, these unassumingly "small" efforts received attention and have since served as a preliminary knowledge base that have supported continuous efforts to create possibilities in developing people's creative potential.

Towards the end of the twentieth century, there were social movements to develop potentials of all people under the policies of "the no child left behind" and "every child matters" and the like in different parts of the world. Creativity as a potential of every person has been a widely accepted belief. Movements of nurturing creativity and developing talents rallied supports. Consequently, the number of policies for creative education and creative industries grew sporadically. Some societies set up collaborative plans and rolled out programs to nurture all including the vulnerable (e.g., in 2004 Singapore declared herself as an inclusive society) and the talented regardless of their backgrounds. In the United Kingdom efforts to encourage imagination and nurture creativity were observed in the classroom (Craft 1999). Creativity was acknowledged as a key to innovation and social-economic transformation in Asia such as China, Hong Kong, Korea, Japan, Singapore, and Taiwan. In these societies, one saw an exponential rate of people investing in the capacity to use creative digital devices in their everyday communication compatible to the rate of marketing electronic devices and smart-phone technologies. The adjective of "creative" has been accepted as an everyday vocabulary. More often than not investing in creativity has been linked to economic development sometimes leading to negative consequences for society, e.g., Korea (Sriraman and Lee 2016).

For nearly 15 years, efforts have been seen in converging knowledge of creativity in the field of psychology. Within the first 6 years of the twenty-first century, two annual reviews on creativity were released. The first annual review by Runco (2004) used the four-Ps framework (Rhodes 1961, person, process, product and press or environment) to organize the contents of the past creativity research. The second annual review by Hennessey and Amabile (2010) followed the line of thought of a systems approach and constructed a model to orientate the continuously increase number of studies on creativity which had blurred the disciplinary and cultural boundaries of the field of psychology of creativity. The renaissance of the studies of creativity overshadowed the previously negative sentiments of uncovering the truth of the humans' potentials. Instead, there have been efforts to outline flourishing conditions for and concise understanding of nurturing creativity. In Kaufman and Beghetto's (2009) four-c framework, there emerged familiar terms such as Big-creativity (or historical creativity), professional creativity (or domain-relevant creativity), little or everyday creativity, and transformational creativity or mini-creativity. Their framework suggested that mini-c and little-c are within all persons. Not all people will persist in pursuing professional c. Rarely, a person has the golden opportunity to experience his (her) own Big-c. The factors and conditions that influence the unfolding of professional c and Big c are complex, and are beyond one person's control. Glaveanu (2011) based on the knowledge of cultural psychology rewrote the language of the four Ps to the five 5As (action, activity, artefact, audience, and affordance). In essence, the person in his(her) social-cultural realm is an actor and an audience, whose action and activity are influenced by the affordance (or what the environment offers). With reference to positive and humanistic psychologies and Zen Buddhist philosophy, Tan (2012) advocates nurturing creativity for constructive growth, ethical practices, and the good.

Creativity is about generating novel and useful ideas (Sternberg and Lubart 1999), bringing something to being (May 1975) and flourishing humanness (Frankl 1984). In the literature, the genre of *novel* best represents life. Novel is constructed in "the zone of direct contact with the inconclusive present day reality" (Bakhtin 1981, p. 39). In life, humans continuously interact with the others, share their knowledge, and experience novelties. Collaboration that generates conceptual conflicts creates ruptures in the existing knowledge systems and opportunities for knowledge innovation. Conceptual convergence in iteration accommodates conflicts and transforms them to shared knowledge. Accommodation is a form of adapting by modifying cognitive structures to fit the otherness. As an incident of learning, developing capacity is converging broad-based sensing, perceiving and feeling of the world and the "correct", systematic, logical and goal-directed processes of representing the world (Ponomerav 2008a).

According to Ponomerav (2008b), throughout our life, we experience development of multiple forms of knowledge. *Contemplative-explanatory* knowledge emerges from the curiosity and philosophical needs of society. It grows out of practice, common-sense, life experience, work of literature and art, and so on. The person contemplates and records everyday knowledge; and describes it with reference to some existing theory. *Empirical knowledge* relates directly with concrete objects and integral events. It assists in solving practical problems. *Active-transformative* knowledge takes empirical models and transforms into abstract-analytical knowledge. It builds systems of modelling. We never cease to grow our coordinated abilities to react, represent (plan), recall, and reproduce meanings and knowledge. The world of knowledge we construct "mirrors" the diversity, order, and creativity of the world in which we live. Creative cultural divergence is based on internal persuasive discourse, which "is freely developed and allows for new voices to join in and participate." (Hsu 2012, p. 108) Creative persons possess creative qualities, abilities or characteristics (see Guilford 1950) which include but are not exclusive to sensitivity to problems, ideational fluency, flexibility of set, ideational novelty, synthesizing ability, reorganizing or redefining ability. Knowledge construction and innovation is a mean and an end of creativity development. In creating, we are motivated to generate and explore (Ward et al. 1999) and coordinate abilities which include motivating, domain- relevant processing (knowledge, techniques) and creativity-relevant processing (ideation, breaking sets) (Amabile 1983). In each phase of creating, there exists a "*continuum*" of sub-processes (Lubart 2001).

8.2.2 Constructing the World of Knowledge

Knowledge construction is a social-historical, cultural, and over-generational endeavour. It is a convergence *in* divergence *for* emergence experience in the human world throughout our life span. In the field of psychology, convergent creativity in the context of knowledge construction has been elaborated by eminent psychologists, Jean Piaget (1896–1980), Lev Vygotsky (1896–1934), and John Dewey (1859–1952), to name but a few. A child prior to language acquisition constructs the world of knowledge by using his(her) senses, feeling, perception, and movement (Ponomerav 2008a). Scaffolding is an example of social-cultural *convergence* when the adult enters into the zone of proximal development of the child and guides his(her) development (Vygotsky 1978). Creative teaching is dependent upon *congruence* in teacher roles and a process of *coherence* in assessment, activity and instruction. Teachers adopt multiple roles in everyday classrooms. They possess multiple role-identities. *Congruence* as an instance of convergence in divergence in creative teaching (Tan 2015a) is about relevant, multiple teacher role-identities that are combined as a co-ordinated competency and that guides the dissemination of knowledge and skills. *Coherence* is another instance of convergence *in* divergence of creative teaching (Tan 2015a) taking the process of delivery of effective instruction as a mean and an end towards creative learning. The teachers design the lessons of the day according to the needs and styles of the learners.

Convergence signifies the readiness to transit to the capacity of constructing a structure, designing a plan, or coordinating abilities. According to Rich et al. (2013), convergent cognition refers to a *common underlying conceptual base* in which the relationship is unified, interconnected, and interdependent. Immediately after a baby is born, s/he is ready to *imitate* movement of gestures that s/he notices (Rizzolatti and Craighero 2004). S/he is curious to feel and touch any objects or people who appear before him(her). The new born is prepared to relate to the caregivers. S/he is motivated to fulfil his(her) needs. S/he has the potential to acquire knowledge in everyday life and in various domains. S/he is intuitively alert to learn about all things that come before him (her). *Imitation* is a significant *convergent* capacity that a child has since s/he is born. Through *imitation* s/he relates to the caregivers and people around him(her). Further, the child has the ability to organize information available and accessible to him(her). If the information fits into his(her) cognitive structures, s/he assimilates it; otherwise, s/he modifies his(her) cognitive structures to accommodate the new information. Organisation and adaptation (assimilation, accommodation) are *convergent* processes in constructing the world of knowledge, mental models, structures, coordinated abilities, or internal plans. "Operation" or structure is a reality referring to the child's deductive capability (Piaget 1928).

8.2.3 Everyday Learning

The authors are in favour of a "*middle way*" (moderate) and a continuum approach to convergence in creativity (see Tan 2014). Observing the practice of moderation in life we adhere to cultivating good conducts and avoiding dogmatism. We are contented with sufficient, optimal and balanced conditions in life. On becoming a person, we make appropriate effort to carry out what we deem as important for ourselves and for the people around us. With an understanding of continuum, we bridge the seemingly dichotomous discourses of divergence versus convergence such as a divergent task (list as many usages as possible for a "paper clip") versus a convergent task (At a cold and icy night a tired man reaching home had his dinner and switched off all the lights before he went to bed. The next morning he woke up and found himself walking through his residence full of dead bodies. What caused deaths of the residents in the man's home? (see e.g., Nielsen et al. 2008). We postulate that the underlying creative process and action of generating and exploring likely common but the transformation of creative structures in interaction with accumulative, coordinated, and integrative information creates variations in forms creativeness and types of knowledge.

Learning in everyday life and classroom shall regard congruence in roles and coherence in contents and process of learning (Tan 2015a). In a newspapers article released on August 24, 2015, a story of twin cubs which were born, with 4 h apart in the Zoo of Washington attracted the attention of a home tutor. According to the article, the mother panda gave birth to the first cub in 2005, the second in 2012, which unfortunately died 6 days later, and the third in 2013. A decade after the first cub was born, twin cubs arrived after the mother panda successful went through the process of inseminated fertilization from frozen and lived sperms of two different male panda living in China and the United States of America.

To understand the information reported in the article, the adult adopted a *convergen*t approach to posing questions related to factual knowledge, accuracy of information, and drawing "correct" conclusions (Cropley 2006). The child displayed some traces of coordinated abilities in reading the text, relating his previous reading on the same theme months ago, and in imagining how life can be better for both the adult and cub pandas in the zoo of Washington and in his home country (Singapore). Variability in generating (Cropley 2006) questions were used to assist the child to grasp accurately factual information of the article: (a) Who was the author of the article? (b) Where did the Panda cub deliver? (c) What were the names of the officials in the zoo who spoke to the reporter? (d) How many cubs have the panda delivered since 2005? (e) How many of her off-springs survived? (f) How far was apart from her first delivery to the present deliveries? Variability in exploring (Cropley 2006) aimed to guide the child to conclude what he read. It went beyond the individual panda in the article and discussed about the endanger species in the world. As a matter of fact, according to the article, there are about 1600 of them in the wild life, and 300 in captivity. Those who live in captivity have a low rate of fertility. The child who read the newspapers article answered the questions and penned down three sentences summarizing the essence of the article. He experienced some disequilibrium especially in understanding the term "twin" and the word "survived". Guided by the adult, he related the number "two" to the word "twin", the operation of "subtraction" to the early dead of the 6 day old panda. Learning is a cycle of rhythm of life, relating abstract ideas (e.g., death, survive) to concrete experiences (e.g., panda in the Singapore zoo), and interests of the child (e.g., reproduction). The joy of discovery in the cycle of romance from free questioning guides the child to the next cycle of learning appreciating the need to learn the language of a discipline such as mathematics (Woodhouse 2012). Improvisation is novel as new behaviour emerged in the in-between space and through collective interaction (Sawyer 1999). The child summarized the essence of the article and improvised the life in the wildlife and in the zoo. He imagined how the cubs played with their mother, and how he observed them playing taking the role as a visitor of the zoo at home and in the bamboo forest far away.

8.3 Classroom (Mathematical) Learning

Convergent thinking plays an important role in mathematical learning. Many early learning processes such as sorting, counting, stacking, categorizing converge into the abstraction of ordinality and cardinality, and the basis for the generalization of number. As students progress through mathematics, the structures they encounter become increasingly abstract (sets, relations and so on) with generalization as a key feature of mathematical thinking. In a sense abstraction and generalization can be viewed as a convergence of thinking of different properties of mathematical objects and the ability to eliminate superficial similarities to focus on structural similarities. Vadim Krutetskii (1976) analyzed the generalization ability of both "normal" and gifted students in a series of experiments. Krutetskii viewed the ability to generalize as one manifestation of the creativity of the individual. He hypothesized that "students with different abilities are characterized by differences in degree of development of both the ability to generalize mathematical material and the ability to remember generalizations" (Krutetskii 1976, p. 84). One of the attributes of students who were able to generalize mathematical ability was the ability to switch from a direct to a reverse train of thought (reversibility), which capable students performed with relative ease. The mathematical context in which this reversibility was observed was in transitions from usual proof to proof via contradiction (reductio ad absurdum), or when moving from a theorem to its converse.

Krutetskii studied 19 students with varying mathematical abilities. The problems used by Krutetskii in his experiments met the following criteria. (1) The problems were of graded difficulty; (2) The problem sets consisted of simple problems as well as some that required “mathematical creativity”; (3) Some problems were simply put to evaluate skill mastery. These problems were based on new material that students had encountered in their curriculum. Based on his experiments with the 19 students, Krutetskii concluded that more “capable” students were able to rapidly and broadly form mathematical generalizations. He noted that these “capable” students were able to discern the general structure of the problems before they solved them. The “average” students were not always able to perceive common elements in problems, and the “incapable” students faired poorly in this task. These results led Krutetskii to examine “gifted” students separately followed by an examination of incapable students. The final experiment was a study of 24 “gifted”, 22 “average” and 8 “incapable” students. Based on these series of experiments Krutetskii identified four levels of generalization as a function of the ability of the students. The researcher will quote directly from Krutetskii’s writings.

Level 1: Cannot generalize mathematical material according to essential features even with help from the experimenter and after a number of intermediate single-type practice exercises.

Level 2: generalizes mathematical material according to the essential features with the experimenter’s help and after a number of single-type practice exercises, with individual inaccuracies and errors.

Level 3: generalizes mathematical material according to essential features, independently, but after some single-type exercises and with insignificant errors. Proper faultless generalization comes with insignificant promptings and leading questions from experimenter.

Level 4: generalizes mathematical material correctly and immediately, “on the spot”, without experiencing difficulties, without help from experimenter, and without special practice in solving problems of a single type (Krutetskii 1976, pp. 254–255).

Krutetskii came to the conclusion that in order for students to correctly formulate generalizations, they had to abstract from the specific content, and single out similarities, the structures and relationships. The ability to generalize consists of two aspects: (1) subsuming a particular case under a known general concept; and (2) the ability to deduce the general from particular cases (in this instance the generality is unknown). The work of Krutetskii has subsequently been extended by Sriraman (2002, 2003, 2004a, b, c) in the contexts of number theory and combinatories where high school students were able to distill convergent properties such as Steiner n-tuples, the Dirichlet principle and Diophantine n-tuples. In all these experiments the principle of “convergence *in* divergence *for* emergence” became exemplified and supports strong evidence to Tan’s (2015b) theory of convergent thinking for fostering creativity. Moreover the case study of Ramanujan presented in the concluding section of this chapter provides a compelling account of convergent creativity that unified the study of infinite series, continued fractions and geometric function theory into "convergent" formulae that still provide number theorists fodder a century later- even though this may be construed as an extreme case, it nonetheless illustrates our argument for fostering convergence in creativity development.

8.4 Conclusion

This chapter elaborates the mechanisms of convergence *in* divergence for emergence with reference to *knowledge-induced* creativity which has its base in *effortful* creativity (Cropley 2006; Tan 2015b). Our assumptions for convergent creativity include learning is a way of life to develop capacity of mankind for the good, ethics, universal values, health, positivity, and possibility. Learning for life goes beyond effortful commitment, knowledge-induced engagement, and domain-relevant processes. Learning *in* practice takes convergence *in* divergence as a core basis for emergence. Throughout our life, we master meta-theoretical strategies (selection, optimization, and compensation, Baltes 1987), meta-awareness (imagination, and imitation), meta-regulation (broaden-and-build theory, Fredrickson 2001), and so on. We construct different types of knowledge (e.g., contemplative, explanatory, and empirical, Ponomerav 2008b) and show varying forms of creativity (e.g., mini-, everday, and professional, Kaufman and Beghetto 2009). Mathematical learning and creativity is a life engagement. It is about thinking and feeling the world through mathematical symbols, representations, and language. The ultimate aim of mathematical learning and creativity is good life, ethical relations, and healthy living. Our understanding of convergence *in* creativity in mathematical learning and creativity is a cultural and disciplinary boundary crossing endeavour. Table 8.1 outlines the main points of convergent creativity for capacity building.

8.4.1 A Novel-Like Legend

This chapter presents our preliminary thoughts on convergent thinking, creativity, and mathematical learning. The chapter regards creativity as an incidence of learning (Guilford 1950), and learning as a way of life to build capacity to discover

Table 8.1 Convergent creativity for building capacity

Mechanism	Principle	Process	Structure
Convergence	Continuity	Self-discovery	Imagination (intuition)
Divergence	Interaction	Collaboration	Tradition
			Cross-domain knowledge and knowing
Emergence	Complementarity		Imitation

meanings in life. Learning is both personalized and collaborative in inquiring into the good (Tan 2012) and in acquiring knowledge from the human society. Particularly, we wish to understand convergent creativity in light of capacity building and knowledge construction in the context of learning a subject matter (e.g., mathematics). Our reflections on convergent creativity orientate around a legend of the world renowned mathematician, *Srinivasa Ramanujan Iyengar* (1887–1920). We regard as fortunate to come across abundant narrations and analyses of Ramanujan's life and work freely accessible in the Internet including those from eminent people such as G. H. Hardy (1877–1947), Bruce C. Berndt (1939–), and George E. Andrews (1938–) (see Berndt n.d.; Andrews n.d.-a, n.d.-b). In this chapter, we specifically make reference to a British mathematician, Hardy's (1937) inspirational account on Ramanujan's mathematical capacity, his unique method of inquiry, and convergent creativity in mathematics.

> It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked … by induction from numerical examples … (W)ith his memory, his patience, and his power of calculation he combined a power of generalization, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often rally startling, and made him, in his own peculiar field, without a rival in his day. (Hardy 1937, 149)

At the early years of development, Ramanujan encountered traditional wisdom (*divergence* in cultural traditions, see Hsu 2012) and showed interests in numbers. According to Hardy (1937), Ramanujan grew up in a Brahmin family of a high caste. He adhered to all observations in his caste, and remained strictly a vegetarian practitioner until the end of his life. Ramanujan, who was once a clerk turned an extraordinary mathematician, showed at his early years (before 10) his exceptionality in mathematics. His talent in mathematics was recognized at the age of 12 and 13. Only at the age of 16 he was exposed to George Schoobridge Carr's volumes (*A synopsis of elementary results in pure and applied mathematics)*. Carr's volumes were an inspiration for Ramanujan to establish formulae. With no other resources, each solution was like a piece of research to him. Ramanujan credited his achievements in arriving at the formulae to the gifts he received from the goodness of Namakkal in his dreams. It was believed that Carr's volumes contributed to the unfolding of full powers in mathematics in Ramanujan. With some encouragement from people in his homeland, India, Ramanujan wrote and sent his voluminous work to great mathematicians in the United Kingdom. In his twenties, Ramanujan left for England and had since worked closely with Hardy. His talents in mathematics flourished further in the new environment in which he engaged in intensive and daily sharing of his discoveries with his mentor: "… he was showing me half a dozen new ones almost every day … ." (Hardy 1937, p. 146).

Travelled to Cambridge and worked with colleagues in the Trinity College, Ramanujan and his mentor(s) combined their personal (imagination) and social resources (collaboration), funds of knowledge (including contemplative-explanatory, empirical, and active-transformative, Ponomerav 2008b), and sources of creativity (e.g., professional and Big, Kaufman and Beghetto 2009). Together they resolved conceptual conflicts and brought their knowledge in mathematics to convergent creativity. Ramanujan's algebraic formulae and transformation of infinite series orientated the direction of his creations, and served as heuristics to design meanings (see Cropley 2006). Ramanujan was a great master of hypergeometric series and continual fractions (Hardy 1937); creating new series and patterns beautifully which was therapeutic, meaningful, and spiritual to him. Life was the beauty of mathematics (*emergence*). In Hardy's (1937) account, Ramanujan's was enthusiastic to generate and share with him novel theorems: "… he was a mathematician anxious to get on with the job … " (p. 146) Proofing his own theorems and inquiring into the process of creating novel theorems were not in his list of priory and interest. Hardy, the British mentor showed Ramanujan the importance to derive proofs to verify his own discoveries. Ramanujan's determination to overcome challenges and to maintain enthusiasm in mathematics served as an exemplified case of inquiry into *convergent creativity* (see Craft 1999). To elaborate we cite Hardy's account on Ramanujan's convergent creativity: "It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing. … He worked, …, by induction from numerical examples; all his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation he combined a power of generalization, a feeling for form, and a capacity for rapid modification of his hypotheses …" (Hardy 1937, p. 149).

Hardy admitted that his discovery of Ramanujan was a romantic incident in his life time. He was able to understand Ramanujan's brilliance, and was ready to openly listen to Ramanujan's generative variabilities daily and to critically build up his capacity in finding proofs. Ramanujan's joy of imagination was substantiated by Hardy's persistence in searching for proofs. "A mathematician usually discovered a theorem by an effort of intuition; the conclusion strikes him as plausible, and he sets to work to manufacture a proof." (Hardy 1937, p. 151) Coordinated abilities of generating theorems and of proving the theorems systematically are essential to remove any fallacy and to confirm accuracy of one's imagination.

8.5 Convergence in Creativity for Good Life

Inquiry into the good requires a continual renewal of *moderate attitudes* towards the external influences (*continuity*). It is about living *in* the tradition and contemporary knowledge, as well as living *with* self-discovery (Sundararajan and Raina 2013) and openness in collaboration (Jarczak 2011). We reflect on *convergence*, a mechanism of creativity and a core of *effortful* endeavour. Convergent thinking exists in *effortful* creativity (Cropley 2006), which is knowledge-i*nduced*. Effort on a *continuum* of the action or process is likely a matter of the degree of intensiveness. Likewise, on a *continuum* of knowing and knowledge is likely the common underlying process and action of creating, while emergence of variations in terms of forms and functions is likely dependent on how accumulative, coordinated, and integrative information interacts with and makes senses to the persons who use it. Tacit knowledge (Polanyi 1968), people's conceptions (Sternberg 1985), what is in the person's mind or the like is a basic of scientific knowledge (see Polanyi 1968). Creative thoughts share commonalities of *goal-directed* thoughts (*complementarity*), which occur during problem solving, planning, reasoning, and decision making, and *spontaneous* thoughts, which includes mind-wandering (Mok 2014) and memory processing (Christoff et al. 2008). Collaboration in knowledge convergence involves the like-minded people with epistemic curiosity in resolving conceptual conflict and in committing to iterative refinement of shared knowledge (Jarczak 2011).

For convergent creativity to emerge, it is essential for a person to possess the structure of creative agency. In learning, humans are agents of change (Bandura 2001), actors innovators of knowledge-based activities. Humans as agents are reactive, pro-active, reflective, and creative. We are able to predict or think ahead of time and space (Bandura 2001). We understand the social cultural world not only when we take conscious and effortful actions but also when we observe how *interactions* unfold when we are with other people or part of shared systems. Born into the living human world, we sense, observe, and model the others. We develop meta-strategies to optimize opportunities and compensate our shortcoming (Baltes 1987). Humans are receivers of accumulative expertise, knowledge, and skills. We create our own world of knowledge. An essence of creativity development is to shape, orientate, and lead life to its fullest (Cropley 2006). Our seeing and knowing orientates our direction to develop humanism within and without. We put in intensive *efforts* to develop coordinated abilities, collaborate with the others, synthesize resources, set up plans, and relate to the good and ethics.

In narrating the great Indian genius in mathematics, Hardy (1937) revealed his wisdom to recognize the invaluable gift of Ramanujan. Ramanujan built up his capacity in multiple aspects in mathematics mainly through self-absorption in and joy of doing mathematics. He enjoyed a prolonged duration in the romantic phase of learning (Woodhouse 2012) and his absorbed mind was at all times engrossed in the stage generating novelties and exploring discoveries (Ward et al. 1999). The pre-inventive structures of mathematics went through rounds of transformations and emerged as novel formulae and series. Hardy admitted the reservoir of Ramanujan's unprecedented imagination. In his narration, Hardy (1937) noted how Ramanujan instantly saw a pattern in a number (e.g., 1729) as "a sum of two cubes in two different ways" (p. 147). It was evident that Ramanujan possessed convergent capacity to generate varying types of *knowing-knowledge* (contemplative, explanatory, empirical, and active-transformative, Ponomerav 2008b). He had the capacity to coordinate abilities (e.g., imagination) and to transform tacit knowledge (e.g., the number plate of a cab) instantly to scientific knowledge (e.g., the series patterns). As a critical contemporary and mentor, Hardy identified Ramanujan inadequate capacity to derive proofs for some of his imaginations particular in the analysis of number theory. Hardy attributed this Ramanujan's incapacity to the insufficient instructions and mentorship he received in his early years of education. Our inquiry into Ramanujan's creativity development seems to concur preliminary to the critical reflections of scientific creativity that formal education can be important to transform creatively some forms of knowledge (e.g., analysis). Freedom of imagination and transformation is essential for other forms of knowledge creation. It is imperative and immediate for educators and scientists to examine the good of schooling and a claim that formal education limits creativity capacity. Likewise it is essential to investigate the importance of freedom of thought that enhances creative imagination in varying incidents. Future reflections shall focus on imagination as a prerequisite of creativity (Vygotsky 2004), its roles in convergent creativity, and its possibilities for knowledge construction.

8.5.1 Final Words

In generating theorems, functions, and series, Ramanujan attained the supreme joy compatible to the spiritual unity with Namakkal. He reached a high level of imagination, self-discovery, and spontaneous cognition. Convergence in creativity unfolds and consolidates after a series of iterations of imagination (divergence) and/or analysis (convergence). We are hopeful that under flourishing conditions knowledge-*induced* convergence in creativity not only a source of domain-relevant capacity but also a reservoir for self-care and wellness. Convergent creativity has its underlying processes in spontaneous and controlled cognition (Mok 2014) as well as in creative imagination and creative collaborations (Hardy 1937). The continuous appreciation of “the loss notebook of Ramanujan” and its subsequent renewals in knowledge of mathematics has remained amazing evidence of convergent creativity. Our next journey of reflections on convergent creativity shall focus on how to re-construct the conditions for imaginative creativity that Ramanujan enjoyed in India and those for analytical creativity that he experienced in England. Reflections on how flourishing conditions enhanced convergent creativity can be a key to support full development of human potentials in the presence of positive affordances (what the environments offer for ethical practices). Our views on convergent creativity can serve as a gentle reminder for a meaningful journey of creativity development not only in the world of mathematical abstractions but also in the relational and humanized world.

8.6 Remarks

The compilation of this chapter is evidence of an interdisciplinary cooperation between a psychologist-educator and a mathematician-educator, going beyond their own fields of specialization into a genuine understanding of creativity and giftedness in learning.

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Chapter 9
The Origin of Insight in Mathematics

Reuben Hersh and Vera John-Steiner

Abstract This paper has four objectives: (1) to address some psychological sources that motivate creative mathematicians to do sustained research, (2) to use case studies and self-reports to identify cognitive and mathematical strategies, (3) to give inspiring examples of creative breakthroughs in the teaching of mathematics, (4) to report on a startling recent discovery in artificial intelligence, with thought-provoking implications for the management of human intelligence: the pursuit of novelty, unrestricted by any other prescribed goal or objective, radically speeds up evolutionary adaptation.

Keywords Creativity • Mathematics • Clarence Stephens • Potsdam model • Moore method

The study of creativity increased dramatically in the last 30 years. It moved from the study of genius, started by Galton in the nineteenth century, to wide-ranging inquiry, both into major innovations and contributions in the arts and sciences, and into creative behavior in everyday life. Most of the studies have been carried out by social scientists, but there is increasing interest in other disciplines to examine insight, discovery, and long-term or transformative contributions.

The leading mathematician Terence Tao challenges the myth that one has to be a genius to make contributions to mathematics.

> In order to make good and useful contributions to mathematics, one does need to work hard, learn one's field well, learn other fields and tools, ask questions, talk to other mathematicians, and think about the "big picture." And yes, a reasonable amount of intelligence, patience, and maturity is also required. But one does **not** need some sort of magic "genius gene" that spontaneously generates *ex nihilo* deep insights, unexpected solutions to problems, or other supernatural abilities.

His comments agree with the psychologist Howard Gardner (*Creating Minds*, 1993), who emphasizes the long apprenticeship of people engaged in creative work, their early attempts to develop a new approach, and their protracted commitment to

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solving interesting problems, in order to achieve significant changes and even transformations (John-Steiner (1997)). Tao points out that

> The number of interesting mathematical research areas and problems to work on is vast – far more than can be covered in detail just by the "best" mathematicians, and sometimes the set of tools or ideas that you have will find something that other good mathematicians have overlooked, especially given that even the greatest mathematicians still have weaknesses in some aspects of mathematical research. As long as you have education, interest, and a reasonable amount of talent, there will be some part of mathematics where you can make a solid and useful contribution. It might not be the most glamorous part of mathematics, but actually this tends to be a healthy thing; in many cases the mundane nuts-and-bolts of a subject turn out to actually be more important than any fancy applications.

9.1 Psychological Sources of Creativity

In a classic study, Getzels and Jackson (1962) found that gifted adolescents had parents who encouraged their offspring's adventurous spirits and enthusiasm for life. Once these students focused on a particular area, they learned to join skill and discipline to their sense of wonder. In the life story of contributors to science and art we often hear of lonely childhoods where reading, solving problems, or studying nature filled a gap caused by illness or social isolation. Such people like to read biographies of "distant teachers," whose experiences replenish a desire to go beyond the known

Closely related to the curiosity of creative people is their intensity, which can make them stay with a problem that others would abandon as too difficult. Einstein recalled how at age 5 he attempted to understand how a compass worked and guessed that "something deeply hidden had to be behind things" (Schillp 1970, p. 5). In a study of successful competitors in the international Mathematical Olympiad, the author Steve Olson found that the competitors revealed "an extraordinary capacity for mental focus; they were willing to stay with a problem for hours, or even days" (Olson 2004, p. 63 in Hersh and John-Steiner 2011, p. 23). If there's one quality young mathematicians share it's this power of concentration. In a book about prodigies Feldman and Goldsmith (1986) write of a similar tendency of intense dedication, self-confidence, and "a mixture of adult and child-like qualities" (p. 12). Ellen Winner (1996), in her book *Gifted Children*, describes young prodigies who already as preschoolers displayed an intense fascination with numbers. One such child, KyLee, looked for numbers wherever he went. His memory for numbers was also astounding. The computer provided him with additional ways to pursue his interest in addition, word problems, and other arithmetic operations. He called himself "the number boy."

Many mathematically gifted children score high on spatial tests. Some are also high-scorers on verbal tests. They enjoy taking toys apart and trying to figure out how machines work. Their memory for numbers is often impressive.

Their persistence and curiosity can lead to early success in mathematical tasks, and to rewards, like honors classes in mathematics, specialized summer camps, and local, state, national, and international competitions. All of these contribute to an individual's self-confidence. Gifted young people spend more time alone than with their peers, and their specialized interests can contribute to isolation, so opportunities for contact with others who share their interests are important.

Some young mathematicians are fortunate in receiving exceptional instruction in their early adolescence. Richard Courant was encouraged by his teacher who preferred to lead his students to discoveries on their own, rather than teaching them formulaic approaches. The exceptional instruction at the Lutheran Gymnasium in Budapest, Hungary, produced some famous mathematicians. The best known of them was John von Neumann. In recollecting their mathematics teacher László Rátz the physicist Eugene Wigner wrote, "He loved teaching; he knew his subject and how to kindle interest in it. He imparted the deepest understanding. Many…teachers had great skill, but no one could invoke the beauty of the subject like Rátz" (Wigner 1992, p. 50).

Independence and willingness to find their own way of solving problems can be the beginning of the lifelong courage, to go beyond the accepted method of solving a mathematical problem (Gustin (1985)). Many young people with mathematical gifts find the ordinary classroom a difficult setting. It concentrates on memorizing algorithms, rather than innovative problem solving and deeper knowledge of mathematical abstraction. Mathematics classes at specialized schools like the Bronx High School of Science in New York City provide for analogical reasoning, discovery of patterns and ways of expanding them. Not all mathematically gifted children become mathematicians; many choose other sciences or engineering as their future profession.

