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Francesco Amato

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To Velia and Lorenzo

Preface

The last thirty years have witnessed an enormous effort in the field of robust control of dynamical systems. The main objective of this book is that of presenting, in a unified framework, the main results appeared in the literature on this topic, with particular reference to the robust stability problem for linear systems subject to time-varying uncertainties.

The book mainly focuses on those problems for which a definitive solution has been found; indeed most of the results we shall present are given in the form of necessary and sufficient conditions involving the feasibility of Linear Matrix Inequalities based problems.

For self-containedness purposes, most of the results provided in the book are proven. We have tried to maintain the development of the proofs as simple as possible, without sacrificing the mathematical rigor.

Some parts of the book (especially those contained in Chaps. 2, 3 and 5) can be teached in advanced control courses; however this work is mainly devoted to both researchers in the field of systems and control theory and engineers working in industries which want to apply the methodologies presented in the book to practical control problems. To this regard, as the various results are derived, they are immediately reinforced with real world examples.

Catanzaro *Francesco Amato*
July 2005

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1. Introduction

According to the celebrated paper [68], linear systems uncertainties can be divided into two big families: *dynamical input-output* uncertainties and *state-space* uncertainties.

Dynamical input-output uncertainties are typically time-invariant and come from approximations related to the system model. Often the system under consideration is either infinite-dimensional or high order finite-dimensional; then a first step toward the controller synthesis procedure is that of associating a reduced order finite-dimensional model to the original system by neglecting the high frequency dynamics.

Model approximations, neglected dynamics, etc., can be lumped into one or more dynamical, finite-dimensional, linear time-invariant systems, which are located in series and/or in parallel to the nominal system. In this case the whole uncertain system is described as in Fig. 1.1, where u and y are the input and the output, and $G(s)$ and $\Delta(s)$ correspond to the nominal and the uncertain part of the system respectively.

The complex matrix function $\Delta(s)$ can be either a full block transfer function matrix (when there is only one source of uncertainty) or a block diagonal transfer function matrix (when there are multiple sources of uncertainties, for example at the input and the output of the plant). At each frequency, a bound on the norm of each uncertainty block is given on the basis of experimental considerations.

In this context, in the last twenty-five years, the problems of checking the system stability and designing a controller for a given system which maintains closed loop stability and, at the same time, guarantees given performance requirements in presence of uncertainties, have been widely investigated and many related issues have been solved in an elegant and computationally appealing way. The main outcomes of this activity have been the $\mathcal{H}_\infty$ and μ control theories; see among others the fundamental works by Zames [181–183], Zames and Francis [184], Doyle [63], Doyle and Stein [67, 68], Lehtomaki et al [115], Safonov and Athans [155, 156], Safonov et al [157], Cruz et al [53], Freudenberg and Looze [76, 77], Francis and Zames [74], Francis et al [73], Doyle et al [65], and, to witness an assessment of the methodologies, the books published on this topic ([64, 72, 121, 166] and [185] among others) and the collection of papers by Dorato [61] and Dorato and Yedavalli [62].

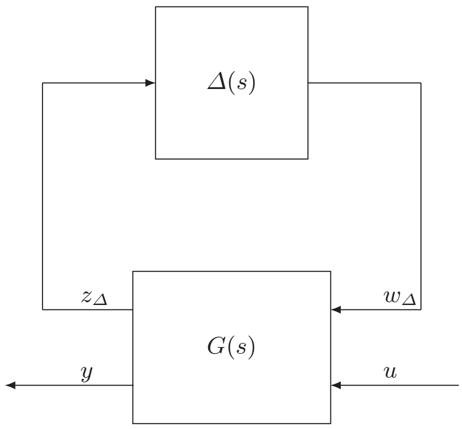


Fig. 1.1. Dynamical input-output uncertainties

Conversely, state-space uncertainties take into account the approximate knowledge of the numerical values of the physical parameters on which the system depends. While dynamical input-output uncertainties appear at a transfer function level and are time-invariant, state-space uncertainties enter directly the system matrices, are memoryless (possibly time-varying) and can be divided into two major classes, that is *parametric* uncertainties

$$\dot{x} = A(p)x + B(p)u \quad p \in R \subset \mathbb{R}^q \tag{1.1a}$$

$$y = C(p)x + D(p)u\,, \tag{1.1b}$$

and *norm bounded* uncertainties

$$\dot{x} = (A + \Delta A)\,x + (B + \Delta B)\,u \tag{1.2a}$$

$$y = (C + \Delta C)\,x + (D + \Delta D)\,u\,, \tag{1.2b}$$

where

$$\begin{pmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \Delta (I - H\Delta)^{-1} \begin{pmatrix} E_1 & E_2 \end{pmatrix}\,, \quad \|\Delta\| \leq 1\,, \tag{1.3}$$

and F_i, E_i, $i = 1, 2$, H are matrices of suitable dimensions.

It can be shown that the uncertain system (1.2)–(1.3) and, under some assumptions on the parameter dependence, the uncertain system (1.1) can be recasted into the $\mathcal{H}_\infty$ and μ frameworks represented in Fig. 1.1. However, as said, $\mathcal{H}_\infty$ and μ theories deal with time-invariant dynamical uncertainties and mainly rely on input-output operator theory methods; conversely, in this book we deal with time-varying memoryless uncertainties. Therefore an approach based on the Lyapunov state-space stability theory will be used. Nevertheless very intriguing connections between the two approaches exist and will be analyzed in this work.

To introduce more precisely the problems we deal with in this book, let us consider a zero-input uncertain system depending on a parameter p (representation (1.1)) in the form

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2+p & -1 \end{pmatrix} x =: A(p)x\,. \tag{1.4}$$

Assume that the exact value of the parameter is unknown, and that we have only an information regarding the minimum and maximum value that the parameter can attain

$$p \in [\underline{p}, \overline{p}]\,. \tag{1.5}$$

At this point we must distinguish between three different situations:

i) The parameter is constant;
ii) the parameter is time-varying, but the time behavior is unknown;
iii) the parameter is time-varying and a bound on the maximum rate of variation is known.

In case i) to establish system stability it is sufficient to check that the eigenvalues of $A(p)$ have negative real part for all $p \in [\underline{p}, \overline{p}]$. If, as in the case of system (1.4), the system depends on only one parameter one can apply the classical Routh Criterion [75]; in the multi-parameter case one can use techniques based on the Kharitonov Theorem [112], on the Edge Theorem [27, 31], on the μ analysis approach [63].

If the parameter is time-varying (case ii), negativeness of the real part of the eigenvalues is no longer sufficient to guarantee system stability [57]. In this case the stability analysis proceeds via the use of Lyapunov techniques, which leads to the definition of *quadratic stability*.

Case iii) can be considered intermediate between the two described above, but differently from these ones it has come to researchers attention more recently and many issues are still open and will be the subject of future research.

The central issue of this book will be the stability analysis and synthesis in presence of time-varying uncertainties and therefore will deal with cases ii) and iii) above. Since stability versus time-varying parameters implies stability versus constant parameters, the techniques developed in case ii) can also be used, at the price of some conservatism, to deal with case i); moreover, the approaches developed in the context of case iii) can apply to case i), by taking a zero rate of variation.

1.1 Book Organization

The book is composed of two introductory chapters (this chapter and Chap. 2 which provides some useful results on linear time-varying systems) and four chapters dealing more specifically with the analysis and design of uncertain linear systems.

The first part of Chap. 3 considers systems depending on parametric uncertainties whose time behavior is unknown. To this end we introduce the concept of *quadratic stability.* A system is said to be quadratically stable when there exists a quadratic Lyapunov function whose derivative, computed along the solutions of the system, is negative definite for all values of the parameters. Quadratic stability guarantees exponential stability versus *all admissible* time realizations of parameters within their bounding set (which is assumed to be a hyper-box). It will be shown that testing the quadratic stability of a given system is equivalent to find a feasible solution to a set of Linear Matrix Inequalities (LMIs) when the dependence of the system matrix $A(\cdot)$ on the parameters can be written as the ratio of a multi-affine matrix-valued function and a multi-affine polynomial. In the case of general nonlinear dependence on parameters some techniques are provided to transform the original nonlinear function into a multi-affine one to apply again (at the price of some conservatism) the previous result.

When the number of uncertain parameters is big, the application of the above-mentioned techniques may demand a prohibitive computational burden. In this case a statistical approach to quadratic stability analysis of uncertain systems can be pursued; roughly speaking, by this approach one can conclude that a given system is quadratically stable *with a certain probability.*

Often, stability is not the only requirement that a system must exhibit; therefore a specific section of Chap. 3 is devoted to deal with the problem of quadratic stability plus performances. In particular quadratic $\mathcal{D}$-stability, where $\mathcal{D}$ is a suitable open domain contained in the left half of the complex plane, quadratic stability with an $\mathcal{L}_2$ performance bound and quadratic stability with the satisfaction of a linear quadratic (LQ) performance criterion (guaranteed cost) are considered.

In the second part of Chap. 3 norm bounded uncertainties are considered. In this case a necessary and sufficient condition for quadratic stability is the existence of a positive definite solution of a Riccati inequality, which can be again converted to an LMI problem by using Schur Complements. Interesting connections existing between quadratic stability and $\mathcal{H}_\infty$ control theory are also investigated.

Throughout the chapter, some examples are illustrated to both clarify the application of the various results presented and to show how the theory can be applied to real-life systems.

As said, quadratic stability is a strong form of stability; it guarantees exponential stability of the given system versus arbitrary fast variation of the parameters. When it is known that some (or all) of the parameters are constant or slowly time-varying, the quadratic stability approach may become conservative. Therefore in Chap. 4 some techniques, which allow the stability analysis by taking into account the information about the rate of variation, are provided. These techniques are based on the use of *parameter dependent* quadratic Lyapunov functions. We shall show, by continuing the examples of Chap. 3, that, in this way, it is possible to obtain less conservative conditions for system stability. This last approach can be used in particular when the uncertainty is constant (the rate of variation is zero).

In Chap. 5 we consider the design problem. A number of conditions for quadratic stabilization via state and output feedback, both for parametric and norm bounded uncertainties, are provided. As in Chap. 3, a section is devoted to discuss quadratic stabilization with performances, namely quadratic $\mathcal{D}$-stabilization, quadratic stabilization with an $\mathcal{L}_2$ performance bound and guaranteed cost control.

The theory developed in previous chapters considered continuous-time systems. More recently researchers have tried to generalize to discrete-time systems (which are becoming more and more important as computer controlled systems are playing a major role in control applications) the results found in the continuous-time case. In Chap. 6 an overview of such results is provided; moreover an interesting real life application is presented. We consider the problem of controlling a plasma wind tunnel, to simulate the re-entry conditions of space vehicles by reproducing desired trajectories in pressure and temperature on a test model. It is shown that the linearized model of the plant can be described by a discrete-time system depending on nine uncertain parameters; eight of such parameters exhibit small excursions and no bound on their rate of variation is available, while the last parameter turns out to be slowly varying and a bound on the rate of variation is known. A controller is designed, on the basis of the theory developed in this chapter, so to robustly stabilize the overall closed loop system versus the above-mentioned uncertain parameters.

For self-containedness purposes, the proofs of the main theorems are provided. Whenever possible we give alternative (and simpler) proofs of those ones available in the literature, while maintaining a rigorous treatment of the matter; in any case a reference is made to the paper where the theorem has been originally stated. In the simpler cases the proofs are left as exercises at the end of the chapter. Moreover each chapter is equipped with a summary which recalls the main topics we have dealt with, outlines a brief history of the development of the research concerning such topics and provides further references for alternative approaches other than the ones considered in the book.

Finally, all numerical computations done in the examples have been performed with the aid of the MATLAB™ software.

2. Linear Time-Varying Systems

In this chapter we consider the qualitative behavior of solutions of the system of linear differential equations

$$\dot{x}(t) = A(t)x(t)\,, \qquad t \in \mathbb{R}^+\,, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$. In particular we shall investigate which hypothesis the matrix function $A(\cdot)$ must satisfy such that *existence* and *uniqueness* of the solution of system (2.1) are guaranteed. Moreover some conditions guaranteeing stability of the equilibrium of system (2.1) will be given.

2.1 Existence and Uniqueness

Our main result of the section is a theorem guaranteeing existence and uniqueness of the solution of system (2.1). In the following $\mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{n\times n})$ denotes the space of the matrix-valued functions of dimension $n \times n$ which are piecewise continuous over $\mathbb{R}^+$. Therefore if $\Theta(\cdot)$ is of class $\mathcal{PC}$, in any compact interval contained in $\mathbb{R}^+$ it has a finite number of discontinuity points; at a discontinuity point the left and right limits of $\Theta(\cdot)$ exist and are finite.

Theorem 2.1 (Existence and uniqueness of the solution). *Let* $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ *and assume that* $A(\cdot) \in \mathcal{PC}\left(\mathbb{R}^+, \mathbb{R}^{n\times n}\right)$*; then system* (2.1) *admits a* unique *solution* $\varphi(\cdot, t_0, x_0) \in \mathcal{C}_0\left([t_0, +\infty), \mathbb{R}^n\right)$ *satisfying* $\varphi(t_0, t_0, x_0) = x_0$.

Proof. Since $A(\cdot)$ is of class $\mathcal{PC}$, for any given x, $A(t)x$ is a bounded and continuous function of t, with the exception of at most a finite number of points in any compact subset of $[t_0, +\infty)$. Moreover, for arbitrary $x, y \in \mathbb{R}^n$, $T > 0$,

$$\begin{aligned} \|A(t)x - A(t)y\| &\leq \|A(t)\|\|x - y\| \\ &\leq k_T\|x - y\| \\ &:= sup_{t\in[t_0,t_0+T]}\|A(t)\|\|x - y\|\,, \qquad t \in [t_0, t_0 + T]\,, \end{aligned} \tag{2.2}$$

that is the RHS of (2.1) satisfies a global Lipschitz condition in x. This allows to apply the existence and uniqueness theorem for differential equations (see [42], pp. 470–471). □

In the following, unless otherwise stated, we shall assume that the hypothesis of Theorem 2.1 hold for $A(\cdot)$.

2.2 The State Transition Matrix

Let us denote by $x(t) = \varphi(t, t_0, x_0)$ the unique solution of system (2.1) starting from x_0 at time t_0. Now it is simple to show (see Exercise 2.1) that, for a given pair $(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, the mapping

$$x_0 \in \mathbb{R}^n \mapsto \varphi(t, t_0, x_0) \in \mathbb{R}^n \tag{2.3}$$

is linear. Hence by the Matrix Representation Theorem [130], p. 188, there exists a matrix $\Phi(t, t_0)$ such that

$$\varphi(t, t_0, x_0) = \Phi(t, t_0) x_0 \,. \tag{2.4}$$

The matrix function $(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \to \Phi(t, t_0)$ is called the *State Transition Matrix*. It plays a fundamental role for the study of linear time-varying systems. The next result is obvious.

Fact 2.1 (Solution of system (2.1)). *The unique solution of system (2.1) starting at time t_0 from x_0 is*

$$x(t) = \Phi(t, t_0) x_0 \,. \tag{2.5}$$

□

Note that, for all $x_0 \in \mathbb{R}^n$, we have $\Phi(t_0, t_0) x_0 = x(t_0) = x_0$. From this follows that

$$\Phi(t_0, t_0) = I \,. \tag{2.6}$$

Equality (2.6) is referred to as the *consistency* property of the State Transition Matrix.

Definition 2.1 (Fundamental Matrix). Any solution $X(\cdot)$ of the matrix differential equation

$$\dot{X}(t) = A(t) X(t) \,, \tag{2.7}$$

satisfying $\det(X(t)) \neq 0$ for all $t \in \mathbb{R}^+$, is called a *Fundamental Matrix* of system (2.1). ◇

Fact 2.2 (Computation of Φ via a Fundamental Matrix). *The following equality holds for all $t, t_0 \in \mathbb{R}^+$, $t \geq t_0$,*

$$\Phi(t, t_0) = X(t) X^{-1}(t_0) \,, \tag{2.8}$$

where $X(\cdot)$ is any Fundamental Matrix of system (2.1).

Proof. The proof follows from the fact that both sides of (2.8) satisfy the same matrix differential equation

$$\dot{\Lambda}(t) = A(t)\Lambda(t)\,, \qquad \Lambda(t_0) = I\,. \tag{2.9}$$

□

The next properties follow directly from Fact 2.2

Fact 2.3 (Composition (Transition)). *For all* $t, t_0, t_1 \in \mathbb{R}^+$, $t \geq t_1 \geq t_0$,

$$\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)\,. \tag{2.10}$$

□

From Fact 2.3 it follows that, letting

$$x_1 = \varphi(t_1, t_0, x_0)\,, \qquad t_1 > t_0\,,$$

we have

$$\varphi(t, t_0, x_0) = \varphi(t, t_1, x_1)\,, \qquad t \geq t_1\,. \tag{2.11}$$

Equality (2.11) is called *Transition Property* of the state; this explains the name given to $\Phi(t, t_0)$.

By (2.8) we can extend the definition of $\Phi(\cdot, \cdot)$ to the case in which the first argument is not greater than the second argument, indeed for $0 \leq t_0 \leq t$

$$\Phi(t_0, t) := X(t_0)X^{-1}(t)\,. \tag{2.12}$$

The next result, which derives directly from (2.8) and (2.12), shows that $\Phi(t_0, t)$ is exactly the inverse of $\Phi(t, t_0)$.

Fact 2.4 (Inversion). *For all* $(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *we have*

$$\Phi(t, t_0)^{-1} = \Phi(t_0, t)\,. \tag{2.13}$$

□

Note that, for all $(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\Phi(t, t_0)$ is always invertible; this means that, for all $0 \leq t_0 \leq t$, it is always possible to go back in time and obtain x_0 starting from $x(t)$:

$$\begin{aligned} x_0 &= \Phi(t, t_0)^{-1}x(t) \\ &= \Phi(t_0, t)x(t)\,. \end{aligned} \tag{2.14}$$

Example 2.1.
Let us consider system (2.1) with

$$A(t) = \begin{pmatrix} -1 - 5\cos t \sin t & -5\cos^2 t + 1 \\ 5\sin^2 t - 1 & -1 + 5\cos t \sin t \end{pmatrix}\,. \tag{2.15}$$

It is simple to verify that, according to Fact 2.2, the State Transition Matrix is given by (2.8), where

$$X(t) = e^{-t} \begin{pmatrix} \cos t & -5t\cos t + \sin t \\ -\sin t & 5t\sin t + \cos t \end{pmatrix} . \tag{2.16}$$

△

In general, the analytical computation of the Transition Matrix is not a simple task. The following theorem, provides a way of computing (with some approximation) $\Phi(t, t_0)$ as the partial sum of the so-called *Peano-Baker* (PB) series.

Theorem 2.2 (Peano-Baker (PB) series, [42], p. 13). *The PB series*

$$\begin{aligned} I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)\left[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\right] d\tau_1 \\ + \int_{t_0}^{t} A(\tau_1)\left[\int_{t_0}^{\tau_1} A(\tau_2)\left[\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3\right] d\tau_2\right] d\tau_1 + \cdots \end{aligned} \tag{2.17}$$

uniformly converges to the State Transition Matrix $\Phi(t, t_0)$ *over any compact interval of* $\mathbb{R}^+$.

The next theorem deals with the important case of linear time-invariant systems.

Theorem 2.3 (State Transition Matrix for LTI systems, [42], p. 71). *Assume that* $A(\cdot) = A \in \mathbb{R}^{n\times n}$. *In this case we have*

$$\begin{aligned} \Phi(t, t_0) &= \Phi(t - t_0) \\ &= \sum_{i=0}^{+\infty} \frac{A^i (t - t_0)^i}{i!} \\ &=: \exp(A(t - t_0)) . \end{aligned}$$

Proof. The proof follows from Theorem 2.2 and the fact that

$$\int_{t_0}^{t} A(\tau_1)\left[\int_{t_0}^{\tau_1} A(\tau_2)\left[\cdots\left[\int_{t_0}^{\tau_{i-1}} A(\tau_i)d\tau_i\right] d\tau_{i-1}\right]\cdots d\tau_2\right] d\tau_1 = \frac{A^i(t - t_0)^i}{i!} . \tag{2.18}$$

□

The evaluation of $\Phi(t, t_0)$ in the time-invariant case can be performed numerically by computing the partial sum of the series $\sum_{i=0}^{+\infty} \frac{A^i(t-t_0)^i}{i!}$.

However, in the time-invariant case, the Transition Matrix can be also evaluated in closed form either via the Laplace transform method or by performing a similarity transformation on A to put it in Jordan form (see [42] Chaps. 3–4).

2.3 Lyapunov Stability of Linear Time-Varying Systems

In the following the various definitions of Lyapunov stability of the equilibrium point $x = 0$ of system (2.1) are recalled; remember that $\varphi(\cdot, t_0, x_0)$ is the solution starting from x_0 at time t_0.

Good sources for Lyapunov stability theory are the books [96,109,154,170] and the paper by Kalman [107].

Definition 2.2 (Lyapunov stability). The equilibrium point $x = 0$ of system (2.1) is said to be

i) *stable if and only if* for all $t_0 \geq 0$ and for all $t \geq t_0$

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon, t_0) > 0 : \|x_0\| < \delta(\epsilon, t_0) \Rightarrow \|\varphi(t, t_0, x_0)\| < \epsilon\,;$$

ii) *uniformly stable if and only if* in i) δ does not depend on t_0;

iii) *uniformly attractive if and only if* for all $t_0 \geq 0$

$$\exists \eta > 0 \;:\; \|x_0\| < \eta \Rightarrow \lim_{t \to \infty} \|\varphi(t, t_0, x_0)\| = 0$$

uniformly with respect to t_0 and x_0;

iv) *uniformly asymptotically stable if and only if* it is uniformly stable and uniformly attractive;

v) *unstable if and only if* it is not stable.

◇

The above definitions apply to linear time-varying systems as well as to general nonlinear systems. On the other hand, when linear systems are dealt with, the equilibrium $x = 0$ is attractive *iff* it is *globally* attractive, that is definition iii) holds for all $x_0 \in \mathbb{R}^n$; this is a direct consequence of (2.4). From this fact it follows that the property of uniform asymptotic stability, when possessed by a linear system, is always global. To this regard, note that for linear systems, as shown in Theorem 2.4 below, uniform asymptotic stability only depends on the State Transition Matrix.

In the sequel we shall write, with slight abuse of language, “system (2.1) is stable (uniformly stable, etc.)” in place of “the equilibrium point $x = 0$ of system (2.1) is stable (uniformly stable, etc.)”.

The next theorem is a necessary and sufficient condition for uniform asymptotic stability in terms of the State Transition Matrix (see Exercise 2.2).

Theorem 2.4. *System* (2.1) *is uniformly asymptotically stable if and only if both the following conditions hold*

i) There exists a scalar $k > 0$ *such that for all* $t_0 \geq 0$ *and for all* $t \geq t_0$

$$\|\Phi(t, t_0)\| \leq k\,;$$

ii)

$$\lim_{t\to\infty} \|\Phi(t,t_0)\| = 0$$

uniformly with respect to t_0.

□

Definition 2.3 (Exponential stability). System (2.1) is said to be *exponentially stable* if and only if there exist positive scalars k and α such that for all $t_0 \geq 0$ and for all $t \geq t_0$

$$\exists \mu > 0 \,:\, \|x_0\| < \mu \Rightarrow \|\varphi(t,t_0,x_0)\| \leq k e^{-\alpha(t-t_0)} \|x_0\| \,.$$

◇

The following theorem is a necessary and sufficient condition for exponential stability of linear systems in terms of the State Transition Matrix; it follows directly from Definition 2.3.

Theorem 2.5. *System* (2.1) *is exponentially stable if and only if there exists positive scalars* k *and* α *such that for all* $t_0 \geq 0$ *and for all* $t \geq t_0$

$$\|\Phi(t,t_0)\| \leq k e^{-\alpha(t-t_0)} \,.$$

□

From the above theorem, it readily follows that, for linear systems, exponential stability, if exhibited by the system, is always global.

Directly from the definitions we have that exponential stability implies uniform asymptotic stability; however, when the system is linear, it can be shown (Exercise 2.3) that the properties of uniform asymptotic stability and exponential stability are equivalent.

Theorem 2.6. *System* (2.1) *is exponentially stable if and only if it is uniformly asymptotically stable.* □

To use the stability definitions given above, one should compute the solutions of system (2.1); this is often a difficult, or even impossible task, unless the system we are considering is time-invariant, that is A does not depend on time. As said in Sect. 2.2, in this last case we can evaluate analytically the solution of the system; in this way it is readily obtained the following well known theorem.

Theorem 2.7 (Stability for linear time-invariant systems [42], p. 185). *Let* $A(\cdot)$ *in equation* (2.1) *be constant, that is* $A(t) = A \in \mathbb{R}^{n\times n}$. *Then system* (2.1) *is exponentially stable* if and only if *all the eigenvalues of* A *are located in the open left half of the complex plane.* □

Remember that a matrix A having all eigenvalues in the open left half of the complex plane is said to be "Hurwitz". In the same way, the statement "the linear time-invariant system $\dot{x}(t) = Ax(t)$ is Hurwitz stable" means that A is an Hurwitz matrix, i. e. that the system is exponentially stable.

Therefore the stability analysis for linear time-invariant systems is reduced to the computation of system eigenvalues. Unfortunately this kind of analysis cannot be extended to time-varying systems, as the following simple example shows.

Example 2.2.
Consider system (2.1) with

$$A(t) = \begin{pmatrix} -1 - 9\cos^2 6t + 12\sin 6t\cos 6t & 12\cos^2 6t + 9\sin 6t\cos 6t \\ -12\sin^2 6t + 9\sin 6t\cos 6t & -1 - 9\sin^2 6t - 12\sin 6t\cos 6t \end{pmatrix}. \tag{2.19}$$

The eigenvalues of $A(t)$ are -1 and -10 for all t. However in [151, 171] it is shown that the the explicit solution of the system under consideration starting at $t_0 = 0$ from x_0 is

$$x(t) = \Phi(t,0)x_0 = \frac{1}{5}\begin{pmatrix} \mathrm{e}^{2t}(\cos 6t + 2\sin 6t) + 2\mathrm{e}^{-13t}(2\cos 6t - \sin 6t) & 2\mathrm{e}^{2t}(\cos 6t + 2\sin 6t) - \mathrm{e}^{-13t}(2\cos 6t - \sin 6t) \\ \mathrm{e}^{2t}(2\cos 6t - \sin 6t) - 2\mathrm{e}^{-13t}(\cos 6t + 2\sin 6t) & 2\mathrm{e}^{2t}(2\cos 6t - \sin 6t) + \mathrm{e}^{-13t}(\cos 6t + 2\sin 6t) \end{pmatrix} x_0\,, \tag{2.20}$$

and therefore the equilibrium $x = 0$ is unstable. △

It is interesting to notice that there exist cases in which some of the eigenvalues of the linear system under consideration are in the right half of the complex plane for all t and still the system is exponentially stable [164, 173]; this is the case of the system in Example 2.1.

These considerations show that, when system (2.1) is time-varying, one cannot infer the stability properties from the eigenvalues location in the complex plane. Note that, in the above example, it was possible to derive the instability of the equilibrium point directly from Definition 2.2, since the expression of the State Transition Matrix was available. In general, if the system matrix is time dependent, it is not possible to analytically express the State Transition Matrix of the system. To perform the stability analysis in the last case we have to use the Lyapunov Stability Theorem, which is stated next (for a proof see [96], p. 199).

Theorem 2.8 (Lyapunov Theorem for exponential stability). *Consider system* (2.1) *and assume there exist a function* $v(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$, $(t,x) \mapsto v(t,x)$, *with* $v \in \mathcal{C}_0\big(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}\big)$ *and three positive constants* a, b *and* c *such that*

i) $v(t,0) = 0\,, \qquad \forall t \in \mathbb{R}^+$ *;*

ii) for all $x \in \mathbb{R}^n$ *and for all* $t \in \mathbb{R}^+$

$$a\|x\|^2 \leq v(t,x) \leq b\|x\|^2 \,;$$

iii) the Dini derivative of v *along the system trajectories, defined as*

$$\dot{v}(t,x) := \limsup_{h\to 0} \frac{v(t+h, x+hA(t)x) - v(t,x)}{h} \,,$$

satisfies for all $x \in \mathbb{R}^n$ *and for all* $t \in \mathbb{R}^+$ *the condition*

$$\dot{v}(t,x) \leq -c\|x\|^2 \,.$$

Then system (2.1) *is exponentially stable.* □

If v satisfies condition ii) it is said to be *positive definite and decrescent*; if $\dot{v}$ satisfies iii) it is said to be *negative definite*.

When v is continuously differentiable the derivative along the solutions becomes

$$\dot{v}(t,x) := \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} A(t)x \,. \tag{2.21}$$

Often we shall use, as candidate Lyapunov function, the quadratic form

$$v(t,x) = x^T P(t) x \,, \tag{2.22}$$

with $P(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n\times n})$ and positive definite bounded, that is satisfying

$$\alpha I \leq P(t) \leq \beta I \,, \quad \forall t \in \mathbb{R}^+ \,, \tag{2.23}$$

for some positive numbers α and β. In the sequel to denote that the matrix function $\Theta(\cdot)$ is positive definite the notation of [158] is used, that is we write

$$\Theta >> 0 \,; \tag{2.24}$$

in the same way to denote that $\Theta(\cdot)$ is negative definite, that is $-\Theta(\cdot)$ is positive definite, we write $\Theta << 0$.

With the choice (2.22) we have

$$\begin{aligned} \frac{v(t+h, x+hA(t)x) - v(t,x)}{h} &= x^T \frac{P(t+h) - P(t)}{h} x \\ &\quad + x^T \left(A^T(t)P(t+h) + P(t+h)A(t)\right) x \\ &\quad + h x^T A^T(t) P(t+h) A(t) x \,. \end{aligned} \tag{2.25}$$

Obviously, if at a given point t, $P(\cdot)$ is differentiable, by letting $h \to 0$ in (2.25), we have that negative definiteness of $\dot{v}$ is guaranteed by imposing

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq -\gamma I \,, \tag{2.26}$$

for some $\gamma > 0$.

Conversely, at the points where $P(\cdot)$ is not differentiable, to guarantee negative definiteness of $\dot{v}$ we must have

$$\frac{P(t+h)-P(t)}{h} + A^T(t)P(t+h) + P(t+h)A(t) \leq -\gamma I\,, \tag{2.27}$$

for all h in a neighborhood of $h=0$.

In the sequel, for the sake of simplicity, we shall capture both (2.26) and (2.27) by writing, according to the notation introduced in (2.24),

$$\dot{P} + A^T P + PA << 0\,, \tag{2.28}$$

remembering that, at the points where $P(\cdot)$ is not differentiable, the extended interpretation given in (2.27) has to be considered.

On the basis of the above discussion, we can state the following corollary of Theorem 2.8.

Corollary 2.1. *System* (2.1) *is exponentially stable if there exists a positive definite bounded matrix-valued function* $P(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n\times n})$ *satisfying the differential Lyapunov inequality* (2.28). □

It is interesting to note that Corollary 2.1 can be reversed (see Exercise 2.4).

Based on Theorem 2.8 we can derive a number of sufficient conditions guaranteeing the stability of a linear time-varying system.

2.4 Sufficient Conditions for Exponential Stability

The next theorem (see [109], Exercise 4.24) is useful to better understand the behavior of time-varying systems.

Theorem 2.9. *Consider system* (2.1) *and assume that* $\lim_{t\to\infty} A(t) = \overline{A} \in \mathbb{R}^{n\times n}$. *Then system* (2.1) *is exponentially stable* if and only if *all the eigenvalues of* $\overline{A}$ *are located in the open left half of the complex plane.*

Now consider system (2.1) and assume that $A(\cdot)$ is continuous, bounded and that the eigenvalues of $A(t)$ are uniformly located in the open left half of the complex plane, that is there exists a scalar $\mu > 0$ such that

$$Re\,(\lambda_i(A(t))) \leq -\mu\,, \quad i = 1,\dots,n\,, \quad \forall t \in \mathbb{R}^+\,. \tag{2.29}$$

In this case, from Theorem 2.9 it follows that a *necessary* condition for such a system to be not exponentially stable is that at least one entry of $A(\cdot)$ is oscillatory in a neighborhood of $+\infty$.

To clarify this point and to introduce the next result, let us consider the following example.

Example 2.3.
Consider system (2.1) with

$$A(t) = \begin{pmatrix} -1 + 5\sin(\omega t)\cos(\omega t) & -5\cos^2(\omega t) \\ 5\sin^2(\omega t) & -1 - 5\sin(\omega t)\cos(\omega t) \end{pmatrix}, \tag{2.30}$$

where $\omega > 0$. The two eigenvalues are -1 for all t. Note that the value of the parameter ω is related to the rate of variation of $A(\cdot)$. Indeed simple computations show that

$$\|\dot{A}(t)\| = 5\omega\,. \tag{2.31}$$

In this case for $\omega \in [0, 5)$ a Fundamental Matrix is given by

$$X(t) = \begin{pmatrix} \mathrm{e}^{(\sqrt{\omega(5-\omega)}-1)t}\big(\cos\omega t + \sqrt{\frac{\omega}{5-\omega}}\sin\omega t\big) & \mathrm{e}^{-(\sqrt{\omega(5-\omega)}+1)t}\big(\cos(\omega t) - \sqrt{\frac{\omega}{5-\omega}}\sin(\omega t)\big) \\ \mathrm{e}^{(\sqrt{\omega(5-\omega)}-1)t}\big(\sin\omega t - \sqrt{\frac{\omega}{5-\omega}}\cos\omega t\big) & \mathrm{e}^{-(\sqrt{\omega(5-\omega)}+1)t}\big(\sin(\omega t) + \sqrt{\frac{\omega}{5-\omega}}\cos(\omega t)\big) \end{pmatrix}. \tag{2.32}$$

Note that for $\omega = \omega^* \cong 0.209$ the exponent of the $(1,1)$ and $(2,1)$ entries is zero. Moreover for $\omega \in [0, \omega^*)$, all the exponential terms are negative, which guarantees exponential stability of the system, while for $\omega \geq \omega^*$ the system is not exponentially stable. △

The lesson learned from Example 2.3 is the following: if we consider the linear time-varying system (2.1) with $A(\cdot)$ continuous, bounded and such that its eigenvalues have uniformly negative real part, the system is exponentially stable if the oscillations of the entries of the system matrix are sufficiently slow. In the next theorem we shall see that this property holds in general for linear time-varying systems.

Theorem 2.10 (Stability of slowly varying systems, [57, 151]). *Consider system* (2.1) *with* $A(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n\times n})$. *Moreover assume that*

i) *the matrix function $A(\cdot)$ is bounded, that is there exists a positive number m such that*

$$\|A(t)\| \leq m\,, \qquad \forall t \in \mathbb{R}^+\,; \tag{2.33}$$

ii) *the eigenvalues of $A(\cdot)$ are uniformly located in the open left half of the complex plane, that is* (2.29) *holds;*

iii) *there exists a scalar $\epsilon > 0$ such that*[1]

$$\|\dot{A}(t)\| \leq \epsilon\,, \quad \forall t \in \mathbb{R}^+\,.$$

[1] Again, at the points where $A(\cdot)$ is not differentiable, we have $\limsup_{h\to 0} \|\big(A(t+h) - A(t)\big)/h\| \leq \epsilon$.

Then if ϵ is sufficiently small, system (2.1) *is exponentially stable*[2].

Proof. Let us consider, for all $t \in \mathbb{R}^+$, the Lyapunov equation

$$A^T(t)P(t) + P(t)A(t) = -Q\,, \tag{2.34}$$

where Q is any positive definite matrix.

By virtue of hypotheses ii) there exists, for all $t \in \mathbb{R}^+$, a unique positive definite solution $P(t)$; moreover $P(\cdot)$ is continuous because $A(\cdot)$ is.

Now inequality (A.15) applied to (2.34) yields

$$\begin{aligned} \lambda_{min}(P(t)) &\geq \frac{\lambda_{min}(Q)}{2\|A(t)\|}\,, \\ &\geq \frac{\lambda_{min}(Q)}{2m}\,, \qquad \forall t \in \mathbb{R}^+\,. \end{aligned} \tag{2.35}$$

Now, from (A.19) and hypothesis i) and ii), we have that there exist positive k and α (which do not depend on t) such that

$$\lambda_{max}(P(t)) \leq \frac{k^2\lambda_{max}(Q)}{2\alpha}\,, \qquad \forall t \in \mathbb{R}^+\,. \tag{2.36}$$

Hence $P(\cdot)$ is positive definite bounded since, for all $t \in \mathbb{R}^+$,

$$\frac{\lambda_{min}(Q)}{2m} I \leq P(t) \leq \frac{k^2\lambda_{max}(Q)}{2\alpha} I\,. \tag{2.37}$$

At the points where $P(\cdot)$ is differentiable we have

$$\begin{aligned} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) &= \dot{P}(t) - Q \\ &\leq \Big(\|\dot{P}(t)\| - \lambda_{min}(Q)\Big) I\,. \end{aligned} \tag{2.38}$$

By deriving (2.34) we obtain that $\dot{P}(\cdot)$ satisfies the following Lyapunov equation

$$A^T(t)\dot{P}(t) + \dot{P}(t)A(t) = -\Big(\dot{A}^T(t)P(t) + P(t)\dot{A}(t)\Big)\,. \tag{2.39}$$

Therefore, by applying (A.18), we have that

$$\begin{aligned} \|\dot{P}(t)\| &\leq \frac{k^2\|\dot{A}^T(t)P(t) + P(t)\dot{A}(t)\|}{2\alpha} \\ &\leq \frac{k^2\lambda_{max}(P(t))\|\dot{A}(t)\|}{\alpha} \\ &\leq \frac{k^4\lambda_{max}(Q)\epsilon}{2\alpha^2}\,. \end{aligned} \tag{2.40}$$

[2] Actually, it is sufficient that condition ii) and iii) hold in a neighborhood of $+\infty$.

Where $P(\cdot)$ is not differentiable, we can repeat the same computations as above, by replacing $\dot{P}(\cdot)$ and $\dot{A}(\cdot)$ by the corresponding incremental ratios.

Finally, from (2.38), we have that for all $t \in \mathbb{R}^+$

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq \left(\frac{k^4 \lambda_{max}(Q)\epsilon}{2\alpha^2} - \lambda_{min}(Q) \right) I\,. \tag{2.41}$$

Therefore, for a sufficient small ϵ, $P(\cdot)$ satisfies (2.28) and Corollary 2.1 guarantees exponential stability of system (2.1). □

Remark 2.1. If we relax Assumption ii) of Theorem 2.10, by requiring that, for all $t \in \mathbb{R}^+$, $Re\big(\lambda_i(A(t))\big) < 0$, the simple asymptotic stability rather than exponential stability is guaranteed (see [7]). ◇

2.5 Input-Output Gain of a Linear Time-Varying System

Let us denote by $\mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m)$ the space of the real vector-valued functions with m components which are square integrable on $\mathbb{R}^+$.

Consider an exponentially stable linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t)\,, \quad t \in [0, +\infty) \tag{2.42a}$$

$$z(t) = C(t)x(t) + D(t)w(t)\,, \tag{2.42b}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are bounded and of class $\mathcal{PC}$ of dimensions $n \times n$, $n \times v$, $s \times n$ and $s \times v$ respectively.

System (2.42) uniquely defines the linear operator

$$\begin{aligned} &\varGamma_{zw} : \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^v) \mapsto \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^s) \\ &\quad w \mapsto z = \varGamma_{zw}(w) := \int_0^t C(t)\varPhi(t,\tau)B(\tau)w(\tau)d\tau + D(t)w(t)\,. \end{aligned}$$

We recall that the $\mathcal{L}_2$ induced norm of the operator $\varGamma_{zw}$ is defined as follows

$$\|\varGamma_{zw}\| := \sup_{w \in \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^v) - \{0\}} \frac{\|z\|}{\|w\|}\,, \tag{2.43}$$

where, for a given vector-valued function $u(\cdot) \in \mathcal{L}_2$, we define the $\mathcal{L}_2$ norm of $u(\cdot)$ as

$$\|u\| := \left(\int_0^{+\infty} u^T(t)u(t)dt \right)^{1/2}\,. \tag{2.44}$$

This section concerns the computation of the $\mathcal{L}_2$ norm of the input-output operator $\varGamma_{zw}$.

First we consider the case in which system (2.42) is strictly proper ($D = 0$). The next result provides a necessary and sufficient condition for the $\mathcal{L}_2$ norm of the operator $\mathit{\Gamma}_{zw}$ to be less than one in terms of the feasibility of a certain Riccati differential inequality.

Lemma 2.1 ([167]). *Let us consider system* (2.42) *and assume that* $D(\cdot)$ *is zero. Then the following statements are equivalent:*

i) System (2.42) *is exponentially stable and* $\|\mathit{\Gamma}_{zw}\| < 1$*;*
ii) there exists a positive definite bounded matrix-valued function $P(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n\times n})$ *such that*

$$\dot{P} + A^T P + PA + PBB^T P + C^T C << 0\,. \tag{2.45}$$

Proof.
i) ⇒ ii). Let us consider the following system with augmented output

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \tag{2.46a}$$

$$\overline{z}(t) = \begin{pmatrix} C(t) \\ \epsilon I \end{pmatrix} x(t)\,, \tag{2.46b}$$

where ϵ is a positive number that will be chosen later.
We have

$$\begin{aligned} \|\mathit{\Gamma}_{\overline{z}w}\|^2 &= \sup_{w\in\mathcal{L}_2(\mathbb{R}^+,\mathbb{R}^v)-\{0\}} \frac{\|\overline{z}\|^2}{\|w\|^2} \\ &= \sup_{w\in\mathcal{L}_2(\mathbb{R}^+,\mathbb{R}^v)-\{0\}} \frac{\|z\|^2 + \epsilon^2\|x\|^2}{\|w\|^2} \\ &\le \|\mathit{\Gamma}_{zw}\|^2 + \epsilon^2\|\mathit{\Gamma}_{xw}\|^2\,, \end{aligned} \tag{2.47}$$

where $\mathit{\Gamma}_{xw}$ is the operator mapping w to x.
Since $\|\mathit{\Gamma}_{zw}\| < 1$, there exists a positive number δ such that

$$\|\mathit{\Gamma}_{zw}\|^2 < 1 - \delta^2\,. \tag{2.48}$$

Moreover, system (2.46a) is exponentially stable, therefore $\|\mathit{\Gamma}_{xw}\|$ is a finite number; pick ϵ such that

$$\epsilon^2\|\mathit{\Gamma}_{xw}\|^2 < \frac{\delta^2}{2}\,. \tag{2.49}$$

From (2.47), (2.48) and (2.49) it follows that

$$\begin{aligned} \|\mathit{\Gamma}_{\overline{z}w}\|^2 &< 1 - \delta^2 + \epsilon^2\|\mathit{\Gamma}_{xw}\|^2 \\ &< 1 - \frac{\delta^2}{2}\,. \end{aligned} \tag{2.50}$$

The last inequality implies that

$$\sup_{w\in\mathcal{L}_2(\mathbb{R}^+,\mathbb{R}^v)-\{0\}} \left[\|\overline{z}\|^2 - \left(1-\delta^2/2\right)\|w\|^2\right] < 0\,. \tag{2.51}$$

This is equivalent to say that the cost functional

$$\int_0^{+\infty} \left[x^T(t)(C^T(t)C(t)+\epsilon I)x(t) - \left(1-\delta^2/2\right) w^T(t)w(t)\right] dt \tag{2.52}$$

subject to (2.46) has a nonpositive (and therefore finite) supremum. From differential game theory applied to linear systems it follows (see [32], Theorem 8.3) that there exists a positive definite bounded matrix-valued function $P(\cdot)\in\mathcal{C}_0(\mathbb{R}^+,\mathbb{R}^{n\times n})$ such that

$$\begin{aligned}\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + P(t)B(t)B^T(t)P(t)&\\ + C^T(t)C(t) + \epsilon I = 0\,, \quad t\in\mathbb{R}^+\,;&\end{aligned} \tag{2.53}$$

since $\epsilon > 0$ the proof follows.
ii) $\Rightarrow$ **i)**. First of all note that system (2.42) is exponentially stable since ii) guarantees the existence of a positive definite bounded continuous $P(\cdot)$ satisfying (see Corollary 2.1)

$$\dot{P} + A^T P + PA << 0\,. \tag{2.54}$$

Now, since $C(\cdot)$ is bounded, condition ii) implies the existence of a positive definite bounded continuous matrix-valued function $P(\cdot)$ and a scalar $\epsilon > 0$ such that

$$\begin{aligned}\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + P(t)B(t)B^T(t)P(t)&\\ + (1+\epsilon)C^T(t)C(t) < 0\,, \quad t\in\mathbb{R}^+\,.&\end{aligned} \tag{2.55}$$

Let $x(\cdot)$ the unique solution of (2.42a) starting from $x(0)=0$ under the input $w(\cdot)\in\mathcal{L}_2$; we have

$$\begin{aligned}\frac{d}{dt}\left(x^T(t)P(t)x(t)\right) &= (A(t)x(t)+B(t)w(t))^T P(t)x(t)\\ &\quad + x^T(t)P(t)(A(t)x(t)+B(t)w(t)) + x^T(t)\dot{P}(t)x(t)\\ &= x^T(t)\left(A^T(t)P(t)+P(t)A(t)+\dot{P}(t)\right)x(t)\\ &\qquad + w^T(t)B^T(t)P(t)x(t) + x^T(t)P(t)B(t)w(t)\\ &< -x^T(t)P(t)B(t)B^T(t)P(t)x(t) - (1+\epsilon)\|z(t)\|^2\\ &\quad + w^T(t)B^T(t)P(t)x(t) + x^T(t)P(t)B(t)w(t)\\ &= \|w(t)\|^2 - \|\overline{w}(t)\|^2 - (1+\epsilon)\|z(t)\|^2\,,\end{aligned} \tag{2.56}$$

where

$$\overline{w}(t) := w(t) - B^T(t)P(t)x(t)\,. \tag{2.57}$$

The last inequality can be rewritten

$$\frac{d}{dt}\big(x^T(t)P(t)x(t)\big) + (1+\epsilon)\|z(t)\|^2 - \|w(t)\|^2 < -\|\overline{w}(t)\|^2 \le 0\,. \tag{2.58}$$

By integrating (2.58) between 0 and $+\infty$, taking into account that the system is exponentially stable and that all the involved signals are of class $\mathcal{L}_2$, we obtain

$$(1+\epsilon)\|z\|^2 - \|w\|^2 < -\|\overline{w}\|^2 \le 0\,. \tag{2.59}$$

From the last inequality the proof follows. □

Next, we consider the general case of a non-zero $D(\cdot)$; also we deal with the more general case of a non-unitary gain. We follow the approach of [34].

To this end let us rescale the output variable as follows

$$\begin{aligned}\tilde{z} &= \gamma^{-1}z\\ &= \gamma^{-1}Cx + \gamma^{-1}Dw\\ &=: \tilde{C}x + \tilde{D}w\,.\end{aligned} \tag{2.60}$$

Note that $\|\Gamma_{zw}\| < \gamma$ implies $\|D(t)\| < \gamma$ for all $t \in \mathbb{R}^+$ and therefore $\|\tilde{D}(t)\| < 1$ for all $t \in \mathbb{R}^+$ (see [167]). Therefore, without loss of generality, in the following we shall assume that the matrix functions $I - \tilde{D}^T(t)\tilde{D}(t)$ and $I - \tilde{D}(t)\tilde{D}^T(t)$ are positive definite.

Now consider the following input-output transformation which relates the original variables w and $\tilde{z}$ to the new variables $\hat{w}$ and $\hat{z}$

$$\begin{pmatrix} w(t)\\ \hat{z}(t)\end{pmatrix} = \begin{pmatrix} \tilde{D}^T(t) & \big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{1/2}\\ \big(I - \tilde{D}(t)\tilde{D}^T(t)\big)^{1/2} & -\tilde{D}(t)\end{pmatrix}\begin{pmatrix}\tilde{z}(t)\\ \hat{w}(t)\end{pmatrix}. \tag{2.61}$$

Since the transformation (2.61) is orthogonal we have that

$$\|w\|^2 + \|\hat{z}\|^2 = \|\tilde{z}\|^2 + \|\hat{w}\|^2\,, \tag{2.62}$$

which can be rewritten

$$\|\tilde{z}\|^2 - \|w\|^2 = \|\hat{z}\|^2 - \|\hat{w}\|^2\,. \tag{2.63}$$

If $\Gamma_{\tilde{z}w}$ and $\Gamma_{\hat{z}\hat{w}}$ denote the operator mapping w to $\tilde{z}$ and the operator mapping $\hat{w}$ to $\hat{z}$ respectively, from (2.63) it follows that $\|\Gamma_{\tilde{z}w}\| < 1$ *iff* $\|\Gamma_{\hat{z}\hat{w}}\| < 1$.

Since $\|\tilde{z}\| = \gamma^{-1}\|z\|$, we can conclude that $\|\Gamma_{zw}\| < \gamma$ *iff* $\|\Gamma_{\hat{z}\hat{w}}\| < 1$.

Next we derive the state space equations of the transformed system. Note that from (2.61) we have

$$\begin{aligned} w(t) &= \tilde{D}^T(t)\tilde{z}(t) + \big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{1/2}\hat{w}(t) \\ &= \tilde{D}^T(t)\big(\tilde{C}(t)x(t) + \tilde{D}(t)w(t)\big) + \big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{1/2}\hat{w}(t)\,; \end{aligned} \tag{2.64}$$

therefore

$$w(t) = (I - \tilde{D}^T(t)\tilde{D}(t))^{-1}\tilde{D}^T(t)\tilde{C}(t)x(t) + (I - \tilde{D}^T(t)\tilde{D}(t))^{-1/2}\hat{w}(t)\,. \tag{2.65}$$

By replacing the expression of w into equation (2.42a) we have

$$\dot{x}(t) = \hat{A}(t)x(t) + \hat{B}(t)\hat{w}(t)\,, \tag{2.66}$$

where

$$\hat{A}(t) = A(t) + \gamma^{-2}B(t)\big(I - \gamma^{-2}D^T(t)D(t)\big)^{-1}D^T(t)C(t) \tag{2.67a}$$

$$\hat{B}(t) = B(t)\big(I - \gamma^{-2}D^T(t)D(t)\big)^{-1/2}\,. \tag{2.67b}$$

Now replacing the expression of w (2.65) into (2.60) we get

$$\begin{aligned} \tilde{z} = \Big(I + \tilde{D}(t)\big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{-1}\tilde{D}^T(t)\Big)\,\tilde{C}(t)x(t) \\ + \tilde{D}(t)\big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{-1/2}\hat{w}(t)\,. \end{aligned} \tag{2.68}$$

Therefore

$$\begin{aligned} \hat{z}(t) &= \big(I - \tilde{D}(t)\tilde{D}^T(t)\big)^{1/2}\tilde{z}(t) - \tilde{D}(t)\hat{w}(t) \\ &= \big(I - \tilde{D}(t)\tilde{D}^T(t)\big)^{1/2}\Big(I + \tilde{D}(t)\big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{-1}\tilde{D}^T(t)\Big)\,\tilde{C}(t)x(t) \\ &\quad + \big(I - \tilde{D}(t)\tilde{D}^T(t)\big)^{1/2}\tilde{D}(t)\big(I - \tilde{D}^T(t)\tilde{D}(t)\big)^{-1/2}\hat{w}(t) - \tilde{D}(t)\hat{w}(t) \\ &= \hat{C}(t)x(t) + \hat{D}(t)\hat{w}(t)\,, \end{aligned} \tag{2.69}$$

where

$$\hat{C}(t) = \gamma^{-1}\Big(I + \gamma^{-2}D(t)\big(I - \gamma^{-2}D^T(t)D(t)\big)^{-1}D^T(t)\Big)^{1/2}C(t) \tag{2.70a}$$

$$\hat{D}(t) = 0\,. \tag{2.70b}$$

Note that to derive (2.69) we have used the fact that (see Exercise 2.6)

$$\big(I - \tilde{D}(t)\tilde{D}^T(t)\big)^{1/2} = \big(I + \tilde{D}(t)(I - \tilde{D}^T(t)\tilde{D}(t))^{-1}\tilde{D}^T(t)\big)^{-1/2}\,. \tag{2.71}$$

Since $\hat{D} = 0$, from Lemma 2.1 we have that $\|\Gamma_{\hat{z}\hat{w}}\| < 1$ *iff* there exists a continuous positive definite $P(\cdot)$ such that

$$\dot{P} + \hat{A}^T P + P\hat{A} + \hat{C}^T\hat{C} + P\hat{B}\hat{B}^T P << 0\,. \tag{2.72}$$

By replacing the expressions of $\hat{A}$, $\hat{B}$ and $\hat{C}$ into (2.72) we obtain the following theorem.

Theorem 2.11. *Let us consider system* (2.42)*; then the following statements are equivalent:*

i) System (2.42) *is exponentially stable and* $\|\Gamma_{zw}\| < \gamma$*;*
ii) $\|D(t)\| < \gamma$ *for all* $t \in \mathbb{R}^+$ *and there exists a positive definite bounded matrix-valued function* $P(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n\times n})$ *such that*

$$\dot{P} + A^T P + PA + \gamma^{-2} C^T C + \left(PB + \gamma^{-2} C^T D\right)\left(I - \gamma^{-2} D^T D\right)^{-1}\left(B^T P + \gamma^{-2} D^T C\right) << 0 \,. \tag{2.73}$$

□

Multiplying both members of (2.73) by γ^2 and rescaling $P(\cdot)$, we have that condition ii) is equivalent to the existence of a positive definite bounded continuous matrix-valued function $P(\cdot)$ such that

$$\dot{P} + A^T P + PA + C^T C + \left(PB + C^T D\right)\left(\gamma^2 I - D^T D\right)^{-1}\left(B^T P + D^T C\right) << 0 \,. \tag{2.74}$$

When system (2.42) is time-invariant we can look, in the statement of Theorem 2.11 and without loss of generality, to a constant positive definite matrix P.

The time-invariant version of Theorem 2.11 is an alternative statement of the famous Bounded Real Lemma, which plays a fundamental role in most of robust control theory and dates back to the work on absolute stability and passivity theory due to Popov [142, 143], Yakubovich [176–178] and Kalman [105, 106].

An alternative proof for the time-invariant version of Theorem 2.11 can be obtained via frequency domain arguments according to [110] which, in turn, is inspired to the work by Willems [172].

2.6 Discrete-Time Systems

In this section we deal with the behavior of the solutions of the discrete-time linear system

$$x(k+1) = A(k)x(k), \qquad k \in \mathbb{N}_0 \,. \tag{2.75}$$

Starting at time k_0 from the initial condition $x(k_0) = x_0$, we iteratively obtain

$$\begin{aligned} x(k_0+1) &= A(k_0)x_0 \\ x(k_0+2) &= A(k_0+1)A(k_0)x_0 \\ &\vdots \\ x(k) &= A(k-1)A(k-2)\cdots A(k_0)x_0 \,. \end{aligned} \tag{2.76}$$

From (2.76) we can readily derive the following result.

