Undergraduate Lecture Notes in Physics

Kjell Prytz

Electrodynamics: The Field-Free Approach

Electrostatics, Magnetism, Induction, Relativity and Field Theory

Springer

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Kjell Prytz
Department of Physics
University of Gävle
Gävle
Sweden

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Till mina barn Nicklas och Jonatan,
inspirationens och glädjens källa

Preface

The intent of this book is to serve as an undergraduate textbook in electrodynamics at a basic or advanced level.

The objective is to attain a general understanding of electrodynamic theory and its basic experiments and phenomena in order to form a foundation for further studies in the engineering sciences as well as in modern quantum physics.

The outline of the book is based on the following principles:

- Introduce each phenomenon with relevant and complete experiments
- Focus on experiments and observations accessible to the student
- Base the theory on the concept of force and mutual interaction
- Present electrodynamics using the same principles as in the preceding mechanics course
- Treat electric, magnetic and inductive phenomena cohesively with respect to force, energy, dipoles and material
- Aim at explaining that theory of relativity is based on the magnetic effect
- Introduce field theory *after* the basic phenomena have been explored in terms of force.

Overview

The book starts by considering the different types of forces that occur between electric charges. These may be directly related to their motion, i.e. charges at rest, in uniform motion and in acceleration. The forces, known as electric, magnetic and inductive, are treated cohesively and formulated through observations and measurements. The subsequent chapters are more or less direct applications of the force formulas.

Chapter 3 introduces the energy concept as a direct consequence of force through the principle of work. The inductive force is then utilized to derive magnetic energy. Neumann's formula for inductance is fully derived and used to express magnetic energy.

Chapter 4 covers macroscopic systems whose characteristics are obtained through a summation of mutual interactions between infinitesimal elements of charge. Calculation techniques for capacitance and inductance are introduced and shown to be useful concepts in case the system is homogeneous.

Chapters 5 and 6 deal with the conductor and electric circuits which constitute the experimental environment from which electrodynamics was developed and technical applications originated. The microscopic description of electric conduction, the origin of resistance and its relation to heat are treated first. Then the resonance circuit which includes the other two circuit components, capacitance and inductance, is introduced.

Chapter 7 introduces electric and magnetic dipoles, which are significant concepts since nature generally may be described in terms of such objects. The expressions for electric and magnetic dipole–dipole interaction energies are then central to providing both force and torque.

Chapter 8 investigates how different electrically and magnetically neutral materials respond to electric and magnetic influences. It is then assumed that the material is composed of dipoles. The material parameters are introduced and techniques for measuring them are described. A mathematically rigorous treatment of the dipole, or generally multipole, interactions is presented in the accompanying Appendices A and B.

In Chap. 9, it is shown conceptually how the magnetic and inductive dynamics arise as motional consequences of the electric force assuming that interactions take time; they are mediated at the speed of light. Alternatively, one may utilize the knowledge of electric and magnetic forces to derive the speed of light. In a special case both the magnetic force and the Faraday-Henry's law of induction are derived. It is also shown how electromagnetic dynamics is related to relativity, using the fact that magnetism is the motional consequence on which the special theory of relativity is based. Since we build the theory upon the concept of force the material is unique to this book. Chapter 9 also introduces Lorentz transformation in the form of a tutorial. Prerequisites of Chap. 9 are only Chaps. 1–3, thus these four chapters may form a concise course in basic electrodynamics and its relation to relativity.

In Chap. 10, electromagnetic field theory is introduced and Maxwell's equations formulated. The fields are indeed already defined by the force formulas, but expressed in Maxwell's equations in terms of their divergence and curl. This is motivated by showing that the boundary conditions of the fields are then defined. Using the fields, the Poynting vector may be formulated corresponding to the power transported from an electrodynamic system.

An important feature of this book is thus that field theory is introduced after the physical phenomena that constitute electrodynamics have been described, interpreted and formulated in terms of fundamental forces.

In Chap. 11, antenna theory is introduced using the principle of retarded interactions, i.e. taking into account that interactions take time. The small loop and the small wire antennas are treated assuming current is uniform and varies harmonically with time. Furthermore, the antenna array is discussed. The basic principles of retarded interactions and array effects are thus developed and may then be applied to natural oscillators as are found in nature. In this way, the reflection law, the refraction law and the phenomenon of Brewster reflection are derived and fully explained. The power delivered by an antenna is also analysed using the Poynting vector derived in a previous chapter.

Appendix D contains solutions to the exercises appearing in the book.

Prerequisites and Target Audience

Although electrodynamics is described in this book from its first principles, prior knowledge of about one semester of university studies in mathematics and physics is required, including vector algebra, integral and differential calculus as well as a course in mechanics, treating Newton's laws and the energy principle. The target groups are teachers, engineering and physics students as well as professionals in the field, e.g. high-school teachers and employees in the telecom industry. Also chemistry and computer science students may benefit from the book.

Study Tips

Learning physics inevitably implies active involvement, especially in problem-solving and experimental studies. We recommend that the discussed experiments also be implemented in practice, not least to avoid tendencies to abstraction.

Some of the exercises, marked with an asterisk, are included in the theory of the book and need to be solved before the chapter that follows them. The exercises marked with a 'C' are more challenging and normally not suitable for independent problem solving.

A solution manual is included in Appendix D.

Website

The book has a website where you will find reader comments, recommended Internet links, videos on relevant experiments/phenomena, further exercises and suggested laboratory work. Please consult the publisher's website to obtain the relevant web address.

Acknowledgment

I want to express my gratitude to all students who contributed so much to the courses I have given over the years, including everything from basic and advanced electromagnetic courses to theory of relativity and application courses in Antenna and Microwave engineering at different levels in different study programs. The reflections, comments and questions I have received from them have been of crucial importance for my own development and the genesis of this book. I would also like to thank my colleagues for many intense and fruitful discussions on Physics in general and its role in society. Many interesting conversations with lecturers Göran Nordström on the subject's didactics and Peter Johansson on the connection between relativity and electrodynamics have been indispensable for me. The latter coined the term 'motional consequence' which is diligently used in this book. Many thanks to Dr. Jenny Ivarsson for a thorough scientific review of a first version of this book and to engineering student Nicklas Bjärnhall Prytz for indispensable advice on pedagogical issues as well as for making this book readable in English.

This book was originally published in Swedish and is available from its publisher Studentlitteratur. I appreciate all their support during the work, in particular with the drawing of the figures.

Finally, I would like to thank the Springer staff for all their support during the translation and revision of the book.

2014 Kjell Prytz

Contents

Symbols

$\bar{F}, \bar{f}$	Force (Newton, N)
U	Energy (Joule, J)
W	Work (Joule, J)
$\bar{s}$, $\bar{L}$, $\bar{d}$	Length (Meter, m)
$\bar{R}$	Distance (Meter)
$\bar{r}$	Position vector (Meter)
$\bar{r}'$	Position vector of source element (Meter)
V	Volume (Cubic meter, m^3)
$\bar{v}$	Velocity (Meter/second, m/s)
γ	Relativistic factor (Dimensionless)
$\hat{n}$	Surface normal (–)
n	Number per volume Density of turns (–)
m	Mass (Kilogram, kg)
m_0	Rest mass (Kilogram, kg)
q, Q	Charge (Coulomb, C)
λ	Charge per length (C/m)
σ	Charge per area (C/m^2)
σ	Conductivity (Siemens, Si)
ρ	Charge per volume (C/m^3)
I, i	Current strength (Ampère, A)
$\bar{K}$	Current per length; surface density (A/m)
$\bar{J}$	Current per area; volume density (A/m^2)
Φ	Electric potential (Volt, V)
$\Delta\Phi$	Electric voltage (V)
ε_j	Induced voltage (V)
C	Capacitance (Farad, F)
R	Resistance (Ohm, Ω)
M_{jk}	Mutual inductance (Henry, H)
L	Self-inductance (H)
N	Number (–)

G	Frictional coefficient (also gravitational constant) (Kg/s)
$\bar{p}$	Electric dipole moment (Cm)
$\bar{m}$	Magnetic dipole moment (Am^2)
κ_e	Dielectric constant (older ε_r) (Dimensionless)
κ_m	Relative magnetic permeability (older μ_r) (Dimensionless)
$\bar{P}$	Polarisation density of electric dipole moment (C/m^2)
$\bar{M}$	Magnetization density of magnetic dipole moment (A/m)
$\bar{E}$	Electric field (N/C, V/m)
$\bar{B}$	Magnetic flux density (Tesla, T)
$\bar{D}$	Electric displacement (C/m^2)
$\bar{H}$	Magnetic field intensity (A/m)
g	Gravitational acceleration (9.82 m/s^2 (latitude dependent))
ε_0	Electric permittivity ($8.85 \cdot 10^{-12}$ C/(Vm))
μ_0	Magnetic permeability ($4\pi \cdot 10^{-7}$ Tm/A)
c	Speed of light in air/vacuum ($3.00 \cdot 10^8$ m/s)
G	Gravitational constant ($6.67 \cdot 10^{-11}$ Nm^2/kg^2)

Formulae

Rectangular Coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z})$

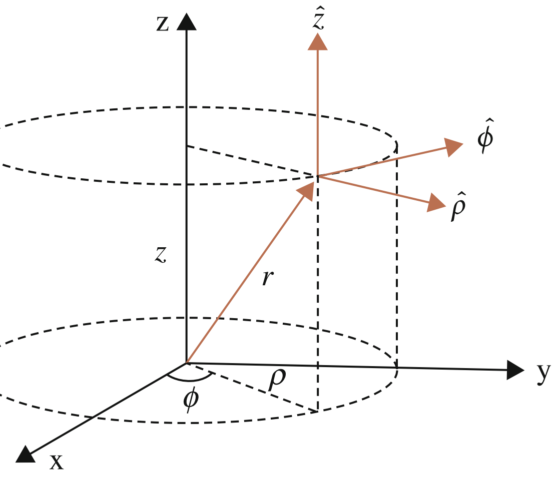
$$\nabla U = \frac{dU}{dx}\hat{x} + \frac{dU}{dy}\hat{y} + \frac{dU}{dz}\hat{z}$$

$$\nabla^2 U = \frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}$$

$$\nabla \cdot \bar{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

$$\nabla \times \bar{A} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}$$

Cylindrical Coordinates (ρ, ϕ, z)



$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z$$

Unit vectors

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{z} = \hat{z}$$

Position vector

$$\bar{r} = \rho\hat{\rho} + z\hat{z}$$

Volume element

$$dV = \rho\, d\rho\, d\phi\, dz$$

Vector operations

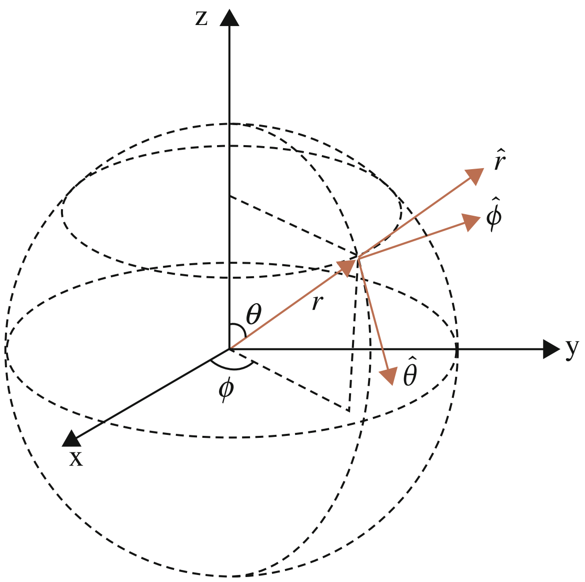
$$\nabla U = \frac{dU}{d\rho}\hat{\rho} + \frac{1}{\rho}\frac{dU}{d\phi}\hat{\phi} + \frac{dU}{dz}\hat{z}$$

$$\nabla^2 U = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2U}{d\phi^2} + \frac{d^2U}{dz^2}$$

$$\nabla \cdot \bar{A} = \frac{1}{\rho}\frac{d}{d\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{dA_\phi}{d\phi} + \frac{dA_z}{dz}$$

$$\nabla \times \bar{A} = \left(\frac{1}{\rho}\frac{dA_z}{d\phi} - \frac{dA_\phi}{dz}\right)\hat{\rho} + \left(\frac{dA_\rho}{dz} - \frac{dA_z}{d\rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{d}{d\rho}(\rho A_\phi) - \frac{dA_\rho}{d\phi}\right)\hat{z}$$

Spherical Coordinates (r, θ, ϕ)



$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

Unit vectors

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

Position vector

$$\bar{r} = r\hat{r}$$

Volume element

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Vector operations

$$\nabla U = \frac{dU}{dr}\hat{r} + \frac{1}{r}\frac{dU}{d\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{dU}{d\phi}\hat{\phi}$$

$$\nabla^2 U = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dU}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2U}{d\phi^2}$$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{d}{dr}(r^2A_r) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{dA_\phi}{d\phi}$$

$$\begin{aligned}\nabla \times \bar{A} = {} & \frac{1}{r\sin\theta}\left[\frac{d}{d\theta}(A_\phi\sin\theta) - \frac{dA_\theta}{d\phi}\right]\hat{r} \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{dA_r}{d\phi} - \frac{d}{dr}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{d}{dr}(rA_\theta) - \frac{dA_r}{d\theta}\right]\hat{\phi}\end{aligned}$$

Vector Algebra

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

$$(\bar{A}\times\bar{B})\times\bar{C} = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{A}(\bar{B}\cdot\bar{C})$$

$$\nabla\cdot(U\bar{A}) = (\nabla U)\cdot\bar{A} + U(\nabla\cdot\bar{A})$$

$$\nabla\times(U\bar{A}) = (\nabla U)\times\bar{A} + U(\nabla\times\bar{A})$$

$$\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$$

$$\nabla\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} - \bar{B}(\nabla\cdot\bar{A}) - (\bar{A}\cdot\nabla)\bar{B} + \bar{A}(\nabla\cdot\bar{B})$$

$$\nabla(\bar{A}\cdot\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} + (\bar{A}\cdot\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$$

$$\nabla \times (\nabla U) = 0$$

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

Integral Theorem

Divergence theorem

$$\int_V \nabla \cdot \bar{A} dV = \oint_S \bar{A} \cdot d\bar{a}$$

Stokes' theorem

$$\oint_C \bar{A} \cdot d\bar{L} = \int_S (\nabla \times \bar{A}) \cdot d\bar{a}$$

$$\oint_C U d\bar{L} = \int_S d\bar{a} \times \nabla U$$

Integration by parts

$$\int g(x) f(x) dx = G(x) f(x) - \int G(x) \frac{df}{dx} dx$$

where $G(x)$ is primitive function to $g(x)$

Useful Integrals

$$\int_{-\infty}^{\infty} \frac{dy}{(y^2 + x^2)^{3/2}} = \frac{2}{x^2}$$

$$\int_a^b \frac{1}{(d^2 + x^2)^{3/2}} dx = \left[\frac{x}{d^2 \sqrt{x^2 + d^2}} \right]_a^b$$

$$\int_a^b \frac{dx}{(c^2 + d^2 - 2xcd)^{1/2}} = \left[-\frac{\sqrt{d^2 + c^2 - 2xcd}}{cd}\right]_a^b$$

$$\int_a^b \frac{xdx}{(c^2 + d^2 - 2xcd)^{1/2}} = \left[-\frac{xcd + c^2 + d^2}{4c^2d^2}\sqrt{-2xcd + c^2 + d^2}\right]_a^b$$

Elliptic Functions

Complete elliptic integral of the first kind

$$F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\alpha}{\left[1 - k^2 \sin^2 \alpha\right]^{1/2}}$$

Complete elliptic integral of the second kind

$$E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \left[1 - k^2 \sin^2 \alpha\right]^{1/2} d\alpha$$

Taylor expansions

$$F(k, \pi/2) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots\right]$$

$$E(k, \pi/2) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \cdots\right]$$

Limit value

$$\lim_{k^2 \to 1} F\left(k, \frac{\pi}{2}\right) = \frac{1}{2}\ln\frac{16}{1 - k^2}$$

Complex Numbers

Imaginary unit

$$j = \sqrt{-1}$$

A complex number z has a real part x and an imaginary part y

$$z = x + jy$$

Polar form of complex numbers

$$z = Ae^{j\theta} = A(\cos\theta + j\sin\theta)$$

$$A = \sqrt{x^2 + y^2}$$

$$x = A\cos\theta, \; y = A\sin\theta$$

Operations on Distance Vector *R*

$$\bar{R} = \bar{r}_2 - \bar{r}_1$$

Gradient with respect to position vector r_1

$$\nabla_1 \frac{1}{R} = \frac{\bar{R}}{R^3}$$

Gradient with respect to position vector r_2

$$\nabla_2 \frac{1}{R} = -\frac{\bar{R}}{R^3}$$

Chapter 1
Basic Principles

I think the facts leave no doubt that the very mightiest among the chemical forces are of electric origin. The atoms cling to their electric charges, and opposite electric charges cling to each other.
Hermann von Helmholtz, 1881

Electrodynamics is the topic on which most of physics is based. The physics subtopics such as mechanics, wave motion, thermodynamics, atomic physics and so on all have their origin in the electric force. Pressure and temperature, sound, material bending, winds, atomic and molecular structure etc. are basically the effects of interactions between electric charges, together with all our senses and our physiology in its entirety, including the genetic code. Observations beyond the senses are made with instruments whose design is based exclusively on electrodynamics. So when physicists study subatomic dynamics and discover two other types of interactions, the strong nuclear force that holds nucleons together and the weak atomic force causing fusion and radioactive decay, this is done with senses and instruments whose function is based on the electric force.

Electrodynamics is, together with gravity, the force controlling everyday life. One major issue of physics is whether these two forces have a common origin. Rephrased: is there a relationship between mass and charge; could it be that the one is generated dynamically from the other? For those who are interested in basic research, issues of this nature are particularly favourable for fruitful research projects.

Also gravity is experienced through the electric force. As you read this, you may sit in a chair and feel the so-called normal forces, which are electric and arise indirectly via your weight. But when gravity acts alone, such as in a free fall, it is not experienced; you are ‘weightless’.

Electrodynamics is based on the concept of electric charge, which has its origin in the electrons and protons of the atom and is the source of the electromagnetic force. All macroscopic charge is composed of these elementary charges and the attraction between them is the condition for the bonded system known as the atom. Consequently, also molecules are formed through the electric interaction.

Throughout this book the mutual interaction between charged objects is emphasized. These appear ‘pairwise’, i.e. they interact two by two, according to Newton’s

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fundamental laws of nature. The dynamics of nature is then a sum of individual pairwise interactions. This summation requires knowledge of the interaction between any two fundamental items such as two electrons, both at rest and in general motion. In electrodynamics there is a good, although not complete, knowledge about this force, which in the latter case, moving charges, essentially has been observed in closed conductors.

As a complement to the force view of nature, physics may be described by the energy concept, defined through the pairwise interaction via the concept of work. The result of work is stored energy U in the system (also known as potential energy). Force and energy are intimately connected via the energy principle from which force F arises as an energy minimization:

$$\bar{F} = -\nabla U \tag{1.1}$$

i.e. force is the negative gradient of the stored energy U. This principle is one cornerstone of physics. Force is in turn defined by Newton's laws of motion:

$$\bar{F} = \frac{d}{dt}(m\bar{v}) \tag{1.2}$$

i.e. force is the time derivative of momentum. Formula (1.2) also defines inertial mass which is a measure of an object's resistance to motional change. Furthermore, the scientific definition of time and space originates from this formula.

Force and energy form the core of physics. Since energy is a scalar it is computationally advantageous to first obtain this quantity and then to use the energy principle (1.1) to achieve the desirable force vector. In this book, this technique is used extensively and a prior knowledge in the two concepts is essential.

1.1 Exercises

1.1 a. Apply the energy principle, formula (1.1), in case of free fall toward the earth's surface. Explain the origin of the potential energy and how it generates the active force.
b. Use (1.1) to determine the force acting on the falling object and the earth.
c. Identify the forms of energy involved in the dynamics. What became of the energy in the final state, i.e. when the object is located on the ground?
d. Why is the motion of the earth usually neglected? Reflect upon how Newton's third law (action and reaction) is affected by this neglect.

1.2 a. Explain why a heavier object has a shorter fall time than a lighter one if they are released from the same high altitude above the earth.
b. There is a famous film recorded on the moon where the principle of equal fall time for different weights is tested and verified. What is the difference

in conditions between the earth and the moon? (Ref: see web link on the book's website).

1.3 a. Explain what is meant by the concepts of gravitational and inertial mass. Give examples of phenomena where the first and second is at work. How are they related to each other?
b. Find out what Mach's principle means and explain in your own words how the inertial mass arises from the gravitational. (Ref: see web link on the book's website).

1.4 a. A space station is located at about 400 km above the earth's surface. Estimate the gravitational force on a person at this height. Explain why a person staying on a space station can be regarded as being in a weightless condition.
b. Check with what amount a measurement with a balance of a 1 kg mass would change if it were measured at an altitude of 100 m compared to at the ground. Would it be possible to perform this measurement and thereby verify the distance dependence of gravity?

1.5 Explain why the coriolis and centrifugal forces must be regarded as fictitious forces. Give examples of phenomena in which these concepts are used.

1.6 What is meant by the term 'relative motion'? Could there be any form of absolute motion?

1.7 How would you define the concept of time? How are clocks made?

1.8 What does the concept 'space' mean to you? What is meant by distance and how is it measured?

1.9 If a movie is playing at a slower speed, any motion in the movie is seen consistently slower. How does that influence your perception of time in the movie?

Further Readings

A.K.T. Assis, *Relational Mechanics*, (Apeiron Montreal, 1999)
A.P. French, *Newtonian Mechanics*, The MIT Introductory Physics Series, (Norton, New York, 1971)

Recommended Mathematical Handbooks

M.R. Spiegel, *Mathematical Handbook*, (McGraw-Hill, 1968)
R.K. Wangsness, *Electromagnetic fields*, (Ch. 1), (Wiley, 1986)

Chapter 2
Electrodynamic Force

Electric and magnetic forces. May they live for ever, and never be forgot, if only to remind us that the science of electromagnetics, in spite of the abstract nature of its theory, involving quantities whose nature is entirely unknown at the present, is really and truly founded on the observations of real Newtonian forces, electric and magnetic respectively.

Oliver Heaviside, 1900

In this chapter the force between charges at rest, in uniform motion, and in acceleration will be formulated. These are termed electric, magnetic and inductive respectively. A minimum set of experiments will be introduced through which the forces may be studied in detail. From the measurements the force formulas are deduced and expressed in terms of the fundamental entities, i.e. charge and its motion.

It must not be forgotten, however, that force formulas in general are strictly valid only for the experimental conditions under which they were deduced. New phenomena are therefore expected at higher speeds of the charges as well as at a smaller distance scale than what is valid for the experiments discussed here. These frontiers of physics are denoted theory of relativity and quantum mechanics respectively for which the classical electrodynamic theory is the basis.

2.1 Electric Charges at Rest–Electric Force

Macroscopically electric charged objects arise apparently in different ways. It is generally, however, a matter of separating unlike elementary charges from each other, usually electrons from their atoms. A simple way to generate electricity is by friction. Thunderclouds, for example, are formed at large temperature differences between wet air masses which create strong air currents whereby friction occurs between the water droplets. Electrons are then transported from one molecule to another, giving rise to an imbalance counteracted by nature. This reaction is known as electric force. Similarly, an electric imbalance occurs when different materials are rubbed against each other, for instance a cloth on an amber stone.

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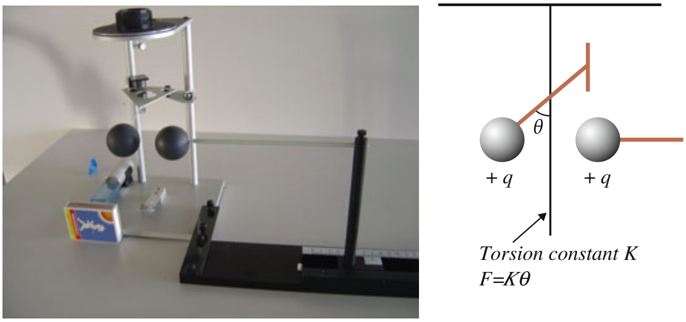


Fig. 2.1 The torsion balance. The *left ball* is attached to a rod which in turn is mounted to the *thin vertical thread*. The *right ball* is fixed at a certain position. The balls can be charged for example by allowing a charged plastic rod touching the balls. When the *left-hand ball* repels it will cause the *vertical wire* to rotate and via its torsion constant K the acting force can be determined from the rotational angle θ. The size of the equipment is illustrated by means of the matchbox

The word electric is derived from the ancient Greek word for amber,[1] which the first recorded studies on this phenomenon revolve around. In the late 1700s Coulomb studied these forces in a systematic manner. By means of a torsion balance, Fig. 2.1, the force between small objects could be determined for varying distance R and charge q. The latter was varied by allowing a charged object to touch another identical uncharged object under the assumption that the excess charge is then distributed equally between the objects. The amount of charge is not directly accessible but defined by the force that arises.

This force, where the charged objects initially are at rest, is called electric or electrostatic. The measurement can be carried out using for example Pasco's equipment in the figure. It is difficult to achieve a precision better than 20 %, so it is reasonable to ask how Coulomb managed to carry out his experiments at such a precision that the law which he formulated in the eighteenth century is still valid today:

$$\bar{F}_{1\to 2} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}\hat{R} \tag{2.1}$$

which is the force on object 2 caused by object 1, Fig. 2.2.

$\hat{R}$ is a unit vector which indicates the direction of the distance vector pointing towards the affected charge. ε_0 is a natural constant called the electric permittivity. The force on charge 1 is opposite in direction but equal in magnitude. By observation, the electric force is known to be either repulsive or attractive, a fact that is taken into account by introducing signs of the charges (a mathematical miracle). Science historians suspect that Coulomb never completely verified this formula experimentally but copied the formula of Newton's law of gravitation, basically replacing mass with charge.

[1] In greek the word is $\eta\lambda\epsilon\kappa\tau\rho o$ transcribed as ilektro.

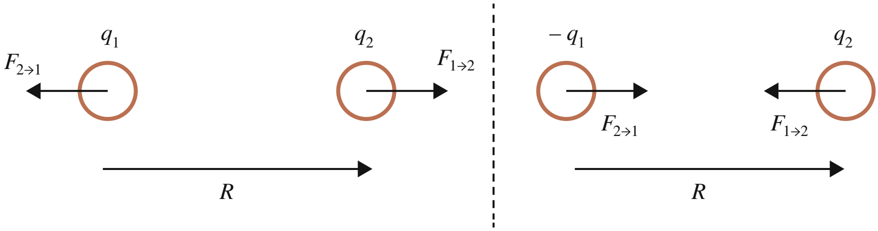


Fig. 2.2 Two electrically charged objects interact. *Left* Repulsive force for like charges. *Right* Attractive force for unlike charges

Nevertheless, the Coulomb force formula is one of the most important discoveries in physics, well-proven over time with different methods. Note that the force formula, like Newton's gravitational force, deals with pairwise interactions. In order to obtain force and energy for general systems, the pairwise forces between its constituents must be summed. This usually requires integration as the number of charges is so large that a macroscopic object may be considered as consisting of charged infinitesimal elements, i.e. a continuous charge distribution. More about this in Chap. 4.

The unit of charge is a Coulomb whose magnitude is defined from formula (2.1).

2.2 Uniform Motion–Magnetic Force

Knowledge of electric charges in motion originates mainly from studies of current-carrying conductors. However, a full understanding requires the study of free charges in motion, such as in an electron tube or a particle accelerator.

To describe the effect of uniform motion, both parallel and perpendicular relative motion must be examined where the latter case requires one party to consist of free charges such as an electron beam. These motional effects should be regarded as a correction to the Coulomb force (2.1), known as magnetic for historical reasons. The word originates from ancient Greece, referring to the city where magnetic stones were found. Today, however, magnetism is known to be about electric charges in motion, explored and systematised using basically the current-carrying conductor.

2.2.1 Electric Current

In principle, an electric current may be generated in a conductor by bringing a charged object to its proximity. The conduction electrons in the metal, which can move almost freely, will then be put into motion. By means of a battery, i.e. a voltage source (see Chap. 3), the current can be maintained in a closed conductor.

The term 'closed conductor' refers to a conductor structure as a closed path where conduction electrons by means of a battery are moving in a closed path, i.e. a circuit.

The battery voltage can typically be varied and because the voltage is proportional to current, described by Ohm's law (see Chap. 5), different strengths of the current may be obtained. Alternatively, for a given voltage, current may be varied by the diversion into different branches of a circuit. Current I corresponds to the amount of charge that passes an area A per unit time:

$$I = nqvA \tag{2.2}$$

where n is the number of charges per volume, q their individual charge and v its velocity, so-called drift velocity in a conductor, see also Chap. 5. The velocity v is parallel to the normal of surface A. The current direction is defined by the motional direction of positive charges, so that the direction of current in a conductor is opposite the direction of the electron drift velocity.

The unit of current is thus coulomb/second, usually referred to as Ampère.

2.2.2 Measurement of Parallel Motion

For the study of force between charges in parallel/antiparallel motion, the straight conductors may be used, Fig. 2.3.

The force between these can be measured with an apparatus shown in Fig. 2.4.

Either a torsion balance may be used as in the previous electric experiments or the force may be balanced by means of weights. Let the parallel conductors have length L, much larger than the distance x between them. For parallel currents as in the left of Fig. 2.3, the measurement gives

$$\bar{F}_{1\to 2} = -\frac{\mu_0 I_1 I_2}{2\pi x} L\hat{x} \tag{2.3}$$

I is current and the natural constant μ_0 is called magnetic permeability. As in the electric case, the force on the other conductor is opposite in direction but equal

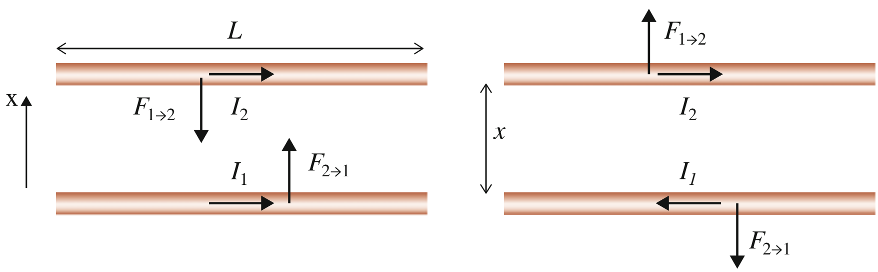


Fig. 2.3 *Left* Attractive force for parallel currents. *Right* Repulsive force for opposite (anti-parallel) currents

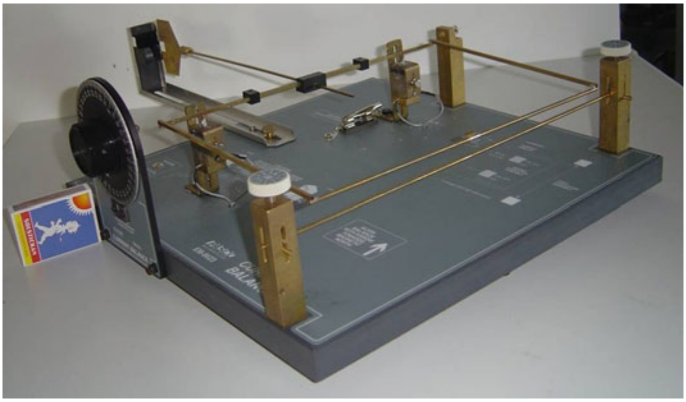


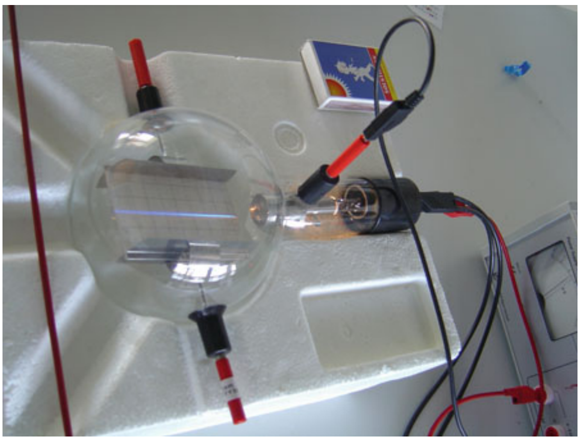
Fig. 2.4 Apparatus for measuring the magnetic force. The two parallel conductors at the forefront are horizontally oriented where the *lower* one is fixed and the *upper* one can move freely

in magnitude. An attractive (repulsive) force is observed for parallel (anti-parallel) currents, included in the force formula using opposite signs for opposite currents.

2.2.3 Measurement of Perpendicular Motion

The force between charges in relative perpendicular motion can be explored using an electron beam incident on a straight current-carrying conductor as in Fig. 2.5. Since the conductor is electrically neutral, the generated force is magnetic, arising from the motion of charge.

Fig. 2.5 Apparatus for measuring the magnetic force between charges in perpendicular motion. An electron beam is incident perpendicularly on a long straight conductor



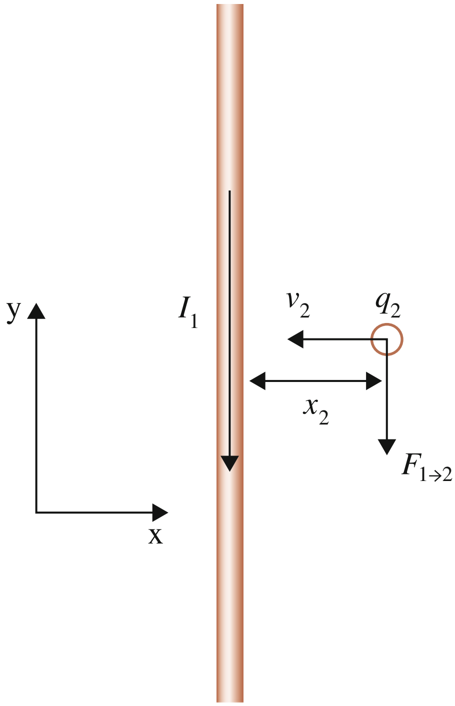


Fig. 2.6 Schematic view of the experiment with $q_2 < 0$

As is shown in Exercise (2.8) the measurement provides the force on a charge q_2 in the beam, Fig. 2.6,

$$\bar{F}_{1\to 2} = \frac{\mu_0 I_1}{2\pi x_2} q_2 v_2 \hat{y} \tag{2.4}$$

where $\hat{y}$ is the unit vector in the y-direction and I_1 is the current in the conductor. q_2, v_2 and x_2 are charge, velocity and distance to the conductor respectively of one particular free electron. The current I_1 flows in the negative y direction and is therefore negative as well as charge q_2 and speed v_2 so that the force becomes negative, i.e. parallel to the conductor current I_1, in accordance with observation.

The measurement assumes a knowledge of the charge to mass ratio of the electron, a quantity usually measured in high-school, see Exercise (5.12). Alternatively, let an electron beam pass through a pair of charged parallel plates using an apparatus such as in Fig. 2.5. By balancing the electric force by a parallel straight current-carrying conductor, the charge to mass ratio of the electron is obtained. The electric force from the two charged plates will be calculated in Chap. 4.

The magnetic force of the perpendicular motion can not be studied with closed conductors because the force then vanishes, see Exercise (2.10).

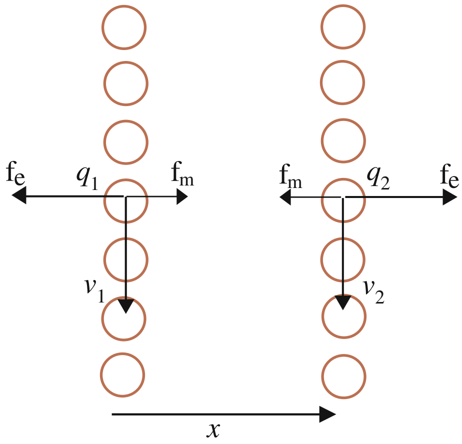


Fig. 2.7 From the two parallel conductors in Fig. 2.3 the electrons corresponding to the electric current are extracted, the so-called conduction electrons. Thus, two parallel fluxes of *free* electrons are obtained

2.2.4 Magnetic Force for General Uniform Motion

Based on the two measurements above, a formula is sought for the magnetic force between two free electric charges in uniform but otherwise arbitrary motion.

Parallel motion. Considering free electrons (or charges in general) in parallel motion, the expected force situation is then as shown in Fig. 2.7.

To determine the magnetic force on charge 2, replace length L with the infinitesimal element dL in formula (2.3). This element corresponds to one and only one electron. Using formula (2.2) the *current element* becomes

$$IdL = nqvAdL = nqvdV = qv \tag{2.5}$$

where dV is the volume occupied by one electron.

Thus, the magnetic force on electron 2 caused by the left flux (no. 1) of Fig. 2.7 is[2]

$$\bar{F}_{1\to el2} = -\frac{\mu_0 I_1}{2\pi x} I_2 dL_2 \hat{x} = -\frac{\mu_0 I_1}{2\pi x} q_2 v_2 \hat{x} \tag{2.6}$$

where flux 1 is assumed to be long in relation to x. This force is the pairwise sum of the magnetic forces between all electrons in flux 1 and the single electron 2.

It will now be shown that the contribution to the magnetic force between *one* current element in conductor 1, equivalent to an electron, and electron 2 may be written

$$\bar{f}_{m||} = -\frac{\mu_0 I_1 dL_1}{4\pi R^2} q_2 v_2 (a \cos\theta \hat{y} + \sin\theta \hat{x}) \tag{2.7}$$

[2] The force is written without infinitesimal sign despite the fact that the right-hand side is infinitesimal. The reason is that the force element is to be regarded as a specific electron.

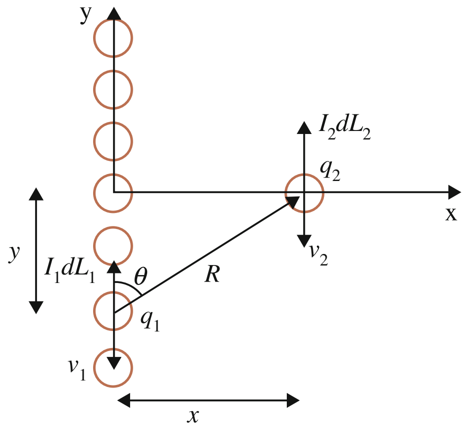


Fig. 2.8 Definition of parameters used in the expression for the force between two charges in parallel motion. The current elements are defined for negative charge q

where R is the distance between the electrons, θ is defined in Fig. 2.8 and a is a constant.

The factors inside the brackets in formula (2.7) is selected with the following criteria:

- The vertical component $\hat{y}$ of formula (2.7) should be cancelled after integration over dL_1.
- The horizontal component $\hat{x}$ of formula (2.7) should coincide with formula (2.6) after integration.

The first criterion gives of course other possible expressions for the y component than that in formula (2.7) (which?). The present selection is based on simplicity. To examine the correctness of formula (2.7), the force is integrated over the entire length of conductor 1, which is assumed to be of infinite length:

$$\bar{F}_{1\to el2} = -\int_{-\infty}^{\infty} \frac{\mu_0 I_1 dy}{4\pi(y^2+x^2)} q_2 v_2 (a\cos\theta\,\hat{y} + \sin\theta\,\hat{x}) \tag{2.8}$$

Since $\cos\theta = y/\sqrt{y^2+x^2}$ and $\sin\theta = x/\sqrt{y^2+x^2}$ we obtain

$$\bar{F}_{1\to el2} = -\int_{-\infty}^{\infty} \underbrace{\frac{\mu_0 I_1 dy}{4\pi(y^2+x^2)^{3/2}} q_2 v_2 (ay\,\hat{y}}_{\text{odd function}} + x\hat{x}) = -\frac{\mu_0 I_1}{2\pi x} q_2 v_2 \hat{x} \tag{2.9}$$

In formula (2.9) the first term in the integral is an odd function which thus vanishes. The second term involves the integral

$$\int_{-\infty}^{\infty} \frac{dy}{(y^2 + x^2)^{3/2}} = \frac{2}{x^2}$$

Thus, formula (2.6) has been recreated. Formula (2.7) is therefore a possible candidate for the magnetic force between charges in parallel motion.

Perpendicular motion is treated similarly to the former parallel case. A general trial formula for the force between two charges is proposed which is then integrated and compared to the measurement result.

According to observation the magnetic force is vertical in this case, parallel to current I_1, Fig. 2.6. Therefore, the formula should be written so that the horizontal component vanishes after integration. As the formulas (2.4) and (2.6) are equal in magnitude, $\hat{y}$ must be multiplied by $\sin\theta$ and $\hat{x}$ is multiplied by $b\cos\theta$ where b is a constant (compare previous case), Fig. 2.9. Hence, the force on charge 2 for perpendicular motion is expressed as:

$$\bar{f}_{m\perp} = \frac{\mu_0 I_1 dL_1}{4\pi R^2} q_2 v_2 (\sin\theta\,\hat{y} + b\cos\theta\,\hat{x}) \tag{2.10}$$

which should be integrated from minus to plus infinity, since the straight conductor is long. This action is technically equal to the previous case (see Exercise 2.8), so that the cosine term disappears after integration and the result is formula (2.4). Formula (2.10) is thus verified as a candidate for the magnetic force between charges in relative perpendicular motion.

General motion. From the two complementary results, formulas (2.7) and (2.10), the general magnetic force formula may be deduced. The formula must be expressed in a coordinate free form and in terms of vectors. In addition, the simplest possible expression should be chosen. With these restrictions it is quite feasible to obtain the formula by trial and error. The only vectors that can be used are the velocities and

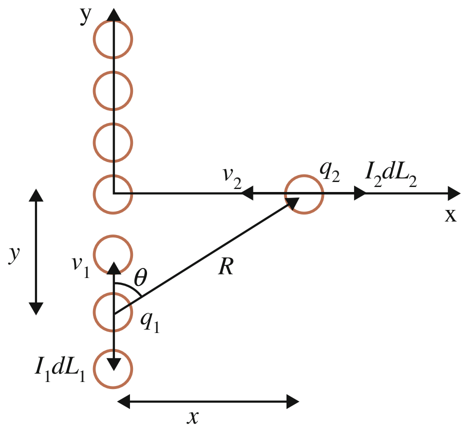
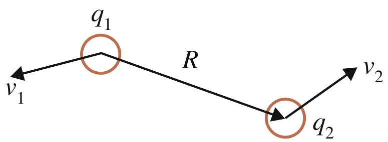


Fig. 2.9 Definition of parameters used in the expression for the force between charges in perpendicular motion. The current elements are defined for negative charge q

the distance. In Appendix E, a systematic approach is given and the result is

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{2.11}$$



which is force on object 2. This formula is valid for both cases $\bar{f}_{m||}$ and $\bar{f}_{m\perp}$ (verify this) and is therefore the general magnetic force formula, i.e. for charges in uniform motion with arbitrary relative motional direction, see figure to the above. The reader is reminded that the measurements are based on electric current in conductors, meaning that charges move slowly.

With $a \neq 0$ longitudinal magnetic forces are allowed which appear when $\theta = 0$ in Fig. 2.8. These forces have been indicated in experiments but not yet been measured with any precision.

In 1902/1903 Tait and Whittaker performed an analysis on the full measurement done in 1820–1825 by Ampère concerning forces between current-carrying conductors. Whittaker and Tait also argued the validity of Newton's third law about action and reaction (see next section) and found the following force formula:

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + (\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{2.12}$$

i.e. (2.11) with $a = 1$.

However, (2.12) was first proposed by Maxwell in 1865 as one of four candidates based on Ampère's measurements.

Another magnetic force formula was proposed by Grassman in 1845. Based on the measurements of Ampère together with the exclusion of longitudinal forces, i.e. $a = 0$ in (2.11), he obtained

$$\bar{f}^G_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1] \tag{2.13}$$

Commonly, (2.13) is taken as the general magnetic force formula. It is usually written

$$\bar{f}^G_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}\bar{v}_2 \times (\bar{v}_1 \times \hat{R}) \tag{2.14}$$

as a consequence of vector algebra.

The reason for the popularity of formula (2.14) is that it correctly describes phenomena appearing in daily life as well as in technical applications. In plaintext it means that it describes such cases where at least one of the interacting objects moves in a closed path, including both permanent magnets and electric circuits. In case both objects are free charges, a certain problem with Grassman's formula (2.13)/(2.14) will appear. This is discussed in the next section.

It may be shown that the general magnetic force formula (2.11) is reduced to (2.13)/(2.14) in case one of the interacting objects is a closed current, see Exercise (2.10).

2.2.5 Evaluation of the Magnetic Force Formulas

Momentum. Grassman's force formula (2.13)/(2.14) is marred by a peculiar property since for free charges it violates Newton's third law: to every force there is a counterforce of equal size and opposite direction. Hence, Grassman's formula violates the law of momentum conservation. This is shown as follows.

The force (2.13)/(2.14) acts on object 2. To obtain the force on object 1, the quantities are interchanged and the sign of the distance vector changed:

$$\bar{f}_{m1}^{G} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(\bar{v}_1 \cdot \bar{v}_2)\hat{R} - (\bar{v}_1 \cdot \hat{R})\bar{v}_2] \neq -\bar{f}_{m2}^{G} \tag{2.15}$$

The problem occurs since formula (2.13) is asymmetric in v_1 and v_2. This is corrected in formula (2.12) which for object 1 becomes

$$\bar{f}_{m1} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(\bar{v}_1 \cdot \bar{v}_2)\hat{R} - (\bar{v}_1 \cdot \hat{R})\bar{v}_2 - (\bar{v}_2 \cdot \hat{R})\bar{v}_1] = -\bar{f}_{m2} \tag{2.16}$$

in agreement with Newton's third law.

In this book, formula (2.11) is considered as the basic magnetic force formula and it will be referred to as Whittaker's formula. The constant a is still today unknown.

However, neither Grassman's nor Whittaker's force is generally directed along the connection line of the charges. Such a force will violate the principle of angular momentum conservation, i.e. rotational motion, discussed in Exercise (2.12).

In technical applications magnetic effects usually appear through *closed* current-carrying conductors. In this case, (2.11) is simplified considerably since the *two* last terms then vanish, a simplification easily obscured if Grassman's force (2.14) is used as guiding formula. This is shown in Exercise (2.10).

Electric charges in parallel motion. Consider the interaction between two identical charges in parallel motion with equal velocity v perpendicular to distance vector R, Fig. 2.10. This phenomenon appears both in particle accelerators and in electron tubes and is therefore well documented for free charges. Both formulas (2.11) and (2.14) give

$$\bar{f}_m = -\frac{\mu_0 q v q v \hat{R}}{4\pi R^2} \tag{2.17}$$

and the total force on q_2 becomes

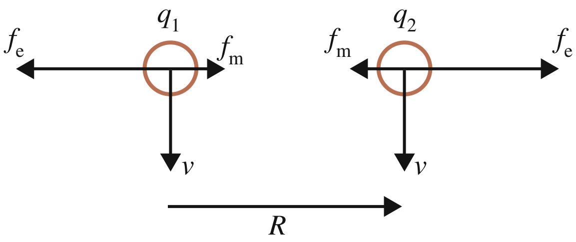


Fig. 2.10 Force between two like charges in parallel motion

$$\bar{f} = \bar{f}_e + \bar{f}_m = \left(\frac{q^2}{4\pi\varepsilon_0 R^2} - \frac{\mu_0 q^2 v^2}{4\pi R^2} \right) \hat{R} = \frac{q^2}{4\pi\varepsilon_0 R^2}(1 - \varepsilon_0\mu_0 v^2)\hat{R} \tag{2.18}$$

Formula (2.18) is well verified for free charges in particle accelerators. Since the formula stems from measurement on closed conductors it provides another verification for the model of electric conductor current which was used above.

Formula (2.2) provides an estimate of the drift velocity of the conduction electrons in ordinary conductors of about 1 mm/s. Using your self-measured values for ε_0 and μ_0 or utilizing tabulated values it is found that at this speed the magnetic force of Fig. 2.10 is a factor 10^{23} *less* than the electric force. Nevertheless, the magnetic force is easily observed using current-carrying conductors because of the large number of conduction electrons all contributing to the magnetic force in the same direction.

Now consider the case when the motion of the two like charges is anti-parallel. From the definition of electric current, formula (2.2), it is clear that this case is, with respect to magnetic force, equivalent to considering two unlike charges in parallel motion, see Fig. 2.11.

Since two parallel conductors with opposite currents repel, the magnetic force on object 2 (right object) is positive and the electric force is negative:

$$\bar{f} = \bar{f}_e + \bar{f}_m = \left(-\frac{q^2}{4\pi\varepsilon_9 R^2} + \frac{\mu_0 q^2 v^2}{4\pi R^2} \right) \hat{R} = \frac{q^2}{4\pi\varepsilon_0 R^2}(-1 + \varepsilon_0\mu_0 v^2)\hat{R} \tag{2.19}$$

in agreement with formulas (2.11) and (2.14).

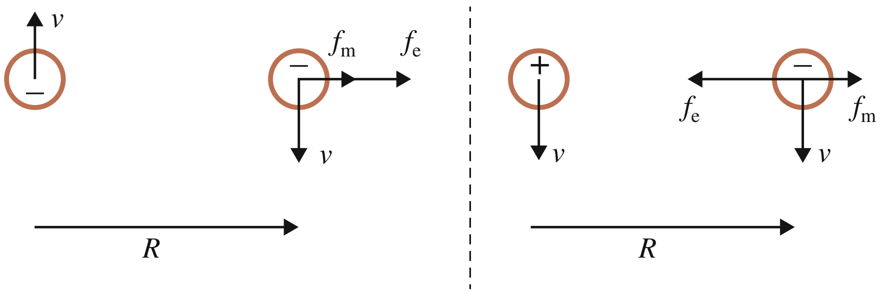


Fig. 2.11 Two like charges in anti-parallel motion (*left*) is, w.r.t. the magnetic force, equivalent to two unlike charges moving in parallel (*right*)

Extrapolating beyond the given measurement range, which is up to a tenth of light speed, will show that at the speed $v = 1/\sqrt{\varepsilon_0 \mu_0}$ the magnetic and electric forces are balanced and the interaction ceases. This phenomenon can be observed in particle accelerators where like charges move in parallel close to this speed.

On the basis of Figs. 2.10 and 2.11 the origin of magnetic force and its relation to theory of relativity will be explored in Chap. 9.

2.3 Accelerated Motion–Inductive Force

Electric charges in acceleration may be examined by connecting and disconnecting a battery included in a circuit, Fig. 2.12.

The current in the loop can be measured using a current meter based on the magnetic force as above. When the battery is connected the current in the circuit increases as in Fig. 2.12. Thus, it takes a certain time for the electrons to reach their final speed. This is a kind of inertia which can be considered as a resistance to the effect of the battery.

Charges in acceleration can be further studied by using two loops, Fig. 2.13. The following is then observed:

A. *The loops are in the same plane, one enclosing the other and with parallel cross sections, Fig.* 2.13 *left*

If and only if the current in one *varies*, a current is induced in the other according to the following:

- Increasing the current in one conductor induces an opposite current in the other.
- Decreasing the current in one conductor induces a same directed current in the other.
- The faster the current variation in one conductor the greater induction current in the other.

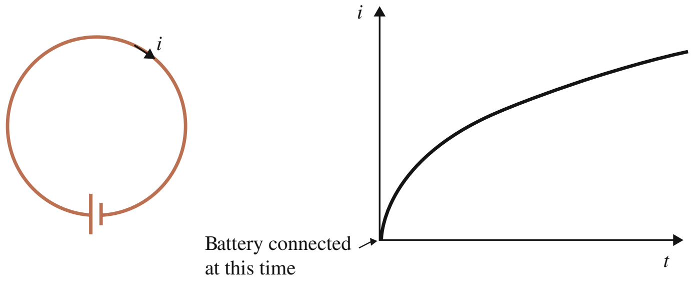


Fig. 2.12 An electric conductor is connected to a battery (*left*). When the battery is turned on, the current in the circuit increases as in the *right figure*

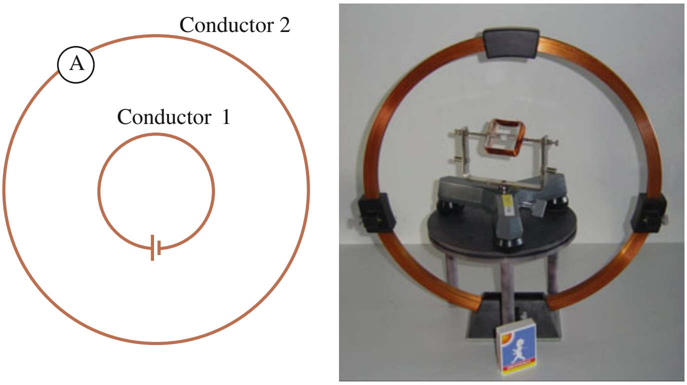
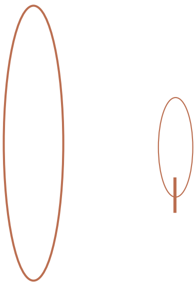


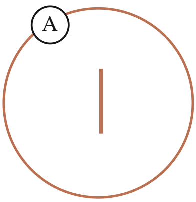
Fig. 2.13 Two closed conductors utilized for investigation of the induction effect. One of them is connected to a battery and the other to a current meter. The *right-hand figure* shows an example of an apparatus

B. *The loops are in parallel planes with common axes*



For larger distances between the loops, examinations as in case A give qualitatively the same result but quantitatively weaker induction current compared to the result in case A.

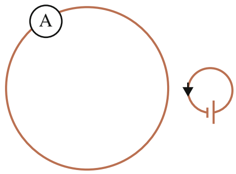
C. *Crossed loops, cross section surface at 90 degrees angle to each other*



Current is *not* induced from examinations as in case A. Note that our current meter only registers current that passes it.

D. *Constant battery current* gives no induction current.

E. *The loops are in the same plane but not enclosing*



A far smaller current is induced compared to case A, which quickly decreases with increasing distance.

The observations may be summarized:

- The response, i.e. the induction current, is such that changes of the system's total current is opposed.
- The response is strictly anti-parallel to the acceleration the battery causes since no induction current is registered for crossed conductors.
- The response decreases with distance between the loops.

To reduce the complexity of the phenomena a second experiment is set up. This consists of two large rectangular circuits, Fig. 2.14.

Since the circuits are large, the two sides closest to each other will control the interaction. Thus, in this way the interaction between two straight and parallel conductors is examined. As in the previous experiment there is no response from circuit 2 in case the battery current is constant as well as when the two straight conductors are perpendicular to each other. The force that appears on the electrons in the straight conductor 2 is therefore either parallel or anti-parallel to the length element dL_1. A

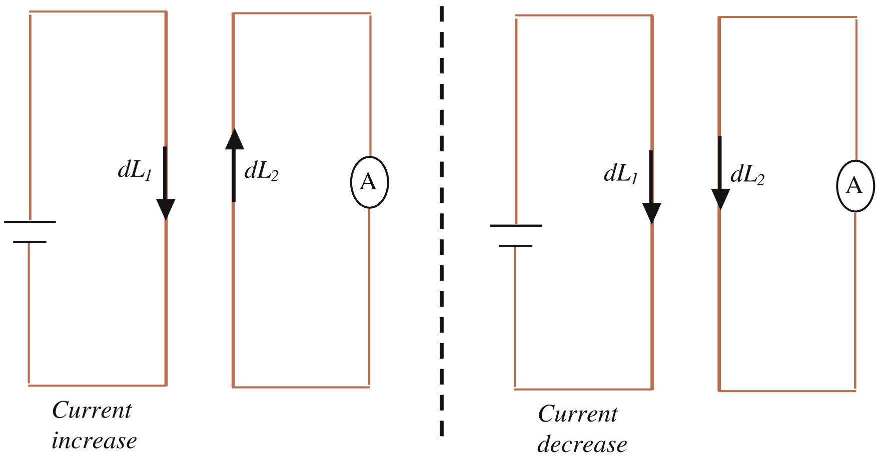


Fig. 2.14 Two large *rectangular circuits* are used to examine the induction phenomena. In the *left* case, the battery current increases whereas in the *right* case the battery current decreases

force appears only if there is an *acceleration* of the electrons in conductor 1, see Fig. 2.14. The force is observed to be proportional to the first time derivative of the battery current as is also observed in the previous experiment.

All observations and measurements from these two experiments may now be summarized in a concise formula. The inductive force on an electron of conductor 2 is deduced as

$$\bar{f}_{ind,2} = -C(R)q_2\frac{dI_1}{dt}d\bar{L}_1 \tag{2.20}$$

which is force on charge q_2 in segment 2 belonging to the conductor without battery. In formula (2.20), the force is really infinitesimal but since the length element dL_1 is considered to contain one concrete electron, the force is denoted without the infinitesimal sign. The formula is motivated as follows:

According to observation, charge 2 is affected even at rest, wherefore it is assumed that the inductive force is proportional to q_2 but independent of the speed of the charge. This assumption is based on the fact that, according to the electric and magnetic cases, the source of the force is either current or charge.

The force is directed along segment 1 which motivates the inclusion of the length element $d\bar{L}_1$ in the formula. The length element is directed along the current.

The time derivative in formula (2.20) reflects the fact that only a variation of current gives an induction current. Higher order derivatives are excluded as it is observed that a linearly increasing current induces a constant current in the other conductor.

The minus sign takes care of the fact that the system opposes current changes; a phenomenon which is referred to as Lenz' law.

$C(R)$ is a quantity including other relevant variables, such as the distance R, see Exercise (2.16). $C(R)$ will be determined in Sect. 3.4.

Formula (2.20) gives the force on a charge q_2 from a charge q_1 contained in a length element dL_1. To obtain the total force on q_2 all length elements of conductor 1 have to be summed up along the conductor

$$\bar{F}_{ind,2} = -\int_{Cond.\ 1} C(R)q_2\frac{dI_1}{dt}d\bar{L}_1 \tag{2.21}$$

Note that q_1 and q_2 may belong to the same conductor, so-called self induction, as in Fig. 2.12.

To get a clearer view of the direct interaction between two charges, formula (2.20) may be rewritten using formula (2.5):

$$\bar{f}_{ind,2} = -C(R)q_2q_1\frac{d\bar{v}_1}{dt} \tag{2.22}$$

If this formula is applied to current growth in a single conductor, the similarity to Newton's force formula $\bar{F} = m\,d\bar{v}/dt$ becomes flagrant, indicating that induction is a sort of electric counterpart to inertia.

Finally, the reader is reminded that these results are obtained from measurements using closed conductors. In analogy to magnetic force, the possibility that further terms arise for interaction between free charges cannot be excluded. There is no experimental data for this case available up to now.

The next chapter will discuss the inductive effect from an energy perspective and based on formula (2.21) the well verified Faraday-Henry induction law is derived.

In Chap. 9 the origin of the inductive force is examined.

2.4 Summary

The force between two charges at rest is called electric and is given by

$$\bar{f}_e = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}\hat{R} \tag{2.1}$$

where the distance vector R is directed towards the affected charge.

A current element is defined as

$$I d\bar{L} = q\bar{v} \tag{2.5}$$

The force between two current elements is called magnetic and is given by

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{2.11}$$

Commonly, the magnetic force is written

$$\bar{f}_{m2}^{G} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1] \tag{2.13}$$

which for free charges violates the principle of momentum conservation, i.e. Newton's third law.

Formulas (2.11) and (2.13) are consistent for closed conductors where only the first term in both formulas remains. They also agree if charge 2 is free but charge 1 is part of a closed conductor.

For charges in acceleration there is also an inductive force. For two interacting charges in closed conductors it is

$$\bar{f}_{ind,2} = -C(R) q_2 q_1 \frac{d\bar{v}_1}{dt} \tag{2.22}$$

which is the force acting on charge 2.

According to observation, inductive force decreases with distance.

The total electrodynamic force is (acting on object 2)

$$\bar{f} = \bar{f}_e + \bar{f}_m + \bar{f}_{ind}$$
$$= \underbrace{\frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R}}_{\text{Electric}} + \underbrace{\frac{\mu_0 q_1 q_2}{4\pi R^2}((-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2)}_{\text{Magnetic}} \underbrace{-C(R) q_2 q_1 \frac{d\bar{v}_1}{dt}}_{\text{Inductive}} \tag{2.23}$$

The magnetic and inductive forces are obtained from measurements using closed conductors and are therefore valid at low speed only.

2.5 Exercises

Exercises marked with '*' are an integral part of the book's curriculum.

Exercises marked with a 'C' are more challenging. It is sufficient to be able to follow the proposed solution found in Appendix D.

2.1 How large an excess/deficit of positive charge of the earth/sun is required in order for the electric force to be of the same magnitude as gravity?

2.2 a. Determine the drift speed in a current-carrying conductor under normal circumstances.

b. Two current-carrying straight conductors are oriented in parallel as in Fig. 2.3. Determine numerically electric and magnetic forces between two conduction electrons, located as q_1 and q_2 in Fig. 2.7.

2.3 What does the amount of charge 1 Coulomb signify? How would you construct a charge meter?

2.4 What is meant by the current strength 1 Ampère? How would you construct a current meter?

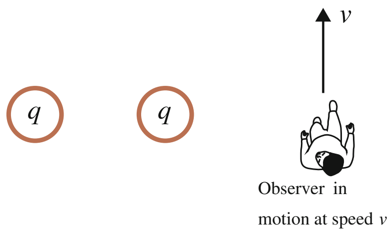
2.5 The apparatuses used to measure electric and magnetic force, described in Figs. 2.1 and 2.4, utilize either a torsional effect or a weight balance.

a. Could you propose some other way to measure force?

b. On what knowledge are the methods based? How, for example, is the torsional constant determined?

c. Reflect upon whether knowledge (i.e. science) is absolute or relative.

*2.6 *Principle of relative motion with observer*



a. Referring to Fig. 2.10, explain what the velocity v is relative to.

b. If the charges are at rest relative to a table, for example, and an observer is moving with velocity v, perpendicular to the connection line between the charges, what force situation would you then expect?

c. What exactly is observed by the person?

d. Imagine that you observe a pair of charges at rest. If you suddenly start to move perpendicular to the connection line of the charges, you will accordingly perceive a slower repulsion than if you were at relative rest. How would you as a scientist interpret this phenomenon?

Compare Exercise (1.6).

2.7 A straight current-carrying conductor interacts with a parallel free electron beam at a distance x.

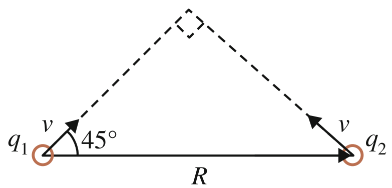
a. Denote the drift speed v_d and let the electron beam be generated by an acceleration voltage $\Delta\Phi$. Determine the force on one of the electrons in the beam.

b. Arrange an experiment to measure the force.

2.8 a. Use formula (2.4) to calculate the magnetic displacement in Figs. 2.5 and 2.6 over a length $x_a - x_b$ assuming the displacement is small.

b. Show that integration of formula (2.10) over L_1 from minus to plus infinity results in formula (2.4).

2.9 Consider a situation as in the figure.



a. Determine the magnetic force (2.11) with $a = 1$.

b. Compare with Grassman's formula (2.13)/(2.14).

c. Check the compatibility with Newton's third law in both cases.

C* 2.10 *Closed conductors and free charges*

a. Show that Grassman's magnetic force, formula (2.13), becomes in terms of current elements

$$\bar{f}^{G}_{m2} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[-\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1) + d\bar{L}_1(d\bar{L}_2 \cdot \hat{R})\right] \tag{2.24}$$

b. Show that Whittaker's force, formula (2.11), becomes in terms of current elements

$$\bar{f}_{m2} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[-\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1) + d\bar{L}_1(d\bar{L}_2 \cdot \hat{R}) + a d\bar{L}_2(d\bar{L}_1 \cdot \hat{R})\right] \tag{2.25}$$

c. Show that Grassman's and Whittaker's formulas agree for closed conductors and is given by

$$\bar{f}^{G}_{m2} = \bar{f}_{m2} = -\frac{\mu_0 I_1 I_2}{4\pi R^2}\left[\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1)\right] \tag{2.26}$$

i.e. the two last terms in (2.25) vanish.

Hint: Integrate (2.25) over closed paths and use Stoke's integral theorem.

d. Show that Grassmann's and Whittaker's formulas also agree if object 1 is a closed current-carrying conductor but object 2 is a free charge.

e. Show that a conductor of infinite length is equivalent to a closed conductor.

f. Explain, based on formula (2.26), why the magnetic interaction between charges in perpendicular relative motion, as in Fig. 2.5, cannot be studied with closed conductors.

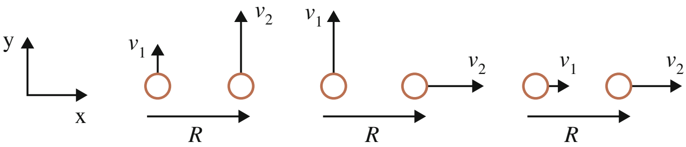
2.11 *Ampère's force law*

The first complete investigation of magnetic force was done by Ampère in 1820–1825. He studied the interaction between all possible *closed* conductor structures and could compile his measurements in the following force formula:

$$\bar{f}^{A}_{m2} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[-2(d\bar{L}_2 \cdot d\bar{L}_1) + 3(\hat{R} \cdot d\bar{L}_2)(\hat{R} \cdot d\bar{L}_1)\right]\hat{R} \tag{2.27}$$

a. Check whether Ampère's force fulfils Newton's third law.

b. Compare Ampère's (2.27), Grassman's (2.13) and Whittaker's (2.12) force in the following three cases of interaction between two point charges q_1 and q_2:

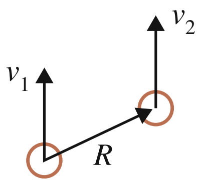


c. Show that for closed conductors the three force formulas agree.

d. Explain why Ampère's force law cannot be valid at high speeds $v \sim 1/\sqrt{\varepsilon_0\mu_0}$ (see also Chap. 9). Hint: Compare formula (2.18).

2.12 *Angular momentum issue*

Consider the interaction in the figure.



a. Determine the magnetic force with formula (2.12). Compare your result with Grassman's formula (2.13). Point out the basic difference between the two results.

b. Both cases generate angular momentum. Could there be any issues with this?

2.13 a. Explain why Ampère's force formula conserves angular momentum.

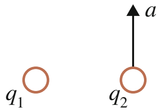
b. Based on Exercise (2.11), give two arguments as to why Ampère's force formula cannot be a candidate for the magnetic force between free charges.

2.14 The names chosen for the forces that have been examined in this chapter, i.e. electric, magnetic and inductive, have historical origins but are otherwise non-informative expressions. Based on the information given in this chapter, could you propose alternative names of these forces?

In Exercises 2.15–2.23 *the induction phenomenon is treated for both free charges and closed conductors. Use the inductive force formula* (2.22) *in both cases although it is strictly valid only for charges in closed conductors at low speed.*

2.15 *Elementary inductive force*

Consider the two charges in the figure.



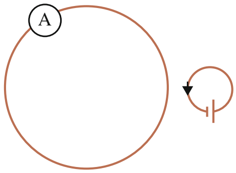
The right one has an acceleration a, the left one is at rest.

Determine the force on the left charge for

a. like charges,

b. unlike charges.

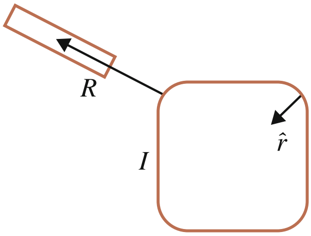
2.16 Let the inner, battery connected, conductor in Fig. 2.13 be placed outside the larger conductor as in the figure below. Let the battery generate a monotonically increasing current.



a. It is observed that the induction current decreases with increasing distance between the loops. What does this imply for the quantity C in formula (2.20)?

b. Make a sketch of the inductive forces acting on the conduction electrons in the left conductor. Estimate qualitatively the expected induction current in comparison with the situation in Fig. 2.13.

2.17 A loop with current I and drift velocity v has rounded corners with curvature radius r. Close to the loop, there is a piece of metal, see figure.

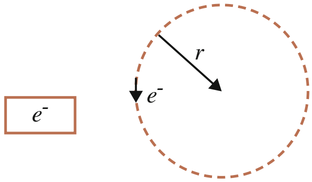


a. Determine the force on an electron in the piece of metal caused by a conduction electron with relative positions according to the figure.

b. Is the force dependent on the current direction?

2.18 *Synchrotron radiation*

A piece of metal is placed outside a circular electron beam made up of one single electron, see figure. If the beam has radius r and the electron speed v, with what frequency will a conduction electron in the metal oscillate?



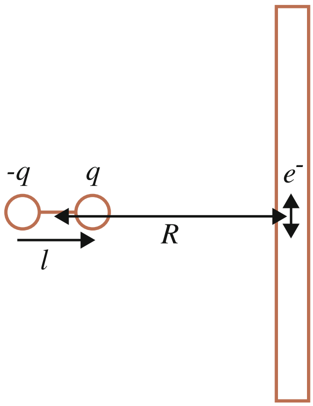
a. vertically?

b. horizontally?

This is a simplified principle for so-called synchrotron radiation.

2.19 *Microwave oven/dipole antenna*

In a simplified model of a microwave oven, water molecules are interacting with a straight current-carrying conductor according to the figure.



The alternating current in the conductor corresponds to a speed of the conduction electrons according to $v = v_0 \sin \omega t$.

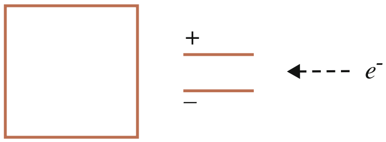
Let the dipole moment of the water molecules be $\bar{p} = q\bar{l}$ where q is the charge of the poles and l is the distance between these.

a. For a molecule with horizontal dipole moment, determine the torque caused by a conduction electron placed as in the figure.

b. Sketch a graph showing how the torque varies with time.

In general distance effects must also be considered. Furthermore, the straight conductor is also an oscillating electric dipole since charge is accumulated at its ends. This will be discussed in Chap. 11.

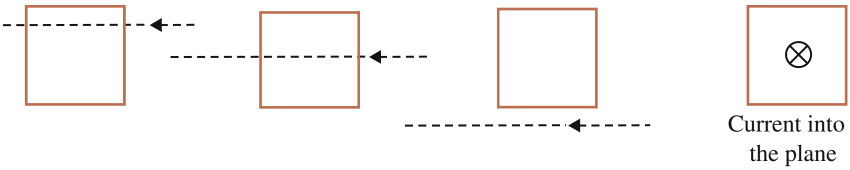
2.20 An electron beam is directed towards a square conducting loop. Just in front of the loop there are two oppositely charged parallel plates, see figure.



a. Sketch the path of the electrons between the plates.

b. How does the loop respond when an electron moves between the plates? State the current direction generated in the loop.

2.21 A beam with constantly *accelerating* electrons interacts with a square conductor loop. Consider the four cases below and state if and in what direction a current is induced in the loop.



2.22 To generate current in a conductor a battery is connected which accelerates the electrons to their final speed, the so-called drift speed.

a. Let the conductor be a circular loop.

Make a sketch showing the inductive forces on a single electron due to interactions with the other electrons in the loop.

Assume that the inductive force decreases with distance.

b. Repeat with a straight conductor.

c. If there were only one single electron in the conductor would then inductive effects arise at all?

C 2.23 *Gravitation*

a. Would you think that magnetic and inductive effects, i.e. effects due to uniform and accelerating motion, also occur in the case of gravitation?

b. The inductive forces generate a type of inertia in a conductor, i.e. they resist the acceleration of charge. Massive objects also exhibit inertia. If this inertia has its origin in an effect similar to that for electric charges in a conductor, what force might be its cause?

c. How does this relate to Mach's principle (compare Exercise 1.3b)?

d. If in the entire universe there were only one object, would it possesses inertia?