In his book *Creativity*, Mihaly Csikszentmihalyi (2009) identified ten characteristics of creative individuals. These characteristics can be contradictory. The first one is high energy coupled with ability to work quietly. This relates to intensity mentioned above. Creativity tests often measure fluency. Creative individuals confronted with a challenging task can produce a lot of associations and problem solving strategies. But the chemist Linus Pauling commented that one of the secrets of effective problem solving is to both come up with a lot of ideas and also to know how to get rid of most of them. In a related vein, Hadamard (1945) wrote, "Good mathematicians, when they make them [errors], which is not infrequent, soon perceive and correct them. I make many more of them than my students do; only I always correct them so that no trace of them remains in the final result. The reason for that is whenever an error has been made, insight…warns me that my calculations do not look as they ought to" (p. 49).

Playfulness is another characteristic listed by Csikszentmihalyi, which he couples with its opposite, ability to concentrate for long periods of time. In an interesting study with male and female physics graduate students, the Danish anthropologist Hasse (2002) found that men engaged in more playful explorations of physical concepts, while females remained more focused on acquiring knowledge in their domain. This difference reaches back to childhood, where boys often like rigorous play activities and joking with each other, while girls may prefer long conversations. Male mathematicians enjoy jokes and stories about their colleagues, one way in which they construct their sense of belonging to a shared tradition.

Innovative work is not always rewarded, particularly in its early stages. So it is important to enjoy the work itself. In speaking about their research, many mathematicians convey a deep sense of joy and aesthetic pleasure in resolving difficult problems with elegant and clear solutions. One of the most fully documented mathematical discoveries is Andrew Wiles' solution of Fermat's Last Theorem. The philosopher of mathematics Emily Grosholz (2011) describes his motivation as follows:

> On the one hand, we have a narrative about an episode in the life of one man (in a community of scholars) who, inspired by a childhood dream of solving Fermat's Last Theorem, and fortified by enormous mathematical talent, a stubborn will, and the best number theoretical education the world could offer, overcame various obstacles to achieve truly heroic success. Indeed, the most daunting and surprising obstacle arose close to the end, as he strove to close a gap discovered in the first draft of his proof. (p. 21)

It took Wiles seven years and the help of a collaborator to reach his final result. He didn't let the mathematical community know that he was working on this problem and that it was taking him such a long time. While much of the work was done in solitude, he was using a great variety of previous results and strategies developed by his predecessors. Csikszentmihalyi characterizes creative work as both deeply rooted in the traditions of established knowledge together with a willingness to go beyond what is known – A trajectory which in Wiles' case was nourished by "a stubborn will."

9.2 Cognitive Strategies in Mathematical Problem Solving

One frequently used strategy in developing new insights is *extending the domain*. Geometers in the nineteenth century confronted the challenge posed by Euclid's parallel axiom. In Göttingen, Germany, Carl Friedrich Gauss, one of the great mathematicians of his era, had a close friend from Hungary, Farkas Bolyai, with whom he shared ideas about this axiom. In 1799 Gauss wrote to Bolyai about his efforts to try to prove this axiom, and bemoaned his difficulty in doing so. Bolyai worked on this problem while he taught mathematics for 47 years at Maros-Vásárhely. We now know that direct attempts to prove this axiom cannot work. Bolyai wrote in his autobiography: "I was obsessed with this problem to the point where it robbed me of my rest, [and] deprived me of my tranquility" (Halsted 1955, p. xi).

Bolyai's son János studied mathematics with his father. His mastery was very rapid, and he pushed his father to present him with increasingly difficult material. At age 15 he went to Vienna to study engineering. Then he entered the military. While posted as a captain at Temesvár, he continued to work on the problems that had preoccupied his father and Gauss. He had the simple but profound insight that in deriving the consequences of denying Euclid's parallel postulate, he was not arriving at the expected contradiction. On the contrary, he was creating a whole new, radically different and unimagined "geometry." (For instance, the sum of the angles of a triangle is not equal to two right angles, as in Euclidean geometry: it can take on ANY value LESS than two right angles.) In presenting this work to his father, who originally warned him against devoting himself to this topic, he expressed his sense of gratitude and closeness to the elder Bolyai, "In certain regards I consider you as a second self" (Halsted, xxviii). His father was eager to have these results published rapidly and he attached them as an Appendix to his own major work the Tentamen. He also added some of his own reflections and a comparison of Bolyai János' work with that of Lobatchevsky, who also developed what he called "imaginary geometry." His ideas were basically the same as those of Bolyai. Gauss responded to Bolyai's work by agreeing to its validity because it corresponded to some of his own, unpublished work.

> While these mathematicians started their work by extending the domain of geometry, in the process of their endeavors they created a whole new subfield, non-Euclidean geometry. The problem itself pushes the thinker to a new paradigm beyond the boundaries of the one dominating the field at the start of their work.

The achievement of Bolyai and Lobachevsky became integrated into three-dimensional Euclidean geometry when the Italian geometer Beltrami discovered that the axioms of non-Euclidean geometry are satisfied by the "geodesics" (lines of shortest connection) on a certain saddle-shaped surface called a "pseudosphere." This interpretation proved that the seemingly anti-intuitive or meaningless non-Euclidean geometry actually is consistent (free of self-contradiction), at least if we are confident that Euclidean geometry itself is consistent.

Still another remarkable interpretation of non-Euclidean geometry was discovered by the great French analyst Henri Poincare. Poincare was motivated to study non-Euclidean geometry by a striking insight he experienced while stepping onto a bus on a geology fieldtrip (an anecdote often repeated, as in Hadamard 1945). He suddenly "saw" that the transformation rules of non-Euclidean geometry are just the ones he needed for a totally different problem: the rules for transforming the "theta-Fuchsian functions."

Poincare's model of non-Euclidean geometry is more exotic even than Beltrami's pseudosphere. In the "upper half of the complex plane" (the complex numbers whose imaginary part is positive) he defines a new kind of "straight line." Of course, this is not "straight" in the usual sense of the word, but it does satisfy the axioms for a "straight line" in non-Euclidean geometry. The key point is to define distance along the "straight lines."

If A and B are two points in this upper half-plane then Poincare's non-Euclidean "straight line" from A to B is defined to be an arc of the semi-circle through A and B which has its center on the "boundary" (the line $y=0$). (There is exactly one such semi-circle.) Now, he wants the distance to be "additive," meaning that if C is any point between A and B, the "distance" (which we are trying to define) from A to C plus the "distance" from C to B should equal the total "distance" from A to B. Also, since he doesn't want to allow points to reach the boundary, he wants the "distance" from any point to an end point of its semicircle to be infinite.

Poincare's distance formula for his model of non-Euclidean geometry is somewhat complicated, unless you have studied projective geometry. In projective geometry, one meets a thing called the “cross ratio”; it is a certain algebraic expression involving four collinear points, and is important because it is a projective “invariant.” (Under a projective transformation, distances and angles may change, but the cross ratio remains the same.) Poincare constructs the semicircle with the center on the real axis through A and B. This semicircle cuts the real axis at two points, call them U and V. A and B have projections on the real axis, call them A′ and B′. The distance between A and B is defined to be half of the logarithm of the cross ratio of A′, B′, U,V. It works; you can check it out. (There is an exception if A and B happen to be on a Euclidean straight line perpendicular to the real axis. In that case, the Poincare non-Euclidean distance is taken to be the same as the regular Euclidean distance.)

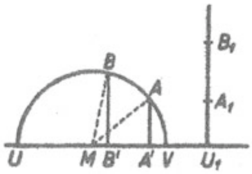

Diagram taken from Meschkowski, “Noneuclidean Geometry” (Meschkowski (1964))

One hard thing to accept or get used to about this half-plane model of Poincare’s is just that it is a half-plane. In order to get this convenient, easily constructed model, he mapped the non-Euclidean plane onto a half-plane. One just feels ill at ease, uncomfortable, cutting off or throwing away the other half, the lower half-plane, which is simply forgotten about; it is not needed or useful in this model. The discomfort is because of a breaking of symmetry, the symmetry in the ordinary complex plane between the upper and lower halves, has been broken, by throwing away the lower half, and defining a new distance in the upper half.

Jacques Hadamard (1945) quoted a remark of Souriau that “in order to invent, one must *think aside*” (p. 48). This strategy applies to the Ph. D. thesis of Reuben Hersh, which deals with a “mixed initial value boundary problem” for signal propagation. A vibrating medium (say, a string or a membrane) is being shaken in a prescribed way at its boundary, and its position and velocity are prescribed at the beginning of the motion (“initially”). The standard approach was to think of such problems as initial value problems, with an extra complication due to the boundary input. Hersh’s breakthrough was to approach it the *opposite* way – to get rid of the initial-value aspect, and then solve a pure boundary problem. In the next few years he found that this trick worked much more generally than in his original assignment. For instance, he could analyze heat diffusion with boundary input just as well as signal propagation.

A widespread strategy in the sciences and in mathematics is to *make sense of anomalous*, *unexpected results*. Hersh’s work on random evolution was motivated by such puzzlement – the need to clarify something obscure and tantalizing. With the probabilist Richard Griego and the functional analyst Einar Hille they came across a little known result of the famous probabilist, Mark Kac. He had found a probabilistic solution of a certain partial differential equation, the telegraph equation – in itself, a rather special result of limited importance, but tantalizing, because it seemed to be anomalous, unexpected. Probabilistic methods had been well developed and well recognized in two of the three main classical types of partial differential equations – the parabolic and the elliptic. The third type, the hyperbolic, did not have a probabilistic version. There seemed to be good reasons why this was so, and nobody was trying to use probability in the field of hyperbolic equations. But the telegraph equation is a hyperbolic equation! It was a very special little example, but it seemed weird and unexpected, like finding a jungle fruit growing in the desert. After serious effort, Hersh saw the basic principle underlying this example, and spelled it out explicitly. This not only gave a different and better explanation of Kac's example, it opened up the road to many other examples, and even better, it opened up the road to using probabilistic tools like the central limit theorem to prove difficult results about partial differential equations. With several different collaborators, it took years of steady work to complete harvesting the fruits of this one creative breakthrough.

A related strategy is "*bisociation*" invented by Arthur Koestler (1964) in an influential work on creativity. He dealt with creativity in general, not just mathematical creativity. The word bisociation is not often used in writing about mathematics education, but it is well known that mathematical advances often come about by linking ideas or methods from different parts of mathematics. An example is "probabilistic potential theory," where a random process such as Brownian motion is brought under potential theory (the theory of equilibria of continuous media). Kakutani, Courant-Friedrichs-Lewy, and Doob are among the most famous contributors. The key is that even though the sample path of a Brownian particle is almost always extremely rough and irregular, its expected value turns out to be beautifully smooth and well-behaved.

The subject of algebraic geometry went through successive transformations, each of which demanded a deep creative insight by a brilliant mathematician. Solomon Lefschetz became famous for work which he described as piercing the great whale of algebraic geometry with the harpoon of algebraic topology (Lefschetz (1924)). Later on Andre Weil and Oscar Zariski put algebraic geometry into axiomatic form, using the methods and concepts of abstract algebra created by Emmy Noether with Emil Artin. And still later the famous conjectures by Weil about the solutions of Diophantine equations were proved through the transformation of the subject by Alexandre Grothendieck, which involved basing it on a new abstract "theory of categories" that had originally been created as an alternative to set theory.

Of course, we cannot attempt here to spell out the details of these mathematical theories. We merely mention them, for those who may already be acquainted with them, as examples of creative advances in mathematics. In the case of probabilistic potential theory, we do give a reference to a popular article by Reuben Hersh and Richard J. Griego (1969).

It is important to recognize that there can be creative leaps in mathematical exposition as well as in research. For example, the whole book by Richard Courant and Herbert Robbins (1979), *What is mathematics*? is a great creative accomplishment, even though all the mathematics it presents was already well-known to experts. Making it accessible to general readers was not a routine task! The Chauvenet Prize of the Mathematical Association of America and the Steele Prize of the American Mathematical Society are awarded annually to recognize significant accomplishments in mathematical exposition.

Creative thinkers use *analogical reasoning*, *intuition*, *metaphors*, *synthesis*, and a combination of different *representational modalities* including graphic approaches. These approaches are used in creative teaching as well as in mathematical research.

9.3 Creativity in Mathematics Teaching

It is essential to realize that every act of problem-solving, even at the elementary level, involves an element of creativity. A mathematical problem starts with some given material, and a request for information related to that material. If the desired information can be obtained by extracting it from the material presented by a simple observation, or by a well-practiced calculation, then the problem is an exercise, rather than a genuine problem. Most routine schoolwork does not require the student to solve problems. But problem contests, from the lowest level to the highest, require creative acts by the contestants. In plane geometry, it is often necessary to add a "construction line" to the given diagram. In many problems, it is required to "bring in" some idea or method not mentioned in the statement of the problem. If the needed construction line or method is very similar to ones that the student has previously seen, we would call it an easy problem. But even an easy problem demands some creativity from its solver.

The alienation of many students from school mathematics is in part the result of constant formulaic approaches rather than problem solving ones. Educators are pushed hard to teach correct outcomes by the present emphasis on prescribed competencies to be achieved by a certain grade level. Under this model of education, students do not learn the cognitive strategies so crucial to discovery. There have been pushes against such a narrow approach at both the K-12 and college levels. Our book *Loving and Hating Mathematics* describes a few of these approaches.

One of the reforms is to use problems taken from everyday experiences, such as shopping, or fixing and dividing food. Jere Confrey, a well-known Piagetian mathematician, uses the mixing of concentrates to make lemonade as a way to teach proportions. Children shift from one kind of drink to another, changing the quantities and learning about ratios. These concrete physical activities are combined with mathematical tools such as data tables and other representations. The Dutch mathematician Hans Freudenthal also used day-to-day experiences: recording the weather, using a calendar, or making frequency distributions of favorite pets. The best known U.S. program using some of these ideas is Everyday

Mathematics, based in Chicago. It uses everyday activities, which may be carried out either by the whole class, by small groups, or by individuals. There are many opportunities for the children to discuss their strategies with each other.

Freudenthal was a leading researcher on Lie groups and algebras, who turned his attention to mathematics history and pedagogy. He called his approach to teaching "realistic mathematics". He created an institute of mathematics education in Utrecht. The emphasis was practical problem solving, rather than drilling of addition and multiplication. This approach was developed by Catherine Fosnot and her collaborators in New York. Their program also emphasizes problem solving, and figuring out how certain procedures work, thereby deepening children's thinking. By rejecting a traditional approach, they encourage analogies and imagination, while also assisting children in finding precise solutions.

The severe underrepresentation of minority youth in mathematical programs in higher education was a challenge for Robert Moses who had been a famous leader in the struggle for voting rights in Mississippi. He also had studied graduate mathematics with Willard Quine at Harvard. He decided that basic algebra skill is a fundamental civil right, and he has developed new teaching methods to elevate the algebra skills of students in the ghettoes of Atlanta and Jackson. His approach includes a focus on the emotional aspects of learning mathematics. His program includes community participation, peer instruction, and the development of a strong self-concept by learners.

> Young tutors…are modeling what they themselves have learned. They have changed by being forced to slip under the rigid requirements of the present federal legislation affecting schools, the Algebra Project is contributing to greater mastery, self-confidence, and self-respect in students who might otherwise have turned off and dropped out of school.
>
> But as Moses warns, "A network of tradition for this, involving teachers, students, and community, isn't established in one fell swoop. You go around it and around it, and you keep going around it and deepening it. You keep returning to it until all of the implications of what you are doing become clear and sink in." (Hersh and John-Steiner 2011, p. 314)

The math educator Uri Treisman used a creative approach to understand the difference between the ways two ethnic groups at the University of California, Berkeley studied calculus. Ethnographic observation revealed that while Black students studied by themselves, Chinese students worked together in group sessions. They got together for meals, asked each other questions, helped each other with homework, and critiqued each other's approaches. Based on these findings, Treisman organized workshop communities where students met in addition to their regular classes. Students were approached with respect. Their groups were referred to as honors programs rather than as remedial intervention. The Treisman model was very successful and has been adopted in several universities.

Group interaction is a neglected feature in the literature on mathematical preparation and discovery. Our book provides several examples of the impact of informal mathematical communities where participants rely on conversation, advice, and sometimes heated arguments to bring the discipline to life.

Loving and Hating Mathematics describes two contrasting unusual approaches to teaching mathematics at the college level, by Robert Lee Moore in Austin, Texas, and Clarence Stephens in Potsdam, New York. The "Moore Method" is famous for producing an amazing number of first-class researchers. Moore's students were assigned to prove a succession of theorems in point-set topology, with no hints or guidance from the professor, no access to texts or references, and no cooperation or communication between students outside of class. It worked wonderfully well for those who could survive it. Clarence Stephens was a veteran of traditionally Black colleges at Prairie View, Texas and Morgan State in Baltimore. He had a fantastically successful career at Potsdam, a state college in far northern New York, almost in Canada. His methodology was the direct opposite of Moore's. He dared to believe that every student who has a sincere desire to learn mathematics can do so, "under the right conditions." His success seems to prove he was right. The right conditions? Allowing the student all the time he/she needs to absorb and master mathematical concepts and methods, and a teacher who has complete confidence that the student will succeed. While Moore focused on the cognitive aspects of discovery, Stephens emphasized the importance of the emotional aspects of learning, particularly mutual respect and self-confidence.

A similar emphasis is advocated by Nel Noddings. She taught mathematics in public schools in New Jersey before she became dean of the college of education at Stanford University. There she advocated a radically different philosophy of education. "We are overly reluctant to face the fact that human interests vary widely and that many highly intelligent people are just not attracted to mathematics…I don't know what talents and interests are lost under such coercion, what levels of confidence are eroded, what nervous habits develop, what rationalizations are concocted, or what evils are visited on the next generation as a result of our benevolent insistence" (Noddings 1993, pp. 8–9).

All of these ventures had features in common: audacity, self-confidence, independent thinking, and courage to follow through and persist in one's vision. These characteristics of creativity are amply illustrated in the growing literature in this field.

In summary, teaching a challenging discipline like mathematics requires new visions and sustained creative innovations by teachers.

We hope that reading about the innovative work of major contributors to mathematics education may help some classroom teachers to see how to experiment and break out of the habitual. New insights could be adapted to the classrooms of young learners. They might join the fun of pursuing innovative solutions, rather than being held to formulaic answers.

9.4 New Insights from Computer Science

The relentless push toward assessment, with "setting of objectives", and "metrics" to measure the approach to an objective, is a plague of modern bureacracy, in education even more than in other realms. . Endless testing and assessing of students, teachers, schools, and school systems inhibits creativity and spontaneity. Many educators and social thinkers object to this strait jacket, and argue that it is often counterproductive. A sensational new development in artificial intelligence adds tremendous support to their cause.

> Ken Stanley and Joel Lehman, professors of computer science, in a series of papers and a book ("Why Greatness Cannot Be Planned: The Myth of the Objective") give fascinating reports of experiments in artificial evolution. When evolution was pushed to maximize the approach to some objective, it failed. Discarding the pressure for that objective, and rewarding novelty – what might, in human terms, be called creativity – dramatically speeded up the improvement.

One experiment was simulating rats trapped in a maze. Trying to get closer to the exit resulted in failure. Trying more and more different directions led quite quickly to escape! The next example is a two-legged robot that must learn to walk and balance. Rewarding efforts that looked like approaching a solution got disappointing results. Rewarding novel and different attempts reached success much faster (Stanley and Lehman (2011, 2015)).

Of course, in situations where the approach to the desired result is not hampered by booby traps or misleading improvements that lead nowhere, reward for decreasing the distance to the objective may be successful. But imporant, difficult tasks, such as improving the outcome of a course of instruction, often don't succeed under such a simplistic strategy. Looking for novelty may bring unexpected progress and breakthroughs.

This welcome support from computer science will improve our chance of restoring sanity to the administration of education in our country and the world.

9.5 Conclusions

In examining the emergence of insight in mathematics, we have relied on some frequently occurring strategies of discovery. These included extending the domain, thinking aside, reversing a habitual approach, making sense of anomalous results, bisociation, analogical reasoning, intuition, metaphors, synthesis, and a combination of different representational modalities. In teaching, recent innovators have broken out of the rigid memorization of algorithms and rely instead of "rely in" it should be "rely on" connecting real life experiences of students with mathematical problem solving. They also consider the student as a whole person whose emotions, self-concept, and reliance on the teacher's confidence in them are as important as the cognitive strategies that they are exposed to.

Contemporary life has introduced computers to all aspects of our existence. They also play a significant role in mathematical discovery and proof. This chapter ended with an example from this newly developing area of mathematics. Contemporary experimentation in many subfields of mathematics attests the vibrancy of this, one of the oldest fields of knowledge.

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Chapter 10
Creativity in Doubt: Toward Understanding What Drives Creativity in Learning

Ronald A. Beghetto and James B. Schreiber

Abstract What propels creativity in learning? In this chapter, we discuss a long-standing—yet often overlooked—form of reasoning that helps address this question. That form of reasoning is called abductive reasoning (introduced by the early American Pragmatist, Charles Sanders Peirce). Abductive reasoning represents a special form of creative reasoning that is triggered by states of genuine doubt. Genuine doubt occurs whenever our everyday habits and beliefs fall short in making sense of a situation. In the context of learning, genuine doubt occurs anytime a learner is unable to inductively or deductively reason through an academic task or situation. As we will discuss, these states of doubt represent opportunities for creative learning. Specifically, our aim in this chapter is to demonstrate, by way of example, how abduction and creativity work together in every day learning. We will also discuss how understanding this link will help clarify efforts aimed at supporting creativity in the classroom, expand current conceptions of creativity, and provide directions for research on creativity in educational settings.

Keywords Creativity • Learning • Motivation

What propels creativity in learning? One way to approach this question is to take a step back and briefly consider a few reasons why people engage in the act of learning. In some cases learning is unintentional (e.g., we learn vicariously from an older sibling that touching a hot stove is a bad idea). In other cases learning is intentional and intrinsically motivated (e.g., we are interested in making fresh ricotta cheese, so we purchase the requisite ingredients, a recipe book, and perhaps watch several *Youtube* videos that walk us through the steps of ricotta cheese making). In still other cases, such as school based learning, learning is intentional but extrinsically or even hedonistically motivated. Put simply, we are motivated to learn what our

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teacher wants us to learn because we fear the punishment of not learning (e.g., low grades, social ridicule, disappointing our parents, guilt) or we seek the perceived benefits (e.g., high marks, social recognition, praise, and some future benefits).

What, if any, role does creativity play in this process? In the above cases, creativity may seem counter productive to learning. When learning about the dangers of a hot stove, how could creativity possibly play a role? Similarly, when learning how to make ricotta cheese, why would someone want to put a new spin on the information if all they want to do is simply reproduce the same taste and texture of that style of cheese? Moreover, if someone wants to get a good grade from a math teacher, why risk the grade by straying away from what the teacher expects?

We agree that in some cases, attempting to demonstrate creativity may not be necessary. Indeed, a sign of an accomplished creator is knowing when and when not to be creative (Kaufman and Beghetto 2013). That said, there is an important difference between attempting to be creative and the kind of creative reasoning that we argue needs to occur at some point in the process of personally meaningful learning. What then are the circumstances that propel creativity in learning? The purpose of this chapter is to address this question.

We open the chapter by providing a brief context for our argument. Next, we discuss how states of doubt trigger "micromoment opportunities" (Beghetto 2013) for creative reasoning and how the concept of abductive reasoning can shed important light on the role that creativity plays in meaningful learning experiences in formal school environments. We then provide classroom examples to help illustrate the process. We close with a discussion of how the ideas presented herein can help clarify efforts aimed at supporting creativity in the classroom, expand current conceptions of creativity, and provide directions for research on creativity in traditional and non-traditional educational settings.

10.1 Creativity in the Context of Learning

Scholars have long recognized a tight association between creativity and learning (see Beghetto 2016a for an overview). In fact, some scholars have argued that creativity and learning represent essentially the same phenomenon (Guilford 1967). We view creativity and learning as separate, but tightly interconnected processes. Along these lines creativity is part of the learning process and learning can result in creative contributions (Beghetto 2016a, b). With respect to the first part of this relationship, the focus is on the more subjective experience of creativity (Beghetto 2007; Guilford 1967; Stein 1953; Vygotsky 1967/2004). Specifically, this is the view that subjective or mini-c creativity involves the new and personally meaningful interpretations of experiences, actions or events (Beghetto and Kaufman 2007). As such, when students learn something new and personally meaningful they are, by definition, engaging in a creative act. This process like other more objective creative processes is a combinatorial process (Mumford et al. 2012; Rothenberg 1996). A full description of the creative learning process has been described elsewhere

(Beghetto 2016a); however, the aspects most germane to the goal of this chapter are worth highlighting.

The creative learning process starts with students attempting to make sense of a new, discrepant learning stimulus in light of what they already know. If successful, the creative combination of the new stimulus and the learner's prior knowledge will result in a new and personally meaningful understanding. This process refers to the more subjective or intra-psychological aspect of creative learning (Beghetto 2016a), wherein creativity serves the act of learning. We would therefore argue that anytime someone learns something new and personally meaningful they have engaged in a creative process (See also Beghetto and Kaufman 2007; Guilford 1950; Littleton and Mercer 2013; Piaget 1973; Sawyer 2012; Vygotsky 1967/2004).

The more subjective or intra-psychological process of creativity in the service of classroom learning is of most interest in the present chapter. That said it is worth noting that a full expression of creative learning also includes the more objective, inter-psychological sphere of the process. This refers to cases where students' subjective insights are shared and make a creative contribution to the learning of others (Beghetto 2016a, 2007). As has been argued elsewhere (Kaufman and Beghetto 2009), all forms of creativity have their genesis in more subjective (or mini-c) insights and interpretations. In the context of classroom learning, the progression from more subjective (mini-c) insights to more objective (little-c) creative contributions usually occurs as a result of receiving feedback from teachers and peers (Beghetto 2007; Beghetto and Kaufman 2014). When this happens, the intra-psychological process of creativity in learning has moved beyond the individual and made a creative an inter-psychological contribution to the learning of others (Beghetto 2016a).

The question at this point is: What triggers or propels the subjective (and even inter-subjective) creative process in learning? We would argue with Charles S. Peirce (and other pragmatists, such as John Dewey) that the creative process is triggered by a state of doubt. We would further argue that the creative process that is triggered in these moments of doubt represents a special form of reasoning called abductive reasoning, which in turn can result in creative resolution and the development of a new and personally meaningful understanding. We represent this logic schematically in Fig. 10.1.

The schematic represented in Fig. 10.1 elaborates on aspects of the creative learning model presented in Beghetto (2016a). Specifically, Fig. 10.1 zeros-in on the more micro-motivational and micro-reasoning process experienced by students engaged in creative learning. As displayed in Fig. 10.1, this motivational process starts with a discrepant learning stimulus (i.e., something that differs from a student's prior understanding and expectations). If the learning stimulus is not discrepant then it will be ignored or simply incorporated into what students already know. If how-

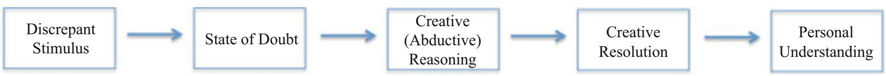

Fig. 10.1 Role of creativity in personal understanding

ever the learner experiences a discrepant event they are moved into state of doubt. This state of doubt serves as the motivational engine for creativity in support of learning. As will be discussed in the following section, a state of doubt triggers a special kind of creative reasoning (called abductive reasoning) that allows learners to resolve their doubt by generating a new and personally meaningful understanding. Importantly, this new understanding is never finalized. It is always open to revision and modification.

Prior to elaborating on this process of creativity in doubt, it is worth stressing a few key aspects of our argument. We assert that all new and personally meaningful understanding results from this creative process. That said we do recognize that there are cases where it may appear (to external observers) that students have learned something new. One example would be a student who memorizes an algorithm for solving a particular type of mathematical problem. Prior to memorizing the algorithm, the student was not able to successfully solve a set of problems. After memorizing the algorithm, the student can solve problems the teacher provides with 100 % accuracy. In many cases this may pass the basic test of behavioral learning. That is, a student who could not solve a particular type of problem can now perform the desired behavior of accurately solving such problems. We would argue along the lines of John Searle's Chinese Room argument (Cole 2014) that simply being able to perform a task is not the same thing as having a meaningful understanding.

As the philosopher John Searle persuasively demonstrated in his thought experiment refuting Alan Turning's test of machine intelligence (i.e., a machine being able to pass as a human), a person who does not speak a word of Chinese could be locked away in a room, receive questions written in Chinese through a slot in the door, and using an algorithm, could appear to understand Chinese by producing accurate written responses written using Chinese characters. The same can be said of the student who memorizes a mathematical algorithm. The appearance of a correct response is not sufficient to make a claim that the student understands the content, task, or procedure (Beghetto and Plucker 2006). One of the best ways for students to demonstrate their understanding is to provide a response that is both original (at least in the context of the classroom) and task appropriate (i.e., meets the contextually specific task constraints). The combination of originality and task appropriateness as defined in a particular context represent the core defining elements of creativity (Beghetto and Kaufman 2014; Plucker et al. 2004).

Our focus in this chapter is on the intra-psychological process that drives the development of a learner's new and personally meaningful understanding. We also recognize that the viability of a learner's personal understanding, particularly in a classroom environment, should go through the process of being tested through interactions with others and the external environment (see Beghetto 2016a, b). In fact, as we mentioned earlier, the process of doing so can result in making a creative contribution to the learning of others.

10.2 Creativity in Doubt

In order to understand the role that we see doubt playing in the creative process, we need to briefly highlight a few concepts that we adapted from the work of Charles Sanders Peirce and the Semioticians following in his footsteps. Specifically, we will highlight sign action, genuine doubt, and reasoning. Each is briefly described in the sections that follow.

10.2.1 Sign Action

In order to understand the role that a state of doubt plays in creative reasoning, we find it helpful to situate it in the semiotic process of sign action. Put simply, sign action as the linking among signs, the object of the sign, and the interpretant (reaction to a sign on the person interpreting it). Houser (1987) provides a nice example of the three components, a sign, its object, and the interpretant (Fig. 10.2).

When you first examined the crude drawing of a bottle with skull and crossbones, you may have made the quick interpretation of death or poison in the bottle before reading further. Sign action occurs quickly and most anything can be a sign. Indeed, anything that stands for or represent something else can be a sign (see Pharies 1985 for examples). Similarly, anything can also be an object of signification. The flexibility of possibilities for signs and objects of interpretation opens up almost endless possibilities. The interpretant is the effect that a sign has on the person interpreting it, which can include feelings, actions, and thoughts.

This semiotic process occurs in everyday teaching and learning. Consider an example from mathematics. Teachers typically provide students with practice problems after having introduced a mathematical concept. A common way of checking understanding is to call on students to share their solutions and procedure for attain-

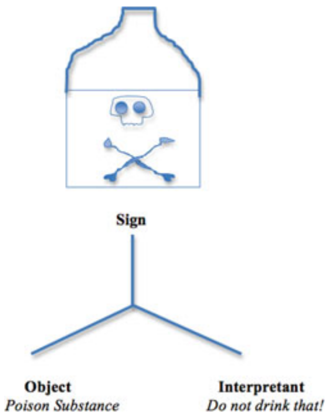

Fig. 10.2 Example of sign action

ing those solutions. The students' solutions and procedures (signs) represents students' understanding (object) and the teacher makes a determination whether to move on or allot more time to providing additional instruction and practice (interpretant). In many cases, this routine is straightforward and the meaning making (or semiosis) moves forward without difficulty. But this is not always the case. Consider a student who provides a correct answer (sign), which initially signifies understanding (object), but the student's unusual explanation raises doubt on the part of the teacher that the student actually understands the procedure or concept (interpretant). In instances where routine semiosis is disrupted by doubt, a more complex sign relation is enacted. As will be discussed, these disruptions are signifiers of creative potential (Beghetto 2013).