Fact 2.5. *The unique solution of system* (2.75) *starting at time* k_0 *from* x_0 *is*

$$x(k) = \Phi(k, k_0)x_0 \,, \tag{2.77}$$

where

$$\Phi(k, k_0) = \begin{cases} A(k-1)A(k-2)\cdots A(k_0) & \text{if } k > k_0 \\ I & \text{if } k = k_0 \end{cases} . \tag{2.78}$$

□

The matrix function $(k, k_0) \in \mathbb{N}_0 \times \mathbb{N}_0 \to \Phi(k, k_0)$ is called the State Transition Matrix of the discrete-time system (2.75). Note that

$$\Phi(k_0, k_0) = I \,; \tag{2.79}$$

this is in accordance with the fact that

$$x(k_0) = x_0 \,. \tag{2.80}$$

For this reason (2.79) is referred to as the consistency property of the State Transition Matrix.

The following result is a direct consequence of the definition of the State Transition Matrix.

Fact 2.6 (Composition (Transition)). *For all* k, k_0, $k_1 \in \mathbb{N}_0$, $k \geq k_1 \geq k_0$,

$$\Phi(k, k_0) = \Phi(k, k_1)\Phi(k_1, k_0) \,. \tag{2.81}$$

□

Now assume that $A(h)$ is nonsingular for $h \in \{k_0, k_0 + 1, \ldots, k-1\}$ and define

$$\Phi(k_0, k) := A^{-1}(k_0)A^{-1}(k_0 + 1)\cdots A^{-1}(k-1) \,. \tag{2.82}$$

Fact 2.7 (Inversion).
Assume that $A(h)$ *is nonsingular for* $h \in \{k_0, k_0 + 1, \ldots, k-1\}$, *then*

$$\Phi(k, k_0)^{-1} = \Phi(k_0, k) \,. \tag{2.83}$$

□

Therefore, while for continuous-time systems the Transition Matrix is always invertible, for discrete-time systems invertibility is guaranteed by the nonsingularity of $A(\cdot)$. This means that, given integer numbers k_0 and k with $0 \leq k_0 \leq k$, it is not always possible to go back in time and obtain x_0 starting from $x(k)$.

When system (2.75) is time-invariant, that is $A(k) = A \in \mathbb{R}^{n\times n}$ for all $k \in \mathbb{N}_0$, we have that

$$\begin{aligned}\Phi(k, k_0) &= \Phi(k - k_0) \\ &= A^{k-k_0}\,. \end{aligned} \tag{2.84}$$

The Transition Matrix in the time-invariant case can be computed in closed form either by using the Z-transform method or by performing a similarity transformation which puts A in Jordan form (see [42], Chaps. 3–4).

2.6.1 Lyapunov Stability of Discrete-Time Systems

Definition 2.2 generalizes in an obvious manner to discrete-time systems. Concerning exponential stability, we have the following definition.

Definition 2.4 (Exponential stability for discrete-time systems). System (2.75) is said to be *exponentially stable* if and only if there exist a scalar $\rho \in [0, 1)$ and a positive scalar m such that, for all $k_0 \in \mathbb{N}_0$ and for all integer $k \geq k_0$,

$$\|x(k)\| \leq m\rho^{k-k_0}\|x_0\|\,.$$

◇

The following result is a necessary and sufficient condition for exponential stability involving the State Transition Matrix; it follows directly from Definition 2.4.

Theorem 2.12. *System* (2.75) *is exponentially stable if and only if there exists a scalar* $\rho \in [0, 1)$ *and a positive scalar* m *such that for all integer* $k_0 \geq 0$ *and all integer* $k \geq k_0$

$$\|\Phi(k, k_0)\| \leq m\rho^{k-k_0}\,. \tag{2.85}$$

□

Exponential stability is equivalent to uniform asymptotic stability when the system is linear.

In order to establish the stability properties of system (2.75) one should analytically compute the State Transition Matrix $\Phi(k, k_0)$. As for continuous-time systems, this is always possible only in the time-invariant case.

Theorem 2.13 (Stability of linear time-invariant discrete-time systems [42], p. 213). *Let* $A(\cdot)$ *in* (2.75) *be constant, that is* $A(k) = A \in \mathbb{R}^{n\times n}$. *Then system* (2.75) *is exponentially stable* if and only if *all the eigenvalues of* A *are located in the open unit disk centered at the origin of the complex plane, that is*

$$|\lambda_i(A)| < 1\,, i = 1, \ldots, n\,. \tag{2.86}$$

□

Like in the continuous-time case, Theorem 2.13 cannot be extended to time-varying systems. Therefore, when the system we deal with is time-varying, we must resort to the following theorem which generalizes Theorem 2.8 to discrete-time systems.

Theorem 2.14 (Lyapunov Theorem for exponential stability of discrete-time systems). *Consider system* (2.75) *and assume there exists a function* $v(\cdot,\cdot) : \mathbb{N}_0 \times \mathbb{R}^n \mapsto \mathbb{R}$, $(k,x) \mapsto v(k,x)$, *with* $v(k,\cdot) \in \mathcal{C}_0\big(\mathbb{R}^n, \mathbb{R}\big)$ *and three positive constants* a, b *and* c *such that*

i) $v(k,0) = 0\,, \qquad \forall k \in \mathbb{N}_0\,;$

ii) for all $x \in \mathbb{R}^n$ *and for all* $k \in \mathbb{N}_0$

$$a\|x\|^2 \le v(k,x) \le b\|x\|^2\,;$$

iii) the first difference of v *along system trajectories defined as*

$$\Delta v(k,x) := v(k+1, A(k)x) - v(k,x)$$

satisfies for all $x \in \mathbb{R}^n$ *and for all* $k \in \mathbb{N}_0$ *the condition*

$$\Delta v(k,x) \le -c\|x\|^2\,.$$

Then system (2.75) *is exponentially stable.* □

The terminology used in the continuous-time case also applies to the discrete-time case.

When we use as Lyapunov function the quadratic form

$$v(k,x) = x^T P(k) x\,, \tag{2.87}$$

with $P(\cdot)$ positive definite bounded

$$\alpha I \le P(k) \le \beta I\,, \quad \alpha, \beta > 0\,, \quad \forall\, k \in \mathbb{N}_0\,, \tag{2.88}$$

we obtain

$$\begin{aligned} \Delta v(k,x) &= x^T A(k)^T P(k+1) A(k) x - x^T P(k) x \\ &= x^T \left(A(k)^T P(k+1) A(k) - P(k)\right) x\,. \end{aligned} \tag{2.89}$$

Therefore we can state the following corollary of Theorem 2.14.

Corollary 2.2. *System* (2.75) *is exponentially stable if there exists a positive definite bounded matrix-valued function* $P(\cdot)$ *such that for some positive scalar* γ *and for all* $k \in \mathbb{N}_0$

$$A(k)^T P(k+1) A(k) - P(k) \le -\gamma I\,. \tag{2.90}$$

□

The next result (see Exercise 2.7) is the counterpart of Theorem 2.10 for discrete-time systems.

Theorem 2.15 (Stability of slowly varying discrete-time systems). *Consider system* (2.75) *and assume that:*

i) *The matrix function $A(\cdot)$ is bounded, that is there exists a positive number m such that*

$$\|A(k)\| \leq m\,, \qquad \forall k \in \mathbb{N}_0\,; \tag{2.91}$$

ii) *the eigenvalues of $A(\cdot)$ are uniformly located in the unit disk for all k, that is there exists a scalar $\mu < 1$ such that*

$$|\lambda_i(A(k))| \leq \mu\,, \quad i = 1, \ldots, n\,, \quad \forall k \in \mathbb{N}_0\,;$$

iii) *there exists a scalar $\epsilon > 0$ such that*

$$\|A(k+1) - A(k)\| \leq \epsilon\,, \quad \forall k \in \mathbb{N}_0\,.$$

Then if ϵ is sufficiently small, system (2.75) *is exponentially stable.* □

Summary

In this chapter we have illustrated some fundamental results on linear time-varying systems, which will be useful in the sequel of the book.

First we have given a sufficient condition for existence and uniqueness of the solution of system (2.1); for an extensive treatment of this topic the reader is referred to the books [51] and [126].

A fundamental role in the study of linear time-varying systems is played by the State Transition Matrix, which, in principle, can be computed as the sum of the Peano-Baker series. In practice, the State Transition Matrix can be evaluated in closed form only if the system is time-invariant. In the last case it reduces to the matrix exponential which can be computed, for example, by means of Laplace Transform methods.

Several stability definitions for system (2.1) have been stated; in particular we have focused on uniform asymptotic stability and exponential stability which, for linear systems, are equivalent concepts.

A classical necessary and sufficient condition for exponential stability requires the computation of the State Transition Matrix, and therefore it cannot be applied in practice unless the system we deal with is time-invariant. In this last case exponential stability is guaranteed if all the eigenvalues of the system matrix are located in the open left half of the complex plane. In the general time-varying case it is not possible to infer the stability properties of the system from the location of the eigenvalues and therefore one has to resort to the Lyapunov approach.

An application of the Lyapunov theorem shows that when the system is sufficiently slowly varying in time the eigenvalues location in the left half of the complex plane is still sufficient to guarantee exponential stability of the system.

A particular attention has been given to the definition of input-output gain of a linear time-varying system, as this concept will be exploited throughout the book to study the performance problem. In this context, the main result that has been stated is the time-varying version of the Bounded Real Lemma.

The chapter is ended by the description of some properties of discrete-time systems; it is interesting to note that the majority of the results are analogous to the continuous-time case.

Exercises

Exercise 2.1. Assume that we are under the hypothesis of Theorem 2.1 and let us denote by $\varphi(t, t_0, x_0)$ the unique solution of system (2.1) starting from x_0 at t_0. Show that, for a given pair $(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, the mapping

$$x_0 \in \mathbb{R}^n \mapsto \varphi(t, t_0, x_0) \in \mathbb{R}^n \,, \tag{2.92}$$

is linear. ◇

Exercise 2.2 ([42], p. 183). Prove the statement of Theorem 2.4. ◇

Exercise 2.3 ([42], p. 184). Prove the statement of Theorem 2.6. ◇

Exercise 2.4 ([109], p. 175). Prove that if system (2.1) is exponentially stable and $A(\cdot)$ is bounded there exists a positive definite, bounded, continuous matrix function $P(\cdot)$ satisfying the hypothesis of Corollary 2.1.
(Hint: let

$$P(t) = \int_t^{+\infty} \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau \,, \tag{2.93}$$

where $Q(\cdot)$ is any continuous, positive definite bounded matrix-valued function.) ◇

Exercise 2.5 ([7]). Assume that we are under the hypothesis i) and ii) of Theorem 2.10. Prove that a quantitative estimate of the maximum allowable rate of variation of the system matrix guaranteeing exponential stability of system (2.1) is given by

$$\|\dot{A}(t)\| \leq \frac{\mu \lambda_{min}(Q)}{2\|Q\|_F} \underline{\sigma}^2\big(A(t) \oplus A(t)\big) \,, \quad \forall t \in \mathbb{R}^+ \,, \tag{2.94}$$

where μ is any number belonging to the interval $[0, 1)$, Q is any positive definite matrix, $\underline{\sigma}(M)$ is the minimum singular value of the matrix M, $\|M\|_F$

denotes the Frobenius norm of M and $M \oplus N$ is the Kronecker sum of the matrices M and N (see Sect. A.2).
(Hint: The solution of (2.34) can be expressed as follows

$$vec(P(t)) = -\left(A^T(t) \oplus A^T(t)\right)^{-1} vec(Q)\,. \quad) \tag{2.95}$$

Note that the RHS in (2.94) is maximized if we choose Q to be the identity matrix. ◇

Exercise 2.6. Prove equality (2.71). ◇

Exercise 2.7 ([58]). Following the same guidelines of Theorem 2.10, prove Theorem 2.15.
(Hint: Use inequalities (A.22) and (A.26).) ◇

Exercise 2.8 ([7]). Prove the discrete-time counterpart of Exercise 2.5; that is show that, under the hypothesis i) and ii) of Theorem 2.15, condition

$$\begin{aligned} &\|A(k+1) \otimes A(k+1) - A(k) \otimes A(k)\| \\ &\qquad \leq \frac{\mu \underline{\sigma}\big(A(k) \otimes A(k) - I\big) \underline{\sigma}\big(A(k+1) \otimes A(k+1) - I\big)}{\sqrt{n}}\,, \quad \forall\, k \in \mathbb{N}_0\,, \end{aligned} \tag{2.96}$$

where μ is any number belonging to the interval $[0,1)$, guarantees exponential stability of system (2.75). ◇

3. Quadratic Stability

Most part of this chapter deals with the Lyapunov stability analysis of a linear system subject to parametric uncertainties as given by

$$\dot{x}(t) = A(p)x(t)\,, \qquad t \in [0, +\infty)\,, \tag{3.1}$$

where, as usual, $x(t) \in \mathbb{R}^n$, $p \in R \subset \mathbb{R}^q$ is the vector of uncertain parameters and $A(\cdot)$ is continuous. We assume that the set R is a hyper-box , that is a set in the form

$$R := [\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \cdots \times [\underline{p}_q, \overline{p}_q]\,. \tag{3.2}$$

We denote the set of the vertices of R by R^v.

Obviously equation (3.1) represents a collection of an infinite number of systems. For any given $p \in R$ equation (3.1) yields a system of differential equations with constant coefficients; conversely if p is a vector-valued function of time belonging to a certain functional space $\mathcal{S}$, for any $p(\cdot) \in \mathcal{S}$ system (3.1) defines a system of differential equations with time-varying coefficients.

A straightforward application of the results of Section 2.3 shows that, if the parameters are time-varying, Hurwitzness of the system matrix $A(p)$ for any $p \in R$ no longer guarantees exponential stability of system (3.1). In this case we need a different approach based on quadratic Lyapunov functions.

3.1 Necessary and Sufficient Conditions for Quadratic Stability

We start with the definition of quadratic stability.

Definition 3.1 (Quadratic Stability [25, 39, 116]). System (3.1) is said to be *quadratically stable (QS) in* R if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that for all $p \in R$

$$A^T(p)P + PA(p) < 0\,. \tag{3.3}$$

$\diamond$

As shown in the next theorem, quadratic stability implies exponential stability of system (3.1) for all admissible (in the sense specified below) time realizations of the parameters.

Theorem 3.1. *Assume that system* (3.1) *is QS. Then for any function* $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$ *the linear time-varying system*

$$\dot{x}(t) = A(p(t))x(t)\,, \qquad t \in [0, +\infty)\,, \tag{3.4}$$

is exponentially stable.

Proof. Let $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$. Then $\tilde{A}(\cdot) := A(p(\cdot)) \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{n\times n})$; therefore, for the given $p(\cdot)$, system (3.4) admits a unique solution according to Theorem 2.1.

Now, since R is compact, (3.3) implies that there exists a positive definite matrix P such that

$$\tilde{A}^T P + P\tilde{A} << 0\,. \tag{3.5}$$

The proof follows from Corollary 2.1 (applied with a constant P) and the arbitrariness of the function $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$. □

Remark 3.1. It is readily seen that behind Definition 3.1 and the proof of Theorem 3.1 is the use of a constant quadratic Lyapunov function in the form $v(x) = x^T Px$. Time invariance is the key point which allows to deal with arbitrarily varying parameters; in Chapter 4 we shall see that in presence of further information about the parameters behavior (i.e. a bound on the rate of variation) it is more convenient to use quadratic Lyapunov functions which depend on parameters. Such Lyapunov functions, along a given parameter vector time realization, are time-varying Lyapunov functions in the form (2.22). ◇

Remark 3.2. The assumption on piecewise continuity of parameters covers almost all cases of interest in the engineering applications. From a mathematical point of view, it should be noted that, using the result in [51], p. 67, it is possible to extend the class of admissible parameters realizations to the set of Lebesgue measurable vector-valued functions. ◇

From Definition 3.1 it follows that system (3.1) is QS *iff* the following problem admits a feasible solution.

Problem 3.1.
Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P > 0 \tag{3.6a}$$

$$A^T(p)P + PA(p) < 0 \qquad \forall p \in R\,. \tag{3.6b}$$

◇

The following result shows that quadratic stability of system (3.1) is equivalent to quadratic stability of the dual system; we shall use this fact in Chap. 5 in the quadratic stabilization context.

Lemma 3.1. *System* (3.1) *is QS* if and only if *there exists a positive definite matrix* Q *such that for all* $p \in R$

$$A(p)Q + QA^T(p) < 0\,. \tag{3.7}$$

Proof. Pre- and post-multiply (3.3) by P^{-1}; then let $P^{-1} = Q$. □

Note that, if Q is a positive definite matrix satisfying the hypothesis of Lemma 3.1, system (3.1) is QS and a suitable Lyapunov function is $x^T Q^{-1} x$.

Problem 3.1 is a feasibility problem in the matrix variable P subject to an *infinite* number of Linear Matrix Inequalities (LMIs) (one for each $p \in R$). The remaining part of this section will be devoted to convert inequality (3.6b) into a *finite* number of LMIs.

In particular we shall show that the kind of parameter dependence specified in the next assumption allows to reduce (3.6b) into a finite number of inequalities.

Assumption 3.1. The system matrix $A(\cdot) : R \to \mathbb{R}^{n\times n}$ is the ratio of a multi-affine matrix-valued function of p and a multi-affine polynomial of p

$$A(p) = \frac{N_A(p)}{d_A(p)} = \frac{\sum_{i_1,i_2,\dots,i_q=0}^{1} A_{i_1,\dots,i_q} p_1^{i_1} p_2^{i_2} \cdots p_q^{i_q}}{\sum_{i_1,i_2,\dots,i_q=0}^{1} a_{i_1,\dots,i_q} p_1^{i_1} p_2^{i_2} \cdots p_q^{i_q}}\,, \tag{3.8}$$

where $d_A(p) \neq 0$ for all $p \in R$ and $N_A(p) \in \mathbb{R}^{n\times n}$. ◇

Note that the parameter dependence considered in Assumption 3.1 is quite general and recovers, as particular cases, the usual affine and multi-affine dependence.

Remark 3.3. Important. The fact that the entries of $A(\cdot)$ are the ratio of multi-affine polynomials does not guarantee the satisfaction of Assumption 3.1; for example the elements of the matrix function

$$A(p) = \begin{pmatrix} \frac{p_1}{p_2} & \frac{p_2}{p_3} \\ 0 & 1 \end{pmatrix}$$

are the ratio of multi-affine polynomials, but $A(p)$ does not satisfy Assumption 3.1 since

$$A(p) = \frac{\begin{pmatrix} p_1 p_3 & p_2^2 \\ 0 & p_2 p_3 \end{pmatrix}}{p_2 p_3}\,.$$

◇

Now let

$$M(p) := A^T(p)P + PA(p)\,. \tag{3.9}$$

If $A(p)$ satisfies Assumption 3.1, the matrix-valued function $M(p)$ is the ratio of a multi-affine matrix valued function and a multi-affine polynomial; therefore we can apply Theorem A.2 and obtain the following result.

Theorem 3.2. *System* (3.1)*, where* $A(\cdot)$ *satisfies Assumption 3.1, is QS* if and only if *there exists a positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *such that*

$$A^T(p_{(i)})P + PA(p_{(i)}) < 0\,, \qquad i = 1,\dots,2^q\,,$$

where $p_{(i)}$ *is the* i*-th vertex of* R*.* □

Remark 3.4. Theorem 3.2 was originally proven in [100] when $A(\cdot)$ depends multi-affinely on parameters. The generalization to the structure considered in Assumption 3.1 is made in [85]. Note that in the paper [100] it is not explicitly recognized that the parameters are allowed to be time-varying. ◇

Remark 3.5. The parameter dependence considered in Assumption 3.1 is the more general polynomial dependence for which the vertices type result of Theorem 3.2 can be stated. To convince about this point consider the first order system

$$\dot{x}(t) = (-p^2+1)x(t) =: a(p)x(t)\,, \quad p \in R := [-2,2]\,. \tag{3.10}$$

By letting $P = 1 > 0$ we have that

$$2a(-2)P = 2a(2)P < 0\,. \tag{3.11}$$

Therefore the hypothesis of Theorem 3.2 are satisfied. On the other hand system (3.10) is not QS since for $p = 0$ it is unstable.

In the same way it is simple to build examples of systems for which $A(\cdot)$ is the ratio of a multi-affine numerator and a quadratic denominator which satisfy the hypothesis of Theorem 3.2 and are not QS. ◇

Note that the application of Theorem 3.2 requires us to check that the denominator of $M(\cdot)$ is never zero in R. Since this denominator is a multi-affine function of p, this check can be done in a simple way by using an obvious corollary of Theorem A.2.

Corollary 3.1. *Let* $d_M(p)$ *the denominator of* $M(p)$*. Then* $d_M(p) \neq 0$ *for all* $p \in R$ if and only if *either* $d_M(p_{(i)}) > 0$*,* $i = 1,\dots,2^q$*, or* $d_M(p_{(i)}) < 0$*,* $i = 1,\dots,2^q$*.* □

Under Assumption 3.1, by virtue of Theorem 3.2, Problem 3.1 is equivalent to the following LMIs based feasibility problem.

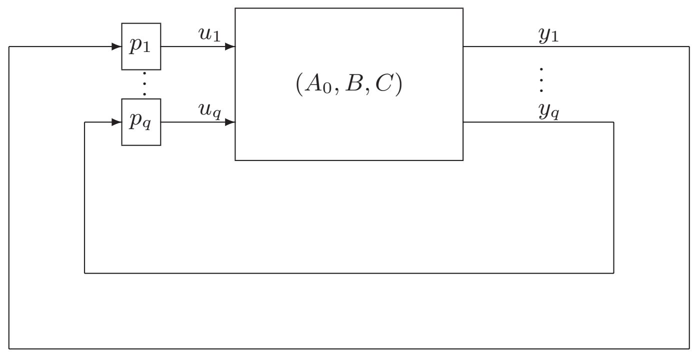


Fig. 3.1. The closed loop system with uncertain gains on the channels

Problem 3.2.
Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P > 0 \tag{3.12a}$$

$$A^T(p_{(i)})P + PA(p_{(i)}) < 0\,, \qquad i = 1, \ldots, 2^q\,. \tag{3.12b}$$

◇

Example 3.1 (Quadratic Stability Margin).
Let us consider the feedback system depicted in Fig. 3.1 and described by the state space equations

$$\dot{x}(t) = A_0 x(t) + Bu(t) \tag{3.13a}$$

$$y(t) = Cx(t) \tag{3.13b}$$

$$u(t) = \Delta(p)y(t)\,, \tag{3.13c}$$

where $A_0 \in \mathbb{R}^{n\times n}$, $B = \begin{pmatrix} b_1 \ldots b_q \end{pmatrix} \in \mathbb{R}^{n\times q}$, $C^T = \begin{pmatrix} c_1 \ldots c_q \end{pmatrix} \in \mathbb{R}^{n\times q}$ and $\Delta(p) \in \mathbb{R}^{q\times q}$ is a diagonal matrix whose elements are the components of the parameter vector $p = \begin{pmatrix} p_1 \ldots p_q \end{pmatrix}^T$. This uncertain system can be described concisely in the form (3.1) with

$$\begin{aligned} A(p) &= A_0 + B\Delta(p)C \\ &= A_0 + \sum_{i=1}^{q} b_i c_i^T p_i \\ &= A_0 + L(p)\,, \end{aligned} \tag{3.14}$$

where $L(p)$ is a linear matrix-valued function. Now, given the unit square in $\mathbb{R}^q$

$$R_u := \{p \in \mathbb{R}^q \,:\, \|p\|_\infty \le 1\}\,, \tag{3.15}$$

and defined θ as the *dilatation factor* of the set R_u

$$\theta R_u := \{p \in \mathbb{R}^q \,:\, \|p\|_\infty \le \theta\} \ , \theta > 0 \, , \tag{3.16}$$

we define the Quadratic Stability Margin (QSM) of system (3.13) as follows

$$\rho_Q := \sup \, \{\theta > 0 \,:\, \text{system (3.13) is quadratically stable in } \theta R_u\} \ . \tag{3.17}$$

Clearly the QSM can be interpreted as an estimate[1] of the supremal allowable amplitude of *time-varying* parameters which guarantees a stable closed loop system; therefore the QSM can be seen as the generalization to time-varying gains of the classical concept of Multivariable Gain Margin (MGM, see [56]).

Examining (3.16), (3.17), it is readily seen that an equivalent definition of the QSM of system (3.13) is the following

$$\rho_Q := \sup \, \{\theta > 0 \,:\, \text{system } \dot{x} = A(\theta p)x \text{ is quadratically stable in } R_u\} \ . \tag{3.18}$$

By noticing from (3.14) that

$$A(\theta p) = A_0 + \theta L(p) \, , \tag{3.19}$$

we conclude that an estimate of the QSM can be computed by solving the following Generalized Eigenvalue Problem (GEVP, see [38], p. 11) in the variables θ and P.

Problem 3.3.

$$\begin{aligned} &\max \theta \\ &\quad \text{s.t.} \\ &\quad \theta > 0 \\ &\quad P > 0 \\ &A_0^T P + P A_0 + \theta \left(L^T(p_{(i)})P + PL(p_{(i)}) \right) < 0 \, , \quad i = 1, \ldots, 2^q \, , \end{aligned}$$

where $p_{(i)}$ is the i-th vertex of R_u. ◇

Now let us compute the QSM for a system in the form (3.13) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \, , \quad b_1 = b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, , \quad c_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \, , \quad c_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \, . \tag{3.20}$$

By solving the GEVP 3.3 with the aid of the LMI toolbox [83], we find $0.583 < \rho_Q < 0.584$.

By weighting the time-varying gains differently, we can give a more general definition of QSM. This is simply obtained by replacing the vector infinity norm in (3.15) by the *weighted* infinity norm $\| \cdot \|_\infty^w$, with $w \in \mathbb{R}^q$. △

[1] Because quadratic stability is sufficient but not necessary for exponential stability versus time-varying parameters (see the Summary at the end of the chapter).

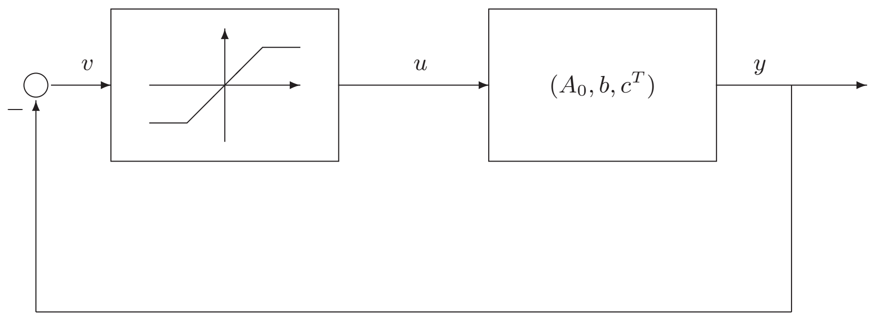


Fig. 3.2. Nonlinear system with saturation

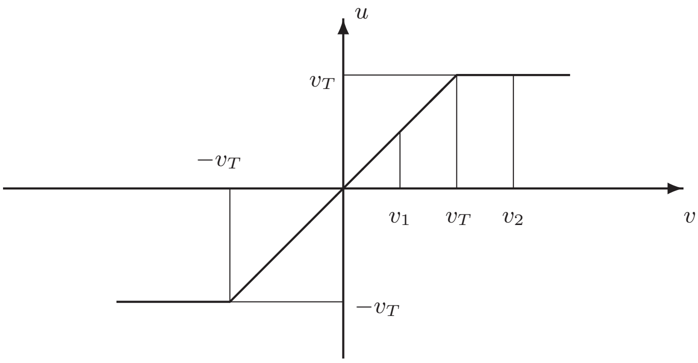


Fig. 3.3. The saturation element

Example 3.2 (Stability of a nonlinear system with saturation). Quadratic Stability can be used to study the stability properties of the nonlinear closed-loop system in Fig. 3.2,[2] which is composed of the SISO linear system

$$\dot{x}(t) = A_0 x(t) + b u(t) \tag{3.21a}$$

$$y(t) = c^T x(t) \tag{3.21b}$$

and the static nonlinear element with equation $u = N(v)$. It is assumed (without loss of generality) that the nonlinear element has unitary slope; moreover we denote by $[-v_T, v_T]$ its linear range (see Fig. 3.3).

If, at a given instant t_1, the value of the input of the nonlinear element is $v_1 \leq v_T$ (see Fig. 3.3) the value of the instantaneous input-output gain associated with the nonlinear element is (assume $v_1 \neq 0$)

[2] The stability analysis can be also performed via the Circle Criterion (see [109], Sect. 5.1.2), however our goal in this example is to illustrate the possible use of quadratic stability methods for nonlinear system analysis; our approach is along the guidelines described in [109], Sect. 5.1.3.

$$g(t_1) := \frac{u(t_1)}{v(t_1)} = \frac{v_1}{v_1} = 1\,, \tag{3.22}$$

that is no saturation is present; note that when $v_1 = 0$, the gain $g(\cdot)$ is not defined. In the same way, if at the time instant t_2 the value of the input is $v_2 > v_T$, the input saturates the nonlinear element and the gain becomes

$$g(t_2) = \frac{v_T}{v_2} < 1\,. \tag{3.23}$$

Defining $g(t) = 1$ when $v(t) = 0$, the instantaneous input-output gain is a continuous function $g(\cdot)$ attaining values into the interval $[g_{min}, 1]$ where $g_{min} \geq 0$.

Now refer to the linear time-varying element in Fig. 3.4. It is simple to recognize that, for each admissible input-output pair $\{v^*(\cdot), u^*(\cdot)\}$ associated with the *nonlinear* time-invariant element in Fig. 3.3, there exists a continuous gain realization $g^* : [0, +\infty) \to [g_{min}, 1]$, such that, given the input $v^*(\cdot)$, the output of the system in Fig. 3.4 is exactly $u^*(\cdot)$.

This fact suggests to replace the nonlinear element described in Fig. 3.3 by the linear time-varying gain described in Fig. 3.4, where the function $g(\cdot)$ is allowed to be any member of the set

$$\left\{ g(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}) \,:\, g(t) \in [g_{min}, 1]\,,\, t \in [0, +\infty) \right\}\,. \tag{3.24}$$

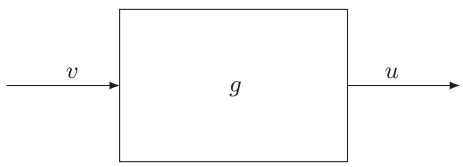


Fig. 3.4. The linear time-varying element

This operation leads to the linear closed loop system in Fig. 3.5, which has the following state-space description

$$\dot{x}(t) = \big(A_0 - bc^T g\big)x(t)\,. \tag{3.25}$$

Now assume that system (3.25) is quadratically stable; this means that there exists a positive definite matrix P such that

$$\big(A_0 - bc^T g\big)^T P + P\big(A_0 - bc^T g\big) < 0\,, \qquad g \in [g_{min}, 1]\,. \tag{3.26}$$

By Theorem 3.1, quadratic stability in $[g_{min}, 1]$ of the linear system (3.25) implies exponential stability of the same system for any time-varying gain $g(\cdot)$

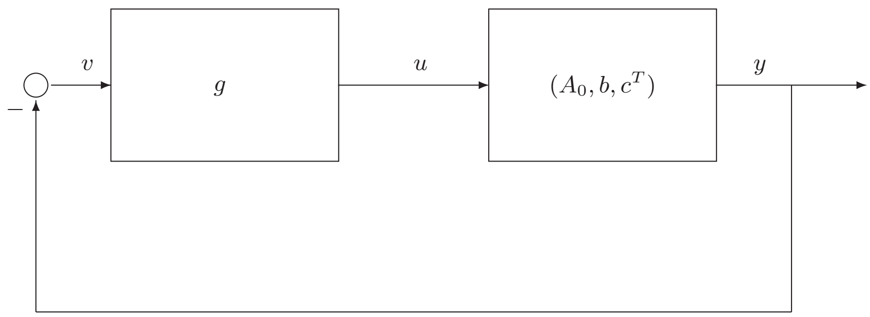


Fig. 3.5. Linear system with the time-varying element replacing the saturation nonlinearity

in the set (3.24); but what can we conclude about the equilibrium $x = 0$ of the original nonlinear system in Fig. 3.2?

It is simple to recognize that quadratic stability of system (3.25) implies the existence of a positive definite quadratic Lyapunov function, namely $v(x) = x^T P x$, whose derivative along the state trajectories of the nonlinear system in Fig. 3.2 is negative definite whenever $|c^T x| = |v| \leq v_T / g_{min}$; therefore local exponential stability of the equilibrium of the nonlinear system is guaranteed.

An estimate of the Region of Attraction ([109], p. 136) can be obtained according to [109], pp. 141–143. Let a any positive number such that

$$R_a := \left\{x \in \mathbb{R}^n \,:\, x^T P x \leq a\right\} \subseteq \left\{x \in \mathbb{R}^n \,:\, |c^T x| \leq v_T / g_{min}\right\} . \qquad (3.27)$$

Any set R_a defined as above represents an estimate of the Region of Attraction; in order to obtain the less conservative estimate of this region (for the given Lyapunov function) one must consider the largest number a, say $\overline{a}$, such that (3.27) is satisfied.

We can conclude that every solution of the nonlinear system in Fig. 3.2 starting from an initial state $x_0 \in R_{\overline{a}}$ exponentially converges to zero.

From (3.27) it is clear that the amplitude of the Region of Attraction depends on g_{min}, the smaller g_{min} the bigger the amplitude. If $g_{min} = 0$, then the nonlinear system is globally exponentially stable.

As for the linear system, since the dependence of the system matrix on g is affine, we have that, from Theorem 3.2, system (3.25) is quadratically stable *iff* there exists a positive definite matrix P such that the following pair of LMIs is satisfied

$$\left(A_0 - bc^T g\right)^T P + P\left(A_0 - bc^T g\right) < 0 \,, \qquad g \in \{g_{min}, 1\} \,. \qquad (3.28)$$

The approach described here has been used in [13] to analyze aircraft proneness to Pilot Involved Oscillations (PIOs). PIOs are due to a misadaptation between the pilot and the aircraft during some tasks in which tight closed-loop control of the aircraft is required, with the aircraft not responding as expected to pilot commands. This situation can trigger a pilot action capable of driving the aircraft out of control, which in some cases can only be recovered by the pilot releasing the column and exiting from the control loop.

PIOs can be of Category I (the closed-loop pilot-vehicle system has a linear behavior), Category II (the closed-loop system has a nonlinear behavior, mainly characterized by the saturation of position or rate limited elements) or Category III (the closed-loop system has a highly nonlinear behavior); in particular, Category II PIOs have been studied through a quadratic stability approach in [13].

As said, Category II PIOs are caused by the saturation of position or rate limiters; such kind of nonlinearities are unavoidably present in every aircraft, because of physical constraints on elements such as stick/column deflections, actuators position and rate limiters, etc. Roughly speaking, PIO occurrence can be identified with closed-loop instability of the aircraft-pilot system; therefore PIOs analysis is reduced to a robust stability analysis of the closed-loop system formed by the pilot and the aircraft dynamics in presence of saturated actuators.

Let us consider the pitch axis model of the X-15 aircraft depicted in Fig. 3.6 (see [113]). The main blocks in Fig. 3.6 are: the pilot gain k_p, the nonlinear actuator, whose rate limiting is provided by the saturation nonlinearity (normalized to be symmetric with unit slope) which precedes the position integrator, and the pitch axis transfer function $\theta(s)/\delta(s)$ from the control surface position to the variable controlled by the pilot; we denote by $\dot{\delta}_{max}$ the maximum output amplitude of the saturation element. The numerical values of the elements in the block diagrams are

$$\begin{aligned}\frac{\theta(s)}{\delta(s)} &= \frac{3.476(s+0.0292)(s+0.883)}{(s^2+0.019s+0.01)(s^2+0.8418s+5.29)}\\ \tau_R &= 0.04\,[\text{sec}]\\ \dot{\delta}_{max} &= 15\,[\text{deg/sec}]\,.\end{aligned}$$

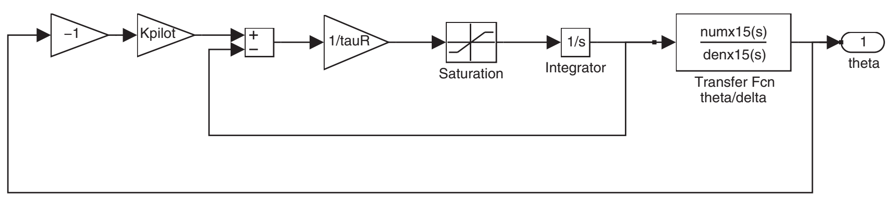


Fig. 3.6. Analysis model of the X-15 Landing Flare PIO

The stability analysis of the system in Fig. 3.6 is performed versus the pilot gain k_p; indeed it is well known that critical full attention manoeuvres, like tracking, aerial refuelling, etc., may require a high pilot gain which can trigger PIO occurrence. It is also assumed that the pilot gain is uncertain but constant, i.e. it is held fixed to some particular value during the maneuver, the actual value maybe depending on the flight phase and the particular pilot.

By block diagram algebra it is clear that the scheme in Fig. 3.6, after the nonlinearity has been replaced by the linear time-varying gain $g(t)$, can be rearranged to yield the scheme in Fig. 3.7. Now let us denote by $(A_0, b, c^T(k_p))$ a state space representation of the linear time-invariant block in Fig. 3.7 (obtained for example using a controllable canonical form). Note that the dependence of $c(\cdot)$ on k_p is affine.

Now, referring to the system in Fig. 3.7, our goal is to estimate the region in the (k_p, g) plane in which Hurwitz stability versus the time-invariant parameter k_p and quadratic stability versus the parameter g is guaranteed; this in turn guarantees exponential stability of the nonlinear system in Fig. 3.6.

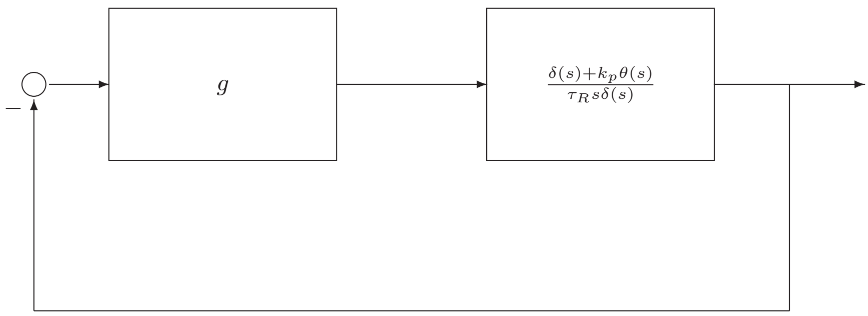


Fig. 3.7. Equivalent system to the one in Fig. 3.6 with the time-varying element replacing the saturation nonlinearity

To this end the following algorithm is used; the algorithm is based on the fact that, for a given k_p, the closed-loop system is quadratically stable (see (3.28)) *iff* there exists a positive definite matrix P such that the following pair of LMIs are satisfied

$$\left(A_0 - bc^T(k_p)g\right)^T P + P\left(A_0 - bc^T(k_p)g\right) < 0\,, \qquad g \in \{g_{min}, 1\}\ . \quad (3.29)$$

Algorithm 3.1.

Step 1
Let $\Delta k = 0.01$, $k_{min} = 0$, $k_{max} = 0 + \Delta k$, $\Delta g = 0.025$, $g_{min} = 0.0015$.
Step 2
Solve the following feasibility problem:

Problem 3.4.
Find $P > 0$ such that

$$\begin{aligned}&\big(A_0 - bc^T(k_p)g\big)^T P + P\big(A_0 - bc^T(k_p)g\big) < 0\\ &\qquad\qquad g \in \{g_{min}, 1\}\ , \qquad k_p \in \{k_{min}, k_{max}\}\ .\end{aligned} \tag{3.30}$$

Step 3

If Problem 3.4 is not feasible **then** plot the box $[g_{min}, 1] \times [0, k_{min}]$ and **let** $g_{min} = g_{min} + \Delta g$, **else let** $k_{min} = k_{max}$, $k_{max} = k_{min} + \Delta k$ **end**.
If $g_{min} < 1$ **then goto** Step 2; **else stop**.

◇

Remark 3.6. Note that the feasibility of the 4 LMIs contained in Problem 3.4 also guarantees quadratic stability versus k_p in the interval $[k_{min}, k_{max}]$; since k_p is actually time-invariant, the conditions expressed by the above-mentioned LMIs could seem conservative. This is not true in practice, because the interval $[k_{min}, k_{max}]$ is very small; indeed, by using continuity arguments, it is readily seen that Hurwitz stability implies quadratic stability if the parameter excursion is sufficiently small. ◇

Remark 3.7. The approach used in Algorithm 3.1 corresponds to consider a quadratic Lyapunov function which is piecewise constant *wrt* the parameter k_p and independent on g. Indeed, in Chapter 4 we shall show that to take into account time-invariant parameters for robust stability analysis it is necessary to use parameter dependent quadratic Lyapunov functions; however in that case we shall look to functions which depend continuously on parameters[3]. ◇

In Fig. 3.8 we have depicted the stability region computed via Algorithm 3.1; the boundary of the same region is plotted in Fig. 3.9 versus the operating region of the aircraft, that is a box of the admissible values of the pair (k_p, g).

Note that there is a non-empty region of the operating envelope which is not contained in the stability region. This means that some admissible combinations of the pilot gain and the nonlinear gain might give rise to instability (that is a PIO). Actually the X-15 aircraft has been subject to a dramatic PIO accident during a landing flare (see [122]).

Finally, it is interesting to notice that, by inspection of Fig. 3.9, we obtain some hints for the design of an actuator free from PIOs; it is clear that the operating envelope box must shrink so to coincide with the dashed box. This requires the use of an actuator with a larger linear range. △

[3] The reason for considering in Chap. 4 continuous parameter dependent Lyapunov functions follows from the fact that we shall treat time-invariant parameters as parameters with zero rate of variation in the more general context of parameters with bounded rate of variation. For bounded rate parameters the stability analysis is performed via Theorem 2.8 which requires continuity of the Lyapunov function.

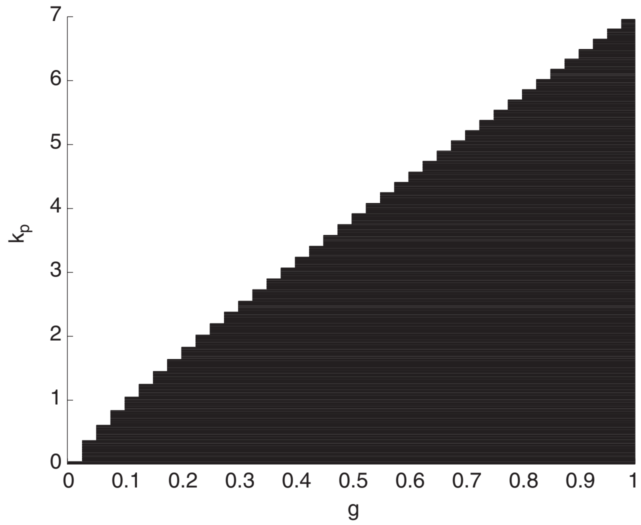


Fig. 3.8. The stability region in the (k_p, g) plane

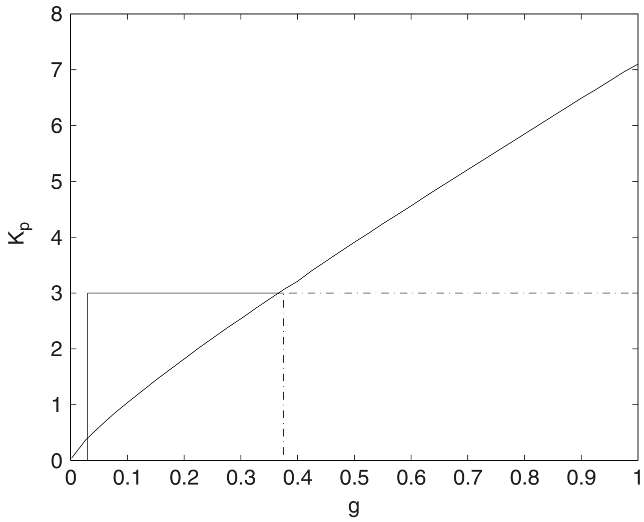


Fig. 3.9. The stability region versus the operating envelope

3.1.1 Polytopic Systems

Consider the uncertain linear time-varying system

$$\dot{x}(t) = A(t)x(t)\,, \qquad A(t) \in \operatorname{conv}\left\{A_{(1)}, A_{(2)}, \ldots, A_{(l)}\right\} =: \mathcal{A},\ t \in [0, +\infty)\,, \tag{3.31}$$

where $A(\cdot)$ is piecewise continuous.

An uncertain system in the form (3.31) is also called a *polytopic system* ([38], p. 53), since it represents a family of linear time-varying systems whose system matrix attains values into the polytope with vertices $A_{(i)}$, $i = 1, 2, \dots, l$; representation (3.31) arises, for example, when we try to capture the dynamics of a (possibly nonlinear) system working at different operating conditions.

The polytopic system (3.31) is said to be QS *if and only if* there exists a positive definite matrix P such that

$$A^T P + PA < 0\,, \quad \forall A \in \mathcal{A}\,, \tag{3.32}$$

which is clearly equivalent to require the existence of a positive definite matrix P such that

$$A_{(i)}^T P + PA_{(i)} < 0\,, \quad i = 1, \dots, l\,. \tag{3.33}$$

It is simple to recognize that quadratic stability of the polytopic system (3.31) guarantees exponential stability of any linear time-varying system

$$\dot{x}(t) = A(t)x(t) \tag{3.34}$$

with $A(\cdot) \in \mathcal{PC}(\mathbb{R}^+, \mathcal{A})$.

In a sense, polytopic systems can be seen as a sub-class of the systems satisfying Assumption 3.1. Assume, for example, that the polytopic system we deal with is obtained by the convex hull of two matrices

$$\dot{x}(t) = A(t)x(t)\,, \qquad A(t) \in \operatorname{conv}\left\{A_{(1)}, A_{(2)}\right\}, \; t \in [0, +\infty)\,. \tag{3.35}$$

Then consider the uncertain system

$$\dot{x}(t) = A(p)x(t) = A_0 + A_1 p\,, \tag{3.36}$$

where $p \in [-1, 1]$ and

$$A_0 = \frac{1}{2}(A_{(1)} + A_{(2)}) \tag{3.37a}$$

$$A_1 = \frac{1}{2}(A_{(2)} - A_{(1)})\,; \tag{3.37b}$$

obviously we have that $A(-1) = A_{(1)}$ and $A(1) = A_{(2)}$.

Since the image of an affine matrix function defined over a hyper-box (an interval in this case) is equal to the convex hull of the image of the vertices (see Theorem A.1) we have that

$$A([-1, 1]) = \operatorname{conv}\left\{A_{(1)}, A_{(2)}\right\}\;; \tag{3.38}$$

therefore the uncertain system (3.36) is equivalent (from the point of view of the quadratic stability analysis) to the polytopic system (3.35).

In general, referring to system (3.31), there always exists an uncertain system in the form (3.1) satisfying

$$A(R) \supseteq \mathcal{A}\,. \tag{3.39}$$

Obviously, when $\mathcal{A}$ is a strict subset of $A(R)$, quadratic stability of system (3.1) is sufficient but not necessary for quadratic stability of system (3.31).

3.2 The General Nonlinear Parameter Dependence Case

In this section we consider the case in which the system matrix $A(\cdot)$ does not satisfy Assumption 3.1 and, therefore, it is no longer immediate to transform inequality (3.6b) into a finite number of LMIs. Also, to simplify the notation, we assume that the denominator of $A(\cdot)$ is unitary, since the ideas of this section can be immediately extended to the general non-unitary denominator case.

3.2.1 Polynomial Dependence on Parameters

In [136,162] and [163] the problem of testing robust (Hurwitz) stability of uncertain characteristic polynomials depending on parameters in a polynomial fashion is considered. The original idea was to transform the polynomial dependence into a multi-affine one, by introducing further fictitious parameters, and then to apply the Edge Theorem [31].

This idea can be extended to test quadratic stability of system (3.1) when $A(\cdot)$ depends polynomially on p, that is

$$A(p) = \sum_{\alpha_1,\ldots,\alpha_q} A_{\alpha_1,\ldots,\alpha_q} p_1^{\alpha_1} \cdots p_q^{\alpha_q}\,, \tag{3.40}$$

where $A_{\alpha_1,\ldots,\alpha_q} \in \mathbb{R}^{n\times n}$ and $\alpha_i = 0, 1, \ldots, \mu_i$, $\mu_i \in \mathbb{N}$, $i = 1, \ldots, q$.

After $A(\cdot)$ has been replaced by a multi-affine matrix function, quadratic stability can be studied via Theorem 3.2; since the image of the new multi-affine matrix function contains that one of the original system matrix, this approach introduces a certain degree of conservatism.

For the sake of clarity we illustrate the idea through a simple single parameter example. Let us consider the matrix function

$$A(p) = \begin{pmatrix} -p^2 & p \\ -1 & -5 \end{pmatrix}, \tag{3.41}$$

with $p \in [\underline{p}, \overline{p}]$, and define

$$\hat{A}(\delta) := \begin{pmatrix} -\delta_1\delta_2 & -\delta_1 \\ -1 & -5 \end{pmatrix}, \tag{3.42}$$

with $\delta = \begin{pmatrix}\delta_1 & \delta_2\end{pmatrix}^T \in [\underline{p}, \overline{p}]^2$; it is obvious that $\hat{A}([\underline{p}, \overline{p}]^2) \supset A([\underline{p}, \overline{p}])$. Obviously, the replacement of p^2 by $\delta_1\delta_2$ is rather conservative because it introduces two parameters which vary independently each other.

In the general multi-parameter case it is possible to reduce the problem to the multi-affine case by generalizing the above idea. Let h_i be the highest degree of p_i. Introduce fictitious variables $\delta_{i1}, \dots, \delta_{ih_i}$, $i = 1, \dots, q$, such that

$$\delta := (\delta_{11} \dots \delta_{1h_1} \dots \delta_{q1} \dots \delta_{qh_q})^T \in [\underline{p}_1, \overline{p}_1]^{h_1} \times \dots \times [\underline{p}_q, \overline{p}_q]^{h_q} \tag{3.43}$$

where $\delta_{i1} = p_i$, and replace in (3.40) each $p_i^{\alpha_i}$ by the product $\delta_{i1}\delta_{i2} \cdots \delta_{i\alpha_i}$. In this way we obtain the multi-affine function defined over the hyper-box $[\underline{p}_1, \overline{p}_1]^{h_1} \times \dots \times [\underline{p}_q, \overline{p}_q]^{h_q}$

$$\hat{A}(\delta) = \sum_{i_{11},\dots,i_{1h_1},\dots,i_{q1},\dots,i_{qh_q}=0}^{1} \hat{A}_{i_{11},\dots,i_{1h_1},\dots,i_{q1},\dots,i_{qh_q}} \delta_{11}^{i_{11}} \cdots \delta_{1h_1}^{i_{1h_1}} \cdots \delta_{q1}^{i_{q1}} \cdots \delta_{qh_q}^{i_{qh_q}} \,. \tag{3.44}$$

Obviously $\hat{A}\left([\underline{p}_1, \overline{p}_1]^{h_1} \times \dots \times [\underline{p}_q, \overline{p}_q]^{h_q}\right) \supset A(R)$.

The conservatism of the technique increases with the number of uncertain parameters q. Such conservatism can be reduced by splitting the hyper-cube $[\underline{p}_1, \overline{p}_1]^{h_1} \times \dots \times [\underline{p}_q, \overline{p}_q]^{h_q}$ into smaller hyper-cubes. This idea is explained in details in the next section where a more general approach to create a multi-affine function starting from a nonlinear one is provided.

3.2.2 The Polytopic Covering Technique

In this section the idea of Sect. 3.2.1 is extended to a quite a general class of nonlinear functions; moreover we propose a different method to replace the original parameters by a new set of fictitious parameters. Such method may obtain a tighter covering of the set $A(R)$ and therefore a less conservative application of the quadratic stability test.

For a given continuous scalar function $f : R \mapsto \mathbb{R}$, we shall show how to build a *multi-affine* function $f^m(p, \delta)$ such that, for each $p \in R$, there is a $\delta \in [0, 1]$ with

$$f^m(p, \delta) = f(p) \,. \tag{3.45}$$

To this end, let $\underline{f}$ and $\overline{f}$ be multi-affine affine functions satisfying for all $p \in R$

$$\underline{f}(p) \le f(p) \le \overline{f}(p) \,. \tag{3.46}$$

Now define $f^m : R \times [0, 1] \mapsto \mathbb{R}$ to be the multi-affine function given by

$$f^m(p,\delta) = (1-\delta)\underline{f}(p) + \delta\overline{f}(p)\,. \tag{3.47}$$

Clearly, f^m is multi-affine. Moreover for a given $p \in R$ there exists $\delta \in [0,1]$ given by

$$\delta = \frac{f(p) - \underline{f}(p)}{\overline{f}(p) - \underline{f}(p)} \tag{3.48}$$

such that $f^m(p,\delta) = f(p)$.

Clearly, for a given $f(\cdot)$, there are several possible choices of the bounding functions $\underline{f}$ and $\overline{f}$. As we shall see later in this section, the choice of the bounding functions may strongly influence the degree of conservatism of the technique; therefore great care must be posed when doing this choice.

If $f(\cdot)$ possesses a complex structure, the application of the above procedure can be greatly simplified if we split the original function f into the product of many functions with a simple structure. This idea leads to the procedure described in the next section.

A General Procedure

Throughout this section we shall consider the following class of matrix-valued functions.

Assumption 3.2. The matrix function $A(\cdot)$ has the following structure

$$A(p) = \sum_{i_1,i_2,\ldots,i_\nu=0}^{1} A_{i_1,i_2,\ldots,i_\nu} f_1(p)^{i_1} f_2(p)^{i_2} \cdots f_\nu(p)^{i_\nu}\,, \tag{3.49}$$

where $f_j : R \to \mathbb{R}$ is continuous, $A_{i_1,i_2,\ldots,i_\nu} \in \mathbb{R}^{n\times n}$, and, for all $j = 1,2,\ldots,\nu$, if f_j is not multi-affine, there exist *known* multi-affine functions $\underline{f}_j(p)$, $\overline{f}_j(p)$ such that

$$\underline{f}_j(p) \le f_j(p) \le \overline{f}_j(p), \quad \forall p \in R. \tag{3.50}$$

◇

Note that the structure of $A(\cdot)$ considered in Assumption 3.2 is very general. In particular it recovers the polynomial dependence considered in Sect. 3.2.1; indeed the matrix function

$$A(p) = \sum_{\alpha_1,\ldots,\alpha_q} A_{\alpha_1,\ldots,\alpha_q} p_1^{\alpha_1} \cdots p_q^{\alpha_q}, \tag{3.51}$$

where $\alpha_i = 0,1,\ldots,\mu_i$, $i = 1,\ldots,q$, is recovered by (3.49) by letting, for example,

$$f_1(p) = p_1, \ldots, f_{\mu_1}(p) = p_1^{\mu_1}, f_{\mu_1+1}(p) = p_2, \ldots\,; \tag{3.52}$$

moreover it is rather simple to find suitable bounding functions for $p_i^{\alpha_i}$, $2 \le \alpha_i \le \mu_i$ (again, this topic is dealt with later in this section).