*2.24 *Current densities.*

In Sect. (2.2.1) electric current I was defined as a flux. If the vector $\bar{J}$ is introduced as the volume density of current, corresponding to current per unit area, current becomes:

$$I = \int_S \bar{J} \cdot d\bar{a}$$

where S is the area through which charges pass and $d\bar{a}$ is accordingly an element of the surface S.

a. Show that the current density J becomes

$$\bar{J} = nq\bar{v}$$

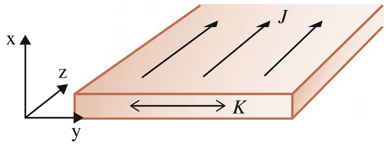
where n is the density of charge carriers, q their charge and $\bar{v}$ the drift velocity.

b. Show that a current element

$$Id\bar{L} = \bar{J}dV$$

where $d\bar{L}$ is an infinitesimal length element of a conductor and dV the corresponding volume.

c. In many cases, e.g. for current-carrying plates and wide cables, it is practical to introduce a surface density of current $\bar{K}$, corresponding to current per unit length where the length is perpendicular to the current density $\bar{J}$. $\bar{K}$ and $\bar{J}$ are parallel.



Consider for example a current carrying plate as in the figure.

Total current is given by

$$I = \int K dy = \iint J dx dy$$

Since

$$I = \int K dy = \iint \frac{dK}{dx} dx dy$$

the infinitesimal surface current density becomes

$$d\bar{K} = \bar{J} dx$$

Hence, show that for an infinitesimal cross section $dxdy$ the current element becomes

$$dId\bar{L} = d\bar{K}dydz$$

C 2.25 A long straight conductor with length T_1, part of a closed conductor, interacts with a beam of free electrons with a shorter length T_2, perpendicular to T_1. See Figs. 2.5 and 2.6. The force on the free electrons is given by formula (2.4).

Determine the force on the *conductor* with

a. Grassman's force formula (2.24).
b. Whittaker's force formula (2.25) with $a = 1$.
c. Is Newton's principle of action and reaction fulfilled?
d. Arrange an experiment to measure the force. How large a current in the electron tube is required to get a measurable force?

Further Readings

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Chapter 3
Electrodynamic Energy

That small word ‘Force’, they make a barber’s block,
Ready to put on
Meanings most strange and various, fit to shock
Pupils of Newton…
The phrases of last century in this
Linger to play tricks
Vis viva and Vis Mortua and Vis Acceleratrix:
Those long-nebbed words that to our text books still
Cling by their titles,
And from them creep, as entozoa will,
Into our vitals.
But see! Tait writes in lucid symbols clear
One small equation;
And Force becomes of Energy a mere
Space-variation.

James Clerk Maxwell, 1876

Maxwell refers here to formula (1.1) as Tait’s equation.

In describing physical processes, the energy concept is complementary to the force description. Being a scalar, it is computationally advantageous to compute the energy of a system and then apply the energy principle, formula (1.1), to obtain all components of force.

Energy exists in two forms that are continuously converted into each other:

- Potential energy, also known as stored energy.
- Kinetic energy, also known as motional energy.

The energy principle explains that the total energy in nature is conserved and that the dynamics, i.e. the forces, result as a minimization process of stored energy.

Energy appears in an interaction and is stored in the system defined by the interacting constituents. Stored energy can not be attributed to a single object.

In this chapter electric and magnetic energy will be defined. The starting point is the concept of work, i.e. force times distance. The active forces are the electric and the inductive generating electric and magnetic energy respectively.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_3

Furthermore, the inductance concept will be introduced through which Faraday-Henry's law of induction is formulated.

3.1 Electric Energy

A system consisting of a number of charges possesses potential energy known as electric energy. The amount of energy corresponds to the work needed to merge these charges.

In Fig. 3.1 a work W is done in bringing together two like charges q_i and q_j. While the charge q_i is fixed, let the external force F bring charge q_j along the R-axis towards charge q_i starting from infinity. The force F must be at least as large as the repulsive electric force and the minimum work required is thus

$$W = \int_{\infty}^{R_{ij}} \bar{F} \cdot d\bar{s} = \int_{\infty}^{R_{ij}} -\frac{q_i q_j}{4\pi \varepsilon_0 R'^2_{ij}} dR'_{ij} = \left[\frac{q_i q_j}{4\pi \varepsilon_0 R'_{ij}} \right]_{\infty}^{R_{ij}} = \frac{q_i q_j}{4\pi \varepsilon_0 R_{ij}} = U_{ij} \quad (3.1)$$

In the first integral $\bar{F} \cdot d\bar{s}$ is positive since ds and F are parallel. The integrand of the second integral is also positive because dR'_{ij} is negative since distance decreases. In formula (3.1), the force F is set equal to the electric force so that kinetic energy does not appear. U_{ij} is then the electric stored energy between two charges.

For a general system a pairwise summation is done:

$$U_e = \frac{1}{2} \sum_{i,j} \frac{q_i q_j}{4\pi \varepsilon_0 R_{ij}} \quad (3.2)$$

where the sum runs over all charges as long as i does not equal j. The factor $1/2$ accounts for double counting. Note that the sign of the charges decides whether the energy is positive or negative. If for example a pair of charges have opposite signs the electric energy is negative indicating that the process to bring one charge to the other does not require any external work but appears naturally.

The concept electric potential Φ_i is defined through the electric energy. At the position of charge q_i the potential is

$$\Phi_i = \sum_{j} \frac{q_j}{4\pi \varepsilon_0 R_{ij}} \quad (3.3)$$

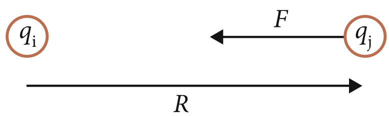


Fig. 3.1 Two like charges are brought together by external force F

such that the electric energy may be written

$$U_e = \frac{1}{2} \sum_i q_i \Phi_i \tag{3.4}$$

The dynamics of a system is governed by the change of the stored energy with respect to the space coordinates. If a charge q is moved from point a to point b and everything else is unchanged, the energy change of the system is

$$\Delta U_e = q(\Phi_b - \Phi_a) \tag{3.5}$$

The potential difference between two points is also referred to as voltage

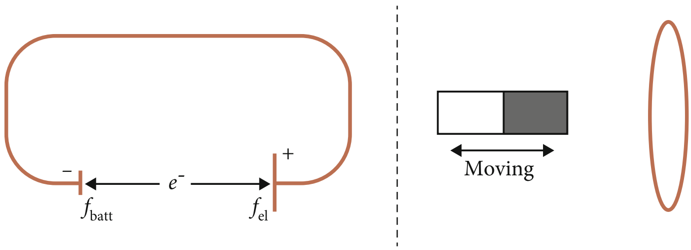
$$\Delta\Phi = \Phi_b - \Phi_a \tag{3.6}$$

a quantity that common voltmeters are calibrated to measure.

3.2 The Voltage Source

Magnetic energy will be defined for *closed* current-carrying conductors. Before doing this, the concept 'voltage source' needs to be defined. In Sect. 2.2 magnetic force was discussed in terms of the electric circuit which is driven by a voltage source. This provides a voltage by which electrons gain kinetic energy. During this process the voltage source is doing work and therefore its stored energy is diminished.

The function of the voltage source may either be based on an electrochemical or a magnetic process, see figure.



An *electrochemical* cell, more commonly known as a battery, transports electrons through chemical processes from the plus to the minus pole of the battery and works accordingly against the internal electric force, see left figure above. The electric energy, of which the electron is one part, increases thereby and will be used to create kinetic energy of the electron in the next step.

A corresponding process in the gravitational case is rolling a ball uphill and letting gravity generate motion downhill.

A *magnetic* voltage source works in principle as illustrated in the right-hand figure, where the moving permanent magnet usually is another current-carrying circuit. In this case there is accordingly no battery voltage but it is customary to introduce a fictitious so-called induction voltage defined by the generated current. This will be discussed in Sect. 3.5.

Henceforth, the word ‘battery’ will be referred to as a voltage source of any kind.

3.3 Magnetic Energy

Magnetic energy for electric circuits is defined by the amount of work a battery has to do in order to accelerate the conduction electrons to a final current I. The force the battery has to overcome is accordingly the inductive force, formula (2.20), since it opposes the external influence as well as being acceleration dependent.

Single loop. Consider a single loop as in Fig. 3.2. Suppose there is a current i in the loop at time t. As the charges are accelerated, the current increases. The work required to accelerate an infinitesimal amount of charge dq through the closed conductor once is

$$dW_b = -\int_{Loop} \bar{F}_{ind} \cdot d\bar{L}_1 \tag{3.7}$$

where formula (2.21) gives

$$\bar{F}_{ind} = -\int_{Loop} C(R)dq\frac{di}{dt}d\bar{L}_2 \tag{3.8}$$

In formula (3.8) the infinitesimal forces are integrated over $d\bar{L}_2$ to sum up the total inductive force that acts on the infinitesimal charge dq. In formula (3.7) the total inductive force is integrated over $d\bar{L}_1$ to sum up the total work required to transport the charge dq through the whole loop. The length elements $d\bar{L}_1$ and $d\bar{L}_2$ are in this case part of the same loop.

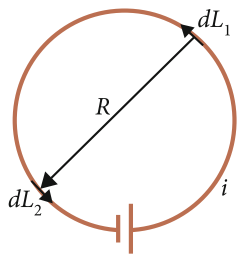


Fig. 3.2 Definition of interacting current elements dL for a single loop

Formula (3.8) is really an infinitesimal force since it acts on the infinitesimal element dq, which however is to be considered as a real electron. Therefore the force is denoted without infinitesimal sign.

The infinitesimal magnetic energy becomes

$$dU_m = dW_b = \iint\limits_{Loop} C(R)dq\frac{di}{dt}d\bar{L}_1 \cdot d\bar{L}_2$$
$$= dq\frac{di}{dt}\iint\limits_{Loop} C(R)d\bar{L}_1 \cdot d\bar{L}_2 \tag{3.9}$$

The double integral is called inductance

$$M = \iint\limits_{Loop} C(R)d\bar{L}_1 \cdot d\bar{L}_2 \tag{3.10}$$

and contains only geometrical properties of the conductor. The integration in (3.10) includes all pairwise interactions between the length elements of the conductor. The scalar product ensures that perpendicular elements do not affect each other, which is experimentally observed (see Sect. 2.3). Applied to only *one* conductor, as in this case, M is called self inductance.

The magnetic energy may now be expressed as:

$$dU_m = Mdq\frac{di}{dt} = Midt\frac{di}{dt} = Mi\,di \tag{3.11}$$

The total energy transferred by the battery when the current increases from 0 to its final value I becomes[1]

$$U_m = \int_0^I Mi\,di = M\frac{I^2}{2} \tag{3.12}$$

Several loops. Consider now a system of several loops, Fig. 3.3. To find the total magnetic energy the interactions between the loops have to be considered. The interaction is governed by the inductance M which can generally be written

$$M_{jk} = \int\limits_k \int\limits_j C(R)d\bar{L}_j \cdot d\bar{L}_k \tag{3.13}$$

[1] Note the similarity between formula (3.12) and mechanical kinetic energy 1/2 mv^2. This is not a coincidence since magnetism is directly connected to the motion of charges. It is interesting that JC Maxwell proposed the name 'electric inertia' for inductance M, see also Sect. 2.3.

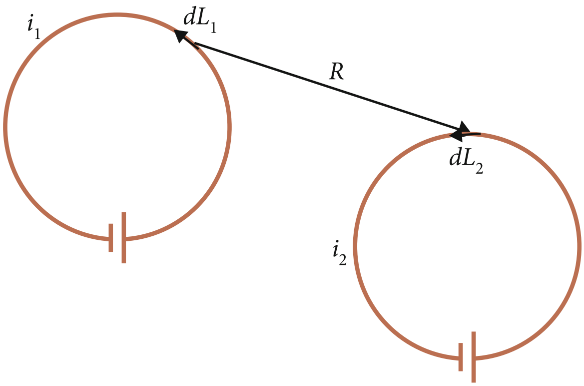


Fig. 3.3 Definition of interacting current elements dL for a pair of loops

For different conductors j and k, M_{jk} is called mutual inductance. It will be shown in Sect. 3.4 that C is solely dependent on the distance between the current elements so that

$$M_{jk} = M_{kj} \tag{3.14}$$

According to formula (3.11), a current increase di_k causes a change of magnetic energy

$$dU_m = \sum_j \sum_k M_{jk} i_j di_k \tag{3.15}$$

which should be integrated to the final current values I_j, I_k. To illustrate principles, consider a system with just two loops:

$$dU_m = M_{11} i_1 di_1 + M_{12} i_1 di_2 + M_{21} i_2 di_1 + M_{22} i_2 di_2 \tag{3.16}$$

Since the energy of the system is independent of the order in which the currents were generated in the loops (why?) it may be done for one loop at a time:

1. Increase the current to I_1 in loop 1: $di_2 = 0$, $i_2 = 0$,

$$U_{m1} = M_{11} \frac{I_1^2}{2} \tag{3.17}$$

2. Increase the current to I_2 in loop 2: $i_1 = const = I_1$, $di_1 = 0$,

$$U_{m2} = M_{12} I_1 I_2 + M_{22} \frac{I_2^2}{2} \tag{3.18}$$

The total energy for this system of two loops is $U_m = U_{m1} + U_{m2}$. Generally, for a system with N loops ($N = 1$ for a single loop), the energy becomes

$$U_m = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} M_{jk} I_j I_k \tag{3.19}$$

where j may also equal k, so-called self inductance.

Formula (3.19) was obtained by assuming fixed inductance. But it is also valid for varying inductances, which will be utilized in the next section.

3.3.1 Magnetic Force from Magnetic Energy

When dealing with magnetic forces, closed circuits are usually considered where the current is maintained by a battery and is therefore (quasi)-constant. When applying the energy force relation, formula (1.1), the battery plays a crucial role in the following way.

Consider an interaction between two current-carrying loops, Fig. 3.3. For constant current, a change of the magnetic interaction energy appears through a change of the mutual inductance, i.e. through a change of the distance between the loops or their relative orientation. Thus, according to formula (3.19),

$$dU_m = \frac{1}{2} I_1 I_2 dM_{12} + \frac{1}{2} I_2 I_1 dM_{21} = \frac{1}{2}[I_1 d(I_2 M_{12}) + I_2 d(I_1 M_{21})] \tag{3.20}$$

where the currents are given at final values, i.e. restored by the batteries. This requires work from the batteries dW_b that will change the energy according to formula (3.16):

$$\begin{aligned} dU_b = -dW_b &= -(I_1 M_{12} dI_2 + I_2 M_{21} dI_1) \\ &= -[I_1 d(M_{12} I_2) + I_2 d(M_{21} I_1)] \end{aligned} \tag{3.21}$$

where the inductances are regarded as final values and the batteries restore the current in that situation. Note that the self inductances are not included here since only the interaction between the loops is considered.

Thus, $dU_b = -2dU_m$ and the change of the total energy becomes

$$dU = dU_m + dU_b = -dU_m \tag{3.22}$$

The force is therefore

$$\bar{F}_m = \nabla U_m \qquad \textit{Constant current!} \tag{3.23}$$

so that in case of constant current the magnetic force acts towards maximizing the magnetic energy.

3.4 General Inductance—Interaction Between Two Current Elements

Using formula (3.23) and the knowledge about the magnetic force between parallel currents as well as the general magnetic energy formula (3.19), the inductance formula (3.13) will now be completed.

It is then first relevant to ask why the magnetic energy, whose definition is based upon the work done by the inductive force, is related to the *magnetic* force according to formula (3.23). Consider two parallel current-carrying conductors, e.g. of circular form. For parallel currents there is an attractive force. During the attraction the currents will however decrease, see Exercise (3.9c), which is exactly compensated by the battery. The work of the battery, related to the work of the inductive force as explained above, is thereby connected to the energy change that arises because of the magnetic force between circuits.

Based on the fact that magnetic energy is related to magnetic force according to formula (3.23), the expression of the factor C in formulas (2.22) and (3.13) may be determined. The known force formula between two parallel current elements is then utilized, obtained from formula (2.11), Fig. 3.4.

The force on element 2 becomes

$$\bar{f}_m = -\frac{\mu_0 I_1 \Delta L_1 I_2 \Delta L_2}{4\pi R^2}\hat{R} \tag{3.24}$$

The magnetic interaction energy for this system is according to formula (3.19)

$$U_m = M_{12} I_1 I_2 \tag{3.25}$$

where from formula (3.13)

$$M_{12} = C(R)\Delta L_1 \Delta L_2 \tag{3.26}$$

so that the magnetic energy may be written

$$U_m = C(R) I_1 \Delta L_1 I_2 \Delta L_2 \tag{3.27}$$

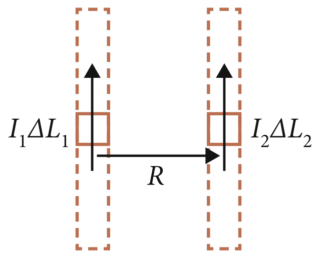


Fig. 3.4 Two parallel current elements

Using formula (3.23), the magnetic force between the current elements becomes

$$\bar{f}_m = \nabla U_m = I_1 \Delta L_1 I_2 \Delta L_2 \frac{dC(R)}{dR}\hat{R} \tag{3.28}$$

which results in

$$C(R) = \frac{\mu_0}{4\pi R} + \text{constant} \tag{3.29}$$

Let the constant equal zero so that the energy tends to zero when distance tends to infinity. The general formula for inductance becomes

$$M_{jk} = \frac{\mu_0}{4\pi}\int_k\int_j \frac{d\bar{L}_j \cdot d\bar{L}_k}{R} \tag{3.30}$$

where R is the distance between the length elements, Figs. 3.2 and 3.3. This is Neumann's formula from 1845.

Further experimental support to the formulas (3.29) and (3.30) is obtained from Faraday-Henry's law of induction, derived in next section.

3.5 Faraday-Henry's Induction Law

Energy is related to voltage through formulas (3.5) and (3.6). These are now generalized to be valid also for magnetic energy. The induction voltage ε is then given by formulas (3.9) and (3.10)

$$\varepsilon = \frac{dU_m}{dq} = \frac{di}{dt}M \tag{3.31}$$

for a single circuit with constant inductance M.

If inductance is time dependent, (3.31) is generalized as

$$\varepsilon_j = \frac{d}{dt}(i_k M_{jk}) \tag{3.32}$$

By summing over all circuits, including circuit j for self induction, the general induction formula becomes

$$\varepsilon_j = \sum_k \frac{d}{dt}(i_k M_{jk}) \tag{3.33}$$

where M_{jk} is given by formula (3.30), i.e. Neumann's inductance formula.

Formula (3.33) is known as Faraday-Henry's induction law and is well verified. It confirms therefore the validity of formulas (2.22) and (3.30).

3.6 Electrodynamic Force—Updated

Utilizing formula (3.29) the inductive force between two charges, formula (2.22), becomes

$$\bar{f}_{ind} = -\frac{\mu_0 q_2 q_1}{4\pi R}\frac{d\bar{v}_1}{dt} \tag{3.34}$$

which is the inductive force acting on charge 2. This is Weber's formula from 1846. For like charges it is oppositely directed to the acceleration of charge 1. Note the distance dependence $1/R$.

The force between two electric charges, formula (2.23), becomes

$$\begin{aligned}\bar{f} &= \bar{f}_e + \bar{f}_m + \bar{f}_{ind} \\ &= \underbrace{\frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R}}_{\text{Electric}} + \underbrace{\frac{\mu_0 q_1 q_2}{4\pi R^2}((-\bar{v}_1\cdot\bar{v}_2)\hat{R} + (\bar{v}_2\cdot\hat{R})\bar{v}_1 + a(\bar{v}_1\cdot\hat{R})\bar{v}_2)}_{\text{Magnetic}} \\ &\quad \underbrace{-\frac{\mu_0 q_2 q_1}{4\pi R}\frac{d\bar{v}_1}{dt}}_{\text{Inductive}}\end{aligned} \tag{3.35}$$

and for parallel motion as in Fig. 2.10:

$$\bar{f} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}(1 - \varepsilon_0\mu_0 v^2)\hat{R} - \frac{\mu_0 q_2 q_1}{4\pi R}\frac{d\bar{v}_1}{dt} \tag{3.36}$$

which in both cases is the force on charge 2.

3.7 Summary

Electric energy for a system of discrete charges is

$$U_e = \frac{1}{2}\sum_{i,j}\frac{q_i q_j}{4\pi\varepsilon_0 R_{ij}} \tag{3.2}$$

Electric potential at a point i is

$$\Phi_i = \sum_j \frac{q_j}{4\pi\varepsilon_0 R_{ij}} \tag{3.3}$$

Potential difference between two points a and b is known as voltage

$$\Delta\Phi = \Phi_b - \Phi_a \tag{3.6}$$

Magnetic energy appears through the work of the inductive force. The inductive force between two charges q_1 and q_2 in a closed conductor is

$$\bar{f}_{ind} = -\frac{\mu_0 q_2 q_1}{4\pi R}\frac{d\bar{v}_1}{dt} \tag{3.34}$$

for force on object 2.

Magnetic energy for a system of closed conductors is

$$U_m = \frac{1}{2}\sum_{j,k} M_{jk} I_j I_k \tag{3.19}$$

where the mutual inductance is

$$M_{jk} = \frac{\mu_0}{4\pi}\int_k\int_j \frac{d\bar{L}_j \cdot d\bar{L}_k}{R} \tag{3.30}$$

The induction voltage derives from magnetic energy and is given by

$$\varepsilon_j = \sum_k \frac{d}{dt}(i_k M_{jk}) \tag{3.33}$$

which is known as Faraday-Henry's induction law.

3.8 Exercises

3.1 When a chlorophyll molecule absorbs energy from the sun it is sometimes said that an electron of the molecule is excited. What is wrong with this description according to the principles outlined in this chapter?

Answer: A single electron cannot be excited. The energy increase is stored in the molecule through an electric interaction between the electron and the rest of the molecule. Thus, it is rather the molecule that is being excited.

3.2 Consider the electric pairwise interaction. Convince yourself that the electric energy decreases for both repulsive and attractive interaction.

3.3 *Electrostatic energy*

a. Determine the electric energy for a system of three charges q_1, q_2 and q_3, located at the corners of a uniform triangle with side a.
b. Determine the force on each charge using the energy principle, formula (1.1).
c. Investigate whether Newton's third law, the principle of action and reaction, is fulfilled.

3.4 a. Show that if the mutual inductance (3.30) is used for two charges in motion the magnetic energy becomes

$$U_m = \frac{\mu_0 q_1 q_2}{4\pi R}\bar{v}_1 \cdot \bar{v}_2 \tag{3.37}$$

b. Investigate whether formula (1.1) gives the magnetic force (2.16) (for constant v). Discuss the discrepancy. Under what conditions is (3.37) valid?

3.5 Consider two conductors. One of them is connected to a battery. Based on formula (3.34), determine the response in one conductor if the current in the other increases/decreases for conductors that are

a. straight and parallel,
b. circular and coaxial, as in Fig. 4.13.

Hint: Draw a figure showing the inductive forces on the conduction electrons in the four cases (compare Exercise 2.22).

*3.6 Two circular loops are oriented in parallel and have a common axis. A current is generated in one of them such that it increases linearly with an amount ΔI during a time interval ΔT. In the other conductor an induced voltage ε is registered.

a. Determine the mutual inductance.
b. The loops are now placed so far from each other that the distance between two current elements may be considered to be independent of their relative orientation. For parallel orientation and linear current growth $\Delta I/\Delta T$, a voltage ε_2 is induced. Determine the mutual inductance for any angle θ between their surface normals.

3.7 *The permanent magnet*

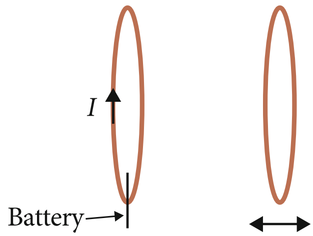
Consider a permanent bar magnet with rectangular cross section. Frequently such an object is described in terms of south and north poles.

a. If you split such an object in the middle, what will happen with the poles? If you continue to split the objects new magnets with north and south poles are obtained. Thus, the permanent magnet may be considered as a collection of small magnetic dipoles.
b. How are these oriented in a homogeneously magnetized material?
c. How are they oriented in a non-magnetic material?
d. What does one of these small dipoles correspond to in terms of current?
e. Draw a figure showing how the dipoles and their corresponding current loops are oriented in a homogeneously magnetized object.
Answer: see Figs. 7.7 and 8.14.
f. Where are the net currents located in a homogeneous permanent magnet, see Fig. 8.14?
g. How does a material become magnetized?

h. How can the stored energy of a magnet be measured? (See also Exercise 8.22)

3.8 A single loop without battery is located close to a circuit connected to a battery. Specify the five different ways to induce a current in the former loop.

*3.9 *Relative motion—Einstein's starting point*
Two identical circular conductor loops are placed in parallel with a common axis.



One of them carries a constant current I, see figure. By external influence, the other loop is now moved along the common axis.

a. Specify the induction law (3.33) for this case.
b. What type of force is acting on the conduction electrons of the moving loop? Show in a figure how the inductive force, formula (3.34), acts in this case.
c. Determine the direction of the induced current in the moving loop when it is approaching and receding the current-carrying loop.
d. What will be the result if instead the current-carrying loop is moving?
e. Discuss the relativity principle of Galilei based on this example.

C 3.10 *Magnetic energy between free charges*

Formula (3.37) gives the magnetic energy between two charges contained in closed conductors. The corresponding expression for free charges has for a long time been sought for.
In (1995), Bueno and Assis combined the four most important proposals in a single formula:

$$U_m = \frac{\mu_0}{4\pi} q_1 q_2 \left[\frac{1+k}{2} \frac{\bar{v}_1 \cdot \bar{v}_2}{R} + \frac{1-k}{2} \frac{(\hat{R} \cdot \bar{v}_1)(\hat{R} \cdot \bar{v}_2)}{R} \right]$$

where k takes the value $1,\ -1,\ 0$ or -5 corresponding to proposals from Neumann (1845) (compare 3.37), Weber (1846), Maxwell (1870), Darwin (1920) and Graneau (1985) respectively.

a. Determine the force from this energy formula and compare it with (2.12) and (2.13).

b. Are any of the expressions compatible with the observation described by formula (2.12)?
c. Decide whether Newton's third law is fulfilled.
d. Show that the second term vanishes for closed conductors as well as for the situation in Fig. (2.10).

Further Reading

M. Bueno, A.T.K. Assis, Equivalence between the formulas for inductance calculation. Can. J. Phys. **78**, 357 (1997)

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J. Henry, On the production of currents and sparks of electricity from magnetism. Am. J. Sci. Arts. **22**, 408 (1832)

F. Neumann, Allgemeine Gesetze der inducirten elektrischen Ströme, Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin, pp. 1–87 (1845)

W. Weber, On the measurement of electrodynamic forces. Ann. Phys. **73**, 193 (1848)

Chapter 4
Macroscopic Systems of Unbound Charges

God does not care about our mathematical difficulties. He integrates empirically.

Albert Einstein, 1942

In this chapter macroscopic systems of individual electric charges will be examined. These are called unbound charges in contrast to the bound charges that materials are usually made up of. The conduction electrons in a metal is the most common example of unbound charges. These are unbound inside the material but cannot normally leave the metal due to effects of surface bonding.

The aim is to examine the interaction between macroscopic objects. The computational method is introduced using specific examples, gradually increasing in complexity. The method is based on the principle that the total force between the two systems can be obtained by summing up the pairwise interactions between its constituents. In practice, this summation becomes an integration where the microscopic constituents are represented by infinitesimal elements. Electric systems consist of infinitesimal point charges $dq = \rho dV$ where ρ is the charge density and dV a volume element. The constituents of magnetic systems are the infinitesimal current elements $Id\bar{L}$, i.e. current times the length element.

Often, however, it is impractical to study individual interactions between charges. The calculations may be facilitated by introducing the concepts of capacitance and inductance by which the force is considered as a collective action. These concepts are particularly useful for homogeneous charge and current distributions.

In the calculations, the position vector for the source is denoted $\bar{r}'$ and that for the influenced element $\bar{r}$. The distance vector between the elements is denoted $\bar{R}$ as before, pointing to the affected element.

Cylindrical and spherical coordinates are prerequisites for this chapter.

In Chap. 8, macroscopic systems will be re-examined, then with emphasis on *bound* charge and current appearing as electric and magnetic dipoles.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_4

4.1 Electric Dynamics

In electrostatic systems the basic interaction is described by Coulomb's force law, formula (2.1):

$$\bar{F}_{1\to 2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R} \tag{2.1}$$

4.1.1 Electrically Charged Wire and Point Charge

The force between a homogeneously charged wire with charge per unit length λ' and a point charge q is examined, Fig. 4.1. An infinitesimal amount of charge of the wire $\lambda' dz'$ is located at a position $\bar{r}' = (x', y', z')$. The position vector $\bar{r}$ points to the point charge q and the distance between q and $\lambda' dz'$ is $\bar{R}$. The force on the charge q from an infinitesimal element on the wire becomes

$$d\bar{F} = \frac{\lambda' dz' q}{4\pi\varepsilon_0 R^2}\hat{R} = \frac{\lambda' dz' q}{4\pi\varepsilon_0 R^3}\bar{R} \tag{4.1}$$

To obtain the total force of all contributions from the elements on the wire have to be summed. Firstly, the distance vector is expressed in terms of coordinates

$$\bar{R} = \bar{r} - \bar{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z} \tag{4.2}$$

Let the wire coincide with the z-axis, i.e. $x' = y' = 0$. To simplify, let the point charge q be located at the point $(x, y, z) = (d, 0, 0)$ so that

$$\bar{R} = d\hat{x} - z'\hat{z} \tag{4.3}$$

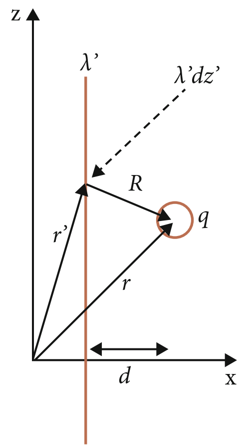


Fig. 4.1 A point charge q interacts with a charged wire

If the length of the wire is L and is centred about the coordinate origin, the total force on q is

$$\bar{F} = \frac{\lambda' q}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{(d\hat{x} - z'\hat{z})dz'}{(d^2 + z'^2)^{3/2}} \tag{4.4}$$

This is an integral over z' consisting of two terms corresponding to the x and z components of the force. The first integral becomes

$$\begin{aligned} F_x &= \frac{\lambda' q d}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{1}{(d^2 + z'^2)^{3/2}} dz' = \frac{\lambda' q d}{4\pi\varepsilon_0} \left[\frac{z'}{d^2\sqrt{z'^2 + d^2}} \right]_{-L/2}^{L/2} \\ &= \frac{\lambda' q d}{4\pi\varepsilon_0} \left(\frac{L/2}{d^2\sqrt{(L/2)^2 + d^2}} + \frac{L/2}{d^2\sqrt{(L/2)^2 + d^2}} \right) \\ &= \frac{\lambda' q}{4\pi\varepsilon_0 d} \frac{L}{\sqrt{(L/2)^2 + d^2}} \end{aligned} \tag{4.5}$$

The z component is an odd function and vanishes (only in this case where the point charge is located halfway along the length of the wire).

For a very long wire, mathematically infinite, end effects are neglected. Practically it means that the charge q is placed close to the wire. The force becomes

$$\lim_{L\to\infty} \bar{F} = \frac{\lambda' q}{2\pi\varepsilon_0 d}\hat{x} \tag{4.6}$$

since d^2 in the square root of the denominator of formula (4.5) may be neglected.

Thus, the force between a long wire and a point charge is inversely proportional to the distance between them.

4.1.2 Force Between Two Electrically Charged Wires

Next consider two parallel homogeneously charged wires with length L located in the xz plane, Fig. 4.2.

If end effects are neglected, i.e. $L \to \infty$, the force on each length element on the right wire is given by formula (4.6), with $q = \lambda dz$. The total force between the wires becomes

$$\bar{F} = \frac{\lambda' \lambda L}{2\pi\varepsilon_0 d}\hat{x} \tag{4.7}$$

If the wires are of finite length, i.e. end effects are included, formula (4.4) is generalized to a double integral. Thus, with one wire coinciding with the z axis and

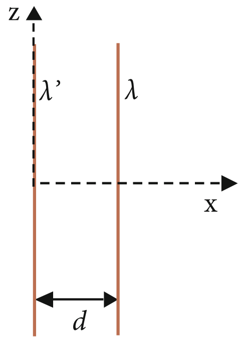


Fig. 4.2 Two parallel charged wires interact

the other placed at a distance d and both centred about $z = 0$ (see Exercise 4.1),

$$\bar{F} = \frac{\lambda'\lambda}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{d\hat{x} + (z - z')\hat{z}}{(d^2 + (z - z')^2)^{3/2}} dz' dz \tag{4.8}$$

4.1.3 Force Between a Point Charge and an Electrically Charged Plate

Consider the force on a point charge from a large homogeneously charged plate, so large that effects at the edge of the plate may be neglected. This approximation is valid if the distance d between the point charge and the plate is small relative to the size of the plate.

The plate is located parallel to the yz plane and the point charge has coordinates (x, y, z), Fig. 4.3. Viewed along the z direction, Fig. 4.4, the plate may be considered as being built up by close wires.

The force on the point charge from *one* wire is given by formula (4.6):

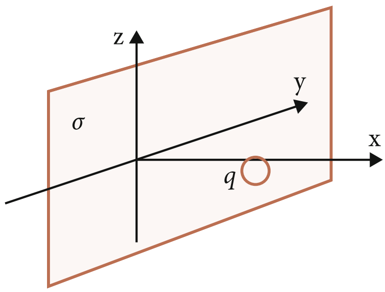


Fig. 4.3 Interaction between a charged plate and a point charge

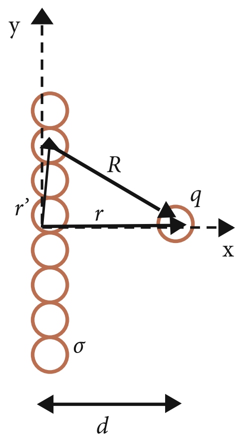


Fig. 4.4 The plate, observed from above, is treated as if it consists of close parallel wires directed into the plane (z direction)

$$d\bar{F} = \frac{\lambda' q}{2\pi\varepsilon_0 R}\hat{R} = \frac{\lambda' q}{2\pi\varepsilon_0 R^2}\bar{R} \tag{4.9}$$

Using (4.9) a summation of the wires along y can be performed. The wire thickness is then considered to be infinitesimal, dy'. A length element of the wire contains a charge $q' = \lambda' dz' = \sigma dy' dz'$ where σ accordingly is charge per unit surface of the plate. Thus, $\lambda' = \sigma dy'$. The coordinates of the point charge are set to $(d, 0, 0)$. With the plate in the yz plane the distance vector becomes $\bar{R} = (\bar{r} - \bar{r}') = d\hat{x} - y'\hat{y} - z'\hat{z}$. The calculation is performed in the xy plane so that $z = z' = 0$. The force on the point charge becomes

$$\bar{F} = \int_{-\infty}^{\infty} \frac{q\sigma dy'}{2\pi\varepsilon_0(d^2 + y'^2)}(d\hat{x} - y'\hat{y}) \tag{4.10}$$

The integral of the y component, the second term, vanishes since the integrand is an odd function. This is also physically reasonable (why?). The force becomes

$$\bar{F} = \frac{\sigma q d}{2\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{dy'}{(d^2 + y'^2)}\hat{x} = \frac{\sigma q d}{2\pi\varepsilon_0}\left[\frac{1}{d}\arctan\frac{y'}{d}\right]_{-\infty}^{\infty}\hat{x} = \frac{\sigma q}{2\varepsilon_0}\hat{x} \tag{4.11}$$

independent of distance from the plate. However, it has been assumed that the distance d is much smaller than the size of the plate. See also Exercise (4.2).

4.1.4 Electric Force Between a Point Charge and a Charged Sphere

Next the force between a point charge and a homogeneously charged sphere is examined. The calculation is done with the point charge inside and outside the sphere. At the intersection, the results should coincide. To utilize the symmetry, spherical coordinates are used. The charge density of the sphere is denoted ρ. The total charge of the sphere is Q and its centre is placed at the coordinate origin (Fig. 4.5).

To obtain the force between the objects, it is in this case preferable to use the energy principle, formula (1.1). The energy is obtained from formula (3.2) with the sum replaced by an integration:

$$U_e = \frac{q}{4\pi\varepsilon_0} \int\limits_V \frac{\rho dV'}{R} \tag{4.12}$$

where V is the volume of the sphere. Note that the correction factor $1/2$ in formula (3.2) is missing since the sum runs over just one of the objects.

Without losing generality the point charge may be placed on the z axis. Using the cosine rule, the distance vector may be written

$$R = (r'^2 + z^2 - 2r'z\cos\theta')^{1/2} \tag{4.13}$$

where θ' is the angle between r and r'. The energy becomes

$$U_e = \frac{q\rho}{4\pi\varepsilon_0} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{a} \frac{r'^2 \sin\theta' dr' d\theta' d\phi'}{(r'^2 + z^2 - 2r'z\cos\theta')^{1/2}} \tag{4.14}$$

where a is the radius of the sphere. Integrating over ϕ' and making the coordinate change $t = cos\,\theta'$ gives:

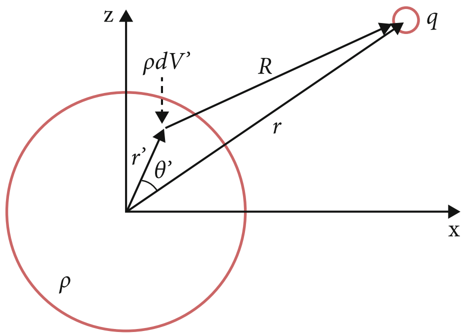


Fig. 4.5 Interaction between a homogeneously charged sphere and a point charge

$$U_e = \frac{q\rho}{2\varepsilon_0} \int_0^a r'^2 dr' \int_{-1}^{1} \frac{dt}{(r'^2 + z^2 - 2r'zt)^{1/2}} \tag{4.15}$$

The integral over t is

$$\begin{aligned} \int_{-1}^{1} \frac{dt}{(r'^2 + z^2 - 2r'zt)^{1/2}} &= \left[-\frac{\sqrt{z^2 + r'^2 - 2zr't}}{zr'} \right]_{-1}^{1} \\ &= \frac{1}{zr'}(|z + r'| - |z - r'|) = \begin{cases} \frac{2}{z} & z > r' \\ \frac{2}{r'} & z < r' \end{cases} \end{aligned} \tag{4.16}$$

For $z > a$,

$$U_e = \frac{q\rho}{z\varepsilon_0} \int_0^a r'^2 dr' = \frac{q\rho a^3}{3z\varepsilon_0} = \frac{qQ}{4\pi\varepsilon_0 z} \tag{4.17}$$

i.e. the same result as between two point charges at a distance z. Thus, outside a homogeneously charged sphere the sphere is equivalent to a point charge with all its charge Q located at the centre.

For $z < a$, the electric energy becomes

$$\begin{aligned} U_e &= \frac{q\rho}{z\varepsilon_0} \int_0^z r'^2 dr' + \frac{q\rho}{\varepsilon_0} \int_z^a r' dr' = \frac{q\rho}{\varepsilon_0} \frac{z^2}{3} + \frac{q\rho}{\varepsilon_0} \frac{a^2}{2} - \frac{q\rho}{\varepsilon_0} \frac{z^2}{2} \\ &= \frac{q\rho}{6\varepsilon_0} \left(3a^2 - z^2 \right) = \frac{qQ}{8\pi\varepsilon_0 a} \left(3 - \frac{z^2}{a^2} \right) \end{aligned} \tag{4.18}$$

At $z = a$ the two results coincide as expected.

The force on a point charge inside the sphere, say at $z = ka$ where $k < 1$ becomes

$$\bar{F} = -\nabla U_e = -\frac{d}{dz} U_e \hat{z} = \frac{q\rho ka}{3\varepsilon_0} \hat{z} = \frac{qQ'ka}{3\varepsilon_0 4\pi/3\,(ka)^3} \hat{z} = \frac{qQ'}{4\pi\varepsilon_0 (ka)^2} \hat{z} \tag{4.19}$$

where Q' is the total charge up to the radius ka. Thus, the force on the point charge is given solely by the charge existing at a radius less than or equal to the distance between the point charge and the centre of the sphere. This phenomenon is known as Gauss' theorem and will be revisited in Exercise (10.3).

4.1.5 Capacitor and Capacitance

The capacitance is a useful quantity when describing electric interactions between macroscopic objects, in particular when the charge distribution is homogeneous. It may be defined through a construction called a capacitor which usually consists of two objects, normally two conductors. These are charged by a voltage source which transports electrons from one object to the other resulting in equal amounts of charge but with opposite signs on the two objects, Fig. 4.8.

The capacitor may be used to accumulate and temporarily store electric energy and is as such used in many different technical applications. The name capacitor refers to having capacity (to store charge). Ideally the imbalance of charge of the capacitor is maintained after disconnecting the voltage source. Practically, however, a constant discharge to the surrounding takes place at a rate depending on the properties of the surrounding environment.

In case the charge distributions on the two objects are homogeneous, it may be concluded using formula (3.3) that the potential from one object and therefore the voltage $\Delta\Phi$ between the two objects is proportional to the stored charge Q, i.e.,

$$\Delta\Phi = \frac{Q}{C} \Rightarrow C = \frac{Q}{\Delta\Phi} \tag{4.20}$$

The capacitance C is therefore a purely geometrical quantity. By measuring the voltage between the two objects, the stored charge on the objects as well as the stored energy are obtained through the capacitance of the system. Before applying the capacitance concept to force and energy calculations the capacitance for two important cases will be derived.

As will be shown in Sect. 5.1, excess charge is collected at the surface of a conductor wherefore such systems are similar to thin shells. This simplifies the calculation of capacitance. The concept of capacitance is however not reserved for conductor systems. In this chapter, the material of the objects will not be specified but will in general be assumed to be thin shells. In Chap. 5 the discussion of capacitance continues specifically for conductors.

4.1.5.1 Capacitance of Parallel Plates

Consider two large parallel plates with area A and distance x, Fig. 4.6. What is the capacitance? To determine the capacitance the relation between voltage and charge is examined. To this end, the work required to transport a point charge between the plates is determined. Assume the plates to have the same charge Q with opposite signs. The force on a point charge q between the plates is twice that of formula (4.11) since two plates now act on the charge.

$$\bar{F}_q = \frac{Qq}{A\varepsilon_0}\hat{x} \tag{4.21}$$

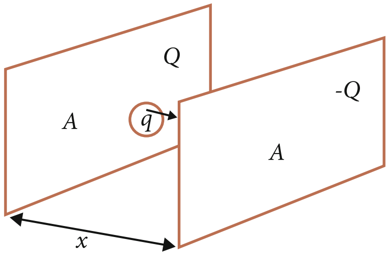


Fig. 4.6 Point charge between two parallel plates

When the charge q is moving in a direction opposite to the electric force from one to the other plate, the work done is [see formulas (3.1) and (3.5)]

$$q\Delta\Phi = \bar{F}\cdot\bar{x} = \frac{Qq}{A\varepsilon_0}x \tag{4.22}$$

so that

$$\Delta\Phi = \frac{Qx}{A\varepsilon_0} \tag{4.23}$$

and the capacitance is

$$C = \frac{A\varepsilon_0}{x} \tag{4.24}$$

4.1.5.2 Capacitance for Two Spherical Shells

Consider two concentric spherical shells. Assume they are charged to Q and $-Q$ respectively, Fig. 4.7. What is the capacitance? Utilizing the method introduced in the previous section, the work to bring a point charge from the inner to the outer shell is determined:

$$W = q\Delta\Phi = \int_a^b \bar{F}\cdot d\bar{s} \tag{4.25}$$

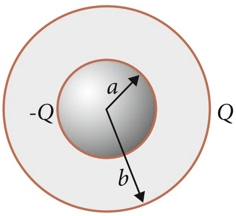


Fig. 4.7 Two concentric spherical shells forming a capacitor

According to Sect. 4.1.4 the force on the point charge is determined solely by the inner shell and is equal to that between two point charges:

$$W = q\Delta\Phi = \int_a^b \bar{F} \cdot d\bar{s} = \int_a^b \frac{qQ}{4\pi\varepsilon_0 r^2} dr = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{4.26}$$

The capacitance becomes

$$C = 4\pi\varepsilon_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \tag{4.27}$$

4.1.6 Electric Energy Stored in a Capacitor

The electric energy stored by a charged capacitor is now determined. The work to generate the energy is provided by a battery.

The battery is connected to the capacitor and its task is to bring electrons from one plate to the other, Fig. 4.8. Assume that at time t the charge on one of the plates is q. The work required to bring an infinitesimal amount of charge dq from one plate to the other is

$$dW_b = dq\Delta\Phi(t) \tag{4.28}$$

where $\Delta\Phi(t)$ is the voltage between the *plates* at time t. This varies with time according to the change of charge on the plates:

$$\Delta\Phi(t) = \frac{q(t)}{C} \tag{4.29}$$

In reality, the charges are transported through the battery but the work that is converted to potential energy is in magnitude the same as if the charges were transported

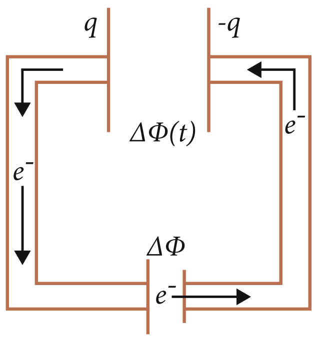


Fig. 4.8 Charging a capacitor using a battery

between the plates. Thus, the battery work will be partly converted into kinetic energy, corresponding to heat in the circuit. The minimum work to charge the capacitor to a charge Q is

$$W_b = \int_0^Q \frac{q}{C} dq$$

where the charge Q is achieved at time T. Let this denote the final state so that $\Delta\Phi(T) = \Delta\Phi$ is the battery voltage. The electric energy stored by the capacitor becomes

$$U_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q\Delta\Phi = \frac{1}{2}C\Delta\Phi^2 \tag{4.30}$$

4.1.7 Electric Force Between Two Charged Plates

As an application of formula (4.30) the electric force between two charged plates is now determined. This will be done in two different ways: firstly without and then with a voltage source connected to the plates. In both cases the energy principle is used, keeping in mind that the complete energy of the system must be considered. In the first case this is the electric energy but in the second case there is also a battery energy, compare the magnetic case in Sect. 3.3.1.

Consider two identical parallel charged plates with opposite charges Q and surface area A, Fig. 4.9 (left). One plate is located at $x = x_1$ and the other at $x = x_2$ where $x_2 > x_1$. The force on the latter becomes

$$\bar{F} = -\nabla U = -\nabla U_e = -\frac{d}{dx_2}\frac{1}{2}\frac{Q^2}{C}\hat{x} = \frac{Q^2}{2}\frac{1}{C^2}\frac{dC}{dx_2}\hat{x} = \frac{1}{2}\Delta\Phi^2\frac{dC}{dx_2}\hat{x} \tag{4.31}$$

For large plates, capacitance formula (4.24) may be used so that

$$\frac{dC}{dx_2} = \frac{d}{dx_2}\frac{A\varepsilon_0}{x_2 - x_1} = -\frac{A\varepsilon_0}{(x_2 - x_1)^2} = -\frac{C^2}{A\varepsilon_0} \tag{4.32}$$

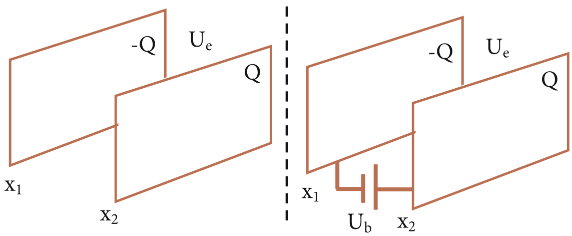


Fig. 4.9 Interaction between two parallel plates without and with a connected battery

and the force is thus

$$\bar{F} = -\frac{1}{2}\frac{\Delta\Phi^2 C^2}{A\varepsilon_0}\hat{x} = -\frac{1}{2}\frac{Q^2}{A\varepsilon_0}\hat{x} \tag{4.33}$$

The force is attractive and constant. The voltage between the plates Q/C decreases during the process but the charge Q remains constant. Measuring the force in this way is challenging (why?).

By connecting the capacitor to a voltage source the measurement is simplified, Fig. 4.9 (right). The total energy in the system is then

$$U = U_e + U_b \tag{4.34}$$

i.e. the electric energy stored by the plates plus the energy of the battery. The battery will now maintain the voltage between the plates. When the plates approach each other the tendency of nature to reduce the voltage between the plates is opposed by the battery by adding charge to the plates. The battery energy decreases by dU_b for each small amount of charge dq that is added to the plates.

$$-dU_b = dW_b = dq\Delta\Phi = dC\Delta\Phi^2 \tag{4.35}$$

The change of the system's total energy then becomes

$$dU = dU_e + dU_b = \frac{1}{2}dC\Delta\Phi^2 - dC\Delta\Phi^2 \tag{4.36}$$

so that force on plate 2 is

$$\bar{F} = -\frac{dU}{dx_2}\hat{x} = \frac{1}{2}\Delta\Phi^2\frac{dC}{dx_2}\hat{x} \tag{4.37}$$

which is the same as in the first case, the only difference being that voltage is maintained. This is a clear advantage when performing electrostatic experiments since discharging effects are then neutralized.

Using the capacitance formula (4.24) results in

$$\bar{F} = -\frac{1}{2}\Delta\Phi^2\frac{\varepsilon_0 A}{(x_2 - x_1)^2}\hat{x} \tag{4.38}$$

which for a voltage of a few kilovolts gives a relatively large and easily measured force. However, the measurement is dangerous since high voltage and large amounts of charge are involved. Be careful!

4.2 Magnetic Dynamics

For magnetic systems, the fundamental interaction is governed by formula (2.11):

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{2.11}$$

If the force on charge 2 originates from the current in a closed conductor the last term in (2.11) vanishes and Grassman's formula results:

$$\bar{f}^G_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1] \tag{2.13}$$

$$= \frac{\mu_0 q_1 q_2}{4\pi R^2}[\bar{v}_2 \times (\bar{v}_1 \times \hat{R})] \tag{2.14}$$

In case the interaction takes place between closed conductors, the second as well as the third term in formula (2.11) vanish:

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} \tag{2.26}$$

4.2.1 Magnetic Force Between a Point Charge and a Long Straight Current

In Sect. 2.2.4 it was shown that the force between a long straight current-carrying conductor and a charge Q with velocity v *parallel* to the current is given by formula (2.6)

$$\bar{F}_{2\to Q} = -\frac{\mu_0 I_2}{2\pi\rho} Q v \hat{\rho} \tag{4.39}$$

where ρ is the shortest distance between the charge and the conductor. Q is negative for an electron. This formula will now be generalised to allow for an arbitrary motion of the charge Q, Fig. 4.10.

Since the straight conductor is considered as being of infinite length and therefore closed (see Exercise 2.10), Grassman's magnetic force law may be used. Expressed in current elements, formula (2.5), the force on element 1 is

$$d\bar{F} = \frac{\mu_0 I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times \hat{R})}{4\pi R^2} \tag{4.40}$$

where the distance vector R is directed from element 2 to element 1.

The current is directed along the z-axis. Using cylindrical coordinates (z, ρ, ϕ) and putting $I_1 d\bar{L}_1 = Q\bar{v}$ the force becomes

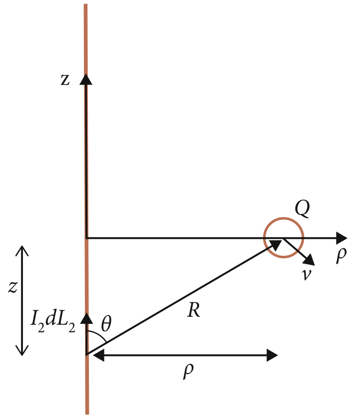


Fig. 4.10 A straight current-carrying conductor interacts with a charge Q in motion

$$d\bar{F} = \frac{\mu_0 Q\bar{v} \times (I_2 d\bar{L}_2 \times \hat{R})}{4\pi R^2} = \frac{\mu_0 Q\bar{v} \times (I_2 dz \sin\theta\hat{\phi})}{4\pi R^2}$$
$$= Q\bar{v} \times \hat{\phi}\frac{\mu_0 I_2}{4\pi}\frac{\rho}{(z^2+\rho^2)^{3/2}}dz \tag{4.41}$$

where $\hat{\phi}$ is a unit vector in cylindrical coordinates, perpendicular to $\hat{z}$ and $\hat{\rho}$.

To get the total force on Q an integration of the complete infinite conductor is performed,

$$\bar{F}_{2\to Q} = Q\bar{v} \times \hat{\phi}\frac{\mu_0 I_2}{4\pi}\int_{-\infty}^{\infty}\frac{\rho}{(z^2+\rho^2)^{3/2}}dz = Q\bar{v} \times \hat{\phi}\frac{\mu_0 I_2}{2\pi\rho} \tag{4.42}$$

where the positive current direction is in the positive z direction. This formula gives formula (2.6) for parallel currents and formula (2.4) for perpendicular currents.

4.2.2 Magnetic Force Between Point Charge and Large Current-Carrying Plate

Several long straight current-carrying conductors are now put together to form a large plate. The force between this array of conductors and a point charge at a distance r from the plate is examined. In Fig. 4.11 the conductors are viewed from above and the z-axis is directed out of the plane. The calculation is performed for $z = 0$.

Formula (4.42) gives the force from one of the infinitesimal straight conductors. Let K denote current per length along y and $\hat{\phi} = (\sin\alpha\hat{x} + \cos\alpha\hat{y})$ so that

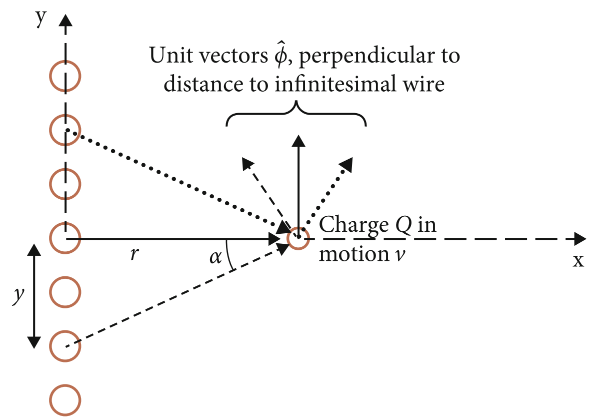


Fig. 4.11 A point charge Q in motion outside a current-carrying plate assumed to be built up by infinitesimally thin wires in the z direction

$$
\begin{aligned}
d\bar{F} &= Q\bar{v} \times \frac{\mu_0}{2\pi}\frac{K\,dy}{\sqrt{y^2+r^2}}(\sin\alpha\hat{x} + \cos\alpha\hat{y}) \\
&= Q\bar{v} \times \frac{\mu_0}{2\pi}\frac{K}{\sqrt{y^2+r^2}}\left(\frac{y}{\sqrt{y^2+r^2}}\hat{x} + \frac{r}{\sqrt{y^2+r^2}}\hat{y}\right)dy
\end{aligned} \tag{4.43}
$$

The total force on Q becomes

$$
\bar{F} = Q\bar{v} \times \hat{y}\frac{\mu_0}{2\pi}Kr\int_{-\infty}^{\infty}\frac{1}{y^2+r^2}dy = Q\bar{v} \times \hat{y}\frac{\mu_0 K}{2}\frac{r}{|r|} \tag{4.44a}
$$

independent of the distance r from the large plate. For positive x values $r/|r| = 1$ and for negative x values $r/|r| = -1$. The x component in (4.43) vanishes after integration since the integrand is an odd function of y. Note that y and therefore also the angle α are negative for the lower half of the plane.

To express formula (4.44a) in a coordinate free form, the surface current density K is used as a vector in the current direction. The surface normal of the plate, $\hat{n}$, is introduced and defined as a unit vector perpendicular to the surface pointing towards the other object. Thus

$$
\bar{F} = Q\bar{v} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.44b}
$$

4.2.3 Magnetic Force Between a Straight Conductor and a Large Plate

To obtain the force between a straight current-carrying conductor and a current-carrying plate, formula (2.5) is used to rewrite (4.44b),

$$d\bar{F} = I d\bar{L} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.45}$$

so that the force on the whole conductor becomes

$$\bar{F} = I\bar{L} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.46}$$

where L is the length of the conductor directed along the current direction. I is the current in the straight conductor and K is the current per unit length in the plate.

4.2.4 Magnetic Force and Energy Between Two Parallel Current-Carrying Plates

Formula (4.46) is now used to determine the force between parallel current-carrying plates. Edge effects are neglected. Let plate 2 be of infinite extent and let the other have side lengths Y and Z. The plates carry constant currents per unit length, K_1 and K_2, in either direction. In Fig. 4.12 the currents have the same sign.

With $\hat{n}$ directed towards plate 1, formula (4.46) gives the force on an infinitesimal wire of plate 1

$$d\bar{F} = K_1 dy Z\hat{z} \times \hat{y}\frac{\mu_0 K_2}{2} \tag{4.47}$$

Since plate 2 is of infinite extent there is no dependence on y so the integration gives

$$\bar{F} = \frac{\mu_0}{2} Y Z K_1 K_2 \hat{z} \times \hat{y} = -\frac{\mu_0}{2} Y Z K_1 K_2 \hat{x} \tag{4.48}$$

The force is accordingly attractive for parallel currents.

The magnetic interaction energy is obtained by calculating the work needed to separate the plates a distance X. For constant current, the force is independent of x. When parallel currents are separated the magnetic energy decreases. Let $U_m(x = x_2)$ denote a reference energy at $x = x_2$. Move plate 1 from $x = x_2$ a distance X so the magnetic energy becomes

$$U_m = U_m(x = x_2) - F_{ext} X = -\frac{\mu_0}{2} X Y Z K_1 K_2 \tag{4.49}$$

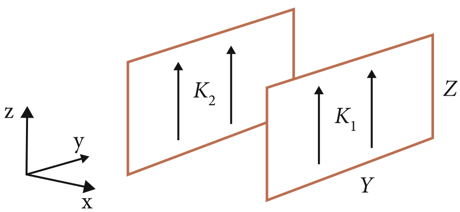


Fig. 4.12 Two parallel current-carrying planes

where in the last step the reference energy is put to zero. For parallel currents K_1 and K_2, the magnetic energy increases with decreasing distance between the plates, in accordance with formula (3.23). For opposite currents, formula (4.49) changes sign.

In Exercise (4.9), the energy is used to determine mutual inductance.

4.2.5 Inductance

The quantity inductance may be used to describe magnetic-inductive interactions between closed circuits. Formula (3.30) defines inductance as

$$M_{jk} = \frac{\mu_0}{4\pi} \int\limits_{Cond\ k} \int\limits_{Cond\ j} \frac{d\bar{L}_j \cdot d\bar{L}_k}{R} \tag{4.50}$$

This formula will now be applied to different closed systems. Having determined inductance, formulas (3.19), (3.23) and (3.33) give energy, force and induction voltage.

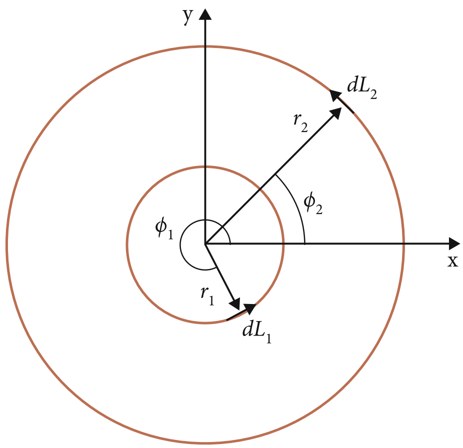
4.2.5.1 Mutual Inductance Between Two Coaxial Loops

Consider two coaxial, circular and homogeneous conductors. What is their mutual inductance? First, let the loops have a common centre with $z_1 = z_2 = 0$, Fig. 4.13.

Denote the radii of the circles $r_1 = a$ and $r_2 = b$. The scalar product between the length elements is

$$d\bar{L}_1 \cdot d\bar{L}_2 = ab \cos{(\phi_2 - \phi_1)} d\phi_1 d\phi_2 \tag{4.51}$$

Fig. 4.13 Two circular coaxial conductors



and the distance between the length elements

$$R = |\bar{r}_2 - \bar{r}_1| = \left(a^2 + b^2 - 2ab\cos(\phi_2 - \phi_1)\right)^{1/2} \tag{4.52}$$

where the cosine rule has been used. This is inserted into Neumann's formula (4.50):

$$M_{12} = \frac{\mu_0 ab}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\phi_2 - \phi_1) d\phi_1 d\phi_2}{\left(a^2 + b^2 - 2ab\cos(\phi_2 - \phi_1)\right)^{1/2}} \tag{4.53}$$

The integral over one of the angles results in a factor 2π so that

$$\begin{aligned} M_{12} &= \frac{\mu_0 ab}{2} \int_0^{2\pi} \frac{\cos\phi d\phi}{\left(a^2 + b^2 - 2ab\cos\phi\right)^{1/2}} \\ &= \mu_0 ab \int_0^{\pi} \frac{\cos\phi d\phi}{\left(a^2 + b^2 - 2ab\cos\phi\right)^{1/2}} \end{aligned} \tag{4.54}$$

If the loops are located a distance d from each other in the z direction, the only change appears in the distance R so that

$$M_{12} = \mu_0 ab \int_0^{\pi} \frac{\cos\phi d\phi}{\left(d^2 + a^2 + b^2 - 2ab\cos\phi\right)^{1/2}} \tag{4.55}$$

This integral may be solved numerically with a computer but may also be analysed using elliptical functions.

To this end, the substitution $\phi = \pi - 2\alpha$ is made so that $\cos\phi = -\cos 2\alpha = -(1 - 2\sin^2\alpha)$ and $d\phi = -2d\alpha$ to obtain:

$$\begin{aligned} M_{12} &= -2\mu_0 ab \int_0^{\pi/2} \frac{(1 - 2\sin^2\alpha)d\alpha}{[(a+b)^2 + d^2 - 4ab\sin^2\alpha]^{1/2}} \\ &= -\frac{2\mu_0 ab}{[(a+b)^2 + d^2]^{1/2}} \int_0^{\pi/2} \frac{(1 - 2\sin^2\alpha)d\alpha}{(1 - k^2\sin^2\alpha)^{1/2}} \end{aligned} \tag{4.56}$$

where $k^2 = \dfrac{4ab}{(a+b)^2 + d^2}$.

Formula (4.56) may be expressed in terms of elliptic functions. The elliptic integral of the first kind F is defined as

$$F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\alpha}{\left[1 - k^2 \sin^2 \alpha\right]^{1/2}} \tag{4.57}$$

and of the second kind

$$E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \left[1 - k^2 \sin^2 \alpha\right]^{1/2} d\alpha \tag{4.58}$$

The first term of the integral in formula (4.56) is exactly F whereas the second term may be expressed in terms of F and E:

$$\begin{aligned} &\int_0^{\pi/2} d\alpha \frac{k^2 \sin^2 \alpha}{(1 - k^2 \sin^2 \alpha)^{1/2}} \\ &= \int_0^{\pi/2} d\alpha \left[\frac{1}{(1 - k^2 \sin^2 \alpha)^{1/2}} - \frac{1 - k^2 \sin^2 \alpha}{(1 - k^2 \sin^2 \alpha)^{1/2}}\right] = F - E \end{aligned} \tag{4.59}$$

Thus, formula (4.56) may be written

$$M_{12} = -\frac{2\mu_0 ab}{[(a+b)^2 + d^2]^{1/2}} \left[\left(1 - \frac{2}{k^2}\right) F + \frac{2}{k^2} E\right] \tag{4.60}$$

a result first obtained by Maxwell (1873). An advantage of this analytical form over numerical calculation of (4.55) is the possibility to make approximations. The elliptic integrals may namely be series expanded in the following way:

$$F(k, \pi/2) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots\right] \tag{4.61}$$

and

$$E(k, \pi/2) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \cdots\right] \tag{4.62}$$

4.2.5.2 Force Between Two Coaxial Circular Loops

Consider two loops at a large distance from each other such that $d \gg a$ and $d \gg b$, so that $k \ll 1$. The lowest order approximation is then obtained if terms up to fourth order in k of the elliptic integrals are kept:

$$\begin{aligned} M_{12} &= -\frac{2\mu_0 ab}{[(a+b)^2+d^2]^{1/2}}\left[\left(1-\frac{2}{k^2}\right)\left(\frac{\pi}{2}+\frac{\pi}{8}k^2+\frac{9\pi}{128}k^4\right)\right. \\ &\qquad\qquad \left. +\frac{2}{k^2}\left(\frac{\pi}{2}-\frac{\pi}{8}k^2-\frac{3\pi}{128}k^4\right)\right] \\ &= -\frac{2\mu_0 ab}{[(a+b)^2+d^2]^{1/2}}\left(\frac{\pi}{8}k^2-\frac{3\pi}{16}k^2\right) = \frac{\mu_0 ab}{[(a+b)^2+d^2]^{1/2}}\frac{\pi k^2}{8} \end{aligned} \tag{4.63}$$

Using $k^2 \approx \dfrac{4ab}{d^2}$ mutual inductance becomes

$$M_{12} = \frac{\mu_0 \pi a^2 b^2}{2d^3} \tag{4.64}$$

This lowest order approximation is also known as the dipole approximation since the dipole is the smallest unit of a magnetic system. This terminology will be further explained in Sect. 7.2.

The force between two loops may now be determined. Assume the currents in the loops to be maintained by a battery. The magnetic force is then given by formula (3.23). The loops are located at z_1 and z_2, where $z_2 \gg z_1$ so that (4.64) may be used:

$$M_{12} = \frac{\mu_0 \pi a^2 b^2}{2(z_2 - z_1)^3} \tag{4.65}$$

The force on loop 2 becomes

$$\bar{F}_2 = I_1 I_2 \frac{d}{dz_2} M_{12}\hat{z} = -I_1 I_2 \frac{3\mu_0 \pi a^2 b^2}{2(z_2-z_1)^4}\hat{z} \tag{4.66}$$

which is an attractive (repulsive) force for parallel (anti-parallel) currents, as expected.

4.2.5.3 Self Inductance for an Idealized Coil

An idealized coil can be thought of as consisting of parallel circular loops whose total mutual and individual inductances become the self-inductance of the coil. The loops are tight with an infinitesimal thickness such that the coil becomes a cylindrical homogeneous surface. The total inductance of the coil is obtained through an infinitesimal summation, i.e. an integration, Fig. 4.14.

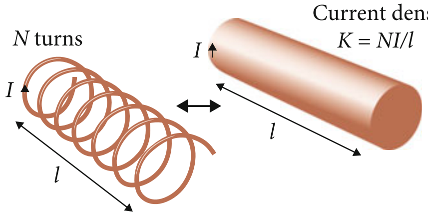


Fig. 4.14 An array of several close concentric identical *circular loops* is approximated with a homogeneous cylindrical surface

In the left-hand figure the total inductance becomes a summation

$$M_{tot} = \sum_{j,k} M_{jk} \tag{4.67}$$

where j may also be equal to k. In the right-hand figure the summation is replaced by an integration

$$M_{tot} = \int\int M'_{jk} dz_j dz_k \tag{4.68}$$

where z is a coordinate along the axis of the coil. M' is inductance density defined in the following way:

Assume the coil has N turns, length l and current I through one loop. Consider the interaction energy between two of the loops

$$U_m = M_{jk} I_j I_k = M_{jk} \frac{NI}{l} \Delta z_j \frac{NI}{l} \Delta z_k \tag{4.69}$$

so that formula (4.68) should be written

$$M_{tot} = \frac{N^2}{l^2} \int_0^l \int_0^l M_{jk} dz_j dz_k \tag{4.70}$$

where M_{jk} is given by formula (4.55), i.e. self inductance for one loop or mutual inductance between two loops.

Based on formula (4.56), with $a = b$ and the distance between two loops $(z_j - z_k)$, the self inductance for an idealized coil becomes

$$M_{tot} = -\frac{N^2}{l^2} \int_0^l \int_0^l \frac{2\mu_0 a^2}{[4a^2 + (z_j - z_k)^2]^{1/2}} \int_0^{\pi/2} \frac{(1 - 2\sin^2\alpha) d\alpha}{(1 - k^2 \sin^2\alpha)^{1/2}} dz_j dz_k \tag{4.71}$$

$$k^2 = \frac{4a^2}{4a^2 + (z_j - z_k)^2} \tag{4.72}$$

For the integration over z, formula (4.71) is rewritten:

$$M_{tot} = -2N^2\mu_0 a^2/l^2 \int_0^{\pi/2} (1 - 2\sin^2\alpha) \times \int_0^l \int_0^l \left[4a^2 + (z_j - z_k)^2 - 4a^2\sin^2\alpha\right]^{-1/2} dz_j dz_k d\alpha \tag{4.73}$$

Make the variable substitution $t = z_j - z_k$,

$$\begin{aligned}
&\int_0^l \int_0^l \left[\frac{1}{4a^2 + (z_j - z_k)^2 - 4a^2\sin^2\alpha}\right]^{1/2} dz_j dz_k \\
&= \int_0^l \int_{-z_k}^{l-z_k} \left[\frac{1}{4a^2\cos^2\alpha + t^2}\right]^{1/2} dt dz_k \\
&= \int_0^l \left[\sinh^{-1}\frac{t}{2a\cos\alpha}\right]_{-z_k}^{l-z_k} dz_k = \int_0^l \left(\sinh^{-1}\frac{l - z_k}{2a\cos\alpha} + \sinh^{-1}\frac{z_k}{2a\cos\alpha}\right) dz_k \\
&= 2\left[z_k \sinh^{-1}\frac{z_k}{2a\cos\alpha} - \left(z_k^2 + 4a^2\cos^2\alpha\right)^{1/2}\right]_0^l \\
&= 2\left(l\sinh^{-1}\frac{l}{2a\cos\alpha} - \left(l^2 + 4a^2\cos^2\alpha\right)^{1/2} + 2a\cos\alpha\right)
\end{aligned} \tag{4.74}$$

which is inserted into (4.73),

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l} \times \int_0^{\pi/2} \left[(1 - 2\sin^2\alpha)\left(\sinh^{-1}\frac{l}{2a\cos\alpha} - \left(1 + \frac{4a^2}{l^2}\cos^2\alpha\right)^{1/2} + \frac{2a}{l}\cos\alpha\right)\right] d\alpha \tag{4.75}$$

Consider now the case when $l \gg a$, a so-called solenoid. Keeping terms up to order a/l the inductance becomes

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l} \int_0^{\pi/2} \left[(1 - 2\sin^2\alpha)\left(\ln\frac{l}{a\cos\alpha} - 1 + \frac{2a}{l}\cos\alpha\right)\right] d\alpha$$

$$= -\frac{N^2 4\mu_0 a^2}{l}\left[\left(1 - \ln\frac{l}{a}\right)\int_0^{\pi/2}(2\sin^2\alpha - 1)d\alpha + \int_0^{\pi/2}(2\sin^2\alpha - 1)\ln(\cos\alpha)d\alpha - \frac{2a}{l}\int_0^{\pi/2}(2\sin^2\alpha - 1)\cos\alpha d\alpha\right] \tag{4.76}$$

There are then three integrals to evaluate. The first vanishes:

$$\int_0^{\pi/2}(2\sin^2\alpha - 1)d\alpha = 0 \tag{4.77}$$

The second integral may be evaluated through integration by parts:

$$\int_0^{\pi/2}(2\sin^2\alpha - 1)\ln(\cos\alpha)d\alpha = -\int_0^{\pi/2}\cos 2\alpha\ln(\cos\alpha)d\alpha = \underbrace{\left[-\frac{\sin 2\alpha}{2}\ln(\cos\alpha)\right]_0^{\pi/2}}_{\sin\pi\ln 0=0} - \int_0^{\pi/2}\frac{\sin 2\alpha\sin\alpha}{2\cos\alpha}d\alpha = -\frac{\pi}{4} \tag{4.78}$$

(note the interesting limit value in (4.78)) and the third integral is

$$\int_0^{\pi/2}(2\sin^2\alpha - 1)\cos\alpha d\alpha = -\frac{1}{3} \tag{4.79}$$

The self-inductance for a solenoid becomes

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l}\left(-\frac{\pi}{4} + \frac{2a}{3l} + O\left(\frac{a}{l}\right)^2 + \cdots\right) = \frac{\mu_0 N^2 \pi a^2}{l}\left(1 - \frac{8a}{3\pi l} + \cdots\right) \tag{4.80}$$

Note that the first order correction is rarely negligible and therefore further correction terms need also be considered, see Exercise (4.18).