10.2.2 Genuine Doubt and Complex Sign Relations

Genuine doubt is a key mechanism in the development of new ideas, whether they are personal and subjective or life altering. Day to day our sign models work, our beliefs are stable, and we happily move from moment to moment. When we encounter a moment where we are in doubt, genuine doubt, we have a chance at developing new signs and, thus, new beliefs. Genuine doubt comes about when a "functioning habit is interrupted" (Burks 1946, p. 303). It arises from experience and is naturally embedded in relevant contexts and situations. When people experience genuine doubt they engage in a more complex process of sign relations (see Fig. 10.3).

When you go to a medical doctor, for instance, the symptoms you arrive with are the signs. The Immediate object might be a common disease that is associated with the symptoms you currently have. The Dynamic object is the actual disease that has caused the condition. The Immediate interpretant is the actual diagnosis, the Dynamic is the prognosis by the doctor and potentially a prescription, and the Final is the true diagnosis (Fig 10.3: adapted from Houser 1987). How does genuine doubt occur in such a system? Consider the following example.

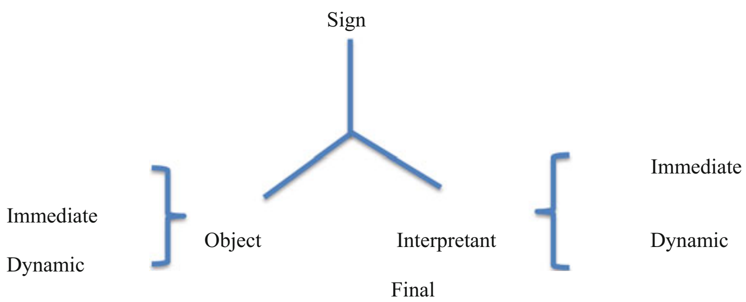

Fig. 10.3 Complex sign system

At 13 months, Jim's daughter started solids and one day while eating boiled carrots, something went wrong and she could not drink or eat anything. This was a sign which led the family doctor to wonder if there was an allergy, but wanted her to go to the hospital. At the hospital, the doctors realized a carrot was stuck in her esophagus. Thus the sign action started again quickly, and they decided the Dynamic was that it was simply stuck and prescribed staying in the hospital a bit longer to wait for it to move. At this point, the complex sign system seemed complete. The carrot did not move (creating doubt) and a new sign action process began from this genuine doubt. The doctors had a new Immediate interpretant of an esophageal stricture for the stuck carrot. This further led to a new Dynamic Interpretant of surgery.

This process of genuine doubt also occurs in the context of teaching and learning. For example, when a student provides an unexpected response during a classroom discussion, it can raise genuine doubt on the part of teachers and the student's peers. This tends to occur in the modal pattern of classroom discourse, which involves the teacher asking known-answer questions, students attempting to provide expected responses, and teachers evaluating students on whether they match what the teacher expects to hear and how the teacher expects to hear it (Beghetto 2013). When this happens, a rupture occurs in the relationship between what was expected and what is now being experienced. These micromoment ruptures occur in everyday teaching and learning situations and represent moments of creative opportunity (Beghetto 2013). In this way, the experience of doubt serves as a signifier of creative potential.

There are a couple of ways to respond to doubt in such situations. One way is to simply ignore it. This often happens when teachers are so focused on attaining some predetermined response or moving along in the lesson plan that they end up dismissing the unexpected and, in turn, inadvertently close off opportunities for creative expression and new meaning making. This "closing off" approach, also called "killing ideas softly" (Beghetto 2013), resolves doubt and moves along a more predetermined path of sense making. Another way is to try to make meaning out of these moments. Doing so requires a special form of reasoning, called Abductive reasoning.

10.3 Abductive Reasoning

Most people are aware of two kinds of reasoning, inductive and deductive (Burks 1946). Deductive reasoning is used to draw conclusions based on premises that we believe to be true (e.g., *All people need water to survive, I am a person; therefore, I need water to survive*). Inductive reasoning is used to make an inference from a sample to a whole (e.g., *Edward likes to read novels by Stephen King. I therefore assume that if I buy him Stephen King's new novel, he'll like it*). In most cases, deductive and inductive reasoning are sufficient for moving through the routines of everyday learning and life. Indeed, deduction and induction predominate in traditional instruction. Students are basically told the basic facts or rules pertaining to some academic subject area (e.g., "any number times zero is zero") and asked repeatedly to apply that (deductive) reasoning to specific examples (e.g., "what is three times zero?"). Similarly, students are encouraged try out and test the viability of their understanding (inductive reasoning).

Peirce, however, argued that there is a third kind of reasoning: Abductive reasoning (CP 5.145). Of the three, abduction is the only one that generates new ideas (Burks 1946) and it is the mode that gets triggered by states of genuine doubt. Abduction typically arises when our existing knowledge is insufficient to explain what we are experiencing. In short, our existing theory of what should happen fails us.

Johannes Kepler, for instance, *expected* that the orbit of planets would be circular since he *believed* that the universe was harmonious and circles are more harmonious than ellipses or any other geometrical form. To his surprise, he could not make the observed orbits conform to circles and was forced to propose a new hypothesis about the harmoniousness of the universe (Abductive reasoning).

Although abductive reasoning is less common than deductive and inductive reasoning, it is important that educators not overlook it. Doing so will come at the expense of opportunities to develop new, creative ideas. This is easier said than done in many classrooms. One reason this happens is because moments of genuine doubt are not seen as opportunities for creative expression (or Abductive reasoning), but rather as disruptions that need to be resolved and eliminated, rather than explored and capitalized on (Beghetto 2013). We would argue that the micromoment ruptures caused by experiences of genuine doubt serve as key opportunities to support abductive reasoning and, in turn, creative expression and deeper learning. In order to understand how this happens, it will be helpful to take a closer look at abductive reasoning and the different forms it can take.

10.3.1 Forms of Abductive Reasoning

As we have discussed, when students (or teachers) experience genuine doubt it is an opportunity to engage in abductive reasoning. Abductive reasoning can take many forms. Shank and Cunningham (1996) have, for instance, described an elaboration of Peirce's types of reasoning by identifying six modes of abduction. The six modes of abduction are part of a larger framework comprised of a total of ten types of reasoning (six modes of abduction, three modes of induction, and one of deduction). In the context of our discussion we will narrow our focus on the six forms of abduction adapted from descriptions in Cunningham and his colleagues (Arici et al. 1998; Cunningham et al. 2002; Shank and Cunningham 1996). The six forms of abductive reasoning are "systematically related to each other" (Shank and Cunningham 1996), but we will discuss them separately to help illustrate each one. We feel it worth noting that although the forms we discuss are in a particular order, abductive reasoning can be (and often is) non-linear in nature.

10.3.2 Six Forms of Abductive Reasoning

Imagine two students. Both are very strong students and continually challenge themselves with complex and difficult problems. One student is struggling with a new, difficult math problem. The other is trying to find information on the internet to write a report on school dropouts. Both students feel stuck and are experiencing a state of genuine doubt about what they know in relation to these two problems. Both engage in abductive reasoning as an effort to resolve this doubt. A person using abductive reasoning need not go through all six forms to resolve doubt, but for our purposes we will use all six to illustrate the similarities and differences.

10.3.2.1 Omen and Hunch

The first form is an omen or hunch. This is a tentative form of reasoning. Our student trying to solve a difficult math problem notices a portion of a problem that seems to point to a possible way to solve the problem. This portion of the problem serves as an omen of possibility. Once she starts work on the problem she may decide to abandon the approach if it is no longer helpful in making progress toward a solution. The omen, however, points to a possible future path of resolution. An omen, therefore, is an observable sign that is resolved in future acts of inquiry, engagement or observation. Sometimes an Abductive inference is more implicit. In such cases it would be considered a hunch. Our student searching the Internet for information about dropouts may have a hunch that clicking a link on the sociology of education might lead to relevant information. Again, a hunch is a tentative form of reasoning, but may payoff in resolving the state of doubt experienced by the student.

10.3.2.2 Symptom

A symptom is the next form of abductive reasoning. Our student who is working the challenging math problem may feel like it is not clear what general strategy applies to solving this specific problem. So she has to make an inference based on the "symptoms" of the problem. She may see a feature of the problem that reminds her of problems she has solved in the past using regrouping strategies. Symptoms are sign-actions that involve making an inference about whether some specific thing in our present experience or observation has possible resemblance to some more general case. The Abductive inference of symptom is based on one's prior experiences (e.g., the student's prior experience with regrouping strategies). Our student searching the Internet for information for a report on school dropouts may come across a specific item of information about school truancy. He will need to decide if truancy is a symptom of dropping out and therefore worth following.

10.3.2.3 Metaphor and Analogy

The next form of abduction is a metaphor or analogy. Our math student may continue to run into dead ends even after reading and re-reading her notes and textbooks, watching explanatory YouTube videos, and asking family members to help. In all cases, the specific problem does not make sense to her. As she thinks about it some more, however, she starts to see a resemblance to her favorite video game. It's a strategic war game that involves the grouping and re-grouping of troops. It is through this resemblance that she is able to develop a new understanding of the problem. In this way, a metaphor/analogy serves as a type of abductive inference that occurs from manipulating a resemblance to create or discover a new way of thinking, a new rule, or new way of understanding. Returning to our example of the student searching the internet for information for his report on school dropouts, at some point he runs into a dead end. He is having little success in locating relevant information. As he thinks about the issue some more he sees dropping out as being related to similar issues impacting runaways. He then uses running away as a metaphor for dropping out and this new connection helps him move forward in his research.

10.3.2.4 Clue

A clue is the next type of abductive reasoning. Our math student flips through her old homework folder and happens upon earlier problems she worked on. The way she solved those previous problems shares a resemblance to the current problem and thereby provides a “clue” or potential path for solving the new problem. Unlike a symptom, the clue is a sign that indicates some past state of affairs that has led to the clue (i.e., the previously solved problem). The student working on the school drop out report may view teen pregnancy as clue to the cause of school dropouts. He may ask himself, “Is there any connection between the two or is it just a coincidence?”

10.3.2.5 Diagnosis/Scenario

The next type of abductive inference is a diagnosis or scenario. The student working on the difficult math problem may start to see a pattern in this problem as compared to other problems she has worked on. She may then reason that this problem represents a larger class of problems that share similar features. She then comes up with a possible rule or strategy for how she might solve the current problem. A diagnosis or scenario is a form of abductive reasoning that involves combining individual observations into a more coherent judgment or scenario. Similarly, our Internet user is now moving toward tentative accounts of the cause of dropouts and is attempting to unite these accounts in a more unified form like a narrative or scenario. He starts to put together a narrative or scenario of dropping out based on the various factors he has identified.

10.3.2.6 Explanation

The final form of abduction is explanation. The student working on the math problem has never seen a problem exactly like this one, but based on her observation of this problem and previous problems she has solved she might be able to put forth a tentative explanation that this problem is part of a larger class of problems that require regrouping strategies to solve. Explanations are therefore a type of abductive reasoning used to form a general rule or explanation. This form of abduction seems closest to what Josephson and Josephson (1996) call "reasoning to the best explanation" (p. 5). Our internet user, may eventually propose an explanation for school dropouts that is consistent, coherent, and parsimonious. Such explanations can then become fodder for inductive testing and deductive elaborations.

Taken together these six forms of Abductive reasoning help generate new possibilities that serve to resolve states of doubt and can promote personal understanding. Abductive reasoning can therefore be thought of as a form of reasoning that helps promote new and personally meaningful insights or what has been called mini-c creativity (Beghetto and Kaufman 2007). Of course, in the context of learning, such insights need to be tested and vetted in the context of the classroom (Beghetto 2016a). The feedback and support provided by this process can result in deeper learning and little-c creative contributions to the learning of students' peers and even their teachers (Beghetto 2013, 2016a).

Recalling Fig. 10.1, our focus in this chapter is to illustrate how learning stimuli can trigger states of doubt that, in turn, can be resolved through abductive (or creative) reasoning which resolve doubt and lead to new and personally meaningful understanding. The next section provides an example of this process.

10.4 An Example of Abductive Reasoning in Creative Learning

In what follows we provide an example of creative learning (adapted from Schreiber 2001). We start by briefly describing the context and then briefly apply each of the elements in Fig. 10.1 to the example.

10.4.1 Context

Early in Jim's (second author) career, he was part of a problem based learning experiment with 30 entering students who took all courses in a PBL format. One of the problems for the Physics component was a twist on the traditional egg drop in elementary school. The students were given the task to protect a can of soda that was going to be flung out of a trebuchet. Each student was provided five pieces of paper, 2 ft of tape, one 6 oz. plastic cup, one can of soda, 2 ft of string, and two paper bags. These were the specific constraints related to the task at hand. Each of these is an object with interpretations related to the students' experience. The contextual constraint was the fact they were told this at the moment they had to complete the task with no prep time to solve the inherent problem in the task.

10.4.2 Element 1: Discrepant Stimulus

In the context of the egg drop problem, students are confronted with a task that uses familiar objects in an unfamiliar context. Students often find these kinds of learning situations compelling because they catch their attention. In this way, ill-defined problems can serve as a "discrepant stimulus" (i.e., something different than what is expected), which can trigger creative learning (Beghetto 2016a). As we have argued, such a stimulus can also put students into a state of genuine doubt.

10.4.3 Element 2: Genuine Doubt

The constraints of the egg drop problem (no prep time to solve the problem) can generate a state of genuine doubt. Genuine doubt is an uncomfortable state that motivates resolution. As we discussed it can be resolved by ignoring it or engaging in Abductive reasoning.

10.4.4 Element 3: Abductive Reasoning

We will focus on one student's reasoning, whom we will call Barry, to illustrate how the six forms of abductive reasoning were used to resolve doubt and resulted in creative resolution and new understanding.

1. Omen/Hunch- "The parachute was like the first thing.." "I was thinking maybe I could crumple up some paper." Here the student has some hunches—some initial possibilities.

2. Symptom- “I was thinking the cup might catch some air [distance and height], so I needed to be concerned about that…[velocity and force as it hit the ground].” The distance and height are symptoms of being flung out of a full size trebuchet and lead to more inferences about what might happen when it lands.
3. Metaphor/Analogy “I was thinking about…I did it in our physics class, …it was an egg thing..we had multiple steps to crack the egg, …separate the yolk [Rube Goldberg contest]..but then went back to parachute. Roll the parachute like a sleeping bag, except not quite.” Here Barry is trying to use other experiences as analogies to move ahead in the process.
4. Clue- “The bags are kind of small and that probably wouldn’t work..I said hey wait I have two bags.” The two bags are a clue for the student as the progress toward a solution occurs.
5. Diagnosis/Scenario- “Paper comes down in the shape of the can so that when it hits [absorbs force]..all the can will slide or something and most of the impact will go to the paper…and the cup will pull the parachute out.” (Fig. 10.4).
6. Explanation “So to protect the can will be protected after landing..by the paper and …the cup will pull the bags out and the parachute will slow it down…This will protect the can.” Here, Barry has a more complete explanation of how it should all work.

At this point, Barry just tested the design—the student was done talking. The can was fired from the trebuchet and the student did have conclusive statement. “I am pretty happy and my idea worked.. [Where was the major impact?]. Bottom of the cylinder and it didn’t really do that much, that ground didn’t really.” Later, students wrote formal research papers on the physics of their designs.

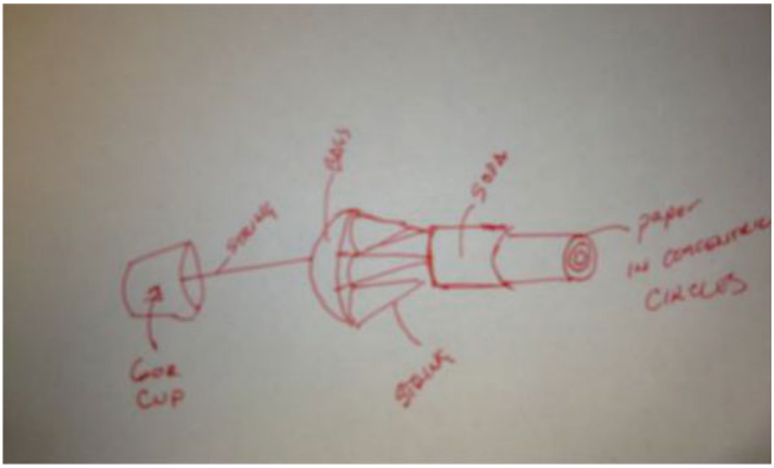

Fig. 10.4 Reproduced drawing of tested soda can protection system

10.4.5 Elements 4 and 5: Creative Resolution and New Understanding

The students were allowed to have their mini-c moments and work through their own understanding of the problem and the constraints without faculty telling them what to do or not to do. This was personally meaningful, and many of the designs would be considered Little-C creativity because the faculty were impressed with the designs. Yet, the rigor and understanding of physics was not ignored or watered down. It was crucial to their written work later and the discussion that occurred post- trebuchet launching.

Taken together, this example illustrates the role that abductive reasoning can play in creative learning. Specifically, how one student (Barry) experienced a state of genuine doubt, which in turn triggered various forms of abductive reasoning and resulted in deeper understanding of physics.

10.5 Concluding Thoughts

In this chapter our goal was to argue that creative learning can be triggered from the experience of doubt. The resolution of doubt is a key motivator in the creative process allowing for abduction to occur. We believe that our description of this process provides creativity researchers and learning theorists with a new way of thinking about the subjective motivational processes that can drive and support creative learning. We encourage researchers and practitioners to explore this process and examine whether and how this conceptualization contributes to existing and new models of supporting creative learning.

We also acknowledge that these creative learning opportunities are sometimes missed or even dismissed in practice. This is particularly the case when moments of doubt are ignored or predetermined paths are forced on such moments. When this happens, the abductive process is short-circuited and opportunities for cultivating creative learning are missed. Yet another direction for future research and practice should therefore focus on helping to identify and cultivate the potential of such opportunities and examine whether doing so simultaneously promotes creative thinking and deeper understanding.

Much of this work can occur in traditional and non-traditional educational settings. In the context of more traditional classrooms, work is needed to empirically examine when and how teachers allow for the moment of doubt to blossom and how they engage with students during these moments. Projects in mathematics and science classrooms are great starting points as well as projects exploring this process in other domains and across all grade levels. There are also ample opportunities for conducting such work in non-traditional educational settings such as workplaces, museums, archeological dig sites, and crimescene investigation. We hope that the ideas shared in this chapter inspire researchers and practitioners to develop and test out these ideas. Doing so can lead to developing much needed insights into how to simultaneously support creativity and learning across disciplines and across the lifespan.

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Part II
Perspectives on Giftedness

Chapter 11
What Is Special About the Brain Activity of Mathematically Gifted Adolescents?

Roza Leikin, Mark Leikin, and Ilana Waisman

Abstract This paper addresses the neuro-cognitive characterization of super mathematically gifted high school students. The research population consisted of three groups of students excelling in mathematics: super mathematically gifted (S-MG), generally gifted students who excel in school mathematics (G-EM), and students who excel in school mathematics but are not identified as being generally gifted (NG-EM). An Event Related Potentials (ERP) research methodology was employed to examine behavioral and electrophysiological measures associated with insight-based and learning-based problem solving. Forty-two male adolescents participated in the study. Analysis of the electrical potentials evoked when solving these two distinct types of problems revealed three types of neuro-efficiency effects, which highlight the different characteristics of electrical activity of super mathematically gifted students. These characteristics are predominantly task-dependent, emerge at different stages of the task and are reflected in different scalp topography.

Keywords Problem solving • Giftedness • Excellence in mathematics • Super-gifted in mathematics • Event Related Potentials (ERP) • Neuro-efficiency effect

11.1 Rationale

Super-gifted (S-G) individuals are considered to be at the extreme end of the intelligence continuum and are significantly more advanced than those individuals identified as generally gifted (Silverman 1989). Because of the great variability within the gifted range, it is important to distinguish between levels of giftedness (Goldstein et al. 1999).

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Extensive neuroscience research has explored brain activity during mathematical processing related mainly to computational skills (eg, Santens et al. 2010) as well as research examining the neurobiological basis of individual differences in general intelligence (eg, Deary et al. 2010). Additionally, several studies discuss the differences in the brain organization of gifted and mid-ability students (eg, O'Boyle 2008). However, to the best of our knowledge, no study has yet examined the brain activity of the gifted student who exhibits superior excellence in mathematics (hereinafter referred to as S-MG) associated with mathematical problem solving.

This study examined brain activity using Event Related Potentials (ERP) during mathematical problem solving. It attempts to ascertain both the behavioral (accuracy and reaction time) and the electrophysiological measures (the strength and scalp distribution of ERPs) that distinguish S-MG students from generally gifted students who excel in school mathematics (G-EM) and from those students excelling in mathematics who are not identified as being generally gifted (NG-EM).

11.2 Background

11.2.1 Gifted and Super-Gifted Students

Giftedness is often defined as an unusually high ability or unusually high intelligence that exceeds a certain cut-off score (Winner 2000). Most of the identification methods for levels of intellectual giftedness are based on Intelligence Quotient (IQ) levels. Most commonly, gifted intelligence is characterized by an IQ that exceeds 2 Standard Deviations (SD) above the population mean (usually 130). Super-gifted students (cf. extremely-gifted, exceptionally gifted or profoundly gifted) are defined by an IQ score above 145 or at least 3 SD above the mean (Feldman 2003). These super- gifted (S-G) individuals appear in the population at a ratio of less than 1:1000 (Gross 2009; Vaivre-Douret 2011; Silverman 2009). Silverman (1989) defined the S-G individuals as those whose performance is significantly beyond the norm of the gifted.

Extremely gifted children, who are reported to have exceptionality in talking, reading and imagination, tend to develop much earlier in these fields than their peers of average ability (Hollingworth 1942; Vaivre-Douret 2011). Highly gifted students display intrinsic motivation during problem solving, whereas students of average ability are motivated extrinsically (Kanevsky 1995; Gross 2009). The highly gifted possess advanced analogical thinking and acquisition of new information, apply different problem solving strategies (Sternberg 1981) and exhibit a faster processing speed on cognitive tasks (Steiner and Carr 2003; Paz-Baruch et al. 2014). All the aforementioned characteristics are heightened in S-G individuals (Gross 2009). The S-G possess a striking ability for "dual processing" – the ability to process two sets of information simultaneously and in parallel (Gross 2009). It is found that S-G students have superior working memory as compared to their moderately gifted and non-gifted counterparts (eg, Leikin et al. 2014). Note, however, that this population of mathematically super-gifted individuals has not been sufficiently studied. We chose to investigate not only the cognitive or affective characteristics of super-gifted individuals but also to compare their ability to solve different types of mathematical problems relative to other groups of mathematically able students.

11.2.2 Studying Mathematics in High School

One of the central topics in the school curriculum is Euclidean geometry, which is concerned with geometric figures and their properties (NCTM 2000). Geometrical reasoning is usually associated with mutually related visual and logical components (Fischbein 1993), and one of the essential topics included in the school geometry program is the concept of area of figures and their measurement (Clements et al. 1997). The students' conceptual understanding of the above-mentioned topic is facilitated by comparing the areas of two figures (Kospentaris et al. 2011). Another significant topic of study in the school mathematics curriculum is function, which is one of the fundamental concepts in mathematics, in general, and in school algebra, in particular (Da Ponte 1992). Through the study of functions, students can perform translations between their different representations (Janvier 1987). At the same time, insight-based problems are not part of the high school mathematics curriculum. These problems have a relatively simple solution but it is difficult to discover (Kershaw and Ohlsson 2004). Furthermore, these problems are usually unfamiliar to solvers and require a great cognitive effort to find a solution (Mayer 1995). Insight is considered to be a central trait of giftedness (Davidson 2003).

In the present study, we examined the behavioral and neuro-cognitive measures related to the solving of both learning-based and insight-based mathematical problems by students with different levels of general giftedness, who all excel in mathematics.

11.2.3 Neuro-Cognitive Activity Associated with Mathematical Thinking and Operations

Considerable research has investigated the neural basis of mathematical cognition. However, the majority of these studies have dealt with the brain areas activated during number processing and simple arithmetic in normal and abnormal populations (eg, Arsalidou and Taylor 2011). The special fronto-parietal brain network associated with mathematical processing has been identified in several fMRI studies (eg Dehaene et al. 2003). It was demonstrated that the parietal cortex plays an important role in tasks involving arithmetic problem solving (eg Zamarian et al. 2009) and that it supports more complex mathematical processing such as algebraic equations, geometry proof generation, and calculus (Anderson et al. 2011; Sohn et al. 2004). The processes of advanced mathematical problem solving were found to be associated with the frontal cortex (eg Anderson et al. 2011). It was shown that as the complexity of the problems increases, more brain areas are involved in the solving process (Zamarian et al. 2009).

As to electrophysiological research, some ERP studies of arithmetic demonstrated a late positive wave peaking between 500 and 1000 ms, which has been explained as retrieving or calculating an answer (eg, Galfano et al. 2004). It was found that its strength intensifies as problem complexity increases (Núñez-Peña et al. 2006). Note, however, that this positive wave appeared after the N400 component, the maximum of which is achieved over the anterior electrode sites. This anterior negativity is frequently interpreted as an indicator of a mismatch between the given answer and the preceding question or is thought to be connected to the involvement of working memory and attention resources (eg, Niedeggen et al. 1999). Additionally, both N400-like and P500-700 components appeared to be related to insight-based problem solving (eg, Dietrich and Kanso 2010).

Though there is extensive research into the neural basis of mathematics, the issue of individual differences in mathematical processing has not been studied sufficiently. Moreover, there has been almost no research on the neuro-cognition of mathematical problem solving among intellectually gifted individuals.

11.2.4 Neuro-Cognitive Research on Giftedness

The brains of gifted individuals demonstrate enhanced development and activation of the right hemisphere, ability to activate task-appropriate regions in a well-orchestrated and coordinated manner, and better brain connectivity (O'Boyle 2008; Prescott et al. 2010).

There is strong empirical evidence showing that individuals with higher intelligence exhibit lower mostly-frontal brain activation (neural efficiency hypothesis) compared with less intelligent individuals (Neubauer and Fink 2009). Note that neural efficiency might appear when individuals solve tasks of low to moderate difficulty. When the complexity of tasks rises to an elevated level of difficulty, highly intelligent individuals seem to devote more cortical resources as compared to less intelligent individuals (Dunst et al. 2014).

EEG studies showed that whereas the latencies of the P3 ERP component were negatively correlated with intelligence (Beauchamp and Stelmack 2006), the amplitudes of P3 were positively correlated (De Pascalis et al. 2008). Gevins and Smith (2000) have also shown that values of amplitude of brain activation as well as topographic patterns of activation are associated with levels of general intelligence. For example, high-ability participants showed greater parietal activity and less prefrontal activity than their low-ability counterparts (Jausovec and Jausovec 2004). Moreover, highly intelligent participants exhibited more brain activity in the early stages of task performance, while average individuals did so in the later stages of task processing (Jausovec and Jausovec 2004). Zhang et al. (2015) demonstrated that mathematically gifted adolescents exhibit a highly integrated fronto-parietal network.

Even so, until recently, most research on the gifted population has used non-mathematical problems and has not differentiated between general giftedness and excellence in mathematics.

11.3 The Study

11.3.1 The Goal of the Study

The goal of the present study was to examine differences in behavioral and electrophysiological measures associated with mathematical problem solving among excelling in mathematics students who vary in their level of general giftedness (NG-EM and G-EM) and exhibit superior performance in mathematics (S-MG).

11.3.2 Participants and Sampling Procedure

In this paper we report findings that focus on ERP measures associated with the problem-solving performance of right-handed male individuals. This was a partial group from an initial sample of about 200 students chosen out of a much larger potential research population of 1200 students from 10th–11th grade (16–18 year olds) who studied mathematics in school in either high-level (HL) or regular-level (RL) classes. The sampling procedure was directed towards implementing the distinctions between G and EM characteristics as well as those of S-MG.

For the purpose of narrowing down the sample, the large preliminary research population of 1200 students was examined using Raven's Advanced Progressive Matrix Test (RPMT) (Raven et al. 2000) for general intelligence and the SAT-M test (Scholastic Assessment Test in Mathematics) for mathematical excellence. Although previous studies showed a high correlation between SAT-M and Raven Tests, a high SAT-M score does not necessarily indicate high intelligence (Frey and Detterman 2004). We used a shortened Raven Test containing 30 items with a 15-min time limit together with a short version of the SAT-M Test that contains 35 items with a time limit of 30 min (Zohar 1990).

G factor: The students were characterized as generally gifted (G) providing they studied in classes for gifted students (identified by a national examination as having IQ > 130 in the third grade). The Raven Test was employed for two purposes: For validation of the G factor in students from the classes for generally gifted students and for exclusion of students with a high score on Raven Test from a NG (identified as non-gifted) group of participants. A Raven score of 28 out of 30 was set as a lower border for inclusion in the G group (to ensure that 1 % of population was included in the G group).

Table 11.1 The groups of students who excel in school mathematics

		G-EM	NG-EM	S-MG
Insight-based test		17	17	7
Learning-based tests	Area related test (geometry)	18	17	7
	Function related test (algebra)	19	16	7

G-EM Gifted Excelling in Mathematics, *NG-EM* Non-Gifted Excelling in Mathematics, *S-MG* Super-Gifted in Mathematics

EM factor: Mathematics is a compulsory subject in Israeli high schools and students can be placed in one of three levels of mathematics: high, regular and low. The level of instruction is determined by students' mathematical achievements in earlier grades. The differences in instruction at the high level (HL) differ from that at the regular level (RL) in terms of the depth of the learning material and the complexity of the mathematical problem-solving involved. All 1200 students studied mathematics at HL or RL. Students who were ultimately included in the EM group learned HL mathematics and had scores higher than 90 out of 100. An SAT-M score above 26 out of 35 was chosen as a control measure for the EM sampling (2 % of general poplation).

S-MG students: An additional fifth group of participants included 7 "super mathematically-gifted" (S-MG) students who were described by mathematics professors as being students with extraordinary mathematical abilities. While learning in high school (10th-12th grades), these students also took mathematics or computer science classes at a university, achieving a mean score above 95, or were members of the International Computer Science Olympiad team. These students comprise 1/20,000 of their age-group in the overall Israeli population.

In line with the goal of the study, we formed three research groups of students excelling in mathematics according to their level of general giftedness (G-EM and NG-EM) and their superior performance in mathematics (S-MG) (Table 11.1).

11.3.3 Tests and Experimental Procedure

There were two tests which corresponded to the school mathematics curriculum and to the skills necessary for attaining/achieving success in mathematical problem solving:

Area-related test: Participants received a drawing of a geometric object. Part of this drawing was shaded. The participants had to determine what area of the drawing was shaded or what the area of the geometric object was in reference to the shaded part.

Function-related test: Participants received a graph of a mathematical function followed by an equation. They had to determine whether the graph and the equation represented the same function

An insight-based test was specially designed to analyse brain activity associated with solving unfamiliar problems (eg, Sternberg and Davidson 1995). This test

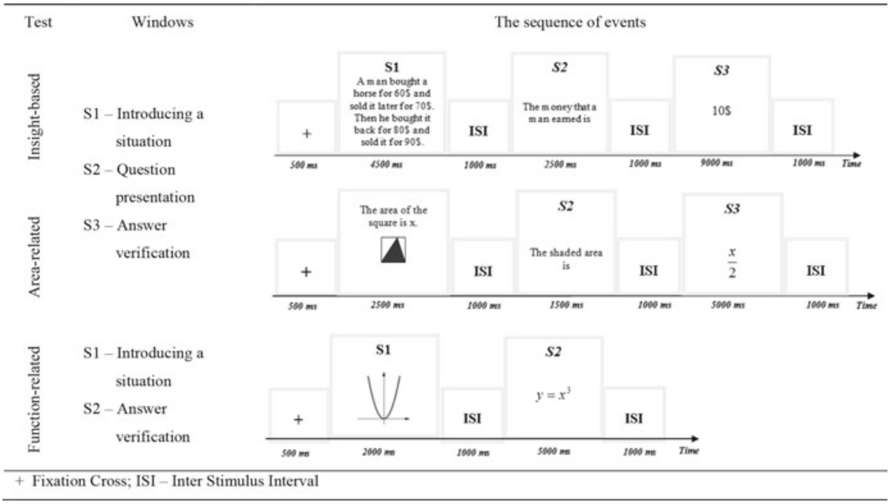

Fig. 11.1 The sequence of events and examples of the tasks

contained problems that had a simple solution that is not easy to discover. Sometimes the answer is counterintuitive and sometimes the solution strategy seems to belong to a different branch of mathematics than that which is required by the problem.