Remark 3.8. As said at the beginning of Sect. 3.2, the structure considered in (3.49) can be generalized so to include a denominator in the form

$$a(p) = \sum_{i_1,i_2,\ldots,i_\eta=0}^{1} a_{i_1,i_2,\ldots,i_\eta} g_1(p)^{i_1} g_2(p)^{i_2} \cdots g_\eta(p)^{i_\eta} , \tag{3.53}$$

where $g_j : R \to \mathbb{R}$ is continuous, $a_{i_1,i_2,\ldots,i_\eta} \in \mathbb{R}$ and, for all $j = 1, 2, \ldots, \eta$, if g_j is not multi-affine, there exist known multi-affine functions $\underline{g}_j(p)$, $\overline{g}_j(p)$ such that a condition analogous to (3.50) holds.

It is readily seen that the first two steps of the procedure below can be easily generalized to take into account the presence of a non-unitary denominator. ◇

Let $\mu \leq \nu$ the number of non-multi-affine functions in (3.49). The following procedure constructs $2^{q+\mu+\gamma}$ points (not necessarily distinct) in $\mathbb{R}^n$, where $0 \leq \gamma \leq q(\nu - 1)$. We will show in Theorem 3.3 that the convex hull of these points includes (that is covers) $A(R)$. The method is named "Polytopic Covering Technique" since the convex hull is a polytope in $\mathbb{R}^{n\times n}$.

Procedure 3.1 ([10, 11]). The procedure is composed of three steps.

Step 1

Construct the matrix function $\Psi(p, \delta)$ obtained from (3.49) by introducing the fictitious parameters

$$\delta_j \in I_j := \begin{cases} [0, 1] & \text{if } f_j \text{ is not multi-affine} \\ \{0\} & \text{if } f_j \text{ is multi-affine} \end{cases} , \quad j = 1, \ldots, \nu , \tag{3.54}$$

and substituting for $f_j(p)$, $j = 1, \ldots, \nu$, the multi-affine function

$$f_j^m(p, \delta_j) := \begin{cases} (1 - \delta_j)\underline{f}_j(p) + \delta_j \overline{f}_j(p) & \text{if } f_j \text{ is not multi-affine} \\ f_j(p) & \text{if } f_j \text{ is multi-affine} . \end{cases} \tag{3.55}$$

Let $\mathcal{D} := I_1 \times I_2 \times \cdots \times I_\nu$; hence $\Psi(p, \delta)$ is defined over $R \times \mathcal{D}$. Observe that $\mathcal{D}$ is a hyper-box with 2^μ vertices and that $\Psi(p, \delta)$ is a multi-affine function of the δ_i's and a polynomial function of the p_i's; moreover, since in (3.49) there are at most ν products, the maximum exponent of each p_i in $\Psi(p, \delta)$ cannot exceed ν.

Step 2

Replace each $p_i^{\alpha_i}$, $\alpha_i > 1$, by a multi-affine function as in (3.55); obviously, in this case, the bounding functions will depend only on p_i. The number of replacements, say γ, may vary between zero (when Ψ is already multi-affine, that is Ψ does not contain powers of $p_1, \ldots, p_q$ greater than 1) and $q(\nu - 1)$.

In this way we introduce a new vector of fictitious variables ϵ ranging in the hyper-box $\mathcal{E}$ defined in the same way as $\mathcal{D}$ and define the new function $\Phi(p, \delta, \epsilon)$. From the previous discussion follows that the hyper-box $\mathcal{E}$ has 2^γ vertices.

Step 3
Define the hyper-box $\Omega := R \times \mathcal{D} \times \mathcal{E}$ and evaluate the points $\Phi_{(i)} := \Phi(\omega_{(i)})$, where $\omega_{(i)}$, $i = 1, \ldots, 2^q 2^\mu 2^\gamma$, are the vertices of Ω.

$\diamond$

Remark 3.9. Note that, in the practical application of Procedure 3.1, there is no need of introducing the fictitious parameter δ_j if f_j is multi-affine. $\diamond$

Remark 3.10. It is worth noting that the points $\Phi_{(i)}$ are not necessarily distinct. In many practical situations the number of distinct points is largely smaller than $2^{q+\mu+\gamma}$. $\diamond$

Remark 3.11. When the dependence on parameters is polynomial, like in (3.51), and we choose the functions f_j according to (3.52), it is simple to recognize that, after the application of Step 1 of Procedure 3.1, the obtained function is already multi-affine. Therefore, in this case, we need not to apply Step 2 of the procedure. $\diamond$

The key point in the proof of the next theorem is that, as one can simply recognize, the function Φ constructed in Step 2 depends multi-affinely on p, δ and ϵ.

Theorem 3.3 ([11]).

$$A(R) \subseteq \mathcal{A} := \mathrm{conv}\{\Phi_{(i)},\, i = 1, \ldots, 2^{q+\mu+\gamma}\}$$

Proof. Define $\Psi(p, \delta)$ replacing in (3.49) $f_j(p)$ with $f_j^m(p, \delta_j)$ for $j = 1, \ldots, \nu$. From the expression of f_j^m we know that for all $p \in R$ there exists $\delta_j \in [0, 1]$ such that $f_j^m(p, \delta_j) = f_j(p)$, and hence for all $p \in R$ there exists $(\delta_1, \ldots, \delta_\nu)^T \in \mathcal{D}$ such that $\Psi(p, \delta) = A(p)$. Therefore we can conclude that

$$A(R) \subseteq \Psi(R \times \mathcal{D}). \tag{3.56}$$

As said, when we apply the procedure described in Step 2 to $\Psi(p, \delta)$, the resulting function $\Phi(p, \delta, \epsilon)$ defined over $R \times \mathcal{D} \times \mathcal{E}$ will be multi-affine. Obviously $\Phi(p, \delta, \epsilon)$ is such that

$$\Psi(R \times \mathcal{D}) \subseteq \Phi(R \times \mathcal{D} \times \mathcal{E})\,. \tag{3.57}$$

Now, let $\Omega := R \times \mathcal{D} \times \mathcal{E}$. Inclusion (3.57) together with (3.56) and the multi-affine nature of Φ, which enables us to apply Lemma A.1, yields

$$\begin{aligned} \mathrm{conv}\{\Phi_{(i)},\, i = 1, \ldots, 2^{q+\mu+\gamma}\} &\supseteq \Phi(\Omega) \\ &\supseteq A(R)\,. \end{aligned} \tag{3.58}$$

$\square$

When the number of non-multi-affine functions is big, the Polytopic Covering technique becomes computationally intractable. In this case we suggest to resort to the approaches described in Sect. 3.3 to check quadratic stability of the uncertain system (3.1).

On the Choice of The Bounding Functions

In general the choice of the bounding functions satisfying (3.50) is not unique; note, however, that a suitable choice can sensibly improve the goodness of the covering. For example if the f_j's are continuous, the multi-affine functions $\underline{f}_j$ and $\overline{f}_j$ can be chosen to be constant and equal to the minimum and the maximum of f_j respectively. On the other hand it should be clear that the better the functions $\underline{f}_j$ and $\overline{f}_j$ fit f_j, the less conservative the covering will be.

For instance consider again matrix $A(\cdot)$ in (3.41) with $p \in [1,2]$. If we replace p^2 by constant bounding functions, that is $\underline{f}(p) = 1$ and $\overline{f}(p) = 4$ we obtain

$$\begin{aligned} \Psi(p,\delta) &= \begin{pmatrix} -\left((1-\delta)+4\delta\right) & p \\ -1 & -5 \end{pmatrix} \\ &= \begin{pmatrix} -(1+3\delta) & p \\ -1 & -5 \end{pmatrix} . \end{aligned} \tag{3.59}$$

Conversely, by using the affine functions $\overline{f}(p) = 3p - 2$ (which connects the extreme points of the graph of p^2) and $\underline{f}(p) = 2p - 1$ (whose gradient is the derivative of p^2 computed for $p = 1$), we obtain

$$\begin{aligned} \Psi(p,\delta) &= \begin{pmatrix} -\left((1-\delta)(2p-1)+\delta(3p-2)\right) & p \\ -1 & -5 \end{pmatrix} \\ &= \begin{pmatrix} -(p\delta+2p-\delta-1) & p \\ -1 & -5 \end{pmatrix} . \end{aligned} \tag{3.60}$$

The latter is a better covering of the image of $A(p)$ in the interval $[1,2]$. To convince about this point, note that the covering (3.59) is equivalent to replace the points of the graph of the function p^2 by the points belonging to the shadowed rectangle in Fig. 3.10; conversely covering (3.60) corresponds to replace the points of the graph of the function p^2 by the points belonging to the triangle evidenced in Fig. 3.11. Finally in Fig. 3.12 the covering obtained by the approach of Sect. 3.2.1 is represented; note that this last covering corresponds to choose as bounding functions $\underline{f}(p) = p$ and $\overline{f}(p) = 2p$ and is more conservative than the covering of Fig. 3.11.

To generalize the choice of the bounding functions used to build the less conservative covering in Fig. 3.11 and to set up a completely *automatic procedure* which generates as output the polytopic covering of a given function, we need the following more strict assumption on the f_j's.

Assumption 3.3. For all $j = 1, \ldots, \nu$ if f_j is not multi-affine then it is convex (or concave) and differentiable. ◇

If f_j is a convex or concave differentiable function it is simple to generate automatically affine functions $\underline{f}_j$ and $\overline{f}_j$ satisfying (3.50). We show this fact in the case of convex functions.

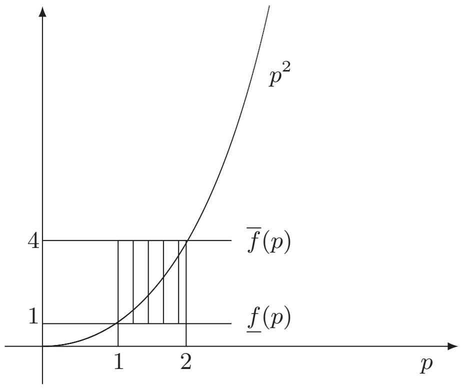


Fig. 3.10. Covering of the function $f(\cdot)$ by constant functions

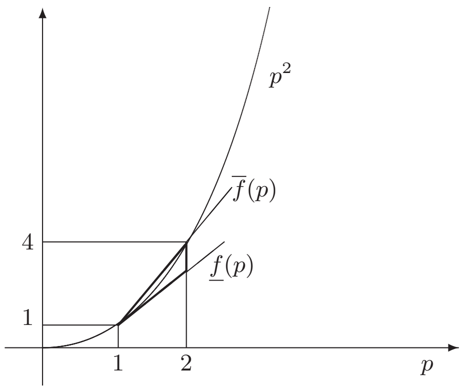


Fig. 3.11. Covering of the function $f(\cdot)$ by the bounding functions $\underline{f}(p) = 2p - 1$ and $\overline{f}(p) = 3p - 2$

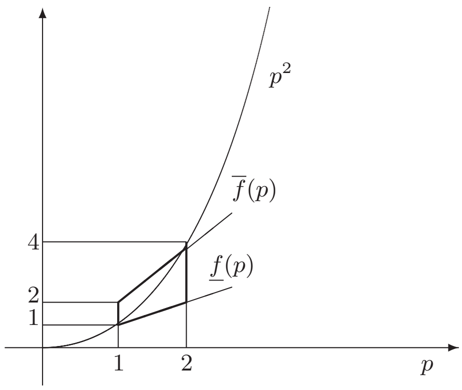


Fig. 3.12. Covering of the function $f(\cdot)$ via the approach of Sect. 3.2.1

Procedure 3.2 (Generation of the bounding functions when f_j is convex [11]).

Construction of $\underline{f}_j$.
Let $p^\star \in R$ and $\pi_j(p^\star)$ the gradient of $f_j(p)$ in $p^\star$. The following inequality holds (see [150], Theorem 25.1)

$$f_j(p) \geq f_j(p^\star) + \pi_j(p^\star)(p - p^\star), \qquad \forall p \in R . \tag{3.61}$$

Then set

$$\underline{f}_j(p) = f_j(p^\star) + \pi_j(p^\star)(p - p^\star), \qquad p \in R . \tag{3.62}$$

Construction of $\overline{f}_j$.
Consider the images of the vertices of R under f_j, namely $f_{j1} := f_j(p_{(1)})$, ..., $f_{j2^q} := f_j(p_{(2^q)})$, and order them with decreasing magnitude, say $f_{jh_1} \geq \cdots \geq f_{jh_{2^q}}$. Simple considerations from linear algebra show that there exists an integer $r \in \{q+1, \ldots, 2^q\}$ such that it is univocally determined the hyper-plane $h_j(p)$ in the space $\mathbb{R}^{q+1}$ connecting the points $(p^T_{(h_1)}, f_{jh_1})$,..., $(p^T_{(h_r)}, f_{jh_r})$. By virtue of the convexity of f_j we have that, for all $p \in R$, $f_j(p) \leq h_j(p)$; hence set

$$\overline{f}_j(p) = h_j(p). \tag{3.63}$$

◇

Note that, in the example considered in Fig. 3.11, the covering has been obtained by choosing $p^\star$ in (3.62) as the extreme left point of the interval. Obviously, in general, the choice of $p^\star$ depends on the nature of the function.

Also it is interesting to note that many functions are convex or concave only if the parameters on which they depend do not change sign. This is often the case since, in many practical situations, the parameters have a physical significance and hence are inherently positive. In any case we can always split, prior to the application of the procedure, the original hyper-box in smaller hyper-boxes, each contained only in one orthant of $\mathbb{R}^q$.

Example 3.3 (Automatic Steering of a Bus).
Let us consider the model of the city bus O 305, augmented with an integrator, considered in [1], p. 17,

$$\dot{x} = A(p)x + bu \tag{3.64a}$$
$$y = c^T x, \tag{3.64b}$$

where $u(t) = \dot{\delta}_f(t) \in \mathbb{R}$, δ_f is the front wheel steering angle, $y(t) \in \mathbb{R}$ is the lateral displacement from the guiding wire measured at the front of the bus, and

$$
\begin{aligned}
A(p) &= \frac{\begin{pmatrix} -(c_r + c_f)p_2 & -p_1 p_2^2 + (c_r l_r - c_f l_f) & 0 & 0 & c_f p_2 \\ \frac{c_r l_r - c_f l_f}{i^2} p_2^2 & -\frac{c_r l_r^2 + c_f l_f^2}{i^2} p_2 & 0 & 0 & \frac{c_f l_f}{i^2} p_2^2 \\ 0 & p_1 p_2^2 & 0 & 0 & 0 \\ p_1 p_2^3 & l_s p_1 p_2^2 & p_1 p_2^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}{p_1 p_2^2} \\
b &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \\
c^T &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
p_1 &= m \,, \qquad p_2 = v \,,
\end{aligned}
$$

m and v being the mass and velocity of the bus respectively, with

$$p = \begin{pmatrix} p_1 & p_2 \end{pmatrix}^T \in [9950, 32000] \times [3, 20] =: R \,.$$

The numerical values of the coefficients are [1], p. 24:

$i = 3.294$ $[m]$	inertial radius
$c_r = 470000$ $[N/rad]$	cornering stiffness (rear wheels)
$c_f = 198000$ $[N/rad]$	cornering stiffness (front wheels)
$l_r = 1.93$ $[m]$	distance between the rear axis and the center of gravity
$l_s = 6.12$ $[m]$	distance between the center of gravity and the sensor antenna
$l_f = 3.67$ $[m]$	distance between the front axis and the center of gravity.

Now the following controller is proposed in [1], p. 327, in order to stabilize system (3.64) with some performance specifications (see Fig. 3.13)

$$k(s) = 25^3 \frac{0.15 s^2 + 0.7 s + 0.6}{(s + 25)(s^2 + 25 s + 625)} \,. \tag{3.65}$$

The proposed controller stabilizes the system in the presence of constant uncertain parameters. However, in our example, we assume that the velocity

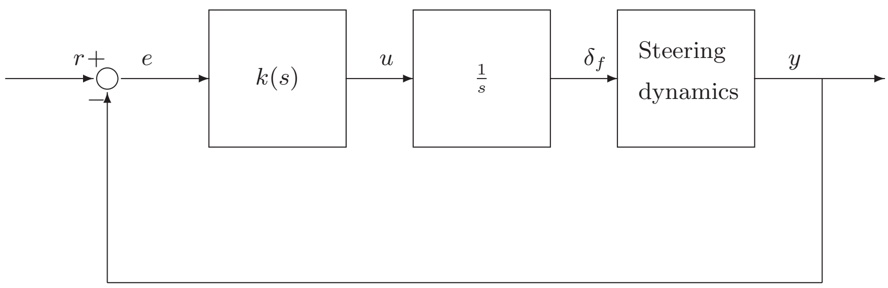


Fig. 3.13. Closed loop control scheme for the city bus O 305

is *time-varying*. This results from reasonable physical considerations. Indeed following the assigned path, the number of passengers in the bus does not change (so the mass is time-invariant); on the other hand, velocity may vary.

Our objective in the following is to verify whether the system in Fig. 3.13 is *robustly stable*, that is whether it is exponentially stable *wrt* the constant parameter p_1 (mass) and the time-varying parameter p_2 (velocity). This objective is accomplished by exploiting the approach of Algorithm 3.1, (see also Remarks 3.6–3.7).

Obviously, exponential stability versus the parameter p_2 is guaranteed by quadratic stability. However, since the numerator and denominator of the system matrices in (3.64) do not depend on p_2 in a multi-affine fashion, we have to use the Polytopic Covering technique to obtain a system depending multi-affinely on parameters. To this end, note that, according to (3.62) and (3.63), the function p_2^n, with $n \in \mathbb{N}$ and $p_2 \in [\underline{p}_2, \overline{p}_2]$, is bounded by the following functions

$$\underline{f}(p_2^n) = n\underline{p}_2^{n-1} p_2 + \underline{p}_2^n (1-n) \tag{3.66a}$$

$$\overline{f}(p_2^n) = \frac{\overline{p}_2^n - \underline{p}_2^n}{\overline{p}_2 - \underline{p}_2} p_2 + \frac{\underline{p}_2^n \overline{p}_2 - \overline{p}_2^n \underline{p}_2}{\overline{p}_2 - \underline{p}_2} \,. \tag{3.66b}$$

For $n = 2, 3$, $\underline{p}_2 = 3$, $\overline{p}_2 = 20$, equations (3.66) yield the following pair of bounding functions

$$\underline{f}(p_2^2) = 6p_2 - 9\,, \quad \overline{f}(p_2^2) = 23p_2 - 60$$
$$\underline{f}(p_2^3) = 27p_2 - 54\,, \quad \overline{f}(p_2^3) = 469p_2 - 1380\,.$$

By introducing the fictitious parameters $\delta = \begin{pmatrix}\delta_1 & \delta_2\end{pmatrix}^T$ and applying Procedure 3.1, system (3.64) can be re-written in the form

$$\dot{x} = \hat{A}(p,\delta)x + bu \tag{3.67a}$$

$$y = c^T x \tag{3.67b}$$

where $(p^T, \delta^T)^T \in R \times [0,1]^2$.

Now let

$$\dot{x}_k = A_k x_k + b_k e \tag{3.68a}$$

$$u = c_k^T x_k \tag{3.68b}$$

be a state-space representation of the controller $k(s)$. Putting together (3.67) with (3.68) we obtain that the closed loop system in Fig. 3.13 is described by the 8-th order uncertain linear system

$$\begin{pmatrix} \dot{x} \\ \dot{x}_k \end{pmatrix} = \begin{pmatrix} \hat{A}(p,\delta) & bc_k^T \\ -b_k c^T & A_k \end{pmatrix} \begin{pmatrix} x \\ x_k \end{pmatrix} =: \hat{A}_{cl}(p,\delta) \begin{pmatrix} x \\ x_k \end{pmatrix}, \tag{3.69}$$

where $\hat{A}_{cl}(p,\delta)$ satisfies Assumption 3.1.

To check robust stability of system (3.69), we propose the following algorithm, which, as said, is similar to Algorithm 3.1.

Algorithm 3.2.

Step 1

Let

$$\begin{aligned} \Delta p_1 &= \frac{32000 - 9950}{100} \\ p_{1_{min}} &= 9950 \\ p_{1_{max}} &= p_{1_{min}} + \Delta p_1 \,. \end{aligned}$$

Step 2

Try to solve the following feasibility problem.

Problem 3.5.

Find $P > 0$ such that

$$\begin{aligned} &\hat{A}_{cl}(p,\delta)^T P + P\hat{A}_{cl}(p,\delta) < 0\,, \\ &\qquad \left(p^T, \delta^T\right)^T \in \left(\left[p_{1_{min}}, p_{1_{max}}\right] \times \left[3, 20\right] \times [0,1]^2\right)^v \,. \end{aligned} \tag{3.70}$$

Step 3

If Problem 3.5 is not feasible **then Stop**: we cannot prove robust stability of system (3.69), **else let** $p_{1_{min}} = p_{1_{max}}$, $p_{1_{max}} = p_{1_{min}} + \Delta p_1$ **end**.
If $p_{1_{min}} < 32000$ **then goto** Step 2; **else stop**: system (3.69) is robustly stable.

◇

The application of Algorithm 3.2 shows that an iteration such that Problem 3.5 is not feasible exists; therefore the application of Algorithm 3.2 does not allow to prove robust stability of the closed loop system. On the other hand, since the covering of parameter p_2 introduces conservatism in the analysis, at this point we cannot establish whether the system is robustly stable or not.

To show that, in this case, the system *is not* robustly stable, it suffices to notice that the LMIs based feasibility problem below does not admit a feasible solution.

Problem 3.6.

Find $P > 0$ such that

$$A_{cl}(p_1,p_2)^T P + PA_{cl}(p_1,p_2) < 0\,, \qquad p_1 = 32000\,, p_2 = 3 : step : 20\,,$$

where $step = \big(20-3\big)/10$ and $A_{cl}(\cdot)$ is the "uncovered" closed loop system matrix

$$A_{cl}(p) = \begin{pmatrix} A(p) & bc_k^T \\ -b_k c^T & A_k \end{pmatrix} . \tag{3.71}$$

◇

Finally, it is interesting to observe that the application of Procedure 3.2 allows us to prove robust stability of the system if $p_2 \in [3, 10]$, that is if the bus speed belongs to the interval $[3, 10]$ m/sec; also, we shall see in Chap. 4 (Exercise 4.2) that the system is robustly stable if we consider a bounded rate parameter p_2. △

Improving the Covering

Following the ideas contained in [56,162,163] and [11], we will show that the goodness of the fitting of the set $A(R)$ by the constructed polytope can be improved by splitting the set R into smaller sub-hyper-boxes. However the computational complexity growths exponentially; therefore the methodology described in this section is only applicable when both the number of the non-multi-affine functions which appear in the expression of $A(p)$ and the number of uncertain parameters is small.

We say that $\mathcal{T} = \{R_1, R_2, \ldots, R_l\}$ is a rectangular covering of the hyper-box R if $\cup_{r=1}^{l} R_r = R$. We define $\mathbf{T}$ as the set of all rectangular coverings of R.

Now let $\mathcal{C}(\mathbb{R}^{n\times n})$ the set composed of all compact sets in $\mathbb{R}^{n\times n}$. We equip this set with the Hausdorff metric

$$d_H(S_1, S_2) := \max\left\{ \max_{x_1\in S_1} d(x_1, S_2), \max_{x_2\in S_2} d(S_1, x_2) \right\}, \quad S_1, S_2 \in \mathcal{C}(\mathbb{R}^{n\times n}), \tag{3.72}$$

where $d(x, S) = d(S, x)$ denotes the usual Euclidean distance of the point x from the set S.

Let $\{\mathcal{T}_h\}_{h\in\mathbb{N}}$, $\mathcal{T}_h \in \mathbf{T}$, a sequence of rectangular coverings, set up as follows. $\mathcal{T}_1 = \{R\}$; $\mathcal{T}_2$ is constructed by splitting R into 2^q hyper-boxes by lines parallel to the coordinate axes; $\mathcal{T}_3$ is obtained applying the same procedure to each hyper-box in $\mathcal{T}_2$ and so on; note that $\mathcal{T}_h$ is composed of $2^{(h-1)q}$ hyper-boxes. We denote by R_{hr} the r-th hyper-box in $\mathcal{T}_h$ and by $\Psi^{(hr)}$, $\Phi^{(hr)}$ the matrix functions constructed according to Procedure 3.1 for $A(\cdot)$ restricted to R_{hr}.

It is rather obvious that, when the bounding functions are constructed according to (3.62) and (3.63), as the splitting is made finer and finer, the union of the convex hull of the images of the functions $\Phi^{(hr)}$, $r = 1, \ldots, 2^{(h-1)q}$, converges to the union of the images of the functions $\Psi^{(hr)}$ which, in turn, converges to the image of $A(\cdot)$; therefore we can state the following result.

Theorem 3.4 ([11]). *Consider a matrix function $A(\cdot)$ satisfying Assumption 3.3 and the sequence $\{\mathcal{T}_h\}_{h\in\mathbb{N}}$. Then*

$$\lim_{h\to\infty} \bigcup_{r=1}^{2^{(h-1)q}} \operatorname{conv} \Phi^{(hr)}(R_{hr} \times \mathcal{D} \times \mathcal{E}) = A(R)$$

in the metric space $(\mathcal{C}(\mathbb{R}^{n\times n}), d_H)$, where $\Phi^{(hr)}$ is the matrix function constructed on R_{hr} according to Procedure 3.1 and (3.62), (3.63). □

Now, based on Theorem 3.4, we provide an iterative algorithm, which tests the quadratic stability of the system under consideration.

A sufficiency and a necessary test constitute the core of the algorithm. The sufficiency test is applied to the system augmented with the fictitious parameters and therefore guarantees quadratic stability. Conversely, following the idea behind Problem 3.6, the necessary test is applied to the original uncovered system evaluated in correspondence of a finite number of points in the set R; if the last test fails the system is not QS.

Given a collection of sets $\mathcal{S}$, we denote by $\mathcal{S}_h$ the h-th element of $\mathcal{S}$ and by $\operatorname{card}(\mathcal{S})$ the number of elements of $\mathcal{S}$.

Algorithm 3.3. The algorithm is composed of three steps.

Step 1
Let $\mathcal{S} = \{R\}$.
Step 2
For $h = 1, \ldots, \operatorname{card}(\mathcal{S})$ repeat the following procedure:
Sufficiency test.
Apply Procedure 3.1 to $A(\cdot)$ restricted to the hyper-box $\mathcal{S}_h$; in this way we obtain a polytope $\mathcal{A}_h$ including $A(\mathcal{S}_h)$. Apply the quadratic stability test to $\bigcup_h \mathcal{A}_h$. **If** the test is successful **then stop**: system (3.1) is *quadratically stable*; **else** carry out the necessary test described below.
Necessary test.
Test the quadratic stability of system (3.1) over the finite set

$$\bigcup_h \left\{p_{(hi)}, i = 1, \ldots, 2^q\right\},$$

where $p_{(hi)}$ is the i-th vertex of $\mathcal{S}_h$. **If** for some pair (h, i) the test fails **then stop**, the system is *not quadratically stable.*
Step 3
Split each element belonging to $\mathcal{S}$ into 2^q hyper-boxes, put them in $\mathcal{S}$ and **goto** Step 2.

◇

Due to Theorem 3.4, if system (3.1) is QS Algorithm 3.3 converges into a finite number of iterations. If system (3.1) is not QS, it can be proven (see [9]) that either Algorithm 3.3 terminates in a finite number of iterations or the sequence of the positive definite matrices, generated from the iterative solution of the necessary test at Step 2 of the algorithm, is unbounded.

3.3 Other Approaches for Quadratic Stability Analysis

When there are many non-multi-affine functions in the expression of $A(\cdot)$ or q, the number of parameters, is big it may be difficult to apply the approaches of Sects. 3.1 and 3.2.

In this case one may follow the alternative approaches proposed in the next sections.

3.3.1 Gridding

Let us consider the following algorithm.

Algorithm 3.4.

Step 1
Define the set (note that the points $p^{(i)}$ are not necessarily the vertices of R)

$$\mathcal{P} = \left\{ p^{(i)}, i = 1, \ldots, \eta \,, \, p^{(i)} \in R \right\} .$$

Step 2
Solve the following feasibility problem, subject to a *finite* number of constraints.

Problem 3.7.
Find a symmetric matrix P such that

$$\begin{aligned} P &> 0 \\ A^T(p)P + PA(p) &< 0 \,, \quad \forall p \in \mathcal{P} \,. \end{aligned}$$

Step 3
If Problem 3.7 is unfeasible **then stop**; system (3.1) is not QS.
If Problem 3.7 is feasible, given a feasible solution P^*, test that

$$A^T(p)P^* + P^*A(p) < 0 \,, \quad \forall p \in R \,. \tag{3.73}$$

If (3.73) is satisfied **then stop**; System (3.1) is QS. **Else goto** Step 4.
Step 4
Define the set

$$\mathcal{P}' = \left\{ p^{(i)}, i = 1, \ldots, \eta' \,, \, p^{(i)} \in R \right\} , \qquad \eta' > \eta \,,$$

such that $\mathcal{P}' \supset \mathcal{P}$; let $\eta = \eta'$, $\mathcal{P} = \mathcal{P}'$ and **goto** Step 2. ◇

In Step 1 of Algorithm 3.4, η points of the set R are collected into the set $\mathcal{P}$; the matrix-valued function $A^T(p)P + PA(p)$ is then evaluated in correspondence of such points in order to obtain the constraints of Problem 3.7. If Problem 3.7 is feasible, $A^T(p)P + PA(p)$ is guaranteed to be negative definite on $\mathcal{P}$. Experience suggests that the first time Step 1 is performed, it is convenient to collect into $\mathcal{P}$ the vertices of R together with its center point. Assuming that Problem 3.7 admits a feasible solution P^*, in Step 3 one must check if condition (3.73) is satisfied; if the LHS of (3.73) does not exhibit convexity properties, the check can be performed via brute force gridding. If (3.73) is not satisfied, it is convenient to collect some of the points where the check has failed (chosen in a suitable strategic way) and join them to the points already contained in $\mathcal{P}$. This way leads to the definition of the set $\mathcal{P}'$ in Step 4. In principle the iterative algorithm terminates when either Problem 3.7 is unfeasible or when the check in Step 3 is satisfied. Actually, since there is an increment of the set $\mathcal{P}$ every time Step 4 is performed, the situation in which, due to numerical difficulties, it becomes impossible to carry out Step 2 of the Algorithm, could be reached. In that case no conclusion can be drawn concerning the quadratic stability of system (3.1).

Obviously, the existence of a solution of Problem 3.7 is only necessary for (3.73). Therefore sufficiency must be checked at Step 3. As said, if the matrix function at the LHS in (3.73) does not exhibit convexity properties, condition (3.73) could be checked by gridding the parameter space (see also [33] for further comments about the choice of the gridding points). In any case, this approach becomes prohibitive, from a computational point of view, in presence of a large number of parameters, since we cannot perform a sufficiently dense sampling of the parameter space. The alternative, as we shall show in the next section, is to make use of a probabilistic approach to generate a set of sample points in the parameter space where to check (3.73); the key point is that the cardinality of such set is independent, in the sense specified later, from the dimension of the parameter space.

3.3.2 A Statistical Approach

While in previous sections the parameter vector p has been considered uncertain but deterministic, in the context of this section we look at p as a random vector attaining values into the set R and described by a probability density function $f(p)$. The theoretical background of what follows is taken by [168].

Let us consider n_S i.i.d. (independent and identically distributed) samples of p, namely $\tilde{p}^{(1)}, \tilde{p}^{(2)}, \ldots, \tilde{p}^{(n_S)}$, generated according to the same probability density $f(p)$.

Denote by $u(\cdot)$ a given Lebesgue measurable performance real scalar function defined over R and by $\hat{u}_{n_S}$ the maximum of $u(\cdot)$ over the n_S samples, that is

$$\hat{u}_{n_s} := \max_{i=1,2,\ldots,n_S} u(\tilde{p}^{(i)})\,. \tag{3.74}$$

The aim is to evaluate

$$Pr\{u(p) > \hat{u}_{n_s}\}\,, \tag{3.75}$$

where $Pr(E)$ denotes the probability of the event E.

Since $u(p)$ is a random variable and the probability density function $f(p)$ is unknown, we can get only an estimate of the probability expressed by (3.75).

In [168] it is shown that, given $\varepsilon \in (0,1)$ (the so-called accuracy) and $\delta \in (0,1)$ (the confidence), the number n_S of samples randomly picked from the parameter set R, according to a Uniform Probability Distribution, which guarantees that

$$Pr\{Pr\{u(p) > \hat{u}_{n_S}\} \leq \varepsilon\} \geq 1-\delta \tag{3.76}$$

is obtained by the following formula

$$n_S \geq \frac{\ln(\frac{1}{\delta})}{\ln(\frac{1}{1-\varepsilon})}\,. \tag{3.77}$$

Roughly speaking, what formula (3.76) says is the following: If ε and δ are sufficiently "small" numbers then there is a "high" probability ($\geq 1-\delta$) that the probability of the event $u(p) > \hat{u}_{N_S}$ is "small" ($\leq \varepsilon$).

In other words we can establish that "the ratio between the Lebesgue measure of the set $\{p \in R \,:\, u(p) > \hat{u}_{n_S}\}$ and the measure of the set R is not greater than ε"; this statement is true with probability not smaller than $1-\delta$.

Remark 3.12. Obviously, whatever the values of δ and ε are, there is no guarantee that the estimated maximum $\hat{u}_{n_S}$ is close to the actual maximum $\max_{p\in R} u(p)$. However if ε and δ are sufficiently small and the function $u(p)$ is sufficiently smooth the estimated and actual maximum may be close. ◇

Remark 3.13. Note that the bound (3.77) depends on ϵ and δ; this is obvious, since as δ and ϵ become smaller and smaller the number of samples needs to increase because we want to guarantee (3.76) with more accuracy and confidence. However, differently from the other approaches, the same bound does not depend on q (the number of uncertain parameters), the size of the set R and the probability density function $f(p)$. Therefore the statistical approach can be very useful when there are many uncertain parameters; indeed in that case the approaches of Sects. 3.1 and 3.2, as well as methods based on gridding, become practically unapplicable (see also the next section). ◇

Remark 3.14. The number n_S in (3.76) is considerably smaller than the number of samples required by the classical *Chernoff* bound (which equals $\ln(\frac{2}{\delta})/(2\varepsilon^2)$ [46]). For example, for $\varepsilon = \delta = 0.001$ the Chernoff bound requires 3800452 samples, while (3.77) requires 6905 samples; the question concerning the number of samples is also discussed in [114]. ◇

On the basis of the above discussion, we modify Algorithm 3.4 as follows.

Algorithm 3.5.

Step 0
Define a given accuracy $\epsilon \in (0,1)$ and confidence $\delta \in (0,1)$.
Step 1
Define the set

$$\mathcal{P} = \left\{ p^{(i)}, i = 1, \dots, \eta \,, \, p^{(i)} \in R \right\} .$$

Step 2
Solve the following LMIs feasibility problem, subject to a *finite* number of constraints.

Problem 3.8.
Find a symmetric matrix P such that

$$\begin{aligned} P &> 0 \\ A^T(p)P + PA(p) &< 0 \,, \quad \forall p \in \mathcal{P} \,. \end{aligned}$$

Step 3
If Problem 3.8 is unfeasible **then stop**; system (3.1) is not QS.
If Problem 3.8 is feasible, given a feasible solution P^*, define

$$u(p) = \lambda_{max}\big(A^T(p)P^* + P^* A(p)\big) \,. \tag{3.78}$$

Then take n_S points, $\tilde{p}^{(1)}$, $\tilde{p}^{(2)}$, ..., $\tilde{p}^{(n_S)}$, where n_S satisfies (3.77), from the parameter set R according to a Uniform Probability Distribution and evaluate $\hat{u}_{n_S}$ on these points (see (3.74)). **If** $\hat{u}_{n_S} < 0$ **then stop** (see the comments below). **Else goto** Step 4.
Step 4
Define the set

$$\mathcal{P}' = \left\{ p^{(i)}, i = 1, \dots, \eta' \,, \, p^{(i)} \in R \right\} , \qquad \eta' > \eta$$

such that $\mathcal{P}' \supset \mathcal{P}$; let $\eta = \eta'$, $\mathcal{P} = \mathcal{P}'$ and **goto** Step 2.

◇

According to Remark 3.12, the response of Algorithm 3.5 has to be carefully interpreted. If, given ε and δ, the number $\hat{u}_{n_S}$ evaluated at Step 3 is negative we cannot conclude that system (3.1) is QS in the deterministic sense. What we can say is that, with probability not smaller than $1-\delta$, the ratio between the measure of the set where (3.78) is nonnegative and the measure of the set R is not greater than ϵ. If δ and ϵ are sufficiently small and the function $u(\cdot)$ defined in (3.78) is sufficiently smooth, the actual maximum of $u(\cdot)$ may be negative and therefore the system may be QS in the deterministic sense. In any case the statistical approach should be used only when the other approaches are not applicable due to an hard computational burden (see also the next section).

Further readings concerning the application of the statistical approach to quadratic stability are the papers [78, 79] and [30]. The reader interested to the more general statistical robustness issues are referred to the works by Ray and Stengel [165] and [149], Barmish and Lagoa [29], Khargonekar and colleagues [111] and [179].

3.3.3 A Comparison Between the Various Methods for Quadratic Stability Analysis

It can be interesting to compare the different techniques for quadratic stability analysis we have discussed so far in this chapter. It is clear that the key factors to decide which way to follow are the kind of dependence on parameters, the number of parameters q and the number of non-multi-affine functions μ involved in the structure of $A(p)$.

To this end we divide the values of q into three categories:

- $q \leq q_{small}$ means that the number of parameters is considered "small";
- $q_{small} < q < q_{big}$ means that the number of parameters is considered "medium";
- $q \geq q_{big}$ means that the number of parameters is considered "big".

Obviously the values of q_{small} and q_{big} are subjective and strongly depend on the power of the computer one is using to perform computations.

At the same time we roughly divide μ into two categories:

- $\mu \leq \mu_{med}$ means that there are a few non-multi-affine functions;
- $\mu > \mu_{med}$ means that there are a lot of non-multi-affine functions.

If $A(\cdot)$ satisfies Assumption 3.1 it is in any case recommended to use Theorem 3.2; however in presence of many parameters (say $q \geq q_{big}$) a viable alternative can be that one of applying Algorithm 3.5 (the statistical approach).

If $A(\cdot)$ does not satisfy Assumption 3.1 and we face with a small number of parameters ($q \leq q_{small}$), in place of using the polytopic covering approach it can be realistic to apply Algorithm 3.4 in which Step 3 is solved by gridding, since in this case we can perform a so fine sampling of the parameter space to guarantee practical certainty that, if (3.73) is satisfied, the system is QS. The drawback of Algorithm 3.4 is that, if we are unlucky, many iterations are required before finding the optimal P^*.

If $A(\cdot)$ does not satisfy Assumption 3.1 and $q > q_{small}$, we have to consider two different situations: if there are a few non-multi-affine functions in $A(p)$ ($\mu \leq \mu_{med}$) it is preferable to use the polytopic covering approach; conversely the only practically applicable approach is Algorithm 3.5.

The above considerations are reported in Table 3.1.

Table 3.1. Choice of the technique for quadratic stability analysis

Dependence	**Number of parameters** q	**Number of non-multi-affine functions** μ	**Method**
Assumption 3.1	Small to medium ($q < q_{big}$)	Not applicable	Theorem 3.2
Assumption 3.1	Big ($q \geq q_{big}$)	Not applicable	Theorem 3.2 or Algorithm 3.5 (statistical)
No Assumption 3.1	Small ($q \leq q_{small}$)	A few ($\mu \leq \mu_{med}$)	Algorithm 3.4 (gridding) or Polytopic Covering
No Assumption 3.1	Small ($q \leq q_{small}$)	A lot ($\mu > \mu_{med}$)	Algorithm 3.4 (gridding)
No Assumption 3.1	Medium to Big ($q > q_{small}$)	A few ($\mu \leq \mu_{med}$)	Polytopic Covering
No Assumption 3.1	Medium to Big ($q > q_{small}$)	A lot ($\mu > \mu_{med}$)	Algorithm 3.5 (statistical)

3.4 Quadratic Stability and Performances

In most engineering applications stability is not the unique requirement that a system should exhibit. Therefore in this section we investigate the conditions the systems matrix has to satisfy in order to ensure at the same time quadratic stability and further performances.

In particular, the first problem we consider is that of ensuring quadratic stability of the system and, at the same time, that the system poles belong to prespecified regions of the complex plane. The second problem concerns the satisfaction of an $\mathcal{L}_2$ performance bound together with quadratic stability. Finally the issue of guaranteeing quadratic stability together with an LQ criterion is discussed.

Throughout this section we assume that the matrix $A(\cdot)$ depends on parameters according to Assumption 3.1. In the other circumstances we can use one of the techniques described in Sects. 3.2 or 3.3.

3.4.1 Quadratic Stability and Pole Placement (Quadratic $\mathcal{D}$-Stability)

According to [48, 49] we refer to the so-called LMI regions of the complex plane.

Definition 3.2 (LMI Region [48]). An LMI region is any subset $\mathcal{D}$ of the complex plane defined as

$$\mathcal{D} := \left\{ z \in \mathbb{C} \, : \, \Lambda + z\Theta + z^* \Theta^T < 0 \right\} , \tag{3.79}$$

where $\Lambda \in \mathbb{R}^{h\times h}$ and $\Theta \in \mathbb{R}^{h\times h}$ are real matrices, Λ is symmetric and the matrix-valued function

$$\Lambda + z\Theta + z^*\Theta^T =: F_{\mathcal{D}}(z) \tag{3.80}$$

is the *characteristic function* of $\mathcal{D}$. ◇

LMI regions include, among others, the strip, the disk and the cone with apex at the origin. For example the half plane $\Re(z) < -\alpha$ is characterized by

$$F_{\mathcal{D}}(z) = z + z^* + 2\alpha\,, \tag{3.81}$$

while the unit disk centered at the origin is characterized by

$$F_{\mathcal{D}}(z) = \begin{pmatrix} -1 & z \\ z^* & -1 \end{pmatrix}\,; \tag{3.82}$$

finally the conic sector with apex at the origin and inner angle 2θ is described by

$$F_{\mathcal{D}}(z) = \begin{pmatrix} \sin\theta(z+z^*) & \cos\theta(z-z^*) \\ \cos\theta(z^*-z) & \sin\theta(z+z^*) \end{pmatrix} < 0\,. \tag{3.83}$$

Definition 3.3 ($\mathcal{D}$-stability). The linear time-invariant system

$$\dot{x}(t) = Ax(t) \tag{3.84}$$

is said to be *$\mathcal{D}$-stable* if and only if all the eigenvalues of A are in the region $\mathcal{D}$. ◇

A generalization of the Lyapunov Theorem for linear systems leads to the following necessary and sufficient condition for $\mathcal{D}$-stability (see Exercise 3.1); in the sequel, given two matrices F and G, the symbol $F \otimes G$ denotes the Kronecker product of F and G [88].

Theorem 3.5 ($\mathcal{D}$-stability [48, 49]). *System* (3.84) *is $\mathcal{D}$-stable if and only if there exists a positive definite matrix P such that*

$$M_{\mathcal{D}}(A,P) := \Lambda \otimes P + \Theta \otimes (PA) + \Theta^T \otimes (A^T P) < 0\,. \tag{3.85}$$

□

The generalization of Definition 3.1 leads to the concept of quadratic $\mathcal{D}$-stability.

Definition 3.4 (Quadratic $\mathcal{D}$-stability [48, 49]). System (3.1) is said to be *quadratically $\mathcal{D}$-stable* if and only if there exists a positive definite matrix P such that for all $p \in R$

$$M_{\mathcal{D}}(A(p),P) = \Lambda \otimes P + \Theta \otimes (PA(p)) + \Theta^T \otimes (A^T(p)P) < 0\,. \tag{3.86}$$

◇

It is simple to show that Definition 3.4 recovers the classical Definition 3.1 when $\mathcal{D}$ is the left half of the complex plane; indeed in this case we have $\Lambda = 0$ and $\Theta = 1$ (see (3.81) with $\alpha = 0$).

It can be shown (see Exercise 3.2) that quadratic $\mathcal{D}$-stability of system (3.1) is equivalent to quadratic $\mathcal{D}$-stability of the dual system. Therefore an equivalent condition for quadratic $\mathcal{D}$-stability is the existence of a positive definite matrix Q such that

$$M_{\mathcal{D},d}(A(p), Q) := \Lambda \otimes Q + \Theta \otimes (A(p)Q) + \Theta^T \otimes (QA^T(p)) < 0\,, \quad \forall\, p \in R\,. \tag{3.87}$$

Obviously, when $\mathcal{D}$ is a subset of the left half of the complex plane, quadratic $\mathcal{D}$-stability guarantees quadratic stability and that the eigenvalues of $A(p)$ are in the region $\mathcal{D}$ for all $p \in R$. However, since the parameter vector p is allowed to be time-varying, this last consideration is of little value, because the concept of "pole" is meaningless for time-varying system. Nevertheless, when $\mathcal{D}$ is a subset of the left half of the complex plane, quadratic $\mathcal{D}$-stability guarantees more than the simple exponential stability in presence of time-varying parameters, as the next result shows.

Theorem 3.6 ([49]). *Assume that system* (3.1) *is quadratically $\mathcal{D}$-stable, that is there exists a positive definite matrix P satisfying* (3.86)*; then, defined $v(x) = x^T P x$ and given any $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$, we have that for all $t \in \mathbb{R}^+$*

$$\frac{1}{2}\frac{\dot v(t,x)}{v(x)} \in \mathcal{D} \cap \mathbb{R}\,, \tag{3.88}$$

where $\dot v(t,x)$ is the derivative of v along the solutions of system $\dot x(t) = A(p(t))x(t)$.

Proof. Multiply the left and right hand side of (3.86) by $I \otimes x^T$ and $I \otimes x$ respectively. We obtain, by virtue of (A.8), that for all $x \neq 0$

$$\Lambda \otimes \left(x^T P x\right) + \Theta \otimes \left(x^T P A(p) x\right) + \Theta^T \otimes \left(x^T A^T(p) P x\right) < 0\,. \tag{3.89}$$

For a given $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$, we have that

$$\begin{aligned} \dot v(t,x) &= x^T \left(A^T(p(t))P + PA(p(t))\right)x \\ &= 2x^T A^T(p(t)) P x \\ &= 2x^T P A(p(t)) x\,. \end{aligned} \tag{3.90}$$

Dividing (3.89) by $v(x)$ and using (3.90) we obtain that for all $x \neq 0$ and $t \in \mathbb{R}^+$

$$\Lambda \otimes 1 + \Theta \otimes \frac{1}{2}\frac{\dot v(t,x)}{v(x)} + \Theta^T \otimes \frac{1}{2}\frac{\dot v(t,x)}{v(x)} < 0\,. \tag{3.91}$$

This last inequality implies that $\frac{1}{2}\dot v(t,x)/v(x)$ belongs to $\mathcal{D}$ for all $t \in \mathbb{R}^+$; the proof follows from the fact that $\frac{1}{2}\dot v(t,x)/v(x)$ is a real scalar. □

Theorem 3.6 implies that if, for example, system (3.1) is quadratically $\mathcal{D}$-stable with $\mathcal{D}$ the half plane described by (3.81) with $\alpha > 0$, we have for all $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$

$$\frac{\dot{v}(t,x)}{v(x)} < -2\alpha \,, \tag{3.92}$$

which in turn implies that, for all admissible parameter realizations, a solution $x(\cdot)$ of system (3.1) starting at x_0 satisfies

$$x^T(t)Px(t) < e^{-2\alpha t} x_0^T P x_0 \,, \qquad \forall t \in \mathbb{R}^+ \,, \tag{3.93}$$

which represents a bound on the decay rate of $x(\cdot)$ for all admissible behaviors of the uncertain parameters.

Note that, for a given p, (3.86) is an LMI in the variable P; by using Assumption 3.1, we have that system (3.1) is quadratically $\mathcal{D}$-stable *iff* the following LMIs problem is feasible.

Problem 3.9.
Find a symmetric matrix P such that

$$P > 0 \tag{3.94a}$$

$$\Lambda \otimes P + \Theta \otimes (PA(p_{(i)})) + \Theta^T \otimes (A^T(p_{(i)})P) < 0 \,, \quad i = 1, \ldots, 2^q \,. \tag{3.94b}$$

◇

Finally, the concept of quadratic $\mathcal{D}$-stability is useful to deal with discrete-time systems. Indeed in Chap. 6 we shall show that quadratic $\mathcal{D}$-stability of the continuous-time system (3.1) is equivalent to quadratic stability of the discrete-time system $x(k+1) = A(p)x(k)$ when $\mathcal{D}$ is the unit disk centered at the origin of the complex plane.

3.4.2 Quadratic $\mathcal{L}_2$ Performance

The concept of quadratic $\mathcal{L}_2$ performance[4] has been introduced in [175] to cope at the same time with quadratic stability and $\mathcal{L}_2$ performance.

Consider the uncertain system

$$\dot{x}(t) = A(p)x(t) + B(p)w(t) \tag{3.95a}$$

$$z(t) = C(p)x(t) + D(p)w(t) \,, \tag{3.95b}$$

where $A(\cdot) \in \mathbb{R}^{n\times n}$, $B(\cdot) \in \mathbb{R}^{n\times v}$, $C(\cdot) \in \mathbb{R}^{s\times n}$ and $D(\cdot) \in \mathbb{R}^{s\times v}$ are continuous matrix functions. Note that equations (3.95) define a family of input-output operators $\Gamma_{zw}\big(p(\cdot)\big)$ where $p(\cdot)$ is any realization of the time-varying parameter vector.

[4] We prefer to use the locution "$\mathcal{L}_2$ performance" rather than the more familiar "$\mathcal{H}_\infty$ performance" since we deal with time-varying parameters which lead to input-output operators corresponding to time-varying systems.

Definition 3.5 ([175]). Given $\gamma > 0$, system (3.95) is said to possess a *quadratic* $\mathcal{L}_2$ *performance bound* γ if and only if $\|D(p)\| < \gamma$ for all $p \in R$ and there exists a positive definite matrix P such that, for all $p \in R$,

$$\begin{aligned} & A^T(p)P + PA(p) + C^T(p)C(p) \\ + & \left(PB(p) + C^T(p)D(p)\right)\left(\gamma^2 I - D^T(p)D(p)\right)^{-1}\left(B^T(p)P + D^T(p)C(p)\right) < 0\,. \end{aligned} \tag{3.96}$$

◇

The next lemma justifies Definition 3.5.

Lemma 3.2. *If system* (3.95) *possesses a quadratic* $\mathcal{L}_2$ *performance bound* γ *then system* (3.95a) *(with* $w = 0$*) is quadratically stable and* $\|\Gamma_{zw}(p(\cdot))\| < \gamma$ *for all* $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$.

Proof. From (3.96) it readily follows that, for all $p \in R$,

$$A(p)^T P + PA(p) < 0\,, \tag{3.97}$$

and hence the quadratic stability of the system $\dot{x} = A(p)x$.

Now let $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$ and $\tilde{A}(\cdot) := A(p(\cdot))$, $\tilde{B}(\cdot) := B(p(\cdot))$, $\tilde{C}(\cdot) := C(p(\cdot))$, $\tilde{D}(\cdot) := D(p(\cdot))$; note that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are of class $\mathcal{PC}$.

Since R is compact (3.96) guarantees that

$$\tilde{A}^T P + P\tilde{A} + \tilde{C}^T\tilde{C} + \left(P\tilde{B} + \tilde{C}^T\tilde{D}\right)\left(\gamma^2 I - \tilde{D}^T\tilde{D}\right)^{-1}\left(\tilde{B}^T P + \tilde{D}^T\tilde{C}\right) << 0\,. \tag{3.98}$$

By applying Theorem 2.11, with a constant P, (3.98) guarantees that $\|\Gamma_{zw}(p(\cdot))\| < \gamma$. □

Now, by applying Fact A.3, we obtain that (3.96) is equivalent to

$$\begin{pmatrix} A^T(p)P + PA(p) + C^T(p)C(p) & PB(p) + C^T(p)D(p) \\ B^T(p)P + D^T(p)C(p) & -\left(\gamma^2 I - D^T(p)D(p)\right) \end{pmatrix} < 0\,. \tag{3.99}$$

Then, by applying again Fact A.3 with

$$Q = \begin{pmatrix} A^T(p)P + PA(p) & PB(p) \\ B^T(p)P & -\gamma^2 I \end{pmatrix}, \quad S = \begin{pmatrix} C(p) & D(p) \end{pmatrix}, \quad R = -I\,, \tag{3.100}$$

we obtain that (3.96) is equivalent to

$$\begin{pmatrix} A^T(p)P + PA(p) & PB(p) & C^T(p) \\ B^T(p)P & -\gamma^2 I & D^T(p) \\ C(p) & D(p) & -I \end{pmatrix} < 0\,. \tag{3.101}$$

Note that, even if $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ separately satisfy Assumption 3.1, it is not guaranteed that the LHS of (3.101) satisfies the hypothesis of Theorem A.2. In order to transform (3.101) into a finite number of inequalities let us consider the following assumption.

Assumption 3.4. The matrices of system (3.95) can be written

$$\begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix} = \frac{N_S(p)}{d_S(p)}, \tag{3.102}$$

where $N_S(p)$ is a multi-affine matrix valued function and $d_S(p)$ is a multi-affine function with $d_S(p) \neq 0$ for all $p \in R$. ◇

Some important cases in which Assumption 3.4 is satisfied are:

- Matrix $A(\cdot)$ satisfies Assumption 3.1 and matrices $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are parameter independent;
- matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ satisfy Assumption 3.1 with denominator equal to one (i.e. they are multi-affine matrix functions).

Under Assumption 3.4, the LHS of (3.101) satisfies the hypothesis of Theorem A.2 and therefore inequality (3.101) can be converted into a finite number of LMIs. In this case we can conclude that system (3.95) possesses a quadratic $\mathcal{L}_2$ performance bound γ *iff* the following LMIs based problem admits a feasible solution.