4.2.5.4 Self-inductance for a Circular Loop

For a thin circular loop the condition is $a \gg l$ and formula (4.72) gives $k^2 \approx 1$. Formula (4.71) may then be written

$$M_{tot} = -\frac{\mu_0 N^2 a}{l^2} \int_0^l \int_0^l k \left[\left(1 - \frac{2}{k^2} \right) F + \frac{2}{k^2} E \right] dz_j dz_k \tag{4.81}$$

According to a prominent handbook[1]

$$\lim_{k^2 \to 1} F\left(k, \frac{\pi}{2}\right) = \frac{1}{2} \ln \frac{16}{1-k^2} \tag{4.82}$$

and

$$\lim_{k^2 \to 1} E\left(k, \frac{\pi}{2}\right) = 1 \tag{4.83}$$

so that

$$\begin{aligned} M_{tot} &= -\frac{\mu_0 N^2 a}{l^2} \int_0^l \int_0^l (2 - F) dz_j dz_k \\ &= -\frac{\mu_0 N^2 a}{l^2} \int_0^l \int_0^l \left(2 - \frac{1}{2} \ln \frac{16}{1-k^2} \right) dz_j dz_k \\ &= -\frac{\mu_0 N^2 a}{l^2} \int_0^l \int_0^l \left(2 - \frac{1}{2} \ln \frac{16(4a^2 + (z_j - z_k)^2)}{(z_j - z_k)^2} \right) dz_j dz_k \\ &\approx -\frac{\mu_0 N^2 a}{l^2} \int_0^l \int_0^l \left(2 - \ln \frac{8a}{z_j - z_k} \right) dz_j dz_k \end{aligned} \tag{4.84}$$

Evaluating the integral gives (see Exercise 4.15):

$$M_{tot} = \frac{\mu_0 N^2 a}{l^2} \left[-\frac{1}{2} l^2 + l^2 \ln \frac{8a}{l} \right] = \mu_0 N^2 a \left[\ln \frac{8a}{l} - \frac{1}{2} \right] \tag{4.85}$$

[1] See e.g. Abramowitz' and Stegun's book available on the www.

Note, however, that the elliptic integral F varies quickly with k so formula (4.85) should be used with caution.

4.2.6 Induction in a Moving Rod Interacting with Current-Carrying Plate

The concept of inductance is useful when only closed currents are involved in the interaction. The concept of a closed conductor implies that the drift motion of the conduction electrons is along the conductor (compare Exercise 3.9c). The following example involves a non-closed conductor and illustrates a case where inductance should not be used.

Let a thin bar-formed conductor, oriented along the x axis, move at a constant speed v in the negative z direction. It interacts with an infinite plate with homogeneous and constant current in the positive z direction, Fig. 4.15. The induced voltage between the end points of the bar is now to be found.

Since the bar is a non-closed conductor the complete magnetic force formula must be considered:

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{2.11}$$

If the infinite plate, equivalent to a closed conductor, is object 1 the third term in (2.11) vanishes.

Since the conduction electrons of the bar are moving in parallel to the conduction electrons of the plate, these will be attracted in accordance with formula (2.11). A current I_2 will then be generated in the bar until enough charge has accumulated at its ends so that the electric force balances the horizontal magnetic force.

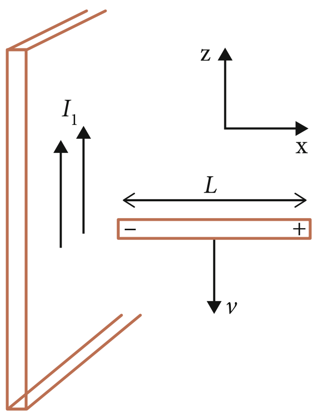


Fig. 4.15 A straight conductor in motion interacts with a large current-carrying plate

The current in the bar, I_2, is directed in the positive x direction. To initially accelerate the bar an external force is needed. This force acts against the vertical magnetic force that I_2 causes according to formula (4.46). To maintain a constant speed, the external force must balance this vertical magnetic force. The external force is then doing work and the change in magnetic energy during a displacement dz becomes, using formulas (3.23) and (4.46),

$$dU_m = -\frac{\mu_0 K_1}{2} I_2 L dz \tag{4.86}$$

where L is the length of the bar and K_1 is current per unit length along y. This magnetic energy is successively converted to electric energy through the accumulation of electric charge at the ends of the bar which according to formula (3.31) creates a voltage

$$\varepsilon_2 = \frac{dU_m}{dq} = -\frac{1}{dq}\frac{\mu_0 K_1}{2}\frac{dq}{dt} L dz = -\frac{\mu_0 K_1 L}{2}\frac{dz}{dt} = \frac{\mu_0 K_1}{2} L v \tag{4.87}$$

where v is the final speed of the bar conductor.

See Exercise (4.17) for an application.

4.3 Summary

The electric force between a point charge q and a long wire with charge density λ' is

$$\bar{F} = \frac{\lambda' q}{2\pi\varepsilon_0 d}\hat{x} \tag{4.6}$$

The electric force between two charged long parallel wires is

$$\bar{F} = \frac{\lambda'\lambda L}{2\pi\varepsilon_0 d}\hat{x} \tag{4.7}$$

The electric force between a large plate with charge surface density σ and a point charge q is

$$\bar{F} = \frac{\sigma q}{2\varepsilon_0}\hat{x} \tag{4.11}$$

A system consisting of two objects with the same charge Q with opposite signs has a capacitance

$$C = \frac{Q}{\Delta\Phi} \tag{4.20}$$

and stores the electric energy

$$U_e = \frac{1}{2}\Delta\Phi Q = \frac{1}{2}C(\Delta\Phi)^2 \tag{4.30}$$

The magnetic force between a point charge Q and a long straight current-carrying wire is

$$\bar{F}_{2\to Q} = Q\bar{v} \times \hat{\phi}\frac{\mu_0 I_2}{2\pi\rho} \tag{4.42}$$

The magnetic force between a point charge Q and a large plate with surface normal $\hat{n}$ and linear current density K is

$$\bar{F} = Q\bar{v} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.44b}$$

The magnetic force between a straight current and a large current-carrying plate is

$$\bar{F} = I\bar{L} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.46}$$

The mutual inductance between two coaxial circular loops with radii a and b at a distance d is

$$M_{12} = -\frac{2\mu_0 ab}{[(a+b)^2 + d^2]^{1/2}}\left[\left(1 - \frac{2}{k^2}\right)F\left(k, \frac{\pi}{2}\right) + \frac{2}{k^2}E\left(k, \frac{\pi}{2}\right)\right] \tag{4.60}$$

where F and E are elliptical functions.

The exact formula for self inductance of a coil with circular cross section, length l and cross section radius a is

$$\begin{aligned} M_{tot} &= -\frac{N^2 4\mu_0 a^2}{l} \\ &\times \int_0^{\pi/2}\left[(1 - 2\sin^2\alpha)\left(\sinh^{-1}\frac{l}{2a\cos\alpha} - \left(1 + \frac{4a^2}{l^2}\cos^2\alpha\right)^{1/2}\right.\right. \\ &\left.\left. + \frac{2a}{l}\cos\alpha\right)\right]d\alpha \end{aligned} \tag{4.75}$$

The approximate formula for self inductance of a coil with a circular cross section is (third term is obtained from Exercise 4.18)

$$M_{tot} = \frac{\mu_0 N^2 \pi a^2}{l}\left(1 - \frac{8a}{3\pi l} + \frac{7}{8}\frac{a^2}{l^2} + O\left(\frac{a}{l}\right)^3 \cdots\right)$$

4.4 Exercises

4.1 Derive formula (4.8), evaluate the integral and show that it reduces to formula (4.7) if $L \rightarrow \infty$.

4.2 Formula (4.11) was obtained by summing long wires of a plate. Instead, determine the force on a point charge at a distance d from the plate by summing interactions with infinitesimal surface elements of the plate. Assume that the surface charge density σ is constant. The charge of an element of the plate is $q' = \sigma dy' dz'$.

C* 4.3 Show that the capacitance for a homogeneously charged system consisting of two long coaxial cylindrical shells with inner radius a and outer radius b is

$$C = \frac{2\pi \varepsilon_0 L}{\ln \frac{b}{a}} \tag{4.88}$$

where L is their common length.

4.4 Determine capacitance of a homogeneously charged system consisting of two long parallel thin cylindrical shells with radius a, length L and distance d from axis to axis.

4.5 How would you proceed to measure the force (4.38)?

C 4.6 Based on the magnetic force formula (2.25) with $a = 1$ in Exercise (2.10), determine the force between two straight and parallel current-carrying conductors separated a distance d in the following cases:

a. They are of equal length T.
b. They have different lengths T_1 and T_2.
c. In Exercise b, a force directed along the length of the conductor is obtained, a so-called longitudinal force. How can it be measured? Does it remain for closed conductors?

4.7 A large current-carrying plate with a constant linear current density $\bar{K} = K\hat{z}$ is located in the yz plane. A straight conductor with length T and current I is placed parallel to the plate at a distance d forming an angle 45° with the y axis.

Determine the force on the straight conductor.

4.8 Two parallel infinite plates carry opposite linear current density $\bar{K} = \pm K\hat{z}$. Determine the force on a point charge q with constant velocity $\bar{v}$ for any position.

4.9 Determine the mutual inductance between two large plates with homogeneous current distributions

a. As in Fig. 4.12.
b. In opposite directions.

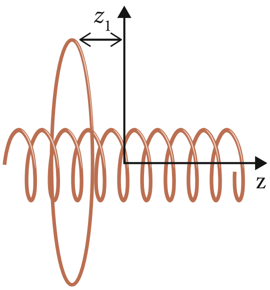
Hint: utilize the energy formula (4.49).

4.10 Consider two long parallel conductors, each one part of a large square circuit. Both have length T and the distance between them is d with $T \gg d$.

a. Determine the mutual inductance between the two straight conductors.
b. Determine the force on one of the conductors and compare with Exercise (4.6).
c. Determine the induced voltage in one of them if the current varies as $I = I_0 \sin \omega t$ in the other.
d. Explain why the inductance method for force calculation works in this case although the straight conductors don't form a closed circuit.

4.11 Using the result in Exercise (4.10a), determine the self inductance for a plate with length T much longer than its width X.

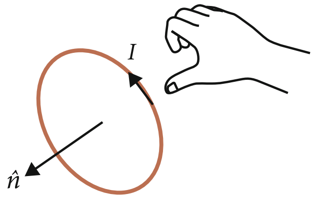
* 4.12 A thin circular loop with radius a encloses a coil with length l, number of turns N and circular cross section with radius b. The loop and the coil are oriented along a common axis. The loop is located at a distance z_1 from the center of the coil, see figure.



a. Determine the mutual inductance.
b. Determine the force on the loop as a function of the distance z_1. The loop and the coil carry currents I_1 and I_2 respectively.
c. Determine induced voltage in the loop if the current in the coil varies as $I = I_0 \sin \omega t$.

C* 4.13 Let the loop in Exercise (4.12) have a radius a much smaller than the radius b of the coil.

a. Determine the mutual inductance.



Introduce the concept dipole moment for a loop, $\bar{m} = IA\hat{n}$, where I is the current in the loop and A its area with surface normal $\hat{n}$. The surface normal is directed according to 'the right hand rule', i.e. when fingers are oriented along the current direction, the thumb points in the direction of the surface normal.

b. Express the interaction energy in terms of the dipole moment.

c. Now assume that the surface normal of the loop forms an angle θ with the coil axis z. Show that the interaction energy becomes

$$U_m = \mu_0 \frac{N}{l} I_2 \hat{z} \cdot \bar{m} \tag{4.89}$$

where $\hat{z}$ is the surface normal of the coil cross section.

d. Determine the force and the torque on the loop and specify the angles for stable and unstable equilibrium of the system.

4.14 Consider two cylindrical coils connected in series. They are of different lengths but have the same density of turns and cross sectional radius.

a. Show that their total inductance is equal to the sum of the individual self-inductances in the lowest order of formula (4.80) but *not* for higher orders.
b. What may be concluded about the interaction between the coils in the lowest and in the higher order?

4.15 a. Show that (4.85) is a consequence of (4.84).
b. Draw a graph of the elliptical function $F(k, \pi/2)$ and evaluate the validity of the limits in (4.82).

*4.16 a. Determine the force between two identical parallel and coaxial current-carrying circular loops when their radius is much greater than the distance between them.

b. Extend the loops to short coils with N number of turns. Calculate the force and compare it with that for straight parallel conductors.
c. How would you proceed to measure the force between the coils? See figure above.

4.17 Estimate the maximum electric voltage between the wings of a typical flying aircraft.
Model geomagnetism as a large plate with linear current density 80 A/m.
Exercise (8.14) discusses a technique that may be used to explore geomagnetism.

4.18 Determine the third term in the expansion of the coil self-inductance (4.80).

Further Readings

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Chapter 5
Conductors and Resistive Effects

Electricity is often called wonderful, beautiful; but it is so only in common with the other forces of nature. The beauty of electricity or of any other force is not that the power is mysterious, and unexpected, touching every sense at unawares in turn, but that it is under law, and that the taught intellect can even govern it largely. The human mind is placed above, and not beneath it, and it is in such a point of view that the mental education afforded by science is rendered super-eminent in dignity, in practical application and utility; for by enabling the mind to apply the natural power through law, it conveys the gifts of God to man.

Michael Faraday, 1858

The electric conductor is one corner stone of modern technology. Its unique properties makes it also suitable for the study of the fundamental dynamics of nature. It has thereby been found how the constituents of nature, in this case electrons, interact under different conditions: at rest, in uniform motion and in acceleration.

With the discovery and completion of the periodic system, new knowledge arose about metals. Its unique properties as a current and thermal conductor originate from the fact that each atom in the metal contributes with one or two electrons from their outermost orbitals to the so-called conduction band where electrons can move relatively freely. However, the motion is not entirely free due to three reasons:

Firstly, the conduction electrons interact electrically with the rest of the material which consists of the atoms from which they originate. The atomic vibrations constitute a barrier, a so-called resistance, to their motion. This generates heat which can be of both advantage and disadvantage in engineering applications.

Secondly, there are inductive forces, i.e. internal interactions between the conduction electrons themselves, which macroscopically are described by the quantity inductance. This is, as previously indicated, the electrodynamic counterpart to inertia in mechanics.

Thirdly, there are capacitive effects which arise due to physical discontinuities of the conductor. This stores electric energy as in a capacitor, see Chap. 4.

In this chapter, the dynamics that arise in the conductor due to resistive effects are investigated. In the next chapter, the examination is extended to include the inductive and capacitive properties of the electric circuit.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
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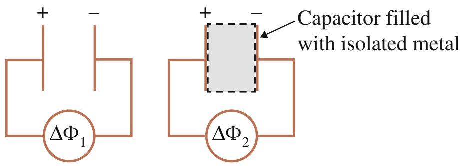


Fig. 5.1 The voltage over a capacitor is measured without material (*left*) and with material (*right*) between its plates

5.1 The Metal as a Conductor

To investigate the conductor properties of a metal, the following experiment can be carried out:

Charge a capacitor consisting of two parallel plates. Disconnect the battery and measure the voltage $\Delta\Phi$ according to Fig. 5.1 (left).[1] The voltmeter basically measures the force between the charges accumulated on the different plates. Then introduce a metallic material between the plates filling the entire volume as in Fig. 5.1 (right). The contact between the metal and the capacitor must be insulated, alternatively leave a small air gap. The voltmeter shows that the voltage between the plates has almost disappeared.

It can be concluded that in the inserted metallic material, a sufficient amount of charge has been transported to precisely neutralize the charge on the capacitor plates. This result occurs for both very small and very large initial voltages between the plates. The metal can thus easily transport electric charge; it is a good conductor, which is its distinguished property.

The final state is static since the voltmeter measures zero voltage. If a charge is placed inside the metallic material, it would not be affected by any force despite the fact that external forces affect the metallic material.

Thus, it can be concluded that under static conditions, there is no net charge inside a metal. The negative charge is neutralized everywhere by positive charge. Excess charge only appears on the metal surface.

5.2 Relaxation Time

The next step in these experiments is to allow the voltage over the capacitor plates to vary with time. This is achieved by constructing a circuit with a battery that generates an alternating voltage, Fig. 5.2.

With an oscilloscope, it is recorded that the voltage across the resistor is the same as the battery voltage, even for high frequencies. The conclusion can be drawn that the metal neutralizes the voltage across the plates instantly. The process in the metal to bring charges to the surface of the capacitor plates in order to neutralize its charge

[1] To prevent discharges through the meter, a so-called electrostatic voltmeter has to be used. This has extremely high inner resistance.

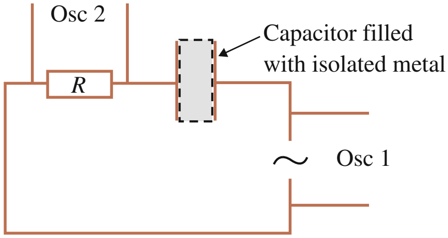


Fig. 5.2 Circuit with a resistor and a capacitor in series. The voltage over the resistor and the voltage source is registered by an oscilloscope

is thus extremely fast, faster than what can be measured. This property explains why one is protected from lightning inside a metal shell such as a car. The charges of the flash are neutralized immediately on the outer surface.

The time-lapse in a conductor may be estimated by establishing a model of its dynamics. When the metal is placed between the charged capacitor plates, an electric force will act on the charges of the metal causing the conduction electrons to move towards one of the plates, leaving behind a deficit of negative charge on the other side. During the motion, they are exposed to a frictional force originating in their interaction with the rest of the conductor material.The frictional force is proportional to the velocity at low speeds as in this case (see Exercises 2.2 and 5.6). Using formulas (1.1), (1.2) and (3.5), the equation of motion for the conduction electrons becomes (neglecting effects of inductance and capacitance):

$$m\frac{d\bar{v}}{dt} = -q\nabla\Phi - G\bar{v} \tag{5.1}$$

where G is a frictional coefficient depending on the nature of the material, m the mass of the charge q, v its velocity caused by the external influence (drift speed) and Φ the electric potential experienced by the charge q.

The relaxation time is the time it takes for the charges to be restored after the external influence has ceased. To find this time, put $\nabla\Phi = 0$, so that

$$m\frac{d\bar{v}}{dt} = -G\bar{v} \tag{5.2}$$

which has the solution

$$v(t) = Ae^{-\frac{G}{m}t} \tag{5.3}$$

where A is the velocity at $t = 0$, i.e. when the voltage source is disconnected. The speed of the charges in the conductor thus decreases exponentially with a time constant

$$t_r = \frac{m}{G} \tag{5.4}$$

called relaxation time. The frictional coefficient G is related to the concept of resistance which will be discussed next.

5.3 Resistance

A conductor is connected to a battery, Fig. 5.3. The battery voltage $\Delta\Phi$ and current I through the conductor are measured. A linear relationship between the two is observed for ordinary values of the voltage. The proportionality is called resistance R and Ohm's law is thus obtained:

$$\Delta\Phi = RI \tag{5.5}$$

Ohm's law is valid after several relaxation times because the measurement is performed a relatively long time after connecting the source. Since relaxation time is also a measure of the time it takes for the electrons to reach terminal speed, a constant drift speed is expected.

Solving Eq. (5.1) for the circuit in Fig. 5.3 gives

$$\bar{v}(t) = -\frac{q}{G}\nabla\Phi\left(1 - e^{-\frac{G}{m}t}\right) \tag{5.6}$$

so that when $t \to \infty$ then $\bar{v} \to -q/G\ \nabla\Phi$, i.e. a constant.

Ohm's law may be used to gain information about G. For simplicity, imagine a homogeneous and straight conductor with length l and cross sectional area S. The voltage $\Delta\Phi = -l\nabla\Phi$ and the current $I = \bar{J} \cdot \bar{S}$ where J is current per unit surface. It follows from formula (2.5) that $\bar{J} = nq\bar{v}$ where n is the number of conduction electrons per unit volume and v is the drift speed of the conduction electrons (see Exercise 2.24). J is oppositely directed to v since $q < 0$. Ohm's law may then be written

$$l\nabla\Phi = -Rnq\bar{v}S \tag{5.7}$$

so that

$$\bar{v} = -\frac{l}{RnqS}\nabla\Phi \tag{5.8}$$

The frictional coefficient becomes

$$G = \frac{q^2 RnS}{l} \tag{5.9}$$

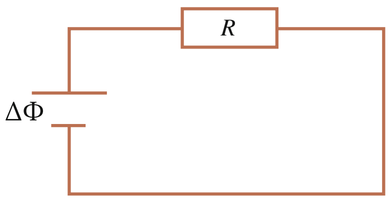


Fig. 5.3 A conductor with resistance R is connected to a battery

which shows that the origin of resistance is friction. The quantity $l/RS = \sigma$ is known as conductivity, which is independent of the conductor geometry. Accordingly, it is measurable through Ohm's law and available in tables. Relaxation time becomes

$$t_r = \frac{m}{G} = \frac{m}{q^2 n}\sigma \tag{5.10}$$

The charge q and the mass m of the electron may be measured as in Exercises (5.12) and (5.13) and their density n determined using Avogadro's number (see Chap. 2). The order of magnitude estimate of the relaxation time becomes

$$t_r \sim 10^{-14}\text{s} \tag{5.11}$$

The drift speed is given by

$$v = \frac{I}{Snq} \tag{5.12}$$

Typical values of cross sectional area and current might be $S \approx 10^{-6}\,\text{m}^2$ and $I \approx 1\,\text{A}$ which gives $v \sim 0.1\,\text{mm/s}$. The electrons also exhibit a random non-directional motion which is many orders of magnitude larger.

5.4 Heat Power

A normal kettle utilizes the heat generated by the electric current of a conductor. This heat power is relatively easy to measure through the temperature change in a water bath in which the conductor is submerged. Through the heat capacity of water the heat power is obtained.

In this way, the model for electric conduction described above may be verified because it predicts the amount of generated heat, whose source is friction. If friction did not exist, the electrons would be continuously accelerated through the conductor and achieve very high speeds. Instead, this imagined kinetic energy is converted into heat which is calculated in the following manner.

The heat energy is the work done by the frictional force, which for *one* electron is

$$W_{fr} = Gvl \tag{5.13}$$

where the homogeneous conductor is utilized once again, with length l, parallel to the drift speed v. Power P is this energy divided by the time needed for the electron to pass through the conductor l/v

$$P_{fr} = Gv^2 = \frac{q^2 RnS}{l}v^2 = \frac{q^2 n v^2}{\sigma} \tag{5.14}$$

To obtain the total heat power per unit volume V this is multiplied by n, the number of electrons per unit volume:

$$\frac{P_{fr}}{V} = \frac{q^2 n^2 v^2}{\sigma} = \frac{J^2}{\sigma} \tag{5.15}$$

a result first obtained by Joule in 1841.

5.5 The Principle of Charge Conservation

In the analysis above, the fundamental law of electric charge conservation has been used implicitly. The strongest evidence for this principle originates from the measurement of electric currents in circuits. The well-verified Kirchoff's law states that in a branch point of a circuit, the sum of the input currents equals the output. With the modern understanding of electric current, the conservation of electric charge follows directly.

This can be expressed mathematically by a continuity equation in the following way. The charge Q in a volume V may be changed only by transportation of charge through the enclosing surface. Thus, if the total current through that surface is denoted by I:

$$I + \frac{dQ}{dt} = 0 \tag{5.16}$$

This equation is valid over an extended volume, but may be localized by introducing current density

$$\bar{J} = \frac{dI}{da}\hat{n} \tag{5.17}$$

where da is an infinitesimal surface and $\hat{n}$ its surface normal. The continuity equation becomes

$$\oint_S \bar{J} \cdot d\bar{a} + \frac{dQ}{dt} = 0 \tag{5.18}$$

where S is the surface enclosing the volume V. Using the divergence theorem and the notation ρ for charge density, (5.18) may be written

$$\int_V \nabla \cdot \bar{J} dV + \int_V \frac{d\rho}{dt} dV = 0 \tag{5.19}$$

which must hold for any volume so that

$$\nabla \cdot \bar{J}(x, y, z) + \frac{d\rho(x, y, z)}{dt} = 0 \tag{5.20}$$

This is the continuity equation for electric charge in its general form, valid for each point in space. It will be revisited in Chap. 10 when Maxwell's equations are derived.

5.6 Summary

Metals are conductors whose electric current consists of conduction electrons which are very loosely bound in the material.

Under static conditions, all positions inside a conductor are electrically neutral. Excess charge only exists on the surface.

The equation of motion for the conduction electrons in a metal is given by

$$m\frac{d\bar{v}}{dt} = -q\nabla\Phi - G\bar{v} \tag{5.1}$$

Ohm's law is a definition of resistance

$$\Delta\Phi = RI \tag{5.5}$$

The drift speed in a conductor is given by

$$\bar{v} = -\frac{l}{RnqS}\nabla\Phi \tag{5.8}$$

The relaxation time for a conductor is

$$t_r = \frac{m}{G} = \frac{m}{q^2 n}\sigma \tag{5.10}$$

The heat power per unit volume in a conductor is given by Joule's law

$$\frac{P_{fr}}{V} = \frac{q^2 n^2 v^2}{\sigma} = \frac{J^2}{\sigma} \tag{5.15}$$

The continuity equation for electric charge is

$$\nabla \cdot \bar{J}(x, y, z) + \frac{d\rho(x, y, z)}{dt} = 0 \tag{5.20}$$

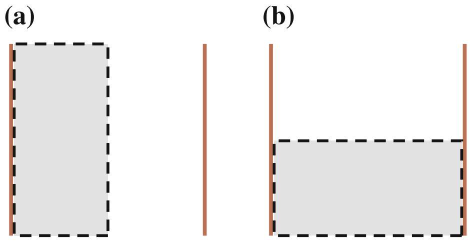
5.7 Exercises

5.1 Determine the capacitance per unit length for a system consisting of two homogeneous long parallel cylindrical conductors of radii a and b respectively and placed at a distance d between their centers.

5.2 Determine the capacitance per unit length for a system consisting of two long coaxial homogeneous cylindrical conductors. The inner conductor has radius a and the outer has inner radius b and outer radius c.

5.3 Determine the capacitance for a system consisting of two homogeneous metallic balls with radii a and b respectively and with a distance d between their centers.

5.4



A capacitor is charged to a voltage $\Delta\Phi$ between its parallel plates, after which the battery is disconnected.
A metal slab with isolated surfaces is introduced between the capacitor plates filling half its volume with two possible orientations as in the figures. How is the voltage affected in each case?

*5.5 *Series coupled capacitors*

A metal plate with thickness $t = x_2 - x_1$ and area A is placed between and parallel to the plates of a large capacitor. The capacitor plates have area A and are separated a distance d.

a. Determine the capacitance.
b. If the capacitor plates are located at $x = 0$ and $x = d$, show that the resulting capacitance may be written

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{5.21}$$

where

$$C_1 = \frac{\varepsilon_0 A}{x_1}, \quad C_2 = \frac{\varepsilon_0 A}{(d - x_2)}$$

c. Show that formula (5.21) is valid for a series coupling of C_1 and C_2.

*5.6 *General motional fluid resistance*

The resistive effects in a conductor are similar to the ones appearing in motion through a gas, e.g. in a fall through air. Generally motional resistance of this kind is modelled by terms which are linear and quadratic in the speed so that formula (5.2) may be generalized to

$$m\frac{dv}{dt} = -Gv - Bv^2 \tag{5.22}$$

where the first term relates to so-called laminar flow and the second to turbulent flow.
For a sphere with radius r, $G = C_1 r$ and $B = C_2 r^2$ where $C_1 = 3.1 \times 10^{-4}\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}$ and $C_2 = 0.87\,\text{kg/m}^3$ for motion through air.

a. Explain why the term quadratic in speed could be neglected for the electric conductor.
b. Determine the terminal speed for a sphere falling through air with mass $m = 1\,\text{kg}$ and $m = 100\,\text{kg}$.
c. Compare with Galilei's famous experiment at the Leaning Tower of Pisa.
d. In October 2012, the Austrian Baumgartner fell freely from an altitude of 39 000 m. It was then claimed that he broke the sound barrier. Explain how such a high final speed could be achieved in this case compared to task b.

5.7 A homogeneous straight conductor with cross sectional area S and length l has a conductivity σ.

a. Express the heat generation per unit time in terms of resistance R and current I.
b. Express Ohm's law (5.5) with the conductivity σ and the current density J in vector form.

5.8 *The mirror miracle*

a. Why does a metallic object become colder than other objects on a cold winter day?
b. Why do you burn yourself if a metal rod is inserted into the fire but not if the rod is made of wood?
c. Why are metals shiny? How does a mirror work?
See also Exercise (11.4)

*5.9 Show that the continuity equation (5.20) implies

$$\frac{d}{dt}(q\bar{v}) = 0$$

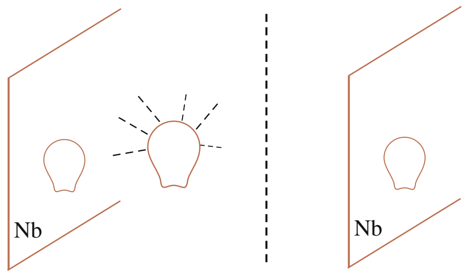
5.10 *Newton's force law from the continuity equation of mass*

a. Formulate the continuity equation for a flux of mass of a liquid or a gas.
b. Show that Newton's force law $\bar{F} = m\bar{a}$ follows from this continuity equation.

5.11 *The long lived mirror image*

The Nobel Prize in Physics in 2012 highlighted an experiment based on a sort of long lived mirror.

The basic idea may be understood in terms of a metal plate and a light bulb, according to the figures. The metal in the experiment was Nubidium, Nb.



After disconnecting the bulb, it could be observed that the mirror was active, i.e. the mirror image was maintained, for 0.13 s.

a. Estimate the resistivity of the Nubidium plate.
b. How can such a low resistivity be achieved?
   In the rewarded experiment the mirror was activated by microwaves.

5.12 *Determination of charge to mass ratio of the electron*

A beam of electrons is directed towards an ideal coil perpendicular to its axis. When inside the coil the electrons exhibit a circular path. The radius of the path is measured and denoted by r. The current per length unit in the coil along the coil axis is K and the high voltage generating the electron beam is $\Delta\Phi$.
Using these measurable quantities find an expression for the charge to mass ratio of the electron.
Hint: In Exercise (7.12) it is shown that the force on a charge inside an ideal coil is twice that from a large plate, formula (4.44b).
The performance of this experiment needs a tube filled with a gas at low pressure. The electrons excite the gas atoms which when de-excited become visible. The track of the electrons may then be observed from which the radius can be measured.

5.13 *Determination of the electron charge—the Millikan experiment*.

In the beginning of 1900 Millikan designed an apparatus for measuring the elementary charge, i.e. the electron charge. It consists of a vertical parallel plate capacitor located in a dark container filled with air. On the upper plate a small hole is drilled. The purpose of this crucial hole is to let small oil drops pass through into the region between the plates. The oil drops are produced by using a simple spray mechanism. While passing through the hole they become charged through the friction that occurs against the wall of the hole. The drops become randomly charged (Fig. 5.4).
The hypothesis is then that the amount of charge is so small that by measuring on several drops the *discreteness* of charge may be revealed. In such

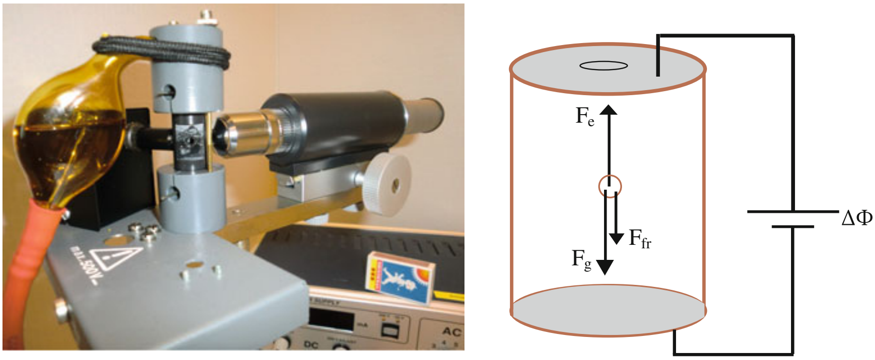


Fig. 5.4 The Millikan apparatus, student version by PHYWE. The oil drops are illuminated and observed by the telescope. Using a built-in distance scale the velocity of an oil drop may be determined through a time measurement. The voltage over the capacitor plates is established by means of the pins on the picture

a case, the measurement should confirm that the charge of the oil drops is always a multiple of a smallest charge corresponding to the electron charge. The measurement consists of determining the speed of the oil drop in two cases:

- Without voltage over the plates. The drop is falling by gravity with speed v_f.
- With high voltage over the plates. The motion is reversed and the drop rises with speed v_r.

The speeds are terminal speeds which are achieved quickly.
When analysing this phenomena it is first realized that basically three forces have to be considered:

- The gravitational F_g
- The air resistance F_{fr}
- The electric F_e

Assuming the oil drops to be spherical, the frictional force is (compare Exercise 5.6):

$$\bar{F}_{fr} = -G\bar{v} = 6\pi\eta a\bar{v}$$

where η is the viscosity of air and a is the radius of the drop. The last expression is called Stokes' law and is obtained from fluid mechanics.

a. Determine the terminal speeds v_f and v_r.
b. Find an expression from which the charge of the oil drop may be determined.

5.14 *The origin of electric charge—the atom*

In the beginning of 1900 Bohr proposed a model of the atom considering its constituents as pointlike particles carrying charge. Basically his model has survived up to the present day. (Compare Exercise (7.19) for a development of the model.)
Consider a hydrogen atom consisting of one proton as the nucleus and one surrounding electron.

a. Assuming the electron orbiting around the proton in a circular path with radius $r = 0.5 \times 10^{-10}$ m, what is the stored electric energy of the system?
b. Due to its relatively large mass, the proton is assumed to be fixed. What is the speed of the electron?
Use tabulated values for the elementary charge and electron mass.

C* 5.15 *Edge effects of parallel plates capacitor*

In Sect. 4.1.5.1 the capacitance for a parallel plate capacitor was derived. The capacitor was then assumed to be so-called ‘ideal’, meaning that the plates are so large that their edges may be disregarded. This is of course in practice never true but ‘edge effects’ are always present. These may be understood by considering the procedure for obtaining capacitance. In Fig. 4.6, external work is done by transporting a point charge against the electric force from one to the other plate.
Consider the following statement.
‘If the transportation of the point charge is done close to the edges of the plates a smaller electric force will act against the external force and thereby a smaller work is done. The calculation of work and consequently capacitance will therefore depend on the path chosen for the point charge.’

a. Is this reasonable? Explain.
b. What is the reason for this wrong statement?
c. What condition must hold for the potential over the plates and for the electric force between the plates?
d. How will the conditions found in task c affect the charge distribution on the plates?
e. Assuming the charge distribution to be known, find an expression for calculating the capacitance.

Solution

a. This is not reasonable since the work should be independent of chosen path. It depends only on the voltage between the plates.
b. The reason is that a homogeneous charge distribution has been assumed, as in the ‘ideal’ capacitor case. At the edges there is a deficit of neighbouring charges to build up the force.

c. The potential is constant over each plate. The electric force parallel to the surface normal of the plates is constant independent of the location along the plate.
d. The charge density will increase when approaching the edges of the plates.
e. Denote surface charge density σ which has been obtained using the conditions as in task c. Plate area is A and voltage $\Delta\Phi = 1\,\text{V}$. Divide the plate into n small area elements. The true capacitance becomes

$$C_{true} = \frac{Q}{\Delta\Phi} = \frac{A}{n}\sum_{k=1}^{n}\sigma_k \; on\, positive\, plate$$
$$= -\frac{A}{n}\sum_{k=1}^{n}\sigma_k \; on\, negative\, plate$$

which may be used in computer calculations.

Ref: IEEE, components, packaging and manufacturing A, vol. 17(3) 1994

Further Reading

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Chapter 6
Electric Circuits

Watson, ... if I can get a mechanism which will make a current of electricity vary in its intensity, as the air varies in density when a sound is passing through it, I can telegraph any sound, even the sound of speech.

Alexander Graham Bell, 1926

The electric current is the basis for modern technology. The applications consist of electric circuits, i.e. closed conductors. Using a voltage source, electric currents are generated which are controlled by the other components of the circuit: resistors, inductors and capacitors. This chapter examines examples of electric dynamics arising when all these components co-exist. The objective is to achieve an understanding of oscillation and resonance circuits.

In circuits, only the collective effect of charges in motion is observed. Therefore, the theoretical treatment is based on the energy concept. The generated energy is thus magnetic in an inductor, electric in a capacitor and heat in a resistor. The key to describing current variation in a circuit is Kirchoff's potential law, being based on the principle of energy conservation.

Familiarity with complex numbers is assumed.

6.1 Measurement of Capacitance Using an RC Circuit

Connect a battery in series with a resistor R and capacitor C, Fig. 6.1. C and R denote the total capacitance and resistance of the circuit, including the internal resistance of the battery. Applying the potential law, the following equation results

$$\Delta\Phi - R\frac{dq}{dt} - \frac{q}{C} = 0 \tag{6.1}$$

where q is the positive charge on one of the capacitor plates. If the source is connected at time $t = 0$ the solution for q becomes

$$q = C\Delta\Phi(1 - e^{-\frac{t}{RC}}) \tag{6.2}$$

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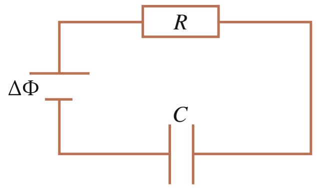


Fig. 6.1 Capacitor, resistor and battery connected in series

so that the current in the circuit is

$$i = \frac{dq}{dt} = \frac{\Delta\Phi}{R} e^{-\frac{t}{RC}} \tag{6.3}$$

Usually, the capacitance is so small that the charging process becomes very quick. Using a pulse generator as a voltage source and an oscilloscope as a voltmeter the current variation may be studied. By generating a square wave, corresponding to an alternating connection and disconnection of the battery, formula (6.3) may easily be explored. It is convenient to determine the time it takes for the current to reach half its maximum value, the so-called half-time

$$T_{1/2} = RC \ln 2 \tag{6.4}$$

which can be used to determine capacitance if R is known.

6.2 Measurement of Inductance Using an RL Circuit

An inductor is now connected in series with a battery and a resistor, Fig. 6.2. In circuit theory, it is customary to denote self-inductance L. L and R represent the circuit's entire inductance and resistance respectively.

When a voltage is applied, the current increases in the circuit, i.e. the conduction electrons are accelerated. The inductive effect is an inertia against this acceleration, i.e. a resistance. From the positive to the negative pole of the battery, the potential across the inductor decreases, described by Faraday-Henry's law of induction, formula (3.31):

$$\Delta\Phi - Ri - L\frac{di}{dt} = 0 \tag{6.5}$$

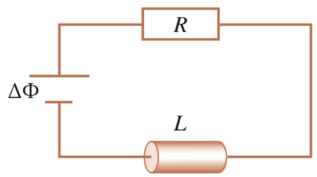


Fig. 6.2 Inductor, resistor and battery connected in series

If the battery is connected at time $t = 0$ the solution becomes

$$i = \frac{\Delta\Phi}{R}(1 - e^{-t\frac{R}{L}}) \tag{6.6}$$

with half-time

$$T_{1/2} = \frac{L}{R}\ln 2 \tag{6.7}$$

from which inductance can be determined if R is known. As in the capacitor case the time-lapse is short making it necessary to use a pulse generator and an oscilloscope.

6.3 The Oscillation Circuit

Many instruments that measure inductance and capacitance utilize the so-called oscillation circuit. Consider the series circuit as in Fig. 6.3. Two cases will be analysed. Firstly, the battery generates a constant voltage and secondly a voltage that varies sinusoidally with time. This latter case is suitable for studying the important resonance phenomena. Oscillatory solutions are sought in both cases.

6.3.1 The RLC Circuit with Constant Input Voltage

The potential equation is formulated for the circuit in Fig. 6.3:

$$\Delta\Phi - Ri - \frac{q}{C} - L\frac{di}{dt} = 0 \tag{6.8}$$

Differentiating with respect to time makes the first term vanish since the input voltage is constant:

$$\frac{1}{LC}i + \frac{R}{L}\frac{di}{dt} + \frac{d^2i}{dt^2} = 0 \tag{6.9}$$

The solution to this homogeneous differential equation of second order is found by first determining the roots m to the characteristic equation

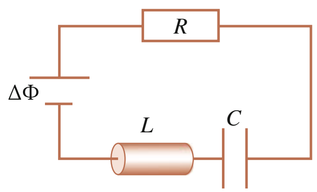


Fig. 6.3 Capacitor, inductor, resistor and battery connected in series

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0 \tag{6.10}$$

This second-order equation has two roots

$$m_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$m_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{6.11}$$

The solution becomes

$$i(t) = Ae^{m_1 t} + Be^{m_2 t} \tag{6.12}$$

so that oscillatory solutions appear only if

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

Then the solution may be written

$$i(t) = \left(Ae^{j\omega t} + Be^{-j\omega t}\right) e^{-\frac{R}{2L}t} \tag{6.13}$$

where

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad and \quad j = \sqrt{-1} \tag{6.14}$$

For a non-resistant system the frequency becomes

$$\frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

which is known as the natural frequency of the system.

If current $i = 0$ at $t = 0$ then $A = -B$ so that

$$i(t) = D\sin(\omega t)e^{-\frac{R}{2L}t} \tag{6.15}$$

The constant D is determined by noting that at $t = 0$ there is neither charge q on the capacitor nor current i in the circuit so that formula (6.8) gives

$$\Delta\Phi = L\frac{di}{dt}\,|_{t=0} = L\omega D \Rightarrow D = \frac{\Delta\Phi}{\omega L} \tag{6.16}$$

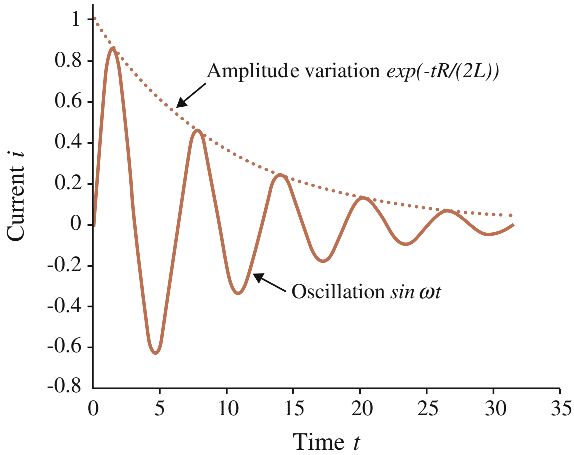


Fig. 6.4 Current in a damped oscillation circuit

Formula (6.15) describes an attenuated oscillating current. The attenuation may be used to determine L and the oscillation to determine C, assuming R is known.

A typical time dependence is shown in Fig. 6.4.

There are many mechanical analogues to this oscillation sequence. Consider for example a weight connected to a spring. At the turning points the energy is purely potential and at the equilibrium point the energy is purely kinetic. In the electric RLC circuit the turning points correspond to a collection of electrons on the capacitor plate storing purely electric energy. The kinetic energy corresponds to magnetic energy stored as current in the circuit (compare formula 3.12). The attenuation arises due to friction, i.e. resistance R. In Exercise (6.7) this analogy is further explored.

6.3.2 Forced Oscillation Circuit

Now let the voltage source generate an oscillating force on the electrons. A resonance phenomenon will then appear similarly to what would happen if an external force acts on the spring system as in the previous section. When the frequency of the external force equals the natural frequency of the spring system very large amplitudes appear known as resonance.

Assume the battery provides a time-harmonic (sinusoidal) voltage with angular velocity ω_s (index s for 'source') which by simplicity is described by a complex exponential function:

$$\Delta\Phi = \Delta\Phi_0 e^{j\omega_s t} \tag{6.17}$$

The solution is then complex and the physical current given by its imaginary part. In analogy with formulas (6.8) and (6.9), the equation becomes (c for complex)

$$\frac{1}{LC}i_c + \frac{R}{L}\frac{di_c}{dt} + \frac{d^2 i_c}{dt^2} = Fe^{j\omega_s t} \tag{6.18}$$

where $F = j\Delta\Phi_0\omega_s/L$

According to the formalism of differential equations, the current is the sum of the solution to the homogeneous equation (6.9), and a particular solution of (6.18). For the latter the simplest possible solution is tested (p for particular):

$$i_c^p = Ae^{j\omega_s t} \tag{6.19}$$

which is introduced in (6.18):

$$\left(\frac{1}{LC} + \frac{R}{L}j\omega_s - \omega_s^2\right)Ae^{j\omega_s t} = Fe^{j\omega_s t} \tag{6.20}$$

so that

$$A = \frac{F}{\frac{1}{LC} + \frac{R}{L}j\omega_s - \omega_s^2} = \frac{\left(\frac{1}{LC} - \omega_s^2 - \frac{R}{L}j\omega_s\right)F}{\left(\frac{1}{LC} - \omega_s^2\right)^2 + \left(\frac{R}{L}\omega_s\right)^2}$$

$$= \frac{\left(j\left(\frac{1}{LC} - \omega_s^2\right) + \frac{R}{L}\omega_s\right)\frac{\omega_s}{L}\Delta\Phi_0}{\left(\frac{1}{LC} - \omega_s^2\right)^2 + \left(\frac{R}{L}\omega_s\right)^2} \tag{6.21}$$

A is now written in polar form:

$$A = |A|\,e^{-j\varphi} \tag{6.22}$$

where

$$|A| = \frac{|F|}{\sqrt{\left(\frac{1}{LC} - \omega_s^2\right)^2 + \left(\frac{R}{L}\omega_s\right)^2}} \tag{6.23a}$$

and

$$\tan\varphi = -\frac{\frac{1}{LC} - \omega_s^2}{\frac{R}{L}\omega_s} = \frac{\omega_s L - \frac{1}{\omega_s C}}{R} \tag{6.23b}$$

The complex particular solution becomes:

$$i_c^p = \frac{\frac{\omega_s}{L}\Delta\Phi_0}{\sqrt{\left(\frac{1}{LC} - \omega_s^2\right)^2 + \left(\frac{R}{L}\omega_s\right)^2}}e^{j(\omega_s t - \varphi)} = \frac{\Delta\Phi_0 e^{j(\omega_s t - \varphi)}}{\sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2}} \tag{6.24}$$

The full solution is formula (6.15) plus the imaginary part of (6.24):

$$i(t) = \underbrace{\frac{\Delta\Phi_0}{\omega L}\sin(\omega t)e^{-\frac{R}{2L}t}}_{\text{Transient}} + \underbrace{\frac{\Delta\Phi_0}{\sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2}}\sin(\omega_s t - \varphi)}_{\text{Stationary}} \tag{6.25}$$

The first term is transient since it is damped and vanishes with time. The final current in the circuit, the so-called stationary current, corresponds to the second term which is utilized in resonance circuits. Note that the current in such a circuit is shifted an angle φ in phase compared to the input voltage.

The current depends on the frequency of the voltage source and reaches a maximum at the natural frequency

$$\frac{\omega_s}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \tag{6.26}$$

known as the resonance frequency at which the current becomes

$$i(t) = \frac{\Delta\Phi_0}{R}\sin(\omega_s t - \varphi) \tag{6.27}$$

i.e. the circuit is purely resistive. Thus, by determining the maximum current for varying frequency, the inductance or the capacitance may be determined if the other is known.

Resonance circuits have wide applications, e.g. as electronic senders and receivers of different kinds. The voltage source in the circuits above may for example be a receiving antenna.

6.3.3 Impedance

For the forced RLC circuit above, the stationary current is

$$i(t \gg 0) = \frac{\Delta\Phi_0}{\sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2}}\sin(\omega_s t - \varphi) \tag{6.28}$$

The expression in the denominator is called impedance Z

$$Z = \sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2} \tag{6.29}$$

and is the generalization of current resistance since the current amplitude is given by

$$i_0 = \frac{\Delta\Phi_0}{Z} \tag{6.30}$$

Reactance is defined as

$$X = \omega_s L - \frac{1}{\omega_s C} \tag{6.31}$$

so that the phase shift φ becomes

$$\tan\varphi = \frac{X}{R} \tag{6.32}$$

Exercise (6.5) further develops the method of using complex numbers when working with AC (alternating current) circuits, the so-called $j\omega$ method.

6.4 Summary

The series RC circuit is described by the equation

$$\Delta\Phi - R\frac{dq}{dt} - \frac{q}{C} = 0 \tag{6.1}$$

The series RL circuit is described by the equation

$$\Delta\Phi - Ri - L\frac{di}{dt} = 0 \tag{6.5}$$

The series RLC circuit with a constant battery voltage is described by the equation

$$\Delta\Phi - Ri - \frac{q}{C} - L\frac{di}{dt} = 0 \tag{6.8}$$

and with a sinusoidal time dependent voltage source in complex representation

$$\frac{1}{LC}i_c + \frac{R}{L}\frac{di_c}{dt} + \frac{d^2 i_c}{dt^2} = Fe^{j\omega_s t} \tag{6.18}$$

The physical solution in the latter case is

$$i(t) = \frac{\Delta\Phi_0}{\omega L}\sin\omega t e^{-\frac{R}{2L}t} + \frac{\Delta\Phi_0}{\sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2}}\sin(\omega_s t - \varphi) \tag{6.25}$$

where the first term is transient and the second is stationary.

Impedance is defined by

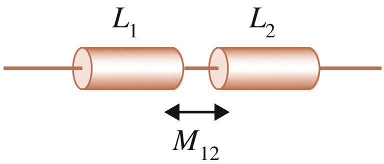
$$Z = \sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2} \tag{6.29}$$

6.5 Exercises

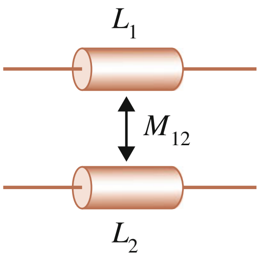
6.1 Consider a discharge of a fully charged capacitor as in Fig. 6.1 without an external voltage source. Determine how the current varies in the circuit. How would you investigate the discharging process in practice?

6.2 Let the voltage source in the RL circuit of Fig. 6.2 generate a square formed alternating voltage. When the voltage level reaches zero the current will start to decrease. Determine the current during this interval.

*6.3 Consider two inductors with self-inductances L_1 and L_2.



a. If they are connected in series, show that the total inductance becomes $L = L_1 + L_2 + 2M_{12}$



where M_{12} is the mutual inductance.

b. What is the total inductance if the two inductors are connected in parallel?

c. Show that the mutual inductance between two conductors can be expressed as
$M_{12} = k\sqrt{L_1 L_2}$ where $-1 \leq k \leq 1$

Hint: Use formula (3.19).

*6.4 For the resonance circuit with a sinusoidal input voltage, Fig. 6.3, find

a. the momentary input power.
b. the average input power.
c. the phase difference between current and voltage at maximum input power.
d. an expression for the frequency of the input voltage in terms of capacitance and inductance at maximum input power.

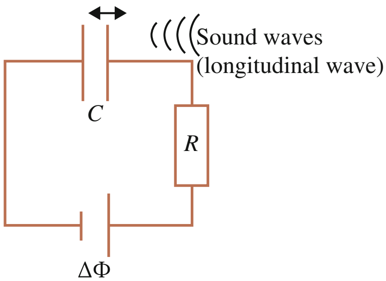
*6.5 For the resonance circuit with a sinusoidal input voltage, Fig. 6.3, consider current and voltage as complex, so-called phasors, see formula (6.17).

a. Represent current and voltage as vectors in the complex coordinate plane and indicate their relative phase difference.
b. If complex impedance is defined as

$$Z_c = \frac{\Delta\Phi_c}{i_c}$$

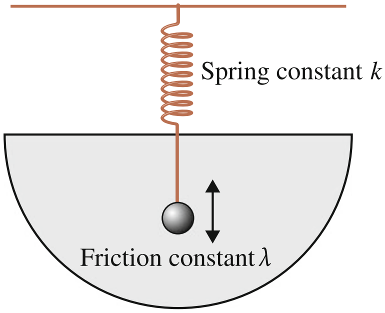
show that $Z_c = Ze^{j\varphi}$ where Z is given by formula (6.29) and φ is defined by formula (6.23a, 6.23b).
c. Represent complex impedance in the complex plane and state the length of the vector as well as its projection on the two axes.

6.6 A capacitor microphone consists of two parallel plates where one is fixed and the other is an easily movable membrane.
Both plates have an area A and are connected to a battery voltage $\Delta\Phi$ and to a load of resistance R.



a. If the distance between the plates is changed from d to $d + s$, how much charge passes through the load R?
b. Explain how this change in distance occurs in the microphone.

6.7 Consider the oscillating circuit in Fig. 6.3. It is quite instructive to perform an analogy to mechanical oscillations by considering a damped oscillation of a ball connected to a spring. The damping may occur due to viscous forces in the medium in which it oscillates, see figure.



Formulate the equation of motion for the ball and identify the correspondence between the mechanical and the electric quantities, defined by Eq. (6.8):

$$\Delta\Phi - Ri - \frac{q}{C} - L\frac{di}{dt} = 0$$

Further Readings

M. Alonso, E.J. Finn, *Fundamental University Physics*, vol. II (Addison-Wesley, Reading, 1983)
H.T. Glisson, *Introduction to Circuit Analysis and Design* (Springer, New York, 2011)

Chapter 7
Electric and Magnetic Dipoles

... every chemical combination is wholly and solely dependent on two opposing forces, positive and negative electricity, and every chemical compound must be composed of two parts combined by the agency of their electrochemical reaction, since there is no third force. Hence it follows that every compound body, whatever the number of its constituents, can be divided into two parts, one of which is positively and the other negatively electrical.

Jöns Jacob Berzelius, 1819

The following two chapters deal with electric and magnetic dipoles. An electric dipole is a neutral system consisting of two electric poles with the same amount of charge but with different signs. A magnetic dipole is described analogously by a north and a south pole, a terminology which originates from geomagnetism. A more basic description is a simple current loop.

Dipoles are important for two main reasons:

- As an approximation to systems, both macroscopic and microscopic, in the far distance.
- As a model for electric and magnetic material properties at a microscopic level.

In this chapter the interaction between a dipole and different objects are discussed. Particularly important is the interaction energy between two dipoles which will be derived for both the electric and the magnetic case. These formulas are fundamental for the dynamics of nature.

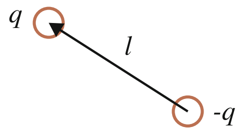
Chapter 8 discusses material properties resulting from dipole dynamics.

7.1 Electric Dipole

The electric dipole is illustrated in Fig. 7.1. Its poles are point-like and the line between them reflects the fact that the charges are bound to each other. The dipole properties are given by the charge q and the distance vector l between the poles. The dipole moment is

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_7

Fig. 7.1 Electric dipole



$$\bar{p} = q\bar{l} \tag{7.1}$$

and is directed from minus to plus pole.

The interaction between an electric dipole and another object stores electric energy which can be derived as follows. The dipole is in the potential Φ, associated with the other object, Fig. 7.2. Let the dipole be so small that the distance between its poles is infinitesimal.

Since $U = q\Phi(r_+) - q\Phi(r_-) = qd\Phi = qd\bar{r} \cdot \nabla\Phi$, the energy becomes

$$U = \bar{p} \cdot \nabla\Phi \tag{7.2}$$

This is the interaction energy between the dipole and the system which generates the potential Φ.

The force on the dipole is

$$\begin{aligned}\bar{F} &= -\nabla U = -\nabla(\bar{p} \cdot \nabla\Phi) \\ &= -\nabla\Phi \times \underbrace{(\nabla \times \bar{p})}_{=0} - \bar{p} \times \underbrace{(\nabla \times \nabla\Phi)}_{=0} - \underbrace{(\nabla\Phi \cdot \nabla)\bar{p}}_{=0} - (\bar{p} \cdot \nabla)\nabla\Phi\end{aligned}$$

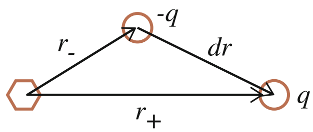
Only the last term is non-vanishing since p is a constant and term no 2 contains the curl of a gradient which vanishes identically. The force becomes

$$\bar{F} = -(\bar{p} \cdot \nabla)\nabla\Phi \tag{7.3}$$

The dipole also experiences a torque τ:

$$\begin{aligned}\bar{\tau} &= -\frac{dU}{d\theta}\hat{\theta} = -\frac{d}{d\theta}(\bar{p} \cdot \nabla\Phi)\hat{\theta} \\ &= p\,|\nabla\Phi|\sin\theta\hat{\theta} \quad \textit{Unconventional direction}!\end{aligned} \tag{7.4}$$

Fig. 7.2 A dipole with length dr interacts with an arbitrary system to the left



where θ is the angle between the dipole moment and the gradient of the potential. The rotation takes place until $\theta = \pi$, i.e. when p and $\nabla\Phi$ are anti-parallel. The torque is then zero and the energy (7.2) is at a minimum. The sign in the first equality of formula (7.4) is not generally valid but has to be chosen according to the specific circumstances. In this case, the minus sign is adequate since the energy decreases when θ increases from 0 to π.

The direction of a rotational motion is conventionally defined as perpendicular to the motion according to the right-hand rule, see figure in Exercise (4.13). The reason for this definition can be understood from formula (7.4) since this formula may then be written as a general cross product

$$\bar{\tau} = -\bar{p} \times \nabla\Phi \tag{7.5}$$

This dynamic will now be investigated in a few examples.

7.1.1 Interaction Between Dipole and Point Charge

Consider the interaction between a dipole and a point charge placed at the origin, Fig. 7.3. What is the energy and the force?

The potential from a point charge is

$$\Phi = \frac{q}{4\pi\varepsilon_0 r} \tag{7.6}$$

and accordingly the energy is

$$U = \bar{p}\cdot\nabla\Phi = -\frac{q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\bar{p} \tag{7.7}$$

The force is obtained using formula (7.3)

$$\begin{aligned}\bar{F} &= -(\bar{p}\cdot\nabla)\nabla\Phi = \frac{q}{4\pi\varepsilon_0}(\bar{p}\cdot\nabla)\frac{\bar{r}}{r^3}\\ &= \frac{q}{4\pi\varepsilon_0}\left(p_x\frac{d}{dx} + p_y\frac{d}{dy} + p_z\frac{d}{dz}\right)\frac{(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)^{3/2}}\end{aligned} \tag{7.8}$$

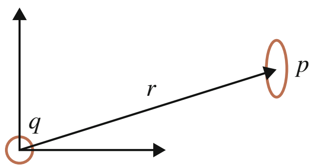


Fig. 7.3 A point charge q interacts with a dipole p

which is the force on the dipole since the point charge is located at the origin. This formula contains a derivative of a product which can be evaluated component-wise so that

$$F_x = \frac{q}{4\pi\varepsilon_0}\left(-3\frac{\bar{p}\cdot\bar{r}}{r^5}x + \frac{p_x}{r^3}\right)\hat{x} \tag{7.9}$$

and analogously for F_y and F_z. Expressed in vector form the force is

$$\bar{F} = \frac{q}{4\pi\varepsilon_0}\left(-3\frac{\bar{p}\cdot\hat{r}}{r^3}\hat{r} + \frac{\bar{p}}{r^3}\right) \tag{7.10}$$

The verification of these two last formulas as well as the determination of torque are left to Exercise (7.2).

7.1.2 Dipole-Dipole Interaction

The next example examines the interaction between two dipoles, e.g. two molecules. To find the energy, the potential generated by a dipole, Φ_{dipole}, is first determined. This is achieved by summing the potentials from the two poles, Fig. 7.4.

The potential from a positive point charge is denoted Φ. Thus,

$$\begin{aligned}\Phi_{dipole} &= \Phi(|\bar{r}_+|) - \Phi(|\bar{r}_-|) = \Phi(|\bar{r}_+|) - \Phi(|\bar{r}_+ + d\bar{r}|)\\ &= -d\bar{r}\cdot\nabla\Phi = \frac{q}{4\pi\varepsilon_0 r^2}d\bar{r}\cdot\hat{r} = \frac{\bar{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}\end{aligned} \tag{7.11}$$

where r is the distance from the pointlike dipole to the point where the potential is given. Denote the position vectors of the two dipoles r_1 and r_2 respectively and the distance between them R, Fig. 7.5. Let the dipole p_2 be located in the potential generated by dipole p_1 so that using formula (7.2) energy becomes

$$U = \bar{p}_2\cdot\nabla\left(\frac{\bar{p}_1\cdot\hat{R}}{4\pi\varepsilon_0 R^2}\right) \tag{7.12}$$

The gradient of a scalar product is found in a handbook, compare formula (7.3). Treating the dipole moments as constant, the energy becomes

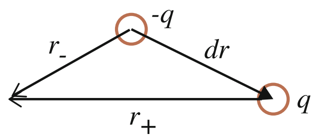


Fig. 7.4 Electric potential generated by a dipole

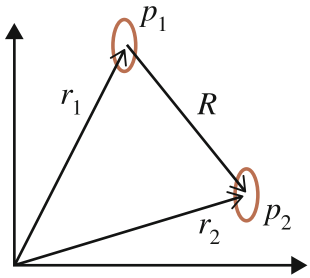


Fig. 7.5 Interaction between two dipoles

$$U = \frac{\bar{p}_2}{4\pi\varepsilon_0} \cdot \left[\bar{p}_1 \times \left(\nabla \times \frac{\hat{R}}{R^2} \right) + (\bar{p}_1 \cdot \nabla) \frac{\hat{R}}{R^2} \right] \tag{7.13}$$

The first term contains

$$\nabla \times \frac{\hat{R}}{R^2} = \nabla \times \nabla \frac{1}{R} = 0 \tag{7.14}$$

and the second term

$$(\bar{p}_1 \cdot \nabla) \frac{\hat{R}}{R^2} = -3 \frac{\bar{p}_1 \cdot \hat{R}}{R^3} \hat{R} + \frac{\bar{p}_1}{R^3} \tag{7.15}$$

in analogy with formula (7.9).

The energy may then be written

$$U = \frac{1}{4\pi\varepsilon_0 R^3} [\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})] \tag{7.16}$$

The evaluation of force and torque is left to Exercise (7.16).

7.1.3 Interaction Between a Charged Plate and a Dipole

In Sect. (4.1.3) the force between a point charge q and a large homogeneously charged plate was found to be $F = \sigma q/(2\varepsilon_0)$ parallel to the surface normal where σ is the surface charge density of the plate. Since $\bar{F} = -\nabla U = -q\nabla\Phi$, the potential from the plate, oriented in the yz plane as in Fig. 7.6, becomes

$$\Phi = -\frac{\sigma}{2\varepsilon_0} x \tag{7.17}$$

up to a constant which is put to zero. The interaction energy between the plate and a dipole becomes

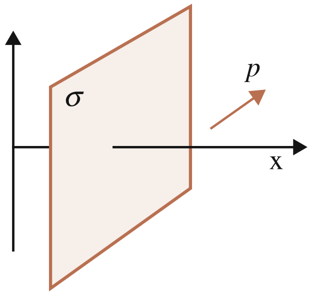


Fig. 7.6 Interaction between a large charged plate and dipole p

$$U = \bar{p} \cdot \nabla \Phi = p_x \frac{d\Phi}{dx} = -\frac{p_x \sigma}{2\varepsilon_0} = -\frac{\bar{p} \cdot \hat{n} \sigma}{2\varepsilon_0} \tag{7.18}$$

where $\hat{n}$ is the surface normal of the plate. If the dipole is oriented such that it lacks dipole moment along the surface normal of the plate, the energy is accordingly zero.

The force on the dipole becomes

$$\bar{F} = -(\bar{p} \cdot \nabla)\nabla\Phi = 0 \tag{7.19}$$

since the force on a point charge from a large plate is independent of distance. For the dipole, the force on the positive and the negative pole are of same magnitude but are oppositely directed.

However, a torque arises

$$\bar{\tau} = -\bar{p} \times \nabla\Phi = \frac{\sigma}{2\varepsilon_0} \bar{p} \times \hat{x} = \frac{\sigma}{2\varepsilon_0} \bar{p} \times \hat{n} \tag{7.20}$$

which is zero when the dipole moment is parallel or anti-parallel to the plate's surface normal $\hat{n}$.

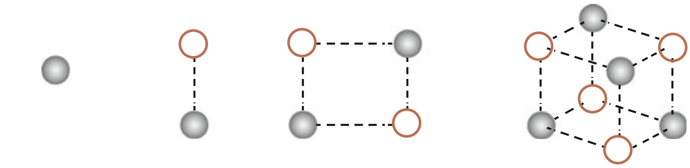
7.1.4 Generalized Electric Dipole Moment

An arbitrary charge distribution may be assigned a dipole moment and an approximate description of its interactions may be obtained using the formulas above.

If a system is regarded as a dipole, its interactions generally have to appear at a rather large distance. Indeed, a system looks like a point far away and is treated accordingly as a point charge. With decreasing distance, the system's structure is revealed with the dipole structure as the first correction to the point character. If the system's total charge is zero, its dipole structure is the first approximation. At the next level the quadrupole structure is revealed, next octupole structure and so on. Based on a series expansion with respect to distance r, the potential can then be written

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\bar{p}\cdot\hat{r}}{r^2} + O\left(\frac{1}{r^3}\right) quadrupole + O\left(\frac{1}{r^4}\right) octupole + \cdots\right) \tag{7.21}$$

where r is the distance from the center of the system to the observation point. O denotes order in the series expansion. The figure below illustrates the first four terms of the multipole expansion. Filled and unfilled poles have the same charge but opposite signs.



The multipole theory is further discussed in Exercise (7.1) and in appendices AB.

The generalized electric dipole moment is the geometrical mean of positive charge minus the negative counterpart. In the discrete case, the formula becomes

$$\bar{p} = \sum_i \bar{r}_i q_i \tag{7.22}$$

where the sum runs over all charges in the system and $\bar{r}_i$ is the position vector for charge i.

Usually, systems are continuous and described by a charge density $\rho(\bar{r}_i)$. An infinitesimal volume then contains charge $\rho(\bar{r}_i)dV$ and the generalization of formula (7.22) becomes

$$\bar{p} = \int_V \bar{r}\rho(\bar{r})dV \tag{7.23}$$

where V is the volume of the system.

7.2 Magnetic Dipole

The theory of magnetic dipoles and their interactions shares many of its features with that for electric dipoles.

The idea of a magnetic dipole moment stems from the permanent magnet which is described in terms of north and south poles together with the effective length vector between them. This is in analogy with the electric dipole moment. As is now well-known, magnetism corresponds to charges in motion and the pole description is fictitious. In fact, the magnetic properties of a conventional permanent bar magnet originate from the electric currents flowing on its surface. This will be studied in detail in Sect. 8.2.1.

The bar magnet may be thought of as a number of closed parallel current loops, Fig. 7.7. An ideal theoretical dipole is small and consists of two point-like poles. Since magnetic charge does not exist the smallest magnetic unit is a dipole. The smallest possible formation of a magnetic dipole is a point-like permanent magnet, corresponding to a single closed loop. For reasons of symmetry it must be circular. In the macrocosm, the exact dipole is really a spherical current distribution and the simple loop is a good dipole approximation at fairly large distance, see Sect. 8.2.2.3.

The question is then what dipole moment should be attributed to this object. The convention is to define the magnetic dipole moment analogously to the electric. It is therefore necessary to investigate how two parallel circular loops interact and then use the equivalence of formula (7.16) (electric dipole-dipole energy) to define the magnetic dipole moment. Let the loops be positioned far from each other so that they are perceived as small. Their interaction energy is given by formulas (3.19) and (4.64):

$$U = \frac{\mu_0 I_1 I_2 \pi a^2 b^2}{2R^3} \tag{7.24}$$

where a and b are the radii of the respective loop and R is the distance, Fig. 7.8.

The direction of a magnetic dipole moment is defined as the surface normal to the loop, according to the right-hand rule with current as reference direction, see figure in Exercise (4.13). For the same current direction, the dipole moments are parallel and directed along the distance vector R. The currents are treated as constant so that the force is given by (4.66). As discussed in Sect. 3.3.1, the total energy is then minimized, since it is understood that the currents are maintained by an external battery.

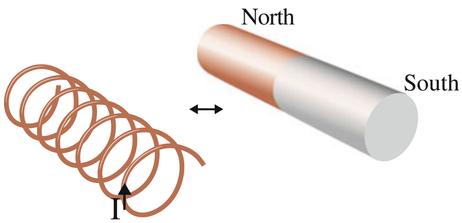


Fig. 7.7 A permanent magnet, on the *right*, corresponds to a set of parallel conducting loops, on the *left*

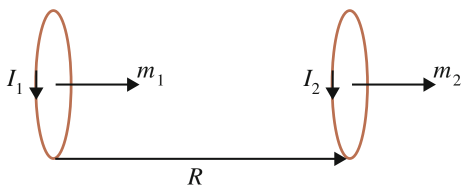


Fig. 7.8 Interaction between two current loops

The magnetic dipole moment is defined analogously to the electric. In formula (7.16) the electric dipole moment is replaced by the magnetic so that the expression for the interaction energy between two magnetic dipoles with dipole moments m_1 and m_2 becomes

$$U = -\frac{\mu_0}{4\pi R^3}[\bar{m}_1 \cdot \bar{m}_2 - 3(\bar{m}_1 \cdot \hat{R})(\bar{m}_2 \cdot \hat{R})] \tag{7.25}$$

where $1/\varepsilon_0$ has been replaced by the magnetic permeability μ_0. Also, a minus sign has been introduced since the magnetic energy is maximized for constant currents. Formula (7.25) is derived in Appendix B.

If the dipole moments are parallel to the distance vector R, as in Fig. 7.8, the energy becomes

$$U = \frac{\mu_0}{2\pi R^3} m_1 m_2 \tag{7.26}$$

The energy is positive, which indicates an attractive force. Comparison with formula (7.24) gives

$$\begin{aligned} \bar{m}_1 &= I_1 \pi a^2 \hat{n}_1 \\ \bar{m}_2 &= I_2 \pi b^2 \hat{n}_2 \end{aligned} \tag{7.27}$$

where $\hat{n}$ is the surface normal. The dipole moment for a circular current loop is thus its current times its area. This is generally valid for closed loops with a fixed surface normal.

7.2.1 Interaction Between a Magnetic Dipole and a Large Current-Carrying Plate

Consider the interaction between a large current-carrying plate and a magnetic dipole. The latter consists of a single loop which is arbitrarily oriented and carries a constant current I.

