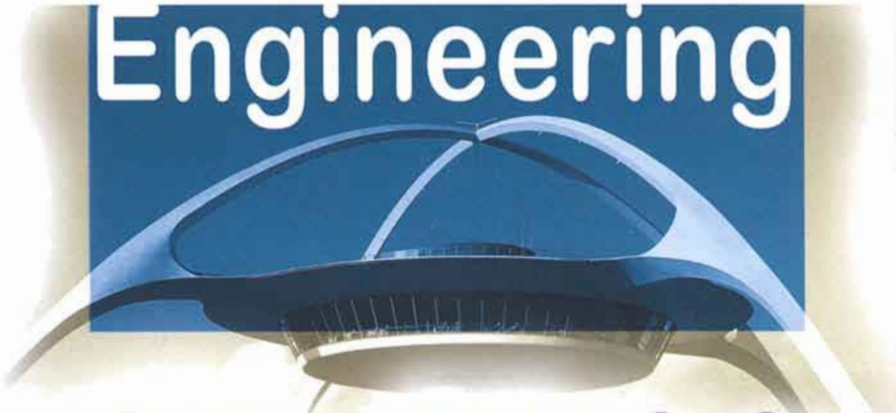
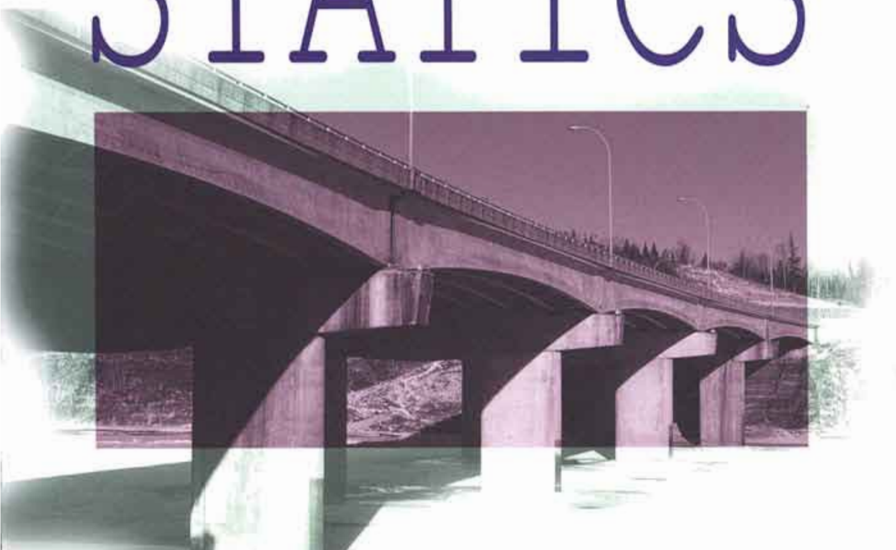
A.S. HALL F.E. ARCHER R.I. GILBERT

Engineering



STATICS



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SECOND EDITION

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STATICS

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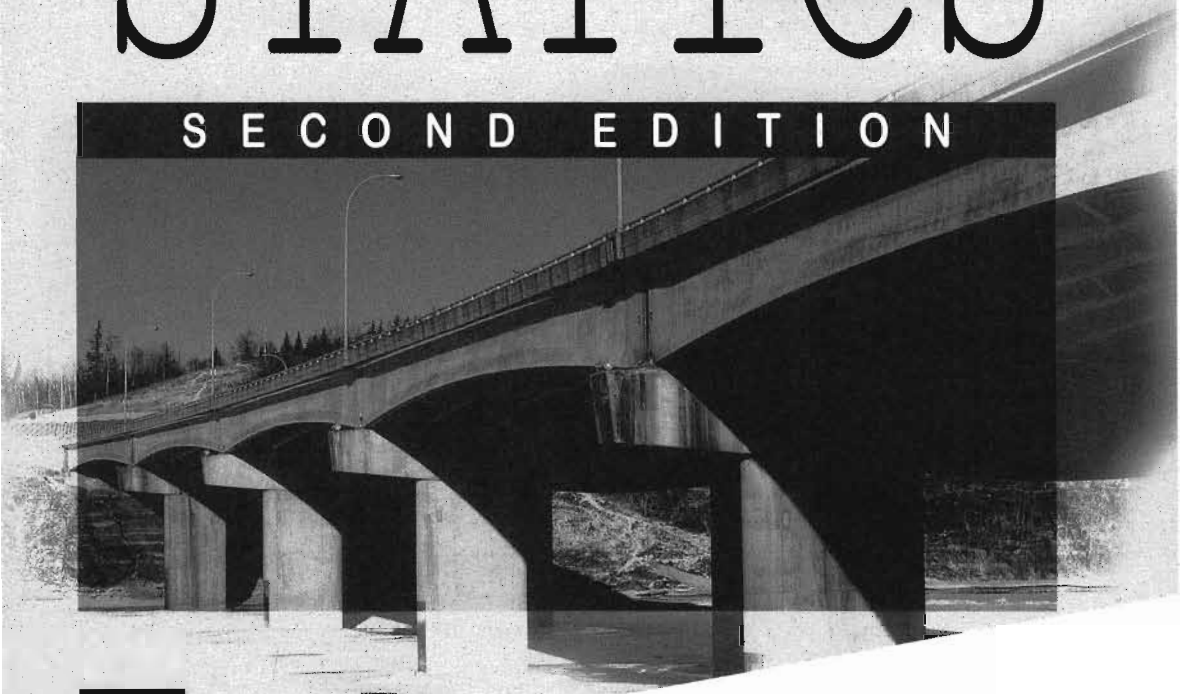

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Contents

Preface

As in the previous editions, this book deals with the fundamentals of *statics* and their application to a broad range of engineering problems. The treatment is intended primarily for undergraduate students of civil, structural and environmental engineering, but we hope that it will also be of use to anyone with an interest in engineering mechanics.

The present edition contains a substantial revision of much of the text of the previous work and several additions to it. We have removed much of the material related to graphical methods of solution, which is now dated and less relevant. But in doing so, we have tried to ensure that students can still gain a physical understanding of the solution to particular problems, as well as an analytical one.

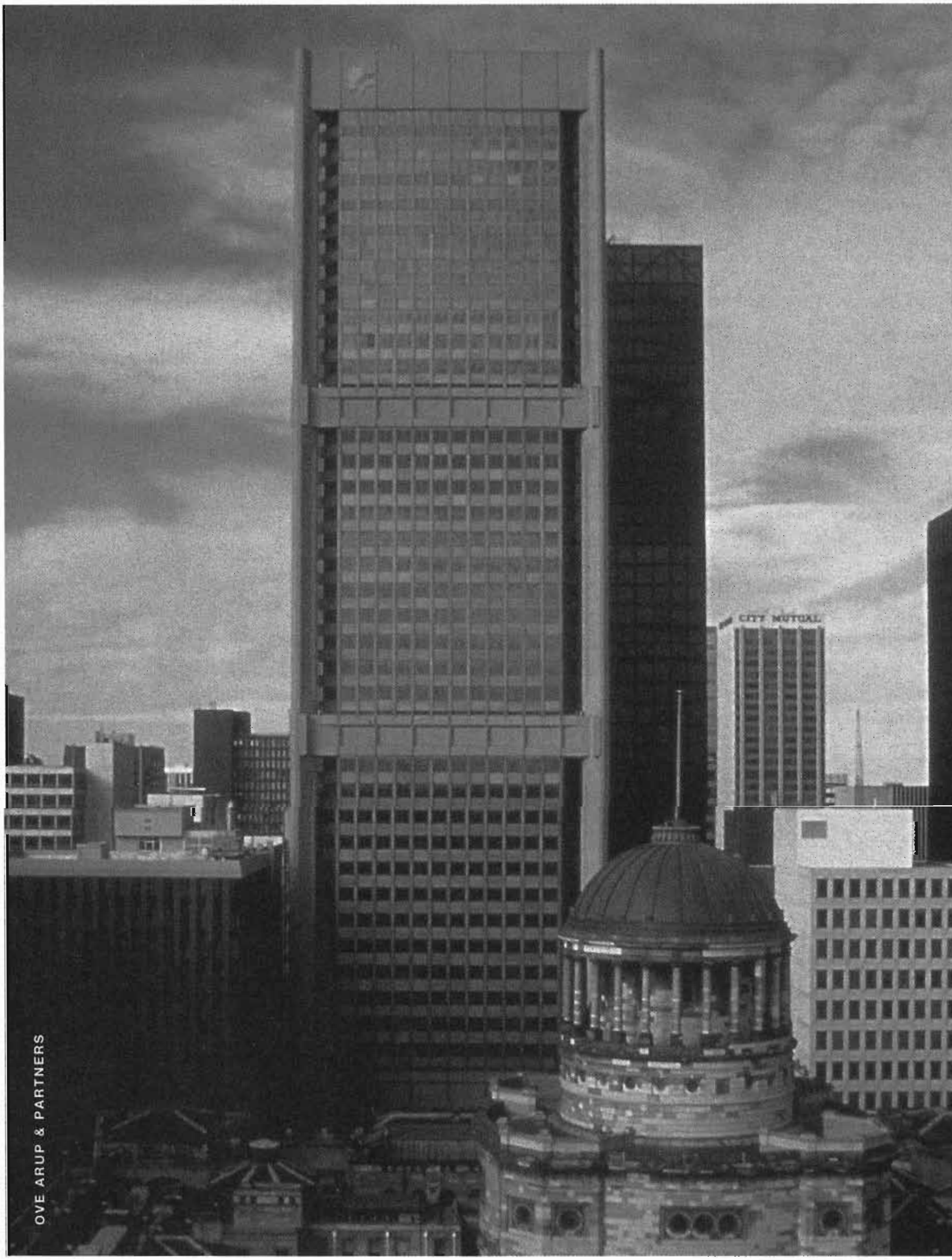
A new chapter on fluid statics (Chapter 14) deals with the forces exerted on submerged bodies, as well as with buoyancy and the stability of floating bodies. The chapter on flexible cables (Chapter 9) contains what is believed to be a new approach to the analysis of the *catenary* which allows for the direct solution of many cable problems that have conventionally been solved using trial and error methods. The Appendix has been added, dealing with the geometrical properties of plane figures. These properties have been described and developed by considering some statics problems involving parallel forces acting normal to a plane figure. This approach is directly applicable to problems in fluid statics and to beam theory that may be studied in subsequent courses. However, the equations are also of general application.

Additional tutorial problems have been added at the end of most chapters.

AS Hall
FE Archer
RI Gilbert

part 1

TWO-DIMENSIONAL STATICS—FORCES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Introduction

1.1 Scope

The subject of mechanics deals generally with the effect of forces on a body from the point of view of the motion of the body. If a group (or system) of forces, when applied to a body, causes no change in its state of motion, the group is said to be in *equilibrium.* The study of the conditions of equilibrium constitutes that branch of mechanics known as *statics,* and it is with this branch that this book deals.

Statics is the cornerstone of structural engineering, but fundamental concepts, analytical methods and analogies from statics are directly applicable in almost every branch of engineering. A sound knowledge of the application of the principles of statics is therefore essential for all engineering students and engineering practitioners.

1.2 Force

A *force* is defined as that which changes, or tends to change, the velocity of a body. Force is a vector quantity, possessing direction as well as magnitude. A force is not completely defined unless its magnitude, direction and line of action are specified. It will be seen later that when effects of a force upon a body other than that relating to its tendency to change the velocity are considered, it is necessary also to specify the point of application of the force.

The basic unit of force is the *newton* (symbol N). The newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2. For many engineering problems, the newton is a rather small unit, and the unit most commonly used is the *kilonewton* (symbol kN) which is 1000 N.

1.3 Body

The term *body* is used to denote the particular section of matter under consideration. If a bowl of water stands upon a table, we may wish to consider the forces acting on the table alone, the table and bowl together with or without the water, the water within the bowl, or possibly one part only of the table. Whatever part is chosen is called the *body.*

The forces dealt with in statics are the forces exerted upon the body from outside, i.e. forces *external* to that body.

Before solving a practical problem it is important to be clear regarding the extent of the body under consideration and the forces acting externally upon it.

1.4 Newton's Laws of Motion

Since the whole subject of mechanics stems from Newton's three Laws of Motion these will be stated:

First Law: A body will remain at rest or continue to move with uniform velocity unless acted upon by an external force.

Second Law: If an external force acts upon a body, the rate of change of momentum is proportional to the force, and takes place in the direction of the force.

Third Law: To every action there is a reaction equal in magnitude and opposite in direction.

Sir Isaac Newton developed these laws in the late seventeenth century from a study of the motion of objects. The application of these laws to engineering problems is the topic of this book.

The first law deals with bodies in equilibrium and is the basis for the study of statics. The second law is concerned with accelerating bodies and is the basis of the branch of mechanics known as *dynamics*. The third law is fundamental to an understanding of the concept of force. In engineering applications, the word 'action' may be taken to mean force and so, if a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

Newton also propounded a Law of Gravitation, which together with his three Laws of Motion enabled him to explain the movement of the planets in the solar system. According to this law, any two bodies of mass m_1 and m_2 exert a force of attraction on each other. This gravitational force is proportional to the masses and inversely proportional to the square of the distance between their centres, d. That is:

$$F = G\frac{m_1 m_2}{d^2} \tag{1.1}$$

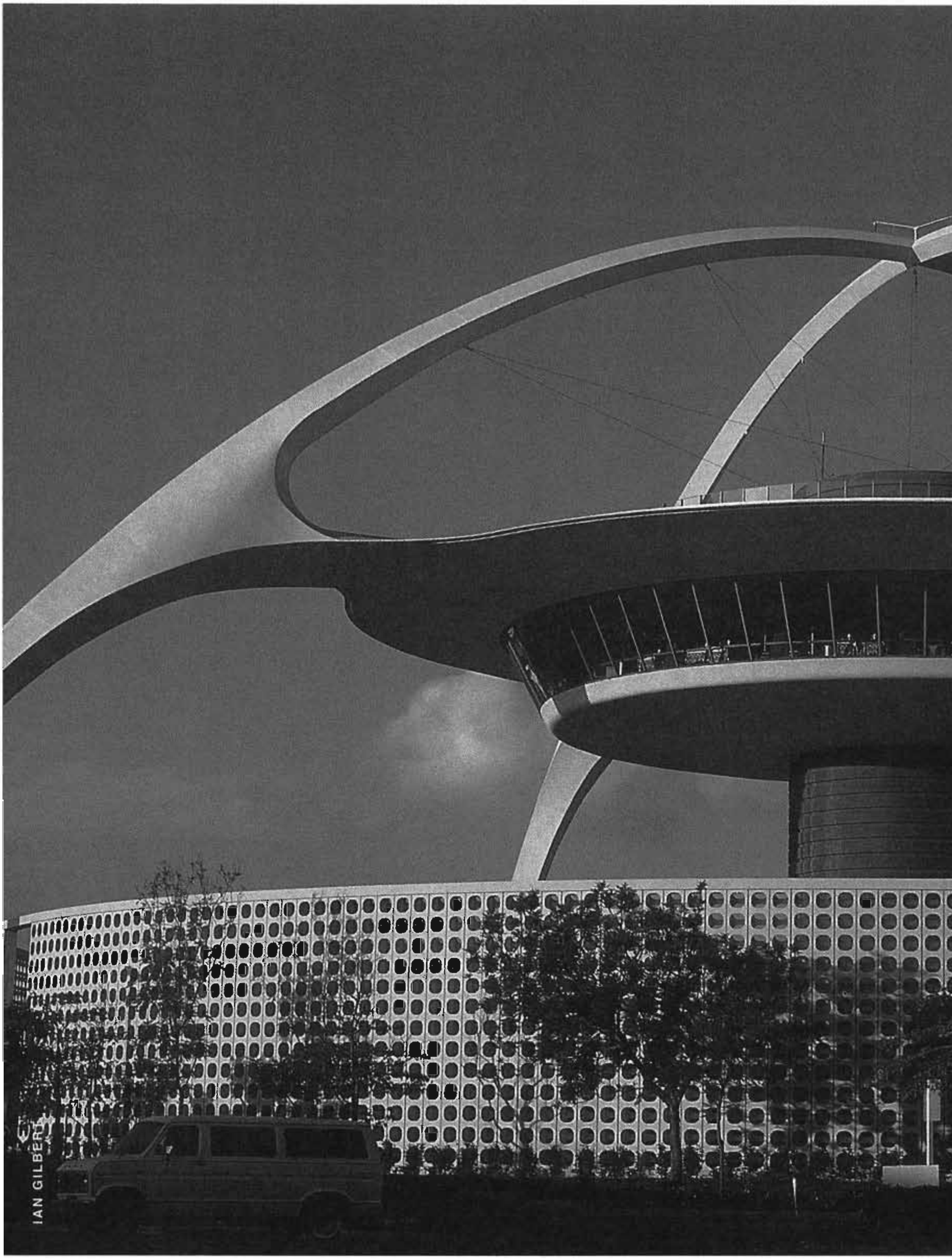
where G is a gravitational factor which according to Newton is constant throughout the universe.

1.5 Mass and weight

The term *mass* is difficult to define precisely. However, for engineering purposes it is sufficient to know that the mass of a body is an absolute quantity, independent of the position of the body and its surroundings.

On the other hand, the *weight* of a body is dependent on its position. For everyday purposes, the weight of a body may be defined as the gravitational force exerted on the body by the earth when the body is situated at the earth's surface. The acceleration due to gravity at the earth's surface is approximately 9.81 m/s^2, and hence the *weight* of a 1 kg mass is a force of 9.81 N acting towards the earth's centre (i.e. vertically downwards). It is often taken as 10 N for approximate calculations.

Since the earth is not quite spherical, the weight of a 1 kg mass will vary slightly from place to place on the earth. If it is moved to a considerable distance above the earth's surface, its weight will be much less and at great distances its weight may become negligibly small. If it approaches another celestial body (the moon or another planet) it will be attracted to that body, and its 'weight' will be greater or less than its weight on earth depending on the size of the celestial body.


IAN GILBERT

Composition and Resolution of Forces

2.1 Resultant

The *resultant* of a group of forces is that single force which, when applied to a body, produces the same effect (as far as the motion is concerned) as the group. The resultant is thus the equivalent single force. It should be noted that it is only equivalent as far as the external motion of the body is concerned but will not be equivalent in other respects. For example, a single large force may cause damage to the body that would not be caused by a large number of small forces. For this reason, the resultant is often said to be *statically equivalent* to the group of forces, i.e. equivalent as far as statics is concerned, but not equivalent in all respects.

2.2 Resultant of two forces

Figure 2.1 shows two forces F_1 and F_2 whose lines of action intersect at A. To find their resultant, the lines AB and AD are set out to represent the forces to scale (Figure 2.2). Upon completion of the parallelogram ABCD, the diagonal AC represents the resultant in magnitude, direction and position. This construction is called the Parallelogram of Forces.

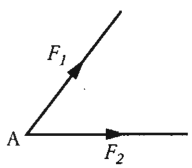


Figure 2.1

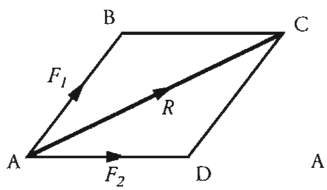


Figure 2.2

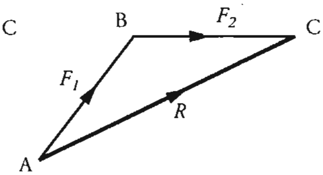


Figure 2.3

Alternatively, the two forces may be represented in magnitude and direction by AB and BC (Figure 2.3). The magnitude and direction of the resultant is then given by the line AC. In this construction the lines AB and BC must be so drawn that the arrows which denote the directions of the forces 'track' in the same sense around the figure. The arrow of the resultant tracks in the opposite sense. This construction is called Vector

Addition. It can be seen that the triangle of Figure 2.3 is identical to the upper half of Figure 2.2.

If the two forces F_1 and F_2 are at right angles to each other, the magnitude of the resultant and its direction θ (with respect to the force F_2) are given by:

$$F = \sqrt{F_1^2 + F_2^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{F_1}{F_2} \tag{2.1}$$

The foregoing constructions apply to all vector quantities. The second construction is easily appreciated if applied to displacements. If AB and BC (Figure 2.3) represent two consecutive displacements, then AC is the single displacement which would bring the body to the same final position, and is thus the resultant.

The fact that the construction shown in Figure 2.2 (or Figure 2.3) leads to the resultant force may be deduced from Newton's Second Law. It may also be demonstrated by a simple laboratory experiment. Consider three strings AB, AC and AD joined together at A such that points A, B, C and D all lie in a vertical plane (Figure 2.4). Strings AB and AC pass around smooth (i.e. frictionless) pulleys. Weights W_1 and W_2 are attached so that the tension in AB is W_1 and the tension in AC is W_2. If a third weight W_3 is suspended from AD, the point A will take up an equilibrium position provided W_3 is less than (W_1 + W_2).

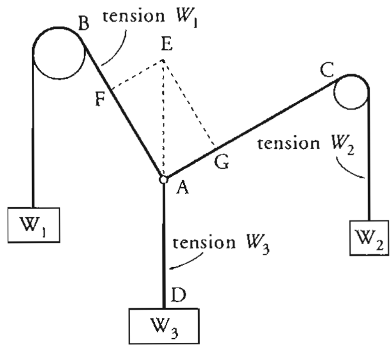


Figure 2.4

Since the point A is in equilibrium, the resultant of the two tensions W_1 and W_2 must be equal and opposite to the tension W_3. DA is produced to E so that the length AE represents W_3 to some scale, EF is drawn parallel to AC, and EG is drawn parallel to AB. It will be found that AF represents W_1 and AG represents W_2 to the same scale as AE represents W_3. Line AE, representing a force equal and opposite to tension W_3, must be the resultant of W_1 and W_2, and it is given by the diagonal of the parallelogram AFEG with sides representing W_1 and W_2.

2.3 Components of a force

A given force may be replaced by two forces provided that their vector sum is equal to the given force. These two forces are known as *components* of the given force.

If a given force F (Figure 2.5a) is to be resolved into components lying in the given directions Ox and Oy, the force is first represented to scale by the line OA (Figure 2.5b). If the parallelogram OBAC is then completed by lines parallel to Ox and Oy, the lines

OB and OC must represent the required components F_y and F_x respectively, since evidently their resultant is the given force (see Figure 2.2). F_x and F_y are calculated readily from Figure 2.5b using the sine rule.

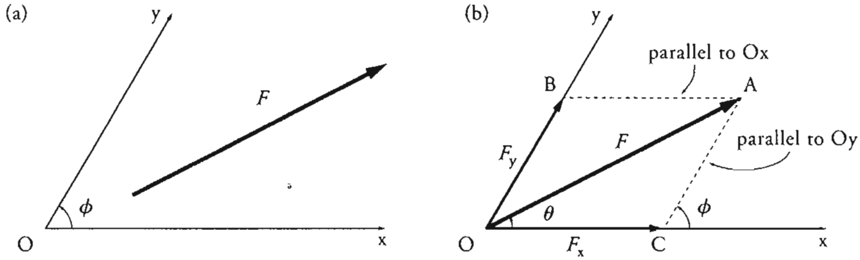


Figure 2.5

The process of breaking the single force into two components is the reverse of finding the resultant of two forces; but whereas the two given forces have only one possible resultant, the single resultant may be broken into many different pairs of components depending on the directions required for the components. Thus, there is an infinite number of pairs of components of the force F, as many as there are ways of drawing a triangle one of whose sides is OA, but only one pair satisfies the given directions Ox and Oy.

EXAMPLE 2.1

Find the components of the force $F = 20$ N in the directions Ox and Oy (Figure 2.6).

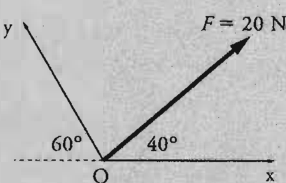


Figure 2.6

SOLUTION

A triangle of forces is drawn, either as Figure 2.7a or 2.7b. If the given force F is represented by the side AB, the components may be thought of as an alternative path from A to B. Using the sine rule:

$$\frac{20}{\sin 60} = \frac{F_x}{\sin 80} = \frac{F_y}{\sin 40}$$

Hence: $F_x = 22.7$ N and $F_y = 14.8$ N

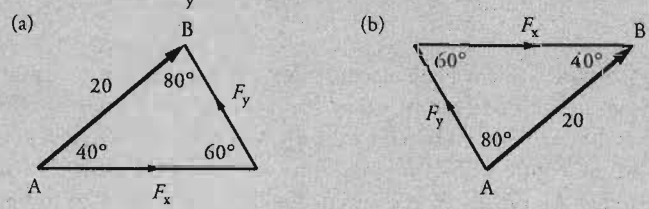


Figure 2.7

The most common problem is that of finding components of F which lie parallel to and perpendicular to a given direction Ox (Figure 2.8). In this case, the axes Oy and Ox are at right angles ($\phi = 90°$) and F is at an angle θ to the direction Ox. From simple geometry:

$$F_x = F\cos\theta \quad \text{and} \quad F_y = F\sin\theta \tag{2.2}$$

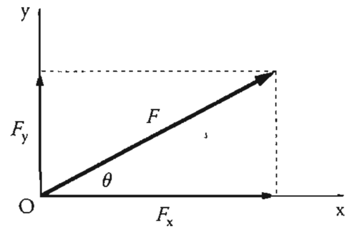


Figure 2.8

It follows that F is the resultant of its components F_x and F_y. In the orthogonal system of Figure 2.8, where the components F_x and F_y are at right angles, the magnitude and direction of F are obtained by substituting F_y for F_2 and F_x for F_1 in Equation 2.1:

$$F = \sqrt{F_x^2 + F_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{F_y}{F_x} \tag{2.3}$$

2.4 Resultant of concurrent forces

If the lines of action of a number of forces pass through a common point, the forces are said to be *concurrent*.

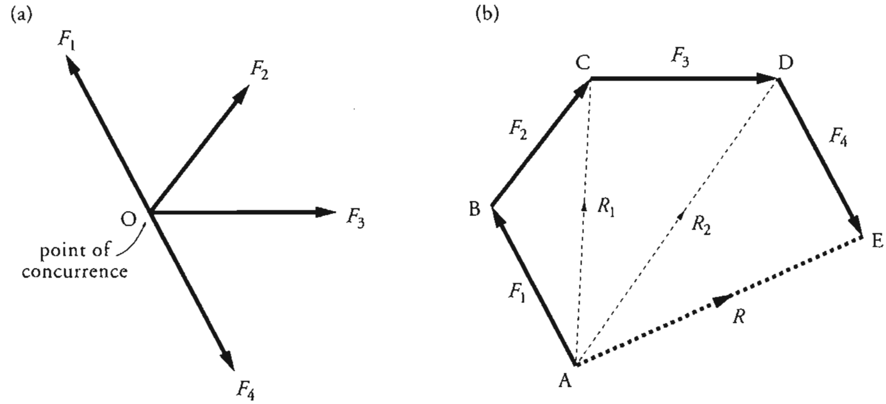


Figure 2.9

Consider the four concurrent forces F_1 to F_4 shown in Figure 2.9a. The resultant of the group of forces can be found from the graphical construction of Figure 2.9b. Forces F_1 and F_2 may be combined by the method of vector addition into a resultant R_1 (triangle ABC in Figure 2.9b). R_1 may then be combined with F_3 (triangle ACD) to give a resultant R_2. Finally R_2 is combined with F_4 (triangle ADE) to give the final resultant R. The line AE gives the magnitude and direction of R. The line of action of the resultant passes through the point of concurrence (point O in Figure 2.9a). By successively adding force vectors using the procedure illustrated in Figure 2.9b, the resultant of any number of concurrent forces can be determined.

Figure 2.9a, showing the actual disposition of the forces is called a *space diagram.* Figure 2.9b is called a *force polygon.*

It is convenient to think of the forces F_1 to F_4 as represented to scale by arrows, or vectors, radiating from O. These vectors may be 'picked up' and 'arranged' to form the force polygon ABCDE. The vectors must be placed parallel to the original direction and the arrows must track around the polygon. Taking the forces in a different order will produce a different polygon but it will end at the same point E. The resultant is AE, the alternative straight line path from A to E.

It would be possible to calculate, by trigonometry, the length and direction of the unknown side AE of the force polygon in Figure 2.9b. A less laborious method is to resolve each force into its components in two chosen directions, and then combine the components.

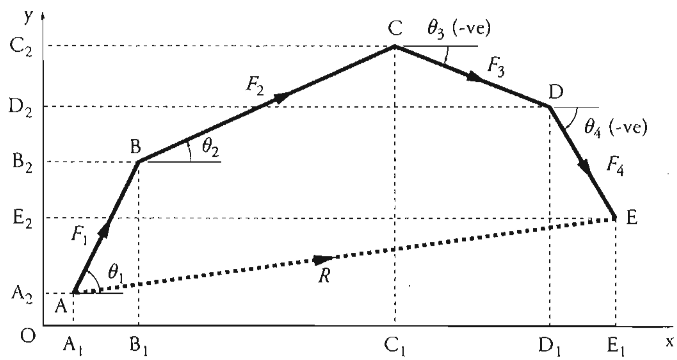


Figure 2.10

Suppose that two perpendicular axes Ox and Oy are chosen in the plane of the force polygon (Figure 2.10), and let the forces F_1, F_2 etc. make angles of θ_1, θ_2 etc. with the x axis, θ being measured positive anticlockwise from Ox. Let the components of any force F in the x and y directions be F_x and F_y respectively.

If the vertices of the polygon are projected onto the x axis at A_1, B_1, C_1 etc., it will be seen that the x component of F_1 is A_1B_1, that of F_2 is B_1C_1 and so on. The algebraic sum of these components is A_1E_1. Similarly, if the vertices of the polygon are projected on to the y axis, the y components of the individual forces are A_2B_2, B_2C_2 etc., their algebraic sum being A_2E_2.

Then A_1E_1 and A_2E_2 are the components of the resultant, and may easily be combined to give the magnitude and direction of R.

For any force, the components F_x and F_y are obtained from Equation 2.2. If R_x and R_y are the components of the resultant in the x and y directions respectively, then:

$$R_x = \sum F_x = \sum F\cos\theta \quad \text{and} \quad R_y = \sum F_y = \sum F\sin\theta \qquad \textbf{(2.4)}$$

and from Equation 2.3:

$$R = \sqrt{R_x^2 + R_y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) \qquad \textbf{(2.5)}$$

EXAMPLE 2.2

Find the resultant of the forces concurrent at O shown in Figure 2.11.

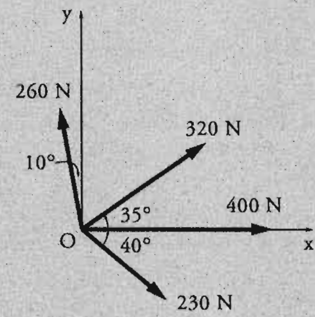


Figure 2.11

SOLUTION

Force (N)	θ	F_x (= $F\cos\theta$)	F_y (= $F\sin\theta$)
230	-40°	176.2	-147.8
400	0°	400.0	0
320	35°	262.1	183.5
260	100°	-45.2	256.1
Summations		793.2	291.8

From Equations 2.4:

$$R = \sqrt{793.2^2 + 291.8^2} = 845.2 \text{ N} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{291.8}{793.2}\right) = 20.20°$$

A force polygon illustrating the graphical determination of the resultant is shown in Figure 2.12.

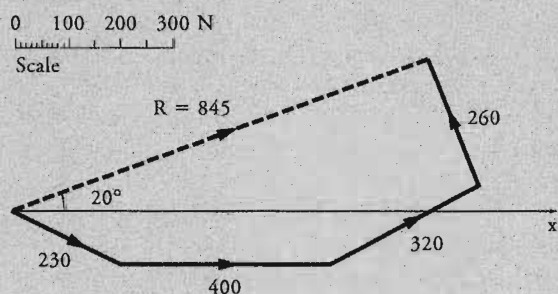


Figure 2.12

Problems

2.1 Determine the x and y components of the forces shown in Figure P2.1.

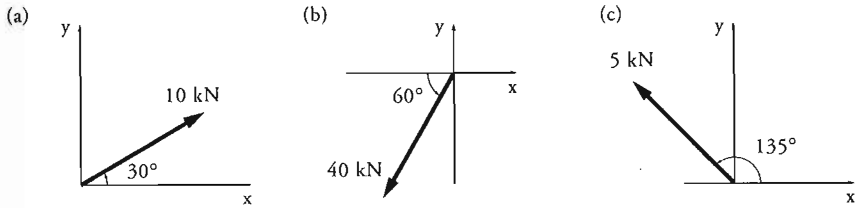


Figure P2.1

2.2 Determine the u and v components of the forces shown in Figure P2.2.

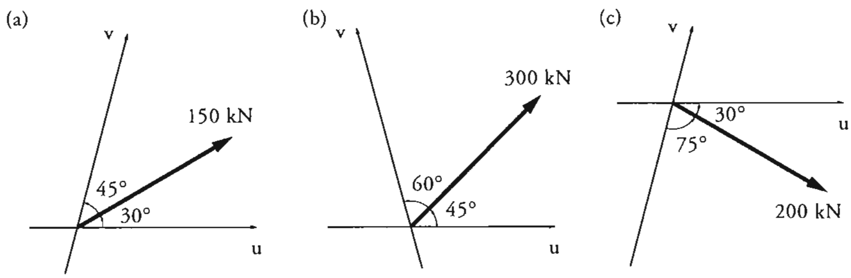


Figure P2.2

2.3 A force of 100 N acts in the direction of the x axis. Resolve this force into two components P and Q which make angles of +30° and −45° respectively with the x axis.

2.4 The following forces are concurrent: 50 N at 0°, 80 N at 25°, 10 N at 85°, and 40 N at 190°, the angles being measured anticlockwise from the positive x direction. Find the resultant analytically. Draw a polygon of forces to check your solution graphically.

2.5 Find the resultant of two concurrent forces P and Q. P has a magnitude of 80 N and acts along a line which makes an angle of 20° with the x direction. The magnitude of Q is 30 N and this force acts along a line which makes an angle of 155° with the x direction. Angles are measured anticlockwise from the positive x direction.

2.6 A boat is towing two water skiers as shown in Figure P2.6. The tow ropes are attached to the boat at A. The tension in AB is 500 N and the tension in AC is 420 N. Find the resultant force acting on the boat and its direction relative to the centreline of the boat.

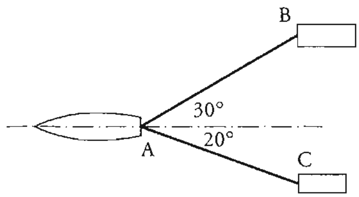


Figure P2.6

2.7 Find the resultant (magnitude and direction) of the systems of forces shown in Figure P2.7.

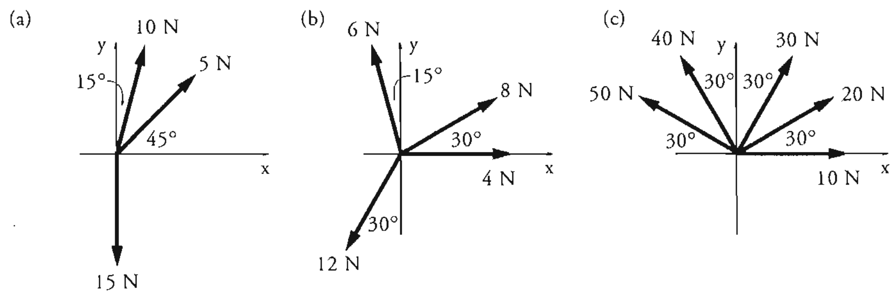


Figure P2.7

2.8 Find the resultant of the system of concurrent forces shown in Figure P2.8. Give the magnitude and direction relative to the x axis.

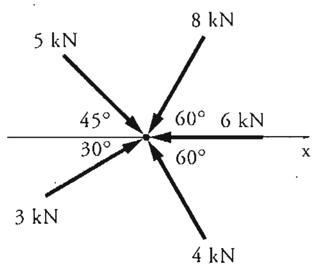


Figure P2.8

2.9 For the force systems shown in Figure P2.9a to c:

(i) Find the value of force A and the direction of C, if C is the resultant of the remaining forces.

(ii) Find the value of A and the direction of C, if A is the resultant of the remaining forces.

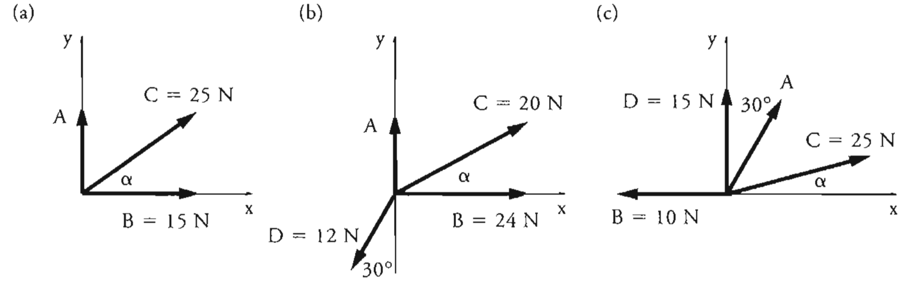


Figure P2.9

2.10 Two men are attempting to lift a crate by means of two ropes attached to a ring (Figure P2.10). They need to exert a vertically upward resultant force of 400 N. If one rope has a tension of 300 N at an angle of 45°, while the other has tension P at an angle θ what must be the values of P and θ?

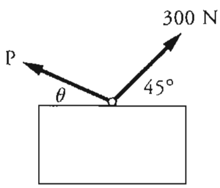


Figure P2.10

2.11 In order to support the 250 kg mass, the resultant of the forces in cables AC and BC must be directed vertically upward (Figure P2.11). Find the forces in each of the cables.

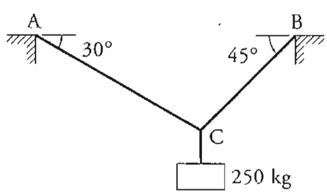


Figure P2.11

2.12 Two forces P and Q act through a point, the angle between their lines of action being θ. Show that the magnitude of the resultant is:

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

and that its line of action is inclined at the angle α to the direction of force P where:

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

2.13 Forces of 50 N and 60 N act through the point O, as shown in Figure P2.13. Find their resultant in magnitude and direction. Calculate the components of the resultant in the directions Ou and Ov. Check the answers by determining separately the components of the 50 N force and the 60 N force in the directions Ou and Ov and adding the results.

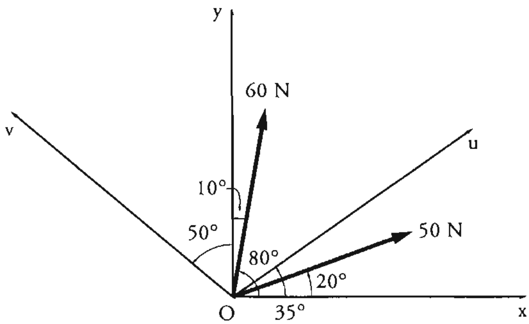


Figure P2.13

2.14 A force of 4 N is the resultant of two concurrent forces, one of 5 N and the other of 7 N. What is the angle between these two forces?

2.15 Three concurrent forces act in the directions shown in Figure P2.15. If the magnitude of the resultant is 14.56 kN, find the angles α and β.

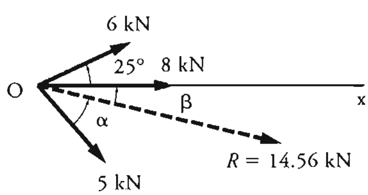


Figure P2.15

2.16 Three forces act through the point O in the directions shown in Figure P2.16. The magnitudes of the forces are 15 kN, 10 kN and P. The resultant R acts at an angle of 10° to Ox. Find the magnitudes of P and R.

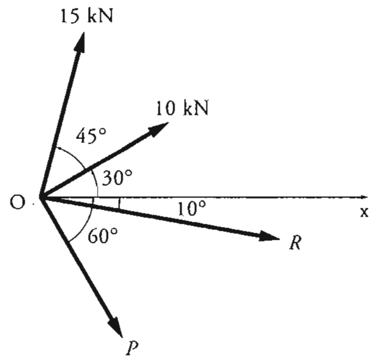
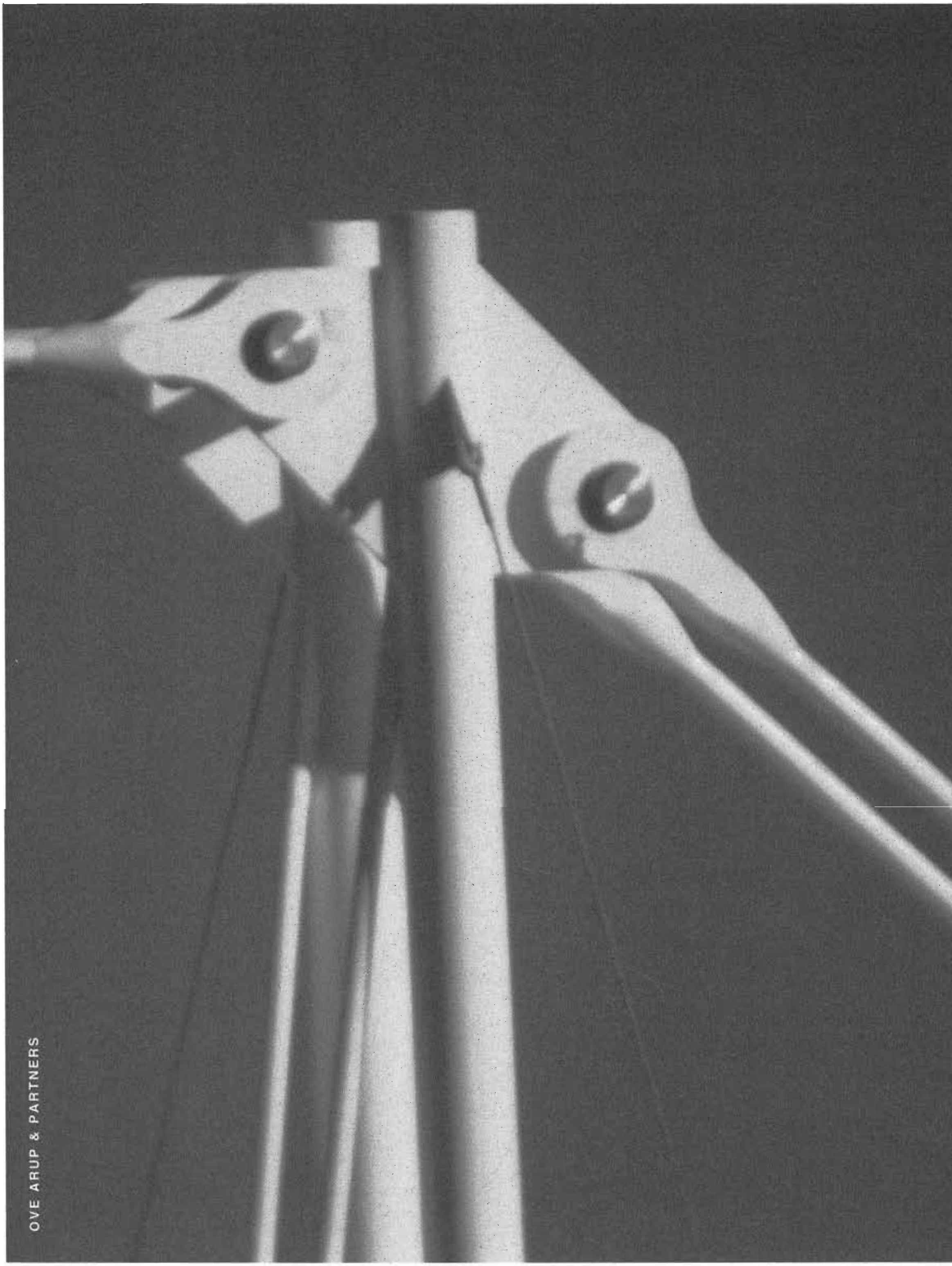


Figure P2.16


OVE ARUP & PARTNERS

Equilibrium of Concurrent Forces

3.1 Equilibrium

If a body is acted upon by a system of forces which has a resultant, this resultant produces a change in the state of motion in accordance with Newton's Second Law.

When the resultant of a group of *concurrent forces* is zero, the motion remains unchanged and the body is in *equilibrium.* The system of forces is also said to be in equilibrium. Conversely, any body which remains at rest (as a large number of engineering structures do) must be acted upon by a system of forces having a zero resultant.

3.2 Conditions of equilibrium

If the resultant of a system is zero, the forces, when added vectorially (as in Figure 2.9b, page 10), must form a closed polygon (i.e. the last point of the force polygon must coincide with the initial point). When all the forces but one are known, the magnitude and direction of the unknown force may be found if the system is known to be in equilibrium, as the unknown force will be represented by the vector required to close the force polygon.

Example 3.1

Find the magnitude and direction of the force Q if the concurrent system of Figure 3.1 is in equilibrium.

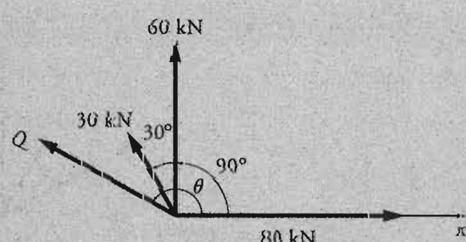


Figure 3.1

SOLUTION

This problem may be solved graphically in the following manner. Combine the known forces 80 kN, 60 kN and 30 kN vectorially by drawing the three sides of the force polygon ABCD in Figure 3.2, in which AB, BC and CD are parallel to the directions of the known forces respectively and the lengths of the sides are proportional to the forces.

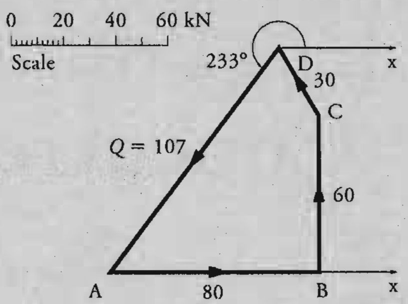


Figure 3.2

Since, for equilibrium of the four forces, the polygon must close, the final force Q must be represented by the line DA. Then by measurement:

$$Q = 107 \text{ kN} \quad \text{and} \quad \theta = +233°$$

The force which must be added to a given system to produce equilibrium is called the *equilibrant*. Thus Q is the equilibrant of the three known forces in Example 3.1. The equilibrant is equal and opposite to the resultant. For any group of concurrent forces in equilibrium, any one force is the equilibrant of the others.

Analytically, the resultant of a system of concurrent forces is defined by Equation 2.4 and Equation 2.5. For the resultant to be zero, both its x and y components must be zero (i.e. $R_x = R_y = 0$). Hence the conditions of equilibrium are:

$$\Sigma F_x = 0 \tag{3.1}$$

$$\Sigma F_y = 0 \tag{3.2}$$

The directions x and y need not be at right angles but they must be different.

EXAMPLE 3.2

Re-solve Example 3.1 analytically.

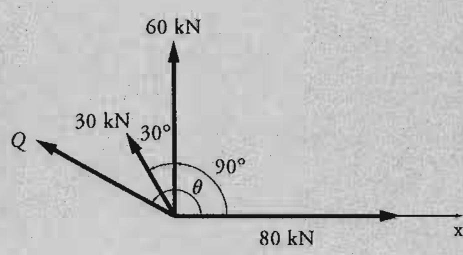


Figure 3.1

SOLUTION

Force (kN)	θ	$F_x = F\cos\theta$	$F_y = F\sin\theta$
80	0°	+80.00	0
60	90°	0	+60.00
30	120°	−15.00	+25.98
Q	θ	$Q\cos\theta$	$Q\sin\theta$
Summations		$Q\cos\theta$ +65.00	$Q\sin\theta$ +85.98

For equilibrium:

$$Q\cos\theta + 65.00 = 0 \quad \text{or} \quad Q\cos\theta = -65.0$$

$$Q\sin\theta + 85.98 = 0 \quad \text{or} \quad Q\sin\theta = -85.98$$

Dividing the second equation by the first to eliminate Q yields:

$$\tan\theta = \frac{-85.98}{-65.98} \quad \text{and} \quad \theta = 52.9^\circ \text{ or } 232.9^\circ$$

Since both the vertical and horizontal components of Q are negative, Q is directed in the third quadrant and therefore $\theta = 232.9°$. Substituting θ into either equation gives $Q = 107.8$ kN.

EXAMPLE 3.3

The members of a truss exert forces on one of the joints as shown in Figure 3.3. Find the two unknown forces P and Q.

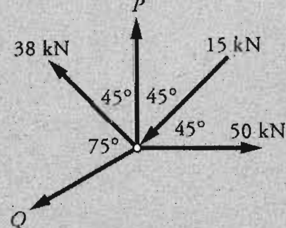


Figure 3.3

SOLUTION

Outward forces are considered to be positive. Hence, the unknown forces P and Q are taken as acting outward at the angles shown. The given force of 15 kN will be taken as negative at an angle of 45°, although it could equally well be taken as +15 kN at 225°.

Force (kN)	θ	$F_x = F \cos \theta$	$F_y = F \sin \theta$
+50	0°	+50	0
−15	45°	−10.61	−10.61
P	90°	0	$+P$
+38	135°	−26.87	+26.87
Q	210°	$-0.886Q$	$-0.5Q$
Summations		12.52 $-0.866Q$	16.26 $+P$ $-0.5Q$

For equilibrium:

$$-0.866Q + 12.52 = 0 \qquad \therefore Q = 14.46 \text{ kN}$$

$$P - 0.500Q + 16.26 = 0 \qquad \therefore P = -9.03 \text{ kN}$$

(i.e. Q acts outward as shown in Figure 3.3, and P acts inward).

It is often advantageous to resolve in directions perpendicular to P and Q, rather than in directions Ox and Oy, in order to obtain two equations each containing only one unknown.

EXAMPLE 3.4

Calculate the forces in the cables AC and BC supporting the 200 kN weight shown in Figure 3.4a.

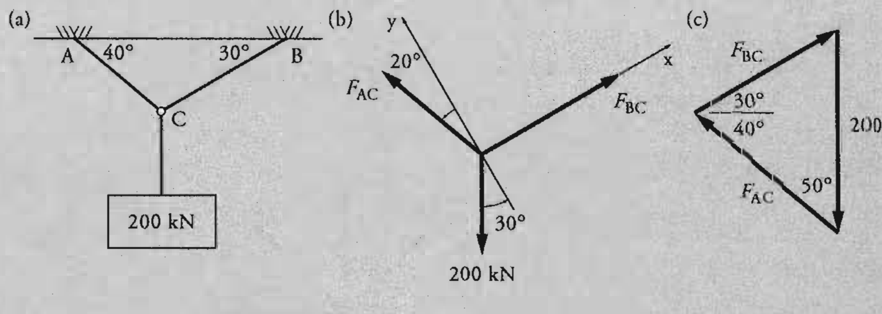


Figure 3.4

SOLUTION

The forces acting on the ring at C are shown in Figure 3.4b. The sum of the forces in the y direction (perpendicular to BC) is zero. Thus:

$$F_{AC} \cos 20 = 200 \cos 30 \qquad \therefore F_{AC} = 184.3 \text{ kN}$$

The sum of the forces in the x direction (along BC) is also zero. Thus:

$$F_{BC} - F_{AC} \sin 20 - 200 \sin 30 = 0 \qquad \therefore F_{BC} = 163.0 \text{ kN}$$

Alternatively the forces F_{AC} and F_{BC} could be determined using the sine rule from the geometry of the force triangle in Figure 3.4c.

The geometry of the problem is specified in Example 3.4. In reality, the geometry may be affected by the deformation of the cables under load. In fact, any structure will deform under load. The calculation of such deformations involves a knowledge of the physical characteristics of the materials from which the structure is made and is beyond the scope of this book. In the remainder of the book, it will be assumed that deformations under load are negligibly small (and in fact they often are) and the solution of problems will be based on the geometry of the unloaded structure.

3.3 Bodies, actions and reactions

We have been concerned up to now with the processes of combining forces into a resultant or of separating a force into components. The forces involved have been clearly specified by means of a diagram. In practical situations the forces are not usually so clearly specified and the first task of the analyst is to identify the forces which must be considered before the foregoing processes can be applied.

Forces occur as a result of the interaction of two bodies. The term *body* is used to specify any material object, or even any arbitrarily chosen grouping of matter. For the time being we shall use the term *body* to mean any easily recognisable object such as a block of wood, a ladder, a bridge, a roof truss and so on.

A commonly occurring force is the *weight* of a body, which was defined in Section 1.5 as the gravitational force acting mutually between the body and the earth. To say that the weight of a table is 400 N is to indicate that the earth pulls the table with a force of 400 N and also that the table pulls the earth with the same force. When two bodies are in contact they exert a force on one another at the contact face. Often this force is a result of the weights of these and other bodies.

Figure 3.5a shows a block of weight W resting on a horizontal floor. By virtue of its weight it exerts a force on the floor and the floor exerts an equal and opposite force on the block. These interactive forces R are shown in Figure 3.5b and since the block is in equilibrium it is clear that R must be equal to W. The floor exerts just enough force to hold the block up. The force exerted by the floor on the block is a *passive* force. For this reason this force is often called the *reaction,* while the force exerted by the block on the floor is called the *action.* According to Newton's Third Law, action and reaction are equal and opposite.

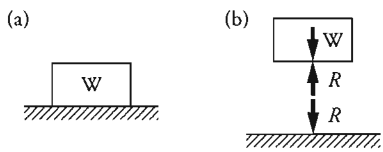


Figure 3.5

However, the terms action and reaction are often interchangeable: it makes little difference to the solution of problems which is called action and which is called reaction. In Figure 3.6a, two planks of wood of weight W_1 and W_2 are resting on a floor and are leaning against one another in such a way that they are both in equilibrium. At the inrerface B they clearly exert mutual forces on one another. If the interface B is smooth and vertical, these forces are horizontal and are called X in Figure 3.6b. It is immaterial which of these is called the action and which is called the reaction. In order to maintain equilibrium of plank AB the floor must supply an upward reaction R_1 equal to W_1 and also a horizontal reaction R_2

equal to X. The force R_1 is called the *normal reaction* (i.e. at right angles to the floor) and R_2 is a *frictional force.* (This will be discussed in the next section.) The plank AB of course exerts equal and opposite actions R_1 and R_2 on the floor.

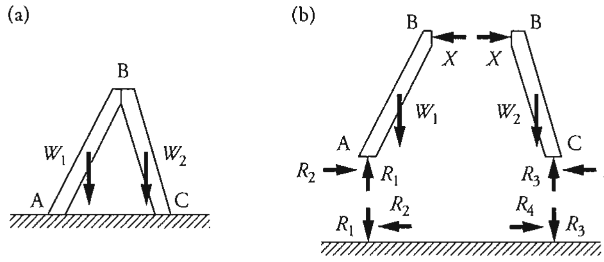


Figure 3.6

In the application of the principles of statics to any practical problem it is essential first to be clear about two things: what is the body whose equilibrium is being discussed, and what are the forces acting on this body?

If it is desired to evaluate the force exerted between the two planks at the interface B in Figure 3.6a, then it is convenient to consider the equilibrium of one of the planks, say AB. It is advisable to draw this plank separately so the force can be shown acting on it. This is often called a *freebody diagram.* (Freebodies are discussed further in Chapter 6.) Freebody diagrams for both planks AB and BC are shown in Figure 3.6b.

The forces on the chosen body will usually comprise the weight of the body and forces which are applied to it by any other bodies which are in contact with it. In the present example, there is the weight W_1, the force X exerted by the other plank and the force R exerted by the floor. The last can be expressed either by the components R_1 and R_2 or by a single inclined force R, whichever is more convenient. Once the freebody is drawn and the forces acting on it are clearly identified there is usually little difficulty in applying the principles of statics to those forces.

It should be noted that the term *body* applies to liquids and gases as well as to solids. One of the forces acting on a piston in an engine will be the force exerted by the gas in the cylinder. One of the forces exerted on a dam is that of the water retained by the dam.

3.4 Friction

When a force or a system of forces is applied to a body in such a way that it tends to cause the body to slide on another surface, a force known as *friction* is called into play at the interface. For instance, if a heavy box is resting on the floor and we attempt to slide it along the floor by applying a moderate force to it, we may find that the box does not in fact move. A frictional force has been evoked sufficient to resist the force which we have applied. If we attempt to move the box in the other direction friction may still defeat us. The force of friction always acts in a direction to oppose motion. However, if we increase our efforts sufficiently the box will eventually move. Evidently there is a limit to the frictional force.

Friction is caused by surface roughness. Even an apparently smooth surface will reveal irregularities under a microscope, and if two such surfaces are in contact the interpenetration of the irregularities tends to resist relative motion and is the cause of the force which we call friction. If either surface is made smoother then the friction force will decrease. The introduction of a layer of oil between the two surfaces may prevent contact and may almost eliminate friction. Although friction is never in reality quite zero it is sometimes small enough to be neglected. A surface is called *smooth* in mechanics if it generates negligible frictional forces.

It can be shown experimentally that the limiting, or maximum, frictional force that can be generated at the interface of two bodies is proportional to the normal force acting at the interface. If this normal force is N and the maximum possible friction is F, then:

$$\frac{F}{N} = \text{constant} = \mu \qquad \textbf{(3.3)}$$

and μ is called the *coefficient of friction.* The force F does not depend on the area of the contact surface.

Figure 3.7a shows a box of weight W resting on a table. It is being pulled by a rope with a force P at an angle θ to the horizontal. The forces acting on the box are shown in the freebody diagram Figure 3.7b.

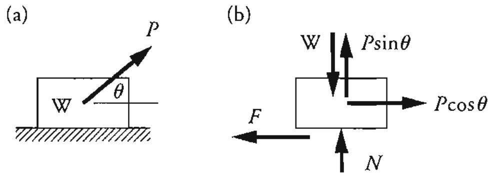


Figure 3.7

Although the forces are not strictly concurrent, since F acts at the bottom of the box and the other forces act at the centre, they will be considered as concurrent in this chapter. The total downward force on the table is $W - P\sin\theta$, and assuming that W exceeds $P\sin\theta$ the table supplies an upward reaction N equal to $W - P\sin\theta$. The vertical components of force are therefore balanced, and the *normal* force at the surface is:

$$N = W - P\sin\theta \qquad \textbf{(3.4)}$$

The maximum friction available is therefore:

$$F_{max} = \mu N = \mu (W - P\sin\theta) \qquad \textbf{(3.5)}$$

If $P\cos\theta$ is less than F_{max}, then the force F will also be less than F_{max} since friction is a passive force. It will be just sufficient to balance $P\cos\theta$ and the box will not move. If $P\cos\theta$ exceeds F_{max} the resultant force will be $P\cos\theta - F_{max}$ and the box will move.

Suppose the box weighs 80 N and the coefficient of friction is 0.4. Consider first the box is pulled with a force of 20 N at $\theta = 30°$ (Figure 3.8a). Figure 3.8b shows the forces on the box expressed in terms of components. The reaction from the table is 70 N and the maximum friction available is therefore 0.4 × 70 N or 28 N. Since the horizontal component of the rope force is only 17.32 N, the actual friction force just balances this and the box is in equilibrium.

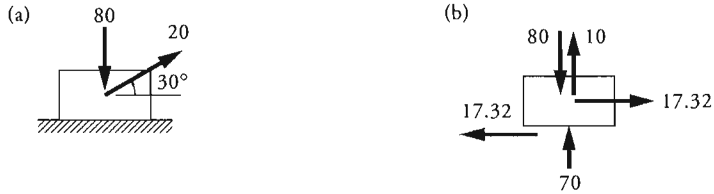


Figure 3.8

In Figure 3.9a, the same box as in Figure 3.8 is pulled with a force of 40 N at $\theta = 30°$. The force components are shown in the freebody diagram Figure 3.9b. Note that the increase in the inclined force has decreased the table reaction and consequently the maximum friction available is now only $0.4 \times 60 = 24$ N. The horizontal component of the 40 N force exceeds this and the box will move in this case: it is not in equilibrium.

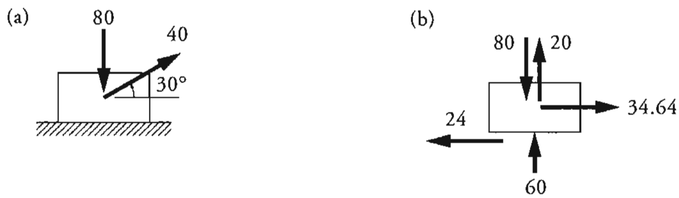


Figure 3.9

For any given angle θ it would be a simple matter to determine the value of P for which the value of $P \cos \theta$ is just equal to the maximum friction force. For this value we say that the box is on the point of moving. Actually, once the body starts to move, the friction force decreases slightly.

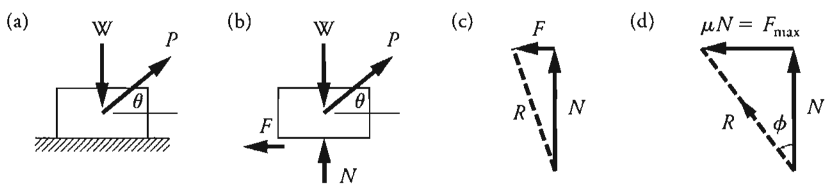


Figure 3.10

It is sometimes convenient to combine the normal reaction N and the friction force F into a resultant inclined reaction R. Figure 3.10a shows a box on a table as before. Figure 3.10b is a freebody diagram showing all forces acting on the box. For a small value of the force P, the friction force called into play is less than F_{max} and the body does not move. The resultant reaction R is shown in Figure 3.10c. As P increases the ratio of F to N increases. When the box is on the point of moving $F = F_{max} = \mu N$ (Figure 3.10d). The force R is then inclined at the maximum possible angle to N. This angle is called the *angle of friction* ϕ, and:

$$\mu = \tan \phi \tag{3.6}$$

In the case of a *smooth* plane, no friction is present and the reaction is always normal to the plane.

A block rests on a rough plane inclined at α to the horizontal as shown in Figure 3.11a. It is acted upon by its own weight W, the normal reaction from the plane N and a friction force F shown in the freebody diagram Figure 3.11b. Since in this case the block tends to slide down the plane, F will act up the plane. Figure 3.11c shows the force triangle which relates these forces provided the block does not slide. Clearly $N = W\cos\alpha$ and $F = W\sin\alpha$. The resultant reaction R is inclined at α to N. When the slope of the plane is increased until the block is on the point of sliding, then $F = \mu N$ and α is equal to the angle of friction ϕ.

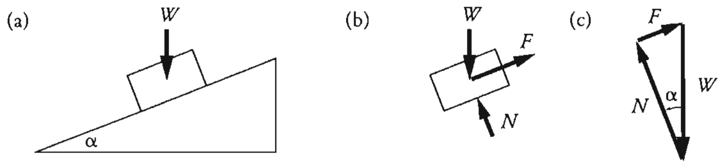


Figure 3.11

If a box is being pulled up an inclined plane by means of a rope (Figure 3.12a) the equilibrium may be examined by resolving forces parallel to and normal to the plane (Figure 3.12b). The friction F will act down the plane when $P\cos\theta$ exceeds $W\sin\alpha$, but if $W\sin\alpha > P\cos\theta$ then F will act up the plane.

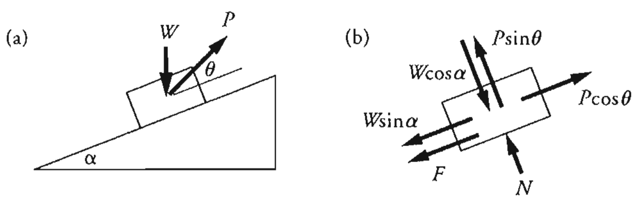


Figure 3.12

EXAMPLE 3.4

Find the range of values for which the horizontal force P will prevent the 2.0 kN box from slipping down or moving up the inclined plane in Figure 3.13. Assume the coefficient of friction between the box and the inclined surface is $\mu = 0.15$.

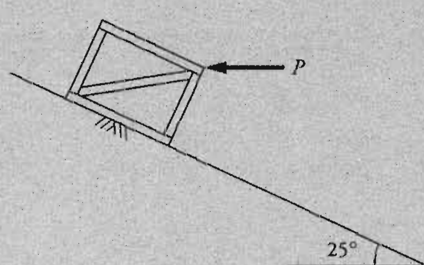


Figure 3.13

SOLUTION

When the box is just about to move up the incline, the force P is at its maximum value within the required range. The freebody diagram of the box is shown in Figure 3.14a.

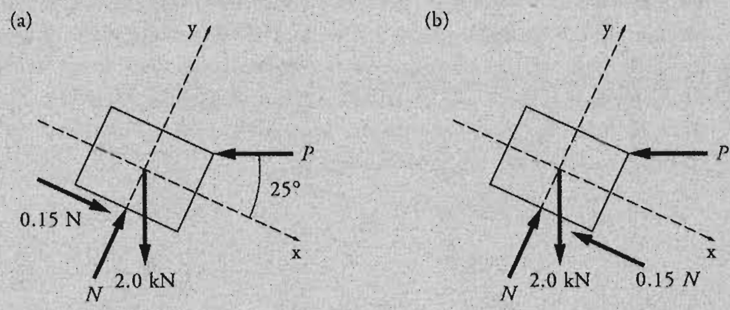


Figure 3.14

As the box has not yet begun to move, the forces shown in Figure 3.14a are in equilibrium, with the frictional force acting down the plane at its maximum value μN. The sums of the forces parallel to and normal to the incline are zero.

$$\Sigma F_x = 0: \quad 0.15N + 2.0 \sin 25 - P \cos 25 = 0$$

$$0.15N - 0.906P = -0.845$$

$$\Sigma F_y = 0: \quad N - 2.0 \cos 25 - P \sin 25 = 0$$

$$N - 0.423P = 1.813$$

Solving these simultaneous equations gives:

$$N = 2.373 \text{ kN} \quad \text{and} \quad P = 1.325 \text{ kN}$$

$$\therefore P_{max} = 1.325 \text{ kN}$$

When the box is just about to slip down the incline, the force P is at its minimum value within the required range and the corresponding freebody diagram is shown in Figure 3.14b. The equilibrium equations are now:

$$\Sigma F_x = 0: \quad -0.15N + 2.0 \sin 25 - P \cos 25 = 0$$

$$-0.15N - 0.906P = -0.845$$

$$\Sigma F_y = 0: \quad N - 2.0 \cos 25 - P \sin 25 = 0$$

$$N - 0.423P = 1.813$$

Solving gives:

$$N = 2.062 \text{ kN} \quad \text{and} \quad P = 0.591 \text{ kN} = P_{min}$$

Therefore the required range is:

$$0.591 \text{ kN} \leq P \leq 1.325 \text{ kN}$$

3.5 Equilibrium of three forces

If three forces acting in the same plane (coplanar) are in equilibrium they must be concurrent. If two of the forces intersect at a point A, then they have a resultant which also passes through A. For equilibrium the third force must be equal and opposite to this resultant and must act in the same line. Consequently the third force also passes through A. It follows that three non-coplanar forces cannot be in equilibrium.

Thus three forces in equilibrium must be coplanar and concurrent. (Note: The forces may be parallel, i.e. concurrent at infinity.) Some problems are simplified by recognising this fact.

Also peculiar to the case of three coplanar forces is the fact that any triangle, whose sides are parallel to the forces taken in order, will serve as a force polygon to some scale. This is because all such triangles are geometrically *similar* and have their sides in the same proportion. This cannot be said of polygons with more than three sides.

Example 3.5

A ladder 6 m long and weighing 220 N rests against a smooth (i.e. frictionless) wall at an angle of 30° to the vertical (Figure 3.15a). Find the reactions at the wall at A and the floor at B. In addition, what is the minimum coefficient of friction between the ladder and the floor such that the ladder will not slip?

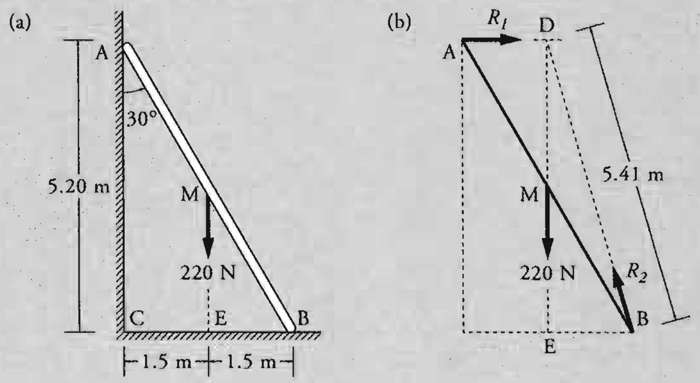


Figure 3.15

Solution

The forces acting on the ladder AB which maintain it in equilibrium are the weight of the ladder, the horizontal reaction R_1 exerted by the wall and the reaction R_2 exerted by the floor. These forces are shown in the freebody diagram Figure 3.15b.

Since only three forces are involved, they must be concurrent. The weight, acting through the mid-point M of the ladder, intersects the reaction R_1, which must be normal to the wall, at D as shown. Therefore R_2 also passes through D.

The sides of the triangle DEB are parallel to the three forces, and therefore proportional to those forces. After calculating the dimensions of triangle DEB in Figure 3.15 we have:

$$\frac{5.20}{220} = \frac{1.5}{R_1} = \frac{5.41}{R_2}$$

From which: $R_1 = 63.5$ N and $R_2 = 228.9$ N

The reaction R_2 has a vertical component V equal to the weight of the ladder, and a horizontal component equal to μV (which is equal to R_1). From the diagram:

$$\frac{\mu V}{V} = \frac{1.5}{5.20} = 0.288$$

Hence the minimum coefficient of friction required is 0.288.

Another example where this theorem may be used is shown in Figure 3.16a. Here, a box of weight W rests against a frictionless wall and is supported by a rope AB. Figure 3.16b shows a freebody diagram of the box acted upon by the weight W, the reaction from the wall R and the tension in the rope T. For equilibrium, these three forces must be concurrent. This will determine the equilibrium position, and the box will slide until this position is attained. It may be noted that the triangle ACD is a triangle of forces for the forces on the box.

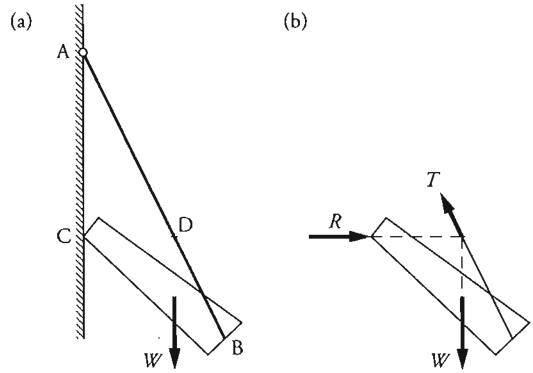


Figure 3.16

Problems

3.1 If the forces shown in Figure P3.1 are in equilibrium, find the magnitude and direction of P.

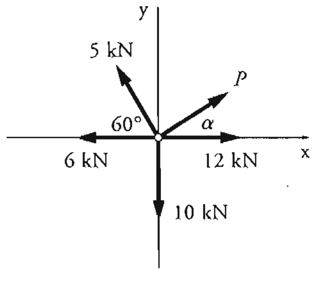


Figure P3.1

3.2 The forces shown in Figure P3.2 are in equilibrium. Calculate the magnitude of the two forces P and Q. (Resolve at right angles to P and Q in turn.)

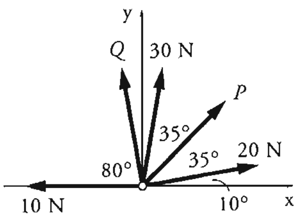


Figure P3.2

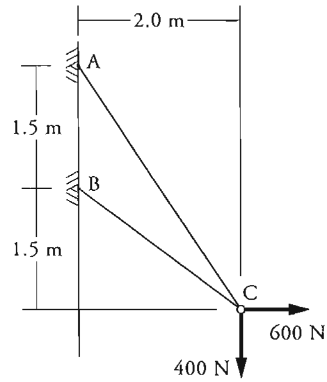


Figure P3.3

3.3 Determine the force in cables AC and BC in Figure P3.3, if the ring at C is in equilibrium.

3.4 Forces of 100 N and 50 N act along the x axis and at +45° respectively. Find the directions of two forces 120 N and 60 N, such that the four forces are in equilibrium. (Give the two possible solutions.)

3.5 A block weighing 10 N rests on a plane inclined at 20° (Figure P3.5). The coefficient of friction μ, between the block and the plane is 0.5. R is the reaction normal to the plane. Figure P3.5 is a freebody diagram of the block. What is the horizontal force H, which is just sufficient to move the block up the plane with uniform motion?

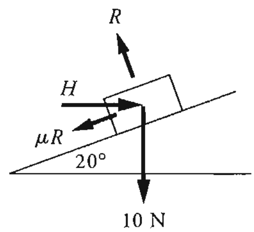


Figure P3.5

3.6 ABC is a roof truss spanning 20 m (Figure P3.6). It is supported at A so that the reaction, or supporting force, at A may act in any direction. The support at C is such that the reaction here must be vertically upward or downward. If a force of 20 kN is applied at the apex of the truss in a direction of 20° to the vertical, find the value of the reactions at A and C.

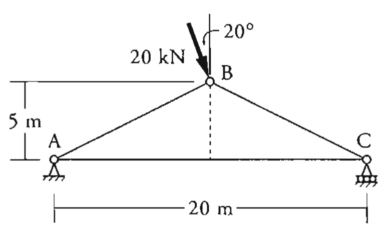


Figure P3.6

3.7 Figure P3.7 shows a reciprocating mechanism. When the crank is in the position shown the piston instantaneously has no acceleration (its velocity is a maximum). Consequently the forces acting upon it are in equilibrium. At this moment the force on the face of the piston is 1 kN and the force exerted by the connecting rod is in the direction of BC. Draw a freebody diagram of the piston, showing the forces acting on it, and hence find the force in the connecting rod and the force exerted by the cylinder walls on the piston.

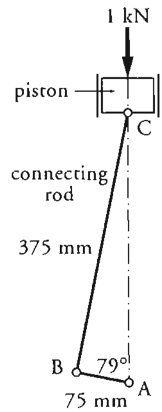


Figure P3.7

3.8 A heavy uniform bar AB of weight W is suspended from a hinge support at a point A above a smooth inclined plane, as shown in Figure P3.8. The lower end of the bar B rests on the inclined plane. Find:

(i) the force exerted by the bar on the inclined plane

(ii) the reaction of the hinge A if the rod is inclined at 30° to the vertical and the inclined plane at 30° to the horizontal.

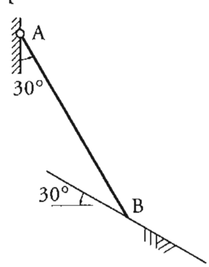


Figure P3.8

3.9 The inclined plane in Problem 3.8 forms the upper surface of a block of negligible weight which rests on a horizontal table. When the bar AB is resting on the inclined plane the block is on the point of sliding on the horizontal table. What is the coefficient of friction between the block and the table?

3.10 Figure P3.10 is a freebody diagram of a block of weight 20 N being pulled up an inclined plane by a force F which makes an angle of 5° with the plane. If the coefficient of friction between the block and the plane is 0.8, find the value of F which is just sufficient to move the block.

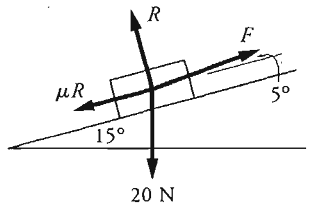


Figure P3.10

3.11 A load of 100 kN is supported by a crane as shown in Figure P3.11. DAE is the cable which passes over a smooth pulley at A. Draw a freebody diagram of the pulley A and hence find the forces in the cable, the tie AC and the jib AB of the crane. (Assume that the diameter of the pulley and the weight of the jib are both negligible.)

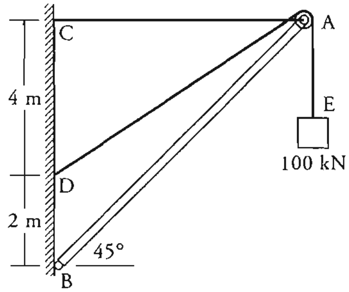


Figure P3.11

3.12 In Figure P3.12, a uniform bar AB is 600 mm long and weighs 40 N. The end A rests against a smooth wall and the end B is supported by a 1 m long rope BC which is fixed to the wall at C. Find the inclination of the bar to the vertical when it is in equilibrium (other than hanging straight down) and the tension in the rope.

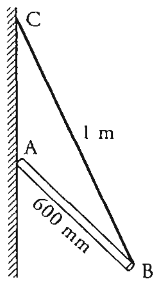


Figure P3.12

3.13 A cylinder weighing 2 kN is supported by two smooth walls. Determine the normal forces exerted by the walls on the cylinder at A and B.

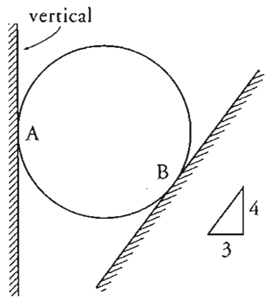


Figure P3.13

3.14 A rod AB weighing 10 kN is supported by a pin at A and a cable BD. The centre of gravity of the rod is at its midpoint C. Determine the horizontal and vertical components of the pin force at A and the tension in the cable.

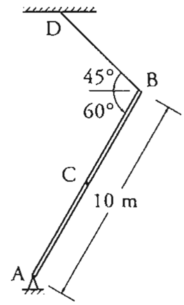


Figure P3.14

3.15 The system of forces shown in Figure P.3.15 is in equilibrium. Find the unknown forces P and Q by considering the components in the x and y directions.

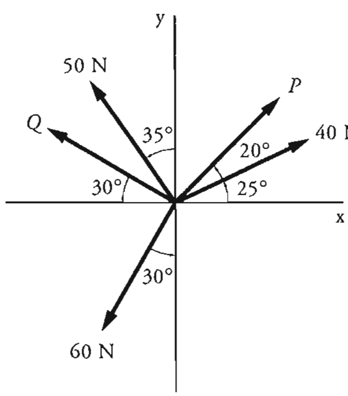


Figure P3.15

3.16 The plate ABC in Figure P3.16 is in the shape of an equilateral triangle whose sides are 1 m in length. It is supported on a frictionless peg at A, while the corner C rests on the smooth plane CD. A force of 10 N is applied at B in the direction shown. Find the reactions at A and C. (The weight of the plate is negligible.)

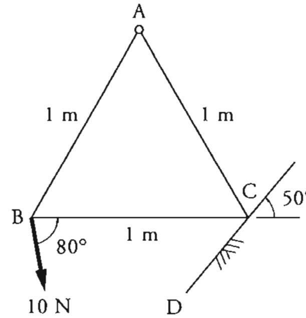


Figure P3.16

3.17* The rectangular plate ABCD is in equilibrium under the action of the forces shown in Figure P3.17. Find the force P and angles θ and ϕ.

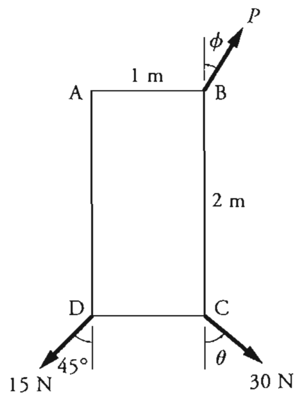


Figure P3.17

3.18* A ladder 6 m long rests against a vertical wall so that it is inclined at 60° to the horizontal floor on which it rests. The ladder weighs 500 N. What is the maximum height at which a man weighing 1200 N may stand on the ladder without it slipping if the coefficient of friction between the ladder and wall and ladder and floor is 0.4?

3.19* A uniform beam AB of length 6 m and weight W, resting with the end A on a rough horizontal plane and with a point C bearing against a smooth fixed horizontal rail, is just on the point of sliding down. The distance AC is 4.5 m. The coefficient of friction between the beam and the plane is 0.5.

(i) What is the angle of inclination of the beam to the horizontal?

(ii) What horizontal force applied at A would cause the beam to slide up?

3.20* A lever of length 2.1 m is used to move a smooth cylinder of 700 mm diameter and weighing 100 N up a plane inclined to the horizontal at an angle of 30°. What force must be exerted on the end of the lever if the lever is inclined at 60° to the inclined plane?

3.21* The bar BC in Figure P3.21 is supported by two rigid bars AB and AC connected to BC through frictionless hinges. AB and AC are connected through a frictionless hinge to a support at A. Forces act on BC as shown in the diagram. Find the unknown load P and the reaction at A for equilibrium.

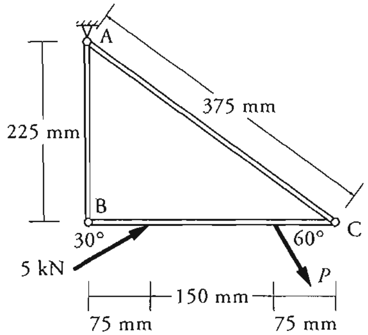


Figure P3.21

3.22* The block B in Figure P3.22 weighs 500 N and rests on a plane inclined at 25° to the horizontal. The coefficient of friction between the plane and block B is 0.15. On top of this block is another block A, which weighs 1500 N. The coefficient of friction between A and B is 0.4. Block A is pulled with a force F which makes an angle α with the plane. If α is greater than a certain critical angle the block A will slide relative to B provided F is large enough. However, if α is less than this critical angle, then a suitable force F will cause both blocks to move together up the plane. What is the critical value of α, and what force F applied at this angle α will just cause the blocks to move?

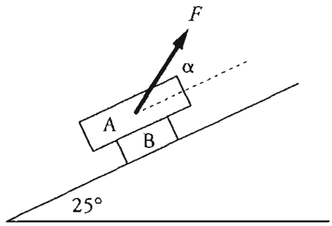


Figure P3.22

3.23* The bent lever in Figure P3.23 has its arms at 90° and is pivoted at C. AC is 375 mm and BC is 150 mm. A force P of 150 N is applied at A at 15° to the horizontal and another force Q is applied at B at 20° to the vertical. Find the magnitude of Q and the magnitude and direction of the reaction at C, if the lever is in equilibrium.

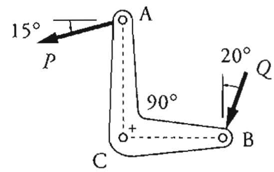


Figure P3.23

3.24* A bar AB, which is acted upon by two forces of $50\sqrt{3}$ N and 100 N in the directions shown in Figure P3.24, is supported by ropes which are attached to a peg C. The total length of the rope is 6 m. Find the lengths AC and CB for the bar to be in equilibrium. What are the tensions in the ropes?

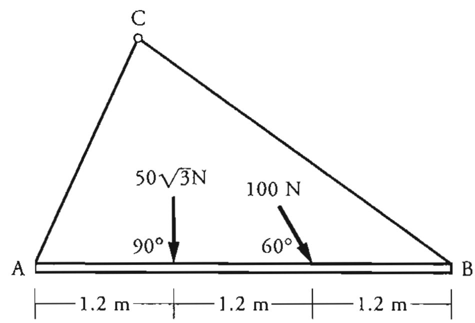


Figure P3.24

3.25* A square plate of 10 N weight is in equilibrium in a vertical plane perpendicular to a smooth vertical wall with one corner of the plate in contact with the wall. An adjacent corner of the plate is attached to a point in the wall by a string whose length is equal to the side of the square. Find the angle of inclination of the string and its tension.

* Difficult problems, suitable for later study.

Non-concurrent Forces

4.1 Moments

If a force acts on a body, then the *moment* of the force about any point in its plane is defined as the product of the force and the perpendicular distance of the point from the line of action of the force.

The moment of the force F about the point O in Figure 4.1 is:

$$(M_F)_O = Fd \qquad \textbf{(4.1)}$$

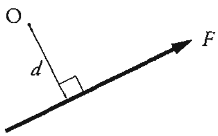


Figure 4.1

The moment M_F may be thought of as a measure of the tendency of the force F to cause rotation about an imaginary axis through the point O and perpendicular to the plane containing the force and the point. If a body is pivoted at O, the force F acting on the body will cause rotation about O, in the absence of any other constraints. However, the moment of a force can be calculated about any point and not just the points about which the body can physically rotate.

The moment of a force F about a point O is the same as the sum of the moments of the components of F about the point O. Consider the force F shown in Figure 4.2a.

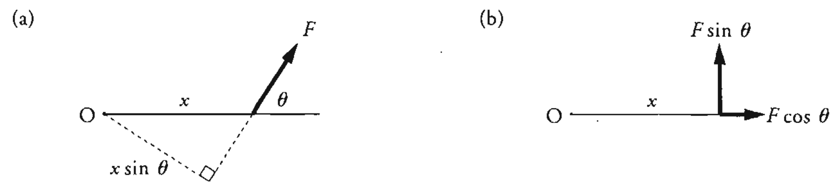


Figure 4.2

The moment of F about the point O is:

$$(M_F)_O = Fx \sin\theta$$

In Figure 4.2b, the force F is resolved into its vertical and horizontal components. The sum of the moments of the two components of F about the point O is:

$$(M_F)_O = F\sin\theta\, x + F\cos\theta \times 0 = Fx\sin\theta$$

In this case, the horizontal component of F passes through the point O (i.e. the distance from O to the line of action of the horizontal component is zero) and the moment of the horizontal component about O is zero.

The unit of moment is the *Newton metre* (Nm) with variations Nmm, kNm, etc. according to the choice of units for force and length respectively. The sign (positive or negative) will depend upon the direction of rotation and may be arbitrarily taken as clockwise or anti-clockwise to suit the particular problem.

4.2 Resultant of non-concurrent forces

By definition, a resultant has to produce the same effect (with respect to the motion it causes) as the group of forces it replaces. This requires that it has the same moment, about any point, as the combined moments of the forces in the group.