Problem 3.10 (Quadratic $\mathcal{L}_2$ performance).
Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P > 0 \tag{3.103a}$$

$$\begin{pmatrix} A^T(p_{(i)})P + PA(p_{(i)}) & PB(p_{(i)}) & C^T(p_{(i)}) \\ B^T(p_{(i)})P & -\gamma^2 I & D^T(p_{(i)}) \\ C(p_{(i)}) & D(p_{(i)}) & -I \end{pmatrix} < 0, \qquad i = 1, \ldots, 2^q. \tag{3.103b}$$

◇

We end the section by noting that system (3.95) possesses a quadratic $\mathcal{L}_2$ performance bound γ *iff* there exists a positive definite matrix Q such that, for all $p \in R$,

$$\begin{pmatrix} QA^T(p) + A(p)Q & B(p) & QC^T(p) \\ B^T(p) & -\gamma^2 I & D^T(p) \\ C(p)Q & D(p) & -I \end{pmatrix} < 0. \tag{3.104}$$

The last condition is obtained from (3.101) by pre- and post-multiplying the LHS by $\mathrm{diag}(P^{-1}, I, I)$.

3.4.3 Quadratic Guaranteed Cost

The definition of guaranteed cost control has been introduced in [45] and then casted into the quadratic stability framework by Petersen in [141]. This definition essentially concerns the satisfaction of an LQ criterion together with quadratic stability.

Let us consider system (3.1) with initial condition $x_0 = x(0)$; consider the cost index

$$J = \int_0^{+\infty} x^T(t)\Pi x(t)dt\,, \tag{3.105}$$

where Π is a positive definite matrix.

Definition 3.6. Let P a positive definite matrix. System (3.1) is said to possess a *quadratic guaranteed cost with associated cost matrix* P wrt *the index* (3.105) if and only if for all $p \in R$

$$A^T(p)P + PA(p) + \Pi < 0\,. \tag{3.106}$$

△

The next theorem justifies Definition 3.6.

Theorem 3.7. *Assume that system* (3.1) *exhibits a quadratic guaranteed cost with associated cost matrix* P wrt *the index* (3.105)*; then system* (3.1) *is QS and, for all* $p(\cdot) \in \mathcal{PC}(\mathbb{R}^+, R)$*,*

$$J < x_0^T P x_0\,. \tag{3.107}$$

Proof. From (3.106) it follows that, for all $p \in R$,

$$A^T(p)P + PA(p) < -\Pi < 0\,; \tag{3.108}$$

therefore system (3.1) is QS.

Now, given any admissible parameter realization $p(\cdot)$, we have that, for all $t \in \mathbb{R}^+$,

$$x^T(t)\Pi x(t) < -\frac{d}{dt}\big(x^T(t)Px(t)\big)\,. \tag{3.109}$$

The proof follows by integrating (3.109) between 0 and $+\infty$ and taking into account that the system is quadratically stable and therefore $x(t) \to 0$ as $t \to +\infty$. □

Under Assumption 3.1, in order to have the minimum guaranteed cost we can solve the following EVP in the variables θ and P.

Problem 3.11 (Minimum quadratic guaranteed cost).

$$\begin{gathered} \min \theta \\ \text{s.t.} \\ P > 0 \\ P < \theta I \\ A^T(p_{(i)})P + PA(p_{(i)}) + \Pi < 0\,, \quad i = 1, \dots, 2^q\,. \end{gathered}$$

◇

Finally, by pre- and post-multiplying (3.106) by P^{-1}, we obtain that system (3.1) exhibits a quadratic guaranteed cost with associated cost matrix Q^{-1} *wrt* the index (3.105) *iff* for all $p \in R$

$$\begin{pmatrix} QA^T(p) + A(p)Q & Q\Pi \\ \Pi Q & -\Pi \end{pmatrix} < 0 \,. \tag{3.110}$$

3.5 Norm Bounded Uncertainties

Let us consider the uncertain linear system

$$\dot{x}(t) = (A + F\Delta E)\, x(t) \,, \tag{3.111}$$

where $A \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{q\times n}$ and $\|\Delta\| \leq 1$. The uncertainty considered in (3.111) is said to be norm bounded.

A system in the form (3.111) arises when we consider uncertainty at the system matrix entries level. Equation (3.111) is a generalization of the more intuitive uncertainty representation

$$\dot{x}(t) = (A + \Delta)x(t) \,, \tag{3.112}$$

where each entry of Δ captures the uncertainty of the corresponding element of A. The advantage of (3.111) is that the uncertain matrix Δ is normalized, while F and E take into account scaling factors.

Note that system (3.111) is equivalent to the classical feedback system given by the connection between the linear system

$$\dot{x}(t) = Ax(t) + Fw_\Delta(t) \tag{3.113a}$$
$$z_\Delta(t) = Ex(t) \,, \tag{3.113b}$$

and the uncertainty Δ, according to

$$w_\Delta(t) = \Delta z_\Delta(t) \,. \tag{3.114}$$

Looking at the representation (3.113)–(3.114) of the uncertain system (3.111), we note that a larger class of uncertainties can be captured by considering a nonzero direct feedthrough matrix $H \in \mathbb{R}^{q\times p}$ in (3.113); therefore in the following of this section we consider the more general uncertain system obtained by the feedback connection of

$$\dot{x}(t) = Ax(t) + Fw_\Delta(t) \tag{3.115a}$$
$$z_\Delta(t) = Ex(t) + Hw_\Delta(t) \,, \tag{3.115b}$$

and the uncertainty Δ given by (3.114).

System (3.115)–(3.114) is depicted in Fig. 3.14; it can be equivalently rewritten

$$\begin{aligned}\dot{x}(t) &= \left(A + F\Delta(I - H\Delta)^{-1}E\right) x(t) \\ &= \left(A + F(I - \Delta H)^{-1}\Delta E\right) x(t)\,, \end{aligned} \tag{3.116}$$

which explains why the uncertainty we deal with in this section is also referred to as *linear-fractional norm bounded*.

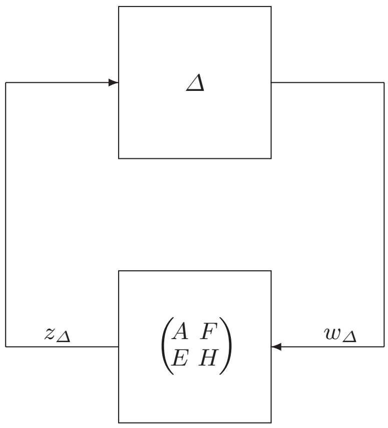


Fig. 3.14. Closed loop for the uncertain system (3.116)

Note that (3.116) (as well as (3.115)–(3.114)) is well posed if $I - H\Delta$ is invertible for all Δ with $\|\Delta\| \leq 1$. This is equivalent to require that $\|H\| < 1$; therefore this assumption will be implicitly done in the following. On the other hand the conditions for quadratic stability we will find later automatically guarantee the satisfaction of such assumption.

According to Definition 3.1, system (3.116) is QS *iff* there exists a positive definite matrix P such that for all Δ with $\|\Delta\| \leq 1$

$$\left(A + F\Delta(I - H\Delta)^{-1}E\right)^T P + P\left(A + F\Delta(I - H\Delta)^{-1}E\right) < 0\,. \tag{3.117}$$

By following the guidelines of Theorem 3.1, it is simple to show that quadratic stability of system (3.116) guarantees exponential stability of the system

$$\dot{x}(t) = \left(A + F\Delta(t)(I - H\Delta(t))^{-1}E\right) x(t) \tag{3.118}$$

for all uncertain matrix valued functions $\Delta(\cdot) \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{p\times q})$[5] with

[5] Again, using the result in [51], p. 67, it is possible to extend the class of admissible uncertainty realizations to the set of Lebesgue measurable matrix-valued functions.

$$\|\Delta(t)\| \leq 1\,, \qquad \forall t \in \mathbb{R}^+\,. \tag{3.119}$$

The next result is a necessary and sufficient condition for quadratic stability of system (3.116). It is proven, according to [38], p. 62, by using an $\mathcal{S}$-Procedure argument (e.g. see [38], p. 24). Essentially the same result (with $H = 0$) was proven, in the earlier literature, by exploiting a direct bound of the uncertain part of the LHS in inequality (3.117) (see [138] and the Summary at the end of the chapter).

Theorem 3.8. *System* (3.116) *is QS* if and only if *there exists a positive definite matrix* P *such that*

$$\begin{pmatrix} A^TP + PA + E^TE & PF + E^TH \\ F^TP + H^TE & -(I - H^TH) \end{pmatrix} < 0\,. \tag{3.120}$$

Proof. First of all note that (3.120) implies that $I - H^TH > 0$; therefore we have

$$\begin{aligned} \|H\Delta\| &\leq \|H\|\|\Delta\| \\ &\leq \|H\| < 1\,; \end{aligned} \tag{3.121}$$

this guarantees that $I - H\Delta$ is nonsingular for all Δ with $\|\Delta\| \leq 1$.

System (3.116) is QS *iff* there exists a positive definite matrix P such that (3.117) holds for all Δ with $\|\Delta\| \leq 1$. Inequality (3.117) is equivalent to require that

$$x^T(A^TP + PA)x + 2x^TPFv < 0\,, \tag{3.122}$$

for all $x \in \mathbb{R}^n$, $x \neq 0$, and $v \in S_x$ where

$$\begin{aligned} S_x &:= \{v \,:\, v = \Delta(Hv + Ex)\,,\ \|\Delta\| \leq 1\} \\ &= \left\{v \,:\, v^Tv \leq (Hv + Ex)^T(Hv + Ex)\right\}\,. \end{aligned} \tag{3.123}$$

We can conclude that system (3.116) is QS *iff* there exists a positive definite matrix P such that

$$\begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} A^TP + PA & PF \\ F^TP & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} < 0 \tag{3.124}$$

for all $x \in \mathbb{R}^n$, $x \neq 0$, and $v \in S_x$.

By applying the $\mathcal{S}$-procedure (see [38], p. 24) we obtain that the last condition is equivalent to the existence of a positive definite matrix P and a nonnegative scalar τ such that

$$\begin{pmatrix} A^TP + PA + \tau E^TE & PF + \tau E^TH \\ F^TP + \tau H^TE & -\tau(I - H^TH) \end{pmatrix} < 0\,. \tag{3.125}$$

Since (3.125) implies $\tau > 0$, without loss of generality we can divide both members of (3.125) by τ and rescale matrix P by the factor τ. From this consideration the proof follows. □

Note that Theorem 3.8 is directly stated in the form of LMIs feasibility problem. An equivalent condition for quadratic stability of system (3.116) is the existence of a positive definite matrix P which satisfies the Riccati inequality (with $\|H\| < 1$)

$$A^T P + PA + E^T E + (PF + E^T H)(I - H^T H)^{-1}(F^T P + H^T E) < 0 \,; \quad (3.126)$$

the last condition is obtained from (3.120) by applying Fact A.3.

Conversely, we have that another application of Fact A.3 to (3.120) leads to the following equivalent condition for quadratic stability (note the analogy with (3.101))

$$\begin{pmatrix} A^T P + PA & PF & E^T \\ F^T P & -I & H^T \\ E & H & -I \end{pmatrix} < 0 \,. \quad (3.127)$$

Finally, note that by pre- and post-multiplying (3.127) by $\mathrm{diag}(P^{-1}, I, I)$ we obtain that an equivalent condition for quadratic stability of system (3.116) is the existence of a positive definite matrix Q such that

$$\begin{pmatrix} QA^T + AQ & F & QE^T \\ F^T & -I & H^T \\ EQ & H & -I \end{pmatrix} < 0 \,; \quad (3.128)$$

the last condition will be exploited in the context of quadratic stabilization via state feedback in Sect. 5.4.1.

The next example illustrates the uncertainty characterization via representation (3.116). A comparison between parametric and norm bounded uncertainties is considered; it is also shown that uncertainty modelling via norm bounded uncertainties may introduce conservatism.

Example 3.4.
Let us consider the linear system

$$\dot{x}(t) = Ax(t) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} x(t) \,. \quad (3.129)$$

Assume that each entry of the system matrix is subject to uncertainties. In particular the 11-entry is subject to an uncertainty which is 10% of the nominal value, the 22-entry is subject to an uncertainty which is 40% of the nominal value and the off-diagonal entries are subject to an uncertainty which is 20% of the nominal value.

If we assume that the uncertainties affecting the elements of A are independent each other we can represent this uncertain system in the following way

$$\dot{x}(t) = A(p)x(t) = \begin{pmatrix} -1 + 0.1p_1 & 1 + 0.2p_2 \\ -1 + 0.2p_3 & -1 + 0.4p_4 \end{pmatrix} x(t) \,, \quad \|p\|_\infty \le 1 \,. \quad (3.130)$$

For following developments, we rewrite system (3.130) by introducing a scaling factor θ as follows

$$\dot{x}(t) = \left[\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} + \theta \begin{pmatrix} 0.1p_1 & 0.2p_2 \\ 0.2p_3 & 0.4p_4 \end{pmatrix} \right] x(t) \,, \quad \|p\|_\infty \leq 1 \,, \tag{3.131}$$

where θ takes into account which percentage of the nominal uncertainty is considered in the quadratic stability analysis; for example for $\theta = 1$ the nominal uncertainty is considered.

System (3.131) is subject to affine parametric uncertainty; therefore it satisfies Assumption 3.1 and we can solve a GEVP to estimate (see Problem 3.3)

$$\theta_{sup,par} := \sup \{\theta \, : \, \text{system (3.131) is QS}\} \,. \tag{3.132}$$

We obtain

$$2.22 < \theta_{sup,par} < 2.23.$$

Now note that system (3.131) can be rewritten

$$\begin{aligned} \dot{x}(t) &= \left[\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} + \theta \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right] x(t) \\ &=: (A + \theta \hat{F} \Delta E) x(t) \,. \end{aligned} \tag{3.133}$$

In order to qualify system (3.133) as a system subject to norm bounded uncertainties, we have to understand how to bound matrix

$$\Delta := \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \tag{3.134}$$

in order to capture the original uncertainty representation. This is not a trivial task; in particular note that the bound $\|\Delta\| \leq 1$ is not sufficient, since, for example, the uncertainty

$$\tilde{\Delta} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.135}$$

is admissible and $\|\tilde{\Delta}\| = 2$.

In order to give an estimate of the required bound, note that

$$\begin{aligned} \max_{\|p\|_\infty \leq 1} \|\Delta\|^2 &\leq \max_{\|p\|_\infty \leq 1} \|\Delta\|_F^2 \\ &= \max_{\|p\|_\infty \leq 1} \sum_{i=1}^{4} p_i^2 \\ &= 4 \,. \end{aligned} \tag{3.136}$$

Therefore a suitable bound is $\|\Delta\| \leq 2$.

In conclusion we can capture the original uncertain system via the norm bounded representation

$$\dot{x}(t) = (A + \theta F \Delta E)x(t)\,, \quad \|\Delta\| \le 1 \tag{3.137}$$

with

$$F := 2\hat{F} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}\,, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\,, \tag{3.138}$$

and θ is the usual scaling factor.

It is important to note that, for a given θ, the class of systems covered by (3.137) strictly contains the original class (3.131). This is the weak point of uncertainty modelling via norm bounded representations: they often introduce conservatism in the quadratic stability analysis.

Now we can apply Theorem 3.8 to estimate

$$\theta_{sup,nb} := \sup\{\theta \,:\, \text{system (3.137) is QS}\}\,. \tag{3.139}$$

We obtain

$$1.76 < \theta_{sup,nb} < 1.77.$$

As expected

$$\theta_{sup,nb} < \theta_{sup,par}\,. \tag{3.140}$$

It is important to remark that the GEVP to evaluate $\theta_{sup,par}$ is subject to $2^4 = 16$ LMIs while the GEVP to evaluate $\theta_{sup,nb}$ is subject to only one LMI; therefore the less conservative approach based on the parametric description of the uncertainty pays a bigger price in terms of computational burden.

A less conservative estimate of the quadratic stability margin via the norm bounded uncertainties approach can be obtained by allowing the uncertain matrix Δ to be structured; this topic is discussed in the next section. △

3.5.1 The Multi-Block Case

Let us consider the uncertain linear system

$$\dot{x}(t) = \left(A + \sum_{i=1}^{n_b} F_i \Delta_i (I - H_i \Delta_i)^{-1} E_i \right) x(t)\,, \tag{3.141}$$

where $A \in \mathbb{R}^{n\times n}$, $F_i \in \mathbb{R}^{n\times p_i}$, $E_i \in \mathbb{R}^{q_i\times n}$, $H_i \in \mathbb{R}^{q_i\times p_i}$, $\|H_i\| < 1$, and $\|\Delta_i\| \le 1$, $i = 1, \ldots, n_b$.

Note that system (3.141) can be put in the form (3.116) with (see also Fig. 3.15)

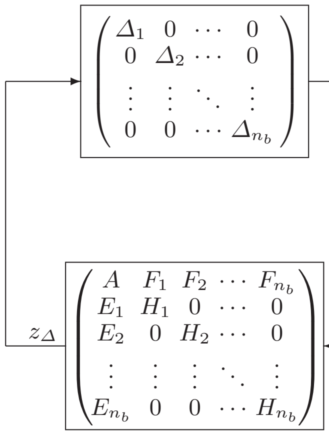


Fig. 3.15. Closed loop form for the uncertain system (3.141)

$$F = \begin{pmatrix} F_1 & F_2 & \cdots & F_{n_b} \end{pmatrix} \tag{3.142a}$$

$$E = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_{n_b} \end{pmatrix} \tag{3.142b}$$

$$H = \operatorname{diag}(H_1, H_2, \ldots, H_{n_b}) \tag{3.142c}$$

$$\Delta = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_{n_b})\,; \tag{3.142d}$$

therefore we say that system (3.141) is subject to structured, norm bounded uncertainties.

According to Definition 3.1, system (3.141) is QS *iff* there exists a positive definite matrix P such that for all Δ defined in (3.142d) with $\|\Delta_i\| \leq 1$, $i = 1, \ldots, n_b$,

$$\left(A + \sum_{i=1}^{n_b} F_i \Delta_i (I - H_i \Delta_i)^{-1} E_i \right)^T P + P \left(A + \sum_{i=1}^{n_b} F_i \Delta_i (I - H_i \Delta_i)^{-1} E_i \right) < 0\,. \tag{3.143}$$

Obviously quadratic stability of system (3.141) guarantees exponential stability of the system

$$\dot{x}(t) = \left(A + \sum_{i=1}^{n_b} F_i \Delta_i(t) (I - H_i \Delta_i(t))^{-1} E_i \right) x(t) \tag{3.144}$$

for all uncertain matrix valued functions $\Delta(\cdot)$ with the structure defined in (3.142d), $\Delta_i \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{p_i \times q_i})$ with

$$\|\Delta_i(t)\| \leq 1\,, \qquad \forall t \in \mathbb{R}^+\,, \qquad i = 1, \ldots, n_b\,. \tag{3.145}$$

The following theorem (see Exercise 3.3), which is stated directly in the form of LMIs feasibility problem, is a sufficient condition for quadratic stability of system (3.141); it can be proven by following the same guidelines of Theorem 3.8. Note that the theorem is not necessary for quadratic stability; indeed the application of the $\mathcal{S}$-Procedure in the multi-block case does not yield an equivalent condition as in the single block case (see (3.124) and (3.125)).

Theorem 3.9. *System* (3.141) *is QS if there exist positive scalars τ_i, $i = 1, \ldots, n_b - 1$, and a positive definite matrix P such that*

$$\begin{pmatrix} A^T P + PA + \sum_{i=1}^{n_b} \tau_i E_i^T E_i & PF_1 + \tau_1 E_1^T H_1 & \cdots & PF_{n_b} + \tau_{n_b} E_{n_b}^T H_{n_b} \\ F_1^T P + \tau_1 H_1^T E_1 & -\tau_1 (I_{p_1} - H_1^T H_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{nb}^T P + \tau_{n_b} H_{n_b}^T E_{n_b} & 0 & \cdots & -\tau_{n_b} (I_{p_{n_b}} - H_{n_b}^T H_{n_b}) \end{pmatrix} < 0\,, \tag{3.146}$$

where $\tau_{n_b} = 1$. □

Example 3.5.
Consider again the uncertain system (3.131); such system can be rewritten, according to representation (3.141), as follows

$$\begin{aligned} \dot{x}(t) = \Bigg[\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} + \theta \Bigg[\begin{pmatrix} 0.1 \\ 0 \end{pmatrix} (1\ 0)\, p_1 + \begin{pmatrix} 0.2 \\ 0 \end{pmatrix} (0\ 1)\, p_2 \\ + \begin{pmatrix} 0 \\ 0.2 \end{pmatrix} (1\ 0)\, p_3 + \begin{pmatrix} 0 \\ 0.4 \end{pmatrix} (0\ 1)\, p_4 \Bigg] \Bigg] x(t)\,, \quad \|p\|_\infty \leq 1\,. \end{aligned} \tag{3.147}$$

Therefore we can use Theorem 3.9 to compute

$$\theta_{sup,mb} := \sup \left\{ \theta \,:\, \text{system}\,(3.147)\,\text{is}\,\text{QS} \right\}\,. \tag{3.148}$$

We obtain

$$2.14 < \theta_{sup,mb} < 2.15\,; \tag{3.149}$$

note that $\theta_{sup,mb} < \theta_{sup,par}$ (see Example 3.4). Although (3.147) is an equivalent representation of the uncertain system (3.131), the fact that the two margins do not coincide is not surprising. Indeed $\theta_{sup,par}$ has been computed by applying Theorem 3.2 which is necessary and sufficient for quadratic stability of system (3.131), while $\theta_{sup,mb}$ has been computed applying Theorem (3.9), which is only sufficient.

We can conclude that, given the uncertain system (3.131), three different approaches for the computation of the quadratic stability margin can be followed; each approach involves a different computational burden and exhibits a different level of conservatism. The first approach (see Example 3.4) looks at the uncertain system as subject to parametric uncertainties; the evaluation of the quadratic stability margin, in this case, is nonconservative and requires the solution of a GEVP with two variables (P and θ) and subject to 16 LMIs. The second approach, pursued in the current example, models the uncertain part in norm bounded, multi-block form; the evaluation of the quadratic stability margin is conservative and requires the solution of a GEVP with five variables (P, θ, τ_i, $i = 1, 2, 3$) subject to one multi-block LMI. Finally, the third approach (see again Example 3.4) captures the uncertainty by a single norm-bounded block; it is the more conservative and less expensive, from a computational point of view, since involves a GEVP with two variables subject to a single LMI. △

Generally speaking, when the uncertain system we deal with depends on parametric uncertainties according to (3.1), rather than using the approaches of Sections 3.1–3.2, the system can be brought, under certain hypothesis, in the multi-block form (3.141) by using, for example, the methodologies proposed in [71] and [128]; then we can use the sufficient condition (3.146) to establish quadratic stability. The drawback is that, for high order systems and/or complex parameter dependencies, it is very difficult to put the given uncertain system in the form (3.141); moreover the usual presence of many blocks and/or repeated blocks may render the condition overly conservative.

3.5.2 Quadratic Stability and Performances

Quadratic $\mathcal{D}$-Stability

When the system is subject to norm bounded uncertainties, Definition 3.4 generalizes as follows.

Definition 3.7. System (3.116) is said to be *quadratically $\mathcal{D}$-stable* if and only if there exists a positive definite matrix P such that for all $\Delta \in \mathbb{R}^{p\times q}$ with $\|\Delta\| \le 1$

$$\begin{aligned} \Lambda \otimes P + \Theta \otimes \big(P(A + F\Delta(I - H\Delta)^{-1}E)\big) \\ + \Theta^T \otimes \big((A + F\Delta(I - H\Delta)^{-1}E)^T P\big) < 0\,. \end{aligned} \tag{3.150}$$

◇

The interpretation of the quadratic $\mathcal{D}$-stability property is the same as that one for parametric uncertainties. Remember that $\Lambda = \Lambda^T$ and Θ are $h \times h$ matrices involved into the definition of the region $\mathcal{D}$ (see (3.79)).

The following result is a *sufficient* condition for quadratic $\mathcal{D}$-stability.

Theorem 3.10 ([49]). *System* (3.116) *is quadratically $\mathcal{D}$-stable if there exist positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{k\times k}$, where $k = rank(\Theta)$, such that*

$$\begin{pmatrix} M_{\mathcal{D}}(A,P) & \Theta_1^T \otimes (PF) & \left(\Theta_2^T Z\right) \otimes E^T \\ \Theta_1 \otimes (F^T P) & -Z \otimes I_p & Z \otimes H^T \\ (Z\Theta_2) \otimes E & Z \otimes H & -Z \otimes I_q \end{pmatrix} < 0 \,. \tag{3.151}$$

where $M_{\mathcal{D}}$ *has been defined in* (3.85)*, and* $\Theta_1 \in \mathbb{R}^{k\times h}$*,* $\Theta_2 \in \mathbb{R}^{k\times h}$ *are full row rank matrices satisfying* $\Theta = \Theta_1^T \Theta_2$*.* □

Proof. The proof resembles that one of Theorem 3.8 with the obvious modifications.

First of all we have that (3.151) implies

$$\begin{pmatrix} Z \otimes I_p & -Z \otimes H^T \\ -Z \otimes H & Z \otimes I_q \end{pmatrix} > 0 \,, \tag{3.152}$$

which in turn guarantees that $I - H^T H$ is positive definite. From this it follows that $I - H\Delta$ is invertible for all Δ with $\|\Delta\| \le 1$.

System (3.116) is quadratically $\mathcal{D}$-stable *iff* there exists a positive definite matrix P such that for all Δ with $\|\Delta\| \le 1$

$$\begin{aligned} \Lambda \otimes P &+ \Theta \otimes \left(P\left(A + F\Delta(I - H\Delta)^{-1}E\right)\right) \\ &+ \Theta^T \otimes \left(\left(A + F\Delta(I - H\Delta)^{-1}E\right)^T P\right) \\ &= M_{\mathcal{D}}(A,P) + \Theta \otimes \left(PF\Delta(I - H\Delta)^{-1}E\right) \\ &\qquad + \Theta^T \otimes \left(E^T (I - H\Delta)^{-T} \Delta^T F^T P\right) \\ &= M_{\mathcal{D}}(A,P) + \left(\Theta_1^T \otimes (PF)\right)\left(\Theta_2 \otimes \left(\Delta(I - H\Delta)^{-1}E\right)\right) \\ &\qquad + \left(\Theta_2^T \otimes (E^T (I - H\Delta)^{-T} \Delta^T)\right)\left(\Theta_1 \otimes (F^T P)\right) < 0 \,, \end{aligned} \tag{3.153}$$

where, in the last equality, we have used (A.8).

Condition (3.153) is equivalent to require that for all vector $x \in \mathbb{R}^{hn}$, $x \neq 0$, and Δ with $\|\Delta\| \le 1$

$$x^T M_{\mathcal{D}}(A,P)x + 2x^T\left(\Theta_1^T \otimes (PF)\right)\left(\Theta_2 \otimes \left(\Delta(I - H\Delta)^{-1}E\right)\right)x < 0 \,, \tag{3.154}$$

which in turn is equivalent to require that, for any given $x \neq 0$,

$$x^T M_{\mathcal{D}}(A,P)x + 2x^T\left(\Theta_1^T \otimes (PF)\right)v < 0 \tag{3.155}$$

for all $v \in S_x$ with

$$S_x := \left\{ v \in \mathbb{R}^{kp} \,:\, v = \left(\Theta_2 \otimes \left(\Delta(I - H\Delta)^{-1}E\right)\right)x \,,\, \|\Delta\| \le 1 \right\} \,. \tag{3.156}$$

We have that $v = \left(\Theta_2 \otimes \left(\Delta(I - H\Delta)^{-1}E\right)\right)x$ is the unique solution of

$$v = (I_k \otimes \Delta) r_{v,x} \,, \tag{3.157}$$

where

$$r_{v,x} := (I_k \otimes H)v + (\Theta_2 \otimes E)x \in \mathbb{R}^{kq}\,. \tag{3.158}$$

By virtue of the last consideration an equivalent characterization of the set S_x is the following

$$S_x = \left\{ v \in \mathbb{R}^{kp} \,:\, v = (I_k \otimes \Delta) r_{v,x}\,,\ \|\Delta\| \le 1 \right\}\,. \tag{3.159}$$

Now, given any $Z > 0$, $Z \in \mathbb{R}^{k\times k}$, define the set

$$\hat{S}_x := \left\{ v \in \mathbb{R}^{kp} \,:\, r_{v,x}^T (Z \otimes I_q) r_{v,x} - v^T (Z \otimes I_p) v \ge 0 \right\}\,. \tag{3.160}$$

It is simple to recognize that $\hat{S}_x \supseteq S_x$. Indeed let $v \in S_x$; this means that there exists Δ with $\|\Delta\| \le 1$ such that $v = (I_k \otimes \Delta) r_{v,x}$. Therefore

$$\begin{aligned} r_{v,x}^T (Z \otimes I_q) r_{v,x} - v^T (Z \otimes I_p) v &= r_{v,x}^T (Z \otimes I_q) r_{v,x} \\ &\quad - r_{v,x}^T (I_k \otimes \Delta)^T (Z \otimes I_p)(I_k \otimes \Delta) r_{v,x} \\ &= r_{v,x}^T (Z \otimes I_q) r_{v,x} - r_{v,x}^T Z \otimes (\Delta^T \Delta) r_{v,x} \\ &= r_{v,x}^T \big(Z \otimes (I_q - \Delta^T \Delta) \big) r_{v,x} \ge 0\,, \end{aligned} \tag{3.161}$$

where the last inequality follows from the fact that the Kronecker product of two positive semidefinite matrices is still positive semidefinite; (3.161) guarantees that $v \in \hat{S}_x$.

On the basis of the last consideration we can conclude that (3.153) is guaranteed if (3.155) holds for all $v \in \hat{S}_x$.

This last condition can be rewritten

$$\begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} M_{\mathcal{D}}(A,P) & \Theta_1^T \otimes (PF) \\ \Theta_1 \otimes (F^T P) & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} < 0 \tag{3.162}$$

whenever

$$\begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} (\Theta_2^T \otimes E^T)(Z \otimes I_q)(\Theta_2 \otimes E) & (\Theta_2^T \otimes E^T)(Z \otimes I_q)(I_k \otimes H) \\ (I_k \otimes H^T)(Z \otimes I_q)(\Theta_2 \otimes E) & (I_k \otimes H^T)(Z \otimes I_q)(I_k \otimes H) - Z \otimes I_p \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \ge 0\,. \tag{3.163}$$

Exploiting, as usual, $\mathcal{S}$-Procedure arguments (see [38], p. 24) we have that the last condition is equivalent to require the existence of a scalar $\tau > 0$ such that

$$\begin{pmatrix} M_{\mathcal{D}}(A,P) & \Theta_1^T \otimes (PF) \\ \Theta_1 \otimes (F^T P) & -\tau Z \otimes I_p \end{pmatrix} + \tau \begin{pmatrix} \Theta_2^T \otimes E^T \\ I_k \otimes H^T \end{pmatrix} (Z \otimes I_q) \big(\Theta_2 \otimes E \ \ I_k \otimes H \big) < 0\,. \tag{3.164}$$

The proof follows dividing by τ both members of the last inequality, rescaling P and applying Fact A.3. □

Note that the condition in Theorem 3.10 is only sufficient. However when Θ has rank one ($k = 1$ in the proof of the theorem) matrix Z reduces to a scalar, which can be set without loss of generality to one, the arguments of the proof can be reversed and the condition becomes also necessary, as in the case of Theorem 3.8, where $\mathcal{D}$ is the left half of the complex plane (see also Exercise 3.4).

Moreover by pre and post-multiplying (3.151) by

$$\begin{pmatrix} I_h \otimes P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

we obtain that an alternative sufficient condition for quadratic $\mathcal{D}$-stability is the existence of positive definite matrices Q and Z such that

$$\begin{pmatrix} M_{\mathcal{D},d}(A,Q) & \Theta_1^T \otimes F & \left(\Theta_2^T Z\right) \otimes (QE^T) \\ \Theta_1 \otimes F^T & -Z \otimes I_p & Z \otimes H^T \\ (Z\Theta_2) \otimes (EQ) & Z \otimes H & -Z \otimes I_q \end{pmatrix} < 0\,, \tag{3.165}$$

where $M_{\mathcal{D},d}(A,Q)$ has been defined in (3.87); note that the last condition, differently from (3.151), *is not* an LMI.

Quadratic $\mathcal{L}_2$ Performance Consider the uncertain system (again assume $\|H\| < 1$ for well posedness)

$$\dot{x}(t) = \left(A + F\Delta(I - H\Delta)^{-1}E\right) x(t) + Bw(t) \tag{3.166a}$$

$$z(t) = Cx(t) + Dw(t)\,, \tag{3.166b}$$

or equivalently

$$\dot{x}(t) = Ax(t) + Fw_\Delta(t) + Bw(t) \tag{3.167a}$$

$$z_\Delta(t) = Ex(t) + Hw_\Delta(t) \tag{3.167b}$$

$$z(t) = Cx(t) + Dw(t) \tag{3.167c}$$

$$w_\Delta(t) = \Delta z_\Delta(t)\,. \tag{3.167d}$$

Note that system (3.166) defines a family of input-output operators $\Gamma_{zw}\big(\Delta(\cdot)\big)$ depending on the time realization of the uncertain matrix Δ.

Definition 3.8. Given $\gamma > 0$, system (3.166) is said to possess a *quadratic $\mathcal{L}_2$ performance bound* γ if and only if $\|D\| < \gamma$ and there exists a positive definite matrix P such that for all Δ with $\|\Delta\| \leq 1$

$$\begin{aligned}\left(A + F\Delta(I - H\Delta)^{-1}E\right)^T P + P\left(A + F\Delta(I - H\Delta)^{-1}E\right) + C^T C \\ + (PB + C^T D)(\gamma^2 I - D^T D)^{-1}(B^T P + D^T C) < 0\,.\end{aligned} \tag{3.168}$$

◇

By using the same arguments of Lemma 3.2 and taking into account Theorem 2.11, it is simple to show that if system (3.166) possesses a quadratic $\mathcal{L}_2$ performance bound γ, then it is quadratically stable and $\|\Gamma_{zw}\big(\Delta(\cdot)\big)\| < \gamma$ for all $\Delta(\cdot) \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{p\times q})$ with $\|\Delta(t)\| \leq 1$ for all t.

Following the same guidelines of Theorem 3.8 we can prove the next result.

Theorem 3.11. *System* (3.166) *possesses a quadratic* $\mathcal{L}_2$ *performance bound* γ if and only if $\|D\| < \gamma$ *and there exists a positive definite matrix* P *and a positive scalar* τ *such that*

$$\begin{pmatrix} A^TP+PA+C^TC+\tau E^TE+(PB+C^TD)(\gamma^2I-D^TD)^{-1}(B^TP+D^TC) & PF+\tau E^TH \\ F^TP+\tau H^TE & -\tau(I-H^TH) \end{pmatrix} < 0\,. \tag{3.169}$$

□

By applying Fact A.3, condition (3.169) can be converted into the usual matrix Riccati inequality

$$\begin{aligned} A^TP + PA + \tau E^TE + \tau^{-1}(PF + \tau E^TH)(I - H^TH)^{-1}(F^TP + \tau H^TE) \\ + C^TC + (PB + C^TD)(\gamma^2I - D^TD)^{-1}(B^TP + D^TC) < 0\,. \end{aligned} \tag{3.170}$$

Remark 3.15. It is interesting to note that in Theorem 3.11, differently from the simple quadratic stability case, even if $D = 0$ the scaling of P does not allow to eliminate the parameter τ from (3.169) or (3.170) (see also Remark 1 in [20]). ◇

Nor (3.170) neither (3.169) are LMIs in the variable P. Starting from (3.170) and applying again Fact A.3, we arrive to the following LMIs based feasibility problem which represents a necessary and sufficient condition for system (3.166) to possess a quadratic $\mathcal{L}_2$ performance bound γ.

Problem 3.12.
Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ and a scalar τ such that

$$P > 0 \tag{3.171a}$$

$$\tau > 0 \tag{3.171b}$$

$$\begin{pmatrix} A^TP + PA + \tau E^TE + C^TC & PB + C^TD & PF + \tau E^TH \\ B^TP + D^TC & -(\gamma^2I - D^TD) & 0 \\ F^TP + \tau H^TE & 0 & -\tau(I - H^TH) \end{pmatrix} < 0\,. \tag{3.171c}$$

◇

By further applying Fact A.3 we have that an equivalent condition to the feasibility of Problem 3.12 is the existence of a positive definite matrix P and a positive scalar τ such that

$$\begin{pmatrix} A^TP + PA & PB & PF & \tau E^T & C^T \\ B^TP & -\gamma^2 I & 0 & 0 & D^T \\ F^TP & 0 & -\tau^2 I & \tau H^T & 0 \\ \tau E & 0 & \tau H & -I & 0 \\ C & D & 0 & 0 & -I \end{pmatrix} < 0\,. \tag{3.172}$$

Note that the last condition, although convex in P and τ, is not an LMI.

The dual, equivalent condition of (3.171) is (see Exercise 3.5) the existence of a positive definite matrix Q and a positive scalar τ such that

$$\begin{pmatrix} AQ + QA^T + \tau FF^T & B + QC^TD & QE^T + \tau FH^T & QC^T \\ B^T + D^TCQ & -(\gamma^2 I - D^TD) & 0 & 0 \\ EQ + \tau HF^T & 0 & -\tau(I - HH^T) & 0 \\ CQ & 0 & 0 & -I \end{pmatrix} < 0\,. \tag{3.173}$$

The last condition will be exploited for state feedback design.

Quadratic Guaranteed Cost

Consider system (3.116) with initial condition $x_0 = x(0)$ and the cost index (3.105). We can give the following definition.

Definition 3.9. Let P a positive definite matrix. System (3.116) is said to possess a *quadratic guaranteed cost with associated cost matrix* P wrt *the index* (3.105) if and only if for all Δ with $\|\Delta\| \le 1$

$$\left(A + F\Delta(I - H\Delta)^{-1}E\right)^T P + P\left(A + F\Delta(I - H\Delta)^{-1}E\right) + \Pi < 0\,. \tag{3.174}$$

$\triangle$

By using the same arguments of Theorem 3.7, we can conclude that if system (3.116) exhibits a quadratic guaranteed cost with associated cost matrix P *wrt* the index (3.105) then it is QS and $J < x_0^T P x_0$ for all $\Delta(\cdot) \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{p\times q})$.

The proof of the next theorem follows the same guidelines of Theorem 3.8.

Theorem 3.12. *System* (3.116) *exhibits a quadratic guaranteed cost with associated cost matrix* P wrt *the index* (3.105) if and only if *there exists a positive scalar* τ *such that*

$$\begin{pmatrix} A^TP + PA + \tau E^TE + \Pi & PF + \tau E^TH \\ F^TP + \tau H^TE & -\tau(I - H^TH) \end{pmatrix} < 0\,. \tag{3.175}$$

$\square$

We can conclude that the minimum guaranteed cost problem can be converted into the following EVP in the variables θ, P and τ.

Problem 3.13 (Minimum quadratic guaranteed cost - Norm bounded uncertainties).

$$\min \theta$$
$$\text{s.t.}$$
$$\tau > 0$$
$$P > 0$$
$$P < \theta I \tag{3.175}$$

◇

The EVP 3.13 admits the optimal solution $P^\star$ *iff* the following EVP admits the optimal solution $Q^\star = P^{\star -1}$ (see Exercise 3.6).

Problem 3.14.

$$\max \theta$$
$$\text{s.t.}$$
$$\tau > 0 \tag{3.176a}$$
$$\theta > 0 \tag{3.176b}$$
$$Q > \theta I \tag{3.176c}$$
$$\begin{pmatrix} AQ + QA^T + \tau FF^T & QE^T + \tau FH^T & Q\Pi \\ EQ + \tau HF^T & -\tau(I - HH^T) & 0 \\ \Pi Q & 0 & -\Pi \end{pmatrix} < 0\,. \tag{3.176d}$$

◇

The formulation of the guaranteed cost problem via Problem 3.14 will be useful in the design context.

3.6 Connections between Quadratic Stability and $\mathcal{H}_\infty$ Control

At the end of the Eighties a number of papers (see among others [98,110,139]) pointed out several connections between quadratic stability and $\mathcal{H}_\infty$ control. Actually the results contained in these papers are based on the work on absolute stability and passivity theory due to Popov [142–144], and Yakubovich [176–178].

The central point is the following result. Remember that for a time-invariant system the $\mathcal{L}_2$ induced norm is equal to the $\mathcal{H}_\infty$ norm of the corresponding transfer function matrix.

Theorem 3.13 ([98, 110]). *Define the transfer function matrix*

$$W(s) = E(sI - A)^{-1}F + H\,. \tag{3.177}$$

Then system (3.116) *is QS if and only if A is an Hurwitz matrix and*

$$\|W\|_\infty < 1\,. \tag{3.178}$$

□

Proof. System (3.116) is QS *iff* there exists a positive definite matrix P which satisfies (3.126). On the other hand condition (3.126) is equivalent, by virtue of Theorem 2.11 applied to time-invariant systems with $\gamma = 1$, to condition (3.178) with A Hurwitz. □

In other words quadratic stability of system (3.116) is equivalent to a *small gain* condition for the system having transfer function matrix given by (3.177).

Now let us consider the closed loop system in Fig. 3.16, where $W(s) \in \mathcal{RH}_\infty^{q\times p}$ and $\Delta(s) \in \mathcal{RH}_\infty^{p\times q}$. Remember that the closed loop system is well posed if $I - W(\infty)\Delta(\infty)$ is invertible (see [185], p. 67).

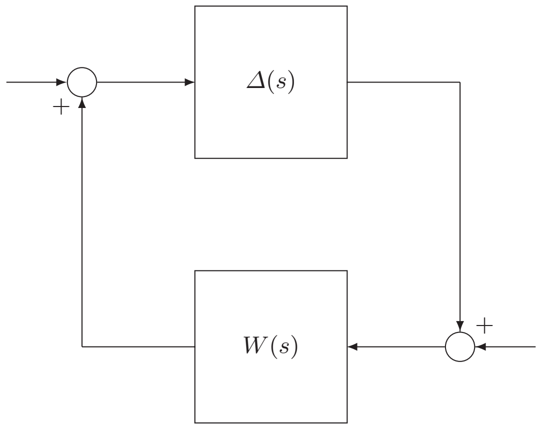


Fig. 3.16. Closed loop scheme for Small Gain Theorem

This closed loop system is said to be *internally stable* if all loop signals are bounded when the exogenous inputs are bounded. Since both W and Δ belong to $\mathcal{RH}_\infty$, a necessary and sufficient condition for internal stability is $(I - \Delta W)^{-1} \in \mathcal{RH}_\infty^{p\times p}$ (see [185], p. 69).

Now it is well known, from classical $\mathcal{H}_\infty$ theory, that (3.178) is necessary and sufficient for well posedness and internal stability of the closed loop system in Fig. 3.16 with respect to dynamic perturbations $\Delta(s) \in \mathcal{RH}_\infty^{p\times q}$ satisfying $\|\Delta\|_\infty \le 1$, which in turn is equivalent to well posedness and internal stability versus memoryless, time-invariant, complex perturbations $\Delta \in \mathbb{C}^{p\times q}$ satisfying $\|\Delta\| \leq 1$ (see [185], p. 139). This is a version of the famous Small Gain Theorem.

Now consider the following definition which extends the quadratic stability definition to complex uncertainties.

Definition 3.10 (Quadratic stability for complex uncertainties [152]). Consider system (3.116) with $\Delta \in \mathbb{C}^{p\times q}$, $\|\Delta\| \leq 1$. Then system (3.116) is quadratically stable *if and only if* there exists a positive definite matrix $P \in \mathbb{C}^{n\times n}$ such that for all $\Delta \in \mathbb{C}^{p\times q}$ with $\|\Delta\| \leq 1$

$$(A + F\Delta(I - H\Delta)^{-1}E)^* P + P(A + F\Delta(I - H\Delta)^{-1}E) < 0\,. \qquad (3.179)$$

◇

It can be proven that (3.178) is also necessary and sufficient for QS versus complex uncertainties [98, 110, 152].

The above facts are resumed in the following theorem.

Theorem 3.14. *Let us consider the closed loop scheme in Fig. 3.16 and denote by (A, F, E, H) a (minimal) realization of $W(s)$; assume that $W(s) \in \mathcal{RH}_\infty^{q\times p}$. Then the following statements are equivalent:*

i) $\|W\|_\infty < 1$.
ii) The closed loop system in Fig. 3.16 is well posed and internally stable for all dynamic perturbations $\Delta \in \mathcal{RH}_\infty^{p\times q}$ satisfying $\|\Delta\|_\infty \leq 1$.
iii) The closed loop system in Fig. 3.16 is well posed and internally stable for all $\Delta \in \mathbb{C}^{p\times q}$ with $\|\Delta\| \leq 1$.
iv) System (3.116) *is QS.*
v) System (3.116) *is QS for $\Delta \in \mathbb{C}^{p\times q}$ with $\|\Delta\| \leq 1$.*
vi) There exists a positive definite matrix P which satisfies the Riccati inequality (3.126).

□

The link between quadratic stability and $\mathcal{H}_\infty$ theory is also useful to determine the amplitude of the largest (in the two-norm sense) uncertainty (open) ball for which system (3.116) is QS. Indeed it is clear that system (3.116) is QS for all Δ such that $\|\Delta\| \leq \gamma$ *iff*

$$\gamma < \frac{1}{\|W\|_\infty}\,. \qquad (3.180)$$

Example 3.6.
Consider again system (3.116) with

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad H = 0\,. \qquad (3.181)$$

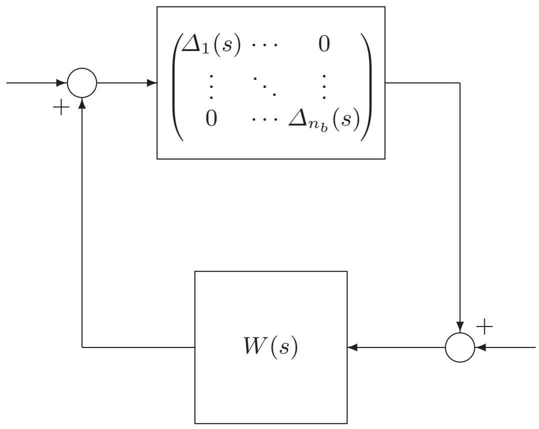


Fig. 3.17. Closed loop scheme for Theorem 3.15

In this case we obtain

$$\|W\|_\infty = \|E(sI - A)^{-1}F\|_\infty \cong 0.567 \tag{3.182}$$

and therefore the system is QS for all Δ with $\|\Delta\| \le \gamma$ and γ any number satisfying

$$\gamma < \frac{1}{\|W\|_\infty} \cong 1.76 \,. \tag{3.183}$$

As expected, the estimate of the uncertainty upper bound computed via the $\mathcal{H}_\infty$ approach is practically coincident with the upper bound $\theta_{sup,nb}$ estimated in Example 3.4 via the LMI approach. △

When we consider the multi-block case (3.141), Theorem 3.14 cannot be extended as a necessary and sufficient condition. Referring to Fig. 3.17 define $W(s) = E(sI - A)^{-1}F + H$ with E, F and H given in (3.142).

We can state the following result.

Theorem 3.15 ([152]). *Consider system* (3.141)*; then the following statements are equivalent:*

i) Matrix A is Hurwitz and there exist $n_b - 1$ positive numbers d_1, d_2, ..., d_{n_b-1} such that

$$\|D_o W D_i^{-1}\|_\infty < 1 \,, \tag{3.184}$$

where

$$D_i = \text{diag}(d_1 I_{p_1}, \ldots, d_{n_b-1} I_{p_{n_b-1}}, I_{p_{n_b}}) \tag{3.185a}$$

$$D_o = \text{diag}(d_1 I_{q_1}, \ldots, d_{n_b-1} I_{q_{n_b-1}}, I_{q_{n_b}}) \,. \tag{3.185b}$$

ii) *There exist* $n_b - 1$ *positive numbers* $\tau_1, \ldots, \tau_{n_b-1}$ *and a positive definite matrix* P *such that* (3.146) *is satisfied.*

Either one of these conditions implies the following:

iii) *System* (3.141) *is QS.*
iv) *System* (3.141) *is QS for complex* $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_{n_b})$ *with* $\Delta_i \in \mathbb{C}^{p_i \times q_i}$*, satisfying* $\|\Delta_i\| \leq 1$*,* $i = 1, \ldots, n_b$*.*
v) *The closed loop system in Fig. 3.17 is internally stable for all dynamic perturbations* $\Delta_i(s) \in \mathcal{RH}_\infty^{p_i \times q_i}$ *satisfying* $\|\Delta_i\|_\infty \leq 1$*,* $i = 1, \ldots, n_b$*.*
vi) *The closed loop system in Fig. 3.17 is internally stable for all* $\Delta_i \in \mathbb{C}^{p_i \times q_i}$ *with* $\|\Delta_i\| \leq 1$*,* $i = 1, \ldots, n_b$*.*

□

In the multi-block case condition iii) and vi) (and therefore iv) and v)) are no longer equivalent; indeed in [152] two two-blocks counter-examples show that iii) does not imply vi) (and therefore iv) and v)) and that, conversely, vi) does not imply iii) (obviously iv) does imply iii)).

Therefore in the multi-block case condition i) and ii) are not necessary for iii), v) and vi); concerning condition iv), when $n_b = 2$ (two blocks) condition i), ii) and iv) are equivalent (see again [152]).

Conditions v) and vi) are still equivalent and can be expressed in terms of an equivalent condition involving the infinity norm of the Structured Singular Value (μ) of $W(s)$ [63, 185]. Indeed we can state the following result (note that the uncertainty set is open).

Theorem 3.16 ([63]). *The following statements are equivalent:*

i) *The closed loop system in Fig. 3.17 is internally stable for all dynamic perturbations* $\Delta_i(s) \in \mathcal{RH}_\infty^{p_i \times q_i}$ *satisfying* $\|\Delta_i\|_\infty < 1$*,* $i = 1, \ldots, n_b$*.*
ii) *The closed loop system in Fig. 3.17 is internally stable for all* $\Delta_i \in \mathbb{C}^{p_i \times q_i}$ *with* $\|\Delta_i\| < 1$*,* $i = 1, \ldots, n_b$*.*
iii) *Matrix* A *is Hurwitz and*

$$\mu_{\boldsymbol{\Delta}}(W(j\omega)) \leq 1\,, \qquad \forall \omega \in \mathbb{R}^+\,,$$

where $\boldsymbol{\Delta}$ *denotes the block structure of the perturbation in Fig. 3.17, that is*

$$\boldsymbol{\Delta} := \big\{\Delta \,:\, \Delta = \mathrm{diag}\big(\Delta_1, \ldots, \Delta_{n_b}\big)\,, \quad \Delta_i \in \mathbb{C}^{p_i \times q_i}\,,\ i = 1, \ldots, n_b\big\}\,, \tag{3.186}$$

and $\mu_{\boldsymbol{\Delta}}(W(j\omega))$ *is the* structured singular value *of* $W(j\omega)$ *computed* wrt *the uncertainty structure* $\boldsymbol{\Delta}$*.*

□

Further readings concerning the connections between quadratic stability and μ theory are the seminal paper [39] and [66, 119].

Finally, it is interesting to remark that, with the obvious modifications, Theorem 3.14 can be extended to illustrate the connections between quadratic $\mathcal{D}$-stability and $\mathcal{H}_\infty$ control [49].

Summary

In this chapter we have considered linear systems subject to parametric uncertainties in the form (3.1). The definition of quadratic stability has been given and it has been shown that, if the system depends on parameters as the ratio of a multi-affine matrix-valued function and a multi-affine polynomial (Assumption 3.1), quadratic stability is equivalent to the feasibility of an LMIs based problem (Theorem 3.2).

Quadratic stability guarantees exponential stability of the system versus arbitrarily (Lebesgue measurable) time realizations of parameters (Theorem 3.1). A natural question is whether quadratic stability is also necessary for exponential stability of the uncertain system. The answer is *not*; indeed there exist uncertain systems which are exponentially stable for all admissible time realizations of parameters, but which are not quadratically stable (see for example [38], p. 73, [133]).

According to [37], a given class of Lyapunov functions is said to be “universal” for system (3.1) if the existence of a Lyapunov function which proves exponential stability of the uncertain system implies the existence of a Lyapunov function belonging to the class. Following this definition, the class of quadratic Lyapunov functions used in the definition of quadratic stability is not universal for systems in the form (3.1). Conversely, as shown in [37], the class of *polyhedral* Lyapunov functions is universal (for an interesting discussion on this topic see the seminal paper [40]).

A polyhedral Lyapunov function has the form

$$v(x) = \max_{i=1,\ldots,s} v_i^T x\,, \tag{3.187}$$

where $v_i^T \in \mathbb{R}^n$, $i = 1, \ldots, s$, are appropriate row-vectors.

For example consider the following system [133]

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1+p & -1 \end{pmatrix} x(t)\,, \tag{3.188}$$

where $|p| \leq \gamma$.

For $\gamma > \sqrt{3}/2 \cong 0.866$ system (3.188) is not QS; in [133] it is shown that a certain piecewise quadratic Lyapunov function proves exponential stability for $\gamma = 0.94$. Finally in [5], by using polyhedral Lyapunov functions, it has been proven that the system is exponentially stable whenever $\gamma < 1$.

The fact that the class of polyhedral Lyapunov functions is universal justifies the effort of searching for a polyhedral Lyapunov function for a system in the form (3.1). Unfortunately, while the search over the quadratic functions in the form $x^T P x$, for a given uncertain system, can be conducted exhaustively, the search of the optimal polyhedral function requires to fix a value for s in (3.187); only for $s \to \infty$ the entire set of polyhedral functions is spanned. Moreover, as s growths, the computational complexity increases.

When system (3.1) does not satisfy Assumption 3.1 we can resort to the polytopic covering technique. It consists of the introduction of fictitious parameters which allow to transform the dependence of $A(\cdot)$ on p into a multi-affine one. A viable alternative is the use of algorithms based on gridding or on statistical methods. This last approach guarantees quadratic stability in a probabilistic sense, and is particularly suited when the system depends on many uncertain parameters.

Then quadratic stability and performance has been considered; in particular three important cases have been discussed: quadratic $\mathcal{D}$-stability, quadratic $\mathcal{L}_2$ performance and quadratic stability with an LQ performance criterion. In all cases necessary and sufficient conditions are provided when the system matrices satisfy Assumptions 3.1 and 3.4.

Another important class of uncertain systems is that one of systems subject to norm bounded uncertainties. In this case, testing quadratic stability is equivalent to check a single LMI: This is Theorem 3.8.