It was shown in Sect. 4.2.3 that the force on a current element IdL from a large current-carrying plate is independent of distance and given by formula (4.45)

$$d\bar{F} = Id\bar{L} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{4.45}$$

With the coordinate system as in Fig. 7.9 (see also figure in Exercise 7.9) formula (4.45) becomes

$$d\bar{F} = Id\bar{L} \times \hat{y}\frac{\mu_0 K}{2} \tag{7.28}$$

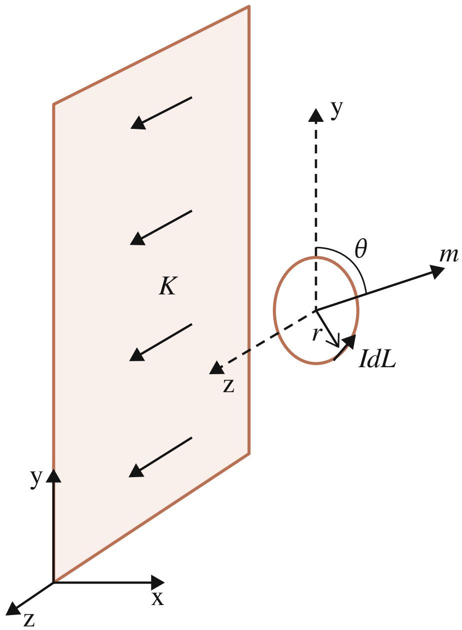


Fig. 7.9 Circular loop with dipole moment m interacts with a large plate

where K is current per unit length in the y direction and the plate is parallel to the yz-plane. The current direction in the plate is parallel to the z axis and its surface normal $\hat{n}$ points in the positive x direction.

For a circular loop close to the plate the translational force vanishes since opposite current elements give opposing forces ($d\bar{L}$ changes sign). Considering the loop as a bar magnet, the forces on the fictitious poles cancel since the force is independent of distance. This is equivalent to the electric dipole's interaction with a large charged plate, Fig. 7.6.

However, as in the electric case a torque τ arises:

$$d\bar{\tau} = \bar{r} \times d\bar{F} = I\frac{\mu_0 K}{2}\bar{r} \times (d\bar{L} \times \hat{y}) = I\frac{\mu_0 K}{2}[d\bar{L}(\bar{r} \cdot \hat{y}) - \hat{y}(\underbrace{\bar{r} \cdot d\bar{L}}_{=0})] \quad (7.29)$$

where the last equality follows from vector algebraic rules. For a circle the radius r is perpendicular to dL so the last term vanishes. Torque becomes

$$\bar{\tau} = I\frac{\mu_0 K}{2}\oint_C (\bar{r} \cdot \hat{y})d\bar{L} = I\frac{\mu_0 K}{2}\int_S d\bar{a} \times \nabla(\bar{r} \cdot \hat{y})$$
$$= I\frac{\mu_0 K}{2}\int_S d\bar{a} \times \hat{y} = \frac{\mu_0 K}{2}\bar{m} \times \hat{y}$$

where S is the area defined by the circle and $d\bar{a}$ is a surface element. In the second equivalence, a version of Stokes' theorem has been used. In coordinate free form the torque becomes

$$\bar{\tau} = \frac{\mu_0}{2}\bar{m} \times (\bar{K} \times \hat{n}) \tag{7.30}$$

and the dipole moment of the loop is

$$\bar{m} = I \int_S d\bar{a} \tag{7.31}$$

Thus, the loop orients with its dipole moment either parallel or anti-parallel to the y axis, depending on the current direction of the plate.

The interaction energy may be determined from the torque since

$$\bar{\tau} = -\frac{dU}{d\theta}\hat{\theta} = \left|\frac{\mu_0 K}{2} m \sin\theta\right| \hat{\theta} \quad \textit{Unconventional direction}! \tag{7.32}$$

where θ is the angle between the dipole moment m and the y axis, Fig. 7.9. Note that the energy principle for torque is applicable only if the direction of the torque is along the rotational motion. For reasons discussed in Sect. 7.1 the direction of torque is redefined to become perpendicular to the motion, in accordance with formula (7.30).

As in the electric case, formula (7.4), the sign of the first equality in formula (7.32) must be determined from the specific conditions. Referring to Fig. 7.9 with the current K in the positive z direction, the magnetic torque acts so as to turn the loop towards $\theta = 0$. Consequently $d\theta < 0$, increasing the magnetic energy to a maximum at the stable equilibrium. Therefore

$$U = \frac{\mu_0 K}{2}\bar{m} \cdot \hat{y} = \frac{\mu_0}{2}\bar{m} \cdot (\bar{K} \times \hat{n}) \tag{7.33}$$

which is generally valid for loops with a fixed surface normal.

Considering magnetism at an atomic level with electron spins and electron orbitals as its source, the currents should be treated as constant. As a consequence, the magnetic interaction energy is maximized. This is why ferro- and paramagnetic materials are attracted by an external influence, see Chap. 8. Which voltage source that maintains the current in these cases is a particularly interesting issue.

7.2.2 Induced Voltage in a Rotating Loop Interacting with a Current-Carrying Plate

In Sect. 3.5 Faraday-Henry's induction law was derived. This will now be applied to a generator construction. Consider two conductors where conductor 1 carries a current. A voltage is induced in conductor 2 according to formula (3.32)

$$\varepsilon_2 = M_{12}\frac{d}{dt}I_1 + I_1\frac{d}{dt}M_{12} \tag{7.34}$$

The second term is a so-called motional induction which arises through a change of the geometrical relation between the two conductors, achieved either by relative translational or rotational motion of some segment or the whole conductor.

For example, let conductor 1 be a large plate with a constant current I_1 in the z direction and let conductor 2 be a loop close to the plate, Fig. 7.9. Assume the dipole approximation to be valid for this loop. If the loop is rotating, a voltage will be induced because of the motional induction term in formula (7.34), which will now be determined. In principle, Neumann's inductance formula (3.30) may be utilized to determine the mutual inductance, which will be time dependent since the angle between two length elements varies with time. Alternatively, a short-cut through the energy may be chosen, using formula (7.33). From formula (3.19), the interaction energy is

$$U = M_{12}I_1I_2 \tag{7.35}$$

Rewriting (7.33) in terms of current such that $I_1 = KY$, where Y is the length of the plate along y, and $\bar{m} = I_2\bar{A} = I_2A\hat{n}_2$ with A being the area of the loop, energy becomes

$$U = \mu_0\frac{I_1}{2Y}I_2A\hat{n}_2\cdot\hat{y} = \mu_0\frac{I_1}{2Y}I_2A\cos\theta \tag{7.36}$$

where θ is the angle between the y axis and the surface normal of the loop. The mutual inductance becomes

$$M_{12} = \frac{\mu_0A\cos\theta}{2Y} \tag{7.37}$$

The induced voltage in the loop (conductor 2) becomes

$$\varepsilon_2 = I_1\frac{d}{dt}M_{12} = -\frac{\mu_0K}{2}A\sin\theta\frac{d\theta}{dt} \tag{7.38}$$

where $d\theta/dt$ is called angular velocity.

7.2.3 *Generalized Magnetic Dipole Moment—Interaction Between Rotating Cylinders*

As a further example of magnetic dipole moment, consider the interaction between two homogeneously charged rotating cylinders in the dipole approximation. Since the magnetic dipole moment formula (7.31) is valid only for closed loops, a generalized formula must first be developed and the dipole moment for this case determined. Then the magnetic energy is derived utilizing formula (7.25). In addition, there is in this case also electric energy.

The dipole moment, formula (7.31), is generalized in the following way (see also Appendix B)

$$\bar{m} = I \int_S d\bar{a} = \frac{1}{2} I \oint_C (\bar{r} \times d\bar{L}) = \frac{1}{2} \int_V (\bar{r} \times \bar{J}) dV \tag{7.39}$$

valid for an arbitrary current distribution with volume V and current density J, defined by the relation $Id\bar{L} = \bar{J}dV$, compare formula (2.5) and Exercise (2.24).

Denote the length l, the radius a, the charge Q and the angular velocity about the cylinder axis ω, Fig. 7.10. The coordinate system is chosen such that its origin is at the centre of the cylinder. The current density is obtained from formula (2.5) as $\bar{J}(r) = \rho_{ch}\bar{v}(r)$ where ρ_{ch} is charge density and $\bar{v}$ is the velocity of a charge element at position r. This motion is due to the rotation of the cylinder so that

$$\bar{J}(r) = \rho_{ch}\bar{\omega} \times \bar{r} \tag{7.40}$$

Using cylindrical coordinates the position vector is $\bar{r} = \rho\hat{\rho} + z\hat{z}$ and the current density becomes $\bar{J}(r) = \rho_{ch}\omega\rho\hat{\phi}$. The integrand of (7.39) may then be written

$$\bar{r} \times \bar{J} = \rho_{ch}\rho^2\omega\hat{z} - \rho_{ch}\omega\rho z\hat{\rho} \tag{7.41}$$

which is introduced in (7.39)

$$\bar{m} = \frac{1}{2} \int_0^a \int_{-l/2}^{l/2} \int_0^{2\pi} \rho_{ch}\rho^2\omega\rho d\phi dz d\rho\hat{z} - \rho_{ch}\omega\rho z\rho d\phi dz d\rho\hat{\rho} \tag{7.42}$$

The ρ component vanishes after the integration over ϕ and the dipole moment becomes

$$\bar{m} = \frac{1}{4}\pi\rho_{ch}\omega l a^4\hat{z} \tag{7.43}$$

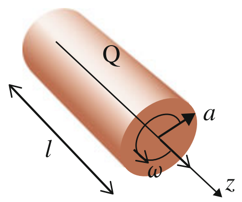


Fig. 7.10 A charged cylinder rotates about its axis of symmetry (the z axis)

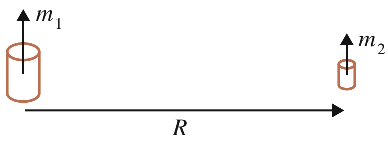


Fig. 7.11 Interaction between charged rotating cylinders

Applying the dipole approximation, valid at large distance, the energy is given by formula (7.25).

Consider for example two rotating charged parallel cylinders at a large distance R from each other, oriented as in Fig. 7.11.

The dipole moments are parallel and perpendicular to the distance R. The magnetic energy becomes

$$U = -\frac{\mu_0}{4\pi R^3} m_1 m_2 \tag{7.44}$$

For equal rotation directions the energy is negative. If the rotational speed is maintained by an external source, i.e. a constant current is provided, a repulsive magnetic force results.

In this case, there is an electric force as well, also repelling for like charges.

7.3 Summary

The electric dipole moment for an ideal dipole is defined as

$$\bar{p} = q\bar{l} \tag{7.1}$$

The general definition for an electric dipole moment is

$$\bar{p} = \int_V \rho(\bar{r})\bar{r}dV \tag{7.23}$$

An electric dipole interacting with a system associated with a potential Φ gives rise to the electric energy

$$U = \bar{p} \cdot \nabla\Phi \tag{7.2}$$

Two interacting electric dipoles give rise to the electric energy

$$U = \frac{1}{4\pi\varepsilon_0 R^3}[\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})] \tag{7.16}$$

The magnetic dipole may approximately be modelled by a current loop with the dipole moment

$$\bar{m} = I \int_S d\bar{a} \tag{7.31}$$

The generalized magnetic dipole moment is

$$\bar{m} = \frac{1}{2} \int_V (\bar{r} \times \bar{J}) dV \tag{7.39}$$

Two interacting magnetic dipoles give rise to the magnetic energy

$$U = -\frac{\mu_0}{4\pi R^3} [\bar{m}_1 \cdot \bar{m}_2 - 3(\bar{m}_1 \cdot \hat{R})(\bar{m}_2 \cdot \hat{R})] \tag{7.25}$$

The interaction energy between a magnetic dipole and a large plate with current density K and surface normal $\hat{n}$ is

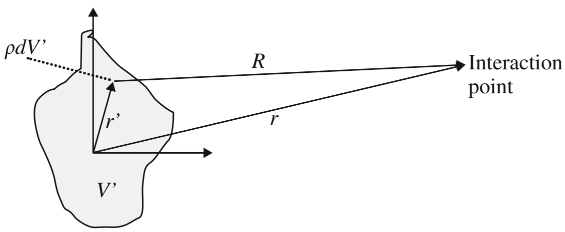
$$U = \frac{\mu_0}{2} \bar{m} \cdot (\bar{K} \times \hat{n}) \tag{7.33}$$

7.4 Exercises

*7.1 *Multipole expansion*

As was discussed in Sect. 7.1.4, an arbitrary charge distribution may be expressed as a series of terms corresponding to different orders of multipoles, starting with the monopole and the dipole.

Consider the charge distribution in the figure. For an interaction at long distances, i.e. $r \gg r'$, it is worthwhile to series expand the potential (also known as Taylor or MacLaurin expansion).



Simplify the notation by putting $1/(4\pi\varepsilon_0) = 1$.

a. Show that the electric potential may be expressed as

$$\Phi(r) = \int\limits_{V'} \frac{\rho(r')dV'}{r(1+\frac{r'^2-2\bar{r}\cdot\bar{r}'}{r^2})^{1/2}}$$

and can be series expanded

$$\Phi(r) = \int\limits_{V'} \frac{\rho(r')dV'}{r}\left(1+\frac{\bar{r}\cdot\bar{r}'}{r^2}-\frac{1}{2}\frac{r'^2}{r^2}+\frac{3}{8}\frac{(r'^4-4r'^2\bar{r}\cdot\bar{r}'+4(\bar{r}\cdot\bar{r}')^2}{r^4}+\cdots\right)$$

b. Keep terms up to order $1/r^3$ and show that

$$\Phi(r) = \frac{1}{r}\int\limits_{V'} \rho(\bar{r}')dV'+\frac{\bar{r}}{r^3}\cdot\int\limits_{V'} \bar{r}'\rho(\bar{r}')dV'+\frac{1}{2}\int\limits_{V'}\left(\frac{3(\bar{r}\cdot\bar{r}')^2}{r^5}-\frac{r'^2}{r^3}\right)\rho(\bar{r}')dV'$$

c. The first term is the potential from a monopole, the second from a dipole (see formulas (7.11) and (7.23)) and the third from a quadrupole.

Show that the quadrupole term Q may be written in coordinate form

$$Q = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{x_i x_j}{r^5}\int\limits_{V'}(3x_i^{'}x_j^{'}-\delta_{ij}r'^2)\rho(\bar{r}')dV'$$

where the integral is known as a quadrupole moment

$$Q_{ij}^{'} = \int\limits_{V'}(3x_i^{'}x_j^{'}-\delta_{ij}r'^2)\rho(\bar{r}')dV'$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

x_1, x_2, x_3 corresponds to x, y, z.

d. How many elements of $Q_{ij}^{'}$ are independent?

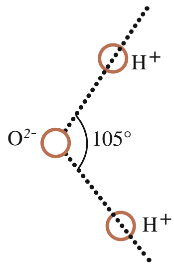
7.2 a. Determine the general formula for the torque acting on the dipole in Fig. 7.3.
b. Verify formulas (7.9) and (7.10).
c. *Project task*: Write a computer program that animates the interaction between a point charge and a dipole.

Take note of the following:

Consider both translational force and torque. The forces are mutual according to Newton's third law. In an animation, the forces should be calculated in time steps. For each step the positions of both objects are altered which have to be calculated. At these actual positions, a new force and torque calculation has to be done.

Use for example Matlab or Matematica.

7.3 A water molecule positioned at the coordinate origin with dipole moment along the z axis interacts with a Sodium ion placed on the x axis at the coordinate $x = 1.0\,\mu\text{m}$. The electric dipole moment of a water molecule is $6.2 \times 10^{-30}\,\text{Cm}$.



a. Determine force and torque on the water molecule.
b. The structure of a water molecule is shown in the figure. Knowing its dipole moment, given above, determine the distance between the hydrogen atoms.

7.4 Show that for both electrically neutral and magnetic systems, the dipole moment is independent of the location of the coordinate system's origin.

Hint: Use formulas (7.23) and (7.39) and make the coordinate transformation $\bar{r} \rightarrow \bar{r} + \bar{a}$.

7.5 Consider a charge distribution where two electrons are located at the two lower corners of a square and two protons in the upper corners. The square has a side length h.

Determine the electric dipole moment of the system using a coordinate system with origin

a. at the lower left corner of the square.
b. at the centre of the square.
   Is the result in accordance with the theorem obtained in Exercise (7.4)?
c. Determine the quadrupole moment of the system for the two different coordinate systems as above.

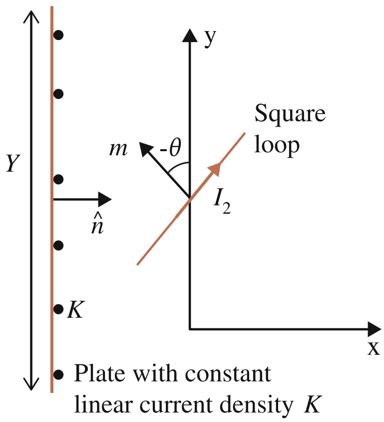
7.6 A sphere of radius a has a homogeneous positive charge q on its upper half and a homogeneous negative charge on its lower half. Total charge is zero.

Determine its electric dipole moment.

7.7 Apply formula (7.22) for an idealized dipole and compare with formula (7.1).

7.8 Determine the magnetic dipole moment for a rectangular loop with sides a and b carrying current I.

7.9 Determine torque and force on a square loop with sides b and current I_2 interacting with a large current-carrying plate with surface current density $K\hat{z}$. The loop is oriented such that its surface normal is in the xy plane, see figure.



7.10 *Motor and generator*

Suggest a construction for a

a. motor based on the principles in Exercise (7.9).
b. generator based on the induction principle discussed in Sect. 7.2.2.

7.11 Consider Exercise (7.9)

a. Is there any difference in induced voltage between a motion such that the dipole moment of the loop rotates around the x axis and that around the z axis (as in Exercise 7.9)?
b. What is the induced voltage for a rotation around the y axis?

*7.12 Explain why an ideal coil (a solenoid) is equivalent to two large parallel plates with opposite homogeneous currents.

Hint: Compare formula (4.89) in Exercise (4.13) with formula (7.33).

*7.13 In this task mass is denoted by μ and angular momentum by L (not to be confused with self inductance).

Consider a discrete system consisting of charges in motion.

a. Show that the magnetic moment can be written

$$\bar{m} = \frac{1}{2} \sum_i \bar{r}_i \times q_i \bar{v}_i$$

where r is the position vector, q is charge and v is velocity.

b. Introduce angular momentum L and show that the magnetic moment may be expressed as

$$\bar{m} = \sum_i \frac{q_i}{2\mu_i} \bar{L}_i$$

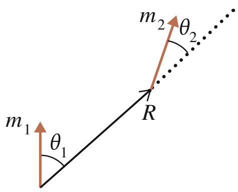
where μ_i is the mass of charge i.

C* 7.14 Two spherical charge distributions with charge densities ρ_1 and ρ_2 and radii a_1 and a_2 respectively are located on the x axis at a distance d from each other. Both rotate with constant angular velocities ω_1 and ω_2 in the z direction.

a. Determine the magnetic force in the dipole approximation.
b. Determine the total force.
c. Next let them rotate in opposite directions and determine for which angular velocity the interaction vanishes. In this case, let the spheres have the same radii, angular speeds and charge densities.
d. For the same conditions as in task c, let the objects move in parallel with velocity $\bar{v} = v\hat{z}$. Determine the resulting force and state the relation between angular and translational velocity for which the interaction vanishes.

e. *Additional task:* Discuss qualitatively the possibility of neutralizing gravitation and thereby facilitating space travel, a subject that is sometimes encountered in science fiction.

*7.15 A magnetic dipole m_1 is fixed at the origin and directed in the positive z direction. Another free dipole m_2 is placed somewhere in the xz plane.



a. Let the distance R be fixed and show that the relation between the angles θ_1 and θ_2, defined in the figure, is

$$\tan\theta_2 = -\frac{1}{2}\tan\theta_1 \qquad (7.45)$$

b. Draw a graph showing how m_2 orients over the whole xz plane.
c. Draw a corresponding graph for the electric case.
d. Suggest how these phenomena may be investigated experimentally. Compare Fig. 10.1.

C* 7.16 *Force and torque between interacting dipoles*

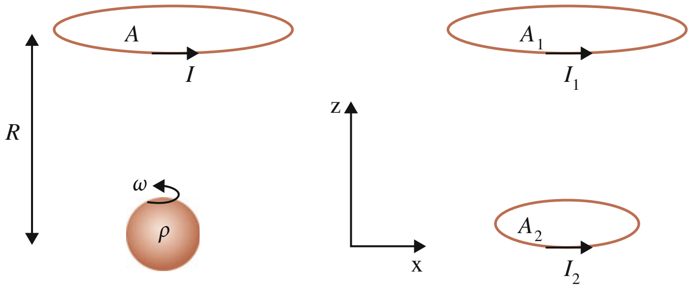
a. Derive general formulas for force and torque between two dipoles for both electric and magnetic interactions.
b. Consider two water molecules, each with dipole moment 6.2×10^{-30} Cm. Let one of them be located at the coordinate origin with its dipole moment along the z axis. The other one is placed on the x axis at the coordinate $x = 1.0\,\mu$m with its dipole moment forming an angle of 45° to the z axis.

Determine torque and force on the molecules.

7.17 Consider the two interactions in the figure. The upper object is a fixed current loop with surface and current given in the figure.

a. Let the loop interact with a constantly rotating sphere as in the left figure. The sphere has radius a, charge density ρ and angular velocity $\bar{\omega} = \omega\hat{z}$. In the dipole approximation, determine the magnetic energy for parallel and anti-parallel dipole moments. Which state is a stable equilibrium?
b. Let the loop interact with a constant current loop as in the right figure.

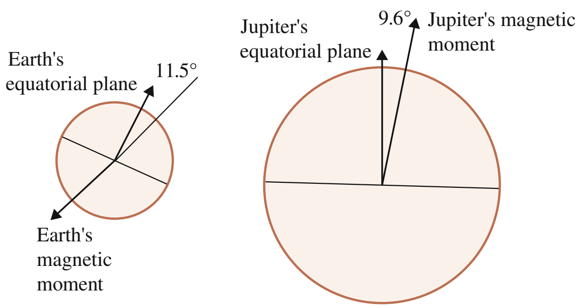
Determine the magnetic energy for parallel, perpendicular and anti-parallel dipole moments.



Task a corresponds to the classical, i.e. non-quantum mechanical, model of electron and nuclear spin resonance. Task b corresponds to the classical model of the Zeeman effect, in which the lower loop mimics an electron orbital.

7.18 *Geomagnetism*

Certain planets in the solar system possess magnetic properties which may be approximated as dipoles (Table 7.1).



a. Using the table and the figure, determine the magnetic force between the Earth and Jupiter, see Exercise (7.16). Compare with the gravitational force.
   Let the Sun, the Earth and Jupiter be located on a common axis and use the average distance as the actual distance.
b. How might geomagnetism arise?

Table 7.1 Data for Excercise (7.18)

Planet	Average distance to Sun (m)	Tilt of rotational plane	Mass (kg)	Magnetic dipole moment (Am^2)	Angle between magnetic dipole and equator plane
Earth	144×10^9	23.5°	5.97×10^{24}	8.8×10^{22}	NS 11.5°
Jupiter	749×10^9	3.1°	1.90×10^{27}	1.5×10^{27}	SN 9.6°

c. The planet Venus does not possess magnetic properties. What could be the reason for this?
d. It has been established that migrating animals in air and at sea utilize geomagnetism as a navigation aid. In several different species it has been discovered that the brain contains specialized cells filled with the permanently magnetic material magnetite. The magnetic dipole moment for such a cell in a salmon trout has been measured to be approximately $10^{-13}\,\mathrm{Am}^2$.

Also cranes are believed to have a magnetic navigation ability. Assume its magneto-sensitive cell to have the same dipole moment as the trout. Determine the torque on such a cell if the crane is flying at an altitude of 10 000 m with its distance vector parallel to the dipole moment of the earth and a relative orientation between the dipole moments of 90°.

Hint: Use the result in Exercise (7.16a).

7.19 *The astronomical H line*

In the 1950s, the so-called H line was discovered in radio-astronomical observations, corresponding to microwaves with a wavelength of 21 cm. Its source was identified as a so-called hyperfine splitting of the energy levels of the hydrogen atom. Since the source is hydrogen, the dominating substance of stars, it could be used for a precise measurement of the density of stars and was utilized, among other things, to establish the spiral structure of the Milky Way.

As is known from high-school physics, atoms are described by quantum mechanics. According to the Bohr model of the atom, an interaction takes place by a change in the atom's energy level. Quantum mechanics has revealed that a registration of a wavelength of 21 cm corresponds to an atomic energy change of 9.4×10^{-25} J. What could be its dynamical source in the hydrogen atom?

It was soon established that its source is the interaction due to the spins of the proton and the electron in the hydrogen atom, i.e. a spin-spin interaction. The concept of spin corresponds in the macrocosm to a self-rotation, see Exercise (7.17). Since the proton and the electron are electrically charged, their spins are associated with a magnetic dipole moment. These have been measured by isolating electrons and protons and letting them interact with an external source, such as a current-carrying coil. The measured values are:

$$m_e = 9.27 \times 10^{-24}\,\mathrm{Am}^2$$
$$m_p = 2.86 \times 10^{-26}\,\mathrm{Am}^2$$

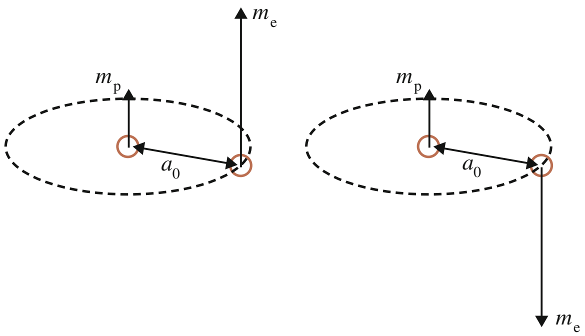


Fig. 7.12 The hydrogen atom is modelled with a proton as the nucleus and an electron orbiting around it. The relative orientations of their magnetic dipole moments can either be parallel (*left*) or anti-parallel (*right*)

From these data, determine the energy change of the hydrogen atom when the relative orientation of the proton and electron spins changes from parallel to anti-parallel. The spins are oriented perpendicular to the distance, which in the Bohr model is $a_0 = 0.5 \times 10^{-10}$ m (Fig. 7.12).

Answer:

From formula (7.25), the energy change is

$$\Delta U_m = 2\frac{\mu_0}{4\pi R^3} m_e m_p = 2.1 \times 10^{-25}\,\mathrm{J}$$

where the factor 2 arises since the two energy levels, corresponding to parallel and anti-parallel spins, differ only by sign.

The obtained value is correct in magnitude, indicating that the dynamics is identified. The deviation is due to so-called quantum mechanical effects. These appear because in microcosm it is not possible to consider the objects as distinct particles; they have an extension in both time and space. This extension is described by a probability function, expressed with a so-called wave function, specifying the functional form of the extension. Practically, it means that the distance R is not unique, but each distance value has to be weighted against a certain probability. In this case, there is a certain probability even for $R = 0$ which will contribute largely to the energy since there is a division by zero in formula (7.25). However, this is done over an infinitesimal interval in space so the final result is finite. This effect is called Fermi's contact term, named after the discoverer and gives rise to the dominating part of the energy difference.

This example illustrates the importance of quantum mechanics and could be a good starting point for its study. Also note that this example connects microcosm to astronomy, the smallest to the largest.

Further Readings

W.D. Parkinson, *Introduction to Geomagnetism* (Scottish Academic Press, Edinburgh, 1983)
J.R. Reitz, F.J. Milford, R.W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, 1993)

Original Papers

H.I. Ewen, E.M. Purcell, Radiation from galactic hydrogen at 1,420 Mc/sec. Nature **168**, 356 (1951)
D. Gubbinsand, T.G. Masters, Driving mechanism for the Earth's dynamo. Adv. Geophys. **21**, 1 (1979)

Chapter 8
Material Properties

One of the most immediate consequences of the electrochemical theory is the necessity of regarding all chemical compounds as binary substances. It is necessary to discover in each of them the positive and negative constituents… No view was ever more fitted to retard the progress of organic chemistry. Where the theory of substitution and the theory of types assume similar molecules, in which some of the elements can be replaced by others without the edifice becoming modified either in form or outward behaviour, the electrochemical theory divides these same molecules, simply and solely, it may be said, in order to find in them two opposite groups, which it then supposes to be combined with each other in virtue of their mutual electrical activity… I have tried to show that in organic chemistry there exist types which are capable, without destruction, of undergoing the most singular transformations according to the nature of the elements.

Jean-Baptiste-André Dumas, 1828

Electric and magnetic material properties are explored via the material's response from an external electric and magnetic influence respectively. The strength and direction of the response may be derived from the inner structure of the material.

In an approximate model, the material is considered to consist of electric dipoles (except metals) and magnetic dipoles. By measuring the responding force, information on the dipole structure of the material is therefore obtained, within this model.

In this chapter, principal experiments are studied and a theoretical description of the dipole model is developed. The objective is to establish a connection between macroscopic (measurable) quantities and the basal microscopic dynamics in the material.

8.1 Electric Response Forces

A material surrounded by air (vacuum) always responds attractively to an external electric influence. The reason is that the external charge attracts opposite charges in the material which therefore are placed closer to the external agent, Fig. 8.1.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_8

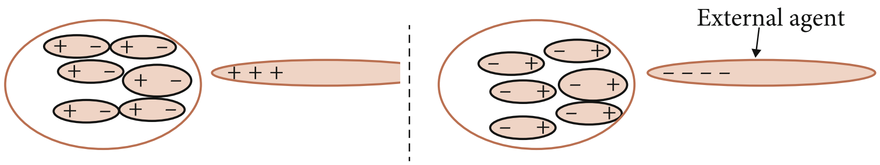
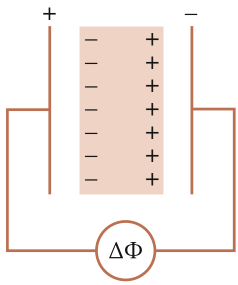


Fig. 8.1 An electric influence on a material in vacuum induces and attracts dipoles resulting in an attractive force

Fig. 8.2 A material is introduced into a charged capacitor



To explore the electric material response, an equivalent method as in Fig. 5.1 may be used, Fig. 8.2. When a material is introduced into the capacitor, it is observed that the voltage between the plates decreases. A non-conductive material, a so-called isolator or dielectric material, affects the voltage much less than a metal. While metals were discussed in Chap. 5, this chapter will focus on the isolator.

According to the discussion in Sect. 5.1, the decrease in voltage must be due to a charge displacement as illustrated in Fig. 8.2. The electric response is quantified by introducing a parameter called the dielectric constant κ_e.[1] If the voltage without an inserted material is denoted $\Delta\Phi_1$ and the voltage with a material that *fully* fills the space between the plates is denoted $\Delta\Phi_2$, the definition is

$$\kappa_e = \frac{\Delta\Phi_1}{\Delta\Phi_2} \tag{8.1}$$

assuming the material is homogeneous.

Since the voltage always decreases when a material is inserted, $\kappa_e > 1$. For an ideal metal κ_e becomes infinite.

The dielectric constant may also be expressed in terms of capacitance, see Sect. 4.1.5. Since the charge Q on the plates is unchanged

$$\kappa_e = \frac{\Delta\Phi_1}{\Delta\Phi_2} = \frac{Q/C_1}{Q/C_2} = \frac{C_2}{C_1} \Rightarrow C_2 = \kappa_e C_1 \tag{8.2}$$

---

[1] The word 'dielectric' refers to the dipole character of an isolator. Older name for dielectric constant is electric relative permittivity with notation ϵ_r.

so that the capacitance for a material filled capacitor increases a factor κ_e compared to an empty capacitor.

8.1.1 Electric Force Between a Charged Capacitor and a Material

The electric response forces are usually very small and difficult to observe. Indeed, in the last section the measurement required a refined meter. An experiment will now be described where this force becomes observable and then the active force is calculated. Figure 8.3 shows the principal process.

An isolator is placed outside a parallel plate capacitor connected to a battery. It is observed that the material is drawn into the volume between the plates all the way to the right edge where the force vanishes. Changing the polarity of the capacitor gives the same result.

Since the force is extremely small the friction has to be minimized for the material to move. This may be put into practice by placing the material on an air flow track (see movie on book's website).

Alternatively, if the apparatus is turned so that the motion of the material occurs vertically, the force may be measured by a balance working at a precision of one milligram. To obtain a measurable force, a high voltage up to 6 kV has to be used. To minimize the influence of the voltage, the plates have to be placed far away from the balance, Fig. 8.4.

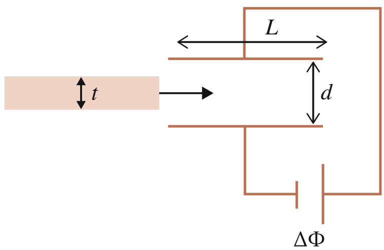
The *average* force on the material will now be calculated using the energy method. The energy stored by the capacitor is $1/2\, C\Delta\Phi^2$, which increases during the process. Including the battery energy, the total energy will decrease.

Using formula (1.1), the average force is

$$\langle F \rangle = -\frac{\Delta U_{TOT}}{L} \tag{8.3}$$

where ΔU_{TOT} is the difference in total energy with and without material between the plates, which becomes

Fig. 8.3 An electrically neutral material interacts with a charged capacitor. An attractive force arises



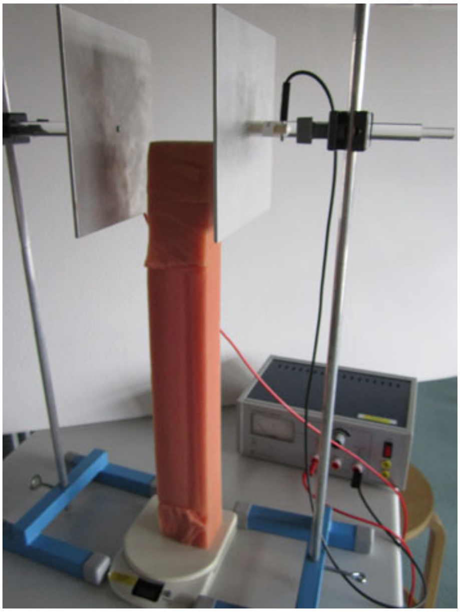


Fig. 8.4 The material is placed on a balance at the *lower* part of the picture. At the *upper* part, the *parallel plates* are seen which are connected to a high voltage source, located in the *rear*

$$\Delta U_{TOT} = U_D - U_0 + \Delta U_B \tag{8.4}$$

where U_D and U_0 are the capacitor energies with and without material respectively. ΔU_B is the change of the battery energy and

$$U_D - U_0 = \frac{1}{2}\Delta C \Delta\Phi^2 \tag{8.5}$$

The task of the battery is to maintain the voltage between the plates. In this way, work is performed by increasing the charge on the plates so that

$$\Delta U_B = \Delta Q \Delta\Phi = -\Delta C \Delta\Phi^2 = -2(U_D - U_0) \tag{8.6}$$

The minus sign in the second equivalence ensures that the stored battery energy decreases while it is doing work. The total energy change becomes

$$\Delta U_{TOT} = -(U_D - U_0) \tag{8.7}$$

which is negative as expected. Using (8.5)

$$\Delta U_{TOT} = -\frac{1}{2}\Delta\Phi^2(C_D - C_0) \tag{8.8}$$

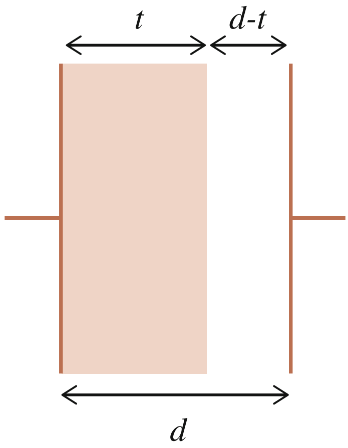


Fig. 8.5 Capacitor filled partly with a material

The task is then reduced to determining the capacitance of a partly material filled capacitor, Fig. 8.5. Assuming an ideal capacitor, it is straight-forward to show that the capacitance is independent of where in the capacitor the material is placed, see Exercise (8.2). Letting the material be located at the edge of one of the plates, the capacitor is equivalent to a series coupling of two capacitors. One of them is fully filled with a material with thickness t, denoted C_1. The other is empty with thickness $d - t$, denoted C_2.

Using formula (4.24) for the capacitance of an ideal plate capacitor, the total capacitance is (see Exercise 5.5)

$$C_D = \frac{C_1 C_2}{C_1 + C_2} = \varepsilon_0 A \left(\frac{\kappa_e/t[1/(d-t)]}{\kappa_e/t + 1/(d-t)} \right) = \varepsilon_0 A \frac{\kappa_e}{\kappa_e(d-t) + t} \tag{8.9}$$

The energy change may then be written

$$\Delta U_{TOT} = -\frac{1}{2} \Delta\Phi^2 \varepsilon_0 A \left(\frac{\kappa_e}{\kappa_e(d-t) + t} - \frac{1}{d} \right) \tag{8.10}$$

The average force on the material becomes

$$\langle F \rangle = -\frac{\Delta U_{TOT}}{L} = \frac{1}{2} \Delta\Phi^2 \varepsilon_0 \frac{A}{L} \left(\frac{\kappa_e}{\kappa_e(d-t) + t} - \frac{1}{d} \right) \tag{8.11}$$

It is worthwhile to reflect upon how this force arises. Consider Fig. 8.3. Since the ideal approximation has been used only vertical forces on the material are expected. What is the origin of the horizontal force? It is in fact so-called edge effects that are active in this case. These were neglected in the calculation and appear because of the finite size of the plates, see Exercise (5.15). The material will be polarised in such a way that its negative charges are turned towards the positive plate and vice versa.

Thus a net horizontal force appears, see Exercise (8.3). Neglecting these effects in the calculation is justified since the *average* force was calculated. The edge effects are approximately equal before and after the process and almost cancel out in the energy difference, formula (8.7).

8.1.2 The Dielectric Constant—Not a Constant

Measurements of the dielectric constant κ_e show that it depends on:

- Material
- Temperature
- The strength of the external influence
- The variation frequency of the external influence.

Within the dipole model, the observations are quite comprehensible. Knowing that the electric dipoles originate in the material's molecules, the external influence acts in the following way, Fig. 8.6:

- It induces a dipole moment by displacing the charge distribution within the molecule
- It creates an alignment of the dipoles through a torque according to formula (7.5).

Referring to Fig. 8.6, it may be concluded that a higher degree of alignment of the dipoles results in a lower voltage between the plates and accordingly a larger dielectric constant. The degree of alignment varies with the factors above since:

- Higher temperature results in more random dipole motion.
- Stronger external influence results in a higher degree of alignment.
- Higher frequency results in a lower degree of alignment since it takes time for the dipoles to align.

Table 8.1 shows the dielectric constant for some materials under different conditions.

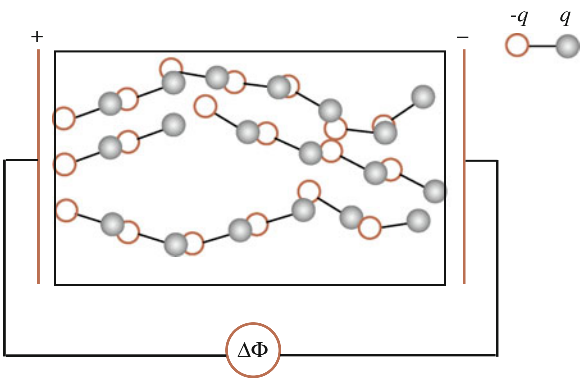


Fig. 8.6 A possible orientation of a material's dipoles under influence of charged *parallel plates*

Table 8.1 Values of the dielectric constant

Substance	Temp (°C)	Pressure (MPa)	κ_e
Water	25	10	78.9
Water	200	10	35.1
Water	550	10	1.1
Ethanol	−60		41.0
Ethanol	25		24.3

	Frequency (Hz)	10^3	10^6	10^8
Substance	Temp (°C)	κ_e	κ_e	κ_e
Plastic/Nylon	25	3.5	3.1	3.0
	84	11.2	4.4	3.4
Vinylite	24	5.6	3.3	2.8
	79	8.1	5.5	3.4

Note the different temperature dependence between liquids and solids

8.1.3 Bound Charges

The measurement of the dielectric constant κ_e provides information about the dipole dynamics within the material. It is in this way a link to the microcosm. To establish the connection between macroscopic (observable) and microscopic (model dependent) quantities, the vector polarisation $\bar{P}(\bar{r})$ is introduced, corresponding to dipole moment per unit volume. The total dipole moment in the volume V is therefore

$$\bar{p} = \int_V \bar{P}(\bar{r}) dV \tag{8.12}$$

The purpose of this section is to establish a relation between the polarisation and the external influence.

Polarisation P may be related to the net charge created by the dipoles, so-called bound charge. Consider a material homogeneously polarised between two charged metallic plates, Fig. 8.7. The voltage between the plates, denoted $\Delta\Phi$, is the sum of the voltages generated by the plates $\Delta\Phi_f$ and that generated by the material $\Delta\Phi_b$, so that

$$\Delta\Phi = \Delta\Phi_f + \Delta\Phi_b \tag{8.13}$$

where the index f refers to free charges and b to bound charges. The free charges are in this case the conduction electrons of the metallic plates. The voltage from the bound charges becomes

$$\Delta\Phi_b = \Delta\Phi - \Delta\Phi_f = \frac{\Delta\Phi_f}{\kappa_e} - \Delta\Phi_f \tag{8.14}$$

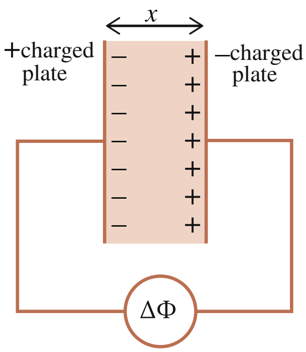


Fig. 8.7 A material filled capacitor

Note that $\Delta\Phi_f$ and $\Delta\Phi_b$ have opposite signs (polarity). This formula may be expressed in terms of surface charge density σ. Using formula (4.23)

$$\frac{\sigma_b}{\varepsilon_0}x = \frac{\sigma_f}{\kappa_e\varepsilon_0}x - \frac{\sigma_f}{\varepsilon_0}x \tag{8.15}$$

where σ_b is the surface charge density for the bound charges and σ_f that for the free charges. Therefore

$$\sigma_b = \sigma_f(\frac{1}{\kappa_e} - 1) \tag{8.16}$$

The next step is to establish the relationship between polarisation P and bound charge density. Assume that x in Fig. 8.7 is the length of a dipole. Denote with A the parallel and equal areas of the plates and the material, the bound total charge q_b and the number of dipoles N to obtain

$$\sigma_b = \frac{q_b}{A} = \frac{q_b x}{V} = \frac{Np}{V} = P \tag{8.17}$$

and

$$\frac{dP}{dx} = \frac{q_b}{V} = \rho_b \tag{8.18}$$

where V is the volume of the material. These two relations are valid in magnitude. Including the vector property of polarisation $\bar{P}$ gives, for each coordinate,

$$\sigma_b = \bar{P} \cdot \hat{n} \quad \rho_b = -\nabla \cdot \bar{P} \tag{8.19}$$

where $\hat{n}$ is the surface normal of the material. The minus sign in front of the divergence of $\bar{P}$ is explained with reference to Fig. 8.9: When the dipole moment density P

increases for increasing z, the negative bound charge will dominate over the positive charge giving a net negative bound charge in the volume.

The relations (8.19) are general and may be used to determine the electric charge distribution of a material when the polarisation is known.

Next the polarisation is related to the external influence. Using formula (7.17) for parallel plates gives $\sigma_f = \left|\varepsilon_0 \nabla \Phi_f\right|$.[2] The bound charge density may then be obtained from formula (8.16) which is introduced in formula (8.19) and the polarisation becomes

$$\bar{P} = \varepsilon_0 \nabla \Phi_f \left(\frac{1}{\kappa_e} - 1 \right) \quad \textit{Special case!} \tag{8.20}$$

where the sign is determined from the fact that polarisation is directed toward decreasing potential. However, formula (8.20) is a special case valid when the external influence is directed parallel to the surface normal of the material $\hat{n}$, i.e. $\nabla \Phi_f \propto \hat{n}$, as in Fig. 8.7. The general case is treated next.

8.1.3.1 Continuity Conditions

The generalization of formula (8.20) may be derived by considering so-called continuity conditions of the potential at the intersection between two materials.

To illustrate basic principles, materials will be considered which are

- Linear—the material response is proportional to the strength of the external influence $\nabla \Phi_f$
- Isotropic—the material response is directed parallel to the external influence (see Exercise 8.12)
- Homogeneous—the dipole structure of the material is independent of position in the material.

Consider a situation as in Fig. 8.8 where the surface normal of the material forms an arbitrary angle to the direction of the external influence. The dipoles of the material align along the direction of the external electric force (effectively) and in this way a row of bound minus charges line up at the surface of the material. Thus, the material response is such that a constant potential appears along its surface while along the surface normal the electric response is such as described previously in formula (8.1).

The following *continuity conditions* are then valid for the gradient of the potential at the intersection between a material and air/vacuum:

$$\hat{n} \cdot \nabla \Phi_1 = \kappa_e \hat{n} \cdot \nabla \Phi_2 \tag{8.21}$$

$$\hat{t} \cdot \nabla \Phi_1 = \hat{t} \cdot \nabla \Phi_2 \tag{8.22}$$

[2] Distinguish between $\Delta\Phi$ and $\nabla\Phi$. For example, in the x direction $\nabla\Phi = \frac{d\Phi}{dx}\hat{x} \approx \frac{\Delta\Phi}{\Delta x}\hat{x}$.

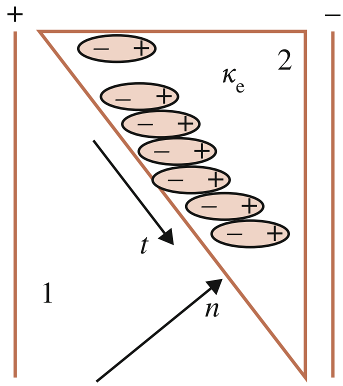


Fig. 8.8 Dipole orientation in material *2* when influenced by charged capacitor

Since formula (8.20) concerns only the normal component it may be written

$$P_n \hat{n} = \varepsilon_0 \left(\frac{1}{\kappa_e} - 1 \right) \nabla \Phi_{fn} = -\varepsilon_0 (\kappa_e - 1) \frac{\nabla \Phi_{fn}}{\kappa_e} = -\varepsilon_0 (\kappa_e - 1) \nabla \Phi_n \quad (8.23a)$$

as the normal component of the gradient of the potential is scaled down a factor κ_e in the material compared to air.

According to formula (8.22), there is no such down-scaling for the tangential component of polarisation, wherefore it may be written

$$P_t \hat{t} = -\varepsilon_0 (\kappa_e - 1) \nabla \Phi_{ft} = -\varepsilon_0 (\kappa_e - 1) \nabla \Phi_t \quad (8.23b)$$

By summing (8.23a) and (8.23b) polarisation becomes

$$\bar{P} = -\varepsilon_0 (\kappa_e - 1) \nabla \Phi \quad (8.24)$$

where all quantities are given in the material. Formula (8.24) is the generalisation of (8.20) and is valid for arbitrary direction of the external influence. It establishes the connection between the microscopic quantity polarisation and the macroscopic quantity potential.

8.1.4 Three Examples of Polarisation

8.1.4.1 Polarised Cylinder

Consider a non-conductive cylindrical homogeneous material polarised by a long negatively charged bar, Fig. 8.9. The cylinder length is L and its plate radius is a. What is its charge distribution?

The potential from the bar is the external influence (see formulas 1.1, 3.5 and 4.6). Introducing a coordinate axis ρ perpendicular to the bar according to Fig. 8.9, the potential and its gradient become

$$\Phi_f = -\frac{\lambda}{2\pi\varepsilon_0} ln\rho$$

$$\nabla\Phi_f = -\frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho} = \frac{\lambda}{2\pi\varepsilon_0(d-z)}\hat{z} \tag{8.25}$$

to the left of the bar. λ is linear charge density, i.e. charge per unit length. Since the electric influence is parallel to the surface normal of the cylinder plate, formula (8.20) may be used and the polarisation becomes

$$\bar{P} = \varepsilon_0\left(\frac{1}{\kappa_e}-1\right)\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{d-z}\hat{z} \tag{8.26}$$

Let the objects be located far away from each other so that

$$\bar{P} \approx \varepsilon_0\left(\frac{1}{\kappa_e}-1\right)\frac{\lambda}{2\pi\varepsilon_0 d}\left(1+\frac{z}{d}\right)\hat{z} \tag{8.27}$$

For negative charge densities λ, the polarisation, i.e. the density of dipole moments, increases with z. Figure 8.9 shows a possible configuration.

Since the dipoles turn their positive side towards the bar, a negative charge is expected on the left plate, a negative net charge in the volume and a positive charge on the right plate. The total amount of bound charge ought to vanish. These expectations may be verified using the formulas above:

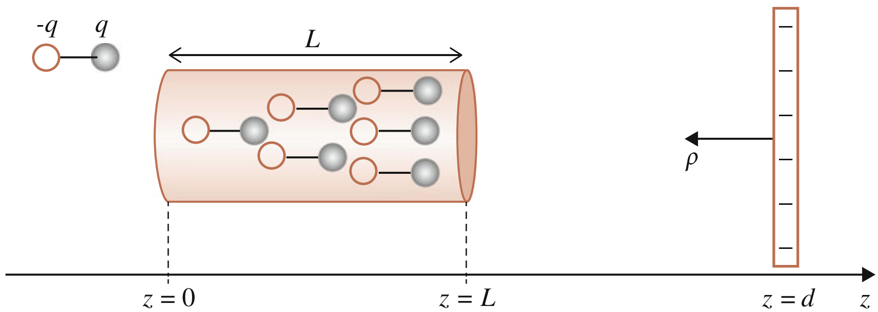


Fig. 8.9 A charged bar at $z = d$ interacts with an uncharged non-conductive *cylinder*. The figure shows qualitatively a possible dipole configuration for a negatively charged bar according to formula (8.27)

The surface charge density is $\sigma_b = \bar{P} \cdot \hat{n}$. For the left plate, $z = 0$ and $\hat{n} = -\hat{z}$:

$$\sigma_b = -\varepsilon_0 \left(\frac{1}{\kappa_e} - 1\right) \frac{\lambda}{2\pi\varepsilon_0 d}$$

which is a negative quantity for negative λ.

For the right plate, $z = L$ and $\hat{n} = \hat{z}$:

$$\sigma_b = \varepsilon_0 \left(\frac{1}{\kappa_e} - 1\right) \frac{\lambda}{2\pi\varepsilon_0 d} \left(1 + \frac{L}{d}\right)$$

which is positive.

Inside the cylinder the volume charge density becomes

$$\rho_b = -\nabla \cdot \bar{P} = -\frac{dP}{dz} = -\varepsilon_0 \left(\frac{1}{\kappa_e} - 1\right) \frac{\lambda}{2\pi\varepsilon_0 d^2}$$

i.e. a homogeneous bound negative charge distribution.

The total charge Q of the cylinder is

$$Q = \rho_b \pi a^2 L + \sigma_b(z = 0)\pi a^2 + \sigma_b(z = L)\pi a^2 = 0 \tag{8.28}$$

as expected.

The calculation of the force is left to Exercise (8.4).

8.1.4.2 A Polarised Sphere in Vacuum

To further illustrate the importance of the material shape when dealing with polarisation the following case is investigated:

A linear, isotropic and homogeneous non-conductive sphere with radius a is placed between two charged plates. To find its polarisation, formula (8.24) is used. As concluded from formula (8.13), the gradient of the potential in the material may be written

$$\nabla\Phi = \nabla\Phi_f + \nabla\Phi_b \tag{8.29}$$

To determine Φ_b consider Fig. 8.10 which shows that its source is the bound charges at the surface of the sphere. Inside the volume, the bound charges cancel out. Using formula (8.19), the surface charges may be expressed in terms of polarisation P:

$$\sigma_b = \bar{P} \cdot \hat{n} = P \cos\theta \tag{8.30}$$

The potential is calculated using formula (3.3). Since the polarisation is homogeneous, the calculation needs to be done only on the x axis. Placing the origin at the

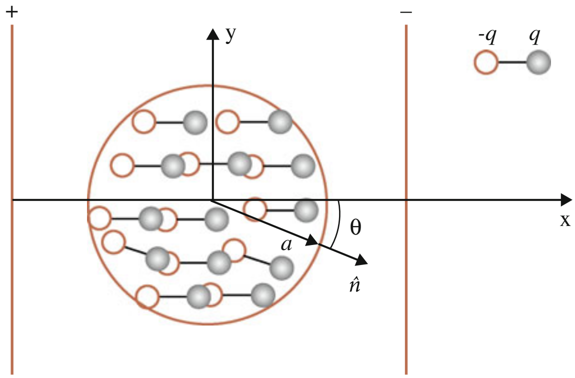


Fig. 8.10 A homogeneous non-conductive *sphere* is placed between two charged plates

centre of the sphere, the potential inside the sphere becomes

$$\Phi_b = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{P\cos\theta a^2\sin\theta d\theta d\phi}{(x^2+a^2-2xa\cos\theta)^{1/2}}$$
$$= \frac{Pa^2}{2\varepsilon_0}\int_{-1}^{1}\frac{tdt}{(x^2+a^2-2xat)^{1/2}} = \frac{Px}{3\varepsilon_0} \quad x \leq a \tag{8.31}$$

where $t = \cos\theta$. Using (8.24), the following relation is obtained

$$\nabla\Phi = \nabla\Phi_f + \frac{\bar{P}}{3\varepsilon_0} = -\frac{\bar{P}}{\varepsilon_0(\kappa_e - 1)} \tag{8.32}$$

so that the polarisation becomes

$$\bar{P} = -3\varepsilon_0\frac{\kappa_e - 1}{\kappa_e + 2}\nabla\Phi_f \tag{8.33}$$

8.1.4.3 Polarisation of a Large Plate with an Arbitrary Surrounding Medium

Previously the surrounding medium has implicitly been vacuum. Consider now the case when the surrounding is an arbitrary substance, e.g. water. Let the object be a plate parallel to the external influence consisting of charged parallel plates, as in Fig. 8.2. Formula (8.20) becomes

$$\bar{P} = \varepsilon_0\nabla\Phi_p - \varepsilon_0\nabla\Phi_w \tag{8.34}$$

where the index p refers to ‘plate’ or ‘particle’ and w stands for ‘water’ (a particle in water is a common scenario in applications). Since the external influence is parallel to the surface normal in this specific case, the gradient of the potential in the object is

$$\nabla\Phi_p = \frac{\nabla\Phi_f}{\kappa_p} \; and \; \nabla\Phi_w = \frac{\nabla\Phi_f}{\kappa_w} \tag{8.35}$$

such that

$$\nabla\Phi_p = \frac{\kappa_w}{\kappa_p}\nabla\Phi_w \tag{8.36}$$

and formula (8.34) becomes

$$\bar{P} = \varepsilon_0 \nabla\Phi_w \left(\frac{\kappa_w}{\kappa_p} - 1\right) \tag{8.37}$$

Note that the sign of the polarisation depends on whether the dielectric constant of the surroundings is greater than or less than that of the object. This phenomenon was first discussed by Mossotti (1850) and Clausius (1879). In Exercise (8.18), the object is a particle in the form of a sphere and thereby the Clausius-Mossotti polarisation formula is obtained. This forms the basis for e.g. the application dielectrophoresis, a popular method for separating and identifying cells and molecules in biological contexts, such as in DNA analysis.

Another practical example is the electric fish’s polarisation of its surroundings, a process equivalent to dielectrophoresis, see Exercise (8.20).

8.2 Magnetic Response Forces

Unlike the electric case in air/vacuum, a material’s response to a magnetic influence may be either attractive or repulsive. Materials are classified into four categories w.r.t. the response: weak and strong attractive and weak and strong repulsive. The latter case appears for metals at low temperatures when the material becomes so-called superconductive.

Let the magnetic influence be a current carrying coil, Fig. 8.11.

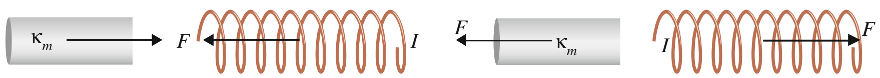


Fig. 8.11 A current-carrying coil and a material interact. Para- and ferromagnetic materials interact attractively (*left*). Diamagnetic materials interact repulsively (*right*)

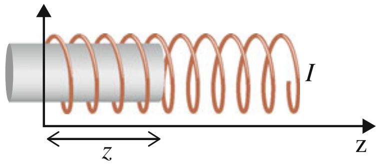


Fig. 8.12 A coil filled partly with a material

To describe the magnetic properties of the material, a parameter called the relative magnetic permeability κ_m is introduced.[3] This is defined as the factor by which the stored magnetic energy in a coil, or generally in an inductor, changes when a material is introduced into the coil and fills it completely, provided the current of the coil is maintained. This definition is equivalent to the electric case for constant voltage which follows from formulas (8.2) and (4.30), see Exercise (8.1).

Consider the situations in Fig. 8.11. The current in the coil is maintained by a battery and the magnetic force acts to increase magnetic energy, compare Sect. 3.3.1. This results in an attractive force for para- and ferromagnetic materials and a repulsive force for diamagnetic materials.

For an inductor fully filled with material with relative magnetic permeability κ_m, the magnetic energy is

$$U_m = \kappa_m U_0 = \frac{1}{2}\kappa_m L_0 I^2 \tag{8.38}$$

where L is the self inductance of the inductor and the index 0 denotes absence of material, i.e. air/vacuum. For a coil fully filled with a material, the self inductance in lowest order becomes, using formula (4.80)

$$L = \frac{\kappa_m \mu_0 N^2 S}{d} \tag{8.39}$$

where N is the number of turns, S the cross sectional area of the coil and d its length. Hence, the inductance changes a factor equal to the relative magnetic permeability.

To measure the permeability, a force formula is needed. Assume the coil is filled by the material up to a length z, Fig. 8.12. The magnetic energy for this system becomes in lowest order

$$U_m = \frac{1}{2}I^2[\kappa_m \mu_0 n^2 S z + \mu_0 n^2 S(d - z)] \tag{8.40}$$

where $n = N/d$ is the density of turns, i.e. number of turns per unit length. For a constant current, formula (3.23) gives the force

$$\bar{F}_m = \nabla U_m = \frac{dU_m}{dz}\hat{z} = \frac{1}{2}I^2\mu_0 n^2 S(\kappa_m - 1)\hat{z} \tag{8.41}$$

[3] Older notation is μ_r.

Table 8.2 Relative magnetic permeability

Substance	Temp (°C)	κ_m
Copper	296	0.9999945
Copper	20	0.9999900
Bismuth	atp	0.9997199
Carbon	atp	0.9999941
Aluminium	atp	1.0000165
Calcium	atp	1.0000400
Water	atp	0.9999870
Oxygen	−182.9	7700.0

which is the force on the *material* since z is the position of its edge. For $\kappa_m > 1$ the force becomes positive and for $\kappa_m < 1$ negative, corresponding to attractive and repulsive force respectively.

If the cross sectional area of the material, S_m, is less than that of the coil, the force becomes

$$\bar{F}_m = \nabla U_m = \frac{dU_m}{dz}\hat{z} = \frac{1}{2} I^2 \mu_0 n^2 S_m (\kappa_m - 1)\hat{z} \tag{8.42}$$

Table 8.2 shows measured values of the relative magnetic permeability.

8.2.1 Magnetization Currents

Equivalent to the electric case, magnetic material properties may be described approximately in terms of dipoles. Thus, under an external magnetic influence magnetic dipoles are induced and aligned in the material. In this section, these dipoles will be expressed in terms of electric current and in the next section the relation between the external influence and the generated magnetization is derived.

To this end, the vector magnetization $\bar{M}(\bar{r})$ (not to be confused with inductance M_{ij}) is introduced as the density of magnetic dipole moment $\bar{m}$:

$$\bar{m} = \int_V \bar{M}(\bar{r}) dV \tag{8.43}$$

where $\bar{r}$ is the position vector. Using formula (7.39), the left-hand side of (8.43) may be replaced with

$$\frac{1}{2}\int_V (\bar{r} \times \bar{J}) dV = \int_V \bar{M}(\bar{r}) dV \tag{8.44}$$

Since the volume V is arbitrary the integrands have to be equal

$$\frac{1}{2}(\bar{r} \times \bar{J}) = \bar{M}(\bar{r}) \tag{8.45}$$

By applying the curl operation on both sides, (8.45) may be solved for J, see Exercise (8.8). Doing this component-wise, magnetization becomes

$$\begin{aligned} M_x &= \frac{1}{2}(yJ_z - zJ_y) \\ M_y &= \frac{1}{2}(zJ_x - xJ_z) \\ M_z &= \frac{1}{2}(xJ_y - yJ_x) \end{aligned} \tag{8.46}$$

so that

$$\begin{aligned} (\nabla \times \bar{M})_x &= -J_x - \frac{1}{2}\left(x\frac{dJ_x}{dx} + y\frac{dJ_x}{dy} + z\frac{dJ_x}{dz}\right) \\ (\nabla \times \bar{M})_y &= -J_y - \frac{1}{2}\left(x\frac{dJ_y}{dx} + y\frac{dJ_y}{dy} + z\frac{dJ_y}{dz}\right) \\ (\nabla \times \bar{M})_z &= -J_z - \frac{1}{2}\left(x\frac{dJ_z}{dx} + y\frac{dJ_z}{dy} + z\frac{dJ_z}{dz}\right) \end{aligned} \tag{8.47}$$

where $\nabla \cdot \bar{J} = 0$ has been used. This follows from formula (5.20) since J, being the current density of the magnetization, is closed. Since dipole moment is independent of location in the coordinate system, see Exercise (7.4), its origin may be placed at the centre of the dipole, Fig. 8.13. In this model, the dipoles are assumed to be point-like, i.e. $\bar{r} \rightarrow 0$, so that

$$\bar{J} = -\nabla \times \bar{M} \tag{8.48}$$

However, the direction of magnetization is defined such that

$$\bar{J} = \nabla \times \bar{M} \quad \textit{In a volume!} \tag{8.49}$$

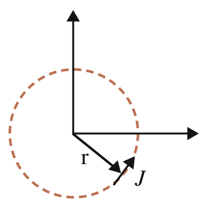


Fig. 8.13 Magnetization current centered at the coordinate origin

meaning that the direction of M is given by the 'right hand rule' as illustrated in the figure of Exercise (4.13). Hence, to define $\bar{J}(\bar{r})$ the magnetization M has to be known in the vicinity of r.

On the surface of a material the concept surface current density K is used, defined in the following way. Figure 8.14 illustrates a bar which is homogeneously magnetised in the negative y direction.

On the surface there is a current I. A current element on the surface in the z direction with the infinitesimal cross sectional area $dxdy$ is defined as (see Exercise 2.24)

$$dId\bar{z} = \bar{J}dxdydz = J_z dxdydz\hat{z} = dK_z dydz\hat{z} \tag{8.50}$$

where K is current per unit length in the y direction. For this vertical current element, the current density becomes

$$J_z = \frac{dM_y}{dx} - \frac{dM_x}{dy} = \frac{dM_y}{dx} \tag{8.51}$$

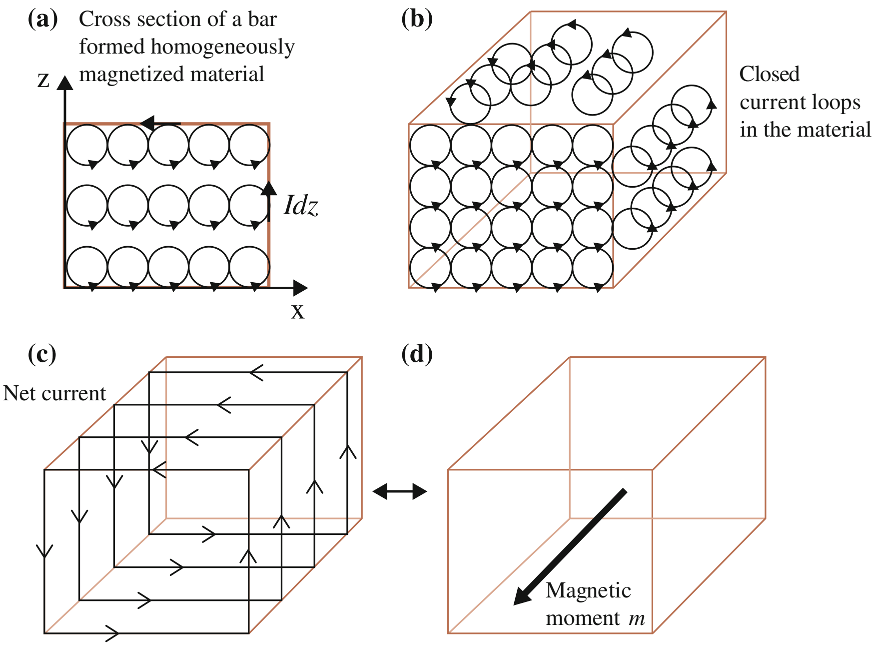


Fig. 8.14 Orientation of magnetic dipoles in a homogeneously magnetized material. The *circles* illustrate individual magnetic dipoles and the *arrows* denote current direction. **a** Cross section of the material. Inside the volume, the currents cancel whereas a net current exists on the surface. **b** Three-dimensional illustration. **c** The net current on the surface is equivalent to that of the current in a coil. **d** The net current corresponds to a magnetic dipole moment in the negative y direction

since $M_x = 0$. The surface density of the current is $dK_z = J_z dx$ so that

$$dK_z = dM_y \Rightarrow K_z = M_y \tag{8.52}$$

up to a constant which must be set to zero to agree with the definition formula (8.45). Since $\bar{K}$ and $\bar{M}$ are perpendicular, it is generally valid that

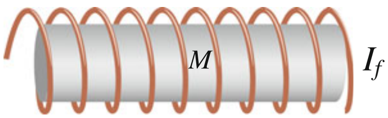
$$\bar{K} = \bar{M} \times \hat{n} \quad \textit{On the surface}! \tag{8.53}$$

where $\hat{n}$ is the surface normal.

8.2.2 Magnetization from a Magnetic Influence

A material is magnetized through an external magnetic influence corresponding to an electric current. Equivalent to the electric case, this current is referred to as *free* while currents arising in the material are referred to as *bound*. The relation between magnetization and the external influence will now be explored in the dipole approximation. While this will be done here within a special case, using the ideal coil as an external influence, a general derivation is presented in Appendix C.

Consider a material completely filling the volume of an ideal coil. The magnetization takes place by increasing the current in the coil from zero to the final value I_f, where f stands for 'free'. The material is successively magnetized during the process until the final magnetization M is obtained. In this case, the magnetization becomes independent of coordinates since the coil affects the material homogeneously.



As is understood from Sect. 8.2.1, net currents I_b (b for bound) will arise only on the surface of a homogeneously magnetized material, Fig. 8.14. The energy of the system is therefore the sum of the coil energy and the interaction energy between the coil current and the surface current of the material. By equating this energy to formula (8.38), i.e. the formula defining κ_m, the sought relation is obtained.

The coil magnetic energy is given by formulas (3.12) and (4.80). Denoting self inductance L the energy becomes

$$U_{coil} = \frac{1}{2}LI_f^2 = \frac{1}{2}\mu_0\frac{N^2}{d}SI_f^2 \tag{8.54}$$

using coil inductance in lowest order. N is the number of turns, d is the coil length and S its cross sectional area. Note that the first equality is obtained by integrating from zero to the final value I_f, thereby the factor $1/2$ in front, see formula (3.12).

The interaction energy U_{int} may also be determined through an integration in the following way. Consider the material as a magnetic dipole with moment $m = i_b S$. In the utilized dipole model, the magnetization vector $\bar{M}$ corresponds to the effective dipole density. The dipole moments of the material are then parallel or anti-parallel to the dipole moment of the external influence. According to formula (4.89, Exercise 4.13) the energy change due to a current change is

$$dU_{int} = i_b S \mu_0 \frac{N}{d} di_f \tag{8.55}$$

which shall now be integrated from zero to the final value I_f. Assuming that i_b increases at the same rate as i_f, so-called linear materials, a function $f(t)$ may be introduced with a range from 0 to 1 such that

$$i_b = f(t) I_b, \quad di_f = I_f df(t) \tag{8.56}$$

where t denotes time. The interaction energy then becomes

$$U_{int} = \int_0^1 S\mu_0 \frac{N}{d} I_b I_f f df = \frac{1}{2} S\mu_0 \frac{N}{d} I_b I_f \tag{8.57}$$

Equating the total magnetic energy $U_m = U_{coil} + U_{int}$ and formula (8.38)

$$U_m = \frac{1}{2}\mu_0 \frac{N^2}{d} S I_f^2 + \frac{1}{2} S\mu_0 \frac{N}{d} I_b I_f = \kappa_m \frac{1}{2}\mu_0 \frac{N^2}{d} S I_f^2 \tag{8.58}$$

results in the following relation between bound and free current

$$I_b = (\kappa_m - 1) N I_f \tag{8.59}$$

where I_b is the total current flowing on the surface of the material and I_f is the current in *one* of the loops of the coil. Since $I_b S = MSd$, (8.59) may be expressed in terms of magnetization M

$$M = (\kappa_m - 1) \frac{N}{d} I_f \tag{8.60}$$

where M is directed along the axis of the coil according to the dipole orientation. For $\kappa_m > 1$ the dipoles of the material align parallel to the dipole moment of the external influence while for $\kappa_m < 1$ they align anti-parallel.

While (8.60) was obtained for a special case, where the material is surrounded by an ideal coil, a general formula will now be presented. Here it will be motivated within the chosen special case. In Appendix C the formula is fully derived.

Using formulas (8.57) and (8.60) magnetization may be written

$$M = (\kappa_m - 1)\frac{2U_{int}}{\mu_0 m_b} \tag{8.61}$$

where $m_b = I_b S$ is the total magnetic dipole moment of the material. This may be vectorized by noting that its direction is given by the dipole moment m_b

$$\bar{M} = (\kappa_m - 1)\frac{2U_{int}}{\mu_0 m_b}\hat{m}_b \tag{8.62}$$

This is a general formula for magnetization in the dipole approximation and should be compared to the electric counterpart (8.24).

The interaction energy, generally given by formula (8.88) and for homogeneous magnetization by formula (8.57), is in the dipole approximation given by formula (7.25):

$$U_{int} = -\frac{1}{2}\frac{\mu_0}{4\pi R^3}[\bar{m}_b \cdot \bar{m}_f - 3(\bar{m}_b \cdot \hat{R})(\bar{m}_f \cdot \hat{R})] \tag{8.63}$$

where in this case there is a factor 1/2 in front since the magnetization is induced. Hence, U_{int} is proportional to m_b meaning that m_b vanishes from the right-hand side of (8.62). However, the direction $\hat{m}_b$ remains, obtained using formula (7.45) in Exercise (7.15).

A few examples will now be discussed.

8.2.2.1 A Material on the Axis of the Influencing Loop

Consider an interaction between a current-carrying loop and a material, placed at a distance x on the extended axis of the loop, Fig. 8.15. What is the interaction energy and the material's magnetization? Let the distance be large enough to ensure that the dipole approximation is valid.