The magnitude and direction of the resultant of a set of non-concurrent forces are determined in the same manner as for a set of concurrent forces (Section 2.4). This assures equivalence as far as translational motion is concerned. To obtain equivalence of rotation the *position* of the resultant is determined so that its moment about any chosen point is equal to the algebraic sum of the moments of the forces. Fortunately, it can be shown that if this condition is satisfied with respect to one point then it is satisfied with respect to all other points.

Using the same notation as in Section 2.4 and in addition letting $(M_F)_A$ denote the moment of the force F about a point A, the magnitude and direction of the resultant are specified, as before, by its x and y components.

$$R_x = \Sigma F_x = \Sigma F\cos\theta \qquad \textbf{(4.2)}$$

$$R_y = \Sigma F_y = \Sigma F\sin\theta \qquad \textbf{(4.3)}$$

where θ is the angle between the axis Ox and the given force.

In order to satisfy the rotation condition about an arbitrary point A:

$$(M_R)_A = \Sigma(M_F)_A \qquad \textbf{(4.4)}$$

which determines the position of R.

As an alternative to expressing the x and y components of F as $F\cos\theta$ and $F\sin\theta$, where θ is the angle between F and the x axis, it is often convenient to let θ_x and θ_y be the angles between F and the x and y axes respectively. The x and y components of F are then $F\cos\theta_x$ and $F\cos\theta_y$. The terms $\cos\theta_x$ and $\cos\theta_y$ are called the *direction cosines* of the vector F and are commonly denoted by l and m. With this notation, the components are Fl and Fm.

The magnitudes of the direction cosines l and m are given by the projection, on the x and y axes, of a unit vector in the direction of F. Clearly $l^2 + m^2 = 1$. (The Fl, Fm

notation has little advantage over the notation $F \cos \theta$ and $F \sin \theta$ in two-dimensional problems, but the advantage is more marked in three-dimensional problems.) In terms of direction cosines, Equations 4.2, 4.3 and 4.4 can be written:

$$R_x = \Sigma Fl \tag{4.2a}$$

$$R_y = \Sigma Fm \tag{4.3a}$$

$$(M_R)_A = \Sigma(M_F)_A \tag{4.4a}$$

EXAMPLE 4.1

Find the resultant of the four forces applied to the horizontal rigid bar ABCD shown in Figure 4.3.

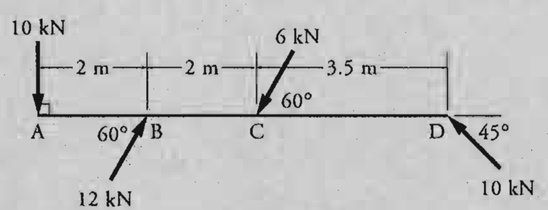


Figure 4.3

SOLUTION

When the positions of the forces are specified by horizontal and vertical measurements, as in this case, it is advisable to break each force into its horizontal and vertical components before taking moments, rather than attempt to draw lever arms perpendicular to the forces as given. The force system of Figure 4.3 is replaced by that of Figure 4.4.

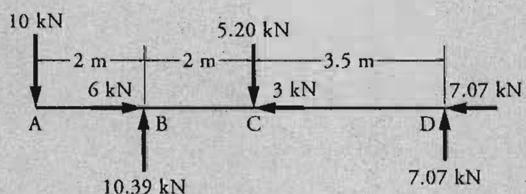


Figure 4.4

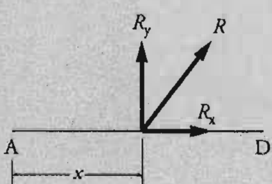


Figure 4..5

Then: $\overset{+}{\rightarrow}$ $R_x = +6 - 3 - 7.07 = -4.07$ kN

$+\uparrow$ $R_y = -10 + 10.39 - 5.20 + 7.07 = 2.26$ kN

Taking moments about A:

$\circlearrowleft+$ $(M_R)_A = \Sigma(M_F)_A = +(10.39 \times 2) - (5.20 \times 4) + (7.07 \times 7.5) = 53.00$ kNm

Note that for those forces passing through A the moment is zero.

The position of the resultant is expressed by stating the point at which it intersects some given line, in this problem the line AD. Let the resultant cut the bar at a point distant x m from A. Replacing the resultant by its components R_x and R_y (Figure 4.5) which are now known, the moment $(M_R)_A$ is equal to $(R_y x)$, since R_x has no moment about A.

Hence: $(R_y x) = 2.26x = 53.00$ and $x = 23.45$ m

Thus: $R_x = -4.07$ kN $R_y = +2.26$ kN and $x = 23.45$ m

If the force F, in Figure 4.6, with components F_x and F_y, acts at the point (x, y), then the moment of the force about the origin is $(F_y x) - (F_x y)$, provided the anticlockwise sense for moments is taken as positive.

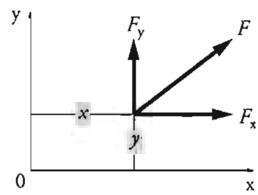


Figure 4.6

EXAMPLE 4.2

Find the resultant of the four forces shown in Figure 4.7. The forces act at points whose co-ordinates are shown.

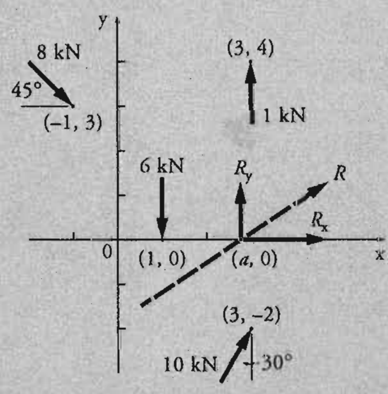


Figure 4.7

SOLUTION

Assume that the resultant cuts the x axis at a distance a (assumed positive) from the origin. Specify the resultant by its x and y components (assumed positive).

Force (kN)	θ	$F_x = Fl$	$F_y = Fm$	Acting at x	Acting at y	Moment about Origin $-F_x y$	Moment about Origin $+F_y x$
8	-45°	+5.657	-5.657	-1	3	-16.971	+5.657
6	-90°	0	-6.000	1	0	0	-6.000
1	+90°	0	+1.000	3	4	0	+3.000
10	+60°	+5.000	+8.660	3	-2	+10.000	+25.981
Summations		+10.657	-1.997			+ 21.667	

$R_x = \Sigma F_x = +10.657$ kN $\qquad R_y = \Sigma F_y = -1.997$ kN

$$\therefore R = \sqrt{10.657^2 + 1.997^2} = 10.84 \text{ kN}$$

$$R_y a = \Sigma(M_F)_O = 21.667 \text{ kNm}$$

$$\therefore a = \frac{+21.667}{-1.997} = -10.85 \text{ m}$$

The diagramatic representation of the resultant is shown in Figure 4.8

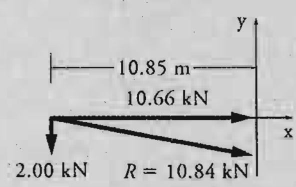


Figure 4.8

In retrospect it can be seen that in the case of a system of forces concurrent at a point A, the sum of the moments about A must be zero. Thus the position equation (Equation 4.4) is automatically satisfied since the resultant also passes through A.

4.3 Parallel forces: Couples

A system of parallel forces is a particular case of non-concurrent forces. Since the directions of all the forces are the same (although the senses might differ) the magnitude of the resultant may be found by algebraic addition. The resultant is parallel to the forces, and its position may be found by equating its moment about any point to the sum of the moments of the forces about the same point.

EXAMPLE 4.3

Find the resultant of the parallel forces of Figure 4.9a.

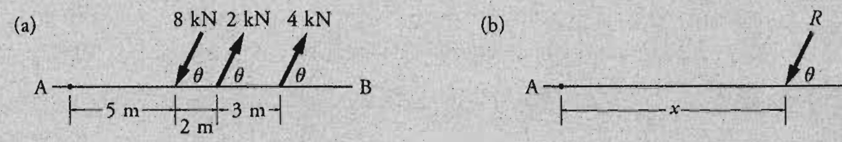


Figure 4.9

SOLUTION

Taking the direction of the 8 kN force as positive, the magnitude of the resultant is:

$$R = +8 - 2 - 4 = +2 \text{ kN}$$

and it acts in the direction θ.

Let R cut the line AB at a point x metres from A, as shown in Figure 4.9b. Taking moments about A:

Moment of resultant = Sum of moments of forces

$$\circlearrowleft + \quad -Rx \sin\theta = -(8 \times 5 \sin\theta) + (2 \times 7 \sin\theta) + (4 \times 10 \sin\theta)$$

$$-Rx = -40 + 14 + 40 = +14$$

$$x = -7 \text{ m}$$

The method breaks down in the case of two parallel forces of equal magnitude but opposite sense (as in Figure 4.10). In this case the magnitude of the resultant is zero (i.e. $F - F$). The moment of the system about such points as B or C is easily seen to be Fd (i.e. non-zero). Such a pair of forces is called a *couple*. It tends to cause rotation without translation. It cannot be replaced by a single force, which must necessarily tend to cause translation as well as rotation. In this context, *translation* means to move from one position to another.

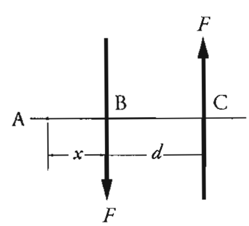


Figure 4.10

The moment of a couple is the same about every point in the plane. If A is any point in Figure 4.10, and ABC is a transversal normal to F, the total moment about A is:

$$M = F(x + d) - Fx = Fd$$

The moment is Fd in an anticlockwise sense and, being independent of x, is evidently the same about every point. The moment Fd is known as the *moment of the couple.*

A couple may be replaced by any other couple having the same moment and the same sense of rotation. The couples shown in Figure 4.11(a), (b) and (c) each have a clockwise moment of 60 kNm and are therefore equivalent. The original couple can also be replaced by any number of couples, the algebraic sum of whose moments is equal to the moment of the given couple.

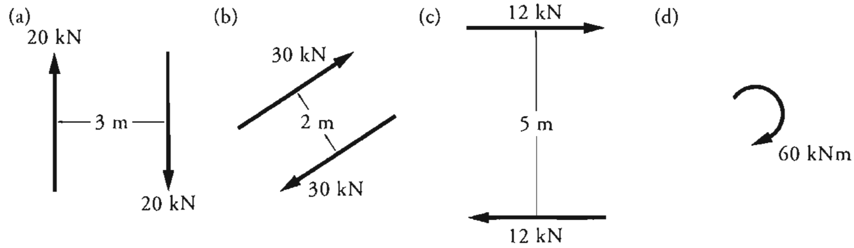


Figure 4.11

A system of forces may be equivalent to a couple. This would be the case if the forces had a resultant of zero magnitude, but yet had a non-zero moment about any point.

Since the actual forces constituting the couple are immaterial it is often represented by a single symbol showing its sense and specifying its magnitude (Figure 4.11d). When this symbol occurs in association with other forces its properties must be remembered:

1. It has no component force in any direction
2. Its moment is the same about all points.

EXAMPLE 4.4

Find the sum of the vertical components of the forces in Figure 4.12 and the total moment of the forces and the couple about A and about D.

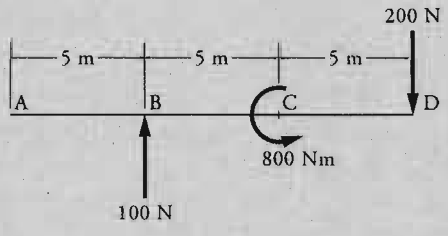


Figure 4.12

SOLUTION

Resolving vertically (and ignoring the couple):

$$+\uparrow\ V_A = +100 - 200 = -100 \text{ N}$$

Taking moments first about A and then about D gives:

$$\circlearrowleft+\ M_A = +(100 \times 5) + (800) - (200 \times 15) = -1700 \text{ Nm}$$

$$\circlearrowleft+\ M_D = -(100 \times 10) + (800) = -200 \text{ Nm}$$

4.4 Distributed forces

In the previous discussions, forces have been represented as line vectors acting at a point. Engineers refer to such forces as *point forces* or *concentrated forces* (or *concentrated loads*). In many engineering problems, forces occur not as point forces but as forces distributed along a length or over an area. For instance, the force exerted by a floor on a supporting beam is distributed along the beam and is not applied at a particular point. Such a force is measured in terms of *force per unit length*, e.g. kN/m. If at a particular location on the beam the floor load is 5 kN/m, this is called the *force intensity*. If this force intensity is constant, then each metre of beam carries 5 kN of load. The load is said to be *uniformly* distributed.

In the real world, forces are always distributed, and the line vectors referred to in statics are their resultants. Sometimes the force intensity and the nature of its variation are well defined, in which case the determination of the magnitude and position of the resultant presents no problem. In other cases, it may be necessary to introduce approximations.

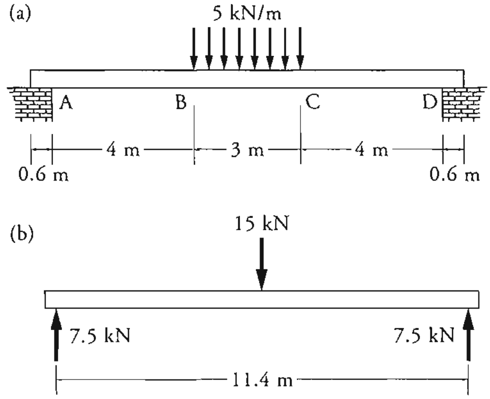


Figure 4.13

Figure 4.13a shows a beam which supports a uniformly distributed load of 5 kN/m intensity acting over the central 3 m of its length (portion BC). The beam is supported at its ends on walls. It is clear by inspection that the resultant of the distributed load on BC is a single force of 15 kN and this acts mid-way between B and C (in this case, the midpoint or mid-span of the beam). Figure 4.13b shows the resultants of all the forces acting on the beam. By symmetry, each of the end reactions is 7.5 kN. However, the distribution of these reactive forces over the 0.6 m length of the support is not well defined. We might assume a uniform distribution and take the resultant to act 0.3 m from the edge of the support. It might be more realistic to suppose that the intensity of the reaction is greater near the edge of the support than at the very end of beam, and to place the resultant at only 0.2 m from the edge of the support. In real situations, forces are rarely known with great accuracy either in magnitude or position. Figure 4.13b is called the *freebody diagram* of the beam. (Freebody diagrams are discussed in some detail in Chapter 6.) The magnitude and position of the resultant of a distributed load of varying intensity may be found as follows.

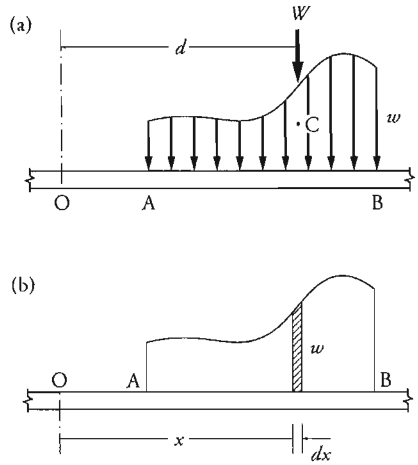


Figure 4.14

Suppose the load distribution is represented graphically as in Figure 4.14a, where the height of the *loading curve* at any point represents the magnitude of the load intensity w at that point. The load on an elemental length of beam dx is $w\,dx$ (see Figure 4.14b). In effect the distributed load consists of a large number of parallel forces of magnitude $w\,dx$. The resultant W is the sum of these forces. That is:

$$W = \int_A^B w\,dx \tag{4.5}$$

and W is the area under the load distribution curve.

To find the position of the resultant, take moments about any convenient point, such as O in Figure 4.14, and equate the moment of the resultant to the sum of the moments of the elemental forces $w\,dx$. That is:

$$Wd = \int_A^B wx\,dx \tag{4.6}$$

The integral $\int wx\,dx$ is the *first moment* about O of the area under the load distribution curve. This means that the resultant acts through the *centroid* of this area. (The concept of the first moment of area and the definition of the centroid of an area is discussed in Appendix, Section A.1.)

Consider the triangular load diagram in Figure 4.15a. The load intensity at $x = L$ is w and at $x = 0$, it is zero. At any intermediate value of x the load intensity w_x is obtained from simple geometry:

$$\frac{w_x}{x} = \frac{w}{L}$$

and therefore:

$$w_x = \frac{wx}{L}$$

The load acting on the small length dx shown in Figure 4.15b is:

$$w_x\,dx = \frac{wx}{L}\,dx$$

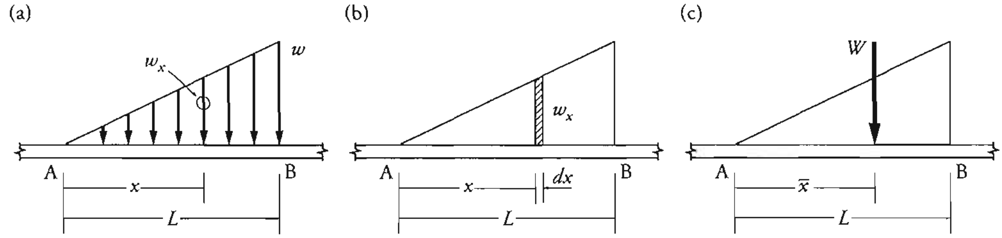


Figure 4.15

The resultant of the linearly varying (triangular) distributed load W is the sum of the loads acting on all the elemental lengths from $x = 0$ to $x = $ L. From Equation 4.5:

$$W = \int_0^L w_x \, dx = \int_0^L \frac{wx}{L} \, dx = \left[\frac{wx^2}{2L} \right]_0^L = \frac{wL}{2}$$

The resultant is the area of the load diagram. The position of the resultant, $\bar{x}$ in Figure 4.15c, is obtained from Equation 4.6:

$$W\bar{x} = \int_0^L w_x x \, dx = \int_0^L \frac{wx^2}{L} \, dx = \left[\frac{wx^3}{3L} \right]_0^L = \frac{wL^2}{3}$$

and with: $$W = \frac{wL}{2}$$

the length: $$\bar{x} = \frac{2L}{3}$$

Evidently, the resultant of a triangular load diagram is located two thirds of the way along the length of the diagram (i.e. through the *centroid* of the triangular load diagram).

For most practical problems, it is sufficient to know that the centroid of a rectangle is its centre (i.e. the intersection of its diagonals) and the centroid of a triangle is at one third of the distance from the base to the apex. All linear load distributions can be divided into rectangles and triangles. (For other shapes see Appendix, Table A1.)

EXAMPLE 4.5

Find the resultant of the distributed load indicated in Figure 4.16a.

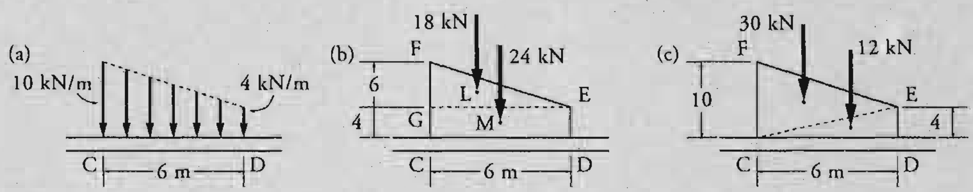


Figure 4.16

SOLUTION

Divide the load diagram CDEF into two components, the centroid of each being known. In Figure 4.16b, the resultant of the triangular part FEG is:

$$(6 \text{ kN/m} \times 6 \text{ m}) \times 1/2 = 18 \text{ kN}$$

and it acts at 2 m from C.

The resultant of the rectangular part CDEG is:

$$(4 \text{ kN/m} \times 6 \text{ m}) = 24 \text{ kN}$$

and it acts at 3 m from C.

These two parallel forces are then combined to give a final resultant of $R = 18 + 24 = 42$ kN acting at x from C where:

$$42x = 18 \times 2 + 24 \times 3 \qquad \therefore x = 2.57 \text{ m}$$

(Figure 4.16c shows an alternative way of dividing up the trapezium CDEF.)

EXAMPLE 4.6

The horizontal wind force acting on the windward face AB of a multistorey building is idealised by the tri-linear load diagram shown in Figure 4.17a. Find the magnitude and position of the resultant wind force on AB.

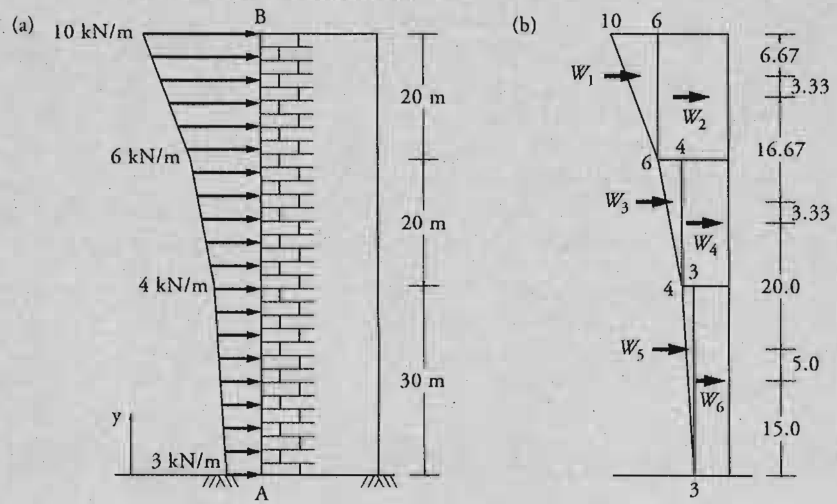


Figure 4.17

SOLUTION

The force intensity diagram is divided into three rectangular and three triangular segments as shown in Figure 4.17b. The resultant of each segment is:

$$W_1 = 0.5 \times 4 \times 20 = 40 \text{ kN} \quad \text{at } y = 63.33 \text{ m}$$

$W_2 = 6 \times 20 = 120$ kN at $y = 60$ m

$W_3 = 0.5 \times 2 \times 20 = 20$ kN at $y = 43.33$ m

$W_4 = 4 \times 20 = 80$ kN at $y = 40$ m

$W_5 = 0.5 \times 1 \times 30 = 15$ kN at $y = 20$ m

$W_6 = 3 \times 30 = 90$ kN at $y = 15$ m

The magnitude of the resultant force R exerted by the wind on surface AB is:

$$R = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 = 365 \text{ kN}$$

and its position is found by equating the moment of the resultant about A with the moment of the six components W_1 to W_6 about A:

$$365 \times \bar{y} = 40 \times 63.33 + 120 \times 60 + 20 \times 43.33 + 80 \times 40 + 15 \times 20 + 90 \times 15$$

$$\therefore \bar{y} = 42.33 \text{ m}$$

4.5 Statically equivalent systems

Much of the work of statics consists of replacing a given system of forces by a statically equivalent system which is more convenient for calculating the effect of the system. Replacing a system of forces by its resultant is one such example.

A given two-dimensional system of forces can be replaced by any other system in which there are at least three independent quantities. If there are just three unknown quantities in the new system, these quantities may be specifically determined such that the new force system is equivalent to the original force system. If the new force system contains more than three unknown quantities, a certain amount of choice exists in assigning magnitudes to these unknowns.

We consider here only the case of three unknown quantities. These three unknowns must then be chosen so that the new system and the given system are:

1. equal as regards their x components
2. equal as regards their y components
3. equal as regards their moments about any chosen point.

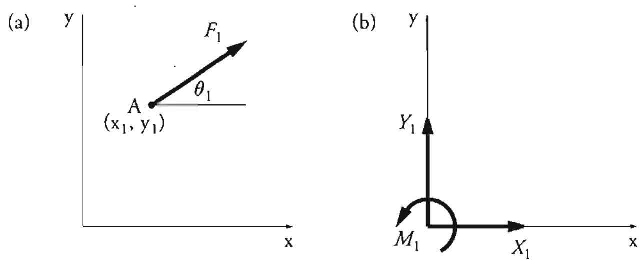


Figure 4.18

Suppose that the given system consists of the single force F_1 acting at A (x_1, y_1) in a direction θ_1 to the x axis (Figure 4.18a). Suppose it is required to replace it by an equivalent system consisting of a force X_1 along the x axis, a force Y_1 along the y axis and a couple M_1 (Figure 4.18b). Equating the x and y components of the new system to the x and y component of the given system, respectively, gives:

$$X_1 = F_1 \cos \theta_1 \qquad \text{and} \qquad Y_1 = F_1 \sin \theta_1$$

Moments can be equated about any point, but the origin is most convenient in this particular problem:

$$M_1 = F_1\, x_1 \sin \theta_1 - F_1\, y_1 \cos \theta_1$$

In a similar way, we could replace a system consisting of any number of forces F_1, F_2 etc. acting in directions θ_1, θ_2 etc. by a new system comprising forces $\overline{X}$ and $\overline{Y}$ along the x and y axes, and a couple $\overline{M}$. By equating the new system to the given system as above, we obtain:

$$\overline{X} = \Sigma F \cos \theta$$

$$\overline{Y} = \Sigma F \sin \theta$$

$$\overline{M} = \Sigma(Fx \sin \theta - Fy \cos \theta)$$

Of course, it is not necessary for the three components of the new system to act at the origin. They can be specified to act anywhere, as in Example 4.7.

EXAMPLE 4.7

Figure 4.19 shows a system of forces acting at the corners of a 300 mm square block. The corner A can be considered as the origin of co-ordinates. It is required to determine a new system of forces which is statically equivalent to the given system. The new system is to consist of force components X and Y acting at B (400, 200) and a couple M (anticlockwise being taken as positive). In Figure 4.19 the forces of the given system are indicated by full lines, while those of the new system are indicated by dotted lines.

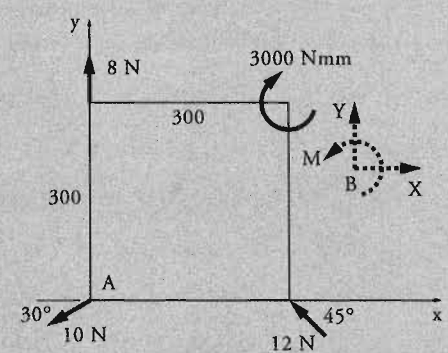


Figure 4.19

SOLUTION

1. Equate the total x components of the two systems:
New system = given system

$$\overset{+}{\rightarrow} \quad X = -10 \cos 30 - 12 \cos 45$$

$$\therefore X = -17.15 \text{ N}$$

2. Equate the total y components of the two systems:

$$+\uparrow \quad Y = +12 \sin 45 - 10 \sin 30 + 8 = +11.48 \text{ N}$$

3. Equate the moments of the two systems about any point. (If a point other than B is chosen, the moment of the new system about the point will involve consideration of the moments of X or Y or both about the point. If we choose B, the moment of the new system is simply M.) Therefore, equating moments of both systems about B gives:

$$\begin{aligned} \circlearrowleft + \quad M &= -(8 \times 400) - 3000 - (12 \cos 45 \times 200) - (12 \sin 45 \times 100) \\ &\quad -(10 \cos 30 \times 200) + (10 \sin 30 \times 400) \\ &= -8478 \text{ Nmm} \end{aligned}$$

The new system need not consist of forces X and Y together with a couple. For instance, when a force system is replaced by its resultant, the new system (the resultant) is a single force, the three unknown quantities being its magnitude, direction and position. We could replace a given force system by three forces acting along the sides of a specified triangle (although not proportional to the sides). In such a case the direction and position of the new forces are known, and the unknown quantities are the three force magnitudes. We could not replace a given system by three forces all parallel to the x axis because such an arrangement would not permit the new system to provide a y component equivalent to that of the original system. In effect, three parallel forces do not represent three *independent* quantities.

A common example of the use of an equivalent system is the replacement of a force by a parallel force and a couple. The force F at Q in Figure 4.20 can be replaced by an equal force F at A together with a couple of moment Fd. The couple Fd has the same moment about A, and is in the same sense, as the original force F at Q.

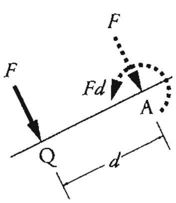


Figure 4.20

Suppose we have a force system expressed in the form of three components X_A, Y_A and M_A in the axes at A (Figure 4.21). We now decide that it would be more convenient to express the system in terms of components in the axes at B, as shown. If the two systems are equivalent, then:

1. resolving parallel to X_B gives: $X_B = X_A \cos\theta + Y_A \sin\theta$
2. resolving parallel to Y_B gives: $Y_B = -X_A \sin\theta + Y_A \cos\theta$
3. taking moments about B gives: $M_B = X_A y - Y_A x + M_A$

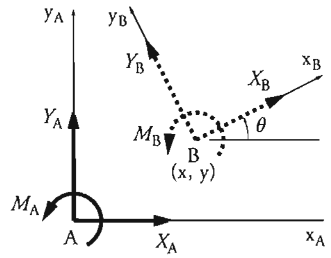


Figure 4.21

Problems

4.1 Figure P4.1 shows a lever pivoted at B. At a certain instant it is horizontal and has forces shown exerted at the ends. The force exerted on the lever by the pivot B is shown in terms of its horizontal and vertical components.

(i) Find the moment of the force system about B.
(ii) Find the moment of the force system about C.
(ii) Which way will the bar rotate?

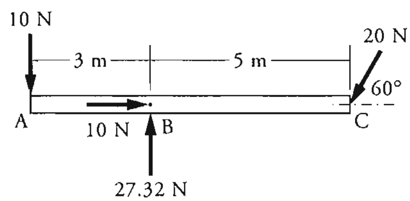


Figure P4.1

4.2 Figure P4.2 shows three forces acting on a bar ABC. Find the resultant.

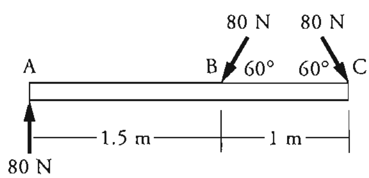


Figure P4.2

4.3 ABC is a frame in the shape of an isosceles triangle (Figure P4.3). A force of 4 kN acts normal to AB at its mid-point. A force of 2 kN acts normal to BC at its midpoint. The 6 kN force at B acts in the direction BA. Find the resultant of these forces expressed in terms of its vertical and horizontal components acting at point D, together with a couple.

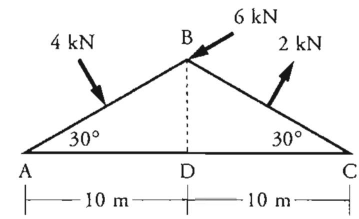


Figure P4.3

4.4 Calculate the resultant of the four parallel forces shown in Figure P4.4.

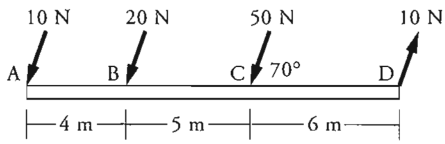


Figure P4.4

4.5 Calculate the resultant of the parallel forces which act at the corners of the 2 m square ABCD (Figure P4.5).

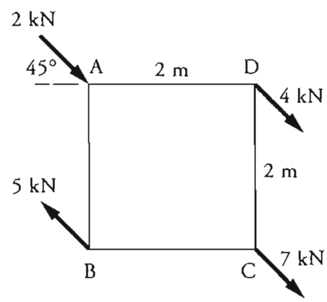


Figure P4.5

4.6 Figure P4.6 shows a triangular plate ABC. Forces of 150 N, 120 N and 180 N act along the sides AB, BC and CA respectively.

(i) Find the resultant.

(ii) Find the resultant if the 180 N force is reversed in direction.

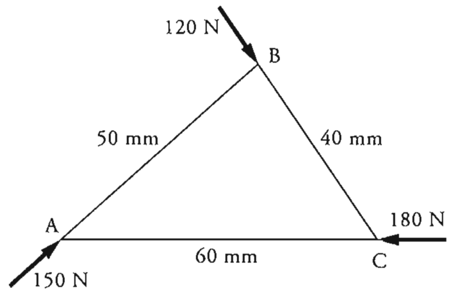


Figure P4.6

4.7 A rigid body ABCD is loaded as shown in Figure P4.7.

(i) Express the resultant of these forces as a horizontal and vertical component at B and a moment.

(ii) Find a point P such that the resultant can be expressed as a horizontal and vertical component only, acting through P.

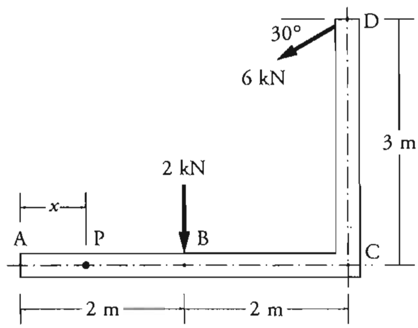


Figure P4.7

4.8 Figure P4.8 shows a 1 m square plate ABCD acted upon by a force of 10 kN along the diagonal BD and an anticlockwise couple of 20 kNm at corner A.

(i) Find the magnitude, direction and point of application of the resultant force.

(ii) Solve the same problem but with the couple applied at B instead of A.

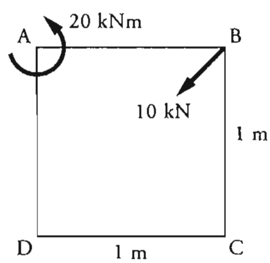


Figure P4.8

4.9 Find the resultant of the five forces shown in Figure P4.9.

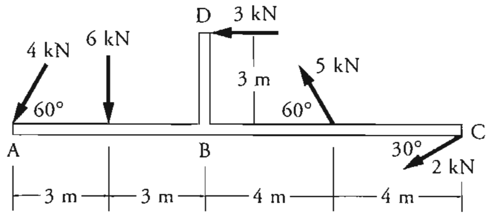


Figure P4.9

4.10 In the parallelogram of Figure P4.10, sides AB and DC are 750 mm while sides AD and BC are 500 mm. The forces shown act along, but are not proportional to, the four sides of the parallelogram.

(i) Find a statically equivalent force system which consists of a force through A and a couple.

(ii) Find a statically equivalent system which consists of a single force.

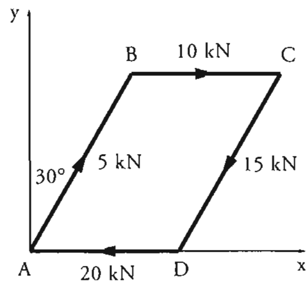


Figure P4.10

4.11 Figure P4.11 shows a square plate acted upon by forces of 8 kN, 12 kN and 15 kN and also a couple of 20 kNm. These are shown by full lines. What forces (shown dashed) acting along the sides of the square will be statically equivalent to the original system, given that $F_1 = F_3$?

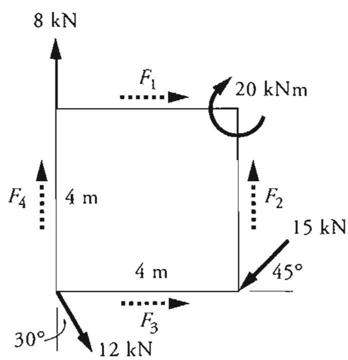


Figure P4.11

4.12 A system of three forces and a couple act on a square plate ADEC (Figure P4.12). Evaluate the statically equivalent system of forces F_1, F_2 and F_3 which act along the sides of the equilateral triangle ABC. (These forces are not proportional to the sides of the triangle.)

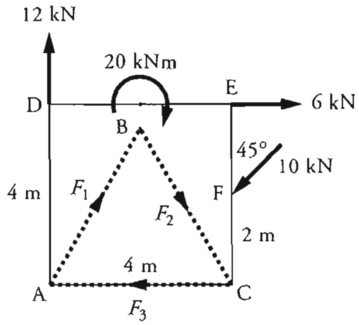


Figure P4.12

4.13 Replace a clockwise couple of 50 kNm by three forces which act along the sides of an equilateral triangle of side 2 m.

4.14 A flat triangular plate ABC is standing so that AC is horizontal and B is above AC. Side AB = 5 m, BC = 5 m and CA = 6 m. Forces of 10 N, 15 N and 20 N act along the sides AB, BC and CA respectively. Replace this system of forces:

(i) by a single force

(ii) by a force through B together with a couple.

4.15 The bent bar ABC is subjected to the loading shown in Figure P4.15. Find the resultant of the forces shown:

(i) as a force through B, and a couple

(ii) as a force through B, and a force at C perpendicular to BC

(iii) as a force at A, and a force at C perpendicular to BC.

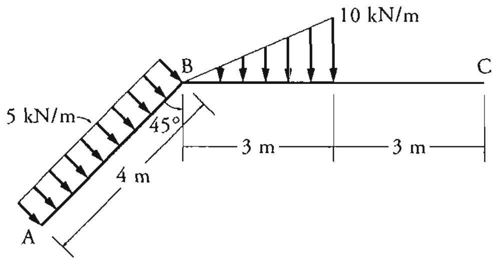


Figure P4.15

4.16 Four coplanar forces act at the points A, B, C and D as shown in Figure P4.16. The co-ordinates of each point are shown in brackets.

(i) Find the resultant of the forces.

(ii) Replace the resultant by the two forces indicated by the dashed lines in Figure P4.16.

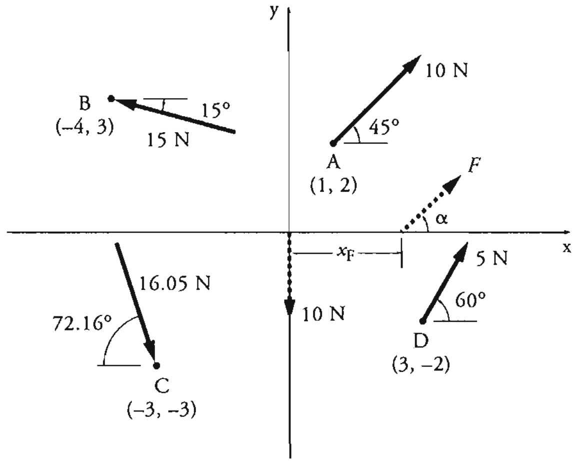


Figure P4.16

4.17 A system of coplanar forces acts as shown in Figure P4.17 along the lines AB and BC. Replace the forces by an equivalent force through D, and a couple at C.

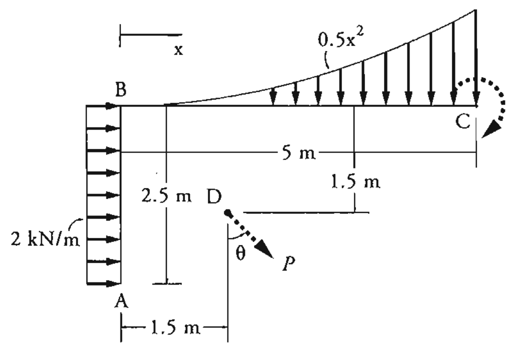


Figure P4.17

4.18 A system of coplanar forces, shown by solid lines, acts as indicated in Figure P4.18. Find the resultant of the system, expressed as:

(i) a force through point C, and a couple

(ii) a force of 60 N at E, and a force F appropriately directed and located.

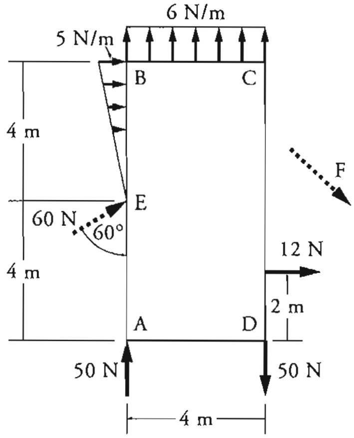


Figure P4.18

4.19 The shape depicted in Figure P4.19 is a regular hexagon of sides 3 m in length. Coplanar forces act along the lines shown. Express the resultant of the forces:

(i) as a force through O, and a couple.

(ii) as a single force through P with no couple. What is the distance OP?

(iii) as two parallel, but unequal forces 3 m apart, one of them passing through O.

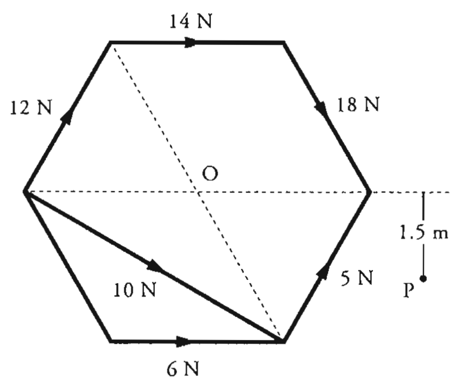


Figure P4.19


OVE ARUP & PARTNERS

Equilibrium of Non-concurrent Forces

5.1 Conditions of equilibrium

A body is in equilibrium only if the forces acting upon it have no resultant force and no resultant couple.

For the resultant force to be zero, the sum of the components of the forces in any two directions must also be zero (Section 3.2).

$$\Sigma F_x = 0 \tag{5.1}$$

$$\Sigma F_y = 0 \tag{5.2}$$

When these conditions are satisfied the system is either in equilibrium or it is equivalent to a couple. If the sum of the moments about any point is zero, i.e.:

$$\Sigma M = 0 \tag{5.3}$$

the system cannot be equal to a couple and must then be in equilibrium. These three equations are the basic expressions of coplanar equilibrium and are used extensively throughout engineering.

Equilibrium can be ensured by other sets of three equations, which are sometimes easier to apply. If the sum of the moments about a point A is zero:

$$\Sigma M_A = 0 \tag{5.4}$$

then the system cannot be equal to a couple, and if it has a resultant, that resultant must pass through A. If then, the sum of the moments about a point B is also zero:

$$\Sigma M_B = 0 \tag{5.5}$$

the resultant is either zero, or passes through B as well as A. To rule out the latter possibility we may now either ensure that the sum of the components along AB is zero, or ensure that the sum of the moments about a point C is zero, where C is not on the line AB:

$$\Sigma M_C = 0 \tag{5.6}$$

EXAMPLE 5.1

Figure 5.1 shows a system of forces consisting of five forces and a couple, the locations of which are defined with reference to a circle of 4 m diameter. If the force system is in equilibrium, calculate the unknown forces F_1, F_2 and F_3. The forces F_2 and F_3, together with the 8 kN force, are parallel to Oy.

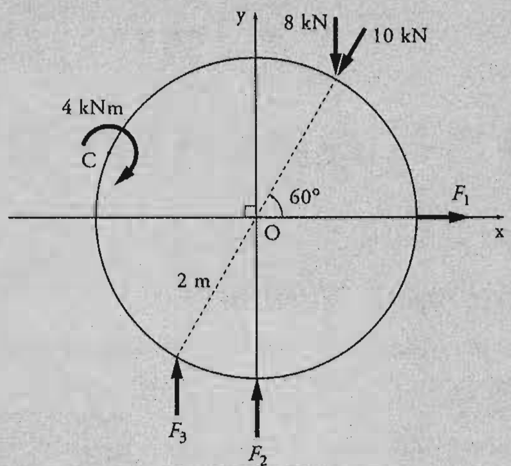


Figure 5.1

SOLUTION

The sum of the forces in the x direction must be zero (Equation 5.1), thus:

$$-10 \cos 60 + F_1 = 0 \qquad \therefore F_1 = +5 \text{ kN}$$

Similarly, in the y direction (Equation 5.2):

$$F_2 + F_3 - 8 - 10 \sin 60 = 0 \qquad \therefore F_2 + F_3 = 16.66 \text{ kN} \qquad \textbf{(A)}$$

Equating moments about O to zero, Equation 5.3 gives:

$$F_3(2 \cos 60) + 8(2 \cos 60) + 4 = 0 \qquad \therefore F_3 = -12 \text{ kN}$$

and from (A): $\therefore F_2 = +28.66$ kN

(Evidently, it would have been quicker to use Equation 5.3 before Equation 5.2.)

5.2 Types of supports for structures

The magnitude and direction of the forces exerted on structures at their supports depend on the type of support. A *hinged* or *pin support*, such as that shown at A in Figure 5.2a, allows rotation but does not permit translation in any direction. The reaction at A (R_A) can act in any direction and is often replaced by its vertical and horizontal components (V_A and H_A) as shown. At a hinged support the reaction cannot include a couple: the reaction must be a force whose line of action passes through the hinge.

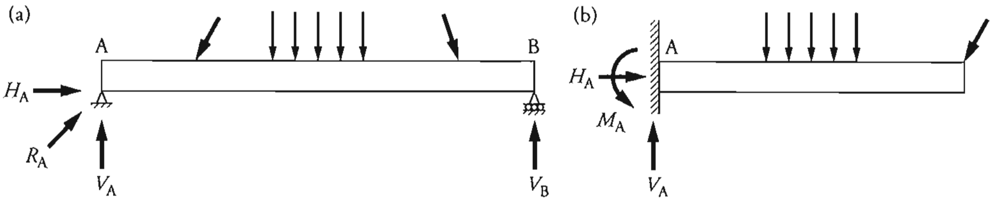


Figure 5.2

The *roller support* at B in Figure 5.2a allows rotation, and translation parallel to the surface on which the roller moves, but does not permit translation perpendicular to the surface. The reaction at B must be a single force at right angles to the surface on which the roller moves. In Figure 5.2a, the reaction at B is a vertical force (V_B) since the bearing surface is horizontal. By convention the roller reaction may be either towards or away from the structure (either *compressive* or *tensile*). The structure cannot lift off the rollers, as it could if the symbol were interpreted physically.

The hinge support and the roller support are often called *simple supports* and a single span beam, with a hinge support at one end and a roller support at the other, such as the beam in Figure 5.2a, is called a *simply-supported beam.*

Some supports, called *built-in* or *fixed supports,* do not permit either translation or rotation and, in planar structures, three reaction components may occur at such a support, as shown at support A of the beam of Figure 5.2b. The hatched symbol for the built-in support at A in Figure 5.2b is commonly used and should signal that the reaction at the support consists of a force, with vertical and horizontal components, V_A and H_A and a couple, M_A. A beam with a built-in support at one end and no support at the other is called a *cantilever.*

5.3 Planar structures in equilibrium

Example 5.2

The cable tower structure ABC in Figure 5.3a is supported on rollers at B and by a pin at A. Determine the reactions under the forces shown.

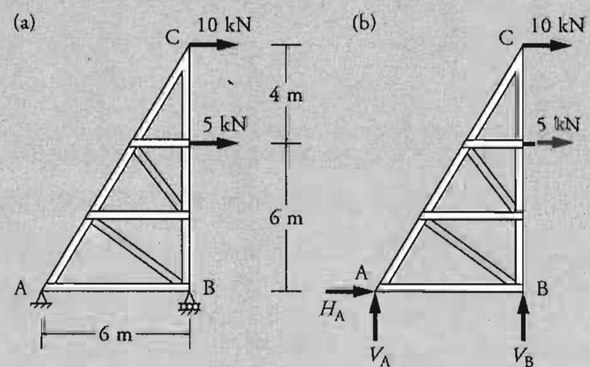


Figure 5.3

SOLUTION

Figure 5.3b is a freebody diagram of the structure showing the applied loads and the reactions exerted by the ground. The body ABC is at rest and the forces shown are all external to ABC and are the only forces acting. Therefore, they must be in equilibrium. The three unknown quantities in this system may be taken as the magnitude of the vertical reaction R_B and the magnitude and direction of the reaction R_A. Alternatively R_A and R_B may be expressed in terms of their horizontal and vertical components, in which case the unknowns are H_A, V_A and V_B.

Three solutions will be demonstrated.

1. Using Equations 5.1, 5.2 and 5.3:

$\Sigma F_x = 0$: $\overset{+}{\rightarrow}$ $H_A + 5 + 10 = 0$ $\quad \therefore H_A = -15$ kN

$\Sigma M_A = 0$: $\circlearrowleft+$ $(V_B \times 6) - (5 \times 6) - (10 \times 10) = 0$ $\quad \therefore V_B = +21.7$ kN

$\Sigma F_y = 0$: $+\uparrow$ $V_A + V_B = 0$ $\quad \therefore V_A = -21.7$ kN

The reactions are therefore as shown in Figure 5.4.

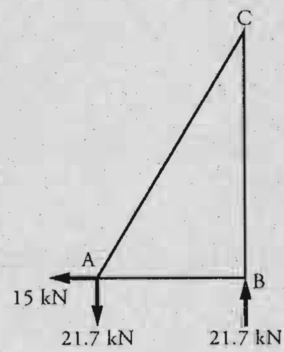


Figure 5.4

2. Using Equations 5.4, 5.5 and 5.6:

$\Sigma M_A = 0$: $\circlearrowleft+$ $(V_B \times 6) - (5 \times 6) - (10 \times 10) = 0$ $\quad \therefore V_B = +21.7$ kN

$\Sigma M_B = 0$: $\circlearrowleft+$ $-(V_A \times 6) - (5 \times 6) - (10 \times 10) = 0$ $\quad \therefore V_A = -21.7$ kN

$\Sigma M_C = 0$: $\circlearrowleft+$ $(H_A \times 10) - (V_A \times 6) + (5 \times 4) = 0$ $\quad \therefore H_A = -15$ kN

It is always theoretically possible to choose the equations such that one unknown is evaluated at a time, and this sometimes simplifies the arithmetic. If moments are taken about the point of intersection of two unknown forces, these two forces do not appear in the equation. If two unknown forces are parallel and the force system is resolved in a direction normal to these, the two unknowns do not appear in the equation.

Hence, if we wish to write an equation which contains only one of the three unknown forces, we examine the other two. If these are parallel we resolve at right angles to them. If they are not parallel we take moments about their point of intersection.

3. The three unknowns will be computed independently. To find H_A, we note that V_A and V_B are parallel. Therefore, resolving normal to V_A and V_B:

$\Sigma F_x = 0$: $\overset{+}{\rightarrow}$ $H_A + 5 + 10 = 0$ $\therefore H_A = -15$ kN

To find V_A, we note that H_A and V_B intersect at B. Therefore, taking moments about B:

$\Sigma M_B = 0$: $\curvearrowright +$ $(V_A \times 6) + (5 \times 6) + (10 \times 10) = 0$ $\therefore V_A = -21.7$ kN

To find V_B we note that H_A and V_A intersect at A. Therefore, taking moments about A:

$\Sigma M_A = 0$: $\curvearrowright +$ $-(V_B \times 6) + (5 \times 6) + (10 \times 10) = 0$ $\therefore V_B = +21.7$ kN

Every precaution should be taken to ensure that the external reactions have been correctly calculated, since they usually form the basis of further work. Consequently, after finding the reactions by one of the above methods it is advisable to check them by reference to independent equations.

EXAMPLE 5.3

A beam rests on a roller support at A and a hinge support at B, 19 m apart, and carries vertical loads as shown in Figure 5.5a. Find the reactions at A and B.

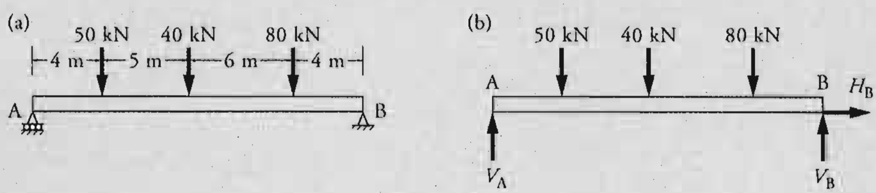


Figure 5.5

SOLUTION

Figure 5.5b is a freebody diagram showing only the forces acting on the beam, including those at the supports. As A is on rollers there is no horizontal reaction at A. Since the given forces are all vertical, then to satisfy the equation $\Sigma F_x = 0$, the reaction component H_B must be zero.

$\Sigma M_A = 0$: $\curvearrowright +$ $(50 \times 4) + (40 \times 9) + (80 \times 15) - (V_B \times 19) = 0$

$\therefore V_B = 92.6$ kN

$\Sigma F_y = 0$: $+\uparrow$ $V_A - 50 - 40 - 80 + 92.6 = 0$ $\therefore V_A = 77.4$ kN

Now using the equation $M_B = 0$ as a check:

$$\circlearrowleft + \quad -(V_A \times 19) + (50 \times 15) + (14 \times 10) + (80 \times 4) = 0 \qquad \therefore V_A = 77.4 \text{ kN}$$

(Alternatively the equation $\Sigma M_B = 0$ may be employed to find V_A and the equation $\Sigma F_y = 0$ used as the check.)

Example 5.4

The beam ABC is uniformly loaded between B and C and carries a concentrated load of 10 kN at B, as shown in Figure 5.6a. It is built in at A. Find the three reaction components at A.

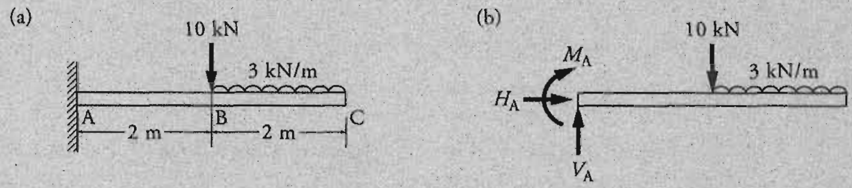


Figure 5.6

Solution

Figure 5.6b is a freebody diagram of the beam. It is necessary first to replace the distributed load by its resultant, which is a force of 6 kN acting mid-way between B and C. Then:

$$\Sigma F_x = 0: \quad H_A = 0$$

$$\Sigma F_y = 0: \quad V_A - 10 - 6 = 0 \qquad \therefore V_A = 16 \text{ kN}$$

$$\Sigma M_A = 0: \quad \circlearrowright + \quad M_A + (10 \times 2) + (6 \times 3) = 0 \qquad \therefore M_A = -38 \text{ kNm}$$

5.4 Summary

If a body is in equilibrium, certain relationships must hold between the forces acting externally on that body. These are expressed analytically by the equations $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$. These relationships enable the determination of any three unknown quantities in the given force system.

If all the forces acting upon the body pass through one point, the third equation is automatically satisfied. Only the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ may then be applied, and only two quantities can be determined.

In applying these equations, care should be taken that *all* forces and couples *external* to the chosen body are taken into account, and equally that no internal forces are included or any forces which do not act on the body.

Problems

5.1 The bell-crank lever ABCD in Figure P5.1 is pivoted at C. The forces shown are at right angles to the lever. If the lever is in equilibrium find the magnitude of the force F and the magnitude and direction of the reaction at the pivot.

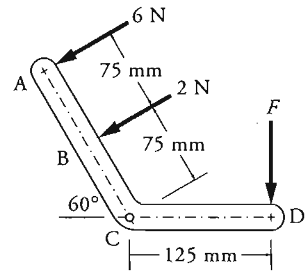


Figure P5.1

5.2 The square plate ABCD in Figure P5.2 is in equilibrium. Find the values of F_1, F_2 and θ.

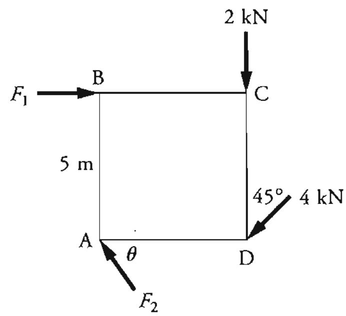


Figure P5.2

5.3 What is the significance of the symbols shown at the supports A and E of the beam shown in Figure P5.3? Find the horizontal and vertical components of the reactions at A and E.

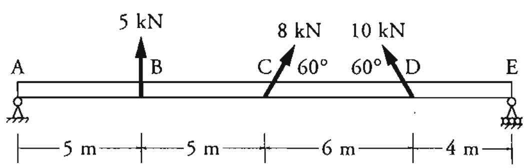


Figure P5.3

5.4 Find the vertical reaction at A, the horizontal reaction at A, and the vertical reaction at D, for the beam loaded as shown in Figure P5.4 (page 68).

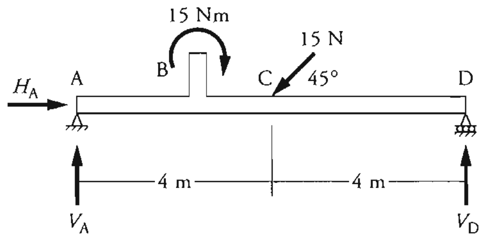


Figure P5.4

5.5 The pulley shown in Figure P5.5 has tangential forces applied to it as indicated. If the pulley is in equilibrium, determine:

(i) the value of P

(ii) the vertical and horizontal components of the reaction at the axle.

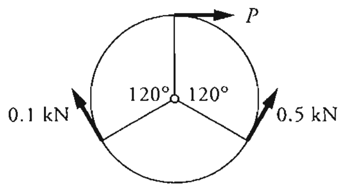


Figure P5.5

5.6 What is the significance of the support symbol shown at A in Figure P5.6? How many reaction components are there at a support of this type? Find the reactions for the loading shown.

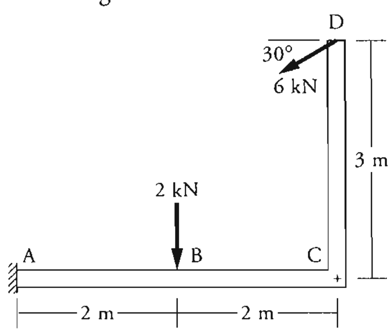


Figure P5.6

5.7 Figure P5.7 shows the same beam as that of the previous problem, but supported in a different manner. What is the total number of reaction components for the beam of Figure P5.7? Determine these reactions.

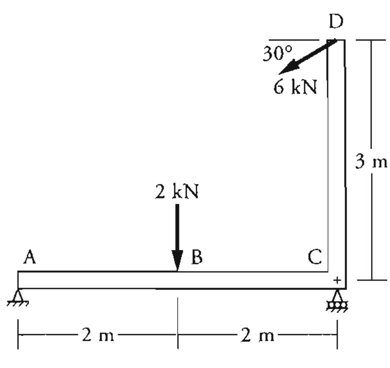


Figure P5.7

5.8 Figure P5.8 is a diagrammatic sketch of a jib crane. The jib weighs 5 kN and the main assembly weighs 40 kN.

(i) When the value of the load L is 10 kN find the reactions at A and B.

(ii) At what value of L will the crane overturn?

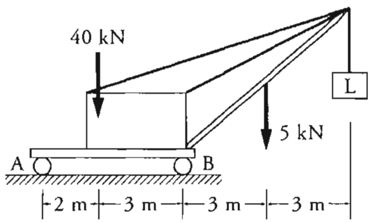


Figure P5.8

5.9 A horizontal beam of length 8 m is supported on rollers at each end, the roller planes being at 60° and 30° respectively to the horizontal (Figure P5.9). At the centre it rests on a smooth peg. For the loading shown, find all the reactions.

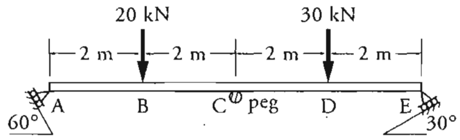


Figure P5.9

5.10 Find the horizontal and vertical components of reaction at A and B for the beam of Figure P5.10.

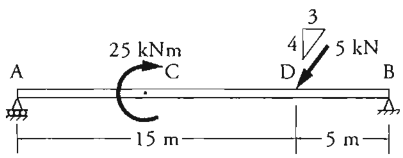


Figure P5.10

5.11 Find the reactions to the truss shown in Figure P5.11. The top inclined chord is bisected by the inclined strut.

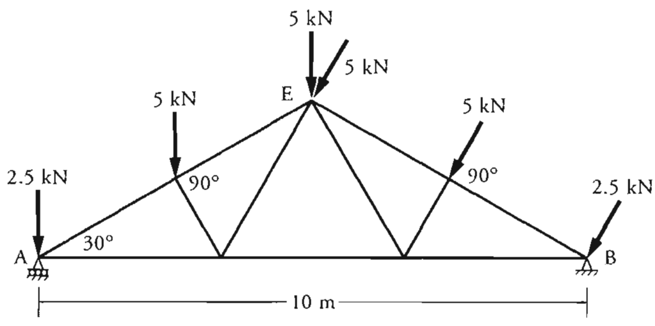


Figure P5.11

5.12 The bar ABC shown in Figure P5.12 is in equilibrium. Evaluate the unknown forces F_1, F_2 and M.

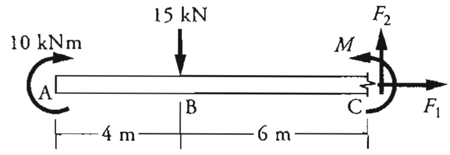


Figure P5.12

5.13 A bar AB is loaded with a uniformly distributed load w, by two forces N and S acting at B, and a couple M (Figure P5.13). If the bar is in equilibrium find the values of N, S and M in terms of w and x.

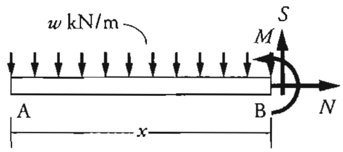


Figure P5.13

5.14 The hollow drum of radius 200 mm in Figure P5.14 rests between two rough surfaces inclined at 30° and 60° respectively to the horizontal. A vertical force P is applied to the drum at a distance e from the centreline. If the coefficient of friction is 0.4 and the weight of the drum is neglected, what is the greatest value of e before the drum slips?

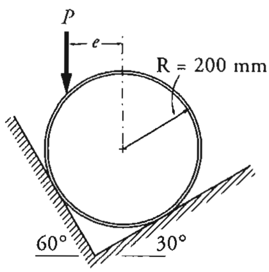


Figure P5.14

5.15 Figure P5.15 shows seven beams loaded in various ways. Find the reactions for each. Attempt to write down the answer either by inspection or with a minimum of calculation.

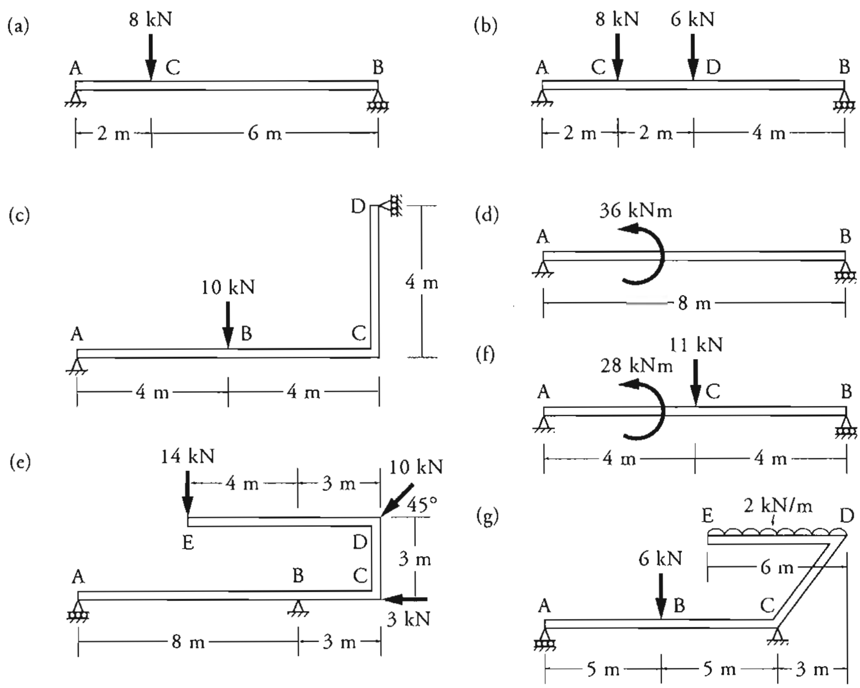


Figure P5.15

5.16 Find the reactions R_1, R_2 and R_3 in the frame of Figure P.5.16. Each triangle is equilateral, with sides 3 m long.

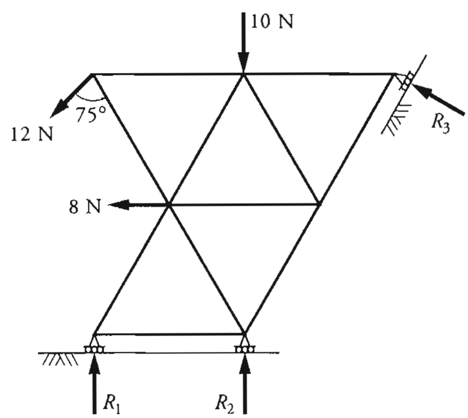


Figure P5.16

5.17 Three beams are supported, one upon another as shown in Figure P5.17. For the loading shown find:

(i) the reactions at each end of beam AB

(ii) the reaction at C of the beam CD.

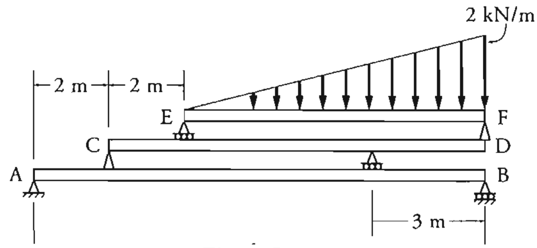


Figure P5.17

5.18 The ladder of weight W in Figure P5.18 is resting on a floor, where the coefficient of friction is 0.3, and against a wall, where the coefficient of friction is 0.2. What is the minimum value of the inclination θ if slipping is not to occur?

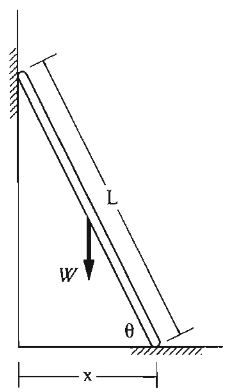


Figure P5.18

5.19 Forces of 8 N, 10 N and 3 N are directed along the edges of the triangular plate ABC shown in Figure P5.19. Find the reaction R_A, and the reaction R_B in magnitude and direction.

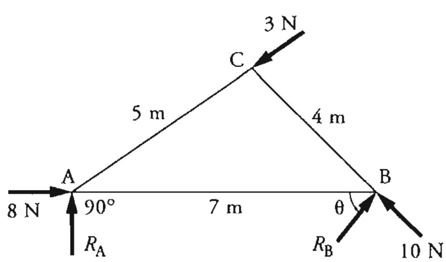


Figure P5.19

5.20 The plank of timber in Figure P5.20 rests with one end on the floor where the coefficient of friction is 0.3, while the other end rests on an inclined wall which slopes at 60° to the horizontal, where the coefficient of friction is 0.4. What is the minimum angle of inclination θ of the plank?

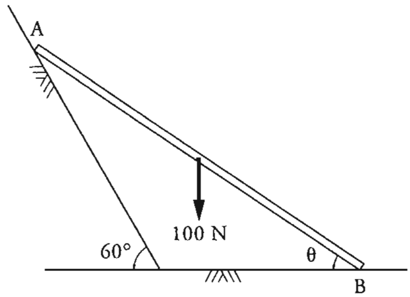
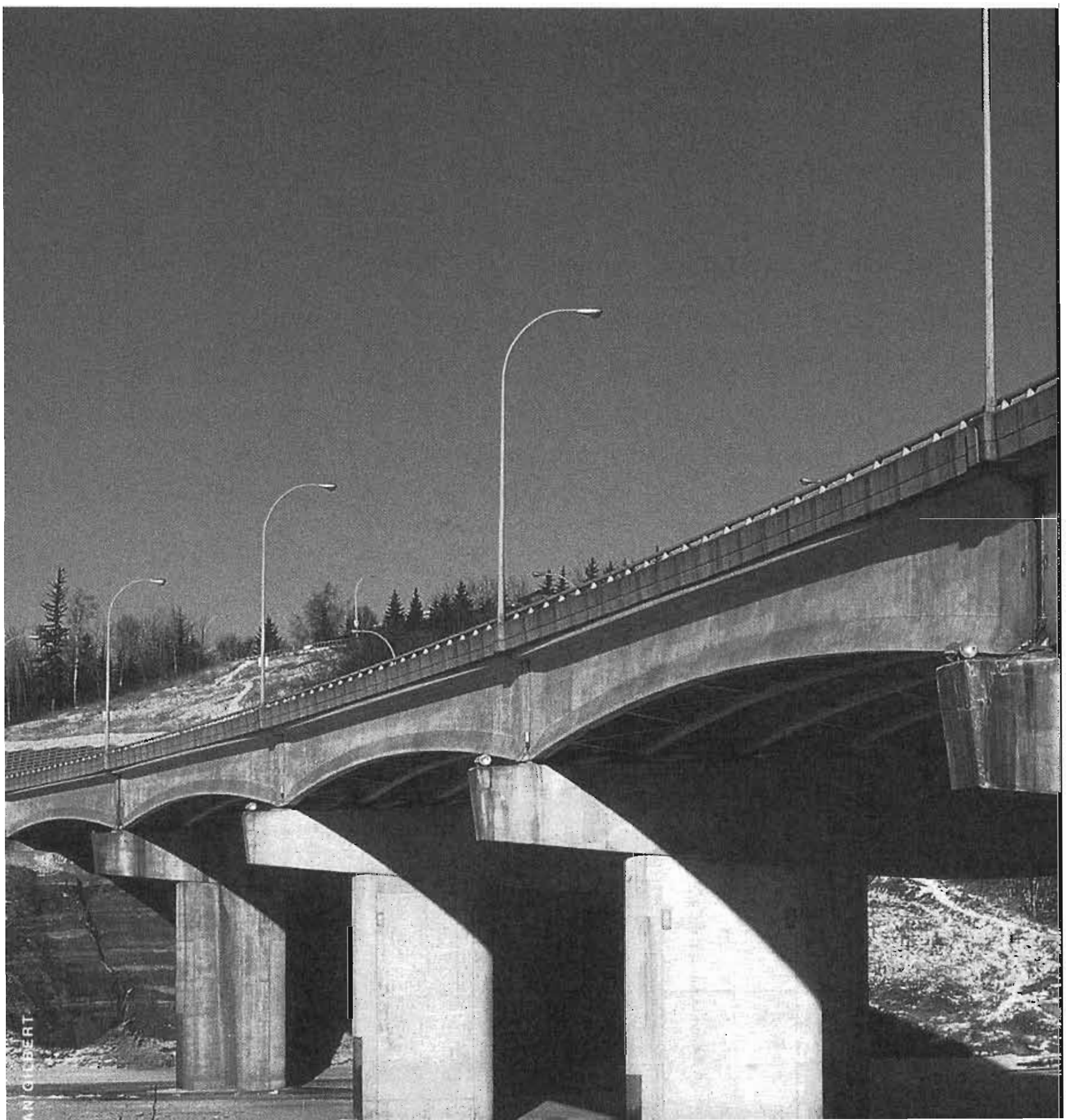


Figure P5.20


GILBERT

part 2

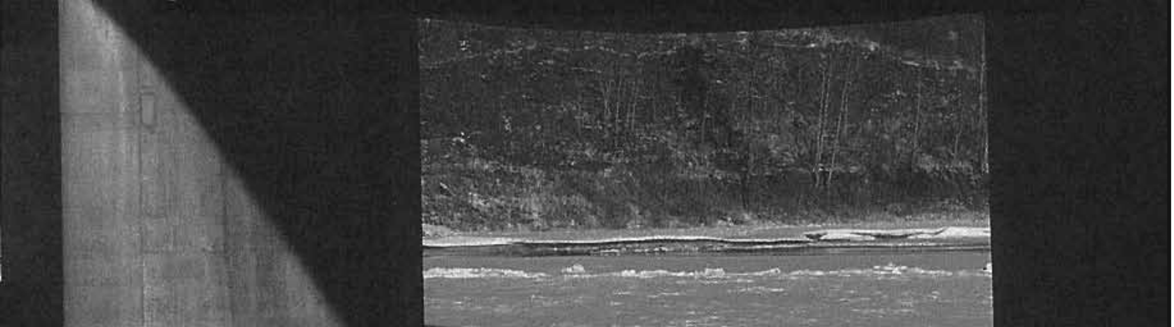
BEAMS AND CABLES

Preamble

In the following chapters, the basic laws of statics will be applied to a number of engineering problems. In nearly all cases the problems discussed have been greatly simplified compared with the real situations. For instance, roof trusses are represented as being composed of weightless bars joined together by frictionless pins, whereas in reality the bars are not weightless and they are jointed by welding, bolting or other means, but they are hardly ever pinned. Beams are also often represented as lines; the depth, and in many problems the weight, is ignored. Support conditions are idealised.

The purpose of the problems is primarily to illustrate the laws of statics. The student should not suppose that having solved these problems he or she is in a position to design roof trusses, beams, suspension bridges and so on. The statics problems are an essential first step, but many other factors must be taken into consideration before practical designs can be made.

The statical problems involved in real situations are frequently too complex to be suitable as elementary illustrations. It is for this reason that simplified problems are used in a first course of statics in order to provide practice in the fundamental concepts.



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Freebodies

6.1 Freebodies and freebody diagrams

In Part 1, the equations of equilibrium were employed to express relationships between the external forces acting upon a single body or structure. In this way, if some of the external forces were known, other forces, usually reactions at supports, could be determined. In general, much more information is required about a structure than merely the support reactions.

Most structures and machines are built up of several components connected together. Such components exert forces upon one another at their junctions, and it is necessary to evaluate these forces. Such forces are *internal* to the structure as a whole, and no information can be obtained about them by considering the equilibrium of the complete structure. According to Newton's Third Law, the force exerted by component X upon component Y is equal and opposite to that exerted by Y upon X. So even if we attempt to include these forces in the equilibrium equations they will cancel out.

The only way we can obtain information about such forces is to consider the equilibrium of one part of the structure. The part is so chosen that the internal force in question becomes *external* to that part. The notion of the *freebody* is of importance here.

In the preceding chapters we introduced the concept of the freebody and the *freebody diagram* when considering the equilibrium of any complete structure, isolated from its supports and acted upon by the given external forces and the reactions exerted by the supports. The concept of the freebody can be extended to apply to any subsection of a structure or machine. The corresponding freebody diagram will be a diagram showing the particular subsection or component, together with the forces which are external to that component. The forces will consist of those which are applied directly to the part being considered as well as those exerted by the remainder of the structure upon the part being considered.

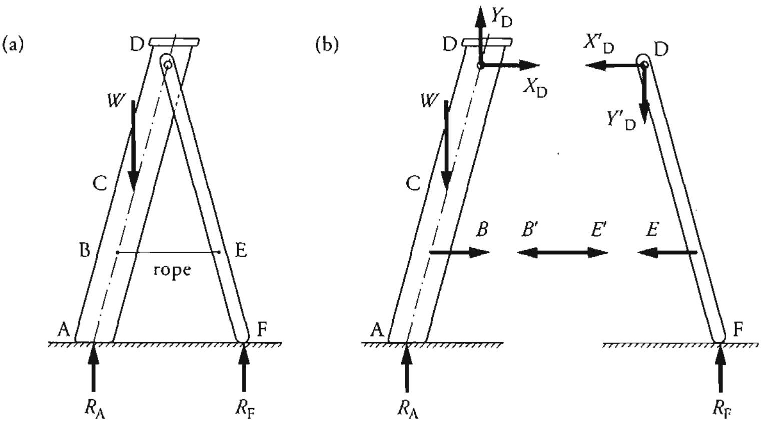


Figure 6.1

Consider the step-ladder shown in Figure 6.1a. The ladder rests on a frictionless surface and a man of weight W is standing on it in a known position. The reactions R_A and R_F are vertical since there is no friction. Figure 6.1a is a freebody diagram of the complete ladder and from this diagram reactions R_A and R_F can be easily found. But equilibrium of the complete structure yields no information about the tension in the rope, or the forces acting on the hinge at D. To study these forces it is necessary to draw freebody diagrams of each of the components as in Figure 6.1b.

Suppose that R_A and R_F have been found from Figure 6.1a. Then by considering the equilibrium of the freebody DF, we can find the unknown forces E, X'_D and Y'_D. The force E' is the reaction to E and is thus equal in magnitude to it. Then by considering the equilibrium of the rope alone we see that $B' = E'$. In this way we can find these forces which are internal as far as the complete ladder is concerned.

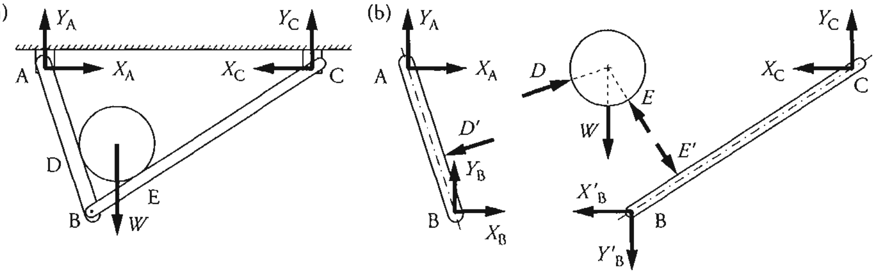


Figure 6.2

As another simple example we may consider the two planks AB and BC of Figure 6.2a, with the cylinder of weight W resting between them. Consideration of the complete assembly will yield no information about the force transmitted by the hinge B. Actually, in this example we cannot even determine the reactions X_A, Y_A, X_C and Y_C since they are four in number and we have only three equilibrium equations for the complete assembly. Both problems are overcome if we dismantle the assembly and consider the equilibrium of individual components as shown in Figure 6.2b. This figure shows three freebody diagrams. Equilibrium of the cylinder will give us the forces *D* and *E*. Equilibrium of AB and BC separately will then give us the other six forces since we have three equations for each component.

In drawing freebody diagrams, we take into account certain properties of structures and their components. In many cases the properties in question are only approximations to those of the real structure. The reaction forces that may develop at idealised supports were discussed in Section 5.2 and in freebody diagrams the supports are replaced by these reactions. For example, if a component of a structure rests against a frictionless surface, the force between the component and the surface must be normal to the surface at the point of contact. Such a support is idealised as a roller support such as that shown at B in Figure 5.2a (page 63).

At a pin-joint, the force exerted by one component on another is assumed to pass through the pin. It is often convenient to express this force in terms of its x and y components. In effect, it is assumed that the structural components are connected by a frictionless pin, although the connection in the real structure will rarely satisfy this condition.

When we come to apply the laws of equilibrium to the various freebodies we must bear in mind all the work of the previous chapters. A few of the most useful points are mentioned.

1. According to Newton's Third Law, action and reaction are equal in magnitude and opposite in direction. Figure 6.3 shows part of two components which in the structure are pinned together at D. If the forces exerted on component 1 by component 2 are $(F_x)_D$ and $(F_y)_D$, then the forces $(F_x)'_D$ and $(F_y)'_D$ exerted on 2 by 1 are equal and opposite.

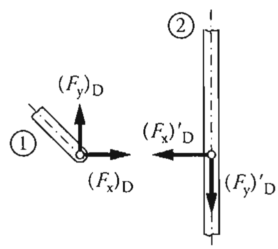


Figure 6.3

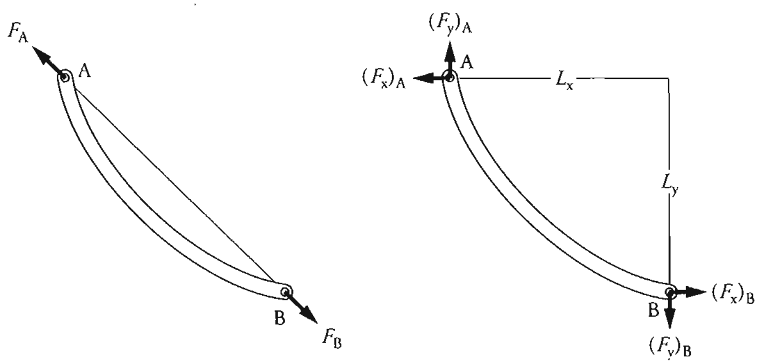


Figure 6.4

2. If any component is acted upon by only two forces, these forces must be equal and opposite and in the same line. Suppose the bar AB (Figure 6.4a) has forces exerted on it through the pins at A and B only. Then $F_A = F_B$ and these act along the line AB. If the forces have been expressed in terms of components (Figure 6.5b), then not only is:

$$(F_x)_A = (F_x)_B \quad \text{and} \quad (F_y)_A = (F_y)_B$$

but also: $$\frac{(F_x)_A}{(F_y)_A} = \frac{L_x}{L_y}$$

as we can see by taking moments about B.

3. If a body is acted on by three forces only, these forces must be concurrent.

The laws of equilibrium may be applied to the complete structure and also to any subsection of the structure. It cannot be too strongly emphasized that before any equations are written it is essential to decide what particular force system is under consideration.

Example 6.1

Three cylinders A, B and C rest between the walls and on the base of a container D (Figure 6.5a). Cylinder A weighs 3.24 kN and has a radius of 90 mm; B weighs 9 kN and has a radius of 150 mm; and C weighs 1 kN and has a radius of 50 mm. Find the forces exerted by the cylinders on the container and on each other. (Neglect friction.)

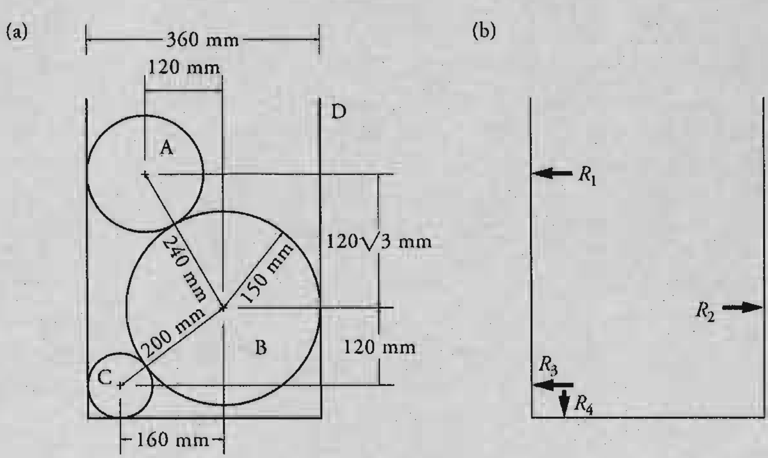


Figure 6.5 (continues)

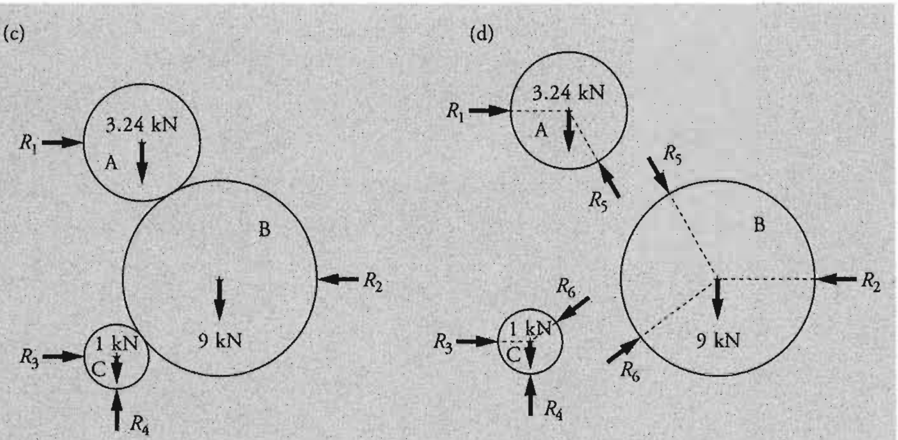


Figure 6.5

SOLUTION

Figure 6.5b is the freebody diagram of the container alone and shows the forces which maintain it in equilibrium. Figure 6.5c is the freebody diagram of the three cylinders taken together. It will be noted that of the seven forces which act on this freebody, four are unknown and therefore cannot be determined directly by applying the three equations of equilibrium to this body.

From the given sizes of the cylinders, the dimensions shown in Figure 6.5a are evaluated. The directions of all the internal forces are thus known. Figure 6.5d shows the freebody diagram of each cylinder individually. By considering the equilibrium of A, the forces R_1 and R_5 are found to be:

$$R_1 = 1.87 \text{ kN} \quad \text{and} \quad R_5 = 3.74 \text{ kN}$$

For cylinder B, R_5 is now known and R_2 and R_6 may then be found:

$$R_2 = 18.19 \text{ kN} \quad \text{and} \quad R_6 = 20.40 \text{ kN}$$

For cylinder C, R_6 is now known and R_3 and R_4 may be found:

$$R_3 = 16.32 \text{ kN} \quad \text{and} \quad R_4 = 13.24 \text{ kN}$$

As a check it will be seen that the forces R_1 to R_4 which act on the freebody of Figure 6.5c are in fact in equilibrium with the weights of the cylinders.

EXAMPLE 6.2

The two bars ABC and DBE shown in Figure 6.6a are continuous past the intersection B where they are connected by means of a frictionless pin. They are supported at C and E by hinged supports. A load of 1 kN is hung from a cable which passes over frictionless pulleys at A and G. The pulleys have a diameter of 1 m. Find the reactions at C and E and the force transmitted through the pin at B. Neglect the weight of the pulleys and bars.