In the earlier literature [138], a different approach was used to prove Theorem 3.8 (with $H = 0$). To this regard, note that (3.126) can be easily derived as a sufficient condition for quadratic stability; indeed, since for any positive scalar τ

$$(\tau^{-1/2} F^T P - \tau^{1/2} \Delta E)^T (\tau^{-1/2} F^T P - \tau^{1/2} \Delta E) \geq 0\,, \tag{3.189}$$

we have that for all Δ with $\|\Delta\| \leq 1$

$$\begin{aligned} E^T \Delta^T F^T P + P F \Delta E &\leq \tau^{-1} P F F^T P + \tau E^T \Delta^T \Delta E \\ &\leq \tau^{-1} P F F^T P + \tau E^T E\,. \end{aligned} \tag{3.190}$$

It follows that the inequality

$$A^T P + P A + \tau E^T E + \tau^{-1} P F F^T P < 0 \tag{3.191}$$

implies the satisfaction of (3.117) (with $H = 0$) for all Δ with $\|\Delta\| \leq 1$.

Without loss of generality we can set $\tau = 1$ in (3.191); this is equivalent to rescale P.

Proving that (3.191) is necessary for quadratic stability, without recurring to the $\mathcal{S}$-Procedure arguments exploited in Theorem 3.8, is not immediate (see [138]).

In the norm bounded uncertainties context it is important to recall the works by Hinrichsen, Pritchard and colleagues [97, 98, 145, 169] concerning the study of the properties of the *complex structured stability radius*, which is defined as

$$\rho_C := sup\left\{\gamma \,:\, A + F\Delta E \,\text{is Hurwitz for all}\, \Delta \in \mathbb{C}^{p\times q}\,,\, \|\Delta\| \leq \gamma\right\}\,. \quad (3.192)$$

Since Hurwitz stability for complex uncertainties is equivalent to quadratic stability for real uncertainties, the complex stability radius can be interpreted as the quadratic stability radius of system (3.116). According to (3.180) we have

$$\rho_C = \frac{1}{\|W\|_\infty}\,. \quad (3.193)$$

As for multi-block norm bounded uncertainties, a sufficient condition for quadratic stability is provided in terms of a suitable LMIs feasibility problem; the condition, however, is not necessary.

Finally the quadratic stability and performances problems (quadratic stability and pole placement, quadratic $\mathcal{L}_2$ performance and quadratic guaranteed cost) for systems subject to norm bounded uncertainties have been discussed.

The last section of the chapter has been devoted to illustrate the connections between quadratic stability of systems subject to norm bounded uncertainties and the $\mathcal{H}_\infty$ control theory.

Exercises

Exercise 3.1. Prove Theorem 3.5. ◇

Exercise 3.2. Show that (3.87) is an equivalent condition for quadratic $\mathcal{D}$-stability of system (3.1).
(Hint: pre- and post-multiply both members of (3.86) by $I \otimes P^{-1}$). ◇

Exercise 3.3 ([152]). By following the same guidelines of Theorem 3.8, prove Theorem 3.9. ◇

Exercise 3.4. Show that condition (3.151) recovers condition (3.120) when $\mathcal{D}$ is the left half of the complex plane. ◇

Exercise 3.5. Prove that system (3.166) possesses a quadratic $\mathcal{L}_2$ performance bound γ *iff* condition (3.173) holds.
(Hint: System (3.166) possesses a quadratic $\mathcal{L}_2$ performance bound γ *iff* $\|D\| < \gamma$ and there exists a positive definite matrix Q such that, for all Δ with $\|\Delta\| \leq 1$,

$$\begin{aligned} Q\left(A + F\Delta(I - H\Delta)^{-1}E\right)^T + \left(A + F\Delta(I - H\Delta)^{-1}E\right)Q + QC^TCQ \\ + (B + QC^TD)(\gamma^2 I - D^TD)^{-1}(B^T + D^TCQ) < 0\,. \end{aligned} \quad (3.194)$$

By following the guidelines of the proof of Theorem 3.8 prove that the last condition is equivalent to the existence of a positive definite matrix Q and a positive scalar τ such that $\|D\| < \gamma$ and

$$\begin{pmatrix} AQ+QA^T+QC^TCQ+\tau FF^T+(B+QC^TD)(\gamma^2 I-D^TD)^{-1}(B^T+D^TCQ) & QE^T+\tau FH^T \\ EQ+\tau HF^T & -\tau(I-HH^T) \end{pmatrix} < 0 \,. \tag{3.195}$$

◇

Exercise 3.6. Prove that system (3.116) exhibits a quadratic guaranteed cost with associated cost matrix Q^{-1} *wrt* the index (3.105) *iff* (3.176d) holds. (Hint: System (3.116) exhibits a quadratic guaranteed cost with associated cost matrix Q^{-1} *wrt* the index (3.105) *iff*, for all Δ with $\|\Delta\| \leq 1$,

$$Q\left(A+F\Delta(I-H\Delta)^{-1}E\right)^T+\left(A+F\Delta(I-H\Delta)^{-1}E\right)Q+Q\Pi Q<0\,. \tag{3.196}$$

By following the guidelines of the proof of Theorem 3.8 prove that the last condition is equivalent to the existence of a positive scalar τ such that

$$\begin{pmatrix} AQ+QA^T+Q\Pi Q+\tau FF^T & QE^T+\tau FH^T \\ EQ+\tau HF^T & -\tau(I-HH^T) \end{pmatrix} < 0\,. \tag{3.197}$$

◇

4. Systems Depending on Bounded Rate Uncertainties

4.1 Quadratic Stability via Parameter Dependent Lyapunov Functions

As stated in Theorem 3.1, quadratic stability of system (3.1) guarantees exponential stability for all time behaviors of parameters which are of interest in the practise (see also Remark 3.2); in particular, exponential stability is guaranteed for discontinuous parameters which exhibit an unbounded rate of variation at the discontinuity points.

Conversely, in many engineering applications, the uncertain parameters are a continuous and slowly varying function of time. In those cases, the quadratic stability approach may result an extremely conservative tool to test system stability.

Assume that $p(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, R)$, where R is the usual hyper-box defined in (3.2), and that a bound on the rate of variation is known

$$|\dot{p}_i(t)| \le h_i \,, \qquad i = 1, \dots, q \,, \qquad t \in \mathbb{R}^+ \,. \tag{4.1}$$

When $p_i(\cdot)$ is not differentiable in place of (4.1) we assume that

$$\max \left\{ |\dot{p}_i(t^-)|, |\dot{p}_i(t^+)| \right\} \le h_i \,, \tag{4.2}$$

where $\dot{p}_i(t^-)$ and $\dot{p}_i(t^+)$ are the left and right limit of $\dot{p}_i(\cdot)$ at t.

Note that condition (4.1) implies that the time derivative parameter vector $\dot{p}(\cdot)$ belongs for all t to the hyper-box $\dot{R}$, defined as follows

$$\dot{R} := [-h_1, h_1] \times [-h_2, h_2] \times \cdots \times [-h_q, h_q] \,, \tag{4.3}$$

which is centered at the origin of the parameter derivative space; as usual we denote by $\dot{R}^v$ the set of the 2^q vertices of $\dot{R}$ and by $h_{(j)}$ the j-th vertex of $\dot{R}$.

To take into account the information on the rate of variation of parameters we have to use a quadratic Lyapunov function *depending on parameters* (see [2, 10, 81]). More precisely consider the following definition.

Definition 4.1. System (3.1) is said to be *quadratically stable via parameter dependent Lyapunov functions in* $R \times \dot{R}$ if and only if there exists a continuously differentiable positive definite matrix valued function $P(\cdot) : p \in R \mapsto P(p)$ such that

$$A^T(p)P(p) + P(p)A(p) + \sum_{i=1}^{q} \frac{\partial P(p)}{\partial p_i} h_{(j)_i} < 0 \tag{4.4}$$

for all $p \in R$ and for all $j = 1, \dots, 2^q$ (note that $h_{(j)_i}$ denotes the i-th component of vector $h_{(j)}$). ◇

Remark 4.1. Behind Definition 4.1 is the use of a parameter dependent quadratic Lyapunov function in the form $v(p,x) = x^T P(p)x$. Such Lyapunov function, along a given parameter vector time realization, is a time-varying Lyapunov function in the form $x^T \tilde{P}(t)x = x^T P(p(t))x$ (see also Theorem 2.10 and Remark 3.1). ◇

Remark 4.2. Condition (4.1) recovers the case in which some of the parameters are constant ($h_i = 0$, for some i). When *all* the parameters are constant, the approach via parameter dependent Lyapunov functions can be a viable alternative to the μ analysis method or to the Routh and Kharitonov based approaches [27, 31, 75, 112] (when such methods are applicable). ◇

We can state the following fundamental result.

Theorem 4.1 ([10, 81]). *Assume that system* (3.1) *is quadratically stable via parameter dependent Lyapunov functions in* $R \times \dot{R}$*; then system* (3.1) *is exponentially stable for all vector valued functions* $p(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, R)$ *satisfying* (4.1) *and* (4.2).

Proof. Let $p(\cdot)$ be any parameter realization satisfying the hypothesis of the theorem; then $\tilde{A}(\cdot) := A(p(\cdot)) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n \times n})$.

Note that, from Theorem A.2, inequality (4.4) is equivalent to

$$A^T(p)P(p) + P(p)A(p) + \sum_{i=1}^{q} \frac{\partial P(p)}{\partial p_i} h_i < 0 \tag{4.5}$$

for all $p \in R$ and for all $h \in \dot{R}$.

Now, since R is compact, inequality (4.5) implies that there exists a positive definite matrix-valued function $\tilde{P}(\cdot) := P(p(\cdot)) \in \mathcal{C}_0(\mathbb{R}^+, \mathbb{R}^{n \times n})$ such that

$$\dot{\tilde{P}} + \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} << 0\,. \tag{4.6}$$

The proof follows from Corollary 2.1 and the arbitrariness of the function $p(\cdot)$. □

Note that, if it is known that $A(p)$ is Hurwitz for some $p = p^* \in R$, the satisfaction of (4.4) automatically guarantees that $P(p)$ is positive definite [81, 82]. Indeed, by letting $h = 0$ in (4.5) we obtain that

$$A^T(p)P(p) + P(p)A(p) < 0 \tag{4.7}$$

for all $p \in R$. In particular we have that (4.7) holds for $p = p^*$; since $A(p^*)$ is Hurwitz, by Lyapunov inequality theory it follows that

$$P(p^*) > 0\,. \tag{4.8}$$

From (4.7) it follows that, for all $p \in R$, $P(p)$ is nonsingular; from (4.8) and the fact that $P(\cdot)$ is continuous we can conclude that $P(p)$ is positive definite for all $p \in R$.

Therefore, if it is known that there exists $p^* \in R$ such that $A(p^*)$ is Hurwitz, without loss of generality we can look for a symmetric (in place of positive definite) continuously differentiable matrix function $P(\cdot)$ which satisfies (4.4); this allows to eliminate, from the feasibility problem involving the search of a suitable $P(\cdot)$, all the constraints coming from the requirement that $P(\cdot)$ is positive definite on R.

On the basis of this consideration, in the sequel of the chapter we shall assume the following.

Assumption 4.1. For some $p^* \in R$, $A(p^*)$ is an Hurwitz matrix. ◇

Finally, let us pre and post-multiply (4.4) by $P^{-1}(p)$; we obtain

$$P^{-1}(p)A^T(p) + A(p)P^{-1}(p) - \sum_{i=1}^{q} \frac{\partial P^{-1}(p)}{\partial p_i} h_{(j)_i} < 0\,. \tag{4.9}$$

Therefore, under Assumption 4.1, system (3.1) is quadratically stable via parameter dependent Lyapunov functions *iff* there exists a symmetric continuously differentiable matrix-valued function $Q(\cdot)$ such that

$$A(p)Q(p) + Q(p)A^T(p) - \sum_{i=1}^{q} \frac{\partial Q(p)}{\partial p_i} h_{(j)_i} < 0\,, \quad j = 1, \ldots, 2^q\,. \tag{4.10}$$

Note that, in the last case, the parameter dependent Lyapunov function proving quadratic stability via parameter dependent Lyapunov functions of system (3.1) is $v(p,x) = x^T Q^{-1}(p) x$.

4.2 Multi-Affine Quadratic Stability

Concerning the choice of a structure for $P(\cdot)$, we start by keeping the dependence on parameters of $P(\cdot)$ as simple as possible; therefore we consider the multi-affine dependence

$$P(p) = \sum_{i_1, i_2, \cdots, i_q = 0}^{1} P_{i_1 i_2 \ldots i_q} p_1^{i_1} p_2^{i_2} \cdots p_q^{i_q}\,. \tag{4.11}$$

Definition 4.2 (Multi-Affine Quadratic Stability). System (3.1) is said to be *multi-affinely quadratically stable (MQS) in* $R \times \dot{R}$ if and only if it is QS via parameter dependent Lyapunov functions in $R \times \dot{R}$ with $P(\cdot)$ any positive definite matrix-valued function in the form (4.11). ◇

Remark 4.3. The existence of a positive definite matrix-valued function $Q(\cdot)$ satisfying (4.10) and having a multi-affine dependence on parameters

$$Q(p) = \sum_{i_1, i_2, \cdots, i_q = 0}^{1} Q_{i_1 i_2 \ldots i_q} p_1^{i_1} p_2^{i_2} \cdots p_q^{i_q} \,, \tag{4.12}$$

guarantees quadratic stability via parameter dependent Lyapunov functions of system (3.1); however, the Lyapunov function proving quadratic stability is $x^T Q^{-1}(p) x$ which exhibits a non-multi-affine dependence on parameters. Therefore, in this case, we cannot conclude that the system is multi-affinely quadratically stable. ◇

It is simple to recognize that, under Assumption 4.1, system (3.1) is MQS *iff* the following problem admits a feasible solution.

Problem 4.1.
Find symmetric matrices $P_{i_1 i_2 \cdots i_q}$ such that, defined $P(p)$ according to (4.11), for all $p \in R$ and $j = 1, \ldots, 2^q$,

$$M(p) + \sum_{i=1}^{q} \frac{\partial P(p)}{\partial p_i} h_{(j)_i} < 0 \,, \tag{4.13}$$

where

$$M(p) = A^T(p) P(p) + P(p) A(p) \,. \tag{4.14}$$

◇

Note that Problem 4.1 cannot be immediately converted into an LMIs feasibility problem. Indeed, *also when* $A(\cdot)$ *satisfies Assumption 3.1*, $M(\cdot)$ (and therefore the matrix function at the LHS in (4.13)) is not the ratio of a multi-affine matrix-valued function and a multi-affine polynomial; conversely the partial derivatives of $P(\cdot)$ are multi-affine. Therefore condition (4.13) cannot be converted into a finite number of inequalities.

Remark 4.4. When $A(\cdot)$ satisfies Assumption 3.1, the matrix function $M(p)$ contains at most quadratic powers of the p_i's, i.e. it is a multi-quadratic function. If we look at each scalar function p_i^2, $i = 1, \ldots, q$, as a single nonlinear function the number of non-multi-affine functions in $M(p)$, say μ, is exactly equal to q. Therefore, looking at the last four rows of Table 3.1, we understand that if $q > \mu_{med}$ it may be convenient to use an algorithm along the lines of Algorithm 3.5 to test feasibility of Problem 4.1. Conversely ($q \leq \mu_{med}$) we can use the Polytopic Covering technique described in Sect. 3.2.2 (this is the approach followed in [8] and [9], where affine parameter dependent Lyapunov functions are considered) or the gridding approach. If the Polytopic Covering approach is used and each function p_i^2, $i = 1, \ldots, q$, is bounded according to the procedure described in Sect. 3.2.2, after the application of Step 1 of Procedure 3.1, the obtained function Ψ is already multi-affine (see also Remark 3.11). $\diamond$

Remark 4.5. To decide the value of the threshold μ_{med} between deterministic and probabilistic methods, it is important to take also into account that, differently from the simple quadratic stability analysis of Chap. 3, the number of vertices which comes out from the covering of the parameter hyper-box R, namely $2^q 2^\mu$ $(= 2^{2q})$ if we proceed according to Remark 4.4, must be multiplied by 2^q, which is the number of vertices of the parameter derivative hyper-box $\dot{R}$. $\diamond$

Therefore, when $A(p)$ satisfies Assumption 3.1 and the Polytopic Covering approach is used, by applying Procedure 3.1 according to the strategy discussed in Remark 4.4, we can transform $M(p)$ into the ratio of a multi-affine matrix-valued function and a multi-affine polynomial, say $\hat{M}(p, \delta)$, by the introduction of fictitious parameters δ_i, $i = 1, \ldots, q$, ranging into the interval $[0, 1]$. The feasibility of the following LMIs problem guarantees multi-affine quadratic stability of system (3.1).

Problem 4.2.
Find symmetric matrices $P_{i_1, i_2, \ldots, i_q}$ such that for all $k = 1, \ldots, 2^q$, $j = 1, \ldots, 2^q$, $l = 1, \ldots, 2^q$,

$$\hat{M}(p_{(k)}, \delta_{(l)}) + \sum_{i=1}^{q} \frac{\partial P}{\partial p_i}(p_{(k)}) h_{(j)_i} < 0 \,. \tag{4.15}$$

$\diamond$

Example 4.1.
Consider again the feedback system depicted in Fig. 3.1, described by the state space equations (3.13)–(3.14), and define R_u and θR_u according to (3.15) and (3.16).

We can associate two different robustness measures with system (3.13): ρ_Q, that is the QSM defined in Example 3.1, and the Multivariable Gain Margin (MGM) defined as follows (see [56])

$$\rho_G := \sup \left\{ \theta > 0 \,,\, \text{matrix}\, A(p)\, \text{is Hurwitz}\, \forall p \in \theta R_u \right\} \,. \tag{4.16}$$

As discussed in the previous chapter, the QSM represents an estimate of the supremal allowable amplitude of time-varying parameters which does not destabilize the system. In the same way the MGM represents the supremal allowable amplitude of time-invariant parameters for which the system is exponentially stable.

Clearly, $\rho_Q \leq \rho_G$; in general this inequality is strict.

In this example, we consider the situation in which $p \in \theta R_u$ and

$$\rho_Q < \theta < \rho_G \,. \tag{4.17}$$

We have the following observations:

i) Exponential stability is not guaranteed for arbitrary time-varying $p(\cdot)$ because $\theta > \rho_Q$;
ii) exponential stability is guaranteed for time-invariant p because $\theta < \rho_G$;
iii) for a given θ, exponential stability is guaranteed for time-varying $p(\cdot)$ which are sufficiently slowly varying in time.

The solution of the following problem provides an estimate of the stability margin defined as the supremal allowable amplitude of the rate of variation of parameters which guarantees an exponentially stable system [9].

Problem 4.3.
Given θ satisfying (4.17) find

$$\sup \,\{\sigma > 0\,,\, \text{system (3.13) is MQS in}\, \theta R_u \times \sigma R_u\}\ . \tag{4.18}$$

◇

To solve Problem 4.3 let us consider a matrix valued function $P(\cdot)$ in the form (4.11), define $M(p)$ according to (4.14) and then introduce a vector of fictitious parameters $\delta \in \mathbb{R}^q$ to obtain a multi-affine matrix valued function $\hat{M}(p, \delta)$ according to Procedure 3.1 (see also Remark 4.4). Then Problem 4.3 can be solved via the following GEVP.

Problem 4.4.

$$\begin{gathered}\max \sigma \\ \text{s.t.} \\ \sigma > 0 \\ \hat{M}(p_{(k)}, \delta_{(l)}) + \sum_{i=1}^{q} \frac{\partial P}{\partial p_i}(p_{(k)}) h_{(j)_i} < 0\,, \quad k, j, l = 1, \ldots, 2^q\,,\end{gathered}$$

where $h_{(j)_i}$ is $+\sigma$ or $-\sigma$ according to the vertex j we are considering and the component i of that vertex (see Fig. 4.1 in the two parameter case). ◇

Now consider again the system with matrices given in (3.20). In this case $0.583 < \rho_Q < 0.584$ (see Example 3.1); moreover simple computations show that $\rho_G = 1$. Problem 4.4 has been solved for some values of θ between 0.6 and 1. The results are presented in Table 4.1. △

When $A(p)$ does not satisfy Assumption 3.1, one should preliminary analyze the structure of $M(p)$ in (4.14) to decide, according to Table 3.1, which kind of approach to follow to establish if the system is MQS. In this case it makes also sense to consider for $P(\cdot)$ more complex structures than the multi-affine one, which leads to the more general definition of *polynomial quadratic stability* (see the next section).

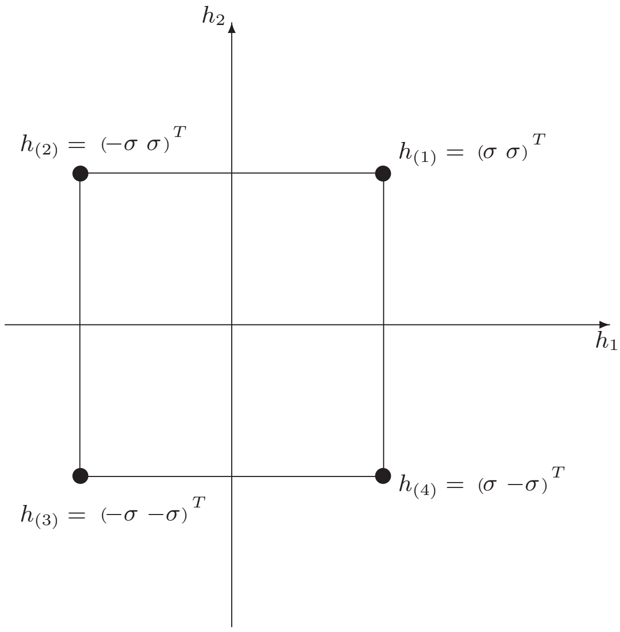


Fig. 4.1. Vertices of the set $\dot{R} = \sigma R_u$ in the two parameter case

Table 4.1. Estimate of the maximum σ for some values of θ

θ	0.6	0.7	0.8	0.9
σ_{max}	2.48	1.45	0.79	0.46

4.3 Polynomial Quadratic Stability

In this section we do not make any particular assumption about the structure of $A(\cdot)$ and look for parameter dependent Lyapunov functions having a polynomial dependence on parameters, that is

$$P(p) = \sum_{\alpha_1,\ldots,\alpha_q} P_{\alpha_1\ldots\alpha_q} p_1^{\alpha_1} \cdots p_q^{\alpha_q} , \tag{4.19}$$

where $\alpha_i = 0, 1, \ldots, \upsilon_i$, $\upsilon_i \in \mathbb{N}$, $i = 1, \ldots, q$.

Definition 4.3 (Polynomial Quadratic Stability). System (3.1) is said to be *polynomially quadratically stable in* $R \times \dot{R}$ if and only if it is QS via parameter dependent Lyapunov functions in $R \times \dot{R}$ with $P(\cdot)$ any symmetric matrix-valued function in the form (4.19). ◇

Again, in this case it is necessary a preliminary analysis of $M(p)$ defined in (4.14) to choose the approach to follow to check feasibility of the following problem.

Problem 4.5.
Find symmetric matrices $P_{\alpha_1,\dots,\alpha_q}$ such that, defined $P(p)$ according to (4.19), condition (4.4) is satisfied. ◇

Example 4.2 (PIO Analysis, cont'd).
Let us consider again the PIO analysis problem defined in Example 3.2.

Assuming an arbitrarily time behavior for the gain $g(t)$ is conservative whenever the nonlinearity input is slowly varying during the saturation regime. To this end remember that, in our example, the saturation output is a pitch rate command with saturation threshold at $15\,deg/sec$; therefore it makes sense to consider a pitch acceleration of a few degrees per square second.

We have that, when the saturation is active,

$$\begin{aligned} |\dot{g}(t)| &= \frac{v_T|\dot{v}(t)|}{v^2(t)} \\ &\leq \frac{|\dot{v}(t)|}{v_T} \,. \end{aligned} \tag{4.20}$$

Therefore, assuming that the amplitude of the derivative of the input to the nonlinear element is bounded by $\dot{v}_M$, we immediately obtain a bound on $|\dot{g}(t)|$.

For a given k_p, we have used a parameter dependent Lyapunov function in the form $x^T P(g) x$ with

$$P(g) = P_0 + P_1 g + P_2 g^2 + P_3 g^3 \,, \tag{4.21}$$

to estimate, with g_{min} and $\dot{v}_M$ given and referring to the system in Fig. 3.7, the region in the (k_p, g) plane which guarantees Hurwitz stability versus the time-invariant parameter k_p and exponential stability with respect to the time-varying parameter $g(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, [g_{min}, 1])$ with

$$|\dot{g}(t)| \leq \frac{\dot{v}_M}{v_T} =: \dot{g}_M \,. \tag{4.22}$$

To this end we propose the following algorithm, which is similar to Algorithm 3.1.

Algorithm 4.1.

Step 1
Let $\Delta k = 0.05$, $k_{min} = 0$, $k_{max} = 0 + \Delta k$, $\Delta g = 0.05$, $g_{min} = 0.05$;
Step 2
Solve the following feasibility problem

Problem 4.6.
Find symmetric matrices P_i, $i = 0, \dots, 3$, such that, defined $P(g)$ according to (4.21), for all $g \in [g_{min}, 1]$

$$\big(A_0 - bc^T(k_p)g\big)^T P(g) + P(g)\big(A_0 - bc^T(k_p)g\big) + \frac{dP(g)}{dg}h < 0 \qquad (4.23)$$
$$h \in \{-\dot{g}_M, \dot{g}_M\}\,, \quad k_p \in \{k_{min}, k_{max}\}\,;$$

Step 3
If Problem 4.6 is not feasible **then** plot the box $[g_{min}, 1] \times [0, k_{min}]$ and **let** $g_{min} = g_{min} + \Delta g$, **else let** $k_{min} = k_{max}$, $k_{max} = k_{min} + \Delta k$ **end**
If $g_{min} < 1$ **then goto** Step 2; **else stop**.

◇

To find a feasible solution to Problem 4.6 we have used an algorithm which follows, with the obvious changes, the same lines of Algorithm 3.4; the test in Step 3 of Algorithm 3.4 is performed by a dense sampling of the interval $[g_{min}, 1]$.

In Fig. 4.2 we have depicted the stability region computed via Algorithm 4.1 for $\dot{v}_M = 0$, which is equivalent to consider a time-invariant g. The curve depicted in the same figure represents the boundary of the (almost) exact Hurwitz stability region of the closed loop system in Fig. 3.7 computed, by considering both g and k_p as static parameters, with the aid of the software ROBAN developed at the Italian Aerospace Research Center (CIRA) [13, 21].

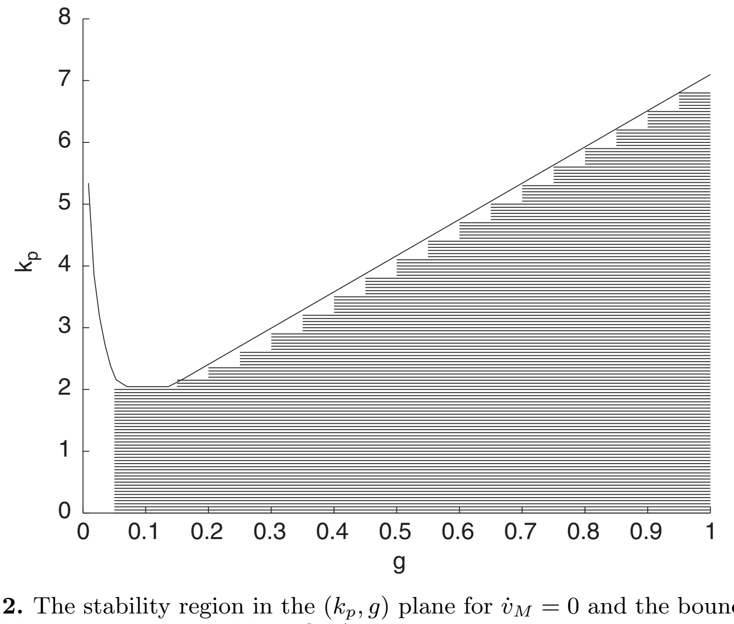


Fig. 4.2. The stability region in the (k_p, g) plane for $\dot{v}_M = 0$ and the boundary of the stability region computed by ROBAN

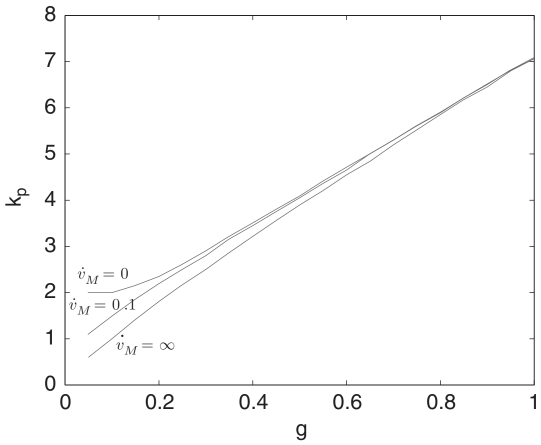


Fig. 4.3. The stability regions in the (k_p, g) plane for $\dot{v}_M = 0, 0.1, \infty$

ROBAN uses a procedure similar to Procedure 3.1 to cover the image of the characteristic polynomial coefficients by a polytope; then Hurwitzness of this polytope[1] is tested by the approach proposed in [43, 44]; the procedure uses a parameter set splitting algorithm which allows to determine the stability region in the parameter space up to the desired resolution.

Note that the boundary of the region computed by Algorithm 4.1 and the ROBAN bound are practically coincident; this shows that the parameter dependent Lyapunov function chosen to study the stability of the closed loop system in Fig. 4.2 works very well in our case, and it is expected that also the results obtained for $\dot{v}_M > 0$ are reliable.

In Fig. 4.3 the boundary of the stability regions for $\dot{v}_M = 0, 0.1, \infty$ are plotted. Obviously, the boundary for $\dot{v}_M = \infty$ is coincident with the boundary of the quadratic stability region depicted in Fig. 3.8.

Note that, to deal with the time-invariant parameter k_p, instead of using Algorithm 4.1, one could have been used a parameter dependent Lyapunov function depending on both k_p and g (see Exercise 4.1). △

4.4 A More General Class of Parameter Dependent Lyapunov Functions

The approach described in the last section may result overlay conservative in the following sense. Note that condition

$$M(p) < 0 \tag{4.24}$$

[1] A family of polynomials is said to be Hurwitz if any member of the family is Hurwitz.

is necessary for (4.13); this follows from the obvious fact that system (3.1) has to be exponentially stable for constant parameters ($\dot{p} = 0$).

Obviously, even when $A(p)$ is Hurwitz for all $p \in R$, there is no guarantee that a polynomial matrix-valued function $P(\cdot)$ in the form (4.19) and satisfying (4.24) exists. Therefore, in some cases, the polynomial quadratic stability approach may have no advantage over the classical quadratic stability approach illustrated in Chap. 3.

Since Hurwitz stability is necessary for quadratic stability via parameter dependent Lyapunov functions, without loss of generality let us assume that system (3.1) is exponentially stable for constant parameters or, equivalently, that matrix $A(p)$ is Hurwitz in the hyper-box R.

In this case we can construct a family of positive definite matrices $P(\cdot)$, such that the matrix function in (4.24) be negative definite, just observing that for all positive definite matrix valued function $S(p)$ and for all $p \in R$ the Lyapunov equation

$$A^T(p)P(p) + P(p)A(p) = -S(p) \tag{4.25}$$

univocally defines a positive definite matrix valued function $P(\cdot)$ which, by construction, *at least satisfies condition* (4.13) for constant parameters. In short, the idea is that of *optimizing over the set of the $S(\cdot)$'s to obtain suitable $P(\cdot)$'s*. We will further discuss this point later, after the proof of Theorem 4.2.

Given $F(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n\times m}$, $p \mapsto F(p)$, we define the Vetter derivative of F with respect to p as follows [41]

$$\frac{dF}{dp} := \begin{pmatrix} \frac{dF}{dp_1} & \frac{dF}{dp_2} & \cdots & \frac{dF}{dp_q} \end{pmatrix}^T .$$

We have the following theorem; for the operator *Lyap* refer to Appendix A.3.

Theorem 4.2 ([6]). *Assume there exists a symmetric, continuously differentiable matrix valued function $S(\cdot) : R \to \mathbb{R}^{n\times n}$ such that for all $p \in R$ and for $j = 1, \ldots, 2^q$*

$$\mathrm{Lyap}(A^T(p), A(p), S(p)) > 0 \tag{4.26}$$

and

$$-S(p) + X(p)\big(h_{(j)} \otimes I_n\big) < 0 \ , \tag{4.27}$$

where

$$\begin{aligned} X(p) = \mathrm{Lyap}\Bigg(& A^T(p), I_q \otimes A(p), \left(\frac{dS}{dp}(p)\right)^T \\ & + \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes \mathrm{Lyap}\big(A^T(p), A(p), S(p)\big)\big) \\ & + \mathrm{Lyap}\big(A^T(p), A(p), S(p)\big) \left(\frac{dA^T}{dp}(p)\right)^T \Bigg) \ ; \end{aligned} \tag{4.28}$$

then system (3.1) *is QS via parameter dependent Lyapunov functions in* $R \times \dot{R}$.

Proof. First of all note that condition (4.26) guarantees that equation (4.25) admits, for all $p \in R$, a unique positive definite solution

$$P(p) = \text{Lyap}(A^T(p), A(p), S(p)) \,. \tag{4.29}$$

Consider the derivative with respect to p^T of both members in (4.25). Using the results in [41] we obtain

$$\frac{dA^T}{dp^T}(p)\big(I_q \otimes P(p)\big) + A^T(p)\frac{dP}{dp^T}(p) + \frac{dP}{dp^T}(p)\big(I_q \otimes A(p)\big) \\ + P(p)\frac{dA}{dp^T}(p) = -\frac{dS}{dp^T}(p) \,, \tag{4.30}$$

which can be rewritten

$$A^T(p)\left(\frac{dP}{dp}(p)\right)^T + \left(\frac{dP}{dp}(p)\right)^T \big(I_q \otimes A(p)\big) \\ = -\left[\left(\frac{dS}{dp}(p)\right)^T + \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes P(p)\big) + P(p)\left(\frac{dA^T}{dp}(p)\right)^T\right] . \tag{4.31}$$

Now note that (4.31) is a generalized Lyapunov equation (see Appendix A.3) which admits a unique solution because the matrix $I_q \otimes A(p)$ has the same eigenvalues of $A(p)$ repeated q times; indeed, since $A(p)$ is such that (4.25) admits a unique solution, the same holds for equation (4.31). Therefore $\frac{dP}{dp}(\cdot)$ is defined for all $p \in R$ as follows

$$\left(\frac{dP}{dp}(p)\right)^T = \text{Lyap}\left(A^T(p), I_q \otimes A(p), \left(\frac{dS}{dp}(p)\right)^T + \right. \\ \left. \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes P(p)\big) + P(p)\left(\frac{dA^T}{dp}(p)\right)^T\right) ; \tag{4.32}$$

moreover $P(\cdot)$ results to be continuously differentiable.

Therefore condition (4.27) can be rewritten

$$A^T(p)P(p) + P(p)A(p) + \left(\frac{dP}{dp}(p)\right)^T \big(h_{(j)} \otimes I_n\big) < 0 \,; \tag{4.33}$$

from the last inequality the proof follows. □

Remark 4.6. Note that Hurwitzness of $A(p)$ for all $p \in R$ (which represents the starting point of our discussion) is implicitly guaranteed from the hypothesis of Theorem 4.2. Indeed (4.27) implies that $S(\cdot)$ is positive definite over R; then (4.26) guarantees that $A(p)$ is Hurwitz for all p in R. ◇

In order to optimize over the set of the $S(\cdot)$'s we need to fix a structure for $S(p)$; as usual we focus on matrix functions $S(\cdot)$ which depend polynomially on p, that is

$$S(p) = \sum_{\alpha_1,\ldots,\alpha_q} S_{\alpha_1,\ldots,\alpha_q} p_1^{\alpha_1} \cdots p_q^{\alpha_q} \,, \tag{4.34}$$

where $\alpha_i = 1, \ldots, \upsilon_i$, $\upsilon_i \in \mathbb{N}$, $i = 1, \ldots, q$.

We propose the following algorithm to find an optimal $S(\cdot)$.

Algorithm 4.2.

Step 1
Define the set

$$\mathcal{P} = \left\{ p^{(i)}, i = 1, \ldots, N \,,\, p^{(i)} \in R \right\} .$$

Step 2
Solve the following feasibility problem, subject to a *finite* number of constraints.

Problem 4.7.
Find symmetric matrices $S_{\alpha_1,\ldots,\alpha_q} \in \mathbb{R}^{n\times n}$ such that, defined $S(p)$ according to (4.34), for all $p \in \mathcal{P}$ and $j = 1, \ldots, 2^q$,

$$\mathrm{Lyap}(A^T(p), A(p), S(p)) > 0 \tag{4.35a}$$

$$-S(p) + X(p)\big(h_{(j)} \otimes I_n\big) < 0 \,, \tag{4.35b}$$

where $X(\cdot)$ has been defined in (4.28). ◇

Step 3
If Problem 4.7 is unfeasible **then stop**; a matrix function $S(\cdot)$ with the structure (4.34) and satisfying Theorem 4.2 does not exist.
If Problem 4.7 is feasible, given a feasible solution $S^*(\cdot)$, test that (4.26) and (4.27) are satisfied for $S(\cdot) = S^*(\cdot)$; in this case **stop**; system (3.1) is QS via parameter dependent Lyapunov functions in $R \times \dot{R}$. **Else goto** Step 4.
Step 4
Define the set

$$\mathcal{P}' = \left\{ p^{(i)}, i = 1, \ldots, N' \,,\, p^{(i)} \in R \right\} , \qquad N' > N \,,$$

such that $\mathcal{P}' \supset \mathcal{P}$, let $N = N'$, $\mathcal{P} = \mathcal{P}'$ and **goto** Step 2.

◇

Now we shall show that the constraints (4.35) define a set of LMIs. We need two auxiliary lemmas.

Lemma 4.1. *Assume that, for a given* p*, the Lyapunov equation* (4.25) *admits a unique solution. Under this assumption define the mapping that to any* $S = S^T \in \mathbb{R}^{n\times n}$ *associates the unique solution of* (4.25)*; then such mapping is linear.*

Proof. Straightforward. □

Lemma 4.2. *Assume that, for a given* p*, the Lyapunov equation* (4.25) *admits a unique solution; then the mapping*

$$\begin{aligned}&\mathcal{L}_{der} : \mathbb{R}^{n\times n} \times \mathbb{R}^{nq\times n} \to \mathbb{R}^{n\times nq}\\ &(S = S^T, \dot S) \mapsto \mathrm{Lyap}\Big(A^T(p), I_q \otimes A(p), \dot S^T\\ &\qquad\qquad + \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes \mathrm{Lyap}\big(A^T(p), A(p), S\big)\big)\\ &\qquad\qquad + \mathrm{Lyap}\big(A^T(p), A(p), S\big)\left(\frac{dA^T}{dp}(p)\right)^T\Big)\end{aligned}$$

is linear.

Proof. We prove that

$$\mathcal{L}_{der}(S_{a1} + S_{a2}) = \mathcal{L}_{der}(S_{a1}) + \mathcal{L}_{der}(S_{a2})\,, \tag{4.36}$$

where

$$S_{a1} = (S_1, \dot S_1)\,, \quad S_{a2} = (S_2, \dot S_2)\,. \tag{4.37}$$

If we let

$$X_i = \mathcal{L}_{der}(S_{ai})\,, \quad i = 1, 2\,, \tag{4.38}$$

from the definition it follows that, for a given p and $i = 1, 2$,

$$\begin{aligned}A^T(p)X_i + X_i\big(I_q \otimes A(p)\big) = -\Big[&\dot S_i^T\\ &+ \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes \mathrm{Lyap}\big(A^T(p), A(p), S_i\big)\big)\\ &+\mathrm{Lyap}\big(A^T(p), A(p), S_i\big)\left(\frac{dA^T}{dp}(p)\right)^T\Big]\,.\end{aligned} \tag{4.39}$$

Summing (4.39) with $i = 1$ and (4.39) with $i = 2$ we obtain

$$A^T(p)(X_1 + X_2) + (X_1 + X_2)\big(I_q \otimes A(p)\big) = -\Big[(\dot{S}_1^T + \dot{S}_2^T) + \left(\frac{dA}{dp}(p)\right)^T \big[I_q \otimes \big(\mathrm{Lyap}\big(A^T(p), A(p), S_1\big) + \mathrm{Lyap}\big(A^T(p), A(p), S_2\big)\big)\big] + \big(\mathrm{Lyap}\big(A^T(p), A(p), S_1\big) + \mathrm{Lyap}\big(A^T(p), A(p), S_2\big)\big)\left(\frac{dA^T}{dp}(p)\right)^T\Big]. \tag{4.40}$$

Now, using Lemma 4.1, equation (4.40) can be rewritten

$$A^T(p)(X_1 + X_2) + (X_1 + X_2)\big(I_q \otimes A(p)\big) = -\Big[(\dot{S}_1^T + \dot{S}_2^T) + \left(\frac{dA}{dp}(p)\right)^T \big(I_q \otimes \mathrm{Lyap}\big(A^T(p), A(p), (S_1 + S_2)\big)\big) + \mathrm{Lyap}\big(A^T(p), A(p), (S_1 + S_2)\big)\left(\frac{dA^T}{dp}(p)\right)^T\Big]. \tag{4.41}$$

From (4.41) we have

$$\begin{aligned} X_1 + X_2 &= \mathcal{L}_{der}(S_{a1}) + \mathcal{L}_{der}(S_{a2}) \\ &= \mathcal{L}_{der}(S_{a1} + S_{a2}). \end{aligned} \tag{4.42}$$

The proof that $\mathcal{L}_{der}(\alpha S_a) = \alpha \mathcal{L}_{der}(S_a)$, with α being any real number, follows the same guidelines. □

Directly from Lemmas 4.1 and 4.2 the next result follows.

Theorem 4.3 ([6]). *The constraints in* (4.35) *define a set of LMIs in the variables* $S_{\alpha_1,\ldots,\alpha_q}$.

Similar comments to those ones of Algorithm 3.4 apply to Algorithm 4.2. In particular the check in Step 3 can be performed either via gridding or probabilistic methods.

The importance of Theorem 4.2 relies in the fact that it allows to evaluate, for a given p, the derivative of $P(p)$ (namely $X(p)$ in (4.27)) *without having explicit knowledge of the analytical expression of such derivative*, which results to be very cumbersome also for low order systems. Indeed, to obtain such expression, one should first compute $P(p)$ using the formula (see Appendix A.3)

$$\mathrm{vec}[P(p)] = -\big(A^T(p) \oplus A^T(p)\big)^{-1}\mathrm{vec}[S(p)], \tag{4.43}$$

(note that the matrix to be inverted at the right hand side in (4.43) is of order $n^2 \times n^2$), and then compute the derivative of such matrix; both these computations should be performed by symbolic calculus.

Summary

In this chapter we have considered the situation in which the parameters have a bounded rate of variation. To exploit this information it is necessary to resort to parameter dependent quadratic Lyapunov functions in the form $x^T P(p)x$.

A key point is the choice of a suitable structure for $P(\cdot)$. When $A(\cdot)$ satisfies Assumption 3.1 it makes sense to optimize over the set of the positive definite matrix-valued functions $P(\cdot)$ which depend multi-affinely on p. In this case the product $P(p)A(p)$, which appears in the expression of the Lyapunov function derivative, is multi-quadratic in p. Also, the number of quadratic functions equals q, the number of uncertain parameters; therefore, according to Table 3.1, if $q \leq \mu_{med}$, the polytopic covering algorithms provided in Chap. 3 can be used to reduce the Lyapunov derivative to a matrix function satisfying Assumption 3.1 and then to apply the vertices result of Sect. 3.1.

An alternative way to eliminate the quadratic terms from the Lyapunov derivative are the convexification methods proposed in [82] and [69].

When $A(\cdot)$ does not satisfy Assumption 3.1 there could be no particular advantage in considering a multi-affine structure for $P(\cdot)$. In this case a more general (polynomial) dependence has been considered; a preliminary analysis of the parameter structure of the Lyapunov derivative is necessary in order to choose the particular approach (polytopic covering, gridding or statistical methods). According to Table 3.1, if $\mu \leq \mu_{med}$, the polytopic covering method can be still suitable; in the other case if $q \leq (>) q_{small}$ the gridding (statistical) approach may represent a viable alternative. Note that in [101], it has been shown that, in the single parameter case ($q = 1$) and for the affine dependence on parameters (i.e. $A(p) = A_0 + A_1 p$), there exists a parameter dependent Lyapunov function $x^T P(p)x$ such that $A^T(p)P(p) + P(p)A(p) < 0$ *iff* there exists a polynomial parameter dependent Lyapunov function of degree m, $P(p) = \sum_{i=0}^{m} P_i p^i$, where m can be computed via a simple formula. Even if [101] does not deal with time-varying parameters, this nice result can be readily applied to the context of Sect. 4.3.

A different approach is proposed in Sect. 4.4. Rather than fixing an a priori structure for $P(\cdot)$, we optimize over the RHS of the parameter dependent Lyapunov equation (4.25). This approach leads to a class of parameter dependent matrix functions $P(\cdot)$ which at least guarantees the satisfaction of the parameter derivative free inequality

$$A^T(p)P(p) + P(p)A(p) < 0\,, \tag{4.44}$$

which, on the other hand, is necessary to guarantee negative definiteness of the Lyapunov derivative in presence of bounded rate parameters. It is interesting to note that, in the context of uncertain systems subject to static parameters, a similar approach has been recently proposed in [47], where, however, a different algorithm is proposed to find the optimal parameter dependent

$P(\cdot)$. Further papers dealing with parameter dependent Lyapunov functions and/or bounded rate uncertain parameters, under various assumptions about the parameter dependence of $A(p)$, are [50, 127, 131].

It is rather straightforward to extend the theory for bounded rate parameters to include some performances. First we have the following result concerning quadratic $\mathcal{D}$-stability via parameter dependent Lyapunov functions (see Exercise 4.3).

Theorem 4.4. *Assume that $A(p)$ satisfies Assumption 4.1 and that there exists a continuously differentiable symmetric matrix-valued function $P(\cdot) : p \in R \to \mathbb{R}^{n\times n}$ such that, for all $p \in R$ and for all $j = 1, \ldots, 2^q$,*

$$\Lambda \otimes P + \Theta \otimes \left[P(p)A(p) + \frac{1}{2}\sum_{i=1}^{q} \frac{\partial P}{\partial p_i} h_{(j)_i} \right] + \Theta^T \otimes \left[A^T(p)P(p) + \frac{1}{2}\sum_{i=1}^{q} \frac{\partial P}{\partial p_i} h_{(j)_i} \right] < 0 \,. \tag{4.45}$$

Then

i) The eigenvalues of $A(p)$ belong to $\mathcal{D}$ for all $p \in R$;
ii) for all $p(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, R)$ satisfying (4.1) and (4.2), defined $v(t,x) = x^T P(p(t))x$, we have that

$$\frac{1}{2}\frac{\dot{v}(t,x)}{v(t,x)} \in \mathcal{D} \bigcap \mathbb{R} \,. \tag{4.46}$$

□

Further, we can state the following result guaranteeing quadratic $\mathcal{L}_2$ performance (see Exercise 4.4).

Theorem 4.5 ([82]). *Assume that $A(p)$ satisfies Assumption 4.1 and that there exists a continuously differentiable symmetric matrix-valued function $P(\cdot) : p \in R \to \mathbb{R}^{n\times n}$ such that, for all $p \in R$ and for all $j = 1, \ldots, 2^q$,*

$$\begin{pmatrix} A^T(p)P(p) + P(p)A(p) + \sum_{i=1}^{q} \frac{\partial P}{\partial p_i} h_{(j)_i} & PB(p) & C^T(p) \\ B^T(p)P & -\gamma^2 I & D^T(p) \\ C(p) & D(p) & -I \end{pmatrix} < 0 \,. \tag{4.47}$$

Then for all vector valued functions $p(\cdot) \in \mathcal{C}_0(\mathbb{R}^+, R)$ satisfying (4.1) and (4.2)

i) System (3.95) is exponentially stable;
ii) $\|\Gamma_{zw}\big(p(\cdot)\big)\| < \gamma$.

□

For what concerns systems subject to norm bounded uncertainties (see (3.116)) the question is much more complicated, since it seems difficult to obtain LMIs conditions guaranteeing exponential stability when we try to take into account the uncertainty rate of variation. In [3] a sufficient condition for exponential stability of an uncertain system in the form (3.116) with $\Delta(t)$ satisfying for all t

$$\dot{\Delta}^T(t)\dot{\Delta}(t) \leq D\,, \tag{4.48}$$

D being a given positive definite matrix, is provided. This condition requires the feasibility of an LMIs based problem; however it may result extremely conservative due to the repetitive application of the $\mathcal{S}$-procedure required by the proof machinery and to the particular structure chosen for the Δ dependent Lyapunov function

$$v(x, \Delta) = x^T(P + N\Delta E)x\,, \tag{4.49}$$

where the matrix N is left free for optimization purposes. Another kind of structure for $v(x, \Delta)$ is proposed in [135], where, however, the analysis is performed for time-invariant uncertainties.

Extensive work in this context has been done by Haddad and Bernstein [90, 91, 93, 95]. The existence of a suitable Δ dependent Lyapunov function is linked to the solvability of a certain Riccati-type equation; an algorithm to solve such equation is proposed in [93]. On the same topic see also the works by Rantzer [102, 148], which generalize to the multi-parameter case the multiplier approach of [129].

Exercises

Exercise 4.1. Study the stability of the closed loop system in Fig. (3.7) with the aid of a parameter dependent Lyapunov function in the form $x^T P(k_p, g)x$ with different structures for $P(k_p, g)$, starting with the multi-affine one, that is

$$P(k_p, g) = P_{00} + P_{10}k_p + P_{01}g + P_{11}k_p g\,.$$

According to Example 4.2, assume that k_p is a constant parameter while g is a time-varying parameter with a bounded rate of variation. △

Exercise 4.2. Let us consider again the analysis of the control system of the Bus O305 described in Example 3.3. Assuming that $\dot{p}_1 = 0$ and that p_2 is arbitrarily time-varying, we found that the system was not robustly stable; on the other hand taking into account arbitrary fast variation of $p_2(t)$ is clearly conservative. Therefore assume that acceleration is bounded ($1m/sec^2$), that is

$$|\dot{p}_2(t)| \leq 1\,.$$

By using the approach of Sect. 4.4 with the simple choice $S(p) = S$ (that is S does not depend on parameters), show that the system is robustly stable. △

Exercise 4.3. Prove Theorem 4.4. △

Exercise 4.4. Prove Theorem 4.5. △

5. Controller Design

In this section we consider an uncertain system in the form

$$\dot{x}(t) = A(p)x(t) + B(p)u(t) \tag{5.1a}$$

$$y(t) = C(p)x(t)\,, \tag{5.1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, $p \in R \subset \mathbb{R}^q$ is the vector of uncertain parameters, R is the hyper-box defined in (3.2) and $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are continuous matrix functions of suitable dimensions. Note that, without loss of generality, we assume that the system is strictly proper ($D = 0$); indeed the techniques we shall present later can be easily modified (it amounts to a change of variable in the controller matrices) in order to capture the case in which there is a non-zero feedthrough matrix.

Note that systems in the form (5.1), subject to time-varying parameters, are also known in the control literature as *Linear Parameter Varying* (LPV) systems [23, 24, 34, 159, 160].

The objective of this chapter is to discuss the stabilization of system (5.1) under various assumptions on the parameters time behavior. Also, we shall consider some performance requirements, along the lines of Chap. 3. Finally, the last section of the chapter is devoted to systems subject to norm bounded uncertainties.

5.1 Quadratic Stabilization

5.1.1 Quadratic Stabilization via State Feedback

In this section we assume that the whole state of system (5.1) is available for feedback, that is

$$\dot{x}(t) = A(p)x(t) + B(p)u(t) \tag{5.2a}$$

$$y(t) = x(t)\,. \tag{5.2b}$$

The following definition introduces the concept of quadratic stabilization via state feedback.

Definition 5.1 (Quadratic stabilizability via state feedback). System (5.2) is said to be *quadratically stabilizable via linear state feedback control* if and only if there exists a matrix $G \in \mathbb{R}^{m \times n}$ such that the closed loop system, obtained from (5.2) by letting $u = Gx$,

$$\dot{x}(t) = \big(A(p) + B(p)G\big)x(t)\,,$$

is quadratically stable. ◇

It is possible to show that quadratic stabilizability via dynamic, time-varying state feedback linear control implies quadratic stabilizability via memoryless, time-invariant, state feedback linear control (see [140]). Hence we do not lose any generality in Definition 5.1 when considering time-invariant memoryless controllers. On the contrary, in [137] an example shows that a nonlinear state feedback controller can quadratically stabilize a linear system subject to parametric uncertainties which is not quadratically stabilizable via linear state feedback. Therefore quadratic stabilizability via state feedback without any other specification does not imply quadratic stabilizability via state feedback linear control, hence this specification in Definition 5.1 is mandatory.

From Lemma 3.1 it follows that the uncertain system (5.2) is quadratically stabilizable via linear control *iff* there exist a positive definite matrix Q and a matrix G such that

$$(A(p) + B(p)G)Q + Q(A(p) + B(p)G)^T < 0\,, \quad \forall p \in R\,. \tag{5.3}$$

As shown in [36, 86], letting

$$V = GQ\,, \tag{5.4}$$

we obtain the following result.

Theorem 5.1. *System* (5.2) *is quadratically stabilizable via linear state feedback control* if and only if *there exist a positive definite matrix Q and a matrix V such that, for all $p \in R$,*

$$A(p)Q + QA^T(p) + B(p)V + V^T B^T(p) < 0\,. \tag{5.5}$$

In this case a linear state feedback controller which quadratically stabilizes system (5.2) *is given by $u = Gx$ with $G = VQ^{-1}$.* □

In order to transform inequality (5.5) into a finite number of LMIs let us assume the following.

Assumption 5.1. The matrices of system (5.2) can be written

$$\big(A(p)\ B(p)\big) = \frac{N_{Sx}(p)}{d_{Sx}(p)}\,, \tag{5.6}$$

where $N_{Sx}(p)$ is a multi-affine matrix-valued function and $d_{Sx}(p)$ is a multi-affine function with $d_{Sx}(p) \neq 0$ for all $p \in R$. ◇

If Assumption 5.1 holds, quadratic stabilizability via linear state feedback control is equivalent to the feasibility of the following LMIs optimization problem.

Problem 5.1.
Find a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a matrix $V \in \mathbb{R}^{m\times n}$ such that

$$Q > 0 \tag{5.7a}$$

$$A(p_{(i)})Q + QA^T(p_{(i)}) + B(p_{(i)})V + V^T B^T(p_{(i)}) < 0\,, \qquad i = 1, \ldots, 2^q\,. \tag{5.7b}$$

◇

When the dependence on parameters of the system matrices in equation (5.2) does not satisfy Assumption 5.1, constraint (5.5) is no longer reducible to a finite number of LMIs as in (5.7b) of Problem 5.1. In this case (see also the next example) we have to resort to the Polytopic Covering technique described in Sect. 3.2.2.