In this case the two dipole moments are parallel/anti-parallel to each other as well as to the distance vector. Formula (8.63) gives the interaction energy for parallel moments

$$U_{int} = \frac{1}{2}\frac{\mu_0}{2\pi x^3}m_b m_f \tag{8.64}$$

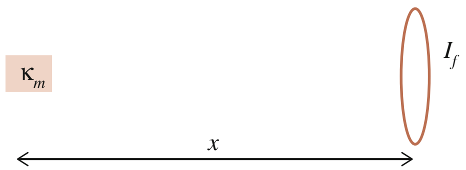


Fig. 8.15 A material interacts with a current loop

The magnetization M is obtained from (8.61) as

$$M = (\kappa_m - 1)\frac{I_f S_f}{2\pi x^3} \tag{8.65}$$

where $m_f = I_f S_f$. Since $m_b = MV$, where V is the volume of the material, the interaction energy becomes

$$U_{int} = \frac{\mu_0}{8\pi^2 x^6}(\kappa_m - 1)\left(I_f S_f\right)^2 V \tag{8.66}$$

Hence, the force between the objects decreases as $1/x^7$, see Exercise (8.9). Note that the direction of the force is independent of the direction of the free current.

8.2.2.2 A Material Off the Axis of the Influencing Loop

Now consider an interaction between a current loop and a material placed at an arbitrary position, Fig. 8.16. As in the previous section, the dipole approximation is assumed.

The interaction energy and the magnetization will be determined using formulas (8.63) and (8.62) respectively. To this end, the relative orientation between $\bar{m}_b$ and $\bar{m}_f$ has to be found. According to formula (7.45) in Exercise (7.15) the relationship between the angles in Fig. 8.16 is

$$\tan\theta_b = -\frac{1}{2}\tan\theta_f \tag{8.67}$$

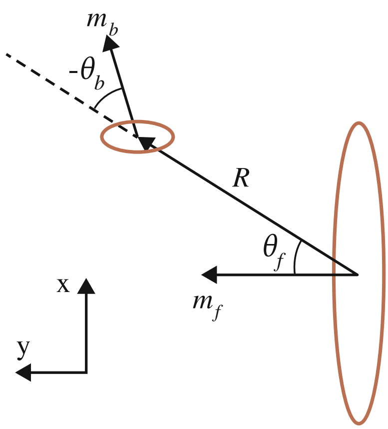


Fig. 8.16 A current loop interacts with dipole m_b

Referring to Fig. 8.16, the dipole moment of the material may be written

$$\begin{aligned}\bar{m}_b &= m_b \cos(\theta_f - \theta_b)\hat{y} + m_b \sin(\theta_f - \theta_b)\hat{x} \\ &= m_b(\cos\theta_f \cos\theta_b + \sin\theta_f \sin\theta_b)\hat{y} \\ &\quad + m_b(\sin\theta_f \cos\theta_b - \cos\theta_f \sin\theta_b)\hat{x}\end{aligned} \tag{8.68}$$

which, using (8.67), becomes

$$\bar{m}_b = -\frac{3}{2} m_b \sin\theta_f \cos\theta_b \left[\left(-\frac{2}{3}\frac{\cos\theta_f}{\sin\theta_f} + \frac{1}{3}\frac{\sin\theta_f}{\cos\theta_f}\right)\hat{y} - \hat{x}\right] \tag{8.69}$$

By dividing this vector by its length, its unit vector is obtained

$$\hat{m}_b = \left(\frac{4}{9}\frac{1}{\sin^2\theta_f} + \frac{1}{9}\frac{1}{\cos^2\theta_f}\right)^{-1/2}\left[\left(-\frac{2}{3}\frac{\cos\theta_f}{\sin\theta_f} + \frac{1}{3}\frac{\sin\theta_f}{\cos\theta_f}\right)\hat{y} - \hat{x}\right] \tag{8.70}$$

The interaction energy (8.63) becomes

$$U_{int} = -\frac{\mu_0}{8\pi R^3} m_b m_f (\cos(\theta_f - \theta_b) - 3\cos\theta_f \cos\theta_b) \tag{8.71}$$

By using formula (8.67) the angle θ_b may be expressed in terms of θ_f

$$U_{int} = -\frac{\mu_0}{8\pi R^3} m_b m_f \left(4\frac{\cos^2\theta_f}{\sin^2\theta_f} + 1\right)^{-1/2}\left[\frac{4\cos^2\theta_f + \sin^2\theta_f}{\sin\theta_f}\right] \tag{8.72}$$

The magnetization of the material is obtained from formula (8.62). The interaction energy U_{int} multiplied by the unit vector $\hat{m}_b$ in this formula is

$$\begin{aligned}U_{int}\hat{m}_b &= -\frac{\mu_0}{8\pi R^3} m_b m_f \left(4\frac{\cos^2\theta_f}{\sin^2\theta_f} + 1\right)^{-1/2}\left[\frac{4\cos^2\theta_f + \sin^2\theta_f}{\sin\theta_f}\right] \\ &\quad \left(\frac{4}{9}\frac{1}{\sin^2\theta_f} + \frac{1}{9}\frac{1}{\cos^2\theta_f}\right)^{-1/2}\left[\left(-\frac{2}{3}\frac{\cos\theta_f}{\sin\theta_f} + \frac{1}{3}\frac{\sin\theta_f}{\cos\theta_f}\right)\hat{y} - \hat{x}\right] \\ &= -\frac{\mu_0}{8\pi R^3} m_b m_f 3\sin\theta_f \cos\theta_f \left[\left(-\frac{2}{3}\frac{\cos\theta_f}{\sin\theta_f} + \frac{1}{3}\frac{\sin\theta_f}{\cos\theta_f}\right)\hat{y} - \hat{x}\right] \\ &= -\frac{\mu_0}{8\pi R^3} m_b m_f \left[\left(-2\cos^2\theta_f + \sin^2\theta_f\right)\hat{y} - 3\sin\theta_f \cos\theta_f \hat{x}\right]\end{aligned} \tag{8.73}$$

and the magnetization is thus

$$\bar{M} = (\kappa_m - 1)\,\frac{m_f}{4\pi R^3}\left[\left(2\cos^2\theta_f - \sin^2\theta_f\right)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}\right] \tag{8.74}$$

The force between the objects is given by the gradient of U_{int} (formula 8.72), where

$$m_b = \left|\bar{M}\right| V \tag{8.75}$$

and V is the volume of the material.

8.2.2.3 Magnetization of a Sphere/Cylinder

In this section the significance of the shape of the material when it comes to magnetization is explored, equivalent to the electric case which was discussed in Sect. 8.1.4.2.

Spherical form

Consider first a permanently homogeneously magnetized sphere. To find its magnetic properties, its interaction with a small current loop, i.e. a point-like magnetic dipole, is investigated. Let the magnetization of the sphere be directed along the z axis, so that $\bar{M} = M\hat{z}$. Denote the dipole moment of the current loop m_2, directed along z and located somewhere on the z axis, Fig. 8.17.

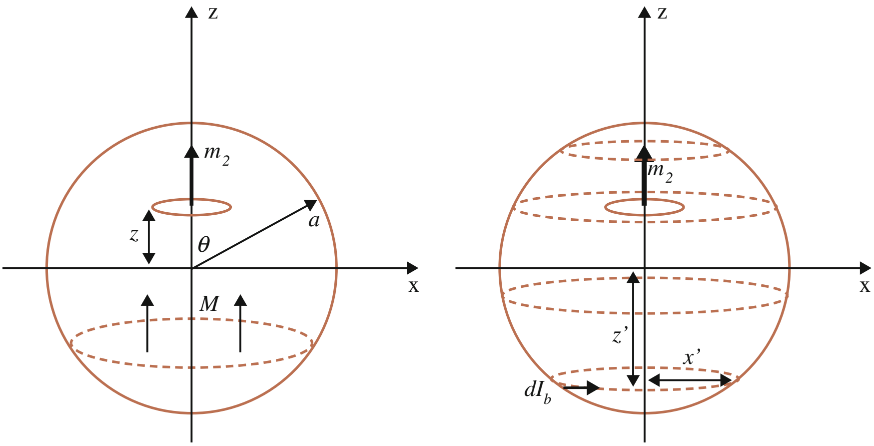


Fig. 8.17 A small magnetic dipole m_2 on the z axis interacts with a homogeneously magnetized *sphere* (*left*). The interaction is modelled as a sum of infinitesimal interactions between the dipole m_2 and *circular rings* on the surface of the *sphere*. The *rings* have radius x' and carry current dI_b (*right*)

The calculation of the interaction energy is performed by dividing the sphere into parallel infinitesimal rings located on the surface of the sphere. Here the surface current density is $\bar{K}_b = K_b\hat{\phi}$ in spherical coordinates. These current rings and the current of the dipole m_2 are accordingly concentric. Note that inside the sphere there are no currents. The task is to sum pairwise interactions between each of these rings and the dipole m_2, equivalent to the case treated in Sect. 4.2.5.1 for an interaction between two circular coaxial currents.

The interaction energy is obtained from the mutual inductance (4.56). Since the dipole m_2 is small, the quantity $k^2 \ll 1$ in formula (4.56). Consequently, formula (4.63) may be used to calculate the mutual inductance.[4] Treating the dipole m_2 as a circular current loop with current I_2 and radius b and using x' as the radii of the rings on the sphere, the infinitesimal interaction energy becomes

$$\begin{aligned} dU_{int} &= I_2 dI_b \frac{\mu_0 x' b}{[(x'+b)^2+(z'-z)^2]^{1/2}} \frac{\pi}{8} \frac{4x'b}{(x'+b)^2+(z'-z)^2} \\ &= \frac{\mu_0}{2} I_2 \pi b^2 \frac{x'^2 dI_b}{[x'^2+(z'-z)^2]^{3/2}} \end{aligned} \tag{8.76}$$

where $x' \gg b$ has been used and z' is the z coordinate for the infinitesimal ring of the sphere, Fig. 8.17.

Making the substitution $x' = a\sin\theta$ and $z' = a\cos\theta$ and from formula (8.53) $dI_b = Kad\theta = M\sin\theta ad\theta$, we obtain

$$dU_{int} = \frac{\mu_0}{2} m_2 \frac{a^2\sin^2\theta Ma\sin\theta d\theta}{[a^2\sin^2\theta + (a\cos\theta - z)^2]^{3/2}} \tag{8.77}$$

so that

$$U_{int} = \frac{\mu_0}{2} m_2 M a^3 \int_0^{\pi} \frac{\sin^3\theta d\theta}{(a^2+z^2-2az\cos\theta)^{3/2}} \tag{8.78}$$

The integral is evaluated making the substitution $t = \cos\theta$. The result is

$$\begin{aligned} U_{int} &= \frac{\mu_0}{2\pi z^3} m_2 \left(\frac{4}{3}\pi a^3 M\right) \qquad z \ge a \\ U_{int} &= \mu_0 \frac{2}{3} m_2 M \qquad z \le a \end{aligned} \tag{8.79}$$

Note that the result for $z \ge a$ implies that the sphere and the dipole m_2 interact like two exact dipoles outside the sphere which is why a homogeneously magnetized sphere is equivalent to an exact dipole, as long as it is viewed from the outside.

[4] Do not confuse magnetization M with mutual inductance M_{12}.

Cylindrical form

For a cylinder of length l, radius a and homogeneous magnetization M along its axis coinciding with the z axis, the infinitesimal interaction energy (8.76) becomes

$$dU_{int} = \frac{\mu_0}{2} I_2 \pi b^2 \frac{a^2 M dz'}{[a^2 + (z' - z)^2]^{3/2}} \tag{8.80}$$

where formula (8.53) has been used. Total energy becomes

$$\begin{aligned} U_{int} &= \frac{\mu_0}{2} m_2 M a^2 \int_0^l \frac{dz'}{[a^2 + (z' - z)^2]^{3/2}} \\ &= \frac{\mu_0}{2} m_2 M \left(\frac{l - z}{\sqrt{a^2 + (l - z)^2}} + \frac{z}{\sqrt{a^2 + z^2}} \right) \end{aligned} \tag{8.81a}$$

Letting $z = l/2$ and $l \to \infty$, the energy becomes $U_{int} = \mu_0 m_2 M$. This is the interaction between a long coil and a magnetic dipole placed inside the coil which was treated in Exercise (4.13). From formula (4.89), it is realised that the magnetization of the long coil is

$$M_C = N I_f / l \tag{8.81b}$$

where N is the number of turns and f denotes 'free'. Formula (8.81b) is in accordance with formula (8.53).

Sphere under external influence

Next consider a sphere that is being homogeneously magnetized by an ideal coil. What is the relation between the magnetization M and the current of the external influence?

For the equivalent calculation in the electric case, Sect. 8.1.4.2, the concept of potential Φ was utilized. For the magnetic case there is no such counterpart, since its source is electric current. Instead, the energy concept is used. While for the electric case the potential from the bound charges, Φ_b, was calculated within the material, for the magnetic case the *interaction energy* between an internal dipole of the material and the rest of the material is considered. This internal dipole also interacts with the external coil, so that the total interaction energy becomes the sum of these two sources. Equating this sum to formula (8.61), fully derived in Appendix C, the sought relation between the material magnetization M and the external influence is found.

Denote the magnetization of the sphere and the coil M_S and $M_C = N I_f / l$ respectively, where the latter is formula (8.81b). For the case $(\kappa_m - 1) > 0$, corresponding to e.g. a paramagnetic material, the material response is such that its current appears in the same direction as the influence, i.e. M_S and M_C are parallel.

Consider now a small internal dipole with moment m_b inside the spherical material located in a small cavity, Fig. 8.18 (upper). For a homogeneous magnetized material

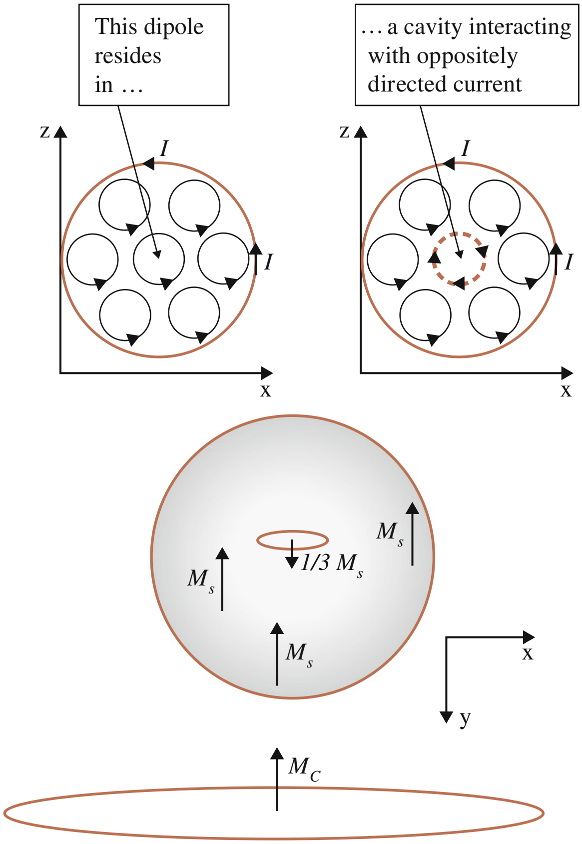


Fig. 8.18 *Top left* Magnetic dipoles in a *circular* cross section of a homogeneously magnetised material. *Top right* A dipole inside the material interacts with a current generated by its neighbours which is oppositely directed to the net current on the surface. *Bottom* A sphere is being magnetized by an ideal coil with the magnetization M_C. In one of the cavities of the *sphere*, a dipole interacts with the rest of the material as well as with the external coil. Note that in the *upper* diagrams the *sphere* is observed from above

there are no net currents inside the volume. However, for the inner surface of the cavity there will be net currents *oppositely* directed to the surface currents of the sphere. Hence, the internal interaction energy of the material is of opposite sign compared to the interaction between the material and the coil.

Consider first the internal interaction energy, i.e. the energy in the interaction between the internal dipole and the rest of the material. Using formula (8.79) for $z \leq a$, it may be obtained in the following way. The coil magnetizes the sphere to the magnetization M_S. Had the object been without any form, i.e. of infinite extent, the energy between the internal dipole and the rest of the material would be $\frac{1}{2}\mu_0 m_b M_S$, according to formula (4.89). The extra factor $1/2$ appears since the energy is *induced* in analogy with formula (8.57). Using formula (8.79), the total energy between the internal dipole and the magnetized sphere is $\frac{1}{2}\mu_0 \frac{2}{3} m_b M_S$, corrected for the inducing process. However, this energy includes the interaction with the magnetizing coil since the magnetization of the sphere is generated by the coil.

The interaction energy between the internal dipole and the material alone is then $-\frac{1}{2}\mu_0\frac{1}{3}m_b M_S$, illustrated in Fig. 8.18, since $\mu_0\frac{2}{3}m_b M_S = \mu_0 m_b M_S - \mu_0\frac{1}{3}m_b M_S$ The interaction energy becomes

$$U_{int} = \frac{1}{2}\mu_0 m_b\left(M_C - \frac{1}{3}M_S\right) \tag{8.82}$$

The first term corresponds to the interaction energy between the internal dipole and the coil, while the second term to that between the internal dipole and the material. Formula (8.82) is equivalent to formula (8.32) for the electric case. The difference in sign of the second term of the right-hand side is due to opposite response directions between electric and paramagnetic dipoles.

This is now to be equated with the interaction energy given by formula (8.61), i.e.

$$U_{int} = \frac{1}{2}\frac{\mu_0 m_b M_S}{\kappa_m - 1} \tag{8.83}$$

so that

$$\frac{1}{2}\frac{\mu_0 m_b M_S}{\kappa_m - 1} = \frac{1}{2}\mu_0 m_b\left(M_C - \frac{1}{3}M_S\right) \tag{8.84}$$

which gives

$$\frac{M_S}{\kappa_m - 1} + \frac{1}{3}M_S = M_C \tag{8.85}$$

and

$$M_S = 3\frac{\kappa_m - 1}{\kappa_m + 2}M_C \tag{8.86}$$

If the external influence is a current loop, (8.86) becomes in the dipole approximation, using formula (7.26)

$$M_S = 3\frac{\kappa_m - 1}{\kappa_m + 2}\frac{m_f}{2\pi d^3} \tag{8.87}$$

where d is the distance between the objects and m_f is the dipole moment of the external influence. This formula should be compared to the electric counterpart, formula (8.33).

The two formulas (8.33) and (8.87) have equivalent structures and are also generalized in the same way for an interaction in arbitrary medium. In the electric case, the resulting formula is known as Clausius-Mossottis formula, see Exercise (8.18). This generalization of formulas (8.86) and (8.87) is used in applications equivalent to dielectrophoresis, i.e. to separate and control microscopic particles, a method accordingly known as magnetophoresis.

8.3 General Multipole Interactions

Up to now, material effects have been examined in the dipole approximation. In case this approximation is not valid, the general formulas have to be considered.

For closed currents, the magnetic interaction energy between systems j and k is given by formula (3.19)

$$U_m = M_{jk} I_j I_k \tag{3.19}$$

Using Neumann's inductance formula (4.50)

$$M_{jk} = \frac{\mu_0}{4\pi} \int\limits_{Cond_k} \int\limits_{Cond_j} \frac{d\bar{L}_j \cdot d\bar{L}_k}{R} \tag{4.50}$$

(3.19) becomes

$$U_m = \frac{\mu_0}{4\pi} \int\limits_{V_j} \int\limits_{V_k} \frac{\bar{J}_j(\bar{r}_j) \cdot \bar{J}_k(\bar{r}_k)}{R} dV_k dV_j \tag{8.88}$$

The general formula for electric energy is obtained from formula (3.2)

$$U_e = \frac{1}{4\pi\varepsilon_0} \int\limits_{V_j} \int\limits_{V_k} \frac{\rho_j(\bar{r}_j)\rho_k(\bar{r}_k)}{R} dV_k dV_j \tag{8.89}$$

A strict derivation of multi-pole interactions originates from these formulas by performing a series expansion w.r.t. the distance between the systems. This is performed in appendices A and B for the electric and magnetic cases respectively.

8.4 Measurement of Electric and Magnetic Material Properties

The dielectric constant κ_e and the relative magnetic permeability κ_m may in principle be measured using the methods outlined above. Practically, however, these should be considered as just qualitative investigations. In the electric case, Fig. 8.2, a high voltage is used since the volt meter requires a high internal resistance and therefore operates at voltages of several kV. As a consequence, discharge effects appear making the conditions unstable. In the magnetic experiment, Fig. 8.11, small forces are active (except for so-called ferromagnetic materials) which are therefore hard to measure. Also in its electric counterpart, Figs. 8.3 and 8.4, the force becomes too small for a precise measurement.

Some practical methods for the measurements of material parameters will now be discussed. First solids are considered and then liquids.

8.4.1 Measurements on Solids

Since the material parameters κ_e and κ_m may be regarded as the factor by which capacitance and inductance respectively change when introducing materials, see formulas (8.2) and (8.39), these quantities may be measured in RC and RL circuits as discussed in Chap. 6.

Consider first the charging and discharging time for a capacitor fully filled with a material connected in series with a resistance R. The half-time, formula (6.4), becomes

$$T_{1/2} = \kappa_e C_0 R \ln 2 \tag{8.90}$$

where C_0 is the capacitance without material. Hence, the measurement of half-time gives a value of the dielectric constant provided the capacitance C_0 is known.

For the magnetic case, the current growth in a material filled coil connected in series with a resistance R is investigated. Formula (6.7) gives the half-time

$$T_{1/2} = \frac{\kappa_m L_0}{R} \ln 2 \tag{8.91}$$

where L_0 is the inductance for an empty coil. Using a good oscilloscope precision measurements may in this way be performed. An interesting investigation is to fill a coil with diamagnetic materials with $\kappa_m < 1$, such as copper and bismuth, and accordingly register a decrease of half-time. The effect is small and challenging to measure.

8.4.2 Measurements on Liquids

In case of measuring on liquids, a direct force action may be utilized.

Consider first the electric case. Let a coaxial cylindrical capacitor be in contact with the surface of a liquid, Fig. 8.19. When a voltage is applied between the conductors, an attractive force acts on the liquid in analogy with Figs. 8.3 and 8.4. The liquid is then drawn into the capacitor to a height h which depends on the dielectric constant of the liquid. Their relation will now be derived using the energy principle.

Three types of energies are involved in the process: gravitational, battery and electric energy. The final height h is found from energy minimization: The energy change becomes

$$\Delta U = \Delta U_e + \Delta U_{gr} + \Delta U_B = -\Delta U_e + \Delta U_{gr} \tag{8.92}$$

since, according to Sect. 4.1.7, the change of battery energy at a constant voltage is twice that of the electric energy change, see formula (4.36).

The change in electric energy is $\Delta U_e = \frac{1}{2} C \Delta\Phi^2 - \frac{1}{2} C_0 \Delta\Phi^2$ where C is the capacitance for the partly liquid filled cylindrical capacitor and C_0 is the capacitance

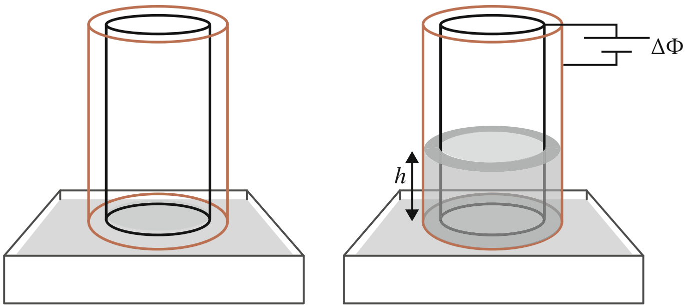


Fig. 8.19 *Left* A cylindrical capacitor is placed on the surface of a liquid. *Right* When a voltage is applied between the conductors the liquid is drawn into the capacitor

of the initially empty capacitor. Referring to Fig. 8.19, neglecting edge effects and using formula (4.88) it is obtained that

$$\Delta U_e = \left[\frac{1}{2}\kappa_e \frac{2\pi\varepsilon_0 h}{\ln(b/a)} + \frac{1}{2}\frac{2\pi\varepsilon_0(L-h)}{\ln(b/a)} - \frac{1}{2}\frac{2\pi\varepsilon_0 L}{\ln(b/a)}\right]\Delta\Phi^2 \tag{8.93}$$

where L is the length of the cylinders and their radii are a and b with $a < b$. In the final state, the capacitor corresponds to a parallel coupling of two capacitors, one is fully filled with liquid and the other is empty. Therefore, the capacitance C is the sum of the individual capacitances.

The change in gravitational energy becomes

$$\Delta U_{gr} = \rho\pi(b^2 - a^2)hg\frac{h}{2} \tag{8.94}$$

where ρ is the density of the liquid and $h/2$ is the height of the liquid's centre of gravity inside the capacitor. The total energy change then becomes

$$\Delta U = -\frac{\pi\varepsilon_0}{\ln(b/a)}[(\kappa_e - 1)h]\Delta\Phi^2 + \rho\pi(b^2 - a^2)g\frac{h^2}{2} \tag{8.95}$$

Since U is minimized, $\Delta U = 0$ which gives

$$-\frac{\pi\varepsilon_0}{\ln(b/a)}(\kappa_e - 1)\Delta\Phi^2 + \rho\pi(b^2 - a^2)g\frac{h}{2} = 0 \tag{8.96}$$

so that

$$(\kappa_e - 1) = \frac{\ln(b/a)(b^2 - a^2)\rho g h}{2\varepsilon_0 \Delta\Phi^2} \tag{8.97}$$

As was discussed in Sect. 8.1.1 the force arises due to edge effects, which are however neglected in the calculation. This is motivated since these effects are unchanged from the initial to the final state and are therefore cancelled in the end.

The magnetic permeability for liquids may be measured similarly, using the so-called Quincke method. A u-tube is filled with the liquid with one leg placed inside a coil, Fig. 8.20. When the coil carries a current it interacts with the liquid in analogy with Fig. 8.11. These figures illustrate the case of an attractive interaction. However, diamagnetic materials will respond with a repulsive force. The height h is directly related to the relative magnetic permeability, now shown using the energy principle.

As in the previous case there are three forms of energies to be considered: the magnetic, the battery and the gravitational. The energy change becomes

$$\Delta U = \Delta U_m + \Delta U_B + \Delta U_{gr} = -\Delta U_m + \Delta U_{gr} \tag{8.98}$$

using the fact that the change of the battery energy is twice that of the magnetic energy change for constant currents, see Sect. 3.3.1.

In the final state, the coil is partly filled with liquid and its inductance is denoted L. The inductance of the initially empty coil is denoted L_0. Using formula (4.80) in lowest order, the magnetic energy change becomes

$$\Delta U_m = \frac{I^2}{2}(L - L_0) = \frac{I^2}{2}\left(\kappa_m \mu_0 n S \frac{h}{l} N + \mu_0 n S \frac{l-h}{l} N - \mu_0 n N S\right) \tag{8.99}$$

where l is the length of the coil, N is the number of turns, S is the cross sectional area and $n = N/l$. Note that this summation of individual inductances is only valid in the lowest order of formula (4.80), see Exercise (4.14).

The total energy change becomes

$$\Delta U = -\frac{I^2}{2}(\kappa_m - 1)\mu_0 n S \frac{h}{l} N + \rho S h g h \tag{8.100}$$

where ρ is the density of the liquid.

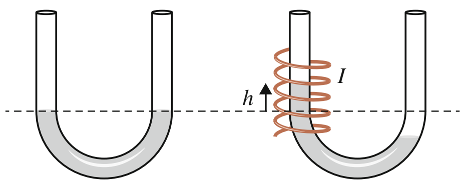


Fig. 8.20 A liquid in a u-tube is inserted in a current-carrying coil

At minimum energy, $\Delta U = 0$, so that

$$(\kappa_m - 1) = \frac{2\rho g l}{I^2 \mu_0 n N} h \tag{8.101}$$

Permeability for solids may also be measured in this way by preparing a solution with e.g. water. By measuring at different concentrations an extrapolation of data can be done down to zero concentration which then corresponds to pure water. In this way the diamagnetic property of water may be observed, see Table 8.2 and Exercise (8.11).

8.5 Summary

Both electric and magnetic material properties may be understood within a model assuming the material to consist of dipoles. In the electric case these dipoles correspond to molecular charge displacements (except for metals). In the magnetic case the dipoles are current loops at the atomic level, including the conduction electrons.

The dielectric constant κ_e describes the electric properties of a material. It is defined as the factor by which the voltage over a capacitor changes when it is filled by the material

$$\Delta\Phi_2 = \frac{\Delta\Phi_1}{\kappa_e} \tag{8.1}$$

Polarisation $\bar{P}$ is defined as the density of dipole moment

$$\bar{p} = \int_V \bar{P}(\bar{r}) dV \tag{8.12}$$

whose source is the bound charges. These are related to polarisation as

$$\sigma_b = \bar{P} \cdot \hat{n} \quad \textit{At the surface!} \qquad \rho_b = -\nabla \cdot \bar{P} \quad \textit{In the volume!} \tag{8.19}$$

Polarisation is related to the electric potential in the material according to

$$\bar{P} = -\varepsilon_0(\kappa_e - 1)\nabla\Phi \tag{8.24}$$

At the intersection between air/vacuum (medium 1) and a material with dielectric constant κ_e (medium 2), the following continuity conditions hold

$$\hat{n} \cdot \nabla\Phi_1 = \kappa_e \hat{n} \cdot \nabla\Phi_2 \tag{8.21}$$

$$\hat{t} \cdot \nabla\Phi_1 = \hat{t} \cdot \nabla\Phi_2 \tag{8.22}$$

The magnetic properties of a material are described by the relative magnetic permeability κ_m. It is defined as the factor by which the magnetic energy for an inductor changes when it is filled with the material:

$$U_m = \kappa_m U_0 = \frac{1}{2}\kappa_m L_0 I^2 \tag{8.38}$$

Magnetization $\bar{M}$ is defined as the density of magnetic dipole moment

$$\bar{m} = \int_V \bar{M}(\bar{r})dV \tag{8.43}$$

whose source is the bound currents related to magnetization as

$$\bar{K} = \bar{M} \times \hat{n} \quad \textit{At the surface!} \tag{8.53}$$

$$\bar{J} = \nabla \times \bar{M} \quad \textit{In the volume!} \tag{8.49}$$

In the dipole approximation a material's magnetization induced by an external magnetic dipole moment m_f is

$$\bar{M} = (\kappa_m - 1)\frac{m_f}{4\pi R^3}[(2\cos^2\theta_f - \sin^2\theta_f)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}] \tag{8.74}$$

where θ_f is the angle between the distance vector $\bar{R}$ and $\bar{m}_f$.

A spherically shaped material surrounded by air is polarised/magnetized by an external influence as

$$\bar{P} = -3\varepsilon_0\frac{\kappa_e - 1}{\kappa_e + 2}\nabla\Phi_f \tag{8.33}$$

in the electric case and

$$M_S = 3\frac{\kappa_m - 1}{\kappa_m + 2}\frac{m_f}{2\pi d^3} \tag{8.87}$$

in the magnetic case. The external influence in the latter case is a dipole with the moment m_f acting in the direction of its moment.

A strict treatment of electric and magnetic (*closed* currents) interactions are based on the formulas

$$U_e = \frac{1}{4\pi\varepsilon_0}\int_{V_j}\int_{V_k}\frac{\rho_j(\bar{r}_j)\rho_k(\bar{r}_k)}{R}dV_k dV_j \tag{8.89}$$

and

$$U_m = \frac{\mu_0}{4\pi}\int_{V_j}\int_{V_k}\frac{\bar{J}_j(\bar{r}_j)\cdot\bar{J}_k(\bar{r}_k)}{R}dV_k dV_j \tag{8.88}$$

respectively. These formulas are series expanded in Appendix A and B respectively to obtain a so-called multi-pole expansion of the interactions.

8.6 Exercises

8.1 a. Show that the dielectric constant (8.1) for a material may be defined as the ratio of a plate capacitor's electric energy when fully filled with material to that without the material.
b. If a battery is connected to the capacitor in Fig. 8.2 how do charge, voltage and capacitance change when a material is introduced between the plates?

8.2 a. Show that the capacitance of the capacitor in Fig. 8.5 is independent of where the material is placed horizontally.
b. Is the result valid also if the material is a metal? Compare Exercise (5.5).

*8.3 Referring to Fig. 8.3, draw a figure showing the principal orientation of the dipoles in the material. Explain from this figure how the attractive force arises.

8.4 a. Referring to Fig. 8.9, determine in the dipole approximation the force between the cylinder and the wire. Denote the dielectric constant by κ_e and the plate radius of the cylinder by a. Hint: Use formula (7.2).
b. If the cylinder is placed vertically and the long wire above at a distance d from the lower end of the cylinder, determine the linear charge density λ of the wire needed to lift the cylinder.
Calculate the force numerically for $a = 1.0\,\text{cm}$ and $d = 10.0\,\text{cm}$.

8.5 Replace the wire in Fig. 8.9 with a positively charged sphere, homogeneously charged with total charge Q. The distance from the left edge of the cylinder to the centre of the sphere is d, plate radius a and the length of the cylinder L, where $d \gg L$.

a. Show qualitatively a possible orientation and distribution of the dipoles in the cylinder.
b. Calculate the induced dipole moment of the cylinder.
c. Determine the charge densities at the surface and inside the volume of the cylinder.
d. Determine energy and force.

8.6 a. An unpolarised homogeneous sphere with radius a and dielectric constant κ_e is placed between two parallel plates with surface charge density σ.
Determine the induced dipole moment of the sphere.

b. Determine the bound charge density inside and at the surface of the sphere.
c. Let the sphere be an electret. This is a material that keeps its induced electric polarisation after the external influence has ceased. Remove the sphere from the plates and place it at a long distance a from a Na^+ ion in the coordinate origin. The dipole moment of the sphere is oriented in the xy plane forming an angle θ with the x axis.
Determine the force on the Na^+ ion and the torque on the sphere.

8.7 Express formula (8.40) with free coordinates and determine the force on the coil and the material.

8.8 Derive (8.48) in the following way:

1. Take the curl of both sides of (8.45).
2. Solve for J with position vector $\bar{r} = 0$. Note that the dipole moment is independent of the location of the coordinate origin, see Exercise (7.4).
3. Utilize the fact that $\nabla \cdot \bar{J} = 0$.

8.9 A small solid cylindrical material interacts with a thin coil, as in Fig. 8.15. The cylinder has radius 0.5 cm and length 3.0 cm. The coil has 100 turns, cross sectional area 0.03 m^2 and carries a current of 10 A. The distance between the centres of the objects is $x = 0.2$ m.
Determine the force in the dipole approximation if the material consists of

a. copper, aluminium or liquid oxygen. Refer to Table 8.2.
b. a superconducting material ($\kappa_m = 0$).

8.10 Propose some experimental arrangements to measure the force in Exercise (8.9). Find the most significant sources of experimental errors.

*8.11 *Magnetic material classes*

As was discussed in Sect. 8.2, magnetic material properties may be categorized into four main classes with respect to its response to an external magnetic influence: weak/strong attractive and weak/strong repulsive. These classes are denoted para-, ferro-, diamagnetic and superconducting materials respectively.

a. State what these classes correspond to in terms of the relative magnetic permeability κ_m.
b. The electron motion in a metal may be categorized into three main classes. Which are these? Compare Exercise (7.17).
c. Assume the electrons to be responsible for the magnetic material properties. Which one of its motional classes is responsible for the diamagnetic property in metals?

d. Superconduction is a material property characterized as maximal diamagnetism. Explain its origin in metals.
Hint: Superconduction occurs at low temperatures. What happens with resistance when temperature decreases?
e. Occasionally in amusement parks, you may find a so-called 'free fall' attraction. The braking at the end of the fall utilizes a diamagnetic effect using induced currents. Propose an arrangement for this effect. Could you imagine other applications of the diamagnetic effect?
f. As was mentioned in Sect. 8.4.2 it is readily demonstrated that water is diamagnetic. How could this property be explained at a molecular level?

8.12 *Anisotropic materials*

There are materials whose response to an external influence is not described as a dipole alignment along the direction of the influence. These materials are called anisotropic. The figure illustrates this phenomenon in two dimensions for the electric case to the left and for the magnetic case to the right (Fig. 8.21).
Thus, the induced dipole moment has a component perpendicular to the direction of the influence.

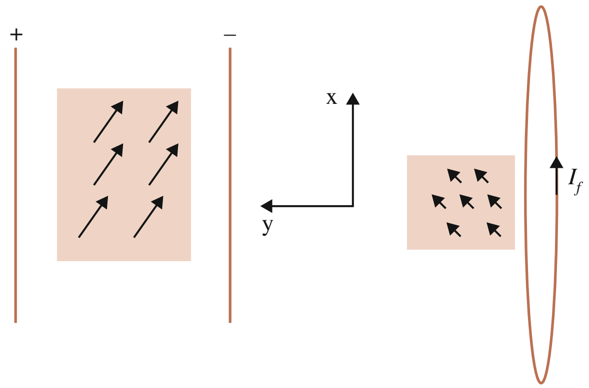
a. What could be the reason for this?
b. Consider formula (8.74) for magnetization

$$\bar{M} = (\kappa_m - 1)\frac{m_f}{4\pi R^3}[(2\cos^2\theta_f - \sin^2\theta_f)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}]$$

and formula (8.20) for electric polarisation

$$\bar{P} = \varepsilon_0 \nabla\Phi_f\left(\frac{1}{\kappa_e} - 1\right)$$

Fig. 8.21 Exercise 8.12



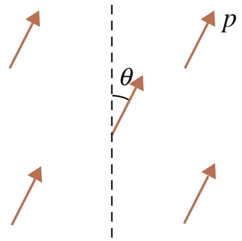


Fig. 8.22 Exercise 8.13

Based on these formulas, suggest a generalization of the material parameters to include these anisotropic effects.

c. How could these effects be registered?

8.13 A certain material influenced externally responds with its dipoles as in (Fig. 8.22). Determine the torque on the centre dipole caused by the other dipoles, placed at the corners of a square with the side a. All dipoles form an angle θ with the vertical and their dipole moment is p.

*8.14 *The Hall sensor*

A Hall sensor is used to measure the magnetic interaction between a magnetized (current-carrying) object and a straight current generated in the sensor. The figure shows the principles of the sensor in an interaction with an idealized coil (Fig. 8.23).

The Hall element, oriented in the xy plane in the figure, consists of a metallic rectangular plate where a voltmeter V is connected over two parallel side plates as in the figure. A Hall current I_H through the plate will interact magnetically with the coil current. This will cause the electrons of the Hall current to be displaced to one side of the Hall element. A voltage builds up over the Hall element whose magnitude depends on the strength of the two currents as well as on the relative orientation between the cross sectional surface normals of the two objects. This voltage gives rise to an electric force that counteracts the magnetic force. At equilibrium between these two forces the voltage is registered by the voltmeter.

Show that this voltage is given by

$$\Delta\Phi = v\mu_0 K l \cos\theta \tag{8.102}$$

where v is the drift velocity in the Hall sensor, K is the current density of the coil, l is the distance between the Hall plates and θ is the angle between the surface normals of the objects.

Hint: According to Exercise (7.12) the idealized coil may be considered as two parallel large plates with opposite current directions so that the force on the Hall current becomes twice that of formula (4.46).

Since the drift velocity and the dimensions of the Hall sensor are known, the measurement provides the current density K of the coil.

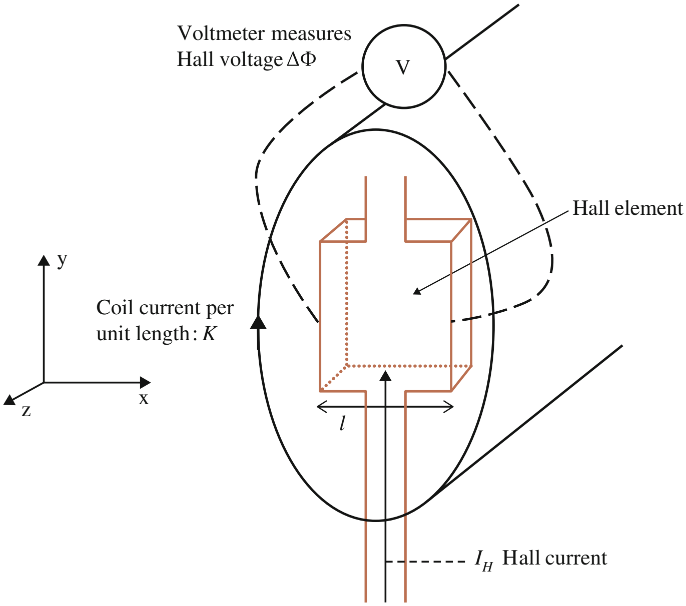


Fig. 8.23 Exercise 8.14

C* 8.15 *Magnetic hysteresis*

Ferromagnetic materials, such as iron, cobalt, and nickel exhibit a strong attractive magnetic response to an external influence. In addition, κ_m is strongly dependent on the strength of the external influence and they may also be magnetized permanently, i.e. the magnetization remains after the influence has ceased. Such a material's magnetic properties may be illustrated through a hysteresis curve (from the greek word *hystera* meaning womb), see figure (Fig. 8.24).
The curve is registered in the following way:
Place the ferromagnetic material in a solenoid (long coil) with an alternating current I_f.

Fig. 8.24 Exercise 8.15

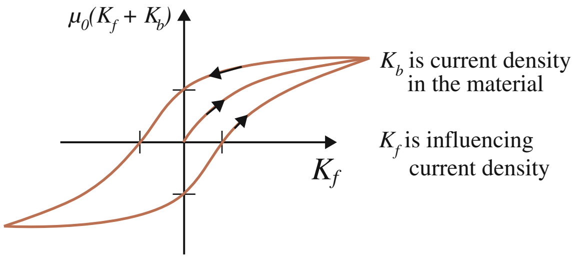


Fig. 8.25 Exercise 8.16

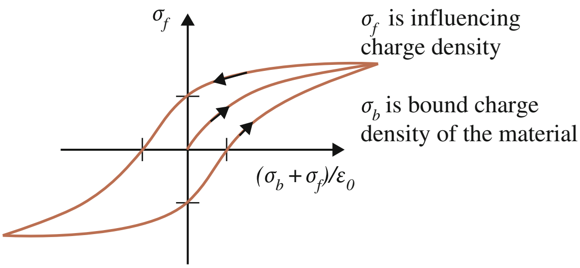
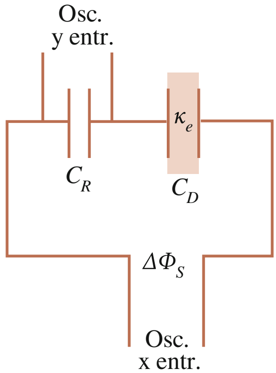


Fig. 8.26 Exercise 8.16



Saw a thin trail in the material parallel to the loops of the coil where a Hall sensor is introduced (see Exercise 8.14).
Connect the Hall voltage $\Delta\Phi_H$ to the y entrance of an oscilloscope. Connect the AC voltage that generates I_f to the x entrance of the oscilloscope.

a. Draw a schematic of the set-up.
b. State the relationship between the Hall voltage and the free current I_f.
c. Express the relative magnetic permeability κ_m in terms of the slope of the curve. Determine κ_m at the maximum K_f in the figure and explain its value.
d. Explain how the remanent magnetization of the material, i.e. the remaining magnetism for $K_f = 0$, may be determined.
e. State some application of this kind of materials.
f. Why is the phenomenon called 'hysteresis'?

C 8.16 *Electric hysteresis*

Ferroelectric materials, such as KNO_3 and $BaTiO_3$, are able to be permanently polarised. Also, their dielectric constants vary with the strength of the external influence. Similarly to the ferromagnetic case, Exercise (8.15), the material exhibits a hysteresis effect according to Fig. 8.25. Note that compared to the magnetic case, the axes are interchanged.

The electric polarisation of a material can be measured using a so-called Sawyer-Tower circuit (1930), see Fig. 8.26.

a. Knowing that the charge on the two capacitors are equal, show that this circuit gives an approximate hysteresis curve on the oscilloscope provided that $\Delta\Phi_S$ is an AC voltage and $C_R \gg C_D$.
b. Convert the oscilloscope voltages to the quantities forming the hysteresis curve above. State the slope of the curve in terms of the dielectric constant and define the remanent (remaining) polarisation.
c. Show that the surface enclosed by the hysteresis curve corresponds to the work done over one cycle.
d. State some applications for ferroelectric materials (search the *www*)

8.17 *The complex electric relative permittivity*

The heat generation in an isolator material may be formalized using a complex expression for the dielectric constant:

$$\kappa_e^c = \kappa_R + j\kappa_I \tag{8.103}$$

provided current, voltage and impedance are all expressed in complex form, see Exercise (6.5).
Consider a material-filled capacitor with capacitance $C = \kappa_e C_0$.

a. Express its complex impedance, introduce a complex dielectric constant and show that a resistive component appear in the impedance.
b. What is the source of the resistive effects in the capacitor?
c. What is the heat power?

*8.18 *Derivation of the Clausius-Mossotti polarisation formula.*

a. For a homogeneous external influence as in Fig. 8.10, determine the polarisation of a sphere with dielectric constant κ_p surrounded by a medium with dielectric constant κ_w.
   Hint: Compare formula (8.24) with (8.37) and generalise (8.33).
b. Show that the dipole moment of a sphere of radius a under the above-mentioned conditions becomes

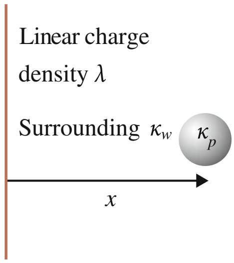
$$\bar{p} = -4\pi\varepsilon_0 a^3 \frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w} \nabla\Phi_w \tag{8.104}$$

*8.19 *Dielectrophoresis*

A small non-conductive sphere of radius a and a long wire with linear charge density λ [C/m] interact at a distance x, see Fig. 8.27.
Determine the force on the sphere if the surrounding medium is

a. air
b. water

Fig. 8.27 Exercise 8.19



This is a principle description of dielectrophoresis, a method by which particles/molecules of different kinds are separated and controlled, e.g. in DNA analysis. In such an application, the particles are influenced by the wire for a certain amount of time. Depending on form, mass and dipole moment the particles are displaced a certain distance during this time interval from which their characteristics can be obtained (Fig. 8.27).

C 8.20 *The electric fish*

The electric fish registers its surroundings through an electrostatic interaction. A realistic model of the fish is an electric dipole with charge 100 nC and length 7.0 cm (Fig. 8.28).
The dipole is oriented along the x axis and a foreign spherical object of radius a and with dielectric constant κ_p is located at $(x = 0, y = 80.0)$ cm. Consider the distance to be large so that the dipole of the fish may be approximated as point-like and the foreign object as small. Determine

a. the polarisation of the foreign object.
b. the force on the object.
c. the change of the potential caused by the foreign object at a point on the body of the fish with the coordinates $(-5.0, 0.0)$ cm.
d. Model the process in a computer program, e.g. Femlab of the Comsol company.

Fig. 8.28 Exercise 8.20

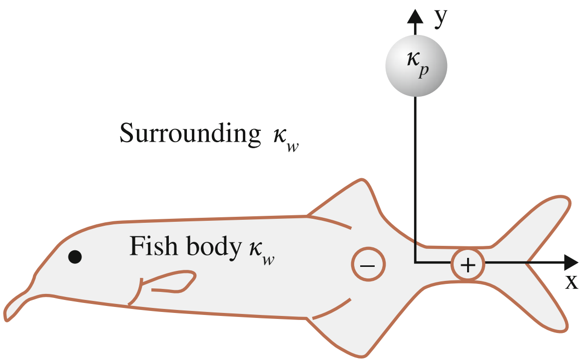


Fig. 8.29 The elephant nose fish senses its environment by means of electric pulses generated by its modified muscle cells. The *white lines* at the rear are marks of this organ. At the *right*, a measurement of the strength and duration of its pulses is performed. The oscilloscope measures 10 mV and 1 ms per division

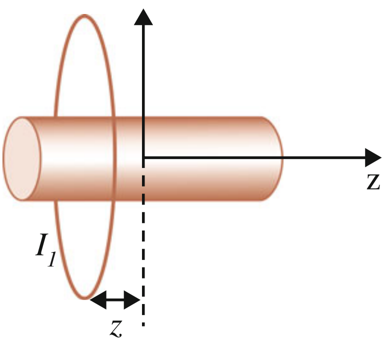
Treat the body of the fish and the surrounding environment as water at normal temperature and pressure (Fig. 8.29).

8.21 *Metamaterials*

At the end of the 1960s, the existence of so-called meta-materials was hypothesized. These are characterized by the fact that the dielectric constant and/or the relative magnetic permeability are/is *negative*. Yet no such material has been discovered.
If a material would be converted to a meta-material at a certain strength of the external influence, what would its virgin curve look like in a hysteresis investigation according to Exercise (8.15) and (8.16)?
See also Exercise (11.10).

8.22 *Measurement of permanent magnetization*

A circular loop of radius a carries current I_1 and is placed at a distance z from the center of a homogeneously magnetized bar, see figure. The bar has a circular cross section with radius b and length l. The measured value of the force is denoted F. Determine the magnetization of the bar.



Further Readings

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Chapter 9
Motional Consequences

I have long held an opinion, almost amounting to conviction, in common I believe with many other lovers of natural knowledge, that the various forms under which the forces of matter are made manifest have one common origin; or, in other words, are so directly related and mutually dependent, that they are convertible, as it were, one into another, and possess equivalents of power in their action.

Michael Faraday, 1845

The previous Chaps. 1–8 form together a summary of classical electromagnetic theory. This theory is based upon three forces: the electric, magnetic and inductive. Since they all originate from electric charge, the question arises whether there is a connection amongst these forces, i.e. are they aspects of one single force? In Chap. 2 it was shown that the magnetic and inductive forces depend on the motion of the charges and it was indicated that these forces should be considered as corrections to the electric force.

To understand why these motional consequences occur, this chapter introduces a model of the electromagnetic interaction based on the assumption that force action is established by an exchange of momentum mediated at a finite speed. In this way, the electric force will depend on the motion of the charges and the magnetic and inductive forces arise naturally. The relation amongst the three forces is then conceptually clarified.

While this analysis is done within a special case, which is the parallel motion as in Fig. 9.1, the general derivation is contained in the theory of special relativity. From Sect. 9.4, this theory will be in focus. Its starting point is the so-called time dilation, meaning that the perception of time depends on the motion of the observer. By defining time through the concept of motion, it is shown that time dilation results as a consequence of the magnetic force.

The exercise section contains a tutorial for additional development of the theory of relativity.

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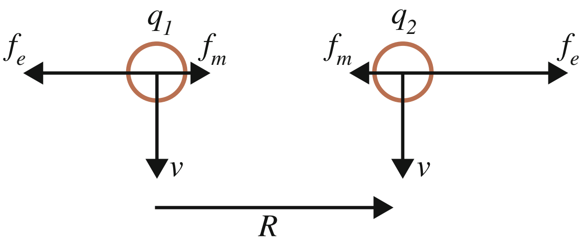


Fig. 9.1 Two identical charges in uniform motion. Electric and magnetic forces are at work

9.1 Modelling the Electrodynamic Interaction

To be able to see the relation between the three electrodynamic forces, the complexity of the phenomena must be reduced to a minimum. In fact, the issue may be explored using the smallest constituent of nature: the pairwise mutual interaction between two fundamental electrically charged objects, e.g. two electrons, Fig. 9.1.

Effects of motion appear if interactions take time, i.e. the momentum is mediated at a finite speed. In addition, to derive the magnetic and inductive forces as motional consequences it turns out that the speed of the mediation must be independent of the speed of the charges.

Thus, the following starting points are obtained:

- The electric force is known.
- Forces are mediated at a finite speed.
- The mediation speed is independent of the objects' motion.

9.2 Magnetism as a Motional Consequence

What is the origin of the magnetic force, i.e. what is its relation to the electric force? In Fig. 9.1, the two forces are separated. In reality only the effect of the resulting net force is observed, so the question may be formulated:

Why is the force between two like electric charges in parallel motion less than that when they are at rest?

Provided the interactions in some way are mediated at a finite speed, i.e. interactions take time, the effective distance over which the charges interact is altered when they move. Figure 9.2 illustrates that the effective distance between the two charges in motion is $R^* > R$.

Let w be the velocity at which the interaction is mediated, independent of the motion of the source and directed along the effective distance R^*. Denote the time it takes to establish the interaction in motion by $T = R^*/w$.

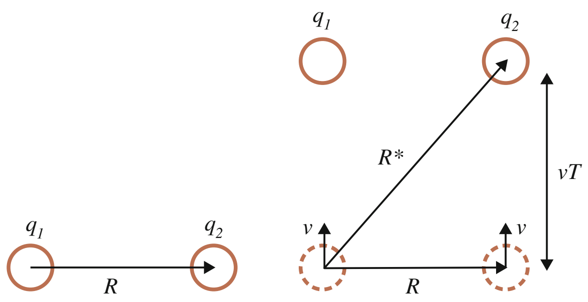


Fig. 9.2 The effective distance over which two charges interact for charges at rest (*left*) and in uniform motion (*right*)

Thus,

$$R^* = \sqrt{R^2 + (vT)^2} = \sqrt{R^2 + \left(\frac{vR^*}{w}\right)^2} \tag{9.1}$$

which gives

$$R^* = \sqrt{\frac{R^2}{1 - v^2/w^2}} \tag{9.2}$$

In Chap. 2 the total force, i.e. electric plus magnetic, was obtained as

$$\bar{f} = \frac{q^2}{4\pi\varepsilon_0 R^2}(1 - \varepsilon_0\mu_0 v^2)\hat{R} \tag{2.24}$$

which may be written

$$\bar{f} = \frac{q^2}{4\pi\varepsilon_0 R^{*2}}\hat{R} \tag{9.3}$$

provided that the mediation speed is $w = 1/\sqrt{\varepsilon_0\mu_0}$ (show this). Conventionally, w is referred to as the speed of light and is denoted by c, i.e. $c = 1/\sqrt{\varepsilon_0\mu_0}$.

Formula (9.3) is identified as the electric force, formula (2.1), acting at the distance R^*. The interaction, which has previously been regarded as an electric plus a magnetic force, is accordingly a sole electric force acting at a distance R^*. This distance increases with the speed v. Note however that the force is directed horizontally, i.e. along R, *not* along the distance of mediation R^*.

The conclusion of this section is thus:

The magnetic force is fictitious, arising as a motional consequence of the electric force which in turn is an effect of interactions taking time.

9.3 Induction as a Motional Consequence

In Sect. 2.3, the source of induction was found to be the accelerated charge described by formula (3.34). The force on charge 2 due to acceleration of charge 1 is

$$\bar{f}_{ind} = -\frac{\mu_0 q_2 q_1}{4\pi R}\frac{d\bar{v}_1}{dt} \tag{3.34}$$

Induction will now be considered in the interaction as in Fig. 9.1. Momentum is transferred from one object to the other along the distance vector R^* according to Fig. 9.3. Consider three moments in time. The time between two of these moments is denoted by T, the speed by v and the distance between the objects by R. The electromagnetic force is known to be horizontal, directed along the distance vector R between the objects. At time $t = t_2$, object 2 (on the right) receives momentum from object 1. Since interactions take time, this momentum is transported along the line R^*. The momentum which is received by object 2 at $t = t_2$ was emitted by object 1 at time t_1 (so-called 'retarded interaction'). Consequently, the received momentum will result in a diagonal upward force, i.e. along R^*. Thus, there is a force which is vertical and the sum of this and the electromagnetic force equals the total force f_{tot}, Fig. 9.3.

At the next instance object 2 emits an impulse to object 1 resulting in a downward recoiling force (not drawn in Fig. 9.3) so that the vertical forces cancel out. This is necessary or else the objects would accelerate vertically which is physically impossible.

The vertical force is presumed to be the inductive f_{ind}, formula (3.34), since it is oppositely directed and proportional to the acceleration of its source, object 1, which

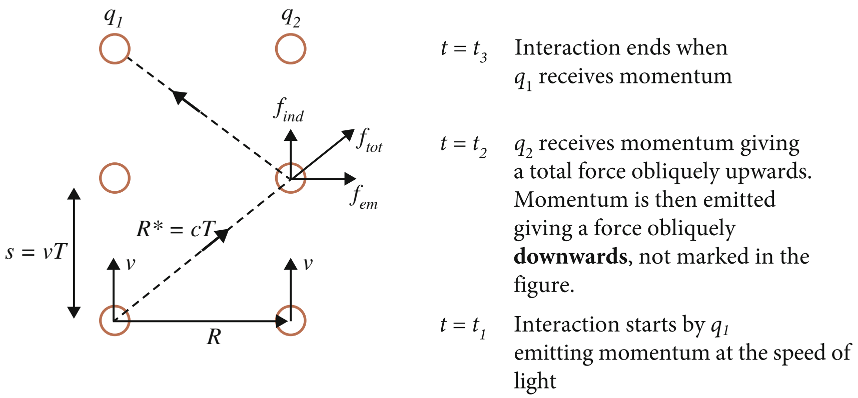


Fig. 9.3 The pairwise interaction at three moments in time. The interaction is established through an exchange of momentum along the *dotted lines*. The interaction from t_2 to t_3 results in a downward force on object 2 which is not drawn in the figure. The objects may be thought of as being in a storage ring so that the distance R is fixed

recoils diagonally downward at $t = t_1$. The purpose is now to derive this force and thereby verifying (3.34).

Since an acceleration dependent force is sought, the question arises: what acceleration would the vertical force cause if it were not cancelled by the corresponding opposite vertical force? As is worked out in Exercise (9.3), a vertical single force acting downwards would decelerate the object from v to 0 during time T, i.e. with acceleration $a = v/T$. From the geometry in Fig. 9.3 it is then established that

$$f_{ind} = \frac{vT}{R} f_{em} = \frac{aT^2}{R} f_{em} = \frac{a}{R} \frac{R^{*2}}{c^2} \frac{q_1 q_2}{4\pi\varepsilon_0 R^{*2}} \tag{9.4}$$

The inductive force on object 2 originates in an impulse from object 1 so that, for identical objects, $\bar{a} = -d\bar{v}_1/dt$ and the force becomes

$$\bar{f}_{ind} = -\frac{q_1 q_2}{4\pi\varepsilon_0 c^2 R} \frac{d\bar{v}_1}{dt} \tag{9.5}$$

which is Weber's inductive force (3.34).

Thus, the inductive force was obtained as a motional consequence of the electric force.

9.4 Special Theory of Relativity

9.4.1 Relative Motion

A parallel motion, Fig. 9.1, will now be utilized to introduce the special theory of relativity. The first step in this theory is to define the concept of relative motion.

The velocity v in Fig. 9.1 has to be related to something. Apparently, the objects are at rest relative to each other so the velocity v must relate to the motion of an observer. Galileo formulated the principle of relative motion: there is no absolute motion, all motion must be expressed relative to something.

Einstein embraced this principle when he examined electrodynamics based on a well-known phenomenon illustrated in Fig. 9.4, compare Exercise (3.9). In the left-hand figure the magnet approaches the non-moving loop and vice versa in the right-hand figure. The induced current depends only on the relative motion.

The principle of relative motion may be extended by introducing an observer who examines the pairwise interaction between two identical charges, Fig. 9.5.

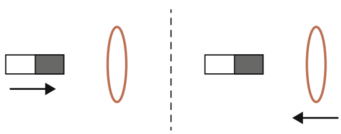


Fig. 9.4 A magnet approaches a coil and vice versa. The same induction current arises in both cases

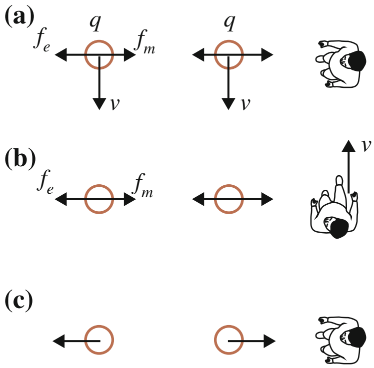


Fig. 9.5 A person observes the pairwise interaction between identical charges in three cases. **a** Observer at rest. **b** Observer in motion. **c** Relative rest

The two upper cases are identical because the relative motion is the same. If the two lower cases are compared it is concluded that a magnetic force may be generated solely by putting oneself in motion. The magnetic force is therefore fictitious because it arises due to the motion of the observer. Compare this to the coriolis and centrifugal forces which are also fictitious as they are generated via motional effects. These three phenomena are therefore not referred to as forces but as ‘effects’ (meaning ‘consequence’). However, the observer in Fig. 9.5 only observes the total force, or rather the change in horizontal motion, i.e. the acceleration. In case b, a slower repulsive motion is observed than in case c.

The total force between the objects thus depends on the motion relative to the observer which can seem peculiar as the observer does not affect the objects. Should not the relative motion of the objects to each other be the sole source of influence on the force with which they interact? It is at this stage that the new physics introduced by relativity theory becomes apparent: the observer is a part of the course of events.

That this is a part of reality is confirmed by the events in a particle accelerator where a collection of identical objects, e.g. electrons, are accelerated in close proximity. It is observed that the repulsion between them becomes vanishingly small for speeds close to the speed of light even though they are at rest relative to each other. However, for the accelerator physicist, i.e. the observer, they are travelling close to the speed of light. The fact that the forces between the particles almost vanish in the final phase of a particle accelerator is also a condition for its function as otherwise it would be difficult to obtain a stable beam of particles.

The effect of the relative motion on the observation itself is the very foundation for the theory of relativity. The interpretation of the result that the force between charges depends on the motion of the observer is in part that the motion affects the experienced course of time. To understand this interpretation one must ponder the concept of time. The basic clock is naturally the motion of the earth; the day and the year. Other clocks are based on other types of repeated motion, such as the pendulum and atomic clocks. Time in physics is therefore nothing else but a measure of a motion in relation to a reference.

That the observer in case b interprets the observation by saying that time runs slower is therefore natural. One can relate to so-called ‘slow motion’-effects in movies. When it is played at a slower pace, all motion is slower and time is experienced as slower. The observer in motion therefore sees the process in 'slow motion' and interprets this as time is passing slower.

When the relative motion reaches the speed of light, i.e. $v = c = 1/\sqrt{\varepsilon_0 \mu_0}$, the magnetic effect and the electric force cancel each other (see formula 2.24) and the repulsive force ceases, time stands still. This conclusion must be valid also for speeds greater than that of light. To understand what happens when the relative motion approaches the speed of light, consider Fig. 9.2. It may be noted that the distance over which the momentum is mediated increases with relative speed to become infinitely large at and above the speed of light. Then no momentum mediation can take place.

9.4.2 Time Dilation

The above discussion will now be quantified by formulating the relativity of time, the so-called time dilation.

An observation is a registration of an object's change in motion in terms of space and time coordinates. Other concepts, such as mass, force and charge are abstract. They are part of the descriptive model that is used to comprehend reality. Since the space coordinates are included in the distance vector R, perpendicular to the motion of the observer in Fig. 9.5, it is assumed that these are not dependent on the observer in this case. The observers at rest and in motion, cases c and b in Fig. 9.5, are doing the same horizontal motion, i.e. none. Thus, only the time coordinate remains. Assume that there is a linear relationship between the time intervals which are perceived by the observer in motion dt and at rest dt_0, i.e. $dt = \gamma dt_0$ where γ depends on v. The repulsive motion that is observed (for like charges) is given by Newton's law of force. In case b in Fig. 9.5, i.e. for the observer in motion, the following repulsion is obtained (see formula 2.24)

$$m\frac{d}{dt}\frac{d\bar{R}}{dt} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\left(1 - \frac{v^2}{c^2}\right)\hat{R} \tag{9.6}$$

and in case c

$$m\frac{d}{dt_0}\frac{d\bar{R}}{dt_0} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\hat{R} \tag{9.7}$$

Thus, in case b the repulsive motion is affected by the magnetic effect whereas in case c only the electric force occurs. m is the mass of the object which at this stage is considered as constant since a description in terms of the coordinates is sought.

Formula (9.6) may be written

$$m\frac{d}{\gamma dt_0}\frac{d\bar{R}}{\gamma dt_0} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\left(1 - \frac{v^2}{c^2}\right)\hat{R} \tag{9.8}$$

Dividing formula (9.7) by formula (9.8) gives $\gamma = 1/\sqrt{1 - v^2/c^2}$ resulting in the time dilation

$$dt = \frac{dt_0}{\sqrt{1 - v^2/c^2}} \tag{9.9}$$

where v is the relative motion between the object and the observer (vertically in Fig. 9.5). The formula reflects the fact that $v < c$.

9.4.3 Relativistic Momentum

The time dilation together with the principle of relative motion cause a violation of momentum conservation as illustrated in the following example.

Consider an interaction between two identical objects in anti-parallel motion, Fig. 9.6. In the upper diagram the observer's motion is such that the two objects have the same speed in opposite directions. In the lower diagram the observer is moving together with the right charge in its vertical motion.

Due to time dilation, the moving observer perceives a slower horizontal (transverse) motion of the leftward charge as compared to the rightward charge. The electromagnetic interaction would thus result in a change of momentum which is less for the leftward charge than for the rightward one. This would violate Newton's third law, the principle of action and reaction, and accordingly the principle of momentum conservation.

To respect this fundamental principle, the expression for the transversal momentum needs to be made invariant, i.e. independent of the vertical motion of the observer. One way of doing this is to redefine the concept of mass such that the diminishing horizontal motion is compensated by a larger mass. The higher the speed relative to an observer the higher the mass becomes. In the lower diagram of Fig. 9.6, the leftward charge has a higher vertical speed than the rightward charge relative to the observer and thereby a larger mass of the leftward charge is anticipated.

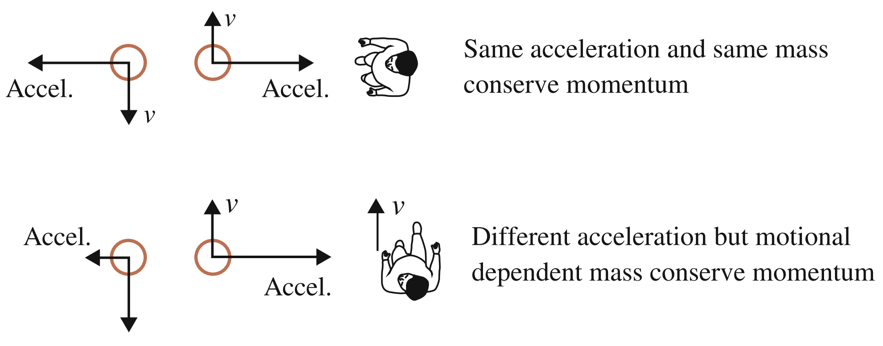


Fig. 9.6 Observation of two like charges in anti-parallel motion

To formalize this discussion formula (9.8) may be rewritten

$$\frac{d}{\gamma dt_0} m \frac{d\bar{R}}{\gamma dt_0} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\left(1 - \frac{v^2}{c^2}\right)\hat{R} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\frac{1}{\gamma^2}\hat{R} \tag{9.10}$$

The momentum is then

$$\bar{p} = m\frac{d\bar{R}}{\gamma dt_0}$$

where m is the mass which is perceived at relative rest and is henceforth denoted by m_0. To achieve invariance of the momentum, i.e. independence of the observer's motion v, it is necessary to multiply the momentum by γ. By multiplying the right-hand as well as the left-hand side of formula (9.10) by γ we obtain

$$\frac{d}{dt}\left(m_0\gamma\frac{d\bar{R}}{dt}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\frac{1}{\gamma}\hat{R} \tag{9.11}$$

where the rest mass m_0 has been introduced and the fact that $dt = \gamma dt_0$ has been used.

The relativistic momentum is thus by definition

$$\bar{p}_{rel} = m_0\gamma\frac{d\bar{R}}{\gamma dt_0} = m_0\gamma\frac{d\bar{R}}{dt} \tag{9.12}$$

It is this quantity that fulfils Newton's principle of action and reaction and is therefore conserved in the processes of nature.

Note that the right-hand side of formula (9.11) is the redefined electromagnetic force. The force formula has been rewritten such that the repulsion which is observed in e.g. Fig. 9.5b is partly caused by the electromagnetic force and partly by a mass change. This reinterpretation is justified since both mass and force are abstract concepts, i.e. not directly observable.

The transverse momentum is now equal for the two observers but not the motion itself. The repulsion in Fig. 9.5 is still slower for the observer in relative motion whereby time dilation arises.

9.4.4 Relativistic Energy

The new definition of momentum, formula (9.12), will result in a new definition of energy. Consider the kinetic energy a single object acquired under the influence of an external force, see Fig. 9.7.

In formula (9.12), $\gamma = 1/\sqrt{1 - v^2/c^2}$ where v is the vertical velocity while $d\bar{R}/dt$ is the horizontal velocity. In Fig. 9.7 the object's direction of motion coincides with the motion relative to the observer wherefore the momentum of the object may be written

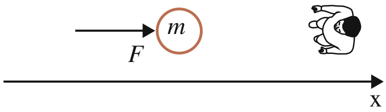


Fig. 9.7 A person observes the acceleration of an object along the x axis

$$\bar{p} = \gamma(u) m_0 \bar{u} \tag{9.13}$$

where u is the speed of the object. This is the general expression for momentum. Newton's force law is thus

$$\bar{F} = \frac{d\bar{p}}{dt} = \frac{d}{dt}\gamma(u) m_0 \bar{u} \tag{9.14}$$

Here, the time coordinate t is time dilated as the process takes place in relative motion. The time coordinate is also included in the speed $\bar{u} = dx/dt\,\hat{x}$.

Through the definition of work as force times distance, relativistic kinetic energy will now be derived. Let the external force F affect the object so that it accelerates from initial speed 0 at $x = 0$ to the final speed v at $x = X$. Consider the case when no other forces are acting so that potential energy can be neglected. All work will then be transformed into kinetic energy, $W = E_{kin}$, which becomes:

$$\begin{aligned}
W &= \int_0^X F dx = \int_0^X \frac{d}{dt}\left[\gamma(u) m_0 u\right] dx = \int_0^X \frac{d}{du}\left[\gamma(u) m_0 u\right] \frac{du}{dt} dx \\
&= \int_0^v \frac{d}{du}\left[\gamma(u) m_0 u\right] u du = \int_0^v \frac{d}{du}\left[\frac{u}{\sqrt{1-u^2/c^2}} m_0\right] u du \\
&= \int_0^v \left[\frac{m_0 u}{\sqrt{1-u^2/c^2}} + \frac{m_0 u^3/c^2}{\left(1-u^2/c^2\right)^{3/2}}\right] du = \int_0^v \frac{m_0 u}{\left(1-u^2/c^2\right)^{3/2}} du \\
&= \left[\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}\right]_0^v
\end{aligned}$$

Thus, relativistic kinetic energy is

$$E_{kin} = \underbrace{\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}}_{\text{Total energy}} - \underbrace{m_0 c^2}_{\text{Rest energy}} \tag{9.15}$$

A crucial difference from the Newtonian expression is that one of the terms does not depend on the velocity v. The conventional interpretation is that m_0c^2 is to be considered as rest energy and $m_0c^2/\sqrt{1-v^2/c^2}$ as the total energy E_{TOT}. It may therefore be written

$$E_{TOT} = E_{kin} + m_0c^2 \tag{9.16}$$

which is the energy expression that is conserved in nature. Note however that in general there is also a potential energy appearing on the right-hand side of formula (9.16).

The energy of an object at rest is

$$E_{TOT} = m_0c^2 \tag{9.17}$$

which is interpreted such as the mass of the object in itself corresponds to a certain amount of energy.

The work W will convert into the classical expression for kinetic energy at low speeds which may be shown by series expanding the expression:

$$E_{kin} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \approx m_0c^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) - m_0c^2 = \frac{m_0v^2}{2} \tag{9.18}$$

The fact that the relativistic expression turns into the classical one for low speeds is a general principle. It also applies to momentum and mass.