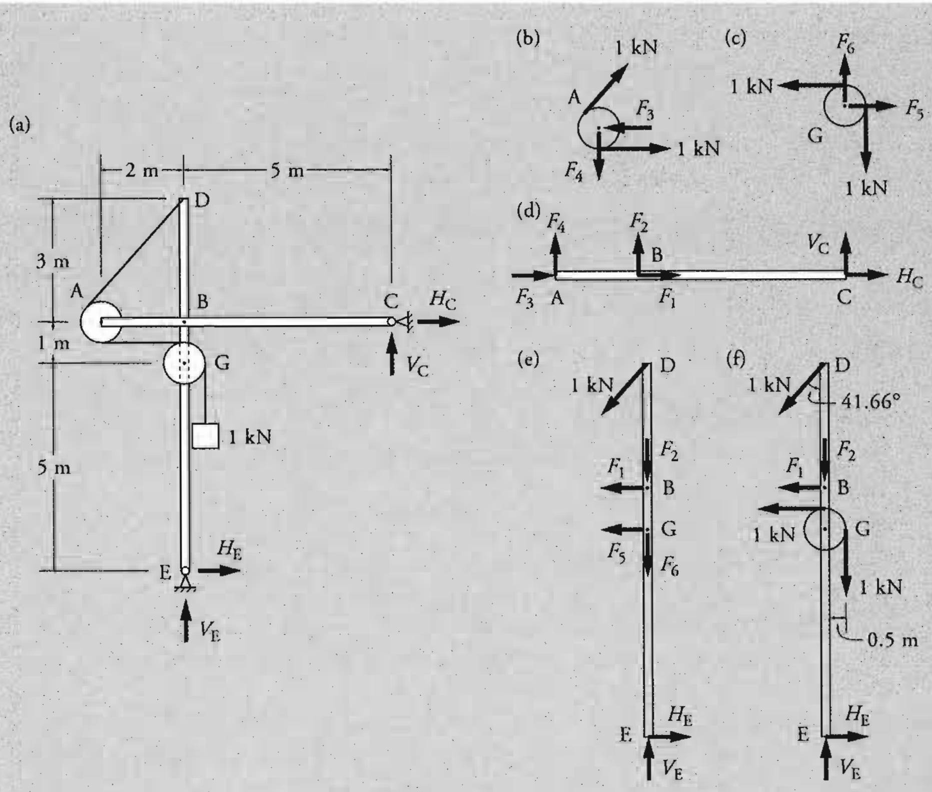


Figure 6.6

Solution

The angle between the cable and DB at D is 41.66°. Considering equilibrium of the freebody of the pulley at A (Figure 6.6b), we get:

$$F_3 = 1.665 \text{ kN} \quad \text{and} \quad F_4 = 0.747 \text{ kN}$$

and for the pulley at G (Figure 6.6c):

$$F_5 = 1.0 \text{ kN} \quad \text{and} \quad F_6 = 1.0 \text{ kN}$$

Considering the freebody ABC (Figure 6.6d):

$$\Sigma F_x = 0: \quad 1.665 + F_1 + H_C = 0$$

$$\Sigma F_y = 0: \quad 0.747 + F_2 + V_C = 0$$

$$\Sigma M_C = 0: \quad (0.747 \times 7) + (5 \times F_2) = 0$$

and hence: $F_2 = -1.046$ kN and $V_C = 0.299$ kN

Considering the freebody DBE (Figure 6.6e):

$$\Sigma F_x = 0: \quad 0.665 - F_1 - F_5 + H_E = 0$$

$$\Sigma F_y = 0: \quad -0.747 - F_2 - 1.0 + V_E = 0 \qquad \therefore V_E = 0.701 \text{ kN}$$

$$\Sigma M_E = 0: \quad (0.665 \times 9) + (6 \times F_1) + (5 \times 1.0) = 0 \quad \therefore F_1 = -1.83 \text{ kN}$$

and hence: $H_E = -0.165$ kN and $H_C = 0.165$ kN.

Checking, we see that $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M_C = 0$ for the complete frame.

The above solution could have been determined using fewer freebodies. Considering the freebody of the complete structure, there are four unknown reaction components V_C, H_C, V_E and H_E. If one of these can be determined by considering the freebody of a substructure, the other three can be evaluated from the equilibrium equations applied to the whole structure. Consider the freebody of the component EBD shown in Figure 6.6f. Taking moments about B gives:

$$0.665 \times 3 - 1 \times 0.5 - 1 \times 0.5 + H_E \times 6 = 0 \qquad \therefore H_E = -0.165 \text{ kN}$$

Considering now the equilibrium of the whole structure:

$$\Sigma F_x = 0: \quad H_E + H_C = 0 \qquad \therefore H_C = 0.165 \text{ kN}$$

$$\Sigma M_E = 0: \quad V_C \times 5 - 0.165 \times 6 - 1 \times 0.5 = 0 \qquad \therefore V_C = 0.299 \text{ kN}$$

$$\Sigma F_y = 0: \quad V_E + 0.299 - 1.0 = 0 \qquad \therefore V_E = 0.701 \text{ kN}$$

The forces at pin B may now be found by considering the freebody EBD again:

$$\Sigma F_x = 0: \quad 0.665 - F_1 - 1.0 - 0.165 = 0 \qquad \therefore F_1 = -1.83 \text{ kN}$$

$$\Sigma F_y = 0: \quad -0.747 - F_2 - 1.0 + 0.701 = 0 \qquad \therefore F_2 = -1.046 \text{ kN}$$

Equilibrium of the bar ABC could have been considered instead of EBD.

A great many practical problems cannot be solved without considering the equilibrium of components as well as the equilibrium of the whole structure. A common case is that of a simple plane frame (Figure 6.7a) consisting of two rigid members ABC and CDE joined together by a frictionless hinge at C and supported at A and E by pin supports.

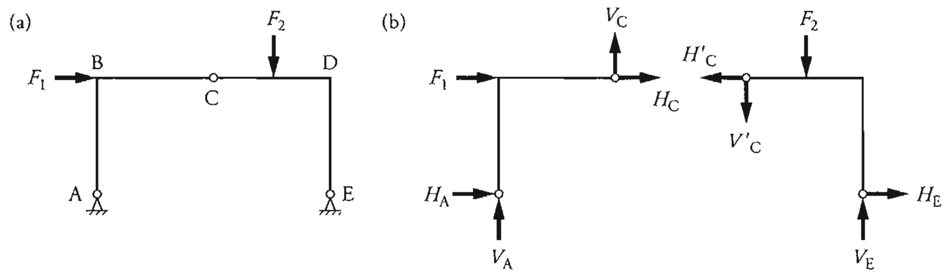


Figure 6.7

Reference to Figure 6.3a indicates that there are four reactions components altogether, namely H_A and V_A at A and H_E and V_E at E. Since there are only three equations of equilibrium for the frame as a whole, the external reactions cannot be evaluated without considering freebodies of the components. Figure 6.7b shows freebody diagrams for each of the components ABC and CDE separately. We now have a total of six unknown force components, but since there are three equilibrium equations for each component, the problem is easily solved. As an alternative we could have drawn a freebody of the whole frame and a freebody of one component. This particular type of problem will be discussed in more detail in Chapter 8.

Many structures are built up of physically identifiable components. For instance the ladder of Figure 6.1 comprises the part AD with the steps, the supporting leg DF and the rope BE. A roof truss comprises a number of bars which are connected together at their ends. In this chapter so far the impression may have been given that a freebody should be separated from the complete structure at the junction between physical components. In fact the components themselves may be subdivided.

A single beam may be arbitrarily divided into two parts and one part considered as a freebody in order that we may find the force transmitted at the interface between the two parts. This particular problem is considered in Chapter 7.

Problems

6.1 For each of the structures shown in Figure P6.1, draw freebody diagrams of the complete system and of each of the components including the pins which connect the various bars.

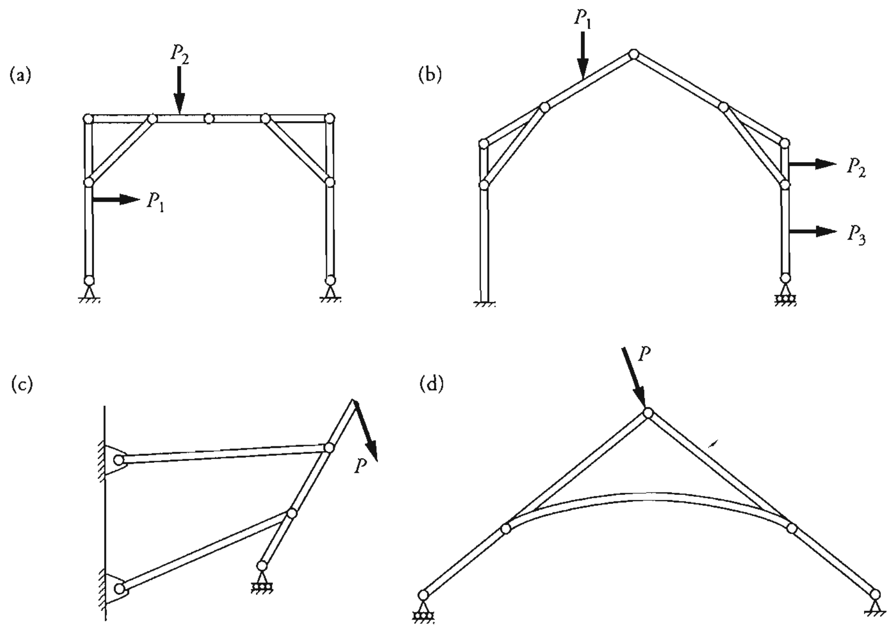


Figure P6.1 (continues)

(e)

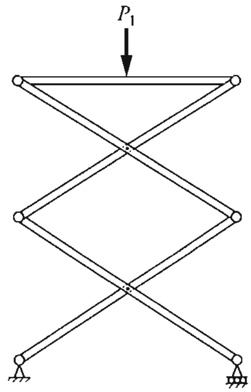
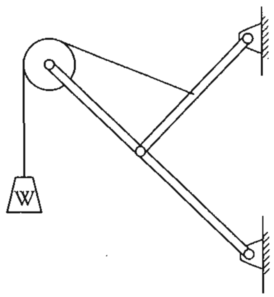


Figure P6.1

6.2 For each of the structures shown in Figure P6.2, draw a freebody diagram for each bar, each pin, the rope and the pulley.

(a)



(b)

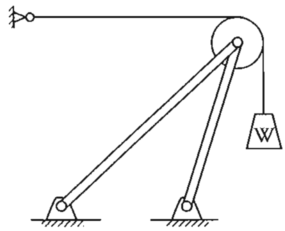


Figure P6.2

6.3 For the pin-jointed truss of Figure P6.3, draw a freebody diagram for each bar and for each pin.

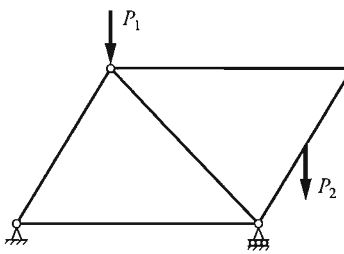


Figure P6.3

6.4 Figure P6.4 shows a plane frame supported on pin supports at A and E. The two rigid components are joined by a frictionless hinge at C. Determine the reaction components at A and E.

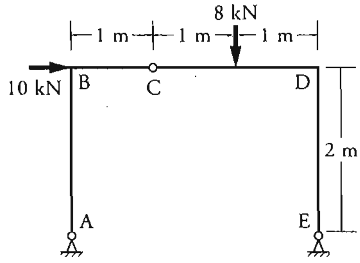


Figure P6.4

6.5 Three cylindrical drums A, B and C rest in a trough, as shown in Figure P6.5. The diameters of the drums are 0.25 m, 0.375 m and 0.500 m respectively. The drums with contents weigh $W_1 = 400$ N, $W_2 = 900$ N and $W_3 = 1600$ N. Determine the reactions R_1 to R_6 between the drums and between the drums and the trough. Also find the reactions R_7, between the trough and the ground, and R_8, between the strut and the ground. (Neglect friction and the weight of the trough.)

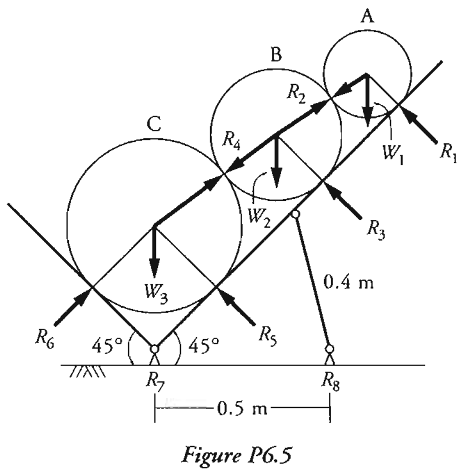


Figure P6.5

6.6 A semicircular trough of weight $2W$, and radius 0.5 m is supported in the cradle shown in Figure P6.6. The members ABC and DBE are continuous past B where they are connected by a pin. Two weights W are supported by a cable which passes over smooth pins at A and D. Find the reactions at C and E and the force exerted on member AC by the pin at B. Ignore friction between the trough and the cradle.

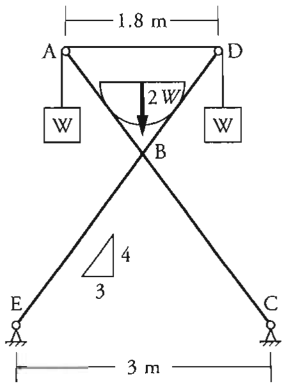


Figure P6.6

6.7 The primitive hoisting frame in Figure P6.7 is made up of two straight bars ABC and DBEF which are connected by a pin at B, and a cable at F. The winding machine is located at G. The member AC is 5 m long, DE is 5 m and EF is 1.5 m. The pulleys C and E are of negligible size. Draw freebody diagrams of the bars ABC and DBEF and write equations of equilibrium for each freebody. Are there sufficient equations to enable the reactions at A and D, the force in the cable and forces across the pins at B, C and E to be found? If not, express these forces in terms of the force X in the cable.

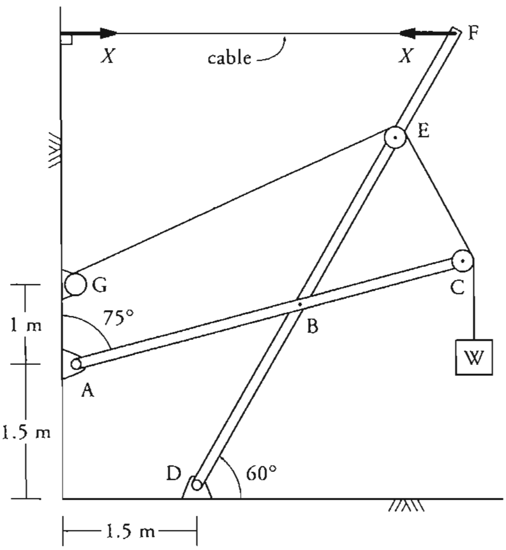
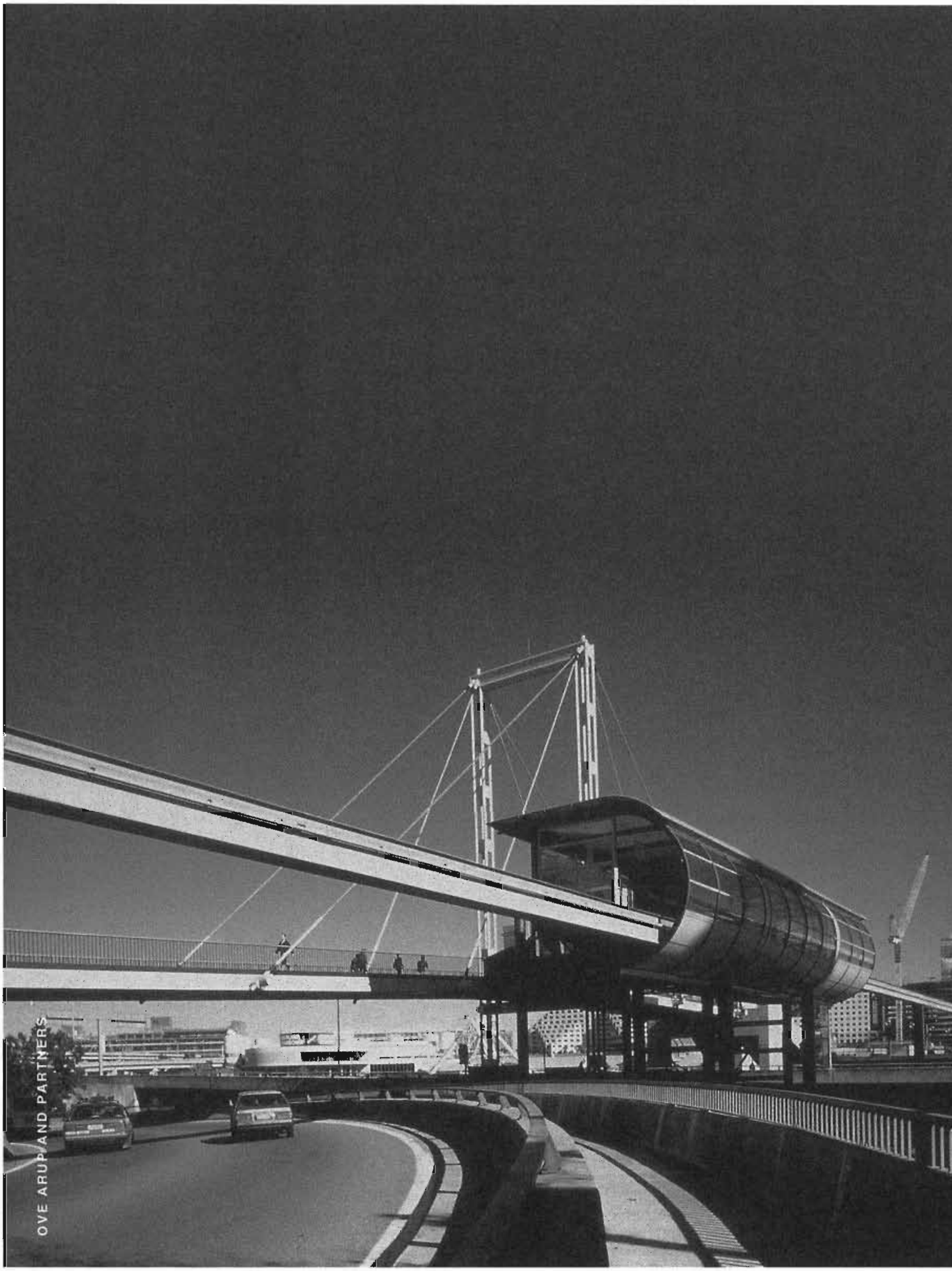


Figure P6.7



OVE ARUP AND PARTNERS

Internal Actions in Beams

7.1 Internal actions at a cross-section

It was pointed out in Chapter 6 that internal forces may be studied by 'cutting' the original body and drawing a freebody diagram of one part. If the cutting surface is suitably chosen, the force in question is then an external force acting on the partial freebody and it can be determined by statics.

This method is of importance when studying the forces and couples acting within a rigid beam by reason of externally applied loads. That such internal actions must exist, can be seen by considering a beam AB in Figure 7.1, supported at each end and carrying a weight W. In this condition, each part of the beam is at rest. Now if the beam is cut through at a cross-section such as C, each part collapses (i.e. neither AC nor CB is in equilibrium). Some force must therefore have been transmitted previously across the section C in order to maintain equilibrium. The details of this force may be ascertained by applying the laws of statics either to the body AC or to the body CB, since the cross-section C is an external surface of both of these bodies.

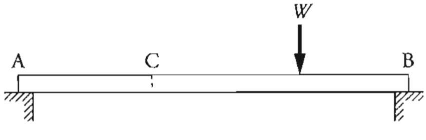


Figure 7.1

The present treatment will be confined to beams lying in one plane and subjected to forces lying in the same plane.

Consider the straight beam ABCD in equilibrium under the action of the forces shown in Figure 7.2a. Suppose that it is required to determine the force transmitted across the section C which is 3 m from A. The bar is cut through at C and the equilibrium of either AC or CD is examined. Figure 7.2b shows the freebody diagram of the portion AC which is acted upon by the given forces at A and B and by an unknown force Q. Since AC is in equilibrium, the unknown force Q must be the equilibrant of the other two and is determined by the method of Section 5.1.

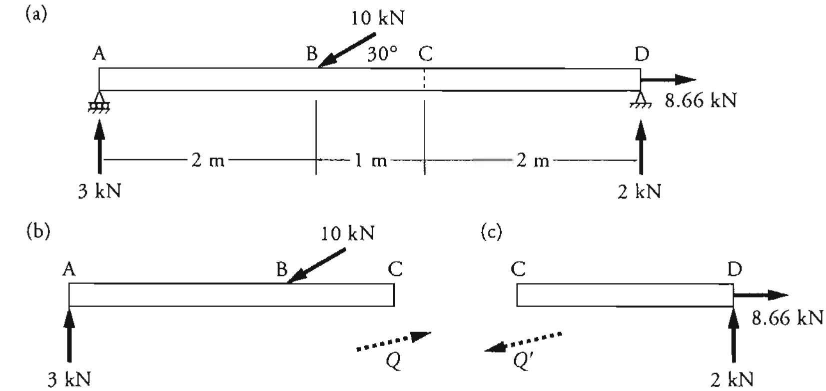


Figure 7.2

This equilibrant Q is the force exerted on body AC by body CD. An equal and opposite force Q' must be exerted upon CD by AC, and this force may be found if the equilibrium of freebody CD is considered (Figure 7.2c).

The fact that the equilibrant force does not actually pass through the cross-section C is of little importance since it may be replaced by a force at C together with a couple (Figure 7.3a). It is convenient in practice to resolve the force into its two components, one parallel to the axis of the bar and one perpendicular to it (Figure 7.3b). Thus in Figure 7.2b the equilibrant Q is expressed by the three quantities: magnitude, direction and position of a single force, whereas in Figure 7.3b it is replaced by a statically equivalent system comprising two forces S and N and a couple M, all acting at the centroid of the cross-section C. It is more convenient to determine these components directly rather than to determine the equilibrant as a single force.

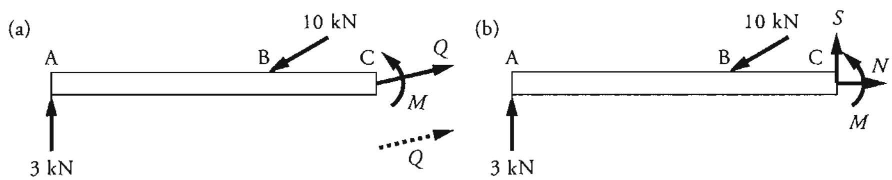


Figure 7.3

This topic deals with beams whose cross-sectional dimensions are small compared with their length. In many examples such beams will be represented by a single line which is the longitudinal axis of the bar. Where this axis is curved, the component forces of the internal action at any section are taken parallel and perpendicular to the tangent to the curve at that section.

The component parallel to the axis is called the *axial force*. This is usually abbreviated to *A.F.* and the force is denoted by N. The component perpendicular to the axis is called the *shear force*. This is abbreviated to *S.F.* and is denoted by S (or sometimes V). The couple is called the *bending moment*. This is abbreviated to *B.M.* and is denoted by M.

DEFINITIONS

At any given cross-section of a beam there may exist an internal action.

The *axial force* is the component of this internal action in a direction parallel to the longitudinal axis of the beam at the section.

The *shear force* is the component of this internal action in a direction normal to the longitudinal axis of the beam at the section.

The *bending moment* is the moment of the internal action about the point where the axis of the bar intersects the given cross-section (i.e. about the centroid of rhe cross-section: see Appendix).

Not only is it convenient to express the internal action at a cross-section of a beam by the axial force, shear force and bending moment but engineers find that this procedure facilitates the design of the beam.

7.2 Sign conventions

Before doing numerical examples we must first discuss sign conventions. Figure 7.2 shows that at any cross-section there are two internal forces: the force Q exerted by CD on AC, and the force Q' exerted by AC on CD. In this illustration, if we determine the bending moment by considering freebody AC we find that M is anticlockwise (Figure 7.3). On the other hand if we consider the equilibrium of CD we find that M is clockwise, being the reaction to the couple exerted on AC. Thus if the terms clockwise and anticlockwise are adopted as criteria of positive and negative, the sign of the bending moment would differ according to whether AC or CD is considered. Besides being inconvenient, this does not reflect the significance of the term bending moment. Similar remarks apply to the axial force and shear force. A more satisfactory sign convention is obtained by considering the effect which these actions have on a small portion of the bar.

Suppose that at the part of the bar under consideration two cuts C_1 and C_2 are made very close together, thus isolating a small element of the bar as indicated by the side elevation shown in Figure 7.4. At each of the cuts C_1 and C_2 are shown the three actions and their reactions. The actions at C_1 and C_2 will be nearly the same since C_1 and C_2 are close together. The element is thus subjected to three *pairs* of actions which tend to distort its shape. It is most important not to confuse these internal actions with forces and couples. A *force* would act either to the left or to the right. For a horizontal member (as in Figure 7.4), an axial force is a *pair* of forces, one of which acts to the left and one to the right. Similarly, a shear force is a *pair* of forces, one up and one down, and a bending moment is a *pair* of couples, one clockwise and one anti-clockwise.

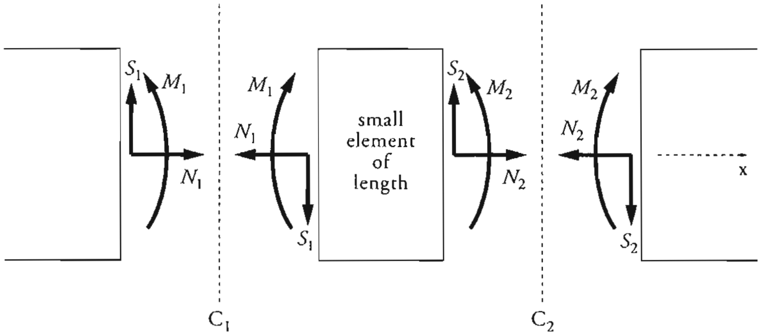


Figure 7.4

The small element in Figure 7.4 is subjected to a pair of forces N_1 and N_2 which in this instance tend to increase its length (i.e. the element is in tension). The axial force N is said to be positive if it puts the element in tension, and negative if it causes compression.

The element is subjected to a pair of forces S_1 and S_2 which tend to cause a shearing type of deformation. If the forces S_1 and S_2 are in the directions shown in Figure 7.4, then the shear force S is said to be positve.

The element is subjected to a pair of couples M_1 and M_2 which tend to bend it. The bending moment M is said to be positive if the element bends concavely upward (or concavely towards a specified positive direction if the bar is not horizontal).

These sign conventions are summarized in Figure 7.5.

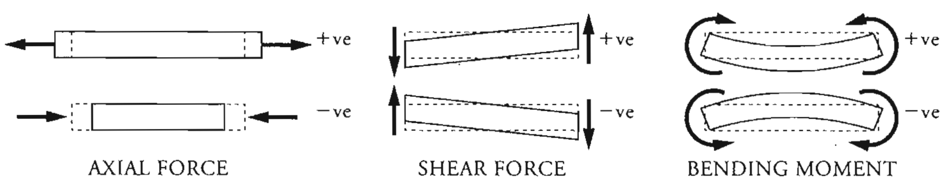


Figure 7.5

The above definition of the sign conventions indicates the physical significance of these signs. In many practical problems the signs may be determined by imagining the nature of the deformation. For instance, a simply supported beam carrying downward loads will bend so that it becomes concave on the top. Provided the x axis is taken from left to right and the y axis upward, such bending will be denoted as positive.

In more general problems, the physical determination of signs is less simple and analytical rules are more convenient. This is important also for computer calculations, since the computer cannot imagine the deformed shape of the beam segment. We define an x axis as running along the beam as in Figure 7.6. Then, if the beam is cut at a section C, the x axis is directed outward on one cut face, the left-hand face in Figure 7.6, and we may call this the *face of positive incidence.* If the forces N, S and M on this face *agree* with the direction of the x and y axes, then they are defined as positive. On the other cut face, the x axis is directed inward as on the right-hand side of Figure 7.6. This face is called *the face of negative incidence* and on this face N, S and M are positive if they *disagree* with the x and y axes.

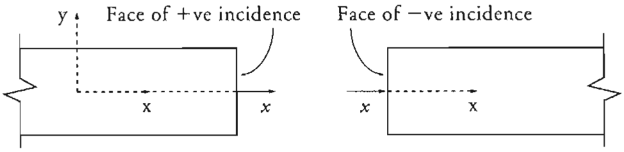


Figure 7.6

Figure 7.7 shows the beam of Figure 7.2a (page 90) cut at C. On each freebody, the internal actions are shown acting in their positive sense as defined above. On the left-hand freebody, which has the positive cut face, N and S are shown in the directions of x and y and M anticlockwise. These indicate the positive directions. In Figure 7.7b the cut face C is negative, so the positive directions of N and S are opposed to x and y and positive M is clockwise.

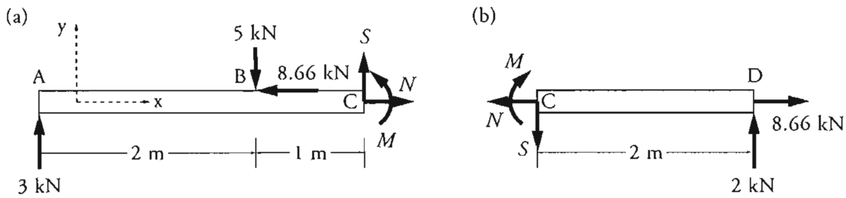


Figure 7.7

Either freebody may now be used, and the equilibrium equations will give N, S and M with the correct signs.

EXAMPLE 7.1

For the beam shown in Figure 7.2a (page 90) determine the internal actions at the cross-section C. First consider the left-hand freebody, then the right-hand freebody (as shown in Figure 7.7 above).

SOLUTION

The arithmetic is simplified if the oblique force is replaced by its x and y components.

1. Figure 7.7a shows the freebody ABC which includes all forces acting on ABC including the internal actions at C. At C, the internal actions are shown as positive: the arrow N is drawn in the x direction, S is drawn in the y direction, and M is anticlockwise. Then for ABC:

$$\Sigma F_x = 0: \quad -8.66 + N = 0 \quad \therefore N = +8.66 \text{ kN}$$

$$\Sigma F_y = 0: \quad +3 - 5 + S = 0 \quad \therefore S = +2 \text{ kN}$$

Taking moments about C:

$$\circlearrowleft\!+ \quad -(3 \times 3) + (5 \times 1) + M = 0 \quad \therefore M = +4 \text{ kNm}$$

2. Figure 7.7b shows the freebody CD. At C, the arrow N is drawn in the negative x direction, S is drawn in the negative y direction, and M is clockwise. Then for CD:

$$\Sigma F_x = 0: \qquad -N + 8.66 = 0 \qquad \therefore N = +8.66 \text{ kN}$$

$$\Sigma F_y = 0: \qquad -S + 2 = 0 \qquad \therefore S = +2 \text{ kN}$$

Taking moments about C:

$$\circlearrowleft + \quad -M + (2 \times 2) = 0 \qquad \therefore M = +4 \text{ kNm}$$

In the case of a cantilever, all reactions are at one end of the beam. It is not necessary to find these reactions before calculating internal actions. One may consider the freebody which includes the free end of the cantilever.

If the beam is not horizontal at the section where N, S and M are required, the x axis is taken along the beam axis and the y axis is also tilted by the same amount. Thus N is always along the beam and S normal to the beam at the section in question (see Example 7.2 below).

For a completely general case, the direction of the x axis at the given section may be defined arbitrarily, but it must be tangential to the beam at that section. For instance, if the beam is vertical at the given section, x may be defined as upward or downward. This will define *member axes* at this section. The choice of member axes will define the meaning of positive B.M. and S.F.

The procedure for finding N, S and M in any planar problem can be summarised as follows:

1. If necessary, determine the reactions at the supports. This will not be necessary if all external forces to one side of the given section are already known.

2. Make a freebody drawing of the portion of the bar to one side of the given cross-section. Include all forces which act on this freebody. Do not include any forces which act on the discarded portion of the bar.

3. At the cut face insert arrows representing the positive directions of N, S and M as explained above.

4. Write the three equations of equilibrium for the freebody and thus evaluate N, S and M. Since N, S and M are the only unknown forces acting on the freebody, they can always be determined using the three equations of equilibrium.

Example 7.2

A bent bar ABCD is loaded as shown in Figure 7.8 and is supported by a pin at A and on rollers at D. Find the reactions at A and D. Then calculate the A.F., S.F., and B.M. at the mid-point K of the inclined portion. Show that the same results are obtained by considering AK as by considering KD.

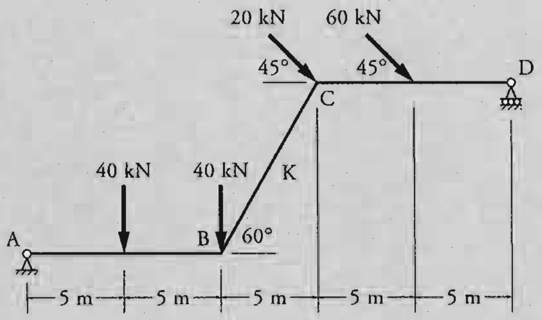


Figure 7.8

SOLUTION

1. **Reactions:** Replace the inclined forces by their vertical and horizontal components. Let the components of the reactions at A be V_A and H_A. Let the reaction at D be V_D (since no horizontal component force can be resisted at D). The freebody diagram of the whole structure is shown in Figure 7.9.

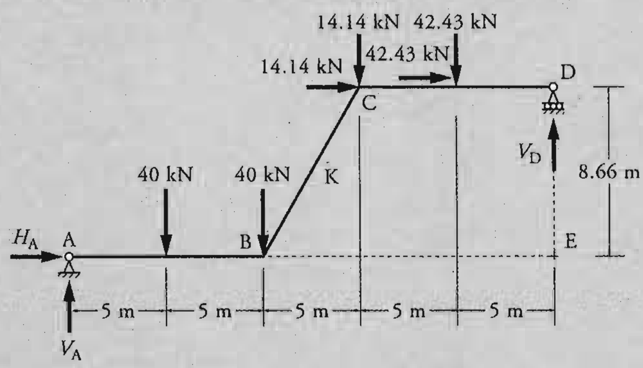


Figure 7.9

When finding reactions it is desirable to calculate each independently as far as possible so that an error in the first does not invalidate the others. Secondly, it is essential to check the reactions by an alternative calculation before proceeding with the problem.

To obtain an equation involving only V_D, take moments about A:

$$\circlearrowleft + \; -(40 \times 5) - (40 \times 10) - (14.14 \times 15) - (14.14 \times 8.66) - (42.43 \times 20) - (42.43 \times 8.66) + (V_D \times 25) = 0$$

$\therefore\; V_D = 86.02$ kN

To obtain an equation involving only V_A, take moments about E, which is the intersection of H_A and V_D:

$$\curvearrowleft + \quad -(V_A \times 25) + (40 \times 20) + (40 \times 15) + (14.14 \times 10) - (14.14 \times 8.66) + (42.43 \times 5) - (42.43 \times 8.66) = 0$$

$\therefore V_A = 50.55$ kN

To obtain an equation involving only H_A, resolve horizontally (i.e. at right angles to V_A and V_D).

$\overset{+}{\rightarrow} \quad +H_A + 14.14 + 42.43 = 0 \qquad \therefore H_A = -56.57$ kN

The reaction H_A is thus in the opposite direction to that assumed.

To check: Taking moments about C yields an equation in which all reactions are involved. Using the above values for H_A, V_A and V_D:

$$\curvearrowleft + \quad (H_A \times 8.66) - (V_A \times 15) + (40 \times 10) + (40 \times 5) - (42.43 \times 5) + (V_D \times 10) = 0$$

$$(-56.57 \times 8.66) - (50.55 \times 15) + (40 \times 10) + (40 \times 5) - (42.43 \times 5) + (86.02 \times 10) = 0$$

This equation checks, and the values are either correct or have compensating errors, which is very unlikely.

2. **Internal actions:** Cut the beam at K and consider the freebody to the left of K (Figure 7.10). Axes x and y are taken at K so that x is in the direction of the beam at K. Since the cut face K is here a face of positive incidence, the positive directions of N, S and M are as shown in Figure 7.10.

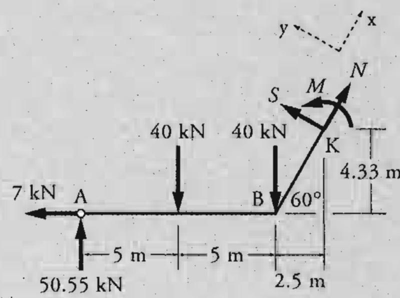


Figure 7.10

Resolving forces parallel to N gives:

$$+\nearrow \quad -(56.57 \cos 60) + (50.55 \cos 30) - (40 \cos 30) - (40 \cos 30) + N = 0$$

$\therefore N = 53.79$ kN

Resolving parallel to S gives:

$$\nwarrow + \quad (56.57 \sin 60) + (50.55 \sin 30) - (40 \sin 30) - (40 \sin 30) + S = 0$$

$\therefore S = -34.26$ kN

Taking moments about K gives:

$$\curvearrowleft + \quad -(56.47 \times 4.33) - (50.55 \times 12.5) + (40 \times 7.5) + (40 \times 2.5) + M = 0$$

$\therefore M = +476.78$ kNm

3. **Internal actions:** Cut the beam at K and now consider the freebody to the right of K (KD in Figure 7.11).

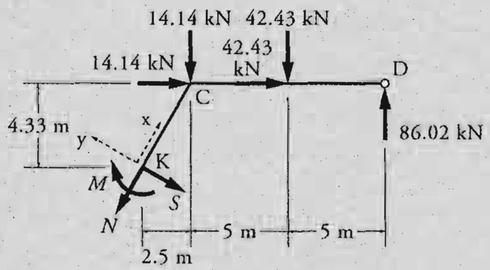


Figure 7.11

Since the face K in this freebody is a face of negative incidence, the positive directions of the internal actions are opposed to the direction of the x and y axes.

Resolving parallel to N gives:

$$+\swarrow \quad -(14.14 \cos 60) + (14.14 \cos 30) - (42.43 \cos 60) + (42.43 \cos 30)$$
$$- (86.02 \cos 30) + N = 0$$

$\therefore N = +53.79$ kN

Resolving parallel to S gives:

$$\searrow + \quad +(14.14 \sin 60) + (14.14 \sin 30) + (42.43 \sin 60) + (42.43 \sin 30)$$
$$- (86.02 \sin 30) + S = 0$$

$\therefore S = -34.26$ kN

Taking moments about K gives:

$$\curvearrowleft + \quad -M - (14.14 \times 4.33) - (14.14 \times 2.5) - (42.43 \times 4.33) - (42.43 \times 7.5)$$
$$+ (86.02 \times 12.5) = 0$$

$\therefore M = +476.75$ kNm

(The slight difference between the two sets of results is due to rounding off.)
The internal actions at a given section K may be computed either from the forces to the left of the section or from the forces to the right, each method giving the same values. This is because the two sets of forces together form a system in equilibrium.

When several forces have to be resolved in order to obtain S and N as in this example, it may be simpler initially to find the vertical and horizontal components (or whatever components are most convenient) of the internal action at the given section. These can then be resolved into directions tangential and normal to the bar. This can be illustrated with reference to Example 7.2.

ALTERNATIVE SOLUTION

Suppose that the reactions have been determined as above. Consider portion AK. Temporarily represent the internal force at K by its vertical and horizontal components V and H and the bending moment M (as shown in Figure 7.12a). The bending moment is found as before by taking moments about K. Then H and V are determined by resolving forces horizontally and vertically, respectively.

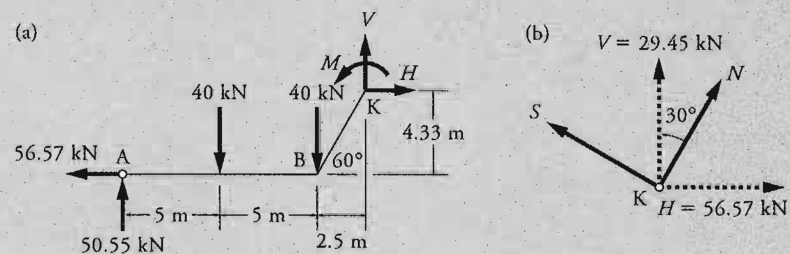


Figure 7.12

That is: $-56.57 + H = 0 \qquad \therefore H = 56.57$ kN

$50.55 - 40 - 40 + V = 0 \qquad \therefore V = 29.45$ kN

These two components are now replaced by the components N and S in the desired direction (Figure 7.12b).

$$N = 29.45 \cos 30 + 56.57 \sin 30 = 53.79 \text{ kN}$$

$$S = 29.45 \sin 30 - 56.57 \cos 30 = -34.27 \text{ kN}$$

Example 7.2 also illustrates the following relationships:

1. The B.M. at any point K of a bar is equal to the algebraic sum of the moments about K of all forces to one side of K. The moment of a particular force is positive or negative according to whether that force tends to cause positive or negative bending at K.

2. The S.F. at any point K of a bar is equal to the algebraic sum of the components, in a direction normal to the longitudinal axis of the bar at K, of all forces to one side of K. The sign of each force is determined according to whether it causes positive or negative shear at K.

3. The A.F. at any point K of a bar is equal to the algebraic sum of the components in a direction tangential to the longitudinal axis of the bar at K, of all forces to one side of K. The sign of each force is determined according to whether it causes positive or negative axial force at K.

When the axis of the beam is curved at the section where it is required to find *N*, *S* and *M*, axes x and y are taken at this particular section such that x is tangential to the beam. This is illustrated in Example 7.3.

EXAMPLE 7.3

The semi-circular bar ABCD of radius 8 m is loaded by forces as shown at A, B and D in Figure 7.13a. Find the components of the internal action at the cross-section C.

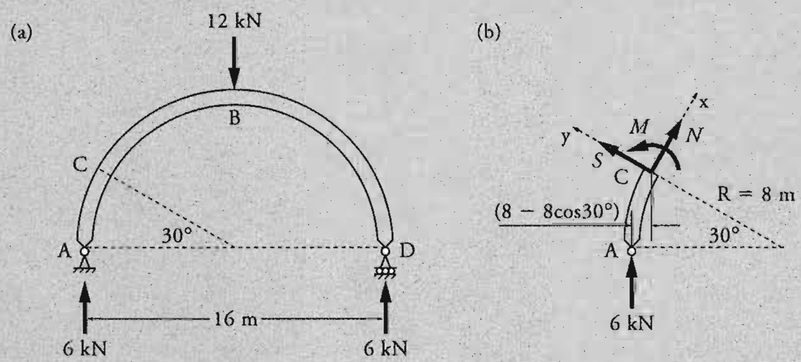


Figure 7.13

SOLUTION

Cut the bar at C and consider the freebody AC to the left of C (Figure 7.13b). At C, take the x axis tangential to the bar. The cut face is here a face of positive incidence, so the positive senses of *N*, *S* and *M* are as shown, with *N* in the x direction, *S* in the y direction and *M* anticlockwise.
Resolving parallel to *N* gives:

$$+\nearrow \quad 6\cos 30 + N = 0 \qquad \therefore N = -5.20 \text{ kN}$$

Resolving parallel to *S* gives:

$$\nwarrow+ \quad 6\sin 30 + S = 0 \qquad \therefore S = -3.0 \text{ kN}$$

Taking moments about C gives:

$$\circlearrowleft+ \quad M - 6(8 - 8\cos 30) = 0 \qquad \therefore M = +6.43 \text{ kNm}$$

7.3 Beams with distributed loading

Distributed loading was discussed in Section 4.4. When a beam supports a floor, the weight of the floor, as well as the weight of the beam itself, constitutes a distributed loading. If each unit length of beam supports the same load, the load is *uniformly distributed*. In order to determine the reactions, the equilibrium of the whole beam is considered, and the distributed load is replaced by its resultant. However, to determine internal actions at a section, the beam is cut at that section. The freebody to one side of the cut may well support only part of the distributed load and this part must be included in the freebody diagram. After the cut has been made and the freebody drawn, the part of the distributed load acting on the freebody is then replaced by its resultant.

EXAMPLE 7.4

A straight beam of length 12 m is supported at A and B and is loaded as shown in Figure 7.14. Find the bending moment at the mid-point C.

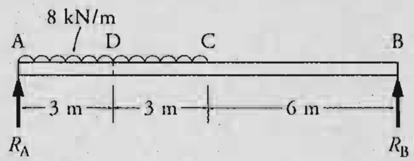


Figure 7.14

SOLUTION

To find the reactions at A and B, the distributed loading is temporarily replaced by its resultant, which is a force of 48 kN at the mid-point of AC.
Taking moments about A:

$$\circlearrowleft + \quad -(48 \times 3) + (R_B \times 12) = 0 \qquad \therefore R_B = 12 \text{ kN}$$

Taking moments about B:

$$\circlearrowleft + \quad -(R_A \times 12) + (48 \times 9) = 0 \qquad \therefore R_A = 36 \text{ kN}$$

Check by resolving vertically:

$$+\uparrow \quad R_A + R_B - 48 = 0 \qquad 36 + 12 - 48 = 0 \qquad \text{O.K.}$$

To find the B.M. at C it is easier to consider the part CB (Figure 7.15) rather than AC. In this case the cut face is one of negative incidence, hence the positive senses of N and S are opposite to x and y and M is clockwise positive.

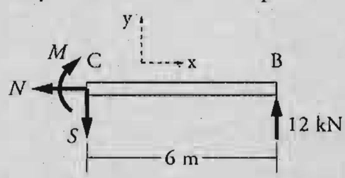


Figure 7.15

Taking moments about C:

$$\circlearrowleft + \quad -M_C + (12 \times 6) = 0 \qquad \therefore M_C = +72 \text{ kNm}$$

EXAMPLE 7.5

In the beam of the previous example find the B.M. and S.F. at the point D.

SOLUTION

The reactions R_A and R_B having been found as before, the beam is cut at D, *the distributed load being left part on each side as it actually occurs.* The part AD is now loaded as in Figure 7.16, while the part DB is loaded as in Figure 7.17.

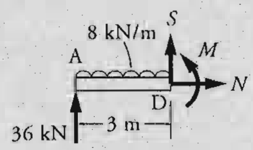


Figure 7.16

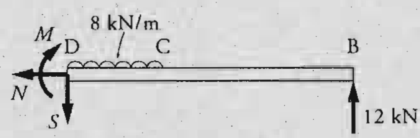


Figure 7.17

Consider the part AD shown in Figure 7.16 and replace the distributed load by its resultant which is 24 kN downward at 1.5 m from A. Taking moments about D gives:

$$\curvearrowright\!+\ (36 \times 3) - (24 \times 1.5) - M = 0 \qquad \therefore\ M = +72 \text{ kNm}$$

and resolving vertically gives:

$$+\uparrow\ 36 - 24 + S = 0 \qquad \therefore\ S = -12 \text{ kN}$$

Alternatively, consider the part DB shown in Figure 7.17. Taking moments about D gives:

$$\curvearrowright\!+\ M + (24 \times 1.5) - (12 \times 9) = 0 \qquad \therefore\ M = +72 \text{ kNm}$$

and resolving vertically gives:

$$+\uparrow\ -S - (8 \times 3) + 12 = 0 \qquad \therefore\ S = -12 \text{ kN}$$

EXAMPLE 7.6

Find the B.M. and S.F. at the mid-span (point C) of the simply-supported beam shown in Figure 7.18 which supports a linearly varying load.

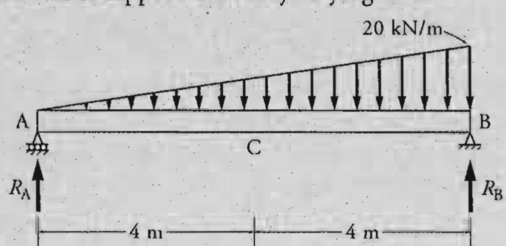


Figure 7.18

SOLUTION

The resultant of the linearly varying load is 0.5 × 20 × 8 = 80 kN acting 5.33 m from A (i.e. through the centroid of the triangular load intensity diagram). Taking moments about A gives:

$$\curvearrowleft + \quad (80 \times 5.33) - (R_B \times 8) = 0 \qquad \therefore R_B = 53.33 \text{ kN}$$

Taking moments about B gives:

$$\curvearrowleft + \quad (R_A \times 8) - (80 \times 2.67) = 0 \qquad \therefore R_A = 26.67 \text{ kN}$$

Checking by resolving vertically gives:

$$+\uparrow \quad R_A - 80 + R_B = 0 \qquad 26.67 - 80 + 53.33 = 0 \qquad \text{O.K.}$$

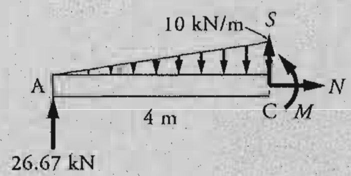


Figure 7.19

By taking moments about C in the freebody shown in Figure 7.19:

$$\curvearrowleft + \quad (26.67 \times 4) - (0.5 \times 10 \times 4 \times \tfrac{4}{3}) - M = 0 \therefore M = +80.0 \text{ kNm}$$

Applying the vertical force equilibrium equation to the freebody of Figure 7.19 gives:

$$+\uparrow \quad 26.67 - (0.5 \times 10 \times 4) + S = 0 \qquad \therefore S = -6.67 \text{ kN}$$

7.4 Variation of A.F., S.F. and B.M. along a beam

The internal actions vary from point to point along a beam. In simple cases it is possible to express this variation in algebraic terms. If the bending moment is computed for a typical point at a distance of x metres from a chosen origin (usually the left-hand end), an expression is obtained for M in terms of x. Similar functions may be derived for the axial force and shear force. Usually such expressions are valid only over a limited part of the beam. This is the case if the loading is discontinuous (i.e. if it consists of concentrated loads, or of distributed loads over some parts only).

It is frequently convenient to illustrate the variation of bending moment by plotting a graph of M against distance along the beam. Such a graph (often called a *bending moment diagram*) may be obtained either by plotting the algebraic functions mentioned above, or by simply calculating M at a number of isolated points along the beam and plotting these values as ordinates.

EXAMPLE 7.7

The beam shown in Figure 7.20 is a main floor beam which supports secondary beams at the points B and C. The loads which these secondary beams exert on the beam AD are considered approximately as point loads, as shown. Express the B.M. and S.F. as functions of x (the distance from end A) for each of the segments AB, BC and CD. Plot graphs of B.M. and S.F. for the complete beam.

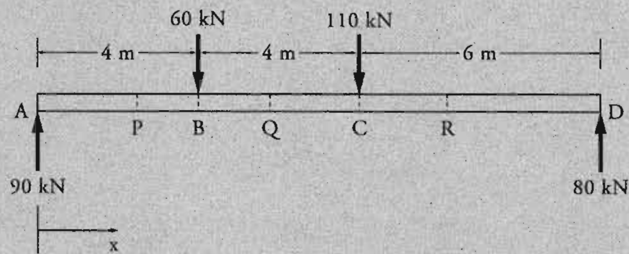


Figure 7.20

SOLUTION

1. First consider a section lying within segment AB at x m from A (shown as P in Figure 7.20). Consider the freebody to the left of P (Figure 7.21a).

At P: $M = 90x$ kNm **(7.1)**

$S = -90$ kN **(7.2)**

Since P is typical of all sections between A and B, these expressions apply over the whole of this segment (i.e. from $x = 0$ to 4 m).

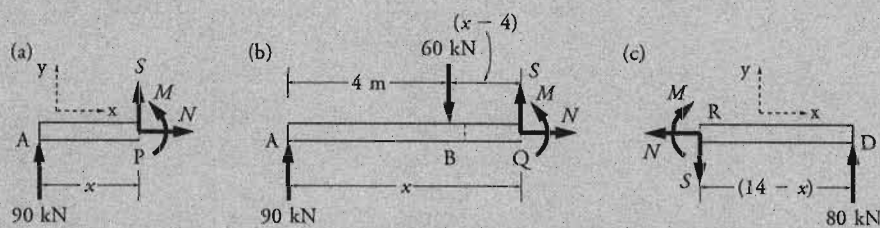


Figure 7.21

2. Now consider a section Q which lies between B and C. The freebody to the left of Q is shown in Figure 7.21b. By equilibrium:

$$\circlearrowright + \quad 90x - 60(x-4) - M = 0 \quad \therefore M = 30x + 240 \text{ kNm} \qquad \textbf{(7.3)}$$

$$+\uparrow \quad 90 - 60 + S = 0 \quad \therefore S = -30 \text{ kN} \qquad \textbf{(7.4)}$$

These expressions apply over segment BC (i.e. from x = 4 m to 8 m).

3. Now consider a section R which lies between C and D. For this case it is easier to consider the freebody to the right of R, as shown in Figure 7.21c.

At R: $M = 80(14 - x) = 1120 - 80x$ kNm **(7.5)**

$S = 80$ kN **(7.6)**

and these expressions are valid for x = 8 m to 14 m.

4. In Figure 7.22, the three expressions for M are plotted against x. It is important to remember that each equation is only valid for a certain range of x.

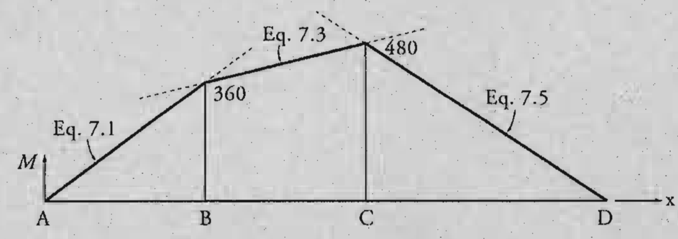


Figure 7.22

Since each of Equations 7.1, 7.3 and 7.5 are linear, it would have been sufficient to compute M at the points B and C. The graph could then have been drawn.

In Figure 7.23, the three expressions for S are plotted against x. These expressions indicate that S is constant between one point load and the next.

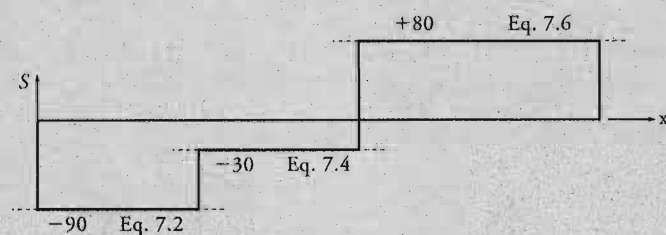


Figure 7.23

EXAMPLE 7.8

For the beam shown in Figure 7.14 (page 100), derive expressions for M and S over the segments AC and CB. Plot graphs of these functions.

SOLUTION

1. Consider a section P within segment AC and x metres from A. Figure 7.24a shows the freebody to the left of P. Replace the distributed load by its resultant. Then by equilibrium:

$\circlearrowleft + \; 36x - 8x(x/2) - M = 0 \qquad \therefore M = 36x - 4x^2$ kNm **(7.7)**

$+\uparrow \; 36 - 8x + S = 0 \qquad \therefore S = 8x - 36$ kN **(7.8)**

These expressions are valid only in the segment AC when $0 \leq x \leq 6$ m.

2. Consider a section Q within segment CB and x metres from A. Figure 7.24b shows the freebody to the right of Q.

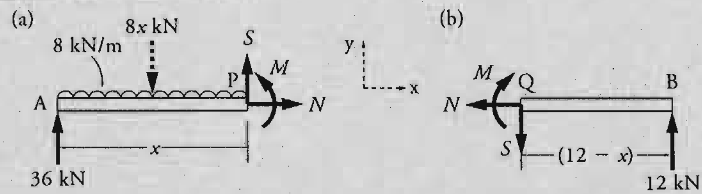


Figure 7.24

At Q: $M = 12\,(12 - x) = 144 - 12x$ kNm **(7.9)**

$S = 12$ kN **(7.10)**

These expressions are valid only in the segment CB when $6 \leq x \leq 12$ m.

3. Graphs of M and S are shown in Figure 7.25.

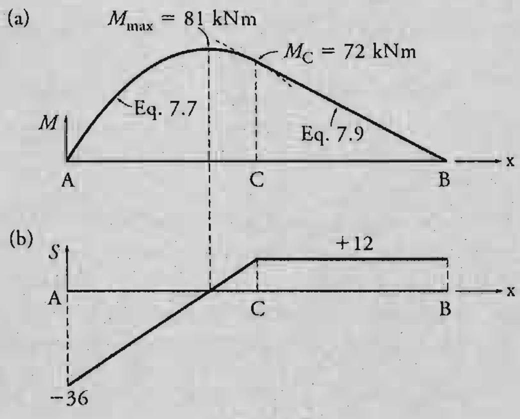


Figure 7.25

It can be seen in Example 7.8 that over the part of the beam which carries uniformly distributed load the shear force varies linearly and the bending moment varies parabolically. Over the unloaded part of the beam, S is constant and M varies linearly. This is discussed again in the next section.

7.5 Equilibrium of a small beam element

For a straight beam loaded normal to its axis, simple relationships exist between the load intensity, the shear force and the bending moment. These relationships can be developed from the equilibrium equations of a small element of the beam. It is assumed that the beam carries distributed load, the intensity of which varies from point to point. The *load intensity*, denoted by w, is the load per unit length of a beam at the particular section considered. The upward direction is taken as positive for all quantities.

Figure 7.26 shows a beam element of length dx isolated by two cuts dx apart. At this location, the load intensity is w kN/m. Hence the total external load on the length dx is wdx kN. At the left-hand cut, the internal actions are M and S, as shown, and at the right-hand cut the internal actions have changed by small amounts dM and dS, respectively.

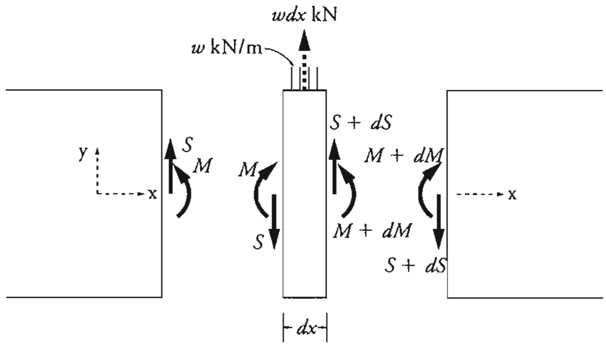


Figure 7.26

For vertical equilibrium of the element:

$$+\uparrow \quad -S + (S + dS) + wdx = 0 \qquad \therefore = w - \frac{dS}{dx} \tag{7.11}$$

Taking moments about the centre of the element, we obtain:

$$\curvearrowleft\!+ \quad S\frac{dx}{2} + (S + dS)\frac{dx}{2} - M + (M + dM) = 0$$

and if the product of the two infinitesimal quantities dS and dx is neglected:

$$S = -\frac{dM}{dx} \tag{7.12}$$

Note that the change in shear from one point to another is numerically equal to the load between the two points. The increment of bending moment balances the couple formed by the shears on each side of the element. The signs in the two equations depend on the sign convention adopted for w, S and M.

It is often more useful to express the equations in the alternative forms:

$$S = -\int w\, dx \tag{7.13}$$

$$M = -\int S\, dx \tag{7.14}$$

The point of maximum moment occurs when the slope of the bending moment diagram is zero. Equation 7.12 indicates that this point coincides with the point of zero shear, i.e. when $S = 0$.

EXAMPLE 7.9

For the beam of Example 7.4 (Figure 7.14 page 100), derive expressions for the S.F. and B.M. over the portions AC and CB, by means of Equations 7.13 and 7.14, and find the position and magnitude of the maximum bending moment.

SOLUTION

For the part AC: $w = -8$ (negative downwards)

From Equation 7.13: $S_{AC} = -\int w\,dx = -\int -8\,dx = +8x + C_1$

At the left-hand support, the value of S is equal to $-R_A$ or -36 kN. Thus when $x = 0$, $S = -36$ and therefore $C_1 = -36$ kN. Thus:

$$S_{AC} = 8x - 36 \text{ kN} \qquad \textbf{(7.15)}$$

This is the same as Equation 7.8.
From Equation 7.14:

$$M_{AC} = -\int S_{AC}\,dx = -\int(8x - 36)dx = -4x^2 + 36x + C_2$$

At A, $M = 0$ because the beam is simply supported. Thus when $x = 0$, $M = 0$ and therefore $C_2 = 0$. Thus:

$$M_{AC} = 36x - 4x^2 \text{ kNm} \qquad \textbf{(7.16)}$$

This is the same as Equation 7.7.
The segment CB is unloaded, so $w = 0$

$$\therefore S_{CB} = -\int 0\,dx = C_3$$

At B, where $x = 12$: $S = R_B = +12$ kN.

Hence: $C_3 = 12$ and $S_{CB} = +12$ kN **(7.17)**

This agrees with Equation 7.10.
From Equation 7.14:

$$M_{CB} = -\int 12dx = -12x + C_4$$

At B, where $x = 12$, $M = 0$ since the beam is simply supported and hence $C_4 = 144$.

Thus: $M_{CB} = 144 - 12x$ kNm **(7.18)**

This agrees with Equation 7.9.
From Equation 7.15, $S = 0$ when $x = 4.5$ m. The maximum bending moment occurs at this point, and from Equation 7.16:

$$M_{max} = (36 \times 4.5) - (4 \times 4.5^2) = 81.0 \text{ kNm}$$

as shown in Figure 7.25a.

In this section we have considered the equilibrium of a small element and then integrated. In the previous section we considered the equilibrium of a freebody of finite size. The two results in each case depend only upon statics.

EXAMPLE 7.10

For the beam of Example 7.6 (Figure 7.18 page 101), derive expressions for the S.F. and B.M. anywhere on the span. Hence, calculate the position and magnitude of the maximum B.M.

SOLUTION

The linearly varying load w in Figure 7.18 can be expressed in terms of the distance x from the support A as:

$$w = -\frac{20x}{8} = -2.5x \qquad \text{(negative downwards)}$$

From Equation 7.13:

$$S = -\int -2.5x \, dx = 1.25x^2 + C_1$$

At the left-hand end, when $x = 0$, $S = -R_A = -26.67$ kN.

Therefore $C_1 = -26.67$ kN and:

$$S = 1.25x^2 - 26.67 \text{ kN} \qquad \textbf{(7.19)}$$

From Equation 7.14:

$$M = -\int (1.25x^2 - 26.67) \, dx = 26.67x - \frac{1.25}{3}x^3 + C_2$$

At A, where $x = 0$, $M = 0$ and therefore $C_2 = 0$.

Thus: $$M = 26.67x - \frac{1.25}{3}x^3 \qquad \textbf{(7.20)}$$

At the mid-point of the span, point C in Figure 7.18, where $x = 4$ m, Equations 7.19 and 7.20 give:

$$S = -6.67 \text{ kN} \quad \text{and} \quad M = 80 \text{ kNm}$$

which agree with the values obtained in Example 7.6.
From Equation 7.19:

$$S = 0 \quad \text{when} \quad x = \sqrt{\frac{26.67}{1.25}} = 4.619 \text{ m}$$

Substituting this value of x into Equation 7.20 gives the magnitude of the maximum bending moment:

$$M_{max} = 26.67 \times 4.619 - \left(\frac{1.25}{3} \times 4.619^3\right) = 82.13 \text{ kNm}$$

7.6 More about shear force & bending moment diagrams

From the discussion and examples in Sections 7.4 and 7.5, the following observations can be made with respect to straight beams.

SHEAR FORCE DIAGRAMS:

1. Where a concentrated load is applied normal (or transverse) to the axis of beam, a step occurs in the shear force diagram equal in magnitude to the concentrated load.
2. Between points of load application, the shear force is constant.
3. In regions of a beam subjected to a uniformly distributed load, the shear force diagram is linear.
4. In regions of a beam subjected to a linearly varying load, the shear force diagram is parabolic.
5. The points on the shear force diagram where the shear force is zero correspond to the points where the bending moment is either a maximum or a minimum.

BENDING MOMENT DIAGRAMS:

1. Between points of transverse load application, the bending moment diagram is linear.
2. In regions of a beam subjected to a uniformly distributed load, the bending moment diagram is parabolic.
3. In regions of a beam subjected to a linearly varying load, the bending moment diagram is cubic.
4. At the points where concentrated transverse loads are applied to a beam, the bending moment diagram changes direction (kinks).
5. At a point where a couple is applied, a step occurs in the bending moment diagram of magnitude equal to the applied couple.
6. The bending moment diagram reaches a maximum or minimum at points where the shear force is zero.

Problems

7.1 ABCD is a horizontal beam carrying vertical loads as shown in Figure P7.1. Find the B.M. and S.F. at the points E and F.

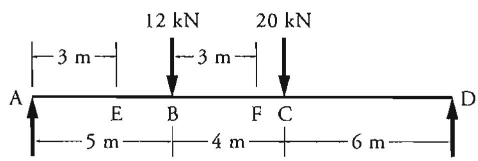


Figure P7.1

7.2 ABCD is a bar inclined at 30° to the horizontal (Figure P7.2), pinned at A, and supported at D on rollers which provide a reaction normal to the bar. It carries a horizontal force of 20 kN at B and a vertical force of 10 kN at C. Find the B.M., S.F. and A.F. at the mid-point of the bar.

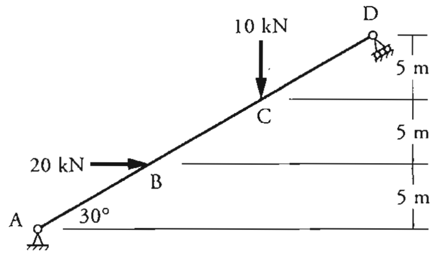


Figure P7.2

7.3 For the beam shown in Figure P7.3 find the B.M. and S.F. at E.

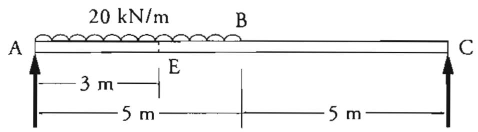


Figure P7.3

7.4 Calculate M, S and N at the mid-points of AB and BC of the beam of Figure P7.4.

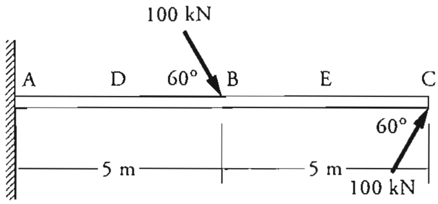


Figure P7.4

7.5 For the beam shown in Figure P7.5, find the bending moment and shear force at the mid-point of segments AB, BC, CD, DE and EF.

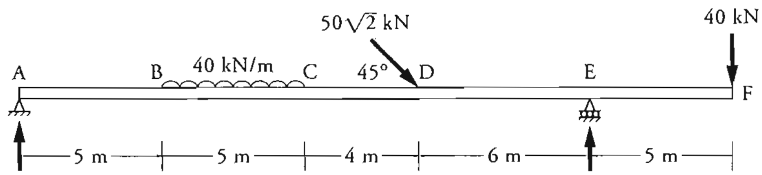


Figure P7.5

7.6 For the beam of Figure P7.6, find N and M at P, Q and R, and S at Q. Why is the S.F. indeterminate at P and R?

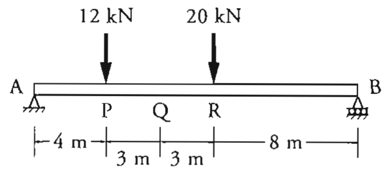


Figure P7.6

7.7 For the beam of Figure P7.7:
(i) find N, S and M at P
(ii) express S and M in each segment of the beam as a function of x, where x is the distance from A.

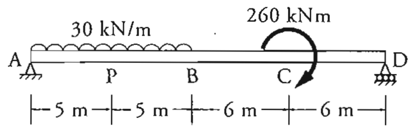


Figure P7.7

7.8 For the beam of Figure P7.8:
(i) find M_A, M_B, M_C, M_D
(ii) find the shear force at A, B, just to the left of C, just above C, just to the left of E, and just to the right of E
(iii) find the axial force in the portions AB, BC, and CD.

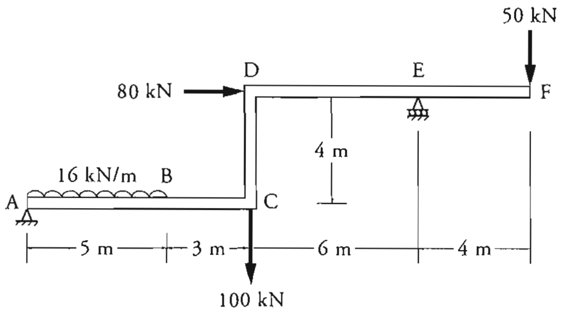


Figure P7.8

7.9 A semi-circular rigid beam is loaded as shown in Figure P7.9. Find the bending moment and axial force at A, B, C and D. Also find the shear force at A and B.

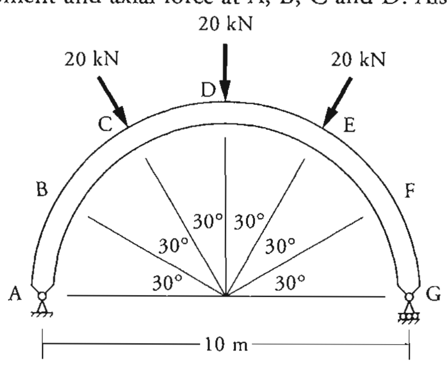


Figure P7.9

7.10 For the beam shown in Figure P7.1 (page 109), draw graphs of the B.M. and S.F.

7.11 For the beam shown in Figure P7.3 (page 110):
(i) express M and S in terms of x, where x is the distance from A
(ii) draw the B.M and S.F. diagrams for this beam.

7.12 Draw the B.M. diagram for the beam of Figure P7.6 (page 111).

7.13 For the beam of Figure P7.7 (page 111) derive expressions for M and S for each of the segments AB, BC and CD.

7.14 AD is a beam hinged at A and on rollers at D (Figure P7.14). Lugs are welded onto the beam at B and C, and horizontal forces are applied as shown.
(i) Find R_A and R_D.
(ii) Find the B.M. at A, just each side of B, just each side of C, and at D.
(ii) Find the S.F. and A.F. at the mid-point of each segment.

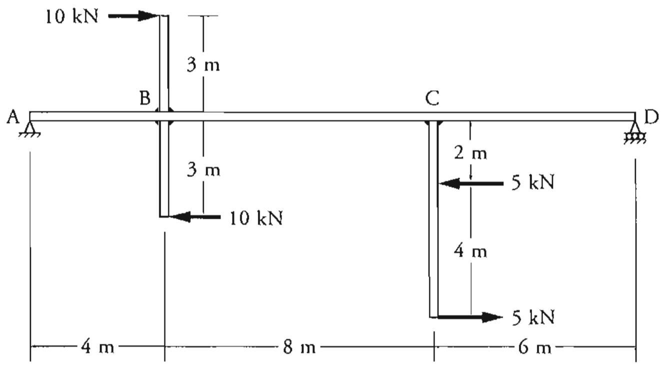


Figure P7.14

7.15* Figure P7.15 shows a bar AD hinged at A and on rollers at D. Couples M_1 and M_2 are applied at points B and C.
(i) Find R_A and R_D.
(ii) Using A as the origin, write equations giving the value of M and S over each segment of the bar.

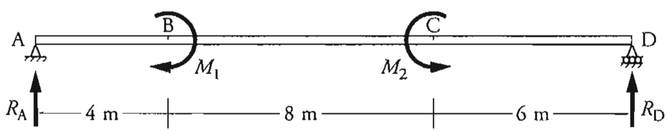


Figure P7.15

7.16* A T-shaped vertical cantilever is used for supporting equipment at a stadium. The loading is shown in Figure P7.16. Find N, S and M at the mid-points of CD and BE, and also at E.

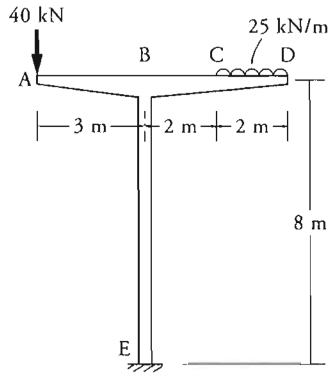


Figure P7.16

7.17* The structure shown in Figure P7.17 is used as a gantry on a wharf. It is required to raise a load of 60 kN at D. Due to wind, a horizontal force of 10 kN is exerted at B. Draw the B.M. diagrams for ABC and CD.

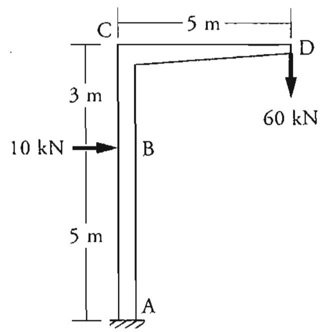


Figure P7.17

7.18* Figure P7.18 shows a bent cantilever lying in the vertical plane. It is cantilevered from A and is required to support a load of 15 kN at D. Find *N*, *S* and *M* at the mid-points of AB, BC and CD.

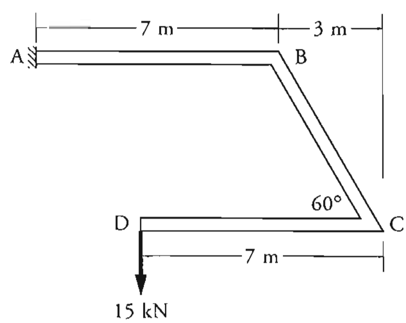


Figure P7.18

7.19* A bent beam ABCDE (Figure P7.19) lies in the vertical plane and forms part of a structure to support a projecting portion of a building. It is pinned at E and has a vertical roller support at A. Calculate *N*, *S* and *M* at the mid-points of AB, CD and DE, and *M* at C and D.

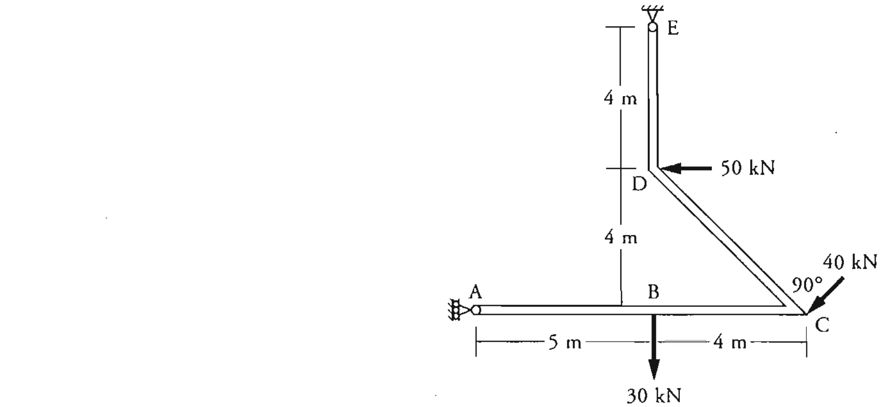


Figure P7.19

7.20* A barge is loaded with both concentrated and distributed loads as shown in Figure P7.20. The cross-section of the barge is uniform along its length. Draw the bending moment and shear force diagrams for this loading.

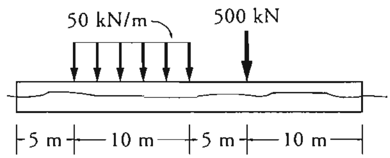


Figure P7.20

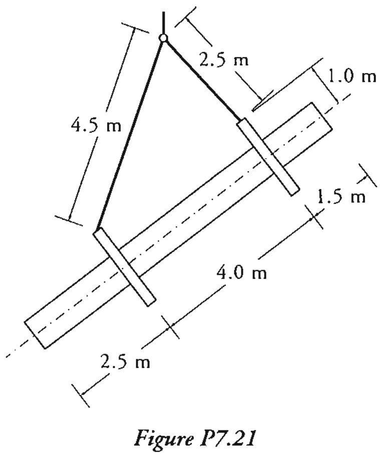


Figure P7.21

7.21* A pipe of 300 mm outside diameter and weighing 750 N/m is held by means of yokes in a sling as shown in Figure P7.21. Plot bending moment and shear force diagrams for the pipe in this position.

7.22* A beam ABC is continuous over two spans and is pinned to the three pin-ended members AD, BE, CF (Figure P7.22). Plot the bending moment diagram for the beam.

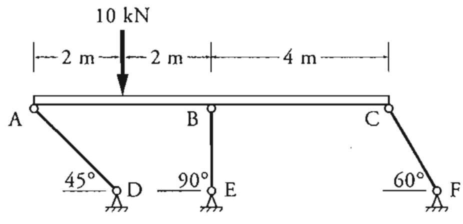


Figure P7.22

7.23* The rigid-jointed frame ABC in Figure P7.23 is fixed at A and C, and is loaded at B. An accurate analysis shows the bending moments at A, B and C to be: $M_A = -0.01PL$, $M_B = +0.01PL$ and $M_C = -0.01PL$, where positive bending moment is defined to produce tension on the lower side of the members. Determine the axial force in the members.

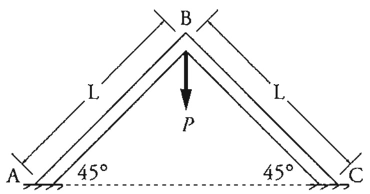


Figure P7.23

7.24 The beam ABC is loaded as shown in Figure P7.24. Using the relationships: $S = -\int w\, dx$ and $M = -\int S\, dx$, derive expression for S and M in the part AB and BC. Find S and M at B.

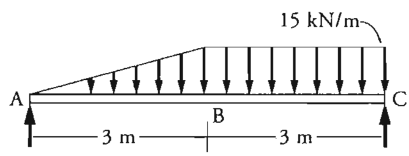
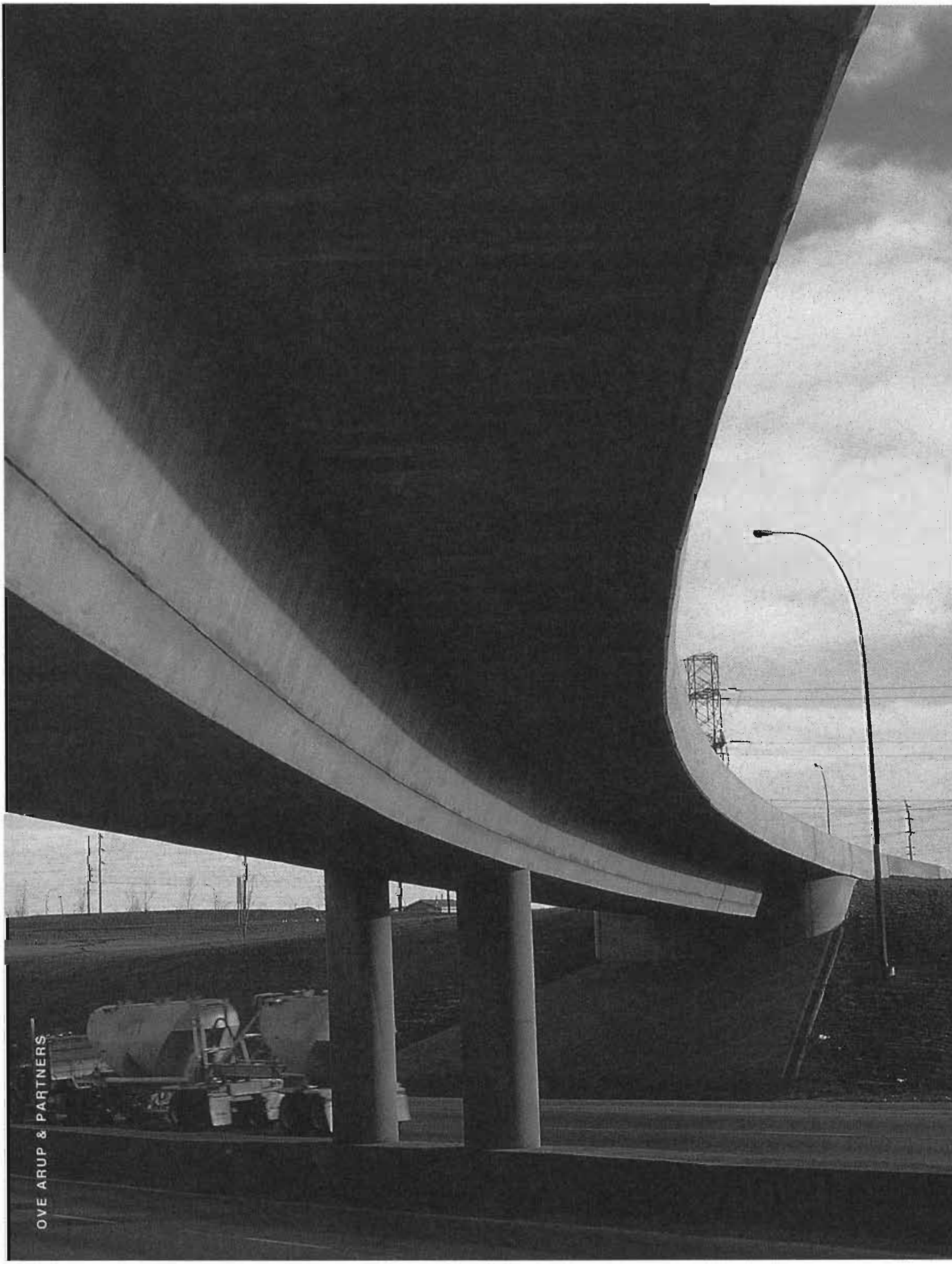


Figure P7.24

* Difficult problems, suitable for later study.



OVE ARUP & PARTNERS

Other Beam Problems

8.1 Conditions of equilibrium

If a beam, or rigid bar, lies in one plane and supports loads in the same plane, it requires at least three components of reaction to maintain it in equilibrium. If it has just three appropriate reaction components, then these can be calculated from the equilibrium equations for a planar system of forces. The problems considered in Chapter 7 were of this type.

If the number of reaction components are too few the structure will be unstable, i.e. it will move under certain types of load. The beam in Figure 8.1 has only two reactions. It will resist the load in Figure 8.1a, but not that in Figure 8.1b, and is therefore unstable.

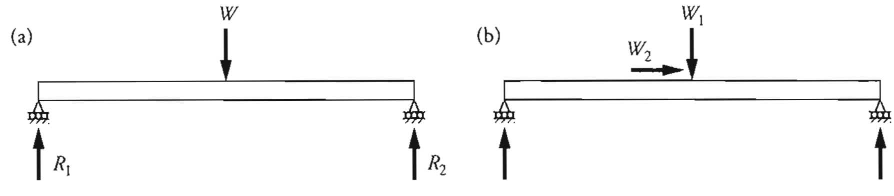


Figure 8.1

Even if there are three reactions they must be *suitable.* They may not be concurrent, nor all parallel (i.e. concurrent at infinity). The structure of Figure 8.2a will rotate about A under the load shown (the three reactions are concurrent). The beam of Figure. 8.2b has three reactions but if subjected to a horizontal load it will not be stable.

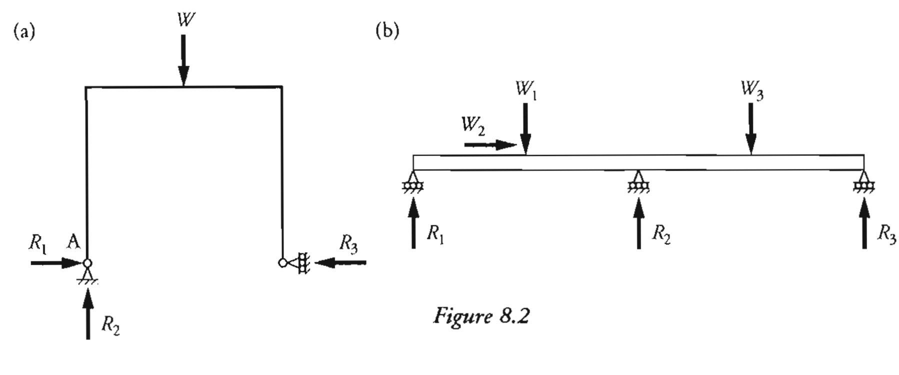


Figure 8.2

If the beam has more than three reaction components, which is quite common in practice, then the three equations of equilibrium applied to the complete beam are not sufficient to evaluate these reactions, and additional information is required. Sometimes this extra information is available in the fotm of a known bending moment at some point other than a support. For instance, the beam may contain one or more internal hinges. If a hinge is assumed to be frictionless, the bending moment must be zero at that location. It is then possible to consider the freebody to one side of the hinge and thus obtain an extra equation.

For all reactions to be determined using only the principles of statics, the number of internal hinges must be the same as the extra number of equations required. For instance a beam with five reactions must contain two hinges so that these two *equations of condition*, together with the three equations of external equilibrium are equal to the number of reactions. If there are more than two hinges (in this case) the beam is not stable and will collapse under certain loads. If there are less than two hinges the reactions cannot be determined by statics alone and the solution is not the subject of this book. Such beams are not statically determinate.

8.2 Beams with one internal hinge

A beam such as that shown in Figure 8.3 may be used in bridge construction. It is often more economical than two simply supported beams AB and BC. There are four reactions, and these cannot be determined unless either the freebody AD or DC is considered in addition to overall equilibrium. To avoid extensive arithmetic, it is important to write the equilibrium equations in a suitable sequence.

Example 8.1

The beam of Figure 8.3a is supported on rollers at A and B, pinned at C and contains a hinge at D. Calculate the reactions for the loading shown.

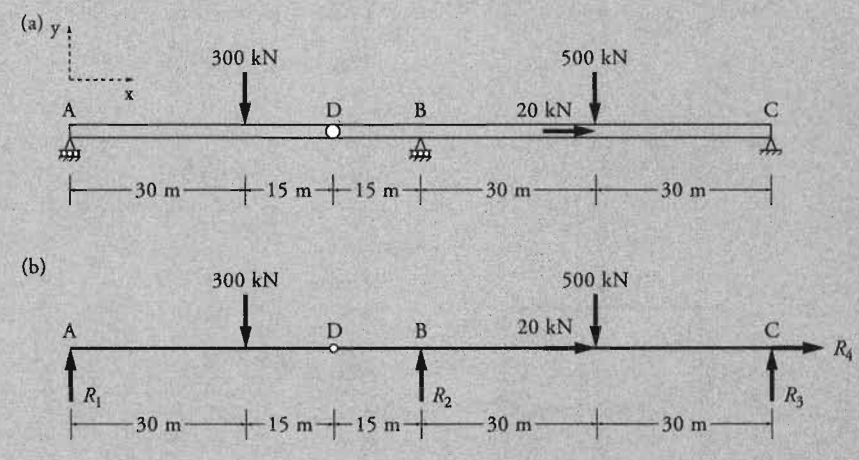


Figure 8.3

SOLUTION

Consider the equilibrium of the freebody of the complete beam (Figure 8.3b).

$$\Sigma F_x = 0: \qquad +20 + R_4 = 0 \qquad \therefore\ R_4 = -20 \text{ kN}$$

The equation $\Sigma F_y = 0$ involves three unknown reactions. An equation of moments about A or B or C involves two unknowns. The best procedure is to consider the freebody AD (Figure 8.4). Since M at the hinge is zero, taking moments about D gives:

$$45R_1 - (300 \times 15) = 0 \qquad \therefore\ R_1 = 100 \text{ kN}$$

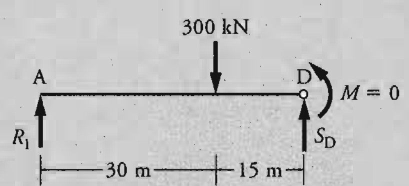


Figure 8.4

Reverting now to the freebody of the complete beam (Figure 8.3b), an equation of moments about B gives:

$$\circlearrowleft + \quad -(100 \times 60) + (300 \times 30) - (500 \times 30) + 60R_3 = 0 \quad \therefore\ R_3 = 200 \text{ kN}$$

Finally:

$$\Sigma F_y = 0: \quad +100 - 300 + R_2 - 500 + 200 = 0 \qquad \therefore\ R_2 = 500 \text{ kN}$$

A common problem is that of a portal frame (Figure 8.5) pinned at each end, and with an internal hinge. A three hinged arch is a similar problem. Each structure is essentially a bent beam. There is a horizontal and a vertical reaction component at each end, making four reactions in all.