Example 5.1 (Automatic steering of a bus, cont'd).
Let us reconsider Example 3.3. We have found that, with the given controller, the system is not robustly stable versus the time-invariant parameter $p_1 \in [9950, 32000]$ and the time-varying parameter $p_2 \in [3, 20]$ (remember that p_1 is the mass and p_2 the bus speed).

In the design phase, it is difficult to take into account the time invariance of p_1; therefore our goal in this example will be that of designing a state feedback controller (assuming that the system state is available) which quadratically stabilizes the closed loop system with respect to both parameters.

Since $A(p)$ does not satisfy Assumption 3.1, we replace the functions p_2^2 and p_2^3 via a pair of multi-affine functions, by the introduction of the fictitious parameters δ_1 and δ_2, and arrive to the state equation (3.67a). Then Problem 5.1 is solved in correspondence of the vertices of the hyperbox $[9950, 32000] \times [3, 20] \times [0, 1]^2$. Additional constraints have been imposed to limit the amplitude of the control input (see [38], p. 103) and of the state variables during the transient (see [4]). The problem is feasible; a solution is

$$G = -\begin{pmatrix} 8.3018 & 5.5491 & 20.9591 & 0.6286 & 5.7631 \end{pmatrix}\,. \tag{5.8}$$

Therefore we can conclude that system (3.64a) is quadratically stabilizable via linear state feedback control. △

5.1.2 Quadratic Stabilization via Output Feedback

In this section we consider the general case in which the full state is not available for feedback. The next definition generalizes Definition 5.1 to the output feedback case.

In order to obtain operative results, the controller is allowed to depend on the parameters; this implies that the parameter vector $p(\cdot)$, although *a priori* uncertain, has to be measurable on-line. For the same reason, full order controllers of order greater than or equal to the plant order are considered. For the sake of simplicity and without loss of generality, we initially focus on controllers of the same order as the plant.

A theory can be developed for parameter independent controllers; however the obtained conditions cannot be converted into tractable optimization problems. As we shall see in Sect. 5.4.1, this is not true for the norm bounded uncertainties case.

Definition 5.2 (Quadratic stabilizability via output feedback). System (5.1) is said to be *quadratically stabilizable via parameter dependent output feedback linear control* if and only if there exists a dynamical controller in the form

$$\dot{x}_c(t) = A_K(p)x_c(t) + B_K(p)y(t) \tag{5.9a}$$
$$u(t) = C_K(p)x_c(t) + D_K(p)y(t)\,, \tag{5.9b}$$

where $x_c(t) \in \mathbb{R}^n$, and $A_K(\cdot)$, $B_K(\cdot)$, $C_K(\cdot)$, $D_K(\cdot)$ are continuous matrix-valued functions, such that the closed loop system obtained by the connection of system (5.1) and controller (5.9) is quadratically stable. $\diamond$

In this chapter we shall present two different approaches for controller design. The first approach considers controllers in the general form (5.9); the second approach looks for controllers in state feedback/state observer form and shows how quadratic stabilization can be obtained via a sort of Separation Property (see [22], Ch. 8). The interesting point is that the two approaches are shown to be equivalent, that is there is no loss of generality in assuming that the controller has a state feedback/state observer structure. To this regard, the last result of the section extends the classical Youla parameterization (see [22], Ch. 8) to the quadratic stabilizability context. We shall see that the class of all quadratically stabilizing controllers is obtained by the lower Linear Fractional Transformation (LFT) between a quadratically stabilizing controller in state feedback/state observer form and any quadratically stable system.

To prove the main result of this section, let us consider the following technical lemma.

Lemma 5.1 ([34]). *Given symmetric matrices $S \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$, the following statements are equivalent.*

i) There exist symmetric matrices $T \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n}$, and nonsingular matrices $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ such that

$$P := \begin{pmatrix} S & M \\ M^T & T \end{pmatrix} > 0\,, \qquad P^{-1} = \begin{pmatrix} Q & N \\ N^T & Z \end{pmatrix}\,. \tag{5.10}$$

ii)

$$\begin{pmatrix} Q & I \\ I & S \end{pmatrix} > 0\,. \tag{5.11}$$

□

Proof. From i) we have

$$\begin{pmatrix} S & M \\ M^T & T \end{pmatrix} \begin{pmatrix} Q & N \\ N^T & Z \end{pmatrix} = I\,, \tag{5.12}$$

and therefore

$$MN^T = I - SQ \tag{5.13a}$$
$$SN + MZ = 0 \tag{5.13b}$$
$$M^T Q + TN^T = 0\,. \tag{5.13c}$$

From (5.13) we have that

$$P\Pi_1 = \Pi_2\,, \tag{5.14}$$

with

$$\Pi_1 = \begin{pmatrix} Q & I \\ N^T & 0 \end{pmatrix}, \qquad \Pi_2 = \begin{pmatrix} I & S \\ 0 & M^T \end{pmatrix}. \tag{5.15}$$

Since N is nonsingular Π_1 is nonsingular and

$$\begin{aligned} 0 &< \Pi_1^T P \Pi_1 \\ &= \Pi_1^T \Pi_2 = \begin{pmatrix} Q & I \\ I & S \end{pmatrix}. \end{aligned} \tag{5.16}$$

Conversely assume ii) holds. Then $I - SQ$ is nonsingular and we can find nonsingular matrices M and N such that (5.13a) holds. Moreover define

$$T = -M^T Q N^{-T} \tag{5.17a}$$
$$Z = -M^{-1} S N\,. \tag{5.17b}$$

Note that T and Z are symmetric; indeed (for T)

$$\begin{aligned} T^T &= -N^{-1} Q M \\ &= -N^{-1} Q (I - SQ) N^{-T} \\ &= -N^{-1} (I - QS) Q N^{-T} \\ &= T\,. \end{aligned} \tag{5.18}$$

Now define P and the nonsingular matrix Π_1 according to (5.10) and (5.15) respectively. We have

$$0 < \begin{pmatrix} Q & I \\ I & S \end{pmatrix} = \Pi_1^T P \Pi_1 \,, \tag{5.19}$$

therefore $P > 0$; finally, it is simple to verify that P^{-1} equals the second of (5.10). □

Example 5.2.
We illustrate the application of Lemma 5.1 for what concerns the construction of P starting from symmetric matrices S and Q satisfying (5.11). Define

$$Q = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \,, \quad S = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \,. \tag{5.20}$$

Let $N = I_2$ and

$$M = I_2 - SQ = \begin{pmatrix} -15 & -8 \\ -7 & -10 \end{pmatrix} \,; \tag{5.21}$$

note that M is nonsingular. Now define T and Z according to (5.17); we have

$$T = \begin{pmatrix} 82 & 50 \\ 50 & 58 \end{pmatrix} \,, \quad Z = \begin{pmatrix} 0.2340 & -0.0638 \\ -0.0638 & 0.2447 \end{pmatrix} \,. \tag{5.22}$$

Finally define P and P^{-1} as in (5.10); we have $PP^{-1} = I_4$, as expected. △

Now we have that System (5.1) is quadratically stabilizable via parameter dependent output feedback linear control *iff* there exists a controller in the form (5.9) and a positive definite matrix $P \in \mathbb{R}^{2n \times 2n}$ such that

$$A_{CL}^T(p) P + P A_{CL}(p) < 0 \,, \quad \forall p \in R \,, \tag{5.23}$$

where

$$A_{CL}(p) = \begin{pmatrix} A(p) + B(p) D_K(p) C(p) & B(p) C_K(p) \\ B_K(p) C(p) & A_K(p) \end{pmatrix} \,. \tag{5.24}$$

Following [48], let us partition P and its inverse according to (5.10) and define Π_1 and Π_2 as in (5.15). Note that there is no loss of generality in assuming that M (and therefore N) is nonsingular. If this was not true, it is always possible to slightly perturb M in order to satisfy the nonsingularity requirement and (5.23) (see [48]).

Since N is nonsingular, Π_1 is nonsingular; pre and post-multiplying (5.23) by Π_1^T and Π_1 and recalling (5.14) we obtain

$$\begin{pmatrix} A(p)Q + QA^T(p) + B(p)\hat{C}_K(p) + \hat{C}_K^T(p)B^T(p) & A(p) + \hat{A}_K^T(p) + B(p)D_K(p)C(p) \\ A^T(p) + \hat{A}_K(p) + C^T(p)D_K^T(p)B^T(p) & SA(p) + A^T(p)S + \hat{B}_K(p)C(p) + C^T(p)\hat{B}_K^T(p) \end{pmatrix} < 0\,, \quad \forall p \in R\,, \tag{5.25}$$

where

$$\hat{B}_K(p) = MB_K(p) + SB(p)D_K(p) \tag{5.26a}$$

$$\hat{C}_K(p) = C_K(p)N^T + D_K(p)C(p)Q \tag{5.26b}$$

$$\begin{aligned}\hat{A}_K(p) = {} & MA_K(p)N^T + SB(p)C_K(p)N^T \\ & + MB_K(p)C(p)Q + S\big(A(p) + B(p)D_K(p)C(p)\big)Q\,. \end{aligned}\tag{5.26c}$$

The change of variable in (5.26), proposed in [80] and [48], is the key step to linearize (5.23) *wrt* the involved matrix-function variables.

From the above discussion and Lemma 5.1 it follows that quadratic stabilizability of system (5.1) is equivalent to the existence of positive definite matrices S and Q, and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that (5.25) and (5.11) hold.

Given Q, S, $\hat{A}_K(p)$, $\hat{B}_K(p)$, $\hat{C}_K(p)$ and $D_K(p)$ satisfying (5.25) and (5.11), by inversion of (5.26) we can recover the controller matrices. Indeed, let M and N be nonsingular matrices satisfying $MN^T = I - SQ$; we have

$$B_K(p) = M^{-1}\left(\hat{B}_K(p) - SB(p)D_K(p)\right) \tag{5.27a}$$

$$C_K(p) = \left(\hat{C}_K(p) - D_K(p)C(p)Q\right)N^{-T} \tag{5.27b}$$

$$\begin{aligned}A_K(p) = {} & M^{-1}\Big(\hat{A}_K(p) - SB(p)C_K(p)N^T \\ & -MB_K(p)C(p)Q - S\big(A(p) + B(p)D_K(p)C(p)\big)Q\Big)N^{-T}\,. \end{aligned}\tag{5.27c}$$

Note that (5.27a) and (5.27b) must be solved before (5.27c).

It is simple to recognize that, concerning the abstract condition for quadratic stabilizability, we can get rid of (5.11); however the satisfaction of condition (5.11) is necessary in order to reconstruct the controller.

Indeed assume there exist positive definite matrices Q and S and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ satisfying (5.25); then, if Q and S does not satisfy (5.11), define $Q_\lambda = \lambda Q$ and $S_\lambda = \lambda S$, with $\lambda > 1$ such that (5.11) is satisfied by Q_λ and S_λ; then define $\hat{B}_{K\lambda}(p) = \lambda\hat{B}_K(p)$ and $\hat{C}_{K\lambda}(p) = \lambda\hat{C}_K(p)$. It readily follows that, since $\lambda > 1$, for all $p \in R$,

$$\begin{pmatrix} A(p)Q_\lambda + Q_\lambda A^T(p) + B(p)\hat{C}_{K\lambda}(p) + \hat{C}_{K\lambda}^T(p)B^T(p) & A(p) + \hat{A}_K^T(p) + B(p)D_K(p)C(p) \\ A^T(p)+\hat{A}_K(p)+C^T(p)D_K^T(p)B^T(p) & S_\lambda A(p)+A^T(p)S_\lambda+\hat{B}_{K\lambda}(p)C(p) + C^T(p)\hat{B}_{K\lambda}^T(p) \end{pmatrix} < 0\,. \tag{5.28}$$

Therefore we can state the following theorem.

Theorem 5.2. *System* (5.1) *is quadratically stabilizable via parameter dependent output feedback linear control* if and only if *there exist positive definite matrices* S *and* Q*, and continuous matrix-valued functions* $\hat{A}_K(p)$*,* $\hat{B}_K(p)$*,* $\hat{C}_K(p)$ *and* $D_K(p)$ *such that* (5.25) *holds. In this case:*

i) If (5.11) *holds, defined* M *and* N *to be any nonsingular matrices such that* $MN^T = I - SQ$*, a quadratically stabilizing controller is given by* (5.9) *where* $A_K(p)$*,* $B_K(p)$*,* $C_K(p)$ *are obtained by* (5.27)*;*
ii) if (5.11) *does not hold, rescale* S *and* Q *by a factor* $\lambda > 1$ *such that* (5.11) *is satisfied, rescale by the same factor* $\hat{B}_K(p)$ *and* $\hat{C}_K(p)$ *and repeat the procedure at point i) to evaluate the controller matrices.*

□

The next theorem shows that the quadratic stabilizability condition of Theorem 5.2 can be further simplified.

Theorem 5.3. *System* (5.1) *is quadratically stabilizable via parameter dependent output feedback linear control* if and only if *there exist positive definite matrices* Q *and* S *and continuous matrix-valued functions* $\hat{C}_K(\cdot)$ *and* $\hat{B}_K(\cdot)$ *such that, for all* $p \in R$*,*

$$A(p)Q + QA^T(p) + B(p)\hat{C}_K(p) + \hat{C}_K^T(p)B^T(p) < 0 \tag{5.29a}$$

$$SA(p) + A^T(p)S + \hat{B}_K(p)C(p) + C^T(p)\hat{B}_K^T(p) < 0\,. \tag{5.29b}$$

In this case:

i) If (5.11) *holds, defined* M *and* N *to be any nonsingular matrices such that* $MN^T = I - SQ$*,* $\hat{A}_K(p) = -A^T(p)$*,* $D_K(p) = 0$*, a quadratically stabilizing controller is given by* (5.9) *where* $A_K(p)$*,* $B_K(p)$*,* $C_K(p)$ *are obtained by* (5.27)*;*
ii) if (5.11) *does not hold, rescale* S *and* Q *by a factor* $\lambda > 1$ *such that* (5.11) *is satisfied, rescale by the same factor* $\hat{B}_K(p)$ *and* $\hat{C}_K(p)$ *and repeat the procedure at point i) to evaluate the controller matrices.*

Proof. Condition (5.25) implies (5.29). Conversely, if (5.29) holds, the quadruple $\hat{A}_K(\cdot) = -A^T(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $D_K = 0$ satisfies (5.25). □

Concerning the practical computation of a quadratically stabilizing controller, it can be performed if the following assumptions hold (we refer to the statement of Theorem 5.3).

Assumption 5.2.

i) $\hat{B}_K(p)$ and $\hat{C}_K(p)$ are restricted to be constant matrices;
ii) matrix-valued functions $A(\cdot)$ and $B(\cdot)$ satisfy Assumption 5.1;

iii) matrix-valued functions $A(\cdot)$ and $C(\cdot)$ satisfy

$$\big(A(p)\ C(p)\big) = \frac{N_{Sy}(p)}{d_{Sy}(p)}\,, \tag{5.30}$$

where $N_{Sy}(p)$ is a multi-affine matrix-valued function and $d_{Sy}(p)$ is a multi-affine function with $d_{Sy}(p) \neq 0$ for all $p \in R$.

◇

If Assumption 5.2 holds, we have that system (5.1) is quadratically stabilizable if the following LMIs problem is feasible.

Problem 5.2.
Find symmetric matrices Q, S, and matrices $\hat{B}_K$, $\hat{C}_K$ such that

$$Q > 0 \tag{5.31a}$$

$$S > 0 \tag{5.31b}$$

$$A(p_{(i)})Q + QA^T(p_{(i)}) + B(p_{(i)})\hat{C}_K + \hat{C}_K^T B^T(p_{(i)}) < 0\,, \quad i = \ldots, 2^q \tag{5.31c}$$

$$SA(p_{(i)}) + A^T(p_{(i)})S + \hat{B}_K C(p_{(i)}) + C^T(p_{(i)})\hat{B}_K^T < 0\,, \quad i = 1, \ldots, 2^q \tag{5.31d}$$

$$\begin{pmatrix} Q & I \\ I & S \end{pmatrix} > 0\,. \tag{5.31e}$$

◇

If Problem 5.2 is feasible, a quadratically stabilizing controller can be computed according to the following procedure:

i) Let $\hat{A}_K(p) = -A^T(p)$ and $D_K(p) = 0$;
ii) Compute nonsingular matrices M and N such that $MN^T = I - SQ$;
iii) Evaluate the controller matrices according to (5.27).

Example 5.3 (Automatic steering of a bus, cont'd).
Let us consider again the bus control example and remove the assumption done in Example 5.1 about the availability of the system state.

With the aid of Theorem 5.3 we design a quadratically stabilizing output feedback controller. First we solve the feasibility Problem 5.2 and find the matrices

$$S = 10^3 \begin{pmatrix} 0.0077 & 0.0019 & -0.0043 & -0.0229 & 0.0110 \\ 0.0019 & 0.0433 & -0.0049 & -0.0243 & -0.0013 \\ -0.0043 & -0.0049 & 0.0051 & -0.0042 & -0.0097 \\ -0.0229 & -0.0243 & -0.0042 & 1.1945 & -0.0632 \\ 0.0110 & -0.0013 & -0.0097 & -0.0632 & 0.0405 \end{pmatrix} \tag{5.32a}$$

$$Q = \begin{pmatrix} 98.8870 & 7.5472 & 13.2077 & -0.3784 & -34.4381 \\ 7.5472 & 42.8816 & -1.8198 & -0.4720 & -40.2997 \\ 13.2077 & -1.8198 & 54.7276 & -5.0165 & -64.5527 \\ -0.3784 & -0.4720 & -5.0165 & 1.6515 & -2.0111 \\ -34.4381 & -40.2997 & -64.5527 & -2.0111 & 584.6124 \end{pmatrix} \tag{5.32b}$$

$$\hat{B}_K = \begin{pmatrix} -29.2007 & -1.8144 & -11.9009 & -8.6387 & 15.7290 \end{pmatrix}^T \tag{5.32c}$$

$$\hat{C}_K = \begin{pmatrix} -32.8245 & -127.0175 & 16.0563 & 136.2669 & -74.5389 \end{pmatrix} . \tag{5.32d}$$

Setting $N = 100 I_5$, $M = (I - SQ)N^{-T}$ and using (5.32), we obtain the controller matrices

$$D_K = 0 \tag{5.33a}$$

$$B_K = M^{-1}\hat{B}_K \tag{5.33b}$$

$$C_K = \hat{C}_K N^{-T} \tag{5.33c}$$

$$A_K(p) = -M^{-1}\left(A^T(p) + SbC_K N^T + MB_K c^T Q + SA(p)Q\right) N^{-T} . \tag{5.33d}$$

In Figs. 5.1 and 5.2 the time behaviors of the control input $u(t)$ and of the output $y(t)$ respectively are reported. The simulation is performed with respect to an initial output displacement of 0.15 $[m]$, to a constant parameter $p_1 = 32000$ and a time-varying parameter p_2 switching between 3 and 20 $[m/sec]$ according to Fig. 5.1. Note that a discontinuous parameter variation between the minimum and maximum attainable values can be considered, according to [146], the worst time-varying parameter realization that can affect the system. △

Note that (5.29a) has the form of a state feedback design condition (see (5.5)); inequality (5.29b) is the dual condition which typically arises in the observer matrix gain design context. While (5.29a) is a quadratic stabilizability condition, inequality (5.29b) can be interpreted as a *quadratic detectability* condition [119, 120]. In other words (5.29) open the doors to the extension of the Separation Property to the quadratic stabilization context. In the sequel of this section we shall investigate this point.

In what follows we shall use the following compact notation to denote the state space realization of controller (5.9)

$$\left(\begin{array}{c|c} A_K(p) & B_K(p) \\ \hline C_K(p) & D_K(p) \end{array}\right) . \tag{5.34}$$

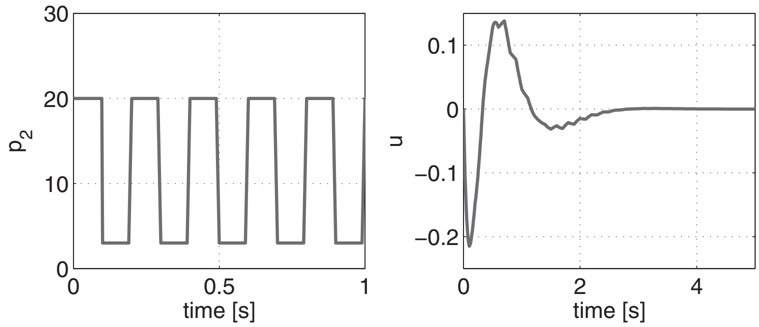


Fig. 5.1. Time behavior of the mass p_2 and of the control input u

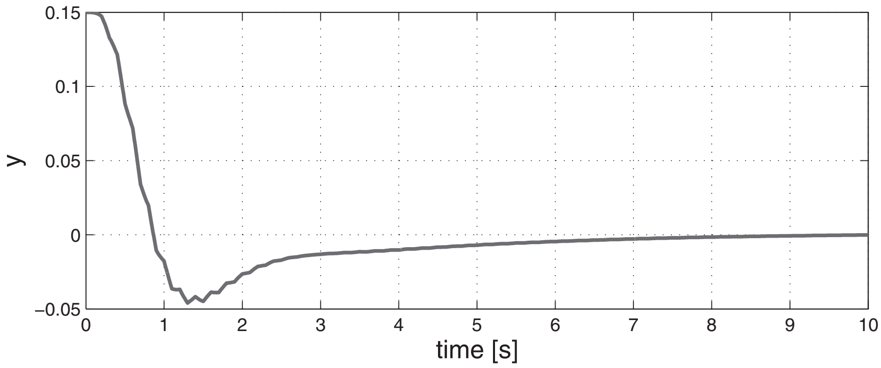


Fig. 5.2. Time behavior of the output $y(t)$

Let us consider output feedback controllers which are in the state feedback/state observer form, that is

$$\dot{x}_c(t) = A(p)x_c(t) + B(p)u(t) + L(p)\left(C(p)x_c(t) - y(t)\right) \tag{5.35a}$$

$$u(t) = G(p)x_c(t)\,, \tag{5.35b}$$

where $L(\cdot)$ and $G(\cdot)$ are continuous matrix-valued functions left free for optimization purposes.

Note that, according to notation (5.34), the state space realization of system (5.35) can be denoted as

$$\left(\begin{array}{c|c} A(p) + B(p)G(p) + L(p)C(p) & -L(p) \\ \hline G(p) & 0 \end{array}\right). \tag{5.36}$$

Now we shall prove that system (5.1) is quadratically stabilizable via an output feedback linear controller in the form (5.35) *iff* the hypothesis of Theorem 5.3 hold; the proof is obtained by following arguments based on the Separation Property. This approach has been first exploited in [35] (for a complete proof of the results of [35] the reader is referred to [120]; see also [34]).

We need some auxiliary results.

Lemma 5.2 ([120]). *Quadratic stability of system* (3.1) *is invariant under a state space nonsingular transformation of the form* $\hat{x} = T^{-1}x$.

Proof. See Exercise 5.1. □

A corollary of Lemma 5.2 is the following result.

Corollary 5.1 ([120]). *Consider the uncertain system* (3.1) *with*

$$A(p) = \begin{pmatrix} A_{11}(p) & A_{12}(p) \\ 0 & A_{22}(p) \end{pmatrix} ; \tag{5.37}$$

then system (3.1), (5.37) *is QS* if and only if *the following systems are both QS*

$$\dot{v}(t) = A_{11}(p)v(t) \tag{5.38a}$$

$$\dot{w}(t) = A_{22}(p)w(t) \,. \tag{5.38b}$$

Proof. See Exercise 5.2. □

Now we have that the connection between (5.1) and (5.35) yields the following system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} = \begin{pmatrix} A(p) & B(p)G(p) \\ -L(p)C(p) & A(p) + B(p)G(p) + L(p)C(p) \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} . \tag{5.39}$$

Let us consider the following state transformation

$$\begin{aligned} \begin{pmatrix} x \\ x - x_c \end{pmatrix} &= \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} \\ &=: T^{-1} \begin{pmatrix} x \\ x_c \end{pmatrix} . \end{aligned} \tag{5.40}$$

By virtue of Lemma 5.2, system (5.39) is QS *iff* the following system is QS

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{x}(t) - \dot{x}_c(t) \end{pmatrix} &= T^{-1} \begin{pmatrix} A(p) & B(p)G(p) \\ -L(p)C(p) & A(p) + B(p)G(p) + L(p)C(p) \end{pmatrix} T \begin{pmatrix} x(t) \\ x(t) - x_c(t) \end{pmatrix} \\ &= \begin{pmatrix} A(p) + B(p)G(p) & -B(p)G(p) \\ 0 & A(p) + L(p)C(p) \end{pmatrix} \begin{pmatrix} x(t) \\ x(t) - x_c(t) \end{pmatrix} . \end{aligned} \tag{5.41}$$

In turn, by virtue of Corollary 5.1, system (5.41) is QS *iff* the systems

$$\dot{v}(t) = (A(p) + B(p)G(p))v(t) \tag{5.42a}$$
$$\dot{w}(t) = (A(p) + L(p)C(p))w(t) \tag{5.42b}$$

are both QS.

From Lemma 3.1 and Definition 3.1 it follows that systems (5.42a) and (5.42b) are QS *iff* there exist positive definite matrices Q and S such that, for all $p \in R$,

$$(A(p) + B(p)G(p))\,Q + Q\,(A(p) + B(p)G(p))^T < 0 \tag{5.43a}$$
$$(A(p) + L(p)C(p))^T\,S + S\,(A(p) + L(p)C(p)) < 0\,. \tag{5.43b}$$

By letting $G(p)Q = \hat{C}_K(p)$ and $SL(p) = \hat{B}_K(p)$ we reobtain (5.29).

Note that, when the Separation Property based approach is followed, it is not necessary to satisfy (5.11) for the controller design.

The following theorem provides a parameterization of all controllers which quadratically stabilizes a given uncertain plant. Note that the parameterization provides all full order controllers of order greater than or equal to n and that the 'central' controller is observer based. This shows, as said before, that there is no loss of generality in looking for controllers of order n. The proof of the theorem can be found in [120] and makes use of the concepts of quadratic stabilizability and detectability.

Theorem 5.4 (Parameterization of all quadratically stabilizing controllers). *Given system* (5.1) *and* $G(\cdot)$, $L(\cdot)$ *satisfying* (5.43)*, the set of all quadratically stabilizing controllers in the form* (5.9) *is given by the lower LFT (see [185], Chap. 9) between the system (see Fig. 5.3)*

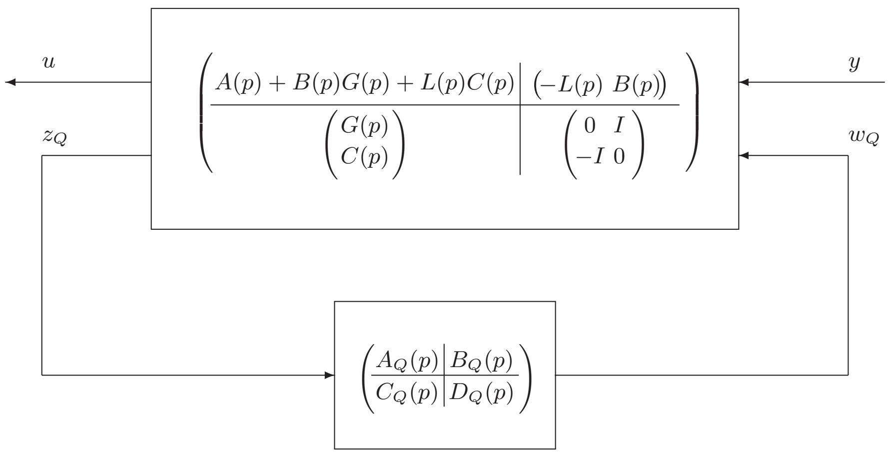


Fig. 5.3. Parameterization of all quadratically stabilizing controllers

$$\left(\begin{array}{c|c} A(p)+B(p)G(p)+L(p)C(p) & \begin{pmatrix} -L(p) & B(p) \end{pmatrix} \\ \hline \begin{pmatrix} G(p) \\ C(p) \end{pmatrix} & \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \end{array}\right), \tag{5.44}$$

and any parameter dependent system

$$\left(\begin{array}{c|c} A_Q(p) & B_Q(p) \\ \hline C_Q(p) & D_Q(p) \end{array}\right), \tag{5.45}$$

with $A_Q(\cdot)$, $B_Q(\cdot)$, $C_Q(\cdot)$ *and* $D_Q(\cdot)$ *continuous matrix-valued functions on the set* R *and* $A_Q(\cdot)$ *such that the system*

$$\dot{x}_Q(t) = A_Q(p)x_Q(t) \tag{5.46}$$

is QS. □

5.2 Quadratic Stabilization with Performances

5.2.1 Quadratic $\mathcal{D}$-Stabilization

We start with quadratic $\mathcal{D}$-stabilization via state feedback. Remember that the region $\mathcal{D}$ has been defined in (3.79).

Definition 5.3 (Quadratic $\mathcal{D}$-stabilizability via state feedback). System (5.2) is said to be *quadratically $\mathcal{D}$-stabilizable via linear state feedback control* if and only if there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the closed loop system, obtained from (5.2) by letting $u = Gx$, is quadratically $\mathcal{D}$-stable. ◇

From (3.87) and Definition 5.3 it follows that system (5.2) is quadratically $\mathcal{D}$-stabilizable via linear state feedback *iff* there exist a positive definite matrix Q and a matrix G such that, for all $p \in R$,

$$\Lambda \otimes Q + \Theta \otimes \left[\big(A(p)+B(p)G\big)Q\right] + \Theta^T \otimes \left[Q\big(A(p)+B(p)G\big)^T\right] < 0\,. \tag{5.47}$$

By letting $GQ = V$ we can state the following result.

Theorem 5.5. *System* (5.2) *is quadratically $\mathcal{D}$-stabilizable via state feedback linear control* if and only if *there exist a positive definite matrix Q and a matrix V such that, for all $p \in R$,*

$$\Lambda \otimes Q + \Theta \otimes (A(p)Q + B(p)V) + \Theta^T \otimes \big(QA^T(p) + V^T B^T(p)\big) < 0\,; \tag{5.48}$$

in this case a state feedback controller which quadratically stabilizes system (5.2) *is given by $u = Gx$ with $G = VQ^{-1}$.* □

If Assumption 5.1 holds, quadratic $\mathcal{D}$-stabilizability is equivalent to the feasibility of the following LMIs based problem.

Problem 5.3.
Find a symmetric matrix Q and a matrix V such that

$$Q > 0 \tag{5.49a}$$

$$\Lambda \otimes Q + \Theta \otimes \big(A(p_{(i)})Q + B(p_{(i)})V\big) + \Theta^T \otimes \Big(QA^T(p_{(i)}) + V^T B^T(p_{(i)})\Big) < 0 \quad i = 1, \ldots, 2^q . \tag{5.49b}$$

◇

Now we consider quadratic $\mathcal{D}$-stabilization via output feedback.

Definition 5.4 (Quadratic $\mathcal{D}$-stabilizability via output feedback). System (5.1) is said to be *quadratically $\mathcal{D}$-stabilizable via parameter dependent output feedback linear control* if and only if there exists a dynamical compensator in the form (5.9) such that the closed loop system obtained by the connection of system (5.1) and controller (5.9) is quadratically $\mathcal{D}$-stable. ◇

In order to prove the main result of the section we need the following result (see Exercise 5.3); consider a block partitioned symmetric matrix in the form

$$X := \begin{pmatrix} X_{11} & \cdots & X_{1i} & \cdots & X_{1j} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1i}^T & \cdots & X_{ii} & \cdots & X_{ij} & \cdots & X_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1j}^T & \cdots & X_{ij}^T & \cdots & X_{jj} & \cdots & X_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1n}^T & \cdots & X_{in}^T & \cdots & X_{jn}^T & \cdots & X_{nn} \end{pmatrix} ; \tag{5.50}$$

now let us denote by X^{ij} the symmetric matrix obtained by permuting the block row i with the block row j and the block column i with the block column j, that is

$$X^{ij} = \begin{pmatrix} X_{11} & \cdots & X_{1j} & \cdots & X_{1i} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1j}^T & \cdots & X_{jj} & \cdots & X_{ij}^T & \cdots & X_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1i}^T & \cdots & X_{ij} & \cdots & X_{ii} & \cdots & X_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{1n}^T & \cdots & X_{jn}^T & \cdots & X_{in}^T & \cdots & X_{nn} \end{pmatrix} . \tag{5.51}$$

Lemma 5.3. *Matrix X is positive (negative) definite* if and only if *matrix X^{ij} is positive (negative) definite.* □

Theorem 5.6. *The following statements are equivalent.*

i) *System* (5.1) *is quadratically $\mathcal{D}$-stabilizable via output feedback linear control.*

ii) *There exist positive definite matrices Q and S and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that, for all $p \in R$,*

$$\begin{pmatrix} \Lambda \otimes Q & \Lambda \otimes I \\ \Lambda \otimes I & \Lambda \otimes S \end{pmatrix} + \begin{pmatrix} \Theta \otimes \big(A(p)Q + B(p)\hat{C}_K(p)\big) & \Theta \otimes \big(A(p) + B(p)D_K(p)C(p)\big) \\ \Theta \otimes \hat{A}_K(p) & \Theta \otimes \big(SA(p) + \hat{B}_K(p)C(p)\big) \end{pmatrix} + \begin{pmatrix} \Theta^T \otimes \big(QA^T(p) + \hat{C}_K^T(p)B^T(p)\big) & \Theta^T \otimes \hat{A}_K^T(p) \\ \Theta^T \otimes \big(A^T(p) + C^T(p)D_K^T(p)B^T(p)\big) & \Theta^T \otimes \big(A^T(p)S + C^T(p)\hat{B}_K^T(p)\big) \end{pmatrix} < 0 \,. \tag{5.52}$$

iii) *There exist positive definite matrices Q and S and continuous matrix-valued functions $\hat{C}_K(\cdot)$ and $\hat{B}_K(\cdot)$ such that, for all $p \in R$,*

$$\Lambda \otimes Q + \Theta \otimes \big(A(p)Q + B(p)\hat{C}_K(p)\big) + \Theta^T \otimes \big(QA^T(p) + \hat{C}_K^T(p)B^T(p)\big) < 0 \tag{5.53a}$$

$$\Lambda \otimes S + \Theta \otimes \big(SA(p) + \hat{B}_K(p)C(p)\big) + \Theta^T \otimes \big(A^T(p)S + C^T(p)\hat{B}_K^T(p)\big) < 0 \,. \tag{5.53b}$$

Proof. According to Definition 3.4, system (5.1) is quadratically D-stabilizable via output feedback linear control *iff* there exists a positive definite matrix P such that, for all $p \in R$,

$$\Lambda \otimes P + \Theta \otimes \big(PA_{CL}(p)\big) + \Theta^T \otimes \big(A_{CL}^T(p)P\big) < 0 \,, \tag{5.54}$$

where $A_{CL}(p)$ has been defined in (5.24).

Now let us partition P and its inverse according to (5.10); note that, without loss of generality, we can assume that M and N are nonsingular matrices [48]. Pre- and post-multiply (5.54) by $I \otimes \Pi_1^T$ and $I \otimes \Pi_1$ respectively, where Π_1 has been defined in (5.15), and use properties (A.8); by virtue of Lemma 5.1 we obtain that (5.54) is equivalent to the existence of positive definite matrices S and Q and and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ satisfying (5.11) and, for all $p \in R$,

$$\Lambda \otimes \begin{pmatrix} Q & I \\ I & S \end{pmatrix}$$
$$+ \Theta \otimes \begin{pmatrix} A(p)Q + B(p)\hat{C}_K(p) & A(p) + B(p)D_K(p)C(p) \\ \hat{A}_K(p) & SA(p) + \hat{B}_K(p)C(p) \end{pmatrix}$$
$$+ \Theta^T \otimes \begin{pmatrix} QA^T(p) + \hat{C}_K^T(p)B^T(p) & \hat{A}_K^T(p) \\ A^T(p) + C^T(p)D_K^T(p)B^T(p) & A^T(p)S + C^T(p)\hat{B}_K^T(p) \end{pmatrix} < 0\,, \tag{5.55}$$

where we have used the change of variable (5.27).

The equivalence between (5.55) and ii) follows from the repetitive application of Lemma 5.3; moreover, by following the same arguments used in the proof of Theorem 5.2, it is readily seen that we can get rid of (5.11).

Now ii) clearly implies iii). Conversely, if iii) holds, given any pair of matrix-valued functions $\hat{A}_K(\cdot)$, $D_K(\cdot)$, either ii) holds or we can rescale Q, S, $\hat{C}_K(\cdot)$, $\hat{B}_K(\cdot)$ by a positive scalar $\lambda > 1$ such that the new matrices satisfy ii). □

By virtue of Theorem 5.6 we have two possible choices for controller synthesis. If we look to condition ii), the controller matrices can be designed according to Theorem 5.2.

Concerning condition iii) it is simple to recognize that it can be directly derived by an approach based on the Separation Property (see Exercise 5.4). Therefore, in that case, a quadratically $\mathcal{D}$-stabilizing controller can be obtained by letting $G(p) = \hat{C}_K(p)Q^{-1}$ and $L(p) = S^{-1}\hat{B}_K(p)$ in (5.35).

When Assumption 5.2 holds the feasibility of the following LMIs problem guarantees quadratic $\mathcal{D}$-stabilizability via output feedback; we refer to the design of controllers in the form (5.35).

Problem 5.4.
Find symmetric matrices Q, S and matrices V, W, such that

$$Q > 0 \tag{5.56a}$$

$$S > 0 \tag{5.56b}$$

$$\Lambda \otimes Q + \Theta \otimes \left(A(p_{(i)})Q + B(p_{(i)})V\right) + \Theta^T \otimes \left(QA^T(p_{(i)}) + V^T B^T(p_{(i)})\right) < 0 \quad i = 1, \ldots, 2^q \tag{5.56c}$$

$$\Lambda \otimes S + \Theta \otimes \left(SA(p_{(i)}) + WC(p_{(i)})\right) + \Theta^T \otimes \left(A^T(p_{(i)})S + C^T(p_{(i)})W^T\right) < 0 \quad i = 1, \ldots, 2^q\,. \tag{5.56d}$$

◇

If Problem 5.4 is feasible, a quadratically $\mathcal{D}$-stabilizing controller has the structure (5.35) with constant gain matrices $G = VQ^{-1}$ and $L = S^{-1}W$.

5.2.2 Quadratic $\mathcal{L}_2$ Performance Control

As usual we deal with the state feedback first; let us consider the following system

$$\dot{x}(t) = A(p)x(t) + B_1(p)w(t) + B_2(p)u(t) \tag{5.57a}$$

$$z(t) = C_1(p)x(t) + D_{11}(p)w(t) + D_{12}(p)u(t)\,, \tag{5.57b}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^v$, $u(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^s$.

Definition 5.5 (Quadratic $\mathcal{L}_2$ performance control via state feedback). Given $\gamma > 0$, the uncertain system (5.57) is said to be *stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via linear state feedback control* if and only if there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the closed loop system, obtained from (5.57) by letting $u = Gx$, possesses a quadratic $\mathcal{L}_2$ performance bound γ. ◇

From (3.104) we know that system (5.57) is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via linear state feedback control *iff* there exist a matrix G and a positive definite matrix Q such that for all $p \in R$

$$\begin{pmatrix} Q\big(A(p)+B_2(p)G\big)^T + \big(A(p)+B_2(p)G\big)Q & B_1(p) & Q\big(C_1(p)+D_{12}(p)G\big)^T \\ B_1^T(p) & -\gamma^2 I & D_{11}^T(p) \\ \big(C_1(p)+D_{12}(p)G\big)Q & D_{11}(p) & -I \end{pmatrix} < 0\,. \tag{5.58}$$

By using the usual machinery we can state the following result.

Theorem 5.7. *System* (5.57) *is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via linear state feedback control* if and only if *there exist a positive definite matrix Q and a matrix V such that*

$$\begin{pmatrix} A(p)Q + QA^T(p) + B_2(p)V + V^T B_2^T(p) & B_1(p) & QC_1^T(p) + V^T D_{12}^T(p) \\ B_1^T(p) & -\gamma^2 I & D_{11}^T(p) \\ C_1(p)Q + D_{12}(p)V & D_{11}(p) & -I \end{pmatrix} < 0\,. \tag{5.59}$$

In this case a state feedback controller which stabilizes system (5.2) *with a quadratic $\mathcal{L}_2$ performance bound γ is given by $u = Gx$ with $G = VQ^{-1}$.* □

Let us consider the following assumption.

Assumption 5.3. The matrix-valued functions $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C_1(\cdot)$, $D_{11}(\cdot)$ and $D_{12}(\cdot)$ are such that

$$\begin{pmatrix} A(p) & B_1(p) & B_2(p) \\ C_1(p) & D_{11}(p) & D_{12}(p) \end{pmatrix} = \frac{N_{Sw}(p)}{d_{Sw}(p)}\,, \tag{5.60}$$

where $N_{Sw}(p)$ is a multi-affine matrix-valued function and $d_{Sw}(p)$ is a multi-affine function with $d_{Sw}(p) \neq 0$ for all $p \in R$. ◇

When Assumption 5.3 holds stabilizability with a quadratic $\mathcal{L}_2$ performance bound γ via state feedback is equivalent to the feasibility of the following LMIs optimization problem.

Problem 5.5.
Find a symmetric matrix Q and a matrix V such that

$$Q > 0 \tag{5.61a}$$

$$\begin{pmatrix} A(p_{(i)})Q + QA^T(p_{(i)}) + B_2(p_{(i)})V + V^T B_2^T(p_{(i)}) & B_1(p_{(i)}) & QC_1^T(p_{(i)}) + V^T D_{12}^T(p_{(i)}) \\ B_1^T(p_{(i)}) & -\gamma^2 I & D_{11}^T(p_{(i)}) \\ C_1(p_{(i)})Q + D_{12}(p_{(i)})V & D_{11}(p_{(i)}) & -I \end{pmatrix} < 0$$

$$i = 1, \dots, 2^q . \tag{5.61b}$$

◇

Next, we proceed with dynamic output feedback. Let us consider the system

$$\dot{x}(t) = A(p)x(t) + B_1(p)w(t) + B_2(p)u(t) \tag{5.62a}$$
$$z(t) = C_1(p)x(t) + D_{11}(p)w(t) + D_{12}(p)u(t) \tag{5.62b}$$
$$y(t) = C_2(p)x(t) + D_{21}(p)w(t) . \tag{5.62c}$$

Definition 5.6 (Quadratic $\mathcal{L}_2$ performance control via output feedback). The uncertain system (5.62) is said to be *stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via parameter dependent output feedback linear control* if and only if there exists a dynamical compensator in the form (5.9) such that the closed loop system obtained by the connection of system (5.62) and controller (5.9) possesses a quadratic $\mathcal{L}_2$ performance bound γ. ◇

Theorem 5.8. *System* (5.62) *is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via parameter dependent output feedback linear control* if and only if *there exist positive definite matrices S and Q and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that* (5.11) *holds and (the dependence on p is omitted to simplify the notation)*

$$\begin{pmatrix} AQ+QA^T+B_2\hat{C}_K + \hat{C}_K^T B_2^T & A + \hat{A}_K^T + B_2 D_K C_2 & B_1 + B_2 D_K D_{21} & QC_1^T + \hat{C}_K^T D_{12}^T \\ A^T + \hat{A}_K + C_2^T D_K^T B_2^T & SA+A^T S+\hat{B}_K C_2+C_2^T \hat{B}_K^T & SB_1 + \hat{B}_K D_{21} & C_1^T + C_2^T D_K^T D_{12}^T \\ B_1^T + D_{21}^T D_K^T B_2^T & B_1^T S + D_{21}^T \hat{B}_K^T & -\gamma^2 I & D_{11}^T + D_{21}^T D_K^T D_{12}^T \\ C_1 Q + D_{12}\hat{C}_K & C_1 + D_{12} D_K C_2 & D_{11}+D_{12} D_K D_{21} & -I \end{pmatrix} < 0 \tag{5.63}$$

where

$$\hat{B}_K(p) = MB_K(p) + SB_2(p)D_K(p) \tag{5.64a}$$

$$\hat{C}_K(p) = C_K(p)N^T + D_K(p)C_2(p)Q \tag{5.64b}$$

$$\begin{aligned}\hat{A}_K(p) &= MA_K(p)N^T + SB_2(p)C_K(p)N^T \\ &\quad + MB_K(p)C_2(p)Q + S(A(p) + B_2(p)D_K(p)C_2(p))Q\,,\end{aligned} \tag{5.64c}$$

and M, N are any nonsingular matrices such that $MN^T = I - SQ$.

In this case a controller which quadratically stabilizes with an $\mathcal{L}_2$ performance bound γ system (5.62) *is given by* (5.9) *with $A_K(p)$, $B_K(p)$ and $C_K(p)$ provided by the inversion of system* (5.64). □

Proof. From (3.101) it follows that system (5.1) is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via output feedback linear control *iff* there exist a controller in the form (5.9) and a positive definite matrix P such that

$$\begin{pmatrix} A_{CL}^T(p)P + PA_{CL}(p) & PB_{CL}(p) & C_{CL}^T(p) \\ B_{CL}^T(p)P & -\gamma^2 I & D_{CL}^T(p) \\ C_{CL}(p) & D_{CL}(p) & -I \end{pmatrix} < 0\,, \tag{5.65}$$

where

$$A_{CL}(p) = \begin{pmatrix} A(p) + B_2(p)D_K(p)C_2(p) & B_2(p)C_K(p) \\ B_K(p)C_2(p) & A_K(p) \end{pmatrix} \tag{5.66a}$$

$$B_{CL}(p) = \begin{pmatrix} B_1(p) + B_2(p)D_K(p)D_{21}(p) \\ B_K(p)D_{21}(p) \end{pmatrix} \tag{5.66b}$$

$$C_{CL}(p) = \begin{pmatrix} C_1(p) + D_{12}(p)D_K(p)C_2(p) & D_{12}(p)C_K(p) \end{pmatrix} \tag{5.66c}$$

$$D_{CL}(p) = D_{11}(p) + D_{12}(p)D_K(p)D_{21}(p)\,. \tag{5.66d}$$

Let us partition P and P^{-1} according to (5.10); again, without loss of generality, we can assume that M and N are nonsingular matrices. Then, pre- and post-multiplying (5.65) by $\mathrm{diag}(\Pi_1^T, I, I)$ and $\mathrm{diag}(\Pi_1, I, I)$ respectively, where Π_1 and Π_2 have been defined in (5.15), recalling that $P\Pi_1 = \Pi_2$ and using the change of variable (5.27), we obtain from Lemma 5.1 that the existence of $P > 0$ and controller in the form (5.9) satisfying (5.65) is equivalent to the existence of $Q > 0$, $S > 0$ and continuous matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $D_K(\cdot)$ such that (5.63) and (5.11) hold. □

An alternative approach for the design of the output feedback controller is provided in [34] under simplifying assumptions on the system matrices (see [185], p. 270).

The practical computation of an output feedback controller can be performed if the following assumptions hold.

Assumption 5.4.

i) $\hat{A}_K(p)$, $\hat{B}_K(p)$, $\hat{C}_K(p)$ and $D_K(p)$ are restricted to be constant matrices (note that in any case, by inversion of the system (5.64), the controller matrices turn out to be parameter dependent).

ii) The system matrices in (5.62) are such that the LHS in (5.63) can be written as the ratio of a multi-affine matrix valued function and a multi-affine polynomial which is nonzero for all $p \in R$.

◇

If Assumption 5.4 holds, we have that system (5.62) is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ if the following LMIs based problem admits a feasible solution.

Problem 5.6.
Find positive definite matrices Q and S and matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ and D_K such that (5.63) and (5.11) hold at the vertices of the hyper-box R. ◇

5.2.3 Guaranteed Cost Control

We only focus on the state feedback case, since the output feedback would require the introduction of some concepts which are behind the scope of this book.

Therefore consider system (5.2). The objective is to find a state feedback controller $u = Gx$ such that the overall closed loop system possesses a quadratic guaranteed cost with associated cost matrix P *wrt* the index (3.105).

According to (3.110), this is equivalent to the existence of a matrix G such that, for $Q = P^{-1}$,

$$\begin{pmatrix} Q(A(p) + B(p)G)^T + (A(p) + B(p)G)Q & Q\Pi \\ \Pi Q & -\Pi \end{pmatrix} < 0\,. \tag{5.67}$$

If Assumption 5.1 holds we can solve the following EVP in the variables θ, Q and V in order to find a state feedback controller with minimum guaranteed cost.

Problem 5.7 (Guaranteed cost via state feedback).

$$\begin{gathered} \max \theta \\ \text{s.t.} \\ \theta > 0 \\ Q > \theta I \\ \begin{pmatrix} A(p_{(i)})Q + QA^T(p_{(i)}) + B(p_{(i)})V + V^T B^T(p_{(i)}) & Q\Pi \\ \Pi Q & -\Pi \end{pmatrix} < 0 \\ i = 1, \ldots, 2^q\,. \end{gathered}$$

◇

If Problem 5.7 is feasible, a state feedback controller which minimizes the guaranteed cost is obtained by letting $G = VQ^{-1}$.

5.3 Robust Stabilization in the Presence of Bounded Rate Parameters

We consider the state feedback case first. Note that, differently from quadratic stabilizability, for tractability reasons the controller gain is allowed to depend on parameters.

Definition 5.7. System (5.2) is said to be *quadratically stabilizable via parameter dependent Lyapunov functions in* $R \times \dot{R}$ *via linear state feedback control* if and only if there exists a continuous matrix-valued function $G(\cdot)$ such that the connection of the control law $u = G(p)x$ with system (5.2) is quadratically stable via parameter dependent Lyapunov functions in $R \times \dot{R}$. ◇

From (4.10) it follows that quadratic stabilizability via parameter dependent Lyapunov functions in the state feedback case is equivalent to the existence of a continuous matrix-valued function $G(\cdot)$ and a positive definite continuous matrix-valued function $Q(\cdot)$ such that for all $p \in R$

$$\big(A(p) + B(p)G(p)\big)Q(p) + Q(p)\big(A(p) + B(p)G(p)\big)^T - \sum_{i=1}^{q} \frac{\partial Q(p)}{\partial p_i} h_{(j)_i} < 0\,, \quad j = 1, \ldots, 2^q\,. \tag{5.68}$$

We can easily state the following theorem.

Theorem 5.9. *System* (5.2) *is quadratically stabilizable via parameter dependent Lyapunov functions in* $R \times \dot{R}$ *via linear state feedback control* if and only if *there exist a continuously differentiable matrix-valued function* $Q(\cdot)$ *and a continuous matrix-valued function* $V(\cdot)$ *such that*

$$A(p)Q(p) + Q(p)A^T(p) + B(p)V(p) + V^T(p)B^T(p) - \sum_{i=1}^{q} \frac{\partial Q(p)}{\partial p_i} h_{(j)_i} < 0\,, \quad j = 1, \ldots, 2^q\,. \tag{5.69}$$

In this case a state feedback controller which quadratically stabilizes system (5.2) *via parameter dependent Lyapunov functions is given by* $u = G(p)x$ *with* $G(p) = V(p)Q^{-1}(p)$. □

In order to reduce the statement of Theorem 5.9 to an LMIs feasibility problem we need:

i) To fix a structure for $Q(\cdot)$; for example multi-affine.
ii) To fix a structure for $V(\cdot)$; for example $V(\cdot)$ can be optimized over the set of constant matrices.

iii) To reduce the LHS of (5.69) to a matrix-valued function satisfying Assumption 3.1.

Now we consider quadratic stabilizability via output feedback.

Definition 5.8. System (5.1) is said to be quadratically stabilizable via parameter dependent Lyapunov functions in $R \times \dot{R}$ via parameter dependent output feedback linear control *if and only if* there exists a dynamical compensator in the form (5.9) such that the closed loop system obtained by the connection of system (5.1) and controller (5.9) is quadratically stable via parameter dependent Lyapunov functions in $R \times \dot{R}$. ◇

Note that Lemma 5.2 and Corollary 5.1 can be extended to the quadratic stability via parameter dependent Lyapunov functions context; then, generalizing the derivation (5.43)–(5.39), it is easy to obtain the following result.

Theorem 5.10. *System* (5.1) *is quadratically stabilizable via parameter dependent Lyapunov functions in* $R \times \dot{R}$ *via parameter dependent output feedback linear control* if *there exist positive definite continuously differentiable matrix-valued functions* $S(\cdot)$ *and* $Q(\cdot)$ *and continuous matrix-valued functions* $V(\cdot)$ *and* $W(\cdot)$ *such that, for all* $p \in R$ *and* $j = 1, \dots, 2^q$,

$$A(p)Q(p) + Q(p)A^T(p) + B(p)V(p) + V^T(p)B^T(p) - \sum_{i=1}^{q} \frac{\partial Q(p)}{\partial p_i} h_{(j)_i} < 0 \tag{5.70a}$$

$$A^T(p)S(p) + S(p)A(p) + W(p)C(p) + C^T(p)W^T(p) + \sum_{i=1}^{q} \frac{\partial S(p)}{\partial p_i} h_{(j)_i} < 0\,. \tag{5.70b}$$

In this case a controller which quadratically stabilizes via parameter dependent Lyapunov functions system (5.1) *has the structure* (5.35) *with* $G(p) = V(p)Q^{-1}(p)$ *and* $L(p) = S^{-1}(p)W(p)$.

The fact that conditions (5.70) are also necessary for quadratic stabilizability via parameter dependent Lyapunov functions via parameter dependent output feedback linear control can be proven by generalizing to the context of the current section the Youla parameterization result stated in Theorem 5.4.

In order to practically apply Theorem 5.10 we need:

i) To fix a structure for $Q(\cdot)$ and $S(\cdot)$ (for example multi-affine);
ii) to fix a structure for $V(\cdot)$ and $W(\cdot)$ (for the sake of computational complexity they can be optimized over the set of constant matrices);
iii) to reduce the LHSs of (5.70) to matrix-valued functions which can be written as the ratio of a multi-affine matrix valued function and a multi-affine polynomial.

5.4 Systems Depending on Norm Bounded Uncertainties

5.4.1 Quadratic Stabilization

Let us consider the following uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t)\,, \tag{5.71}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and

$$\begin{pmatrix} \Delta A & \Delta B \end{pmatrix} = F\Delta(I - H\Delta)^{-1} \begin{pmatrix} E_1 & E_2 \end{pmatrix}\,, \tag{5.72}$$

where $\|H\| < 1$ and $\Delta \in \mathbb{R}^{p\times q}$ is any uncertain matrix with $\|\Delta\| \leq 1$.

As usual, the uncertain system (5.71) can be obtained as the feedback loop connection between system

$$\dot{x}(t) = Ax(t) + Fw_\Delta(t) + Bu(t) \tag{5.73a}$$
$$z_\Delta(t) = E_1x(t) + Hw_\Delta(t) + E_2u(t) \tag{5.73b}$$

and $w_\Delta = \Delta z_\Delta$.