In classical physics, the relationship between momentum and kinetic energy is $E = p^2/(2m)$. In the relativistic case $p = m_0v/\sqrt{1-v^2/c^2}$ and $E_{TOT} = m_0c^2/\sqrt{1-v^2/c^2}$ so that

$$p^2c^2 + m_0^2c^4 = \frac{m_0^2v^2c^2}{1-v^2/c^2} + m_0^2c^4 = \frac{m_0^2c^4}{1-v^2/c^2} = E_{TOT}^2 \tag{9.19}$$

Thus

$$E_{TOT}^2 = p^2c^2 + m_0^2c^4 \tag{9.20}$$

which is a useful formula in many contexts, e.g. in quantum mechanics.

In Exercise (9.5), formula (9.20) is used as a starting point for the definition of so-called four-vectors.

9.5 Summary

Both the magnetic and the inductive force are motional consequences of the electric force and can be derived from the fact that interactions take time; they are mediated at the speed of light. These motional consequences are commonly known as ‘relativistic effects’.

The special theory of relativity is based on the principle of relative motion and Newton's principle of action and reaction.

Time dilation

$$dt = \frac{dt_0}{\sqrt{1 - v^2/c^2}} \tag{9.9}$$

is a direct consequence of the magnetic effect.

For an object with rest mass m_0, relativistic momentum is

$$\bar{p} = \gamma(u) m_0 \bar{u} \tag{9.13}$$

by which the principle of action and reaction is fulfilled so that momentum is conserved.

The relativistic kinetic energy is

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 \tag{9.15}$$

9.6 Exercises

9.1 A football match is replayed on TV at a five times slower pace compared to actual time. How quickly do you have to move relative to the football pitch to obtain the same effect?

9.2 The American physicist Feynman once said: 'some claim that the special theory of relativity is derivable from the relationship between the electric and the magnetic force but nothing could be more wrong'.
Discuss this statement.

9.3 In Sect. 9.3 the acceleration from the inductive force in Fig. 9.3 was claimed to be $a = v/T$. Motivate this by considering the initial acceleration brought about by an external force to put charges in motion v.
Hint: Let this acceleration take place during the interaction time T and convince yourself that the inductive force must equal the external force.

*9.4 *Length contraction*

The muon is a non-stable particle decaying into an electron and a so-called neutrino. The life time for a muon at rest is $\tau_0 = 2.2\,\mu$s.
Muons are part of the cosmic radiation towards the earth and can be identified using e.g. scintillation detectors. In a classic experiment, the number of muons at a certain height above the earth was investigated and compared to the number of muons at the surface. During a certain time interval, 5.0×10^4 muons were identified at the altitude 3.0 km and 1.1×10^4 were identified at the surface of the earth. The speed of the muons was determined to be $v = 0.95\,$c.

The formula for radioactivity is assumed to be valid for the decay of the muons:

$$N(t) = N(t = 0)e^{-t/\tau}$$

where τ is the life time of the muon.
Investigate how many muons that are *expected* at the surface of the earth during the same time interval

a. Without the relativistic effect, i.e. no time dilation.
b. With the relativistic effect.
Now consider the issue in the rest system of the muon. The number of muons at the surface of the earth should be the same as before, but the life time of the muon for this observer is that at rest $\tau_0 = 2.2\,\mu\text{s}$.
c. Utilizing this example, show that length contraction is a direct result of time dilation:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{9.21}$$

where L_0 is the distance perceived by an observer at rest.

Exercises 9.5–9.11 are written as a tutorial. Lorentz transformations and the concept of 4-dimensional space-time will be introduced.

*9.5 *Four-vectors—four-dimensional space-time*

Relativistic effects, i.e. motional consequences originating in time dilation (equivalent to the magnetic effect), were interpreted by Minkowski as evidence for a four-dimensional world with time as the fourth dimension. The starting point for this interpretation is the identification of so-called *invariants*. These are quantities independent of the motion of the observer and are formed from the magnitude of a four-dimensional vector. In this exercise and the next one, two such *four-vectors* shall be defined.

a. Starting from (9.20)

$$E_{TOT}^2 = p^2c^2 + m_0^2c^4$$

where p is relativistic momentum, an invariant may be formed:

$$\frac{E_{TOT}^2}{c^2} - p^2 = m_0^2c^2$$

which has the same value for any observer in arbitrary motion.

The magnitude squared of a four-vector is defined in the following way:

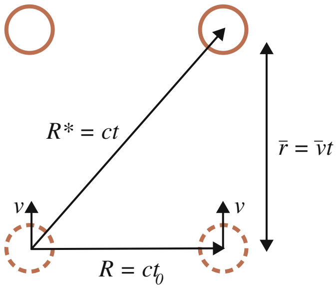
$$A^2 = A_1^2 + A_2^2 + A_3^2 - A_4^2 \tag{9.22}$$

Thereby, show that the four-vector

$$p_\mu = \left(p_x, p_y, p_z, \frac{E_{TOT}}{c}\right) \tag{9.23}$$

has an invariant magnitude. The greek letter as the index of a four-vector is standard notation.

*9.6 The figure describes two like electric charges in parallel motion.



With the aid of the figure, show that

$$r^2 + c^2t_0^2 = c^2t^2 \Leftrightarrow x^2 + y^2 + z^2 + c^2t_0^2 = c^2t^2$$

Form the invariant

$$x^2 + y^2 + z^2 - c^2t^2 = -c^2t_0^2 \tag{9.24}$$

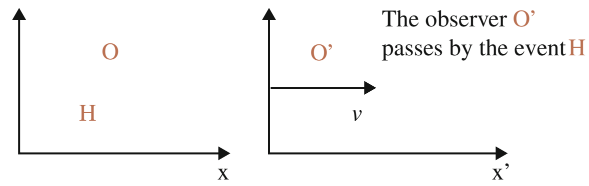
and show that the four-vector

$$r_\mu = (x, y, z, ct) \tag{9.25}$$

has an invariant magnitude.

*9.7 *Lorentz transformations*

The special theory of relativity deals with how observers in uniform motion relative to each other perceive events. This exercise explores how the coordinates in these systems are related.
Let the observer O be at rest whereas the observer O' is moving with the velocity v in the x direction according to the figure below. Assume that they observe an event H (e.g. a spider that spins its web) and wish to describe the coordinates of the event in time and space. Let their coordinate systems coincide at $t = 0$.



In classical physics, the following so-called Galilei transformations apply:

$$\begin{aligned} x' &= x - vt \\ t' &= t \\ y' &= y \\ z' &= z \end{aligned}$$

Because of time dilation and length contraction, different transformations are expected within the theory of relativity. Provided that these are linear and approach Galilei transformations for $c \rightarrow \infty$, the general transformations become

$$\begin{aligned} x' &= A(x - vt) \\ t' &= Bx + Dt \\ y' &= y \\ z' &= z \end{aligned}$$

The factors A, B and D are determined by the invariant (9.24), i.e.

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \tag{9.26}$$

a. Using (9.26), derive the Lorentz transformations:

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ y' &= y \\ z' &= z \end{aligned} \tag{9.27a}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

b. Show that the Lorentz transformations convert to Galilei transformations for $c \rightarrow \infty$. This is important as it shows that the difference between classical and relativistic physics is the fact that 'interactions take time'.

9.8 a. Show that length contraction and time dilation are results of the Lorentz transformations. Start from (9.27a) in differential form:

$$dx' = \gamma(dx - vdt)$$
$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$
$$dy = dy'$$
$$dz = dz' \tag{9.27b}$$

Answer: The time dilation is a direct result since $dx = 0$.

The distance dx' should however be expressed for $dt' = 0$ since the end points of this distance are to be measured simultaneously.

$$dx' = \gamma(dx - vdt) = \gamma\left(dx - v\left(\frac{dt'}{\gamma} + \frac{v}{c^2}dx\right)\right)$$
$$dx' = \gamma\left(1 - \frac{v^2}{c^2}\right)dx = \frac{dx}{\gamma}$$

b. Based on the Lorentz transformations, show that the concept of simultaneity is relative, i.e. dependent on the motion of the observer.

Answer: Assume that two events with the positions x_1 and x_2 occur at time $t = t_1$ in the system of the observer O. In a moving system, the times for these two events are:

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$
$$t_2' = \gamma\left(t_1 - \frac{v}{c^2}x_2\right)$$

Thus, $t_1' \neq t_2'$, i.e. no simultaneity in the system of O′.

c. Try to find a conceptual explanation for the relativity of simultaneity.

*9.9 Derive the Lorentz transformations for the four-vector momentum

$$p_\mu = \left(p_x, p_y, p_{z,} \frac{E_{TOT}}{c}\right)$$

Answer: Since the Lorentz transformations were obtained from the condition of invariance (9.26), p_μ obeys the same transformations as the coordinate vector, i.e.:

$$p_x' = \gamma\left(p_x - v\frac{E_{TOT}}{c^2}\right)$$
$$\frac{E_{TOT}'}{c^2} = \gamma\left(\frac{E_{TOT}}{c^2} - \frac{v}{c^2}p_x\right)$$
$$p_y' = p_y$$
$$p_z' = p_z \tag{9.28}$$

9.10 *Velocity transformations*

a. Derive the Lorentz transformations for velocity utilizing (9.27b).

Answer: Velocity in the moving system becomes

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$\frac{dz'}{dt'} = \frac{dz}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{\frac{dz}{dt}}{\gamma(1 - \frac{v}{c^2}\frac{dx}{dt})}$$

b. Let an object be travelling at 0.90 c in the negative x direction in the system of O. If O′ is travelling at 0.50 c, what speed will be perceived in the system of O′ in classical and relativistic physics respectively?
O and O′ are observers in relative motion according to Exercise (9.7).

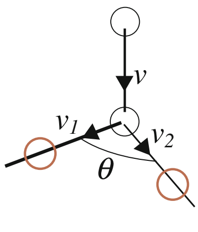
Answer: In classical physics the speed would be 1.4 c. In relativistic physics

$$\frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{-0.90c - 0.50c}{1 + \frac{0.50c}{c^2}0.90c} = \frac{-1.40c}{1.45}$$

i.e. less than c. Nothing can move faster than light in the theory of relativity.

9.11 *Relativistic kinematics*

In a particle physics experiment an interaction between two protons is studied. In the rest frame of the observer one proton is at rest and the other is incoming with speed $\bar{v}$. Assume that the interaction may be viewed as a collision, such as between two billiard balls.



Determine the angle θ between the two paths of the protons after the ‘collision’ using

a. classical physics
b. relativistic physics. Express the angle in terms of the outgoing velocities.

Hint: Apply the principles of energy and momentum conservation.

Answer:

a. Momentum and energy conservation

$$m\bar{v} = m\bar{v}_1 + m\bar{v}_2$$

$$\frac{mv^2}{2} = \frac{mv_1^2}{2} + \frac{mv_2^2}{2}$$

give

$$v^2 = v_1^2 + v_2^2 + 2\bar{v}_1 \cdot \bar{v}_2$$
$$v^2 = v_1^2 + v_2^2$$

such that

$$\bar{v}_1 \cdot \bar{v}_2 = 0 \Rightarrow \cos\theta = 0$$
$$\theta = \frac{\pi}{2}$$

b. Momentum and energy conservation

$$\gamma m_0 \bar{v} = \gamma_1 m_0 \bar{v}_1 + \gamma_2 m_0 \bar{v}_2$$
$$m_0 c^2 + \gamma m_0 c^2 = \gamma_1 m_0 c^2 + \gamma_2 m_0 c^2$$

give

$$\gamma \bar{v} = \gamma_1 \bar{v}_1 + \gamma_2 \bar{v}_2 \qquad (i)$$
$$1 + \gamma = \gamma_1 + \gamma_2 \qquad (ii)$$

Squaring the first equation gives

$$\gamma^2 v^2 = \gamma_1^2 v_1^2 + \gamma_2^2 v_2^2 + 2\gamma_1\gamma_2 \bar{v}_1 \cdot \bar{v}_2$$

The left-hand side may be replaced using Eq. *(ii)*:

$$\gamma^2 = (\gamma_1 + \gamma_2)^2 - 2(\gamma_1 + \gamma_2) + 1$$
$$\frac{1}{1 - \frac{v^2}{c^2}} = (\gamma_1 + \gamma_2)^2 - 2(\gamma_1 + \gamma_2) + 1$$
$$1 = (\gamma_1 + \gamma_2)^2 - 2(\gamma_1 + \gamma_2) + 1 - [(\gamma_1 + \gamma_2)^2 - 2(\gamma_1 + \gamma_2) + 1]\frac{v^2}{c^2}$$
$$\gamma^2 v^2 = c^2[(\gamma_1 + \gamma_2)^2 - 2(\gamma_1 + \gamma_2)]$$

so that

$$2\gamma_1\gamma_2\bar{v}_1 \cdot \bar{v}_2 = c^2[(\gamma_1+\gamma_2)^2 - 2(\gamma_1+\gamma_2)] - \gamma_1^2 v_1^2 - \gamma_2^2 v_2^2$$

and

$$\cos\theta = \frac{c^2[(\gamma_1+\gamma_2)^2 - 2(\gamma_1+\gamma_2)] - \gamma_1^2 v_1^2 - \gamma_2^2 v_2^2}{2\gamma_1\gamma_2 v_1 v_2}$$

9.12 *The metric tensor* $g_{\mu\nu}$

a. Let the space-time coordinates be infinitesimal intervals and express the invariant (9.24) as

$$\sum_{\mu=1}^{4}\sum_{\nu=1}^{4} g_{\mu\nu}\, dr_\mu dr_\nu \tag{9.29}$$

Determine thereby the elements of the metric tensor $g_{\mu\nu}$.
The metric tensor has a fundamental role in the general theory of relativity dealing with so-called curved space-time. In special theory of relativity, space-time is flat.

b. Convince yourself that on a curved surface, such as the surface of an orange, a different metric tensor is obtained than that for a flat surface.
Hint: When forming the invariant (9.24), the starting point was Pythagora's theorem. Is this valid on a curved surface?

Starting from this exercise, one may begin to examine the general theory of relativity dealing with general motions of observers, i.e. including acceleration.

*9.13 *Magnetic force*

a. Whittaker showed in 1910 that the general magnetic force formula between two electric charges based on the measurements of Ampère 1820–1825 is

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}\Big[3(b+1)(\bar{v}_1 \cdot \hat{R})(\bar{v}_2 \cdot \hat{R}) \times \hat{R} - (2+b)(\bar{v}_1 \cdot \bar{v}_2)\hat{R} - b(\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2\Big]$$

where a and b are constants.

Show with the aid of formula (9.3) that $b = -1$ and compare with formula (2.11).

b. Under what conditions were Ampère's measurements carried out?

Answer: Closed conductors and low speed relative that of light. Whittaker's formula is thus valid only for low speed of the charges.

9.14 The magnetic effect between two charges in parallel motion, Fig. 9.2, has a natural conceptual explanation according to Sect. 9.2.
Consider the perpendicular motion, Fig. 2.9. Is it possible to find a natural explanation also in this case within the model that has been used for the parallel motion?

Answer: The problem here is that it is not possible to find a reference system where both charges are at rest which is necessary for the Coulomb force formula to be valid.

9.15 In this chapter it has been established that the magnetic and inductive dynamics are motional consequences of the electric force.

With this in mind, consider Exercise (2.23) once more.

Ref: K. Prytz, Sources of inertia in an expanding universe, Open Physics, 13 (2015) 130

9.16 Is the quantity temperature relative, i.e. is the observed temperature for a system dependent on the motion of the observer?

What is the limit value of the temperature when the observer approaches light speed?

Answer: The impossibility of travelling at or above the speed of light may also be realised using thermodynamic principles. To this end, the simple pairwise interaction may be generalised into a real system. As has been mentioned before, all real systems are constructed via the sum of simple pairwise interactions. At the speed of light, the interactions cease, i.e. all motional change in the system ceases. Because temperature corresponds to microscopic motion, this would mean that a system has been generated at absolute zero, which is impossible according to thermodynamic principles. Thus, there is a relation between light speed and absolute zero temperature.

9.17 In September 2011 the research group OPERA reported a surprising measurement of the speed of so-called neutrinos, a kind of elementary particles which interact only via weak force and also possibly via gravitation.
The experiment utilized neutrinos that were produced in an accelerator at Cern in Geneva and travelled below the earth's surface to a subterranean facility in central Italy. The distance covered was 731.278 km and the journey time, measured using the satellite-based GPS-system, was determined to be on average 2.43922 ms.

a. Determine the average speed of the neutrinos.
b. Compare this to the speed of light in vacuum, 299.792 km/s.
c. How does this result compare to the conclusion from the theory of relativity that nothing can travel faster than light?

This measurement was however declared to be faulty in March 2012. (Otherwise this chapter had to be rewritten.)

Further Readings

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B. Hoffman, *Albert Einstein: Creator and Rebel* (Plume Books, New York, 1972)
P. Lorrain, D.R. Corson, F. Lorrain, *Electromagnetic Fields and Waves* (Freeman and Company, New York, 1988)

Original Papers

K. Prytz, Force between electric charges: a new approach to relativity theory. Galilean Electrodyn. Spec. Issue **1**, 11 (2007)
K. Prytz, The origin of electromagnetic induction. Galilean Electrodyn. **23**, 99 (2012)

Chapter 10
Field Theory

In fact, whenever energy is transmitted from one body to another in time, there must be a medium or substance in which the energy exists after it leaves one body and before it reaches the other ... and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise.

James Clerk Maxwell, 1873

10.1 The Concept of a Field

The notion of an electric and magnetic field originates in the issue of conceptually interpreting action at a distance. By letting a field be emitted from one object and received by the other, the mechanism of the force action was assumed to be clarified. Such a description of electromagnetic phenomena may have practical advantages (often apparent) since the complexity of the problem is then divided into two parts associated with each object respectively. The field from one object is first calculated and secondly the force on the other object located in the field from the first object. The strength of this procedure appears when the field generating object is fixed which is often the case in technical applications. However, in case of mutual interactions, i.e. basal physics, the field concept is less useful.

Within electromagnetic signal theory, involving for example antennas and wave guides with belonging transmission and reflection effects, the electromagnetic field theory is predominant.

Therefore, in this chapter these fields will be defined with the aim to formulate the four equations describing their dynamics, constituting the basis for the science of electromagnetic engineering. These are called Maxwell's equations and were developed by Maxwell and Heaviside during the second half of the nineteenth century.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_10

10.2 The Electric and the Magnetic Fields

The electric field $\bar{E}$ from a point charge is defined through the electric force formula (2.1):

$$\bar{F}_{1\to 2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R} = q_2\bar{E}_{1\to 2} \tag{10.1}$$

such that

$$\bar{E}_{1\to 2} = \frac{q_1}{4\pi\varepsilon_0 R^2}\hat{R} \tag{10.2}$$

is the electric field generated by charge 1 and received by charge 2.

Similarly, a magnetic field $\bar{B}$ from a point charge in motion is defined from the force formula (2.14), i.e. Grassman's formula, valid if charge 1 is part of a closed conductor

$$\bar{F}^G_{1\to 2} = \frac{\mu_0 q_1 q_2}{4\pi}\frac{\bar{v}_2 \times (\bar{v}_1 \times \hat{R})}{R^2} = q_2\bar{v}_2 \times \bar{B}_{1\to 2} \tag{10.3}$$

such that

$$\bar{B}_{1\to 2} = \frac{\mu_0}{4\pi}\frac{q_1\bar{v}_1 \times \hat{R}}{R^2} \tag{10.4}$$

which is referred to as Biot-Savart's formula.

From Chap. 4 the field from different configurations may be extracted. Formula (4.6) gives the electric field from a long straight charged wire

$$\bar{E} = \frac{\lambda}{2\pi\varepsilon_0 d}\hat{\rho} \tag{10.5}$$

where $\hat{\rho}$ is a unit vector perpendicular to the wire.

Formula (4.11) gives the electric field from a large charged plate

$$\bar{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} \tag{10.6}$$

where $\hat{n}$ is the surface normal of the plate.

The magnetic field from a straight current is obtained from formula (4.42)

$$\bar{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} \tag{10.7}$$

and that from a large current-carrying plate from formula (4.44b)

$$\bar{B} = \frac{\mu_0}{2}(\bar{K} \times \hat{n}) \tag{10.8}$$

10.3 Dipoles

An inspiring experiment is to illustrate the electric and magnetic field lines using macroscopic dipoles, Fig. 10.1.

Put some semolina in oil and let it be influenced by an external charge distribution. The elongated grains will align with the external electric field. Similarly, iron filings will align along an external magnetic field. The reason is that the external influence induces dipoles in the grains and the filings which experience a torque according to the description in Chap. 7. This will rotate the dipoles until an extreme energy value is achieved, compare Exercise (7.15). From Chap. 7, the torque and the energy for these processes may be extracted in terms of the fields. From the definition of electric potential, formula (3.3), it is first established that

$$\bar{E} = -\nabla\Phi \tag{10.9}$$

so that, using formula (7.2),

$$U_e = -\bar{p} \cdot \bar{E} \tag{10.10}$$

which accordingly is the energy arising in an *interaction* between the dipole and the charged object that generates the electric field E, nothing else.

The torque is obtained from formula (7.5)

$$\bar{\tau} = \bar{p} \times \bar{E} \tag{10.11}$$

In the magnetic case, the energy is given by formula (7.33) (or generally by formula (C5))

$$U_m = \bar{m} \cdot \bar{B} \tag{10.12}$$

without a minus sign since the magnetic energy is maximized in case of a permanent dipole m in a constant magnetic field.

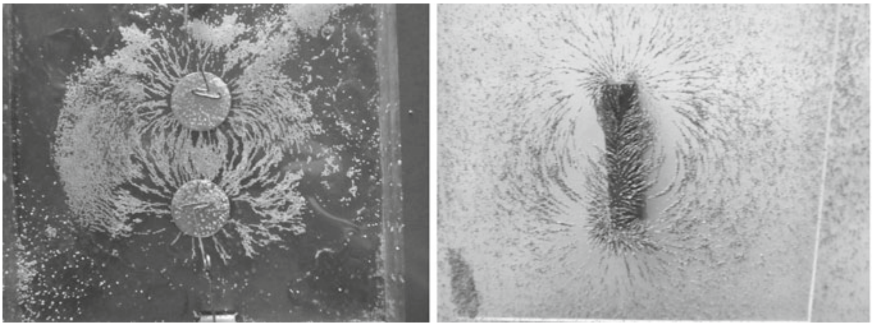


Fig. 10.1 *Left* The metallic plates are oppositely charged by a van de Graaf generator. Semolina in castor oil is sprinkled around. *Right* Iron filings are sprinkled around a permanent bar magnet

The torque (7.30) may then be written

$$\bar{\tau} = \bar{m} \times \bar{B} \tag{10.13}$$

These four last formulas describe the discussed experiments.

In addition, the field lines generated by a dipole is obtained by comparing formulas (10.10) and (10.12) to the dipole-dipole interaction energy, formulas (7.16) and (7.25):

$$\bar{E}_{dipole} = -\frac{1}{4\pi\varepsilon_0 R^3}[\bar{p} - 3(\bar{p} \cdot \hat{R})\hat{R}] \tag{10.14}$$

$$\bar{B}_{dipole} = -\frac{\mu_0}{4\pi R^3}[\bar{m} - 3(\bar{m} \cdot \hat{R})\hat{R}] \tag{10.15}$$

giving the field lines as illustrated in Fig. 10.1.

10.4 Material Effects

Material effects were discussed in Chap. 8. To describe these phenomena, three vector quantities are needed: the external influence, the material response and the total effect in the material. Table 10.1 states how these are represented in field theory.

Note that the magnetic field B is in this context named magnetic flux density (sometimes also referred to as magnetic induction). This terminology will become clear in Sect. 10.6.1.

In the electric case, the relation between the field vectors is obtained using formula (8.24):

$$\bar{P} = -\varepsilon_0(\kappa_e - 1)\nabla\Phi = \bar{D} - \varepsilon_0\bar{E} \tag{10.16}$$

so that the electric displacement vector becomes

$$\bar{D} = -\varepsilon_0\kappa_e\nabla\Phi = \varepsilon_0\kappa_e\bar{E} \tag{10.17}$$

where the electric field inside the material is

$$\bar{E} = -\nabla\Phi \tag{10.18}$$

In the magnetic case, formula (C6) is utilized to express magnetization

Table 10.1 Vector fields needed to treat electromagnetic material properties

	Influence	Material response	Total field in the material
Electric	Electric displacement D	Polarisation P	Electric field E
Magnetic	Magnetic field intensity H	Magnetisation M	Magnetic flux density B

$$\mu_0 \bar{M} = (\kappa_m - 1)\frac{\mu_0}{4\pi}\int_{V_f} \frac{\bar{J}_f \times \bar{R}}{R^3} dV_f = \bar{B} - \mu_0 \bar{H} \tag{10.19}$$

so that the magnetic field intensity becomes

$$\bar{H} = \frac{1}{4\pi}\int_{V_f} \frac{\bar{J}_f \times \bar{R}}{R^3} dV_f \tag{10.20}$$

and the magnetic flux density

$$\bar{B} = \mu_0 \kappa_m \frac{1}{4\pi}\int_{V_f} \frac{\bar{J}_f \times \bar{R}}{R^3} dV_f \tag{10.21}$$

Thus, the following relations between total field and the external influence have been found

$$\bar{E} = \frac{\bar{D}}{\varepsilon_0 \kappa_e} \tag{10.22}$$

$$\bar{B} = \mu_0 \kappa_m \bar{H} \tag{10.23}$$

Based on formulas (10.22) and (10.23), materials may be classified with respect to properties of the material parameters κ_e and κ_m:

- *Linear[non-linear] material:* material parameters are *independent [dependent]* on the strength of the external influence. The ferromagnetic and ferroelectric materials are examples of non-linear materials, see Exercises (8.15), (8.16) and (10.6).
- *Isotropic [anisotropic material]:* material parameters are *scalars [matrices]*, see Exercise (8.12).
- *Homogeneous [non-homogeneous material]:* the material parameters are coordinate *independent [dependent]*.

Consider now the interaction energy between a material and an external influence. In the magnetic case, this is obtained from formula (C5):

$$U_{int}^m = \frac{\mu_0}{4\pi}\frac{1}{2}\int_{V_f}\int_{V_b} \bar{M} \cdot \frac{\bar{J}_f \times \bar{R}}{R^3} dV_b dV_f = \frac{1}{2}\mu_0 \bar{H} \cdot \int_{V_b} \bar{M} dV_b \tag{10.24a}$$

To find out the energy stored in the material, (10.19) and (10.23) are used to rewrite (10.24a) as

$$\frac{dU_{int}^m}{dV_b} = \frac{1}{2}\mu_0 \bar{H} \cdot \bar{M} = \frac{1}{2}\mu_0 \bar{H}(\kappa_m - 1)\bar{H} = \frac{1}{2}\bar{B} \cdot \bar{H} - \frac{1}{2}\mu_0 H^2 \tag{10.24b}$$

where the first term corresponds to the stored energy density in the material and the second to the negative free energy density, generated by the external influence.

In the electric case, the interaction energy becomes (see Exercise 10.2)

$$U_{int}^{e} = -\frac{1}{2}\frac{\bar{D}}{\varepsilon_0} \cdot \int\limits_{V_b} \bar{P} dV_b \tag{10.25a}$$

Following the procedure from the magnetic case, the total interaction energy (10.25a) may be written as the difference of the energy in the material and the free energy. Using formulas (10.16) and (10.17) the energy density becomes

$$\frac{dU_{int}^{e}}{dV_b} = -\frac{1}{2}\frac{\bar{D}}{\varepsilon_0} \cdot \bar{P} = -\frac{1}{2}\frac{\bar{D}}{\varepsilon_0} \cdot (\bar{D} - \varepsilon_0 \bar{E}) = \frac{1}{2}\bar{E} \cdot \bar{D} - \frac{1}{2}\frac{D^2}{\varepsilon_0} \tag{10.25b}$$

where the first term corresponds to the energy density stored in the material and the second to the negative free energy density generated by the external influence.

10.5 Boundary Conditions

One essential advantage using field theory lies in the simplified usage of so-called boundary conditions. These define specific conditions on the fields at discontinuities of the surrounding media giving substantial information on the fields globally. For all types of electromagnetic signal transfers, the boundary conditions are decisive. Applications include wave guides, antennas, filters and resonance cavities.

Boundary conditions were discussed for the electric case in Sect. 8.1.3.1. This is now extended to general intersections as well as the magnetic case. Firstly it will be shown that the boundary conditions require knowledge of the divergence and curl of the fields. These are then determined for the electric and the magnetic fields, in total four formulas.

10.5.1 General Vector Field

Consider first a general vector field F traversing a discontinuity of the surroundings, Fig. 10.2. The normal component of the discontinuity is directed from medium 1 to 2. What is the change of the vector's components parallel (tangential) and perpendicular (normal) to the discontinuity?

1. *Normal component* $\bar{F}_n // \hat{n}$
Form a small cylinder with surface S and volume V around the discontinuity, Fig. 10.3. Using the divergence theorem

Fig. 10.2 Vector field at a discontinuity

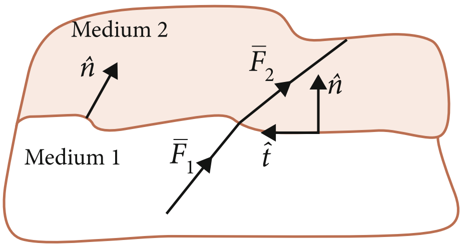
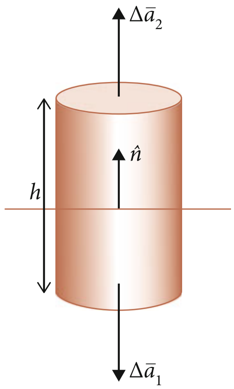


Fig. 10.3 A cylindrical surface is formed around the discontinuity



$$\oint_S \bar{F} \cdot d\bar{a} = \int_V \nabla \cdot \bar{F} dV \tag{10.26}$$

a condition for the normal component may be obtained in the following way. Evaluate the left-hand side

$$\oint_S \bar{F} \cdot d\bar{a} = \bar{F}_1 \cdot \Delta\bar{a}_1 + \bar{F}_2 \cdot \Delta\bar{a}_2 + \textit{contribution from the mantle} \tag{10.27}$$

Let the height $h \to 0$ so that the contribution from the mantle vanishes. Using (10.26), we then obtain

$$(\bar{F}_2 - \bar{F}_1) \cdot \hat{n} = \lim_{h\to 0} h\nabla \cdot \bar{F} \tag{10.28}$$

indicating that the normal component is continuous provided the divergence of the vector is finite at the intersection surface.

2. *Tangential component* $\bar{F}_t \perp \hat{n}$
Form a small rectangle C around the discontinuity, Fig. 10.4. Stokes' theorem is now utilized to gain information at the discontinuity

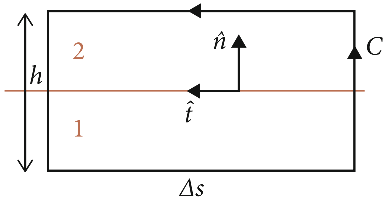


Fig. 10.4 A rectangular path is formed around the discontinuity

$$\oint_C \bar{F} \cdot d\bar{s} = \int_S \nabla \times \bar{F} \cdot d\bar{a} \tag{10.29}$$

where $d\bar{a} = \hat{n}' da$ is a surface element of the rectangle. For a small rectangle we obtain

$$\hat{t} \cdot (\bar{F}_2 - \bar{F}_1)\Delta s + \ contribution\, from\, verticals\ = \nabla \times \bar{F} \cdot \hat{n}' h \Delta s \tag{10.30}$$

Let $h \to 0$ so that the contribution from the vertical sides vanishes:

$$\hat{t} \cdot (\bar{F}_2 - \bar{F}_1)\Delta s = \nabla \times \bar{F} \cdot \hat{n}' h \Delta s \tag{10.31}$$

Since $\hat{t} = \hat{n}' \times \hat{n}$

$$(\hat{n}' \times \hat{n}) \cdot (\bar{F}_2 - \bar{F}_1)\Delta s = \nabla \times \bar{F} \cdot \hat{n}' h \Delta s \tag{10.32}$$

which may be written

$$\hat{n}' \cdot (\hat{n} \times (\bar{F}_2 - \bar{F}_1))\Delta s = \nabla \times \bar{F} \cdot \hat{n}' h \Delta s \tag{10.33}$$

giving

$$\hat{n} \times (\bar{F}_2 - \bar{F}_1) = \lim_{h \to 0} h \nabla \times \bar{F} \tag{10.34}$$

which may also be expressed directly in terms of tangential components (show this)

$$\bar{F}_{2t} - \bar{F}_{1t} = \lim_{h \to 0} h[\nabla \times \bar{F} \times \hat{n}] \tag{10.35}$$

indicating that the tangential component of a vector field is continuous at a discontinuity provided its curl is finite at the intersection surface.

Formulas (10.28) and (10.35) show that the electromagnetic boundary conditions are defined by the divergence and curl of the electric and the magnetic fields. This is the main motivation for introducing Maxwell-Heaviside's field equations.

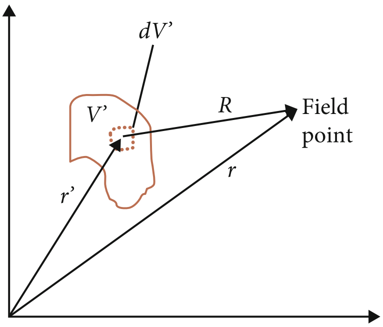


Fig. 10.5 An infinitesimal element *dV'* of volume *V'* contributes to the field at the field point

10.5.2 Divergence and Curl for Static Electric and Magnetic Fields

The boundary conditions are now to be specified for the electric and magnetic fields by determining their divergence and curl. Consider first the static (time independent) case. The time varying case will be treated in Sect. 10.6. Primed coordinates are used for the position of the source elements, $\bar{r}' = (x', y', z')$, and unprimed coordinates for the position of the field, $\bar{r} = (x, y, z)$, Fig. 10.5.[1]

The electric field is given by formula (10.2) and may be expressed as

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int\limits_{V'}\frac{\rho(\bar{r}')}{R^2}\hat{R}dV' \tag{10.36}$$

where $\bar{R} = \bar{r} - \bar{r}'$ and ρ is the charge density.

For the divergence of E, $\nabla\cdot\bar{E}$, ∇ acts on the unprimed coordinate so that

$$\nabla\cdot\frac{\hat{R}}{R^2} = \nabla\cdot\frac{\bar{R}}{R^3} = \bar{R}\cdot\nabla\frac{1}{R^3} + \frac{1}{R^3}\nabla\cdot\bar{R} = 0 \quad \text{for} \quad \bar{R} \neq 0$$

and

$$\nabla\cdot\bar{E}(\bar{r}) = 0 \quad \text{for} \quad \bar{R} \neq 0$$

For $\bar{R} = 0$, the following procedure may be used

$$\int\limits_{V'}\nabla\cdot\frac{\hat{R}}{R^2}dV' = \oint\limits_{S'}\frac{\hat{R}}{R^2}\cdot d\bar{a}' = \oint\limits_{S'}\frac{\cos\theta'}{R^2}da' = \oint\limits_{S'}d\Omega = 4\pi$$

[1] This section contains relatively advanced mathematics. For the reader who is not familiar with this it is recommended to read briefly.

where $d\Omega$ is the infinitesimal solid angle which, over a closed surface, is 4π. Since $\int_{V'} \nabla \cdot \frac{\hat{R}}{R^2} dV'$ vanishes everywhere apart from a point at $\bar{R} = 0$ where it is 4π, the divergence of E may be written

$$\nabla \cdot \bar{E}(\bar{r}) = \frac{1}{\varepsilon_0} \int_{V'} \rho(\bar{r}')\delta(\bar{r} - \bar{r}')dV' \tag{10.37}$$

where the delta function is defined as

$$\begin{aligned} &\int_{V'} f(\bar{r}')\delta(\bar{r} - \bar{r}')dV' = 0 \quad \text{if } \bar{r} \text{ outside the volume } V' \\ &\int_{V'} f(\bar{r}')\delta(\bar{r} - \bar{r}')dV' = f(\bar{r}) \quad \text{if } \bar{r} \text{ inside the volume } V' \end{aligned} \tag{10.38}$$

It is then obtained

$$\nabla \cdot \bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0} \tag{10.39}$$

Hence, the divergence of the electric field is non-vanishing only at a point where charge exists.

The magnetic field is given by formula (10.4)

$$\bar{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\bar{J} \times \bar{R}}{R^3} dV' = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\bar{L}' \times \bar{R}}{R^3} \tag{10.40}$$

for a closed conductor C'. Here it has been used that $Id\bar{L} = \bar{J}dV$, see Exercise (2.24). The divergence of B becomes

$$\nabla \cdot \bar{B} = \frac{\mu_0 I}{4\pi} \int_{C'} [-d\bar{L}' \cdot \underbrace{\nabla \times \frac{\bar{R}}{R^3}}_{=0} + \frac{\bar{R}}{R^3} \cdot \underbrace{\nabla \times d\bar{L}'}_{=0}] \tag{10.41}$$

The second term vanishes since the differential operator acts on the unprimed coordinates. The first term also vanishes since

$$\nabla \times \frac{\bar{R}}{R^3} = \nabla \times \nabla \left(-\frac{1}{R}\right) = 0 \tag{10.42}$$

because it generally holds that the curl of a gradient vanishes. Thus

$$\nabla \cdot \bar{B} = 0 \tag{10.43}$$

The curl of the *electric field*

$$\nabla \times \bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{V'} \rho(\bar{r}')\nabla \times \frac{\bar{R}}{R^3} dV' = 0 \tag{10.44}$$

according to formula (10.42).

The curl of the *magnetic field*

$$\nabla \times \bar{B}(\bar{r}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \nabla \times \left(\bar{J}(\bar{r}') \times \frac{\bar{R}}{R^3} \right) dV' \tag{10.45}$$

Since

$$\bar{J}(\bar{r}') \times \frac{\bar{R}}{R^3} = -\bar{J} \times \nabla \frac{1}{R} = \nabla \times \frac{\bar{J}}{R} \tag{10.46}$$

curl of B becomes

$$\nabla \times \bar{B}(\bar{r}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \nabla \times \nabla \times \frac{\bar{J}}{R} dV' = \frac{\mu_0}{4\pi} \int\limits_{V'} \left(\nabla \left(\nabla \cdot \frac{\bar{J}}{R} \right) - \nabla^2 \frac{\bar{J}}{R} \right) dV' \tag{10.47}$$

The first integral

$$\frac{\mu_0}{4\pi} \int\limits_{V'} \nabla \left(\nabla \cdot \frac{\bar{J}}{R} \right) dV' = \frac{\mu_0}{4\pi} \nabla \int\limits_{V'} \bar{J} \nabla \frac{1}{R} dV' = -\frac{\mu_0}{4\pi} \nabla \int\limits_{V'} \bar{J} \nabla' \frac{1}{R} dV' \tag{10.48}$$

where ∇' means differentiation w.r.t. $\bar{r}'$.

The integral in formula (10.48) may be integrated by parts

$$\int\limits_{V'} \bar{J} \nabla' \frac{1}{R} dV' = \left[\frac{\bar{J}}{R} \right] - \int\limits_{V'} \frac{\nabla' \cdot \bar{J}(\bar{r}')}{R} dV' = 0 \tag{10.49}$$

since such a large volume may be chosen so that the first term vanishes. The second term vanishes by formula (5.20) for *static* conditions.

The second term in (10.47) is evaluated using

$$\nabla^2 \frac{\bar{J}}{R} = -\bar{J} \nabla \cdot \frac{\hat{R}}{R^2} \tag{10.50}$$

so that

$$\frac{\mu_0}{4\pi}\int_{V'} -\left(\nabla^2 \frac{\bar{J}}{R}\right) dV' = \frac{\mu_0}{4\pi}\int_{V'} \bar{J}\left(\nabla \cdot \frac{\hat{R}}{R^2}\right) dV'$$
$$= \frac{\mu_0}{4\pi}\int_{V'} \bar{J}(\bar{r}')(4\pi\delta(\bar{r}-\bar{r}'))dV' = \mu_0 \bar{J}(\bar{r}) \quad (10.51)$$

where the introduction of the delta function in the third step parallelizes the procedure for obtaining the divergence of the electric field, formula (10.37). We then obtain

$$\nabla \times \bar{B}(\bar{r}) = \mu_0 \bar{J}(\bar{r}) \quad (10.52)$$

10.5.3 Boundary Conditions for Static Electric and Magnetic Fields

The boundary conditions for a general vector field, obtained in Sect. 10.5.1, are now specified for the electric and magnetic fields in case of static conditions. The results from Sects. 10.5.1 and 10.5.2 are first compiled in Table 10.2:

1. *Normal component of* E
   Table 10.2 gives

$$(\bar{E}_2 - \bar{E}_1)\cdot\hat{n} = \lim_{h\to 0} h\nabla\cdot\bar{E} = \lim_{h\to 0} h\frac{\rho}{\varepsilon_0} \quad (10.53)$$

Referring to Fig. 10.3 it is understood that $Q = h\Delta a\rho$ is the total charge contained in the cylinder. When $h \to 0$ this may be written $Q = \Delta a\sigma$ where σ is charge per unit surface. Therefore

$$(\bar{E}_2 - \bar{E}_1)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0} \quad (10.54)$$

2. *Tangential component of* E

$$\bar{E}_{2t} - \bar{E}_{1t} = 0 \quad (10.55)$$

3. *Normal component of* B

$$(\bar{B}_2 - \bar{B}_1)\cdot\hat{n} = 0 \quad (10.56)$$

4. *Tangential component of* B

$$\bar{B}_{2t} - \bar{B}_{1t} = \mu_0 \lim_{h\to 0} h[\bar{J}\times\hat{n}] \quad (10.57)$$

Table 10.2 Conditions used for deriving boundary conditions

General	Electric	Magnetic
$(\bar{F}_2 - \bar{F}_1)\cdot\hat{n} = \lim_{h\to 0} h\nabla\cdot\bar{F}$	$\nabla\cdot\bar{E}(\bar{r}) = \rho(\bar{r})/\varepsilon_0$	$\nabla\cdot\bar{B} = 0$
$\bar{F}_{2t} - \bar{F}_{1t} = \lim_{h\to 0} h[\nabla\times\bar{F}\times\hat{n}]$	$\nabla\times\bar{E}(\bar{r}) = 0$	$\nabla\times\bar{B}(\bar{r}) = \mu_0\bar{J}(\bar{r})$

$\bar{J}$ is current density, i.e. current per area. Referring to Fig. 10.4, the total current through the rectangle's area becomes $I = \bar{J} \cdot h\Delta s\hat{n}'$ so that

$$\bar{B}_{2t} - \bar{B}_{1t} = \mu_0 \bar{K} \times \hat{n} \tag{10.58}$$

where $\bar{K} = I/\Delta s\ \hat{n}'$, i.e. current per length unit. The current is parallel to the surface normal of the rectangle.

10.6 Maxwell's Equations

The four equations defining the boundary conditions of the electromagnetic fields are called Maxwell's equations and given in the static case by Table 10.2:

$$\nabla \cdot \bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0} \tag{10.59}$$

$$\nabla \cdot \bar{B}(\bar{r}) = 0 \tag{10.60}$$

$$\nabla \times \bar{E}(\bar{r}) = 0 \tag{10.61}$$

$$\nabla \times \bar{B}(\bar{r}) = \mu_0 \bar{J}(\bar{r}) \tag{10.62}$$

However, there are known electromagnetic phenomena not included in the formulas above. These are Faraday-Henry's induction effect, formulas (3.33) and (3.34), and the continuity equation for electric charge, formula (5.20). These include the time variation of the fields.

10.6.1 Accelerating Charges—The Time Variation of the Magnetic Field

Induction effects occur for time varying current, i.e. accelerating charges corresponding to time varying magnetic field. The underlying force on charges in acceleration is the Weber formula (3.34)

$$\bar{f}_{ind} = -\frac{\mu_0 q_2 q_1}{4\pi R} \frac{d\bar{v}_1}{dt} \tag{10.63}$$

for force on charge 2 giving an induced voltage in circuit 2, formula (3.33),

$$\varepsilon_2 = \frac{d}{dt}(i_1 M_{12}) = \frac{d}{dt}\left(i_1 \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\bar{L}_1 \cdot d\bar{L}_2}{R} \right) \tag{10.64}$$

where $\bar{R} = \bar{r}_2 - \bar{r}_1$. The right-hand side may be expressed in terms of the magnetic flux density B in the following way. Consider the integrand in the expression for the magnetic field, formula (10.40):

$$\frac{d\bar{L}_1 \times \bar{R}}{R^3} = -d\bar{L}_1 \times \nabla_2 \frac{1}{R} = \nabla_2 \times \frac{d\bar{L}_1}{R} \tag{10.65}$$

since $\nabla_2 \times d\bar{L}_1 = 0$. The magnetic field becomes

$$\bar{B}(\bar{r}_2) = \nabla_2 \times i_1 \frac{\mu_0}{4\pi} \oint_{C_1} \frac{d\bar{L}_1}{R} \tag{10.66}$$

The *magnetic flux* through the surface S_2 is

$$\int_{S_2} \bar{B}(\bar{r}_2) \cdot d\bar{a}_2 = \int_{S_2} \nabla_2 \times \left(i_1 \frac{\mu_0}{4\pi} \oint_{C_1} \frac{d\bar{L}_1}{R} \right) \cdot d\bar{a}_2 = i_1 \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\bar{L}_1 \cdot d\bar{L}_2}{R} \tag{10.67}$$

where Stokes' theorem was used in the last equality. Thus, the magnetic flux through the surface S_2 generated by circuit 1 is

$$\Phi_{1\to 2} = i_1 \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\bar{L}_1 \cdot d\bar{L}_2}{R} \tag{10.68}$$

whose time derivative is the right-hand side of formula (10.64). Accordingly, induction voltage becomes

$$\varepsilon_2 = \frac{d}{dt} \int_{S_2} \bar{B}(\bar{r}_2) \cdot d\bar{a}_2 \tag{10.69}$$

This is the voltage induced in circuit 2, which according to formula (10.9) together with the definition of voltage, formula (3.6), may be written

$$\varepsilon_2 = \oint_{C_2} \bar{E} \cdot d\bar{L}_2 = \int_{S_2} (\nabla \times \bar{E}) \cdot d\bar{a}_2 \tag{10.70}$$

so that

$$\nabla \times \bar{E} = -\frac{d\bar{B}}{dt} \tag{10.71a}$$

where a minus sign has been introduced on the right-hand side by the following reason. By combining (10.69) and (10.70) including the minus sign from (10.71a) the voltage becomes

$$\oint_{C_2} \bar{E} \cdot d\bar{L}_2 = -\frac{d}{dt} \int_{S_2} \bar{B}(\bar{r}_2) \cdot d\bar{a}_2 \tag{10.71b}$$

so that an increasing (decreasing) magnetic field gives a negative (positive) current change in accordance with observation. This effect is known as Lenz's law, see Exercise (3.5). Note that the direction of current is defined to be parallel to the electric field.

10.6.2 The Continuity Equation—Time Variation of Electric Field

The continuity equation for electric charge, formula (5.20)

$$\nabla \cdot \bar{J} + \frac{d\rho}{dt} = 0 \tag{10.72}$$

constitutes implicit information on the dynamics of free charges based on the hypothesis that electric charge is conserved. The equation deals with charges in motion and accordingly a time varying electric field. This effect may be included in the field static formula (10.62) by considering the divergence of current

$$\nabla \cdot \bar{J} = \frac{1}{\mu_0} \nabla \cdot \nabla \times \bar{B} = 0 \tag{10.73}$$

For a *time varying* electric field, (10.73) becomes using (10.72)

$$\nabla \cdot \bar{J} + \frac{d\rho}{dt} = \frac{1}{\mu_0} \nabla \cdot \nabla \times \bar{B} \tag{10.74}$$

Using formula (10.59), the charge density may be replaced by

$$\varepsilon_0 \nabla \cdot \bar{E} = \rho \tag{10.75}$$

so that

$$\nabla \cdot \bar{J} + \varepsilon_0 \frac{d\nabla \cdot \bar{E}}{dt} = \frac{1}{\mu_0} \nabla \cdot \nabla \times \bar{B} \tag{10.76}$$

or

$$\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{d\bar{E}}{dt} \tag{10.77}$$

This may be interpreted that the sources to the magnetic field are both an electric current and a time varying electric field.

10.7 Potentials

The mathematical structure of Maxwell's equations allows for an introduction of so-called potentials by which the fields may be expressed. The purpose of this procedure is to simplify the computational technique.

Consider first the field static case. Since $\nabla \times \bar{E}(\bar{r}) = 0$ and the curl of a gradient vanishes, the electric field may be expressed by a scalar potential Φ:

$$\bar{E} = -\nabla \Phi \tag{10.78}$$

which is formula (10.9).

In the magnetic case, since $\nabla \cdot \bar{B}(\bar{r}) = 0$ and the divergence of a curl vanishes, the magnetic field may be expressed as

$$\bar{B} = \nabla \times \bar{A} \tag{10.79}$$

where A is called the magnetic vector potential. Using formula (10.66), the vector potential is

$$\bar{A} = I \frac{\mu_0}{4\pi} \oint_C \frac{d\bar{L}}{R} \tag{10.80}$$

For time varying fields, the electric field is modified according to Faraday-Henry's induction law (10.71a)

$$\nabla \times \bar{E}(\bar{r}, t) = -\frac{d}{dt} \nabla \times \bar{A}(\bar{r}, t) \tag{10.81}$$

so that

$$\bar{E}(\bar{r}, t) = -\nabla \Phi(\bar{r}, t) - \frac{d\bar{A}(\bar{r}, t)}{dt} \tag{10.82}$$

Since

$$\nabla \cdot \bar{B}(\bar{r}, t) = 0 \tag{10.83}$$

also for time varying fields, formula (10.79) is unaltered

$$\bar{B}(\bar{r}, t) = \nabla \times \bar{A}(\bar{r}, t) \tag{10.84}$$

10.8 Power Transportation—The Poynting Vector

The power evaluated in an electrodynamic system may be expressed exclusively in terms of fields. The stored energy of a system changes by reasons of heat generation and energy transportation in or out of the system. It is the objective here to express these energy forms in terms of fields in order to identify the power transportation in/out of the volume. This is of crucial importance when dealing with signal transfer between electrodynamic systems such as antennas.

The analysis starts from the expression of heat power, i.e. Joule's law (5.15). Using the principle of energy conservation, heat power equals the change of stored energy minus the power transported out from the system. Consider a volume V. The heat power density is

$$\frac{P_{fr}}{V} = \frac{J^2}{\sigma} \tag{10.85}$$

where J is the current density in the volume and σ is the conductivity. The index fr stands for friction, the origin of heat power. According to Exercise 5.7, Ohm's law may be written

$$\bar{J} = \sigma \bar{E} \tag{10.86}$$

where E is the electric field generating the current J. Since the current is generated in the direction of the electric field, the heat power density may be written as a scalar product

$$\frac{P_{fr}}{V} = \bar{J} \cdot \bar{E} \tag{10.87}$$

The heat power becomes

$$P_{fr} = \int\limits_V \bar{J} \cdot \bar{E} dV \tag{10.88}$$

which is now to be expressed in terms of fields only. Using Maxwell's equation (10.77) together with (10.22) and (10.23), current density becomes

$$\bar{J} = \nabla \times \bar{H} - \frac{d\bar{D}}{dt} \tag{10.89}$$

so that heat power may be written

$$P_{fr} = \int\limits_V \left(\nabla \times \bar{H} - \frac{d\bar{D}}{dt} \right) \cdot \bar{E} \tag{10.90}$$

Using vector algebra, the first term of the integrand

$$(\nabla \times \bar{H}) \cdot \bar{E} = \bar{H} \cdot (\nabla \times \bar{E}) - \nabla \cdot (\bar{E} \times \bar{H}) = -\bar{H} \cdot \frac{d\bar{B}}{dt} - \nabla \cdot (\bar{E} \times \bar{H}) \tag{10.91}$$

where the Faraday-Henry induction law (10.71a) has been used in the last equality. Rearranging terms and utilizing the divergence theorem we obtain

$$-\int_V \left(\bar{E}\cdot\frac{d\bar{D}}{dt}+\bar{H}\cdot\frac{d\bar{B}}{dt}\right)dV = P_{fr} + \oint_A (\bar{E}\times\bar{H})\cdot d\bar{a} \tag{10.92}$$

where A is the surface enclosing the volume V.

The left-hand side is then rewritten so that

$$-\frac{d}{dt}\int_V \left(\frac{1}{2}\bar{E}\cdot\bar{D}+\frac{1}{2}\bar{H}\cdot\bar{B}\right)dV = P_{fr} + \oint_A (\bar{E}\times\bar{H})\cdot d\bar{a} \tag{10.93}$$

According to formulas (10.24b) and (10.25b) the left-hand side corresponds to the time change of stored electric and magnetic energy. The second term of the right-hand side is therefore the power transported in or out of the volume. The integrand of this term is called the Poynting vector

$$\bar{S} = \bar{E}\times\bar{H} \tag{10.94}$$

corresponding to the power transported per unit area. In Chap. 11 the Poynting vector will be applied to find the power transmitted by an antenna.

10.9 Summary

In static conditions, the electric field E is defined by the electric force formula

$$\bar{F}_{1\to 2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R} = q_2\bar{E}_{1\to 2} \tag{10.1}$$

$$\bar{E}_{1\to 2} = \frac{q_1}{4\pi\varepsilon_0 R^2}\hat{R} \tag{10.2}$$

In Maxwell-Heaviside field theory, the static magnetic field B is defined by Grassman's force formula

$$\bar{f}^{G}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi}\frac{\bar{v}_2\times(\bar{v}_1\times\hat{R})}{R^2} = q_2\bar{v}_2\times\bar{B}_{1\to 2} \tag{10.3}$$

$$\bar{B}_{1\to 2} = \frac{\mu_0}{4\pi}\frac{q_1\bar{v}_1\times\hat{R}}{R^2} \tag{10.4}$$

Using the continuity equation and the Faraday-Henry induction law, the time variation of the fields may be derived to obtain the general Maxwell-Heaviside field equations

$$\nabla\cdot\bar{E}(\bar{r},t) = \frac{\rho(\bar{r},t)}{\varepsilon_0} \tag{10.59}$$

$$\nabla \cdot \bar{B}(\bar{r}, t) = 0 \tag{10.60}$$

$$\nabla \times \bar{E}(\bar{r}, t) = -\frac{d\bar{B}(\bar{r}, t)}{dt} \tag{10.71}$$

$$\nabla \times \bar{B}(\bar{r}, t) = \mu_0 \bar{J}(\bar{r}, t) + \mu_0 \varepsilon_0 \frac{d\bar{E}(\bar{r}, t)}{dt} \tag{10.77}$$

The boundary conditions of the electric and magnetic fields are

$$(\bar{E}_2 - \bar{E}_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0} \tag{10.54}$$

$$\bar{E}_{2t} - \bar{E}_{1t} = 0 \tag{10.55}$$

$$(\bar{B}_2 - \bar{B}_1) \cdot \hat{n} = 0 \tag{10.56}$$

$$\bar{B}_{2t} - \bar{B}_{1t} = \mu_0 \bar{K} \times \hat{n} \tag{10.58}$$

The power transported in or out of a volume is

$$P_t = \oint_A (\bar{E} \times \bar{H}) \cdot d\bar{a}$$

where A is the surface enclosing the volume.

Table 10.3 summarizes the field theoretical description of material effects.

Table 10.3 Summary of field theory with respect to material effects

	Electric	Magnetic
Components	$C = \kappa_e C_0$	$L = \kappa_m L_0$
Dipole moment	$\bar{p} = \int_V \bar{P}(\bar{r}) dV$	$\bar{m} = \int_V \bar{M}(\bar{r}) dV$
Charge and current densities	$\rho_b = -\nabla \cdot \bar{P}$	$\bar{J}_b = \nabla \times \bar{M}$
	$\sigma_b = \bar{P} \cdot \hat{n}$	$\bar{K}_b = \bar{M} \times \hat{n}$
	$\sigma_b = \sigma_f (1/\kappa_e - 1)$	$I_b = (\kappa_m - 1) N I_f$
Polarisation from external influence	$\bar{P} = (1 - 1/\kappa_e)\bar{D}$	$\bar{M} = (\kappa_m - 1)\bar{H}$
Total field in the material	$\varepsilon_0 \bar{E} = \bar{D} - \bar{P}$	$\bar{B}/\mu_0 = \bar{H} + \bar{M}$
Fields from free charge/current	$\nabla \cdot \bar{D} = \rho_f$	$\nabla \times \bar{H} = \bar{J}_f$
Energy density in the material	$\frac{dU_e}{dV} = \frac{1}{2}\bar{E} \cdot \bar{D}$	$\frac{dU_m}{dV} = \frac{1}{2}\bar{B} \cdot \bar{H}$

10.10 Exercises

10.1 Determine the magnetic field intensity H far away from a current loop. Hint: Utilize formula (8.74).

C 10.2 Derive the electric energy for the interaction between free (unbound) charge and a material, formula (10.25a).

*10.3 Derive the following integral forms of Maxwell-Heaviside equations in the field static case:

a.

$$\oint_S \bar{E} \cdot d\bar{a} = \frac{Q}{\varepsilon_0} \tag{10.95}$$

where S is a closed surface and Q is the charge enclosed by this surface.

b.

$$\oint_C \bar{B} \cdot d\bar{l} = \mu_0 I \tag{10.96}$$

where C is a closed curve and I is the current enclosed by this curve.

c.

$$\oint_S \bar{B} \cdot d\bar{a} = 0 \tag{10.97}$$

where S is a closed surface.

*10.4 Use formula (10.95) to obtain the electric field outside the following homogeneously charged objects:

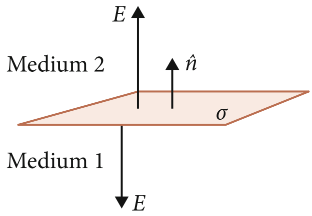
a. Large plate, surface charge density σ
b. Long cylinder, charge per meter λ
c. Sphere with total charge Q

*10.5 Use formula (10.96) to determine the magnetic field B generated by the following homogeneous current distributions:

a. Outside a long straight conductor carrying current I
b. Outside a large plate carrying current per meter K
c. Inside and outside a long coil with n close turns per meter and carrying current I.

*10.6 Express (a) the magnetic and (b) the electric hysteresis curve in terms of the magnetic fields B, H and the electric fields E, D respectively. Utilize Exercises (8.15) and (8.16).
c. Express the work done over one ferroelectric cycle in terms of the fields E and D. Utilize Exercise (8.16c).

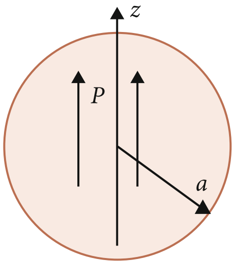
10.7 Apply the continuity condition (10.54) to determine the magnitude of the electric field from a large plate with a constant surface charge density σ. The surrounding is air/vacuum.



C 10.8 A sphere with radius a is homogeneously magnetized with magnetization $\bar{M} = M\hat{z}$. Its center is located in the coordinate origin as in Fig. 8.17.

a. Determine the magnetic flux density $\bar{B}$ on the z axis inside and outside the sphere.
b. Determine the magnetic field intensity $\bar{H}$ on the z axis inside and outside the sphere.
c. Give the continuity condition for the normal component of $\bar{H}$ and verify that this is fulfilled in this case.
   Hint: Use formula (8.79) and apply formulas (10.23) and (10.56).
d. According to formula (10.20), the source of the magnetic field intensity $\bar{H}$ is free current $\bar{J}_f$. Explain why $\bar{H}$ even so is finite in this case.

C 10.9 A sphere with its center in the coordinate origin and with radius a is homogeneously electrically polarised with polarisation $\bar{P} = P\hat{z}$.

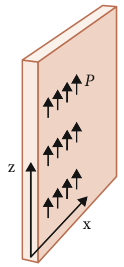


a. Determine the electric field $\bar{E}$ on the z axis inside and outside the sphere.
b. Determine the electric displacement $\bar{D}$ on the z axis inside and outside the sphere.
c. Give the continuity condition for the normal component of the displacement vector $\bar{D}$ and verify that this is fulfilled in this case.

Hint: Utilize formula (8.31) and apply formulas (10.22) and (10.54).

d. Show that $\nabla \cdot \bar{D} = \rho_f$. Accordingly, the source of $\bar{D}$ is free charge. Explain why $\bar{D}$ is finite despite the fact that only a bound charge exists in this case.

10.10 A large plate consists of parallel vibrating electric

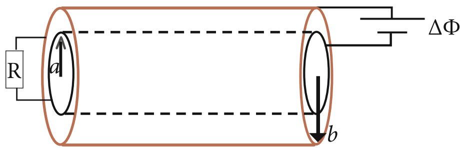


dipoles homogeneously distributed with a density of dipole moment as

$$\bar{P} = P_0 \sin \omega t \hat{z}$$

where ω is the angular velocity of the vibration and t is the time coordinate. Determine the magnetic flux density generated by these dipoles.

10.11 A coaxial cable is connected to a voltage source $\Delta\Phi$ in one end and a resistive load R in the other.



a. Calculate the Poynting vector between and outside the conductors.
b. Determine the direction of the Poynting vector inside the conductors.
c. Determine the power transfer to the load and compare with the circuit theoretical expression.

10.12 A cylindrical parallel plate ideal capacitor with plate radius a and with a distance d between the plates is being charged with a voltage source $\Delta\Phi$.

a. Determine the power transportation into the volume between the plates.
b. Show that the volume density of this power equals the time derivative of formula (10.25b).

Further Readings

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Chapter 11
Antenna Theory—The Loop and the Dipole

It was shortly after midday on December 12, 1901, [in a hut on the cliffs at St. John's, Newfoundland] that I placed a single earphone to my ear and started listening. The receiver on the table before me was very crude - a few coils and condensers and a coherer - no valves [vacuum tubes], no amplifiers, not even a crystal. I was at last on the point of putting the correctness of all my beliefs to test. … [The] answer came at 12:30. … Suddenly, about half past twelve there sounded the sharp click of the 'tapper' … Unmistakably, the three sharp clicks corresponding to three dots sounded in my ear. 'Can you hear anything, Mr. Kemp?' I asked, handing the telephone to my assistant. Kemp heard the same thing as I. … I knew then that I had been absolutely right in my calculations. The electric waves which were being sent out from Poldhu [Cornwall, England] had travelled the Atlantic, serenely ignoring the curvature of the earth which so many doubters considered a fatal obstacle. … I knew that the day on which I should be able to send full messages without wires or cables across the Atlantic was not far distant.

Guglielmo Marconi

(Extracted from the book 'Marconi, my father' by D. Marconi (2000). Used by kind permission of Guernica Editions.)

In Chap. 9 it was shown that the magnetic and inductive forces may be interpreted as motional consequences of the electric force, an interpretation based on the fact that interactions take time, Figs. 9.1 and 9.2.

Figure 9.2 shows that charge 2 receives the momentum at $t = t_2$, emitted by charge 1 at time t_1. These times are related according to

$$t_1 = t_2 - \frac{R^*}{c} \tag{11.1}$$

where time t_1 is called 'retarded time'. In e.g. Exercise (2.19), the retardation effect was not considered since the distance is small and the two times t_1 and t_2 are almost equal. To describe a general interaction the retardation effects must be considered.

In this chapter, two basic examples will be analysed showing the importance of retardation. These are the loop and dipole antennas, corresponding to a closed circular

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9_11

and a straight (non-closed) conductor respectively, both fed by an AC voltage and assumed to be small so that the current is uniform. The principles of an antenna array are then discussed by analysing a two-element system.

In the exercise section it is shown that antenna theory is also applicable to natural processes. By modelling matter as vibrating dipoles, equivalent to the dipole antenna, the reflection and refraction laws may be derived and thoroughly understood. In particular, the interesting phenomenon called Brewster reflection is clarified.

11.1 The Loop Antenna

Consider a closed circular current-carrying conductor interacting with a point-like charge q at rest and at a large distance from the loop, Fig. 11.1.

From the electrodynamical force formula (3.35) it is first noted that an interaction occurs only through the inductive force, i.e. the acceleration dependent Weber force (3.34). For an interaction at a distance R,

$$d\bar{F} = -\frac{q d\bar{L}}{4\pi\varepsilon_0 c^2 R}\frac{dI(t)}{dt} \tag{11.2}$$

which becomes in retarded form

$$d\bar{F} = -\frac{q d\bar{L}}{4\pi\varepsilon_0 c^2 R}\frac{dI(t - R/c)}{dt} \tag{11.3}$$

Let the current I vary harmonically with time so that in complex form:

$$I(t) = I_0 e^{j\omega t} \tag{11.4}$$

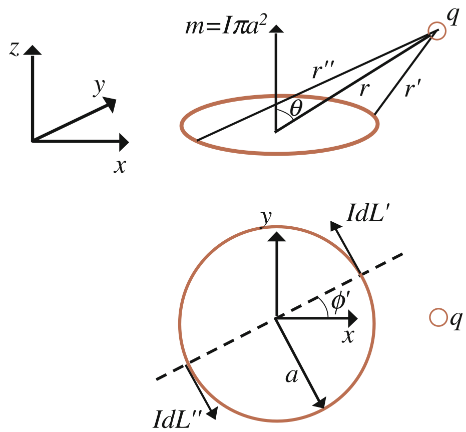


Fig. 11.1 A circular current-carrying loop with radius a and located in the xy plane interacts with charge q. The vector m is its dipole moment

giving

$$\frac{dI(t-R/c)}{dt} = j\omega I_0 e^{j\omega t} e^{-j\omega R/c} \tag{11.5}$$

The infinitesimal force becomes

$$d\bar{F} = -\frac{j\omega q d\bar{L}}{4\pi\varepsilon_0 c^2 R} I_0 e^{j\omega t} e^{-j\omega R/c} \tag{11.6}$$

Consider now the circular loop, Fig. 11.1. By symmetry reasons, the total force is directed along the spherical coordinate ϕ of the point charge q. Let it be located at $y = 0$, i.e. on the x axis in the xy plane as in Fig. 11.1. The length element dL' contributes to the infinitesimal ϕ component of the force as

$$dF_\phi = -\frac{j\omega q a d\phi' \cos\phi'}{4\pi\varepsilon_0 c^2 r'} I_0 e^{j\omega t} e^{-j\omega r'/c} \tag{11.7}$$

where $dL' = ad\phi'$.

Consider two diametrically opposite length elements of the loop. These will contribute to the force in opposite directions but due to their different distance to the charge q, a net effect will appear:

$$dF_\phi = -\frac{j\omega q a d\phi' \cos\phi'}{4\pi\varepsilon_0 c^2} I_0 e^{j\omega t} [\frac{e^{-j\omega r'/c}}{r'} - \frac{e^{-j\omega r''/c}}{r''}] \tag{11.8}$$

Using the cosine theorem, the distances are given by

$$r'^2 = a^2 + r^2 - 2ar\sin\theta\cos\phi'$$

$$r''^2 = a^2 + r^2 + 2ar\sin\theta\cos\phi'$$

where the last two terms have different signs since the elements are diametrically opposite and the angle ϕ' refers to the primed element.

Since $r \gg a$, these distances may be Taylor expanded. Keeping terms up to the order a/r gives

$$r' \approx r(1 - \frac{a}{r}\sin\theta\cos\phi')$$

$$r'' \approx r(1 + \frac{a}{r}\sin\theta\cos\phi')$$

$$\frac{1}{r'} \approx \frac{1}{r}(1 + \frac{a}{r}\sin\theta\cos\phi')$$

$$\frac{1}{r''} \approx \frac{1}{r}(1 - \frac{a}{r}\sin\theta\cos\phi')$$

which are introduced in formula (11.8)

$$dF_\phi = -\frac{j\omega qad\phi' \cos\phi'}{2\pi\varepsilon_0 c^2 r} I_0 e^{j\omega t} e^{-\frac{j\omega r}{c}}$$
$$\times [j \sin(\frac{\omega a}{c} \sin\theta\cos\phi') + \frac{a}{r} \sin\theta\cos\phi' \cos(\frac{\omega a}{c} \sin\theta\cos\phi')] \quad (11.9)$$

In case $\omega a \ll c$, formula (11.9) is simplified:

$$dF_\phi = -\frac{j\omega qad\phi' \cos\phi'}{2\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)} [j\frac{\omega a}{c} \sin\theta\cos\phi' + \frac{a}{r} \sin\theta\cos\phi']$$
$$= -\frac{j\omega q a^2}{2\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)} [j\frac{\omega}{c} + \frac{1}{r}] \sin\theta\cos^2\phi' d\phi' \quad (11.10)$$

To obtain total force, all elements along the loop have to be summed up. Let the loop be small so that the current is uniform, i.e. it does not vary along the loop:

$$F_\phi = -\frac{j\omega q a^2}{2\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)} [j\frac{\omega}{c} + \frac{1}{r}] \sin\theta \int_0^\pi \cos^2\phi' d\phi'$$

so that

$$F_\phi = \frac{\omega q\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)} [\frac{\omega}{c} - \frac{j}{r}] \quad (11.11)$$

The real part is the physical force:

$$F_\phi^R = \frac{\omega q\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^2 r} I_0 [\cos(\omega(t-r/c))\frac{\omega}{c} + \sin(\omega(t-r/c))\frac{1}{r}] \quad (11.12)$$

The first term, varying as $1/r$, is predominant at large distance and is known as ‘radio waves’. In the direction along the loop axis there is no force on the resting charge q.

11.2 The Dipole Antenna

The dipole antenna consists of a straight AC fed wire. The feed gap will here be neglected, Fig. 11.2.

For this antenna there is in addition to the Weber force an electric force, the reason being that charges accumulate at the ends of the wire, motivating its name. The forces will be treated separately, starting with the electric interaction.