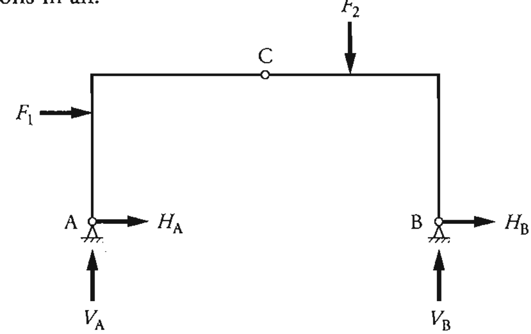


Figure 8.5

Provided A and B are at the same level, an equation of moments about A for the whole structure will yield the value of V_B. An equation of moments about B will yield the value of V_A. Next we consider the freebody AC (or BC) to obtain the value of H_A (or H_B). Finally, $\Sigma F_x = 0$ for the whole structure then gives H_B (or H_A).

For each of the structures shown in Figures 8.3 and 8.5, there are four reaction components and one internal hinge. We have seen that these four reactions can be evaluated using the three equations of equilibrium and a fourth equation resulting from the knowledge that the moment at the hinge must be zero.

From the point of view of calculation, a three-hinged arch is the same as the portal of Figure 8.5. Arches are usually curved but may be polygonal like that of Figure 8.6a.

EXAMPLE 8.2

For the arch of Figure 8.6a, find the reactions at A and E. Also find the bending moment at D.

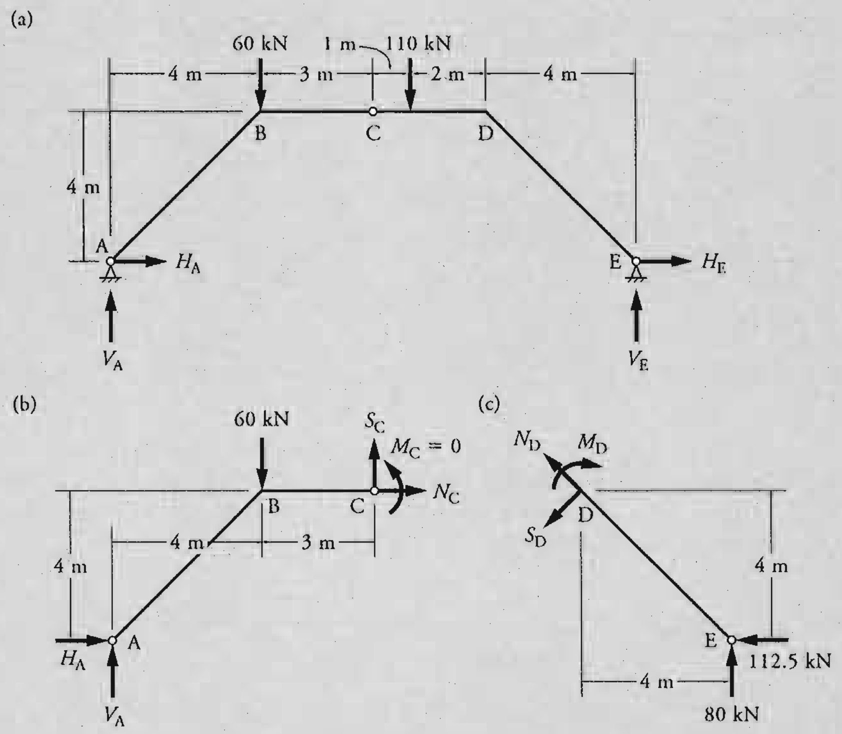


Figure 8.6

SOLUTION

1. For the whole structure (Figure 8.6a), take moments about A:

$$\curvearrowright + \quad (60 \times 4) + (110 \times 8) - (V_E \times 14) = 0 \qquad \therefore V_E = 80 \text{ kN}$$

$$\Sigma F_y = 0: \quad V_A - 60 - 110 + 80 = 0 \qquad \therefore V_A = 90 \text{ kN}$$

Consider freebody AC (Figure 8.6b) and take moments about hinge C:

$$\curvearrowright + \quad (90 \times 7) - (H_A \times 4) - (60 \times 3) = 0 \qquad \therefore H_A = 112.5 \text{ kN}$$

For the complete arch (Figure 8.6a):

$$\Sigma F_x = 0: \qquad \therefore H_E = -112.5 \text{ kN}$$

2. From the freebody ED (Figure 8.6c), the B.M. at D is found by taking moments about D:

$$\curvearrowright + \quad M_D + (112.5 \times 4) - (80 \times 4) = 0 \qquad \therefore M_D = -130 \text{ kNm}$$

A three-pinned portal (or arch) may have the bases (supports) at different levels. The analysis of such problems is essentially the same as in the previous examples.

EXAMPLE 8.3

Find the reactions to the portal frame of Figure 8.7a.

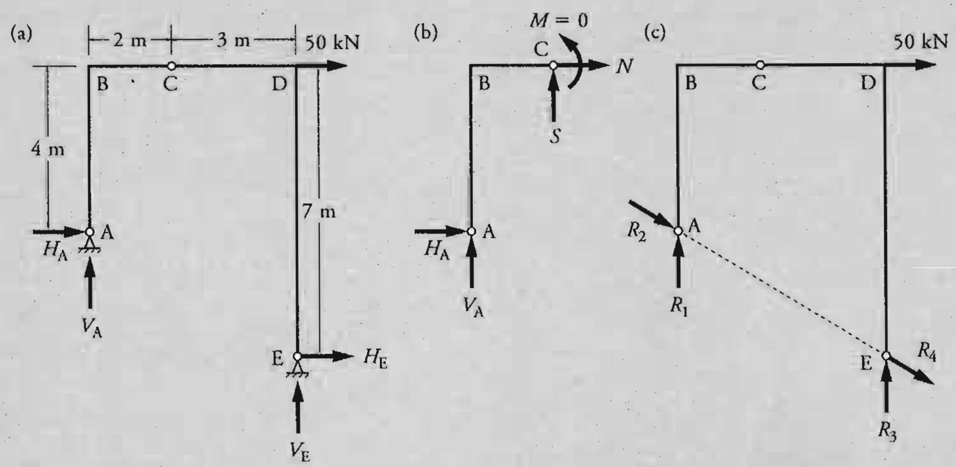


Figure 8.7

SOLUTION

For the complete frame, take moments about E:

$$\curvearrowright + \quad (H_A \times 3) + (V_A \times 5) + (50 \times 7) = 0 \qquad \therefore 3H_A + 5V_A + 350 = 0$$

For freebody AC (Figure 8.7b), take moments about C:

$$\curvearrowright + \quad -4H_A + 2V_A = 0$$

These two equations may now be solved to give:

$H_A = -26.9$ kN and $V_A = -53.8$ kN

Then for the complete structure:

$\Sigma F_x = 0$: $\quad H_A + H_E + 50 = 0 \quad \therefore H_E = -23.1$ kN

and taking moments about A:

$(50 \times 4) - (V_E \times 5) + (23.1 \times 3) = 0 \quad \therefore V_E = +53.8$ kN

In Example 8.3, simultaneous equations can be avoided by resolving reactions at A and E into oblique components, as in Figure 8.7c, so that R_2 and R_4 act along the line AE. Reactions R_1 and R_3 are now easily found from external equilibrium. However, the equation of moments to find R_2 (or R_4) now involves an oblique lever arm. For this reason, it may be easier to use the solution given above.

8.3 Further examples

We have seen that a beam with four reaction components can be solved if it contains an internal hinge. More generally it can be solved if the B.M. at any point is known. The hinge is a special case when the B.M. is known to be zero.

Example 8.4

Find the four reactions of the frame of Figure 8.8a, if it is known that the B.M. at C is 20 kNm.

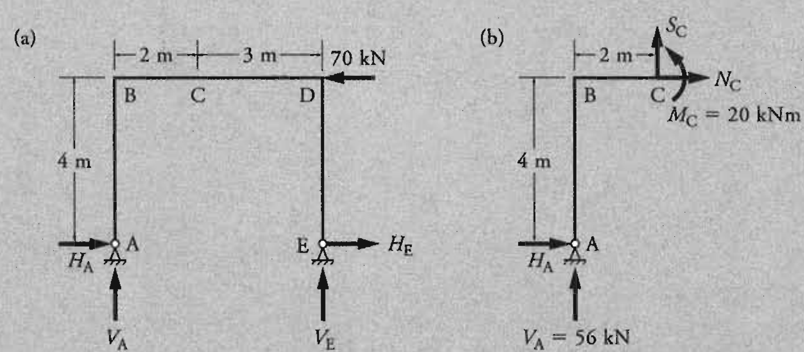


Figure 8.8

Solution

For the complete frame, take moments about A:

$(70 \times 4) + (V_E \times 5) = 0 \quad \therefore V_E = -56$ kN

$\Sigma F_y = 0$: $\quad V_A + V_E = 0 \quad \therefore V_A = +56$ kN

Now consider the freebody ABC (Figure 8.8b). Take moments about C:

$$\circlearrowleft + \quad (H_A \times 4) - (56 \times 2) + 20 = 0 \qquad \therefore H_A = +23 \text{ kN}$$

For the complete frame:

$$\Sigma F_x = 0: \quad H_A + H_E - 70 = 0 \qquad \therefore H_E = +47 \text{ kN}$$

Figure 8.9a shows a beam continuous over three spans and containing hinges (M = 0) at E and F. The part EF is referred to as a *suspended span* in bridge construction. We now have five reaction components and two hinges. There are three equations of overall equilibrium and two equations expressing the fact that $M = 0$ at E and F (the latter two are sometimes called *equations of condition*). These five equations are sufficient to determine the reactions.

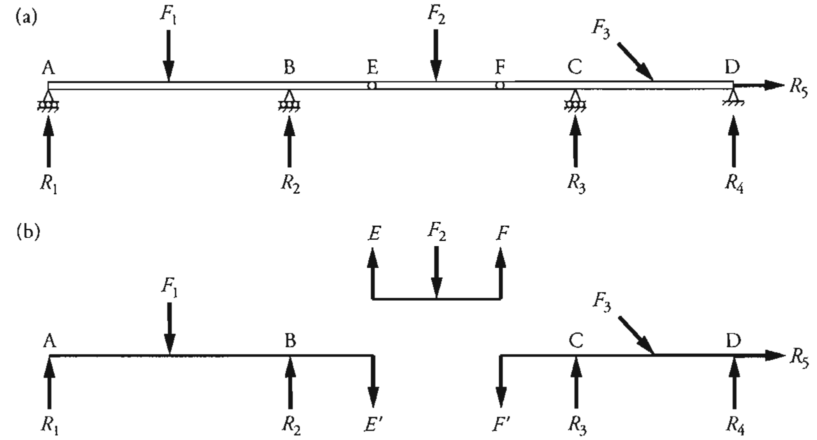


Figure 8.9

First we may find R_5 from $\Sigma F_x = 0$. Consideration of freebodies AE and AF yields two equations for R_1 and R_2. Finally R_3 and R_4 may then be found from equilibrium of the complete beam.

Alternatively we may draw the three freebodies AE, EF and FD (Figure 8.9b). The span EF may be analysed separately first. From this we find the forces which EF exerts on the ends of the cantilevers BE and CF. Beams ABE and FCD are now independent simply-supported beams which are quite easy to analyse.

The above concepts can be extended further. A beam loaded in one plane, and having n reaction components, can be solved by the laws of statics if it contains $n - 3$ internal hinges, provided not more than two occur in any straight length between supports. It may also be solved if the bending moments are known at $n - 3$ locations, even if the known values are not zero.

The portal of Figure 8.10 has five reactions. It is important to note that at the direction-fixed support A, there is a couple reaction M_A, as well as force reactions H_A and V_A. There are two internal hinges, and the evaluation of reactions follows from three equations of overall equilibrium and two equations of condition. The latter are obtained from two freebodies, i.e. AB and CE, or CE and BE, and so on.

EXAMPLE 8.5

Find the five reactions of the portal frame of Figure 8.10 and draw the axial force, shear force and bending moment diagrams for the frame.

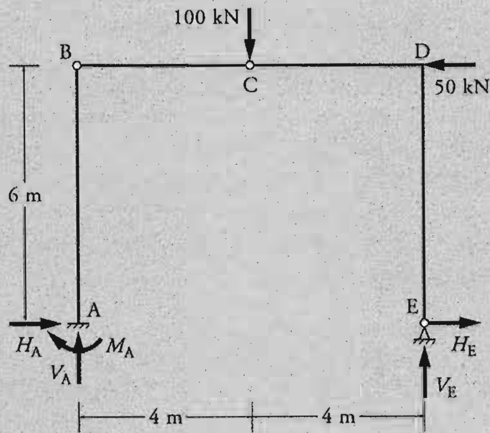


Figure 8.10

SOLUTION

Taking moments about C in the freebody of Figure 8.11a gives:

$$\circlearrowleft + \quad (V_E \times 4) + (H_E \times 6) = 0 \qquad\qquad 2V_E + 3H_E = 0$$

and taking moments about B in the freebody of Figure 8.11b gives:

$$\circlearrowleft + \quad (V_E \times 8) + (H_E \times 6) - (100 \times 4) = 0 \qquad 4V_E + 3H_E = 200$$

Solving these two simultaneous equations gives:

$$V_E = 100 \text{ kN} \qquad \text{and} \qquad H_E = -66.7 \text{ kN}$$

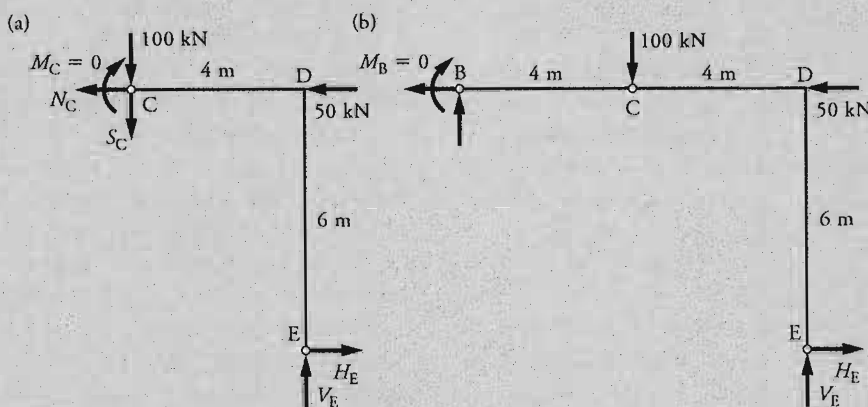


Figure 8.11

With V_E and H_E determined, the three reactions at A can be determined from the equilibrium of the entire structure (Figure 8.10):

$\Sigma F_y = 0$: $V_A - 100 + 100 = 0$ $\therefore V_A = 0$

$\Sigma F_x = 0$: $H_A - 50 - 66.7 = 0$ $\therefore H_A = 116.7$ kN

$\Sigma M_A = 0$: $(100 \times 8) + (50 \times 6) - (100 \times 4) - M_A = 0$ $\therefore M_A = 700$ kNm

A check of these results can be made by considering the freebodies AB and AC, taking moments about B and C, respectively.

The A.F., S.F. and B.M. at any point on the frame can be determined by considering the freebody on either the left or the right of the point in question. Appropriate freebodies for each segment of the frame are shown in Figure 8.12.

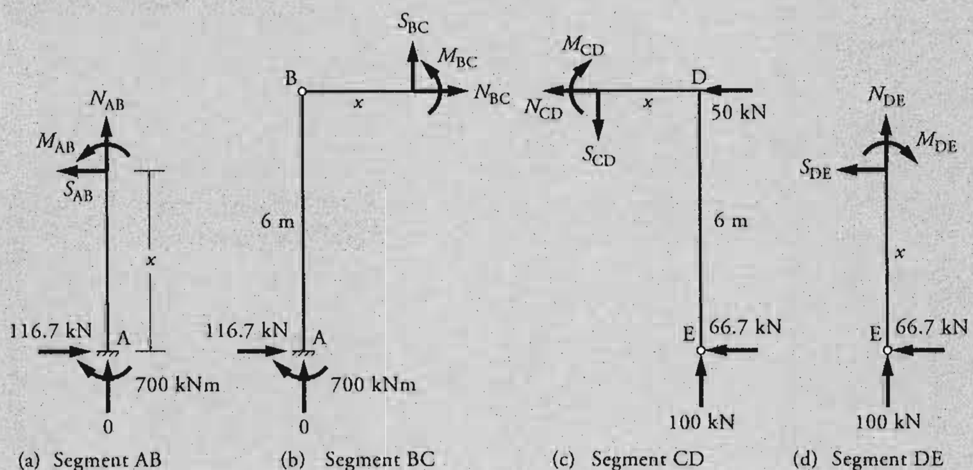


Figure 8.12

From Figure 8.12a:
$N_{AB} = 0$
$S_{AB} = 116.7$ kN
$M_{AB} = 700 - 116.7x$ ($x = 0$ at A to $x = 6$ at B)

From Figure 8.12b:
$N_{BC} = -116.7$ kN
$S_{BC} = 0$
$M_{BC} = 700 - 116.7 \times 6 = 0$

From Figure 8.12c:
$N_{CD} = -116.7$ kN
$S_{CD} = 100$ kN
$M_{CD} = 100x - 66.7 \times 6$ ($x = 4$ at C to $x = 0$ at D)

From Figure 8.12d:
$N_{DE} = -100$ kN
$S_{DE} = -66.7$ kN
$M_{DE} = -66.7x$ ($x = 6$ at D to $x = 0$ at E)

The A.F., S.F. and B.M. diagrams are shown in Figure 8.13(a) to (c) respectively.

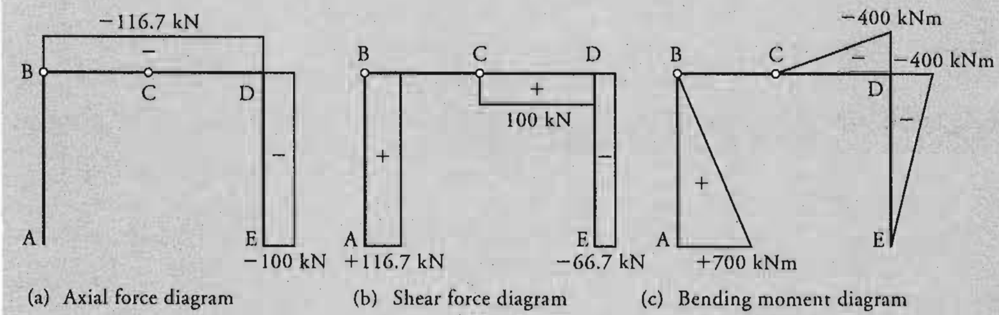


Figure 8.13

Problems

8.1 The beam of Figure P8.1 is pinned at A and supported on rollers at C and F. There is a hinge at D. Find the four reaction components.

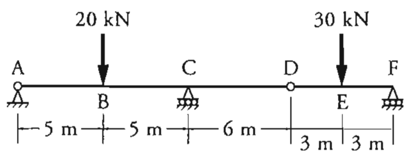


Figure P8.1

8.2 The circular bar of Figure P8.2 is pinned at A and C and has a hinge at B. Find the reactions at A and C.

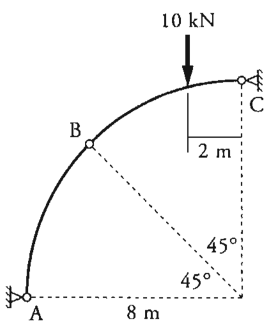


Figure P8.2

8.3 Find the reactions of the structure shown in Figure P8.3.

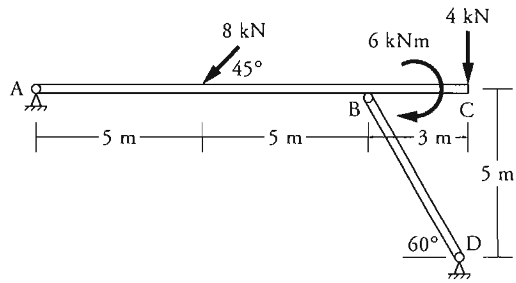


Figure P8.3

8.4 A solid hexagonal plate of sides 2 m is supported as shown in Figure P8.4 and loaded in its own plane. Find the reactions at A and B.

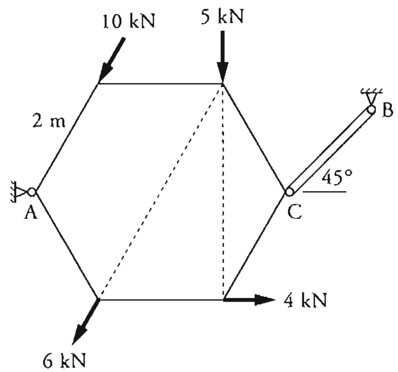


Figure P8.4

8.5 The bent beam ABCD (Figure P8.5) lies in a vertical plane and is loaded as shown. If it is known that the bending moment at C is −10 kNm (i.e. tension at the top of the member), find the reactions at A and D.

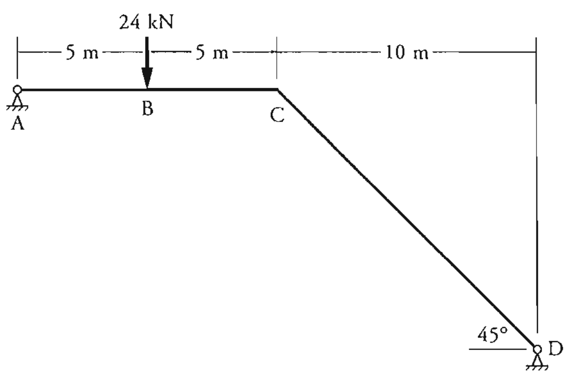


Figure P8.5

8.6 The two-span beam of Figure P8.6 has a hinge at D. Find the vertical reactions at A, B and C for the loading shown and draw the S.F. and B.M. diagrams.

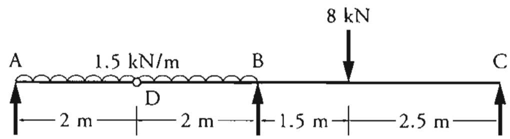


Figure P8.6

8.7 Find the reactions at A and E for each of the frames shown in Figure P8.7 and draw the axial force, shear force and bending moment diagrams.

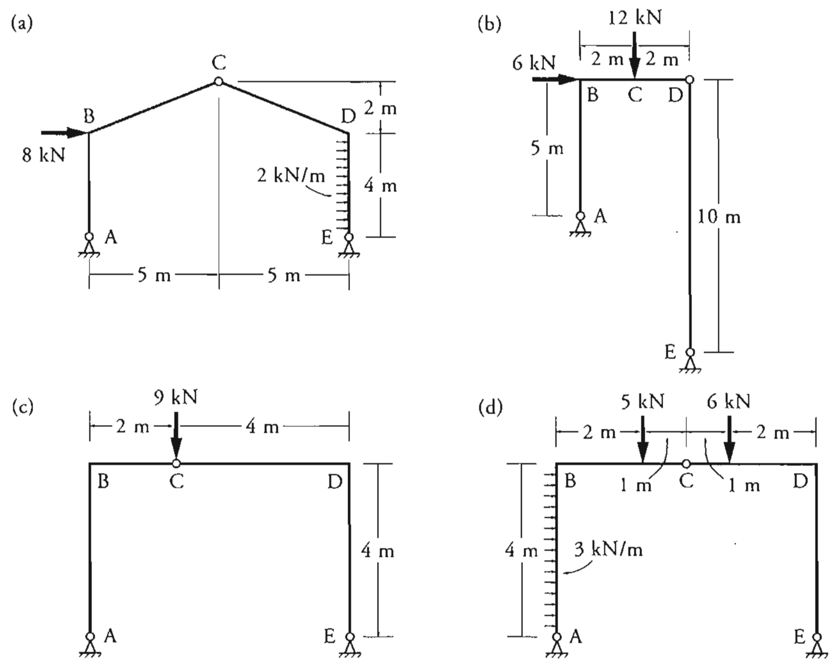


Figure P8.7

8.8 Find the reactions of the three-hinged semi-circular arch of Figure P8.8.

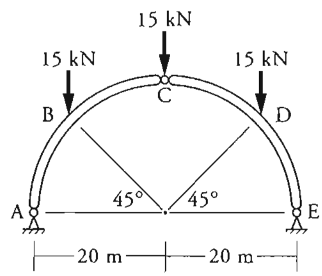


Figure P8.8

8.9 (i) Find the reactions of the three-hinged semi-circular arch of Figure P8.9.
(ii) Find the resultant force transmitted through the pin at D.

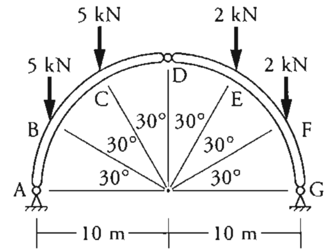


Figure P8.9

8.10* For the portal frame shown in Figure P8.10, find the reactions at A and G and draw the A.F., S.F. and B.M. diagrams.

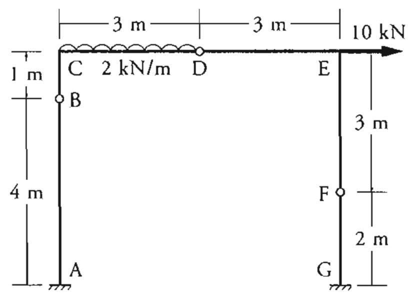


Figure P8.10

8.11* The portal frame of Figure P8.11 is pinned at A and E and has a roller support at D. For the loading shown find the five reactions, given that $M_B = -25$ kNm and $M_D = -20$ kNm (i.e. there is tension on the outside of the frame at B and D).

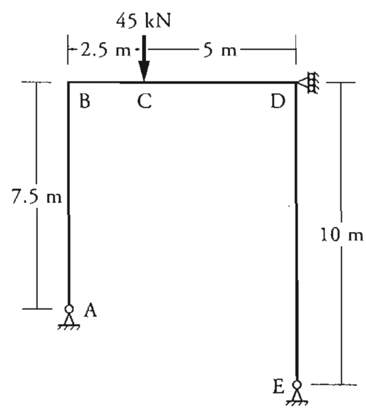


Figure P8.11

8.12* The frame of Figure P8.12 has a pinned support at G and a roller support at A. It has an internal hinge at D and is prevented from collapse by a flexible tie BF. For the loading shown, find the reactions at A and G and the tension, T, in the tie BF.

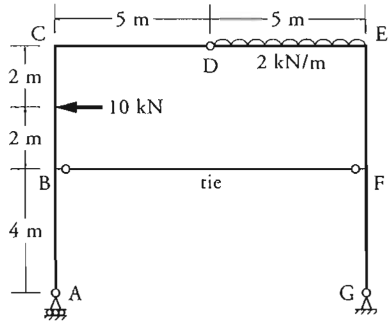


Figure P8.12

8.13* The structure shown in Figure P8.13 has a built-in support at A and a roller support at F. The free end B is vertically above A. There is an internal hinge at D. A 6 kN load is applied successively at B, C, D and E. Find the reactions for each load separately.

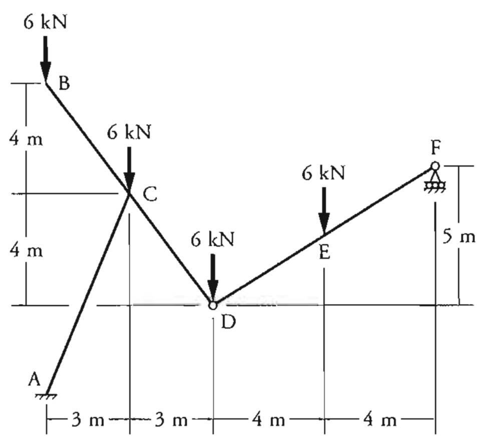


Figure P8.13

8.14* The structure of Figure P8.14 has built-in supports at A and J. There are internal hinges at C, D and F, the hinge at C occurring only in the sloping member CD. An 8 kN load is applied successively at B, E and H. Find the reactions for each load separately.

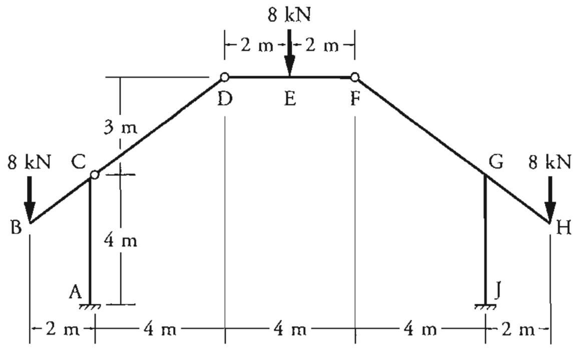
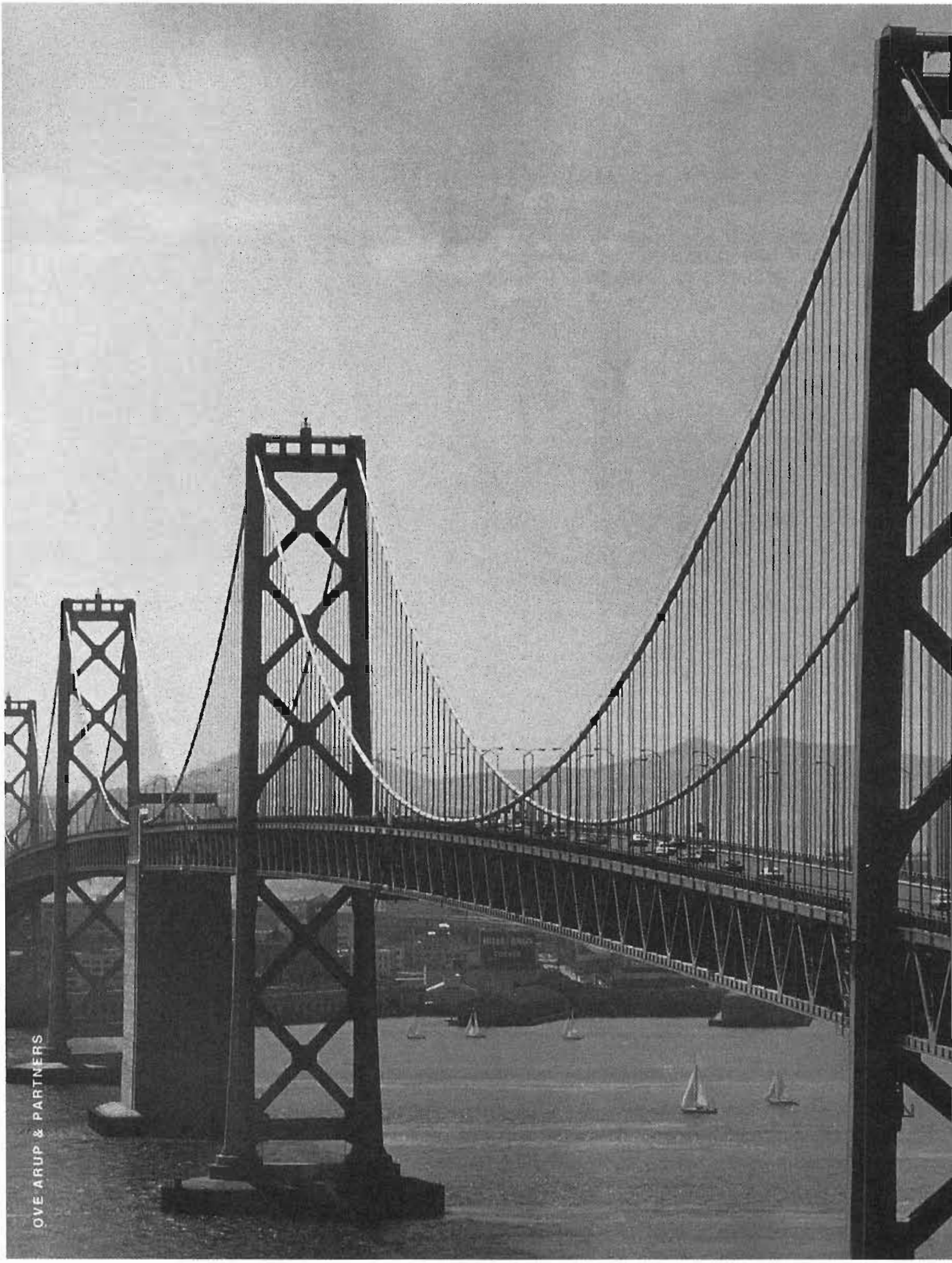


Figure P8.14

* Difficult problems, suitable for later study.



OVE ARUP & PARTNERS

Flexible Cables

Many instances could be cited of structures in which one of the main elements is a *flexible cable*. The essential characteristics of such an element is its inability to resist any actions other than tensile forces. In other words a flexible cable cannot resist bending moment, shear force or a compressive axial force. This does not mean, however, that a cable cannot carry external loads which have a component normal to the direction of the cable. A cable can in fact support such lateral loads if it is firmly attached to supports. It does so by taking up a shape to suit the loading, the shape being such that the bending moment at every point along the cable is zero.

9.1 Cables supporting point loads

As a very simple case, consider a cable of negligible weight supported at A and B and carrying a single weight W mid-way between the supports. The supports at A and B are at the same elevation and are located at a distance L apart, as shown in Figure 9.1. The sag DC at the point of application of the weight W is Y.

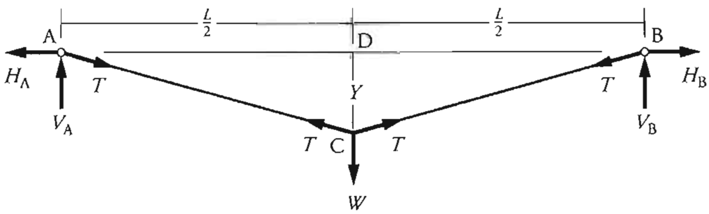


Figure 9.1

By taking moments of all the external forces about B, we find that:

$$V_A = \frac{W}{2}$$

As the bending moment is everywhere zero, the bending moment at C is zero. Hence, by considering the freebody AC, we have:

$$M_C = \frac{V_A L}{2} - H_A Y = 0$$

$$\therefore H_A = \frac{WL}{4Y}$$

From equilibrium of the complete cable, we find that:

$$V_B = \frac{W}{2} \quad \text{and} \quad H_B = H_A$$

The tension T in the cable is the same in both portion AC and CB and can be obtained by considering equilibrium at one of the supports. For equilibrium of the three concurrent forces at A:

$$T = \sqrt{H_A^2 + V_A^2} = \frac{W}{2}\sqrt{\frac{L^2}{4Y^2} + 1} = \frac{Wl_{AC}}{2Y}$$

where l_{AC} is the length of the cable from A to C.

The shape of a weightless cable under any system of loads is the same as the shape of the bending moment diagram that would be obtained if the same loads were applied to a simply supported beam having a span equal to the distance between the cable supports L. The particular scale adopted must be such that the distance around the bending moment graph is equal to the given length of the cable.

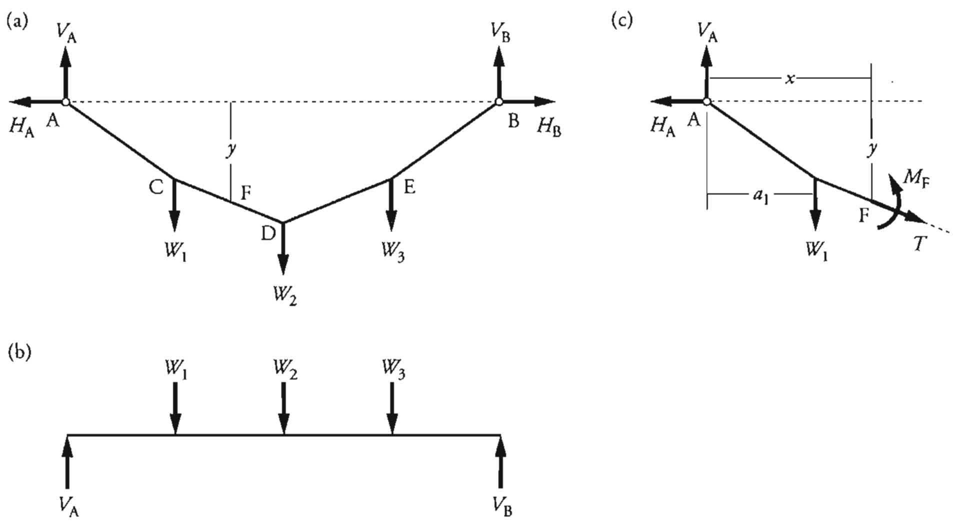


Figure 9.2

Figure 9.2a shows a cable fixed to supports A and B and supporting point loads W_1, W_2 and W_3 at C, D and E. Considering the equilibrium of the complete structures, we can find reaction V_A by taking moments about B, and V_B by taking moments about A. Provided A and B are at the same level (in which case H_A and H_B do not enter into the equations) V_A and V_B will have the same value as the reactions of a corresponding straight beam which carries the same loads (Figure 9.2b).

Consider now the fact that at any point along the cable the bending moment must be zero. At a typical point F (Figure 9.2c) the bending moment is:

$$M_F = (V_A x) - W_1 (x - a_1) - H_A y \tag{9.1}$$

The first two terms may be thought of as the bending moment in a simply supported beam (Figure 9.2b). This is called the *free-span* B.M. and is denoted by M_O. Thus:

$$M_F = M_O - H_A y \tag{9.2}$$

Since $M_F = 0$:

$$H_A y = M_O \tag{9.3}$$

or:

$$y = \frac{M_O}{H_A} \tag{9.4}$$

This shows that the shape of the cable is the same as the free-span bending moment diagram M_O drawn to a particular scale (determined by the magnitude of the horizontal reaction $H_A = H_B = H$). The actual sags at the points C, D and E are related geometrically to the length of the cable.

In the present treatment, it will be assumed that the cables are inextensible, so that the length of the cable is known beforehand. In practice, cables stretch or elongate under load, and since the tension differs from one part of the cable to another, the elongation is not uniform. In many practical cable structures it is necessary to take this extension into account. In such cases, although the shape still corresponds to that of the bending moment diagram, the problem is complicated by the fact that the final length of the cable is initially unknown.

In the cable of Figure 9.2, the weight of the cable was ignored. In consequence the cable was straight between load points, just as the bending moment diagram would be for a weightless beam supporting point loads. If the cable weight is taken into account, it would be found that the cable profile is curved.

EXAMPLE 9.1

An inextendible cable of negligible weight is suspended from points A and D, 14 m apart and at the same level. It carries two loads; one of 60 kN at a horizontal distance of 4 m from A, and one of 110 kN at 8 m from A. Find the shape of the cable and the maximum tension when:

(i) the cable length is 20 m

(ii) the sag at C is 4.2 m.

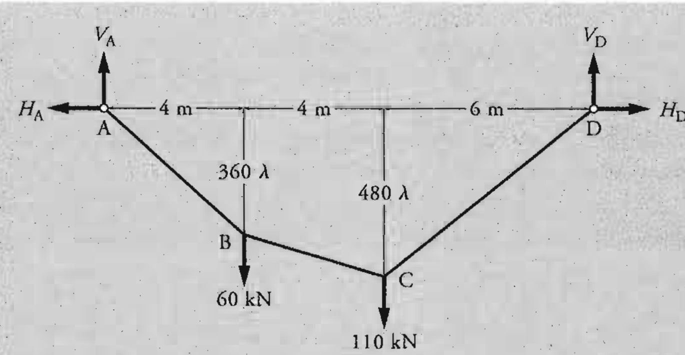


Figure 9.3

SOLUTION

The corresponding free-span problem, with the same loads on a horizontal beam, was solved in Example 7.7 (Figure 7.20 page 103). The free-span B.M. diagram is shown in Figure 7.22 (page 104). It will be noted in that diagram that the bending moment is 360 kNm at the 60 kN load, and 480 kNm at the 110 kN load. As the shape of the hanging cable is similar to the free-span B.M. diagram, the sag at C and D must be 360λ and 480λ (Figure 9.3), where λ is a constant determined by the geometrical specification.

(i) For the case where the cable length is 20 m, AB + BC + CD = 20; and:

$$AB = \sqrt{4^2 + (360\lambda)^2} \qquad BC = \sqrt{4^2 + (120\lambda)^2} \quad CD = \sqrt{6^2 + (480\lambda)^2}$$

From these relationships an equation in λ can be obtained. This is best solved by trial and error. It is found that $\lambda = 0.0143$ Thus: $y_B = 0.0143 \times 360 = 5.15$ m and $y_C = 0.0143 \times 480 = 6.86$ m.

The horizontal reaction H (= $H_A = H_D$) may be evaluated for this case by employing the relationship:

$$H = \frac{M_O}{y}$$

at any point along the cable. At B, for example, $M_O = 360$ kNm, $y = 5.15$ m and

hence: $H = \dfrac{360}{5.15} = 69.9$ kN

At any point along the cable, the horizontal component of tension is equal to the horizontal reaction $H = 69.9$ kN. The actual tension will therefore be greatest where the cable slope is greatest. In this example, this is the part AB.

The length of AB is:

$$\sqrt{4^2 + 5.15^2} = 6.52 \text{ m} \quad \text{and} \quad \therefore T_{AB} = 69.9 \times \frac{6.52}{4} = 114 \text{ kN}$$

(ii) For this case, y_C is specified as 4.2 m. Therefore $480\lambda = 4.2$, and hence $\lambda = 0.00875$. The sag at B is $y_B = 360\lambda = 3.15$ m and the length of the cable is then:

$$\sqrt{4^2 + 3.15^2} + \sqrt{4^2 + 1.05^2} + \sqrt{6^2 + 4.2^2} = 16.55 \text{ m}$$

From Equation 9.2: $H = \dfrac{480}{4.2} = 114.3$ kN

Now the length of AB is $\sqrt{4^2 + 3.15^2} = 5.09$ m:

$\therefore T_{AB} = 114.3 \times \dfrac{5.09}{4} = 145.4$ kN

In the case of a cable which supports specified loads, but is supported at points not on the same level, the reactions may be found by resolving each reaction into a vertical component and a component along the line joining the supports (Figure 9.4). The components R_A and R_D are obtained by taking moments about D and A respectively. They are the same as those of a simply supported beam of span equal to the *horizontal projection* of AD and carrying the loads in the same horizontal location. If the B.M. in this beam is M_O, then the B.M. in the cable at any point is:

$$M = M_O - F(y\cos\phi) = M_O - (F\cos\phi)\,y = M_O - Hy \qquad \textbf{(9.5)}$$

where ϕ is the slope of AD. Since $M = 0$, Equation 9.5 reduces to:

$$Hy = M_O \qquad \text{or} \qquad y = \frac{M_O}{H} \qquad \textbf{(9.6)}$$

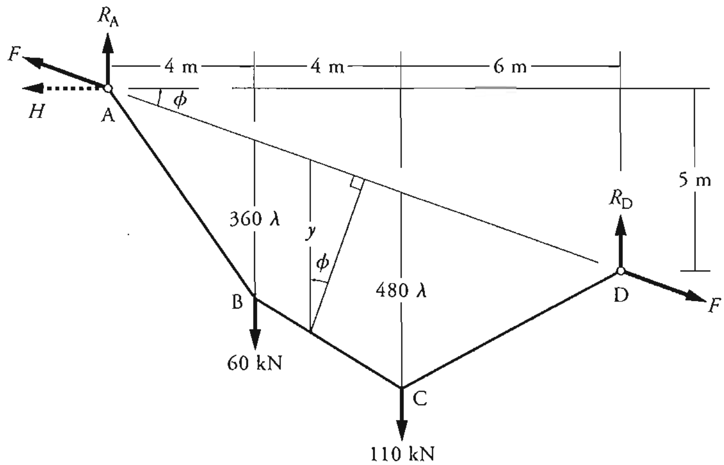


Figure 9.4

The shape of the cable is thus the same as that of the free-span B.M. diagram, with ordinates plotted vertically (i.e. not at right angles to AD). The total vertical reactions are $(R_A + H\tan\phi)$ and $(R_D - H\tan\phi)$. The cable tension is greater at A (the higher end) than would be the case if A and D were level, and less at D. As for a cable with supports at the same level, the maximum tension in the cable occurs where it is steepest.

9.2 Cables supporting uniformly distributed loads

The above method of analysis also applies to a cable which supports a uniformly distributed load (i.e. uniform per length of horizontal projection), or a very large number of point loads of equal magnitude and equally spaced.

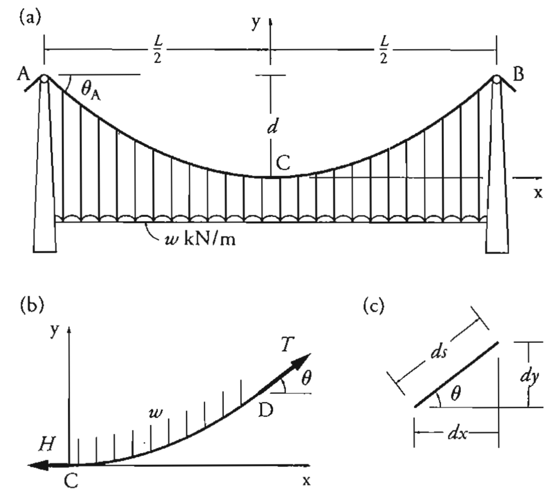


Figure 9.5

The free-span bending moment diagram for a uniformly distributed load is a parabola of maximum ordinate $wL^2/8$ where L is the horizontal span (Figure 9.5a). It follows that a cable loaded in this way takes up a parabolic shape, the sag at the centre depending upon the cable length.

From Equation 9.3, the horizontal reaction is:

$$H = \frac{M_O}{d} = \frac{wL^2}{8d} \tag{9.7}$$

If the origin of co-ordinates is taken at C (Figure 9.5b), the equation of the parabola is:

$$y = \left(\frac{4d}{L^2}\right)x^2 \tag{9.8}$$

and its slope at x is given by:

$$\tan\theta = \frac{dy}{dx} = \left(\frac{8d}{L^2}\right)x \tag{9.9}$$

For equilibrium of the portion CD (Figure 9.5b), $T\cos\theta = H$ and hence:

$$T = H\sec\theta = H\sqrt{1 + \left(\frac{8dx}{L^2}\right)^2} \tag{9.10}$$

The maximum cable tension occurs at the supports where $x = \pm L/2$ and therefore:

$$T_{max} = \frac{wL^2}{8d}\sqrt{1 + \frac{16d^2}{L^2}} = \frac{wL}{2}\sqrt{\frac{L^2}{16d^2} + 1} \tag{9.11}$$

The length of the cable can be determined by considering the elemental length of cable shown in Figure 9.5c:

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + a^2x^2} \tag{9.12}$$

where from Equation 9.9: $a = \dfrac{8d}{L^2}$

For the cable shown in Figure 9.5a, the total length l_{AB} is obtained by integrating Equation 9.12:

$$l_{AB} = 2\int_0^{\frac{L}{2}}\sqrt{1 + a^2x^2}\, dx$$

$$= \left[x\sqrt{1 + a^2x^2} + \frac{1}{a}\ln(x + \frac{1}{a}\sqrt{1 + a^2x^2})\right]_0^{\frac{L}{2}}$$

$$= L\left[\frac{2d}{L}\sqrt{1 + \frac{L^2}{16d^2}} + \frac{L}{8d}\ln\left\{\frac{4d}{L}\left(1 + \sqrt{1 + \frac{L^2}{16d^2}}\right)\right\}\right] \tag{9.13}$$

A simpler expression could be obtained by making use of the binominal expansion of $\sqrt{1 + a^2x^2}$ and then integrating the series term by term. In this way the length of the cable is expressed as:

$$l_{AB} = L\left[1 + \frac{1}{6}\left(\frac{16d^2}{L^2}\right) - \frac{1}{40}\left(\frac{16d^2}{L^2}\right)^2 + \frac{1}{112}\left(\frac{16d^2}{L^2}\right)^3 \cdots\right]$$

$$= L\left[1 + \frac{8}{3}\left(\frac{d}{L}\right)^2 - \frac{32}{5}\left(\frac{d}{L}\right)^4 + \frac{256}{7}\left(\frac{d}{L}\right)^6 \cdots\right] \tag{9.14}$$

This method will be valid (the series will be convergent) provided $L/d \geqslant 4$. Most practical cases are within this range of validity. For most purposes it will be sufficiently accurate to take only two or three terms of the expansion. For instance, for $L/d = 4$, three terms of the expansion give $l_{AB} = 1.1416L$, whereas the closed form solution gives $l_{AB} = 1.1477L$. When $L/d = 8$ three terms of the series give $l_{AB} = 1.04016L$ compared with the closed form solution $l_{AB} = 1.04022L$.

In cases where the maximum sag d is specified, the cable length can be obtained from Equations 9.13 or 9.14. If the cable length is specified these equations can be used to obtain the maximum sag, but a trial and error approach would be most appropriate for solving the equations.

For convenience, the values of l_{AB}/L obtained using Equation 9.13 for parabolic cables similar to the cable shown in Figure 9.5a are given in Table 9.1 for the practical range of sag to span (d /L).

Table 9.1

$\frac{d}{L}$	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	$\frac{l_{AB}}{L}$
0.000	1.0000	0.150	1.0571	0.300	1.2044	0.450	1.4047
0.025	1.0017	0.175	1.0765	0.325	1.2349	0.475	1.4414
0.050	1.0066	0.200	1.0982	0.350	1.2644	0.500	1.4789
0.075	1.0148	0.225	1.1221	0.375	1.2996		
0.100	1.0261	0.250	1.1478	0.400	1.3337		
0.125	1.0402	0.275	1.1753	0.425	1.3688		

EXAMPLE 9.2

The cable shown in Figure 9.6 carries a uniformly distributed load of 8 kN per horizontal metre.

(i) If $d = 10$ m, determine the maximum tension in the cable and the cable's length.

(ii) If the cable length is 120 m, determine the maximum tension in the cable and the sag d.

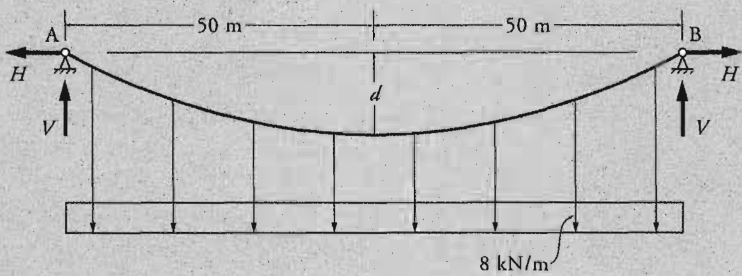


Figure 9.6

SOLUTION

(i) The horizontal reaction is obtained from Equation 9.7:

$$H = \frac{8 \times 100^2}{8 \times 10} = 1000 \text{ kN}$$

The maximum tension in the cable occurs at the supports where:

$$V = \frac{8 \times 100}{2} = 400 \text{ kN}$$

and is obtained from:

$$T_{max} = \sqrt{V^2 + H^2} = 1077 \text{ kN}$$

Since $L/d > 4$, Equation 9.14 may be used to determine the cable length:

$$l_{AB} = 100\left[1 + \frac{8}{3}\left(\frac{10}{100}\right)^2 - \frac{32}{5}\left(\frac{10}{100}\right)^4 + \frac{256}{7}\left(\frac{10}{100}\right)^6\right] = 102.61 \text{ m}$$

Alternatively, from Table 9.1, for $d/L = 0.1$:

$$l_{AB} = 1.0261 \times 100 = 102.61 \text{ m}$$

(ii) With $\frac{l_{AB}}{L} = \frac{120}{100} = 1.20$, interpolation in Table 9.1 gives:

$\frac{d}{L} = 0.296 \quad \therefore\ d = 29.62$ m

The horizontal reaction is obtained from Equation 9.7:

$$H = \frac{8 \times 100^2}{8 \times 29.62} = 337.6 \text{ kN}$$

and the vertical reaction is 400 kN as in part (i). The maximum tension is therefore:

$$T_{max} = \sqrt{337.6^2 + 400^2} = 523.4 \text{ kN}$$

When the supports are at different levels, the maximum sag of a cable subjected to a uniformly distributed load will not occur at the centre of the span. Consider a cable AB with the support B a distance h above the level of A. The loading is w per unit length of horizontal projection (Figure 9.7).

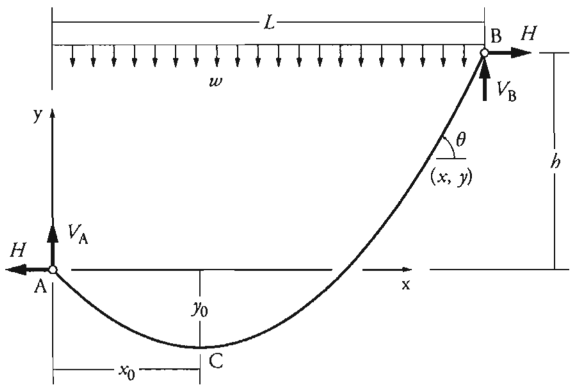


Figure 9.7

By taking moments of all the forces about B we have:

$$V_A L + Hh - \frac{wL^2}{2} = 0 \qquad \textbf{(9.15)}$$

As the bending moment at any point on the cable (x, y) is zero:

$$V_A x + Hy - \frac{wx^2}{2} = 0 \qquad \textbf{(9.16)}$$

Eliminating V_A between Equations 9.15 and 9.16, we find that the shape of the cable is given by:

$$y = \frac{wx^2}{2H} + \left(\frac{h}{L} - \frac{wL}{2H}\right)x \qquad \textbf{(9.17)}$$

Clearly the curve is a parabola. The cable may or may not sag below the support A, and whether it does will depend upon the value of H. The lowest point on the cable is found by setting $dy/dx = 0$ and solving for x. Performing this operation yields the coordinates of the lowest point, (x_0, y_0):

$$x_0 = \frac{L}{2} - \frac{Hh}{wL} \tag{9.18}$$

$$y_0 = \frac{h}{2} - \frac{Hh^2}{2wL^2} - \frac{wL^2}{8H} \tag{9.19}$$

If $H \geqslant \dfrac{wL^2}{2h}$, the cable will not sag below A.

The cable tension at any point can be obtained from $T_x = H \sec \theta$. The maximum tension will occur at the higher support, and is:

$$T_B = H\sqrt{1 + \left(\frac{h}{L} + \frac{wL}{2H}\right)^2} \tag{9.20}$$

In a problem in which w, L, h and y_0 are specified, H may be found from Equation 9.17, and then x_0 from Equation 9.18.

The length of the cable AB can now be found as the sum of the lengths AC and CB. The length AC will be half the length of the symmetrical parabola of base $2x_0$ and sag y_0, while the length CB will be half the length of the parabola of the base $2(L - x_0)$ and the sag $(h + y_0)$.

EXAMPLE 9.3

A cable of negligible weight is suspended between two points A and B. The horizontal projection of AB is 50 m and B is 15 m above the level of A. The maximum sag below A is to be 5 m and the cable carries a uniformly distributed load of 15 N/m. Find the maximum tension and the length of the cable.

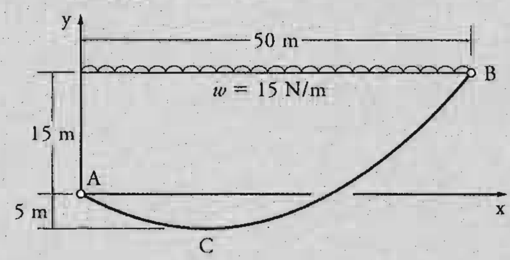


Figure 9.8

SOLUTION

In this example, $L = 50$ m, $h = 15$ m, $y_0 = -5$ m and $w = 15$ N/m. From Equation 9.19:

$$-5 = 7.5 - \frac{H(15)^2}{30(50)^2} - \frac{15(50)^2}{8H} \qquad \therefore H = 3750 \text{ N or } 417 \text{ N}$$

The larger value of H corresponds to a negative value of x_0, hence $H = 417$ N.

From Equation 9.20:

$$T_B = 417\sqrt{1 + \left(\frac{15}{50} + \frac{15 \times 50}{2 \times 417}\right)^2} = 651\text{ N}$$

and from Equation 9.18, the point of maximum sag is:

$$x_0 = 25 - \frac{417 \times 15}{15 \times 50} = 16.66\text{ m}$$

The cable length AB is equal to AC + CB. In calculating the length of the half parabola AC, Equation 9.14 may be used since $L/d > 4$.

Hence: $$l_{AC} = \frac{33.32}{2}\left[1 + \frac{8}{3}\left(\frac{5}{33.32}\right)^2 - \frac{32}{5}\left(\frac{5}{33.32}\right)^4\right] = 17.60\text{ m}$$

More conveniently, for a sag to span ratio of $\frac{d}{L} = \frac{5}{33.32} = 0.1501$, then from Table 9.1:

$$l_{AC} = 0.5 \times 1.0575 \times 33.32 = 17.62\text{ m}$$

To calculate l_{CB}, Equation 9.14 is not appropriate since $L/d < 4.0$. Equation 9.13 must be used. For $d = 20$ m and $L = 2 \times 33.34 = 66.68$ m, (i.e. $d/L = 0.2999$), Table 9.1 gives:

$$l_{CB} = 0.5 \times 1.2041 \times 66.68 = 40.15\text{ m}$$

Therefore the total length of the cable is:

$$l_{AB} = l_{AC} + l_{CB} = 57.77\text{ m}$$

9.3 The catenary

A uniform cable carrying its own weight hangs in the shape of a curve called a *catenary* which is a hyperbolic cosine curve. This problem is not of great engineering interest, since cables are rarely used simply to carry their own weight. Moreover, unless the ratio of sag to span is quite large, the catenary does not differ greatly from a parabola, since for fairly flat curves the weight per metre along the cable is much the same as the weight per horizontal metre.

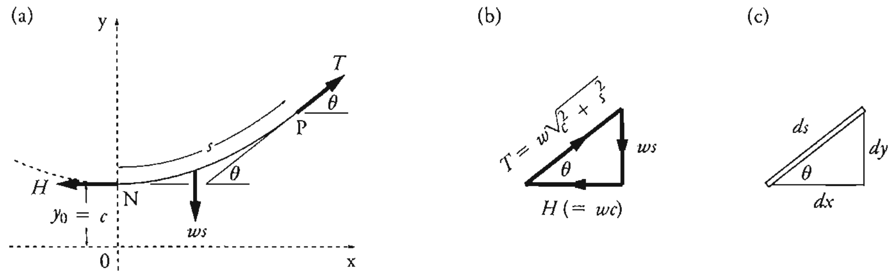


Figure 9.9

Consider the segment of a cable of weight w per unit length between the lowest point, N and any other point P as shown in Figure 9.9a. The segment length is s. The segment is in equilibrium under its self-weight ws, the tension T at point P and the horizontal tension H at the lowest point N. Since there are only three forces they must be concurrent and their relative magnitudes are given by the triangle of forces in Figure 9.9b. The same triangle relates ds, dx and dy in Figure 9.9c where ds is an infinitesimal length of cable at point P and dx and dy are its horizontal and vertical projections.

If the horizontal force H in Figure 9.9a is expressed in terms of the cable weight w such that:

$$H = wc \qquad \textbf{(9.21)}$$

then from Figure 9.9b it can be seen that the tension at any point is given by:

$$T = w\sqrt{c^2 + s^2} \qquad \textbf{(9.22)}$$

Comparison of Figures 9.9b and 9.9c shows that:

$$\frac{dx}{ds} = \frac{H}{T} = \frac{c}{\sqrt{c^2 + s^2}} \qquad \textbf{(9.23)}$$

By integration:

$$x = c\sinh^{-1}\left(\frac{s}{c}\right) + a_1 \qquad \textbf{(9.24)}$$

and since $s = 0$ when $x = 0$, then $a_1 = 0$.

Rearranging Equation 9.24 gives: $\frac{s}{c} = \sinh\left(\frac{x}{c}\right)$ **(9.25)**

From Figures 9.9b and 9.9c: $\frac{dy}{dx} = \frac{s}{c}$ $\quad \therefore \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$

By integration: $y = c\cosh\left(\frac{x}{c}\right) + a_2$

If the origin is taken at a distance c below the lowest point (Figure 9.9a), then when $x = 0$, $y = c$ and $a_2 = 0$. The shape of the curve (the catenary) is therefore:

$$y = c\cosh\left(\frac{x}{c}\right) \qquad \textbf{(9.26)}$$

and, from Equation 9.25, the distance round the cable from N to a point with abscissa x, is:

$$s = c\sinh\left(\frac{x}{c}\right) \qquad \textbf{(9.27)}$$

where $c = \frac{H}{w}$

The tension at any point (Figure 9.9b) from Equation 9.27 is:

$$T = H\frac{ds}{dx} = wc\cosh\frac{x}{c} = wy \qquad \textbf{(9.28)}$$

For a cable hanging under its own weight between two supports A and B, at the same level and located a distance L apart, the length of the cable l_{AB} is obtained from Equation 9.27:

$$l_{AB} = \frac{2H}{w}\sinh\left(\frac{wL}{2H}\right) \quad \textbf{(9.29)}$$

and the sag d below the supports is obtained from Equation 9.26:

$$d = y - c = c\cosh\left(\frac{L}{2c}\right) - c = \frac{H}{w}\left(\cosh\frac{wL}{2H} - 1\right) \quad \textbf{(9.30)}$$

If we define $\alpha = \frac{wL}{2H}$, we can re-express Equation 9.29 as:

$$\frac{l_{AB}}{L} = \frac{1}{\alpha}\sinh\alpha \quad \textbf{(9.31)}$$

and Equation 9.30 as:

$$\frac{d}{L} = \frac{1}{2\alpha}(\cosh\alpha - 1) \quad \textbf{(9.32)}$$

For all practical situations, α will vary between 0 and 2. For convenience, values of l_{AB}/L and d/L (calculated using Equations 9.31 and 9.32) are given in Table 9.2 for these value of α.

Table 9.2 Length to span and sag to span ratios for a catenary.

α	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	α	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	α	$\frac{l_{AB}}{L}$	$\frac{d}{L}$	α	$\frac{l_{AB}}{L}$	$\frac{d}{L}$
0.00	1.0000	0.0000	0.55	1.0512	0.1410	1.10	1.2142	0.3039	1.65	1.5197	0.5150
0.05	1.0004	0.0125	0.60	1.0611	0.1546	1.15	1.2355	0.3206	1.70	1.5563	0.5377
0.10	1.0017	0.0250	0.65	1.0719	0.1683	1.20	1.2579	0.3378	1.75	1.5945	0.5612
0.15	1.0038	0.0376	0.70	1.0837	0.1823	1.25	1.2815	0.3554	1.80	1.6345	0.5854
0.20	1.0067	0.0502	0.75	1.0964	0.1965	1.30	1.3064	0.3734	1.85	1.6764	0.6104
0.25	1.0104	0.0628	0.80	1.1101	0.2109	1.35	1.3327	0.3920	1.90	1.7201	0.6362
0.30	1.0151	0.0756	0.85	1.1248	0.2256	1.40	1.3602	0.4110	1.95	1.7657	0.6629
0.35	1.0205	0.0884	0.90	1.1406	0.2406	1.45	1.3892	0.4306	2.00	1.8134	0.6905
0.40	1.0269	0.1013	0.95	1.1574	0.2559	1.50	1.4195	0.4508			
0.45	1.0341	0.1144	1.00	1.1752	0.2715	1.55	1.4514	0.4716			
0.50	1.0422	0.1276	1.05	1.1941	0.2875	1.60	1.4847	0.4930			

EXAMPLE 9.4

The cable of a high voltage power line is suspended between two points at the same level and 400 m apart. If the cable weighs 60 N/m and the maximum sag is 24 m find the maximum tension and the length of the cable.

SOLUTION

The maximum sag is given by the difference between y at $x = L/2$ and y at $x = 0$. An examination of Equation 9.26 shows that H cannot be found by one operation and an iterative solution may be used. A first approximation for H may be found by assuming that the load w is uniformly distributed in the horizontal projection of the length between the supports, i.e. by assuming the curve to be a parabola of sag $d = 24$ m.

Then: $H = \frac{M_{centre}}{d} = \frac{wL^2}{8d} = \frac{60 \times 400^2}{8 \times 24} = 50 \text{ kN}$

With this as an approximate value for H, the sag can be checked using Equation 9.30:

$$d = \frac{50 \times 10^3}{60}\left(\cosh\frac{60 \times 400}{2 \times 50 \times 10^3} - 1\right) = 24.12 \text{ m}$$

For a value of $H = 51$ kN the maximum sag is found to be 23.64 m. By interpolation the value of H required to give a sag of 24 m is 50.25 kN.

The maximum tension occurs at the end and from Equation 9.28:

$$T_{max} = H\cosh\left(\frac{wL}{H2}\right) = 50.25\cosh\left(\frac{60 \times 400}{50250 \times 2}\right) = 51.7 \text{ kN}$$

The total length of the cable is calculated from Equation 9.29:

$$l = \frac{2 \times 50250}{60}\sinh\left(\frac{60 \times 400}{2 \times 50250}\right) = 404 \text{ m}$$

Unless the cable sag is large compared with the span, the shape of the catenary differs little from that of a parabola. For practical purposes, Example 9.3 could have been solved on the basis of a parabolic shape.

Example 9.4 can be solved more conveniently using Table 9.2.

With $L = 400$ m, $w = 60$ N/m and $d = 24$ m then $d/L = 0.060$.

By interpolation in Table 9.2:

$$\alpha = 0.2389 \quad \text{and} \quad \frac{l_{AB}}{L} = 1.0096 \qquad \therefore l_{AB} = 1.0096 \times 400 = 404 \text{ m}$$

$$H = \frac{wL}{2\alpha} = \frac{60 \times 400}{2 \times 0.2389} = 50.23 \text{ kN}$$

The vertical reaction at each support is $\frac{60 \times 404}{2} = 12.12$ kN and the cable tension at each support is:

$$T_{max} = \sqrt{(12.12)^2 + (50.23)^2} = 51.7 \text{ kN}$$

EXAMPLE 9.5

A cable weighing 60 N/m is to be drawn across a valley. One end of the cable is held at a point B on one side of the valley. The cable passes over a pulley at A on the other side of the valley. A and B are at the same level, 400 m apart and 160 m above the valley floor. What force is required to completely lift the cable from the floor of the valley? What force is required when the length of the cable between A and B is 440 m?

SOLUTION

1. When the cable is just lifted from the valley floor $d = 160$ m.

Therefore: $\frac{d}{L} = \frac{160}{400} = 0.40$

From Table 9.2, by interpolation:

$$\alpha = 1.3711 \text{ and } \frac{l_{AB}}{L} = 1.3443$$

Then: $H = \frac{wL}{2\alpha} = \frac{60 \times 400}{2 \times 1.3711} = 8750 \text{ N} = 8.75 \text{ kN}$

and $l_{AB} = 1.3443 \times 400 = 538$ m

The vertical reaction at A is $\frac{60 \times 538}{2} = 16.14$ kN and therefore the tension in the cable at A is:

$$T = \sqrt{16.14^2 + 8.75^2} = 18.36 \text{ kN}$$

2. When the cable length l_{AB} is reduced to 440 m, $\frac{l_{AB}}{L} = 1.100$ and from Table 9.2, $\alpha = 0.763$.

Therefore: $H = \frac{60 \times 400}{2 \times 0.763} = 15.73$ kN

and the vertical reaction at A is: $V = \frac{60 \times 440}{2} = 13.2$ kN.

The corresponding tension in the cable at A is:

$$T = \sqrt{13.2^2 + 15.73^2} = 20.53 \text{ kN}$$

The sag at this stage may also be obtained from Table 9.2 where $d/L = 0.2003$:

$$\therefore d = 0.2003 \times 400 = 80.1 \text{ m}$$

Problems

9.1 Determine the maximum tension in the cable shown in Figure P9.1. Assume the cable is weightless.

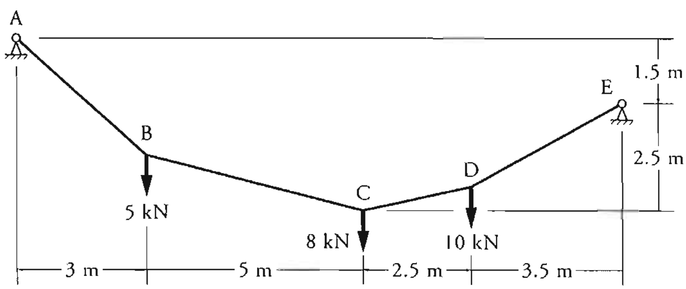


Figure P9.1

9.2 The supports A and B of the cable shown in Figure P9.2 are at the same level and the maximum tension in the cable is 20 kN. Find the sag of the cable at its lowest point.

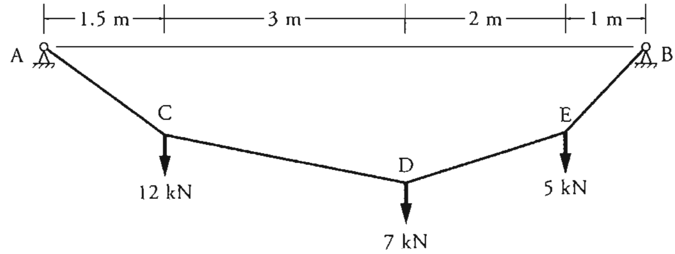


Figure P9.2

9.3 The cable AD in Figure P9.3 is shortened until segment AB is horizontal. Determine the tension in CD if AB is 2 m and BC is 3 m.

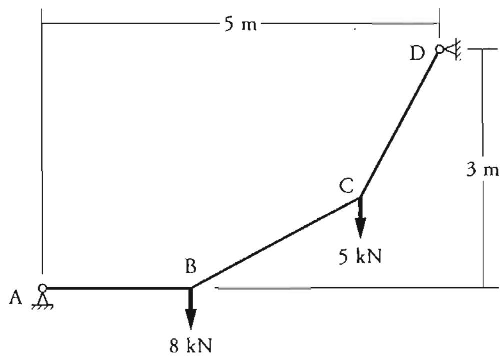


Figure P9.3

9.4 A uniform cable AB is 130 m long and weighs 5 N/m. It is suspended from two points 120 m apart and at the same level. Assume that the cable hangs in the shape of a parabola, i.e. that it carries 5 N per *horizontal* metre.

(i) Find the sag at the centre.
(ii Find the tension at the centre and at the end.
(iii) Find the sag at a point 40 m from A.

9.5 Solve Problem 9.4 if the weight is correctly taken as 5 N/m along the cable, i.e. if the cable shape is a catenary.

9.6 The cable of Problem 9.4 is to be used to support a load suspended above a ravine. If the load of 60 kN is suspended at C, 40 m (horizontally) from A. Find:

(i) the sag at C (neglect the cable weight)
(ii) the increase in sag at C (compared with the freely hanging cable in Problem 9.5)
(iii) the maximum cable tension.

9.7 Solve Problem 9.6 but with a load of 15 kN at D, 80 m from A, *instead of* the load at C.

9.8 The cable of the foregoing problems supports a load of 60 kN at C (see Problem 9.6). A load of 15 kN is now suspended at D (80 m from A) *in addition* to the load at C. Find:

(i) the sag at C and D (neglect the cable weight)

(ii) the additional sag at D when the 15 kN is applied.

(iii) Why is this additional sag at D smaller than when the 15 kN was applied (see Problem 9.7)?

9.9 Find the maximum and minimum tensions in the parabolic cable AB shown in Figure P9.9.

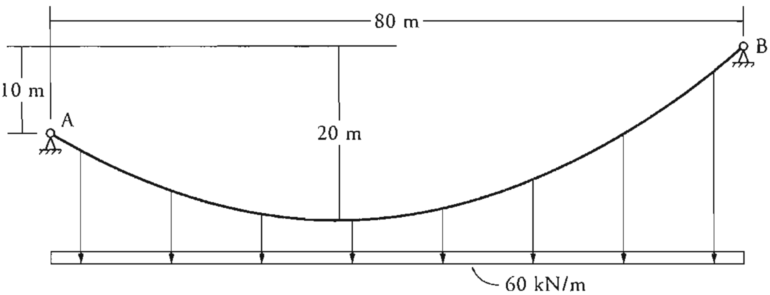
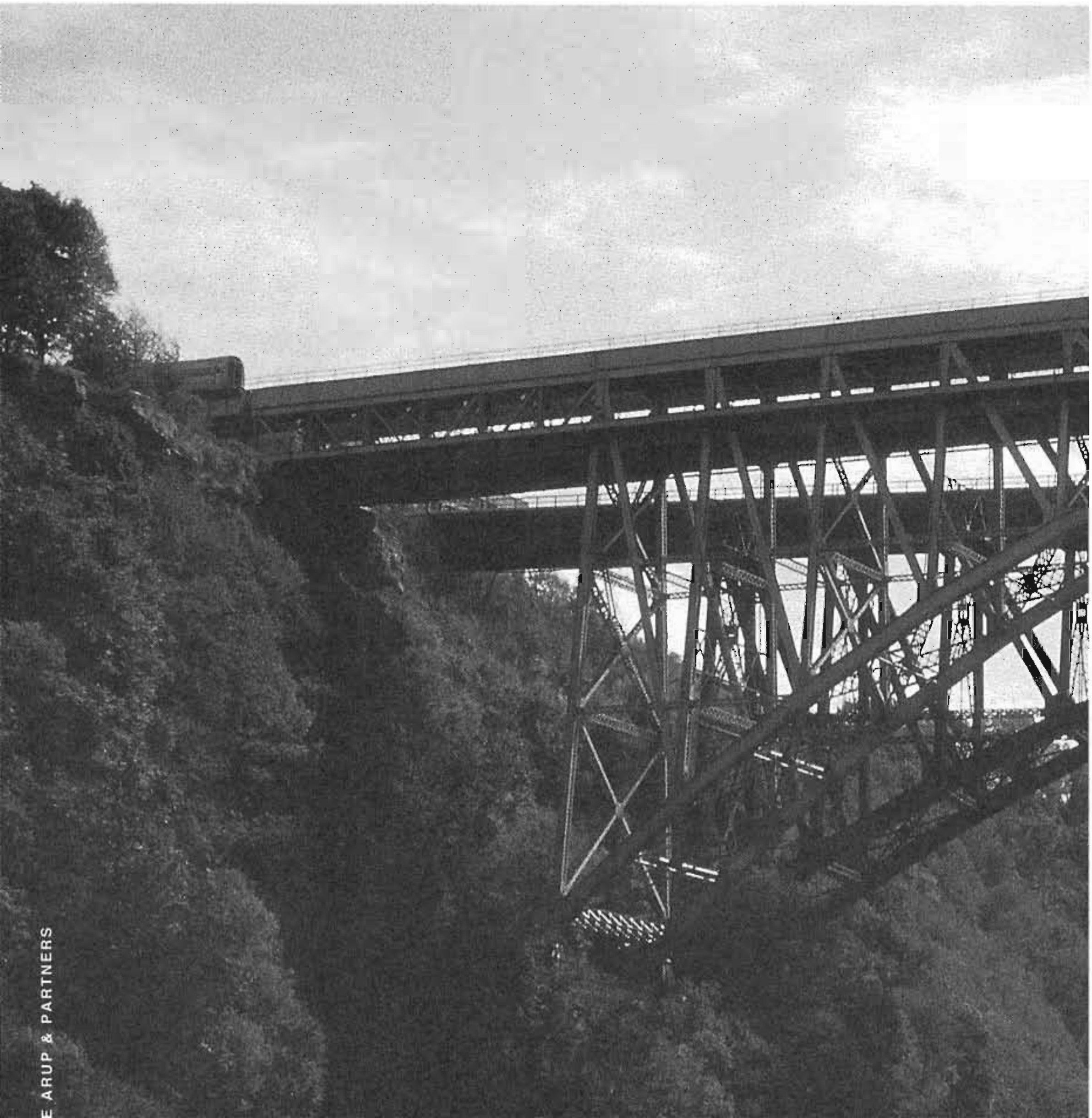


Figure P9.9

9.10 A cable ABCD of negligible weight is suspended between A and D, which are 300 m apart and at the same level. The length of the cable is 400 m. A load of 200 kN is hung at B (AB = 100 m) and another load of 400 kN at C (BC = 150 m). Find the sag at B and C and the maximum tension.

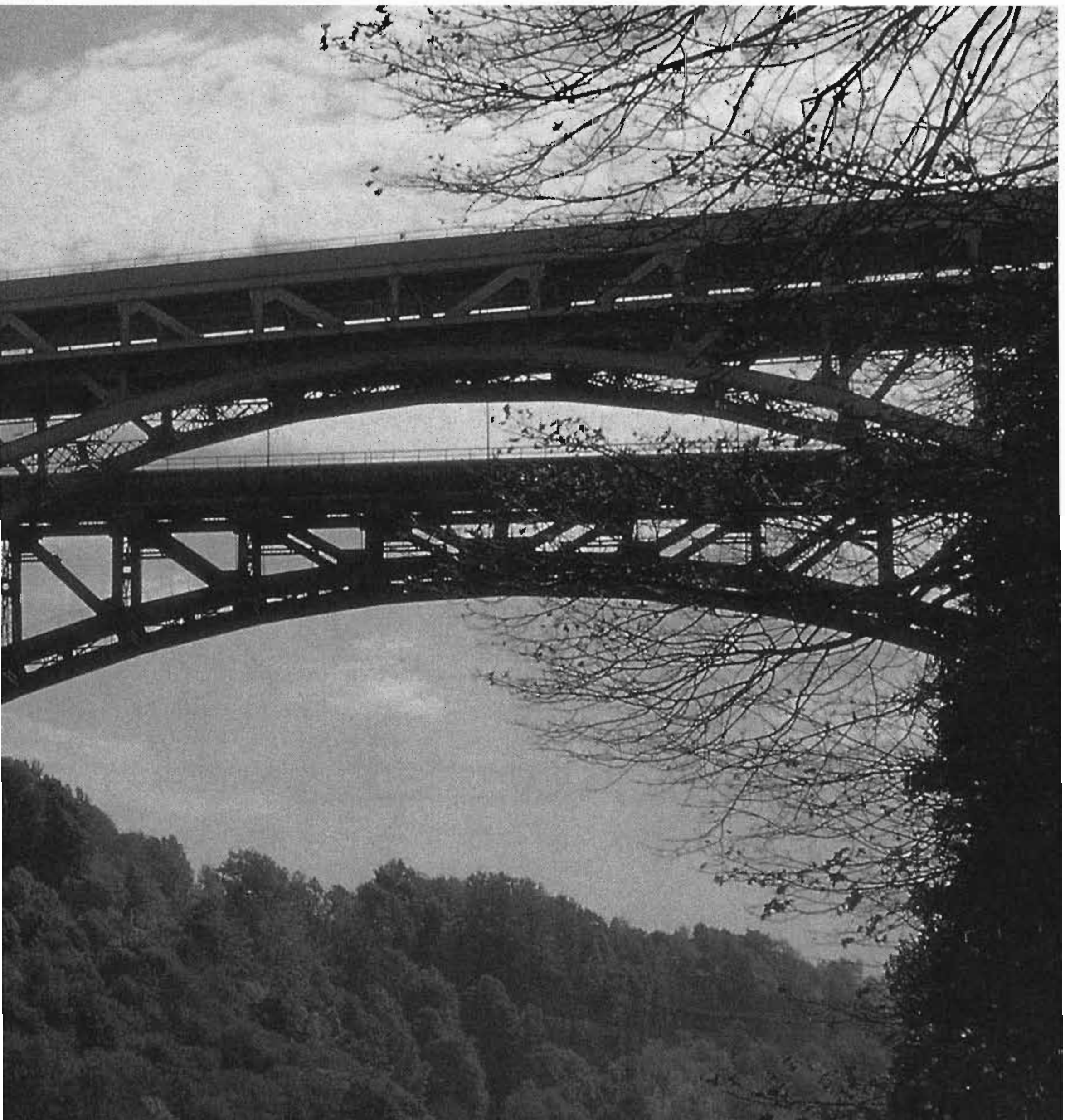
9.11* A cable ABC has a total length of 440 m and weighs 100 N/m. The cable is fixed at points A and C and passes over a frictionless pulley at B. The three supports are at the same level. The distance AB is 180 m and BC is 200 m. Find the maximum sag in each section of the cable and the maximum tension.

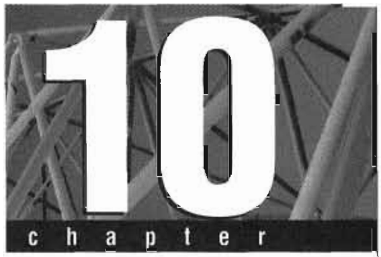
* Difficult problems, suitable for later study.



part 3

PIN-JOINTED TRUSSES





Trusses

10.1 Definitions

A structure which consists of a number of rigid bars fastened together at their ends (see Figure 10.1) is called a *frame* or *truss.* The individual bars are called *members* of the frame. For the purpose of calculating the forces in such a structure, the joints are considered to be either rigid or pinned.

At a *rigid* joint no relative rotation is possible between the ends of the jointed members. Such a state exists in mosr reinforced concrete framed structures, and also in sreel strucrures if the ends of the sreel bars are welded togerher.

At a *pinned* joint, the end of each member is free to rotate (in the plane of the frame) independently of its neighbours. It is imagined thar the members are joined by frictionless pins around which they are free to rotate. Such a srructure is usually called a *truss.*

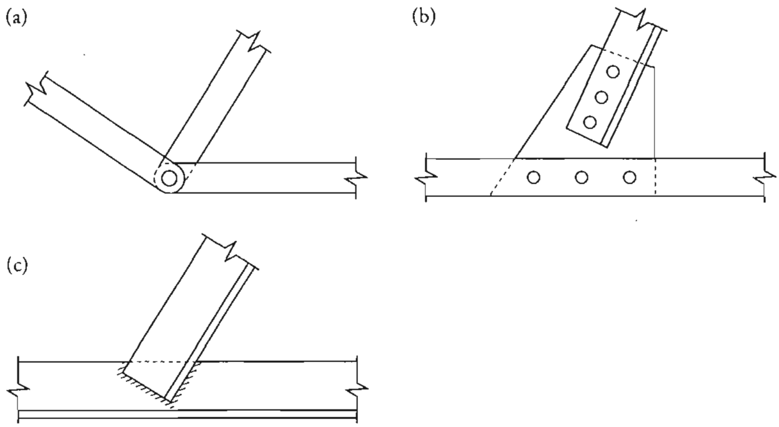


Figure 10.1

In reality, very few structures are built with pinned connections as shown in Figure 10.1a. The connections in practical steel trusses are made by either bolting or welding, as shown in Figures 10.1b and 10.1c. As shown in these figures, some members may even be continuous through the joint while other members are so connected that little if any relative rotation can occur between the members meeting at a joint. However, if the members are arranged in a pattern of triangles, it is found that the axial force is the predominant internal action and there is little tendency for the ends of the members to rotate relative to each other. Furthermore, in the majority of trusses the members are comparatively slender and the joint fixity has only a minor effect upon the internal force system. For these reasons a truss is usually analysed as if the joints are pinned. This assumption simplifies the analysis considerably, and results in a reasonably accurate assessment of the forces in each bar.

The bar forces determined on the assumption that the joints are pinned are sometimes referred to as the *primary* forces, while the *secondary* forces are those arising from the joint fixity. The design engineer must decide if it is necessary to evaluate the secondary forces in any particular design situation.

10.2 Limitations

Only pin-jointed trusses will be considered in this book. The treatment is further limited to trusses lying in one plane and acted upon by loads in the same plane. The principles of solution may be extended to three dimensional trusses provided the statics of three-dimensional force systems is employed (see Part 5).

In order to resist a general type of loading, the truss must have supports capable of supplying reactions which will equilibrate such loading. If the reaction components are just sufficient for this purpose they can be determined by the laws of statics applied to the truss as a whole.

We consider here only statically determinate trusses (i.e. trusses such that the bar forces and reactions can be determined by the laws of statics alone). For a plane truss we may write two equations of equilibrium for each joint, so if the number of joints is j, the number of equilibrium equations is $2j$. The quantities to be determined are the axial forces in the bars and the reaction components. If there are m members and r reaction components, the total number of unknowns is $m + r$. For the truss to be statically determinate it is necessary that:

$$m + r = 2j \tag{10.1}$$

Several other conditions are also necessary. The number of reaction components must be at least three, and these must be arranged to ensure overall stability for all types of loading.

Also, the bars of the truss must be suitably arranged. The bars in a *stable* truss are usually arranged so that the bars form a series of triangular units, with bars connected together at each joint. An arrangement of pin-ended bars that forms a rectangular or quadrilateral unit is unstable, unless externally restrained, and will collapse under load, as illustrated in Figure 10.2a. The insertion of the diagonal bar AC in Figure 10.2b, to form two triangular units, produces a stable truss capable of carrying load. Thus the truss of Figure 10.3a is stable while that of Figure 10.3b is not, although Equation 10.1 is satisfied in both cases. Sometimes the fact that the truss is *not* stable cannot be seen by inspection. For instance, the truss of Figure 10.4, in which the bars are not connected where they cross, satisfies Equation 10.1. However it is not stable. Investigation of such cases of stability will not be dealt with here.