According to the previous definitions, system (5.71), (5.72) is quadratically stabilizable via state feedback control *iff* there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the closed loop system, obtained by (5.71) by letting $u = Gx$ is quadratically stable. Note that, as shown in [153], when dealing with norm bounded uncertainties, and differently from parametric uncertainties, quadratic stabilizability implies quadratic stabilizability via linear control (see also [186]); therefore the use of the adjective "linear" is not necessary.

From Fig. 5.4 it is clear that the closed loop is itself a system subject to norm bounded uncertainties described by the following equations

$$\dot{x}(t) = (A + BG)x(t) + Fw_\Delta(t) \tag{5.74a}$$
$$z_\Delta(t) = (E_1 + E_2G)x(t) + Hw_\Delta(t) \tag{5.74b}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,. \tag{5.74c}$$

From (3.128) we readily obtain that a necessary and sufficient condition for quadratic stabilizability via state feedback of system (5.71) is the existence of a positive definite matrix Q and a matrix G such that

$$\begin{pmatrix} Q(A+BG)^T + (A+BG)Q & F & Q(E_1+E_2G)^T \\ F^T & -I & H^T \\ (E_1+E_2G)Q & H & -I \end{pmatrix} < 0\,. \tag{5.75}$$

By the usual change of matrix variable $GQ = V$ we obtain that quadratic stabilizability of system (5.71) is equivalent to the feasibility of the following LMIs problem.

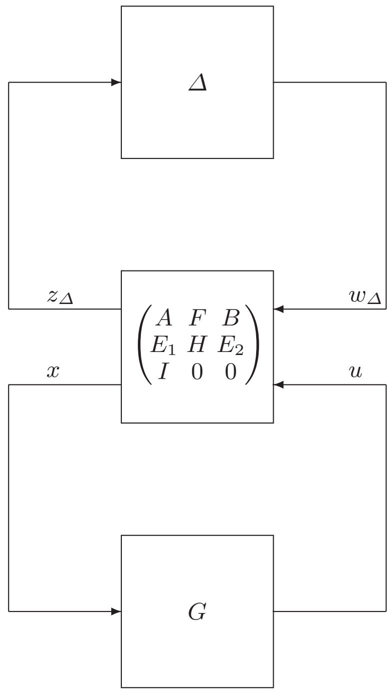


Fig. 5.4. The uncertain system (5.71) and the state feedback controller $u = Gx$

Problem 5.8.
Find a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a matrix $V \in \mathbb{R}^{m\times n}$ such that

$$Q > 0 \tag{5.76a}$$

$$\begin{pmatrix} QA^T + AQ + V^TB^T + BV & F & QE_1^T + V^TE_2^T \\ F^T & -I & H^T \\ E_1Q + E_2V & H & -I \end{pmatrix} < 0\,. \tag{5.76b}$$

◇

If Problem 5.8 is feasible a state feedback controller which quadratically stabilizes system (5.71)–(5.72) is given by $u = Gx$ with $G = VQ^{-1}$.

Next we move to quadratic stabilizability via output feedback. Let us consider the uncertain system

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) + \big(B + \Delta B\big)u(t) \tag{5.77a}$$

$$y(t) = \big(C + \Delta C\big)x(t) + \big(D + \Delta D\big)u(t)\,, \tag{5.77b}$$

where $y(t) \in \mathbb{R}^r$ and

$$\begin{pmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \Delta(I - H\Delta)^{-1} \begin{pmatrix} E_1 & E_2 \end{pmatrix}\,. \tag{5.78}$$

The feedback interpretation of system (5.77) is represented by the equations

$$\dot{x}(t) = Ax(t) + F_1 w_\Delta(t) + Bu(t) \tag{5.79a}$$
$$z_\Delta(t) = E_1 x(t) + H w_\Delta(t) + E_2 u(t) \tag{5.79b}$$
$$y(t) = Cx(t) + F_2 w_\Delta(t) + Du(t) \tag{5.79c}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,. \tag{5.79d}$$

In the following we assume, for the sake of simplicity but without loss of generality, that $D = 0$.

One possible approach would be that of following, along the lines of [120], the same machinery of the parametric uncertainties case described in Sect. 5.1.2.

This approach would lead to uncertainty dependent controllers designed according to the Separation Property. However, in the norm bounded uncertainties case, it is possible to follow another approach which, differently from the parametric case, allows to design *uncertainty independent* dynamical controllers.

This peculiarity is essentially due to the fact that quadratic stability in presence of norm bounded uncertainties is equivalent to an $\mathcal{H}_\infty$ condition for a suitable *certain* system (see Sect. 3.6).

Therefore let us consider a dynamical output feedback controller in the form

$$\dot{x}_c(t) = A_K x_c(t) + B_K y(t) \tag{5.80a}$$
$$u(t) = C_K x_c(t) + D_K y(t)\,. \tag{5.80b}$$

Definition 5.9. System (5.77) is said to be *quadratically stabilizable via output feedback linear control* if and only if there exists a dynamical controller in the form (5.80) such that the resulting closed loop system is QS. ◇

The closed loop connection between system (5.77)–(5.78) and controller (5.80) is depicted in Fig. 5.5.

Again, from this figure it is clear that the closed loop is itself an unforced system subject to norm bounded uncertainties and described by the equations

$$\dot{x}_a(t) = A_{CL} x_a(t) + F_{CL} w_\Delta(t) \tag{5.81a}$$
$$z_\Delta(t) = E_{CL} x_a(t) + H_{CL} w_\Delta(t) \tag{5.81b}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,, \tag{5.81c}$$

where $x_a = \begin{pmatrix} x^T & x_c^T \end{pmatrix}^T$ and

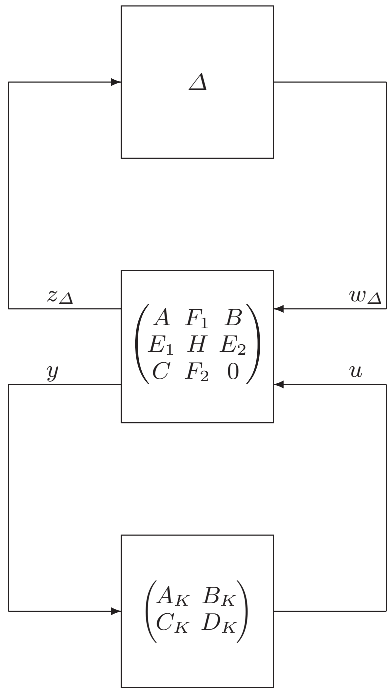


Fig. 5.5. The uncertain system (5.77) and the controller (5.80)

$$A_{CL} = \begin{pmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{pmatrix} \tag{5.82a}$$

$$F_{CL} = \begin{pmatrix} F_1 + BD_K F_2 \\ B_K F_2 \end{pmatrix} \tag{5.82b}$$

$$E_{CL} = \begin{pmatrix} E_1 + E_2 D_K C & E_2 C_K \end{pmatrix} \tag{5.82c}$$

$$H_{CL} = H + E_2 D_K F_2 \,. \tag{5.82d}$$

According to condition (3.127) the closed loop system is QS *iff* the following matrix inequality

$$\begin{pmatrix} A_{CL}^T P + PA_{CL} & PF_{CL} & E_{CL}^T \\ F_{CL}^T P & -I & H_{CL}^T \\ E_{CL} & H_{CL} & -I \end{pmatrix} < 0 \tag{5.83}$$

is satisfied.

By partitioning P and P^{-1} according to (5.10), where, without loss of generality we can assume that M and N are nonsingular matrices, and pre- and post-multiplying (5.83) by $\mathrm{diag}(\Pi_1^T, I, I)$ and $\mathrm{diag}(\Pi_1, I, I)$ respectively, where Π_1 and Π_2 have been defined in (5.15), recalling that $P\Pi_1 = \Pi_2$, we obtain

$$\begin{pmatrix} AQ + QA^T + B\hat{C}_K + \hat{C}_K^T B^T & A + \hat{A}_K^T + BD_K C & F_1 + BD_K F_2 & QE_1^T + \hat{C}_K^T E_2^T \\ A^T + \hat{A}_K + C^T D_K^T B^T & SA + A^T S + \hat{B}_K C + C^T \hat{B}_K^T & SF_1 + \hat{B}_K F_2 & E_1^T + C^T D_K^T E_2^T \\ F_1^T + F_2^T D_K^T B^T & F_1^T S + F_2^T \hat{B}_K^T & -I & H^T + F_2^T D_K^T E_2^T \\ E_1 Q + E_2 \hat{C}_K & E_1 + E_2 D_K C & H + E_2 D_K F_2 & -I \end{pmatrix} < 0\,, \tag{5.84}$$

where

$$\hat{B}_K = MB_K + SBD_K \tag{5.85a}$$

$$\hat{C}_K = C_K N^T + D_K CQ \tag{5.85b}$$

$$\hat{A}_K = MA_K N^T + SBC_K N^T + MB_K CQ + S(A + BD_K C)Q\,. \tag{5.85c}$$

Note the similarity between conditions (5.84) and (5.63); this is not surprising, in view of the equivalence between $\mathcal{H}_\infty$ control and quadratic stabilization in presence of norm bounded uncertainties (see Sect. 3.6). However the LHS of (5.84) does not depend on parameters.

We can state the following result.

Theorem 5.11 ([49]). *System* (5.77) *is quadratically stabilizable via output feedback linear control* if and only if *there exist positive definite matrices* S *and* Q *and matrices* $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ *and* D_K *such that* (5.84) *and* (5.11) *hold.* □

Therefore the procedure for designing a quadratically stabilizing controller consists of the following steps:

i) Find symmetric matrices S and Q and matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ and D_K satisfying the hypothesis of Theorem 5.11;
ii) Find nonsingular matrices M and N satisfying $MN^T = I - SQ$;
iii) Resolve system (5.85) for A_K, B_K and C_K.

5.4.2 Quadratic Stabilization and Performances

Quadratic $\mathcal{D}$-Stabilization

As usual, let us consider quadratic $\mathcal{D}$-stabilization via state feedback first. Obviously, system (5.71), (5.72) is quadratically $\mathcal{D}$-stabilizable via linear state feedback control *if and only if* there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the closed loop system (5.74) is quadratically $\mathcal{D}$-stable.

By using condition (3.165) it follows that system (5.71), (5.72) is quadratically $\mathcal{D}$-stabilizable via state feedback if there exist positive definite matrices $Q \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{k\times k}$ and a matrix G such that

$$\begin{pmatrix} M_{\mathcal{D},d}(A + BG, Q) & \Theta_1^T \otimes F & \left(\Theta_2^T Z\right) \otimes \left(Q(E_1 + E_2 G)^T\right) \\ \Theta_1 \otimes F^T & -Z \otimes I_p & Z \otimes H^T \\ (Z\Theta_2) \otimes ((E_1 + E_2 G)Q) & Z \otimes H & -Z \otimes I_q \end{pmatrix} < 0\,; \tag{5.86}$$

recall that k is the rank of Θ and Θ_1 and Θ_2 are full row rank matrices satisfying $\Theta = \Theta_1^T \Theta_2$.

In the following developments we let $Z = I_k$. This is necessary in order to transform (5.86) into an LMI. In this context, we recall that, when $k = 1$ (this happens for example when $\mathcal{D}$ is the left half of the complex plane or a disk centered at the origin of the complex plane), matrix Z turns out to be a scalar which can be chosen without loss of generality to be unitary; in this case condition (3.165) is also necessary for quadratic $\mathcal{D}$-stability. Therefore the following derivation is also necessary when the region $\mathcal{D}$ is such that Θ has rank one.

By letting $GQ = V$ and $Z = I_k$, we can state the following theorem, which immediately leads to an LMI feasibility problem for the design of the state feedback controller.

Theorem 5.12. *System* (5.71), (5.72) *is quadratically $\mathcal{D}$-stabilizable via state feedback control if there exist a positive definite matrix Q and a matrix V such that*

$$\begin{pmatrix} M_{\mathcal{D},LIN}(A,Q,V) & \Theta_1^T \otimes F & \Theta_2^T \otimes (QE_1^T + V^T E_2^T) \\ \Theta_1 \otimes F^T & -I_{kp} & I_k \otimes H^T \\ \Theta_2 \otimes (E_1 Q + E_2 V) & I_k \otimes H & -I_{kq} \end{pmatrix} < 0 \,, \tag{5.87}$$

where

$$M_{\mathcal{D},LIN}(A,Q,V) := \Lambda \otimes Q + \Theta \otimes (AQ + BV) + \Theta^T \otimes (QA^T + V^T B^T) \,. \tag{5.88}$$

In this case a state feedback controller which quadratically $\mathcal{D}$-stabilizes system (5.71) *is given by $u = Gx$ with $G = VQ^{-1}$.*

The condition is also necessary if Θ has rank one. □

Now we consider the output feedback problem. System (5.77), (5.78) is said to be quadratically $\mathcal{D}$-stabilizable via output feedback linear control *if and only if* there exists a dynamical controller in the form (5.80) which makes the closed loop system quadratically $\mathcal{D}$-stable.

From (3.151) it follows that a sufficient condition for quadratic $\mathcal{D}$-stabilizability via output feedback is the existence of positive definite matrices $P \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{k\times k}$ and matrices A_K, B_K, C_K and D_K such that

$$\begin{pmatrix} M_{\mathcal{D}}(A_{CL},P) & \Theta_1^T \otimes (PF_{CL}) & \left(\Theta_2^T Z\right) \otimes E_{CL}^T \\ \Theta_1 \otimes (F_{CL}^T P) & -Z \otimes I_p & Z \otimes H_{CL}^T \\ (Z\Theta_2) \otimes E_{CL} & Z \otimes H_{CL} & -Z \otimes I_q \end{pmatrix} < 0 \,, \tag{5.89}$$

where A_{CL}, F_{CL}, E_{CL} and H_{CL} are given by (5.82).

Again we let $Z = I_k$ to arrive to LMIs condition.

By following the same machinery of Sect. 5.4.1 we obtain the following result.

Theorem 5.13 ([49]). *System (5.77), (5.78) is quadratically $\mathcal{D}$-stabilizable via output feedback linear control if there exists positive definite matrices S, Q and matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ and D_K such that (5.11) holds and*

$$\begin{pmatrix} X_{11} & \Theta_1^T \otimes \begin{pmatrix} F_1 + BD_K F_2 \\ SF_1 + \hat{B}_K F_2 \end{pmatrix} & \Theta_2^T \otimes \begin{pmatrix} QE_1^T + \hat{C}_K^T E_2^T \\ E_1^T + C^T D_K^T E_2^T \end{pmatrix} \\ \Theta_1 \otimes \begin{pmatrix} F_1^T + F_2^T D_K^T B^T & F_1^T S + F_2^T \hat{B}_K^T \end{pmatrix} & -I_{kp} & I_k \otimes (H^T + F_2^T D_K^T E_2^T) \\ \Theta_2 \otimes \begin{pmatrix} E_1 Q + E_2 \hat{C}_K & E_1 + E_2 D_K C \end{pmatrix} & I_k \otimes (H + E_2 D_K F_2) & -I_{kq} \end{pmatrix} < 0\,, \tag{5.90}$$

where

$$X_{11} := \Lambda \otimes \begin{pmatrix} Q & I \\ I & S \end{pmatrix} + \Theta \otimes \begin{pmatrix} AQ + B\hat{C}_K & A + BD_K C \\ \hat{A}_K & SA + \hat{B}_K C \end{pmatrix} + \Theta^T \otimes \begin{pmatrix} QA^T + \hat{C}_K^T B^T & \hat{A}_K^T \\ A^T + C^T D_K^T B^T & A^T S + C^T \hat{B}_K^T \end{pmatrix}. \tag{5.91}$$

In this case a quadratically $\mathcal{D}$-stabilizing controller has the structure (5.80) where A_K, B_K and C_K can be obtained by solving (5.85).

The condition is also necessary if Θ has rank one. □

Quadratic $\mathcal{L}_2$ Performance Control

Let us consider the following uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + B_1 w(t) + (B_2 + \Delta B)u(t) \tag{5.92a}$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t)\,, \tag{5.92b}$$

where, as usual, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^v$, $z(t) \in \mathbb{R}^s$ and the uncertain part satisfies (5.72).

The closed loop interpretation of system (5.92) is described by the equations

$$\dot{x}(t) = Ax(t) + Fw_\Delta(t) + B_1 w(t) + B_2 u(t) \tag{5.93a}$$
$$z_\Delta(t) = E_1 x(t) + Hw_\Delta(t) + E_2 u(t) \tag{5.93b}$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{5.93c}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,. \tag{5.93d}$$

System (5.92) is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via state feedback *if and only if* there exist a matrix G such that the connection between system (5.92) and $u = Gx$ possesses a quadratic $\mathcal{L}_2$ performance bound γ.

By using representation (5.93), the closed loop system can be seen as an uncertain system subject to norm bounded uncertainties (see Fig. 5.6)

$$\dot{x}(t) = (A + B_2 G)x(t) + Fw_\Delta(t) + B_1 w(t) \tag{5.94a}$$
$$z_\Delta(t) = (E_1 + E_2 G)x(t) + Hw_\Delta(t) \tag{5.94b}$$
$$z(t) = (C_1 + D_{12} G)x(t) + D_{11} w(t) \tag{5.94c}$$
$$w_\Delta = \Delta z_\Delta\,. \tag{5.94d}$$

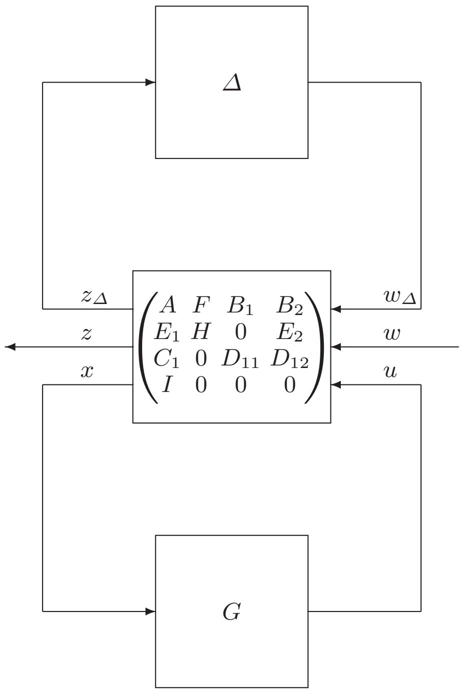


Fig. 5.6. The uncertain system (5.93) and the state feedback controller $u = Gx$

From (3.173) we obtain that a necessary and sufficient condition for stabilizability with a quadratic $\mathcal{L}_2$ performance bound γ via state feedback is the existence of a positive definite matrix Q, a matrix G and a positive scalar τ such that

$$\begin{pmatrix} (A+B_2G)Q+Q(A+B_2G)^T+\tau FF^T & B_1+Q(C_1+D_{12}G)^TD_{11} & Q(E_1+E_2G)^T+\tau FH^T & Q(C_1+D_{12}G)^T \\ B_1^T+D_{11}^T(C_1+D_{12}G)Q & -(\gamma^2 I - D_{11}^TD_{11}) & 0 & 0 \\ (E_1+E_2G)Q+\tau H^TF & 0 & -\tau(I-HH^T) & 0 \\ (C_1+D_{12}G)Q & 0 & 0 & -I \end{pmatrix} < 0 \,. \tag{5.95}$$

As usual we let $GQ = V$ and arrive to the following LMIs based problem whose feasibility is necessary and sufficient for stabilizability with a quadratic $\mathcal{L}_2$ performance bound γ via state feedback.

Problem 5.9 (Stabilizability with a quadratic $\mathcal{L}_2$ performance bound via state feedback).
Find a positive definite matrix Q, a matrix V and a positive scalar τ such that

$$\begin{pmatrix} AQ+QA^T+B_2V+V^TB_2^T+\tau FF^T & B_1+QC_1^TD_{11}+V^TD_{12}^TD_{11} & QE_1^T+V^TE_2^T+\tau FH^T & QC_1^T+V^TD_{12}^T \\ B_1^T+D_{11}^TC_1Q+D_{11}^TD_{12}V & -(\gamma^2I-D_{11}^TD_{11}) & 0 & 0 \\ E_1Q+E_2V+\tau HF^T & 0 & -\tau(I-HH^T) & 0 \\ C_1Q+D_{12}V & 0 & 0 & -I \end{pmatrix} < 0\,. \tag{5.96}$$

◇

If Problem 5.9 is feasible a quadratically stabilizing controller is given by $u = Gx$ with $G = VQ^{-1}$.

Now consider stabilizability with quadratic $\mathcal{L}_2$ performance via output feedback. Let us consider the uncertain system

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) + B_1w(t) + \big(B_2 + \Delta B)u(t) \tag{5.97a}$$
$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \tag{5.97b}$$
$$y(t) = \big(C_2 + \Delta C\big)x(t) + D_{21}w(t) + \big(D_{22} + \Delta D)u(t)\,, \tag{5.97c}$$

where $y(t) \in \mathbb{R}^r$ and the uncertainties satisfy (5.78).

The closed loop interpretation of system (5.97) is described by the following equations

$$\dot{x}(t) = Ax(t) + F_1w_\Delta(t) + B_1w(t) + B_2u(t) \tag{5.98a}$$
$$z_\Delta(t) = E_1x(t) + Hw_\Delta(t) + E_2u(t) \tag{5.98b}$$
$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \tag{5.98c}$$
$$y(t) = C_2x(t) + F_2w_\Delta(t) + D_{21}w(t) + D_{22}u(t) \tag{5.98d}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,. \tag{5.98e}$$

As usual, to simplify computations in the sequel we shall assume $D_{22} = 0$.

Obviously system (5.97) is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via output feedback *if and only if* there exists a dynamical controller in the form (5.80) such that the overall closed loop system possesses a quadratic $\mathcal{L}_2$ performance bound γ.

The connection between system (5.98) and the controller (5.80) (see Fig. 5.7) leads to the following unforced system subject to norm bounded uncertainties

$$\dot{x}_a(t) = A_{CL}x_a(t) + F_{CL}w_\Delta(t) + B_{CL}w(t) \tag{5.99a}$$
$$z_\Delta(t) = E_{CL}x_a(t) + H_{CL}w_\Delta + E_2D_KD_{21}w(t) \tag{5.99b}$$
$$z(t) = C_{CL}x_a(t) + D_{12}D_KF_2w_\Delta(t) + D_{CL}w(t) \tag{5.99c}$$
$$w_\Delta(t) = \Delta z_\Delta(t)\,, \tag{5.99d}$$

where

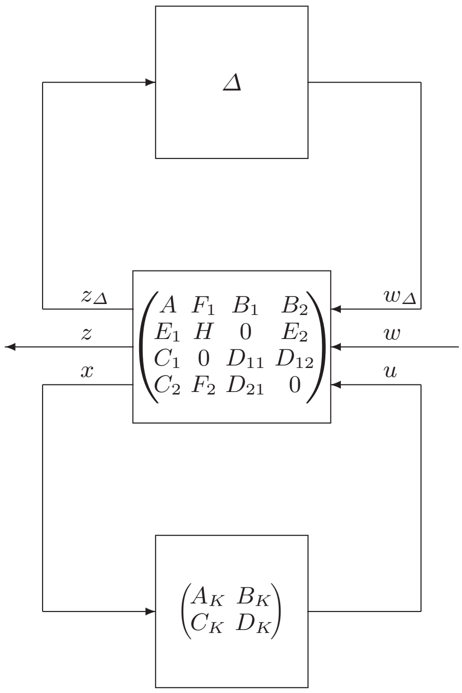


Fig. 5.7. The uncertain system (5.97) and the output feedback controller

$$A_{CL} = \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix} \tag{5.100a}$$

$$F_{CL} = \begin{pmatrix} F_1 + B_2 D_K F_2 \\ B_K F_2 \end{pmatrix} \tag{5.100b}$$

$$B_{CL} = \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix} \tag{5.100c}$$

$$E_{CL} = \begin{pmatrix} E_1 + E_2 D_K C_2 & E_2 C_K \end{pmatrix} \tag{5.100d}$$

$$H_{CL} = H + E_2 D_K F_2 \tag{5.100e}$$

$$C_{CL} = \begin{pmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{pmatrix} \tag{5.100f}$$

$$D_{CL} = D_{11} + D_{12} D_K D_{21} \,. \tag{5.100g}$$

In order to avoid the dependence of z_Δ on w and z on w_Δ in (5.99), we also assume that $D_K = 0$; conversely (5.99) would not be in the form (3.167).

From (3.172) we have that a necessary and sufficient condition for stabilizability via output feedback (with $D_K = 0$) with a quadratic $\mathcal{L}_2$ performance bound γ is the existence of a positive definite matrix P and a positive scalar τ such that

$$\begin{pmatrix} A_{CL}^T P + P A_{CL} & P B_{CL} & P F_{CL} & \tau E_{CL}^T & C_{CL}^T \\ B_{CL}^T P & -\gamma^2 I & 0 & 0 & D_{CL}^T \\ F_{CL}^T P & 0 & -\tau^2 I & \tau H_{CL}^T & 0 \\ \tau E_{CL} & 0 & \tau H_{CL} & -I & 0 \\ C_{CL} & D_{CL} & 0 & 0 & -I \end{pmatrix} < 0 \,. \tag{5.101}$$

As usual, we partition P and P^{-1} according to (5.10) where, without loss of generality, M and N can be assumed to be nonsingular, pre- and post-multiply (5.101) by $\mathrm{diag}(\Pi_1^T, I, I, I, I)$ and $\mathrm{diag}(\Pi_1, I, I, I, I)$ respectively, where Π_1 and Π_2 has been defined in (5.15); recalling that $P\Pi_1 = \Pi_2$, we obtain

$$\begin{pmatrix} M_{LIN_B}(Q, \hat{C}_K) & A + \hat{A}_K^T & B_1 & F_1 & \tau(QE_1^T + \hat{C}_K^T E_2^T) & QC_1^T + \hat{C}_K^T D_{12}^T \\ A^T + \hat{A}_K^T & M_{LIN_C}(S, \hat{B}_K) & SB_1 + \hat{B}_K D_{21} & SF_1 + \hat{B}_K F_2 & \tau E_1^T & C_1^T \\ B_1^T & B_1^T S + D_{21}^T \hat{B}_K^T & -\gamma^2 I & 0 & 0 & D_{11}^T \\ F_1^T & F_1^T S + F_2^T \hat{B}_K^T & 0 & -\tau^2 I & \tau H^T & 0 \\ \tau(E_1 Q + E_2 \hat{C}_K) & \tau E_1 & 0 & \tau H & -I & 0 \\ C_1 Q + D_{12}\hat{C}_K & C_1 & D_{11} & 0 & 0 & -I \end{pmatrix} < 0 \,, \tag{5.102}$$

where

$$M_{LIN_B}(Q, \hat{C}_K) := AQ + QA^T + B_2\hat{C}_K + \hat{C}_K^T B_2^T \tag{5.103a}$$

$$M_{LIN_C}(S, \hat{B}_K) := A^T S + SA + C_2^T \hat{B}_K^T + \hat{B}_K C_2 \,, \tag{5.103b}$$

and $\hat{B}_K$, $\hat{C}_K$ and $\hat{A}_K$ can be obtained by (5.85) setting $D_K = 0$, $B = B_2$ and $C = C_2$.

Therefore, we can state the following result.

Theorem 5.14. *System* (5.97) *is stabilizable with a quadratic $\mathcal{L}_2$ performance bound γ via output feedback (with $D_K = 0$)* if and only if *there exist positive definite matrices S and Q, matrices $\hat{A}_K$, $\hat{B}_K$ and $\hat{C}_K$ and a positive scalar τ such that* (5.11) *and* (5.102) *hold.* □

Note that (5.102) is an LMI if τ is fixed. Concerning the choice of τ the reader is referred to [138, 141].

Therefore the procedure for designing a stabilizing controller with a quadratic $\mathcal{L}_2$ performance bound γ consists of the following steps:

i) Find symmetric matrices S and Q, matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ and a positive scalar τ satisfying the hypothesis of Theorem 5.14;
ii) find nonsingular matrices M and N satisfying $MN^T = I - SQ$;
iii) resolve system (5.85), with $D_K = 0$, $B = B_2$, $C = C_2$, for A_K, B_K and C_K.

Guaranteed Cost Control Consider system (5.71). The goal is to find a state feedback controller $u = Gx$ such that the overall closed loop system possesses a quadratic guaranteed cost with associated cost matrix P *wrt* the index (3.105).

Since, according to (5.74), the closed loop system can be seen as an unforced system subject to norm bounded uncertainties, we obtain (see Problem 3.14) that the solution of the following EVP provides the matrix gain $G = VQ^{-1}$ of a state feedback controller ensuring the minimum quadratic guaranteed cost.

Problem 5.10.

$$\max \theta$$

s.t.

$$\tau > 0 \tag{5.104a}$$

$$\theta > 0 \tag{5.104b}$$

$$Q > \theta I \tag{5.104c}$$

$$\begin{pmatrix} AQ + QA^T + BV + V^T B^T + \tau F F^T & QE_1^T + V^T E_2^T + \tau F H^T & Q\Pi \\ E_1 Q + E_2 V + \tau H F^T & -\tau(I - HH^T) & 0 \\ \Pi Q & 0 & -\Pi \end{pmatrix} < 0\,. \tag{5.104d}$$

◇

Summary

Most part of this chapter has dealt with the quadratic stabilization problem via both state and output feedback for linear systems depending on parametric uncertainties.

Sufficient conditions for state feedback quadratic stabilizability via nonlinear controllers [52, 89, 116, 117], were provided at the end of the Seventies; such conditions were known as *matching conditions*. In [25] a result is provided which allows to dispense with matching conditions on $B(\cdot)$ as far as linear controllers are considered; this is accomplished by converting the original system (5.2) into an augmented system for which the input matrix does not depend on parameters.

Later, in [140], it was proven that, in the context of quadratic stabilizability via state feedback, it is not restrictive to consider memoryless time-invariant controllers in place of dynamical time-varying controllers. On the other hand, the existence of a state feedback nonlinear controller which quadratically stabilizes a given linear system subject to parametric uncertainties does not imply the existence of a *linear* state feedback controller with the same property (for a counterexample see [137]).

However, when the input matrix $B(\cdot)$ is parameter independent, quadratic stabilizability via state feedback implies quadratic stabilizability via state feedback linear control [28, 99]; to this regard, note that, in the example provided in [137], the input matrix does depend on the parameter vector.

Finally, under some assumptions on the system matrices, the necessary and sufficient condition for quadratic stabilizability via state feedback linear control (5.3) can be converted into the feasibility of the LMIs Problem 5.1 via the classical change of variable proposed in [86].

Then we have considered quadratic stabilizability via parameter dependent output feedback. We have shown that the classical Separation Property extends to the quadratic stability setting. Indeed quadratic stabilizability via parameter dependent output feedback is equivalent to the solvability of a pair of parameter dependent Riccati inequalities, one involving the state feedback gain and one the observer gain; such conditions can be turned, under some assumptions on the system matrices, into LMIs.

Since the output feedback controller turns out to be parameter dependent, the basic assumption is that the parameter vector is measurable on-line. Note that, in contrast with the controller structure considered in this book, in [123] sufficient conditions for the existence of a controller guaranteeing quadratic stabilization and having a multivariable PID structure are provided.

Finally, in the context of parametric uncertainties and when the parameters have a bounded rate of variation, we have considered quadratic stabilization via parameter dependent Lyapunov functions. Concerning the state feedback case, for tractability reasons, the feedback gain is allowed to depend on the parameters; regarding the output feedback case, it is shown that the problem can be reduced to the feasibility of two distinct Riccati inequalities, one for the state feedback gain design and the other one for the observer gain design. A probabilistic approach for the state feedback design is proposed in [132].

As for the norm bounded case, the output feedback controller turns out to be parameter independent; an interpretation of this fact is that quadratic stability for norm bounded uncertainties is equivalent to an $\mathcal{H}_\infty$ control problem for a certain system.

For both parametric and norm bounded uncertainties the quadratic stabilization plus performance problem has been considered. Necessary and sufficient conditions for stabilization with quadratic $\mathcal{L}_2$ performance and guaranteed cost control (only the state feedback case) and sufficient conditions for quadratic $\mathcal{D}$-stabilization have been provided.

As for quadratic stabilization and performances with parameter dependent Lyapunov functions, operative conditions can be obtained, by following the usual machinery, starting from Theorems 4.4 and 4.5. Regarding quadratic $\mathcal{L}_2$ performance control the interested reader is also referred to [59] and [174] (note that in [59] only the state feedback case is considered, however the feedback gain is parameter independent, while in [174] the output feedback controller also depends on the parameter derivative); guaranteed cost control issues are treated in [60].

Exercises

Exercise 5.1. Consider the uncertain system (3.1) and a nonsingular matrix T. Show that the system is QS *iff* the transformed system

$$\frac{d\hat{x}(t)}{dt} = T^{-1}A(p)T\hat{x}(t) \tag{5.105}$$

is QS.
(Hint: Pre- and post-multiply both members of (3.3) by T^T and T respectively.) △

Exercise 5.2. Prove Corollary 5.1.
(Hint: Use Lemma 5.2 with

$$T = \begin{pmatrix} \frac{1}{\epsilon}I & 0 \\ 0 & I \end{pmatrix} \tag{5.106}$$

and ϵ sufficiently small.) △

Exercise 5.3. Prove Lemma 5.3.
(Hint: Remember that a symmetric matrix Q is positive (negative) definite *iff*, for all $x \neq 0$, $x^T Q x > (<) 0$.) △

Exercise 5.4. Prove that condition iii) in Theorem 5.6 can be directly derived following a Separation Property approach (as done in Sect. 5.1.2 for quadratic stabilization). △

6. Discrete-Time Systems

In this chapter we deal with the discrete-time linear uncertain system in the form

$$x(k+1) = A(p)x(k)\,, \tag{6.1}$$

where $x(k) \in \mathbb{R}^n$ and, given the usual hyper-box $R \subset \mathbb{R}^q$ defined in (3.2), $p = \begin{pmatrix} p_1 \; p_2 \; \dots \; p_q \end{pmatrix}^T \in R$ is the vector of (possibly time-varying) uncertain parameters.

Also, we shall consider systems subject to norm bounded uncertainties

$$x(k+1) = \left(A + F\Delta(I - H\Delta)^{-1}E\right) x(k)\,, \tag{6.2}$$

where $F \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{q\times n}$, $H \in \mathbb{R}^{q\times p}$, $\|H\| < 1$, and $\|\Delta\| \leq 1$.

6.1 Quadratic Stability

6.1.1 Parametric Uncertainties

Throughout this section we assume that Assumption 3.1 holds; in the other cases one of the alternative approaches described in Chap. 3 have to be used.

Definition 6.1. System (6.1) is said to be *quadratically stable in* R if and only if there exists a positive definite matrix P such that for all $p \in R$

$$A^T(p)PA(p) - P < 0\,. \tag{6.3}$$

◇

From Corollary 2.2 it follows that quadratic stability of system (6.1) guarantees exponential stability of the time-varying system

$$x(k+1) = A(p(k))x(k) \tag{6.4}$$

versus any time-varying realization of parameters $p(\cdot)$ ranging in R.

Now we will show that, under Assumption 3.1 and for a given P, it is necessary and sufficient to check the satisfaction of (6.3) on the vertices of the hyper-box R. This will allow us to state a *necessary and sufficient* condition for quadratic stability in terms of the solvability of a feasibility problem with LMIs constraints.

In order to prove the main result of this section, note that the condition

$$A^T(p)PA(p) - P < 0 \tag{6.5}$$

can be equivalently re-written, by using Fact A.3,

$$\begin{pmatrix} -P & PA(p) \\ A^T(p)P & -P \end{pmatrix} < 0\,. \tag{6.6}$$

Note that condition (6.6) can be directly determined from (3.86) when $\mathcal{D}$ is the unit disk centered at the origin of the complex plane (see Exercise 6.1).

This fact allows to state the following result, which will be exploited in the sequel of this chapter.

Theorem 6.1. *Quadratic stability of the discrete-time system* (6.1) *is equivalent to quadratic $\mathcal{D}$-stability of the continuous-time system* (3.1) *when $\mathcal{D}$ is the unit disk centered at the origin of the complex plane.* □

Since Assumption (3.1) holds, the left hand side of (6.6) is negative definite for all $p \in R$ *iff* it is negative definite at the vertices of R (Theorem A.2). Therefore (6.6) is equivalent to

$$\begin{pmatrix} -P & PA(p_{(i)}) \\ A^T(p_{(i)})P & -P \end{pmatrix} < 0\,. \tag{6.7}$$

Finally applying again Fact A.3 we can conclude that system (6.1) is QS *iff* the following LMIs based feasibility problem admits a feasible solution.

Problem 6.1.
Find a symmetric matrix P such that

$$P > 0 \tag{6.8a}$$

$$A^T(p_{(i)})PA(p_{(i)}) - P < 0\,, \qquad i = 1, \ldots, 2^q\,. \tag{6.8b}$$

◇

Example 6.1 (Quadratic Stability Margin (QSM)).
We consider the extension of the concept of QSM (Example 3.1) to the discrete-time case.

Let us consider the feedback system (we can refer again to Fig. 3.1) described by the state space equations

$$x(k+1) = A_0 x(k) + Bu(k) \tag{6.9a}$$

$$y(k) = Cx(k) \tag{6.9b}$$

$$u(k) = \Delta(p)y(k)\,, \tag{6.9c}$$

where $A_0 \in \mathbb{R}^{n\times n}$, $B = \begin{pmatrix} b_1 \dots b_q \end{pmatrix} \in \mathbb{R}^{n\times q}$, $C^T = \begin{pmatrix} c_1 \dots c_q \end{pmatrix} \in \mathbb{R}^{n\times q}$ and $\Delta(p) \in \mathbb{R}^{q\times q}$ is a diagonal matrix whose elements are the components of the parameter vector $p = \begin{pmatrix} p_1 \dots p_q \end{pmatrix}^T$. This uncertain system can be described concisely in the form (6.1) with $A(p)$ given by (3.14).

Now, given the unit square $R_u \subset \mathbb{R}^q$ defined by (3.15) and defined θ as the dilatation factor of the set R_u (see (3.16)), we define the QSM of system (3.13) as follows

$$\rho_Q := \sup\{\theta > 0 : \text{ system } x(k+1) = A(\theta p)x(k) \text{ is quadratically stable in } R_u\}\,. \tag{6.10}$$

Since $A(\theta p)$ can be written in the form (3.19) and looking at (6.7), we conclude that the computation of the QSM can be performed by solving the following GEVP in the variables θ and P.

Problem 6.2.

$$\begin{aligned}
&\max \theta \\
&\quad \text{s.t.} \\
&\quad \theta > 0 \\
&\quad P > 0 \\
&\begin{pmatrix} -P & PA_0 \\ A_0^T P & -P \end{pmatrix} + \theta \begin{pmatrix} 0 & PL(p_{(i)}) \\ L^T(p_{(i)})P & 0 \end{pmatrix} < 0\,, \quad i = 1, \dots, 2^q\,,
\end{aligned}$$

where $p_{(i)}$ is the i-th vertex of R_u. ◇

For a numerical example see Exercise 6.2. △

6.1.2 Norm Bounded Uncertainties

According to Definition 6.1, the statement *system* (6.2) *is QS* means that there exists a positive definite matrix P such that for all Δ with $\|\Delta\| \le 1$

$$\big(A + F\Delta(I - H\Delta)^{-1}E\big)^T P\big(A + F\Delta(I - H\Delta)^{-1}E\big) - P < 0\,. \tag{6.11}$$

Again, it is simple to recognize that the above definition is equivalent to Definition 3.7 when $\mathcal{D}$ is the unit circle centered at the origin of the complex plane; therefore we can state the following result.

Theorem 6.2. *Quadratic stability of the discrete-time system* (6.2) *is equivalent to quadratic $\mathcal{D}$-stability of the continuous-time system* (3.116) *when $\mathcal{D}$ is the unit circle centered at the origin of the complex plane.* □

Using Theorems 6.2 and 3.10 we can readily obtain a necessary and sufficient condition for quadratic stability of system (6.2) which is directly stated in the form of LMIs based feasibility problem (see Exercise 6.3).

Theorem 6.3. *System* (6.2) *is QS* if and only if *there exists a positive definite matrix P such that*

$$\begin{pmatrix} A^T PA - P + E^T E & A^T PF + E^T H \\ F^T PA + H^T E & F^T PF - (I - H^T H) \end{pmatrix} < 0\,. \tag{6.12}$$

□

6.1.3 Connections Between Quadratic Stability and $\mathcal{H}_\infty$ Control

In this section we generalize to discrete-time systems the results of Sect. 3.6.

Let us denote by $l_2(\mathbb{N}_0, \mathbb{R}^m)$ the space of the real vector-valued sequences with m components which are square summable on $\mathbb{N}_0$.

Remember that the exponentially stable discrete-time system

$$x(k+1) = Ax(k) + Fw(k) \tag{6.13a}$$

$$z(k) = Ex(k) + Hw(k)\,, \tag{6.13b}$$

uniquely defines the linear operator

$$\begin{aligned} &\Gamma_{zw} : l_2(\mathbb{N}_0, \mathbb{R}^p) \mapsto l_2(\mathbb{N}_0, \mathbb{R}^q) \\ &\quad w \mapsto z = \Gamma_{zw}(w) := \sum_{h=0}^{k-1} EA^{k-h-1} Fw(h) + Hw(k)\,. \end{aligned}$$

The l_2 induced norm of the operator Γ_{zw} is defined as follows

$$\|\Gamma_{zw}\| := \sup_{w \in l_2(\mathbb{N}_0, \mathbb{R}^p) - \{0\}} \frac{\|z\|}{\|w\|}\,, \tag{6.14}$$

where, for a given vector-valued sequence $v(\cdot) \in l_2$, we define the l_2 norm as

$$\|v\| := \left(\sum_{k=0}^{+\infty} v^T(k) v(k) \right)^{1/2}\,. \tag{6.15}$$

For linear time-invariant systems we have that the l_2 induced norm equals the $\mathcal{H}_\infty$ norm of the corresponding transfer function

$$\|\Gamma_{zw}\| = \|E(zI - A)^{-1} F + H\|_\infty := sup_{\theta \in [0,\pi]} \|E(e^{j\theta} I - A)^{-1} F + H\|\,. \tag{6.16}$$

Now let us consider the closed loop system in Fig. 6.1, where $W(z) \in \mathcal{RH}_\infty$ and $\Delta(z) \in \mathcal{RH}_\infty$.

We can state the following theorem which generalizes Theorem 3.14 to discrete-time systems.

Theorem 6.4 ([49,134]). *Let us consider the closed loop scheme in Fig. 6.1 and denote by (A, F, E, H) a (minimal) realization of $W(z)$; assume that $W(z) \in \mathcal{RH}_\infty$. Then the following statements are equivalent:*

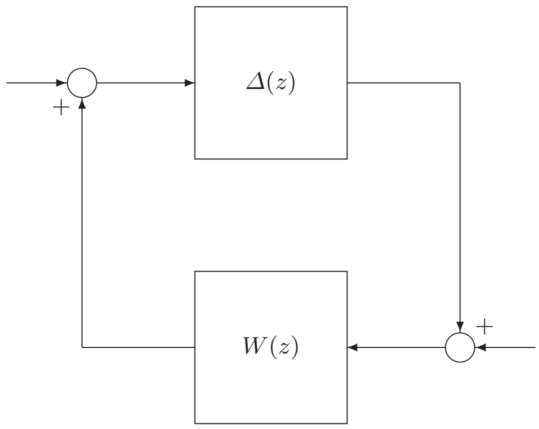


Fig. 6.1. Closed loop scheme for the Small Gain Theorem applied to discrete-time systems

i) $\|W\|_\infty < 1$.
ii) *The closed loop system in Fig. 6.1 is well posed and internally stable for all dynamic perturbations* $\Delta \in \mathcal{RH}_\infty$ *satisfying* $\|\Delta\|_\infty \leq 1$.
iii) *The closed loop system in Fig. 6.1 is well posed and internally stable for all* $\Delta \in \mathbb{C}^{p \times q}$ *with* $\|\Delta\| \leq 1$.
iv) *System* (6.2) *is QS.*
v) *System* (6.2) *is QS for* $\Delta \in \mathbb{C}^{p \times q}$ *with* $\|\Delta\| \leq 1$.
vi) *There exists a positive definite matrix* P *which satisfies* (6.12).

□

6.2 Systems Subject to Bounded Rate Parameters

We consider the class of discrete-time, real vector-valued sequences $p(\cdot) : k \in \mathbb{N}_0 \to p(k)$, satisfying

$$p(k) \in R\,, \quad \forall k \in \mathbb{N}_0\,, \tag{6.17}$$

and

$$|\Delta\big(p_i(k)\big)| \leq h_i\,, \quad i = 1, 2, \ldots, q\,, \quad \forall k \in \mathbb{N}_0\,, \tag{6.18}$$

where $\Delta\big(p_i(k)\big) := p_i(k+1) - p_i(k)$ denotes the first difference of the scalar function $p_i(\cdot)$.

The following theory has been developed in [17] and [19] for the one parameter and the multi-parameter cases respectively. It is interesting to note that, while there is a complete analogy between the quadratic stability theory for continuous-time systems developed in Chap. 3 and that one for discrete-time systems illustrated in Sect. 6.1, the approach to deal with bounded rate parameters is rather different in the two cases. Indeed we shall show that, while for continuous-time systems the approach via parameter dependent continuously differentiable Lyapunov functions has been followed, in the discrete-time context it is more suitable to use piecewise constant parameter dependent functions.

Let us split, according to Fig. 6.2, each interval $[\underline{p}_i, \overline{p}_i]$ into ν_i sub-intervals of equal length ρ_i[1]

$$\begin{aligned}[\underline{p}_i, \overline{p}_i] &= \bigcup_{l=1}^{\nu_i} [p_{i,l}, p_{i,l+1}] \\ &=: \bigcup_{l=1}^{\nu_i} \mathcal{I}_{i,l}\,, \quad p_{i,l+1} - p_{i,l} = \rho_i > 0\,, \quad i = 1, \dots, q, \quad l = 1, \dots, \nu_i\,;\end{aligned} \tag{6.19}$$

note that $p_{i,1} = \underline{p}_i$ and $p_{i,\nu_i+1} = \overline{p}_i$.

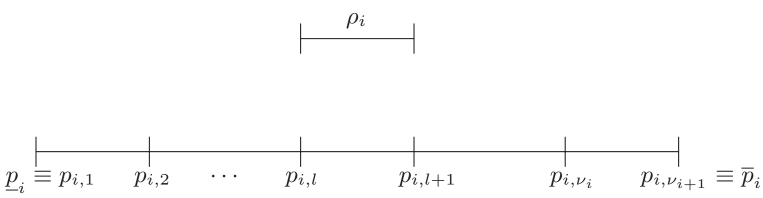


Fig. 6.2. Partition of the interval $[\underline{p}_i, \overline{p}_i]$

Now let us define the following vector function

$$j \,:\, p \in R \mapsto j(p) = \big(j_1(p_1)\; j_2(p_2) \cdots j_q(p_q)\big)^T \in \mathbb{N}^q\,, \tag{6.20}$$

where

$$p_i \in [p_{i,l}, p_{i,l+1}[\mapsto j_i(p_i) = l\,, \quad l = 1, \dots, \nu_i - 1\,, \tag{6.21}$$

and

$$p_i \in [p_{i,\nu_i}, \overline{p}_i] \mapsto j_i(p_i) = \nu_i\,. \tag{6.22}$$

Note that, at a given instant k, $j(p(k))$ univocally individuates the sub-hyper-box

[1] The sub-intervals are considered of equal length for the sake of presentation simplicity.

$$R_j := \mathcal{I}_{1,j_1} \times \mathcal{I}_{2,j_2} \times \cdots \times \mathcal{I}_{q,j_q} \tag{6.23}$$

to which $p(k)$ belongs.

Now define the integers $m_i \leq \nu_i - 1$, $i = 1, 2, \ldots, q$, as the maximum number of sub-intervals that the parameter p_i can jump in one discrete-time step compatibly with (6.18). In other words the numbers m_i are defined in such a way that the following inequalities are satisfied

$$j_i(p_i(k)) \,\widehat{\oplus}\, (-m_i) \leq j_i(p_i(k+1)) \leq j_i(p_i(k)) \,\widehat{\oplus}\, m_i\,, \quad i = 1, 2, \ldots, q\,, \tag{6.24}$$

where, for any relative integer number x,

$$j_i \,\widehat{\oplus}\, x := \begin{cases} j_i + x & \text{if } j_i + x \in [1, \nu_i] \\ \nu_i & \text{if } \quad j_i + x > \nu_i \\ 1 & \text{if } \quad j_i + x < 1 \end{cases} . \tag{6.25}$$

From the definition of m_i and (6.18) it follows that, in one discrete-time step, p_i can move to a sub-interval which is at a *distance* (measured in number of sub-intervals of $[\underline{p}_i, \overline{p}_i]$) which is less than or equal to m_i (see Fig. 6.3 in the case $m_i = 2$).

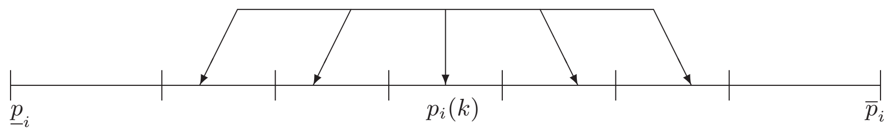


Fig. 6.3. Possible transitions of parameter p_i in the interval $k, k+1$ in the case $m_i = 2$

Therefore, if the parameter vector p belongs at time k to the hyper-box R_j at time $k+1$ it will belong to the set

$$\bigcup_{v_1 \in V_1, \ldots, v_q \in V_q} \mathcal{I}_{1, j_1 \widehat{\oplus} v_1} \times \mathcal{I}_{2, j_2 \widehat{\oplus} v_2} \times \cdots \times \mathcal{I}_{q, j_q \widehat{\oplus} v_q} = \bigcup_{v \in V_1 \times \cdots \times V_q} R_{j \widehat{\oplus} v}\,, \tag{6.26}$$

where

$$V_i := \{-m_i, \ldots, 0, \ldots, m_i\} \tag{6.27}$$

and

$$j \,\widehat{\oplus}\, v := \left(j_1 \,\widehat{\oplus}\, v_1 \; j_2 \,\widehat{\oplus}\, v_2 \; \cdots \; j_q \,\widehat{\oplus}\, v_q \right)^T . \tag{6.28}$$

The following is the main result of the section; it is directly stated in form of LMIs feasibility problem.

Theorem 6.5 ([19]). *Assume there exist $\nu_1 \cdot \nu_2 \cdot \dots \cdot \nu_q$ positive definite matrices $P(j) = P(j_1, j_2, \dots, j_q)$, such that for $j_i = 1, 2, \dots, \nu_i$, $i = 1, 2, \dots, q$,*

$$A^T(p)P(j \mathbin{\widehat{\oplus}} v)A(p) - P(j) < 0\,, \quad \forall p \in R_j\,, \quad \forall v \in V_1 \times \cdots \times V_q\,; \tag{6.29}$$

then system (6.1) *is exponentially stable for any parameter realization* $p(\cdot)$ *satisfying* (6.17) *and* (6.18).

Proof. Let $p(\cdot)$ any discrete-time vector function satisfying (6.17) and (6.18); then consider the positive definite bounded matrix-valued function $k \mapsto P\big(j(p(k))\big)$.

Now, assume that $p(k) \in R_j$; by virtue of how the partition of the set R is realized, there exists some q-tuple $v = \big(v_1\ v_2\ \cdots\ v_q\big)^T \in V_1 \times \cdots \times V_q$ such that $j(p(k+1))$ coincides with $j \mathbin{\widehat{\oplus}} v$. Therefore the satisfaction for all $p \in R_j$ of inequality (6.29) guarantees the existence of a positive scalar γ such that, for all $k \in \mathbb{N}_0$,

$$A^T(p(k))P(j(p(k+1)))A(p(k)) - P(j(p(k))) \le -\gamma I\,. \tag{6.30}$$

The proof follows from Corollary 2.2 and the arbitrariness of $p(\cdot)$. □

Remark 6.1. As said, in Theorem 6.5 we use piecewise constant, parameter dependent Lyapunov functions. ◇

Remark 6.2. Note that, when applying Theorem 6.5, the parameter to be chosen is the number of sub-intervals ν_i in which we partition the interval $[\underline{p}_i, \overline{p}_i]$, $i = 1, \dots, q$, which gives place to the following partition of R

$$R = \bigcup_{l=1}^{\nu_1} \mathcal{I}_{1,l} \times \bigcup_{l=1}^{\nu_2} \mathcal{I}_{2,l} \times \cdots \times \bigcup_{l=1}^{\nu_q} \mathcal{I}_{q,l}\,. \tag{6.31}$$

It is evident that, for a given partitioning of the original hyper-box R, the faster the parameters are allowed to vary in time, the greater are the integers m_i, $i = 1, \dots, q$, which satisfy (6.24) and hence the larger is the set (6.26) to which they can jump in one time step. Since the integers ν_i and m_i, $i = 1, \dots, q$, increase the computational burden of the problem to be solved, one should try to keep the numbers ν_i as low as possible, compatibly with the rate of variation of the parameters. ◇

When $A(p)$ satisfies Assumption 3.1, according to Theorem 6.5, we conclude that system (6.1) is exponentially stable for any parameter realization $p(\cdot)$ satisfying (6.17) and (6.18) if the following LMIs problem admits a feasible solution.

Problem 6.3.
Find positive definite matrices $P(j_1, \dots, j_q)$ which satisfy condition (6.29) for $j_i = 1, 2, \dots, \nu_i$, $i = 1, \dots, q$, with R_j^v replacing R_j. ◇

6.2.1 Connections with the Quadratic Stability Approach

The approach proposed for bounded rate parameters can be applied to obtain sufficient conditions, that are less conservative than those ones proposed in the standard quadratic stability approach, even in the case of parameters with unknown bounds on the rate of variation. The key point is that, in the discrete-time context, the rate of variation of the i-th parameter p_i is in any case bounded if such parameter is assumed to belong to a finite interval. Indeed p_i cannot vary in the unit time step more than the length $|\underline{p}_i - \overline{p}_i|$ of the interval. Hence we have that inequality (6.18) is always satisfied with $h_i = |\underline{p}_i - \overline{p}_i|$.

Therefore if there is no explicit bound on the parameters rate of variation we can establish the following corollary of Theorem 6.5 (stated directly in terms of LMIs feasibility problem) which guarantees exponential stability of system (6.1) for any time realization of the unknown parameter vector $p(\cdot)$. Such a corollary is obtained by letting $m_i = \nu_i - 1$ in Theorem 6.5; indeed if no bound on the rate of variation of the parameters is available, one has to consider that in one step the i-th parameter p_i can reach every sub-interval of the interval $[\underline{p}_i, \overline{p}_i]$.