Each test included 60 tasks. The tests proved to be reliable (Cronbach's alpha >0.7) (Leikin et al. 2013). Computerized tests were designed and administered using E-Prime software (Schneider et al. 2002). All tasks were presented in the center of the computer screen and displayed in black characters on a white background. According to Polya's theory (Polya 1957) of strategies of problem solving, each task was presented in several windows (stages) that appeared consecutively. Each task on the function-related test was presented in two windows (S1 – introduction of task condition, and S2 – answer verification). Each task on the area-related and insight-based tests was presented in three windows (S1 – introduction of task condition, S2 – question presentation, and S3 – answer verification).

The sequence of events and an example of the tasks are presented in Fig. 11.1.

11.3.4 Electrophysiological Recording and Analysis

Scalp EEG data were continuously recorded using a 64-channel BioSemi ActiveTwo system (BioSemi, Amsterdam, The Netherlands) and ActiveView recording software. Pin-type electrodes were mounted on a customized Biosemi head-cap, arranged according to the 10–20 system. Two flat electrodes were placed on the sides of the eyes in order to monitor horizontal eye movement. A third flat electrode was placed underneath the left eye to monitor vertical eye movement and blinks.

Data were recorded using an average-reference on-line. During the session, electrode offset was kept below 50 μV. The EEG signals were amplified and digitized with a 24 bit AD converter. A sampling rate of 2048 Hz (0.5 ms time resolution) was employed. Figure 11.2 depicts the location of the electrodes.

ERPs were analyzed offline using the Brain Vision Analyzer software (Brainproducts). ERPs were Zero Phase Shift filtered offline (bandpass: 0.53 Hz–30 Hz) and referenced to the common average of all electrodes. Epochs with amplitude changes exceeding ±80 μV on any channel were rejected. Ocular artifacts were corrected using the Gratton et al. (1983) method. The ERP waveforms were time-locked to the onsets of S1, S2 and S3. The averaged epochs for ERP (for which only the correct answers were averaged) included a 200 ms pre-stimulus baseline. Each condition resulted in at least 40 trials per participant.

11.3.5 Data Analysis and Statistics

Due to the small number of S-MG participants (N = 7), we performed a non-parametric (Kruskal-Wallis) test in order to examine specific characteristics of S-MG students (as compared to G-EM and NG-EM) with consequent Mann-Whitney tests for pair-wise comparison between the groups. For the pair-wise comparisons, *p*-values were adjusted for multiple comparisons according to the Bonferroni adjustment.

Behavioral data analysis was applied to Accuracy (Acc) (in %) and Reaction time correct responses (RTc). RTc was calculated as the mean time spent for verification of an answer (stage S3 for Area-related and Insight-based tests or S2 for Function-related test) in all trials on the test per person.

In *electrophysiological data analysis* the scalp surface was divided into nine topographical regions (see Fig. 11.2 for details) and the mean amplitude of the single electrodes within each electrode site was averaged in each of the defined time

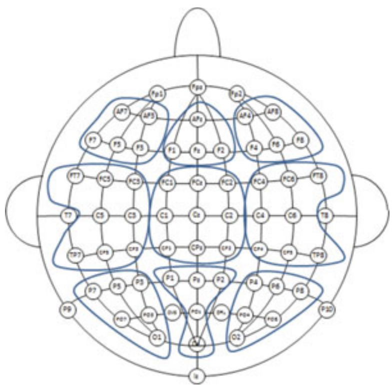

Fig. 11.2 Location of the electrodes and selected electrode sites

Table 11.2 Electrophysiological data analysis

Test	Stage	Time frames for statistical analysis (ms)	Between group comparisons	Measures
Insight-based	S1, S2, S3	250–500, 500–700, 700–900	S-MG vs. G-EM vs. NG-EM	Mean Amplitude in: AL, AM, AR, and PL, PM, and PR, separately
Area-related	S1, S2, S3			
Function-related	S1, S2	300–500, 500–700, 700–900		

AL anterior left, *AM* anterior middle, *AR* anterior right, *PL* posterior left, *PM* posterior middle, *PR* posterior right

frames (Table 11.2). We report herein the significant results associated with late potentials in anterior and posterior sites which were found to be involved in mathematical processing (eg, Anderson et al. 2011): *Anterior left* (AL); *anterior middle* (AM); *anterior right* (AR); *posterior left* (PL); *posterior middle* (PM); *posterior right* (PR) (Fig. 11.2).

11.4 Results

11.4.1 Behavioral Findings

Generally speaking, S-MG students demonstrated the highest Acc and the lowest RTc as compared to their G-EM and NG-EM counterparts (Fig. 11.3). Non-parametric analysis revealed between-group differences in Acc and RTc on all the tests. Subsequent between–group comparisons found that S-MG students had significantly higher Acc and significantly lower RTc as compared to NG-EM students on all the tests. At the same time, the differences in behavioral measures between the two groups of gifted students (S-MG and G-EM) were found only on the Area-related and Function-related tests, which are based mostly on school curriculum (learning-based tests). In this case, S-MG students were significantly more accurate and quicker (even if marginally) than the G-EM participants. In turn, the differences between G-EM and NG-EM groups were found only in the accuracy measure on the Insight-based test, which is not a learning-based test. The Acc in G-EM was marginally significantly higher as compared to NG-EM students. Note that in this context, the insight-based test appeared to be very sensitive to the factor of general giftedness: S-MG and G-EM individuals performed this test better than NG-EM.

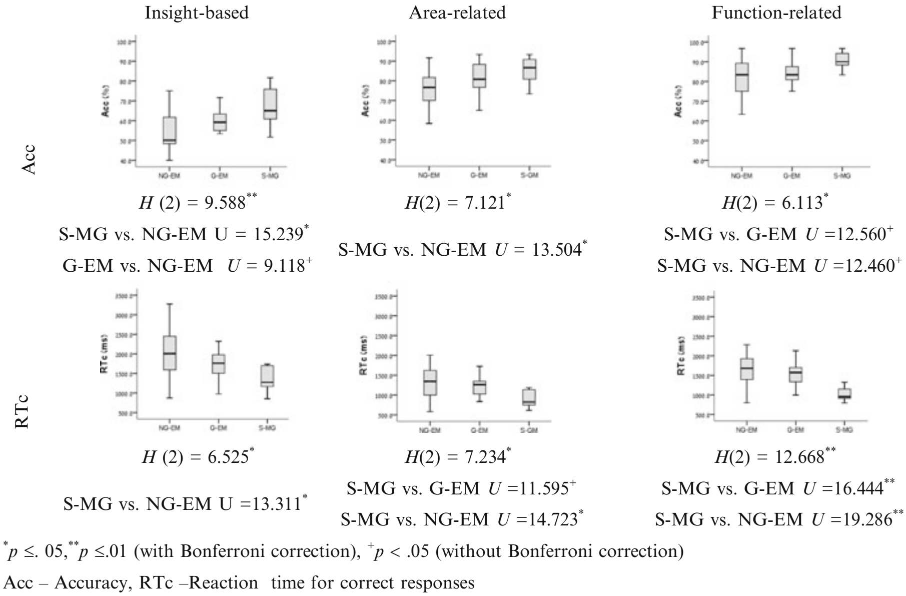

Fig. 11.3 Between-group differences for Acc and RTc revealed by mathematical tests

11.4.2 Electrophysiological Findings

The significant differences in the mean amplitudes at the anterior and posterior sites between the study groups appeared in all three tests reported in this study but at different stages. Pair-wise comparisons showed that there were significant differences mostly between S-MG and the other two groups and to a lesser extent between the G-EM and NG-EM groups. On the other hand, generally speaking, these differences varied considerably in accordance with such variables as the type of test, stage, time interval and electrode sites.

On the insight-based test the lowest electrical potentials were found for S-MG students compared to the other study groups. The differences between the mean amplitudes of the electrical potentials of the S-MG and of the NG-EM students were obtained at S2 and S3. For example, a notable effect was obtained at S3 in the posterior left site (PL) during the time period of 250–500 ms ($H(2)=6.904$, $p<.05$ $U=12.672$, $p<.05$). At the same time, the significant differences in mean amplitudes between the S-MG and G-EM groups appeared only at S2. For example, the effect was shown in the anterior right site (AR) during the time period of 500–700 ms ($H(2)=6.253$, $p<.05$ for S-MG vs. G-EM $U=12.050$, $p<.05$ and for S-MG vs. NG-EM $U=12.756$, $p<.05$). The G-EM participants did not significantly differ from the NG-EM students.

On the area-related test, the results obtained at S2 and S3 were noticeably different. At S2, the results were mostly similar to those on the insight-based test: S-MG participants demonstrated significantly lower electrical potentials compared to the two other groups, while the G-EM group did not differ significantly from the NG-EM group. For example, the effect was shown in the posterior right site (PR) during the time period of 500–700 ms ($H(2) = 10.172$, $p < .01$ for S-MG vs. G-EM $U = 16.167$, $p < .01$ and for S-MG vs. NG-EM $U = 16.235$, $p < .01$). At the same time, at S3, S-MG participants did not differ from the G-EM participants, but both "gifted" groups (S-GM and G-EM) significantly differed from the NG-EM group. For example, the effect was achieved in the posterior middle site (PM) during the time period of 700–900 ms ($H(2) = 7.291$, $p < .05$ for S-MG vs. NG-EM $U = 13.126$, $p < .05$ and for G-EM vs. NG-EM $U = 8.690$, $p < .05$).

On the function-related test, the differences in mean amplitudes were found only at S1 and related to the distinction between the G-EM and NG-EM students and between the S-MG and NG-EM students. For example, these differences were obtained in the posterior middle site (PM) during the time period of 500–700 ms ($H(2) = 9.583$, $p < .01$ for G-EM vs. NG-EM $U = 10.151$, $p < .05$ and for S-MG vs. NG-EM $U = 15.196$, $p < .05$). More interestingly, in the anterior left site (AL), the S-MG group demonstrated relatively high electrical potentials as compared to the G-EM group in the early time interval (300–500 ms) ($H(2) = 6.826$, $p < .05$, $U = 13.729$, $p < .05$).

Finally, note that there was one more interesting finding: Generally, the S-MG participants demonstrated the lower mean electrical potentials compared to the G-EM group, while the G-EM students were characterized by lower activation compared to the NG-EM group. These last differences, however, were significant only on the learning-based tests; ie, function-related and area-related tests.

11.5 Discussion

11.5.1 Three Types of Neuro-Efficiency

The results of the study show that, in general, there was a neuro-efficiency effect related to both superior mathematical abilities and general giftedness. The electrical potentials evoked in S-MG participants were lower compared to those of the G-EM and NG-EM groups. The electrical activity in G-EM students was often lower than that of the NG-EM students. These differences were significant in many cases. These findings are consistent with the neuro-efficiency hypothesis (Neubauer and Fink 2009), whicha brighter individuals show more efficient brain functioning (mostly frontal) than less intelligent individuals on tasks with the same cognitive demands.

Our study contributes meaningfully to the understanding of the connection between expertise in solving mathematical problems, general giftedness and the neuro-efficiency effect which is characterized by an inverse relationship between the accuracy of task performance and the strength of brain activation. While several previous studies examined the strength of brain activity as a function of the level of expertise (eg, Kelly and Garavan 2005), they demonstrated inconsistent results. We

Table 11.3 Three types of neuro-efficiency effects

Characteristic	Insight-based test	Area-related test	Function-related test
Accumulative	**S3**	**S2**	
S-MG>NG-EM	PL 250–500 ms	AR 250–500 ms	
S-MG~G-EM		AL 500–700 ms	
G-EM~NG-EM			
Unique	**S2**	**S2**	
S-MG>G-EM	AR 500–700, 700–900 ms	AR 500–700 ms	
S-MG>NG-EM		AM 500–700 ms	
G-EM~NG-EM		PR 500–700 ms	
		PM 250–500, 500–700 ms	
Giftedness related		**S3**	**S1**
S-MG>NG-EM		PM 250–500, 500–700,	PM 500–700, 700–900 ms
G-EM>NG-EM		700–900 ms	
S-MG~G-EM		AM 700–900 ms	

AL anterior left, *AM* anterior middle, *AR* anterior right, *PL* posterior left, *PM* posterior middle

distinguished between levels of intelligence and levels of expertise (excellence in mathematics). This distinction allowed us to refine our understanding of the neuro-efficiency effect.

At some of the electrode sites and problem-solving stages, there were three different types of neuro-efficiency related to G and EM factors (see Table 11.3).

The first type appears as *an accumulative characteristic* of S-MG students. In these cases, non-significant differences were found between the strength of the electrical potential of G-EM and NG-EM students and between S-MG and G-EM students. These differences, however, only become significant when comparing the S-MG and the NG-EM groups. For example, this type of neuro-efficiency effect was found for the insight-based test at S3, and for the area-related test at S2.

The second type is *a unique characteristic* of S-MG students. In these cases, significant differences are obtained when comparing the S-MG and G-EM groups as well as the S-MG and NG-EM groups. The G-EM and NG-EM students exhibit very similar values on the examined measures. This type of neuro-efficiency effect, which indicates behavioral and/or electrophysiological characteristics that are specific for the S-MG students, was found on the insight-based and the area-related tests at S2.

The third type of neuro-efficiency effect is *a general giftedness-related characteristic*, in which significant differences are revealed when comparing the S-MG and NG-EM as well as the G-EM and NG-EM students. The G-EM and S-MG groups exhibit very similar values on the examined measures. This type of neuro-efficiency effect is associated with G factor. The general giftedness-related characteristics were found on the function-related task at S1 and on the area-related test at S3.

11.5.2 *Task-Dependency of the Effects*

Leikin et al. (2014) described the relationship between the strength and the distribution of electrical activity and the topic of the mathematical tests. In this paper, we demonstrate that the types of neuro-efficiency effects were task-dependent (Fig. 11.4 and Table 11.3).

Giftedness-related characteristics were found in the learning based (area- and function-related) tasks: Both G-EM and S-MG groups evoked significantly lower electrical potentials as compared to NG-EM group in the middle electrode sites. Furthermore, we show that in order to excel in mathematics students do not have to be generally gifted, although excellence in mathematics is related to and enhanced by general giftedness (c.f. Grabner et al. 2006 for expert chess players).

Three groups of participants achieved high accuracy scores on the learning-based tests that were relatively easy for EM students. No significant differences between G-EM and NG-EM groups were found in the behavioral measures, since the students in these two groups exhibited similar performance. However, the electrical potentials evoked in NG-EM students were larger compared to those of G-EM students. Thus, our findings demonstrate the possibility of distinguishing between gifted students and non-gifted students in the EM group on the basis of neuro-cognitive measures.

On the learning-based tasks, extending the excellence continuum did not affect the strength of the brain activation. On the contrary, insight-based tasks revealed differences between S-MG and G-EM.

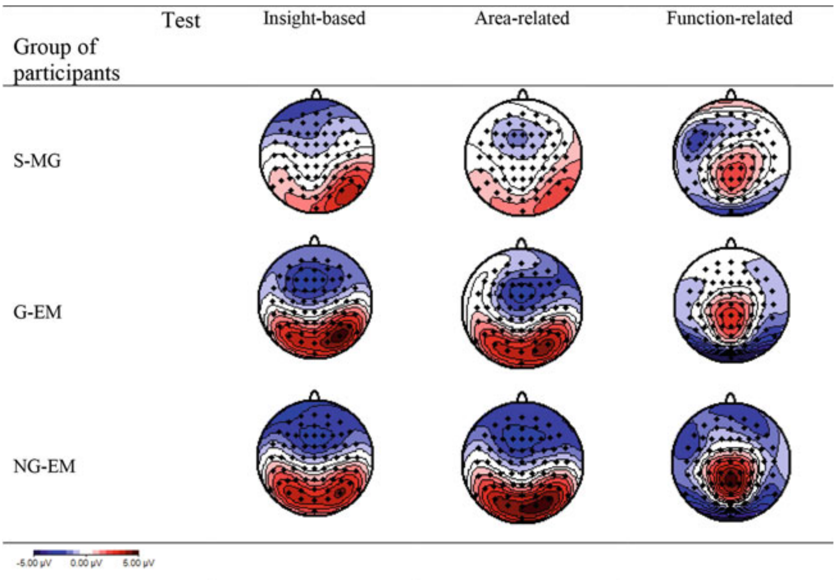

Fig. 11.4 Topographical maps of electrical potentials in S-MG, G-EM and NG-EM students at S2 in the 500–700 ms

The unique characteristic of S-MG was found on the insight-based and area-related tests only during the presentation of a question (ie, S2) in the right and middle electrode sites. According to this characteristic, the mean amplitude for S-MG was significantly lower compared to both G-EM and NG-EM. Accordingly, we speculate that being gifted is not sufficient for possessing efficiency in brain functioning. Compared to learning-based tasks, the insight-based ones seemed to exemplify more difficult tasks, since they are not familiar to the solver and require a specific, non-learned approach to the solving (Sternberg and Davidson 1995).

Previous studies demonstrated the relationship between neuro-efficiency and the difficulty of the task (Dunst et al. 2014; Neubauer and Fink 2009). Dunst et al. (2014) showed that more intelligent individuals exhibited higher brain activity in the more difficult tasks than less bright ones did. Our study refines these findings. We found that on the insight-based test, S-MG showed the lowest mean amplitudes compared to the other two groups, while there were no significant differences between G-EM and NG-EM participants. The relative difficulty of insight-based tasks for G-EM students compared to the S-MG group may be interpreted such that G-EM students did not achieve the highest level of giftedness and expertise in mathematics.

Additionally, the unique characteristic of S-MG was found only at the question presentation stage. Note that some studies have reinforced the issue of the relationships between brain activity, intelligence and the stage of task performance (eg Jausovec and Jausovec 2004). We hypothesize that the lowest electrical activity in S-MG students appeared during S2 because they started solving the problem at S1.

To summarize: Our study presents three types of neuro-efficiency effects associated with insight-based and learning-based problem solving in three groups of excelling in mathematics students. These three types of neuro-efficiency effects relate to such variables as the type of test, stage of the test, giftedness and level of excelling in mathematics. Seemingly, these findings allow us to distinguish mathematically gifted individuals from the general pool of students who excel in mathematics and to identify this group as individuals who are super-gifted in mathematics.

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Chapter 12
Psychological and Neuroscientific Perspectives on Mathematical Creativity and Giftedness

David H. Cropley, Martin Westwell, and Florence Gabriel

Abstract Creativity and giftedness have long been linked together in the literature, particularly where giftedness is conceived, not in the analytically focused sense of schoolhouse giftedness (e.g. Renzulli 1978), but in the sense of creative-productive giftedness that emphasizes the generation and production of ideas. Creativity has a well-established foundation in the psychological literature, and a growing body of work derived from neuroscientific approaches. How do these contrasting psychological and neuroscientific approaches inform our understanding of creativity as a component of giftedness in general? How is giftedness manifest in mathematics in the creative-productive sense? What do psychology and neuroscience tell us about the process of fostering mathematical giftedness specifically?

In this chapter, we examine first general aspects of creativity and giftedness, noting that Treffinger's (2004) *five themes* provide a framework for understanding the connection between creativity and giftedness. Having established that creativity and giftedness are connected through these five themes, we then turn attention first to a psychological view of factors that are important for understanding mathematical creativity and giftedness, followed by a neuroscientific examination of the same.

The chapter concludes with the notion that *mathematical* creativity and giftedness can be thought of as a special case of the intersection of creativity and giftedness more generally, and that creativity and giftedness – mathematical or otherwise – can be characterized by a series of *dualities*. Elements of the person, the cognitive processes employed, the outcome and the environment associated with mathematical creativity and giftedness are unique to this domain, and the blending of psychological and neuroscientific approaches offers the best means for understanding and fostering this ability.

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Keywords Creativity • Giftedness • Mathematics • Psychology • Neuroscience

12.1 Creativity and Giftedness

To understand *mathematical* creativity and giftedness, it is first necessary to examine the long association between general creativity and giftedness that has left its mark on the research literature. Treffinger (2004) outlines the parallel development of interest in creativity research, usually attributed to Guilford (1950), and giftedness, but also explains how these two topics have intersected over the last 60 years. This intersection is, again, attributable to the influence of Guilford and his work on intelligence (Guilford 1959), as well as the contributions of key creativity researchers such as E. Paul Torrance (e.g. Torrance 1972), who worked at the interface of creativity and giftedness.

Treffinger (2004) makes the case that the association of creativity and giftedness is more than simply a result of the overlapping interests of researchers active in both fields, as might be expected. In fact, he argues that there are five *themes* that provide the rationale for the close association between creativity and giftedness (p. xxiv):

1. **Definition**: not only have both topics attracted significant attention in their own right, as researchers seek to define them, but the process of defining both creativity and giftedness gives rise to questions of how they are related to each other, and how the two fields intersect;
2. **Characteristics**: both creativity and giftedness prompt research into how they are manifest in people, and how these indicators might be measured;
3. **Justification**: both give rise to questions of how they contribute to education;
4. **Assessment**: both generate wider questions of measurement – not only how they are manifest in people, but also, for example, what measureable outcomes they lead to. Do both, for example, result in measureable improvements to student outcomes in school?
5. **Nurture**: both creativity and giftedness provoke questions of development – can they be nurtured deliberately and successfully?

Treffinger (2004) also makes the important point that these questions and issues are, by no means, settled. While a great deal of progress has been made across all five themes, for both creativity and giftedness there remain, nevertheless, many unresolved issues. Not least among these are questions of the roles that creativity and giftedness play in specific application domains such as mathematics (see, for example, Sriraman 2008), as well as their more general, and domain-independent, characteristics.

In order to lay a foundation for a discussion of mathematical creativity and giftedness, we will use Treffinger's five themes as a framework for reviewing more recent research on the intersection of creativity and giftedness. The material used by Treffinger (2004) to address the five themes, and contained in the edited volume

(Treffinger and Reis 2004), generally covers research up to 1993. In the present chapter, we will focus also on more recent works to summarize key findings on creativity and giftedness. We also constrain our review of material in one other important way. The purpose of the present chapter is to provide *psychological* and *neuroscientific* perspectives on mathematical creativity and giftedness. Accordingly, we draw primarily from the *psychological* literature of creativity and giftedness, as opposed to other sources, such as educational research.

12.1.1 Definition

Treffinger (2004) drew attention to two contrasting aspects of how creativity is defined, in particular in relation to giftedness. Citing Delcourt (1993), he noted "the importance of non-cognitive dimensions of creative productivity" (p. xxv), while Sternberg and Lubart (1993) also highlight similar dimensions, such as motivation. Conversely, he also incorporated elements of cognitive processes in his discussion of creativity and giftedness, noting that both Sternberg and Lubart (1993) and Runco (1993), link divergent thinking to discussions of creativity and giftedness.

Almost concurrently, Cropley (1994) had noted a shift in conceptualizations of giftedness that strengthened the relationship to creativity. In particular, this reinforced the importance of non-cognitive factors (e.g. motivation) in conceptualizations of giftedness, in addition to cognitive factors. Cropley (1994) argued that "true giftedness" could only be seen in the interaction of creativity with *conventional* intelligence. This model of giftedness was built around Renzulli's (1986, 2011) "three-ring conception" of giftedness, which highlights the *interaction* of: (a) above-average ability; (b) task commitment, and; (c) creativity as contributors to giftedness. What stands out in Renzulli's model, as in the other definitional research (e.g. Preckel et al. 2006), is that both giftedness, and creativity, involve elements of cognition, elements of motivation, elements of the influence of the environment, and an understanding of what is produced, and to what purpose.

The importance of cognitive processes as a component of both creativity and giftedness was also explored by Naglieri and Kaufman (2001), who examined the PASS (Planning, Attention, Simultaneous and Successive) Theory – a cognitive processing approach to children's abilities – as an explanatory tool for aspects of creativity. They note that planning, in particular, plays an important role in creativity, and also draw attention to another important element of the cognitive process. Naglieri and Kaufman (2001) cite the Finke et al. (1992) *Geneplore* model that recognizes *two aspects* of cognition – a generative phase followed by an exploratory phase.

These contrasting definitional approaches to creativity and giftedness highlight the fact that both cognitive and non-cognitive factors are important in each, providing multiple points of intersection between creativity and giftedness.

12.1.2 Personal Characteristics

The second of Treffinger's (2004) five themes concerned the characteristics – in particular, the personal characteristics – that serve as indicators of both creativity and giftedness. Treffinger, however, also noted an important shift in emphasis that strengthens the link between creativity and giftedness. Like Cropley (1994) he saw a shift away from more quantitative conceptualizations of creativity (e.g. "How creative are you?", p. xxvi) to a more qualitative conceptualization, i.e. "How are you creative?" (p. xxvi). In many respects, this shift reflects the distinction already discussed as an element of the definition of creativity, and therefore giftedness. Both, in other words, are not defined, or manifest, simply in terms of a quantified process – divergent thinking, for example – but instead are characterized by personal characteristics such as motivation, feelings and other personal properties such as a tolerance for ambiguity. Treffinger (2004) makes a further, crucial, point. This characterization of both creativity and giftedness may involve not only cognitive abilities and personality traits, but may be represented better by *style preferences* (p. xxvi), implying a more malleable and *dispositional* nature to creativity and giftedness, as well as factors that are specific to particular kinds of applications.

Creativity and giftedness intersect therefore not only in the kinds of characteristics that they share in common, but also in the fact that these characteristics may be open to development, improvement and enhancement.

12.1.3 Justification

Treffinger (2004) makes a compelling case for the interrelationship of creativity and giftedness. Not only are they linked in the way each can be characterized – in terms of themes – but also the content of those themes further deepens the links. Thus, while both are defined in terms of the same cognitive processes and personal qualities, creativity is, according to Renzulli (e.g. 2011), one of the *components of* giftedness. From the point of view of the *justification* of the importance of creativity and giftedness in education specifically, Renzulli's three-ring conception – above average ability, task commitment and creativity – is especially critical. If giftedness is *not* tied to a restrictive definition of high IQ – a facility for doing well in standardized tests – but instead is more broadly conceived of in terms of abilities that include, for example, artistic or psychomotor ability, then the justification for an interest in giftedness in education becomes similarly broadened. In the same way, the inclusion of creativity and task motivation expands *who* can be gifted, and *how* they are gifted, meaning that giftedness in education is no longer an issue of a small minority of those students with exceptional IQs. Creativity makes giftedness a topic of broad importance in education.

Even if the focus is only on creativity, as a component of giftedness, there is a great deal of research that examines the value of creativity in an educational context.

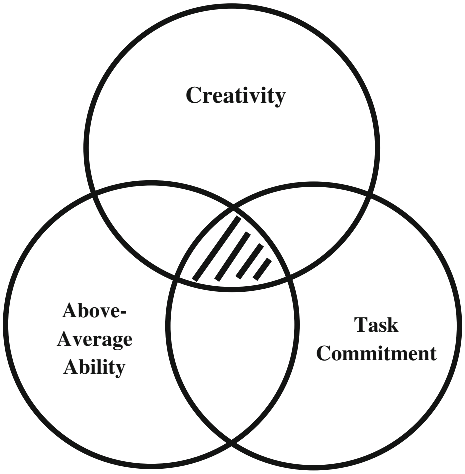

Fig. 12.1 Renzulli's three-ring conception of giftedness

Creativity is important in education not simply as a teachable skill, or as a vital component of giftedness, but because it engenders qualities that are of wider benefit to students. Sternberg (2007) provides perspectives as to why this might be the case. Conceiving creativity as a *habit*, he discusses the keys to developing that habit. Many of these are essential not only to the ability to think creatively, but as characteristics of resilient, effective and successful individuals. Whether the ability to identify and overcome obstacles, for example, drives, or is driven by, creativity, individuals who possess this ability are not only in a better position to be creative, but are inherently more resistant to the ups and downs that they will encounter in life. Therefore, an educational process that fosters creativity, and the keys that help develop the habit, not only gives the individual the best chance of expressing his or her giftedness across a broad range of application domains including mathematics, but also prepares them better for life outside of the educational process.

12.1.4 Assessment

Renzulli's three-ring model (1986, 2011) is emerging in this discussion as a key framework for understanding the relationship between creativity and giftedness. So much so, that it is worth illustrating diagrammatically (Fig. 12.1).

Notwithstanding the fact that the model, as well as the thrust of research referred to earlier, represents a shift from more purely quantitative characterizations of giftedness and creativity, to more qualitative and less restrictive notions, there is still considerable interest in the assessment – that is, the measurement – of the components of giftedness.

It will be no surprise that approaches to assessment follow a similar pattern to the definitions of creativity and giftedness reported earlier. In other words, there is a considerable body of research in creativity, for example, that addresses assessment by drilling down to the level of cognitive processes, or personal qualities. This has been tackled by many researchers in the field of creativity (for a summary of methods see, for example, Cropley 2015). However, at the intersection of creativity and giftedness, Kaufman et al. (2012) give the topic of measurement a comprehensive treatment, while Reis and Renzulli (1991) tackle the topic from a more specific point of view, in relation to the *products* – i.e. the outputs – of student creativity.

Assessment, in relation to creativity, and therefore giftedness, is about much more than just the cognitive aspects or process. To quantify creativity and giftedness requires us to quantify not only how people *think*, but also how they feel and who they are, and, what they produce. In psychology, it is no surprise that the cognitive processes and personal characteristics have received a great deal of attention. Less attention has been given to the outcomes of creativity, although it is understood that, as a minimum, a product must be both novel and effective if it is to be regarded as creative (see Cropley et al. 2011, Cropley and Kaufman 2012 for discussions of the assessment of creative products).

Neuroscience, and its associated tools such as electroencephalography (EEG) and functional magnetic resonance imaging (fMRI), is emerging as a game-changer in relation to questions of assessment in creativity and giftedness, in particular for cognitive aspects of each. Coupled with improved measures of non-cognitive factors, researchers have an expanding toolkit of measures with which to gain a deeper understanding of creativity, giftedness, and their intersection, not least in mathematics.

12.1.5 Nurture

Treffinger (2004) raises the issue of development in relation to creativity and giftedness. This is frequently discussed in creativity literature through the question "Can creativity be taught?" In fact, there seems to be widespread and general agreement in the mainstream literature that creativity is *not* (Cropley 2015) the "expression of a divine spark", but is "best thought of as an accessible, although statistically uncommon, characteristic of people and products." (p. 228). Pioneers of the field, for example Torrance (1972), were in no doubt that creativity can be nurtured, and Cropley (2015) suggests that the correct question is not "can creativity be taught?", but "are we teaching it right?" (p. 229).

The issue of nurture and creativity has been covered extensively. Different programs and strategies have been reviewed (e.g. Cropley and Urban 2000), while the effectiveness of creativity training has been dissected in various ways. Wallach (1985), for example, writing in the context of creativity testing and giftedness, argues that the effects of training programs are very narrow and specific, and difficult to generalize, while Treffinger et al. (1993) suggested that there is scant evidence to support beneficial effects of creativity training. Despite this, there is mounting evidence in more recent studies, (e.g. Scott et al. 2004a, b) to support the efficacy of deliberate nurturing of creativity. A more extensive discussion of the issues of creativity training is given in Cropley (2015).

From the point of view of giftedness, and the three-ring model, while it may be true that above-average ability cannot be taught, per se, nurture remains important. If teaching creativity is a more causal, quantitative expression of nurture, then *encouraging* and *supporting* an above-average ability represents a more *qualitative* expression in relation to giftedness. Neither creativity nor giftedness is fixed and unchangeable.

Before leaving the issue of nurture, one other important, contributing factor is worth noting. Many researchers in creativity, for example Sternberg and Lubart (1993), Isaksen et al. (1999, 2001), stress the importance of the organizational and/or social *climate* – the culture, in other words – in supporting the development of creativity. Nurture is not only an issue of *how* creativity and giftedness are nurtured, but also the *context*, or circumstances, in which they are nurtured.