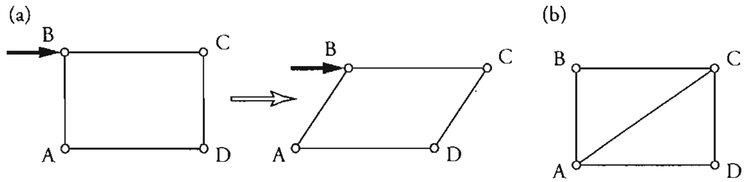


Figure 10.2

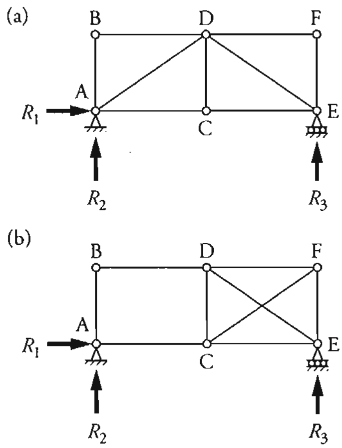


Figure 10.3

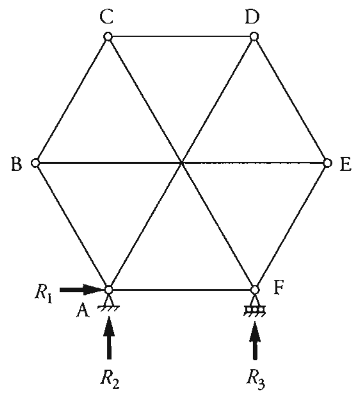


Figure 10.4

In many practical trusses, it is possible to write the equilibrium equations joint by joint and solve for the bar forces as we go along. Such trusses may be solved by hand calculation. Only this type of truss will be dealt with here.

Finally we note that when the truss is loaded each bar will undergo a slight change in length and this will cause a small change in geometry. Such changes are usually negligible and the bar forces are calculated on the basis of the geometry of the unloaded truss.

10.3 Principles of solution

The forces in the various members may be found by applying the laws of statics, thus making it possible to select suitable sizes for the members. Since the truss is assumed to be permanently at rest, the equations of equilibrium may be applied to the whole truss or to any part of it.

By considering the equilibrium of the truss as a whole, the external reactions are found in the same way as they were for rigid beams. The laws of equilibrium are then applied both to the pins and to the bars of which the truss is composed.

Suppose the structure in Figure 10.5 rests on supports at A and E and carries a load W at the mid-point C. Imagine that the bar BD is removed. The frame will collapse and the joints B and D will move towards one another. The function of bar BD is to keep these joints apart. The bar is in *compression.* If the bar CE is removed, collapse would involve C and E moving apart. The function of this bar is to hold joints C and E together, and the bar is in *tension.*

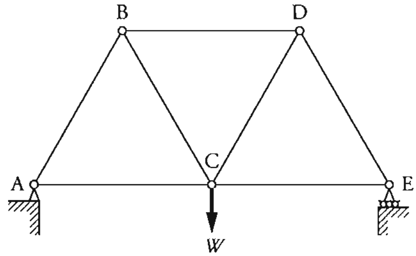


Figure 10.5

In Chapters 11 and 12, we shall consider trusses in which all members are straight and external loads are applied only at the joints. In such trusses, the individual members are subjected only to *axial force.* In cases where the members of the truss are curved, or when loads are applied to the bars themselves, the truss members may be subjected to bending moment and shear force, in addition to axial force. These cases are dealt with in Chapter 13.

If the forces in all bars are required, then it is necessary to consider the equilibrium of every bar and every joint individually. This procedure is known as the *method of joints* and is described in Chapter 11. Sometimes it is required to find the forces in only a few bars. Frequently it is possible to do this without the need to solve the whole truss. The procedure for analyzing the forces in individual bars is known as the *method of sections* and is described in Chapter 12.

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OVE ARUP & PARTNERS

The Method of Joints

11.1 External reactions

In most trusses, an analysis starts with the determination of the external reactions by a consideration of the equilibrium of the complete truss. The procedures for finding the reactions were discussed in Chapter 5.

It should be noted that in some trusses it is possible to determine the bar forces without first calculating the reactions.

11.2 Freebodies

In the method of analysis known as the method of joints, each bar and each joint of the truss is considered in turn as a freebody. Figure 11.1a shows a small truss with joints A, B, C and D. The external loads P_1 and P_2 together with reactions R_1, R_2 and R_3 form a system of forces in equilibrium.

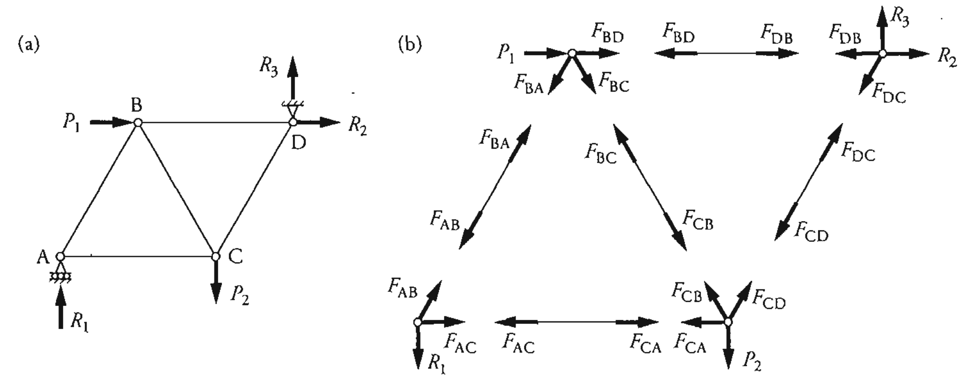


Figure 11.1

Figure 11.1b shows freebody diagrams of the four bars and the four joints which comprise the truss. Each bar is in equilibrium under the action of the forces exerted upon it by the joints at each end. Each pin is in equilibrium under the action of the forces exerted upon it by the adjacent bars and the external loads (applied forces or reactions).

11.3 Equilibrium of members

In the cases considered here, where the external loads are all applied at the joints, each bar is in equilibrium under the action of the two forces exerted on it by the joints at its ends. These two forces must therefore be equal and opposite and must act along the line joining the joints, as shown in Figure 11.2.

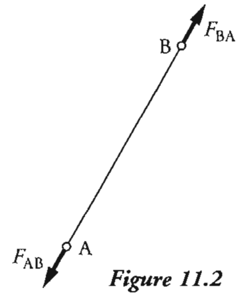


Figure 11.2

11.4 Equilibrium of joints

Each pin in a truss is in equilibrium under the action of the forces applied to it by the surrounding bars together with the external loads, if any, acting at that joint. For the present case (trusses in which all external loads act at joints), the forces exerted by the bars upon a given pin act in the direction of the lines joining the given pin with its neighbours, so that the directions of all forces are known. The forces at a given pin form a concurrent system, implying two equilibrium conditions.

In the method of joints, each joint in turn is considered as a freebody. The two equations of equilibrium are used to express the relationships between the known and unknown forces at the joint. The unknown forces at a joint can be determined provided that there are not more than two such forces.

This is exactly the problem discussed in Section 3.2 (see Example 3.3 page 21). Note that a bar in tension exerts an outward force on the joint, while a bar in compression exerts an inward force. In the freebody diagram of any particular joint, known forces are shown with the appropriate direction. Unknown forces are assumed to be tensile and shown as outward arrows. Solution of the equilibrium equations will then result in a positive sign (confirming tension) or a negative sign (indicating compression).

EXAMPLE 11.1

Find the forces in the members of the truss shown in Figure 11.3 under the given loading. The triangles are all equilateral.

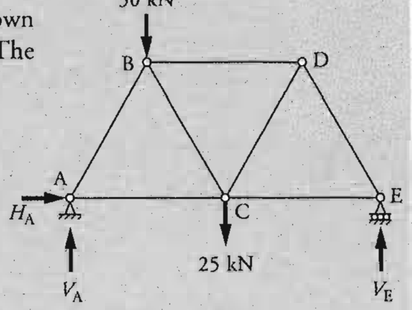


Figure 11.3

SOLUTION

1. The first step is to find the reactions. The horizontal reaction component at A must be zero in this case (since no horizontal load components exist). Taking moments about joints E and A in turn gives:

$$V_A = 35 \text{ kN} \quad \text{and} \quad V_E = 20 \text{ kN} \quad \text{(both upwards)}$$

Checking $\Sigma V = 0$: $35 + 20 - 30 - 25 = 0 \therefore$ O.K.

2. Select a joint at which there are only two unknown forces. Joints A and E satisfy this criterion, joints B, C and D do not. Consider the freebody of joint E shown in Figure 11.4a. The two unknown bar forces F_{ED} and F_{EC} are shown as outward arrows acting at the joint. Also shown is the known reaction force V_E. Although the magnitudes of F_{ED} and F_{EC} are unknown, their lines of action are known, being defined by the geometry of the truss. Writing the equations of vertical and horizontal equilibrium, we have:

$$\Sigma V = 0: \quad F_{ED} \sin 60 + 20 = 0 \qquad \therefore F_{ED} = -23.09 \text{ kN}$$

$$\Sigma H = 0: \quad F_{ED} \cos 60 + F_{EC} = 0 \qquad \therefore F_{EC} = -F_{ED} \cos 60 = +11.55 \text{ kN}$$

Evidently, F_{ED} is compressive (and acts towards the joint), while F_{EC} is tensile, as assumed.

3. Consider the freebody diagram of joint D, shown in Figure 11.4b. The bar force F_{DE} is now known and equals the force F_{ED} calculated in Step 2. Since it is known to be a compressive force (i.e. acting inward on the joint), it is shown acting in that direction. There are therefore just two unknown forces acting at joint D, F_{DB} and F_{DC}, and these may be determined from the two equilibrium equations:

$$\Sigma V = 0: \quad 23.09 \sin 60 - F_{DC} \sin 60 = 0 \qquad \therefore F_{DC} = +23.09 \text{ kN}$$

$$\Sigma H = 0: \quad -F_{DB} - F_{DC} \cos 60 - 23.09 \cos 60 = 0 \quad \therefore F_{DB} = -23.09 \text{ kN}$$

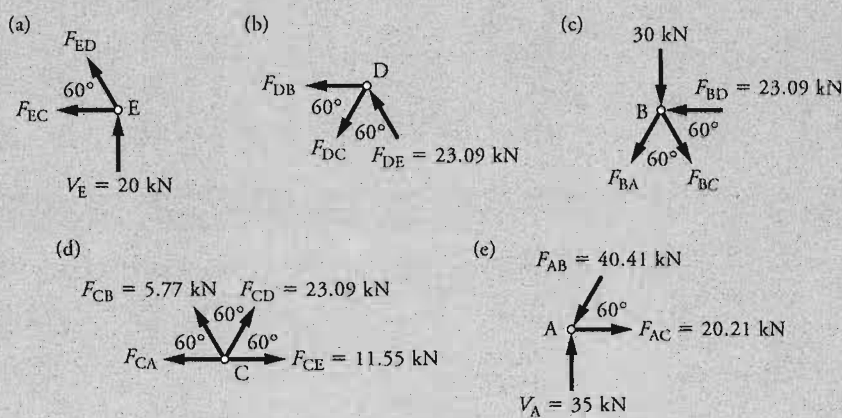


Figure 11.4

4. At joint B (Figure 11.4c), with $F_{BD} = F_{DB} = -23.09$ kN (i.e. 23.09 kN acting inward on the joint), and including the vertical external load of 30 kN, equilibrium gives:

$$\Sigma V = 0: \quad -F_{BA} \sin 60 - F_{BC} \sin 60 - 30 = 0$$

$$\therefore 0.866\, F_{BA} + 0.866\, F_{BC} = -30.0 \text{ kN} \qquad \textbf{(11.1)}$$

$$\Sigma H = 0: \quad -F_{BA} \cos 60 + F_{BC} \cos 60 - 23.09 = 0$$

$$\therefore -0.5\, F_{BA} + 0.5\, F_{BC} = 23.09 \text{ kN} \qquad \textbf{(11.2)}$$

Solving Equations 11.1 and 11.2 gives:

$$F_{BA} = -40.41 \text{ kN} \quad \text{and} \quad F_{BC} = +5.77 \text{ kN}$$

5. At joint C (Figure 11.4d), all bar forces except F_{CA} are known.

$$\Sigma H = 0: \quad -F_{CA} - 5.77 \cos 60 + 23.09 \cos 60 + 11.55 = 0$$

$$\therefore F_{CA} = +20.21 \text{ kN}$$

6. Finally, the equilibrium equations at joint A (Figure 11.4e) may be used to check that the known bar forces and the external load are in fact in equilibrium.

$$\Sigma V = 0: \quad -40.41 \sin 60 + 35 = 0 \qquad \therefore \text{O.K.}$$

$$\Sigma H = 0: \quad -40.41 \cos 60 + 20.21 = 0 \qquad \therefore \text{O.K.}$$

This provides a check on the previous work.

EXAMPLE 11.2

Find the forces in the members of the truss shown in Figure 11.5.

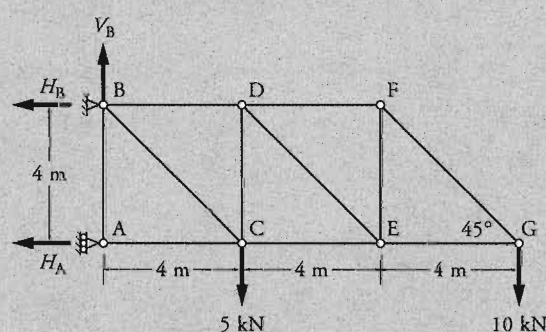


Figure 11.5

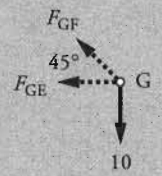
SOLUTION

1. This truss is a cantilever with all reactions at one end. We can start at joint G and it is therefore not necessary to calculate the reactions before finding the bar forces.

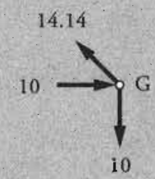
2. As in the previous example, the equilibrium of each joint in turn is considered. In the following, freebody diagrams are shown at the left-hand side of the page with unknown forces acting outward. If the equilibrium equations yield a value with a negative sign, it is then known that the bar is in compression and the force on the joint acts inwards. The true freebody diagram is shown at the right-hand side.

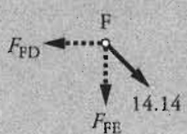
Assumed Forces	**Equations**	**True Forces**

Assumed Forces — *Equations* — *True Forces*

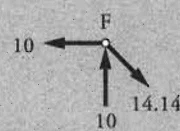


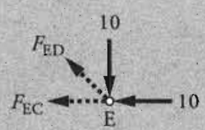
$\Sigma V = 0$: $F_{GF} \sin 45 - 10 = 0$
$\therefore F_{GF} = +14.14$ kN
$\Sigma H = 0$: $-F_{GF} \cos 45 - F_{GE} = 0$
$\therefore F_{GE} = -10$ kN



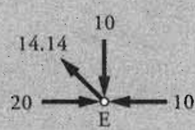


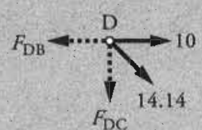
$\Sigma H = 0$: $14.14 \cos 45 - F_{FD} = 0$
$\therefore F_{FD} = +10$ kN
$\Sigma V = 0$: $-14.14 \sin 45 - F_{FE} = 0$
$\therefore F_{FE} = -10$ kN



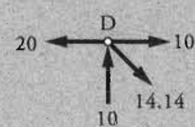


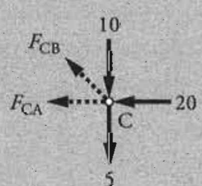
$\Sigma V = 0$: $F_{ED} \sin 45 - 10 = 0$
$\therefore F_{ED} = +14.14$ kN
$\Sigma H = 0$: $-F_{EC} - 14.14 \cos 45 - 10 = 0$
$\therefore F_{EC} = -20$ kN



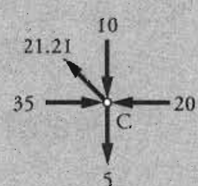


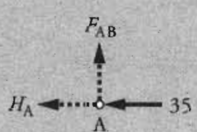
$\Sigma V = 0$: $-F_{DC} - 14.14 \sin 45 = 0$
$\therefore F_{DC} = -10$ kN
$\Sigma H = 0$: $-F_{DB} + 10 + 14.14 \cos 45 = 0$
$\therefore F_{DB} = +20$ kN



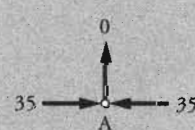


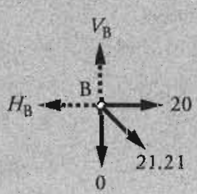
$\Sigma V = 0$: $F_{CB} \sin 45 - 10 - 5 = 0$
$\therefore F_{CB} = +21.21$ kN
$\Sigma H = 0$: $-F_{CA} - 21.21 \cos 45 - 20 = 0$
$\therefore F_{CA} = -35$ kN



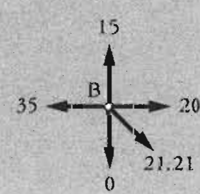


$\Sigma H = 0$: $-H_A - 35 = 0$
$\therefore H_A = -35$ kN
$\Sigma V = 0$: $F_{AB} = 0$





$\Sigma H = 0$: $20 + 21.21 \cos 45 - H_B = 0$
$\therefore H_B = +35$ kN
$\Sigma V = 0$: $-V_B - 21.21 \cos 45 = 0$
$\therefore V_B = +15$ kN



In this problem, two equilibrium equations have been used at every joint, and these equations have provided the values of all the bar forces and also the reactions. The equations of equilibrium of the complete structure may now be used to check the reactions. Regarding the complete truss as a freebody (Figure 11.5), we take moments about B. Then:

$$\curvearrowright\!+\quad (H_A \times 4) + (5 \times 4) + (10 \times 12) = 0 \qquad \therefore\ H_A = -35\ \text{kN}$$

Considering horizontal equilibrium:

$$\Sigma H = 0:\quad H_B - 35 = 0 \qquad \therefore H_B = 35\ \text{kN}$$

and from vertical equilibrium:

$$\Sigma V = 0:\quad V_B - 5 - 10 = 0 \qquad \therefore V_B = 15\ \text{kN}$$

These values agree with those determined earlier.

The results are summarised in Figure 11.6 in which tensile forces are denoted as positive and compressive forces are denoted as negative.

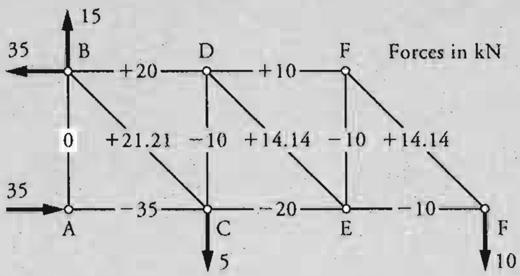


Figure 11.6

It is helpful to imagine how this truss would collapse if a particular member was removed. This leads to the concept of the function of the member (for the given loading) as either keeping two joints apart (compression) or holding them together (tension).

11.5 Use of components of bar forces

For any joint of a truss, the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ involve not the bar forces themselves but the x and y components of these forces. It is often quicker to use these components as the unknown quantities and to calculate the actual bar forces later.

The x and y components of a bar force are proportional to the x and y projections of the bar length, and these dimensions are usually the ones given on the drawing of the truss. In Figure 11.7, the bar GM has projections of 2 m and 5 m. Suppose the x component of the force exerted by GM on joint G has been determined as 16 kN, acting away from the joint. It follows that the y component is 16 × 5/2 = 40 kN also acting away from the joint. At the other end M the force components are now also known.

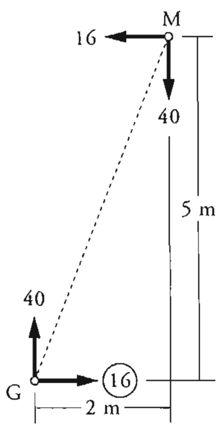


Figure 11.7

EXAMPLE 11.3

Find the components of the bar forces in the truss of Figure 11.8. Hence find the bar forces.

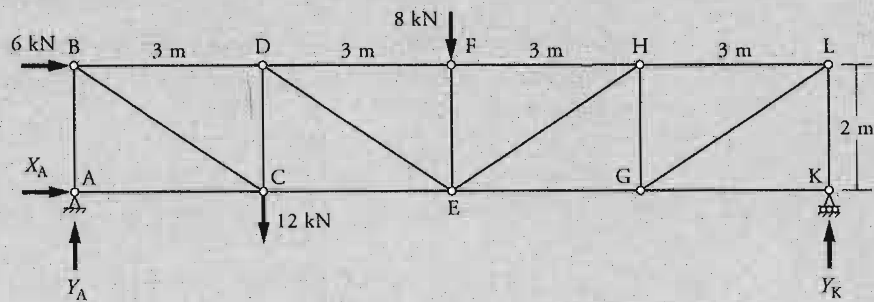


Figure 11.8

SOLUTION

1. First find the reaction components by considering the equilibrium of the complete structure. This gives:

$$X_A = -6 \text{ kN} \qquad Y_A = +12 \text{ kN} \qquad Y_K = +8 \text{ kN}$$

2. Consider the force components acting on joint A (Figure 11.9a). The two circled values are known. Without actually writing the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, it can be seen that bar AC exerts an outward force of 6 kN and bar AB exerts an inward force of 12 kN on the pin A. These forces are shown in Figure 11.9a, and the reactive forces at the far ends of the bars are also shown.

3. Consider the force components at joint B (Figure 11.9b). The known forces are the external force of 6 kN and the upward force of 12 kN in member AB calculated in Step 2. These are shown circled. The unknown forces are F_{BD}, and the components of F_{BC} namely X_{BC} and Y_{BC}, which at first we draw as arrows.

For vertical equilibrium, Y_{BC} must be 12 kN acting downward. From the slope of the member BC, X_{BC} must be $(3/2)Y_{BC}$. Hence $X_{BC} = 18$ kN acting to the right.

Horizontal equilibrium of the joint now gives $F_{BD} = 24$ kN acting to the left. The reactions to these components are shown at C and D. The resultant force in BC is $F_{BC} = \sqrt{X_{BC}^2 + Y_{BC}^2}$, but this need not be computed at this stage.

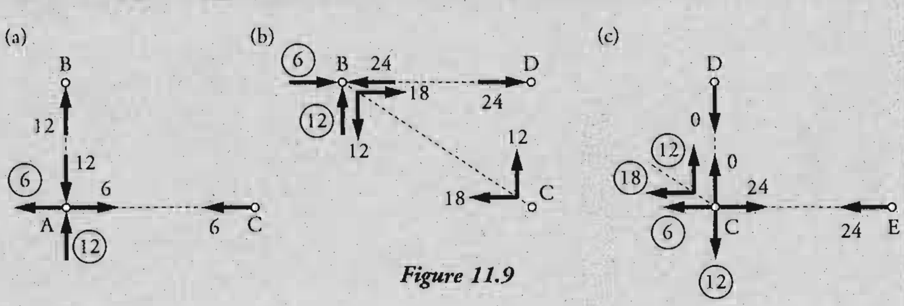


Figure 11.9

4. Consider joint C (Figure 11.9c). The known forces are F_{CA} from Step 2, the components of F_{BC} from Step 3 (all shown circled) and the external force of 12 kN. The unknown forces are F_{CD} and F_{CE}. Vertical equilibrium gives F_{CD} = 0. Horizontal equilibrium gives F_{CE} =24 kN to the right. Reactive forces at D and E are now inserted.

5. Continue in the same way with the remaining joints. It is not really necessary to draw a separate diagram for each joint. The complete solution can be written on a single outline of the truss (Figure 11.10)

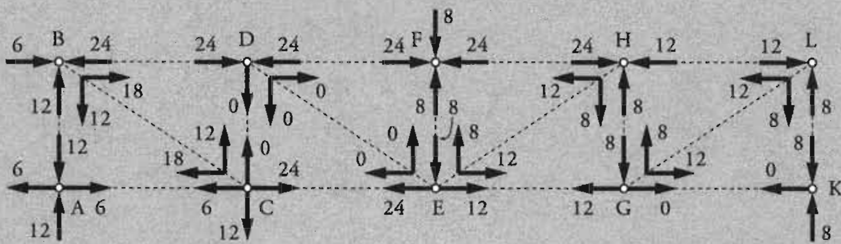


Figure 11.10

6. The final bar forces may now be found from Figure 11.10. In the case of diagonal bars the components are combined. For instance:

$$F_{BC} = +\sqrt{12^2 + 18^2} = +21.63 \text{ kN}$$

The positive sign (tension) is used because the forces exerted by BC act outward on the joints.

For trusses with parallel chords, this method allows the truss to be solved without the need to write the equations down. For a truss with an inclined chord, such as a roof truss, the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ at a joint will produce two simultaneous equations. Even so, it may still be quicker to write these in terms of components rather than the actual bar forces.

Problems

11.1 For the truss of Figure P11.1, find the reactions and determine the forces in the members of the truss.

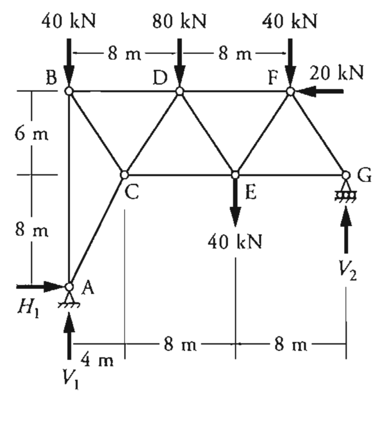


Figure P11.1

11.2 For the truss shown in Figure P11.2:

(i) Find the magnitude and direction of the reactions.

(ii) Determine the bar forces.

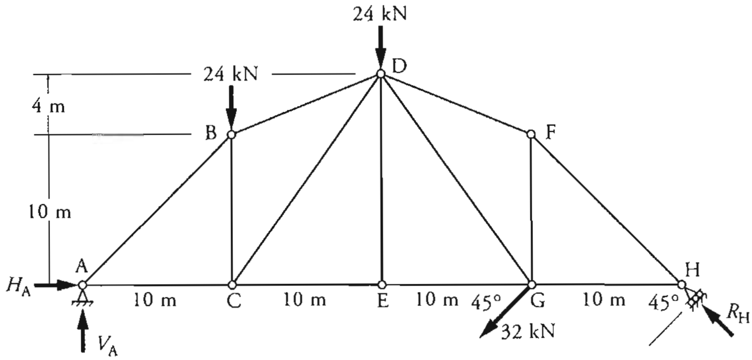


Figure P11.2

11.3 Find the forces in all the members of the truss in Figure P11.3.

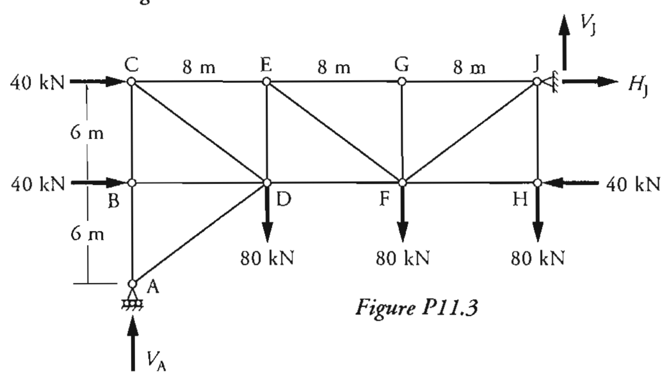


Figure P11.3

11.4 Find the reactions and the bar forces of the truss shown in Figure P11.4.

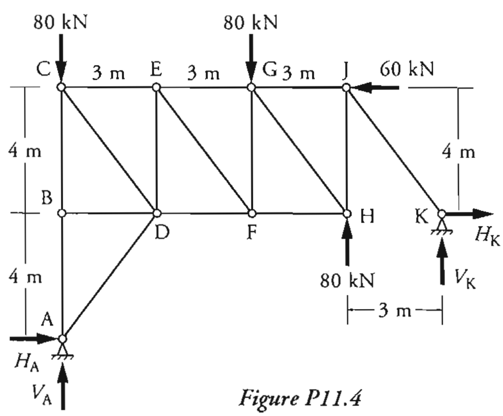


Figure P11.4

11.5 For the truss shown in Figure P11.5, find the forces in all the bars.

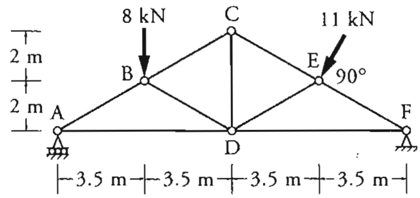


Figure P11.5

11.6 Find the forces in the members of the truss in Figure P11.6 by resolution at joints, working in terms of components.

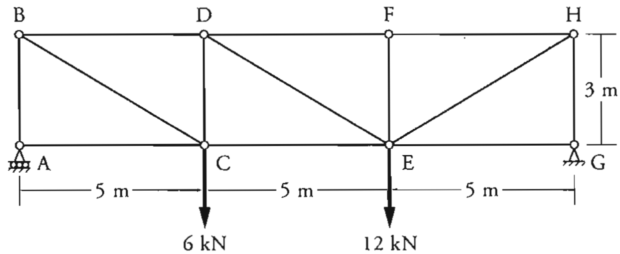


Figure P11.6

11.7 Working in terms of the x and y components of bar forces, solve the truss of Figure P11.7.

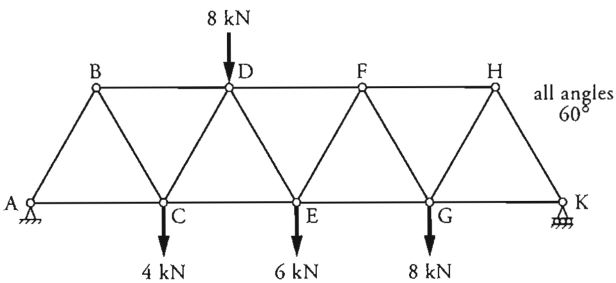


Figure P11.7

11.8 Solve the truss of Figure P11.8 analytically, working in terms of the components of the bar forces.

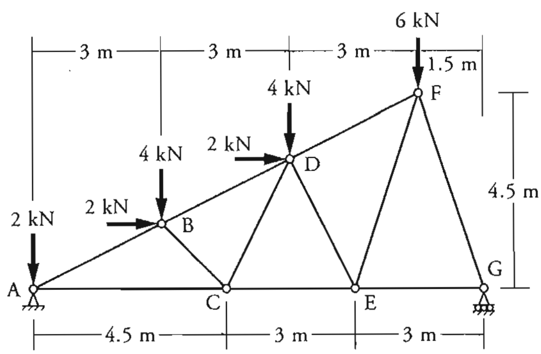


Figure P11.8

11.9 Find the forces in the bars of the truss of Figure P11.9.

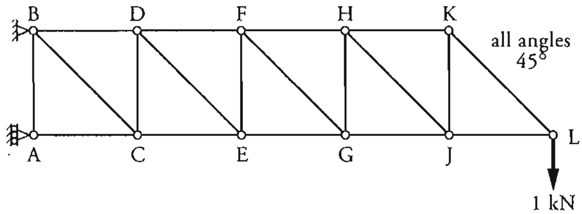


Figure P11.9

11.10 Find the bar forces in the truss of Figure P11.10

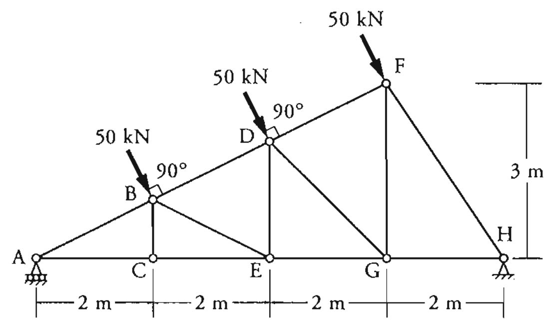


Figure P11.10

11.11 Find the forces in the bars of the truss of Figure P11.11

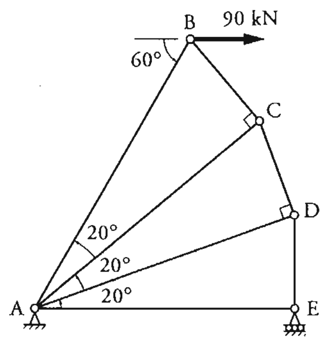


Figure P11.11


OVE ARUP & PARTNERS

The Method of Sections

12.1 Method of Sections

In the methods of the last chapter, the equilibrium of each joint is considered individually. If the force in particular members is required it may be often more convenient to use the method of sections. This procedure is rather similar to that used for finding the internal actions at a particular cross-section of a beam.

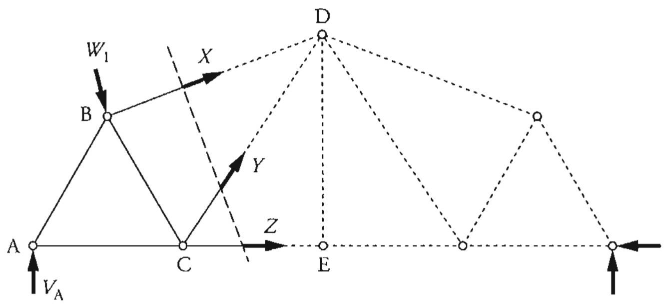


Figure 12.1

Suppose that in the truss of Figure 12.1 the force in CE is required. The truss is cut by a plane which passes through CE and the equilibrium of one part of the truss is considered. The freebody to the left of the cut for instance, is acted upon by the known forces V_A and W_1 and the unknown forces in the cut bars, which may be denoted by X, Y and Z. It has been shown that when the external loads act at the joints, the force in each member acts along the straight line joining the ends of the member, hence the lines of action of X, Y and Z are known, and the forces can be found from the equations of equilibrium.

The method is subject to certain limitations. If the number of cut members exceeds three, some of the unknowns cannot be found. Even in this case, one unknown can be found if all of the others happen to be concurrent. If three members only are cut their forces can be found provided the members are not concurrent, for in this case only two equations are available.

When practicable, each force should be calculated independently of the other unknowns. If Y and Z intersect, the force X is found by taking moments about the intersection point of Y and Z, so that these unknowns are not involved in the resulting equation. If Y and Z are parallel, X is found by resolving at right angles to Y and Z. The same considerations are used when finding Y and Z.

Example 12.1

Find the force in the member CD of the roof truss shown in Figure 12.2.

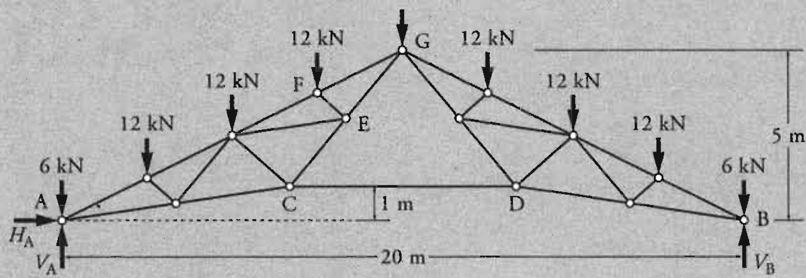


Figure 12.2

Solution

The reactions are first calculated. From the symmetry of the frame and the loading, it is clear that $H_A = 0$ and $V_A = V_B = 48$ kN.

The truss is cut by a plane passing through CD, EG and FG. (Note that some cutting planes would sever more than three bars and are therefore unsatisfactory). Consider the part of the truss to the left of the cutting plane. The freebody diagram is shown in Figure 12.3.

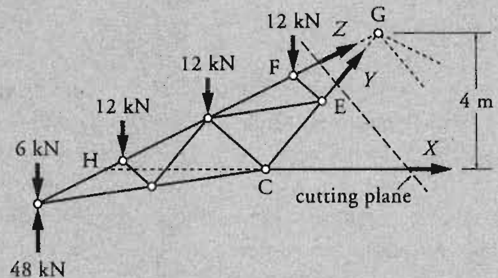


Figure 12.3

Let the forces in the cut bars be X, Y and Z. Assume these are all tensile forces in which case they will act outwards from the cut ends. To find X, take moments about the point of intersection of Y and Z, namely G. *Only the forces acting on the freebody on one side of the cutting plane are considered.*

$\Sigma M_G = 0$: ↶+ $-(48 \times 10) + (6 \times 10) + (12 \times 7.5) + (12 \times 5)$

$$+ (12 \times 2.5) + (X \times 4) = 0 \qquad \therefore X = +60 \text{ kN}$$

The assumed direction is thus correct and the force in CD is 60 kN tension.

If required, the force Y could be found by taking moments about H, the intersection of X and Z. The force Z could similarly be found by taking moments about C, the intersection of X and Y. For each of these equations, the lever arms of the various forces are found by calculation.

12.2 Parallel chorded trusses

Trusses and beams have much in common both as regards their function and their analysis. A beam is used to span a gap between supports and to carry loads. A truss may be employed for the same purpose, being usually chosen when the span becomes too large for a beam to be economical. Certain points of similarity can be seen with reference to trusses with parallel chords.

In a truss of the type shown in Figure 12.4 the horizontal members are known as *chords*, and the remainder are called *web members*. In the following example, the truss is compared with a solid beam spanning between the points A and N, which will be referred to as the analogous beam AN.

EXAMPLE 12.2

In the truss shown in Figure 12.4, find the forces in the members DF, FE, FG and EG.

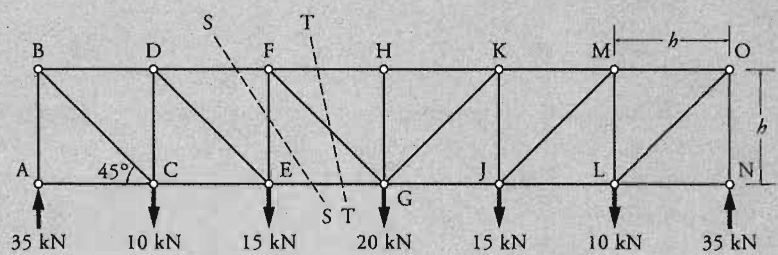


Figure 12.4

SOLUTION

1. **Top chord member DF:** The truss is cut by the plane SS and the freebody to the left of SS is considered (Figure 12.5). Let the forces in the cut bars be X, V and Z and assume that all the cut bars are in tension.

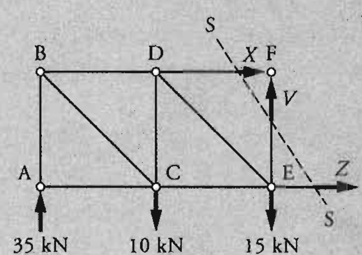


Figure 12.5

To obtain X, take moments about E, which is the intersection of V and Z. Use the panel length h as the unit of length:

$$\circlearrowleft + \quad -(X \times 1) - (35 \times 2) + (10 \times 1) + (15 \times 0) = 0 \; \therefore \; X = -60 \text{ kN}$$

The force in DF is 60 kN compression.

This calculation is very similar to the one which would be made if the truss were replaced by the beam AN and it was required to find the bending moment at the vertical section EF. It is significant that if the truss member DF were removed, the truss would collapse about the point E.

2. **Bottom chord member EG:** Use the same cutting plane SS, and refer again to Figure 12.5. To find the force Z, take moments about F, the intersection of X and V:

$$\circlearrowleft + \quad (Z \times 1) - (35 \times 2) + (10 \times 1) + (15 \times 0) = 0 \qquad \therefore \; Z = +60 \text{ kN}$$

The force in EG is 60 kN tension.

The member EG prevents the truss from collapsing about joint F which is vertically above E. The force in this member is again related to the bending moment in the analogous beam. It might be noted that the chord forces X and Z form a couple which is equal to the bending moment in the analogous beam at the section EF.

3. **The vertical web member EF:** Use the same cutting plane SS, and refer to Figure 12.5. Since the forces X and Z are parallel, V is obtained by resolving at right angles to X and Z, i.e. resolving vertically in this case:

$$+\uparrow \quad 35 - 10 - 15 + V = 0 \qquad \therefore \; V = -10 \text{ kN}$$

The assumed direction of V was incorrect. The force in EF is 10 kN compression.

The above equation is similar to that used to find the shear force in the analogous beam between the points E and G. The force V is equal to the beam shear force in the panel EG.

4. **The diagonal web member FG:** To cut this member the cutting plane TT must be used. After removing one portion of the truss the freebody of the remainder is as shown in Figure 12.6. To find W resolve vertically:

$$+\uparrow \quad 35 - 10 - 15 - W \cos 45 = 0 \qquad \therefore \; W = +14.14 \text{ kN}$$

The force in FG is 14.14 kN tension.

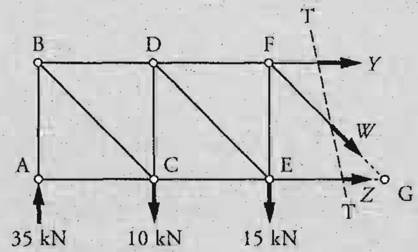


Figure 12.6

The vertical component of the force W is equal to the shear force in the segment EG of the analogous beam.

In a truss with parallel chords, the forces in the chord members are related to the bending moment in the analogous beam, while the forces in the web members are related to the shear force in the analogous beam. A relationship between trusses and solid structures exists even when the chords of the truss are not parallel, but in such a case the similarity is less clearly defined.

Problems

12.1 Figures P12.1a and P12.1b are freebody diagrams of portions of trusses. Find the forces X_1, X_2 etc. in each case.

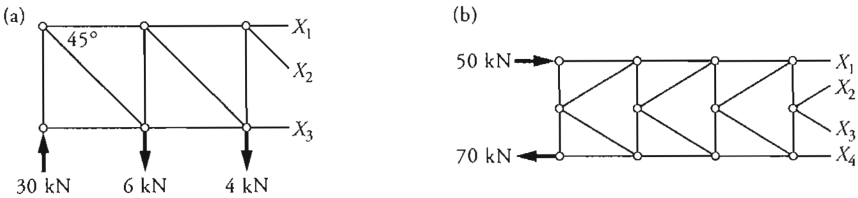


Figure P12.1

12.2 For the truss shown in Figure P12.2, find the reactions and determine the forces in the web members by the method of sections. Also determine the forces in the chord members by considering the equilibrium of the top and bottom joints.

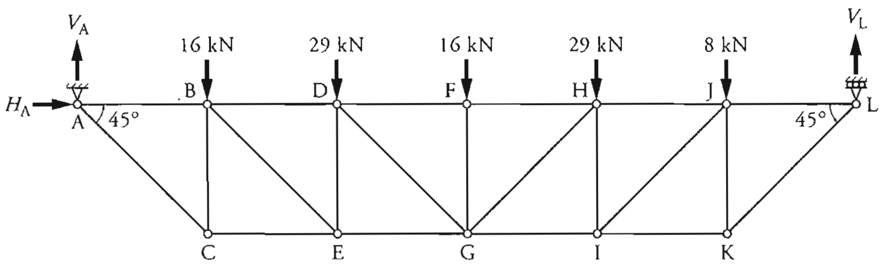


Figure P12.2

12.3 For the truss of Figure P12.3 determine the forces in the members EG, EF and DF by the method of sections.

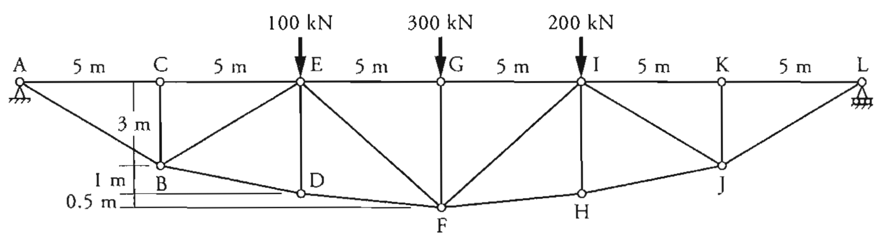


Figure P12.3

12.4 The crane shown in Figure P12.4 carries a load of 50 kN. The jib of the crane is in the same plane as the tower truss. Find the maximum force in any of:

(i) the verticals of the tower

(ii) the diagonals of the tower due to this crane load.

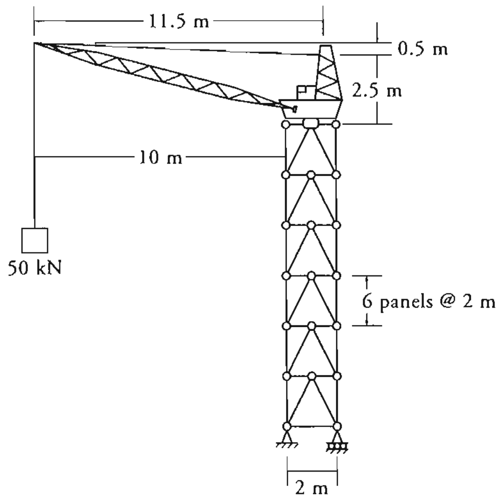


Figure P12.4

12.5 In the truss of Figure P12.5 find the forces in BD, BE and CE by the method of sections.

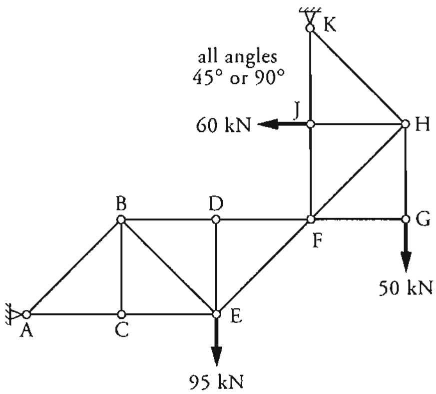


Figure P12.5

12.6 In the truss shown in Figure P11.2 (page 167), find the forces in members CD and CE by the method of sections.

12.7 In the truss shown in Figure P11.6 (page 168), find the forces in bars CE, DE and DF by the method of sections.

12.8 In the truss shown in Figure P11.8 (page 168), find the forces in bars CD and CE by the method of sections.

12.9 The roof structure shown in Figure P12.9 rests on hinged supports at A and G, and the two trusses are connected by a hinge at F. The member CE = 15 m, while AE = EF and AB = BC = CD = DE.

(i) Find the reactions at A and G.

(ii) Calculate the forces in the members of the left-hand truss.

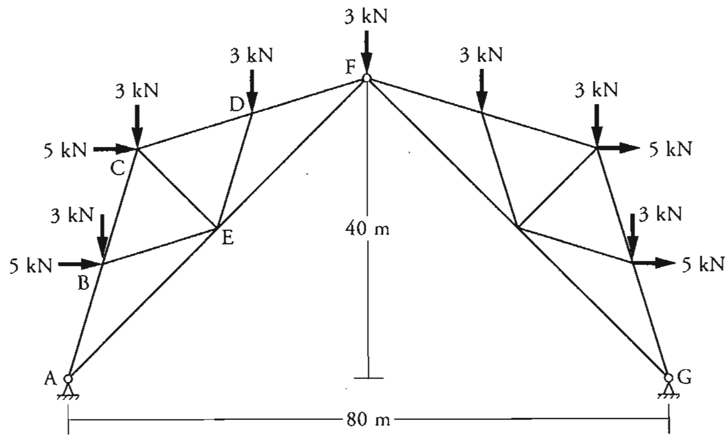


Figure P12.9

12.10 The roof truss shown in Figure P12.10 is supported on rollers at B, and by a hinge at A. The vertical loads P, are equally spaced. By employing both the method of joints and the method of sections, where appropriate, find the forces in the members in the left half of the truss. Assume AJ = JD and DK = KC.

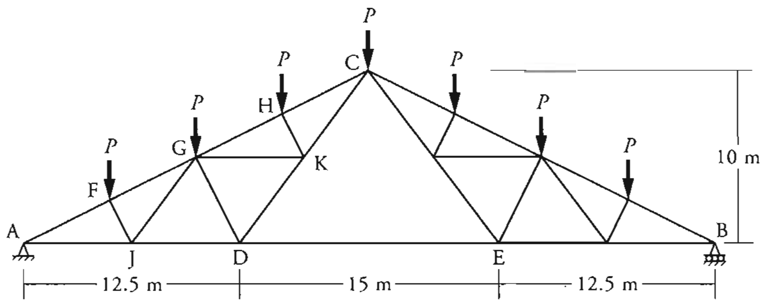
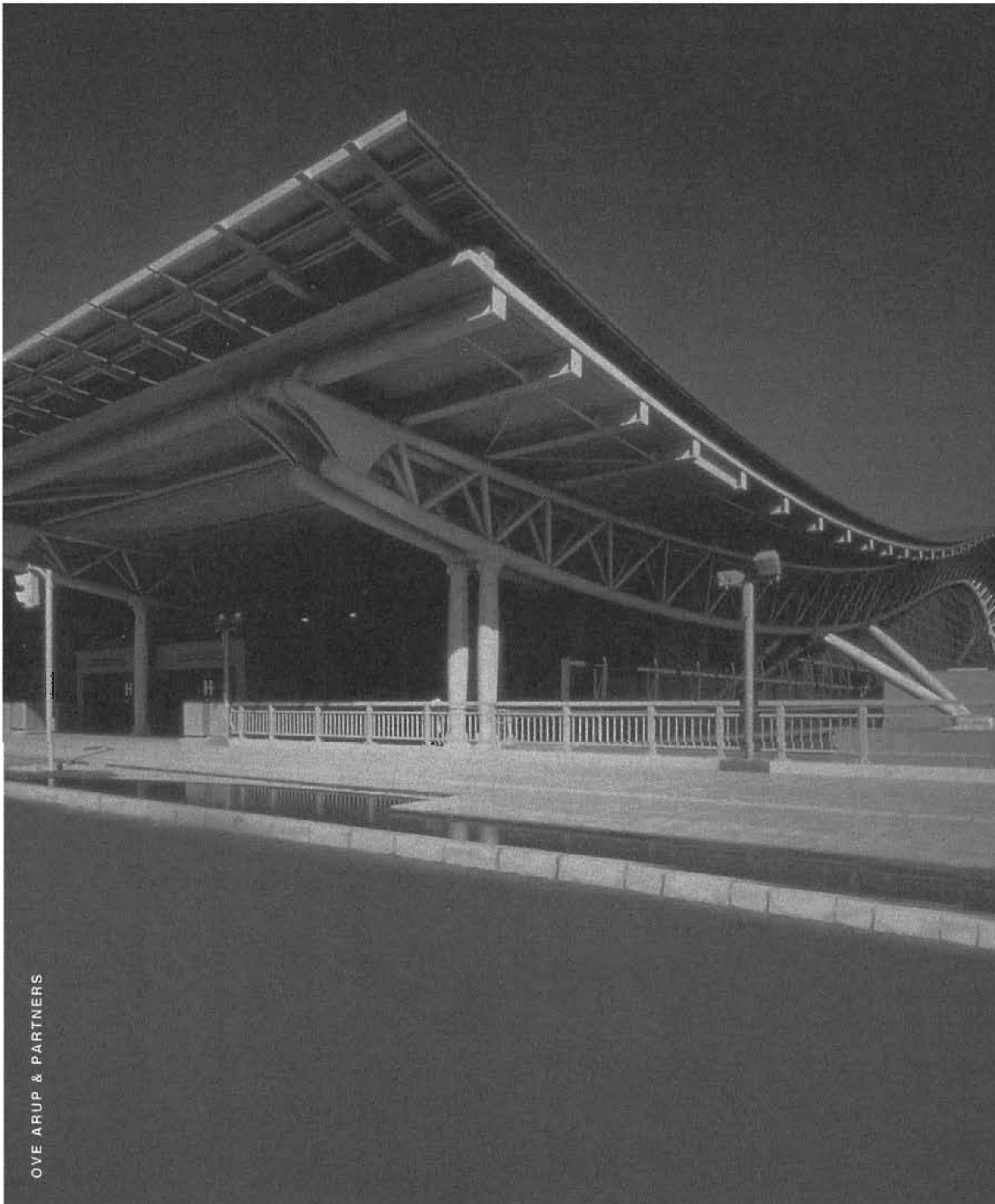


Figure P12.10


OVE ARUP & PARTNERS

Trusses with Loaded or Curved Members

13.1 Equivalent joint loads

If external loads are not applied directly to the joints but to a member of the truss, then that member performs a dual function. Firstly, it acts as a *beam* spanning between the joints at each end of the member, and serves to transfer the loads to these adjacent joints. In this capacity it will be subjected to bending moments, shear forces and in some cases axial forces. Secondly, it acts as a member of the truss. In this capacity it sustains an axial force which is determined by the methods of Chapters 11 and 12.

We note that if the loads on a particular bar in a truss are replaced by a statically equivalent system, then all bar forces outside the region of change will be unaffected.

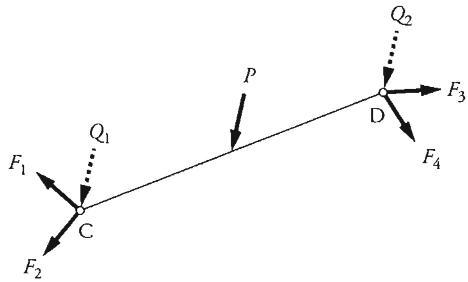


Figure 13.1

For example, consider the bar CD of Figure 13.1 loaded by a single force P. The bar is in equilibrium under the action of the load P and the bar forces F_1, F_2, F_3 and F_4 exerted by adjacent bars. If the force P is replaced by forces Q_1 and Q_2, whose resultant is P, then the forces F_1, F_2 etc. remain unchanged. If Q_1 and Q_2 act at the joints C and D, the truss is now loaded at the joints only and may be analysed by the methods of the previous chapters.

If an analysis of bar CD alone is required, it is often possible by using the method of sections. This will not give the values of the individual forces F_1, F_2 etc. but it will give their resultant, which is all that is required in this case. The procedure will be illustrated by two examples.

EXAMPLE 13.1

In the truss of Figure 13.2 find the axial force, shear force and bending moment in the bar CE.

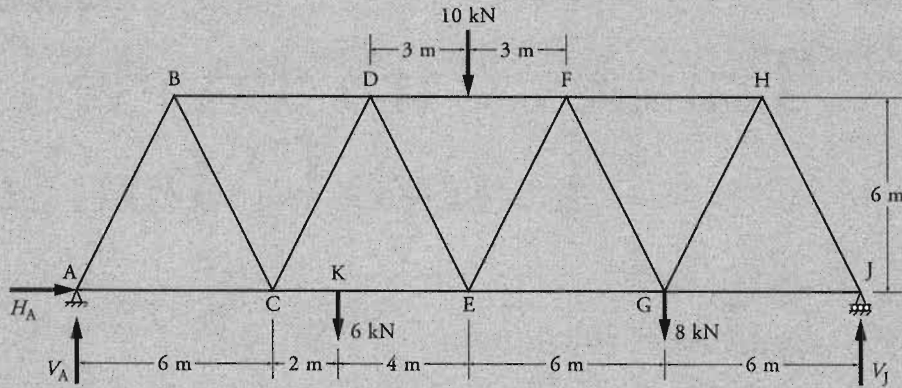


Figure 13.2

SOLUTION

1. **Find the reactions:**

$\Sigma H = 0$: $\therefore H_A = 0$

$\Sigma M_J = 0$: $(V_A \times 24) - (6 \times 16) - (10 \times 12) - (8 \times 6) = 0$

$\therefore V_A = 11$ kN

$\Sigma M_A = 0$: $(V_J \times 24) - (8 \times 18) - (10 \times 12) - (6 \times 8) = 0$

$\therefore V_J = 13$ kN

Checking:

$\Sigma V = 0$: $13 + 11 = 6 + 10 + 8$ $\therefore$ O.K.

2. **Replace the 6 kN force by statically equivalent forces at C and E:** By inspection the replacement forces are 4 kN at C and 2 kN at E both vertically downward.

3. **Find the force in bar CE:** Use the method of sections. Figure 13.3a shows the freebody to the left of the cut section. Taking moments about D, the point of intersection of the two forces whose values are not needed, gives:

$\Sigma M_D = 0$: $(11 \times 9) - (4 \times 3) - (F_{CE} \times 6) = 0$

$\therefore F_{CE} = +14.5$ kN

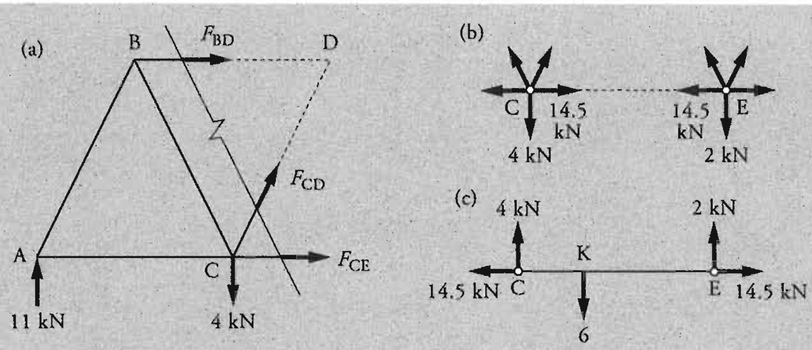


Figure 13.3

4. **Find the forces acting at C and E:** These forces are shown in Figure 13.3b. Although three of the forces at C have not been evaluated, we know that they must equilibrate the two known forces. Hence they must be equivalent to the two components shown in Figure 13.3c. The same applies at E.

5. **Find the internal actions *N*, *S*, *M*:** In Figure 13.3c, the two equivalent loads at C and E are now replaced by the original load at K. The member CE is now a beam. The internal actions are found as in Chapter 7.

The axial force is $N = +14.5$ kN; the shear force is $S = -4$ kN in segment CK and $S = +2$ kN in segment KE. The maximum bending moment occurs at K and is $M_{max} = 4 \times 2 = 8$ kNm.

EXAMPLE 13.2

In the truss of Figure 13.4, find the axial force, shear force and bending moment in the bar DE.

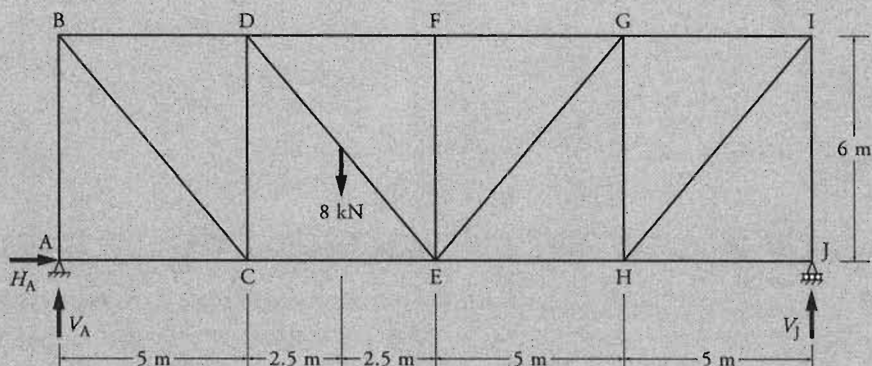


Figure 13.4

SOLUTION:

1. **Find the reactions:** As before we find $H_A = 0$; $V_A = 5$ kN and $V_J = 3$ kN.

2. Replace the 8 kN force by a statically equivalent system of 4 kN at each of joints D and E.

3. **Find the force in bar DE:** Use the method of sections. Figure 13.5a shows the freebody to the left of the cut. Since F_{DF} and F_{CE} are both horizontal we write an equation of vertical equilibrium to find F_{DE} (noting that $\theta = \tan^{-1}(6/5) = 50.2°$):

$$\sum V = 0: \quad 5 - 4 - F_{DE} \sin\theta = 0 \qquad \therefore F_{DE} = 1.3 \text{ kN}$$

The other bar forces at D (and E) together must form a system equal and opposite to the two known forces and are therefore as shown in Figure 13.5b.

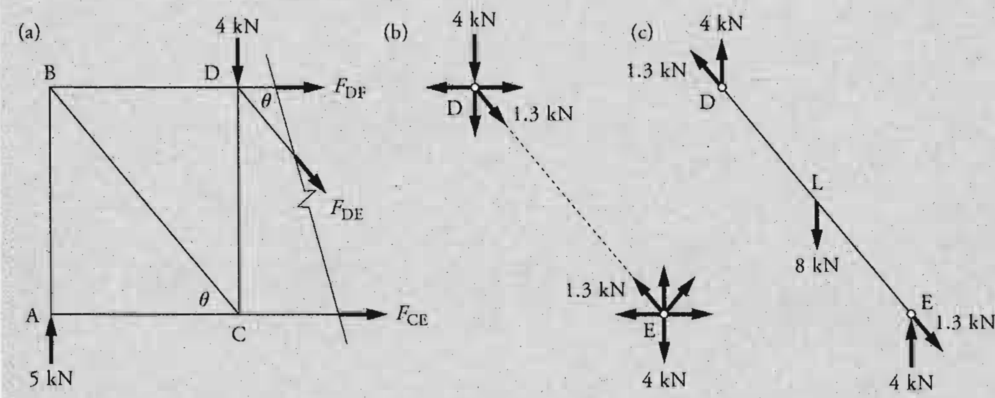


Figure 13.5

4. **Analyse the bar DE:** Replace the two temporary end forces of 4 kN by the force of 8 kN at L (Figure 13.5c).
The *axial force* in segment DL:

$$N_{DL} = 1.3 + 4\sin\theta = +4.37 \text{ kN}$$

and in segment LE:

$$N_{LE} = 1.3 - 4\sin\theta = -1.77 \text{ kN}$$

As a check, we note that the difference between these values, 4.37 + 1.77 = 6.14 kN, is equal to the longitudinal component of the 8 kN load at L (= 8 sin θ).
The *shear force* in segment DL:

$$S_{DL} = -4\cos\theta = -2.56 \text{ kN}$$

and in segment LE:

$$S_{LE} = +4\cos\theta = +2.56 \text{ kN}$$

The signs of the shear forces depend on the direction chosen for x along the bar. The difference of the shears (2.56 + 2.56 = 5.12 kN) is equal to the transverse component of the 8 kN load at L (= 8 cos θ).

The maximum bending moment at L is $M_{max} = 4 \times 2.5 = 10$ kNm

13.2 Trusses with non-straight members

Very rarely it may be necessary to analyse a truss having a member which is either curved or kinked, such as AB in Figure 13.6. Since the member is in equilibrium under the action of the two end forces, these forces must act in the direction of the straight line AB. If the actual member does not lie along this line then these end forces will cause bending moments and shear forces within the member, in addition to axial force. These internal actions can be calculated by the methods of Chapter 7. The out-of-straightness of member AB does not affect the analysis of the truss as a whole.

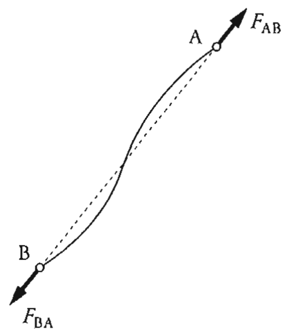


Figure 13.6

Problems

13.1 Find the axial force in each member of the truss shown in Figure P13.1. In the case of the loaded members find also the bending moment at the point of application of the load.

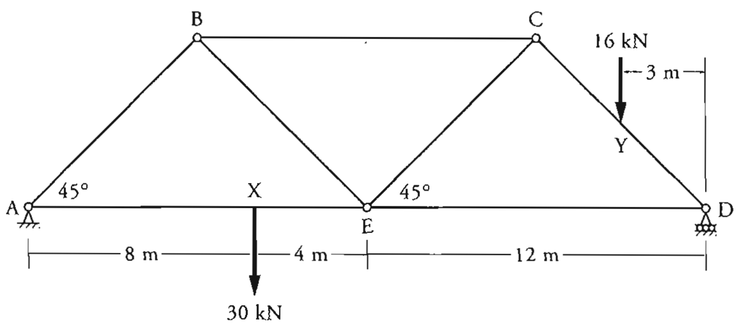


Figure P13.1

13.2 In Figure P13.2, the 16 kN load acts vertically at the mid-point of AB. Find the A.F., S.F. and B.M. at P, which is 1 m from A.

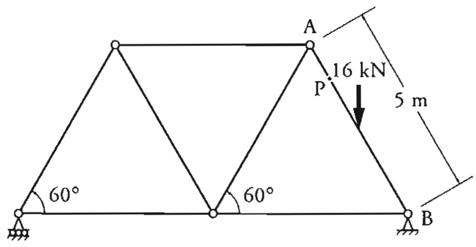


Figure P13.2

13.3 Figure P13.3 shows a roof truss for a small factory. To accommodate the roof sheeting, purlins are placed not only at the panel points A, B, C, D and E but also mid-way between them. If the top chord members AB, BC, CD and DE are to be all the same size, what are the values of the bending moment, shear force and axial force for which these members must be designed.

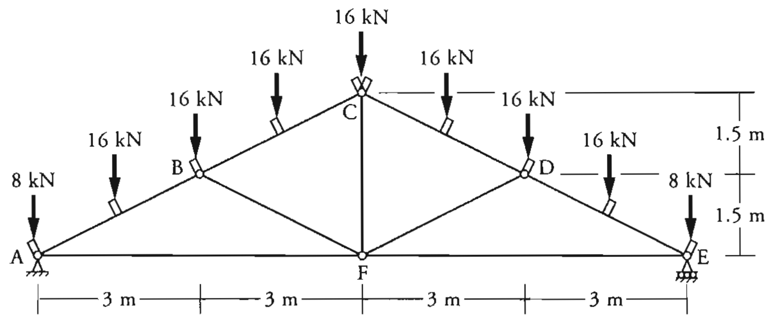


Figure P13.3

13.4 The truss of Figure P13.4 is pin-jointed and the top member is curved. Find N, S and M at the point X.

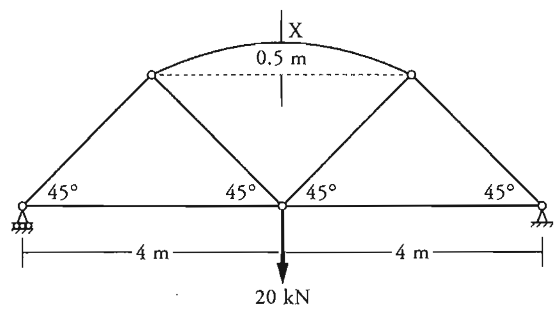


Figure P13.4

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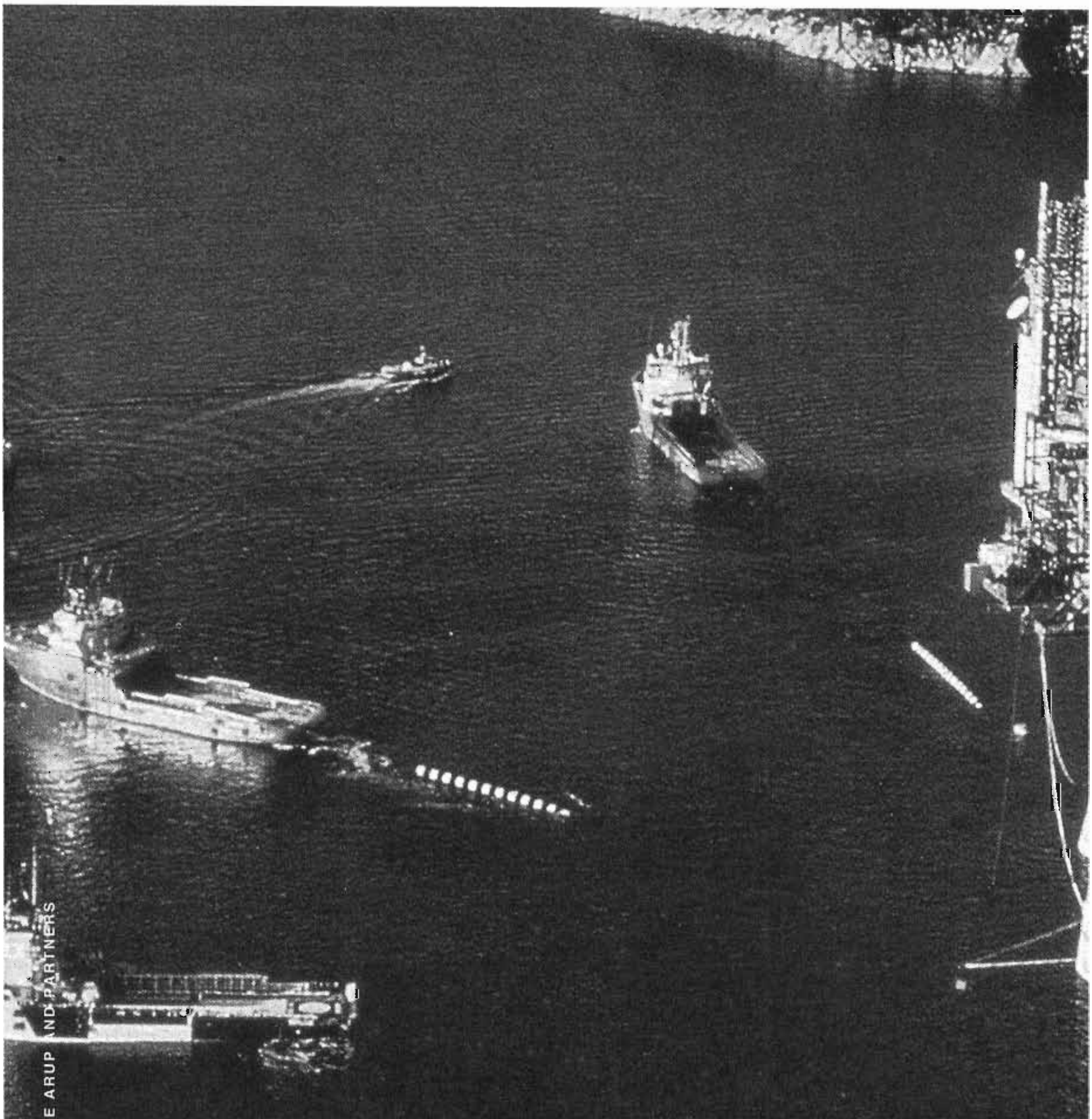
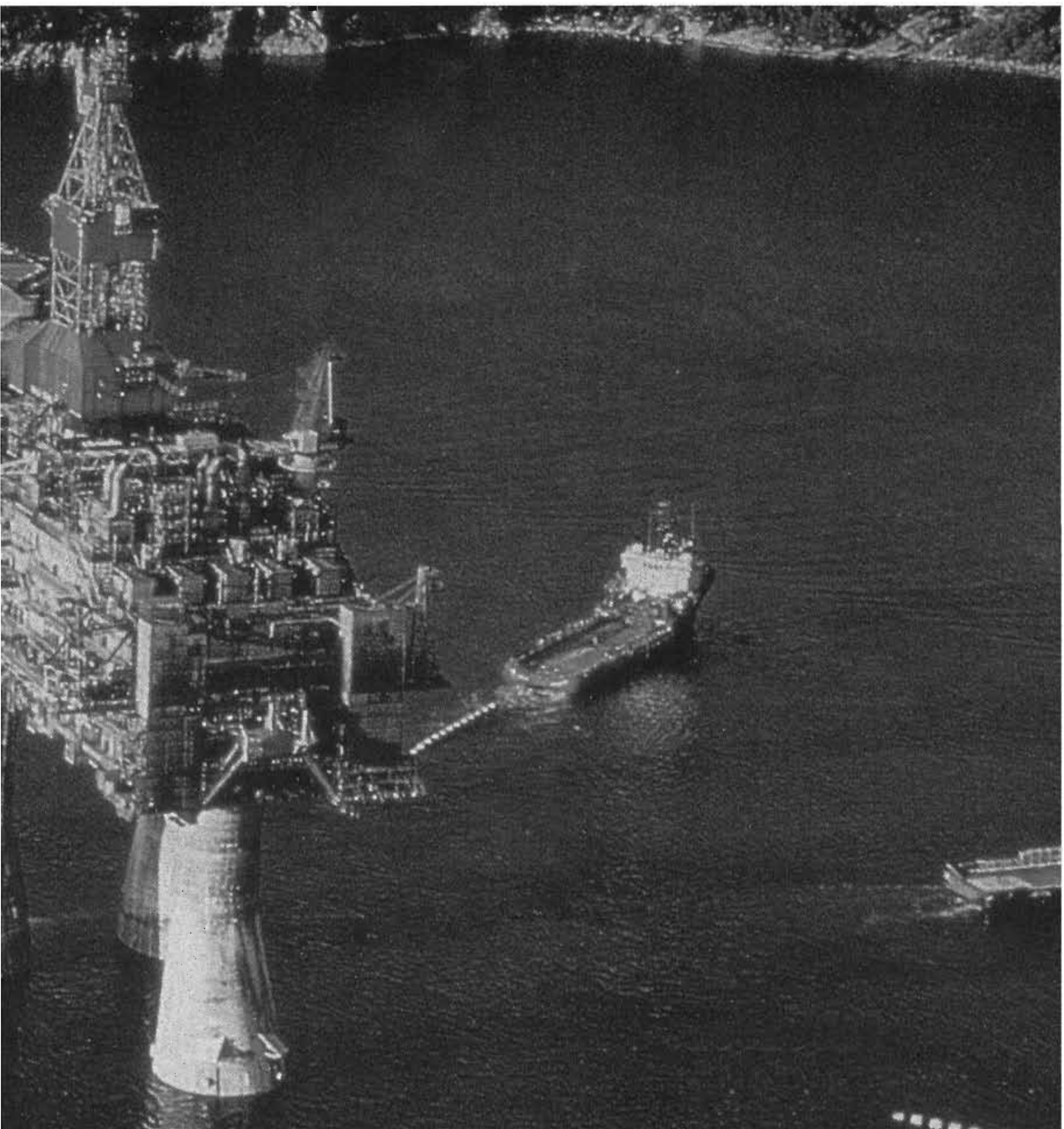
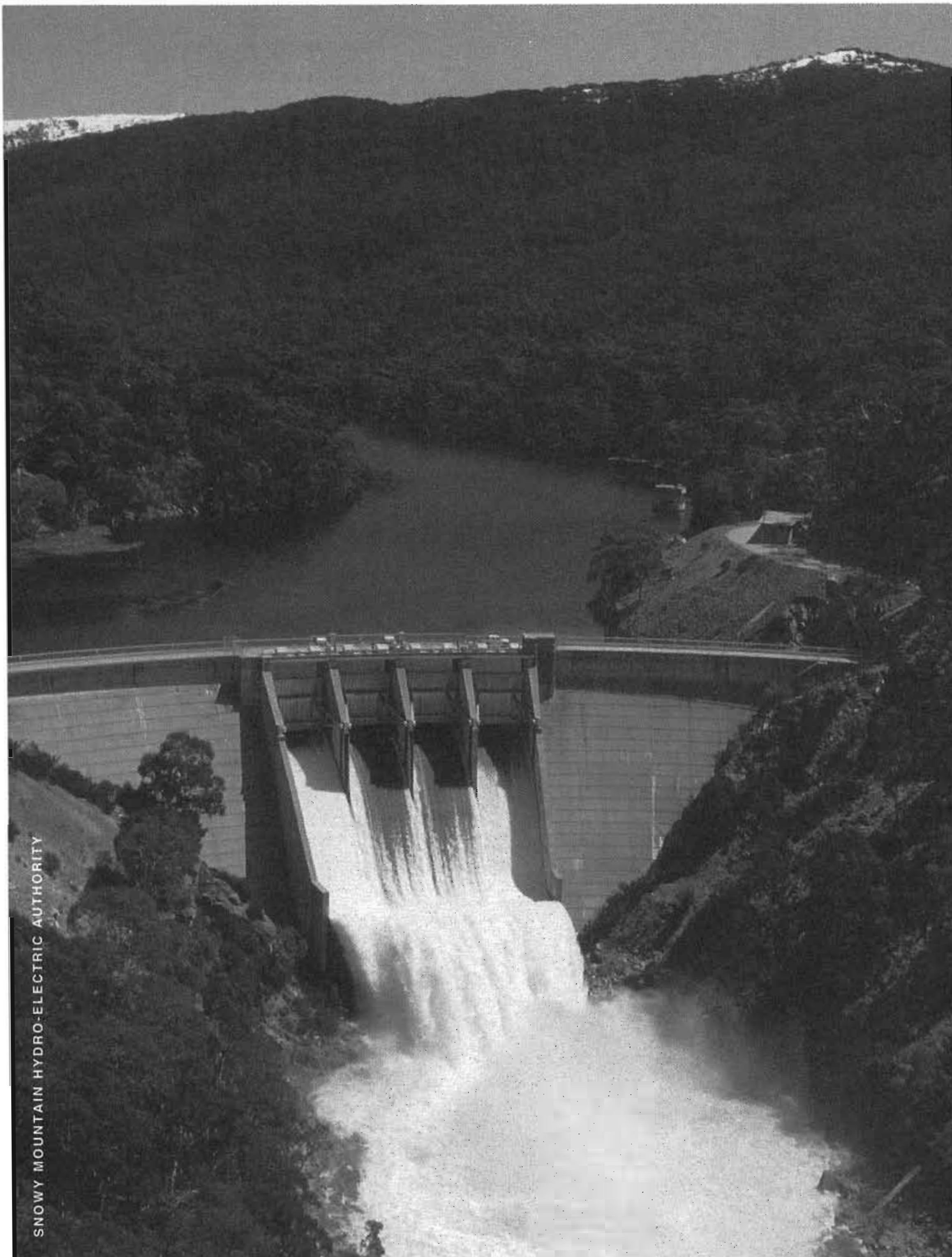

E ARUP AND PARTNERS

part 4

FLUID STATICS





SNOWY MOUNTAIN HYDRO-ELECTRIC AUTHORITY

14
chapter

Fluid Statics

14.1 Introduction

The branch of mechanics dealing with the behaviour of fluids at rest or in motion is known as *fluid mechanics*. That part of fluid mechanics which is concerned with fluids at rest (or in equilibrium) is called *fluid statics*. Fluid statics involves the study of pressure and its variation throughout a fluid. It also involves the calculation of forces exerted by a fluid on the surfaces of structures with which it is in contact. In the case of a fluid at rest, the property that affects the pressure variation is the specific weight or weight density γ which is the weight per unit volume.

The previous parts of this book have dealt with two-dimensional statics. Both the forces and the structures have been essentially coplanar. It is not possible to consider fluids as two-dimensional, but the force systems considered in this chapter are such that there should be no difficulty in applying the principles of previous chapters. Three-dimensional systems of a more general nature will be dealt with in Part 5.

In problems associated with civil and environmental engineering, the fluid most commonly encountered is water. The specific weight of fresh water is 9.81 kN/m^3. The remainder of this chapter will be written in terms of water, but the same principles apply to any other fluid (using an appropriate value of γ).

14.2 Hydrostatic pressure

A *fluid* may be defined as a substance that deforms continuously when subjected to a shear force. In a fluid at rest, therefore, no shear forces exist. This implies that no forces will exist tangential to a submerged surface. The only forces acting on submerged surfaces are forces normal to these surfaces.

This definition leads to the direct solution of many problems. In the first place, it allows us to determine the water pressure at any depth. Consider a prism of water with vertical sides and with horizontal cross-section of area A, which is part of a larger body of water (Figure 14.1). The prism extends from the free surface of the water (i.e. the surface where the water pressure is zero) down to a depth y. The volume of water enclosed within this imaginary boundary is Ay and therefore its weight is:

$$W = \gamma A y \qquad (14.1)$$

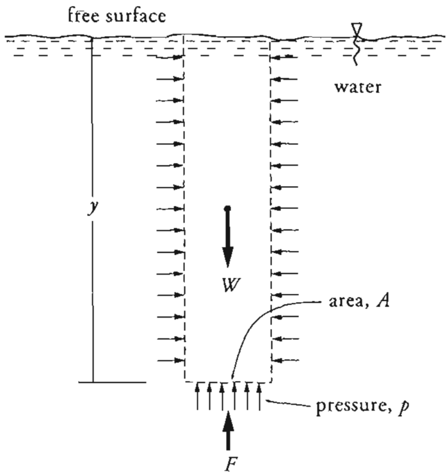


Figure 14.1

Since the whole body of water is at rest, the forces on the vertical sides of the prism are everywhere horizontal. Hence the weight W must be equilibrated by the upward pressure on the base. If the pressure at depth y is p, the resultant force on the base is:

$$F = pA \tag{14.2}$$

Hence: $$pA = \gamma Ay$$

and $$p = \gamma y \tag{14.3}$$

Pascal formulated a law which states that at any point within a fluid at rest the pressure is the same in all directions. This law, together with Equation 14.3 and the definition of a fluid, tells us that the pressure on any submerged body at depth y below the free surface is γy and acts normal to the surface of the body.

In the remainder of this chapter, frequent reference will be made to the Appendix of this book, and it is essential that this Appendix be read in conjunction with the present chapter.

14.3 Hydrostatic pressure on plane surfaces

14.3.1 Pressure on horizontal surfaces

Consider first the pressure on a horizontal surface of area A. Every part of the surface is at the same depth below the free water surface and the pressure is equal to γy. The total force on the surface is $F = \gamma y A$. This resultant force acts at the centroid of the area A (see Appendix, Section A.1).

14.3.2 Pressure on vertical surfaces

As an example of water pressure on a vertical plane, we consider the force exerted by water on the vertical face of a dam (Figure 14.2).

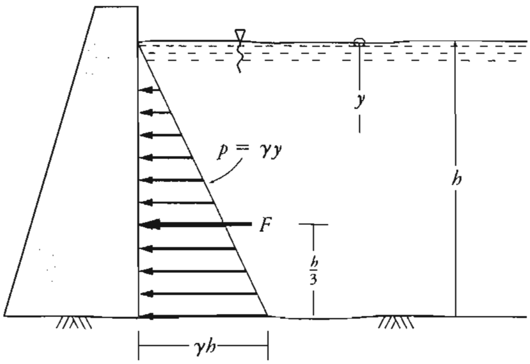


Figure 14.2

The resultant force on a 1 m length of the dam may be calculated as the volume of the pressure intensity diagram, shown in side elevation in Figure 14.2. Thus:

$$F = \gamma h \times \frac{h}{2} \times 1 = 0.5\gamma h^2 \tag{14.4}$$

This force acts not at the centroid of the dam wall but at the level of the centroid of the triangular pressure intensity diagram of Figure 14.2, i.e. at $2h/3$ below the free water surface, as shown.

EXAMPLE 14.1

A large fish tank is 2.0 m long, 0.8 m wide and 0.9 m deep, and is completely filled with fresh water.

(i) Find the resultant hydrostatic force on the bottom surface of the tank.
(ii) Find the resultant force on the 0.8 m wide side surface of the tank.

SOLUTION

(i) From Equation 14.3, the water pressure at the bottom of the tank is:

$$p = 9.81 \times 0.9 = 8.83 \text{ kN/m}^2$$

and the resultant force on the bottom surface is:

$$F = pA = 8.83 \times 2.0 \times 0.8 = 14.13 \text{ kN}$$

acting vertically downwards through the centroid of the rectangular bottom surface.

(ii) The water pressure varies linearly from zero at the water surface to 8.83 kN/m^2 at the bottom of the tank. The resultant force on the 0.8 m wide side surface is therefore the volume of the triangular pressure block shown in Figure 14.3.

$$F = 0.5 \times 0.9 \times 8.83 \times 0.8 = 3.18 \text{ kN}$$

and is located $2h/3 = 0.6$ m below the top water surface and vertically below the centroid of the side surface area of the tank.

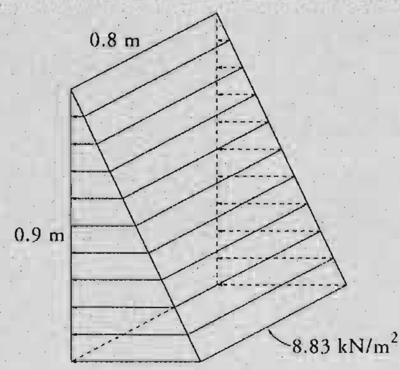


Figure 14.3

EXAMPLE 14.2

An outlet for the discharge of stormwater into a harbour is covered by a gate AB hinged at A as shown in Figure 14.4a. The gate is rectangular, 1.5 m deep and 1.2 m wide, and prevents seawater from entering the outlet at high tide. If the high water level is 2 m above A, as shown, calculate the magnitude and location of the resultant hydrostatic force acting on the gate at high tide. The specific weight of seawater is 10.06 kN/m^3.

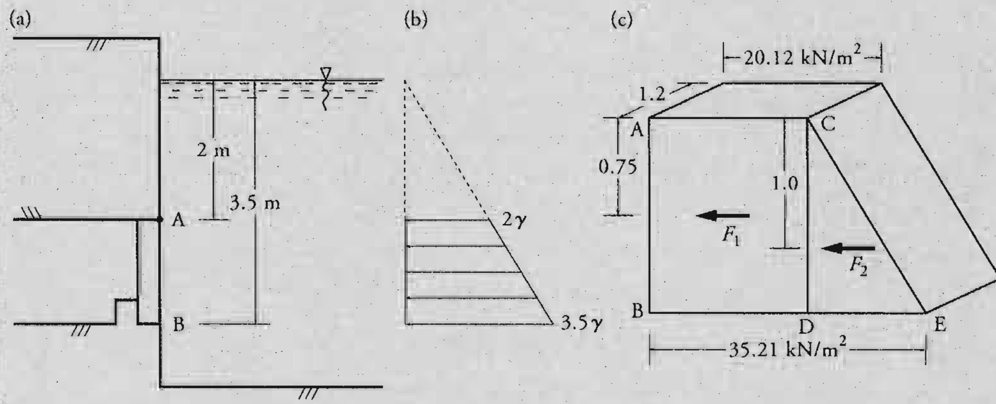


Figure 14.4

SOLUTION

Figure 14.4b shows the variation of pressure from top to bottom of the gate and Figure 14.4c shows the three-dimensional pressure block acting on the gate.