Corollary 6.1 ([18,19]). *System* (6.1) *is exponentially stable for all vector-valued sequences* $p(\cdot) \, : \, k \in \mathbb{N}_0 \mapsto R$ *if there exist* $\nu_1 \cdot \nu_2 \cdot \cdots \cdot \nu_q$ *positive definite matrices* $P(j) = P(j_1, j_2, \ldots, j_q)$, *such that for* $j_i = 1, 2 \ldots, \nu_i$, $i = 1, 2, \ldots, q$

$$A^T(p)P(j \widehat{\oplus} v)A(p) - P(j) < 0 \,, \quad \forall p \in R_j \,, \quad \forall v \in V_1 \times \cdots \times V_q \,, \tag{6.32}$$

where $V_i := \{-\nu_i + 1, \ldots, 0, \ldots, \nu_i - 1\}$, $i = 1, \ldots, q$. □

Note that, by assuming $\nu_i = 1$, $i = 1, \ldots, q$, Corollary 6.1 recovers the classical quadratic stability condition; indeed in this case the Lyapunov function reduces to a fixed quadratic function in the form $x^T P x$, where P is a constant positive definite matrix. Conversely by assuming, for some i, $\nu_i > 1$, we can obtain a less conservative condition guaranteeing exponential stability of the uncertain system.

6.3 A Real World Example: Control of a Plasma Wind Tunnel

In this section we consider the problem of controlling a plasma wind tunnel, to simulate the re-entry conditions of space vehicles by reproducing desired profiles of pressure and temperature on a test model.

The whole control strategy proposed in [14, 15] consists of two terms: a feedforward control action, obtained *off-line* via a classical receding horizon technique [124], to guarantee the trajectory following in absence of external disturbances and an output feedback control action to compensate for possible misalignments between the desired trajectory and the actual one.

Here we focus on the feedback controller which, to take into account the numerical implementation, is designed in the discrete-time setting. The design is based on the theory developed in the previous sections.

In [14,15] the plasma wind tunnel (PWT) available at the Italian Aerospace Research Center (CIRA), is considered. It is shown that the linearized model of the system under consideration can be represented by the uncertain system

$$x(k+1) = A(p,\gamma)x(k) + B(p,\gamma)u(k) \tag{6.33a}$$

$$y(k) = Cx(k)\,, \tag{6.33b}$$

where $p \in \mathbb{R}^8$ and $\gamma \in \mathbb{R}$. The input to the system is $u(k) = \big(I_a(k)\ \dot{m}(k)\big)^T$, where I_a and $\dot{m}$ are the sampled arc current and mass flow rate; the outputs is $y(k) = \big(T_s(k)\ p_s(k)\big)^T$, where T_s and p_s are the sampled stagnation temperature and pressure on the test model. Moreover we have

$$A(p,\gamma) = \begin{pmatrix} [(k_7+k_8)\gamma + p_7 + p_8 + \hat{p}_{70} + \hat{p}_{80}]\, I_2 & I_2 & 0_2 \\ -(k_7\gamma + p_7 + \hat{p}_{70})(k_8\gamma + p_8 + \hat{p}_{80}) I_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 \end{pmatrix} \tag{6.34a}$$

$$B(p,\gamma) = \begin{pmatrix} k_1\gamma + p_1 + \hat{p}_{10} & k_2\gamma + p_2 + \hat{p}_{20} \\ k_3\gamma + p_3 + \hat{p}_{30} & k_4\gamma + p_4 + \hat{p}_{40} \\ (k_1\gamma + p_1 + \hat{p}_{10})(1 - k_5\gamma - p_5 - \hat{p}_{50}) & (k_2\gamma + p_2 + \hat{p}_{20})(1 - k_5\gamma - p_5 - \hat{p}_{50}) \\ (k_3\gamma + p_3 + \hat{p}_{30})(1 - k_6\gamma - p_6 - \hat{p}_{60}) & (k_4\gamma + p_4 + \hat{p}_{40})(1 - k_6\gamma - p_6 - \hat{p}_{60}) \\ -(k_1\gamma + p_1 + \hat{p}_{10})(k_5\gamma + p_5 + \hat{p}_{50}) & -(k_2\gamma + p_2 + \hat{p}_{20})(k_5\gamma + p_5 + \hat{p}_{50}) \\ -(k_3\gamma + p_3 + \hat{p}_{30})(k_6\gamma + p_6 + \hat{p}_{60}) & -(k_4\gamma + p_4 + \hat{p}_{40})(k_6\gamma + p_6 + \hat{p}_{60}) \end{pmatrix} \tag{6.34b}$$

$$C = \big(I_2\ 0_2\ 0_2\big)\ . \tag{6.34c}$$

The parameters $p_1, \dots, p_8$ are time-varying; moreover $p \in R$ where

$$\begin{aligned} R := &[-10^{-3}, 10^{-3}] \times [-1.15, 1.15] \\ &\times [-3\cdot 10^{-4}, 3\cdot 10^{-4}] \times [-5.2\cdot 10^{-3}, 5.2\cdot 10^{-3}] \\ &\times [-1.7\cdot 10^{-3}, 1.7\cdot 10^{-3}] \times [-10^{-3}, 10^{-3}] \\ &\times [-1.7\cdot 10^{-3}, 1.7\cdot 10^{-3}] \times [-2.4\cdot 10^{-3}, 2.4\cdot 10^{-3}]\,. \end{aligned}$$

The parameter $\gamma \in [0,1]$ is slowly varying and is subject to

$$|\gamma(k+1) - \gamma(k)| \le 0.0096. \tag{6.35}$$

Finally

$$\begin{array}{ll}
\hat{p}_{10} = 6.5 \cdot 10^{-3} & k_1 = 2.50 \cdot 10^{-3} \\
\hat{p}_{20} = -0.04 & k_2 = 15.0 \\
\hat{p}_{30} = -1.0 \cdot 10^{-4} & k_3 = 0.00 \\
\hat{p}_{40} = 0.022 & k_4 = 0.0172 \\
\hat{p}_{50} = 0.8146 & k_5 = -0.0189 \\
\hat{p}_{60} = 0.997 & k_6 = -0.0073 \\
\hat{p}_{70} = 0.9883 & k_7 = -0.025 \\
\hat{p}_{80} = 0.2461 & k_8 = -0.042 \,.
\end{array}$$

Our goal is to solving the following problem.

Problem 6.4.
Design a dynamical output feedback controller guaranteeing the closed loop robust exponential stability of system (6.33) versus

i) the time-varying parameters $p(\cdot) : \mathbb{N}_0 \to R$;
ii) the slowly varying parameter $\gamma(\cdot) : \mathbb{N}_0 \to [0,1]$ (whose rate of variation is bounded by (6.35)).

◇

Note that the matrices of system (6.33) depend multi-affinely on p and quadratically on γ.

6.3.1 Controller Design

Now consider the following output feedback γ-gain scheduled controller of order n_c

$$x_c(k+1) = A_c(\gamma)x_c(k) + B_c(\gamma)y(k) \tag{6.36a}$$
$$u(k) = C_c(\gamma)x_c(k) + D_c(\gamma)y(k)\,, \tag{6.36b}$$

where, by partitioning the interval $[0,1]$ into 10 sub-intervals of length 0.1 according to (6.19), we obtain the following correspondence for the controller matrices

$$A_c : \gamma \in [\gamma_j, \gamma_{j+1}] \mapsto A_{cj} \in \mathbb{R}^{n_c \times n_c} \tag{6.37a}$$
$$B_c : \gamma \in [\gamma_j, \gamma_{j+1}] \mapsto B_{cj} \in \mathbb{R}^{n_c \times 2} \tag{6.37b}$$
$$C_c : \gamma \in [\gamma_j, \gamma_{j+1}] \mapsto C_{cj} \in \mathbb{R}^{2 \times n_c} \tag{6.37c}$$
$$D_c : \gamma \in [\gamma_j, \gamma_{j+1}] \mapsto D_{cj} \in \mathbb{R}^{2 \times 2}\,, \tag{6.37d}$$

with $j = 1, \dots, 10$, $\gamma_1 = 0$, $\gamma_{11} = 1$. Note that, by virtue of (6.35), the maximum number of sub-intervals that the parameter γ can jump in one time step is equal to one.

The closed loop connection of (6.33) and (6.36) gives rise to the following system

$$\begin{pmatrix} x(k+1) \\ x_c(k+1) \end{pmatrix} = A_{CL}(p,\gamma) \begin{pmatrix} x(k) \\ x_c(k) \end{pmatrix}$$
$$:= \begin{pmatrix} A(p,\gamma)+B(p,\gamma)D_{cj}C & B(p,\gamma)C_{cj} \\ B_{cj}C & A_{cj} \end{pmatrix} \begin{pmatrix} x(k) \\ x_c(k) \end{pmatrix}, \quad (6.38)$$

for all $\gamma \in [\gamma_j, \gamma_{j+1}]$, $j = 1, \dots, 10$.

By inspection of (6.34) and (6.38) it is readily seen that the matrix function A_{CL} is multi-affine in p and quadratic in γ. Let $\tilde{A}_j(p,\omega^j)$ and $\tilde{B}_j(p,\omega^j)$, $\omega^j = \begin{pmatrix} \gamma & \delta \end{pmatrix}^T$, be the multi-affine matrix functions obtained by replacing, according to Procedures 3.1 and 3.2, in the matrices of system (6.33), restricted to $[\gamma_j, \gamma_{j+1}]$, $j = 1, \dots, 10$, the term $f(\gamma) = \gamma^2$ by $f_j^m(\gamma, \delta)$; correspondingly define as $\tilde{A}_{CL}(p,\omega^j)$ the matrix obtained by substituting in (6.38) the matrix functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ with $\tilde{A}_j(\cdot,\cdot)$ and $\tilde{B}_j(\cdot,\cdot)$. Consider the following problem.

Problem 6.5.

Find $P_j > 0$, A_{cj}, B_{cj}, C_{cj}, D_{cj}, $j = 1, \dots, 10$, such that for $l = 1, \dots, 256$, $i = 1, \dots, 4$

$$\left.\begin{array}{l} \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^1)P_1\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^1) - P_1 < 0 \\ \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^1)P_2\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^1) - P_1 < 0 \end{array}\right\} \quad (6.39a)$$

$$\left.\begin{array}{l} \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^j)P_j\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^j) - P_j < 0 \\ \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^j)P_{j+1}\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^j) - P_j < 0 \\ \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^j)P_{j-1}\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^j) - P_j < 0 \end{array}\right\}, \quad j = 2, \dots, 9, \quad (6.39b)$$

$$\left.\begin{array}{l} \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^{10})P_{10}\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^{10}) - P_{10} < 0 \\ \tilde{A}_{CL}^T(p_{(l)},\omega_{(i)}^{10})P_9\tilde{A}_{CL}(p_{(l)},\omega_{(i)}^{10}) - P_{10} < 0 \end{array}\right\}. \quad (6.39c)$$

◇

Since $\tilde{A}_{CL}(\cdot,\cdot)$ is multi-affine in its arguments we can conclude, by using the results of Sect. 6.1 and Theorem 6.5, that there exists an output feedback dynamical, gain scheduled controller in the form (6.36) solving Problem 6.4 if there exist $P_j > 0$, A_{cj}, B_{cj}, C_{cj} and D_{cj}, $j = 1, \dots, 10$, such that Problem 6.5 admits a feasible solution.

6.3.2 Implementation Aspects and Numerical Results

Since Problem 6.5 is nonlinear in the optimization variables and due to the large number of constraints, some difficulties arise when practically solving this problem; in this section we consider a computationally tractable problem, whose solution also satisfies the original Problem 6.5.

The basic observation, which allows to reduce the computational burden in Problem 6.5, is the fact that the plant is, in the first approximation, diagonal dominant which means that the stagnation temperature mainly depends on the arc current while the stagnation pressure depends on the mass flow-rate. Therefore it is possible to separately design two Single-Input Single-Output (SISO) controllers on the channels 1–1 and 2–2 on low order plants and to verify *a posteriori* the robust stability of the two input-two output whole plant. The following procedure is proposed for the controller synthesis.

Procedure 6.1.

Step 1
A state space model of the 1-1 and 2-2 channels, say $(A_i(p,\gamma), b_i(p,\gamma), c_i^T)$, $i = 1, 2$, is computed.

Step 2
The interval $[0, 1]$, in which γ takes values, is partitioned into 10 sub-intervals of length 0.1.

Step 3
In order to decrease the number of parameters of the controller and for the sake of implementation simplicity, a standard PID controller, gain scheduled with the parameter γ, is assumed. In this way it is also guaranteed the complete rejection of the additive constant errors in the computation of the feedforward control law.

Step 4
For each one of the two input-output channels a reduced optimization problem is solved in order to achieve nominal performance plus robust stability by means of the fixed structure gain scheduled PID controllers.

Step 5
Once the values of the two SISO controller gains have been obtained, the solvability of Problem 6.5 for the whole plant is checked by means of the solution of the LMI problem (6.39), where the only optimization variables are the matrices P_j, $j = 1, \ldots, 10$, the controller matrices being already known from Step 4.

◇

As for the choice of the PID controllers, for the arc current–stagnation temperature channel it is assumed a complete PID controller in the form

$$\frac{k_{c1}z^2 + k_{c2}z + k_{c3}}{z^2 - z} \tag{6.40}$$

whose state space realization is

$$\begin{aligned} x_{c1}(k+1) &= A_{c1}x_{c1}(k) + b_{c1}u_{c1}(k) \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x_{c1}(k) + \begin{pmatrix} k_{c3} \\ k_{c2} + k_{c1} \end{pmatrix} u_{c1}(k) \end{aligned} \tag{6.41a}$$

$$\begin{aligned} y_{c1}(k) &= c_{c1}x_{c1}(k) + d_{c1}u_{c1}(k) \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_{c1}(k) + k_{c1}u_{c1}(k)\,, \end{aligned} \tag{6.41b}$$

while for the mass flow rate–stagnation pressure channel an integral action was sufficient. In state space form we have

$$\begin{aligned} x_{c2}(k+1) &= a_{c2}x_{c2}(k) + b_{c2}u_{c2}(k) \\ &= x_{c2}(k) + k_{c4}u_{c2}(k) \end{aligned} \tag{6.42a}$$

$$\begin{aligned} y_{c2}(k) &= c_{c2}x_{c2}(k) + d_{c2}u_{c2}(k) \\ &= x_{c2}(k)\,. \end{aligned} \tag{6.42b}$$

In the above expressions, according to (6.37), the scalars k_{ci}, $i = 1, \dots, 4$, are gain scheduled with the parameter γ.

As for the optimization problems of Step 4, for the $i - i$ input-output channel ($i = 1, 2$) we have to solve the following problem.

Problem 6.6.
Find symmetric matrices $P_j > 0$ and scalar PID gains (k_{c1j}, k_{c2j}, k_{c3j} for the $1 - 1$ channel and k_{c4j} for the $2 - 2$ channel, $j = 1, \dots, 10$) which minimize the following cost function

$$J_i = \sum_{k=1}^{N_i} \left[y_i(k) - y_{mi}(k)\right]^2 + \rho_i u_i^2(k) \tag{6.43}$$

subject to the constraints (6.39) properly rewritten considering the closed loop system related to the $i - i$ channel. In (6.43) N_i is the optimization time horizon, $y_i(\cdot)$ is the step response of the i-th closed loop nominal model (obtained by letting $p_i = 0$, $i = 1, \dots, 8$, $\gamma = 0.5$), $y_{mi}(\cdot)$ is the step response of the i-th channel reference model

$$\left(1 + \frac{2\zeta_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)^{-1}, \tag{6.44}$$

$u_i(\cdot)$ is the i-th input of the system and ρ_i is a weighting scalar parameter. $\diamond$

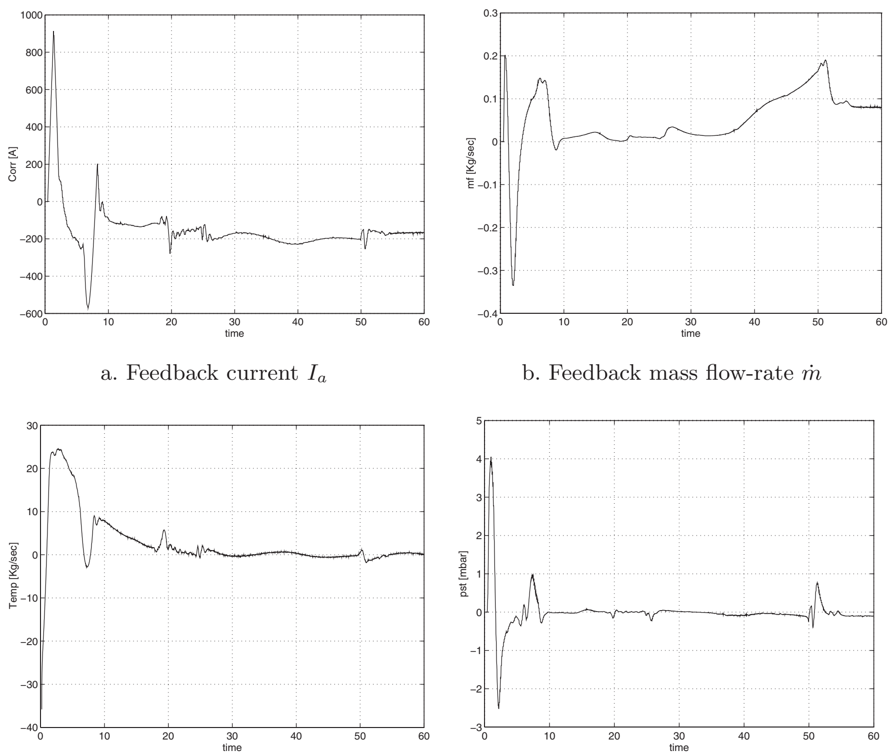
For the solution of Problem 6.6 we assumed $N_1 = 500$, $N_2 = 200$, $\zeta_1 = 0.7$, $\zeta_2 = 0.7$, $\omega_{n1} = 2$, $\omega_{n2} = 10$, $\rho_1 = 0.001$, $\rho_2 = 10^7$.

In Table 6.1 the numerical values of the scheduled gains are given. The solution has been obtained numerically by using the standard algorithms implemented in the MATLAB$^{\mathrm{TM}}$ optimization toolbox [87], and by imposing the LMI constraints with the aid of the LMI toolbox [83].

In Fig. 6.4 the feedback control signals I_a, $\dot{m}$ and the tracking errors T_s, p_s time behaviors corresponding to the closed loop system response to a step disturbance on the stagnation heat flux of $50\,kW/m^2$, which is about the 20% of the initial nominal stagnation heat flux, are shown. The simulation concerns the tracking of trajectory

Table 6.1. Controller Gains

γ	k_{c1}	k_{c2}	k_{c3}	k_{c4}
[0.0, 0.1]	31.000346	-27.390696	-2.9595254	4.0100000
[0.1, 0.2]	30.690535	-27.117538	-2.9230392	3.7000000
[0.2, 0.3]	30.385297	-26.846731	-2.8975177	3.9000000
[0.3, 0.4]	30.076711	-26.577080	-2.8648736	3.8000000
[0.4, 0.5]	29.760077	-26.297461	-2.8340359	3.9000000
[0.5, 0.6]	29.453834	-26.025840	-2.8020896	3.9000000
[0.6, 0.7]	29.140668	-25.748988	-2.7787466	3.9000000
[0.7, 0.8]	28.834175	-25.472897	-2.7503296	4.0000000
[0.8, 0.9]	28.526868	-25.200378	-2.7133745	4.0000000
[0.9, 1.0]	28.215890	-24.931375	-2.6863734	3.9000000



a. Feedback current I_a

b. Feedback mass flow-rate $\dot{m}$

c. Stagnation temperature error T_s

d. Stagnation pressure error p_s

Fig. 6.4. The feedback control action and the tracking errors

$$T_{sd}(t) = \begin{cases} 1270\,K & \text{for} \quad t \in [0,7) \quad \text{and} \quad t \in [53,60] \\ 1270 + 50(t-7)\,K & \text{for} \quad t \in [7,20) \\ 1920\,K & \text{for} \quad t \in [20,27) \\ 1920 - 25(t-27)\,K & \text{for} \quad t \in [27,53) \end{cases}$$

$$p_{sd}(t) = \qquad 50\,mb \quad t \in [0,60]\,,$$

and is performed in presence of an initial state displacement on the temperature of the test model of $40K$.

The perturbation is recovered by the control system and the perfect tracking is guaranteed in a few seconds; moreover the control laws are feasible with respect to the capabilities of the power supply system and the air supply system of the plant.

We conclude the section remarking that an accurate analysis must be performed regarding the gain scheduling mechanism of the controller with the parameter γ. For more details on this point the interested reader is referred to [14].

In the following section, by applying the results of Sects. 5.2.1 and 5.4.2, we shall consider a more systematic approach to the controller design which, as usual, will require the solution of LMIs based feasibility problems.

6.4 Quadratic Stabilization

6.4.1 Parametric Uncertainties

Consider the uncertain discrete-time system in the form

$$x(k+1) = A(p)x(k) + B(p)u(k)\,, \tag{6.46}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$.

Definition 6.2 (Quadratic stabilizability via state feedback). The uncertain system (6.46) is said to be *quadratically stabilizable via linear state feedback control if and only if* there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the closed loop system, obtained from (6.46) by letting $u = Gx$,

$$x(k+1) = \big(A(p) + B(p)G\big)x(k)$$

is QS. ◇

By virtue of Theorem 6.1 quadratic stabilizability via state feedback of the discrete-time system (6.46) is equivalent to quadratic $\mathcal{D}$-stabilizability via state feedback of the continuous-time system (5.2) with $\mathcal{D}$ the unit disk centered at the origin of the complex plane. Therefore, from Theorem 5.5, we can state the following result.

Theorem 6.6. *System* (6.46) *is quadratically stabilizable via linear state feedback control* if and only if *there exist a positive definite matrix Q and a matrix V such that*

$$Q > 0 \tag{6.47a}$$

$$\begin{pmatrix} -Q & A(p)Q + B(p)V \\ QA^T(p) + V^T B^T(p) & -Q \end{pmatrix} < 0\,; \tag{6.47b}$$

in this case a state feedback controller which quadratically stabilizes system (6.46) *is given by $u = Gx$ with $G = VQ^{-1}$.* □

If Assumption 5.1 holds a necessary and sufficient condition for quadratic stabilizability via state feedback of system (6.46) is the feasibility of the following problem involving LMIs.

Problem 6.7.
Find a symmetric matrix Q and a matrix V such that

$$Q > 0 \tag{6.48a}$$

$$\begin{pmatrix} -Q & A(p_{(i)})Q + B(p_{(i)})V \\ QA^T(p_{(i)}) + V^T B^T(p_{(i)}) & -Q \end{pmatrix} < 0\,, \qquad i = 1, \ldots, 2^q\,. \tag{6.48b}$$

◇

Now we proceed with the output feedback case; consider the uncertain system

$$x(k+1) = A(p)x(k) + B(p)u(k) \tag{6.49a}$$

$$y(k) = C(p)x(k)\,, \tag{6.49b}$$

where $y(k) \in \mathbb{R}^r$.

Definition 6.3 (Quadratic stabilizability via output feedback). The uncertain system (6.49) is said to be *quadratically stabilizable via parameter dependent output feedback linear control* if and only if there exists a dynamical controller in the form

$$x_c(k+1) = A_K(p)x_c(k) + B_K(p)y(k) \tag{6.50a}$$

$$u(k) = C_K(p)x_c(k) + D_K(p)y(k)\,, \tag{6.50b}$$

where $x_c(k) \in \mathbb{R}^n$, and $A_K(\cdot)$, $B_K(\cdot)$, $C_K(\cdot)$, $D_K(\cdot)$ are continuous matrix-valued functions, such that the closed loop system obtained by the connection of system (6.49) and controller (6.50) is QS. ◇

From Theorem 5.6, with $\mathcal{D}$ the unit disk centered at the origin of the complex plane, we obtain the following result.

Theorem 6.7. *System* (6.49) *is quadratically stabilizable via parameter dependent output feedback linear control* if and only if *there exist positive definite matrices Q and S and matrix-valued functions $V(\cdot) \in \mathcal{C}_0(R, \mathbb{R}^{m\times n})$ and $W(\cdot) \in \mathcal{C}_0(R, \mathbb{R}^{n\times r})$ such that, for all $p \in R$,*

$$\begin{pmatrix} -Q & A(p)Q + B(p)V(p) \\ QA^T(p) + V^T(p)B^T(p) & -Q \end{pmatrix} < 0 \tag{6.51a}$$

$$\begin{pmatrix} -S & SA(p) + W(p)C(p) \\ A^T(p)S + C^T(p)W^T(p) & -S \end{pmatrix} < 0\,. \tag{6.51b}$$

In this case a quadratically stabilizing controller has the structure (5.36) *with $G(p) = V(p)Q^{-1}$ and $L(p) = S^{-1}W(p)$.* □

When Assumption 5.2 holds the feasibility of the following LMIs problem guarantees quadratic stabilizability via output feedback.

Problem 6.8.
Find symmetric matrices Q, S and matrices V, W, such that

$$Q > 0 \tag{6.52a}$$

$$S > 0 \tag{6.52b}$$

$$\begin{pmatrix} -Q & A(p_{(i)})Q + B(p_{(i)})V \\ QA^T(p_{(i)}) + V^T B^T(p_{(i)}) & -Q \end{pmatrix} < 0\,, i = 1, \ldots, 2^q \tag{6.52c}$$

$$\begin{pmatrix} -S & SA(p_{(i)}) + WC(p_{(i)}) \\ A^T(p_{(i)})S + C^T(p_{(i)})W^T & -S \end{pmatrix} < 0\,, i = 1, \ldots, 2^q. \tag{6.52d}$$

◇

If Problem 6.8 is feasible, a quadratically stabilizing controller has the structure (5.36) with constant gain matrices $G = VQ^{-1}$ and $L = S^{-1}W$.

6.4.2 Norm Bounded Uncertainties

Necessary and sufficient conditions for quadratic stabilizability can be readily obtained by applying the corresponding results for quadratic $\mathcal{D}$-stabilizability of continuous-time systems with $\mathcal{D}$ the unit disk centered at the origin of the complex plane.

As for quadratic stabilizability via state feedback we have the following result. Let us consider the uncertain system

$$x(k+1) = (A + \Delta A)x(k) + (B + \Delta B)u(k)\,, \tag{6.53}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and the uncertainty satisfies (5.72).

From Theorem 5.12 we have the following result.

Theorem 6.8. *System* (6.53), (5.72) *is quadratically stabilizable via state feedback control* if and only if *there exist a positive definite matrix* Q *and a matrix* V *such that*

$$\begin{pmatrix} -Q & AQ + BV & F & 0 \\ QA^T + V^T B^T & -Q & 0 & QE_1^T + V^T E_2^T \\ F^T & 0 & -I_p & H^T \\ 0 & E_1 Q + E_2 V & H & -I_q \end{pmatrix} < 0\,. \tag{6.54}$$

In this case a state feedback controller which quadratically stabilizes system (6.53), (5.72) *is given by* $u = Gx$ *with* $G = VQ^{-1}$. □

Now let us consider the output feedback case; consider the uncertain system

$$x(k+1) = \big(A + \Delta A\big)x(k) + \big(B + \Delta B\big)u(k) \tag{6.55a}$$

$$y(k) = \big(C + \Delta C\big)x(k) + \big(D + \Delta D\big)u(k)\,, \tag{6.55b}$$

where $y(k) \in \mathbb{R}^r$ and the uncertainty satisfies (5.78).

From Theorem 5.13 we have the following result.

Theorem 6.9. *System* (6.55), (5.78) *is quadratically stabilizable via output feedback linear control in the form*

$$x_c(k+1) = A_K x_c(k) + B_K y(k) \tag{6.56a}$$

$$u(k) = C_K x_c(k) + D_K y(k) \tag{6.56b}$$

if and only if *there exists positive definite matrices S, Q and matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$ and D_K such that* (5.11) *holds and*

$$\begin{pmatrix} -Q & -I_n & AQ + B\hat{C}_K & A + BD_K C & F_1 + BD_K F_2 & 0 \\ -I_n & -S & \hat{A}_K & SA + \hat{B}_K C & SF_1 + \hat{B}_K F_2 & 0 \\ QA^T + \hat{C}_K^T B^T & \hat{A}_K^T & -Q & -I_n & 0 & QE_1^T + \hat{C}_K^T E_2^T \\ A^T + C^T D_K^T B^T & A^T S + C^T \hat{B}_K^T & -I_n & -S & 0 & E_1^T + C^T D_K^T E_2^T \\ F_1^T + F_2^T D_k^T B^T & F_1^T S + F_2^T \hat{B}_K^T & 0 & 0 & -I_p & H^T + F_2^T D_K^T E_2^T \\ 0 & 0 & E_1 Q + E_2 \hat{C}_K & E_1 + E_2 D_K C & H + E_2 D_K F_2 & -I_q \end{pmatrix} < 0\,. \tag{6.57}$$

In this case a quadratically stabilizing controller has the structure (6.56) *where A_K, B_K and C_K can be obtained by solving* (5.85). □

Summary

In this chapter we have considered discrete-time linear systems subject to uncertainties. A key point is that quadratic stability of a given discrete-time linear systems is equivalent to quadratic $\mathcal{D}$-stability, with $\mathcal{D}$ the unit disk centered at the origin of the complex plane, of the continuous-time system having the same system matrix.

A necessary and sufficient condition for quadratic stability when the system depends on parametric uncertainties is derived; it is turned into an LMIs based feasibility problem when the matrix $A(\cdot)$ depends on parameters as the ratio of a multi-affine matrix-valued function and a multi-affine polynomial. This result was first proven in [16] without using Shur Complements arguments. When the dependence on parameters does not satisfy Assumption 3.1 we can resort to one of the approaches suggested in Chap. 3.

Concerning the systems depending on uncertainties in norm bounded form, again the equivalence between quadratic $\mathcal{D}$-stability for continuous and quadratic stability for discrete-time systems is exploited; indeed a necessary and sufficient condition for quadratic stability is obtained via the direct application of Theorem 3.10. Moreover the connections between quadratic stability and $\mathcal{H}_\infty$ theory found in the continuous-time case still hold, with the obvious changes, for discrete-time systems.

It is interesting to note that the parallel between continuous and discrete-time systems is also valid for the multi-block case. In [134] a pair of counter-examples shows that, in the multi-block case, quadratic stability for real uncertainties and robust stability for complex uncertainties are no longer equivalent concepts.

Then we have considered systems depending on bounded rate parameters. A sufficient condition for stability has been provided; as usual, this condition is given in terms of the solvability of a feasibility problem involving LMIs. Differently from the continuous-time case, piecewise constant parameter dependent quadratic Lyapunov functions have been used in order to establish the main result of Sect. 6.2.

It is worth to notice that, in the hybrid piecewise affine (PWA) systems context, an approach similar to the one described in Sect. 6.2 has been proposed in [70] and [125]. The main difference between the robust control context, considered in this chapter, and the hybrid system context is that in the first case the focus in on the vector of uncertain parameters, whose jumps in a discrete time interval are bounded by its rate of variation, while in the second case the focus is on the events that possibly drive the state from a certain region to another at each time-step. In [70] and [125] sufficient conditions for the stability and stabilizability with $\mathcal{H}_2/\mathcal{H}_\infty$ performance via state feedback of PWA systems are proposed.

An approach based on parameter dependent Lyapunov functions has been used in [54] and [55] where, however, the parameters rate of variation is not taken into account and the parameters bounds are given in terms of the vector 1-norm instead of the vector ∞-norm (note, however, that the two norms are equivalent when the uncertain system depends on just one parameter).

Finally, in [147] a sufficient condition for the existence of a parameter dependent Lyapunov function for a polytopic discrete-time system is provided. For a general discussion on parameter dependent Lyapunov functions in the discrete-time context the interested reader is referred to [92] (for the specific application to systems depending on bounded rate parameters see [94]); for an LPV perspective of the topics discussed in this chapter see [161].

An important point is that, differently from continuous-time systems, discrete time-varying parameters are always bounded rate [18, 19]. This because a discrete-time parameter cannot vary in the unit time step more than the length of the interval to which it belongs. Hence we have that inequality (6.18) is always satisfied with at least $h_i = |\underline{p}_i - \overline{p}_i|$. Therefore, also when no information on the parameter rate of variation is known, the approach of Sect. 6.2 can be applied to obtain less conservative conditions for system stability (obviously at the price of a greater computational burden with respect to the classical quadratic stability approach of Sect. 6.1).

The practical application of some results established in the previous sections has been illustrated in Sect. 6.3, where the robust stabilization of a plasma wind tunnel is considered. The interesting point is that this system is subject to both fast and slowly varying parameters; the designed controller, which is scheduled with respect to the slowly varying parameter, is shown to exhibit good performances.

Finally, the equivalence between quadratic stability of systems (6.1), (6.2) and quadratic $\mathcal{D}$-stability of systems (3.1), (3.116) respectively, with $\mathcal{D}$ the unit disk, allows to readily find necessary and sufficient conditions for quadratic stabilizability via both state and output feedback, both for parametric and norm bounded uncertainties.

Exercises

Exercise 6.1. Show that condition (6.6) is equivalent to (3.86) when $\mathcal{D}$ is the unit disk centered at the origin of the complex plane. ◇

Exercise 6.2. Consider system (6.9) with

$$A_0 = \begin{pmatrix} 0.9979 & -0.01 \\ 0.01 & 1 \end{pmatrix} \qquad B = b = \begin{pmatrix} 0.008 \\ 0 \end{pmatrix} \qquad C = c^T = \begin{pmatrix} 0 & 1 \end{pmatrix} . \quad (6.58)$$

Show that $\rho_Q \cong 0.249$. ◇

Exercise 6.3. Prove Theorem 6.3. ◇

Exercise 6.4. In this example we extend the concept of gain margin when the parameter has a bounded rate. Consider again the uncertain discrete-time system described by (6.58). We can associate two different robustness measures with such system; one is the QSM defined in Example 6.1 and computed in Exercise 6.2, the other is the gain margin, defined as follows[2]

$$\rho_G := \sup \left\{ \theta > 0 \, : \, \text{matrix } A_0 + bpc^T \text{ is Shur} \, \forall p \in [-\theta, \theta] \right\} . \quad (6.59)$$

In this case the gain margin can be computed with the aid of the Jury Stability criterion [103]. As in the continuous-time case, the gain margin represents the amplitude of the smallest constant parameter which destabilizes the system. In the current case, we have that $\rho_G \cong 1.25$; as expected, $\rho_Q < \rho_G$.

[2] A square matrix is said to be Shur if all its eigenvalues are strictly enclosed in the unit disk centered at the origin of the complex plane.

Next we consider the situation in which p is time-varying and satisfies

$$|p(k)| \leq \overline{p} := 1 \quad \text{for all} \quad k \in \mathbb{N}_0 \,. \tag{6.60}$$

By using Theorem 6.5, compute an estimate of the supremal allowable h such that the system is exponentially stable for all $p(\cdot)$ satisfying (6.60) and (6.18). $\diamond$

A. Appendix

A.1 Definite Matrix Sets

Definition A.1 (Polytope). Given a linear space $\mathcal{V}$ over $\mathbb{R}$ and μ points $v_{(i)} \in \mathcal{V}$, $i = 1, \ldots, \mu$, a polytope of vertices $v_{(i)}$ is a set in the form

$$\Pi = \left\{ v \in \mathcal{V} \,:\, v = \sum_{i=1}^{\mu} \lambda_i v_{(i)}, \sum_{i=1}^{\mu} \lambda_i = 1, \lambda_i \geq 0\,, i = 1, \ldots, \mu \right\}. \tag{A.1}$$

We denote the set of the vertices of the polytope by $\Pi^v := \{v_{(i)}, i = 1, \ldots, \mu\}$. ◇

Note that a hyper-box $R \subset \mathbb{R}^q$ is a particular polytope with 2^q vertices.

Definition A.2 (Convex hull). Given a linear space $\mathcal{V}$ over $\mathbb{R}$ and a set $E \subset \mathcal{V}$ the convex hull of E is defined as the subset of $\mathcal{V}$ composed of all vectors obtained via convex combination from the elements of E, namely

$$\begin{aligned} \operatorname{conv}(E) := \Bigg\{ v \in \mathcal{V} \,:\, v = \sum_{i=1}^{h} \lambda_i v^{(i)}, \\ \sum_{i=1}^{h} \lambda_i = 1, \lambda_i \geq 0, v^{(i)} \in E, i = 1, \ldots, h\,,\, h = 1, 2, \ldots \Bigg\}. \end{aligned} \tag{A.2}$$

◇

Obviously $\operatorname{conv}(E) \supseteq E$, and the equality holds if E is a polytope. Moreover, note that the convex hull of a *finite* set is a polytope.

The next result is known in the literature as the "Mapping Theorem" [180].

Lemma A.1. *Let $F : R \to \mathbb{R}^{n \times n}$, $p \to F(p)$, be a multi-affine matrix-valued function, where $R \subset \mathbb{R}^q$ is a hyper-box. Then*

$$\operatorname{conv}\left(F(R)\right) = \operatorname{conv}\left(F(R^v)\right)\,.$$

□

Originally Lemma A.1 was stated for vector-valued functions; however its extension to the matrix case is immediate, taking into account that we can establish an isomorphism between $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n^2}$.

When the dependence is affine (rather than multi-affine), the image (rather than the convex hull of the image) of a matrix function defined over a hyper-box coincides with the convex hull of the images of the vertices of the hyper-box. This can be simply proved by using Lemma A.1.

Theorem A.1. *Let $F : R \to \mathbb{R}^{n\times n}$, $p \to F(p)$, be an affine matrix-valued function, where $R \subset \mathbb{R}^q$ is a hyper-box. Then*

$$F(R) = \mathrm{conv}\,(F(R^v))\ .$$

Proof. Note that $H \in F(R)$ implies that $H \in \mathrm{conv}\,(F(R))$, which, by virtue of Lemma A.1, in turn implies that H also belongs to $\mathrm{conv}\,(F(R^v))$.

Conversely, assume that $H \in \mathrm{conv}\,(F(R^v))$. Therefore there exist $\lambda_i \geq 0$, $i = 1, \ldots, 2^q$, $\sum_{i=1}^{2^q} \lambda_i = 1$, such that

$$\begin{aligned} H &= \sum_{i=1}^{2^q} \lambda_i F(p_{(i)}) \\ &= F\left(\sum_{i=1}^{2^q} \lambda_i p_{(i)}\right)\ , \end{aligned} \tag{A.3}$$

where the last equality is a consequence of the fact that $F(\cdot)$ is affine. Since R is a hyper-box, we have that $\sum_{i=1}^{2^q} \lambda_i p_{(i)} \in R$; from this the proof follows. □

Now, given a set of symmetric matrices Γ, we write $\Gamma > 0$ meaning that $G > 0$ for all $G \in \Gamma$.

Lemma A.2. *A set of symmetric matrices $\Gamma \subset \mathbb{R}^{n\times n}$ is positive definite* iff $\mathrm{conv}(\Gamma)$ *is positive definite.*

Proof. The fact that $\mathrm{conv}(\Gamma) > 0$ implies $\Gamma > 0$ is obvious. Conversely, assume that $\Gamma > 0$. Now any matrix $G \in \mathrm{conv}(\Gamma)$ can be written, according to the definition of convex hull, as

$$G = \sum_{i=1}^{h} \lambda_i G_i, \tag{A.4}$$

where h is some integer, $\sum_{i=1}^{h} \lambda_i = 1$, $\lambda_i \geq 0$, and $G_i \in \Gamma$, $i = 1, \ldots, h$. Since $G_i > 0$, $i = 1, \ldots, h$, we conclude that G is positive definite. From the arbitrariness of G, the proof follows. □

From Lemmas A.1 and A.2 we obtain the following fundamental result.

Theorem A.2 ([12, 85]). *Let us consider the matrix-valued function* $F : R \to \mathbb{R}^{n\times n}$, $p \to F(p)$, *where* $R \subset R^q$ *is a hyper-box and*

$$F(p) = \frac{N_F(p)}{d_F(p)},$$

with $N_F(\cdot)$ *and* $d_F(\cdot)$ *multi-affine functions of* p *and* $d_F(p) \neq 0$ *for all* $p \in R$.

Then we have that $F(R) > (<)0$ iff $F(R^v) > (<)0$.

Proof. That $F(R) > 0$ implies $F(R^v) > 0$ is obvious; conversely, assume that $F(R^v) > 0$.

Since $d_F(p)$ cannot change sign in R assume that

$$d_F(p) > 0\,, \qquad \forall p \in R\,. \tag{A.5}$$

Condition (A.5) implies that $d_F(p) > 0$ for all $p \in R^v$; since $F(R^v) > 0$, we have that $N_F(R^v) > 0$.

We have the following chain of statements

$$\begin{aligned} N_F(R^v) > 0 &\Rightarrow \mathrm{conv}(N_F(R^v)) > 0 && \text{by Lemma A.2}\\ &\Rightarrow \mathrm{conv}(N_F(R)) > 0 && \text{by Lemma A.1}\\ &\Rightarrow N_F(R) > 0 && \text{by Lemma A.2}\,. \end{aligned} \tag{A.6}$$

The last inequality together with condition (A.5) guarantee that $F(R) > 0$. The proof is analogous when $d_F(p) < 0$ in R. □

A.2 Kronecker Product and Sum

Given two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix} \in \mathbb{R}^{np\times mq}\,. \tag{A.7}$$

Some properties of the Kronecker product are (all matrices are intended to be of compatible dimensions)

$$1 \otimes A = A \tag{A.8a}$$
$$(A+B)\otimes C = A\otimes C + B\otimes C \tag{A.8b}$$
$$(AB)\otimes(CD) = (A\otimes C)(B\otimes D) \tag{A.8c}$$
$$(A\otimes B)^T = A^T \otimes B^T \tag{A.8d}$$
$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}\,. \tag{A.8e}$$

If A and B are square matrices of dimensions $n \times n$ and $p \times p$ respectively, the matrix $A \otimes B$ is square of dimension $np \times np$. In this case the eigenvalues of $A \otimes B$ are the pairwise products of the eigenvalues of A and B; in other words the eigenvalues of $A \otimes B$ are the elements of the set $\{\lambda_i(A)\lambda_j(B)\,,\, i = 1, \ldots, n,\, j = 1, \ldots, p\}$.

In the same way, given two matrices of compatible dimensions A and B, the Kronecker sum of A and B is given by

$$A \oplus B := A \otimes I + I \otimes B\,. \tag{A.9}$$

If A and B are square matrices of dimensions $n \times n$ and $p \times p$ respectively, we have $A \oplus B = A \otimes I_p + I_n \otimes B \in \mathbb{R}^{np \times np}$; in this case the eigenvalues of $A \oplus B$ are the pairwise sums of the eigenvalues of M and N; in other words the eigenvalues of $M \oplus N$ are the elements of the set $\{\lambda_i(A) + \lambda_j(B)\,,\, i = 1, \ldots, n,\, j = 1, \ldots, p\}$.

A.3 The Lyapunov Equation

In this section we deal with the generalized Lyapunov equation

$$MP + PN = -Q\,, \tag{A.10}$$

where $M \in \mathbb{R}^{m \times m}$ and $N \in \mathbb{R}^{n \times n}$.

In [41] it is shown that equation (A.10) admits a unique solution P *iff* $M \oplus N$ in nonsingular, that is *iff* (see Appendix A.2)

$$\lambda_i(M) + \lambda_j(N) \neq 0\,, \quad i = 1, \ldots, m\,, \quad j = 1, \ldots, n\,; \tag{A.11}$$

in this case we have

$$\mathrm{vec}\,(P) = -(N^T \oplus M)^{-1}\,\mathrm{vec}\,(Q)\,, \tag{A.12}$$

where $\mathrm{vec}\,(F)$ denotes the operation of stacking the columns of matrix F.

We also use the notation

$$P = \mathrm{Lyap}\big(M, N, Q\big)\,. \tag{A.13}$$

When $M = A^T$ and $N = A \in \mathbb{R}^{n \times n}$ we obtain the classical Lyapunov equation

$$A^T P + PA = -Q\,. \tag{A.14}$$

Note that if A is Hurwitz, condition (A.11) is satisfied.

A.3.1 Some Useful Inequalities

Let us consider the Lyapunov equation (A.14). Assume that A is Hurwitz and Q is positive definite; in this case P is also positive definite and (see [108])

$$\lambda_{min}(P) \geq \frac{\lambda_{min}(Q)}{2\|A\|} . \tag{A.15}$$

We now recall the following lemma.

Lemma A.3. *Consider the linear time-invariant system*

$$\dot{x}(t) = Ax(t) . \tag{A.16}$$

Then system (A.16) *is exponentially stable if and only if there exist positive scalars* α *and* k *such that for all* $t \geq 0$

$$\| \exp(At)\| \leq k\mathrm{e}^{-\alpha t} .$$

Proof. It immediately follows from Theorem 2.5. □

When A is Hurwitz the unique solution of equation (A.14) can be written (see [42], p. 188)

$$P = \int_0^{+\infty} \exp(A^T t) Q \exp(At) dt ; \tag{A.17}$$

in this case from Lemma A.3 we can derive the following upper bound for the norm of P

$$\begin{aligned} \|P\| &\leq \int_0^{+\infty} \| \exp(At)\|^2 \|Q\| dt \\ &\leq k^2 \int_0^{+\infty} e^{-2\alpha t} dt \|Q\| \\ &= \frac{k^2\|Q\|}{2\alpha} , \end{aligned} \tag{A.18}$$

for some positive scalars α and k.

When Q is positive definite, the solution P also turns out to be positive definite and (A.18) can be rewritten

$$\lambda_{max}(P) \leq \frac{k^2 \lambda_{max}(Q)}{2\alpha} . \tag{A.19}$$

A.3.2 Discrete-Time Lyapunov Equation

Consider the discrete-time Lyapunov equation

$$A^T PA - P = -Q\,, \tag{A.20}$$

where $A \in \mathbb{R}^{n\times n}$ is Shur and $Q \in \mathbb{R}^{n\times n}$ is positive definite; in this case P is also positive definite and (see [84])

$$\lambda_k(Q) \le \lambda_k(P)\,, \quad k = 1, 2, \dots, n\,, \tag{A.21}$$

where, for a given positive definite matrix $S \in \mathbb{R}^{n\times n}$, we have ordered the eigenvalues as $\lambda_1(S) \ge \lambda_2(S) \ge \cdots \ge \lambda_n(S)$.

In particular we have

$$\lambda_{min}(P) = \lambda_n(P) \ge \lambda_n(Q) = \lambda_{min}(Q)\,. \tag{A.22}$$

The following result is the discrete-time counterpart of Lemma (A.3); it is a straightforward consequence of Theorem 2.12.

Lemma A.4. *Consider the linear time-invariant discrete-time system*

$$x(k+1) = Ax(k)\,. \tag{A.23}$$

Then system (A.23) *is exponentially stable if and only if there exist positive scalars* $\rho \in [0, 1)$ *and* m *such that for all* $k = 0, 1, \dots, n, \dots$

$$\|A^k\| \le m\rho^k\,.$$

□

When A is Shur, it is simple to recognize that the unique solution of equation (A.20) has the following expression

$$P = \sum_{k=0}^{+\infty} \left(A^k\right)^T QA^k\,. \tag{A.24}$$

Therefore, from Lemma A.4, there exist positive scalars $\rho \in [0, 1)$ and m such that

$$\begin{aligned} \|P\| &\le \sum_{k=0}^{+\infty} \|A^k\|^2 \|Q\| \\ &\le m^2 \sum_{k=0}^{+\infty} \rho^{2k} \|Q\| \\ &= \frac{m^2\|Q\|}{1-\rho^2}\,. \end{aligned} \tag{A.25}$$

When Q is positive definite, the solution P also turns out to be positive definite and (A.25) can be rewritten

$$\lambda_{max}(P) \le \frac{m^2 \lambda_{max}(Q)}{1-\rho^2}\,. \tag{A.26}$$

A.4 Shur Complements

In the following we present a fundamental result of LMIs theory, that is how to transform a Riccati type inequality into an LMI. First note that we have

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \begin{pmatrix} I & 0 \\ S^T Q^{-1} & I \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & R - S^T Q^{-1} S \end{pmatrix} \begin{pmatrix} I & Q^{-1} S \\ 0 & I \end{pmatrix} . \tag{A.27}$$

Since the left and right multipliers at the RHS in (A.27) are full rank matrices, the LHS in (A.27) is positive definite *iff* the matrix

$$\begin{pmatrix} Q & 0 \\ 0 & R - S^T Q^{-1} S \end{pmatrix} \tag{A.28}$$

is positive definite. Therefore we can state the following result.

Fact A.1. *The matrix*

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \tag{A.29}$$

is positive definite iff Q *is positive definite and* $R - S^T Q^{-1} S$ *is positive definite.* □

The matrix $R - S^T Q^{-1} S$ is called the *Shur Complement* of Q.
In the same way we can write

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \begin{pmatrix} I & SR^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} Q - SR^{-1} S^T & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} I & 0 \\ R^{-1} S^T & I \end{pmatrix} , \tag{A.30}$$

from which we obtain the following alternative condition for the positive definiteness of matrix (A.29).

Fact A.2. *The matrix*

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \tag{A.31}$$

is positive definite iff R *is positive definite and* $Q - SR^{-1} S^T$ *is positive definite.* □

The matrix $Q - SR^{-1} S^T$ is called the Shur Complement of R.
Since

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} < 0 \tag{A.32}$$

can be equivalently rewritten

$$\begin{pmatrix} -Q & -S \\ -S^T & -R \end{pmatrix} > 0 , \tag{A.33}$$

we have the following conditions concerning negative definiteness.

Fact A.3. *The following conditions are equivalent each other*

i) *Matrix*

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \tag{A.34}$$

is negative definite;

ii) *both matrices* Q *and* $R - S^T Q^{-1} S$ *are negative definite;*

iii) *both matrices* R *and* $Q - S R^{-1} S^T$ *are negative definite.* □

Notation

Matrices and vectors are denoted by capital and small letters respectively; scalars are denoted by small letters.

Abbreviations

iff	if and only if
wrt	with respect to
GEVP	Generalized Eigenvalue Problem
LFT	Linear Fractional Transformation
LHS	Left Hand Side
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
LQ	Linear Quadratic
MGM	Multivariable Gain Margin
MQS	Multi-Affine Quadratically Stable
QS	Quadratically Stable
QSM	Quadratic Stability Margin
RHS	Right Hand Side

Mathematical Symbols

$:$	such that
$\forall$	for all
$\exists$	there exists
$:=$	equal by definition
$p \Rightarrow q$	p implies q

Set Theory

$x \in A$	The element x belongs to the set A
$S_1 \bigcup S_2$	The union of the sets S_1 and S_2
$S_1 \bigcap S_2$	The intersection of the sets S_1 and S_2
$S_1 \subseteq S_2$	The set S_1 is a subset of the set S_2

$S_1 \subset S_2$	The set S_1 is a *strict* subset of the set S_2
$S_1 \times S_2$	Cartesian product between the sets S_1 and S_2
$[t, T]$	Closed interval $t \leq \tau \leq T$
$\mathrm{card}(S)$	Number of elements in the set S
$\mathrm{conv}(E)$	Convex hull of the set E
Π^v	Set of the vertices of the polytope Π
$\underline{p} : step : \overline{p}$	The vector $\left(\underline{p}\ \underline{p} + step \cdots \overline{p} - step\ \overline{p}\right)^T$

Numerical Sets

$\mathbb{N}_0$ ($\mathbb{N}$)	Nonnegative (positive) integer numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{R}^+$	Nonnegative real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{R}^n$	Set of the n-tuple of real numbers
$\mathbb{R}^{m \times n}$ ($\mathbb{C}^{m \times n}$)	Real (complex) matrices with m rows and n columns

Function spaces

$\mathcal{PC}(\Omega, S)$	Space of the piecewise continuous vector (matrix)-valued functions defined over Ω and attaining values into the set S
$\mathcal{C}_0(\Omega, S)$	Space of the continuous vector (matrix)-valued functions defined over Ω and attaining values into the set S
$\mathcal{L}_2(\Omega, S)$	Space of the square integrable vector-valued functions defined over Ω and attaining values into the set S
$l_2(\Omega, S)$	Space of the square summable vector-valued sequences defined over Ω and attaining values into the set S
$\mathcal{RH}_\infty^{n \times m}$	Space of the real-rational proper stable transfer functions of dimension $n \times m$; the dimension is omitted if clear from the context

Vector and matrix operators

x_i	The i-th element of the vector x
a_{ij}	The ij-th element of the matrix A
$\det(A)$	Determinant of a square matrix A
$\mathrm{trace}(A)$	Trace of a square matrix A
A^{-1}	Inverse of a square matrix A
A^T	Transpose of matrix A
A^*	Conjugate transpose of matrix A
$diag(A_1, A_2, \cdots, A_r)$	Block diagonal matrix with A_1, A_2, $\ldots$, A_r on the diagonal

$\lambda_i(A)$	i-th eigenvalue of a square matrix A
$\lambda_{max}(A)$ ($\lambda_{min}(A)$)	The maximum (minimum) eigenvalue of the positive definite matrix A
$\underline{\sigma}(A)$	Smallest singular value of the matrix A
$A > 0$	A is (symmetric) positive definite
$A(\geq)0$	A is (symmetric) positive semidefinite
$A < 0$	A is (symmetric) negative definite
$A \leq 0$	A is (symmetric) negative semidefinite
$A >> (<<)0$	The matrix-valued function $A(\cdot)$ is positive (negative) definite
rank(A)	The rank of matrix A
$A \otimes B$	Kronecker product of matrices A and B

Special matrices

I_n	Identity matrix of dimension $n \times n$; n is omitted if the dimension is clear from the context
0_n	Zero matrix of dimensions $n \times n$; n is omitted if the dimension is clear from the context

Norms

$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$; $\mathcal{L}^2$ norm of the square integrable vector-valued function $x(\cdot)$
$\|x\|_\infty$	Infinity norm of the vector $x \in \mathbb{R}^n$ ($= \max\{\|x_1\|, \ldots, \|x_n\|\}$)
$\|x\|_\infty^w$	Infinity norm of the vector x weighted by the vector w
$\|A\|$	Spectral norm of the matrix A (i. e. the maximum singular value of A)
$\|A\|_F$	Frobenius norm of matrix A ($= \sqrt{Trace(A^T A)}$)
$\|\Gamma_{zw}\|$	$\mathcal{L}_2$ (l_2) induced norm of the linear operator mapping the input vector w to the output vector z
$\|H\|_\infty$	Infinity norm of the transfer function $H(s)$.

Scalar operators

$\ln(a)$	The logarithm of $a \in \mathbb{R}^+$
$\|\alpha\|$	Absolute value of $\alpha \in \mathbb{R}\,(\mathbb{C})$
$Re(\alpha)$	Real part of $\alpha \in \mathbb{C}$

Statistical

$Pr\{E\}$	Probability of the event E

Other symbols

$\left(\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right)$	Compact notation (also used without internal delimiters) to denote the state space realization of the system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$
$\square$	End of theorems, lemmas, corollaries, facts and proofs
$\triangle$	End of examples
$\diamond$	End of assumptions, definitions, problems, procedures, exercises and remarks.

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