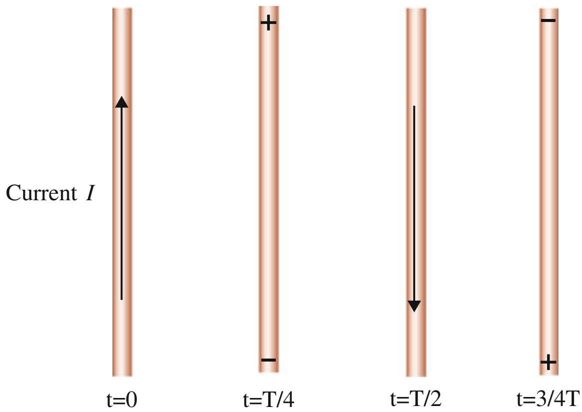


Fig. 11.2 The flow of charge in a dipole antenna. The time period of the AC voltage is denoted T. The AC source is usually connected at the center of the wire

11.2.1 The Oscillating Electric Dipole

The charge accumulation at the ends of the wire is treated as a harmonically oscillating electric dipole:

$$Q(t) = Q_0 e^{j\omega t} \tag{11.13}$$

and the retarded charge becomes

$$Q(t - R/c) = Q_0 e^{j\omega(t-R/c)} \tag{11.14}$$

Using formulas (10.1) and (10.9), the force between point charge Q and another point charge q may be written:

$$\bar{F} = -\frac{1}{4\pi\varepsilon_0} q\nabla\Phi = -\frac{q}{4\pi\varepsilon_0}\frac{d}{dR}\frac{Q(t-R/c)}{R}\hat{R} = -\frac{qQ_0e^{j\omega t}}{4\pi\varepsilon_0}\frac{d}{dR}\frac{e^{-j\omega R/c}}{R}\hat{R}$$
$$= \frac{qQ_0e^{j\omega(t-R/c)}}{4\pi\varepsilon_0}[\frac{j\omega}{cR} + \frac{1}{R^2}]\hat{R} \tag{11.15}$$

The forces on the point charge q from the dipole in Fig. 11.3 becomes

$$\bar{F}^{upper} = \frac{qQ_0e^{j\omega(t-r'/c)}}{4\pi\varepsilon_0}[\frac{j\omega}{cr'} + \frac{1}{r'^2}]\hat{r}' \tag{11.16}$$

$$\bar{F}^{lower} = \frac{qQ_0e^{j\omega(t-r''/c)}}{4\pi\varepsilon_0}[\frac{j\omega}{cr''} + \frac{1}{r''^2}]\hat{r}'' \tag{11.17}$$

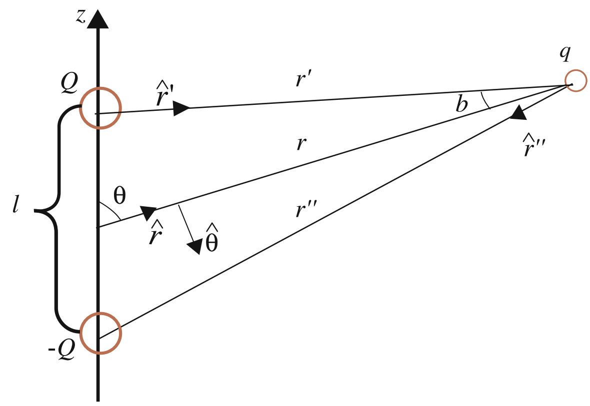


Fig. 11.3 The electric dipole as a model for the charge accumulation at the ends of an AC fed wire

From the geometry of Fig. 11.3, the distance magnitudes are

$$r' \approx r - \frac{l}{2}\cos\theta = r(1 - \frac{l}{2r}\cos\theta)$$

$$r'' \approx r + \frac{l}{2}\cos\theta = r(1 + \frac{l}{2r}\cos\theta)$$

Using the approximation $l \ll r$ and keeping only first order terms in l/r we obtain

$$\frac{1}{r'} \approx \frac{1}{r}(1 + \frac{l}{2r}\cos\theta)$$

$$\frac{1}{r''} \approx \frac{1}{r}(1 - \frac{l}{2r}\cos\theta)$$

which is introduced in formulas (11.16) and (11.17)

$$\begin{aligned} \bar{F}^{upper} &= \frac{qQ_0e^{j\omega(t-r'/c)}}{4\pi\varepsilon_0}[\tfrac{j\omega}{cr}(1+\tfrac{l}{2r}\cos\theta)+\tfrac{1}{r^2}(1+\tfrac{l}{r}\cos\theta)]\hat{r}' \\ \bar{F}^{lower} &= \frac{qQ_0e^{j\omega(t-r''/c)}}{4\pi\varepsilon_0}[\tfrac{j\omega}{cr}(1-\tfrac{l}{2r}\cos\theta)+\tfrac{1}{r^2}(1-\tfrac{l}{r}\cos\theta)]\hat{r}'' \end{aligned} \quad (11.18)$$

The direction vectors are obtained as follows. Referring to Fig. 11.3 it is seen that

$$\hat{r}' = \hat{r}\cos b + \hat{\theta}\sin b$$

The sinus theorem gives

$$\frac{r'}{\sin\theta} = \frac{l/2}{\sin b}$$

so that up to order l/r

$$\cos b = (1 - \sin^2 b)^{1/2} = (1 - \sin^2\theta(\frac{l}{2r'})^2)^{1/2} \approx 1$$

and

$$\sin b = \frac{l}{2r'} \sin\theta \approx \frac{l}{2r} \sin\theta$$

Thus,

$$\hat{r}' = \hat{r} + \hat{\theta}\frac{l \sin\theta}{2r} \tag{11.19}$$

Equivalently

$$\hat{r}'' = -\hat{r} + \hat{\theta}\frac{l \sin\theta}{2r} \tag{11.20}$$

taking into account the opposite signs of the two charges.

For a *small* dipole the current is uniform along the wire. The current becomes

$$I = \frac{d}{dt}Q(t) = \frac{d}{dt}Q_0 e^{j\omega t} = j\omega Q_0 e^{j\omega t} \tag{11.21}$$

and the forces

$$\begin{aligned}
\bar{F}^{upper} &= \frac{qIe^{-j\omega r'/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1+\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1+\frac{l}{r}\cos\theta)]\hat{r}' \\
&= \frac{qIe^{-j\omega r(1-\frac{l}{2r}\cos\theta)/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1+\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1+\frac{l}{r}\cos\theta)]\hat{r}' \\
&= \frac{qIe^{-j\omega r(1-\frac{l}{2r}\cos\theta)/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1+\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1+\frac{l}{r}\cos\theta)] \\
&\quad \times (\hat{r}+\hat{\theta}\frac{l\sin\theta}{2r}); \\
\bar{F}^{lower} &= \frac{qIe^{-j\omega r''/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1-\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1-\frac{l}{r}\cos\theta)]\hat{r}'' \\
&= \frac{qIe^{-j\omega r(1+\frac{l}{2r}\cos\theta)/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1-\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1-\frac{l}{r}\cos\theta)] \\
&\quad \times (-\hat{r}+\hat{\theta}\frac{l\sin\theta}{2r})
\end{aligned} \tag{11.22}$$

The radial components become

$$\begin{aligned}
F_r^{upper} &= \frac{qIe^{j\omega\frac{l}{2c}\cos\theta}e^{-j\omega r/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(1+\frac{l}{2r}\cos\theta)+\frac{1}{r^2}(1+\frac{l}{r}\cos\theta)] \\
F_r^{lower} &= \frac{qIe^{-j\omega\frac{l}{2c}\cos\theta}e^{-j\omega r/c}}{j\omega 4\pi\varepsilon_0}[\frac{j\omega}{cr}(-1+\frac{l}{2r}\cos\theta)-\frac{1}{r^2}(1-\frac{l}{r}\cos\theta)]
\end{aligned} \tag{11.23}$$

Putting

$$A = \frac{qIe^{-j\omega r/c}}{j\omega 4\pi\varepsilon_0}$$

(11.23) is written

$$\begin{aligned} F_r^{upper} &= Ae^{j\omega\frac{l}{2c}\cos\theta}[\frac{j\omega}{cr} + \frac{1}{r^2} + \frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta] \\ F_r^{lower} &= Ae^{-j\omega\frac{l}{2c}\cos\theta}[-\frac{j\omega}{cr} - \frac{1}{r^2} + \frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta] \end{aligned} \tag{11.24}$$

Summing the two forces

$$\begin{aligned} F_r &= A(\frac{j\omega}{cr} + \frac{1}{r^2})(e^{j\omega\frac{l}{2c}\cos\theta} - e^{-j\omega\frac{l}{2c}\cos\theta}) \\ &\quad + A(\frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta)(e^{j\omega\frac{l}{2c}\cos\theta} + e^{-j\omega\frac{l}{2c}\cos\theta}) \\ &= A(\frac{j\omega}{cr} + \frac{1}{r^2})2j\sin(\omega\frac{l}{2c}\cos\theta) + A(\frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta)2\cos(\omega\frac{l}{2c}\cos\theta) \end{aligned} \tag{11.25}$$

For $\omega l \ll c$ Taylor expansion gives

$$\begin{aligned} \sin(\omega\frac{l}{2c}\cos\theta) &\approx \omega\frac{l}{2c}\cos\theta \\ \cos(\omega\frac{l}{2c}\cos\theta) &\approx 1 \end{aligned}$$

so that the radial component of force becomes

$$\begin{aligned} F_r &= A(\frac{j\omega}{cr} + \frac{1}{r^2})2j\omega\frac{l}{2c}\cos\theta + A(\frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta)2 \\ &= 2A\cos\theta(\frac{j\omega}{cr}j\omega\frac{l}{2c} + \frac{1}{r^2}j\omega\frac{l}{2c} + \frac{j\omega l}{2cr^2} + \frac{l}{r^3}) \\ &= \frac{qIe^{-j\omega r/c}}{2\pi\varepsilon_0}\cos\theta(\frac{j\omega l}{2c^2 r} + \frac{l}{cr^2} - \frac{jl}{\omega r^3}) \\ &= \frac{qIe^{-j\omega r/c}l}{4\pi\varepsilon_0 c}\cos\theta(\frac{j\omega}{cr} + \frac{2}{r^2} - 2\frac{jc}{\omega r^3}) \end{aligned} \tag{11.26}$$

The θ component becomes

$$F_\theta^{upper} = Ae^{j\omega\frac{l}{2c}\cos\theta}[\frac{j\omega}{cr} + \frac{1}{r^2} + \frac{j\omega l}{2cr^2}\cos\theta + \frac{l}{r^3}\cos\theta]\frac{l\sin\theta}{2r}$$

$$\approx Ae^{j\omega\frac{l}{2c}\cos\theta}[\frac{j\omega}{cr}+\frac{1}{r^2}]\frac{l\sin\theta}{2r}$$

$$F_{\theta}^{lower}\approx Ae^{-j\omega\frac{l}{2c}\cos\theta}[\frac{j\omega}{cr}+\frac{1}{r^2}]\frac{l\sin\theta}{2r} \tag{11.27}$$

and total force in the theta direction is

$$F_{\theta}\approx A\frac{l\sin\theta}{2r}[\frac{j\omega}{cr}+\frac{1}{r^2}](e^{-j\omega\frac{l}{2c}\cos\theta}+e^{j\omega\frac{l}{2c}\cos\theta})$$

$$\approx A\frac{l\sin\theta}{2r}[\frac{j\omega}{cr}+\frac{1}{r^2}]2$$

$$=\frac{qIe^{-j\omega r/c}}{j\omega 4\pi\varepsilon_0}\frac{l\sin\theta}{r}[\frac{j\omega}{cr}+\frac{1}{r^2}]=\frac{qIe^{-j\omega r/c}l\sin\theta}{4\pi\varepsilon_0 c}[\frac{1}{r^2}-\frac{jc}{\omega r^3}] \tag{11.28}$$

11.2.2 The Inductive Force

For the dipole antenna there is also an inductive force acting on the point charge q. This is the acceleration dependent Weber force, formula (11.3):

$$d\bar{F}=-\frac{qd\bar{L}}{4\pi\varepsilon_0 c^2 r}\frac{dI(t-r/c)}{dt}$$

For a small antenna with uniform current in the z direction the force becomes

$$\bar{F}=-\frac{j\omega ql}{4\pi\varepsilon_0 c^2 r}I_0 e^{j\omega t}e^{-j\omega r/c}\hat{z} \tag{11.29}$$

The direction vector is in spherical coordinates

$$\hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta$$

and the inductive force is

$$\bar{F}=-\frac{Ij\omega ql}{4\pi\varepsilon_0 c^2 r}e^{-j\omega r/c}(\cos\theta\hat{r}-\sin\theta\hat{\theta}) \tag{11.30}$$

11.2.3 Total Force

From (11.28) and (11.30), the θ component of the total force becomes

$$F_{\theta}=\frac{qIe^{-j\omega r/c}l\sin\theta}{4\pi\varepsilon_0 c}[\frac{j\omega}{cr}+\frac{1}{r^2}-\frac{jc}{\omega r^3}] \tag{11.31}$$

and the r component, using (11.26) and (11.30),

$$F_r = \frac{qIe^{-j\omega r/c}l}{4\pi\varepsilon_0 c}\cos\theta(-\frac{j\omega}{cr} + \frac{j\omega}{cr} + \frac{2}{r^2} - 2\frac{jc}{\omega r^3})$$
$$= \frac{qIe^{-j\omega r/c}l}{4\pi\varepsilon_0 c}\cos\theta(\frac{2}{r^2} - \frac{2jc}{\omega r^3}) \quad (11.32)$$

The first term in (11.31) predominates the total force in the far distance from the source, commonly known as 'radio waves'. In this case there is no interaction along the wire, $\theta = 0$, compare Exercise (11.6).

11.3 Antenna Array

An array of antenna elements may be used to create a focused directed force action. Consider a two-element array of two small (point-like) dipole antennas, Fig. 11.4. The dipoles are now oriented along the x axis. To simplify, consider the interaction in the xz plane only so that $\sin\theta$ in formula (11.31) is just replaced by $\cos\theta$. In the far distance the total force becomes

$$F_\theta = \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}[\frac{e^{-j\omega r_1/c}}{r_1}\cos\theta_1 + \frac{e^{-j\omega r_2/c}}{r_2}\cos\theta_2] \quad (11.33)$$

where θ is the angle between the distance vector and the z axis. In the far distance the following approximations may be done:

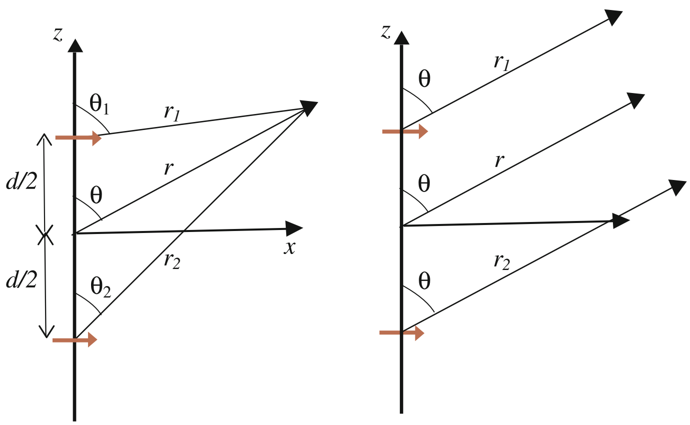


Fig. 11.4 An antenna array consisting of two dipole elements. *Left* Exact geometry. *Right* The far distance approximation

1. For phase variation, i.e. for the exponent in the exponential

$$r_1 \approx r - \frac{d}{2}\cos\theta$$

$$r_2 \approx r + \frac{d}{2}\cos\theta$$

2. For the amplitude, i.e. the $1/r_1$ and $1/r_2$ factors

$$r_1 \approx r_2 \approx r$$

3. For the angle θ

$$\theta_1 \approx \theta_2 \approx \theta$$

The force becomes

$$\begin{aligned} F_\theta &= \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}[\frac{e^{-j\omega(r-\frac{d}{2}\cos\theta)/c}}{r}\cos\theta + \frac{e^{-j\omega(r+\frac{d}{2}\cos\theta)/c}}{r}\cos\theta] \\ &= \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}\frac{e^{-j\frac{\omega}{c}r}}{r}\cos\theta[e^{j\omega\frac{d}{2}\cos\theta/c} + e^{-j\omega\frac{d}{2}\cos\theta/c}] \\ &= \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}\frac{e^{-j\frac{\omega}{c}r}}{r}2\cos\theta\cos(\frac{d\omega}{2c}\cos\theta) \end{aligned} \tag{11.34}$$

Formula (11.34) is valid in the far distance. It also assumes that the antennas oscillate in phase. The force action may be focused in different directions by introducing a phase shift β between the element oscillations. The force then becomes:

$$F_\theta = \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}[\frac{e^{-j\omega(r-\frac{d}{2}\cos\theta)/c+\frac{j\beta}{2}}}{r}\cos\theta + \frac{e^{-j\omega(r+\frac{d}{2}\cos\theta)/c-\frac{j\beta}{2}}}{r}\cos\theta]$$

so that

$$F_\theta = \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}\frac{e^{-j\frac{\omega}{c}r}}{r}2\cos\theta\cos(\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2}) \tag{11.35}$$

11.4 Power Transmission

The antenna power transmission will now be formulated using the Poynting vector (10.94). In field theory the force acting on a point charge at rest is due to an electric field. Both the electrostatic and inductive forces are therefore associated with an electric field, obtained by dividing the force formulas by q, see Exercise (11.11).

The power transmitted per unit area is

$$\bar{S} = \bar{E} \times \bar{H} \tag{11.36}$$

where the magnetic field intensity H in the far distance is generated solely by the electric field E and vice versa.

Since the time variation is usually quicker than what can be measured, the time averaged power is considered. As is shown in Exercise (11.12), when working with complex quantities varying harmonically with time (so-called phasors), the time average of the real Poynting vector is

$$\langle \bar{S} \rangle = \frac{1}{2} \bar{E} \times \bar{H}^* \tag{11.37}$$

where the star denotes complex conjugate.

The magnetic field is given by Maxwell's equation (10.71), the induction law

$$\nabla \times \bar{E} = -\frac{d\bar{B}}{dt} \tag{11.38}$$

The time dependence of the magnetic field is the same as that for the electric field so that

$$\nabla \times \bar{E} = -j\omega \bar{B} \tag{11.39}$$

11.4.1 The Dipole Antenna

Consider first the small dipole antenna with uniform current. Far away from the source, the so-called far field, the electric field is obtained from (11.31):

$$E_\theta \approx \frac{I e^{-j\omega r/c} l \sin\theta}{4\pi\varepsilon_0 c} \frac{j\omega}{cr} \tag{11.40}$$

The coordinates are spherical and the curl becomes

$$\begin{aligned} \nabla \times \bar{E} &= \frac{1}{r}[\frac{d}{dr}(rE_\theta)]\hat{\phi} = \frac{1}{r}\frac{d}{dr}\frac{I e^{-j\omega r/c} l \sin\theta}{4\pi\varepsilon_0 c}\frac{j\omega}{c}\hat{\phi} \\ &= \frac{I\omega^2 e^{-j\omega r/c} l \sin\theta}{4\pi\varepsilon_0 c^3}\hat{\phi} = -\frac{j\omega}{c}E_\theta\hat{\phi} \end{aligned} \tag{11.41}$$

Thus, the magnetic field is

$$\bar{B} = \frac{E_\theta}{c}\hat{\phi} \tag{11.42}$$

The time-averaged real Poynting vector becomes

$$\langle \bar{S} \rangle = \frac{1}{2}\bar{E} \times \bar{H}^* = \frac{1}{2}\bar{E} \times \frac{\bar{B}^*}{\mu_0} = \frac{|E_\theta|^2}{2\mu_0 c}\hat{r}$$
$$\langle \bar{S} \rangle = \frac{1}{2\mu_0 c}(\frac{I_0 \omega l \sin\theta}{4\pi\varepsilon_0 c^2 r})^2\hat{r} \tag{11.43}$$

which is the power per unit area transmitted by the antenna. It propagates in the radial direction perpendicular to both the electric and the magnetic fields. The fields themselves are perpendicular to each other.

11.4.2 The Loop Antenna

The electric far field associated with a loop antenna with uniform current is given by formula (11.11):

$$E_\phi = \frac{\omega\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)}\frac{\omega}{c} \tag{11.44}$$

To find the magnetic far field, the curl of E has to be evaluated

$$\nabla \times \bar{E} = \frac{1}{r\sin\theta}[\frac{d}{d\theta}(E_\phi \sin\theta)]\hat{r} - \frac{1}{r}[\frac{d}{dr}(rE_\phi)]\hat{\theta} \tag{11.45}$$

The first term varies as $1/r^2$ and is neglected in the far field. The second term gives

$$\nabla \times \bar{E} = -\frac{1}{r}[\frac{d}{dr}(rE_\phi)]\hat{\theta} = -\frac{1}{r}\frac{\omega^2\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^3} I_0 \frac{d}{dr}(e^{j\omega(t-r/c)})\hat{\theta}$$
$$= \frac{j\omega^3\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^4 r} I_0 e^{j\omega(t-r/c)}\hat{\theta} \tag{11.46}$$

The magnetic field becomes

$$\bar{B} = j\frac{\nabla \times \bar{E}}{\omega} = -\frac{\omega^2\pi a^2 \sin\theta}{4\pi\varepsilon_0 c^4 r} I_0 e^{j\omega(t-r/c)}\hat{\theta} \tag{11.47}$$

and the time averaged Poynting vector

$$\langle \bar{S} \rangle = \frac{1}{2}\bar{E} \times \bar{H}^* = \frac{1}{2}\bar{E} \times \frac{\bar{B}^*}{\mu_0}$$
$$\langle \bar{S} \rangle = \frac{1}{2\mu_0 c}(\frac{I_0 \omega^2 \pi a^2 \sin\theta}{4\pi\varepsilon_0 c^3 r})^2\hat{r} \tag{11.48}$$

suppressed by a factor $(\omega a/c)^2$ compared to the dipole antenna.

11.5 The Wave Concept

The basic antenna theory outlined above is actually a wave theory. A wave is a concept defined by the following three criteria:

- A wave causes a periodic disturbance in space and time propagating at a finite speed
- A wave exhibits interference effects
- An undisturbed wave has an infinite range

In antenna theory item 1 is true by virtue of the resulting formulas (11.11) and (11.31) which shows periodic disturbances in both space and time.

Item 2 is proved in Exercise (11.2) where formula (11.34) is applied to find maxima and minima in the force pattern.

Item 3 demands a distance dependence as $1/r$, i.e. the far field terms of the formulas. The reason is that the power through a closed sphere is then distance independent. Integrating for example formula (11.43) around a spherical shell of radius r, the power becomes

$$\langle P \rangle = const \cdot \int_0^{2\pi} \int_0^{\pi} \sin^2\theta (\frac{1}{r})^2 r^2 \sin\theta d\theta d\phi$$

independent of distance. The far field approximation is accordingly equivalent to a wave propagation.

An important approximation in wave theory is the concept of a 'plane wave'. This corresponds to a wave with an extended wave front propagating in one particular direction. This may be achieved in two ways:

1. Far away from an oscillating point source where the circular wave fronts are approximately straight over an extended distance.
2. Close to a straight row of pointlike oscillators, i.e. an antenna array.

11.6 Summary

The force on a point charge at rest from a small loop antenna with uniform and harmonically time varying current is

$$F_\phi = \frac{\omega q \pi a^2 \sin\theta}{4\pi\varepsilon_0 c^2 r} I_0 e^{j\omega(t-r/c)} [\frac{\omega}{c} - \frac{j}{r}] \tag{11.11}$$

in spherical coordinates where the loop is located in the xy plane.

The force on a point charge at rest from a small dipole antenna located along the z axis with uniform and harmonically time varying current is

$$F_\theta = \frac{qIe^{-j\omega r/c} l \sin\theta}{4\pi\varepsilon_0 c}\left[\frac{j\omega}{cr} + \frac{1}{r^2} - \frac{jc}{\omega r^3}\right] \tag{11.31}$$

$$F_r = \frac{qIe^{-j\omega r/c} l}{4\pi\varepsilon_0 c}\cos\theta\left(\frac{2}{r^2} - \frac{2jc}{\omega r^3}\right) \tag{11.32}$$

in spherical coordinates.

An array consisting of two small dipole antennas on the z axis at a distance d oriented along the x axis creates a force on a point charge at rest in the far distance

$$F_\theta = \frac{qIj\omega l}{4\pi\varepsilon_0 c^2}\frac{e^{-j\frac{\omega}{c}r}}{r} 2\cos\theta\cos\left(\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2}\right) \tag{11.35}$$

where currents vary harmonically in time and β is the oscillating phase difference between the two elements.

The time average power from an antenna is given by

$$\langle\bar{S}\rangle = \frac{1}{2}\bar{E} \times \bar{H}^* \tag{11.37}$$

where in the far field the magnetic field is given by the Faraday-Henry induction law

$$\nabla \times \bar{E} = -j\omega\bar{B} \tag{11.39}$$

for harmonically time varying fields.

The force generated by oscillating charges may also be considered in wave theory. The generated far distance action fulfils all three criteria of wave propagation.

11.7 Exercises

11.1 *Force magnitude pattern*

Show in a graph the relative force magnitude pattern on a point charge at rest in the far region as a function of the angle θ to the z axis in the following two cases:

a. For a single dipole antenna oriented along the z axis as in Fig. 11.1.
b. For a single loop antenna with surface normal along the z axis as in Fig. 11.3.

What is the direction of maximum and vanishing force in the two cases?

11.2 *Interference*

For a two-element array oscillating in phase what is the condition

a. for interference minimum and maximum?
b. on the distance between the elements in order to avoid destructive interference?

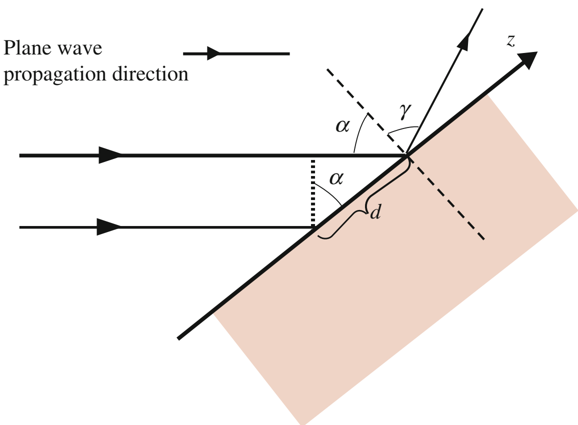
11.3 *Directed interaction*

Consider a two-element dipole array oriented as in Fig. 11.4. The distance between the elements is $d = \frac{c}{4f}$, where c is the speed of mediation and f is the frequency of the oscillation.
What is the direction of maximum force on a point charge at rest in the xz plane for the following cases of oscillation phase difference:

a. $\beta = 0$
b. $\beta = -\frac{\pi}{2}$
c. $\beta = \frac{\pi}{2}$

*11.4 *Reflection law*

a. An antenna interacts in the far distance with a polished metal surface such as a mirror. The figure shows the propagation direction of the plane wave associated with the antenna. Find a relation between the angles α and γ defined in the figure, where the dashed line is surface normal of the material.
   Compare with Exercise (5.8).
b. Repeat with a non-conductive material.



Hint: Use the two element antenna array as a principal model for a general antenna array and determine the angle γ for which there is maximum force on a hypothetical point charge outside the material. The natural oscillators at the surface are activated by the incoming wave to form an antenna array with an oscillating phase shift β. In task a, the natural oscillators are the free conductions electrons whereas in task b these are the molecular dipoles.

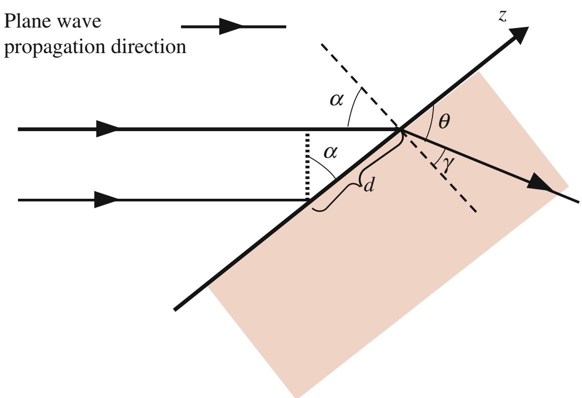
Explain why this phenomenon is better described by the term 're-radiation', rather than 'reflection'.

*11.5 *Snell's refraction law*

Consider an antenna interacting with a non-conductive material in the far distance. Show that the relation between the incident angle α and the 'refraction' angle γ is given by

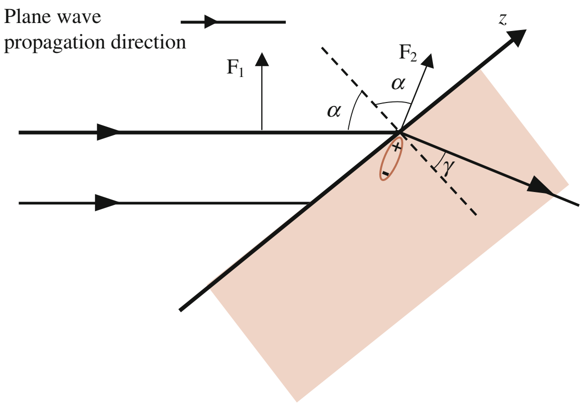
$$c \sin\alpha = c_0 \sin\gamma$$

where c is the mediation speed in the material and c_0 that in the incident medium.
Hint: Utilize the same model as in Exercise (11.4b).



*11.6 *Brewster reflection*

In the far distance from the source, the force is perpendicular to the lines of interaction, see formulas (11.11) and (11.31). Accordingly, there are two possible directions of the force.
Consider an interaction between an antenna and a material as in Exercise (11.5). The two possible direction of force is divided into a direction perpendicular to surface normal, i.e. out of the plane in the figure, and a direction in the plane of the figure. The latter case involves a phenomenon known as 'Brewster reflection'. At a certain incident angle the reflection vanishes.



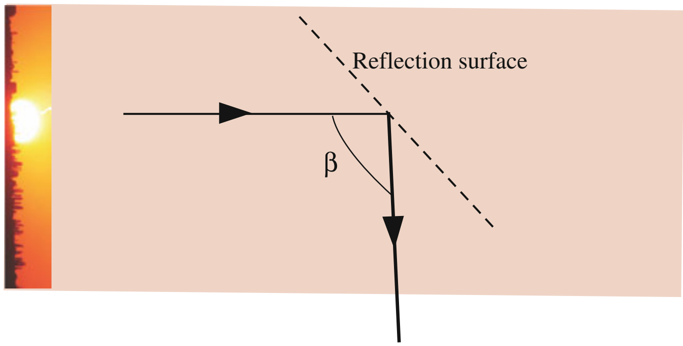
Show that the relation between this angle and the speeds of mediation is

$$\tan \alpha_B = \frac{c_0}{c}$$

Hint: Use the model as in Exercise (11.4b) and utilize the force pattern for a dipole obtained in Exercise (11.1).

*11.7 *Spider vision*

Many species of spider have evolved a vision sense that is sensitive to the direction of force from a source, so-called polarisation vision. Of their usually four pair of eyes, one pair is used for this purpose. The two eyes are structured such that they are sensitive to direction of force in relative perpendicular directions. The spider uses this sense to orient and navigate during dawn and morning when the sun is too close to the horizon to be visible for the spider.



In this case the sun is the source antenna and the spider looks towards heaven. Determine the angle β in the figure for which the spider experiences only one direction of force. This will be the spot on heaven most usable for navigation aid (Fig. 11.5).

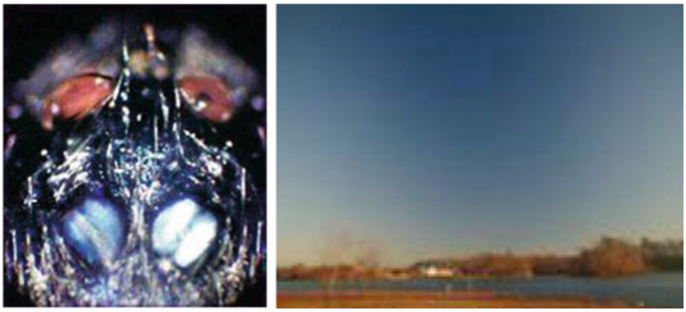


Fig. 11.5 *Left* The pair of this spider's eight eyes which are used for polarisation vision. Used by permission of Marie Dacke, Lund University. *Right* Heaven observed through polarisation sun glasses

*11.8 *Refractive index and speed of light*

Snell's refraction law is frequently expressed with the so-called refractive index n, a material quantity defined by writing the refraction law in the following way:

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2$$

where index 1 refers to the incident material and index 2 to the transmitted material. The refractive index is given in relation to 'vacuum' such that $n_{vacuum} = 1$ exactly.
Find an expression for the refractive index in terms of the material parameters dielectric constant κ_e and relative magnetic permeability κ_m.

11.9 Refractive index for water is around 1.6 whereas the dielectric constant is around 80 in the electrostatic case. Since $\kappa_m \approx 1$ for water there seem to be an inconsistency. Could you explain this?

11.10 *Metamaterials*

Metamaterials are characterized as having a negative refractive index meaning that the angle γ in Exercise (11.5) is negative.
Prove that this violates the principle of cause and effect.
What is the angle of re-radiation for a meta-material?

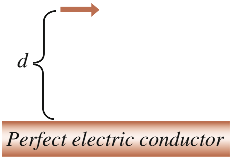
*11.11 Using formulas (10.80) and (10.82) show that the Weber inductive force is associated with an electric field in field theory.

*11.12 *Time average of Poynting vector*

Show that the time average of the Poynting vector is given by formula (11.37).

11.13 Determine the Poynting vector in the far field for an antenna array consisting of two parallel small dipole antennas.

11.14 *Mirror theory*



Consider a small dipole antenna above a sheet of a perfect conductor. Practical examples are an antenna placed on a metal roof, a car or a mobile phone. If the antenna is oriented parallel to the sheet at a distance d as in the figure, determine the generated far field averaged power per unit area.
Hint: Consider the re-radiated (so-called reflected) power as originating from a mirror image of the antenna.

Further Readings

C.A. Balanis, *Antenna Theory* (Wiley, New York, 1997)
P. Moon, D.E. Spencer, A new electrodynamics. J. Frankl. Inst. **257**, 369 (1954)

Original Paper

First dipole antenna:
H.R. Hertz, Ueber sehr schnelle electrische Schwingungen. Ann. der Phys. **267**, 421 (1887)

Appendix A
Electric Multipoles

In Sects. 7.1.4 and 8.3, the general multipole expansion was discussed. In Exercise (7.1), a multipole expansion of the electrostatic potential is performed. In this and the next appendix, the expansion will be obtained from a general interaction picture in the electric and the magnetic case respectively. The starting point is the basic energy formulas (8.89) and (8.88).

The electric energy is given by (8.89), Fig. A.1,

$$U_e = \frac{1}{4\pi\varepsilon_0} \int\limits_V \int\limits_{V'} \frac{\rho(\bar{r})\rho'(\bar{r}')}{R} dV' dV \tag{A.1}$$

where $\bar{R} = \bar{r} - \bar{r}'$. If the systems interact at a large distance, the energy may be expressed through the multipole expansion. The energy is then series-expanded (also known as Taylor or Maclaurin expansion) with respect to the distance between the systems.

In the first step, let the system V' interact with a pointlike system V with the position vector $\bar{r}$, where $r \gg r'$, and expand the energy arisen about the centre of V'. This means that the system V' is expressed in multipole terms. The second step involves a series expansion about the centre of the system V, for each multipole order of V', thereby expressing the system V in terms of multipoles.

The calculation will be made up to the order $1/d^5$ corresponding to a quadrupole–quadrupole interaction, where d is the distance between the centres of the systems. The following definitions will be used:

For a system with volume τ, the total charge is

$$q = \int\limits_\tau \rho(x, y, z) d\tau \tag{A.2}$$

the dipole moment

$$p_i = \int\limits_\tau r_i \rho(x, y, z) d\tau \tag{A.3}$$

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9

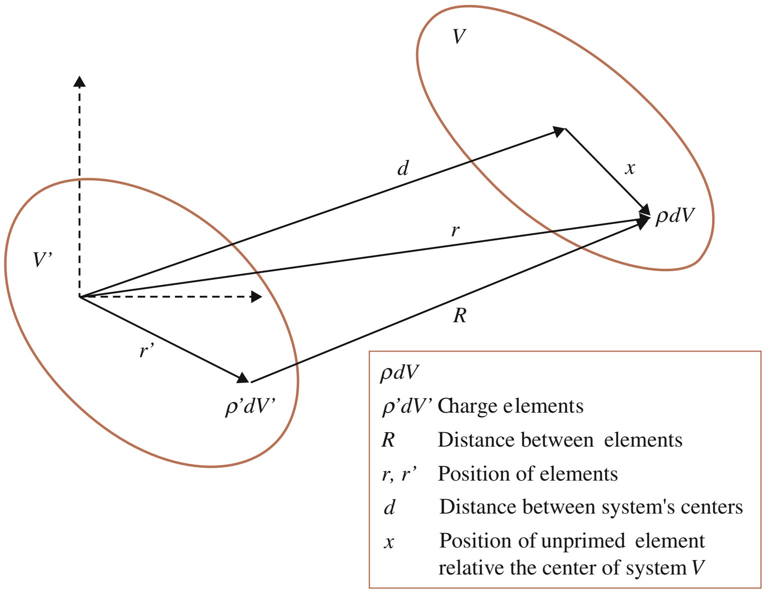


Fig. A.1 Two systems interact electrically

the quadrupole moment

$$Q_{ij} = \int_{\tau} r_i r_j \rho(x, y, z) d\tau \tag{A.4}$$

and so on for higher order moments, where $r_1 = x$, $r_2 = y$, $r_3 = z$. Note that formula (A.4) differs from the definition of quadrupole moment in Exercise (7.1). The latter is known as reduced quadrupole moment and is the more common form. Here, formula (A.4) is chosen as this definition is simpler and results in more transparent formulas.

A.1 Multipole Expansion of System V'

In analogy with Exercise (7.1), the series expansion up to quadrupole order of the primed system gives

$$U_e \approx \frac{1}{4\pi\varepsilon_0}\left[\int_V\int_{V'} \frac{\rho(\bar{r})\rho'(\bar{r}')}{r} dV'dV + \int_V\int_{V'} \frac{\rho\rho'\bar{r}\cdot\bar{r}'}{r^3} dV'dV \right.$$

$$\left. + \frac{1}{2}\int_V\int_{V'} \rho\rho'\left(\frac{3(\bar{r}\cdot\bar{r}')^2}{r^5} - \frac{r'^2 r^2}{r^5}\right) dV'dV\right] \tag{A.5}$$

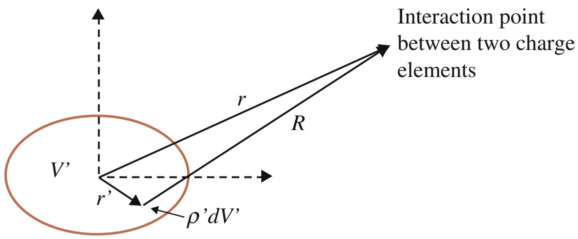


Fig. A.2 Interaction point

These terms correspond to monopole, dipole and quadrupole structures of the primed system (Fig. A.2).

A.2 Multipole Expansion of System *V*

Now let the multipoles of the primed system interact individually with the different multipoles of the system V.

A.2.1 V′ Monopole

The first term in (A.5) corresponds to a monopole of system V':

$$U_e^{m-} = \frac{1}{4\pi\varepsilon_0}\int\limits_V\int\limits_{V'}\frac{\rho(\bar{r})\rho'(\bar{r}')}{r}dV'dV \tag{A.6}$$

A series expansion is performed about the unprimed system's centre in order to include its structure, Fig. A.1. Let the vector $\bar{x}$ be the distance between the centre of the system V and an element of charge, i.e. $\bar{r} = \bar{x} + \bar{d}$, where $d \gg x$. Then

$$\frac{1}{r} = \frac{1}{d\left(1+\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2}\right)^{1/2}} \approx \frac{1}{d}\left(1 - \frac{1}{2}\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2} + \frac{3}{8}\left(\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right) \tag{A.7}$$

Terms which are linear and quadratic in x correspond to dipole and quadrupole of the system V. The first term gives the monopole structure, i.e. the point charge. Thus,

Monopole–monopole

$$U_e^{mm} = \frac{1}{4\pi\varepsilon_0}\frac{1}{d}\int\limits_V\int\limits_{V'}\rho(\bar{r})\rho'(\bar{r}')dV'dV = \frac{qq'}{4\pi\varepsilon_0 d} \tag{A.8}$$

Monopole-dipole

$$U_e^{md} = \frac{1}{4\pi\varepsilon_0} \int_V \int_{V'} \rho(\bar{r})\rho'(\bar{r}') \frac{1}{d} \left(-\frac{1}{2}\frac{2\bar{d}\cdot\bar{x}}{d^2}\right) dV' dV = -\frac{q'\hat{d}\cdot\bar{p}}{4\pi\varepsilon_0 d^2} \tag{A.9}$$

which is formula (7.7).

Monopole-quadrupole

$$U_e^{mq} = \frac{1}{4\pi\varepsilon_0} \int_V \int_{V'} \rho(\bar{r})\rho'(\bar{r}') \frac{1}{d} \left(-\frac{1}{2}\frac{x^2}{d^2} + \frac{3}{8}\left(\frac{2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right) dV' dV$$
$$= -\frac{q'}{8\pi\varepsilon_0 d^3} \left(\sum_i Q_{ii} - \frac{3}{d^2}\sum_{i,j} d_i d_j Q_{ij}\right) \tag{A.10}$$

Compare with Exercise (7.1).

A.2.2 V' Dipole

Now consider the next term in the V'-structure, formula (A.5), corresponding to a dipole. Similarly to the procedure above, a series expansion is performed about the centre of the unprimed system. The integrand of the second term in formula (A.5) becomes

$$\frac{\rho\rho'\bar{r}\cdot\bar{r}'}{r^3} = \rho\rho'\frac{(\bar{x}+\bar{d})\cdot\bar{r}'}{r^3} = \rho\rho'\frac{\bar{x}\cdot\bar{r}'}{r^3} + \rho\rho'\frac{\bar{d}\cdot\bar{r}'}{r^3}$$
$$\approx \rho\rho'\frac{\bar{x}\cdot\bar{r}'}{d^3}\left(1 - \frac{3}{2}\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2} + \frac{15}{8}\left(\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right)$$
$$+ \rho\rho'\frac{\bar{d}\cdot\bar{r}'}{d^3}\left(1 - \frac{3}{2}\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2} + \frac{15}{8}\left(\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right) \tag{A.11}$$

The dipole-monopole term is equivalent to the monopole-dipole term, which was calculated above. The next order is therefore dipole–dipole.

Dipole–dipole
This order corresponds to terms which are linear in r' and x:

$$U_e^{dd} = \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho\rho'\bar{r}\cdot\bar{r}'}{r^3} dV' dV$$
$$\approx \frac{1}{4\pi\varepsilon_0} \iint \rho\rho'\left[\frac{\bar{x}\cdot\bar{r}'}{d^3} + \frac{\bar{d}\cdot\bar{r}'}{d^3}\left(-\frac{3}{2}\frac{2\bar{d}\cdot\bar{x}}{d^2}\right)\right] dV' dV$$

$$= \frac{1}{4\pi\varepsilon_0 d^3}((\bar{p}\cdot\bar{p}') - 3(\hat{d}\cdot\bar{p})(\hat{d}\cdot\bar{p}')) \tag{A.12}$$

which is formula (7.16).

Dipole-quadrupole
In this order, terms which are linear in r' and quadratic in x contribute:

$$U_e^{dq} \approx \frac{1}{4\pi\varepsilon_0}\iint \rho\rho'\left[\frac{\bar{x}\cdot\bar{r}'}{d^3}\left(-\frac{3}{2}\frac{2\bar{d}\cdot\bar{x}}{d^2}\right) + \frac{\bar{d}\cdot\bar{r}'}{d^3}\left(-\frac{3}{2}\frac{x^2}{d^2} + \frac{15}{8}\left(\frac{2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right)\right]dV'dV$$

$$= \frac{3}{4\pi\varepsilon_0 d^5}\left[-\sum_{i,j} Q_{ij}p'_i d_j - \frac{1}{2}\bar{d}\cdot\bar{p}'\sum_i Q_{ii} + \frac{5}{2d^2}\bar{d}\cdot\bar{p}'\sum_{i,j} d_i d_j Q_{ij}\right] \tag{A.13}$$

A.2.3 V' Quadrupole

Quadrupole-monopole and quadrupole-dipole were calculated above but with reversed systems. The energy is of course independent of which system is primed or unprimed. Therefore, there is only one remaining interaction at this level:

Quadrupole–quadrupole
The quadrupole–quadrupole interaction corresponds to the quadratic terms with respect to both r' and x. The quadrupole term in (A.5) is

$$U_e^{q-} = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{2}\iint \rho\rho'\left(\frac{3(\bar{r}\cdot\bar{r}')^2}{r^5} - \frac{r'^2 r^2}{r^5}\right)dV'dV\right] \tag{A.14}$$

$$\begin{aligned}
&\frac{\rho\rho'(3(\bar{r}\cdot\bar{r}')^2 - r'^2r^2)}{r^5}\\
&\quad\approx \rho\rho'(3(\bar{x}+\bar{d})\cdot\bar{r}')^2 - r'^2(\bar{x}+\bar{d})^2)\frac{1}{d^5}\left(1 - \frac{5}{2}\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2} + \frac{35}{8}\left(\frac{x^2+2\bar{d}\cdot\bar{x}}{d^2}\right)^2\right)\\
&\quad= \rho\rho'[3((\bar{x}\cdot\bar{r}')^2 + 2(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}') + (\bar{d}\cdot\bar{r}')^2) - r'^2x^2 - r'^2 2(\bar{x}\cdot\bar{d}) - r'^2d^2]\frac{1}{d^5}\\
&\qquad+ \rho\rho'[3((\bar{x}\cdot\bar{r}')^2 + 2(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}') + (\bar{d}\cdot\bar{r}')^2) - r'^2x^2 - r'^2 2(\bar{x}\cdot\bar{d}) - r'^2d^2]\left(-\frac{5}{2}\frac{x^2}{d^7}\right)\\
&\qquad+ \rho\rho'[3((\bar{x}\cdot\bar{r}')^2 + 2(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}') + (\bar{d}\cdot\bar{r}')^2) - r'^2x^2 - r'^2 2(\bar{x}\cdot\bar{d}) - r'^2d^2]\left(-5\frac{\bar{d}\cdot\bar{x}}{d^7}\right)\\
&\qquad+ \rho\rho'[3((\bar{x}\cdot\bar{r}')^2 + 2(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}') + (\bar{d}\cdot\bar{r}')^2) - r'^2x^2 - r'^2 2(\bar{x}\cdot\bar{d}) - r'^2d^2]\frac{35}{2}\frac{(\bar{d}\cdot\bar{x})^2}{d^9}
\end{aligned} \tag{A.15}$$

where terms which are up to quadratic in r' and x have been kept.

The quadratic terms result in

$$\rho\rho'[3(\bar{x}\cdot\bar{r}')^2 - r'^2x^2]\frac{1}{d^5} + \rho\rho'[3(\bar{d}\cdot\bar{r}')^2 - r'^2d^2]\left(-\frac{5}{2}\frac{x^2}{d^7}\right)$$
$$+\ \rho\rho'[3(2(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}')) - r'^2 2(\bar{x}\cdot\bar{d})]\left(-5\frac{\bar{d}\cdot\bar{x}}{d^7}\right)$$
$$+\ \rho\rho'[3(\bar{d}\cdot\bar{r}')^2 - r'^2d^2]\frac{35}{2}\frac{(\bar{d}\cdot\bar{x})^2}{d^9}$$
$$=\ \rho\rho'\left[\frac{3(\bar{x}\cdot\bar{r}')^2 + \frac{3}{2}r'^2x^2}{d^5} - \frac{15}{2}(\bar{d}\cdot\bar{r}')^2\frac{x^2}{d^7} - \frac{30(\bar{x}\cdot\bar{r}')(\bar{d}\cdot\bar{r}')(\bar{d}\cdot\bar{x})}{d^7}\right.$$
$$\left. -\ \frac{\frac{15}{2}r'^2(\bar{d}\cdot\bar{x})^2}{d^7} + \frac{105}{2}\frac{(\bar{d}\cdot\bar{r}')^2(\bar{d}\cdot\bar{x})^2}{d^9}\right]$$
$$=\ \frac{3}{d^5}\sum_{i,j}\left(\tilde{Q}_{ij}\tilde{Q}'_{ij} + \frac{1}{2}\tilde{Q}_{ii}\tilde{Q}'_{jj}\right) - \frac{15}{2d^7}\sum_{i,j}d_id_j\tilde{Q}'_{ij}\sum_i\tilde{Q}_{ii}$$
$$-\ \frac{30}{d^7}\sum_{i,j,k}\tilde{Q}_{ik}\tilde{Q}'_{ij}d_jd_k - \frac{15}{2d^7}\sum_i\tilde{Q}'_{ii}\sum_{i,j}d_id_j\tilde{Q}_{ij}$$
$$+\ \frac{105}{2d^9}\sum_{i,j}d_id_j\tilde{Q}'_{ij}\sum_{i,j}d_id_j\tilde{Q}_{ij} \tag{A.16}$$

where $\tilde{Q}$ denotes the density of quadrupole moment.

The quadrupole–quadrupole energy is

$$U_e^{qq} = \frac{1}{4\pi\varepsilon_0}\frac{3}{4}\left[\frac{1}{d^5}\sum_{i,j}(2Q_{ij}Q'_{ij} + Q_{ii}Q'_{jj}) - \frac{5}{d^7}\sum_{i,j}d_id_jQ'_{ij}\sum_i Q_{ii}\right.$$
$$-\ \frac{20}{d^7}\sum_{i,j,k}Q_{ik}Q'_{ij}d_jd_k - \frac{5}{d^7}\sum_i Q'_{ii}\sum_{i,j}d_id_jQ_{ij}$$
$$\left. +\ \frac{35}{d^9}\sum_{i,j}d_id_jQ'_{ij}\sum_{i,j}d_id_jQ_{ij}\right] \tag{A.17}$$

The total energy in quadrupole order becomes

$$U_e = U_e^{mm} + U_e^{md} + U_e^{mq} + U_e^{dm} + U_e^{dd} + U_e^{dq} + U_e^{qm} + U_e^{qd} + U_e^{qq}$$
$$= U_e^{mm} + U_e^{dd} + U_e^{qq} + 2U_e^{md} + 2U_e^{mq} + 2U_e^{dq}$$

Exercises

A.1 Use formula (A.10) to determine the electric potential from a pure quadrupole.

Answer:

$$\Phi^q = -\frac{1}{8\pi\varepsilon_0 d^3}\left(\sum_i Q_{ii} - \frac{3}{d^2}\sum_{i,j} d_i d_j Q_{ij}\right)$$

In matrix form

$$\Phi^q = -\frac{1}{8\pi\varepsilon_0 d^3}\left(Tr\,\boldsymbol{Q} - \frac{3}{d^2}\bar{d}\cdot(\boldsymbol{Q}\bar{d})\right)$$

A.2 Express the electric field associated with a pure quadrupole system.

Hint: Use formula (A.13)

Answer:

$$\bar{E} = \frac{3}{4\pi\varepsilon_0 d^5}\left[\boldsymbol{Q}\bar{d} + \frac{1}{2}\bar{d}\sum_i Q_{ii} - \frac{5}{2d^2}\bar{d}\sum_{i,j} d_i d_j Q_{ij}\right]$$

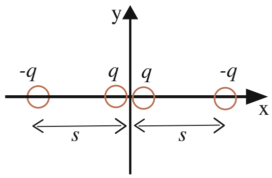
where $\boldsymbol{Q}$ is the quadrupole moment matrix so that

$$\boldsymbol{Q}\bar{d} = (Q_{11}d_1 + Q_{12}d_2 + Q_{13}d_3,\ Q_{21}d_1 + Q_{22}d_2 + Q_{23}d_3,\\ Q_{31}d_1 + Q_{32}d_2 + Q_{33}d_3)$$

Expressing also the second and third terms in matrix form the electric field may be written

$$\bar{E} = \frac{3}{4\pi\varepsilon_0 d^5}\left[\boldsymbol{Q}\bar{d} + \frac{1}{2}\bar{d}\,Tr\,\boldsymbol{Q} - \frac{5}{2d^2}\bar{d}\bar{d}\cdot(\boldsymbol{Q}\bar{d})\right]$$

A.3 A simple system for which the monopole and the dipole moments vanish is given in the figure.



Therefore, the quadrupole approximation is the lowest order term in the multipole expansion.

Use Exercise (A.2) to determine the electric field from this system in the far field, i.e. taking into account the lowest order (quadrupole) term only. The two positive charges in the center may be taken as one single object with charge $2q$ and placed in the origin.

Solution:

$$Q_{ij} = \sum_n r_{i,n} r_{j,n} q_n$$

$$Q_{xx} = x_1^2 q_1 + x_2^2 q_2 + x_3^2 q_3 = -2qs^2$$

$$Q_{yy} = Q_{zz} = Q_{xy} = Q_{xz} = Q_{yz} = 0$$

$$\boldsymbol{Q} = \begin{pmatrix} -2qs^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{Q}\bar{d} = \begin{pmatrix} -2qs^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2qs^2 x \\ 0 \\ 0 \end{pmatrix}$$

$$Tr\ \boldsymbol{Q} = -2qs^2$$

$$\bar{d} \cdot \boldsymbol{Q}\bar{d} = -2qs^2 x^2$$

$$\bar{E} = \frac{3}{4\pi\varepsilon_0 d^5} \left[\begin{pmatrix} -2qs^2 x \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} (-2qs^2) - \frac{5}{2d^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} (-2qs^2 x^2) \right]$$

$$= -\frac{3qs^2}{2\pi\varepsilon_0 d^5} \left[\begin{pmatrix} \frac{3}{2}x - \frac{5}{2d^2}x^3 \\ \frac{1}{2}y - \frac{5}{2d^2}x^2 y \\ \frac{1}{2}z - \frac{5}{2d^2}x^2 z \end{pmatrix} \right]$$

where

$$d^2 = x^2 + y^2 + z^2$$

Appendix B
Magnetic Multipoles

The general formula for the magnetic energy between two systems of *closed* current distributions is given by (8.88), Fig. B.1,

$$U_m = \frac{\mu_0}{4\pi} \int\limits_V \int\limits_{V'} \frac{\bar{J}(\bar{r}) \cdot \bar{J}'(\bar{r}')}{R} dV' dV \tag{B.1}$$

where R is the distance between two current elements and J is the current density. If the systems interact at a large distance, the multipole expansion of the energy is useful. Denoting volume with τ, the first moments of magnetic multipoles are defined as

$$\begin{aligned} m_0 &= \int\limits_\tau \bar{J}(x, y, z) d\tau \\ \bar{m} &= \frac{1}{2} \int\limits_\tau \bar{r} \times \bar{J}(x, y, z) d\tau \\ m_{ij} &= \frac{2}{3} \int\limits_\tau (\bar{r} \times \bar{J}(x, y, z))_i r_j d\tau \end{aligned} \tag{B.2}$$

and are named monopole, dipole, and quadrupole moment respectively.

In the first step, let the system V' interact with a pointlike system V with the position vector r, where $r \gg r'$, and expand the arisen energy about the centre of V'. This means that the system V' is expressed in terms of multipoles. Secondly, series-expand for each multipole order of V' about the centre of the system V, thereby expressing the system V in multipoles (equivalent to Appendix A).

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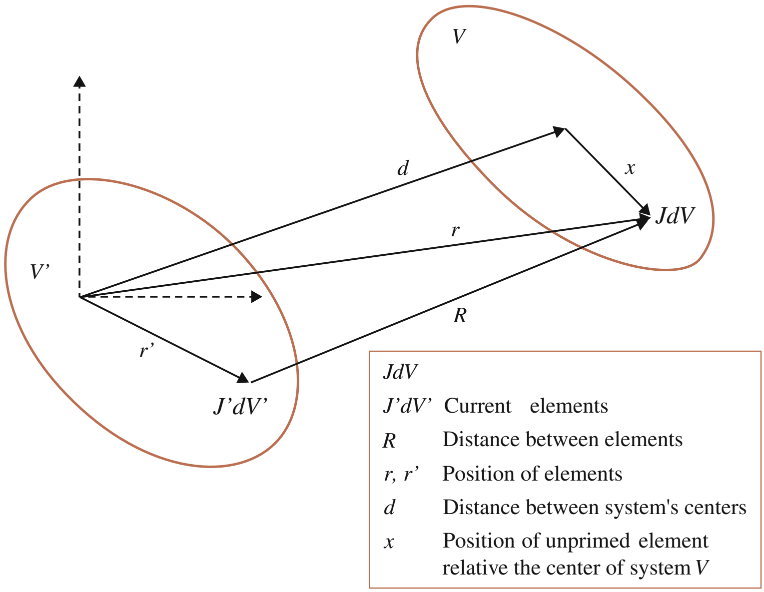

Fig. B.1 Two systems interact magnetically. The figure is the same as Fig. A.1 except that charge elements have been replaced by current elements

B.1 Multipole Expansion of the System V'

The magnitude of $\bar{R} = \bar{r} - \bar{r}'$ can be written

$$R = r\left(1 + \frac{r'^2 - 2\bar{r}\cdot\bar{r}'}{r^2}\right)^{1/2} \tag{B.3}$$

As the second term under the square root sign is small, series expansion is applied

$$\begin{aligned}\frac{1}{R} &= \frac{1}{r}\left[\frac{1}{(1+\frac{r'^2-2\bar{r}\cdot\bar{r}'}{r^2})^{1/2}}\right] \\ &= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'^2 - 2\bar{r}\cdot\bar{r}'}{r^2} + \frac{3}{8}\left(\frac{r'^2 - 2\bar{r}\cdot\bar{r}'}{r^2}\right)^2 + \cdots\right]\end{aligned} \tag{B.4}$$

Up to quadrupole order, i.e. terms which are quadratic in r', we obtain

$$\frac{1}{R} \approx \frac{1}{r} + \frac{\bar{r}\cdot\bar{r}'}{r^3} - \frac{1}{2}\frac{r'^2}{r^3} + \frac{3}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^5} \tag{B.5}$$

The energy becomes

$$U_m \approx \frac{\mu_0}{4\pi} \int_V \int_{V'} \bar{J}(\bar{r}) \cdot \bar{J}'(\bar{r}') \left[\frac{1}{r} + \frac{\bar{r} \cdot \bar{r}'}{r^3} + \frac{1}{2} \frac{(3(\bar{r} \cdot \bar{r}')^2 - r^2 r'^2)}{r^5} \right] dV' dV \quad \text{(B.6)}$$

equivalent to formula (A.5). The first term corresponds to a monopole and vanishes since $\int_{V'} \bar{J}'(\bar{r}') dV' = 0$ for closed currents.

B.2 Multipole Expansion of the System *V*

A series expansion is now performed about the centre of V in order to include its structure. The vector $\bar{x}$ is the distance between the centre of the system V and a current element in the volume V, i.e. $\bar{r} = \bar{x} + \bar{d}$, where $d \gg x$. Consider the term corresponding to a dipole for V'. Series expansion of V up to quadrupole order results in

$$\begin{aligned} \frac{\bar{J} \cdot \bar{J}' \bar{r} \cdot \bar{r}'}{r^3} &= \bar{J} \cdot \bar{J}' \frac{(\bar{x} + \bar{d}) \cdot \bar{r}'}{r^3} = \bar{J} \cdot \bar{J}' \frac{\bar{x} \cdot \bar{r}'}{r^3} + \bar{J} \cdot \bar{J}' \frac{\bar{d} \cdot \bar{r}'}{r^3} \\ &\approx \bar{J} \cdot \bar{J}' \frac{\bar{x} \cdot \bar{r}'}{d^3} \left(1 - \frac{3}{2} \frac{x^2 + 2\bar{d} \cdot \bar{x}}{d^2} + \frac{15}{8} \left(\frac{x^2 + 2\bar{d} \cdot \bar{x}}{d^2} \right)^2 \right) \\ &+ \bar{J} \cdot \bar{J}' \frac{\bar{d} \cdot \bar{r}'}{d^3} \left(1 - \frac{3}{2} \frac{x^2 + 2\bar{d} \cdot \bar{x}}{d^2} + \frac{15}{8} \left(\frac{x^2 + 2\bar{d} \cdot \bar{x}}{d^2} \right)^2 \right) \end{aligned} \quad \text{(B.7)}$$

B.3 Dipole–Dipole Interaction

Terms which contain only the one variable, either r' or x, such as

$$\bar{J} \cdot \bar{J}' \frac{\bar{d} \cdot \bar{r}'}{d^3}$$

vanish since they correspond to a monopole structure. The lowest order is therefore linear in x and r', corresponding to a dipole–dipole interaction. The energy is

$$U_m^{dd} = \frac{\mu_0}{4\pi} \int_V \int_{V'} (\bar{J}(\bar{r}) \cdot \bar{J}'(\bar{r}') \bar{x} \cdot \bar{r}' \frac{1}{d^3} - 3 \bar{J}(\bar{r}) \cdot \bar{J}'(\bar{r}') \bar{d} \cdot \bar{r}' \bar{d} \cdot \bar{x} \frac{1}{d^5}) dV' dV \quad \text{(B.8)}$$

which should be expressed in terms of dipole moment, formula (B.2).

The first term in formula (B.8) is expanded in the following manner:
According to vector algebra

$$(\bar{J}\cdot\bar{J}')(\bar{x}\cdot\bar{r}') = (\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}') + (\bar{J}\cdot\bar{r}')(\bar{J}'\cdot\bar{x}) \tag{B.9}$$

Since for closed currents $\nabla'\cdot\bar{J}' = 0$ we obtain

$$\int\limits_{V'}\int\limits_{V}\nabla'(\bar{x}\cdot\bar{r}')(\bar{r}'\cdot\bar{J})\bar{J}'dVdV'$$

$$= \int\limits_{V'}\int\limits_{V}[(\bar{x}\cdot\bar{J}')(\bar{r}'\cdot\bar{J}) + (\bar{x}\cdot\bar{r}')(\bar{J}\cdot\bar{J}')]dVdV' = 0 \tag{B.10}$$

The expression above vanishes due to the divergence theorem:

$$\int\limits_{V'}\int\limits_{V}\nabla'(\bar{x}\cdot\bar{r}')(\bar{r}'\cdot\bar{J})\bar{J}'dVdV' = \int\limits_{S'}\int\limits_{V}(\bar{x}\cdot\bar{r}')(\bar{r}'\cdot\bar{J})dV\bar{J}'\cdot d\bar{A}' = 0 \tag{B.11}$$

since the volume V' may be chosen greater than the volume for which J' exists so that J' disappears on the surface S'.

The results are compiled:

$$\int\limits_{V'}\int\limits_{V}(\bar{J}\cdot\bar{J}')(\bar{x}\cdot\bar{r}')dVdV' = \int\limits_{V'}\int\limits_{V}[(\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}') + (\bar{J}\cdot\bar{r}')(\bar{J}'\cdot\bar{x})]dVdV'$$

$$\int\limits_{V'}\int\limits_{V}[(\bar{x}\cdot\bar{J}')(\bar{r}'\cdot\bar{J}) + (\bar{x}\cdot\bar{r}')(\bar{J}\cdot\bar{J}')]dVdV' = 0 \tag{B.12}$$

from which it is clear that

$$\int\limits_{V'}\int\limits_{V}(\bar{J}\cdot\bar{J}')(\bar{x}\cdot\bar{r}')dVdV' = \frac{1}{2}\int\limits_{V'}\int\limits_{V}(\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}')dVdV'$$

$$= 2\int\limits_{V'}\int\limits_{V}\bar{M}(\bar{x})\cdot\bar{M}'(\bar{r}')dVdV' \tag{B.13}$$

where M is the density of magnetic dipole moment, that is

$$\bar{m} = \int\limits_{V}\bar{M}dV,\ \ \bar{m}' = \int\limits_{V'}\bar{M}'dV'$$

The second term of formula (B.8) is evaluated in the following manner:
Because $\bar{x}$ was treated as a constant during the calculation of the first step of (B.13), it is also valid that

$$\int_{V'}\int_{V}(\bar{J}\cdot\bar{J}')(\bar{d}\cdot\bar{r}')(\bar{d}\cdot\bar{x})dVdV' = \frac{1}{2}\int_{V'}\int_{V}(\bar{J}\times\bar{d})\cdot(\bar{J}'\times\bar{r}')(\bar{d}\cdot\bar{x})dVdV' \quad \text{(B.14)}$$

since the factor $(\bar{d}\cdot\bar{x})$ is also treated as a constant during the calculation, which is made via the primed coordinate. The right-hand side may be rewritten

$$\begin{aligned}(\bar{J}\cdot\bar{J}')(\bar{d}\cdot\bar{r}')(\bar{d}\cdot\bar{x}) &\Leftrightarrow \frac{1}{2}(\bar{J}\times\bar{d})\cdot(\bar{J}'\times\bar{r}')(\bar{d}\cdot\bar{x})\\ &= \frac{1}{2}(\bar{J}\times\bar{d})\cdot\bar{x}(\bar{J}'\times\bar{r}')\cdot\bar{d} + \frac{1}{2}(\bar{J}\times\bar{d})\times\bar{d}\cdot(\bar{J}'\times\bar{r}')\times\bar{x}\end{aligned} \quad \text{(B.15)}$$

where the equivalence signs indicate that the relationship is valid after integration over the volume V'. This notation is used in the following.

Consider the second term of the right-hand side

$$\begin{aligned}(\bar{J}\times\bar{d})\times\bar{d}\cdot(\bar{J}'\times\bar{r}')\times\bar{x} &= (\bar{d}\cdot(\bar{J}\cdot\bar{d}) - \bar{J}d^2)(\bar{r}'(\bar{J}'\cdot\bar{x}) - \bar{J}'(\bar{r}'\cdot\bar{x}))\\ &= (\bar{d}\cdot\bar{r}')(\bar{J}\cdot\bar{d})(\bar{J}'\cdot\bar{x}) - (\bar{d}\cdot\bar{J}')(\bar{J}\cdot\bar{d})(\bar{r}'\cdot\bar{x})\\ &\quad - (\bar{J}\cdot\bar{r}')(\bar{J}'\cdot\bar{x})d^2 + (\bar{J}\cdot\bar{J}')(\bar{r}'\cdot\bar{x})d^2\\ &= (\bar{d}\cdot\bar{r}')(\bar{J}\cdot\bar{d})(\bar{J}'\cdot\bar{x}) - (\bar{d}\cdot\bar{J}')(\bar{J}\cdot\bar{d})(\bar{r}'\cdot\bar{x})\\ &\quad + d^2(\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}')\end{aligned} \quad \text{(B.16)}$$

The following relations will now be used,

$$\nabla'\cdot(\bar{d}\cdot\bar{r}')(\bar{x}\cdot\bar{r}')\bar{J}' = (\bar{d}\cdot\bar{J}')(\bar{x}\cdot\bar{r}') + (\bar{d}\cdot\bar{r}')(\bar{J}'\cdot\bar{x}) \Leftrightarrow 0 \quad \text{(B.17)}$$

$$\begin{aligned}&\nabla\cdot(\bar{d}\cdot\bar{x})(\bar{d}\cdot\bar{r}')(\bar{J}'\cdot\bar{x})\bar{J}\\ &\quad = (\bar{d}\cdot\bar{J})(\bar{d}\cdot\bar{r}')(\bar{J}'\cdot\bar{x}) + (\bar{d}\cdot\bar{x})(\bar{d}\cdot\bar{r}')(\bar{J}'\cdot\bar{J}) \Leftrightarrow 0\end{aligned} \quad \text{(B.18)}$$

Formula (B.16) then becomes

$$(\bar{J}\times\bar{d})\times\bar{d}\cdot(\bar{J}'\times\bar{r}')\times\bar{x} \Leftrightarrow -2(\bar{d}\cdot\bar{x})(\bar{d}\cdot\bar{r}')(\bar{J}'\bar{J}) + d^2(\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}') \quad \text{(B.19)}$$

Formula (B.15) thus becomes

$$\begin{aligned}&(\bar{J}\cdot\bar{J}')(\bar{d}\cdot\bar{r}')(\bar{d}\cdot\bar{x})\\ &\quad\Leftrightarrow \frac{1}{2}(\bar{J}\times\bar{d})\cdot\bar{x}(\bar{J}'\times\bar{r}')\cdot\bar{d} - (\bar{d}\cdot\bar{x})(\bar{d}\cdot\bar{r}')(\bar{J}'\cdot\bar{J}) + \frac{1}{2}d^2(\bar{J}\times\bar{x})\cdot(\bar{J}'\times\bar{r}')\end{aligned} \quad \text{(B.20)}$$

so that

$$(\bar{J} \cdot \bar{J}')(\bar{d} \cdot \bar{r}')(\bar{d} \cdot \bar{x}) \Leftrightarrow \frac{1}{4}(\bar{J} \times \bar{d}) \cdot \bar{x}(\bar{J}' \times \bar{r}') \cdot \bar{d} + \frac{1}{4}d^2(\bar{J} \times \bar{x}) \cdot (\bar{J}' \times \bar{r}')$$
$$= -\frac{1}{4}(\bar{J} \times \bar{x}) \cdot \bar{d}(\bar{J}' \times \bar{r}') \cdot \bar{d} + \frac{1}{4}d^2(\bar{J} \times \bar{x}) \cdot (\bar{J}' \times \bar{r}') \tag{B.21}$$

B.4 Results

The dipole–dipole energy, formula (B.8), becomes, using formulas (B.13) and (B.21),

$$U_m^{dd} = \frac{\mu_0}{4\pi} \int_V \int_{V'} \left(\frac{2\bar{M} \cdot \bar{M}'}{d^3} - 3\left[-\frac{1}{4}(\bar{J} \times \bar{x}) \cdot \bar{d}(\bar{J}' \times \bar{r}') \cdot \bar{d} + \frac{1}{4}d^2(\bar{J} \times \bar{x}) \cdot (\bar{J}' \times \bar{r}') \right] \frac{1}{d^5} \right) dV' dV$$
$$= \frac{\mu_0}{4\pi} \int_V \int_{V'} \left(\frac{2\bar{M} \cdot \bar{M}'}{d^3} - 3[-(\bar{M} \cdot \hat{d})(\bar{M}' \cdot \hat{d}) + \bar{M} \cdot \bar{M}'] \frac{1}{d^3} \right) dV' dV$$
$$= -\frac{\mu_0}{4\pi} \left(\frac{\bar{m} \cdot \bar{m}'}{d^3} - \frac{3(\bar{m} \cdot \hat{d})(\bar{m}' \cdot \hat{d})}{d^3} \right) \tag{B.22}$$

which is formula (7.25).

Appendix C
Magnetic Energy in the Presence of a Material

In Sect. 8.2.2 formula (8.61) is derived within a special case. In this appendix, formula (8.61) is derived from general principles.

The interaction between an influencing current J_f and a magnetisation current J_b is expressed using formula (8.88)

$$U_{int} = \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b}\frac{\bar{J}_b(\bar{r}_b)\cdot\bar{J}_f(\bar{r}_f)}{R}dV_b dV_f \tag{C.1}$$

where the factor $1/2$ is introduced since J_b is induced by J_f and $\bar{R} = \bar{r}_b - \bar{r}_f$

Formula $(C1)$ may be rewritten using formula (8.49):

$$U_{int} = \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b}\frac{(\nabla_b \times \bar{M})\cdot\bar{J}_f}{R}dV_b dV_f \tag{C.2}$$

With the aid of vector algebra, the integrand may be expressed as:

$$(\nabla_b \times \bar{M})\cdot\frac{\bar{J}_f}{R} = \nabla_b\cdot\left(\bar{M}\times\frac{\bar{J}_f}{R}\right) + \bar{M}\cdot\nabla_b\times\frac{\bar{J}_f}{R} \tag{C.3}$$

The first term of the right-hand side vanishes at integration because of the divergence theorem:

$$\int\limits_{V_b}\nabla_b\cdot\left(\bar{M}\times\frac{\bar{J}_f}{R}\right)dV_b = \oint\limits_{S_b}\left(\bar{M}\times\frac{\bar{J}_f}{R}\right)\cdot d\bar{s}_b = 0 \tag{C.4}$$

since the volume may be chosen large enough so that M vanishes on the surface S_b.

K. Prytz, *Electrodynamics: The Field-Free Approach*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-13171-9

The energy is then

$$U_{int} = \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b} \bar{M}\cdot\nabla_b \times \frac{\bar{J}_f}{R} dV_b dV_f = \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b} \bar{M}\cdot\frac{\bar{J}_f \times \bar{R}}{R^3} dV_b dV_f \tag{C.5}$$

since $\nabla_b \times \bar{J}_f = 0$

The second factor of formula (C.5),

$$\frac{\mu_0}{4\pi}\int\limits_{V_f}\frac{\bar{J}_f \times \bar{R}}{R^3} dV_f$$

may be interpreted as the influence which generates and interacts with the bound current. The numerator has the structure of a magnetic dipole and for isotropic and linear materials, i.e. materials which respond in the same dimension as the external influence as well as in proportion to its strength, it may be stated as

$$\frac{\mu_0}{4\pi}\int\limits_{V_f}\frac{\bar{J}_f \times \bar{R}}{R^3} dV_f = \frac{\mu_0}{\kappa_m - 1}\bar{M} \tag{C.6}$$

The constant in front of the magnetisation M is chosen such that the result will be congruent with formula (8.61). The energy is then

$$\begin{aligned} U_{int} &= \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_b} \bar{M}\cdot\int\limits_{V_f}\frac{\bar{J}_f \times \bar{R}}{R^3} dV_b dV_f \\ &= \frac{\mu_0}{4\pi}\frac{1}{2}\int\limits_{V_b} dV_b \bar{M}\cdot\int\limits_{V_f}\frac{\bar{J}_f \times \bar{R}}{R^3} dV_f = \frac{1}{2}\frac{\mu_0}{\kappa_m - 1} m_b M \end{aligned} \tag{C.7}$$

which is formula (8.61).