The water pressure at the top of the gate at A is $p_A = 2\gamma = 20.12$ kN/m^2 and at the bottom of the gate at B is $p_B = 3.5\gamma = 35.21$ kN/m^2. The resultant force on the gate is the volume of the trapezoidal pressure block of Figure 14.4c:

$$F = 0.5 \times (20.12 + 35.21) \times 1.5 \times 1.2 = 49.80 \text{ kN}$$

To calculate the location of the resultant force, it is convenient to represent the trapezoidal pressure block as the sum of a rectangular block (with side ACDB in Figure 14.4c) of uniform intensity 20.12 kN/m^2 and a triangular block (side CDE in Figure 14.4c) with intensity varying from zero at the top of the gate to 35.21 − 20.12 = 15.09 kN/m^2 at the bottom of the gate.

The resultant force F_1 of the rectangular pressure block is:

$$F_1 = 20.12 \times 1.5 \times 1.2 = 36.22 \text{ kN}$$

and is located at the mid-point of the gate (at $y_1 = 0.75$ m below A).

The resultant force F_2 of the triangular pressure block is:

$$F_2 = 0.5 \times 15.09 \times 1.5 \times 1.2 = 13.58 \text{ kN}$$

and is located two thirds the way down the gate (at $y_2 = 1.0$ m below A).

The moments of the components F_1 and F_2 about any point is the same as the moment of the resultant F about the same point. If F is located y_R below A, then equating moments of F and its components about A gives:

$$Fy_R = F_1y_1 + F_2y_2$$

$$49.80 \times y_R = (36.22 \times 0.75) + (13.58 \times 1.0)$$

$$\therefore y_R = 0.818 \text{ m}$$

The resultant force on the gate is $F = 49.80$ kN and it acts 0.818 m below A.

For surfaces of simple shapes such as those of Examples 14.1 and 14.2, the resultant force is easily found by calculating the volume of the pressure *block*. For more general shapes the reader is referred to Section A2 of the Appendix. There it is shown that if a linearly varying pressure acts upon a plane surface of area A, the resultant force is equal to the pressure at the centroid of the surface times the area A. Thus if the centroid is at a depth y_c below the free surface, the resultant force is (see Equation A.11):

$$F = \gamma y_c A \qquad \textbf{(14.5)}$$

The point through which the resultant acts is called the *centre of pressure* and its depth is denoted by y_{cp}. From Equation A.27, we have:

$$y_{cp} = \frac{I_{xx}}{Ay_c} + y_c \qquad \textbf{(14.6)}$$

where I_{xx} is the second moment of area of the surface calculated about a horizontal axis through its centroid.

EXAMPLE 14.3

Re-solve Example 14.2 using Equations 14.5 and 14.6.

SOLUTION

The width of the gate b is 1.2 m and its depth D is 1.5 m. For this rectangular shape (see Table A2, page 271):

$$I_{xx} = \frac{bD^3}{12} = \frac{1.2 \times 1.5^3}{12} = 0.3375 \text{ m}^4$$

The depth to the centroid of the gate is $y_c = 2 + 0.75 = 2.75$ m and the surface area $A = 1.2 \times 1.5 = 1.8 \text{ m}^2$. The resultant force on the gate (Equation 14.5) is:

$$P = \gamma y_c A = 10.06 \times 2.75 \times 1.8 = 49.8 \text{ kN}$$

and from Equation 14.6:

$$y_{cp} = \frac{0.3375}{1.8 \times 2.75} + 2.75 = 2.818 \text{ m}$$

The centre of pressure is therefore 2.818 m below the free water surface or 0.818 m below the top of the gate (as calculated in Example 14.2).

EXAMPLE 14.4

This example is similar to Example 14.2 except that the gate is circular with a diameter of 2.5 m. The centre of the gate is 5.6 m below the free water surface.

SOLUTION

The area of the gate is:

$$A = \frac{\pi \times 2.5^2}{4} = 4.909 \text{ m}^2$$

From Table A.2, for a circular area:

$$I_{xx} = \frac{\pi \times 2.5^4}{64} = 1.917 \text{ m}^4$$

and from Equations 14.5 and 14.6:

$$F = 10.06 \times 5.6 \times 4.909 = 277 \text{ kN}$$

$$y_{cp} = \frac{1.917}{4.909 \times 5.6} + 5.6 = 5.67 \text{ m}$$

Equation 14.6 shows that the centre of pressure always occurs at a greater depth than the centroid of the surface on which the hydrostatic pressure acts. The distance between the centre of pressure and the centroid I_{xx}/Ay_c decreases as the depth of water increases.

14.3.3 Pressure on inclined surfaces

With slight modification Equations 14.5 and 14.6 may be used to find the resultant fluid pressure on any submerged surface. Consider the inclined surface AB in Figure 14.5a. The hydrostatic pressure acts at right angles to the surface, with $p_A = \gamma\, y_A$ and $p_B = \gamma y_B$ Equation 14.5 needs no modification. For Equation 14.6, the position of the centroid C and the centre of pressure P are best defined by their distance from the line of intersection of the submerged surface with the free water surface, which we may call the *water line*. This line (O in Figure 14.5b) is the line of zero pressure. The distances from C and P to this line are L_c and L_{cp}, respectively. Equation 14.6 then becomes:

$$L_{cp} = \frac{I_{xx}}{AL_c} + L_c \qquad \textbf{(14.7)}$$

where I_{xx} is the second moment of area of the inclined surface about a horizontal axis through the centroid.

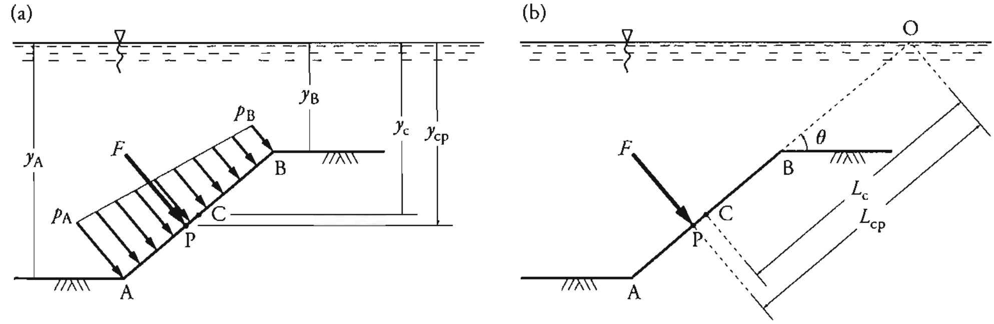


Figure 14.5

The calculation of the fluid force on a plane surface of any shape and inclination may be summarized as follows:

1. The resultant force is equal to the pressure at the centroid times the area of the surface (Equation 14.5).
2. The point P at which this resultant force acts is distant L_{cp} from the water line, where L_{cp} is given by Equation 14.7.

If the submerged surface is vertical, L_{cp} is identical to y_{cp} and L_c is identical to y_c. If the surface is inclined at θ to the horizontal:

$$L_{cp} = \frac{y_{cp}}{\sin\theta} \quad \text{and} \quad L_c = \frac{y_c}{\sin\theta}$$

If the surface is horizontal, P and C coincide.

It remains to locate P in the direction parallel to the water line. Usually there is an axis of symmetry normal to the water line, in which case P lies on this axis. The case where there is no axis of symmetry is dealt with in Appendix, Section A.2.3.

EXAMPLE 14.5

Figure 14.6a shows the cross-section of a steel tank which is 3 m long. If the tank is completely filled with fresh water, find the magnitude and position of the resultant force due to water pressure acting on the surface AB.

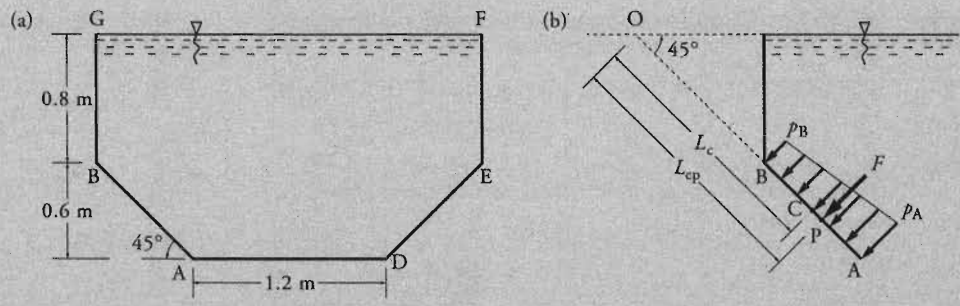


Figure 14.6

SOLUTION

The inclined surface is a rectangle of width $0.6\sqrt{2} = 0.849$ m and length 3 m. The area is $A = 0.849 \times 3 = 2.546$ m^2 and the depth to the centroid, $y_c = 0.8 + 0.3 = 1.1$ m. The inclined distance from the centroid to the water surface level is:

$$L_c = \frac{y_c}{\sin 45} = 1.556 \text{ m}$$

The second moment of area of the surface about the horizontal axis through the centroid is:

$$I_{xx} = \frac{3 \times 0.849^3}{12} = 0.153 \text{ m}^4$$

The resultant force is:

$$F = \gamma y_c A = 9.81 \times 1.1 \times 2.546 = 27.47 \text{ kN}$$

The position of the resultant is obtained from Equation 14.7:

$$L_{cp} = \frac{0.153}{2.546 \times 1.556} + 1.556 = 1.595 \text{ m}$$

The centre of pressure P is located $1.595 - 0.8\sqrt{2} = 0.464$ m from B and half way along the tank.

14.4 Hydrostatic pressure on curved surfaces

As for a plane surface, the hydrostatic pressure on a curved surface acts at right angles to the surface as shown in Figure 14.7a. The resultant force F has horizontal and vertical components, F_H and F_V respectively. F_H is determined by projecting the curved surface onto a vertical plane and calculating the horizontal force on this projected vertical plane, as shown in Figure 14.7b. If y_c is the depth from the free fluid surface to the centroid of the projected area A, then:

$$F_H = \gamma y_c A \tag{14.8}$$

The vertical component F_V is the weight of the volume V of fluid above the curved surface (Figure 14.7c):

$$F_V = \gamma V \tag{14.9}$$

The magnitude and direction of F are calculated from its two components and are given by:

$$F = \sqrt{F_H^{\,2} + F_V^{\,2}} \quad \text{and} \quad \theta = \tan^{-1}\frac{F_V}{F_H} \tag{14.10}$$

where θ is the inclination of F to the horizontal.

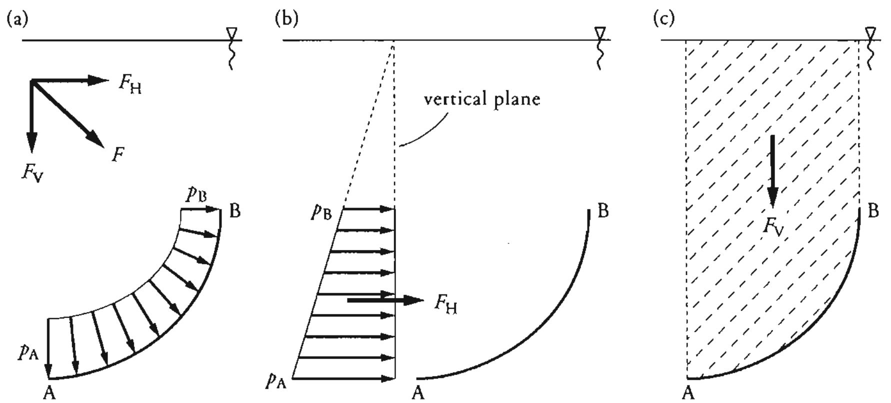


Figure 14.7

EXAMPLE 14.6

For the dam shown in Figure 14.8 the upstream face BD is parabolic with B being the vertex. Find:

(i) the magnitude and direction of the resultant hydrostatic force on the upstream face BD

(ii) the moment per metre of wall about the toe of the wall (point A) caused by the 10 m depth of water.

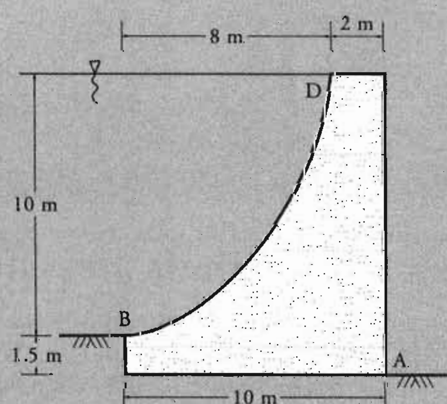


Figure 14.8

Solution

(i) The vertical projected area of the curved wall surface is rectangular 10 m deep and the length of the wall wide. The horizontal component of the hydrostatic force on each 1 m of wall is obtained from Equation 14.8 as:

$$F_H = 9.81 \times 5 \times 10 \times 1 \qquad = 490.5 \text{ kN/m}$$

This acts 6.67 m below the water surface or 4.83 m above A, as shown in Figure 14.9. The vertical component of the hydrostatic force F_V per metre of wall is the weight of the volume of water in the hatched region shown in Figure 14.9. By reference to Table A.1 in the Appendix:

$$F_V = 9.81 \times \tfrac{2}{3} \times 10 \times 8 \times 1 = 523.2 \text{ kN/m}$$

and this acts through the centroid C of the hatched parabolic area, i.e. at $3/8 \times 8 = 3$ m from the axis BE, or 7 m to the left of A.

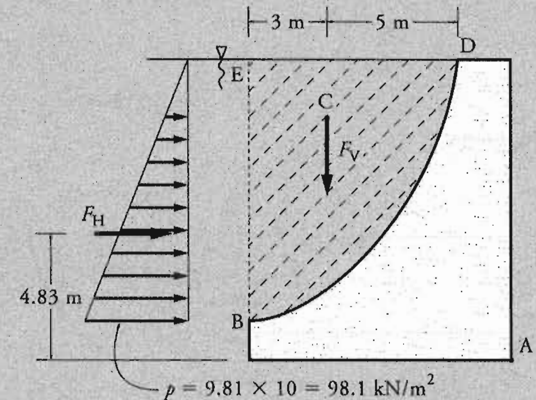


Figure 14.9

From Equations 14.10:

$$F = \sqrt{490.5^2 + 523.2^2} = 717.2 \text{ kN/m}$$

and

$$\theta = \tan^{-1}\frac{523.2}{490.5} = 46.85^\circ$$

(ii) The moment about A caused by F is the same as the sum of the moments caused by the components of F:

$$\circlearrowleft + \quad M_A = 523.2 \times 7 - 490.5 \times 4.83 = 1293 \text{ kNm/m}$$

Example 14.7

Find the horizontal and vertical components of the hydrostatic force exerted on the 2.4 m diameter cylinder shown in Figure 14.10. The cylinder is 8 m long.

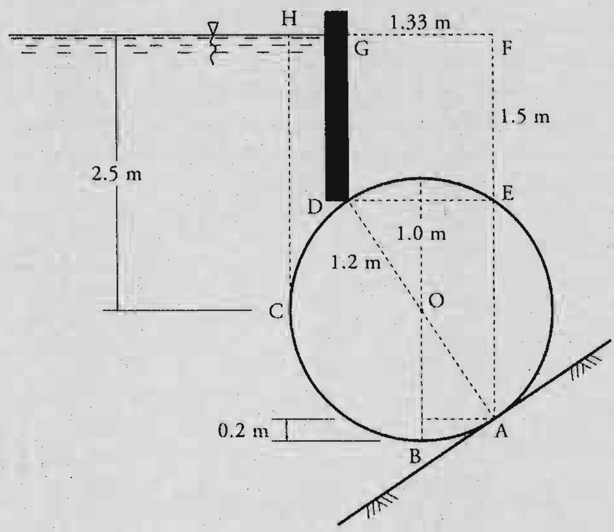


Figure 14.10

Solution

The horizontal component force F_H is the force on the vertical projection of the surface DCB (which acts to the right) minus the force on the vertical projection of the surface AB (which acts to the left).

$$\overset{+}{\rightarrow} \quad F_H = 9.81 \times 2.6 \times 2.2 \times 8 - 9.81 \times 3.6 \times 0.2 \times 8 = 392.4 \text{ kN}$$

This is equivalent to the horizontal force acting to the right on the vertical projection EA. Hence:

$$\overset{+}{\rightarrow} \quad F_H = 9.81 \times 2.5 \times 2.0 \times 8 = 392.4 \text{ kN}$$

The vertical component force F_V is the upward force on the surface CBA, which is the weight of the volume of water displaced by the volume HCBAFH, minus the downward force on the surface DC, which is the weight of the volume of water in HCDG in Figure 14.10.

$\therefore F_V$ = weight of the volume of water GDCBAFG
= 9.81 × 8 × (area of semicircle DCBAD + area of triangle DAE + area of rectangle GDEF)
$= 9.81 \times 8 \times (0.5 \times \pi \times 1.2^2 + 0.5 \times 1.33 \times 2.0 + 1.33 \times 1.5)$
= 438.5 kN

14.5 Buoyancy

When a rigid body is immersed either partly or completely in water, the water exerts an upward force on it. Its weight is then offset, either partly or completely by the water pressure, which is called the *buoyancy* force.

Suppose that it is intended to immerse a solid body in an expanse of water. Before the immersion occurs the space which will later be occupied by the body is occupied by water (Figure 14.11a). This block of water has a volume V and a weight γV which acts at the centre of gravity B of the block. Equilibrium of the block is maintained by the pressure from the surrounding water. Without the need for calculation we can therefore say that the resultant of these pressures is an upward force of magnitude γV acting through B, the centre of gravity of the block of water.

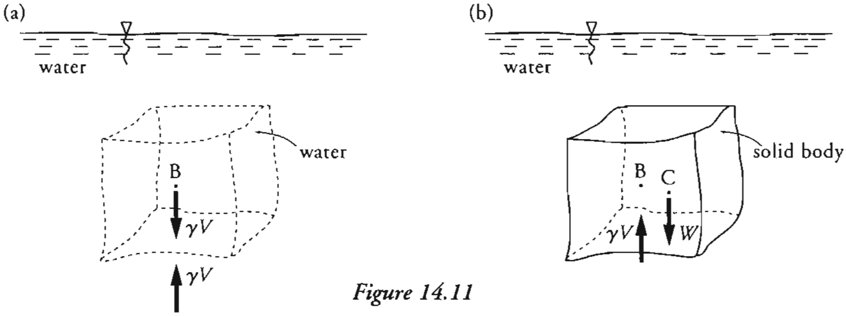


Figure 14.11

We now replace this water by a rigid body of the same size and shape (Figure 14.11b). The resultant of the water pressures acting on the body is a force of γV acting at the centre of gravity of the displaced water.

This upward force is the *buoyancy force*, and the centre of gravity B of the displaced water is called the *centre of buoyancy*. The weight of the immersed body is W acting at its *centre of gravity* C, which may not coincide with B.

If W is greater than γV the body will sink. If W is less than γV the body will rise. If C is not vertically above or below B the body will rotate.

Whether the body floats or is totally immersed, the buoyancy force γV is equal to the weight of the displaced water and acts through the centre of gravity of the displaced water.

Example 14.8

A sealed container is rectangular in plan and cross-section, with a uniformly distributed weight of 900 kN. Its length is 10 m, its width is 4 m and its height is 3 m, as shown in Figure 14.12. Find whether the container will float or sink in sea water and, in the former case determine the depth d of the container below the sea water surface.

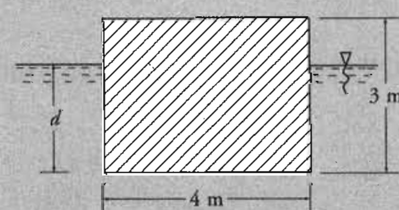


Figure 14.12

Solution

The volume of the container is $10 \times 4 \times 3 = 120\ m^3$. This volume of sea water weighs $120 \times 10.06 = 1207.2$ kN. Since this is greater than the weight of the container the latter will rise until the volume of displaced water is 900 kN. Then:

$$10 \times 4 \times d \times 10.06 = 900\ \text{kN} \quad \therefore\ d = 2.236\ \text{m}$$

14.6 Stability

In Example 14.8, it was assumed that the rectangular container floated with its shortest dimension vertical. How do we know that it does not rotate so that the 4 m or the 10 m dimension is vertical? In both of these orientations, the buoyancy force would exactly balance the weight of the container and thus satisfy equilibrium. We shall see that of these three equilibrium positions two are unstable. If the container is in one of the unstable conditions of equilibrium, the slightest movement will cause it to rotate to one of the other positions. If it is in *stable equilibrium,* then after a slight disturbance, such as that caused by a small wave on the sea's surface, it will return to its former position.

In Figure 14.13 the full line represents the container considered in Example 14.8. The rectangle JKLM is the submerged portion. The centre of gravity of the container is C and the centre of buoyancy (i.e. the centre of gravity of the displaced water) is B, which in this case is 0.382 m below C.

We now give the container a *small* rotation θ. The displaced water is now the quadrilateral JKL'M' (times 10 m, the length of the container). The area of JKL'M' must be the same as that of JKLM but the centroid is not obvious. However, JKL'M' may be regarded as J'K'L'M' plus the triangle NKK' minus the triangle NJJ'. The centroids of these figures are readily determined. In effect we are considering the buoyancy force W as the sum of an upward force W acting at B', plus an upward force F_1 at Y minus a downward force F_2 at Z. For equilibrium F_1 and F_2 must be equal and of opposite sign.

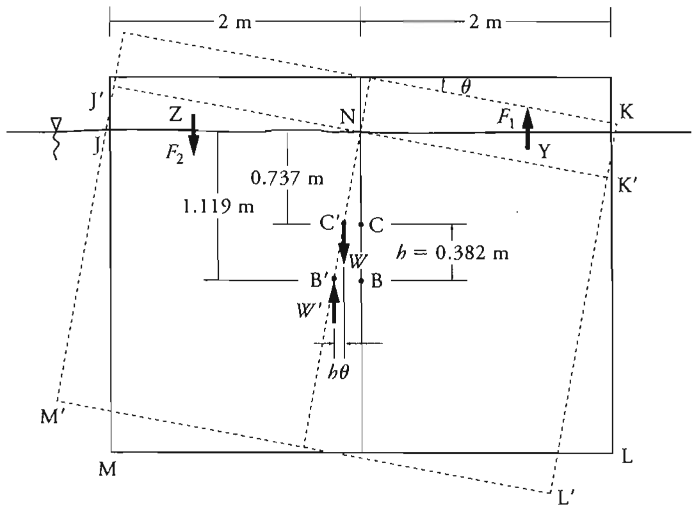


Figure 14.13

The container is acted upon by two couples; a couple F_1F_2 which tends to restore the container to its original position, and a couple $W'W$ which tends to increase the displacement. Evaluating, we have:

$$W = 900 \text{ kN} \quad \text{and} \quad h = 0.382 \text{ m}$$

The couple $W'W$ is:

$$W'W = 900 \times 0.382\theta = 343.8\theta \text{ kNm}$$

The force F_1 (= F_2) is γ times the triangle NKK' times 10 (the length of the block).

$$F_1 = F_2 = 10.06 \times \tfrac{1}{2}(NK \times KK') \times 10$$
$$= 100.6 \times 0.5 \times 2 \times 2\ \theta$$
$$= 201.2\ \theta\ \text{kN}$$

The distance YZ is:

$$YZ = 4 \times \tfrac{2}{3} = 2.67\ \text{m}$$

and the restoring couple F_1F_2 is:

$$F_1F_2 = 201.2\theta \times 2.67 = 536.5\theta\ \text{kNm}$$

The resultant couple is:

$$(536.5 - 343.8)\ \theta = 192.7\theta\ \text{kNm}$$

This couple will restore the container to its original position, hence that position appears to be stable. The reader should now check that the container is also stable in respect of a small rotation about its other horizontal axis.

The foregoing numerical example may easily be generalised. Figure 14.14a shows a floating body of general shape in equilibrium, with the C.G. at a distance h below the centre of buoyancy. The plane of intersection of the body with the free water surface is here called the *water level plane* (JK in Figure 14.14a).

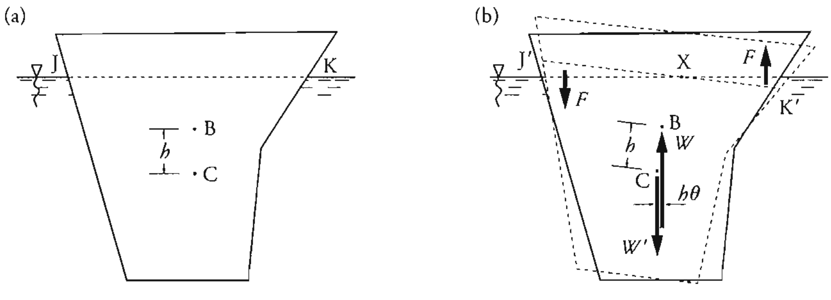


Figure 14.14

If the body is given a small rotation θ, the new system of forces will always consist of two couples:

1. A couple M_1 comprising the two forces F arising from the rotation of the water level plane from JK to $J'K'$.

From Equation A.15 (see Appendix) this couple has a moment:

$$M_1 = M_x = w_1 I_{xx} \qquad \textbf{(14.11)}$$

where I_{xx} is the second moment of area of the water level plane about the axis X through its centroid, and w_1 is the water pressure on this plane at unit distance from X. The vertical displacement at this unit distance is $1 \times \theta$ and the resulting water pressure is $\theta\gamma$. Hence the couple formed by FF is:

$$M_1 = \theta\gamma I_{xx} \qquad \textbf{(14.12)}$$

This couple always acts to restore the body to its original equilibrium position.

2. A couple M_2 formed by the buoyancy force and the weight of the body.

In the original equilibrium position these forces were in the same vertical line. The distance between them is now $h\,\theta$ hence they form a couple:

$$M_2 = Wh\theta \tag{14.13}$$

Taking h as positive if C is below B, M_2 is a restoring couple if h is positive. The total restoring couple is thus:

$$M = M_1 + M_2 = \theta(\gamma I_{xx} + Wh) \tag{14.14}$$

If C is above B, h is negative and if Wh is greater than γI_{xx} the body will overturn.

Problems

14.1 A lock gate on a canal is 5 m wide and is perpendicular to the sides of the canal. On one side of the gate the water is 3.8 m deep and on the other side 2.1 m deep as shown in Figure P14.1. Find the magnitude of the resultant hydrostatic force on the gate and its height above the bottom of the canal.

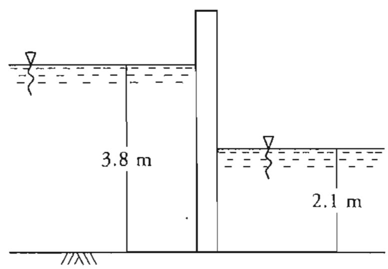


Figure P14.1

14.2 The dam wall shown in Figure 14.2 has a strut BD located every 4 m. Determine the compressive force in each strut BD, neglecting the weight of the dam wall.

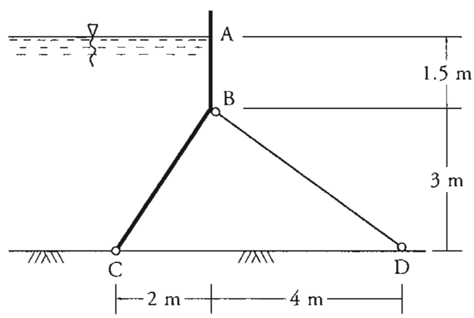


Figure P14.2

14.3 Determine the resultant force F due to water acting on the 1.1 by 2.0 m rectangular gate AB shown in Figure P14.3 and its line of action.

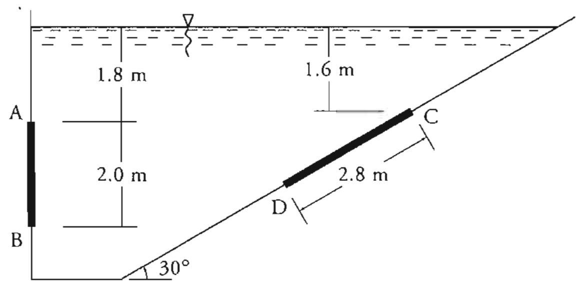


Figure P14.3

14.4 Determine the resultant force due to water acting on the 1.2 by 2.8 m triangular gate CD shown in Figure P14.3. The apex of the triangle is at C, and the gate width at D is 1.2 m.

14.5 A rectangular gate ABC in a vertical tank wall is pivoted at B and rests against a stop at A as shown in Figure P14.5. If the gate is 10 m long perpendicular to the plane of the figure, find:

(i) the reactions at A and B when the free water surface is level with the top of the gate

(ii) the height of the water surface above C when the gate opens (i.e. overturns about B).

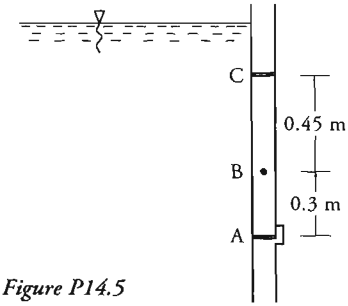


Figure P14.5

14.6 Find the magnitude and location of the horizontal and vertical components of the hydrostatic force per metre acting on curved area AB in Figure P14.6. The line AB in Figure P14.6 is the quadrant of a circle radius 2.5 m. What is the moment of the hydrostatic force on AB about point C.

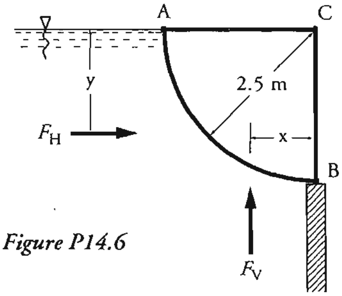


Figure P14.6

14.7 The 2.4 m diameter cylinder shown in Figure 14.7 weighs 45 kN and is 2.0 m long. It retains oil on one side as shown. Neglecting friction at A and B, determine the reactions at A and B. (Assume the specific weight of the oil is $\gamma_{oil} = 7.85 \text{ kN/m}^3$.)

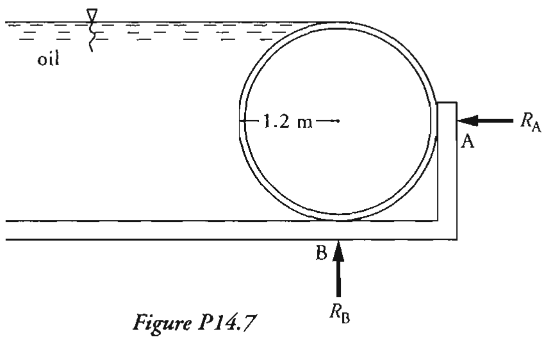


Figure P14.7

14.8 An object weighs 1.0 kN in air and 0.6 kN in water. Find its volume and specific weight.

14.9 A hollow cylinder 1 m in diameter and 2 m long weighs 4 kN.

(i) What weight of lead must be fastened to the outside of the bottom of the cylinder to make it float vertically with 1.2 m submerged?

(ii) What weight of lead is required if the lead is placed inside the cylinder? (Assume $\gamma_{lead} = 110 \text{ kN/m}^3$.)

14.10 An iceberg weighing 8.96 kN/m^3 floats in the ocean (10.06 kN/m^3) with a volume of 14000 m^3 above the surface. What is the total volume of the iceberg?

14.11 The cylindrical tank shown in Figure 14.11 floats in the position shown. Neglecting the thickness of the tank walls, find the weight of the tank. (Take $\gamma = 9.81 \text{ kN/m}^3$.)

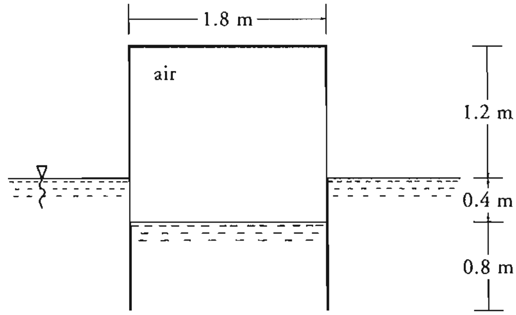


Figure P14.11

14.12 A buoy (Figure P14.12) is to be constructed of a hemisphere diameter 1.2 m surmounted by a cone and is placed into seawater (γ = 10.06 kN/m^3). The total weight of the buoy is 10.5 kN.

(i) If the height H of the cone is 2 m, will the buoy float?

(ii) What is the height H when the buoy is on the verge of sinking?

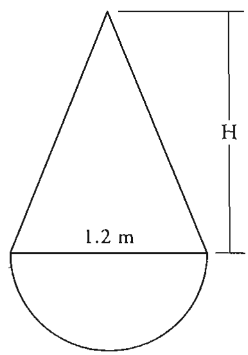


Figure P14.12

14.13 A prism weighing 2 kN has a length of 1.7 m and its cross-section is an equilateral triangle of side 0.6 m. It is placed in fresh water (γ = 9.81 kN/m^3) with the longitudinal axis vertical.

(i) What is the exposed length of prism above the water?

(ii) Is this position stable?

(iii) If not, what is the orientation of the prism in a stable position?

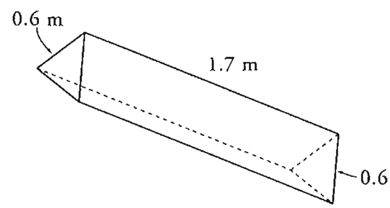


Figure P14.13

14.14 A solid cylinder has a length l, a diameter d and a density of 8 kN/m^3. If it is to float in salt water (γ = 10.06 kN/m^3) with the circular face horizontal what is the maximum value of l/d?

14.15* A solid oblate spheroid (of specific weight 8 kN/m^3) has two major axes of 4 m and the minor axis is 3 m. When it is floating in fresh water (γ = 9.81 kN/m^3), what is the height of the top of the spheroid above the water surface?

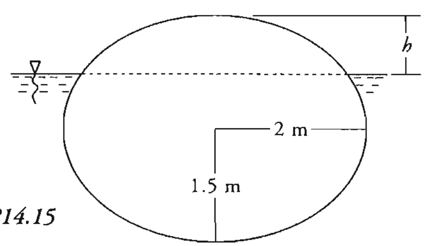
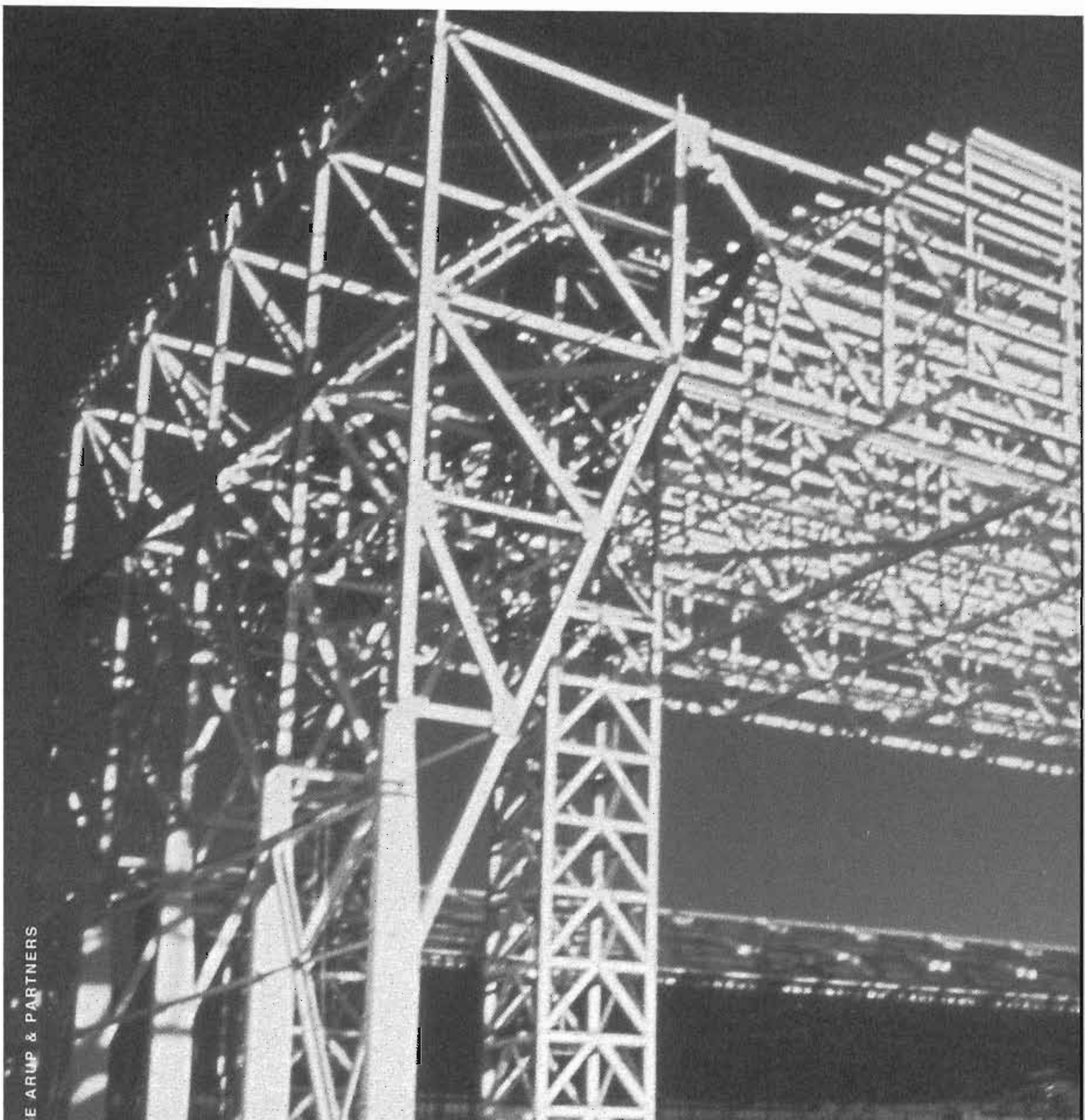


Figure P14.15

* Difficult problem, suitable for later study.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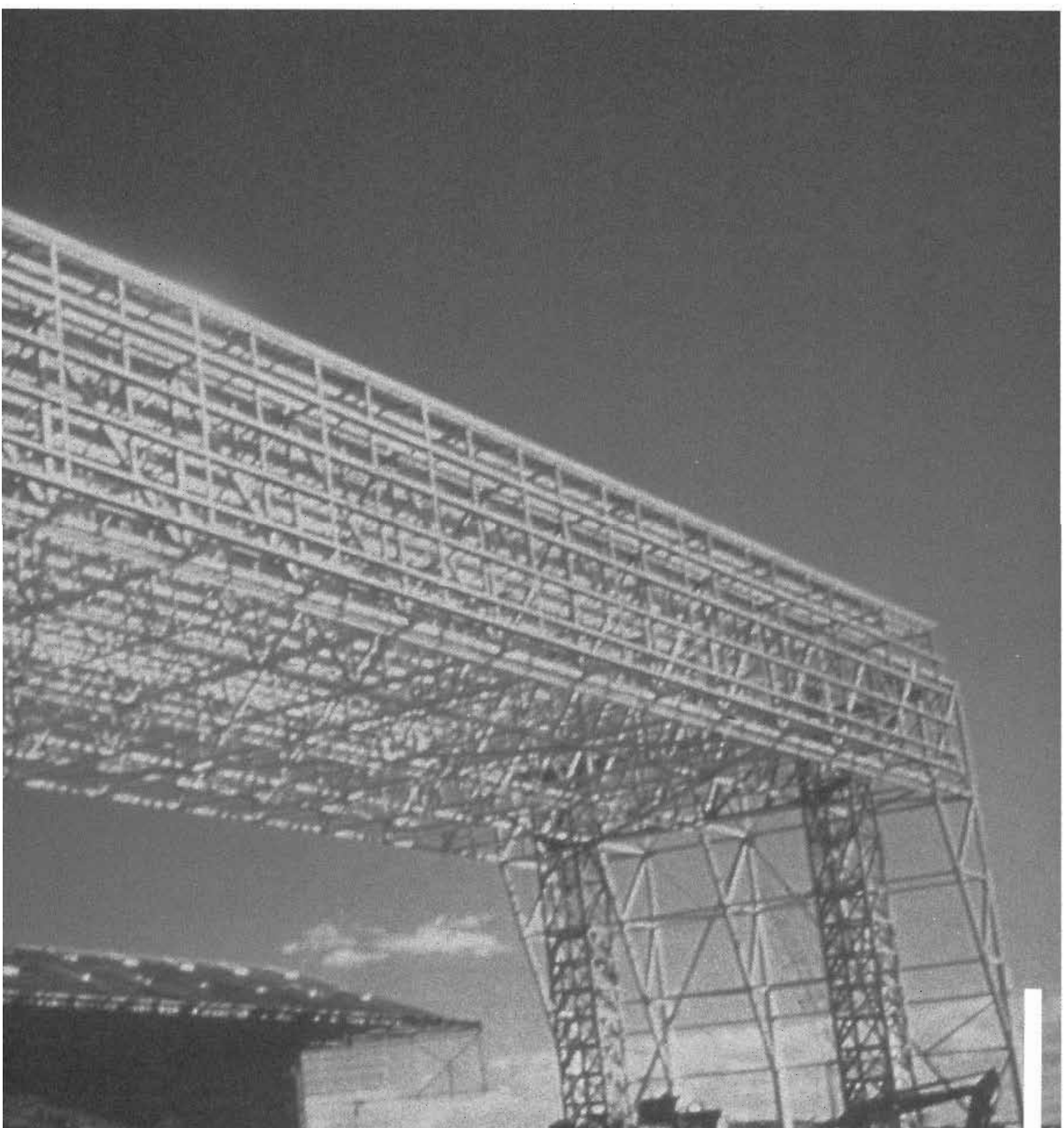
part 5

THREE-DIMENSIONAL STATICS



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Composition and Resolution of Forces

15.1 Components of a force

In considering coplanar forces, it was shown in Section 2.3 that any force could be resolved into components parallel to and perpendicular to a given direction. As any force may be resolved into two components (the force and components being coplanar), each component may be further resolved into sub-components in planes not necessarily containing the original force. Generally we concern ourselves with components in three mutually perpendicular directions.

Let F be a force through O, its line of action being inclined to three mutually perpendicular axes Ox, Oy and Oz at angles θ_x, θ_y and θ_z, respectively (Figure 15.1). The components could be obtained by first resolving F into a force Z in direction Oz and a force R in the plane Oxy, as illustrated in Figure 15.1. R could then be replaced by two forces X and Y in directions Ox and Oy, respectively. For this method, it is necessary first to express the angle between F and R, and also the angles which R makes with Ox and Oy, in terms of θ_x, and θ_y.

It is easier to consider first the converse problem of finding the resultant of component forces. Let X, Y and Z be three forces acting along Ox, Oy and Oz, respectively. The resultant F may be regarded as the diagonal of a rectangular parallelepiped whose sides are X, Y and Z respectively.

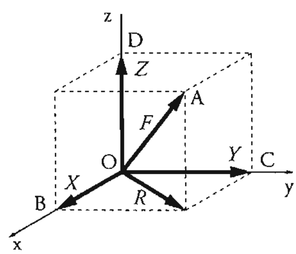


Figure 15.1

The resultant of X and Y is $R = \sqrt{X^2 + Y^2}$ acting in the Oxy plane. The resultant of R and Z, which are perpendicular to each other, is F, where:

$$F = \sqrt{R^2 + Z^2} = \sqrt{X^2 + Y^2 + Z^2} \qquad \textbf{(15.1)}$$

If we consider the triangle OAB (Figure 15.1) which is right-angled at B, we see that $X = F \cos \theta_x$. Similarly from triangles OAC and OAD we see that $Y = F \cos \theta_y$ and $Z = F \cos \theta_z$. Expressed in terms of direction cosines: ($l = \cos \theta_x$; $m = \cos \theta_y$; $n = \cos \theta_z$) we have:

$$X = lF \qquad Y = mF \qquad Z = nF \qquad \textbf{(15.2)}$$

15.2 Resultant of concurrent forces

The methods used for the composition of concurrent coplanar forces can be extended to deal with the composition of concurrent forces in three dimensions (i.e. forces which are not necessarily coplanar).

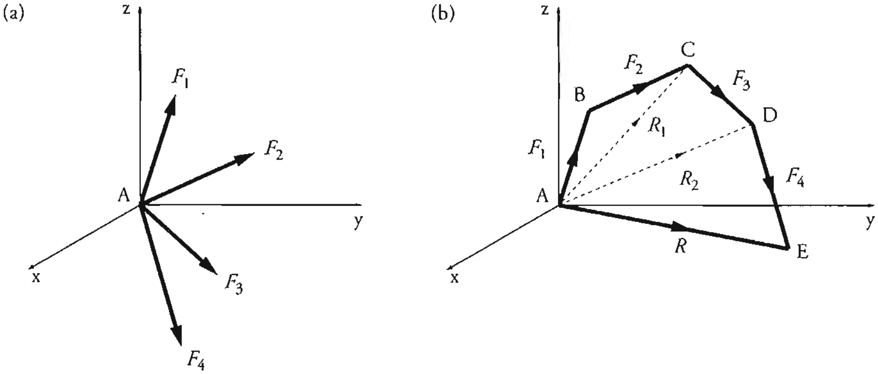


Figure 15.2

Consider the four concurrent forces F_1, F_2, F_3 and F_4 shown in Figure 15.2a. The forces F_1 and F_2 may be added vectorially to give the resultant R_1 (triangle ABC forming a plane, Figure 15.2b). Forces R_1 and F_3, not necessarily in the plane of ABC, may now be added together to give R_2 (triangle ACD forming a plane). The force R_2 may now be added to F_4, which is not in the plane of ACD, to give the resultant R (triangle ADE forming a plane). The process can be used for any number of concurrent forces and the resultant, which must pass through A, is given in magnitude and direction by the line joining the first to the last point of the force polygon drawn in space.

As the force polygon in this case is not a plane figure a single diagram of course cannot deal with the problem. The force polygon can, however, be projected on to two mutually perpendicular reference planes, and the final ray on each projection gives the projection of the resultant on the two reference planes. The true length of this line and its direction give the magnitude and direction of the resultant. Although graphical solutions are possible they are not considered here since analytical methods are more convenient in practice.

Consider any number of concurrent forces F_1, F_2, F_3 ... F_n whose directions are known with respect to three co-ordinate axes Ox, Oy and Oz. The components of each force in the three directions Ox, Oy and Oz may be written down using Equation 15.2:

$$X_1 = l_1F_1 \qquad Y_1 = m_1F_1 \qquad Z_1 = n_1F_1$$

$$X_2 = l_2F_2 \qquad Y_2 = m_2F_2 \qquad Z_2 = n_2F_2 \quad \text{and so on.} \tag{15.3}$$

The sum of the components in the direction Ox is then:

$$\Sigma X = X_1 + X_2 + X_3 + ... + X_n = X_R \tag{15.4a}$$

The sum of the components in the direction Oy is:

$$\sum Y = Y_1 + Y_2 + Y_3 + ... + Y_n = Y_R \quad \textbf{(15.4b)}$$

and the sum of the components in the direction Oz is:

$$\sum Z = Z_1 + Z_2 + Z_3 + ... + Z_n = Z_R \quad \textbf{(15.4c)}$$

$\sum X$, $\sum Y$ and $\sum Z$ are the components of R, the resultant of F_1, F_2 ... F_n. From the previous section then, the magnitude of the resultant is:

$$R = \sqrt{(X_R)^2 + (Y_R)^2 + (Z_R)^2} \quad \textbf{(15.5)}$$

and its direction is defined by the direction cosines l_R, m_R and n_R, where:

$$l_R = \frac{X_R}{R} \qquad m_R = \frac{Y_R}{R} \qquad n_R = \frac{Z_R}{R} \quad \textbf{(15.6)}$$

It is a simple matter to show from the trigonometry of the Parallelogram of Forces; that if two forces F_1 and F_2 (concurrent) are inclined to each other at the angle α then:

$$R = \sqrt{F_1^{\,2} + F_2^{\,2} + 2F_1F_2 \cos \alpha} \quad \textbf{(15.7)}$$

In a similar manner, the student may prove by expanding the expression for $(\sum X)^2$, $(\sum Y)^2$ and $(\sum Z)^2$ and making use of the relationships:

$$l^2 + m^2 + n^2 = 1 \quad \text{and} \quad \cos \alpha_{12} = l_1l_2 + m_1m_2 + n_1n_2$$

that for the general case:

$$R = \sqrt{F_1^{\,2} + F_2^{\,2} + ... + F_n^{\,2} + 2F_1F_2 \cos \alpha_{12} + 2F_1F_3 \cos \alpha_{13} + ... \ etc.} \quad \textbf{(15.8)}$$

where α_{ij} is the angle between F_i and F_j.

EXAMPLE 15.1

Find the resultant of the three forces shown separately in Figures 15.3a, b and c. The angle between the axis Ox and each force is not given but it is less than 90°.

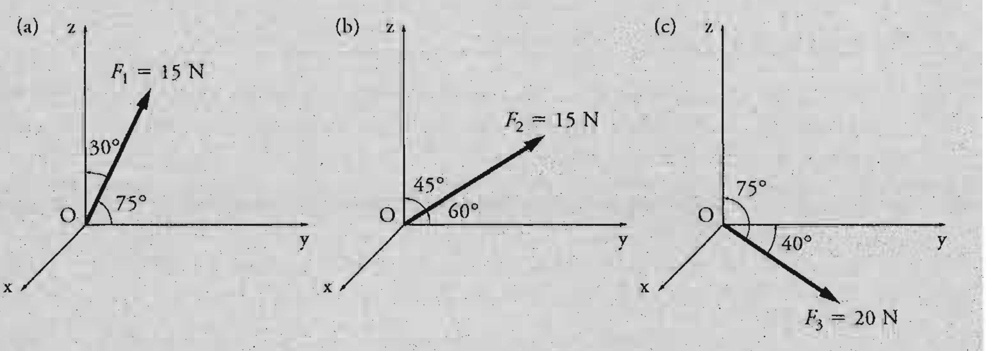


Figure 15.3

SOLUTION

The direction cosines l_1, l_2 and l_3 are found by using the relationship:

$$l^2 + m^2 + n^2 = 1$$

Thus:

$$l_1 = \sqrt{1 - \cos^2 30 - \cos^2 75} = 0.428$$

$$l_2 = \sqrt{1 - \cos^2 60 - \cos^2 45} = 0.500$$

$$l_3 = \sqrt{1 - \cos^2 40 - \cos^2 75} = 0.588$$

The components of each force are then calculated in Table 15.1.

Table 15.1

Force (*N*)	$X = lF$	$Y = mF$	$Z = nF$
$F_1 = 15$	6.42	3.88	12.99
$F_2 = 15$	7.50	7.50	10.61
$F_3 = 20$	11.77	15.32	5.18
Summation	25.69 $= X_R$	26.70 $= Y_R$	28.78 $= Z_R$

The magnitude and direction of the resultant are obtained from Equations 15.5 and 15.6 respectively:

$$R = \sqrt{(25.69)^2 + (26.70)^2 + (28.78)^2} = 46.91 \text{ N}$$

and:

$$l_R = 25.69/46.91 = 0.548 \quad \text{or} \quad \theta_x = 56.8°$$

$$m_R = 26.70/46.91 = 0.569 \quad \text{or} \quad \theta_y = 55.3°$$

$$n_R = 28.78/46.91 = 0.613 \quad \text{or} \quad \theta_z = 52.2°$$

Problems

15.1 A force of 90 N acts along a line passing through the origin of co-ordinates (0, 0, 0) towards the point whose co-ordinates are (4, −4, 7). Resolve this force into three components along the x, y and z axes.

15.2 A force *F* acts along a line passing through the origin of co-ordinates (0, 0, 0) towards a point A, having co-ordinates x_1, y_1 and z_1. Find the components along the x, y and z axes for each of the sets of values of *F*, x_1, y_1 and z_1 shown in Table P15.2.

Table P15.2

	F	x_1	y_1	z_1
(i)	250 N	3	4	10
(ii)	180 N	6	6	-6
(iii)	300 N	-5	9	12
(iv)	520 N	8	-5	$4\sqrt{5}$
(v)	280 N	-2	3	-6

15.3 Table P15.3 shows the magnitudes and directions of a set of concurrent forces. The angles from the x, y and z axis to the respective forces are θ_x, θ_y and θ_z respectively. Find the magnitude and direction of the resultant.

Table P15.3

F	θ_x	θ_y	θ_z
10 N	135°	90°	135°
20 N	45°	45°	90°
30 N	54.74°	54.74°	54.74°

15.4 Table P15.4 gives the values of three sets of concurrent forces. Each set refers to the diagram in Figure P15.4. Find the resultant of each set (i) to (iii).

Table P15.4

	Force	Direction
(i)	500 N	A to D
	280 N	A to G
	200 N	A to C
(ii)	420 N	B to H
	300 N	D to B
	100 N	A to B
(iii)	200 N	K to A
	440 N	K to H
	130 N	G to K

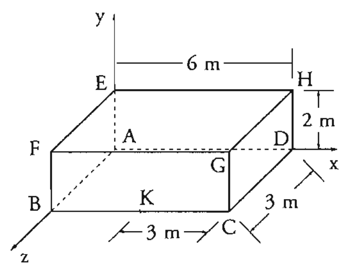
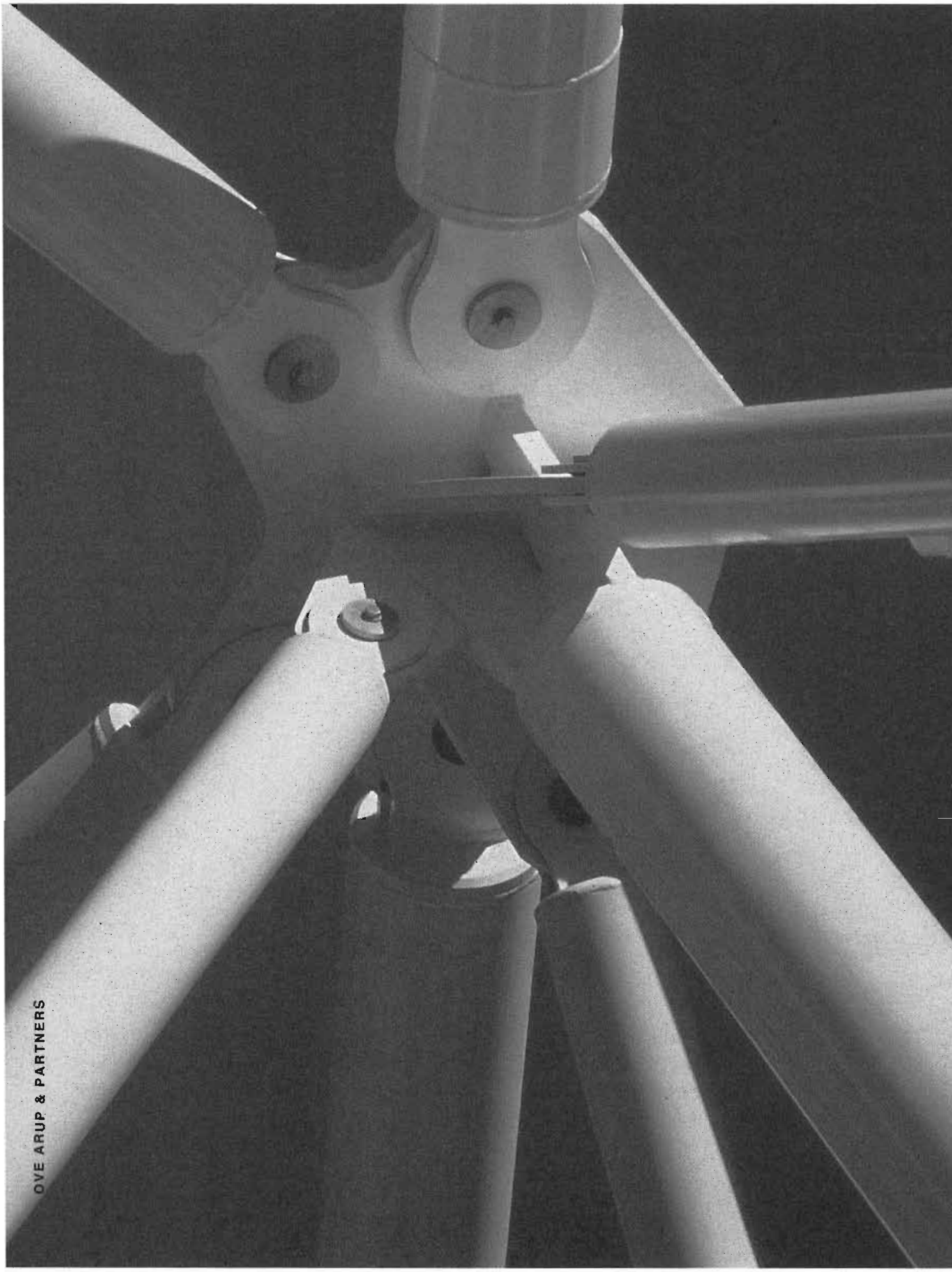


Figure P15.4

15.5 A block weighing 30 N lies on a smooth plane inclined at 20° to the horizontal. The block is pulled with a force of 12 N by a rope parallel to the plane and in a direction making an angle of 60° with the line of greatest slope. The 12 N pull has a component up the plane. Find the resultant force on the block.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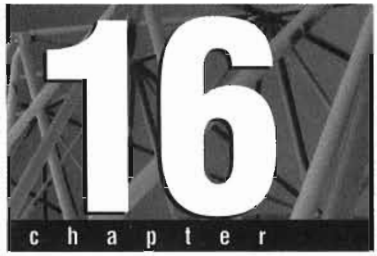


Equilibrium of Concurrent Forces

16.1 Equilibrium

In Section 3.1, it was shown that as a consequence of Newton's Second Law, any body at rest must be acted upon by a system of forces with a zero resultant. This of course applies equally to three dimensional systems. If the resultant of a three dimensional system of concurrent forces is zero, the forces when added vectorially must form a closed polygon in space (i.e. the last point of the force polygon must coincide with the initial point).

16.2 Conditions of equilibrium

The resultant of a system of concurrent forces has been shown to have three components given by Equations 15.4. That is:

$$\sum X = \sum l_i F_i \qquad \sum Y = \sum m_i F_i \qquad \sum Z = \sum n_i F_i$$

If the resultant is zero, each of the components $\sum X$, $\sum Y$ and $\sum Z$ must be zero. Hence, the conditions of equilibrium are:

$$\sum X = 0 \tag{16.1}$$

$$\sum Y = 0 \tag{16.2}$$

$$\sum Z = 0 \tag{16.3}$$

If a concurrent system of forces is known to be in equilibrium and all forces except one are known in magnitude and direction, the above three equations may be used to determine the unknown force in magnitude and direction. This problem is simply the reverse of finding the resultant of the known forces.

EXAMPLE 16.1

Five bars AB, AC, AD, AE and AG are connected together at A. They are shown in Figure 16.1 by their projections on the xy and yz planes. The forces in AB, AC, AD and AE acting at A are as shown in the figure. Find the direction of AG and the magnitude of the force in AG if the system is in equilibrium.

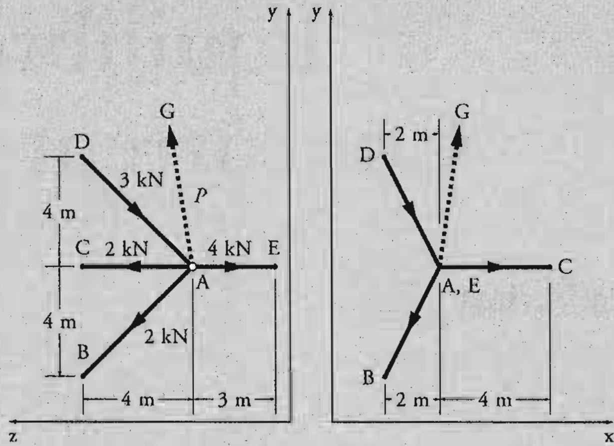


Figure 16.1

SOLUTION

As we did when considering two-dimensional systems, we find the x components of all forces and equate the sum to zero (Equation 16.1). This gives the x component of the unknown force. We then proceed similarly for the y and z components.

It is necessary first to calculate the length of each bar and hence the direction cosines for that bar. For example, for the bar AB we have:

$$L_{AB} = \sqrt{2^2 + 4^2 + 4^2} = 6 \text{ m}$$

Then: $l_{AB} = \dfrac{-2}{6} \qquad m_{AB} = \dfrac{-4}{6} \qquad n_{AB} = \dfrac{+4}{6}$

(The value of l_{AB} is negative since the projection of the force AB on the x axis is in the negative x direction.)

The direction cosines of the other known forces are calculated in the same way and are entered in Table 16.1. The direction cosines of the unknown force P in bar AG are entered as l_1, m_1 and n_1. The components of the known forces are calculated numerically, while the components of P are l_1P, m_1P and n_1P.

Table 16.1

Bar	F (kN)	L (m)	l	m	n	$X = lF$	$Y = mF$	$Z = nF$
AB	2	6	-2/6	-4/6	+4/6	-0.667	-1.333	+1.333
AC	2	$4\sqrt{2}$	$+4/(4\sqrt{2})$	0	$+4/(4\sqrt{2})$	+1.414	0	+1.414
AD	3	6	+2/6	-4/6	-4/6	+1.000	-2.000	-2.000
AE	4	3	0	0	-3/3	0	0	-4.000
AG	P		l_1	m_1	n_1	l_1P	m_1P	n_1P
Summations						$+1.747 + l_1P$	$-3.333 + m_1P$	$-3.253 + n_1P$

From the three equations of equilibrium:

$$\Sigma X = 0: \quad \therefore\ l_1 P = -1.747 \text{ kN}$$

$$\Sigma Y = 0: \quad \therefore\ m_1 P = +3.333 \text{ kN}$$

$$\Sigma Z = 0: \quad \therefore\ n_1 P = +3.253 \text{ kN}$$

Hence:

$$P = \sqrt{1.747^2 + 3.333^2 + 3.253^2} = 4.974 \text{ kN}$$

The direction cosines of P are:

$$l_1 = \frac{-1.747}{4.974} = -0.351$$

$$m_1 = \frac{+3.333}{4.974} = +0.670$$

and
$$n_1 = \frac{3.253}{4.974} = +0.654$$

As a check we note that:

$$l_1^2 + m_1^2 + n_1^2 = (-0.351)^2 + (0.670)^2 + (0.654)^2 = 1.00 \quad \therefore \text{ O.K.}$$

EXAMPLE 16.2

Figure 16.2 shows three bars AB, AC and AD joined together at A. The projections of the bars on the xy and yz planes are given. A force of 10 kN acts at A parallel to the z axis. Find the force in each bar.

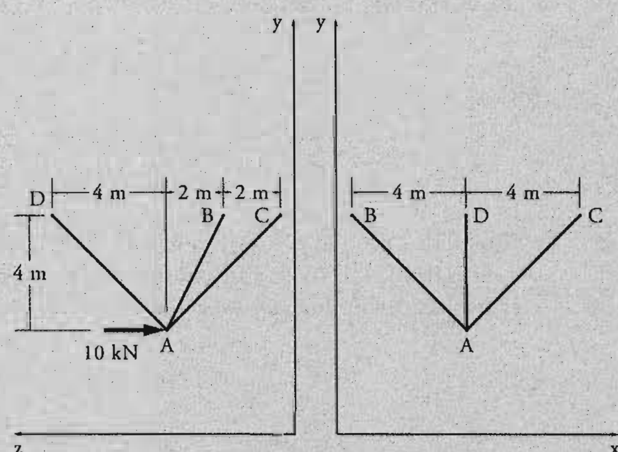


Figure 16.2

SOLUTION

As in the previous example, the direction cosines of each force are the same as those of the bar in which the force acts. Since the bar forces are unknown, it will be assumed that they act away from A. The direction cosines are found as in Example 16.1 and entered in Table 16.2.

Table 16.2

Bar	F (kN)	L (m)	l	m	n
AB	N_{AB}	6	-4/6	+4/6	-2/6
AC	N_{AC}	$4\sqrt{3}$	$+4/4\sqrt{3} = \sqrt{3}/3$	$+4/4\sqrt{3} = \sqrt{3}/3$	$-4/4\sqrt{3} = -\sqrt{3}/3$
AD	N_{AD}	$4\sqrt{2}$	0	$+4/4\sqrt{2} = \sqrt{2}/2$	$+4/4\sqrt{2} = \sqrt{2}/2$
10 kN load	10		0	0	-1

The components of each force are obtained by multiplying the force by its direction cosines. For equilibrium:

$$\Sigma X = 0: \quad -\frac{2}{3}N_{AB} + \frac{\sqrt{3}}{3}N_{AC} = 0 \tag{16.4}$$

$$\Sigma Y = 0: \quad +\frac{2}{3}N_{AB} + \frac{\sqrt{3}}{3}N_{AC} + \frac{\sqrt{2}}{2}N_{AD} = 0 \tag{16.5}$$

$$\Sigma Z = 0: \quad -\frac{1}{3}N_{AB} - \frac{\sqrt{3}}{3}N_{AC} + \frac{\sqrt{2}}{2}N_{AD} - 10 = 0 \tag{16.6}$$

Solving these simultaneous equations gives:

$$N_{AB} = -4.28 \text{ kN} \quad N_{AC} = -4.95 \text{ kN} \quad \text{and} \quad N_{AD} = +8.08 \text{ kN}$$

Example 16.3

In Figure 16.3, A is the origin and the plane xAy is horizontal. The pole AB is fixed by a ball and socket joint to a foundation at A. A vertical load of 3 kN is suspended at B. The end B (co-ordinates 20, 10, 10) is supported by three ropes, BC having a tension of 1 kN and passing through C (5, 0, 15), BD having a tension of 1.5 kN and passing through D (0, 5, 15), and BE carrying a tension of 4 kN. What is the direction of BE for equilibrium and what is the force in AB?

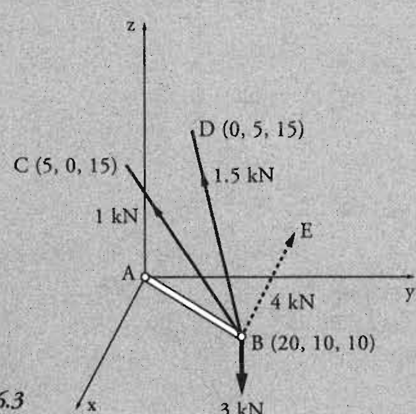


Figure 16.3

SOLUTION

Let the force in AB be P and assume that it acts away from B. Let the direction cosines of the force in BE be l_1, m_1 and n_1.

As in the previous examples, we first calculate the direction cosines of those forces whose directions are known. Equilibrium is then expressed by equating to zero the sum of the components in each of the directions x, y and z.

Consider rope BD for example. In terms of the co-ordinates of B and D, the direction cosines of BD may be found as follows. Let L_x, L_y and L_z be the projections (with appropriate signs) of BD. Then:

$$L_x = x_D - x_B = 0 - 20 = -20$$

$$L_y = y_D - y_B = 5 - 10 = -5$$

$$L_z = z_D - z_B = 15 - 10 = +5$$

and

$$L_{BD} = \sqrt{L_x^2 + L_y^2 + L_z^2} = 21.21$$

$$l_{BD} = \frac{-20}{21.21} = -0.943 \qquad m_{BD} = \frac{-5}{21.21} = -0.236 \qquad n_{BD} = \frac{+5}{21.21} = +0.236$$

Hence the components of F_{BD} are:

$$X_{BD} = l_{BD}F_{BD} = -0.943 \times 1.5 = -1.414 \text{ kN}$$

$$Y_{BD} = m_{BD}F_{BD} = -0.236 \times 1.5 = -0.354 \text{ kN}$$

$$Z_{BD} = n_{BD}F_{BD} = +0.236 \times 1.5 = +0.354 \text{ kN}$$

The components of the other forces are calculated similarly, using symbols where necessary for unknown forces or direction cosines, as tabulated below.

Table 16.3

Member	F (kN)	l	m	n	$X = lF$	$Y = mF$	$Z = nF$
BA	P	-0.816	-0.408	-0.408	-0.816P	-0.408P	-0.408P
BC	1	-0.802	-0.535	+0.267	-0.802	-0.535	+0.267
BD	1.5	-0.943	-0.236	+0.236	-1.414	-0.354	+0.354
BE	4	l_1	m_1	n_1	$4\,l_1$	$4\,m_1$	$4\,n_1$
3 kN load	3	0	0	-1	0	0	-3.000

The equilibrium equations may now be written as:

$$\Sigma X = 0: \quad -0.816P - 2.216 + 4l_1 = 0$$

$$\Sigma Y = 0: \quad -0.408P - 0.889 + 4m_1 = 0$$

$$\Sigma Z = 0: \quad -0.408P - 2.379 + 4n_1 = 0$$

These may be re-arranged to give:

$$l_1 = 0.204P + 0.554 \qquad \textbf{(16.7)}$$

$$m_1 = 0.102P + 0.222 \qquad \textbf{(16.8)}$$

$$n_1 = 0.102P + 0.595 \qquad \textbf{(16.9)}$$

A fourth equation is provided by the fact that:

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad \textbf{(16.10)}$$

Substitution of Equations 16.7, 16.8 and 16.9 into 16.10 gives a quadratic in P from which we obtain:

$$P = -6.958 \text{ kN} \quad \text{or} \quad P = +0.667 \text{ kN}$$

Corresponding to the first value of P, Equations 16.7, 16.8 and 16.9 give:

$$l_1 = -0.865 \quad m_1 = -0.488 \quad n_1 = -0.115$$

For the second value of P we find that:

$$l_1 = +0.690 \quad m_1 = +0.290 \quad n_1 = +0.663$$

Thus there are two positions of the rope BE which will maintain equilibrium with the given forces. The reader should try to picture the two positions from the values of the direction cosines. It will be noted that in one case the pole AB is in compression and in the other case it is in tension.

EXAMPLE 16.4

The vertical pole AB (20 m long) is loaded at B by the loads shown in Figure 16.4. The ropes BD and BC lie in the vertical yz plane, BC being parallel to the y axis. The point E lies on the x axis. Another guy rope (BF) 25 m long is available. Find the coordinates of the point F on the xy plane, to which the guy should be attached to maintain equilibrium. Also find the forces in the guy and in the pole.

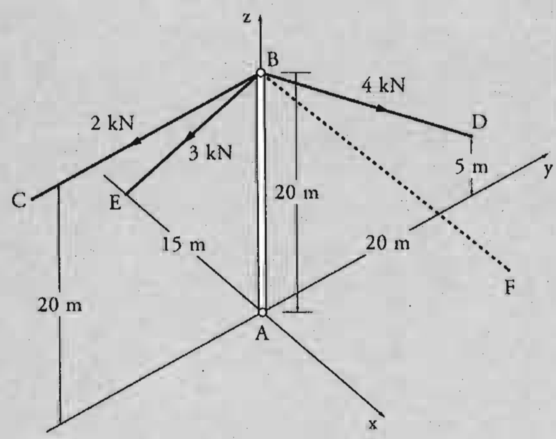


Figure 16.4

SOLUTION

Let the direction cosines of the guy rope BF be l_1, m_1 and n_1. Let the force in BF be P and the force in the pole AB be Q, and assume that both these forces act away from B. Since the length BF is 25 m and F is on the xy plane:

$$n_1 = \frac{-20}{25} = -0.8$$

The direction cosines of BC, BD and BE are calculated as in Example 16.3.

The three equations of equilibrium at B are:

$$\sum X = 0: \quad (-0.6 \times 3) + l_1 P = 0 \qquad \textbf{(16.11)}$$

$$\sum Y = 0: \quad -2 + (0.8 \times 4) + m_1 P = 0 \qquad \textbf{(16.12)}$$

$$\sum Z = 0: \quad (-0.8 \times 3) - (0.6 \times 4) - Q - 0.8P = 0 \qquad \textbf{(16.13)}$$

In addition we have the equation:

$$l_1^2 + m_1^2 + (-0.8)^2 = 1 \qquad \textbf{(16.14)}$$

Equations 16.11 and 16.12 may be re-arranged as:

$$l_1 P = 1.8 \qquad \textbf{(16.15)}$$

$$m_1 P = -1.2 \qquad \textbf{(16.16)}$$

Elimination of P from these two equations gives:

$$l_1 = -1.5m_1$$

and substitution into Equation 16.14 gives $(1.5m_1)^2 + m_1^2 + 0.64 = 0$:

$$\therefore\ m_1 = \pm\, 0.333$$

As in the previous example, there are two positions of the force BF which will maintain equilibrium with the given forces. However, in this problem P must be positive (tension) since a rope cannot sustain compression. Equation 16.16 shows that P is of opposite sign to m_1, hence we must take $m_1 = -0.333$. Therefore, from Equations 16.15 and 16.16:

$$P = 3.6 \text{ kN} \quad \text{and} \quad l_1 = +0.500$$

The co-ordinates of F (which is in the xy plane) are:

$$x_F = 25\, l_1 = +12.5 \text{ m}$$

$$y_F = 25m_1 = -8.33\text{m}$$

The force in the pole is given by Equation 16.13:

$$Q = -7.68 \text{ kN}$$

Alternatively, we may approach the problem in a rather more physical manner. Figure 16.5 shows a plan view of forces at B. Ropes BC and BD both have zero x components, and their y components together are equal to 1.2 kN. Rope BE has a zero y component and its x component is −1.8 kN. Since the rope BF is 25 m long and the pole AB is 20 m long, the point F must be $\sqrt{25^2 - 20^2} = 15$ m from A (Figure 16.5).

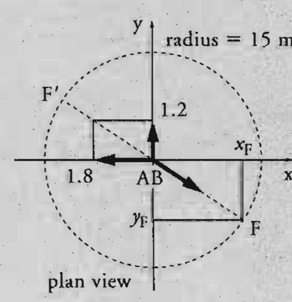


Figure 16.5

Since the rope BF is in tension, the point F must be in the position shown in Figure 16.5 and not at F'. With:

$$\frac{x_F}{y_F} = \frac{1.8}{1.2} \quad \text{and} \quad x_F^2 + y_F^2 = 15$$

we have:

$$x_F = +12.5 \text{ m} \quad \text{and} \quad y_F = -8.33 \text{ m}$$

The component of the rope force P parallel to Ax is 1.8 kN. Since $x_F = 12.5$ m and the length of the rope is 25 m:

$$P = \frac{1.8 \times 25}{12.5} = 3.6 \text{ kN}$$

As before, vertical equilibrium at B leads to Equation 16.13 which gives $Q = -7.68$ kN.

16.3 Further discussion

In the previous section, we considered problems involving a number of concurrent forces in equilibrium and used the three equations of equilibrium to find information on unspecified force magnitudes and directions. In such problems, each force is specified by its magnitude and the signed direction cosines of its line of action, and for each line the relationship between direction cosines is $l^2 + m^2 + n^2 = 1$

In Examples 16.1 to 16.4, the number of unknown forces and directions was such that they could be determined using only the three equilibrium equations. The question arises as to the necessary condition to be met in order to be able to solve a problem involving concurrent forces using only the conditions of equilibrium.

If p is the number of concurrent forces of known or unknown magnitude, there will then be $3p$ direction cosines, known or unknown, describing the lines of action of the forces. If q is the number of forces of specified magnitude and r the number of specified direction cosines, it follows that the number of unknown variables is $(4p - q - r)$ and of these $(p - q)$ are force magnitudes and $(3p - r)$ are direction cosines.

For a given problem to be solved by consideration of statics alone, there must be a total of $(4p - q - r)$ equations available, comprising three equations of equilibrium, and the remaining equations of the form $l^2 + m^2 + n^2 = 1$. In any problem where it is necessary to use equations of the latter type, results will be ambiguous and additional information may be needed to obtain the correct answers to the particular problem.

Problems

16.1 Three forces of magnitude 10 N, 20 N and 30 N act at A in the directions shown in Figure P16.1. Find the forces in the directions AB, AC and AD which would maintain equilibrium.

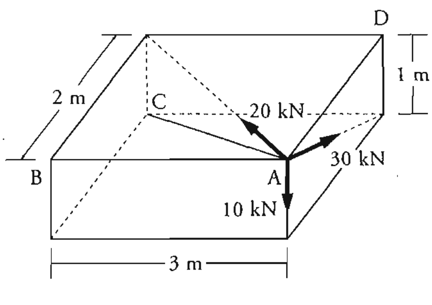


Figure P16.1

16.2 A weight of 100 N is to be supported by a rope below a point A to which four other ropes are attached (see Figure P16.2). One rope lies in the zy plane, and another in the zx plane and each of these has a tension of 50 N. A rope with tension P lies in the plane xy. In what direction must the fourth rope, carrying 60 N tension, lie in order to maintain equilibrium, and what is the value of P?

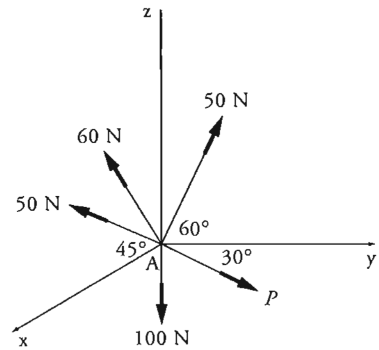


Figure P16.2

16.3 A power line cable has a tension of 500 N at the insulators, where it is supported (Figure P16.3). At each side of the point of support it makes an angle of 10° with the horizontal and lies in a vertical plane. The insulators are suspended from a point A which is restrained by two equal struts AD and AE lying in a horizontal plane, and by two equal ties AB and AC. Find the tension in the ties and the compression in the struts.

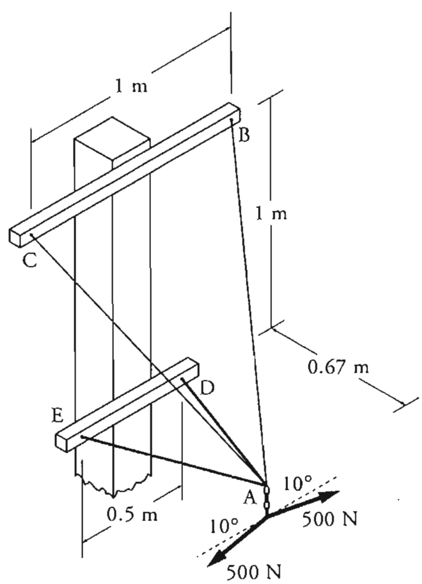


Figure P16.3

16.4 A load of 10 kN is supported by three ropes as shown in Figure P16.4. OA lies along the x axis, and OB lies along the (negative) z axis. C is the point (–3, 5, 2). Find the tension in each rope.

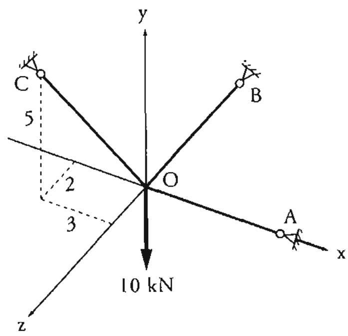


Figure P16.4

16.5 In a pin-jointed space frame there are three bars only (not in the same plane) which are joined at the node A. No external load is applied to the node A. What can be said about the forces in these three bars?

16.6 Figure P16.6 shows a mast AB, which is supported at A in a socket (no moment resistance) and is held in a vertical position by three guy ropes BC, BD and BE, where C(−4, 0, −4) and D(−4, 0, 4). Turnbuckles in these guys are used to put initial tensions in them so that they exert forces at B in the directions shown. The initial tension in BC and BD is 1800 N. What is the initial tension in BE, and what is the compression in the mast?

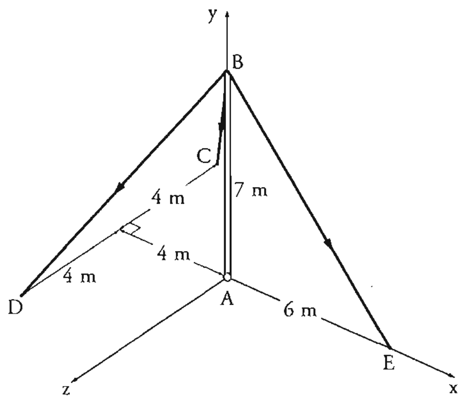


Figure P16.6

16.7 The guy rope BD of the previous problem is changed so that the co-ordinates of point D are (−4, 0, 6) while the initial tension remains as 1800 N. The guy BE must now be changed so that the co-ordinates of E are (6, 0, z). Find the position of E, the initial tension in BE and the compression in the mast.

16.8 A crane is lifting a crate weighing 10 kN off the ground (Figure P16.8). Three slings OA, OB and OC are attached to the crate, as shown in Figure P16.8, so that A, B and C are in a horizontal plane, the slings being connected to the ring O which is 1 metre above the centre of the crate. Initially, the crate is not located directly below the crane hook so that the rope connecting the ring O to the hook is inclined at 10° to the vertical and lies in the yz plane. The edges of the crate run parallel to the x and z directions. When the tension in this lifting rope reaches 6 kN, what are the tensions in the three slings?

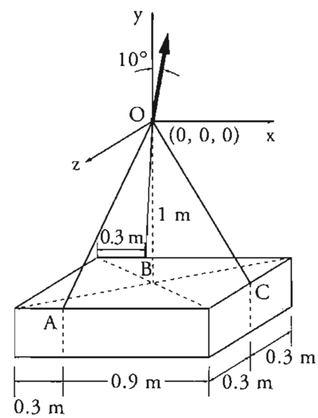


Figure P16.8

16.9 What are the tensions in the three slings when the crate in Problem 16.8 is lifted clear of the ground?

16.10 Four forces are concurrent at the origin of co-ordinates O (0, 0, 0). The first is a force of −10kN acting along the y axis; the second is 5 kN acting along the line from O to a point whose co-ordinates are (2, 3, 5); the third is 5 kN acting in an unknown direction (direction cosines *l*, *m*, *n*); while the fourth is an unknown force *F* acting along a line from O to a point with co-ordinates (–3, 3, −3). Find the values of *F*, *l*, *m* and *n* if the forces are in equilibrium.

OVE ARUP & PARTNERS

Non-concurrent Forces

17.1 Moment of a force

In a planar system of forces, the moment of a force about a point in the plane was defined as the product of the force and the perpendicular distance from the point to the line of action of the force (Section 4.1, Figure 4.1). In fact, the moment of the force is not about the point but about an axis through the point and normal to the plane. In the case of a three-dimensional system of forces, a single point does not adequately define the axis of moments and it is essential therefore to always specify the axis clearly.

Consider first the moment of a force F about an axis A which is normal to F but does not intersect it, as shown in Figure 17.1a. Let plane AA be the plane which contains axis A and is parallel to the force F. Plane FF is parallel to plane AA and contains the force F, as shown. There is only one line normal to these planes which intersects both the axis A and the force F. This is called the *common perpendicular* to F and A. In Figure 17.1a, the common perpendicular intersects A at point O and F at point O′. Let O′O be the y axis and let axis A be the z axis. The normal to these axes through O is then Ox, which is parallel to the force F. Thus axis A lies in the plane xOz (plane AA) and the force F lies in the parallel plane x′O′z′ (plane FF) which is distant d from xOz.

The moment of force F about axis A is $F \times d$, where d is the length of the common perpendicular (i.e. the distance between planes AA and FF).

In effect this is the case considered in Section 4.1, where the force F and the point O all lay in the xy plane.

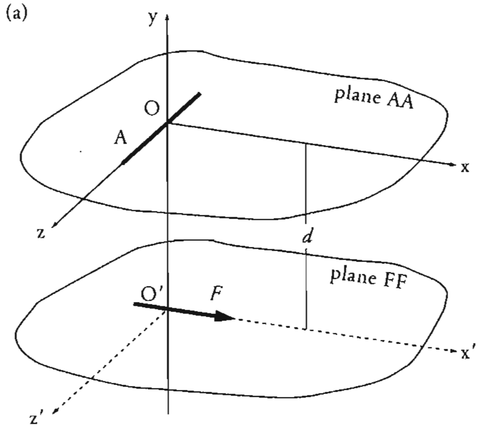


Figure 17.1

For any force F and any axis A, there is in general one and only one pair of parallel planes, one containing F and one containing A. When the force F and the axis A intersect, they share a common plane. The common perpendicular d then equals zero and the moment of F about A is zero.

When F and A are parallel, every plane containing A is parallel to F. One of these planes, AF, is common to both F and A. In this case, we may replace the force F by a parallel force F along A together with a couple lying in the plane AF. If the couple is now replaced by a pair of forces each of which intersect axis A, the original force is replaced by a statically equivalent system consisting of three forces each of which intersects A. The moment of F about A is clearly zero.

In summary, we may say that if there is a plane which contains both the force F and the axis A, the moment of F about A is zero.

We now consider the general case where the force F is not normal to the axis A. Figure 17.1b is similar to Figure 17.1a except that the force F is at the angle θ to the direction of O′x′. F may be resolved into components $F \cos \theta$ along O′x′ and $F \sin \theta$ along O′z′. The component $F \sin \theta$ is parallel to axis A and therefore has no moment about it. Hence the total moment of the force F about the axis A is $F \cos \theta \times d$.

The moment of a force about an axis is a vector quantity and may be denoted as a double-headed arrow along the axis of moment, as illustrated in Figure 17.2a. It is necessary to ascribe a sign (positive or negative) to the vector quantity (i.e. a direction for the arrow). In Figure 17.2a, the vector is drawn in the direction G to D. This is because the moment produces a clockwise rotation when the axis is viewed looking from G to D. This is known as the right-hand screw rule. The arrow points in the direction a right-hand threaded screw will move if activated by the moment. With this convention, positive rotation around the co-ordinate axes Ox, Oy, Oz are as shown in Figure 17.2b. For a two-dimensional system lying in the xy plane, and drawn so that the axis Oz is towards the viewer (Figure 17.2c), positive rotation about the z axis becomes anticlockwise since we are looking in the *negative* direction along Oz.