12.2 A Psychological Perspective on Mathematical Creativity and Giftedness

12.2.1 The Four Ps of Creativity

The literature of creativity and giftedness that was explored earlier suggested that the two topics have a shared history, not only because they were of parallel interest to pioneering researchers, but also because they are subject to a set of common concerns – Treffinger's five *themes* (2004) – that bind them together in terms of how they are defined, what characterizes them in people, why they are important, how they are assessed, and how they are nurtured. In addition, Renzulli's three-ring model (1986, 2011) linked creativity and giftedness together more explicitly, recognizing the former as a key contributor to the latter.

Throughout that discussion, we noted certain characteristics that defined aspects of both creativity and giftedness. These can be restated now:

- The **Person**: aspects of personality – motivation for example – were a recurring feature of definitions, characteristics and assessment;
- The **Process**: cognitive processes, in particular divergent thinking, featured as important elements of the different themes;
- The **Product**: the nature of the output of the creative process was noted as an important element of assessment;
- The **Climate**: the organizational and social environments that surround the gifted individual were recognized as critical to the success of creative endeavors, and efforts to nurture both.

The characteristics noted above are, in fact, the elements of a widely used framework, developed in the field of psychology, for understanding creativity (e.g.

Kaufman 2009). The so-called "4Ps" – Person, Process, Product and Press (Climate) – were defined by Rhodes (1961), building on the work of Barron (1955). The 4Ps describe the psychological resources of the individual (Person) that support creativity, and therefore, giftedness; the special forms of thinking that give rise to creativity (Process); the novel and effective outputs and outcomes of creativity and giftedness (Product), and; the influence of the environment in which the creative and gifted individual operates and is taught (Press).

12.2.2 Domains of Creativity

A second important concept derived from the literature of psychological creativity research is that of *domain*. A long-standing debate in the psychological literature examines the question of whether creativity is *domain-general* – namely, independent of the particular area of application – or *domain-specific* – that is, strongly tied to the area of application. While this debate continues to flourish, there is a broad consensus that creativity is, in fact, both! This has important implications for the question of mathematical creativity and giftedness.

One way of making sense of the question of domain is to examine the implications for training. This has particular ramifications for mathematical creativity and giftedness in an educational context.

In relation to creativity, Baer (1998) made a distinction between *cognitive content domains*, into which he placed mathematics, and *task content domains*, for example, poetry. He found evidence that creative performance in these domains was highly variable, and in fact, found that there is frequently a negative correlation – in other words, a person who is creative in a task content domain may be uncreative in a cognitive content domain. This suggests not only that creativity is domain-specific, but that mechanisms for developing it must be domain-specific as well.

Ludwig (1998) made a similar point finding that creativity can be divided with respect to the psychological demands of the field. He categorized these fields, in general terms, as either *investigative* or *artistic*, and characterized mathematics as *impersonal*, in contrast to *emotive*, in terms of the psychological demands placed on the individual. Ludwig's categorizations also support the idea that training must be similarly differentiated – the way in which mathematical creativity should be developed, is different from the way in which artistic creativity should be developed.

Whether domain-general, or domain-specific, the guiding framework for understanding the salient issues, and for formulating particular strategies, remains the 4Ps. If mathematical creativity is distinct from artistic creativity, for example, then the manner in which it is understood, developed and expressed nevertheless remains characterized in terms of the person, the process, the product and the press. The uniqueness of mathematical creativity therefore exists, if it exists at all, at the level of differences in certain aspects of personality, cognition, output and environment, and not, as sometimes seems to be suggested, in a wholly different model.

In contrast to domain-specific views, Plucker (1998) defended a more balanced approach to creativity and domains. He suggested that domains may be better understood through a distinction between quantity and quality. The amount of originality, in other words, may be general in nature, while the quality of that originality may be domain-specific. This mixed concept is supported by more recent discussions such as Baer (2010, 2012) or the so-called *Amusement Park Theory* (Baer and Kaufman 2005). Importantly, this mixed view remains informed by the 4Ps. Thus, while the generation of novelty may be general in nature, the *exploration* of that novelty – recalling the *Geneplore* model (Finke et al. 1992) – may be far more specific in nature, impacted by values and attitudes that are specific to the particular task, and specific to the particular setting.

Psychological research in creativity, and giftedness, therefore offers a specific domain such as mathematics a conceptual framework – the 4Ps – that should play a key role in organizing and guiding research and understanding.

As we turn our attention to neuroscientific perspectives on mathematical creativity and giftedness, we will focus on one particular slice of the 4Ps framework – *process*. In doing so, we will suggest that the cognitive processes associated with creativity have two features that are important for the present discussion.

1. *Process* is a combination of both *divergent* thinking and *convergent* thinking;
2. *Divergent thinking* is more domain-general in nature, while *convergent thinking* is more specific to a particular domain such as mathematics.

From a neuroscientific perspective, cognitive processes, and especially, the oscillation between divergent and convergent thinking in creativity, offers particular opportunities for expanding understanding of core psychological concepts of creativity and giftedness, not least in an application domain such as mathematics.

12.3 A Neuroscientific Perspective on Mathematical Creativity and Giftedness

12.3.1 Neuroscience and Creativity

The application of cognitive neuroscience is a relatively recent yet increasingly popular methodology for the study of creativity (Dietrich and Kanso 2010; Gonen-Yaacovi et al. 2013; Sawyer 2011). There are huge potential benefits to embracing this approach; however, currently the lack of consensus and the inconsistency of definitions, measures, and experimental procedures represent a stumbling block (Beaty et al. 2014; Fink et al. 2014).

Obviously, the multiple definitions of creativity (i.e. a cognitive state or event; a personality trait; the creative expertise one has; or a measure lifetime creative achievement; Fink and Benedek 2014) pose problems when trying to compare studies. Further to this, the lack of agreement over how to measure creativity and the diversity of experimental procedures presents a direct challenge to the use of neuroscience within this sphere. There are a large number of measures used when testing creativity, including: divergent thinking tasks in which participants have to come up with original ideas for open problems; insight tasks which involve misleading problem representations that need to be restructured; remote association problems requiring loose associations to find semantic relations that are non-obvious; and the production of creative stories, paintings, metaphors, melodies (Dietrich and Kanso 2010; Fink and Benedek 2014). In their paper published in 2010, Arden and her colleagues (Arden et al. 2010) reviewed 45 published neuroscientific studies of creativity, and they found nearly as many tests as there were studies.

These teething troubles are to be expected from a young field such as this; nevertheless, neuroscience in creativity is already showing its worth. A notable example is the debunking of the neuro-myth that the right hemisphere is responsible for creative thought (Yoruk and Runco 2014). Time and again, studies have shown that creativity is not a localized function, but is an emergent feature arising from a diffuse network of neurons across both hemispheres (Sawyer 2011). Furthermore, the brain regions that are activated when engaged in creative tasks overlap with regions that are activated during everyday tasks, demonstrating that creativity is not a special, discrete brain function, but is related to and reliant upon many everyday cognitive skills (Sawyer 2011).

The main neuroscientific procedures that have been applied to creativity are electroencephalography (EEG) and functional magnetic resonance imaging (fMRI). There are well-documented benefits and drawbacks to these techniques, and they are broadly complementary. Both techniques are non-invasive procedures for assessing brain activity, while fMRI has higher spatial resolution than EEG, and EEG has better temporal resolution. An fMRI machine uses a magnetic field to measure the ratio of oxygenated to deoxygenated blood in a given region – the blood oxygen level dependent (BOLD) signal. An increase in the BOLD signal indicates an increase in brain activity. This is because neuronal activation stimulates an increase in blood flow, which is faster than the rate at which neurons can use the oxygen being, supplied, thus the BOLD signal to rises. This signal has a high spatial resolution, which allows brain activity to be accurately mapped. EEG is a method for recording electrical activity of the brain, particularly oscillations in electrical activity. These neural oscillations can be observed over a wide range of frequencies, with different frequency bands (e.g. the "alpha" band at 8–12 Hz and the "beta" band at 13–30 Hz) corresponding to particular types of brain activity. To interpret the EEG output, the signal is typically transformed to reveal the power associated with each frequency band. This information is then commonly used to determine if there has been an event-related synchronization (ERS; power increases relative to the baseline level) or resynchronization (ERD; power decreases) of the neural oscillations.

Cognitive psychology studies have shown that creativity involves many cognitive processes, including defocused attention, cognitive control, flexibility, fluency and working memory (Dietrich 2004). EEG and fMRI research lends neuroscientific support to the behavioral evidence that higher cognitive abilities and executive functions play an important role in creativity (Beaty et al. 2014). It has been well established from EEG-based studies that the power of neural oscillations in the alpha band decreases (i.e. ERD) during tasks that require higher cognitive abilities (Neubauer and Fink 2009). Creativity has also been linked to variations in alpha power. A recently conducted review on divergent thinking showed that alpha power varies as a function of creativity-related task demands, the originality of ideas and the general level of creativity of the participants (Fink and Benedek 2014). In addition, fMRI-based studies have shown that executive functions and creativity activate the same brain areas, namely the prefrontal and parietal regions of the neocortex (Beaty et al. 2014; Gonen-Yaacovi et al. 2013). Benedek and his colleagues (2014) further highlighted the role of the prefrontal cortex in the creative process by showing that the activation of the left inferior frontal gyrus increased with the creative quality of divergent thinking responses.

12.3.2 Divergent Thinking: Convergent Thinking

The processes of creative thinking, transfer of knowledge, and problem solving share certain characteristics. In familiar situations, these processes are not necessarily useful unless there is a change and a different approach is warranted. In unfamiliar situations, creative thinking, transfer and problem solving become the difference between being disabled by “not knowing” and being willing to “have a go” at a task. From each starting point, there are a number of possibilities that may be followed. Some of these pathways might immediately be obvious but others will be obscured by too much knowledge, insufficient knowledge or a range of other reasons, such as the individual’s appetite for risk, that prevent the identification and pursuit of a particular direction.

In unfamiliar situations, the individual or group will generate possible options and then test each idea for its utility. For example, when solving a mathematical problem, it will not be obvious which knowledge and expertise should be applied to the task. It is also true of reading between the lines of a novel or film where the unspoken themes start to become apparent to the reader/viewer, or a nascent theory about “whodunit” emerges – a possible pathway for the narrative tested out as new information becomes available.

Prevalent definitions of the creative process emphasize the two steps of generative, novel thinking followed by a consideration of the value or appropriateness of that novel idea. “Creativity is the ability to produce work that is both novel (i.e. original, unexpected) and appropriate (i.e. useful, adaptive concerning task constraints).” (Sternberg and Lubart 1999).

Divergent thinking can help provide insight to reveal the elegant solution to a mathematical problem as the alternative to a graceless but effective sledgehammer arithmetic approach. If there is really only one way of completing the task then divergent thinking may contribute to inefficient thinking and this is often seen in gifted individuals who may be slower to respond to simple questions than their classmates (Gross 2004).

Divergent thinking is the first step in a creative response to a problem where there are multiple possibilities, as is the case in most real-world problems. It provides students with different options when they do not immediately see the algorithm(s) to deploy. It promotes students to ask themselves “How might I do this?” rather than “Can I do this?” and then answer with “Well, it depends…” – a starting point for divergent thinking. This is in line with findings from the research of Carol Dweck at Stanford University: when educators focus on whether or not students can successfully undertake a task, they engender a fixed mindset and the students focus on their performance (e.g., “Can I do this?”) and their grades tend to stagnate or decrease over time. When educators focus on students’ development, they engender a growth mindset and students focus on their effort (e.g., “How might I do this?”) and their grades tend to improve (Blackwell et al. 2007; Murphy and Dweck 2009; Paunesku et al. 2015).

The divergent thinking will produce some ideas that will be quickly discounted and others that will stand up to testing through critical, convergent thinking that applies the new thoughts to the problem at hand. A particular option may be seem promising but might fall at the last hurdle. The creative, gifted mathematician does not throw it all in and start again. They might undertake more divergent thinking to explore ideas about where they might have gone wrong or what might need to be added or changed to get beyond the current impasse.

This apparent two-step model of creative thinking (i.e., divergent thinking followed by convergent thinking) hides a third subtle step that is potentially crucial – the switch.

Students who step into divergent thinking may be trapped there, continuing to generate so many possibilities but not able to turn their own convergent thinking onto the problem. They continue to produce suggestion after suggestion, meeting the novelty criteria of creativity without determining utility and value. They may find the plethora of options they have generated to be over whelming. To stop thinking in this way and to start thinking in a more critical and convergent way may require an intentional switch to take place. That is for students to recognize that they need to turn their attention to the completion of the task and try out some of the ideas they have generated. Of course, it is likely that in solving an unfamiliar and complex problem, and individual or group may go through a number of cycles of divergent thinking followed by convergent thinking, back to divergent thinking and so on. The judgment as to when and whether to switch from divergent to convergent and back again becomes an important part of the overall process. Too much switching would be ineffective but too little could lead to excessive ineffective persistence in the convergent phase as well as the over exuberant idea generation in the divergent phase.

The divergent, generative step involves “both the retrieval of existing knowledge from memory and the combination of various aspects of existing knowledge into novel ideas” (Paulus and Brown 2007). Common ideas are generated readily after which cognitive processes support the generation of more novel ideas (Beaty and

Silvia 2012; Benedek et al. 2012). While there is still much research to be done (e.g., see Fink and Benedek 2014) it appears that divergent thinking is associated with regions of the brain known to control retrieval and selection of semantic concepts (for example the left inferior frontal gyrus) and areas of the medial temporal lobe and superior temporal gyrus known to play an essential role in the recollection of facts and events. Decreased activation in the right temporoparietal region is also observed which helps focus attention and prevent distraction. This is in line with findings from the EEG that indicate increased power in the alpha band in frontal regions during divergent thinking tasks, indicating increased internal attention (Fink and Neubauer 2006). This enhanced internal attention is considered to be at its greatest during mental imagery, making floating associations and imagination (Cooper et al. 2003, 2006).

Convergent thinking brings the many strands and associations to bear on the task at hand. Inappropriate or useless ideas are eliminated and ideas that are meaningful and useful are selected for retention. Testing ideas and putting them into practice requires quite different thinking and changes where the emphasis on brain activity takes place. There is a switch from an internal to an external focus of attention that can be seen in changes in power of the EEG in regions of the brain with less in alpha and more beta bands. There is greater activation in control mechanisms of the dorsolateral prefrontal cortex as students demonstrate more directed, convergent thinking and employ more of their executive functions. In addition, monitoring for performance (e.g., that the idea is meaningful for the task and is likely to lead to the desired outcome) is apparent through increased activity in the anterior cingulate (Howard-Jones et al. 2005). This is quite different to divergent thinking in which control must be released to some extent.

12.4 Concluding Thoughts, Future Research

The literature of psychological research on creativity and giftedness supports a robust conceptual framework for understanding these topics both in general terms, and in relation to specific areas of application. The 4Ps framework – Person, Process, Product and Press – serves as a means for understanding domain-specific research in creativity and giftedness, for directing lines of enquiry, and, most importantly for allowing a comparison of, so to speak, *apples with apples*.

Mathematical creativity and giftedness can be thought of as a *special case* of the broader intersection of creativity and giftedness. Certain attributes, behaviors, values and conditions support creativity, and by extension, giftedness, regardless of domain. Thus, the personality trait *openness to experience* is consistently linked to higher levels of creativity among individuals, whereas the trait *conscientiousness* shows differences that seem to depend on whether the domain is more *investigative* or *artistic*.

How creativity is manifest in outcomes – the product – also exhibits the same blend of domain-generality and domain-specificity. The literature consistently supports novelty and effectiveness as core characteristics of creative product, however, what constitutes *novel*, or *effective*, may be far more dependent on the specific domain.

The environment, or Press, also plays a dual role. That the press itself is a factor in helping or hindering creativity is clear, however, the specifics of what constitutes a facilitatory environment may differ for mathematicians in comparison to artists.

Finally, the cognitive processes that are central to creativity show the same duality. Divergent thinking is unequivocally a core of creativity – the ability to generate ideas, measured in terms of fluency, flexibility and elaboration, is critical – however, the manner in which those ideas are evaluated and analyzed – convergent thinking – depends for more explicitly on domain-specific factors. Process, furthermore, lends itself to neuroscientific perspectives on mathematical creativity and giftedness, with techniques such as functional magnetic resonance imaging (fMRI) lending themselves to peering inside the mind of the creative mathematician.

Future research directions in mathematical creativity and giftedness can be considered from a range of complementary perspectives offered not only by the psychology of creativity and the discipline neuroscience, but also by the additional strands of cognitive science and education. For example, the systematic nature of 4Ps framework offered by psychological creativity research gives researchers the tools and techniques needed to deepen the understanding of creative and gifted cognition, as well as a deeper, and possibly more causal, understanding of the impact of personal (e.g. motivation) and environmental factors on the development of creative and gifted outcomes in mathematics. However, perhaps a more significant and productive avenue for future research in mathematical creativity and giftedness will be the *integration* not only of psychological and neuroscientific tools and techniques, but also the application of these across all 4Ps as a single, interacting system of factors that contribute to creative/gifted outcomes in mathematics. In other words, future research will not study cognition, for example, *in isolation*, even from the two intersecting perspectives, but instead will study creativity and giftedness as an emergent property of the interaction of elements of the person, cognitive process, and the environment that lead to specific outcomes.

At a more discipline-specific level, imaging studies offer the potential to develop a deeper understanding of the nature of the "switch" referred to earlier, along with the mechanisms that might control it. While researchers might expect some dorsolateral prefrontal cortex activation to take control of the dampening of divergent thinking and to promote convergent thinking, there is a chance that this phenomenon might be difficult to catch in functional MRI because of the temporal resolution of the technique (often over hundreds of milliseconds).

There are also more specific questions that address the cognitive aspects of problem solving and creativity. What are the shared characteristics and/or processes of effective versus creative problem solvers, especially in the domain of mathematics? Is creativity the same as problem solving, or does the latter require the addition of some factor (e.g. knowledge)?

Finally, from the point of view of education: What if we let students in on the secret? Does sharing the structure of creativity with mathematics students help them to be more creative (and/or better problem solvers in complex, unfamiliar and non-routine situations)? Are some frameworks more helpful than others? Would knowledge of the frameworks give teachers, students (and parents) a metacognitive language that enables or improves their learning conversations?

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Chapter 13
What Have We Learned About Giftedness and Creativity? An Overview of a Five Years Journey

Demetra Pitta-Pantazi

Abstract The aim of this chapter is to offer an overview of a series of studies conducted at the University of Cyprus, regarding the definition and identification of mathematically gifted students, the relation between mathematical creativity, practical and analytical abilities, as well as the relation between giftedness, creativity and other cognitive factors such as, intelligence and cognitive styles. During our research in the field of giftedness and creativity we developed material for nurturing primary school mathematically gifted students and also explored the possibilities that technology may offer in the development of mathematical creativity. Although our research is still evolving, this chapter offers a glimpse of some of our most important findings.

Keywords Mathematical giftedness • Mathematical creativity • Identification of mathematical giftedness • Cognitive factors • Intelligence • Cognitive styles

13.1 Introduction

Despite the increased interest in the field of mathematical giftedness and creativity, a number of issues are still open to debate. In this chapter we will present and discuss the findings from 5 years of research in the field of mathematical giftedness and creativity. All of the studies that we present were conducted by a group of researchers from the University of Cyprus, the majority of whom looked at primary school students between the ages of 10 to 12 years old. These studies addressed the issue of defining mathematical giftedness; suggested a process for the identification of mathematically gifted students; investigated the different characteristics of gifted and non-gifted students; looked into the self-perceptions of mathematically gifted students; explored the relationship between creativity and giftedness; studied the

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relationship between giftedness, creativity and intelligence; and studied the impact of a computer environment in the development of creativity. A smaller number of studies investigated university students' cognitive styles and their relation to mathematical creativity. In the light of the findings from all of the aforementioned studies, implications for researchers, policy makers, curriculum developers, teachers and teacher educators will be discussed.

13.2 Mathematical Giftedness

13.2.1 Description and Identification of Giftedness

13.2.1.1 Theoretical Background

The identification of gifted students has long been an issue causing great debate in the field of research on giftedness (Van Tassel-Baska 2000; Ziegler 2009). The lack of a commonly accepted theoretical foundation of giftedness caused a delay in the progress of understanding giftedness. As a result, the absence of a clear definition of mathematical giftedness, along with the heterogeneous nature of gifted students concerning the range of abilities they demonstrate, make the identification of mathematically gifted students extremely challenging (Hoeflinger 1998).

Traditionally, researchers defined giftedness as high general intelligence, measured by high IQ scores (Hollingworth 1942; Terman 1924; Winner 1997). According to Preckel et al. (2006), these uni-dimensional approaches define giftedness exclusively on the basis of high general intelligence. The first definition of giftedness, given by Terman in 1925, suggested as an identification criterion, the use of IQ scores above 140. It was, Leta Hollingworth who invented the concept of above-level testing for the profoundly gifted (Stanley 1990) and for many years, she used the Stanford-Binet Intelligence Scale (Terman 1916). Leta, set the bar for the profoundly gifted at 180 IQ, 5 standard deviations (s.d.) above the mean (Silverman 1991). Some years later, there was a shift, with researchers defining giftedness based on social needs by taking into account societal or educational needs. For instance, Sternberg and Davidson (1986) stated that giftedness is "something we invent, not something we discover. It is what one society or another wants it to be, and hence its conceptualisation can change over time and place" (p.3).

Contemporary conceptualisations of giftedness acknowledge the multidimensionality of the phenomenon (Gardner 1991; Sternberg and Davidson 2005) and debate in favor of a broader definitions beyond traditional notions of IQ (Lohman 2009). Multidimensional definitions of giftedness integrate several factors to describe the concept. In particular, Renzulli (1978) proposed as basic components of giftedness: above average ability; task commitment; and creativity. In addition to this, in Sternberg's (2003) WICS model, giftedness is viewed as a combination of wisdom, intelligence and creativity. Tannenbaum's (2003) Star Model refers to five factors: superior intelligence; exceptional special aptitude; non-intellective facilitators; environmental influences; and chance, or luck – which are considered to interact to produce high levels of productivity, and which are all necessary for becoming a “gifted” individual (Al-Shabatat 2013).

In line with the multi-dimensional definition of giftedness is also the Gagné’s (1991) differentiated model of giftedness and talent. According to Gagné (2003), giftedness describes the possession and use of untrained and spontaneously expressed natural abilities in five aptitude domains: intellectual, creative, socioaffective, perceptual/motor, and others. The degree of these abilities needs to place the individual in the top 10 % of people of the same age. The development and level of expression of these natural abilities are to some extent inherent, and can be observed in many of the tasks that a child may be occupied with. They can be observed more easily and directly in young children, where the influence of environmental factors and formal schooling has not yet had a great impact on them. However, they may still be apparent in older children who have not been given the same opportunities or appropriate schooling as their more fortunate peers. Talent designates the superior performance of the individual in one or more fields of human activity, in our case mathematics. Talent emerges from natural abilities, and is a consequence of the students’ learning experience. Therefore, according to this theory, natural abilities could be developed in the right circumstances, and be transformed into talents in a specific domain. Again, the individual needs to be in the top 10 % of age-peers who have been receiving training in mathematics. Although this model is comprehensive and detailed, there is not much empirical data to support it. Gagné (2005) suggested that future research should try to explore the applicability of this model in specific fields of talent.

The variety of definitions of giftedness proposed over the years resulted in the development of various identification methods and means of measuring giftedness. Hoeflinger (1998) suggested that four principles should guide the selection or development of identification tools: (a) Multiple criteria approach; (b) the context and purpose of the identification tool; (c) inclusiveness; and (d) flexibility and continuity in the process of identification. In the identification tool that we developed, we tried to address these four principles.

The first principle suggests that multiple criteria should be used in the identification tool. This is based on the fact that most definitions (eg Renzulli’s definition) include diverse dimensions to describe giftedness. It is therefore impossible to sample all behaviours using only one test (Salvia and Ysseldyke 2001). For this reason, a combination of valid, reliable, objective and sensitive tools should be used to collect information about students (Coleman 2003; Davis and Rimm 2004). Similarly, recent studies suggest that giftedness models should use different tools to measure cognitive aspects of learning (Ziegler 2009) and creativity (Mann 2006). As a result, intelligence tests, achievement tests, creativity tests, school grades, rating scales, past accomplishments, portfolios, interviews, teacher nominations, parent nominations, peer nominations, and self-reports are included in the list of multiple criteria used for identification purposes.

The context and purpose of identification tools are considered to be important factors in the development of appropriate tools. For instance, in our research program the purpose was to identify mathematically gifted students and not generally gifted students. Therefore, the identification should focus on mathematical abilities, performance and interest (Coleman 2003). In the development of the identification system, special attention should be given to the context of selection instruments in order to ensure that they highlight the mathematical strengths and weaknesses, cognitive abilities and patterns of behaviour of mathematically gifted students.

Inclusiveness is the third principle, which prevents what Birch (1984) called a "narrow identification". This principle ensures that identification tools are not biased in any way by gender, race, colour, socioeconomic status, or geographical location.

The final principle for developing such tools refers to the flexibility and continuity of the process for identifying gifted students whose abilities may not be immediately apparent, but who are likely to develop such abilities in the right circumstances (Davis and Rimm 2004).

Recent trends regarding the identification of giftedness turn to psychological models to explain and describe the construct of giftedness (Shavinina 2009). Considering giftedness as a psychological construct results in the proposal of new models of giftedness that incorporate cognitive processes instead of traits found in gifted individuals. The field of cognitive psychology has produced theories that map cognitive structures and processes in an effort to understand the human mind. According to experiential structuralism theory (Demetriou et al. 2002), the human mind is organized into three levels. The first level involves a set of environment-oriented Specialized Capacity Systems (SCSs) (the qualitative–analytic, the quantitative–relational, the causal–experimental, the spatial–imaginal, and the verbal–propositional). The second level involves a set of higher-order control structures governing self-understanding, self-monitoring, and self-regulation, which is called a hypercognitive system. The third level involves processes and functions underlying the processing of information (speed of processing, control of processing and storage). Shavinina (2009) suggests that scientists should study individuals' cognitive experiences, referring to a system of available psychological mechanisms that form the basis of our understanding of the world, and which determines the specificity of our intellectual activity. Such psychological mechanisms could be considered the specialized capacity systems as described in experiential structuralism theory (Demetriou et al. 2002).

Another significant issue that is involved in the identification of mathematical giftedness concerns the nature of the relationship between giftedness and creativity. A number of researchers argued about the relationship between mathematical giftedness and creativity (eg Sriraman 2005), and many stressed the importance of this relationship in the identification of mathematically gifted. However, due to the problematic nature of empirical evidence, this relationship is still ambiguous (Klavir and Gorodetsky 2009). Various researchers consider creativity as an essential component of giftedness (eg Renzulli 1978). As it was mentioned earlier on, Renzulli's model is comprised by three overlapping circles: creativity, above-average ability and task commitment, stipulating that the intersections denote giftedness. On the contrary, some researchers view giftedness as a prerequisite of creativity (Usiskin 2000), and suggest that mathematical giftedness does not necessarily imply mathematical creativity (Leikin 2008; Sriraman 2008; Usiskin 2000). A third perspective regarding the relationship of giftedness and creativity is suggested by researchers who use the terms giftedness and creativity as synonyms (eg Krutetskii 1976). This perspective is grounded on the fact that gifted and creative individuals share common personality and intellectual characteristics (Piirto 1998).

The relationship between creativity and giftedness is also expressed in other models. The Triarchic Theory of Intelligence is comprised of analytical, creative and practical abilities (Sternberg 2005) and suggests that creativity is one of the central components of intelligent human behavior (Leikin and Pitta 2013). Furthermore, the comprehensive model of giftedness (Milgram and Hong 2009) considers creative talent as one of two distinct types: expert talent and creative talent (Leikin and Pitta 2013). While these share different types of abilities, expert talent involves more analytical or practical thinking ability than creative thinking ability (Leikin and Pitta 2013). In addition, the Actiotope Model and the ideas of Csikszentmihalyi and Wolfe (2000) suggest that the location of creativity is not limited to an individual's mind, but rather that it is also embedded in a system where an individual interacts with a cultural domain and a social field (Leikin and Pitta 2013). It should be noted that, despite of the differences between the models and the impact they make on different factors that contribute to the development of talent, all researchers agree that interaction between personality traits and environmental factors determine, to a great extent, the realization of creative talent (Leikin and Pitta 2013).

Consequently, a shift of research on giftedness should lead to a new approach to the identification of a gifted population, one that will have as its basis both models of giftedness and contemporary cognitive models. As well as encompassing high level mathematical abilities, this approach should also incorporate other factors such as natural abilities (specialized cognitive abilities), cognitive functions (eg, memory and processing), intelligence and creative ability (see Fig. 13.1).

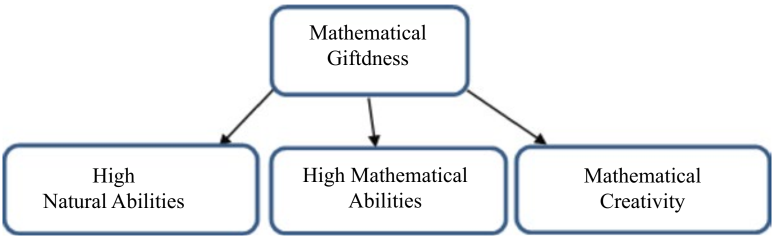

Fig. 13.1 Proposed components of mathematical giftedness

13.2.1.2 Our Results

From the studies that we have carried out (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2011, 2012) we found guidance mainly in three theoretical frameworks: (a) Gagné's (2003) differentiated model of giftedness and talent, (b) Renzulli's (1978) model of giftedness and (c) experiential structuralism theory (Demetriou et al. 2002). These models were not adopted in their entirety, but we selected and combined some of their main aspects. This resulted in the creation of a new theoretical domain specific model for mathematical giftedness, which we then tested (Fig. 13.2).

The model that we suggest in our studies (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2011, 2012) is fairly simple and serves three purposes: It describes what mathematical giftedness is; proposes its constituent factors; and indicates which natural/cognitive abilities may predict or influence it. The model is also fairly economical, since it rests on a rather small number of operational factors whose relationship is also clearly stated.

The proposed model suggests that the fundamental components of mathematical giftedness are mathematical abilities and mathematical creativity. Mathematical abilities are defined by the SCSs: (a) spatial abilities; (b) quantitative abilities; (c) qualitative abilities; (d) verbal abilities; and (e) causal abilities (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2011, 2012). Mathematical creativity is defined in terms of fluency, flexibility and originality. Although all our studies suggest that mathematical giftedness is constituted by mathematical abilities and mathematical creativity, our analysis of the results showed that mathematical abilities contribute more to mathematical giftedness than mathematical creativity (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2011, 2012). Furthermore, all of our studies confirmed that certain natural/cognitive abilities, and specifically fluid intelligence and self-perceptions, are necessary but not sufficient conditions for predicting mathematical giftedness (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2011, 2012). This means, that an individual with high fluid intelligence and strong self-perceptions may or may not be mathematically gifted. On the other hand, all individuals who are mathematically gifted have high fluid intelligence and self-perceptions.