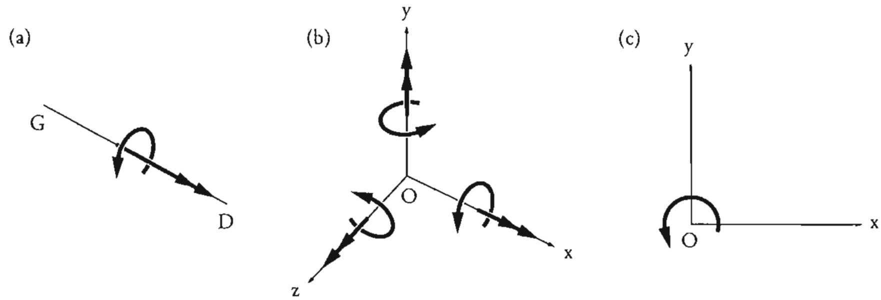


Figure 17.2

Note that if the xy plane (see Figure 17.2b) is rotated positively about Oz, the x axis moves towards Oy. Similarly, a positive rotation of the yz plane moves Oy towards Oz and a positive rotation of the zx plane moves Oz towards Ox.

For the present, we will consider problems in which the forces are either parallel to or perpendicular to the axis of moments. (The more general situation will be considered in Section 17.7.)

EXAMPLE 17.1

Figure 17.3 shows a rectangular box acted upon by forces parallel to Ox, Oy or Oz. Find the total moment of the force system about the three axes and also about axis EF.

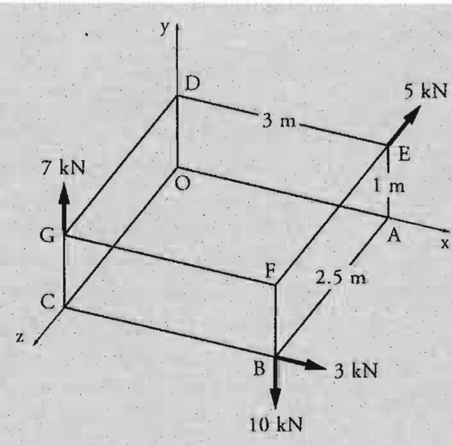


Figure 17.3

SOLUTION

Summing the moments of each force about the x axis gives:

$$M_x = +(10 \times 2.5) - (7 \times 2.5) - (5 \times 1) = +2.5 \text{ kNm}$$

The 3 kN force has no moment about Ox since it is parallel to Ox.

Summing the moments about the y axis gives:

$$M_y = +(5 \times 3) + (3 \times 2.5) = +22.5 \text{ kNm}$$

Neither the 10 kN force nor the 7 kN force has a moment about Oy since they are both parallel to Oy.

Summing the moments about the z axis gives:

$$M_z = -(10 \times 3) = -30 \text{ kNm}$$

The 5 kN force is parallel to Oz, and the 3 kN and 7 kN forces each intersect Oz. Hence none of these forces has any moment about Oz.

The edge EF, being parallel to Oz will be taken as positive from E to F. Then:

$$M_{EF} = -(7 \times 3) + (3 \times 1) = -18 \text{ kNm}$$

The 10 kN force intersects EF and the 5 kN force is parallel to EF, so neither has a moment about EF.

17.2 Couples

In Section 4.3, a *couple* was defined as a pair of parallel forces of equal magnitude but opposite sense. The same applies to couples in a three dimensional system.

In a two-dimensional system, the only quantity which needed to be specified was the *moment of the couple*. This was defined as the product of one of the forces and the distance between them. In a three-dimensional system it is necessary also to define the *direction of the axis* of rotation. This axis may be taken as any line normal to the plane which contains the forces. The direction of such a normal is taken such that the couple has a positive moment about the axis.

The statical effect of a couple on a rigid body is unchanged by:

1. rotating the couple in its plane
2. transferring it to another position in its plane
3. transferring the couple to a plane parallel to the original plane
4. changing the magnitude of the forces and distance between them, provided the product of the force and the distance remains constant.

These properties are illustrated in Figure 17.4, where A and A′ are parallel planes and LM is a line normal to them. Each of the five couples shown has a magnitude Fa and each has an axis in the direction of LM.

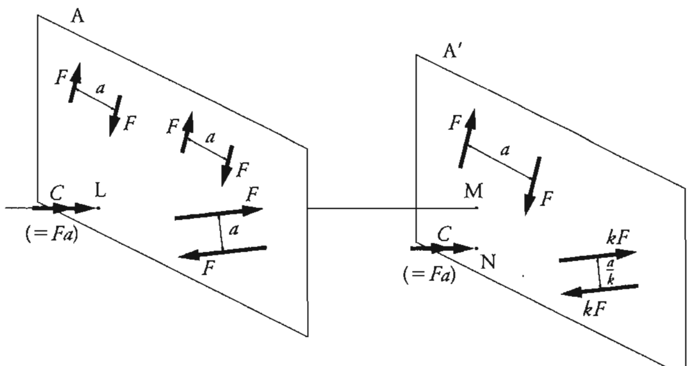


Figure 17.4

A couple, having magnitude and direction, is a vector quantity. A force, which requires magnitude, direction and its line of action for its complete specification, is sometimes called a *localised vector.* It is not necessary to specify the line of action in the case of a couple. It is often convenient to denote a couple by means of a double-headed arrow, the length of which corresponds to the magnitude. In Figure 17.4, any of the five couples could be represented by the arrow at L or alternatively by the arrow at N as shown.

17.3 Resultant of two couples

Two couples may be replaced by a single couple which has the same statical effect as the two couples combined.

If the two couples lie in the same plane, or in parallel planes, then their magnitudes may simply be added algebraically, their directions being the same. This was the situation dealt with in two-dimensional systems (Chapter 4). If they do not lie in parallel planes (i.e. if they do not have the same direction) an expression for their resultant may be obtained from the work of earlier chapters, as follows.

Figure 17.5a shows a couple consisting of two forces P lying in a plane A. The moment of this couple is $M_A = Pe$. Another couple consists of two forces Q lying in plane B which is at an angle α to plane A. The moment of this couple is $M_B = Qf$. In order to combine them, the couple of moment M_A is replaced by a pair of forces R at distance a apart in the plane A, such that one of the forces R acts along the line of intersection of the two planes (line IJ in Figure 17.5a). The couple of moment M_B is similarly replaced by an equivalent couple Rb in plane B, arranged so that one of the forces R acts along IJ and opposes the force R from the couple M_A.

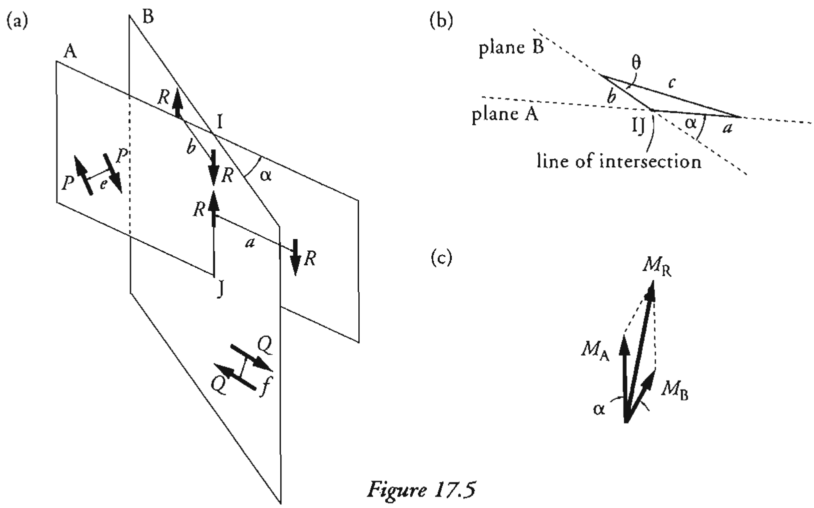


Figure 17.5

Of the four forces R, two cancel out and the remaining two constitute a couple lying in a plane oblique to A and B. To find the magnitude of this couple we need the distance c between the forces. Figure 17.5b is a view looking directly along the line of intersection of planes A and B (i.e. line IJ). It can be seen that:

$$c^2 = a^2 + b^2 + 2ab\cos\alpha \quad \text{or} \quad R^2c^2 = R^2a^2 + R^2b^2 + 2(Ra)(Rb)\cos\alpha$$

The resultant couple M_R is equal to Rc. Hence:

$$M_R = \sqrt{(M_A)^2 + (M_B)^2 + 2M_AM_B\cos\alpha} \tag{17.1}$$

It acts in a plane C which makes an angle θ to plane B, such that:

$$\tan\theta = \frac{a\sin\alpha}{b + a\cos\alpha} = \frac{M_A\sin\alpha}{M_B + M_A\cos\alpha} \tag{17.2}$$

These equations show that the two couples can be combined by vector addition. In Figure 17.5c, the couples M_A and M_B are represented by vectors normal to planes A and B respectively. The resultant couple is then obtained, both in magnitude and direction, by the Parallelogram Law.

17.4 Components of a couple

It follows from the work of the previous section that a single couple may be replaced by two component couples using the usual vector procedures.

Example 17.2

Figure 17.6 shows a plan view of a cantilever beam AB. At B it is acted upon by a couple of magnitude 40 kNm with an axis at 20° to the axis of the beam and in the horizontal xz plane. Find the twisting moment and the bending moment in the cantilever.

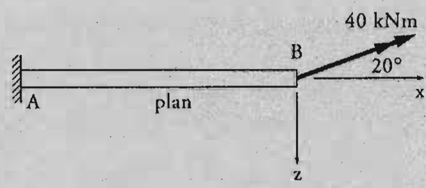


Figure 17.6

Solution

The couple is replaced by its components as shown in Figure 17.7a. The component couple along the beam is:

$$T = 40\cos 20 = 37.6 \text{ kNm}$$

This component causes twisting. It is illustrated in Figure 17.7b by a curly arrow acting around the member axis.

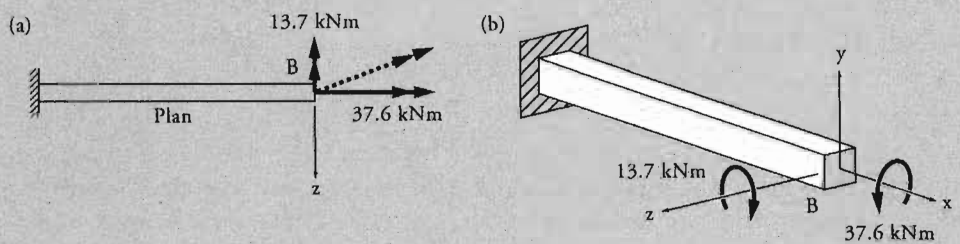


Figure 17.7

The component couple normal to the beam is:

$$M = 40 \sin 20 = 13.7 \text{ kNm}$$

This component causes bending. It is illustrated in Figure 17.7b by a curly arrow acting about the horizontal axis at B. With the usual directions of the axes, M is negative.

More generally, if a couple has a magnitude M and its axis has direction cosines l, m and n then it may be replaced by couples of magnitude lM, mM and nM acting around the axes Ox, Oy and Oz respectively.

17.5 Resultant of a number of couples

The resultant of a number of couples may be determined by first resolving each couple into its components around the axes x, y and z. If the resultant couple M_R, has components, M_{Rx}, M_{Ry} and M_{Rz} then:

$$M_{Rx} = \Sigma M_x \qquad M_{Ry} = \Sigma M_y \qquad M_{Rz} = \Sigma M_z$$

and finally:

$$(M_R)^2 = (M_{Rx})^2 + (M_{Ry})^2 + (M_{Rz})^2$$

EXAMPLE 17.3

The cube OABCDEFG is acted upon by three couples as shown in Figure 17.8a (over). Find the magnitude and direction of the resultant couple.

SOLUTION

Replace the curly arrows of Figure 17.8a by double-headed arrows (Figure 17.8b). The vector OF is now replaced by its x, y, z components. Since the figure is a cube the direction cosines of OF are all equal to $1/\sqrt{3}$, and the components of the 3 kNm couple are thus all equal to $\sqrt{3}$ kNm. Then:

$$M_{Rx} = \Sigma M_x = \sqrt{3} \text{ kNm}$$

$$M_{Ry} = \Sigma M_y = (\sqrt{3} - 2) \text{ kNm}$$

$$M_{Rz} = \Sigma M_z = (\sqrt{3} + 1) \text{ kNm}$$

$$M_R = \sqrt{3 + (\sqrt{3} - 2)^2 + (\sqrt{3} + 1)^2} = 3.25 \text{ kNm}$$

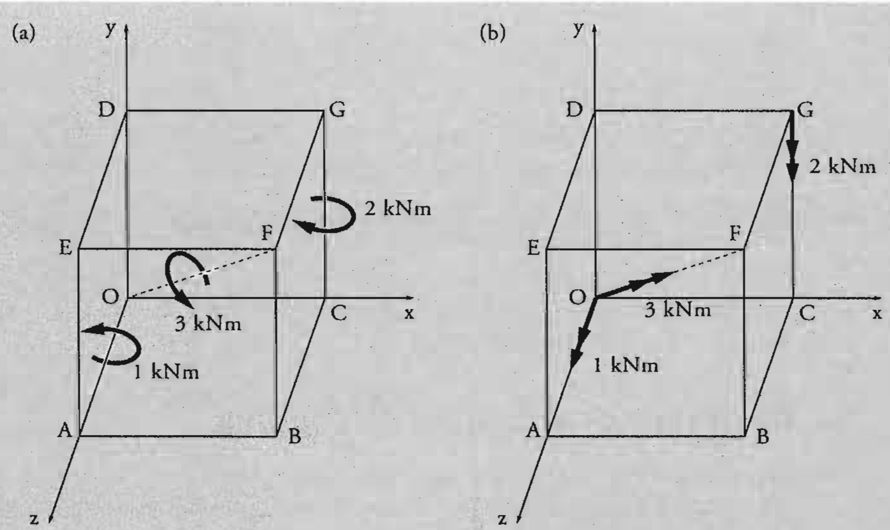


Figure 17.8

The direction cosines of M_R are given by:

$$l = \frac{\sqrt{3}}{3.25} = +0.533 \qquad m = \frac{\sqrt{3} - 2}{3.25} = -0.082 \qquad n = \frac{\sqrt{3} + 1}{3.25} = +0.841$$

17.6 The concept of a resultant

For a two-dimensional system of forces, lying in the xy plane and concurrent at the point A, the resultant is the single force which has the same x component as the combined forces and the same y component as the combined forces. The resultant also passes through the point A. In Chapter 15, we saw that this concept could readily be extended to the three-dimensional situation merely by requiring that the resultant also has the same z component as the combined forces. Again it passes through A.

In Section 4.2, the determination of the resultant of a two-dimensional system of non-concurrent forces, such as that shown in Figure 17.9, was discussed. The magnitude and direction of the resultant are found as for a concurrent system, but the position also needs to be established. Provided the resultant force is non-zero, it can be located so that its moment about any arbitrarily chosen point (say B in Figure 17.9) is the same as the combined moments of the force system. Fortunately, if R has the same moment about an axis through B as do the original forces, then its moment about an axis through any other point C will also be equal to that of the original forces. Of course, the various axes considered are all normal to the plane of the force system. However, if the resultant force is zero, it is necessary to express the resultant as a couple, unless the system is in equilibrium.

This concept of a single resultant force cannot be extended to the case of a three-dimensional system of non-concurrent forces except in special cases. In three dimensions it is possible to imagine the moment of a force about axes in various directions. We may find the magnitude and direction of R by the methods of Chapter 15. By moving R (but retaining the same direction) we may cause R to have the same moment as the original forces about any two axes (say Ox and Oy). But R is then completely defined, and if its moment about Oz is not equal to that of the original forces, then we cannot express the resultant as a single force. It must be accompanied by a suitable couple.

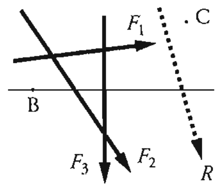


Figure 17.9

In other words, we can replace a given system of forces by a statically equivalent system, but this system cannot be a single force except in special cases.

17.7 Statically equivalent systems

In Section 4.5 we saw that a given system of forces in two-dimensions can be replaced by another system provided that the new system contains three undetermined quantities. These three quantities can be evaluated so that the new system is statically equivalent to the original system.

In a similar way, a three-dimensional system can be replaced by a statically equivalent system provided the new system contains *six* undetermined quantities. As in the two-dimensional case there are certain restrictions on the choice of the undetermined quantities.

We consider first the problem of replacing any given force by three component forces acting along specified axes Ox, Oy and Oz together with three component couples around these axes. Note that in the new system the directions and positions of six vectors (three forces and three couples) are specified, but six magnitudes are undetermined. These may then be determined to ensure equivalence.

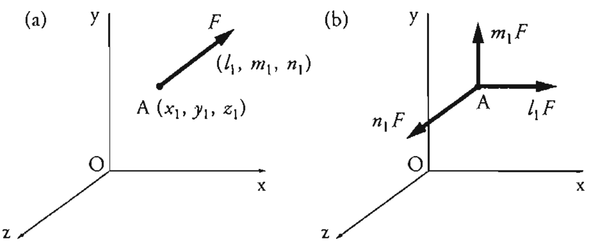


Figure 17.10

Let the given force F act through the point A (x_1, y_1, z_1) in a direction such that the direction cosines are (l_1, m_1, n_1) as shown in Figure 17.10a. The force is first replaced by its three orthogonal components at A. These are l_1F, m_1F and n_1F (Figure 17.10b). We now have to find a new system consisting of three forces X, Y and Z and three couples M_x, M_y and M_z at the origin, which is statically equivalent to the system of Figure 17.10b.

By resolving parallel to each axis in turn and equating the original and the new systems, we see that the force components of the new system are respectively equal to those at A:

$$X = l_1 F \qquad Y = m_1 F \qquad Z = n_1 F$$

To find the couples we must take moments about each axis in turn. The process of taking moments about the x axis is easier if we re-draw Figure 17.10b looking directly along the x axis from the positive x direction back towards the origin (Figure 17.11a).

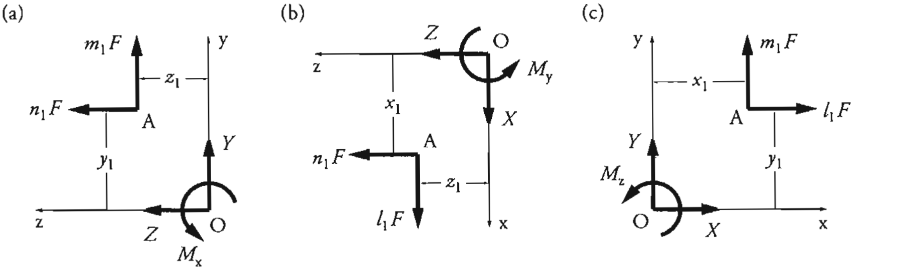


Figure 17.11

The component l_1F at A is parallel to the x axis and therefore has no moment about it. Hence the moment of the system at A about the x axis is $(n_1Fy_1 - m_1Fz_1)$. The moment of the new system at O is simply M_x. If the two systems are to be equivalent, then:

$$M_x = F(n_1y_1 - m_1z_1)$$

Similarly, Figure 17.11b is a view looking normal to the xz plane. Equating the moments of the old and new systems about the y axis we get:

$$M_y = F(l_1z_1 - n_1x_1)$$

Finally, Figure 17.11c is a view normal to the xy plane. Equating moments about the z axis we obtain:

$$M_z = F(m_1x_1 - l_1y_1)$$

Summarising these results, the three forces and three couples at the origin, expressed in terms of the magnitude direction and position of the original force, are:

$$X = l_1 F \qquad Y = m_1 F \qquad Z = n_1 F$$

$$M_x = F(n_1y_1 - m_1z_1) \quad M_y = F(l_1z_1 - n_1x_1) \quad M_z = F(m_1x_1 - l_1y_1) \qquad \textbf{(17.3)}$$

The given force F acting at A can be replaced by three forces acting at any point, not necessarily the origin, and three couples. Suppose it is required to replace F by components at a point B whose co-ordinates are (x_0, y_0, z_0). In Figure 17.11, it is only necessary to re-label point O as point B and to replace x_1, y_1 and z_1 by the distances $(x_1 - x_0)$, $(y_1 - y_0)$ and $(z_1 - z_0)$. Equations (17.3) then become:

$$X = l_1 F \qquad Y = m_1 F \qquad Z = n_1 F$$

$$M_x = F[n_1(y_1 - y_0) - m_1(z_1 - z_0)]$$

$$M_y = F[l_1(z_1 - z_0) - n_1(x_1 - x_0)]$$

$$M_z = F[m_1(x_1 - x_0) - l_1(y_1 - y_0)] \qquad \textbf{(17.4)}$$

The three component forces X, Y and Z at B can be re-combined into a single force at B which will obviously be equal to F. The three couples M_x, M_y and M_z at B can also be combined into a single couple, the axis of which will not, in general, be the same as the line of action of F. It has therefore been shown that a force can be replaced by a parallel force at any other point together with a couple. We note that this system again contains six quantities to be determined, namely a magnitude and two direction cosines to define the force, and a magnitude and two direction cosines to define the couple.

It is not necessary for the replacement system to comprise three concurrent component forces and three component couples. To be valid, the replacement system must contain six independent undetermined quantities. It must be capable of providing a force component in every direction and must have a moment about every axis. It follows that at least three of the unknown quantities must be forces. Indeed, all six may be forces, but in such a case care must be taken to see that the system does have a moment about *every* axis in space.

In general we can say that any two systems are statically equivalent if both:

1. in each of three directions (not all in the same plane), the sum of the components of one system is equal to the sum of the components of the other; and
2. about any three non-parallel axes (not all in the same plane) the sum of the moments of one system is equal to the sum of the moments of the other.

EXAMPLE 17.4

The rectangular box shown in Figure 17.12 is acted upon by a system of forces shown by full lines. Replace these forces by a statically equivalent system indicated by the dashed lines.

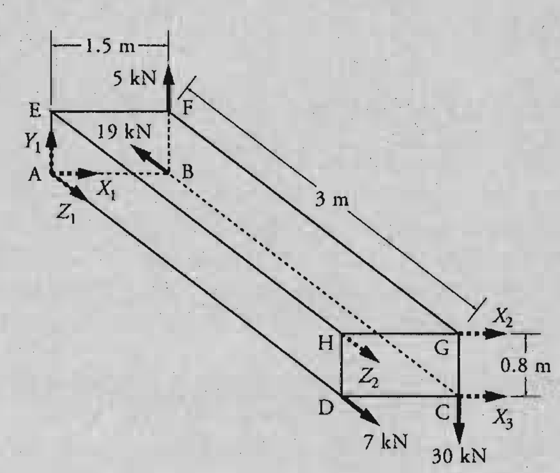


Figure 17.12

SOLUTION

As far as possible, it is desirable to write equations of equivalence such that each equation contains only one of the unknown values in the new system. This avoids the need to solve simultaneous equations. In this case, the force X_2 is the only member of the new system which has a moment about AD. All other forces either intersect this line or are parallel to it. Equating the moments of the new system and the original system about AD gives:

$$\Sigma M_{AD}: \ (X_2 \times 0.8) = (30 \times 1.5) - (5 \times 1.5) \qquad \therefore X_2 = 46.88 \text{ kN}$$

Summing forces in the y direction we obtain:

$$\Sigma Y: \quad Y_1 = 5 - 30 = -25 \text{ kN}$$

Proceeding in a similar manner:

$$\Sigma M_{DH}: \ -3X_1 = 19 \times 1.5 \qquad \therefore X_1 = -9.5 \text{ kN}$$

$$\Sigma X: \quad -9.5 + 46.88 + X_3 = 0 \qquad \therefore X_3 = -37.38 \text{ kN}$$

(The zero on the right-hand side of this equation occurs because the original force system has no x component.)

$$\Sigma M_{AB}: \ 0.8Z_2 = 30 \times 3 \qquad \therefore Z_2 = +112.5 \text{ kN}$$

$$\Sigma Z: \quad Z_1 + 112.5 = 7 - 19 \qquad \therefore Z_1 = -124.5 \text{ kN}$$

A check may be obtained by taking moments about axes other than those used above. In each case it will be found that the moment of the new system is equal to that of the original system.

In the foregoing problems, moments have always been taken about axes parallel either to Ox, Oy or Oz. In problems of this sort, even if the forces are oblique, it is appropriate to resolve them into their x, y, z components before taking moments.

17.8 Moment of a force about any axis

Occasionally it is necessary to take moments about an oblique axis. To deal with this problem a more general approach to moments is required. An expression for the moment of a force about *any* axis can be obtained from the results of Section 17.7. In Figure 17.13, the force F acts through the point A (x_1, y_1, z_1) and has direction cosines l_1, m_1 and n_1, while the axis S passes through B (x_0, y_0, z_0) and has direction cosines l_0, m_0 and n_0. The force and the axis do not intersect and are not parallel, so that F has a moment about S.

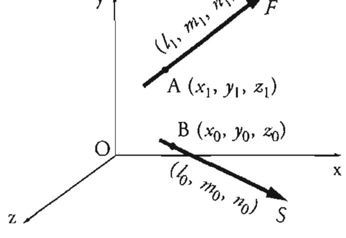


Figure 17.13

We first replace F by a parallel force through B together with three component couples. The new force has no moment about S, and the moment of the couples is readily obtained. From Equations 17.4, the couple about the x axis is:

$$M_x = F[n_1(y_1 - y_0) - m_1(z_1 - z_0)]$$

The component of this couple along axis S (i.e. the moment of this couple about S) is:

$$M_s = M_x l_0 = F[n_1(y_1 - y_0) - m_1(z_1 - z_0)]\, l_0$$

The components of the other couples M_y and M_z are obtained in a similar way, and upon summation we have:

$$M_s = F[n_1(y_1 - y_0) - m_1(z_1 - z_0)]\, l_0 + F[l_0(z_1 - z_0) - n_1(x_1 - x_0)]\, m_0 + F[m_1(x_1 - x_0) - l_1(y_1 - y_0)]\, n_0$$

This expression is conveniently given in the form of a determinant:

$$M_s = F \begin{vmatrix} (x_1 - x_0) & l_1 & l_0 \\ (y_1 - y_0) & m_1 & m_0 \\ (z_1 - z_0) & n_1 & n_0 \end{vmatrix} \qquad \textbf{(17.5)}$$

EXAMPLE 17.5

In Figure 17.14, the 10 kN force acts along the edge CG of the hexagonal pyramid ABCDEFG. Find the moment of this force about the axis FB.

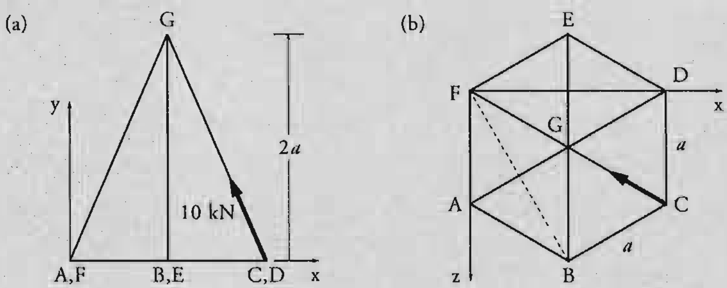


Figure 17.14

SOLUTION

Consider C as the point on the line of action of the force F (10 kN). The co-ordinates of C are:

$$x_1 = a\sqrt{3} \qquad y_1 = 0 \qquad z_1 = a$$

The direction cosines of CG are:

$$l_1 = \frac{x_G - x_C}{L} \qquad m_1 = \frac{y_G - y_C}{L} \qquad n_1 = \frac{z_G - z_C}{L}$$

where L is the length of CG and is given by:

$$L = \sqrt{(x_G - x_C)^2 + (y_G - y_C)^2 + (z_G - z_C)^2}$$

Now: $x_G - x_C = -\frac{\sqrt{3}}{2}a$ $\quad y_G - y_C = 2a$ $\quad z_G - z_C = -\frac{a}{2}$

Hence: $L = a\sqrt{5}$

and: $l_1 = -\frac{\sqrt{3}}{2\sqrt{5}}$ $\quad m_1 = \frac{2}{\sqrt{5}}$ $\quad n_1 = \frac{-1}{2\sqrt{5}}$

For the axis FB, we take F as the point on the axis, since it has co-ordinates (0, 0, 0). The direction cosines of FB are:

$$l_0 = \frac{1}{2} \qquad m_0 = 0 \qquad n_0 = \frac{\sqrt{3}}{2}$$

Then from Equation 17.5:

$$M = 10\begin{vmatrix} a\sqrt{3} & \frac{-\sqrt{3}}{2\sqrt{5}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{5}} & 0 \\ a & \frac{-1}{2\sqrt{5}} & \frac{\sqrt{3}}{2} \end{vmatrix} = 4a\sqrt{5}\ \text{kNm}$$

assuming that a is measured in metres.

Sometimes reference is made to the moment of a force F about a point O in space distant d from F. The moment M_O really means the moment of F about an axis through O normal to the plane containing F and the point O. This situation is the one considered in Chapter 4 and again in Figure 17.1a.

The moment of F about any oblique axis through O is necessarily less than M_O as shown in Figure 17.1b.

EXAMPLE 17.6

Figure 17.15 shows an axis OA through the origin O with direction cosines $l_0 = m_0 = n_0 = 1/\sqrt{3}$. A force of 10 kN acts along a line BC through B (5, 0, 6) with direction cosines $l_1 = 0.488$, $m_1 = 0.781$ and $n_1 = -0.390$.

(i) Find the moment of the force about point O (i.e. the moment about axis OD which is perpendicular to the plane containing the force and the point O).

(ii) Use this moment to find the moment of the force about axis OA.

(iii) Find the moment of the force about axis OA by direct calculation.

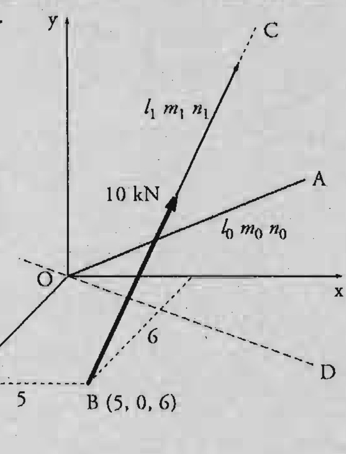


Figure 17.15

SOLUTION

(i) Replace the 10kN force along BC by a parallel force through O and a couple. The x, y and z components of the couple are given by Equation 17.3 in which the coordinates (x_1, y_1, z_1) are those of point B, namely (5, 0, 6). Then:

$$M_x = 10\,[(-0.390 \times 0) - (0.781 \times 6)] = -46.86 \text{ kNm}$$

$$M_y = 10\,[(0.488 \times 6) - (-0.390 \times 5)] = +48.78 \text{ kNm}$$

$$M_z = 10\,[(0.781 \times 5) - (0.488 \times 0)] = +39.05 \text{ kNm}$$

The resultant of these components is:

$$M_{OD} = \sqrt{(-46.86)^2 + (48.78)^2 + (39.05)^2} = 78.10 \text{ kNm}$$

This is often referred to as M_O, the moment of F about O.

(ii) The direction of cosines of M_O (and hence of axis OD) are:

$$l_2 = \frac{-46.86}{78.10} = -0.60 \qquad m_2 = \frac{48.78}{78.10} = 0.625 \qquad n_2 = \frac{39.05}{78.10} = 0.50$$

If α is the angle between OD and OA, then:

$$\cos\alpha = (l_2 l_0) + (m_2 m_0) + (n_2 n_0)$$

$$= (-0.60/\sqrt{3}) + (0.625/\sqrt{3}) + (0.5/\sqrt{3})$$

$$= 0.303$$

The moment of F about axis OA is:

$$M_{OA} = M_{OD}\cos\alpha = 78.10 \times 0.303 = 23.67 \text{ kNm}$$

(iii) M_{OA} may be calculated directly from Equation 17.5:

$$M_{OA} = 10 \begin{vmatrix} 5 & 0.488 & 1/\sqrt{3} \\ 0 & 0.781 & 1/\sqrt{3} \\ 6 & -0.390 & 1/\sqrt{3} \end{vmatrix} = 23.67 \text{ kNm}$$

Problems

17.1 The two couples M_x and M_y shown in Figure P17.1 act on the perpendicular faces of a body as shown. What is the magnitude (moment) and direction of the resultant couple?

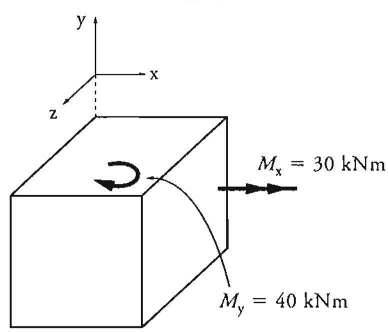


Figure P17.1

17.2 What are the components of the couple shown in Figure P17.2 about the x, y and z axes? What is the moment of the couple about the axis AB?

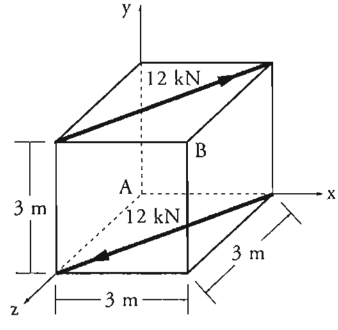


Figure P17.2

17.3 The forces AB and CD shown in Figure P17.3 form a couple. Resolve each force into three components in the directions of the co-ordinate axes, and by taking moments about these axes, determine the three components of the couple. What is the moment of the couple? Use this result to find the distance between AB and CD.

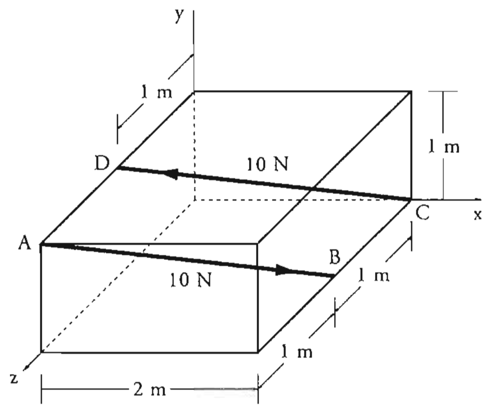


Figure P17.3

17.4 Couples are applied about the axes OA and OB as shown in Figure P17.4. Determine the resultant couple. What is the moment of this couple about the axis OC?

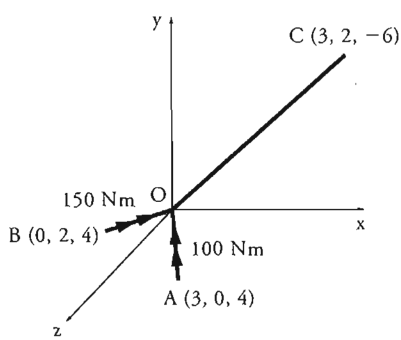


Figure P17.4

17.5 The solid shown in Figure P17.5 is a 1 m cube. Find the resultant of the three couples shown. What is the moment of these couples about the axis AB? What is the moment about the axis through O parallel to AB?

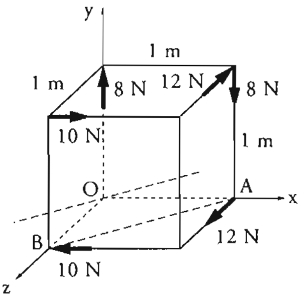


Figure P17.5

17.6 Replace the forces shown in Figure P17.6 by:
(i) a force passing through the centre of the cube plus a couple
(ii) two forces, one passing through the centre of the cube, and one lying in the face BCDE.
In each case state the forces and couple in terms of their x, y and z components.

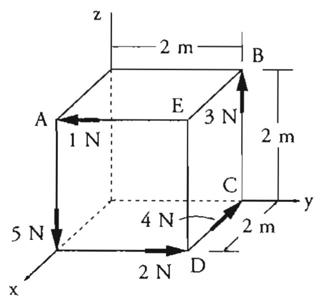


Figure P17.6

17.7 A 1 m cube is acted upon by the five forces shown in Figure P17.7. Replace the forces:
(i) by a force through A, and a couple
(ii) by a force through F, and a couple.

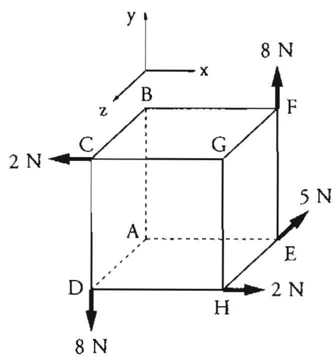


Figure P17.7

17.8 (i) Replace the forces shown in Figure P17.8 by a force through the origin O together with a couple.
(ii) Replace the forces by a single force and a couple acting about an axis parallel to the direction of the force. Define the line of action of the force and the value of the couple.

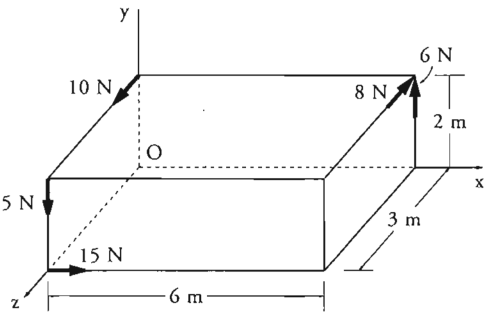


Figure P17.8

17.9 In general it is not possible to replace a given system of forces by the system of forces shown as dashed arrows in Figure P17.9. Why is this?

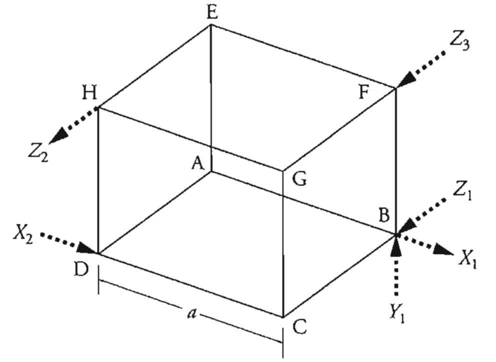


Figure P17.9

17.10 An equilateral tetrahedron ABCD with edges 3 m in length is oriented so that ABC is in the xz plane with A at the origin. Figure P17.10 shows a system of forces expressed in terms of forces along, and couples around, the x, y and z axes. Replace this system by a statically equivalent system comprising forces along each of the sides of the tetrahedron.

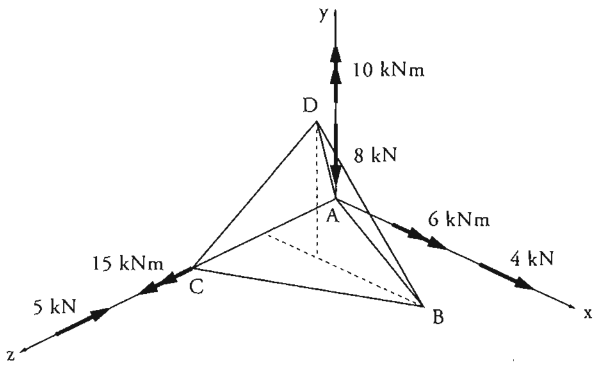


Figure P17.10

17.11 What is the moment of the 150 N force shown in Figure P17.11:
(i) about the axis AB?
(ii) about the axis CD?

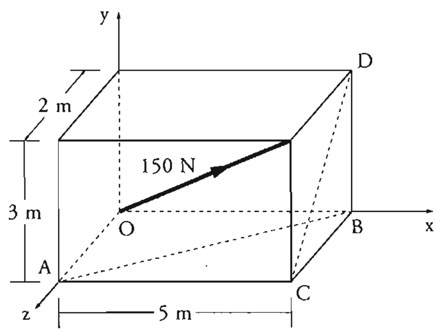


Figure P17.11

17.12 An equilateral tetrahedron ABCD with 100 mm edges is acted upon by a force of 20 N from C to D. What is the moment of this force about AB?

17.13 In the cranked member ABCDE shown in Figure P17.13, AB is parallel to the x axis, BC is parallel to the z axis, CD is parallel to the y axis, and DE is parallel to the x axis. Replace the forces shown by a force and a couple acting on the support at A.

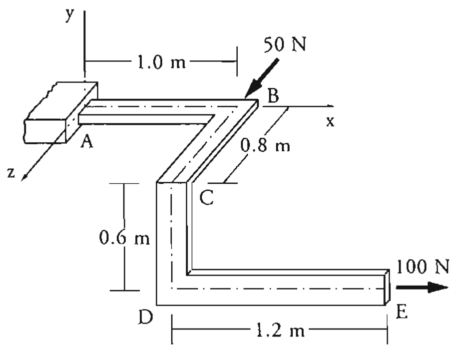


Figure P17.13

17.14 The cube shown in Figure P17.14 has 2 m edges. It is acted upon by the forces and couples shown. Replace the system by a statically equivalent system comprising forces along the x, y and z axes, and couples around these axes.

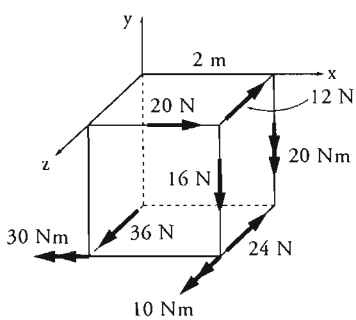


Figure P17.14

17.15 The cantilever ABC (Figure P17.15) lies in the horizontal plane, and the angle ABC is 140°. The load system at C consists of a vertical force of 2 kN, a force of 4 kN in the horizontal plane and normal to BC, and a couple of 22 kNm whose axis coincides with the 4 kN force. Find the moment of the force system at C about axes normal to the bar at D and E respectively. These axes also lie in the horizontal plane and D and E are the mid-points of BC and AB respectively.

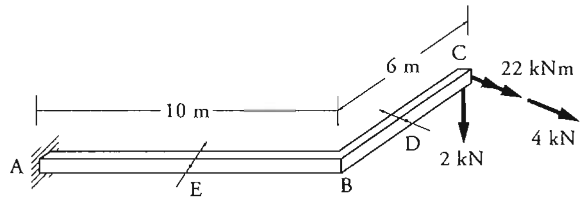


Figure P17.15

17.16 Figure P17.16 shows a bar ACC_1B bent into the quadrant of a circle centre O and radius 6 m, and lying in the horizontal plane. It is cantilevered from A and is free at B. At B it is subjected to a vertical 10 N force and a couple of 40 Nm whose axis is tangential to the circle at B. At C (the angle BOC is 60°) the axis CD is radial, CE is tangential and CF is vertical.

(i) Find the moments of the force system at B, about the axes CD, CE and CF.

(ii) Find the moments of the system about the corresponding three axes at C_1 where the angle BOC_1 is θ.

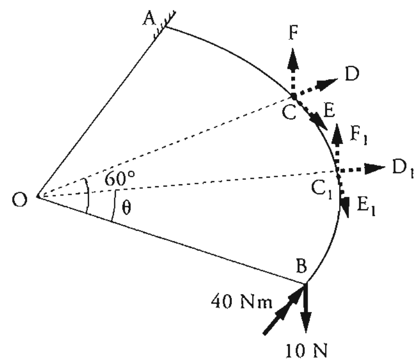


Figure P17.16

17.17 A force F of 50 N acts through the point (2, 8, 4) in a direction for which $l = 0.4$ and $m = 0.5$, the z component of F being positive. An axis S passes through the point (1, 5, 0) and its direction cosines are $l = -0.2$ and $n = 0.6$, and m is positive. If the co-ordinates are given in metres, find:

(i) the moment of F about S;

(ii) the component of F in the direction of S.

17.18 Find the forces in the struts OA, OB and OC when a force $F = 15$ kN acts at O. (The co-ordinates of the points A, B and C are as shown in Figure P17.18, lengths are in metres.) What is the moment of the force F about a line through A and C? Compare this moment with the moment of the force acting from O to B about the same line AC.

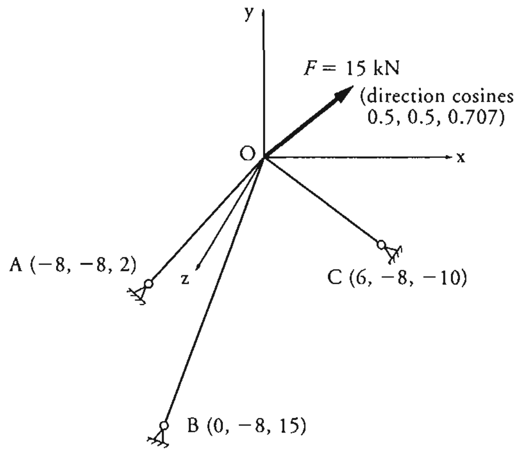
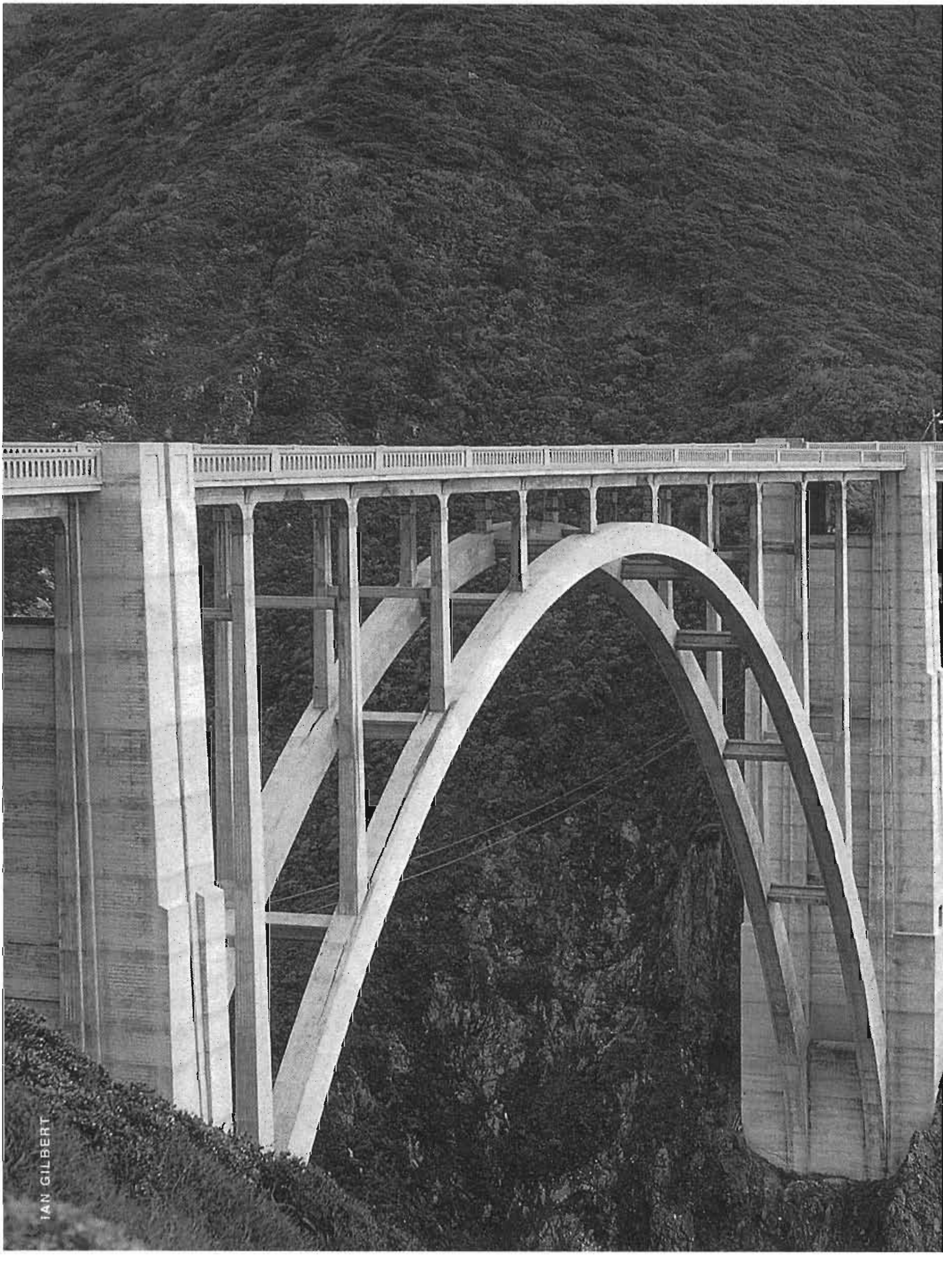


Figure P17.18

17.19* A force of 30 kN acts through a point whose co-ordinates are x = 2, y = 3, z = 4 (lengths are in metres) and in a direction given by the direction cosines $l = 0.4$, $m = 0.4$ and $n = 0.825$. Find the moment of the force about the origin of co-ordinates (i.e. about the axis through O normal to the plane containing the force and the origin).

* Difficult problem, suitable for later study.



IAN GILBERT

Equilibrium of Non-concurrent Forces

18.1 Conditions of equilibrium

Any system of forces may be replaced by a single force through a given point, together with a couple. The single force has components in three mutually perpendicular directions which are equal to the sums of the components of the separate forces in these directions. The couple has components about the three given axes which are equal to the sums of the moments of all the forces in the system about these axes.

A body is in equilibrium only if the forces acting upon it have no resultant force and no resultant couple. For the resultant force to be zero, the sum of the components in each of three mutually perpendicular directions must be zero.

$$\Sigma X = 0 \tag{18.1}$$

$$\Sigma Y = 0 \tag{18.2}$$

$$\Sigma Z = 0 \tag{18.3}$$

In order that the resultant couple should be zero, the sum of the moments of all of the forces about each of three mutually perpendicular axes must also be zero.

$$\Sigma M_x = 0 \tag{18.4}$$

$$\Sigma M_y = 0 \tag{18.5}$$

$$\Sigma M_z = 0 \tag{18.6}$$

These six equations are the general conditions of equilibrium of forces in space. In particular problems a judicious choice of axes about which to take moments will often shorten the solution. For instance any particular force is eliminated from the moment calculation if moments are taken about an axis intersecting this force or an axis parallel to it. This is because a force has no moment about an axis if it either intersects the axis or is parallel to it.

In the solution of problems, it is often convenient to consider what motion will be permitted if a certain reaction is removed. This reveals what equilibrium equation can be used to evaluate this reaction. For instance, if removal of a given reaction would leave the body free to rotate about the y axis, then an equation of moments about the y axis will enable that reaction to be calculated.

EXAMPLE 18.1

Figure 18.1 shows two floor beams. The first is supported at A, where three components of reaction are available, and frames into the second beam at C. The second beam is supported at D and E and is cantilevered beyond E to F. Two reactions are available at D and one at E. Calculate the six force reactions R_1 to R_6 for the loading shown.

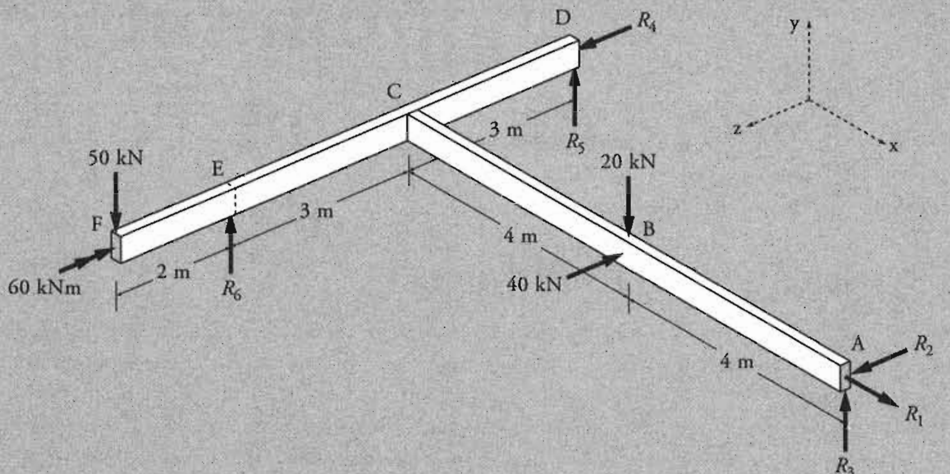


Figure 18.1

SOLUTION

For convenience of reference, three axes are shown in Figure 18.1 The signs of all forces and moments will be taken with respect to these axes.

Summing forces in the x direction we have:

$$\Sigma X = 0: \qquad \therefore R_1 = 0$$

since no external load has a component in this direction.

If we remove the reaction R_3 the whole structure will rotate about FD. Hence, the reaction R_3 may be found by taking moments about FD.

$$-60 - (20 \times 4) + (R_3 \times 8) = 0 \qquad \therefore R_3 = +17.5 \text{ kN}$$

We find R_2 by taking moments about a vertical axis through E:

$$(40 \times 4) - (R_2 \times 8) = 0 \qquad \therefore R_2 = +20 \text{ kN}$$

We can now evaluate R_4 from $\Sigma Z = 0$:

$$-40 + R_2 + R_4 = 0 \qquad \therefore R_4 = +20 \text{ kN}$$

It is not easy to see how R_5 and R_6 can be evaluated separately. If we remove R_5 we see that the structure will rotate about the line AE. This means that R_5 could be found by taking moments about AE, but this equation would be tedious to write. In a similar manner we discover that R_6 could be obtained by an equation of moments about AD. Rather than use these equations we shall use two simultaneous equations for R_5 and R_6.

An equation of moments about AC gives:

$$(50 \times 5) - (R_6 \times 3) + (R_5 \times 3) = 0 \quad \textbf{(18.7)}$$

The equation $\sum Y = 0$ leads to:

$$17.5 - 20 - 50 + R_5 + R_6 = 0 \quad \textbf{(18.8)}$$

Solving these two equations then gives:

$R_5 = -15.4$ kN $\qquad R_6 = +67.9$ kN

The reactions to the structure of Example 18.1 were all parallel to axes Ox, Oy and Oz. Many engineering problems are of this nature, but others involve oblique reactions. In such cases it is often convenient to resolve such reactions into their orthogonal components.

EXAMPLE 18.2

The pole AB of Figure 18.2 is hinged to the wall at B and supported by ties AC and ED, C and D being in the same vertical plane as B. The weight of the pole is 600 N. The 1000 N force acts at right angles to the pole at F and at 30° to the vertical as shown. Find the forces in the ties.

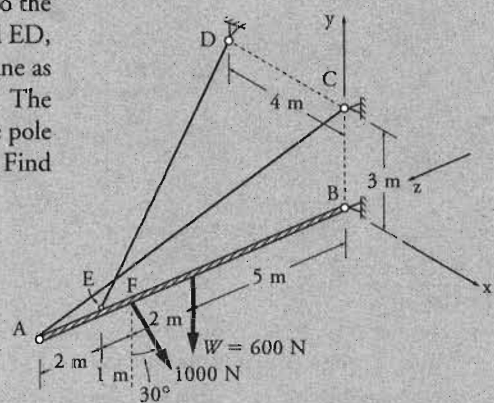


Figure 18.2

SOLUTION

The forces F_{AC} and F_{ED} will be replaced by their components. We may either consider the pole alone as a freebody, in which case the components of F_{AC} and F_{ED} will act at A and E, or we may consider the pole and the ties as a freebody and determine the reactions at C and D. It makes little difference. The former method will be used.

The length of AC: $\qquad L_{AC} = \sqrt{10^2 + 3^2} = 10.44$ m

Since AC is in tension, the force F_{AC} acts from A towards C and its direction cosines are therefore:

$$l = 0 \qquad m = \frac{+3}{10.44} = +0.287 \qquad n = \frac{-10}{10.44} = -0.958$$

The length of ED and the direction cosines of F_{ED} are found in a similar way:

$$L_{ED} = 9.434 \text{ m}$$

$$l = \frac{-4}{9.434} = -0.424 \qquad m = \frac{+3}{9.434} = +0.318 \qquad n = \frac{-8}{9.434} = -0.848$$

Figure 18.3 shows a freebody diagram of the pole with all forces expressed in terms of their orthogonal components.

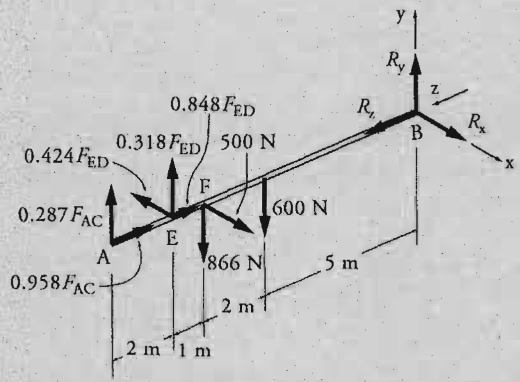


Figure 18.3

Taking moments about axis By we have:

$$-(0.424F_{ED} \times 8) + (500 \times 7) = 0 \qquad \therefore\ F_{ED} = 1032\ \text{N}$$

Taking moments about axis Bx we have:

$$-(0.287F_{AC} \times 10) - (0.318 \times 1032 \times 8) + (866 \times 7) + (600 \times 5) = 0$$

$$\therefore\ F_{AC} = 2243\ \text{N}$$

(If required the reactions at B could now be obtained from the equations $\sum X = 0$, $\sum Y = 0$ and $\sum Z = 0$.)

It will be noted that there are only five reactions to this pole, namely three components at B and the tensions in the two ties. These happen to be sufficient to equilibrate the particular forces given in this problem. Evidently, since there are less than six reaction components, they will not equilibrate all applied forces even if the ties AC and ED were capable of resisting compression. It is left to the teader to note what type of force could not be reacted by the system of support shown.

EXAMPLE 18.3

The shaft AB (Figure 18.4a) transmits power from a belt-driven pulley C of 500 mm diameter, to a gear of 160 mm pitch circle diametet at D. The belt forces are 3000 N and 1800 N. The force at the gear wheel acts tangentially to the pitch circle in a direction at 30° to the horizontal. An end view of the assembly is shown in Figure 18.4b. If friction in the bearings at A and B is neglected, find the reactions at A and B in the x and y directions.

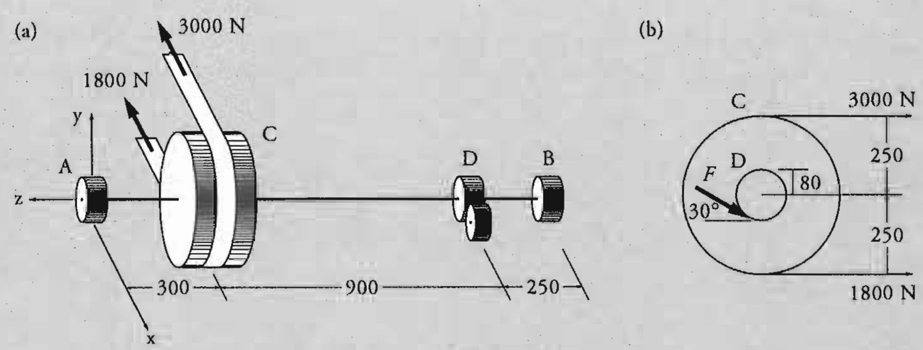


Figure 18.4

SOLUTION

Take moments about the axis of the shaft:

$$(80 \times F) - (3000 - 1800) \times 250 = 0 \qquad \therefore F = 3750 \text{ N}$$

This force may be resolved into x and y components of 3248 N and −1875 N respectively. Taking moments about the y axis at A gives:

$$-(B_x \times 1450) + (3248 \times 1200) + (4800 \times 300) = 0 \qquad \therefore B_x = 3681 \text{ N}$$

$\Sigma X = 0$ then gives:

$$3681 - 3248 - 4800 + A_x = 0 \qquad \therefore A_x = 4367 \text{ N}$$

By taking moments about the x axis at A, we have:

$$(B_y \times 1450) - (1875 \times 1200) = 0 \qquad \therefore B_y = 1552 \text{ N}$$

$\Sigma Y = 0$ then gives:

$$1552 - 1875 + A_y = 0 \qquad \therefore A_y = 323 \text{ N}$$

Problems

18.1 The bar ABC in Figure P18.1 lies in the horizontal plane and the angle ABC is 120°. The support at A can supply reaction components A_1 vertically, A_2 in the direction AB and A_3 normal to A_1 and A_2. The support at B can supply a reaction component B_1 vertically. The support at C can supply a reaction component C_1 vertically and C_2 in the direction of BC. Find the reactions due to the force system shown, where the forces of 5 kN and 16 kN are vertical, and the 12 kN force is horizontal and normal to AB.

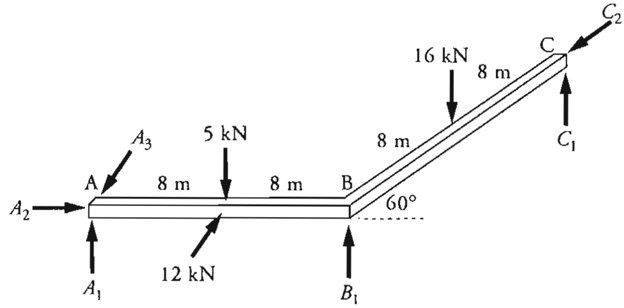


Figure P18.1

18.2 A horizontal rectangular platform weighing 400 N is supported on hinges at A and B and by a chain CD. What are the reactions at the hinges and the tension in the chain when a load of 200 N is located as shown? Assume that the hinges cannot exert reactions in the x direction.

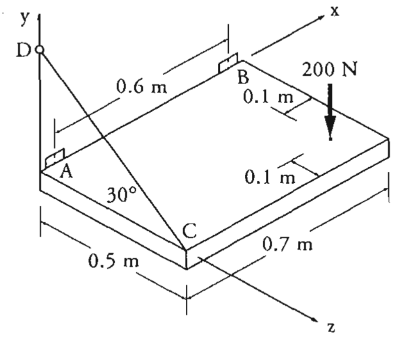


Figure P18.2

18.3 A heavy uniform circular plate of weight W is supported in a horizontal position on three steel spheres A, B and C placed under the circumference of the plate. If the arc AB subtends an angle of 60° at the centre of the plate, and AC subtends an angle of 150°, find the load on each sphere.

18.4 The tripod ABCD in Figure P18.4 rests on horizontal ground. Reaction components can be exerted at A in the x and y directions, at B in the y and z directions, and at C in the x and y directions. ABC is an equilateral triangle of sides 6 m. Bars AD, BD and CD are equal in length and D is 10 m above the ground. Find the reactions due to a 60 N force at D, acting in a horizontal plane and in a direction making 30° with the x direction.

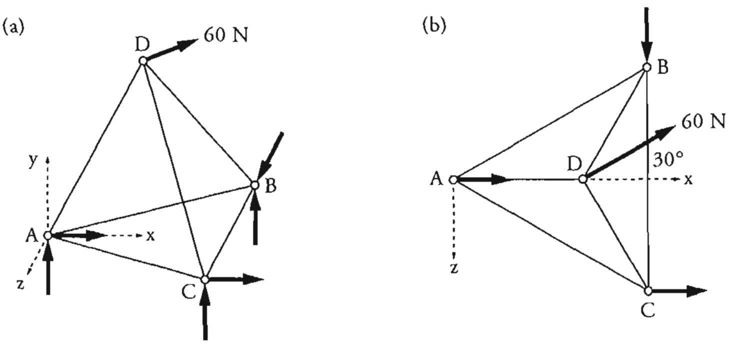


Figure P18.4

18.5 The bent bar ABCDE in Figure P18.5 lies in a vertical plane. The support at A can exert a reaction in any direction; the support at E cannot exert a reaction in the direction of DE, and the support ar C can exert a reaction only normal to the plane of the bar. The uniformly distributed load acts vertically, as does the load at B. The 30 kN force at D and the reactions R_3, R_4, and R_6 all act normal to the plane of the bar. Find the reaction components.

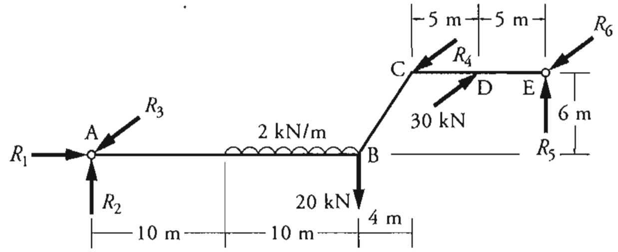


Figure P18.5

18.6 Figure P18.6 shows a triangular table. Determine the forces in the three legs A, B and C of the table if a weight of 9 kN is placed on it as shown in Figure P18.6.

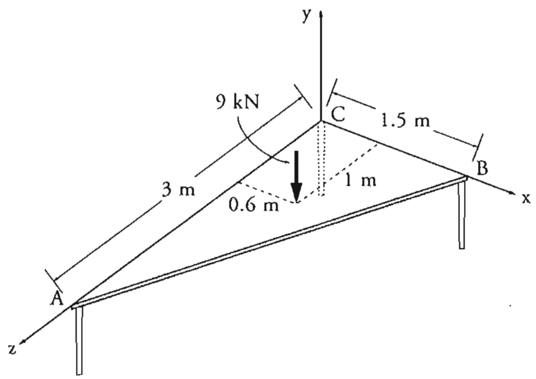


Figure P18.6

18.7 The beam ABC shown in Figure P18.7 has a right-angled bend at B and lies in the horizontal plane. AB = BC = 4 m. The support at C can exert a reaction in any direction. The support at B cannot exert a reaction in the direction BC, and the support at A can exert a reaction in the vertical direction only. Loads are applied as shown at the mid-points of AB and BC. The 6 kN and 9 kN loads and the reactions R_3 and R_5 are all horizontal and normal to the beam. Find the six reaction components.

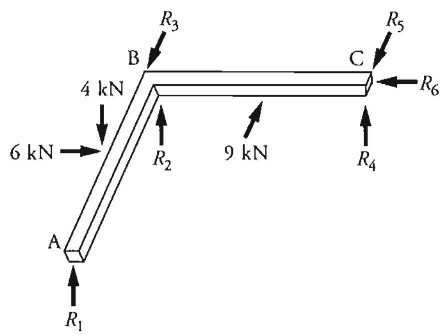


Figure P18.7

18.8 The beam ABCDEF shown in Figure P18.8 lies in a horizontal plane. All joints are 90°. The length of each portion is 3 m. There is a ball joint at F, providing reactions R_1, R_2 and R_3. Vertical reactions are also provided at A and C, and a horizontal reaction at A. A vertical load of 16 kN and a horizontal load of 12 kN are applied at D. Determine the six reactions.

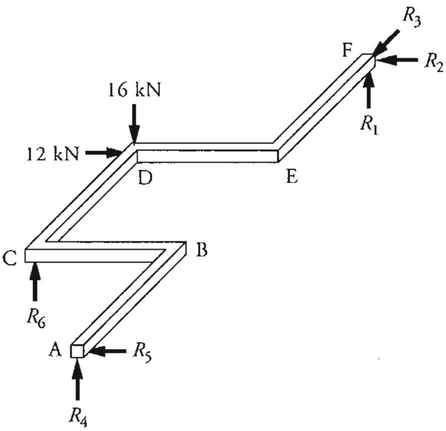


Figure P18.8

18.9 The body ABCDEFGH in Figure P18.9 is supported by cables or struts which provide reactions R_1 to R_6 acting along lines AE, DC, CG, BC, DH and EF respectively. The body is acted upon by a force of 490 N acting along the line DF and one of 200 N acting along FB. Find the six reactions.

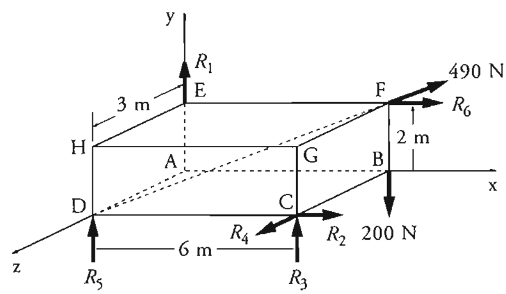


Figure P18.9

18.10 Figure P18.10 shows a forked cantilever lying in the horizontal plane. Calculate the reactions at A due to the applied forces at C and D.

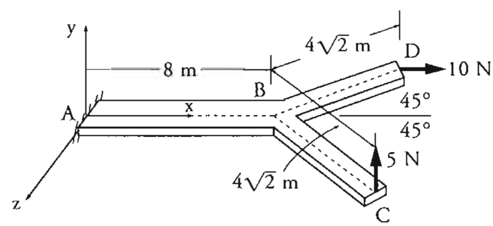


Figure P18.10

18.11 Figure P18.11 shows a bent cantilever lying in the horizontal plane. Find the six reactions at A due to the applied forces at C and D.

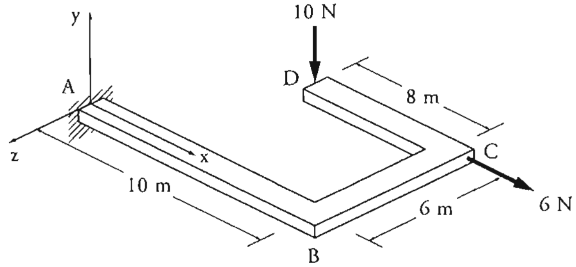


Figure P18.11

18.12 Figure P18.12 shows a vertical cantilever DB to the top of which are attached horizontal arms AB and BC. Forces of 5 kN (in the direction C to B) and 20 kN (vertical) and also a couple of 30 kNm (about a horizontal axis normal to CB) are applied at C. The arm AB carries a distributed loading of 5 kN/m which acts vertically downward. Calculate the six reacrions at D due to these loads. The arm AB is parallel to the x axis and the arm BC is parallel to the z axis.

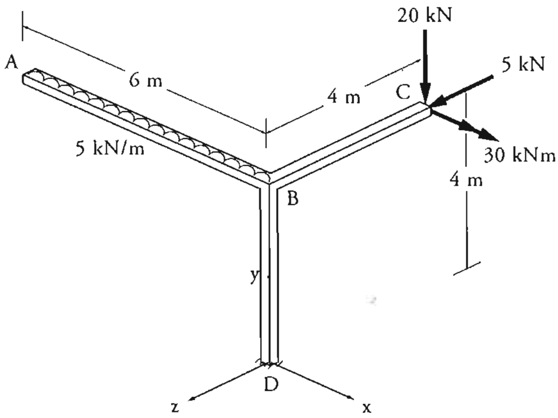


Figure P18.12

18.13 Figure P18.13 shows the freebody of that part of a bent beam which lies to the right of a section A. Find the stress resultants (or internal forces) at the cross-section A. The positive directions of these forces are as indicated in the figure.

18.14

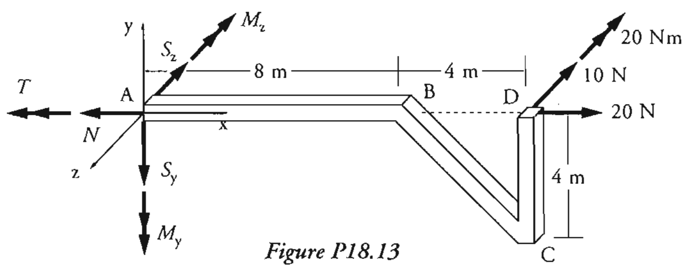


Figure P18.13

The freebody BCDE shown in Figure P18.14 is isolated from a bent beam by a cut at section B. DE is horizontal, CD is vertical, BC is horizontal and at right angles to the plane CDE. Find the three force components N, S_y and S_z at B and the three component couples T, M_y and M_z, if the freebody is in equilibrium. (N is in the direction CB, S_y is parallel to CD and S_z is parallel to ED.)

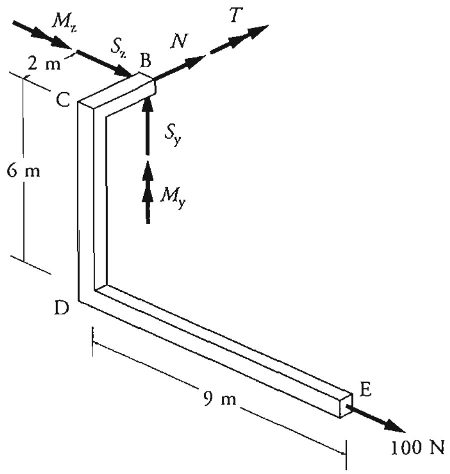


Figure P18.14

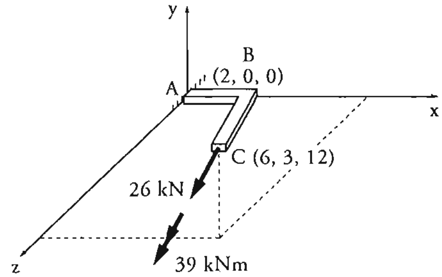


Figure P18.15

18.15 A bent cantilever ABC is located in space as shown in Figure P18.15. An axial force of +26 kN is applied at C in the direction of BC. A couple of +39 kNm is also applied at C about the longitudinal axis of BC. Find the equilibrating reactions at A.

18.16 Figure P18.16 shows a cantilever ABCD, of which ABC lies in the horizontal plane and CD is vertical. The part AB lies along the x axis and BC is a quadrant of a circle of radius r. A horizontal force P is applied at D in the x direction, together with a vertical force Q, as shown. Find the six reactions at A due to the two loads applied at D.

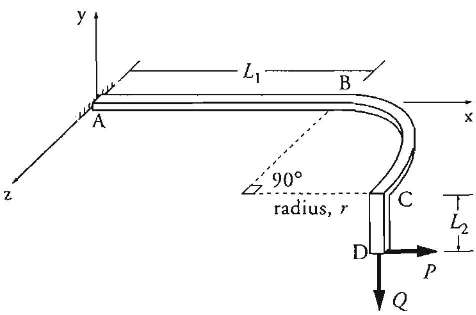


Figure P18.16

18.17 The cranked beam ABCDEF in Figure P18.17 lies in the xz plane. The supports are such that support A can only resist X, Y and M_x components, while support F can only resist Y, Z and M_y components. What are the values of the six reactions due to the three loads shown.

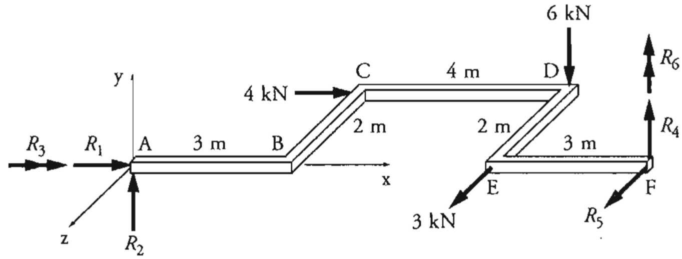


Figure P18.17

18.18 A door of uniform thickness and of weight 300 N measures 2.5 m × 1 m. The door is supported on hinges A and B, A being 0.25 m from the bottom of the door and capable of supplying horizontal and vertical reactions. The hinge B is 0.25 m from the top of the door and cannot supply any vertical support. When the door is erected, the hinge B is not placed vertically above A, so that when the door is closed the top corner C is 20 mm out of the vertical through corner D. What force, acting perpendicular to the door at the mid-height E of the front edge, is required to hold the door open after it has turned through 45° from the closed position?

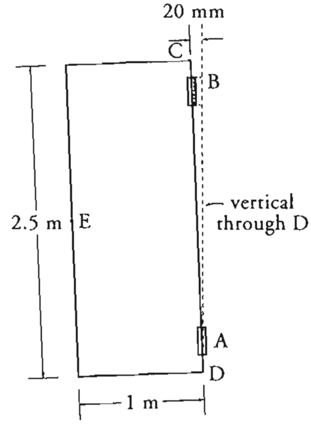


Figure P18.18

18.19 A steel plate ABCD is 1.5 m square and weighs 2 kN. It is suspended by three chains from hooks A′, B′and M′in the walls of a room which is 4 m square. Hooks A′and B′are in adjacent corners of the room and are nearest to corners A and B respectively of the plate. Hook M′is in the middle of the wall opposite A′B′, and is opposite the point M in the plate which is the mid-point of CD. The hooks are all at the same height above the floor. Chains connect A, B and M with A′, B′and M′respectively. Chains AA′ and BB′ are each 2 m long, while MM′ is 2.5 m long. Find the tension in each chain. (Students who are interested in computational methods may like to solve the problem when chain AA′ is increased in length to 2.5 m.)

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Geometrical Properties of Plane Figures

Many problems in engineering and physics require a knowledge of certain geometrical properties of plane figures. In this book, it is appropriate to consider the problem of finding the resultant of a distributed load acting on a plane surface. The results will then be applicable to problems involving fluid pressure on submerged surfaces. They will also be applicable to problems involving stresses, and to more abstract problems.

A.1 Area – First Moment of Area – Centroid

We consider first the case of a load of uniform intensity w (N/mm^2) acting on a plane figure (Figure A.1).

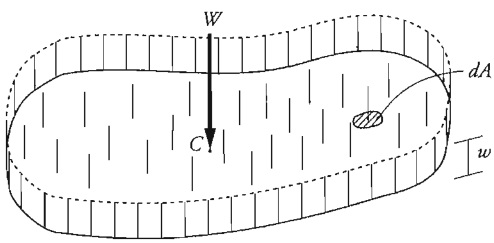


Figure A.1

The load acting on an elemental area dA is:

$$dW = wdA \tag{A.1}$$

and the total load is:

$$W = \int wdA = w\int dA = wA \tag{A.2}$$

where A is the area of the figure. For figures of simple shape, such as those occurring in engineering, the value of A is known from earlier studies. Only for unusual shapes is integration required. W is the resultant of the parallel forces acting on the elemental areas. It may be thought of as the volume of the *pressure block* (Figure A.1) of area A and thickness w.

The position at which the resultant acts is called the *centroid* and is denoted by C in Figure A.2 with co-ordinates x_C, y_C' with respect to arbitrary axes Ox' and Oy'. To evaluate y_C', the sum of the moments of the elemental forces about Ox' is equated to the moment of the resultant *W*. In the same way, x_C' is found by taking moments about Oy'

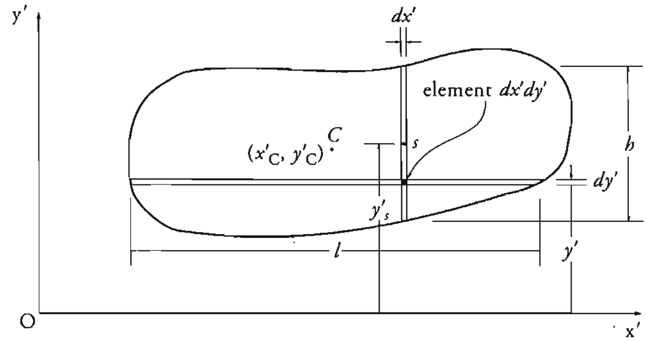


Figure A.2

Consider first the evaluation of y_C'. The force on the element of area $dx'dy'$ at (x', y') is $wdx'dy'$ and its moment about Ox' is $(wdx'dy')y'$. The total moment of the elemental forces is:

$$M = w\int_{y'}\int_{x'} y'dx'dy' \tag{A.3}$$

The term $\int_{y'}\int_{x'} y'dx'dy'$ is a geometrical property of the figure. It is called the *first moment of area* about Ox' and is denoted by $Q_{x'}$. The total moment may then be written as $wQ_{x'}$. The moment of the resultant is wAy'_C. Equating these, we have:

$$wAy'_C = wQ_{x'}$$

hence: $$y'_C = \frac{Q_{x'}}{A} \tag{A.4}$$

The evaluation of $Q_{x'}$ involves the double integration of y' with respect to x' and y'. In practice, one integration is avoided by considering elemental strips rather than elements of area $dx'dy'$.

We may take a strip parallel to the axis Ox' of length l and width dy' (Figure A.2). Since the force on the element is uniform the resultant is clearly $w(ldy')$ and acts at the centre of the strip. The moment of this force about Ox' is:

$$dM = w\,(ldy')y'$$

and the total moment is:

$$M = w\int_{y} ly'dy' = wQ_{x'}$$

In effect, the adoption of the strip instead of the element $dx'dy'$ is equivalent to having integrated with respect to x'. When l has been expressed in terms of y', the value of $Q_{x'}$ is found from:

$$Q_{x'} = \int_{y'} ly'dy' \tag{A.5}$$

Alternatively, we may take the elemental strips parallel to Oy' (Figure A.2). If the length of the strip is h and its centre is y_s' from Ox', then the moment about Ox' of the force on the strip is:

$$dM = w\,(h\,dx')y_s'$$

and the total moment is:

$$M = w\int_{x'} hy_s'dx' = wQ_{x'}$$

When h and y_s' have been expressed in terms of x' then:

$$Q_x' = \int_{x'} hy_s'dx' \quad \textbf{(A.6)}$$

Similarly, by taking moments about Oy' we find that:

$$x'_C = \frac{Q_{y'}}{A} \quad \textbf{(A.7)}$$

EXAMPLE A.1

Find the distance of the centroid of the triangle ABD of Figure A.3 from the base AB.

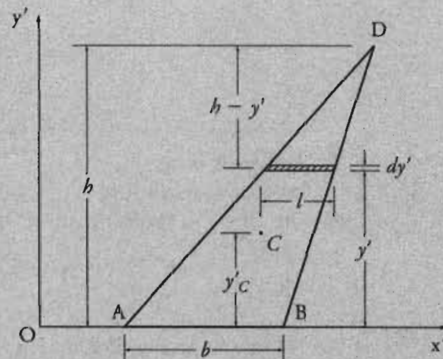


Figure A.3

SOLUTION

Choose the base of the triangle AB as axis Ox'. The length of an element of area distant y' from Ox' is:

$$l = b\left(\frac{h-y'}{h}\right) = b\left(1-\frac{y'}{h}\right)$$

From Equation A.5:

$$Q_{x'} = \int_{y'} ly'dy'$$

$$= \int_0^h b\left(1-\frac{y'}{h}\right)y'dy'$$

$$= \frac{bh^2}{6}$$

We know that the area of the triangle $A = \frac{bh}{2}$, and from Equation A.4:

$$y'_C = \frac{Q_{x'}}{A} = \frac{h}{3}$$

It is usual to use the symbols Cx, Cy for axes with origin at the centroid. The initial convenient axes are therefore called Ox′ and Oy′.

If a plane figure has an axis of symmetry, the centroid must be on that axis. In the plane figure shown in Figure A.4, if the area is divided into elements at right angles to the axis of symmetry, every element is centred on that axis, and the first moment about the axis is zero. Hence the first moment of the total area is zero about the axis of symmetry and the centroid must lie thereon.

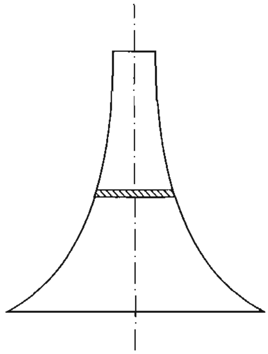


Figure A.4

If the figure has two axes of symmetry, the centroid must be the point of intersection of these two axes.

Since the resultant force W acts at the centroid, the first moment Q about any axis passing through the centroid must be zero. Also the first moment about any axis AA is (see Figure A.5):

$$Q_{AA} = Ae \qquad \textbf{(A.8)}$$

where e is the perpendicular distance of axis AA from the centroid.

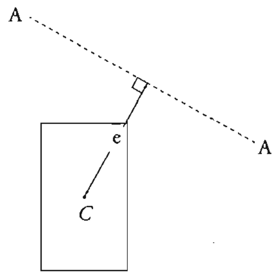
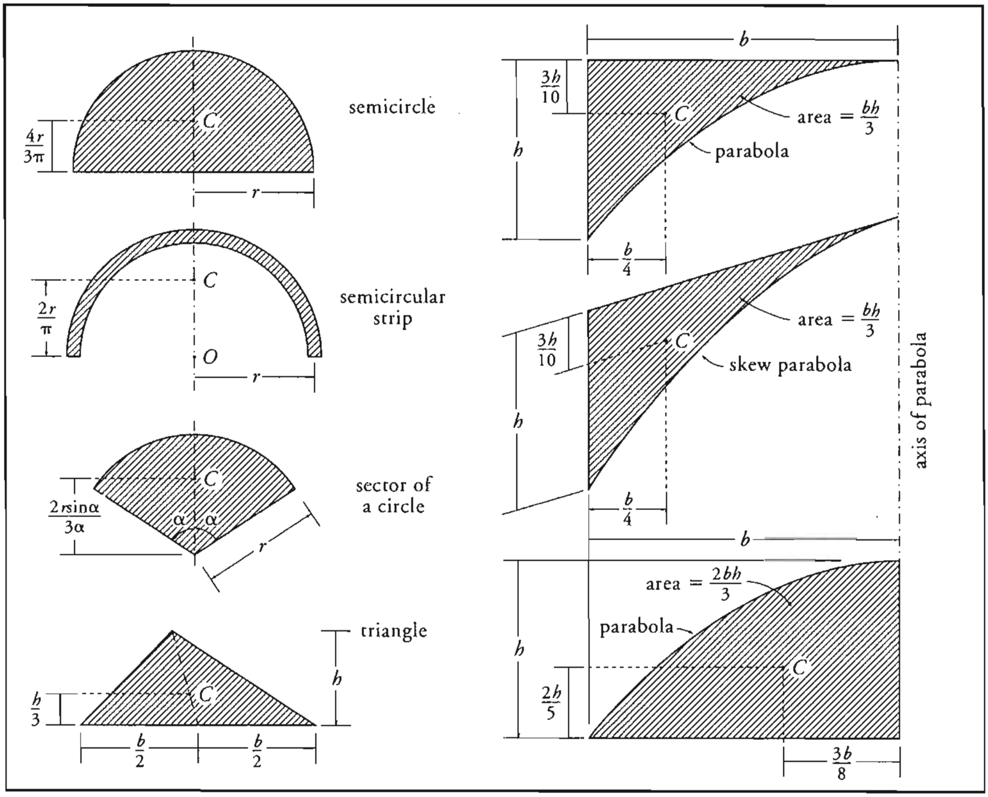


Figure A.5

Most figures which occur in engineering are very simple, or can be sub-divided into simple elements whose centroids are known. Integration may then be replaced by summation, using Equation A.8 for each element. The centroid and properties of some simple shapes are given in Table A.1.

Table A.1: Centroids of figures (courtesy of Jacaranda Wiley Ltd)



EXAMPLE A.2

Find the centroid of the cross-section in Figure A.6.

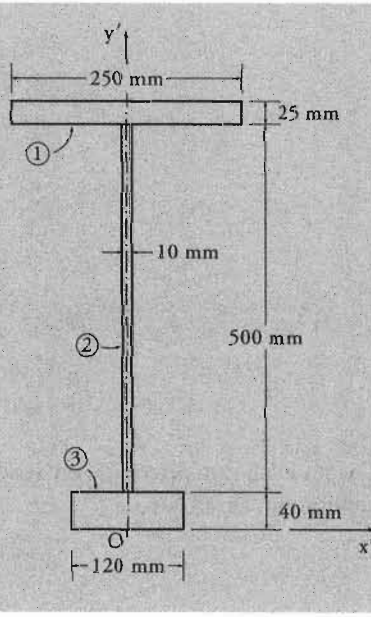


Figure A.6

Solution

Arbitrary axes Ox' and Oy' are chosen (note that the figure is symmetrical about $O_{y'}$). The figure is sub-divided into the rectangular elements 1, 2 and 3. The area of each element and the distance of its centroid from Ox' are tabulated:

Element	A	y'	Ay'
1	6 250	552.5	3 453 125
2	5 000	290	1 450 000
3	4 800	20	96 000
Sum	16 050		4 999 125

Then: $y'_C = \dfrac{\Sigma Ay'}{\Sigma A} = 311.5$ mm

The centroid is located on the axis of symmetry at the point 311.5 mm above Ox'.

A.2 Second Moment of Area – Product Moment of Area

We consider now the case of a plane surface acted upon by a linearly varying distributed load. Figure A.7 shows a figure ABCD subjected to a load which varies linearly in the y direction and is uniform in the x direction. The axis Ox' is the line of zero load. If the load intensity at unit distance from Ox' is denoted by w_1, then the intensity at any point is $w_1 y'$. The axis Oy' is normal to Ox' but is otherwise arbitrary. The load intensity is represented by the block ABCDA'B'C'D'.

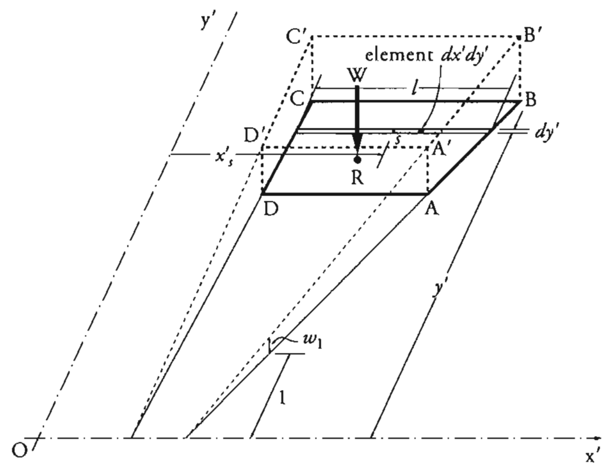


Figure A.7

The problem is to determine the resultant force W, and the co-ordinates of R, its point of application, i.e. (x'_R, y'_R).

A.2.1 Resultant force

Consider an elemental area distant y' from Ox′. The load intensity here is $w_1 y'$. The force on the element is:

$$dW = (w_1 y')dx'dy'$$

and the total load is:

$$W = \iint w_1 y'dx'dy' = w_1 \iint y'dx'dy' \quad \textbf{(A.9)}$$

The integral $\iint y'dx'dy'$ is the first moment of area about Ox′. Hence:

$$W = w_1 Q_{x'} \quad \textbf{(A.10)}$$

If the distance from the centroid to axis Ox′ is y'_C, then:

$$Q_{x'} = y'_C A \quad \text{and} \quad W = w_1 y'_C A \quad \textbf{(A.11)}$$

But $w_1 y_C'$ is the intensity of loading at the centroid, which may be denoted by w_C. So the total load is the intensity of load at the centroid times the area of the figure:

$$W = w_C A \quad \textbf{(A.12)}$$

A.2.2 Determination of y'_R

Since the load intensity varies in one direction and not in the other, the determination of x'_R and y'_R will be different. To find the distance of R from Ox′, we take moments about Ox′. The moment of the element force $(w_1 y')\ dx'dy'$ about Ox′ is:

$$dM_{x'} = (w_1 y'dx'dy')y' = w_1 y'^2 dx'dy' \quad \textbf{(A.13)}$$

Hence the sum of the moments of all the element forces about Ox′ is:

$$M_{x'} = \int_{y'}\int_{x'} w_1 y'^2 dx'dy' = w_1 \int_{y'}\int_{x'} y'^2 dx'dy'$$

The term $\iint y'^2 dx'dy'$ is a geometrical property of the figure. It is called the *second moment of area* about Ox′ and is denoted by $I_{x'x'}$. That is:

$$I_{x'x'} = \iint y'^2 dx'dy' \quad \textbf{(A.14)}$$

and:

$$M_{x'} = w_1 I_{x'x'} \quad \textbf{(A.15)}$$

The moment of the resultant about Ox′ is Wy'_R and from Equation A.11:

$$Wy'_R = (w_1 y'_C A)y'_R$$

Hence:

$$(w_1 y'_C A)y'_R = w_1 I_{x'x'}$$

$$y'_R = \frac{I_{x'x'}}{y'_C A} \quad \textbf{(A.16)}$$

A.2.3 Determination of $I_{x'x'}$

The evaluation of $I_{x'x'}$ involves the double integration of $(y')^2$ with respect to x' and y'. As in the case of the calculation of the first moment of area, one integration may be avoided by considering a strip parallel to the axis Ox' of length l and width dy' (Figure A.8). The force on the strip is uniform and equal to $w_1 y'(l dy')$. The moment of this force about Ox' is:

$$dM_{x'} = w_1 y'(l dy')y' = w_1 l y'^2 dy'$$

The total moment is:

$$M_{x'} = w_1 \int l y'^2 dy' \qquad \textbf{(A.17)}$$

When l has been expressed in terms of y', then:

$$I_{x'x'} = \int l y'^2 dy' \qquad \textbf{(A.18)}$$

The use of an elemental strip parallel to Oy' will rarely be beneficial in this case because the load on such a strip would not be uniform. The position of the resultant force on the element would therefore not be obvious.

EXAMPLE A.3

Find the second moment of area of the rectangle of Figure A.8 about the base AA.

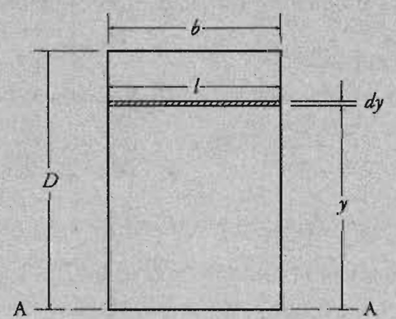


Figure A.8

SOLUTION

The length of every elemental area is $l = b$, and the area of the element is therefore $b dy$. From Equation A.18:

$$I_{AA} = \int_0^D y^2 b\, dy = b\int_0^D y^2 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^D$$

$$\therefore I_{AA} = \frac{bD^3}{3}$$

A.2.4 Determination of x'_R

To find the distance of the resultant *W* from Oy′ in Figure A.7 (page 266), we take moments about Oy′. The force on an element $(dx'dy')$ is $w_1y'(dx'dy')$ and the moment of this force about Oy′ is:

$$dM_{y'} = w_1 y'(dx'dy')x' \qquad \textbf{(A.19)}$$

The total moment is:

$$M_{y'} = w_1\iint y'x'dx'dy' \qquad \textbf{(A.20)}$$

The term $\iint y'x'dx'dy'$ is a geometrical property called the *product moment of area* and is denoted by $I_{x'y'}$:

$$I_{x'y'} = \int_{y'}\int_{x'} y'x'dx'dy' \qquad \textbf{(A.21)}$$

and: $$M_{y'} = w_1 I_{x'y'} \qquad \textbf{(A.22)}$$

The moment of the resultant about Oy′ is:

$$Wx'_R = (w_1y'_CA)x'_R$$

Hence: $$(w_1y'_CA)x'_R = w_1 I_{x'y'}$$

$$x'_R = \frac{I_{x'y'}}{y'_CA} \qquad \textbf{(A.23)}$$

A.2.5 Determination of $I_{x'y'}$

Considering the strip element as before (Figure A.7, page 266), the moment of the force on this area about Oy′ is:

$$dM_{y'} = w_1 y'(ldy')x'_s$$

where x'_s is the distance of the mid-point of the element from Oy′. Thus:

$$M_{y'} = w_1\int y'lx'_sdy' = w_1 I_{x'y'}$$

$$I_{x'y'} = \int y'lx'_s\, dy' \qquad \textbf{(A.24)}$$

The dimensions *l* and x'_s must be expressed in terms of y′ before integration.

EXAMPLE A.4

For the right-angled triangle of Figure A.9, find the value of $I_{x'y'}$ with respect to the axes shown.

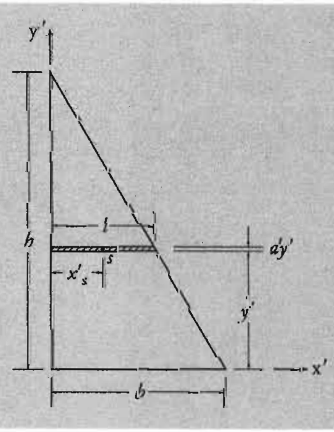


Figure A.9

SOLUTION

For the typical element:

$$l = b\left(1 - \frac{y'}{h}\right) \qquad x'_s = \frac{l}{2} = \frac{b}{2}\left(1 - \frac{y'}{h}\right)$$

and hence from Equation A.24:

$$I_{x'y'} = \frac{b^2}{2}\int_0^h \left(1 - \frac{y'}{h}\right)^2 y'dy'$$

$$= \frac{b^2h^2}{24}$$

Summary

For the case of a load varying linearly in the y direction, the resultant force on the area is the load intensity at the centroid times the area,

$$W = w_1 y'_C A$$

This acts at the point (x'_R, y'_R) where:

$$x'_R = \frac{I_{x'y'}}{y'_C A} \qquad \text{and} \qquad y'_R = \frac{I_{x'x'}}{y'_C A}$$

In these expressions the axis Ox′ must be the line of zero load and the axis Oy′ is any axis normal to Ox′.

A.2.6 Load varying in the x direction

A similar analysis may be made for a load which varies linearly in the x direction and is constant in the y direction, Oy′ being the axis of zero load. Such a load would have a resultant:

$$W = w_1 x'_C A$$

acting at (x'_R, y'_R) where:

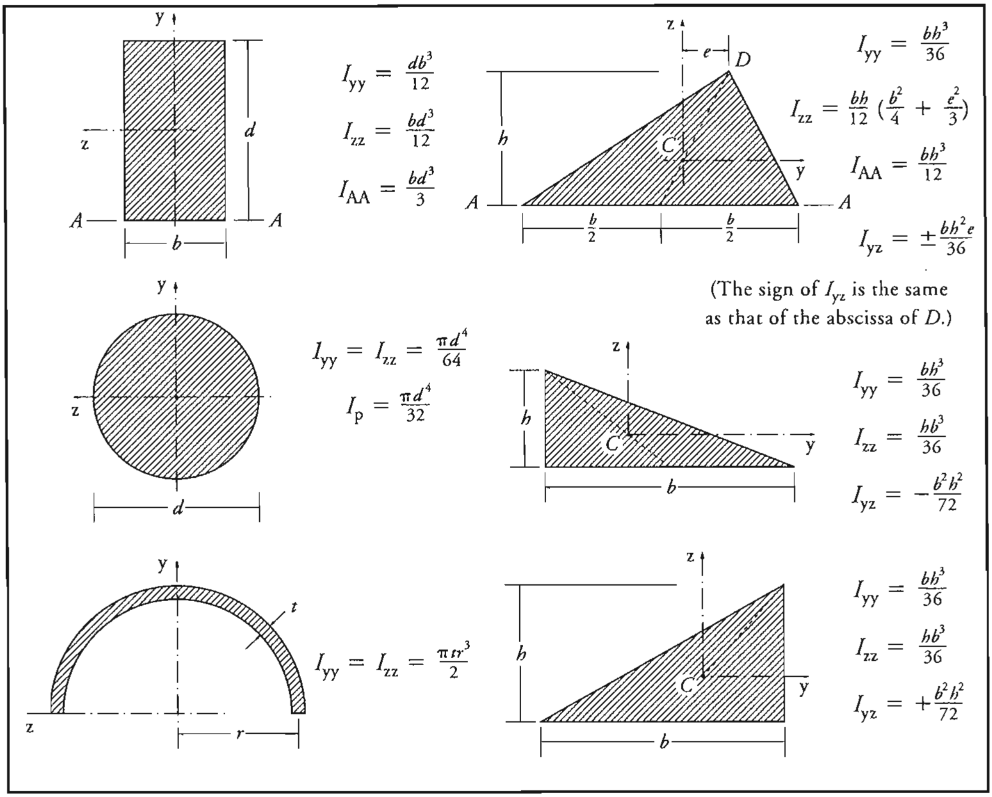
$$x'_R = \frac{I_{y'y'}}{x'_C A} \qquad \text{and} \qquad y'_R = \frac{I_{x'y'}}{x'_C A}$$

The second moment $I_{y'y'}$ is given by:

$$I_{y'y'} = \int x'^2 dx'dy' \tag{A.25}$$

The second moments of area and product moment of area of some simple shapes are given in Table A.2.

Table A.2: Second moments of area and product moments of area of figures (courtesy of Jacaranda Wiley Ltd)



A.3 Shift of axes – Parallel axis theorems

Fairly simple relationships exist between the quantities $I_{x'x'}$ and $I_{y'y'}$ expressed in one set of axes and the same quantities expressed in a parallel set of axes with origin at the centroid. It is often convenient to compute the values in one set of axes and then to derive the values in other axes where direct computation may be more complex.

A.3.1 Effect of axis change on $I_{x'x'}$

Since $I_{x'x'} = \int y'^2 dA$, it is independent of x' and hence a shift of the axis Oy' has no effect on this quantity. Moreover $(y')^2 dA$ is a positive quantity whether y' is positive or negative. Therefore $I_{x'x'}$ will be positive with respect to any position of Ox'. We note also that a shift of axis Ox' implies a shift of the line of zero pressure.

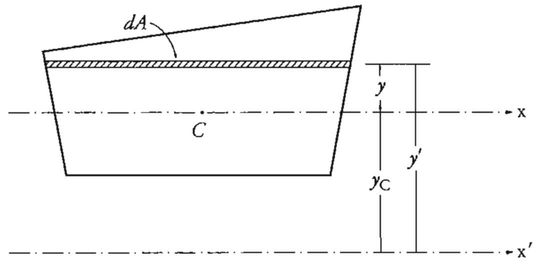


Figure A.10

Figure A.10 shows a typical element of area dA at distance y' from axis x'. Its distance from the parallel axis Cx through the centroid is y, hence:

$$y' = y + y'_C$$

The second moment of area about Cx is:

$$I_{xx} = \int y^2 dA$$

The second moment of area about Ox' is:

$$I_{x'x'} = \int y'^2 dA$$

$$= \int (y + y'_C)^2 dA$$

$$= \int y^2 dA + 2y'_C \int y dA + y'^2_C \int dA$$

Now: $I_{xx} = \int y^2 dA$; $\int y dA$ is the first moment of A about the axis Cx through the centroid and is therefore zero; and $\int dA$ is the area A. Hence:

$$I_{x'x'} = I_{xx} + (y'_C)^2 A \qquad \textbf{(A.26)}$$

The term $(y'_C)^2 A$ is positive whether y'_C is positive or negative. Therefore the second moment of area about an axis through the centroid is less than that about any other parallel axis.

Equation A.26 is often called the *Theorem of Parallel Axes.* It enables us to re-state Equation A.16 in a more convenient form. From Equation A.16, we have:

$$y'_R = \frac{I_{x'x'}}{y'_C A}$$

and using Equation A.26:

$$y'_R = \frac{I_{xx} + (y'_C)^2 A}{y'_C A}$$

$$y'_R = \frac{I_{xx}}{y'_C A} + y'_C \qquad \textbf{(A.27)}$$

It is of interest to interpret the parallel axis theorem in terms of force systems. Figure A.11a shows (in side elevation) a plane figure of area A (represented in side elevation by the line segment EF), with centroid C, acted upon by a linearly varying pressure. The total pressure block EFGH may be regarded as a block of uniform pressure (EFKL) plus a couple represented by HLKG. The uniform pressure block has a resultant $W = w_1 y'_C A$ (Equation A.11) acting at the centroid. The figure HLKG is a couple of moment $M = w_1 I_{xx}$ (Equation A.15).

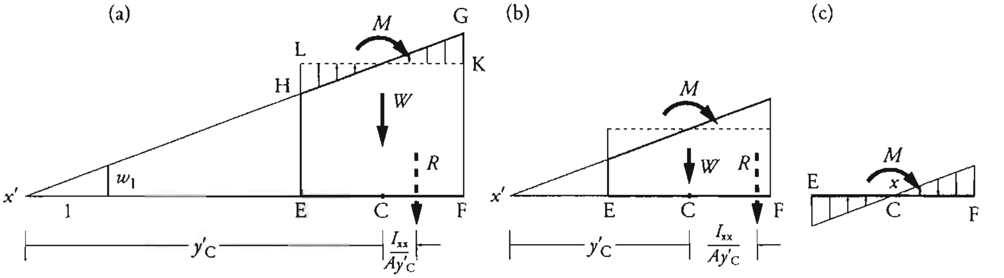


Figure A.11

The force W and the couple M may be replaced by a single resultant force R at a distance M/W from C (see Chapter 4, Section 4.5). In this case:

$$\frac{M}{W} = \frac{I_{xx}}{Ay'_C} \quad \textbf{(A.28)}$$

The total moment of the pressure system about axis x is:

$$M + Wy'_C = w_1 I_{xx} + w_1 A y'^2_C$$

Hence: $$I_{x'x'} = I_{xx} + Ay'^2_C \quad \textbf{(A.29)}$$

which is the same as Equation A.26. The significance of the two terms on the right-hand side of this equation may be seen from Figure A.11b in which the axis of zero load moves nearer to the centroid C, and Figure A.11c in which this axis coincides with C. Clearly the moment of the uniform pressure block, $w_1(Ay'^2_C)$ diminishes as y'_C decreases, while the moment of the couple $w_1 I_{xx}$ remains constant. The distance from the centroid to the position of the resultant force increases as W becomes smaller. In the limit, as y'_C tends to zero, this distance tends to infinity, since the resultant force is now zero.

The Parallel Axis Theorem could be used to find the second moment of area of the rectangle of Figure A.8 about an axis xx through the centroid and parallel to AA. The axis xx is $D/2$ from AA. From Equation A.26:

$$I_{xx} = I_{AA} - A\left(\frac{D}{2}\right)^2$$
$$= \frac{bD^3}{3} - \frac{(bD)D^2}{4}$$
$$= \frac{bD^3}{12}$$

The theorem is particularly useful for finding the second moment of composite figures, a very common problem in engineering.

A.3.2 Effect of axis change on $I_{x'y'}$

The value of the product moment is affected by changes both of Ox′ and of Oy′. Note that for a given element of area dA the product $x'y'$ will be positive if x' and y' have the same sign (i.e. dA lies in the first or third quadrants) and negative if they have opposite signs (i.e. dA lies in the second or fourth quadrants).

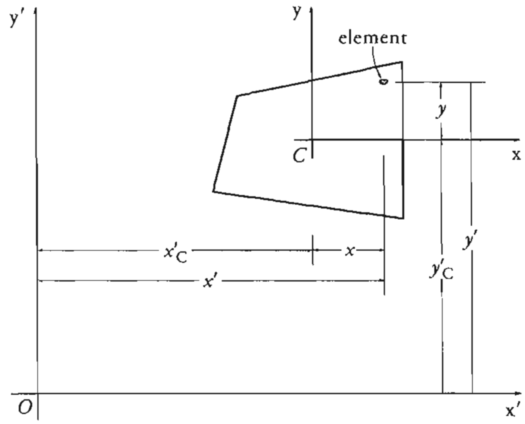


Figure A.12

Figure A.12 shows a typical element with co-ordinates x', y' with respect to the origin O and x, y with respect to the origin at the centroid.

$$x' = x + x'_C$$

$$y' = y + y'_C$$

The product moment about C is:

$$I_{xy} = \int xy\, dA$$

The product moment about O is:

$$I_{x'y'} = \int x'y'\, dA$$

$$= \int (x + x'_C)(y + y'_C)\, dA$$

$$= \int xy\, dA + x'_C \int y\, dA + y'_C \int x\, dA + x'_C\, y'_C \int dA$$

$$= I_{xy} + x'_C\, Q_x + y'_C\, Q_y + Ax'_C\, y'_C$$

Since axes x and y pass through the centroid, $Q_x = Q_y = 0$ and therefore:

$$I_{x'y'} = I_{xy} + Ax'_C\, y'_C \qquad \textbf{(A.30)}$$

The term I_{xy} may be regarded as the product moment of the figure about its ‘own’ axes and the second term $Ax'_C y'_C$ as the product moment about axes $x'y'$ with the area A concentrated at the centroid.

Figure A.13 shows a figure which, with respect to its 'own' axes at C, lies predominantly in the first and third quandrants and thus has a positive value of I_{xy}. With respect to axes Ox' and Oy', the figure lies entirely in the second quadrant and $I_{x'y'}$ is negative. Similarly a shift of origin to O_1 would give a positive value of the product moment considerably greater than I_{xy}.

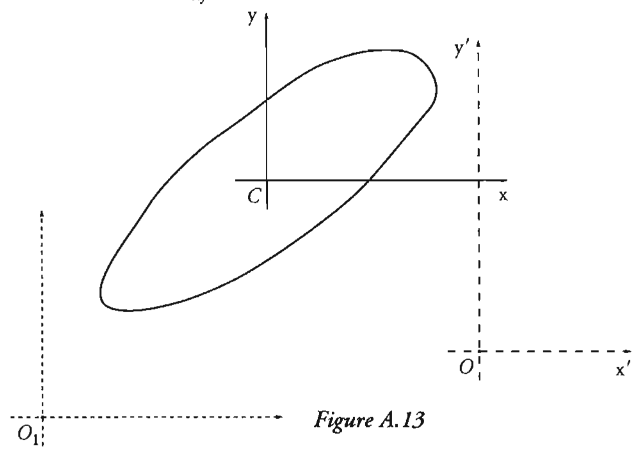


Figure A.13

EXAMPLE A.5

For the figure shown in Figure A.14, determine the position of the centroid and the values of I_{xx}, I_{yy} and I_{xy}. All dimensions are shown in millimetres.

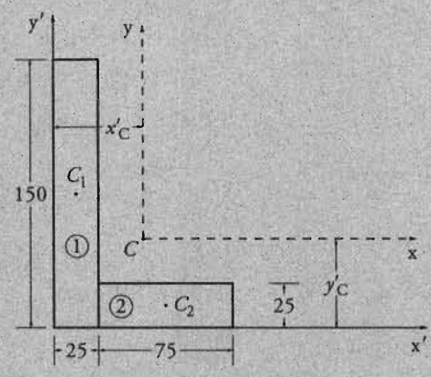


Figure A.14

SOLUTION

1. **Find the centroid:** The figure is divided into two rectangular elements.

Element	b	D	A(mm²)	x'	$Q_{y'} = Ax'$	y'	$Q_{x'} = Ay'$
1	25	150	3.750×10^3	12.5	47×10^3	75	281×10^3
2	75	25	1.875×10^3	62.5	117×10^3	12.5	23×10^3
Sum			5.625×10^3		164×10^3		304×10^3

The co-ordinates of the centroid of the whole figure are:

$$x'_C = \frac{164 \times 10^3}{5.625 \times 10^3} = 29 \text{ mm} \quad \text{and} \quad y'_C = \frac{304 \times 10^3}{5.625 \times 10^3} = 54 \text{ mm}$$

The co-ordinates of the element centroids with respect to C are:

Element 1: $x_1 = 12.5 - 29 = -16.5$

$y_1 = 75 - 54 = 21$

Element 2: $x_2 = 62.5 - 29 = 33.5$

$y_2 = 12.5 - 54 = -41.5$

2. Calculate I^0_{xx}, I^0_{yy}, I^0_{xy} for each element for axes through its own centroid:

Element	b	D	$I^0_{xx} = \frac{bD^3}{12}$	$I^0_{yy} = \frac{Db^3}{12}$
1	25	150	7031×10^3	195×10^3
2	75	25	98×10^3	879×10^3

Because of symmetry, $I^0_{xy} = 0$ for both elements.

3. Calculate I_{xx}, I_{yy} and I_{xy} for the composite figure: Using the parallel axes theorems (Equations A.26 and A.30):

(a) I_{xx} : Element $I_{xx} = I^0_{xx} + Ay^2$

1 $= (7031 + (3.750 \times 21^2)) \times 10^3 = 8685 \times 10^3$

2 $= (98 + (1.875 \times -41.5^2)) \times 10^3 = 3327 \times 10^3$

Total: $I_{xx} = 12\,012 \times 10^3 \text{ mm}^4$

(b) I_{yy} : Element $I_{yy} = I^0_{yy} + Ax^2$

1 $= (195 + (3.750 \times -16.5^2)) \times 10^3 = 1216 \times 10^3$

2 $= (879 + (1.875 \times 33.5^2)) \times 10^3 = 2983 \times 10^3$

Total: $I_{yy} = 4199 \times 10^3 \text{ mm}^4$

(c) I_{xy} : Element $I_{xy} = I^0_{xy} + Axy$

1 $= (0 + (3.750 \times -16.5 \times 21)) \times 10^3 = -1299 \times 10^3$

2 $= (0 + (1.875 \times 33.5 \times -41.5)) \times 10^3 = -2607 \times 10^3$

Total: $I_{xy} = -3906 \times 10^3 \text{ mm}^4$

A.4 Rotation of axes – Principal axes

Suppose that the figure shown in Figure A.14 (page 275) is subjected to a linearly varying pressure with zero pressure along either Cx or Cy. There will be no resultant force but there will be a moment about both Cx and Cy since I_{xy} is not zero.

Rotation of the axes relative to the figure will change the values of I_{xx}, I_{yy} and I_{xy}. In particular I_{xy} will change sign if the axes are rotated through 90° since elements previously in the first quadrant will now be in the second or fourth quadrant. Some axis rotation less than 90° must therefore result in a zero value of I_{xy}. These axes are called principal axes and are denoted by C1 and C2. The second moments about these axes are denoted by I_{11} and I_{22}. One of these is the maximum and one the minimum second moment about any axis. A pressure system varying parallel to one of these axes will have a moment only about the axis of zero pressure.

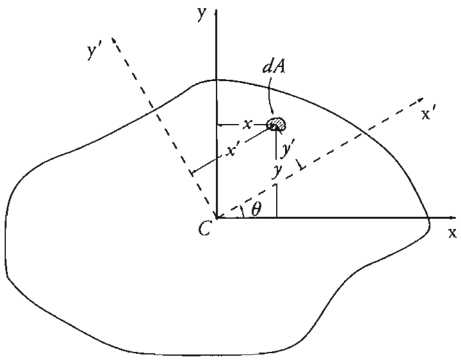


Figure A .15

Suppose that for the figure shown in Figure A.15 the values of I_{xx}, I_{yy} and I_{xy} have been calculated. Consider axes Cx' and Cy' at an angle θ to Cx and Cy respectively.

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

Then:

$$\begin{aligned} I_{x'x'} &= \int y'^2 dA \\ &= \int (-x\sin\theta + y\cos\theta)^2 dA \\ &= \sin^2\theta \int x^2 dA + \cos^2\theta \int y^2 dA - 2\sin\theta\cos\theta \int xy\, dA \\ &= I_{yy}\sin^2\theta + I_{xx}\cos^2\theta - I_{xy}\sin 2\theta \end{aligned}$$

or: $$I_{x'x'} = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta \qquad \textbf{(A.31)}$$

Similarly:

$$I_{y'y'} = \tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta \qquad \textbf{(A.32)}$$

and: $$I_{x'y'} = \tfrac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta \qquad \textbf{(A.33)}$$

To locate the principal axes, we set $I_{x'y'}$ to zero in Equation A.33. This gives:

$$\tan 2\theta = \frac{-I_{xy}}{\tfrac{1}{2}(I_{xx} - I_{yy})} \qquad \textbf{(A.34)}$$

which results in two values of 2θ which differ by 180°, or two values of θ which differ by 90°.

For the figure examined in Example A.5 (page 275), we have:

$$\tan 2\theta = \frac{+3906 \times 10^3}{0.5(12012 - 4199) \times 10^3} = +\ 1.00$$

Hence $2\theta = 45°$ and $\theta = 22.5°$, or $2\theta = 225°$ and $\theta = 112.5°$

The values of I_{11} and I_{22} are obtained from Equations A.31 and A.32:

$$I_{11} = 0.5(12012 + 4199) \times 10^3 + 0.5(12012 - 4199) \times 10^3 \cos 45 + 3906 \times 10^3 \sin 45$$

$$= 8105 \times 10^3 + 2762 \times 10^3 + 2762 \times 10^3$$
$$= 13629 \times 10^3 \text{ mm}^4$$

$$I_{22} = 8105 \times 10^3 - 2762 \times 10^3 - 2762 \times 10^3$$
$$= 2581 \times 10^3 \text{ mm}^4$$

ANSWERS TO PROBLEMS

CHAPTER 2

2.1 (a) $F_x = 8.66$ kN; $F_y = 5.0$ kN.
(b) $F_x = -20.0$ kN; $F_y = -34.64$ kN.
(c) $F_x = -3.54$ kN; $F_y = +3.54$ kN.

2.2 (a) $F_u = 109.8$ kN; $F_v = 77.65$ kN.
(b) $F_u = 269.0$ kN; $F_v = 219.6$ kN.
(c) $F_u = 200.0$ kN; $F_v = -103.5$ kN.

2.3 $P = 73.21$ N and $Q = 51.76$ N.

2.4 $R = 91.70$ N at $\theta = 23.67°$.

2.5 $R = 62.5$ N at $\theta = 39.84°$.

2.6 $R = 834.5$ N at $\theta = 7.32°$.

2.7 (a) $R = 6.38$ N at $\theta = -16.42°$. (b) $R = 3.427$ N at $\theta = -10.01°$.
(c) $R = 97.90$ N at $\theta = 102.4°$.

2.8 $R = 8.04$ kN at $\theta = 223.2°$.

2.9 (i) (a) $\alpha = 53.1°$ (or $-53.1°$) and $A = 20$ N (or -20 N).
(b) $\alpha = 25.84°$ (or $-25.84°$) and $A = 19.11$ (or 1.67 N).
(c) $\alpha = 100.3°$ and $A = 11.1$ N or $\alpha = 199.7°$ and $A = -27.1$ N.
(ii) (a) $\alpha = 126.9°$ (or $-126.9°$) and $A = 20$ N (or -20 N).
(b) $\alpha = 154.2°$ (or $-154.2°$) and $A = -1.67$ N (or -19.10 N).
(c) $\alpha = 19.7°$ and $A = +27.1$ N or $\alpha = -79.7°$ and $A = -11.1$ N.

2.10 $\theta = 41.54°$ and $P = 283.4$ N.

2.11 $F_{AC} = 1795$ N and $F_{BC} = 2199$ N.

2.13 $R = 95.39$ N at 53.01°; $F_{uR} = 98.62$ N and $F_{vR} = 30.53$ N.

2.14 $\theta = 146.0°$.

2.15 $\alpha = 69.92°$ and $\beta = 9.39°$.

2.16 $R = 26.90$ kN and $P = 27.90$ kN.

CHAPTER 3

3.1 $P = 6.66$ kN and $\alpha = 121.7°$.

3.2 $Q = -15.63$ N and $P = -24.92$ N.

3.3 $F_{AC} = -120.2$ N and $F_{BC} = 833.3$ N.

3.4 169.4° and 253.1° respectively, or 219.8° and 136.1° respectively.

3.5 $H = 10.56$ N

3.6 $R_A = 10.29$ kN at 48.33° to the horizontal and $R_C = 11.11$ kN (upwards).

3.7 Force in connecting rod, $F_{BC} = 1.02$ kN compression; Force on cylinder wall $= 0.20$ kN.

3.8 (i) $R_B = 0.289\,W$.
(ii) $R_A = 0.764\,W$ at 10.9° to the vertical.

3.9 $\mu = 0.577$.

3.10 $F = 19.36$ N.

3.11 $F_{AB} = 219.9$ kN compression; $F_{AC} = 72.3$ kN tension; $F_{DA} = 100$ kN.

3.12 $\theta = 39.66°$ and $F_{CB} = 43.3$ N.
3.13 $R_A = 2.67$ kN and $R_B = 3.33$ kN.
3.14 $H_A = 1.83$ kN; $V_A = 8.17$ kN; $T_{BD} = 2.59$ kN.
3.15 $P = 6.32$ N and $Q = -20.74$ N.
3.16 $R_A = 12.85$ N at $-10°$ to the horizontal and $R_C = 18.79$ N at 140° to the horizontal.
3.17 $P = 40.48$ N; $\phi = 7.53°$; $\theta = 10.18°$.
3.18 4.3 m above the ground.
3.19 (i) $\theta = 45°$ (ii) $0.667W$
3.20 5.56 N perpendicular to the lever.
3.21 $P = 14.11$ kN and $R_A = 14.97$ kN ($V_A = 9.72$ kN $+\uparrow$ and $H_A = 11.38$ kN $\overleftarrow{+}$).
3.22 $\alpha = 12.64°$ and $F = 1108$ N.
3.23 $R_C = 487.1$ N at $= 55.39°$ to horizontal and $Q = 385.3$ N. ($H_C = 276.7$ N and $V_C = 400.9$ N)
3.24 $L_{AC} = 2.595$ m; $L_{CB} = 3.405$ m; $F_{BC} = 126.1$ N and $F_{AC} = 96.1$ N.
3.25 $\theta = 18.43°$ to the vertical and $T = 10.54$ N.

CHAPTER 4

4.1 (i) $\Sigma M_B = 56.6$ Nm clockwise.
(ii) $\Sigma M_C = 56.6$ Nm clockwise.
(ii) Clockwise.
4.2 Resultant is a vertical force $R = 58.56$ N $+\downarrow$ which cuts AC 4.732 m to right of A.
4.3 Resultant at D: $R_H = 2.196$ kN($\underleftarrow{+}$), $R_V = 4.732$ kN($+\downarrow$), $M_D = 17.32$ kNm anticlockwise.
4.4 Resultant is 70 N at 70° cutting AD 5.43m to right of A.
4.5 Resultant is 8 kN at $-45°$ cutting BD at 3.01m from B.
4.6 (i) Resultant is a clockwise couple of 5953 Nmm.
(ii) Resultant is a 360 N horizontal force at 16.54 mm above AC.
4.7 (i) $R_x = -5.196$kN; $R_y = -5.0$ kN; $M_B = 9.59$ kNm anticlockwise.
(ii) $x = 0.082$ m.
4.8 (i) Resultant $R = 10$ kN acting in direction BD cutting BA $2\sqrt{2}$ m from B.
(ii) Same as (i).
4.9 Resultant is 11.08 kN at 33.6° to the horizontal and cutting ABC 9.31 m to the left of B.
4.10 (i) Force = 17.32 kN at 210° to direction Ax and couple = 12.99 kNm clockwise.
(ii) Force is 17.32 kN at 210° to direction Ax cutting Ax 1.5 m from A.
4.11 $F_1 = F_3 = -2.303$ kN; $F_2 = -17.91$ kN; $F_4 = 4.91$ kN.
4.12 $F_1 = 22.48$ kN; $F_2 = 16.78$ kN; $F_3 = 20.70$ kN.
4.13 28.87 kN along each side.
4.14 (i) Single force is 6.403 N at 38.66° to the horizontal cutting the line through AC 18m to the right of A.
(ii) Force is 6.403 N at 38.66° to the horizontal; couple is 80 kNm clockwise.

4.15 (i) $F_B = 32.39$ kN at 64.11° below horizontal and couple is 10 kNm anticlockwise.
(ii) $F_c = 1.67$ kN $+\uparrow$ and $F_B = 33.90$ kN acting 65.35° below horizontal.
(iii) $F_c = 12.73$ kN $+\downarrow$ and $F_A = 21.66$ kN acting 49.25° below horizontal.

4.16 (i) Resultant is an anticlockwise couple of 99.44 Nm.
(ii) $F = 10$N, $\alpha = 90°$ and $x_F = 9.944$ m.

4.17 $P = 21.42$ kN; $\theta = 13.50°$ and couple at C = 48.12 kNm.

4.18 (i) Force at C = 32.56N at 47.49° above horizontal and couple is 162.67 Nm clockwise.
(ii) $F = 30.57$ N at $\theta = 168.68°$ intersecting line through BC 41.75 m to right of C.

4.19 (i) Force through O is 46.53 N at −7.24° (i.e. 7.24° below Ox) and couple = 70.74 Nm clockwise.
(ii) Force is 46.53 N at $\theta = -7.24°$. Distance $Ox = 23.91$ m.
(iii) Two forces are 22.95 N through O and 23.58 N 3 m from O both at $\theta = -7.24°$.

CHAPTER 5

5.1 $F = 8.4$ N and $R_C = 14.21$ N at $\theta = 60.81°$.

5.2 $F_1 = -4.83$ kN, $F_2 = 9.05$ kN, $\theta = 147.8°$.

5.3 $V_A = 8.95$ kN $+\downarrow$, $H_A = 1.0$ kN $\overset{\rightarrow}{+}$ and $V_E = 11.64$ kN $+\downarrow$.

5.4 $V_D = 10.30$ N, $V_A = 0.30$ N and $H_A = 10.61$ N.

5.5 (i) $P = 0.4$ kN. (ii) $V_O = 0.52$ kN $+\downarrow$ and $H_O = 0.60$ kN $\underset{\leftarrow}{+}$.

5.6 $V_A = 5.0$ kN($+\uparrow$), $H_A = 5.20$ kN($\overset{\rightarrow}{+}$) and $M_A = -0.41$ kNm (i.e. anticlockwise).

5.7 $V_A = 4.90$ kN, $H_A = 5.20$ kN and $V_C = 0.10$ kN.

5.8 (i) $R_A = 9$ kN and $R_B = 46$ kN. (ii) $L = 17.5$ kN.

5.9 $R_A = 5.0$ kN, $R_C = 40.0$ kN and $R_E = 8.66$ kN.

5.10 $V_A = -0.25$ kN, $V_B = +4.25$ kN and $H_B = +3.0$ kN.

5.11 $V_A = 13.08$ kN, $V_B = 10.25$ kN and $H_B = 6.25$ kN.

5.12 $F_1 = 0$, $F_2 = +15$ kN and $M = -80$ kNm.

5.13 $N = 0$, $S = wx$ and $M = -wx^2/2$.

5.14 $e = 104.3$ mm.

5.15 (a) $H_A = 0$, $V_A = 6$ kN ($+\uparrow$), $V_B = 2$ kN ($+\uparrow$).
(b) $H_A = 0$, $V_A = 9$ kN ($+\uparrow$), $V_B = 5$ kN ($+\uparrow$).
(c) $H_A = 10$ kN ($\underset{\rightarrow}{+}$), $V_A = 10$ kN ($+\uparrow$), $H_D = 10$ kN ($\underset{\leftarrow}{+}$).
(d) $H_A = 0$, $V_A = 4.5$ kN ($+\uparrow$), $V_B = 4.5$ kN ($+\downarrow$).
(e) $V_A = 7$ kN ($+\uparrow$), $H_B = 10.07$ kN ($\underset{\rightarrow}{+}$), $V_B = 14.07$ kN ($+\uparrow$).
(f) $H_A = 0$, $V_A = 9.0$ kN ($+\uparrow$), $V_B = 2.0$ kN ($+\uparrow$).
(g) $V_A = 3$ kN ($+\uparrow$), $H_C = 0$, $V_C = 15$ kN ($+\uparrow$).

5.16 $R_1 = -7.97$ kN; $R_2 = 35.98$ kN; $R_3 = -19.04$ kN .

5.17 (i) $R_A = 1.78$ kN and $R_B = 6.22$ kN. (ii) $R_C = -0.38$ kN.

5.18 $\theta = 57.45°$.

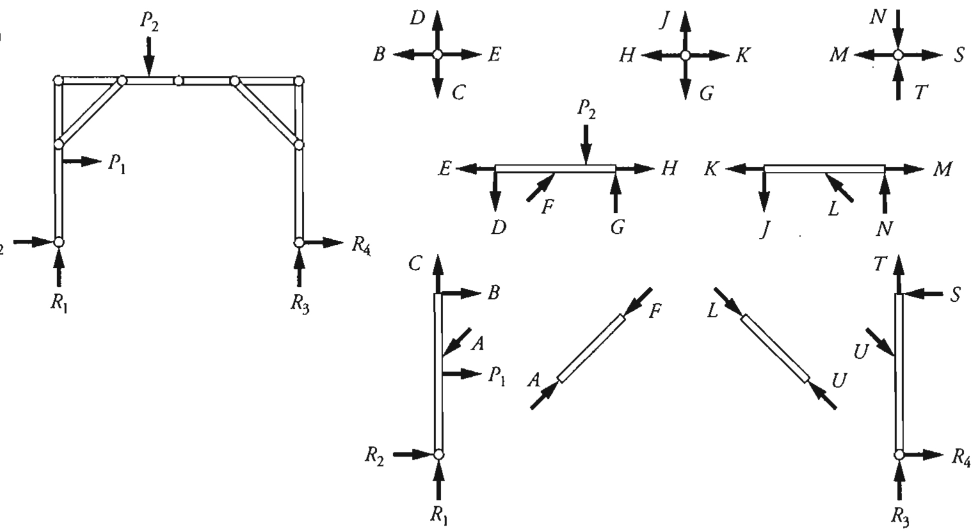
5.19 $R_A = 1.68$ N (+↑), $R_B = -7.19$ N at $\theta = 103.1°$.

5.20 $\theta = 45.9°$.

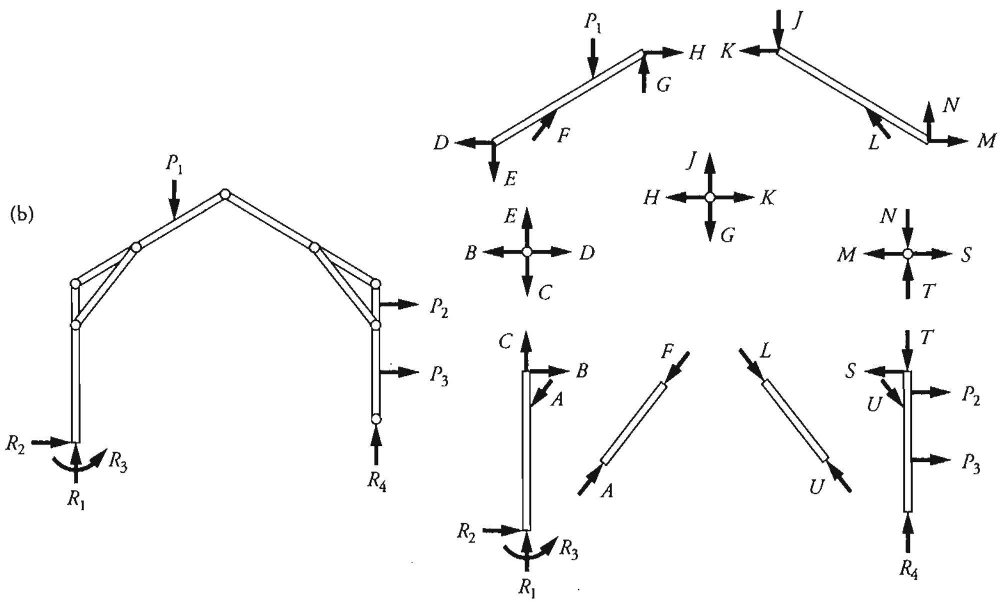
CHAPTER 6

6.1

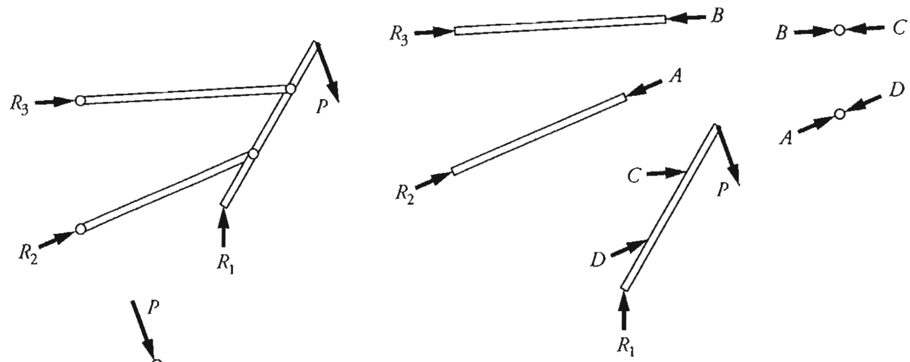
(a)



(b)



(c)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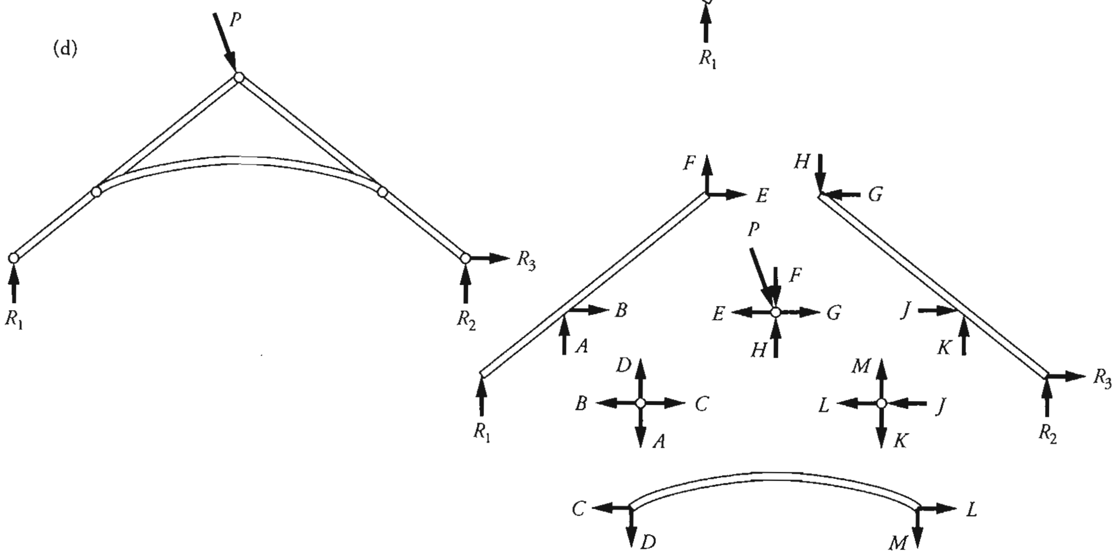
R2
R1
B
R3
A
R2
B
C
D
A
C
P
D
R1

(d)


P
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R2
R3
F
E
H
G
P
F
E
G
H
B
A
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K
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M
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C
L
J
A
K
R1
R2
R3
C
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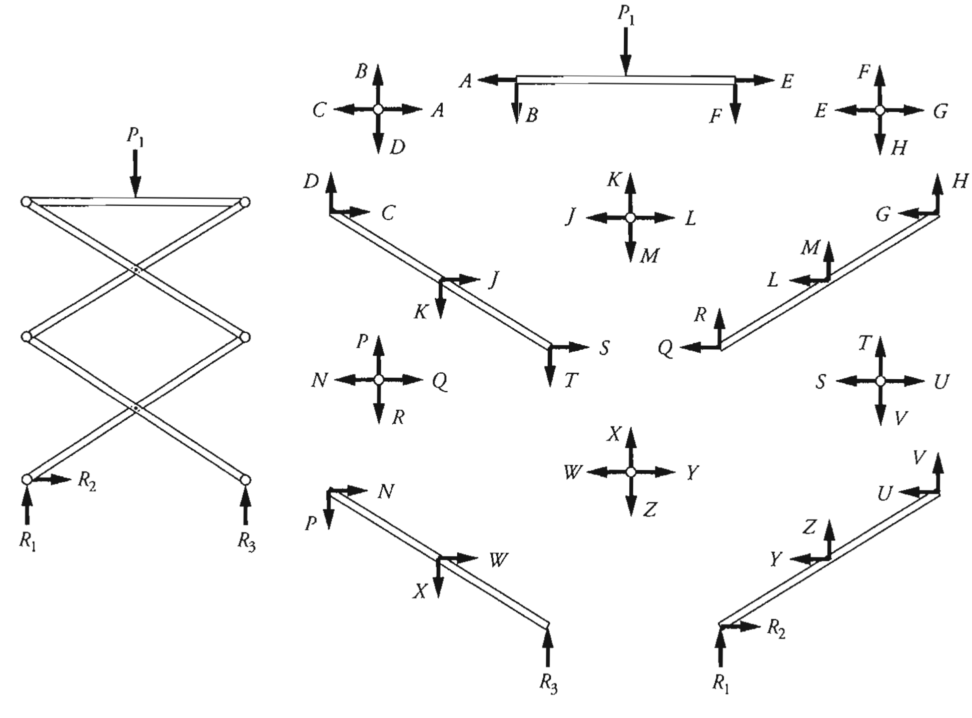
D
M

(e)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R2
R1
R3
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S
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V
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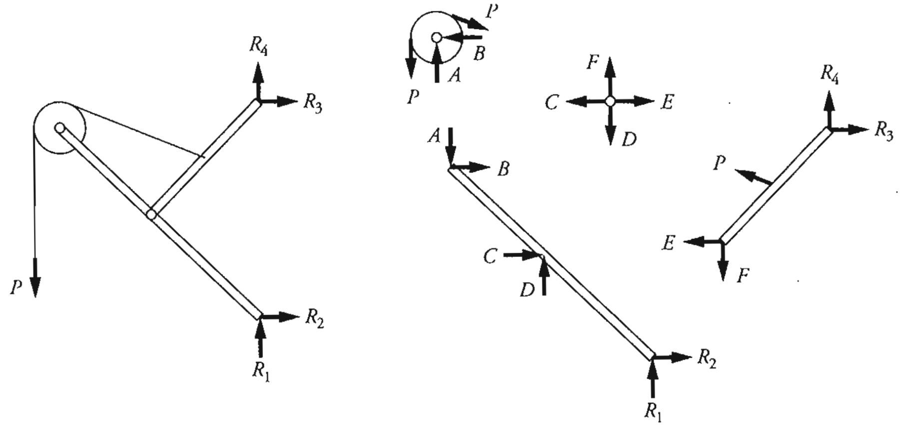
Y
R2
R1

6.2

(a)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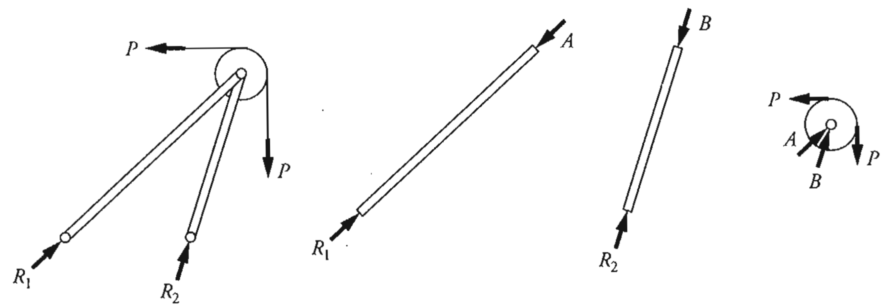
R3
P
A
B
E
F
C
D
P
R2
R1
R2
R1

(b)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P
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B
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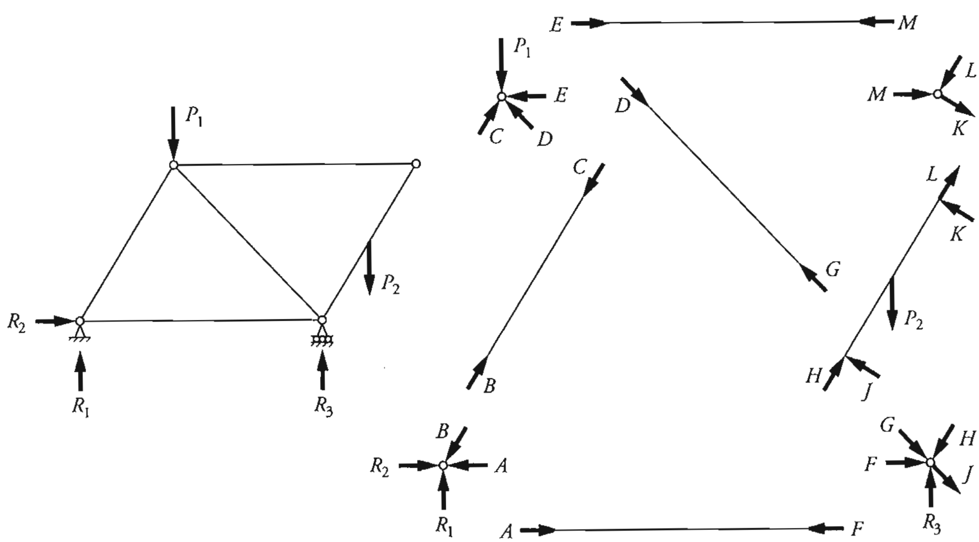
A
B
P
R1
R2
R1
R2

6.3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D
D
M
L
K
P1
C
L
K
G
P2
R2
R1
R3
B
P2
H
J
B
R2
A
R1
G
H
F
J
R3
A
F

6.4 $V_A = 4$ kN $(+\downarrow)$, $H_A = 2$ kN $(\overset{+}{\leftarrow})$, $V_E = 12$ kN $(+\uparrow)$ and $H_E = 8$ kN $(\overset{+}{\leftarrow})$.

6.5 $R_1 = 340.6$ N, $R_2 = 288.7$ N, $R_3 = 711.3$ N, $R_4 = 928.8$ N, $R_5 = 998.7$ N, $R_6 = 2050.6$ N, $R_7 = 1813.6$ N at $\theta = 79.05°$ and $R_8 = 1171$N at $\theta = 107.1°$.

6.6 $V_C = V_E = 2W$, $H_E = -H_C = 1.906W$, $F_{BH} = 2.24W(\overset{+}{\rightarrow})$ and $F_{BV} = 0$.

6.7 $H_A = -0.233W + 2.96X$; $V_A = -0.058W + 0.796X$;
$H_D = 1.142W - 1.96X$; $V_D = 1.474W - 0.796X$;
$F_{BH} = 0.708W - 2.96X$; $F_{BV} = 0.178W - 0.796X$;
$F_{EH} = -0.434W$; $F_{EV} = -1.296W$;
$F_{CH} = -0.476W$; $F_{CV} = -0.121W$.

CHAPTER 7

7.1 $S_E = -16$ kN, $M_E = 48$ kNm; $S_F = -4$ kN, $M_F = 92$ kNm.

7.2 $N = -5.0$ kN, $S = +0.447$ kN, $M = 93.3$ kNm.

7.3 $S_E = -15$ kN and $M_E = 135.0$ kNm.

7.4 $N_D = 100$ kN; $S_D = 0$; $M_D = 433.0$ kNm; $N_E = 50$ kN; $S_E = 86.6$ kN; $M_E = 216.5$ kNm.

7.5 At mid-point of AB: $M = 325$ kNm and $S = -130$ kN.
" BC: $M = 850$ kNm and $S = -30$ kN.
" CD: $M = 660$ kNm and $S = 70$ kN.
" DE: $M = 160$ kNm and $S = 120$ kN.
" EF: $M = -100$ kNm and $S = -40$ kN.

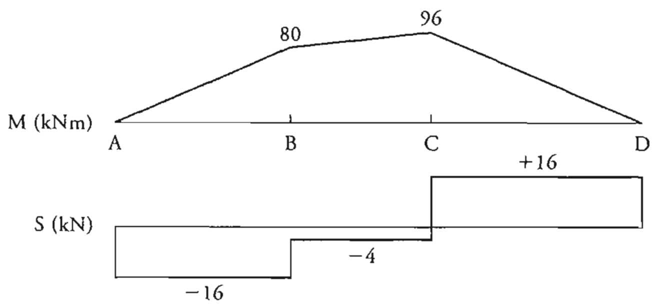
7.6 $N_P = N_Q = N_R = 0$; $M_P = 72.89$ kNm; $M_Q = 91.56$ kNm; $M_R = 110.2$ kN; and $S_Q = -6.22$kN.

7.7 (i) At P: $N = 0$, $S = -70$ kN and $M = 725$ kNm.
(ii) In AB: $S_x = 30x - 220$ kN and $M_x = 220x - 15x^2$ kNm.
In BC: $S_x = +80$ kN and $M_x = 1500 - 80x$ kNm.
In CD: $S_x = +80$ kN and $M_x = 1760 - 80x$ kNm.

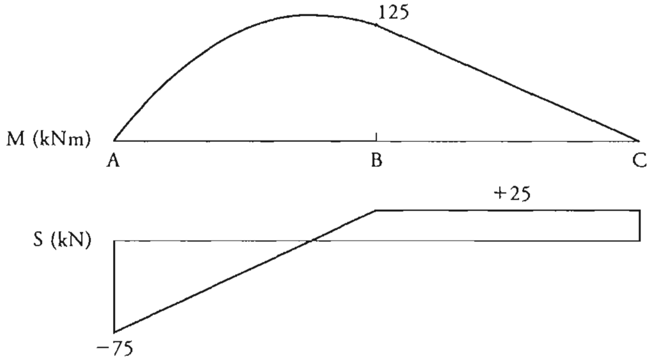
7.8 (i) $M_A = 0$; $M_B = 157.2$ kNm; $M_C = 131.4$ kNm; and $M_D = 451.4$ kNm.
(ii) $S_A = -71.43$ kN; $S_B = 8.57$ kN; $S_{CB} = 8.57$ kN; $S_{CD} = -80$ kN; $S_{ED} = 108.6$ kN; and $S_{EF} = -50$ kN.
(iii) $N_{AB} = 80$ kN, $N_{BC} = 80$ kN and $N_{CD} = 108.6$ kN.

7.9 $N_A = -27.32$ kN; $S_A = 0$; $M_A = 0$; $N_B = -23.66$ kN; $S_B = -13.66$ kN; $M_B = 18.30$ kNm; $N_C = -13.66$ kN; $M_C = 68.3$ kNm; $N_D = -10.0$ kN; and $M_D = 86.6$ kNm.

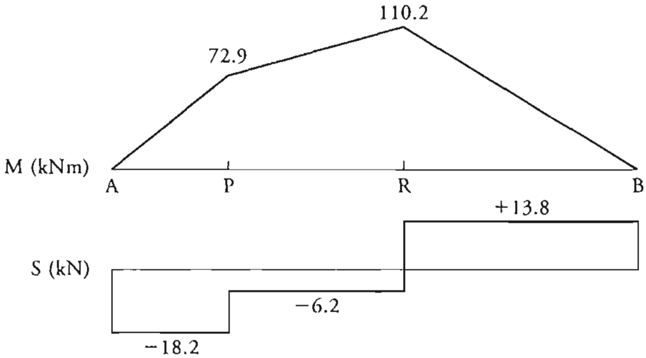
7.10



7.11 (i) In AB: $S = 20x - 75$ and $M = 75x - 10x^2$.
In BC: $S = 25$ and $M = 250 - 25x$.
(ii)



7.12



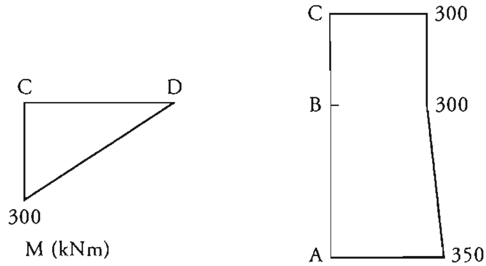
7.13 In AB: $S = 30x - 220$ and $M = 220x - 15x^2$.
In BC: $S = 80$ and $M = 1500 - 80x$.
In CD: $S = 80$ and $M = 1760 - 80x$.

7.14 (i) $H_A = 0$; $V_A = 2.22$ kN(+↓); $V_D = 2.22$ kN (+↑).
(ii) $M_A = 0$; $M_{BA} = -8.89$ kNm; $M_{BC} = 51.11$ kNm;
$M_{CB} = 33.33$ kNm; $M_{CD} = 13.33$ kNm; $M_D = 0$;
(iii) $S_{AB} = S_{BC} = S_{CD} = 2.22$ kN; $N_{AB} = N_{BC} = N_{CD} = 0$

7.15 (i) $R_A = (M_2 - M_1)/18$ +↑ and $R_D = (M_1 - M_2)/18$ +↑ .
(ii) In AB: $S = (M_1 - M_2)/18$ and $M = (M_2 - M_1)x/18$.
In BC: $S = (M_1 - M_2)/18$ and $M = (M_2 - M_1)x/18 + M_1$.
In CD: $S = (M_1 - M_2)/18$ and $M = (M_2 - M_1)x/18 + M_1 - M_2$.

7.16 At midpoint CD: $N = 0$; $S = -25$ kN; and $M = -12.5$ kNm.
At midpoint BE: $N = -90$ kN; $S = 0$; and $M = -30$ kNm.
At E: $N = -90$ kN; $S = 0$; and $M = -30$ kNm.

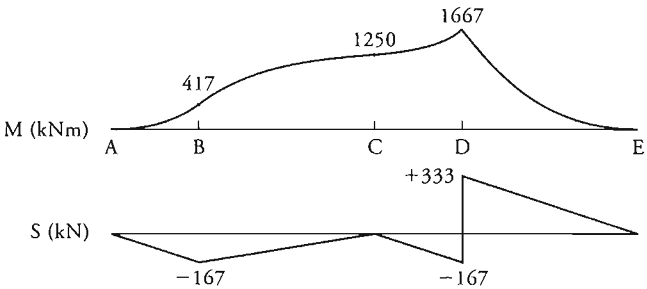
7.17



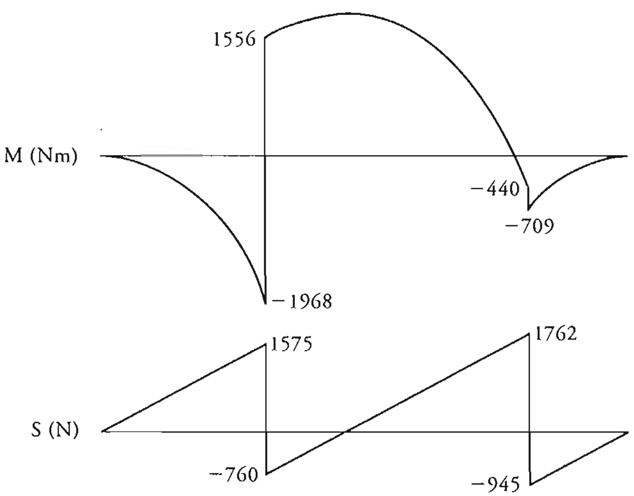
7.18 Midpoint AB: $N = 0$; $S = -15$ kN; $M = +7.5$ kNm.
Midpoint BC: $N = 12.99$ kN; $S = -7.5$ kN $M = 82.5$ kNm.
Midpoint DC: $N = 0$; $S = 15$ kN; $M = 52.5$ kNm.

7.19 Midpoint AB: $N = -67.43$ kN; $S = 0$; $M = 0$.
Midpoint CD: $N = 68.89$ kN; $S = -13.54$ kN; $M = 81.72$ kNm.
Midpoint DE: $N = 58.28$ kN; $S = -10.86$ kN; $M = 21.72$ kNm.
At D: $M_D = 43.43$ kNm.
At C: $M_C = -120.0$ kNm.

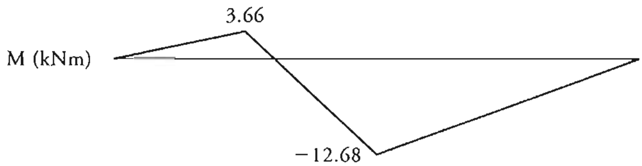
7.20



7.21



7.22



7.23 Thrust = $0.687P$.

7.24 In AB: $S = 2.5x^2 - 26.25$ and $M = 26.25x - 5x^3/6$.
In BC: $S = 15x - 48.75$ and $M = -7.5x^2 + 48.75x - 22.50$
At B: $S = -3.75$ kN and $M = 56.25$ kNm.

CHAPTER 8

8.1 $H_A = 0$; $V_A = 1$ kN; $V_C = 34$ kN; $V_F = 15$ kN.
8.2 $H_A = 1.77$ kN; $V_A = 4.27$ kN; $H_C = -1.77$ kN; $V_C = 5.73$ kN.
8.3 $H_A = 10.64$ kN; $V_A = 1.03$ kN; $H_D = -4.98$ kN; $V_D = 8.63$ kN.
8.4 $H_A = -0.62$ kN; $V_A = 14.24$ kN; $H_B = 4.62$ kN; $V_B = 4.62$ kN.
8.5 $H_A = 14$ kN; $V_A = 11$ kN; $H_D = -14$ kN; $V_D = 13$ kN.
8.6 $V_A = 1.5$ kN; $V_B = 11.0$ kN; $V_C = 1.5$ kN.
8.7 (a) $H_A = -6.67$ kN; $V_A = -4.8$ kN; $H_E = -9.33$ kN; and $V_E = 4.8$ kN.
(b) $H_A = -6$ kN; $V_A = -1.5$ kN; $H_E = 0$; and $V_E = 13.5$ kN.
(c) $H_A = 3$ kN; $V_A = 6$ kN; $H_E = -3$ kN; and $V_E = 3$ kN.
(d) $H_A = -6.25$ kN; $V_A = 1.33$ kN; $H_E = -5.75$ kN; and $V_E = 9.67$ kN.
8.8 $H_A = 11.9$ kN; $V_A = 22.5$ kN; $H_E = -11.9$ kN; and $V_E = 22.5$ kN.
8.9 (i) $H_A = 2.20$ kN; $V_A = 9.03$ kN; $H_G = -2.20$ kN; and $V_G = 4.97$ kN.
(ii) Resultant force through D = 2.41 kN.
8.10 $H_A = -5.25$ kN; $V_A = 1.25$ kN; $M_A = 21$ kNm; $H_G = -4.75$ kN; $V_G = 4.75$ kN; and $M_G = 9.5$ kNm
8.11 $H_A = 3.33$ kN; $V_A = 30.67$ kN; $H_D = -1.33$ kN; $H_E = -2.0$ kN; and $V_E = 14.33$ kN
8.12 $V_A = 8.5$ kN; $H_G = 10$ kN; $V_G = 1.5$ kN; $T = 15.6$ kN
8.13

	V_A(kN)	H_A(kN)	M_A(kNm)	V_F(kN)
Load at B	6	0	0	0
Load at C	6	0	18	0
Load at D	6	0	36	0
Load at E	3	0	18	3

8.14

	H_A(kN)	V_A(kN)	M_A(kNm)	H_J(kN)	V_J(kN)	M_J(kNm)
Load at B	0	8	−16	0	0	0
Load at E	5.33	4	−21.33	−5.33	4	21.33
Load at H	0	0	0	0	8	16

CHAPTER 9

9.1 $T_{max} = 20.75$ kN.
9.2 Sag at D = 1.506 m.
9.3 $T_{CD} = 15.94$ kN.
9.4 (i) 21.94 m. (ii) 410.1 N, 508.1 N, respectively. (iii) 19.5 m.
9.5 (i) 21.82 m. (ii) 429.6N, 538.7 N, respectively. (iii) 19.5 m.
9.6 (i) 23.78 m. (ii) 4.27 m. (iii) 78.27 kN.
9.7 (i) 11.89 m. (ii) −7.64 m. (iii) 19.56 kN
9.8 (i) 23.43 m and 15.62 m. (ii) 3.73 m.
9.9 $T_{max} = 7537$ kN and $T_{min} = 6800$ kN.
9.10 Sag at B = 74.8 m; Sag at C = 119.6 m; $T_{max} = 390$ kN.
9.11 Maximum sag in AB =37.1 m and in BC =59.3 m ; and T_{max} =15.2 kN.

CHAPTER 11

11.1 $H_1 = 20$ kN; $V_1 = 126$ kN; and $V_2 = 74$ kN.

Member	AB	AC	BC	BD	CD	CE	DE	DF
Force (kN)	−86	−44.7	55.3	−30.7	−103.4	68.0	7.2	−92
Member	EF	EG	FG					
Force (kN)	40.9	49.3	−88.9					

11.2 (i) $R_1 = 57.60$ kN; $R_2 = 35.66$ kN; $R_3 = 49.46$ kN.
(ii)

Member	AB	AC	BC	BD	CD	CE	DE	EG
Force (kN)	−50.4	−21.9	−2.6	−38.4	3.2	−23.8	0	−23.8
Member	DG	DF	FG	GH	FH			
Force (kN)	2.0	−37.7	21.0	0	−49.5			

11.3

Member	AB	AD	BD	BC	CD	CE	ED	DF
Force (kN)	−80	0	−40	−80	133.3	−146.7	0	66.7
Member	EF	EG	GF	GJ	FJ	FH	HJ	
Force (kN)	0	−146.7	0	−146.7	133.3	−40	80	

11.4 $R_1 = 24$ kN; $R_2 = 128$ kN; $R_3 = 36$ kN; $R_4 = -48$ kN.

Member	AB	AD	BD	BC	CD	CE	ED	DF
Force (kN)	−96	−40	0	−96	20	−12	−48	−12
Member	EF	EG	GF	FH	GH	GJ	HJ	JK
Force (kN)	60	−48	−48	24	−40	−24	−48	60

11.5 $V_1 = 9.17$ kN; $V_2 = 8.39$ kN; $H_2 = 5.46$ kN.

Member	AD	AB	BD	BC	CD	CE	DE	DF
Force (kN)	16.1	−18.5	−8.1	−10.4	10.3	−10.4	−12.8	20.1
Member	EF							
Force (kN)	−16.9							

11.6

Member	AB	AC	BC	BD	DC	CE	DE	DF
Force (kN)	−8	0	15.5	−13.3	−2.0	13.3	3.9	−16.7
Membe	EF	EG	EH	FH	GH			
Force (kN)	0	0	19.4	−16.7	−10			

11.7

Member	AB	AC	BC	BD	CD	CE	DE	DF
Force (kN)	−15.0	7.5	15.0	−15.0	−10.4	20.2	1.1	−20.8
Member	EF	EG	FG	FH	GH	GK	HK	
Force (kN)	5.8	17.9	−5.8	−15.0	15.0	7.5	−15.0	

11.8

Member	AC	AB	BC	BD	CD	CE	DE	DF
Force (kN)	13.1	−10.2	−4.7	−8.7	3.7	8.1	−6.7	−5.7
Member	EF	EG	FG					
Force (kN)	6.3	3.1	−9.9					

11.9

Member	AB	BD	BC	AC	CD	DF	DE	CE
Force (kN)	0	4	$\sqrt{2}$	−5	−1	3	$\sqrt{2}$	−4
Member	FG	EG	GH	HK	HJ	GJ	JK	KL
Force (kN)	$\sqrt{2}$	−3	−1	1	$\sqrt{2}$	−2	−1	$\sqrt{2}$
Member	EF	FH	JL					
Force (kN)	−1	2	−1					

11.10

Member	AC	AB	BC	CE	BE	BD	DE	EG
Force (kN)	100.6	−112.5	0	100.6	−62.5	−75.0	28.0	44.7
Member	DG	DF	FG	GH	FH			
Force (kN)	−79.0	−37.5	55.9	−11.2	−100.8			

11.11

Member	AE	AD	AC	AB	BC	CD	DE
Force (kN)	0	32.1	30.2	73.4	−82.9	−88.3	−88.3

CHAPTER 12

12.1 (a) $X_1 = -74$ kN; $X_2 = 20\sqrt{2}$ kN; and $X_3 = 54$ kN.
(b) $X_1 = -50$ kN; $X_2 = 0$; $X_3 = 0$; and $X_4 = 70$ kN.

12.2 $R_1 = 0$; $R_2 = 51.67$ kN; and $R_3 = 46.33$ kN.

Member	BC	BE	DE	DG	FG	GH	HI	IJ
Force (kN)	−51.7	50.4	−35.7	9.4	−16.0	13.2	−38.2	54.2
Member	JK							
Force (kN)	−46.3							

Member	AB	BD	DF	FH	HJ	JL	AC	CE
Force (kN)	−51.7	−87.3	−94.0	−94.0	−84.7	−46.3	73.1	51.7
Member	EG	GI	IK	KL				
Force (kN)	87.3	84.7	46.3	65.5				

12.3 $F_{EG} = -833.3$ kN; $F_{EF} = 168.2$ kN; and $F_{DF} = 711.9$ kN.
12.4 (i) Force in vertical members nearest load = −300 kN.
Force in vertical members away from load = +250 kN.
(ii) Force in diagonal members = 0.
12.5 $F_{BD} = -100$ kN; $F_{BE} = 50\sqrt{2}$ kN; and $F_{CE} = -5$ kN.
12.6 $F_{CD} = 3.2$ kN; $F_{CE} = -23.8$ kN.
12.7 $F_{CE} = 13.3$ kN; $F_{DE} = 3.9$ kN; $F_{DF} = -16.7$ kN
12.8 $F_{CD} = 3.7$ kN; $F_{CE} = 8.1$ kN.
12.9 (i) $V_A = 4.76$ kN; $H_A = 5.59$ kN; $V_G = 16.24$ kN;
$H_G = 14.41$ kN.
(ii) $F_{AB} = -15.62$ kN; $F_{AE} = 14.39$ kN; $F_{BE} = -6.84$ kN;
$F_{BC} = -10.39$ kN; $F_{CE} = 4.83$ kN; $F_{CD} = -11.99$ kN;
$F_{DE} = -3.46$ kN; $F_{DF} = -13.05$ kN; and $F_{EF} = 11.41$ kN.
12.10 $F_{AF} = -7.826P$; $F_{AJ} = 7.0P$; $F_{FJ} = -0.894P$; $F_{FG} = -7.379P$;
$F_{GJ} = 1.0P$; $F_{JD} = 6.0P$; $F_{GD} = -1.789P$; $F_{GH} = -6.932P$;
$F_{GK} = 1.0P$; $F_{DK} = 2.0P$; $F_{HK} = -0.894P$; $F_{HC} = -6.485P$;
$F_{CK} = 3.0P$; and $F_{DE} = 4.0P$.

CHAPTER 13

13.1 $M_X = 80$ kNm and $M_Y = 24$ kNm.

Member	AB	AE	BE	BC	CE	DE	CY	YD
Force (kN)	$-12\sqrt{2}$	12	$12\sqrt{2}$	−24	$8\sqrt{2}$	16	$-12\sqrt{2}$	$-20\sqrt{2}$

13.2 At P: $N = 0$; $S = 4$ kN; $M = 4$ kNm.
13.3 $N = -110.9$ kN; $S = 7.2$ kN; $M = 12.0$ kNm.
13.4 $N = -20$ kN; $S = 0$; $M = -10$ kNm.

CHAPTER 14

14.1 $F = 246.0$ kN at 1.516 m above bottom of canal.
14.2 $F_{BD} = 220.0$ kN.
14.3 $F = 60.43$ kN acting 2.919 m below the free water surface on the vertical centreline of the gate.
14.4 $F = 41.75$ kN.
14.5 (i) $R_A = 4.598$ kN and $R_B = 22.99$ kN (both horizontal).
(ii) 0.15 m.
14.6 $F_H = 30.66$ kN; $y = 1.67$ m; $F_V = 48.15$ kN; $x = 1.061$ m; and $M_C = 0$ kNm.
14.7 $R_A = 45.22$ kN and $R_B = 9.49$ kN.
14.8 $V = 0.0408$ m^3 and $\gamma = 24.53$ kN/m^3.
14.9 (i) 5.759 kN. (ii) 5.246 kN.
14.10 128 040 m^3.
14.11 9.99 kN.
14.12 (i) Yes. (ii) $H = 1.569$ m.
14.13 (i) 0.39 m. (ii) No.
(iii) With one longitudinal face parallel to and above the free water surface.
14.14 0.876.
14.15 $h = 0.822$ m.

CHAPTER 15

15.1 $X = 40$ N; $Y = -40$ N; $Z = 70$ N.

15.2

	X	Y	Z
(i)	67.1 N	89.4 N	223.6 N
(ii)	103.9 N	103.9 N	−103.9 N
(iii)	−94.9 N	170.8 N	227.7 N
(iv)	320.0 N	−200.0 N	357.8 N
(v)	−80.0 N	120.0 N	240.0 N

15.3 $R = 41.11$ N in a direction given by $\theta_x = 53.60°$, $\theta_y = 40.06°$, and $\theta_z = 75.56°$.

15.4

	R	*l*	*m*	*n*
(i)	945.8 N	0.9715	0.0846	0.2214
(ii)	160.4 N	0.5715	0.7480	0.3376
(iii)	439.5 N	0.0724	0.2628	−0.9621

15.5 11.2 N at 67.7° to the line of greatest slope with a component down the plane.

CHAPTER 16

16.1 $F_{AB} = 53.6$ N; $F_{AC} = -87.6$ N; and $F_{AD} = 8.9$ N.
16.2 $P = 13.7$ N; the 60 N force acts in the direction with $\theta_x = 134.74°$, $\theta_y = 127.95°$, and $\theta_z = 69.16°$

16.3 Ties: $T_{AB} = T_{AC} = 113.1$ N; Struts: $C_{AD} = C_{AE} = 61.9$ N.
16.4 $T_{OA} = 6.0$ kN; $T_{OB} = 4.0$ kN; and $T_{OC} = 12.3$ kN.
16.5 All three bar forces must be zero.
16.6 $T_{BE} = 2459$ N and compression in mast = 4667 N.
16.7 $z = -1.087$ m; $T_{BE} = 2346$ N; compression in mast = 4423 N.
16.8 $T_{OA} = 0.25$ kN; $T_{OB} = 4.03$ kN; and $T_{OC} = 2.30$ kN.
16.9 $T_{OA} = T_{OB} = 3.62$ kN; and $T_{OC} = 3.89$ kN.
16.10 $F = 10.32$ kN; $l = 0.8670$; $m = 0.3219$; and $n = 0.3804$: or
$F = 4.97$ kN; $l = 0.2500$; $m = 0.9389$; and $n = -0.2366$.

CHAPTER 17

17.1 $M_R = 50$ kNm; $l = 0.6$; $m = -0.8$; $n = 0$.
17.2 $M_x = -25.46$ Nm; $M_y = 0$; $M_z = -25.46$ Nm; $M_{AB} = -29.39$ Nm.
17.3 $M_x = 4.08$ Nm; $M_y = 8.16$ Nm; $M_z = 0$; $M = 9.13$ Nm;
d = 0.913 m.
17.4 $M_x = -60$ Nm; $M_y = -67.08$ Nm; $M_z = -214.2$ Nm;
$M_R = 232.3$ Nm; $M_{OC} = 138.7$ Nm.
17.5 $M_x = -12$ Nm; $M_y = 0$; $M_z = -18$ Nm; $M_R = 21.63$ Nm
about axis with $l = -0.554$; $m = 0$; $n = -0.831$; $M_{AB} = 4.243$ Nm;
moment about axis through O parallel to AB = −4.243 Nm.
17.6 (i) $X = -4$ kN; $Y = 1$ kN; $Z = -2$ kN; $M_x = 11$ kNm;
$M_y = 12$ kNm; $M_z = 5$ kNm.
(ii) At centre: $X = 1$ kN; $Y = 1$ kN; $Z = -13$ kN. In BCDE:
$X = -5$ kN; $Y = 0$; $Z = 11$ kN (the force intersecting CD at 4/11 m from C).
17.7 (i) $F = 5$ N in direction DA; $C = 13.75$ Nm about axis $\theta_x = 54.40°$,
$\theta_y = 68.70°$, and $\theta_z = 43.30°$.
(ii) $F = 5$ N in direction GF; $C = 16.40$ Nm about axis $\theta_x = 37.60°$,
$\theta_y = 90°$, and $\theta_z = 52.40°$.
17.8 (i) $R = 15.17$ N; $l = 0.9891$; $m = 0.0659$; $n = 0.1319$;
$M_R = 101.5$ Nm; $l = 0.1872$; $m = 0.9161$; $n = 0.3546$.
(ii) Force passes through the point (32.09, 0, 10.35) and $M = 29.67$ Nm.
17.9 The system of forces shown in Figure P17.9 does not have a moment about the axis BC.
17.10 $F_{DA} = 10.88$ kN; $F_{AB} = 6.74$ kN; $F_{CA} = 1.83$ kN; $F_{BD} = 7.07$ kN;
$F_{DC} = 5.98$ kN; $F_{CB} = 8.20$ kN.
17.11 (i) 135.5 Nm. (ii) 202.7 Nm.
17.12 1.414 Nm
17.13 $R = 111.8$ N in direction $\theta_x = 26.56°$, $\theta_y = 90°$, and $\theta_z = 63.43°$.
$M_R = 67.1$ Nm about an axis with $\theta_x = 90°$, $\theta_y = 63.43°$, and $\theta_z = 26.56°$.
17.14 $X = 20$ kN; $Y = -16$ kN; $Z = 0$; $M_x = -22$ Nm; $M_y = 92$ Nm;
and $M_z = -22$ Nm.
17.15 −16 kNm; 2.34 kNm
17.16 (i) 86.6 Nm; 10 Nm; and 0.
(ii) $100 \sin \theta$ Nm; $(60 - 100 \cos \theta)$ Nm; and 0.

17.17 (i) 8.18 Nm (ii) 38.41 N
17.18 $F_{OA} = 13.78$ kN ; $F_{OB} = -10.68$ kN ; $F_{OC} = 5.06$ kN; $(M_{AC})_F = 75.80$ kNm; $(M_{AC})_{OB} = -75.80$ kNm.
17.19 28.80 kNm.

CHAPTER 18

18.1 $A_1 = 2.5$ kN; $A_2 = 3.46$ kN; $A_3 = 6.0$ kN; $B_1 = 10.5$ kN; $C_1 = 8.0$ kN; $C_2 = 6.93$ kN.
18.2 $A_y = -173.3$ N; $A_z = 676$ N; $B_y = 413.3$ N; $B_z = -52$ N; $T_{chain} = 720$ N.
18.3 $A = B = 0.268\ W$; $C = 0.464\ W$.
18.4 $X_A = -69.28$ N; $Y_A = -100$ N; $Y_C = 0$; $X_C = 17.32$ N; $Y_B = 100$ N; $Z_B = 30$ N.
18.5 $R_1 = 0$; $R_2 = 19.4$ kN; $R_3 = 0$; $R_4 = 15$ kN; $R_5 = 20.6$ kN; $R_6 = 15$ kN.
18.6 $R_A = 3.0$ kN; $R_B = 3.6$ kN; $R_C = 2.4$ kN.
18.7 $R_1 = 2$ kN; $R_2 = 2$ kN; $R_3 = 1.5$ kN; $R_4 = 0$; $R_5 = 7.5$ kN; $R_6 = 6$ kN.
18.8 $R_1 = 5.33$ kN; $R_2 = 8$ kN; $R_3 = 0$; $R_4 = -5.33$ kN; $R_5 = 4$ kN; $R_6 = 16$ kN.
18.9 $R_1 = 200$ N; $R_2 = 0$; $R_3 = 60$ N; $R_4 = 210$ N; $R_5 = -200$ N; $R_6 = -420$ N.
18.10 $R_x = -10$ N; $R_y = -5$ N; $R_z = 0$; $M_x = 20$ Nm; $M_y = 40$ Nm; $M_z = -60$ Nm.
18.11 $R_x = -6$ N; $R_y = 10$ N; $R_z = 0$; $M_x = 60$ Nm; $M_y = 36$ Nm; $M_z = 20$ Nm.
18.12 $R_x = 0$; $R_y = 50$ kN; $R_z = -5$ kN, $M_x = 30$ kNm; $M_y = 0$; $M_z = -90$ kNm.
18.13 $N = 20$ N; $S_y = 0$; $S_x = -10$ N; $T = 0$; $M_y = 120$ Nm; $M_z = -20$ Nm.
18.14 $N = 0$; $S_y = 0$; $S_z = -100$ N; $T = 600$ Nm; $M_y = -200$ Nm; $M_z = 0$.
18.15 $X = -8$ kN; $Y = -6$ kN; $Z = -24$ kN; $C_x = -12$ kNm; $C_y = 39$ kNm; $C_z = -48$ kNm.
18.16 $X = -P$; $Y = Q$; $Z = 0$; $C_x = -Qr$; $Cy = -Pr$; $C_z = Q(L_1 + r) - PL_2$.
18.17 $R_1 = -4$ kN; $R_2 = 1.8$ kN; $R_3 = 12$ kNm; $R_4 = 4.2$ kN; $R_5 = -3$ kN; $R_6 = -1$ kNm.
18.18 0.849 N.
18.19 $AA' = BB' = 0.927$ kN; $MM' = 1.073$ kN.

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ENGINEERING STATICS second edition, deals with the fundamentals of statics and their application to a broad range of engineering problems. It is primarily intended as a textbook for undergraduate engineering students, and should also be of value to anyone involved or interested in engineering mechanics.

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