In a further study (Kontoyianni et al. 2013) we extended this investigation by examining whether the identification of mathematically gifted students by means of an IQ test or a self-produced mathematics and creativity test would produce the same group of students. To achieve this, we examined 359 4^{th}, 5^{th} and 6^{th} grade students with the WASI IQ test and with a domain specific self-produced mathematics and creativity test. Our analysis once more confirmed that fluid intelligence is a predictor of mathematical giftedness; however different groups of students were identified by the two types of test. It appears that although some students may have a high fluid intelligence, they do not exhibit mathematical giftedness. On the other hand, all students who exhibited mathematical giftedness had a high fluid intelligence. The variance between the two groups of students, those with high fluid intelligence scores and those with high scores in the mathematical giftedness and

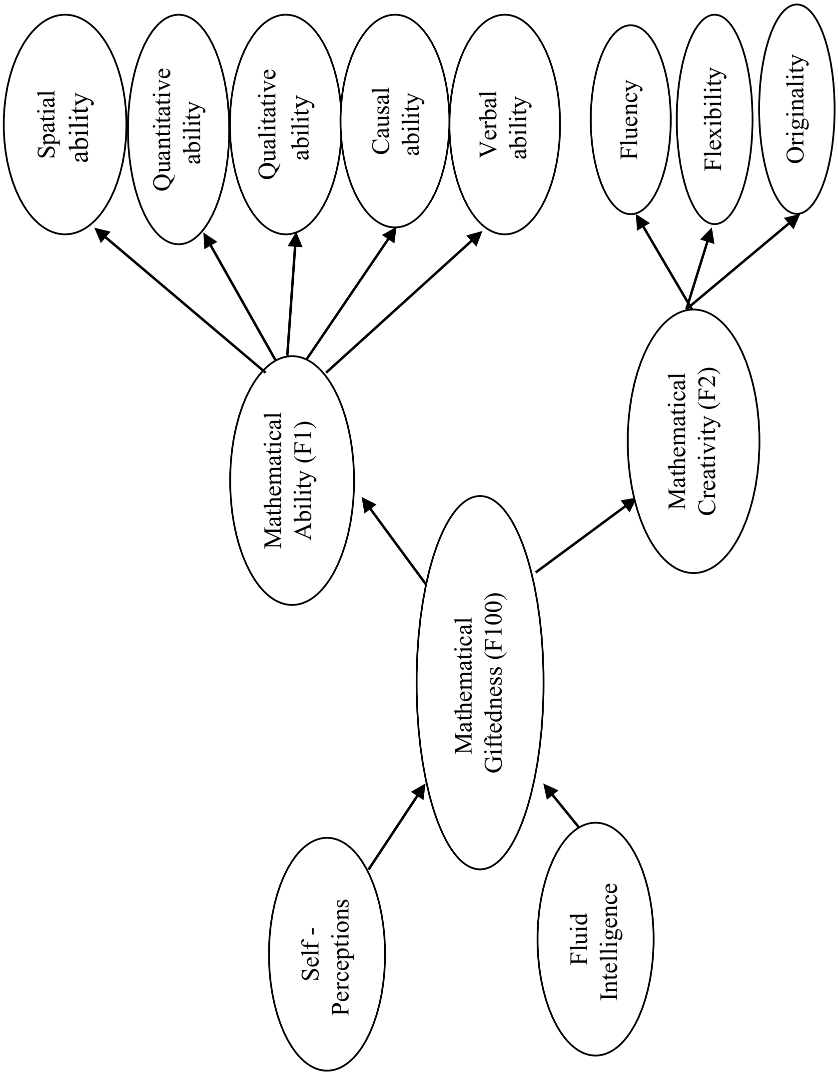

Fig. 13.2 The structure of the proposed model

creativity test, could be explained by students' performance in specific task categories.

Apart from the predictive power of IQ testing, in another study we investigated the predictive power of a self-report questionnaire which investigated students' self-perceptions (Kontoyianni et al. 2011; Pitta-Pantazi et al. 2012). The study was conducted again with 9 to 12 year old students who had to complete a self-report questionnaire with 20 statements on a 5-point Likert type scale. The results of the study suggested that students' self-perceptions of mathematical behavior can be described across five dimensions: (a) learning characteristics; (b) interest or curiosity; (c) social-emotional characteristics; (d) creativity; and (e) mathematical reasoning. Most importantly, the results revealed that the self-report questionnaire that we had developed could predict mathematical giftedness, which was defined based on the theory explained earlier, in terms of mathematical ability and creativity.

Part of our research also included the comparison of creative ability in mathematically gifted and mathematically non-gifted students (Kontoyianni et al. 2011). Gifted and non-gifted students were asked to respond to multiple solution tasks and were assessed based on the number of correct responses provided (fluency), the different mathematical ideas employed (flexibility) and the originality of their responses (uniqueness). The qualitative analysis of the results revealed that both groups of students were able to provide more than one correct solution. However, gifted students were able to provide more advanced, sophistigated, complicated and unique solutions.

13.2.2 Programs to Support the Gifted

Nowadays, there is a wide range of provisions to be considered when catering for mathematically gifted students in elementary school classrooms. These include acceleration, enrichment, differentiation, curriculum compacting, mentorships, and competitions. However, in Cyprus, where we have been carrying out our research studies, very limited programs for mathematically gifted elementary school students are provided. It can be argued that promising Cypriot students are prevented from realizing their full potential, and consequently our country does not benefit from their potential capabilities.

For this reason, one of the aims of the project was to design an enrichment program made up of inviting and challenging tasks that require analytical, creative and practical skills. In this manner, gifted students would have the opportunity to explore topics in more depth, draw out generalizations, and create new problems and solutions related to the topics under investigation. Therefore, we developed a mathematical curriculum specifically for the gifted, in which the activities are organized across the following five strands: Number; Algebraic Reasoning; Geometry; Measurement; Data Analysis; and Probability. These activities were facilitated with the use of technology. Nevertheless, the effectiveness of this enrichment program has not yet been tested.

13.3 Mathematical Creativity

Creative performance is an essential part of doing mathematics (Pehkonen 1997), and mathematical creativity has recently come to be considered as a necessary skill that may and should be enhanced in all students (Mann 2005; Pelczer and Rodríguez 2011). As Lev-Zamir and Leikin (2011) point out: "we consider developing mathematical creativity in each student to be one of the purposes of school mathematics education" (p. 1). However, mathematical creativity is a complex construct and, as such, it has been defined and measured in various ways. According to Mann (2005), myriad definitions of creativity in mathematics have been promulgated (Chamberlin and Mann 2014). Many researchers have argued that to date there is no single well-accepted definition of mathematical creativity, nor any means by which it can best be measured (Mann 2006; Sriraman 2009).

Yet, a widely accepted definition of creativity is provided by Torrance (1994). Torrance (1974) defines fluency, flexibility and originality as the main characteristics of creative individuals. Fluency is identified as the ability to produce many ideas, while flexibility refers to the number, degree and focus of methods/approaches observed in a solution. Originality refers to the likelihood of holding extraordinary, new and unique ideas (Gil et al. 2007). Several definitions regarding mathematical creativity (e.g Ervynck 1991; Gil et al. 2007; Krutetskii 1976; Silver 1997) are based on the Torrance's (1974) concepts of fluency, flexibility and originality (Torrance 1995).

13.3.1 The Relationship Between Mathematical Creativity and Mathematical Ability

13.3.1.1 Theoretical Background

According to Silver (1997) "creativity is closely related to deep, flexible knowledge in content domains" (p. 750). Considering that (a) the creative application of knowledge in various circumstances (Sternberg 1999), (b) the suggestion of original solutions (Shriki 2010) and (c) the ability to find numerous and distinctively different answers in mathematical tasks (Sriraman 2005), are amongst important external behaviors that an individual may exhibit that indicate deep mathematical understanding, then, it is not surprising that mathematical creativity is closely related to deep knowledge in the specific domain (Mann 2005). However, what is not clear (and research results are conflicting) is the structure of the relationship between mathematical creativity and mathematical ability.

The results of a number of studies that examined this relationship varied depending on the instruments used, the populations studied and the specific domain that was examined (Bahar and Maker 2011; Mann 2005). Some of these studies found no correlation between the two constructs (Baran et al. 2011; Haylock 1987), while others provided evidence that the two constructs were related (Bahar and Maker 2011; Sak and Maker2006). Furthermore, a number of studies (eg Bahar and Maker 2011; Sak and Maker 2006) examined the nature of this relationship. According to the results of some of these studies (Bahar and Maker 2011; Sak and Maker 2006), regression analysis verified that the fluency, flexibility, elaboration and originality or the mathematical creativity total score were significant predictors of mathematical ability. On the other hand, other researchers suggested that mathematical knowledge is vital for the development of mathematical creativity (eg Mann 2005; Nakakoji et al. 1999). For example, according to Mann (2005), mathematical achievement is the most significant predictor of creative mathematical performance.

Therefore, there is no agreement as to whether there is a correlation between mathematical creativity and ability (eg Haylock 1997; Sak and Maker 2006) or in which way the two concepts are related. As a result, two relative questions arise: Is there a correlation between mathematical ability and mathematical creativity? Are mathematical abilities prerequisites of mathematical creativity, or vice versa?

13.3.1.2 Our Results

In a few of our studies (Kattou et al. 2011, 2013) we investigated the relationship between mathematical ability and mathematical creativity, and examined the structure of this relationship. Mathematical ability was considered as a multidimensional construct comprised of: (a) quantitative ability (number operations and pre-algebra), (b) causal ability (cause and effect relationships), (c) spatial ability (paper folding, perspective and rotation abilities), (d) qualitative ability (processing similarities and differences) and (e) inductive and deductive abilities. Mathematical creativity was measured with the use of mathematical multiple solution tasks and students' fluency, flexibility and originality in these tasks. Therefore, mathematical creativity was seen as a domain specific type of creativity.

The first step in our investigation was to examine whether there was a correlation between mathematical ability and mathematical creativity. This was triggered by the contradictory results regarding the relationship between these two constructs (Haylock 1997; Jensen 1973 in Haylock 1987). For this, we used Confirmatory Factor Analysis to investigate the relationship and structure of mathematical ability and mathematical creativity. The first quantitative analysis we carried out showed that there was a statistically significant positive correlation between mathematical creativity and mathematical ability. In other words, if a student's mathematical ability is high, then his mathematical creativity is also high; as mathematical ability decreases, mathematical creativity also decreases and vice versa. Once this conclusion was reached, it was important to investigate the relationship between and the structure of these two constructs. Two Models were tested, Model 1 was based on Balka's idea (1974, in Mann 2005) which implies that mathematical ability is a subcomponent of mathematical creativity, and Model 2 was guided by the Integrated Thinking Model (Iowa Department of Education 1989), which implies the reverse relationship, that mathematical creativity is one of the subcomponents of mathematical ability. The statistical analyses that we conducted revealed that the model that best fitted our data was Model 2, which suggests that mathematical creativity is a subcomponent of mathematical ability. This result is in line with the results of other researchers who suggest that mathematical creativity is a prerequisite of mathematical ability (Leikin 2007; Mann 2005), and that mathematical creativity may predict mathematical ability (Bahar and Maker 2011; Sak and Maker 2006).

In another one of our studies (Kattou et al. 2011, 2013) the reverse relationship was also examined, specifically whether mathematical creativity constitutes a predictor of mathematical ability. To do this we identified groups of 4th, 5th and 6th grade students which were characterized by different levels of mathematical ability, and examined whether these groups also presented different levels of mathematical creativity. The results revealed that mathematical ability may be predicted by students' mathematical creativity, and also that the level of mathematical ability depends on the level of mathematical creativity.

These results were further supported by another of our studies (Cleanthous et al. 2010) which explored the differences in mathematical abilities amongst three groups of 9–12 year old students: High IQ and Low Creativity students ($H_{IQ}L_C$), Low IQ and High Creativity students ($L_{IQ}H_C$), and High IQ and High Creativity students ($H_{IQ}H_C$). Students' IQ was measured with the WASI test, their creativity with mathematics multiple solution tasks test, and mathematical ability with a mathematics test. The results revealed that $H_{IQ}H_C$ students had the highest scores in all tests and were able to explain their answers. Interestingly enough, $H_{IQ}H_C$ and $H_{IQ}L_C$ students did not only have statistically significant differences in their mathematical creativity test, but they also had statistically significant differences in all aspects of their mathematics ability test.

Apart from the relationship between mathematical creativity and mathematical ability, we also wanted to explore the relationship of mathematical ability to other types of abilities, such as practical and analytical, which were described by Sternberg's (1997, 1999, 2005) theory of successful intelligence, and we turn now to these issues.

13.3.2 *The Relationship Between Creative, Practical and Analytical Abilities*

13.3.2.1 Theoretical Background

Sternberg (1997, 1999, 2005), in his theory of successful intelligence, considers creativity as one of three intellectual components, along with analytic and practical thinking. Examining students' analytical abilities involves investigating their abilities to analyze, judge, compare and contrast, evaluate and assess. Examining students' practical abilities entails investigating their abilities to apply, use, put into practice, implement, employ and render practical what they know. Finally, examining students' creative abilities implies looking at students' abilities to create, discover, imagine if…, predict and invent (Sternberg and Grigorenko 2004). Although, this theory has long been established, the number of studies conducted specifically in mathematics is quite limited (Sternberg et al. 2006).

The studies that have been carried out to validate Sternberg's triarchic model (Sternberg et al. 1996, 1999, 2001) used Confirmatory Factor Analysis to compare alternative models of the data. These studies suggest that analytical, practical and creative abilities are relatively distinct. Sternberg (2003) claims that although they derive from the same information processing-component there is no reason to expect these three types of thinking to be completely independent. However, they appear to be sufficiently different that one should not conclude that traditional intelligence (which is often measured through analytic tasks) implies high levels of creative or practical abilities, or vice versa. Nonetheless, it is suggested that in order to survive in the world, everyone has to have at least some ability to think analytically, practically and creatively.

Findings about the relationship of these three kinds of abilities – analytical, practical and creative – are conflicting. In particular, in Sternberg and his colleagues' (1999) earlier studies, the researchers used Confirmatory Factor Analysis and found that the three factors of abilities were not correlated. In another study, Sternberg and his colleagues (2001) suggested that there is an intercorrelation among the analytical, practical and creative factors. The researchers claim that this conflict appears because different instruments were used in these studies.

13.3.2.2 Our Results

In one of the studies (Pitta-Pantazi et al. 2010) that we conducted, we investigated 6th grade students' analytical, practical and creative abilities with nets and three-dimensional rectangular arrays of cubes. The confirmatory analysis that we conducted showed that the mathematical tasks may be categorized based on Sternberg's triarchic theory as analytical, practical and creative abilities. These three types of abilities were not hierarchically related. Our analysis also illustrated that students appear to have stronger analytical and practical abilities, while their creative abilities appear to be the weakest of the three.

Apart from the relationship of creative ability to other types of abilities such as analytical and practical abilities, another issue we considered to be worth exploring was the relationship between creative ability and other cognitive factors such as intelligence, memory, information processing and cognitive styles. We will address these matters in the following section.

13.3.3 The Relationship Between Creativity and Several Other Cognitive Factors (Intelligence, Memory, Information Processing, Cognitive Styles)

13.3.3.1 Theoretical Background

In recent years, researchers have investigated the relationship between general creative ability and several cognitive and psychological factors, such as intelligence, memory, information processing, prior knowledge and abilities (eg Sheffield 2009; Sternberg and O'Hara 1999). However, there is a lack of corresponding research on domain specific creativity and in particular, mathematical creativity. Furthermore, in these studies no attempt has been made to ascertain the effect of a combination of cognitive factors on mathematical creativity. Therefore, despite the increased research attention on mathematical creativity during the last 10 years, to date it is not yet clear which cognitive variables affect the appearance of mathematical creativity.

Regarding the relationship between creativity and intelligence, contrasting results have been proposed. On the one hand, a statistically significant relationship was found between the two concepts (eg Ripple and May 1962; Srivastava and Thomas 1991), whereas on the other hand creativity was found to be independent of intelligence (eg Getzels and Jackson 1962). Recently, research interest has focused on the structure of the relationship between creativity and intelligence. Gardner (1993) proposed that intelligence constitutes a superset of which creativity is a subset, whereas, Sternberg and Lubart (1995) considered intelligence to be a subset of creativity. Furthermore, other researchers considered intelligence and creativity as to be overlapping sets (eg Sternberg 1985), or even disjoint sets (eg Torrance 1974).

The importance of memory in creative thinking is also highlighted by a number of researchers (eg Guilford 1962, in Mann 2006). Guilford (1962, in Mann 2006) stressed the importance of organizing, retrieving and applying information where appropriate, in an effort to emphasize the importance of memory in creative thinking. In addition to this, information processing has been proposed as a characteristic of creative thinkers, given that it involves flexibility on switching between conceptual systems (Sternberg and O'Hara 1999).

Research results regarding the relationship between cognitive styles and creativity appear also to be contradictory. Cognitive styles are defined as "an individual's characteristic and consistent approach to organising and processing information" (Tennant, cited in Riding 1997). Different cognitive styles have been identified and proposed by mathematics educators and cognitive psychologists. Some researchers claim that there is no relationship between cognitive styles and creativity (eg, Kirton 1989), while other researchers argue that cognitive styles are associated with creativity, or even predict it (eg, Martinsen and Kaufmann 1999; Sternberg 2012; Woodman and Schoenfeldt 1990). This discrepancy may be due to a number of reasons, such as the cognitive style distinction used and the type of creative behavior under investigation. Thus, there appeared to be a need to further examine this relationship and this was a line of research that we pursued.

Our choice to adopt a certain cognitive psychologist's approach and more specifically Kozhevnikov's (2007) cognitive style distinction, was dictated by the fact that there is recent neuropsychological evidence which supports the existence of this distinction, thus strengthening the validity of this cognitive style construct (Kozhevnikov 2007). More specifically, Blazhenkova and Kozhevnikov (2009) suggests that there are three different dimensions of cognitive style: a verbal style and two types of visual cognitive style (spatial imagery and object imagery cognitive styles). These three dimensions of cognitive style are consistent with current neuroscience research data which suggest that apart from the verbal areas of the brain, the visual areas of the brain are further divided into two functionally and anatomically independent systems: one concerned with the appearance of individual objects and the other with spatial relations between objects and components of objects (Anderson et al. 2008).

Some research evidence has connected object imagery to creative performance. Kunzdorf (1982) found that object imagery facilitated performance in creative production. Other researchers have connected this performance to spatial visualization. Blazhenkova and Kozhevnikov (2009) argues that Einstein's creativity relies on his spatial visualization. They also argue that spatial visualizers' ability to analyse an object part-by-part makes it clearer and more explicit in their mind. This allows them to manipulate spatial images flexibly and make numerous transformations (Kozhevnikov et al. 2005). In addition to this, the same researchers support the idea that spatial visualizers have simple images, free of any detail. On the other hand, it is argued that object visualizers' detailed images may be an obstacle to flexible transformations. In addition to this, there have been some hints in the literature that creativity may also be connected to verbal cognitive style. For example, Blazhenkova and Kozhevnikov (2009) report the case of the famous mathematician Poincare' who "demonstrated an abundance of clearly written sequential text and formulas, without any lines being crossed out or any diagrams drawn" (p. 658). Therefore, we felt that further research was needed to investigate the relationship between these cognitive styles and creative mathematical performance.

13.3.3.2 Our Results

In one of our studies (Pitta-Pantazi et al. 2013) we investigated the relationship between prospective primary school teachers' creative process and their spatial, object and verbal cognitive styles. Participants' mathematical creativity was measured in terms of fluency, flexibility and originality through the administration of a mathematical test composed of mathematical multiple solution tasks. The participants' cognitive styles were measured using the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) (Blazhenkova and Kozhevnikov 2009). Our results suggested that whereas visual cognitive styles (spatial and object imagery) were statistically significant predictors of the participants' mathematical creativity, verbal cognitive style was not. Furthermore, the analysis also yielded the fact that spatial imagery cognitive style was related to all three aspects that make up creativity- fluency, flexibility and originality- whereas the verbal cognitive style was negatively related to flexibility.

In another study (Pitta-Pantazi and Christou 2010), we investigated the relationship of 11 years old students' spatial and object visualization to their analytical, creative and practical abilities in three dimensional geometry. The analysis we conducted showed that object visualization was related to the students' creative abilities in nets, whereas their spatial visualization was related to their practical abilities in three-dimensional arrays of cubes. Furthermore, high and low spatial visualisers differed in their practical abilities in three-dimensional arrays whereas high and low object visualisers differed in their creative abilities in nets

Despite research studies which looked into various cognitive factors that may affect mathematical creativity, a crucial issue that is still open to debate is whether, and in what way, teaching approaches may enhance mathematical creativity. Of course teaching method may be of various types. In one of our studies we investigated the influence of the use of technology in enhancing students' mathematical creativity.

13.3.4 Interventions to Enhance Creativity Through the Use of Technology

13.3.4.1 Theoretical Background

In the literature we come across two conflicting views regarding the relationship between creativity and technology (Clements 1995). One view is that technology enhances only uncreative, mechanistic thinking. The second view is that technology is a valuable tool in creative production (Clements 1995). This second view of the role of technology agrees with the argument, proposed by the National Advisory Committee on Creative and Cultural Education (1999) that technology enables students to find new modes of creativity. According to Yushau et al. (2005), "a proper use of various technologies especially computers in the teaching and learning of mathematics has the potential of helping learners to develop their creativity" (p. 1).

Similarly, several researchers argue that technology can promote students' mathematical creative abilities (Clements 1995; Dunham and Dick 1994; Mevarech and Kramarski 1992; Pardamean and Evelin 2014; Sinclair et al. 2013; Subhi 1999) and that technology may construct a suitable educational environment to enhance the emergence of creative behaviour (eg Betz 1996). According to Dodge (1991), creative computing provides learners the opportunities for fluency, flexibility, association and testing. Opportunities for fluency include the generation of numerous ideas, knowing that only a few will be valuable (Dodge 1991). According to Yang and Chin (1996), instant feedback, speed, range of information, interactivity and personalization are some of the facilities that new technologies offer, motivating learners to think creatively in a short time. With regard to flexibility, technology offers the opportunity to shift between different perspectives; in other words to exchange representations or views of the same construct (Dodge 1991). According to Guilford, the ability to redefine a situation, that is to see and give alternative interpretations of familiar objects, is considered as a trait of creative behaviour (Guilford 1959). The potential of association refers to the ability of putting disparate elements together in order to make new combinations (Dodge 1991). Yushau et al. (2005) argue that lacking the ability to recognize and connect mathematical concepts in different situations is an obstacle that inhibits mathematical creativity (Yushau et al. 2005). Hence, technology enables learners to develop complex ideas by connecting information about the same concept from various sources (Loveless 2003).

The results of empirical studies indicate that technological environments enhance students' creative abilities. In particular, Mevarech and Kramarski (1992) found that students who participated in problem solving activities using the Logo environment had higher creative scores in specific parts of the Torrance Test of Creative Thinking (TTCT) than students who participated in a Guided Logo environment. Subhi's (1999) research extended these results and indicated that problem solving via the Logo environment can enhance creativity in all figural and verbal domains of TTCT. Moreover, Pardamean and Evelin (2014) found that after 8 weeks of Logo programming learning, the experimental group had significantly higher scores compared to the control group on all figural creativity factors, especially in flexibility and originality. Furthermore, in a study by Dunham and Dick (1994) students who used graphing calculators appeared to be more flexible problem solvers.

However, it appears that most of the previous studies conducted about the impact of technology on students' creative abilities are "results oriented". In other words, they focused on whether a certain software environment could or could not enhance students' creative abilities. Therefore, although much work has been done in this area, little attention has been given to the *ways in which* technology can enhance mathematical creative abilities and processes.

13.3.4.2 Our Results

In one of our research studies (Sophocleous and Pitta-Pantazi 2011) we offered primary school students the opportunity to work in a technologically- enhanced three-dimensional geometry software environment. The qualitative results that we collected showed that as a result of the students' interaction with the 3D software their creative ability in terms of fluency, flexibility and originality was enhanced. What was interesting in this study was the specific processes that facilitated creativity. Specifically students' appeared to be more able to imagine, synthesize and elaborate. These three processes of imagination, synthesis and elaboration seemed to have enabled students to provide more creative solutions.

13.4 Concluding Remarks

Our research journey in mathematical giftedness and creativity has offered some interesting results but at the same time left or even led to a large number of open issues. For instance a new model and tool for the identification of mathematical giftedness has been proposed. This model complies with Davidson's (2009) principles for the development of identification tools: (a) it is based on previous research studies and empirical data; (b) its components are clearly specified; (c) it explains, describes and predicts giftedness; (d) it is fairly economical and comprehensible; and is (e) useful and appropriate for education. However, future research could actually improve this model and identification tool even further. For instance, the identification tool may become even more economical and new tasks may be identified which will allow the identification of mathematically gifted students of other ages, either younger (possibly from 5 year olds) or older (possibly 15 and 18 year olds) students. We believe that on the one hand, the early prognosis of such students may give us the possibility to offer them greater support sooner while identification of older students may allow us to support these students before taking decisions for their professional life. Another direction of future research regarding the form of the identification tool could be its modification to a more economical version. Such version would be even more easily applicable in school classrooms and supportive to teachers' work.

Regarding mathematical creativity, our work has yielded some very interesting results regarding its relationship with giftedness as well as with other cognitive factors such as intelligence and cognitive styles. What is still unclear is the most effective way to facilitate mathematical creativity and this is one of the future research directions that we may take. The utilization of various methods, tools and technologies will have a serious role to play in this investigation.

The need of today's society for highly creative people who will do not simply have access or hold a large amount of information which they conceptually understand, but who can actually put this knowledge and understanding in use to produce new knowledge has to be one of the main aims of our educational systems. For this, we will have to ensure the effectiveness of any programs which proclaim that they enhance mathematical creativity.

Lastly but not least, the work we have done regarding the development of materials and methods of teaching mathematically promising students has not yet been completed (and we do not feel that it will be ever completed). We will continue investigating ways that may facilitate promising students to reach their true potential with a variety of approaches and tools. These findings should also be put into teacher education courses (both pre-service and in-service) so that teachers are more educated and prepared to identify and train mathematically promising students.

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Chapter 14
The Interplay Between Excellence in School Mathematics and General Giftedness: Focusing on Mathematical Creativity

Miriam Lev and Roza Leikin

Abstract Observation that the interrelations between mathematical creativity, mathematical expertise and general giftedness are vague is what motivated a large-scale study that explores the relationship between mathematical creativity and mathematical ability. The study employs Multiple Solution Tasks (MSTs) as a tool for the evaluation of mathematical creativity in high-school students. We discuss the links between mathematical creativity, excellence in school mathematics and general giftedness as reflected in an empirical study of senior high-school students in Israel, which implemented the MST tool. The study demonstrated that between-group differences are task-dependent and are a function of mathematical insight integrated in the mathematical task.

Keywords Mathematical creativity • Multiple Solution Tasks (MST) • General giftedness • Excellence in mathematics

14.1 Introduction

The study brought forth in this paper is part of large-scale multidimensional examination of mathematical giftedness (e.g. Leikin et al. 2014a, b, c; Paz-Baruch et al. 2014; Waisman et al. 2014) which introduced a distinction between general giftedness (G factor) and excellence in school mathematics (EM factor) in order to deepen understanding into the construct of mathematical ability. The part of the study presented herein explored the construct of high mathematical ability from the viewpoint of mathematical creativity. The study is based on the observation that questions

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pertaining to the nature of mathematical creativity and its link to mathematical abilities are still open. Since Krutetskii's (1976) seminal study of mathematical abilities, only a small number of empirical studies have been conducted on the characterization of high mathematical abilities as they relate to mathematical creativity (e.g., Leikin 2013). The study presented in this paper incorporates the views of Ervynck (1991), Krutetskii (1976), Polya (1973), and Silver (1997), who claimed that solving mathematical problems in multiple ways is closely related to mathematical creativity and that mathematical insight is an integral component of mathematical creation. The present study employed a model for the evaluation of mathematical creativity using Multiple Solution Tasks (MSTs) that explicitly require solving a mathematical problem in different ways (Leikin 2009, 2013).

14.2 Theoretical Background

The words "Intelligent", "Gifted", and "Talented" are used almost synonymously and are described in terms of remarkable achievement. The definition of general giftedness has transformed over the years. Existing approaches to general giftedness can be divided into two main types: *quantitative* and *qualitative*. Generally speaking, individuals are considered to be gifted if their IQ score is higher than the average by two standard deviations, or more, for a given age (e.g., IQ $\geq$ 130) (Piirto 1999). Identifying gifted students by means of psychometric tools is useful because these tools are objective, inexpensive and easy to apply. Interestingly, a high IQ is also an accepted measure of intelligence in the fields of mathematics, logic and insight (Piirto 1999). This observation ultimately led to our decision to include general giftedness (G factor) as one of the independent variables in our study.

In the 1980s and 1990s a number of researchers strongly criticized the psychometric tool for the identification of giftedness. This critique noted the lack of a relationship between creativity and giftedness. Some suggested an extension of the concept of giftedness, claiming that a gifted student is different from other students not only in quantitatively but qualitatively as well. For example, Marland (1972) proposed a broad, multidimensional definition of giftedness, which used other measures besides IQ alone. He referred to the gifted student as a well-rounded achiever with potential in numerous spheres including: general intellectual ability, specific academic aptitude (shown in academic subjects such as math, science and language), creative thinking, and leadership ability, talent in the visual arts, musical ability and the ability to dance. Renzulli (1978) included in his definition of giftedness non-intellectual elements as well. According to the theory of the three rings, he suggested that giftedness is the combination of high cognitive ability, perseverance and creativity in performing tasks. A person is defined as gifted if, and only if, all these characteristics are found together.

Mathematical giftedness is a specific personal characteristic related to high ability in mathematics, though both constructs do not have a precise definition. High ability in mathematics is usually reflected in the high level of mathematical performance which yields high mathematical achievements (Krutetskii 1976; Piirto 1999).

Krutetskii (1976) defined mathematical ability as that aspect of a person's character that allows him to cope with tasks better and faster. Children with high abilities in mathematics are different from other children by virtue of their cognitive efforts such as intake and processing of mathematical information, logical thinking and thinking with mathematical symbols, simplification of mathematical concepts, quick and expanded inclusion of objects, mathematical relationships and operations, the ability to understand the structure of a problem before solving it and the ability to reverse the sequence of thought (Krutetskii 1976; Paz-Baruch, et al. 2014; Waisman, et al. 2014cc; Leikin et al. 2014a, b, c). High ability in mathematics is usually related to solving complex mathematical problems, whereas mathematical achievement tests are often an insufficient indicator of high ability in mathematics, as they do not contain such problems (Davis and Rimm 2004). Children with high ability in mathematics use more efficient strategies for solving complex problems than children with normal ability in mathematics (Geary and Brown 1991).

High ability in mathematics is connected to mathematical creativity which is also (like high mathematical ability or mathematical giftedness) not well defined (Leikin 2009; Leikin and Pitta-Pantazi 2013; Sriraman 2005). Leikin (2009b) suggested that analysis of creativity in schoolchildren requires the distinction between *relative* and *absolute* creativity: absolute creativity relates to historical achievements and discoveries at a global level ("historical works" in terms of Vygotsky 1930/1984) while relative creativity refers to mathematical creativity exhibited by school students when evaluated in relation to their own previous experiences and to the performance of their peers who have similar educational histories (Leikin 2009, 2013). Our study explored relative creativity, which refers to mathematical creativity exhibited by school students; the evaluation of the originality of their solutions was done through a relative perspective on creativity.

Mathematical creativity in school mathematics is usually associated with problem solving or problem posing (e.g., Silver 1997). Following Torrance (1974), Silver (1997) suggested developing creativity through problem solving as follows. *Fluency* is developed by generating multiple mathematical ideas, multiple answers to a mathematical problem (when such exist), and exploring mathematical situations. *Flexibility* is advanced by generating new mathematical solutions when at least one has already been produced. *Originality* is advanced by exploring many solutions to a mathematical problem and generating a new one. Ervynck (1991), who considered creativity to be a critical component of problem solving, pointed to three different levels of creativity: Level 1 contains an algorithmic solution to a problem, Level 2 involves modeling a situation, and Level 3 makes use of the problem's internal structure. Ervinck's level 3 of creativity actually refers to the ability of a person to perform original, non-algorithmic and, often, insight-based solutions.

As mentioned earlier, the current study utilizes a model for evaluation of creativity using MSTs (Leikin 2009, 2013). Based on Torrance (1974), the model considers three components of creativity – fluency, flexibility, and originality. For evaluation of originality, the model combines Ervynck's (1991) insight-related levels of creativity with conventionality of the solutions; together they comprise students' educational history in mathematics.

14.3 The Model for Evaluation of Creativity with MSTs

A multiple solution task (MST) is an assignment in which a student is explicitly required to solve a mathematical problem in different ways. Solutions to the same problem are considered to be different if they are based on: (a) different representations of some mathematical concepts involved in the task, (b) different properties (definitions or theorems) of mathematical objects within a particular field, or (c) different properties of a mathematical object in different fields (see the definition and various examples of MSTs in Leikin 2007, 2009). See example of the task and its solution in Fig. 14.1.

Figure 14.3 presents three tasks from the test reported in this paper. We used model for the evaluation of mathematical creativity by means of MSTs. This model was introduced in Leikin (2009) and then employed and validated in Levav-Waynberg and Leikin (2012a, b) and in Guberman and Leikin (2012). The exact description of the scoring scheme can be found in Leikin (2013). Figure 14.2 summarizes the scoring scheme.

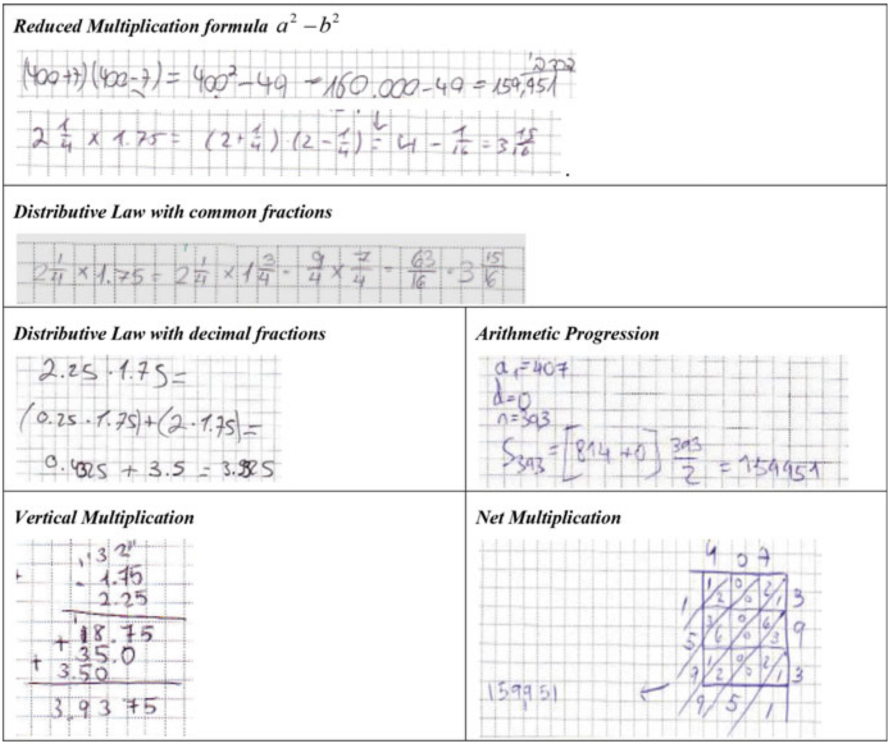

Fig. 14.1 Examples of solution methods used by the students for a calculation problem

	Fluency	Flexibility	Originality	Creativity
Scores per solution	1	$Flx_1 = 10$ - for the first solution $Flx_i = 10$ - solutions from a different group of strategies $Flx_i = 1$ - similar strategy but a different representation $Flx_i = 0.1$ - the same strategy, the same representation	$Or_i = 10$ $P < 15\%$ or for insight/ unconventional solution $Or_i = 1$ $15\% \le P < 40\%$ or for model-based/ partly unconventional solution $Or_i = 0.1$ $P \ge 40\%$ or for algorithm-based/ conventional solution	$Cr_i = Flx_i \times Or_i$
Total score	$Flu = n$	$Flx = \sum_{i=1}^{n} Flx_i$	$Or = \sum_{i=1}^{n} Or_i$	$Cr = \sum_{i=1}^{n} Flx_i \times Or_i$
n is the total number of appropriate solutions $P = (m_j / n) \cdot 100\%$ where m_j *is the number of students who used strategy* j				

Fig. 14.2 Scoring scheme for evaluation of creativity (Based on Leikin 2013)

Tasks: Solve the problem in at least 3 different ways	**Possible solutions**	**Originality score**	**Task characterization**
P1. Calculation: $2\frac{1}{4} \times 1.75 =$	1.1 The distributive law: ✓ in decimal numbers; ✓ in common fraction. 1.2 Vertical multiplication in decimal numbers. 1.3 Arithmetic progression. 1.4 Net multiplication 1.5 Reduced multiplication (see Figure 1)	0.1 0.1 1 1 10	Routine arithmetic problems Solution 1.5 is an insight-based solution: $2\frac{1}{4} \times 1.75$ $= (2 + \frac{1}{4})(2 - \frac{1}{4})$
P2. **Jam problem:** Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly 1/4 of the previous amount to each of the jars. How many jars did she prepare at the start?	2.1 System of equations with two variables. 2.2 Equation with one variable. 2.3 Numerical: Fractions/ Percentages. 2.4 Logical - insight-based solutions. 2.5 Solutions with a diagram. See Appendix 1.2	0.1 0.1 or 10 (depending on type of the equation) 1 10 1 or 10 (depending on type of the diagram)	Routine word problem Solutions 2.3, 2.4, 2.5 are non-routine for secondary mathematics (no use of variables) Solution 2.4 is based on the view of the problem structure: *4 jars include 1/4 of all the jam. Thus there were 20 jars.*
P3. $\begin{cases} 3x + 4y = 14 \\ 4x + 3y = 14 \end{cases}$	3.1 Algebraic combination. 3.2 Substitution. 3.3 Comparison/ subtraction/ addition/ division of equations 3.4 Graphing. 3.5 Solutions with determinants. 3.6 Symmetry based consideration 3.7 Trial and error (as the first solution). See Appendix 1.3	0.1 0.1 0.1 1 1 10 0.1	Routine tasks - System of equations Solutions 3.1, 3.2, 3.3, 3.4 are learned in school. Solution 3.6 is an insight-based one: *The exchange of variables does not change the system which has only one solution: x=y=2*

Fig. 14.3 Three tasks from the test reported in this paper

14.4 The Study

The study aims to examine relationships between mathematical ability and mathematical creativity. For this purpose we distinguish between general gifted (G factor) and excellence in mathematics (EM factor) to examine mathematical creativity in four groups of students differing in the combination of G and EM factors. We asked: How are different combinations of G and EM factors related to mathematical creativity?

The test items were validated during the pilot study (Leikin and Lev 2013), with the aim of examining the construct validity of the test. There were two variants of the test. The validation procedure demonstrates that "parallel" problems in the different variants of the tests provided us with equivalent information in terms of correctness and creativity components within each group. The between-group comparison was performed across the "parallel" tasks after examination of their equivalence.

A research sample of 184 10th–12th grade students (aged 16–18) was selected from a population of ~1200 adolescents who took part in the sampling procedure. The procedure was directed towards investigation of the combined effect of excellence in school mathematics (EM factor) and general giftedness (G factor) on mathematical creativity.

Students for G groups were mainly chosen from classes for gifted students (IQ > 130) and based on their scores on a shortened Raven's Advanced Progressive Matrix Test (RPMT) containing 30 items with a time limit of 15 min (Raven et al. 2000; for the short version used in the study – Zohar 1990). A score of 27 out of 30 affirmed that G group participants represent 1 %, at the most, of any given age group.

Mathematics is a compulsory subject in Israeli high schools, and students can be placed in one of three levels of mathematics: high level (HL), regular level (RL) and low level (LL). Students who were sampled as EM studied mathematics at HL with scores above 92. NEM students studied mathematics at RL with scores above 90 or at HL with scores below 85. Additionally, EM was examined using the shortened version of the SAT-M exam (Scholastic Assessment Test in Mathematics, version taken from Zohar 1990) containing 35 items with a time limit of 30 min. Scores above 26 ensured that EM group students could be regarded as representing approximately 1 % of the general population. The detailed description of the sampling procedure can be found in Paz-Baruch, et al. (2014) and Waisman, et al. (2014).

Sampling procedure led to the construction of four experimental groups (totaling 184 students) determined by a combination of EM and G factors: *G-EM group* included students who were identified as generally gifted and excelling in mathematics (N = 38); *G-NEM group* included students who were identified as generally gifted but did not excel in mathematics (N = 38); *NG-EM group*: students excelling in mathematics who are not identified as generally gifted (N = 51); *NG-NEM group*: students excelling in mathematics who are not identified as generally gifted (N = 57). All participants were paid volunteers. They and their parents signed consent forms.

The study received the approval of the Israel Ministry of Education and of the ethical committee of the University of Haifa.

The MST creativity test was administrated at the schools attended by students from the research sample. If a student from the research sample studied in a particular class, all students in the class took the test. Hence, the test was administered to 665 11th–12th grade students (aged 16–18) in order to address the concept of relative creativity. Of 665 students, 184 belonged to the research sample and 481 served as a reference group for the research sample. As stated above, the differences among the four groups of participants were designed to examine the effects of G and EM factors on students' problem-solving performance and their creativity in solving the problems.

The time allotted for the test was an hour and a half. Creativity components were evaluated according to the model described above. Accordingly, the originality score was evaluated on a relative basis with respect to solutions produced by each of the participants in the reference groups and based on the level of insight embedded in the solution. Correctness of the solution to a problem was evaluated according to the complete solution produced by the student to the problem. For each complete solution a student received 25 points. The fact that other solutions to the problem appeared to be incomplete did not affect the correctness score. The students were asked explicitly to solve each problem in as many ways as possible.

Between-group differences were examined for each problem and each of the creativity components for G factor, EM factor and interactions between G × EM factors using the MANOVA test. Pair-wise differences were examined with comparison of column means test.

14.5 Findings and Discussion

Not surprisingly, for all of the problems (P1, P2 and P3) the highest fluency and flexibility were exhibited by students from the G-EM group and the lowest fluency and flexibility appeared in the group of NG-NEM students. G-NEM and NG-EM revealed similar fluency and flexibility. All students from the G-EM group solved all three problems correctly, and 36 out of 38 students produced at least *two solutions* each. Table 14.1 demonstrates correctness and creativity scores attained by students from the different groups, and between-group differences found on the different examined criteria. We first describe the findings related to each one of the problems and then discuss the problem-dependency of the results and provide possible explanations for our findings.

Students from all research groups except the NG-NEM group solved the arithmetic calculation problem (P1) correctly and produced at least two solutions. In the G-EM, NG-EM and G-NEM groups, over 70 % of the students produced at least two different solutions to the problem. The difference appeared in the ability to successfully produce three different solutions by students from these three research groups: 16 % in G-EM group, 10.5 % in G-NEM group and 2 % in NG-EM group.

Table 14.1 Scores on creativity components and correctness of solutions to Problems 1, 2, 3 in the four groups of participants

		G-EM N=38	G-NEM N=38	NG-EM N=51	NG-NEM N=57	G factor		EM factor		Interaction G×EM	
		Mean (SD)				F (1180) Partial Eta Squared					
P1	**Cor**	25 (0)	25 (0)	25 (0)	24.12 (4.64)	1.279	.007	1.279	.007	1.279	.007
	Flu	3.18 (.77)	2.97 (.54)	2.9 (.54)	2.58 (.75)	11.618**	.061	7.612**	.039	.321	.002
	Flx	11.26 (1.92)	10.96 (.57)	11.02 (1.38)	10.49 (3.13)	1.263	.007	1.765	.010	.128	.001
	Or	3.23 (4.65)	2.38 (4.11)	1.67 (3.43)	0.79 (2.25)	8.666**	.046	2.587	.014	.000	.000
	Cr	30.04 (46.02)	21.94 (40.92)	14.87 (34.36)	6.42 (22.34)	8.312**	.044	2.418	.013	.001	.000
	F (4177)					3.646**		1.791		.579	
P2	**Cor**	23.68 (5.66)	16.58 (11.46)	18.14 (11.27)	11.23 (12)	11.642**	.061	19.253***	.097	.004	.000
	Flu	2.29 (.93)	1.39 (1.22)	1.49 (1.17)	0.81 (.97)	18.496***	.093	23.939***	.117	.430	.002
	Flx	12.92 (5.46)	7.65 (5.2)	8.11 (5.54)	5.09 (5.23)	21.105***	.105	26.703***	.129	1.971	.011
	Or	2.86 (4.43)	1.44 (3.41)	0.93 (2.71)	0.6 (2.24)	8.548**	.045	3.364	.018	1.326	.007
	Cr	27.58 (44.52)	9.1 (27.17)	8.58 (27.1)	4.16 (18.5)	7.347**	.039	6.716	.036	2.532	.014
	F (4177)					5.115***		6.981***		1.565	
P3	**Cor**	25 (0)	23.68 (5.66)	24.51 (3.5)	23.42 (5.01)	.355	.002	3.617	.020	.032	.000
	Flu	3.13 (.53)	3.13 (1.21)	2.88 (.65)	2.56 (.95)	9.897**	.052	1.518	.008	1.518	.008
	Flx	16.39 (6.96)	12.51 (4.76)	11.51 (2.57)	11.01 (2.04)	25.555***	.124	12.066**	.063	7.167**	.038
	Or	2.42 (3.98)	1.17 (2.71)	0.32 (.19)	0.26 (.09)	21.049***	.105	4.072*	.022	3.284	.018
	Cr	22.72 (40.14)	9.78 (27.23)	1.5 (1.97)	1.1 (0.2)	20.528***	.102	4.091*	.022	3.612	.020
	F (4177)					8.894***		3.091*		3.146*	

Cor Correctness, *Flu* Fluency, *Flx* Flexibility, *Or* Originality, *Cr* Creativity
*p<.05; **p<.01; ***p<.001

Students from the NG-NEM group received the lowest scores on all the examined criteria while two students did not succeed in solving this problem correctly and only 31 % of these students produced at least two different solutions. These differences are reflected in different flexibility scores achieved by the students from these three groups. We found a significant effect of G factor on fluency, originality and creativity associated with solving P1, whereas EM factor affected fluency only (Table 14.1). Analysis of the pair-wise differences (Table 14.2) showed that for the originality and creativity criteria G-EM students' scores were significantly higher than those of NG-NEM students in fluency, originality and creativity.

The Jam problem appeared to be the most complex one for study participants. There were students from all four research groups who were unable to solve the Jam problem correctly, so that the mean correctness score on this problem was lower than 25. The G-EM students received the highest mean scores on all of the parameters examined: most of them solved P2 correctly. At least two different solutions (which is an indicator of flexibility in problem-solving) were produced by 65.5 % of G-EM students, about 26 % of G-EM and NG-EM students and about 16 % of NG-NEM students. NG-EM students received higher scores than G-NEM students for correctness, fluency and flexibility, whereas G-NEM received higher scores on originality and creativity of the solutions. Both G and EM factors had significant effects on correctness, fluency and flexibility, while originality and creativity associated with solving Jam problem were affected significantly by G factor only. The G-EM group exhibited significantly higher fluency, flexibility, originality and overall creativity than students from the other three groups. G-EM students were significantly more accurate than their peers from the two NEM groups. Additionally, among NG students NG-EM students were significantly more fluent and flexible than NG-NEM students when solving P2.

When solving the system of equations all G-EM students solved this problem correctly and thus received the highest mean score of 25, as well as the highest scores on all of the creativity components criteria. In the other research groups 1–2 students did not succeed in solving the problem. The mean fluency score in the two groups of G students (G-EM and G-NEM) was higher than 3, whereas in the research groups of non-gifted students the mean fluency score was lower than 3. Two different solutions to this problem were produced by 95 % of G-EM students, about 89 % of G-NEM and NG-EM students and 70 % of NG-NEM students. Both G and EM factors had significant effects on the flexibility, originality and creativity, though the effects of G factor were stronger than those of EM factor. Moreover, a significant interaction between EM and G factors in students' flexibility was related to solving the system of equations: EM factor strengthened the effect of G factor, so that gifted students who excel in school mathematics are significantly more flexible than their non-gifted counterparts, while no significant differences appear in the flexibility of EM and NEM students among NG students. This interaction demonstrates that excellence in mathematics strengthens the effect of giftedness. Consequently, among EM and NEM students, G-group students were more fluent when solving P3; G-EM students were more flexible than students from the other

Table 14.2 Significant effects of G and EM factors and pair-wise group differences related to the three MSTs

	Correctness	Fluency			Flexibility		Originality, Creativity		
G factor	P2**	P1**	P2***	P3**	P2***	P3***	P1**	P2**	P3***
EM factor	P2***	P1**	P2***		P2***	P3**			P3*
Interaction G × EM						P3**			
G-EM vs. G-NEM	P2*		P2*		P2***			P2*	
G-EM vs. NG-EM			P2*		P2**			P2*	
G-EM vs. NG-NEM	P2***	P1**	P2***	P3***	P2**	P3***	P1**	P2***	P3***
G-NEM vs. NG-NEM		P1*		P3*		P3*			P3*
NG-EM vs. NG-NEM	P2*		P2*						
G-NEM vs. NG-EM									

*p < .05; **p < .01; ***p < .001

three study groups; and G-EM students were more original than students from the two groups of NG students.

Naturally, G students had higher scores on all the examined criteria than NG students, and EM students scored higher than NEM students on all the criteria. However, these differences were not always statistically significant and, moreover, G and EM factors had different effects on correctness, fluency, flexibility, originality, and creativity associated with solving these three problems. Both G and EM factors had a significant effect on the correctness of solutions produced for P2, students' fluency when solving P1 and P2, their flexibility when solving P2 and P3 and their originality and creativity associated with P3. The effect of G factor on originality and creativity was stronger than that of EM factor. Furthermore, only G factor significantly affected fluency related to P3, and originality and creativity related to P1 and P2. Additionally, we found a significant interaction between G and EM factors related to flexibility of solving P3. Table 14.2 summarizes significant effects of G and EM factors on all the examined criteria as well as pair-wise group differences on the three problems.

Effects of G factor differed from the effects of EM factor in originality and creativity related to the ability to produce non-algorithmic (original) solutions. In all the cases in which EM factor had significant effects on correctness, fluency, flexibility and originality, G factor also affected these characteristics of problem solving. Additionally to the common effects of G and EM factors, G factor only affected students' fluency related to solving P3 as well as originality and creativity associated with solving P1 and P2. Additionally, we found significant interaction between G and EM factors on the flexibility associated with solving P3 in different ways: while the flexibility scores of NG-EM and NG-NEM students were similar, the flexibility scores of G-EM students were significantly higher than those of G-NEM, NG-EM and NG-NEM students. That is, EM factor significantly strengthened the effect of G factor on students' flexibility when solving P3.

Examination of the pair-wise between-group differences demonstrated that these differences are task dependent. All three tasks – P1, P2, P3 – revealed significant differences between G-EM and NG-NEM students in all the creativity components (except flexibility associated with P1). P3 showed that among students who did not excel in mathematics generally gifted students were more fluent, flexible and original when solving symmetrical systems of equations. P1 revealed differences between G-NEM and NG-NEM students in fluency only. Consistent with the pilot study (Leikin and Lev 2013), P2 appeared to be the most powerful MST in revealing between-group differences. In contrast to the effects revealed by P1 and P3 together, P2 revealed differences only in accuracy between G-EM and G-NEM students, G-EM and NG-NEM students, NG-EM and NG-NEM students. P2 demonstrated that mathematical expertise (expressed in excellence in school mathematics) increases fluency, flexibility and originality among gifted students (differences between G-EM vs. G-NEM students), increases fluency among NG students (differences between NG-EM vs. NG-NEM students); whereas G factor increases fluency, flexibility and originality among EM students (differences between G-EM vs.

NG-EM students). No differences were revealed between G-NEM and NG-EM students for all the tasks in all the examined criteria.

Differences between the effects of G and EM factors, revealed when the students solved P1, P2 and P3, are related to the differences in the possibility to produce algorithmic or insight-based solutions to the problems. P1, which was the simplest task (all participants solved it correctly), could be easily solved in several ways by students in all the study groups; and only 3 (of 30 examined) pair-wise between-group differences appeared to be significant. Solving systems of linear equations (P3) is taught in school using several algorithms, while solving symmetrical systems of equations using symmetry is not included in the curriculum. P3 showed 6 pair-wise between-group differences and was the only task that showed differences in flexibility and originality of G and NG students who did not excel in mathematics. In contrast, P2 was less algorithmic and required modeling activities for solving the problem. The problem had several insight-based solutions that required an understanding of the structure of the problem (Ervynck 1991). Thus, this problem demonstrated significant differences between G-EM and G-NEM students, and between G-EM and NG-EM students.

Consistent with our pilot study (Leikin and Lev 2013), this research demonstrated the complex relationship between knowledge, giftedness and creativity (emphasized by Vygotsky 1930/1984). This study demonstrates that the distinction between excellence in school mathematics and general giftedness, which is the core idea of the examination of mathematical creativity in this study, confirms that expertise in mathematics is a prerequisite for a person to be creative (effects of EM factor). At the same time, we demonstrate that general giftedness has a meaningful effect on students' mathematical creativity and especially on originality in problem solving. Our study shows that correctness in solving mathematical problems is an insufficient criterion when examining students' mathematical abilities and that fluency and flexibility in solving mathematical problems are affected by both giftedness and expertise. In turn, originality and overall creativity are mainly influenced by general giftedness, while mathematical expertise can strengthen the effect of general giftedness.

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Chapter 15
Mathematically Gifted Education: Some Political Questions

Alexander Karp

Abstract This chapter is devoted to political questions in mathematics education. The practice of recognizing certain children as more gifted than others and selecting them accordingly becomes inevitably a focus of public attention, frequently giving rise to disagreements, finding itself at the heart of political discussions, sometimes instigating such discussions, and sometimes reflecting already existing conflicts. Without attempting an exhaustive analysis, the author describes certain episodes, aspects, and slogans of such political battles, while posing some questions for further study.

Keywords The politics of mathematics education • Equity • Mathematically gifted education • Mathematics for all • Elite

15.1 Introduction

Lenin once made a remark that became famous: "It is impossible to live in society and to be free from society" (Lenin 1905). What he had in mind, and what his followers or enemies took this statement to mean for over half a century, did not necessarily coincide, and certain interpretations of this statement are difficult to agree with. It cannot be denied, however, that educational processes that influence thousands and tens of thousands of people cannot simply ignore what is happening in society and in their country as a whole. The identification of the mathematically gifted in education is in this sense unfree, but reflects the influence of certain forces, groups, and social classes—and the way in which influence is embodied in practice and in the life of a country and society is precisely what is usually called politics. Consequently, the history of teaching the mathematically gifted turns out to be intertwined with politics and even with political maneuvering.

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The literature on mathematics education contains frequent complains to the effect that "there is not a long tradition within the mainstream of mathematics education of both critically and rigorously examining the connections between mathematics as an area of study and the larger relations of unequal economic, political and cultural power" (Apple 2000, p. 243). It is indeed noteworthy that, although studies that are in one way or another devoted to the politics of mathematics education are not few in number, they usually do not deal with teaching the mathematically gifted. Gervasoni and Lindenskov (2011), who devote a chapter to students with "special rights" to mathematics education, diligently list whom they have in mind: these groups include the historically disadvantaged and the visually and hearing impaired and in general all those who underperform in mathematics due to their exclusion from quality mathematics teaching and learning. But the mathematically gifted do not make it onto their list. No special rights are relegated to them. Nor, indeed, is it commonly asserted in ordinary conversation that the mathematically gifted have a right to a different kind of education because they are different. Their special education is usually justified at most by the needs of society and the state, but not by their human rights.

The understanding of the aims of mathematically gifted education, its place and its role, as well as all the procedures associated with it, are shaped by the interaction of existing political forces, reflect the outcomes of this interaction, and not infrequently give rise to explicitly political discussions and conflicts. Perhaps the most controversial questions today surround the understanding of how mathematically gifted education is related to and coexists with mathematics education in general. Tannenbaum (2000) not without irony noted "that American society is never interested in teaching both the most and the least successful achievers at the same time." Changes in interests should likely also be explained by changes in the political situation, but in any case the beginning of the third millennium can hardly be considered a period of special attention to the gifted.

It is impossible and probably useless to discuss all of the significant details and episodes pertaining to this issue that have taken place over the course of decades, if not centuries, here. Below, we will attempt to examine only a few aspects of the problem, without claiming to carry out a comprehensive analysis of what has transpired. Let us repeat that literature on the politics of mathematically gifted education is very scarce; consequently, the aim of this paper is to pose questions rather than to provide exhaustive answers to them. In most cases below, the discussion will focus on Russia and the United States, with which the author is most familiar.

15.2 Certain General Considerations

Although mathematically gifted individuals were valued centuries, if not millennia ago, the actual idea of teaching the mathematically gifted appeared relatively recently. The reason for this, of course, is that specifically mathematical talent was not differentiated from a general conception of talent, and also the fact that the aims of education were by no means always seen as they are now; but probably an even more important factor is that previously the structure of society and the understanding of this structure were very different from what they are today (Karp 2009).

The class of which the Russian poet Alexander Pushkin (1799–1837) was a part has probably been studied better than any other in nineteenth century Russia (Rudensky and Rudensky 1976). It is not difficult to determine that among its 29 graduates, in addition to Pushkin, there were at least two other significant poets (Delvig and Küchelbecker), a prominent diplomat (Gorchakov), a noteworthy participant of the opposition movement (Pushchin), several very high-ranking government and military officials (Korf, Kornilov Lomonosov, Maslov, Matyushkin, Steven), and several other individuals who simply rose through the ranks to become generals or attained some equivalent rank (Volkhovsky, Danzas, Komovsky, Yudin, Yakovlev). We need not even mention those not listed above, who also included highly gifted individuals: it is clear that even a very selective class in our own time would unlikely be able to boast anything comparable. And yet there was almost no formal selection of students for this class (to be sure, it was the first class in a new educational institution, and of course those who decided to send their children there were in some way different from everyone else). Many of the teachers, as well as the curriculum of the school as a whole, were by all appearances quite decent, but still the remarkable outcome of this class will become more comprehensible only once we take note of the fact that other classes in this or other privileged institutions also graduated individuals who subsequently occupied prominent positions in society. Society was structured in such a way that such positions were distributed among a very narrow group of individuals. Persons who did not belong to this group simply had no possibility of attaining the rank of general, and even to become a published poet was something that very many were unable to do if only because of their own illiteracy (which of course does not mean that had that not been the case, there would have been many poets of Pushkin's caliber). Nor was there any evident need to attract gifted individuals on a mass scale: they might manage to break into the ranks, as did the Ukrainian poet Taras Shevchenko, for example, but this could happen only in exceptional cases—it was usually not a deliberately sought objective.

The appearance of mathematically gifted education in itself presupposes a social-political revolution, with a different view of the need for an elite and its formation. Indeed, it would be more precise to speak of a series of revolutions: although the French Revolution was an important milestone, certain steps were taken much later, and all of them took place within the context of intense political struggles.

Below, we will discuss certain goals of mathematically gifted education that are directly connected with politics; for now, let us merely note the importance both of the overall worldview and of the conceptions of society and education that are prevalent in the country in which mathematically gifted education arises, and among particular groups within such a country; such views are often inseparable from political views.

Gallagher (2008) lists the basic "engines of change" thanks to which gifted education develops (or fails to develop). Among them he includes:

- Legislation, that is, laws and statutes enacted by the state;
- Court decisions, more important for children with disabilities than for gifted children, but still present and exerting a certain influence;
- Administrative rule making—a category in which he includes practical rules formulated in response to court decisions or enacted laws;
- Professional initiatives—a category in which he includes scientific and other publications by individual professionals or groups of professionals, such as the NCTM.

In addition, he singles out what he calls "advocacy initiatives," a category in which he includes all conceivable reports prepared by various interest groups.

All of these are indeed possible channels through which political influences are transformed into real changes. Naturally, in different countries these "engines of change" may also look different, and even more importantly, there may be others in addition to the ones listed above. The resolutions of the Central Committee of the Communist Party of the Soviet Union or even simply a word from the head of state was no less important in the USSR than any law.

Researchers of the political struggle surrounding the education of the gifted, of course, must study documents that pertain to all of the aforementioned categories; but such researchers must not confine themselves to these categories. The goal of such researchers is to understand why certain aspects of educational practices attract special public attention, how hot political issues are transformed in the world of the school, and what mechanisms interconnect the political and the educational. Consequently, it may be necessary also to analyze documents that have only a partial relation to mathematically gifted education, as well as to bear in mind that far from everything in politics is entrusted to paper and thus preserved in the form of a written document. In some cases, researchers must place their trust merely in indirect evidence, and in others they must analyze multiple varied sources, attempting to reconstruct the characteristics of a bygone era, and only then to try to understand what happened with mathematically gifted education.

15.3 Mathematically Gifted Education as a Means for Creating a New Elite

The usual rhetoric employed in promoting education of the mathematically gifted, as indeed education of the gifted in general, emphasizes the need to prepare a new labor force capable of meeting the challenges of the future. Today's world has become used to the fact that innovations in technology come very rapidly, and consequently it is natural to desire that new people should appear who are capable of handling it.

This was not the case even a half-century ago: what likely sounded more persuasive then were arguments about competing with other countries, which one hears today as well. Economic, and in large measure also military rivalry, motivated the search for individuals who would be capable, for example, of forging a "nuclear shield for the homeland," as they said in the USSR. Schools with an advanced course of study in mathematics in the USSR were created in large part precisely in order to prepare such a labor force and the influence of scientists who were directly connected with the military-industrial sphere was very substantial (Karp 2011a; Kukulin and Maiofis 2015). Nor is it a coincidence that reforms in American education (even if they did not pertain exclusively to the education of the gifted) are associated with the Sputnik, that is, with a direct military threat from the USSR (Fey and Graeber 2003).

The actual idea of preparing a technological elite appeared much earlier (Belhoste and Chatzis 2007). The schools created by the revolution in France on the one hand reflected the developments already attained toward this objective, and on the other hand spurred and accelerated them. Other countries followed suit and started to concern themselves with the selection (even if it was not always so broad) and preparation of mathematically educated specialists. They pertained first and foremost to the preparation of military experts: examples of mathematically advanced educational institutions dedicated to this task can be found around the world, from the United States to Tunisia (Abdeljaouad 2014; Rickey 2001). Lyceums, gymnasia, and later schools with other names, too, began preparing personnel for such specialized education.

Although such educational institutions cannot be said to belong entirely to the category of schools dedicated to mathematics gifted education (if only because students were admitted to these schools by no means on the basis of their mathematical giftedness alone), they nonetheless facilitated the formation of a tradition that was important for mathematics gifted education. To a certain degree, it became traditional to use workers who had been prepared using advanced mathematics curricula for carrying out general administrative functions, and not just for solving technical and technological problems.

An analysis of surviving documents (Karp 2011a, b, c) shows that in establishing Russian schools with an advanced course of study in mathematics, Nikita Khrushchev, who was then at the head of the country, was thinking not only about competing with a foreign adversary, but also about strictly domestic objectives. He saw a danger in the emerging new class (Djilas 1957) of party workers (which ultimately overthrew him) and was thinking about ways to limit its influence, and at the very least not to allow it to transfer its influence by inheritance. This was what gave rise to his numerous arguments—including those which he articulated at meetings of the Presidium of the Central Committee of the Communist Party of the Soviet Union—about the fact that too many children of top officials were becoming top officials, and obtaining a higher education before doing so. Schools with advanced study of mathematics were clearly seen as sources of new personnel, which would thus be capable of helping him in the emerging struggle with the party elite.

To be sure, subsequently these good intentions did not entirely pan out. Restrictions established for various reasons on different categories of citizens (non-party members, Jews, and others) thwarted the careers of the graduates of the newly created schools, sometimes at their earliest stages. Nor could the manner in which people were promoted because of their giftedness but provoke irritation on the part of those who had achieved their positions as a result of having spent many years playing by the rules of the workplace. Consequently, there emerged grounds for political opposition, and for an attitude of suspicion toward the education of the mathematically gifted. Tellingly, however, these schools were never actually shut down.

In the United States, schools for the gifted also (at least in theory) serve as an alternative to private schools in preparing a future elite. Admission to New York City's specialized high schools, for example, is based only and exclusively on the results of a test whose purpose is to determine students' readiness for a heightened level of education. At the same time, researchers note that private schools remain the main "feeders" for top colleges (Morgan 2002): among schools that enroll that greatest percentages of graduates in the most prestigious colleges, private schools are clearly in the lead, while the best schools for the gifted rank twentieth to thirtieth and even below. Moreover, one regularly hears the charge, which will be discussed below, that test-based admissions are in fact biased (in addition, there also exists a principled anti-elitism, according to which it is not the manner in which an elite is formed that's not right—whether on the basis of birth or achievements—nor the potential for a monopoly on power or property in the hands of an elite, but the very acknowledgement of the fact that there are tasks at which not everyone is equally successful).

In any case, it is clear that proclaiming the aim of forming a new elite on the basis of giftedness (including mathematical giftedness), while signaling a certain step forward in the public consciousness, does not in itself mean that the formulated aim is achieved. For our purposes, however, it is important to note that this is evidently one of the most important areas of political struggle.

15.4 On the Politics of Developing Mathematics Curricula

While the politicized nature of the formation of an elite in society is not open to doubt, the politicization of the development of mathematics curricula might seem more questionable. Yet discussions about the mathematics for the gifted turn out to be a part of discussions about mathematics for all people in general. Schubring (2012) thoroughly analyzes the arguments of those who believe that the study of pure mathematics is in general appropriate only for a narrow elite. As an example of such views, he cites Damerow et al. (1986, p. 4):

> Traditionally, mathematics curricula were developed for an elite group of students who were expected to specialize in the subject, and to study mathematics subsequently at higher levels in a tertiary institution.

Thus, the arguments construct the following chain of reasoning: pure mathematics is only needed by a small group of people (which largely coincides with the mathematically gifted), so that by giving pure mathematics a dominant role in education, we neglect the interests of the overwhelming majority—to which only the applications of mathematics are important—for the sake of this small group. And at this point, of course, the issue becomes a political one.

Schubring (2012) demonstrates that it is erroneous to regard German gymnasia, which established many traditions in the teaching of mathematics, as strictly elitist institutions, and he also asks whether mathematics for all can ever ignore general questions, confining itself to minor applied ones:

> Given all the emphasis on improving the quality of mathematics teaching it is the more astonishing that it is taken for granted in the entire movement of "mathematics for all" that mathematics has to be one of the key pillars of school education. One speaks there of "universalisation" of primary schooling and likewise of universalisation of secondary schooling, but it is never reflected that mathematics is presupposed to constitute a basic element of that universe. (p. 446)

Schubring (2012) points out that the appearance of organized mathematics education was connected precisely with professional needs. Discussions of mathematically gifted education should not lose sight of the fact that such education did not by any means arise as a study of abstract mathematical concepts. We have already mentioned the French Grandes Écoles, which played a crucial role in the development of mathematics education, including the education of the gifted. It is evident from its name that the École Polytechnique was not an institution aimed exclusively at pure mathematics.

In more recent times, Soviet schools with an advanced course of study in mathematics developed in the context of a movement toward "polytechnization," that is, the establishment of a link between education and actual labor: schoolchildren had to spend part of the time working in factories or laboratories, acquiring a professional, and not just a general education. Schools with an advanced course of study in mathematics grew out of classes devoted to preparing computer programmers (Karp 2011a).

It is likewise not difficult to name contemporary American programs and schools for the mathematically gifted in which enormous attention is paid to mathematical modeling and to applied mathematics (to name just one example, see the COMAP website www.comap.org, which provides useful information about events for schoolchildren). But the fact is that in order to study mathematical modeling, a student must possess sufficiently strong knowledge of mathematics, and not just some specialized field of mathematics, but mathematical methods and techniques of thought, analysis, and deduction in general.

Thus, there are no grounds for equating mathematically gifted education with pure mathematics. To be sure, any curriculum for the mathematically gifted must be advanced and challenging, and not just "for all", but specifically for them (which likely renders it too difficult for certain other students); but this is a completely different issue.

15.5 Politics in Schools for the Mathematically Gifted

We have already discussed the Soviet government's equivocal attitude toward schools with an advanced course of study in mathematics. Fields Medal winner Sergey Novikov (1996) wrote that "it is no secret that… the powers that be, often not without reason, found a spirit of dissent within the student population of special schools," which they attributed to "international imperialism and Zionism" (p. 34). And indeed, while these schools had been envisioned as institutions in which Soviet skilled workers would be forged, they produced skilled workers whom it was difficult to describe as Soviet.

Of course, the restrictions mentioned above contributed to the students' discontent. But they were hardly its only source. A teacher from a school with an advanced course of study in mathematics, whom we interviewed, and who had once been a student at the same school himself, said of his school: "It was a territory of freedom" (Karp 2010). The territory was free because free discussion was permitted on it. It was expected that the discussion would concern only mathematics and physics, but things did not always work out that way. Yet general discussions, especially discussions about historical-political topics, could lead to confrontations with the authorities. Naturally, this did not happen automatically—schools with an advanced course of study in mathematics produced many people who faithfully served the regime, without entering into any conflicts with it. But such conflicts did sometimes occur.

In connection with this, two questions may be posed. The first question—which may even be posed specifically about Soviet schools for the mathematically gifted—concerns all open expressions of discontent with the political regime at such schools. Something about this topic is detailed, for example, in Sossinsky (2010). We know of notable figures in the Soviet dissident movement who worked at schools with advanced study of mathematics, for example, Anatoly Yakobson, one of the editors of the illegal periodical "Chronicle of Current Events" and a teacher at Moscow's School No. 2. On the other hand, certain events are clearly not sufficiently well known, for example, the shutting down of Leningrad's School No. 121, whose graduates distributed flyers calling for the creation of a democratic (and not a Soviet) form of socialism, for which they received long prison sentences (Alexeeva 2012).

The second question is more general: how is the formation of mathematical talent connected with non-mathematical subjects, including political-historical interests? Graduates from schools with an advanced course of study in mathematics (Bunimovich 2012; Pakhomov 2013) point out the important role played in their lives and educations by the humanities and, if not by direct involvement in political movements, then at least by political interests and contacts with people involved in politics. On the other hand, Pakhomov (2013), who describes the admiration and enthusiasm with which he listened to a reading by the later famous songwriter and dissident Yuli Kim, who was a teacher at Kolmogorov's boarding school at the time and was later fired from there for political reasons, adds that such feelings were shared by approximately 10–15 % of the students; and he even observes that approximately such a percentage of the scientific intelligentsia feels a need to study the humanities in general.

It is unlikely that any quantitative estimates are possible (at least, at present)—evidently, among mathematically gifted people, there have been and are both absolutely apolitical individuals and ones who are quite politically active. We can also observe periods when political activity among mathematicians, including beginning mathematicians, is growing, and periods when it is waning. All of this, in our view, deserves special study.

15.6 On Equity, Genuine and Fictive

And yet probably the most controversial political issue today concerns the relationship between education for all and mathematically gifted education. Should the gifted be singled out at all? If yes, then how? Does not singling out the gifted mean declaring the rest ungifted, with the consequence that the ungifted will not receive any education at all? Does the term "mathematically gifted" merely cover up old racist approaches, and is the selection of the mathematically gifted merely a polite expression of segregation? These are just the first, although perhaps the most important, questions around which political discussions flare up.

These discussions are not new and have accompanied mathematics gifted education over the whole course of its history. Already in 1978, the decision to create the North Carolina School for Science and Mathematics was accompanied by debates in the state senate, which culminated in a vote that ended up in a tie: exactly half of the senators supported the creation of the school, and the same number were against it, considering it an infringement of the principle of equity. The decision to create the school was finally made by the lieutenant governor, who presides over the state senate in North Carolina (Vogeli 1997). But this did not, of course, diminish the importance of the struggle for equity.

Equity in education and specifically in mathematics education in today's world is both an economic issue, since failure to prepare a work force automatically diminishes the effectiveness of a country's economy, and a political issue, since of course people cannot put up with the fact that they are effectively being assigned to second-rate place in life. As Croom (1997) explains, "Equity in mathematics education implies fairness, justice, and equality for all students so that they may achieve their full potential, regardless of race, ethnicity, gender, or socioeconomic status" (p. 2).

This requires further comment, since, although people have been talking about equity in mathematics education for decades and it is unlikely that anyone is openly against it, equity is by no means necessarily understood in the same way by different people, and consequently debates and conflicts stem not only from the fact that agreement over words does not always translate into agreement over deeds, but also from the fact that agreement over words may merely cover up fundamental differences in the way in which these words are understood.

Here it must again be reiterated that the vast contemporary literature on equity in mathematics education is largely concerned with discussing how students cannot obtain even a basic education and how they may be helped to do so. Meanwhile, students' potentials vary, and it is necessary to discuss precisely the possibility of attaining all levels. Less has been written on this topic (this has begun to change only in relatively recent years), and different ideas are often misleadingly interchanged, as when, for example, the notion that everyone should have access to high-quality mathematics education and that practically everyone can and should obtain it at a certain level is equated with the idea that everyone can attain the highest level and "be like Einstein." Alas, there are no grounds whatsoever for believing that, even given unlimited financial resources, it will be possible to produce as many Einsteins as we wish.

In general, as Linchevski et al. (2011) correctly point out, equity and quality often appear to some to be in conflict with each other. Furthermore, quality is often equated with the amount of material studied or with the speed with which students complete assignments. Consequently, we can say—exaggerating, alas, only slightly—that from such a perspective, the rote memorization of "recipes" for solving differential equations becomes high-quality education, while the rote memorization of how to find the y-intercept of the graph of a linear function becomes education of a lower quality. And the struggle for equality turns into a struggle against teaching differential equations, and for making everyone concentrate on the rote memorization of the y-intercept rule—even while allowing certain individual students to help those who cannot wrap their heads around this rule (which will supposedly lead the helpers also to grasp this rule—which is not especially complicated to begin with—even better).

One can fight for equity at different levels, and to break up education at the upper levels is easier than to rebuild the lower ones—but this is unlikely to help in any way to attain the aforementioned goal of all students achieving their full potential. At one time, the political objective was formulated differently: to offer education to all at least formally. This objective was in a certain sense quantitative. It may be said (with a number of qualifications) that in many countries (although, again, far from everywhere) it has been realized. Schubring (2012) notes:

> The process of universalising schooling in general and the teaching of mathematics in particular is continuing – in industrialised and in developing countries. After primary schooling has been universalised – for Western countries basically already during the 19th century, and for developing countries quite recently, now secondary schooling is becoming universalised, in the sense of extending the age limit of compulsory schooling. (p. 457)

Niss (2015) points out that until relatively recently Danish high schools still admitted only 6–7 % of students from their age groups. Such critical problems in mathematics education as the recognition of the possibility of mathematical talent in women (recall, for example, the biographies of Sofia Kovalevskaya and Emmy Noether and the difficulties they experienced) have at least formally been resolved (although of course with caveats and not everywhere).

This process, however, does not mean that everything has been expanded and improved. The Hungarian educators Halmos and Varga (1978) made the following remark about the consequences of the establishment of general rather than selective schools: "It goes without saying that the last 4 years of this general school could offer less to every pupil than what the first 4 years of the earlier 8-grade secondary school could give to a highly selected population of the same age" (p. 225). Instruction at general schools "for all" obviously differs from instruction at schools attended by 6–7 % or even 10–15 % of the students. It probably cannot be otherwise, but it absolutely does not mean that we must become reconciled to the fact that quantitative growth has not necessarily been accompanied by qualitative improvement.

The current period is precisely a time when what is important is quality, understood precisely as the realization of the students' potential. The importance of the new task consists in the fact that educators must struggle less against formal prohibitions than against factors that are much more difficult to change. We are not talking merely about economic or educational inequality. It is evident that schoolchildren from poor and poorly educated families have, generally speaking, fewer possibilities for developing their mathematical talents than children from families that are rich and educated. It is clear that the struggle (including the political struggle) for achieving a certain minimum level of wealth and education for all citizens is useful for mathematically gifted education also. But its development also faces obstacles whose overcoming calls for reforms that are less global in character.

Walker (2003) cites the following episode:

> A Black student excelled in her general level Algebra course. At the end of each quarter students are evaluated, on the basis of grades, to determine if they should be moved into a higher or lower level course. When an administrator asked why the student had not been moved to a higher level course her teacher replied that she needed the student to remain in the course because she was a good influence on, and a good role model for, the other students in the class (who were predominantly black and Latino/a). (p.16)

This episode is far from unique, and such episodes occurred not only half a century ago (Walker 2014), but also in much more recent times (Martin 2006). In such cases, students are regarded not as separate individual personalities with their own peculiarities and needs, but merely as representatives of their group and race (in the instance above, the student was a *useful* representative). Such a form of racism (usually delicately referred to as accepting stereotypes) stands in the way of education over and above economic and purely educational factors. It is important to struggle against it and to overcome it—both by working with teachers and, above all, by creating and promoting programs and examples that undermine such stereotypes.

Ebanks et al. (2012) cite numerous statistics about racial imbalances in elite public schools, noting that while African-Americans constitute 32 % of the student population in New York City's public schools, at Stuyvesant High School (one of the most prestige specialized high schools in the country) only 1.2 % of the students are African-American, while 72 % are Asian, and 24 % are White. This issue was already brought up 20 years ago, in 1996, when the Association of Community

Organizations for Reform Now (ACORN) began publishing its *Secret Apartheid* reports. These reports cited statistics about racial imbalances and argued in particular that "developing the skills and academic competence to compete successfully for admission to Stuyvesant or Bronx Science requires course work which is not available to most black and latino students in the public schools." They also proposed certain measures for rectifying the situation, the most important of which was the following:

> Suspend the competitive testing for the specialized high schools. Until the Board of Education can show that the students of each middle school in the system have had access to curricula and instruction that would prepare them for this test regardless of their color or economic status, the current test for the specialized high schools must remain permanently suspect as the product of an institutional racism inappropriate to an educational system in a democracy (ACORN n.d.)

These proposals gave rise to equally passionate objections. A paper by Hart (1997), entitled "Destroying Excellence," is representative. Unambiguously calling his opponents bandits and comparing them to the ebola virus, which had just recently been discovered, the author states that when he was a student at Stuyvesant, there were almost no Asians among the students, and there were not many children from Catholic and Protestant families, either. The overwhelming majority were Jewish—and why? Because they performed best on exams: "the schools wanted the best students, and the entrance exams don't lie."

Without entering into a discussion of whether it is true that "exams don't lie," we will note that it is impossible to overlook the fact that these honest exams produced completely different results at different times—Jewish students became substantially fewer, and Asian students substantially more numerous, even proportionally to the population of the city, which indicates that the results of the exams were determined not only by students' ability to study well going forward, but also by other factors.

In the late 1990s, despite an intense campaign, the proposition to abolish entrance exams for New York City's elite public schools—until the system should become fair in the opinion of those who opposed it—fell through. The idea of now depriving Asian students from comparatively poor families who have prepared well for an entrance exam of the possibility of receiving an education at an advanced level truly does not seem to be the best means of fighting for equity in the education of the gifted. Reducing the number of programs for the gifted, or eliminating them altogether, seems even less well suited to this purpose: the destruction of these frequently—and as many believe, rightly—criticized programs would merely mean that there would be no alternative at all to private schools.

At the same time, the fact that the curriculum and support for high quality teaching at many schools with a predominantly African-American or Latino population are inadequate to the existing challenges, and thus do not allow students to prepare for admission to elite schools, is confirmed by numerous studies (Kozol 2005, is just one example of such works). Ebanks et al. (2012) discuss the effects of special interventions that help students better to prepare for the entrance exams to such schools. This and numerous other examples of interventions aimed at helping the potentially mathematically gifted undoubtedly deserve to be championed, although of course one can hardly hope that they will quickly and completely remedy the existing state of affairs. The ongoing debates, however, are not limited to purely methodological or organizational considerations.

One vivid manifestation of the politicized nature of mathematics education may be seen in the wish to create a separate mathematics for the oppressed—"*mathematics as a weapon of struggle*," as Gutstein (2012) puts it. The Palestinian mathematics educator Fasheh (1997) expresses perplexity at the fact that, regardless of who has ruled the Palestinian territories—Great Britain, Jordan, Israel, or the Palestinian Authority—mathematics curricula have not differed appreciably. He concludes: "I personally believe that the math we teach or study has lost its life, its soul, and its connectedness to the realities in both the immediate and wider worlds" (p. 24). After exclaiming, "What is the purpose of teaching geometry, in particular, if ordering the place in which people live is not one of our primary concerns?!" (p. 25), he concludes: "My basic argument here is that every curriculum should have—as part of its design—enough room to include the realities and personal experiences and expressions of the people following the curriculum as part of the on-going discussions in math classes" (p. 26).

The key words here are "enough room," because *some* room for the personal experiences and expressions of the people will inevitably be found in almost any curriculum. What we have in this case is specifically a call for the creation of a special course in mathematics based on personal experiences, that is, for every individual group and every separate people.

Similar ideas have been expressed in different countries. One of the fundamental and undoubtedly correct ideas of mathematics (and any other) education—that one must make use of students' personal experience—has been blown up to such proportions that it has given rise to the notion the children should be taught cardinally different courses in mathematics depending on their culture (and this word has also been understood in different ways). Meanwhile, strangely enough no one argues that students of Anglo-Saxon descent should necessarily focus on Alfred the Great when studying trigonometry, or that Jewish students should necessarily concentrate on the discussion of gefilte fish recipes during algebra classes, even though arguments similar to these have been made about students of African or Latin American descent with relative frequency. There is no evidence that this has helped the development and education of the mathematically gifted, however.

There has also been talk about a cultural struggle of sorts taking place within schools themselves. Fordham and Ogbu (1986) published a highly influential paper in which they contended that an important obstacle to the improvement of African-American students was the students' fear that their schoolmates would accuse them of "acting white" and betraying their own culture. However, the conclusions reached in this paper, which gave rise to numerous responses and discussions, have not been confirmed or have been confirmed only in small part by other studies. Tyson et al. (2005), for example, demonstrated that African-American teenagers are quite oriented toward good results and that peer pressure that might prevent them from achieving such results was by no means prevalent in schools. Walker (2012) also gave examples of how, on the contrary, peer groups facilitated the achievement of good results.

Discussions concerning this topic have taken place mainly in the academic sphere, although they have spilled out to the broader public. However, they reflect the controversial nature of questions about education (including mathematics education) and the importance of equity in education for political struggles in contemporary society.

15.7 Conclusion

The aim of this article has been to list and elucidate existing problems, without trying to offer solutions to them, which in any case cannot be simple. If we look at statistics about the sizes of various demographic groups in general and those involved in one way or another in mathematically gifted education and translate them into plain language, we can say that the giftedness of thousands of talented children is not recognized and not supported, and consequently suppressed. This state of affairs cannot be considered acceptable, and the fury directed at claims about the fairness of the existing system, and the intensity of the political fights in general, is understandable. At the same time, revolutionary changes of this situation—such as would rectify everything overnight—are hardly possible. In the words of a Russian poet, “labor and persistence are truer” (Mandel’shtam 1917). To this end, formulating slogans and determining a direction for this labor seem especially important. No new segregation, no matter what terms are used to cover it up, can be beneficial for discovering every child’s potential. The problem is, precisely, that many capable children, despite all kinds of rhetoric, in practice are given no access even to comparatively elementary mathematics classes, such as algebra, and additional explanations that they have no need for algebra—that what they need instead is something special, or that they themselves do not want to study algebra, as it were, as a sign of protest—are hardly useful.

The tendency to see gifted education as being in a certain sense an impediment to education for all seems profoundly misguided. In reality, high-quality education—that is, education that is oriented toward students and the discovery of their potential—is possible only given the existence of a sufficient number of qualified teachers, whose preparation in fact commences when they themselves are schoolchildren (or more precisely, their preparation can be irremediably doomed at this stage). The mere demand that children should be taught well, or even the publication of a sensible curriculum or set of standards (which is no simple matter), cannot create the conditions under which such a demand might be realized or such a curriculum implemented. Given the existence of a sensible curriculum, a highly-professional teacher will be able to create conditions under which capable students will be able to prepare for more advanced mathematics education and, most importantly, will be able to understand themselves and to recognize their own interests in and possibilities for such an education. Without fixing general mathematics education, it is impossible to achieve a genuinely successful mathematically gifted education, to which thousands of potentially highly-gifted schoolchildren have no access at present; but general education must be fixed by relying on those institutions—including institutions involved in mathematically gifted education—which are best prepared mathematically and pedagogically.

Having sketched a picture of the political fights surrounding mathematically gifted education above, we have not given an answer to the most important questions, since we have not demonstrated how mathematically gifted education is connected with various social-economic groups and their positions on other issues. Of course, there is little reason to expect, in the spirit of so-called vulgar sociology, such as was widespread in the USSR during the 1920s, that we will be able to identify certain relations that will hold true at all times, for example, that workers support advanced study of mathematics, while merchants do not (or vice versa). Nonetheless, we can make certain observations and draw certain conclusions. It is clear, for example, that the so-called defense ministries (that is, those responsible for manufacturing weapons) in the USSR supported the creation of specialized physics-mathematics boarding schools for the gifted (Pakhomov 2013). It is clear that even the structuring of mathematics classes—in terms of how much attention they devote to actual mathematics vs. how much attention they devote to ideological discussions—depends on a state's general stance and ambitions of world dominance (Karp and Lee 2010).

In our opinion, there is a need for more and more detailed clarification of the views of all the participants of contemporary discussions about the politics of advanced study of mathematics (and, indeed, all study of mathematics): the views expressed among teachers, teacher educators, business representatives, and various sectors of the parent population in general. This, however, is a goal for other studies.

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Part III
Commentary

Chapter 16
Commentary on Interdisciplinary Perspectives to Creativity and Giftedness

Bharath Sriraman and Roza Leikin

Abstract In this commentary, we analyze the 14 chapters of the book for interdisciplinary themes that unravel and are applicable to mathematics education. In particular, attention is given to interdisciplinary perspectives on the constructs of creativity and giftedness. Dominant themes from clusters of chapters are highlighted.

Keywords Mathematical giftedness • Creativity • Psychology of creativity • Creativity in mathematics education • Mathematical creativity • Research advances

16.1 Interdisciplinary Perspectives to Creativity

A watershed moment for the domain of creativity research in psychology was the speech by Joy Guilford at the American Psychological Association meeting in 1950 (Kaufman and Sternberg 2007), because it spawned research that continues to grow six decades later. Even though creativity has been unfairly criticized by some in educational circles as the "wastelands of psychology", it is a legitimate domain of research within the APA and continues to inform both education and educational psychology today. Indeed numerous curricular documents and national reports (e.g., NACCCE 1999) extol the need to incorporate creativity given the knowledge based economy that their countries find themselves in, and the need to capitalize on human capital in ways different from industrial societies.

Unlike psychology, mathematics education does not as of yet have a "watershed" moment which has created growing interest in creativity. There have been sporadic papers that have reported on creativity in some journals and books, but not any

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sustained efforts to study it systematically. Creativity is often relegated as the purview of professional mathematicians or for students with extraordinary ability interested in mathematical contests. In fact many teachers view it as a form of extra-curricular activity, separate from daily academic subjects (Aljughaiman and Mowrer-Reynolds 2005). In the U.S. and elsewhere "creativity" is used as a psychometric marker for the identification of students with high abilities in mathematics, science, the arts etc. The label "gifted" is very often is applied to such students since "creativity" has continued to be viewed as one possible indicator of high potential following Sidney Marland's (1972) landmark report. The use and/or misuse of these terms as synonyms and numerous writings addressing their relationships is already found in the existing literature (e.g., Haavold 2016; Leikin et al. 2009; Sriraman 2005; Sriraman and Haavold in press).

Given this disclaimer, this book does not attempt any further differentiation or expositions of subtlety in how the terms "high ability", "gifted", "creativity" or talent ought to be used in mathematics education research. What the book does is to present current research on these constructs and their inter-relationships as interdisciplinary and informative to mathematics education. In doing so two major strands are unraveled, the first being the construct of creativity through nine chapters that interweave both general notions as well as particular notions applicable to mathematics education from the perspectives of psychology, philosophy, mathematics and mathematics education.

Haught and Stokes explore the idea of domain constraints which call for developing competencies to overcome the basic constraints in the domain. These competencies take the form of novel problem solving. This idea is explored through young children in math and college students in composition. The ideas in this chapter can be further stretched to what Robert Root-Bernstein calls n-epistemological awareness, i.e., the awareness of those at the frontiers of their field of the constraints that need to be overcome to solve problems (Root-Bernstein 1996). Mathematics provides numerous examples that such domain constraints spurs creativity in the form of new tool development. For example homological algebra was developed to answer basic questions in number theory which young students can comprehend, but solving these problems requires sophistication that is only obtained in graduate level education. Many of the early work on integrals required the likes of John Wallis, Lord Brouncker and Fermat to use interpolation techniques that overcome the constraint of not having the binomial theorem available (Sriraman and Lande in press). Today these integrals are easily solved due to the tools that were subsequently developed. John-Steiner and Hersh continue this vein of thought by addressing some of the psychological factors that spur mathematicians to pursue research for extended time periods. They identify both coping skills and report on how artificial intelligence has implications for the pursuit of novelty and may help "manage" or "navigate" human intelligence in fruitful directions. Their chapter also has implications for teaching in mathematics classrooms. Beghetto and Schreiber pursue a different motivating strand from the perspective of pragmatist philosophy of Charles Saunders Peirce. Their focus is on abductive reasoning which often is investigated in mathematics education studies but in this chapter the reader is asked to view it as an aspect of creative reasoning. In other words when conditioned responses to situations fall short of being able to tackle a new "situation" (academic or otherwise), genuine doubt sets it which can propel creative learning. In their chapter current conceptions of creativity are expanded and some recommendations are made for further research on creativity in educational settings.

The five other chapters in the section of creativity have more of a mathematics education flavor to them. Karwowski and Dziedziewicz address creativity in early mathematics learning. They do so by drawing on a typological model of creativity consisting of the triad of creative abilities, openness and independence. Heuristic rhymes are given as an example of enhancing mathematical creativity. Their chapter occurs at the start of the book because it gives a very in depth overview of the history of creativity research as well as clears up misconceptions about the "watershed" moment widely considered in the U.S. They point out that it was Francis Galton who studied individual differences in the imagination and representations of scientists in the late nineteenth century. It is ironic that Galton's name is synonymous with psychometrics when in fact he initiated the study of human imagination! Mann and Chamberlin provide an overview of affective studies and its connection to creativity. They propose the new sub construct: iconoclasm which helps researchers better understand mathematical problem solving episodes. Both these chapters extend the categories of fluency, flexibility, originality and elaboration which are invoked frequently due to the ubiquity of divergent thinking tests as a measure of creativity.

Two other chapters address pre-service and in-service teachers in Romania and Iceland respectively. Both these chapters report on task based activities used with these particular groups and implications for teacher educators on how creativity is understood. In the former case, Voica and Singer find a relationship between expertise, creativity and the processes of problem solving and posing, in that they develop in tandem with each other. Palsdottir and Sriraman argue modeling tasks provide an avenue for in service teachers to foster creativity in the classroom. Yet their study reports that teachers do not use the modeling activities as intended by the textbooks. The views of teachers reported in this chapter indicate that a dialogic and practical approach to modeling activities is preferred to a strictly mathematical approach. Interestingly enough these findings from the Icelandic context corroborate the social-communicative nature of creativity described by Voica and Singer in their work. Given the excessive emphasis on divergent thinking in creativity research, Tan and Sriraman propose convergence as equally important in the context of mathematics. In their chapter they summarize psychological theories of development and creativity to argue how people develop their capacity in convergence (e.g., collaboration), through mathematical learning (e.g., with coherence, congruence), and for creativity (e.g., imagination). Taken as a whole this section of the book provides a reader with numerous avenues for further development which are commented on in the concluding section.

16.2 Interdisciplinary Perspectives on Giftedness

Five chapters of the book address the notion of giftedness from the viewpoints of neuroscience, classrooms, political landscapes, and its relationship to creativity. The neuroscience perspective is quite new to the field of mathematics education and would help support advocacy for mathematical enrichment. Again in the U.S. a quagmire for mathematics gifted education has been the findings of the ongoing Study of Mathematically Precocious Youth (SMPY) started at Johns Hopkins in 1971, which introduced the idea of above-level testing for the identification of highly gifted youth, labeled as “mathematically precocious” (Sriraman and Haavold in press). For too long this label has been loaded with connotations and associated with psychometric testing. A different and much needed topography is provided by Leikin, Leikin and Waissman where the neuro-cognitive characterization of super mathematically gifted high school students, generally gifted students who excel in school mathematics and students who excel in school mathematics but are not identified as being generally gifted (NG-EM) are compared to highlight differences when solving two distinct types of problems. These authors report on three types of neuro-efficiency effects, which highlight the different characteristics of electrical activity of super mathematically gifted students. More importantly they relate these three types of neuro-efficiency to variables such as type and stage of the test and giftedness. This chapter presents a different way of distinguishing mathematically gifted individuals from the general pool of students who excel in mathematics and the identification of “super gifted” individuals based on their performance. Neuroscience has the potential to inform mathematics education in ways beyond traditional performance testing, which are typically a function of academic exposure.

Three of the chapters in this section address the relationship of giftedness and creativity. Cropley, Westwell and Gabriel explore how contrasting psychological and neuroscientific approaches inform our understanding of creativity as a component of giftedness in general? In doing so a large body of literature synthesizing these constructs from a psychological and neuroscience viewpoint is undertaken. Even though their conclusion might seem to be “obvious”, it is important to note that they present a very coherent psychological view of the “intersecting” nature of these constructs in the domain of mathematics. An important duality is teased out to provide a robust framework for researchers interested in this paradigm. Pitta-Pantazi gives an overview of 5 years of studies conducted at the University of Cyprus that have addressed definitions of mathematical giftedness and its relationship to both ability and creativity. This chapter highlights the possibilities of technology for developing mathematical creativity in primary classrooms. Lev and Leikin provide the results of a large scale study which investigates the link between ability and creativity in the domain of mathematics by using Multiple Solution Tasks as an evaluative tool.

The final chapter in this section takes on the political perspective in the education of the mathematically gifted. It is deliberately placed at the end of the book since it tackles broader issues of equity that are pertinent to school systems worldwide. The reader learns about the political landscape in the U.S. as well as how this landscape was shaped in the former Soviet Union. Karp calls for the need of a better clarification for issues of equity that surround the advanced study of mathematics.

16.3 Concluding Thoughts

The book attempts to bring in chapters from different parts of the world from researchers who have spent substantial time in investigating questions of mathematical ability, mathematical creativity and mathematical giftedness. In doing so, some interdisciplinary perspectives from the neurosciences, psychology and philosophy emerge for consideration by mathematics educators. Do mathematical giftedness and mathematical creativity exist per se? The answer to this question is yes- and conclusively supported by a cross section of studies from psychological, psychometric and neuroscientific viewpoints. In a study by Leikin and Lev (2013) gifted students scored much higher than other students on all measured criteria on all tasks in the mathematics creativity test. Similarly strong correlational relationships exist between mathematical ability and mathematical creativity with the latter subsumed as an aspect of the former (Kattou et al. 2013). In an attempt to validate a theoretical model for optimizing mathematical creativity in the classroom proposed by Sriraman (2005), Haavold (2016) found that internal motivation and an aesthetic sense of mathematics predicted creativity, when controlled for mathematical achievement. In terms of further research, divergent thinking need not become solely associated with creativity research as often seen in psychology. Secondly, both affect and reasoning are broad categories which can be focused towards the study of mathematical creativity as posited in some of the chapters. For example ongoing studies in mathematics such as those conducted in Cyprus focus on the relationship between visualization and creativity (Pitta-Pantazi et al. 2013). Given the domain specific nature of mathematics and the broad nature of both reasoning and affect (e.g., Feldhusen and Westby 2003), we view mathematics education as an appropriate field within which more systematic and longitudinal studies can be carried out. Interdisciplinary frameworks offer newer possibilities for further studies and it is hoped that this book is simply a start.

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