Exercises

C1. Show that if the volumes are pointlike, formula (C.5) is reduced to the dipole–dipole energy.

C2. Show that compass needles are oriented in a circle about a straight current-carrying conductor.

Appendix D
Solutions to Exercises

The mind has its illusions as the sense of sight; and in the same manner that the sense of feeling corrects the latter, reflection and calculation correct the former.
Pierre-Simon Laplace 1749–1827

D.1 Basic Principles

1.1 a. A force appears as a minimization of the gravitational energy U. The work W done by the gravitational force is

$$dW = \bar{F} \cdot d\bar{s} = -dU$$

so that a decrease of energy U gives a positive work meaning that the force is parallel to the displacement ds. Thus, the energy principle gives the direction of the force.

b. If R denotes distance between a falling object with mass m and the center of the earth, the force becomes

$$\bar{F} = -\nabla U = -\frac{d}{dR} mgR\hat{R} = -mg\hat{R}$$

where the distance vector is directed towards the object for which the force is calculated and g is the gravitational acceleration. The forces on the objects have accordingly the same magnitude but opposite direction.

c. The concept 'free fall' implies neglect of air resistance. The energy forms are then the stored gravitation appearing in the interaction between the objects and kinetic energy of both objects.

d. The kinetic energy of the earth may be neglected since its mass is so large. As a consequence, one object in the mutual interaction appears as being fixed which is fundamentally wrong.

1.2 a. Provided the altitude is high enough, the air resistance results in a higher terminal speed for the heavier object. The reason is that air resistance increases

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with speed, implying that its balance with the gravitational force occurs at a higher speed for the heavier object.

b. The moon has negligible atmosphere so that the principle of equal falling times for objects with different masses may be tested and verified.
This phenomenon is closer analysed in Exercise (5.6).

1.3 a. Gravitational mass is the source of gravitational force. The acceleration caused by the force is inversely proportional to inertial mass. The observation of falling times for objects with different masses is one of many possibilities to investigate the relation between gravitational and inertial mass. To a very high degree of accuracy they have been found to be equal.

b. Mach's principle is a hypothesis on the origin of inertial mass. It says that the equality between gravitational and inertial mass is due to the fact that inertia is generated by gravitation. It explains that when an object is accelerated, the rest of the mass in the universe will interact with it so as to oppose the acceleration, i.e. an inertial effect appears.
See also Exercises (2.23) and (9.15).

1.4 a. At this position the only relevant gravitation is that from the earth. The force is

$$F = G\frac{m_1 m_2}{R^2} = 6.67 \times 10^{-11} \frac{5.98 \times 10^{24} \times 100}{(6.38 \times 10^6 + 4.00 \times 10^5)^2} N$$
$$= \frac{3988 \times 10^{13}}{(6.78 \times 10^6)^2} N = 868\,\mathrm{N}$$

Hence, the word 'weightless' is misleading. The floating state the person experiences is due to being in a free fall where the gravitational force acts alone. Thus, the electromagnetic forces, so-called normal forces, are absent.

b. The change of weight would be 31 mg which is easily measurable with a modern precision balance.

1.5 These occur due to the motion of the observer and are not caused by any natural dynamics. A coriolis effect occurs for example because of the rotation of the earth affecting direction of winds and water flow. A centrifugal effect occurs in curved motion, e.g. in a merry-go-round. Through the active force directed towards the circular center, an inertia is generated known as the centrifugal effect.

1.6 All motion must be related to something. There is no absolute motion.

D.2 Electrodynamic Force

2.3 The unit Coulomb C is defined through the basic electric force formula:

$$\bar{F}_{1\to 2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{R}$$

in SI units.
Two charges of 1 C at a distance of one meter interact with a force $1/4\pi\varepsilon_0$.

2.4 The force formula for two parallel currents is

$$\bar{F}_{1\to 2} = -\frac{\mu_0 I_1 I_2}{2\pi x} L\hat{x}$$

in SI units.
Two parallel currents of 1 A at a distance of one meter interact with a force per meter equal to $\mu_0/2\pi$.

*2.6 a. v is the relative velocity between the observer and the charges.
b. Same as in Fig. 2.10 where the charges are in motion.
c. The viewer observes in the horizontal direction the motional change, i.e. the acceleration of the two objects, nothing else.
d. Either that the force is less or that the mass is greater or that time is running slower or all these possibilities as in Einstein's theory of relativity.

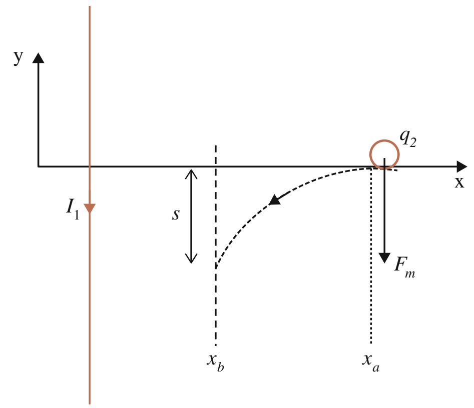
2.7 a. Use formula (2.6)

$$\bar{F}_{1\to el2} = -\frac{\mu_0 I_1}{2\pi x} I_2 dL_2 \hat{x} = -\frac{\mu_0 I_1}{2\pi x} q_2 v_2 \hat{x}$$

where $I_1 = nqv_d A$ and $m_e v_2^2/2 = q\Delta\Phi$
b. Utilize an ordinary electron tube and a current-carrying straight conductor.

The force may be measured by for example letting the electron beam pass through two plates connected to a variable voltage source. The voltage between the plates is adjusted so that the electric force balances the magnetic force. The electric force may then be obtained according to Sect. 4.1.

2.8



a. Formula (2.4):

$$\bar{F}_m = \frac{\mu_0 I_1}{2\pi x} q_2 v_2 \hat{y}$$

$$m\frac{d^2y}{dt^2} = \frac{\mu_0 I_1}{2\pi x} q_2 v_2 = \frac{\mu_0 I_1 q_2}{2\pi} \frac{v_2}{x_a - v_2 t}$$

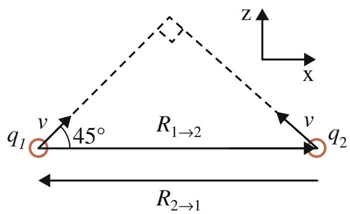
since $x = x_a - v_2 t$ where $t = 0$ at $x = x_a$.
Velocity in the y direction becomes

$$v_y(x_b) = -\frac{\mu_o I_1 q_2}{2\pi m} [\ln(x_a - v_2 t)]_{t_a}^{t_b} = \frac{\mu_o I_1 q_2}{2\pi m} \ln\left(\frac{x_a - v_2 t_a}{x_a - v_2 t_b}\right)$$
$$= \frac{\mu_o I_1 q_2}{2\pi m} \ln\left(\frac{x_a}{x_b}\right)$$

For a small displacement s the path may be approximated by a straight line:

$$\frac{s}{x_b - x_a} = \frac{v_y(x_b)}{v_2} = \frac{\mu_o I_1 q_2}{2\pi m v_2} \ln\left(\frac{x_a}{x_b}\right)$$

2.9



a. Magnetic force on charge 2 is

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}\left[(-\underbrace{\bar{v}_1 \cdot \bar{v}_2}_{=0})\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + (\bar{v}_1 \cdot \hat{R})\bar{v}_2\right]$$
$$= \frac{\mu_0 q_1 q_2}{4\pi R^2} \frac{v}{\sqrt{2}}(-\bar{v}_1 + \bar{v}_2) = -\frac{\mu_0 q_1 q_2}{4\pi R^2} v^2 \hat{x}$$

and on charge 1

$$\bar{f}_{m1} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(\bar{v}_1 \cdot \bar{v}_2)\hat{R} - (\bar{v}_2 \cdot \hat{R})\bar{v}_1 - (\bar{v}_1 \cdot \hat{R})\bar{v}_2] = \frac{\mu_0 q_1 q_2}{4\pi R^2} v^2 \hat{x}$$

since the distance vector then changes direction.
Hence, there is accordance with Newton's third law.

b. From Grassman's formula, force on charge 2 becomes

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1] = -\frac{\mu_0 q_1 q_2}{4\pi R^2} \frac{v^2}{\sqrt{2}} \frac{\bar{v}_1}{v}$$

and force on charge 1 becomes

$$\bar{f}_{m1} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(\bar{v}_1 \cdot \bar{v}_2)\hat{R} - (\bar{v}_1 \cdot \hat{R})\bar{v}_2] = -\frac{\mu_0 q_1 q_2}{4\pi R^2}\frac{v^2}{\sqrt{2}}\frac{\bar{v}_2}{v}$$

which violates Newton's third law.
However, there is also an electric force which at ordinary speeds dominates over the magnetic force. Only at speeds close to light speed is the magnetic force comparable to the electric one.

*2.10 c. $\bar{f}_{m2} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[-\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1) + d\bar{L}_1(d\bar{L}_2 \cdot \hat{R}) + ad\bar{L}_2(d\bar{L}_1 \cdot \hat{R})\right]$
For closed conductors the infinitesimal elements are summed up in closed path integrals.
Consider term 3 integrated over conductor 1 and 2.

$$\oint_1 \oint_2 d\bar{L}_2 \frac{(d\bar{L}_1 \cdot \hat{R})}{R^2}$$

Apply Stokes' law

$$\oint_2 d\bar{L}_2 \oint_1 \frac{\hat{R} \cdot d\bar{L}_1}{R^2} = \oint_2 d\bar{L}_2 \int_{S_1} \left(\nabla_1 \times \frac{\hat{R}}{R^2}\right) \cdot d\bar{S}_1$$

where S_1 is the surface enclosed by conductor 1.

$$\nabla_1 \times \frac{\hat{R}}{R^2} = \nabla_1 \times \frac{\bar{R}}{R^3} = \left(\nabla_1 \frac{1}{R^3}\right) \times \bar{R} + \frac{1}{R^3}\nabla_1 \times \bar{R} = 0$$

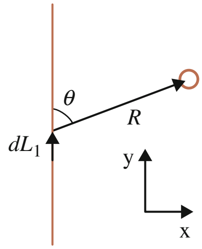
In the same manner it is shown that also term 2 vanishes for closed conductors.

d. According to Exercise c, if charge 1 is part of a closed conductor the last term of Whittakers formula (2.25) vanishes so that this formula becomes identical to Grassman's formula.

e. If charge 1 is part of an infinitely long straight conductor the last term in (2.25) becomes

$$d\bar{L}_2 \int_{-\infty}^{\infty} \frac{\hat{R} \cdot d\bar{L}_1}{R^2} = d\bar{L}_2 \int_{-\infty}^{\infty} \frac{\cos\theta dy}{(y^2 + x^2)}$$
$$= d\bar{L}_2 \int_{-\infty}^{\infty} \frac{y}{(y^2 + x^2)^{3/2}} dy = 0$$

equivalent to a closed conductor.



f. Since the *two* last terms in (2.25) vanish.

2.11 a. Yes

b.

$$\bar{f}^A_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[-2(\bar{v}_2 \cdot \bar{v}_1) + 3(\hat{R} \cdot \bar{v}_2)(\hat{R} \cdot \bar{v}_1)]\hat{R}$$

$$\bar{f}^G_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1]$$

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + (\bar{v}_1 \cdot \hat{R})\bar{v}_2]$$

Leaving out the constant gives

Grassman

Left $\bar{F}_{1\to 2} = -\frac{v_1 v_2}{R^2}\hat{R} \quad \bar{F}_{2\to 1} = \frac{v_1 v_2}{R^2}\hat{R}$

Middle $\bar{F}_{1\to 2} = \frac{v_1 v_2}{R^2}\hat{y} \quad \bar{F}_{2\to 1} = 0$

Right $\bar{F}_{1\to 2} = 0 \quad \bar{F}_{2\to 1} = 0$

Ampère

Left $\bar{F}_{1\to 2} = -2\frac{v_1 v_2}{R^2}\hat{R} \quad \bar{F}_{2\to 1} = 2\frac{v_1 v_2}{R^2}\hat{R}$

Middle $\bar{F}_{1\to 2} = 0 \quad \bar{F}_{2\to 1} = 0$

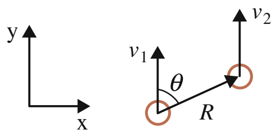
Right $\bar{F}_{1\to 2} = \frac{v_1 v_2}{R^2}\hat{R} \quad \bar{F}_{2\to 1} = -\frac{v_1 v_2}{R^2}\hat{R}$

Whittaker

Left $\bar{F}_{1\to 2} = -\frac{v_1 v_2}{R^2}\hat{R} \quad \bar{F}_{2\to 1} = \frac{v_1 v_2}{R^2}\hat{R}$

Middle $\bar{F}_{1\to 2} = \frac{v_1 v_2}{R^2}\hat{y} \quad \bar{F}_{2\to 1} = -\frac{v_1 v_2}{R^2}\hat{y}$

Right $\bar{F}_{1\to 2} = \frac{v_1 v_2}{R^2}\hat{R} \quad \bar{F}_{2\to 1} = -\frac{v_1 v_2}{R^2}\hat{R}$



2.12 a. Leaving out the constant gives

Grassman

$$\bar{F}_{1\to 2} = -\frac{v_1 v_2}{R^2}\sin\theta\hat{x} \quad \bar{F}_{2\to 1} = \frac{v_1 v_2}{R^2}\sin\theta\hat{x}$$

Whittaker

$$\bar{F}_{1\to 2} = -\frac{v_1 v_2}{R^2}\sin\theta\hat{x} + \frac{v_1 v_2}{R^2}\cos\theta\hat{y}$$

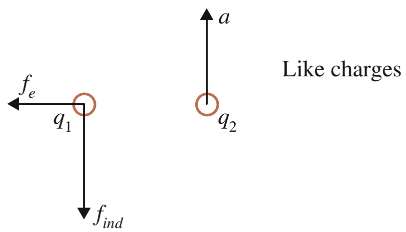
$$\bar{F}_{2\to 1} = +\frac{v_1 v_2}{R^2}\sin\theta\hat{x} - \frac{v_1 v_2}{R^2}\cos\theta\hat{y}$$

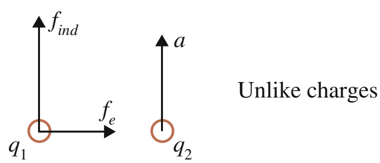
b. Yes, this is still an unsolved problem, as is the non-conservation of momentum for Grassman's and Whittaker's ($a \neq 1$) formulae. However, note that this might be related to the inductive force which is not fully known for free charges.

2.13 a. Ampère's force formula conserves both linear and angular momentum since the force acts along the connection line between the two objects.

b. It doesn't reduce to formula (2.18) for the parallel motion and it gives zero force for the perpendicular motion.

2.15

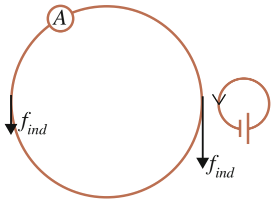




where $\bar{f}_{ind} = -C(R)q_2 q_1 \bar{a}$

2.16 a. The distance dependent factor $C(R)$ must decrease with increasing distance and tend to zero when distance tends to infinity.

b. On the small loop, the conduction electron on the left-hand side will affect the large loop more since it is closer. On the large loop, the right-hand side will be more affected since it is closer. As in Fig. 2.13, there will be an induction current but much smaller since the two forces counteract each other.



2.17 a. The active force is here the inductive, formula (2.22).

$$\bar{f}_{ind} = -C(R)q_2q_1\frac{d\bar{v}_1}{dt}$$

The acceleration of a conduction electron in the loop is

$$\frac{d\bar{v}_1}{dt} = \frac{v^2}{r}\hat{r}$$

so that the force on an electron of the metal piece becomes

$$\bar{f}_{ind} = -C(R)q_2q_1\frac{v^2}{r}\hat{r}$$

b. The force is independent of current direction since the acceleration is always directed inwards.

2.18 The active force is here solely the inductive

$$\bar{f}_{ind} = -C(R)q_2q_1\frac{d\bar{v}_1}{dt} = -C(R)q_2q_1\frac{v^2}{r}\hat{r}$$

so that diametrically opposite positions in the beam will give oppositely directed forces and the electron in the metal will oscillate with frequency $v/(2\pi r)$ vertically as well as horizontally.

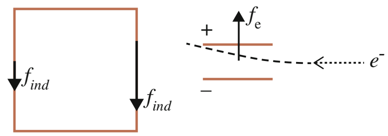
2.19 The poles of the water molecule are affected by oppositely directed vertical forces. Since the length of the dipole may be taken as much less than the distance to the conductor, the distances to the poles are equal and denoted by R. The torque becomes

$$\bar{\tau} = 2\frac{\bar{l}}{2} \times \bar{f}_{ind} = -C(R)q_2 q_1 \bar{l} \times \frac{d\bar{v}_1}{dt} = -C(R)p_{H_2O} e\omega v_0 \cos\omega t \hat{z}$$

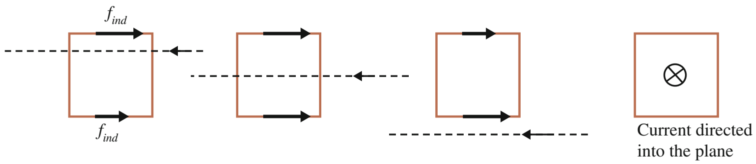
where e is the charge of the electron (negative) and $\hat{z}$ is directed into the paper. Hence, in a microwave oven the water molecules oscillate perpendicular to their own axes which in turn affects the surroundings such that heat is generated.

In general one must here also consider distance effects as well as the fact that the straight conductor is also an oscillating dipole since charge is accumulated at its ends, see Chap. 11.

2.20 Since the right leg is closer to the beam, the inductive force will be larger on this one than on the left leg. Hence, the electrons are driven clockwise, i.e. the current direction becomes anti-clockwise. The inductive force acts also on the horizontal sides but will here not contribute to the current.



2.21



First case from left: the inductive force will be largest on the upper side since it is closest to the beam.

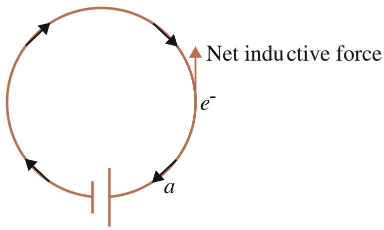
Second case: there is no induction current since the inductive force on the two horizontal sides cancel each other.

Third case: equivalent to the first case.

Fourth case: There is no force in the plane of the loop and thus no induction current.

These experiments are simple to perform since an ordinary straight conductor current may be used as the electron beam.

2.22 a. Consider the electron in the figure. Its net acceleration is clockwise along the tangent of the loop. The inductive force between two electrons is a response oppositely directed the acceleration of the *affecting* object. The horizontal inductive forces cancel. The net inductive force will be directed upwards since the electrons closest to the considered electron accelerate downwards.

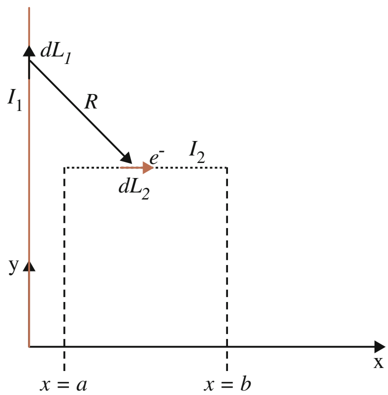


The inductive force acts accordingly as a resistance to the acceleration caused by the battery.

b. In the straight conductor, the acceleration on the electrons is everywhere the same. The inductive force is oppositely directed to this direction and accordingly acts a resistance to the battery acceleration.
c. The inductive force, as all kind of force, occurs in interactions.

2.23 It might be fruitful to consider this phenomena already at this stage. However, this issue will be returned to in Chap. 9, Exercise (9.15).

2.25



a. Grassman's formula for force on object 1 is

$$\bar{f}_{m1}^{G} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1) - d\bar{L}_2(d\bar{L}_1 \cdot \hat{R})\right]$$

Since dL_1 and dL_2 are perpendicular the first term vanishes. Since object 1 is a closed conductor the second term vanishes too, see Exercise (2.19).

Hence

$$\bar{f}_{m1}^{G} = 0$$

b. Whittaker's force for $a = 1$ is

$$\bar{f}_{m1} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[\hat{R}(d\bar{L}_2 \cdot d\bar{L}_1) - d\bar{L}_1(d\bar{L}_2 \cdot \hat{R}) - d\bar{L}_2(d\bar{L}_1 \cdot \hat{R})\right]$$

The first and the third terms vanish as in Exercise a. The second term involves

$$\int_a^b \int_{-\infty}^{\infty} \frac{dy\hat{y}(d\bar{L}_2 \cdot \hat{R})}{R^2} = \int_a^b \int_{-\infty}^{\infty} \frac{dy\hat{y}x dx}{R^3}$$

$$= -\int_{-\infty}^{\infty} dy \left[\frac{1}{(x^2 + y^2)^{1/2}}\right]_{x=a}^{x=b} \hat{y}$$

$$= -\int_{-\infty}^{\infty} dy \left[\frac{1}{(b^2 + y^2)^{1/2}} - \frac{1}{(a^2 + y^2)^{1/2}}\right]\hat{y}$$

$$= -2\int_0^{\infty} dy \left[\frac{1}{(b^2 + y^2)^{1/2}} - \frac{1}{(a^2 + y^2)^{1/2}}\right]\hat{y}$$

$$= -2\left[\ln \frac{y + (b^2 + y^2)^{1/2}}{y + (a^2 + y^2)^{1/2}}\right]_0^{\infty} \hat{y} = 2\ln \frac{b}{a}\hat{y}$$

so that force becomes

$$\bar{f}_{m1} = -\frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{b}{a}\hat{y}$$

c. The force on a charge in the beam is given by formula (2.4)

$$\bar{f}_{m2} = \frac{\mu_0 I_1}{2\pi x_2} q_2 v_2 \hat{y}$$

Converting charge to a current element gives

$$d\bar{F}_{1\to 2} = \frac{\mu_0 I_1}{2\pi x_2} I_2 dx \hat{y}$$

whose direction should agree with the observation described in Fig. 2.6 (compare Exercise 2.8).

Total force on all electrons in the beam becomes

$$\bar{F}_{m2} = \int_a^b \frac{\mu_0 I_1}{2\pi x} I_2 dx \, \hat{y} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{b}{a} \hat{y}$$

Newton's third law is fulfilled for Whittaker's force but not for Grassman's conventional force.

d. One way is to place a square circuit with small mass on a torsion balance and let the electron beam impinge on one of its sides. With this arrangement a smallest force of approximately 5 µN may be measured. How large a force is expected?
For $I_1 = 20$ A, the force becomes

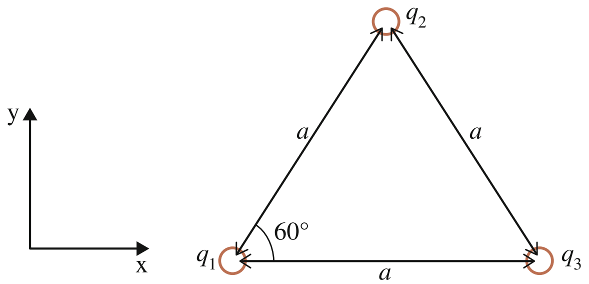
$$F_{m1} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{b}{a} \hat{y} = I_2 40 \cdot 10^{-7} \ln \frac{0.2}{0.002} \approx I_2 20 \, \mu\text{N}$$

so that $I_2 > 250\,\text{mA}$. Electron tubes with this current strength exist on the market but are unusual in student laboratories.

Nevertheless, this simple experiment is sufficient to disprove Grassman's force formula.

D.3 Electrodynamic Energy

3.3



a.

$$U_e = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q_1 q_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}} + \frac{q_1 q_3}{x_3 - x_1} + \frac{q_2 q_3}{[(x_3 - x_2)^2 + (y_3 - y_2)^2]^{1/2}} \right\}$$

b. The force on charge 1 becomes

$$\bar{F}_1 = -\nabla_1 U_e = -\frac{d}{dx_1} U_e \hat{x} - \frac{d}{dy_1} U_e \hat{y}$$

$$= -\frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2((x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y})}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{3/2}} + \frac{q_1 q_3}{(x_3 - x_1)^2} \hat{x} \right]$$

$$= -\frac{1}{4\pi\varepsilon_0} \left[q_1 q_2 \frac{\frac{a}{2}\hat{x} + \frac{\sqrt{3}a}{2}\hat{y}}{a^3} + q_1 q_3 \frac{\hat{x}}{a^2} \right]$$

$$= -\frac{1}{4\pi\varepsilon_0 a^2} \left[\frac{q_1 q_2}{2} (\hat{x} + \sqrt{3}\hat{y}) + q_1 q_3 \hat{x} \right]$$

and equivalently for force on charges 2 and 3.

c. Yes

3.4 a. The inductance

$$M_{jk} = \frac{\mu_0}{4\pi} \int_k \int_j \frac{d\bar{L}_j \cdot d\bar{L}_k}{R}$$

gives the magnetic energy for two charges in motion as

$$U_m = M_{12} I_1 I_2 = \frac{\mu_0}{4\pi} I_1 I_2 \frac{d\bar{L}_1 \cdot d\bar{L}_2}{R} = \frac{\mu_0 q_1 q_2}{4\pi R} \bar{v}_1 \cdot \bar{v}_2$$

where formula (2.5), i.e. $IdL = qv$, has been used in the last equality.

b. Assume the charges are placed on the x axis with distance vector R directed towards charge 2. For constant speed, force becomes

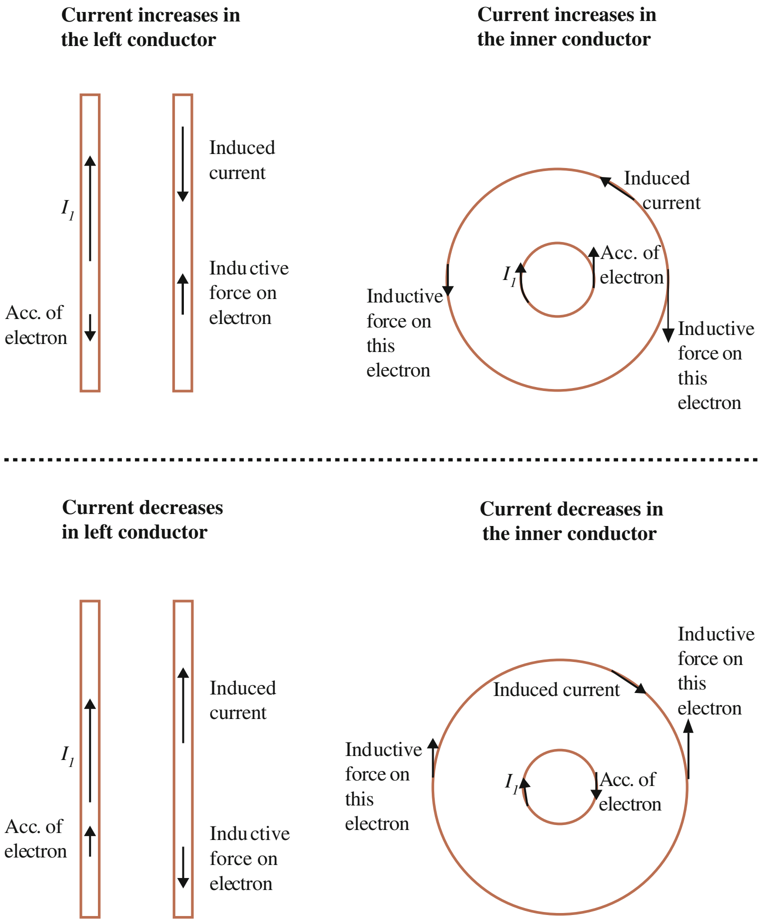
$$\bar{F}_1 = \nabla_1 U_m = \frac{\mu_0 q_1 q_2}{4\pi} \bar{v}_1 \cdot \bar{v}_2 \frac{d}{dx_1} \frac{1}{x_2 - x_1} = \frac{\mu_0 q_1 q_2}{4\pi R^2} \bar{v}_1 \cdot \bar{v}_2 \hat{R}$$

This formula is valid only in the following two cases:

1. Both charges reside in closed conductors, formula (2.26).
2. Parallel motion as in Fig. 2.10.

Thus, the magnetic energy (3.37) is not generally valid.

3.5



Thus, the inductive response acts to oppose changes, which is called the Lenz law.

3.6 a. Faraday-Henry's induction law

$$\varepsilon_j = \sum_k \frac{d}{dt}(i_k M_{jk})$$

becomes

$$\varepsilon_j = M_{jk}\frac{d}{dt}i_k \Rightarrow M_{jk} = \frac{\varepsilon_j}{\Delta I/\Delta T}$$

where ε_j is the induced voltage.

b.

$$M_{jk}(\theta = 0) = \frac{\varepsilon_2}{\Delta I / \Delta T}$$

From the general inductance formula

$$M_{jk} = \frac{\mu_0}{4\pi} \int_{Cond\ k} \int_{Cond\ j} \frac{d\bar{L}_j \cdot d\bar{L}_k}{R}$$

We conclude in this case that, for constant R, inductance becomes

$$M_{jk}(\theta) = \frac{\varepsilon_2}{\Delta I / \Delta T} \cos\theta$$

3.7 a. New north and south poles are formed for each new extracted part.
- b. All in the same direction.
- c. Randomly
- d. A small current loop.
- e. See Figs. 7.7 and 8.14.
- f. Only on the surface.
- g. By external influence, e.g. a current-carrying coil that aligns the magnetic dipoles.
- h. Let the magnet interact with a current loop. At a long distance the magnet may be modelled as an exact dipole and the equivalent current can be determined through the measurement of the force. By treating the magnet as a coil, see Fig. 7.7, its self inductance can be determined and thus the stored magnetic energy.
  Compare with Exercise (8.22).

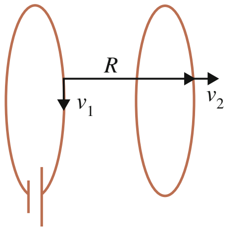
3.8 1. Rotate the loop
2. Translate the loop
3. Rotate the battery circuit
4. Translate the battery circuit
5. Vary the current in the battery circuit

3.9 a. The induction law

$$\varepsilon_j = \sum_k \frac{d}{dt}(i_k M_{jk})$$

becomes

$$\varepsilon_2 = I \frac{d}{dt} M_{12}$$



b. The force is magnetic with the motion of the conduction electrons horizontally.
The inductive force is centripetal and generates no current along the loop.

c. The magnetic force, formula (2.11), is

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}\left[(-\underbrace{\bar{v}_1 \cdot \bar{v}_2}_{=0})\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\underbrace{\bar{v}_1 \cdot \hat{R}}_{=0})\bar{v}_2\right]$$

Consider the neighbouring elements 1 and 2 marked in the figure.
The first and third term vanish and a force in the direction of $\bar{v}_1$ occurs.
On the other hand, if $\bar{v}_2$ is oppositely directed, a force opposite to $\bar{v}_1$ occurs, i.e. a current is induced in loop 2 oppositely directed to the current in loop 1.

d. The same result occurs as in Exercise c since the relative motion is the same.

e. The relativity principle of Galilei means that motion is relative; it has to be related to some other motion. Einstein applied this principle as the basis for the special theory of relativity. He extended the principle, showing its validity also for electromagnetic phenomena as this example indicates. However, instead of the battery connected loop, Einstein preferred a permanent magnet, see Chap. 9.

3.10 a. For force on object 2 the distance vector is $\bar{R} = \bar{r}_2 - \bar{r}_1$.

$$\bar{f}_{m2} = \nabla_2 U_m$$

$$\nabla_2 \frac{\bar{v}_1 \cdot \bar{v}_2}{R} = \frac{1}{R}\nabla_2(\bar{v}_1 \cdot \bar{v}_2) - \frac{1}{R^2}(\bar{v}_1 \cdot \bar{v}_2)\hat{R}$$

$$\nabla_2 \frac{(\hat{R} \cdot \bar{v}_1)(\hat{R} \cdot \bar{v}_2)}{R} = \nabla_2 \frac{(\bar{R} \cdot \bar{v}_1)(\bar{R} \cdot \bar{v}_2)}{R^3}$$

$$= -3\frac{\hat{R}(\bar{R} \cdot \bar{v}_1)(\bar{R} \cdot \bar{v}_2)}{R^4} + \frac{(\bar{R} \cdot \bar{v}_1)\nabla_2(\bar{R} \cdot \bar{v}_2)}{R^3}$$

$$+ \frac{(\bar{R} \cdot \bar{v}_2)\nabla_2(\bar{R} \cdot \bar{v}_1)}{R^3}$$

For constant velocity

$$\nabla_2(\bar{R}\cdot\bar{v}_1) = \bar{v}_1$$
$$\nabla_2(\bar{R}\cdot\bar{v}_2) = \bar{v}_2$$

so that force becomes

$$\bar{f}_{m2} = \frac{\mu_0}{4\pi}q_1q_2\left[\frac{1+k}{2}\left(-\frac{1}{R^2}(\bar{v}_1\cdot\bar{v}_2)\hat{R}\right) + \frac{1-k}{2}\left(-3\frac{\hat{R}(\bar{R}\cdot\bar{v}_1)(\bar{R}\cdot\bar{v}_2)}{R^4} + \frac{(\bar{R}\cdot\bar{v}_1)\bar{v}_2}{R^3} + \frac{(\bar{R}\cdot\bar{v}_2)\bar{v}_1}{R^3}\right)\right]$$
$$= \frac{\mu_0 q_1 q_2}{4\pi R^2}\left[\frac{1+k}{2}(-\bar{v}_1\cdot\bar{v}_2)\hat{R} + \frac{1-k}{2}(-3(\hat{R}\cdot\bar{v}_1)(\hat{R}\cdot\bar{v}_2)\hat{R} + (\hat{R}\cdot\bar{v}_1)\bar{v}_2 + (\hat{R}\cdot\bar{v}_2)\bar{v}_1)\right]$$

b. None of the alternatives agree with formula (2.11)/(2.13), indicating that the magnetic energy between free charges is still an unsolved problem.

D.4 Macroscopic Systems of Unbound Charges

4.2 Force between a point charge q and an infinitesimal charge element q' of the plate is

$$d\bar{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{R^2}\hat{R} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{R^3}\bar{R}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q\sigma dy'dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \times [(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}]$$

so that total force becomes

$$\bar{F} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q\sigma dy'dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \times [(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}]$$

The y and z components vanish since these form odd functions:

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} q\sigma d \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dy'dz'}{[d^2 + (y - y')^2 + (z - z')^2]^{3/2}} \hat{x}$$

where $d = x - x'$.
Put $t = y' - y$ and $u = z' - z$ so that the integral becomes

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} q\sigma d \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dtdu}{[d^2 + t^2 + u^2]^{3/2}} \hat{x} = \frac{1}{4\pi\varepsilon_0} q\sigma d 2 \int_{-\infty}^{\infty} \frac{du}{d^2 + u^2} \hat{x}$$

$$= \frac{1}{4\pi\varepsilon_0} q\sigma d 2 \frac{\pi}{d} \hat{x} = \frac{q\sigma}{2\varepsilon_0} \hat{x}$$

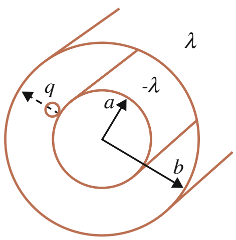
*4.3 Capacitance is defined as

$$C = \frac{Q}{\Delta\Phi}$$

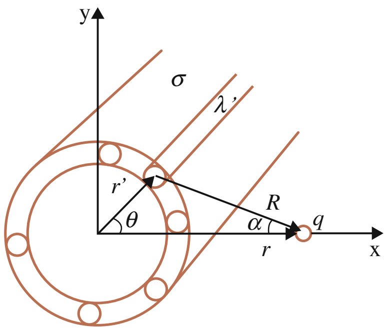
where $Q = \lambda L$ with L as the common length of the cylinders.
Voltage is obtained from

$$W = U_e = q\Delta\Phi$$

To determine the capacitance, the work required to bring a charge against the electric force from one to the other shell is determined, see figure.



The interaction between the point charge and the cylindrical shell is first explored after which the work is calculated with methods developed in Sect. 4.1.5.
Consider a thin shell in cross-section as in the figure below. The force on a point charge located on the x axis should be determined. By symmetry it is noted that the force is directed radially from the shell, i.e. along the x axis in this case.



The cylindrical shell is considered as consisting of infinitesimally thin wires as indicated in the figure. The contribution from one such wire is given by formula (4.6):

$$\lim_{L\to\infty} \bar{F} = \frac{\lambda' q}{2\pi\varepsilon_0 R}\hat{R}$$

Denote with σ the surface charge density of the shell. Since charge per length of the infinitesimal wire is $\lambda' \Leftrightarrow \sigma\, r' d\theta$ it is obtained

$$dF_x = q\frac{\sigma\, r' d\theta\; R\cos\alpha}{2\pi\varepsilon_0 R^2}$$

It is concluded from the figure that

$$R\cos\alpha = r - r'\cos\theta$$

and that

$$R^2 = r'^2 + r^2 - 2rr'\cos\theta$$

so that

$$dF_x = q\frac{\sigma\, r' d\theta (r - r'\cos\theta)}{2\pi\varepsilon_0 (r'^2 + r^2 - 2rr'\cos\theta)}$$

This is now to be integrated over the full circle. The integral may be found in a handbook (e.g. Mathematical Handbook 15.49 1st ed.) and the result depends on whether r is greater or less than r', i.e. if the point charge is placed inside or outside the shell.

1. $r < r'$

$$F_x = q\frac{\sigma}{2\pi\varepsilon_0}\frac{1}{r'}2\int_0^{\pi}\frac{r - r'\cos\theta}{1 + \frac{r^2}{r'^2} - 2\frac{r}{r'}\cos\theta}d\theta$$

$$= 2q\frac{\sigma}{2\pi\varepsilon_0}\frac{1}{r'}\left[\frac{r\pi}{1 - \frac{r^2}{r'^2}} - \frac{r'\pi\frac{r}{r'}}{1 - \frac{r^2}{r'^2}}\right] = 0$$

2. $r > r'$

$$F_x = q\frac{\sigma}{2\pi\varepsilon_0}\frac{r'}{r^2}2\int_0^\pi \frac{r - r'\cos\theta}{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta}d\theta$$

$$= 2q\frac{\sigma}{2\pi\varepsilon_0}\frac{r'}{r^2}\left[\frac{r\pi}{1 - \frac{r'^2}{r^2}} - \frac{r'\pi\frac{r'}{r}}{1 - \frac{r'^2}{r^2}}\right]$$

$$= q\frac{\sigma}{\varepsilon_0}\frac{r'}{r} = q\frac{\lambda}{2\pi\varepsilon_0 r}$$

since the total charge on the complete cylindrical shell is $Q = \lambda L = \sigma 2\pi r' L$.

Thus, inside the shell no force occurs on the point charge while outside the force if appears that all charge is collected on a wire in the center of the cylinder. Compare this result with that of a sphere, Sect. 4.1.4.

Thus, the outer shell has no influence on the point charge. Therefore, the work to bring the charge q from the inner to the outer shell is

$$W = \int_a^b \bar{F}\cdot d\bar{s} = \frac{q\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r} = \frac{qQ/L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

The voltage between the shells becomes

$$\Delta\Phi = \frac{Q/L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

and capacitance

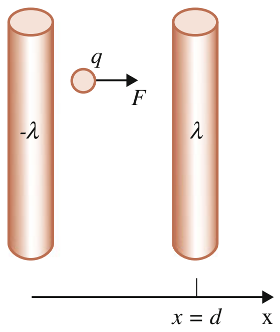
$$C = \frac{Q}{\Delta\Phi} = \frac{2\pi\varepsilon_0 L}{\ln\frac{b}{a}}$$

4.4 Capacitance is defined as

$$C = \frac{Q}{\Delta\Phi}$$

where $Q = \lambda L$ with L as the common length of the cylinders. The voltage is obtained by determining the work needed to bring a plus charge from the negatively charged left cylinder to the positive right cylinder. This work equals the electric energy added to the system.

$$W = U_e = q\Delta\Phi$$



The electric force on the point charge, which has to be overcome during the process, occurs through interactions with both the left and the right cylinder:

$$W = \int_0^d \bar{F} \cdot d\bar{s} = \int_a^{d-a} \frac{q\lambda}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{d-x}\right)\hat{x} \cdot d\bar{s}$$

according to formula (4.6) and Exercise (4.3). The work becomes

$$W = \frac{q\lambda}{2\pi\varepsilon_0}\left[\ln\frac{d-a}{a} - \ln\frac{a}{d-a}\right] = \frac{q\lambda}{\pi\varepsilon_0}\ln\frac{d-a}{a}$$

and the voltage

$$\Delta\Phi = \frac{Q}{L\pi\varepsilon_0}\ln\frac{d-a}{a}$$

and capacitance

$$C = \frac{Q}{\Delta\Phi} = \frac{L\pi\varepsilon_0}{\ln\frac{d-a}{a}}$$

4.5 Consider means as springs, torsional wire, balance, etc.

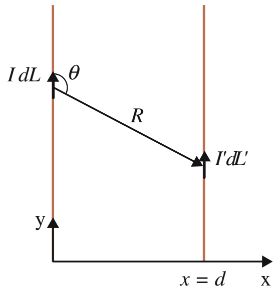
4.6 Infinitesimal force between two current elements is

$$d\bar{F}_m' = \frac{\mu_0 I I'}{4\pi R^2}\left[-\hat{R}(d\bar{L}' \cdot d\bar{L}) + d\bar{L}(d\bar{L}' \cdot \hat{R}) + d\bar{L}'(d\bar{L} \cdot \hat{R})\right]$$

$$\bar{R} = \bar{r}' - \bar{r} = d\hat{x} + y'\hat{y} - y\hat{y}$$

$$d\bar{L} = dy\hat{y}$$

$$d\bar{L}' = dy'\hat{y}$$



The constant $\mu_0 II'/(4\pi)$ is temporarily ignored. Term 3 gives

$$d\bar{f}_3 = \frac{dydy'\hat{y}(\hat{y}\cdot\hat{R})}{R^2} = \frac{dydy'\hat{y}\cos\theta}{R^2}$$

$$\cos\theta = \frac{y'-y}{R}$$

$$d\bar{f}_3 = dydy'\hat{y}\frac{y'-y}{R^3} = \frac{y'-y}{[d^2+(y'-y)^2]^{3/2}}dydy'\hat{y}$$

Term 2 gives the same contribution as term 3. Term 1 gives

$$d\bar{f}_1 = -\frac{dydy'}{R^2}\hat{R}$$

$$\hat{R} = \hat{x}\cos(\theta-\frac{\pi}{2}) - \hat{y}\sin(\theta-\frac{\pi}{2}) = \hat{x}\frac{d}{R} - \hat{y}\frac{y-y'}{R}$$

$$d\bar{f}_1 = -\frac{dydy'}{R^3}(d\hat{x}-(y-y')\hat{y}) = -dydy'\frac{d\hat{x}-(y-y')\hat{y}}{[d^2+(y-y')^2]^{3/2}}$$

Total force becomes

$$\bar{F}'_m = -\int_0^{T_1}\int_0^{T_2} dy'dy\frac{d\hat{x}+(y-y')\hat{y}}{[d^2+(y-y')^2]^{3/2}}$$

where T_1 is the length of the left conductor.
Put $t = y - y'$ so that $dy' = -dt$

$$\bar{F}'_m = \int_0^{T_1}\int_y^{y-T_2} dtdy\frac{d\hat{x}+t\hat{y}}{[d^2+t^2]^{3/2}}$$

$$= \int_0^{T_1} dy\left[\frac{t}{d[d^2+t^2]^{1/2}}\hat{x} - \frac{1}{[d^2+t^2]^{1/2}}\hat{y}\right]_y^{y-T_2}$$

$$= \int_0^{T_1} dy \Bigg[\left(\frac{y - T_2}{d[d^2 + (y - T_2)^2]^{1/2}} - \frac{y}{d[d^2 + y^2]^{1/2}} \right) \hat{x} - \left(\frac{1}{[d^2 + (y - T_2)^2]^{1/2}} - \frac{1}{[d^2 + y^2]^{1/2}} \right) \hat{y} \Bigg]$$

$$= \frac{1}{d}[(d^2 + (y - T_2)^2)^{1/2} - [d^2 + y^2]^{1/2}]_0^{T_1} \hat{x} - [[\ln(t + (t^2 + d^2)^{1/2})]_{-T_2}^{T_1 - T_2} - [\ln(y + (y^2 + d^2)^{1/2})]_0^{T_1}]\hat{y}$$

a. For equal lengths $T_1 = T_2 = T$ the y component vanishes. The force becomes

$$\bar{F}_m' = \frac{1}{d}((d^2 + (T_1 - T_2)^2)^{1/2} - (d^2 + T_2^2)^{1/2} - (d^2 + T_1^2)^{1/2} + d)\hat{x}$$
$$= \frac{1}{d}(2d - 2(d^2 + T^2)^{1/2})\hat{x}$$

and if $T \gg d$, i.e. for long conductors

$$\bar{F}_m' = -\frac{2T}{d}\hat{x}$$

Returning the ignored constant gives then

$$\bar{F}_m' = -\frac{\mu_0 I I'}{2\pi d} T \hat{x}$$

as it should.

b. For different lengths the x component will equal that obtained in Exercise a:

$$\bar{F}_{mx}' = \frac{\mu_0 I I'}{4\pi d}((d^2 + (T_1 - T_2)^2)^{1/2} - (d^2 + T_2^2)^{1/2} - ((d^2 + T_1^2)^{1/2} + d)\hat{x}$$

The y component becomes

$$\bar{F}_{my}' = -\frac{\mu_0 I I'}{4\pi}[[\ln(t + (t^2 + d^2)^{1/2})]_{-T_2}^{T_1 - T_2} - [\ln(y + (y^2 + d^2))^{1/2}]_0^{T_1}]\hat{y}$$
$$= -\frac{\mu_0 I I'}{4\pi} \left(\ln \frac{T_1 - T_2 + ((T_1 - T_2)^2 + d^2)^{1/2}}{-T_2 + (T_2^2 + d^2)^{1/2}} - \ln \frac{T_1 + (T_1^2 + d^2)^{1/2}}{d} \right) \hat{y}$$

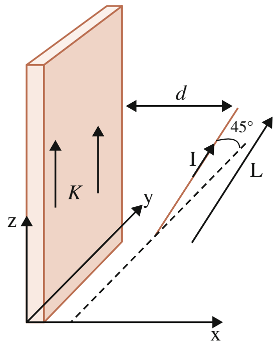
4.7 The force is given by formula (4.45)

$$d\bar{F} = I d\bar{L} \times \hat{y} \frac{\mu_0 K}{2}$$

Since the conductor is straight, formula (4.46) may be used:

$$\bar{F} = I\bar{L} \times \hat{y}\frac{\mu_0 K}{2} = -IL\frac{\mu_0 K}{2}\sin 45^\circ\ \hat{x}$$

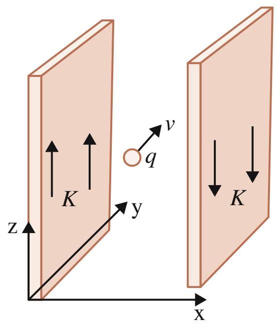
where I is positive if directed parallel to $\bar{L}$, as in the figure.



4.8 Formula (4.44b) may be used:

$$\bar{F} = q\bar{v} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n})$$

The total force is the sum of the force from the left and right plates:



1. Between the plates

$$\bar{F}_l = q\bar{v} \times \frac{\mu_0}{2}K(\hat{z} \times \hat{x}) = q\bar{v} \times \frac{\mu_0}{2}K\hat{y}$$
$$\bar{F}_r = q\bar{v} \times \frac{\mu_0}{2}K(-\hat{z} \times -\hat{x}) = q\bar{v} \times \frac{\mu_0}{2}K\hat{y}$$
$$\bar{F}_{tot} = \bar{F}_l + \bar{F}_r = q\bar{v} \times \mu_0 K\hat{y}$$

2. Outside the plates
   a. Positive x

$$\bar{F}_l = q\bar{v} \times \frac{\mu_0}{2} K(\hat{z} \times \hat{x}) = q\bar{v} \times \frac{\mu_0}{2} K\hat{y}$$
$$\bar{F}_r = q\bar{v} \times \frac{\mu_0}{2} K(-\hat{z} \times \hat{x}) = -q\bar{v} \times \frac{\mu_0}{2} K\hat{y}$$
$$\bar{F}_{tot} = \bar{F}_l + \bar{F}_r = 0$$

   b. Negative x

$$\bar{F}_l = q\bar{v} \times \frac{\mu_0}{2} K(\hat{z} \times -\hat{x}) = -q\bar{v} \times \frac{\mu_0}{2} K\hat{y}$$
$$\bar{F}_r = q\bar{v} \times \frac{\mu_0}{2} K(-\hat{z} \times -\hat{x}) = q\bar{v} \times \frac{\mu_0}{2} K\hat{y}$$
$$\bar{F}_{tot} = \bar{F}_l + \bar{F}_r = 0$$

4.9 The inductance may be obtained using the definition of energy, formula (3.19). The interaction energy for a system with two conductors is

$$U_m = M_{12} I_1 I_2$$

a. Using formula (4.49) for the interaction energy between two parallel large current-carrying plates

$$U_m = -\frac{\mu_0}{2} XYZK_1 K_2 = -\frac{\mu_0}{2} XYZ \frac{I_1}{Y} \frac{I_2}{Y}$$

the mutual inductance becomes

$$M_{12} = -\frac{\mu_0}{2} \frac{XZ}{Y}$$

b. For oppositely directed currents the energy and the inductance change sign:

$$M_{12} = \frac{\mu_0}{2} \frac{XZ}{Y}$$

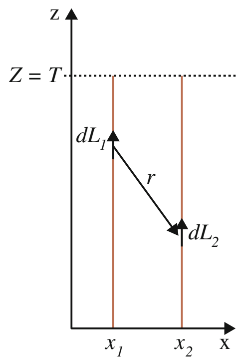
In a direct application of Neumann's formula (3.30)

$$M_{jk} = \frac{\mu_0}{4\pi} \int_k \int_j \frac{d\bar{L}_j \cdot d\bar{L}_k}{R}$$

the current directions are included in the length elements $d\bar{L}$ so that inductance becomes sign sensitive. The currents are then given in magnitude.

Alternatively, it is customary that in some obvious cases, as in Sect. 4.2.5.2, the inductance is given in magnitude and the sign of the energy is regulated by the sign of the currents.

4.10 a. The mutual inductance is obtained from Neumann's formula (3.30)



$$M_{12} = \frac{\mu_0}{4\pi} \int_{Cond\ 1} \int_{Cond\ 2} \frac{d\bar{L}_1 \cdot d\bar{L}_2}{r}$$

where

$$d\bar{L}_1 = dz_1 \hat{z}$$

$$d\bar{L}_2 = dz_2 \hat{z}$$

$$r = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}$$

so that mutual inductance becomes

$$M_{12} = \frac{\mu_0}{4\pi} \int_0^T \int_0^T \frac{dz_1 dz_2}{\sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}}$$

Change variables $t = z_2 - z_1$ so that

$$M_{12} = -\frac{\mu_0}{4\pi} \int_0^T \int_{z_2}^{z_2 - T} \frac{dt\, dz_2}{\sqrt{(x_2 - x_1)^2 + t^2}}$$

$$= -\frac{\mu_0}{4\pi} \int_0^T \left[\sinh^{-1} \frac{t}{|x_2 - x_1|} \right]_{z_2}^{z_2 - T} dz_2$$

$$= \frac{\mu_0}{4\pi} 2 \int_0^T \sinh^{-1} \frac{z_2}{|x_2 - x_1|} dz_2$$

$$= \frac{\mu_0}{2\pi} \left[T \sinh^{-1} \frac{T}{|x_2 - x_1|} - \sqrt{T^2 + (x_2 - x_1)^2} + |x_2 - x_1| \right]$$

When distance tends to infinity the inductance tends to zero as it should. Note also that $M_{12} = M_{21}$.

b. Force is obtained from formulas (3.19) and (3.23)

$$\bar{F}_{1 \to 2} = I_1 I_2 \frac{d}{dx_2} M_{12}$$
$$= \frac{\mu_0}{2\pi} I_1 I_2 \left(T \frac{1}{(T^2 + (x_2 - x_1)^2)^{1/2}} \left(-\frac{T}{(x_2 - x_1)} \right) - \frac{(x_2 - x_1)}{(T^2 + (x_2 - x_1)^2)^{1/2}} + 1 \right) \hat{x}$$
$$= -\frac{\mu_0}{2\pi} I_1 I_2 \left(\frac{T^2 + d^2}{(T^2 + d^2)^{1/2} d} - 1 \right) \hat{x}$$
$$= -\frac{\mu_0}{2\pi} I_1 I_2 \left(\frac{(T^2 + d^2)^{1/2} - d}{d} \right) \hat{x}$$

in accordance with Exercise (4.6).

c. The induction voltage is obtained from formula (3.32)

$$\varepsilon_j = \frac{d}{dt} (i_k M_{jk})$$

For this case (using also formula 3.14)

$$\varepsilon_2 = M_{21} \frac{d}{dt} i_1 = M_{12} \omega I_0 \cos \omega t$$

and

$$\varepsilon_1 = M_{12} \frac{d}{dt} i_2 = M_{12} \omega I_0 \cos \omega t$$

d. According to Exercise (4.6), there is no contribution from terms no. 2 and 3 in formula (2.11). The conductor sides perpendicular to the straight conductor don't contribute to the force. The distal parallel sides may be placed far away to give negligible contribution. Thus, the two long parallel conductors of equal length is in this respect equivalent to a closed conductor, as was pointed out in Exercise (2.10).

4.11 The plate may be considered as consisting of parallel infinitesimal wires. Its self inductance is therefore the sum of the mutual inductances between a pair of wires plus the self inductance of each wire.
From Exercise (4.10)

$$M_{12} = \frac{\mu_0}{2\pi}\left[T \sinh^{-1}\frac{T}{|x_2 - x_1|} - \sqrt{T^2 + (x_2 - x_1)^2} + |x_2 - x_1|\right]$$
$$\approx \frac{\mu_0}{2\pi}T\left(\ln\frac{2T}{x_2 - x_1} - 1\right)$$

since $T \gg X$. In analogy with Sect. 4.2.5.3, the integral is formulated as follows. The interaction energy between two wires is

$$dU_m = M_{12}\frac{I_1}{X}\frac{I_2}{X}dx_1dx_2$$

so that total inductance, i.e. self inductance, becomes

$$M_{tot} = \frac{1}{X^2}\frac{\mu_0}{2\pi}T\int_0^X\int_0^X\left[\ln\left|\frac{2T}{x_2 - x_1}\right| - 1\right]dx_1dx_2 = \frac{\mu_0}{2\pi}T\left(\ln\frac{2T}{X} + \frac{1}{2}\right)$$

*4.12 a. The result for the force between two current loops may here be used by considering the coil as a collection of current loops. The total force is obtained by summing the individual contributions. The mutual inductance between two loops is given by (4.60):

$$M = -\frac{2\mu_0 ab}{[(a+b)^2 + d^2]^{1/2}}\left[\left(1 - \frac{2}{k^2}\right)F + \frac{2}{k^2}E\right]$$

where d is the distance in the z direction between the loops. F and E are elliptic functions defined as

$$F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2}\frac{d\alpha}{\left[1 - k^2\sin^2\alpha\right]^{1/2}}$$
$$E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2}\left[1 - k^2\sin^2\alpha\right]^{1/2}d\alpha$$

and

$$k^2 = \frac{4ab}{(a+b)^2 + d^2}$$

In an infinitesimal loop of the coil the current is $dI_2 = N/lI_2dz_2$. Total mutual inductance between the loop and the coil becomes (see Sect. 4.2.5.3):

$$M_{12} = \int_{-l/2}^{l/2} -\frac{N}{l}\frac{2\mu_0 ab}{[(a+b)^2+(z_2-z_1)^2]^{1/2}}\left[\left(1-\frac{2}{k^2}\right)F+\frac{2}{k^2}E\right]dz_2$$

where

$$k^2 = \frac{4ab}{(a+b)^2+(z_2-z_1)^2}$$

In this case the radius of the loop a is much greater than the coil radius b so that k becomes small. The elliptic functions are

$$F(k,\pi/2) = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\cdots\right]$$

$$E(k,\pi/2) = \frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2k^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3}-\cdots\right]$$

so that

$$M_{12} = \int_{-l/2}^{l/2}\frac{N}{l}\frac{2\mu_0 ab}{[(a+b)^2+(z_2-z_1)^2]^{1/2}}\frac{\pi k^2}{16}dz_2$$

$$= \mu_0\frac{N\pi a^2b^2}{2l}\int_{-l/2}^{l/2}\frac{dz_2}{[(a+b)^2+(z_2-z_1)^2]^{3/2}}$$

b. The force on the loop is obtained by differentiating M_{12} with respect to z_1. The integration in the next step then gives back the integrand up to the sign, i.e.

$$\bar{F}_{2\to 1} = \mu_0 I_1 I_2\frac{N\pi a^2b^2}{2l}\frac{d}{dz_1}\int_{-l/2}^{l/2}\frac{dz_2}{[(a+b)^2+(z_2-z_1)^2]^{3/2}}\hat{z}$$

$$= -\mu_0 I_1 I_2\frac{N\pi a^2b^2}{2l}\left(\frac{1}{\left[(a+b)^2+(\frac{l}{2}-z_1)^2\right]^{3/2}} - \frac{1}{\left[(a+b)^2+(-\frac{l}{2}-z_1)^2\right]^{3/2}}\right)\hat{z}$$

which for negative z_1 (as in the figure) and for the same current direction is a positive force, as it should be.

c. The induction voltage is obtained equivalently to Exercise (4.10c). If the current varies in object 2 the induced voltage becomes

$$\varepsilon_1 = M_{12}\frac{d}{dt}i_2 = M_{12}\omega I_0 \cos\omega t$$

4.13 a. Utilizing formula (4.56), equivalent to formula (4.60), mutual inductance becomes

$$M_{12} = -2\mu_0 ab \int_0^{\pi/2} \frac{(1-2\sin^2\alpha)d\alpha}{[(a+b)^2 + d^2 - 4ab\sin^2\alpha]^{1/2}}$$

which may be written

$$M_{12} = -\frac{N}{l}2\mu_0 ab \int_0^{\pi/2} (1-2\sin^2\alpha) \times \int_{-l/2}^{l/2} \frac{1}{[(a+b)^2 - 4ab\sin^2\alpha + (z_2-z_1)^2]^{1/2}} dz_2 d\alpha$$

$$= -\frac{N}{l}2\mu_0 ab \int_0^{\pi/2} (1-2\sin^2\alpha) \times \left[\sinh^{-1}\frac{l/2 - z_1}{\left((a+b)^2 - 4ab\sin^2\alpha\right)^{1/2}} - \sinh^{-1}\frac{-l/2 - z_1}{\left((a+b)^2 - 4ab\sin^2\alpha\right)^{1/2}}\right] d\alpha$$

Since $l \to \infty$, z_1 may be neglected and the mutual inductance becomes

$$M_{12} = -\frac{N}{l}4\mu_0 ab \int_0^{\pi/2} (1-2\sin^2\alpha) \times \left[\sinh^{-1}\frac{l/2}{\left((a+b)^2 - 4ab\sin^2\alpha\right)^{1/2}}\right] d\alpha$$

$$\approx -\frac{N}{l}4\mu_0 ab \int_0^{\pi/2} (1-2\sin^2\alpha) \ln\left(\frac{l}{\left((a+b)^2 - 4ab\sin^2\alpha\right)^{1/2}}\right) d\alpha$$

$$= -\frac{N}{l}4\mu_0 ab \int_0^{\pi/2} (1 - 2\sin^2\alpha) \ln \frac{l}{\left[(a+b)^2\left(1 - \frac{4ab\sin^2\alpha}{(a+b)^2}\right)\right]^{1/2}} d\alpha$$

$$= \frac{N}{l}2\mu_0 ab \int_0^{\pi/2} (1 - 2\sin^2\alpha) \ln \left(1 - \frac{4ab\sin^2\alpha}{(a+b)^2}\right) d\alpha$$

$$\approx -\frac{N}{l}8\mu_0 \frac{a^2b^2}{(a+b)^2} \int_0^{\pi/2} (1 - 2\sin^2\alpha) \sin^2\alpha d\alpha$$

since $a \ll b$.

The integral is picked from a handbook so that

$$M_{12} \approx -\frac{N}{l}8\mu_0 \frac{a^2b^2}{(a+b)^2}\left(\frac{\pi}{4} - \frac{3\pi}{8}\right) = \frac{N}{l}\mu_0 \frac{a^2b^2}{(a+b)^2}\pi \approx \frac{N}{l}\mu_0 \pi a^2$$

independent of where the loop is placed inside the long coil.

b. The interaction energy becomes

$$U_m = I_1 I_2 M_{12} = \frac{N}{l}\mu_0 I_2 \pi a^2 I_1 = \frac{N}{l}\mu_0 I_2 m$$

where $m = I_1 \pi a^2$ is called the dipole moment of the loop, which will be discussed in more detail in Chap. 7.

c. In Exercise a, the case when the surface normal of the loop is parallel to the axis of the coil was treated. How is the mutual inductance affected if the surface normal forms an angle θ to the coil axis? The coil axis is chosen as the z axis and the direction of the loop is taken into account by forming a vector of the dipole moment:

$$\bar{m} = I_1 \pi a^2 \hat{n}$$

where $\hat{n}$ is the direction of the loop's surface normal. According to the basic definition of inductance, formula (3.30), it is concluded that the scalar product between the infinitesimal elements results in an additional factor which is $\cos\theta$, independent of the object's position so that

$$d\bar{L}_1 \cdot d\bar{L}_2 = ab\cos(\phi_2 - \phi_1)\cos\theta d\phi_1 d\phi_2$$

Since $\cos\theta$ is a constant, nothing is changed in the calculation apart from this factor so that energy becomes

$$U_m = n\mu_0 I_2 m \cos\theta = \frac{N}{l}\mu_0 I_2 \bar{m} \cdot \hat{z}$$

d. The force vanishes since the energy is independent of translational coordinates. However, a torque occurs as

$$\tau = \frac{dU_m}{d\theta} = -\frac{N}{l}\mu_0 I_2 m \sin\theta$$

which turns the loop until the angle is 0 or π.

Stable equilibrium occurs for $\theta = 0$. For $\theta = \pi$ the currents of the loop and the coil are oppositely directed resulting in a repulsive force. As a consequence, unstable equilibrium is attained since the slightest disturbance will turn the loop. On the other hand, for $\theta = 0$, a disturbance will be counteracted since the currents then attract.

Note that the magnetic energy is maximized.

4.14 a. The inductance of a coil is

$$\begin{aligned}
M_{tot} &= -\frac{N^2 4\mu_0 a^2}{l}\left(-\frac{\pi}{4} + \frac{2a}{3l} + O\left(\frac{a}{l}\right)^2 + \cdots\right) \\
&= \mu_0 n\pi a^2\left(N - \frac{8a}{3\pi}n + \cdots\right) \\
L_1 &= \mu_0 n\pi a^2\left(N_1 - \frac{8a}{3\pi}n + \cdots\right) \\
L_2 &= \mu_0 n\pi a^2\left(N_2 - \frac{8a}{3\pi}n + \cdots\right) \\
L_1 + L_2 &= \mu_0 n\pi a^2\left(N_1 + N_2 - \frac{16a}{3\pi}n + \cdots\right)
\end{aligned}$$

Total inductance is

$$L = \mu_0 n\pi a^2\left(N_1 + N_2 - \frac{8a}{3\pi}n + \cdots\right)$$

so that $L = L_1 + L_2$ only in lowest order.

b. In lowest order there is no interaction between the coils, i.e. the mutual inductance vanishes. In higher order there is an interaction so that mutual inductance is finite. See also Exercise (6.3).

4.15 a. The following integral has to be solved:

$$\int_0^l\int_0^l [2 - \ln 8a + \ln(z_j - z_k)]dz_j dz_k$$

$$\int_0^l \ln(z_j - z_k)dz_j = (l - z_k)\ln(l - z_k) - l + z_k \ln z_k$$

$$\int_0^l [(l - z_k)\ln(l - z_k) - l + z_k \ln z_k]\, dz_k = l^2 \left(\ln l - \frac{1}{2}\right) - l^2$$

so that

$$\int_0^l \int_0^l [2 - \ln 8a + \ln(z_j - z_k)] dz_j dz_k = (2 - \ln 8a) l^2 + l^2 \left(\ln l - \frac{3}{2}\right)$$

$$M_{tot} = -\frac{\mu_0 N^2 a}{l^2} l^2 \left(\frac{1}{2} - \ln \frac{8a}{l}\right)$$

*4.16 a. The force on conductor 2 is given by (3.23)

$$\bar{F} = \nabla U_m = \frac{d}{dz_2} U_m \hat{z} = I_1 I_2 \frac{d}{dz_2} M_{12} \hat{z}$$

The exact form of the inductance is given by (4.60)

$$M_{12} = -\frac{2\mu_0 ab}{[(a+b)^2 + d^2]^{1/2}} \left[\left(1 - \frac{2}{k^2}\right) F + \frac{2}{k^2} E\right]$$

where

$$k^2 = \frac{4ab}{(a+b)^2 + d^2}$$

$d = z_2 - z_1$ and $a = b$.
Since $a \gg d$ and $k^2 \to 1$ the force is given by (4.82)

$$\lim_{k^2 \to 1} F\left(k, \frac{\pi}{2}\right) = \frac{1}{2} \ln \frac{16}{1 - k^2}$$

and

$$\lim_{k^2 \to 1} E\left(k, \frac{\pi}{2}\right) = 1$$

Thus, inductance becomes

$$M_{12} = -\mu_0 a (2 - F) = -\mu_0 a \left(2 - \frac{1}{2} \ln \frac{16}{1 - k^2}\right)$$

so that

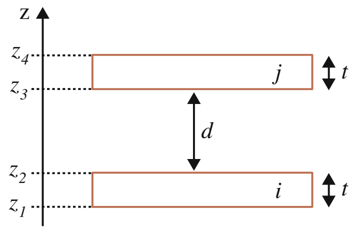
$$\frac{dM_{12}}{dz_2} = \frac{dM_{12}}{dk^2} \frac{dk^2}{dz_2} = -\frac{\mu_0 a}{(z_2 - z_1)} = -\frac{\mu_0 L}{2\pi (z_2 - z_1)}$$

where L is the circumference of the loops. Force becomes

$$\bar{F}_{1\to 2} = -I_1 I_2 \frac{\mu_0 L}{2\pi(z_2 - z_1)}\hat{z}$$

which is equivalent to the result for straight conductors, formula (2.3).

b. The force between single loops are usually small and hard to measure. If instead short coils are used, the current is multiplied by a factor N, the number of turns. Since the current appears quadratically the force is magnified with the product of number of turns from each coil. Two identical coils with 100 turns each will then give a force magnification of a factor 10,000. However, the force formula must then be corrected since the different loops interact at different positions, see figure.



The force between two infinitesimal loops of each conductor is

$$d\bar{F}_{i\to j} = -\hat{z}\frac{\mu_0 L}{2\pi(z_j - z_i)} K_i dz_i K_j dz_j$$

where K is current per length in z direction so that

$$K_i = \frac{N_i I_i}{z_2 - z_1}$$

and

$$K_j = \frac{N_j I_j}{z_4 - z_3}$$

The total force becomes

$$\bar{F}_{i\to j} = -\hat{z}\frac{\mu_0 L K_i K_j}{2\pi(z_2 - z_1)(z_4 - z_3)} \int_{z_3}^{z_4}\int_{z_1}^{z_2} \frac{dz_i dz_j}{z_j - z_i}$$

$$= -\hat{z}\frac{\mu_0 L N_i I_i N_j I_j}{2\pi} \frac{2(d+t)\ln(d+t) - (d+2t)\ln(d+2t) - d\ln d}{t^2}$$

where $t = z_2 - z_1 = z_4 - z_3$ and $d = z_3 - z_2$

For $d \gg t$ force becomes

$$\bar{F}_{i\to j} = -\hat{z}\frac{\mu_0 L N_i I_i N_j I_j}{2\pi(d+t)}$$

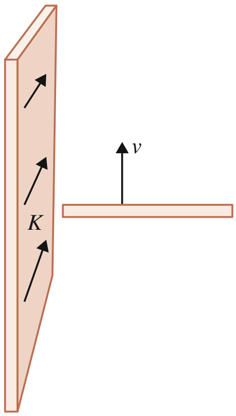
so that the coils act as two straight conductors each with current NI magnifying force a factor N^2.
With this apparatus it is quite feasible to perform a precision measurement of μ_0.

4.17 Maximal voltage is induced when the motion is perpendicular to the surface normal of the plate, as is seen from formula (4.46):

$$\bar{F} = I\bar{L} \times \frac{\mu_0}{2}(\bar{K} \times \hat{n})$$

The voltage is then given by (4.87). Using the length of the two wings as 100 m and a speed of 250 m/s the induced voltage becomes

$$\varepsilon_2 = \frac{\mu_0 K}{2} L v \approx 2\pi \cdot 10^{-7} \times 80 \times 50 \times 250\,\text{V} = 0.6\,\text{V}$$



4.18 The exact formula (4.75) is:

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l} \times \int_0^{\pi/2} \left[(1-2\sin^2\alpha)\left(\sinh^{-1}\frac{l}{2a\cos\alpha} - \left(1+\frac{4a^2}{l^2}\cos^2\alpha\right)^{1/2} + \frac{2a}{l}\cos\alpha\right)\right] d\alpha$$

where terms up to order $(\frac{a}{l})^2$ are kept.

Mutual inductance becomes

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l} \int_0^{\pi/2} \Big[(1 - 2\sin^2\alpha) \times \left(\ln\frac{l}{a\cos\alpha} + \frac{1}{4}\frac{a^2}{l^2}\cos^2\alpha - (1 + 2\frac{a^2}{l^2}\cos^2\alpha) + \frac{2a}{l}\cos\alpha\right)\Big] d\alpha$$

$$= -\frac{N^2 4\mu_0 a^2}{l}\Big[\left(1 - \ln\frac{l}{a}\right)\int_0^{\pi/2}(2\sin^2\alpha - 1)d\alpha + \int_0^{\pi/2}(2\sin^2\alpha - 1)\ln(\cos\alpha)d\alpha - \frac{2a}{l}\int_0^{\pi/2}(2\sin^2\alpha - 1)\cos\alpha d\alpha\Big]$$

$$+ \frac{N^2\mu_0 a^4}{l^3} 7\int_0^{\pi/2}\cos^2\alpha(1 - 2\sin^2\alpha)d\alpha$$

The last term is

$$\frac{N^2\mu_0 a^4}{l^3} 7\int_0^{\pi/2}\cos^2\alpha(1 - 2\sin^2\alpha)d\alpha = \frac{7N^2\mu_0 a^4}{l^3}\left[\frac{\alpha}{2} + \frac{\sin 2\alpha}{4} - 2\left(\frac{\alpha}{8} - \frac{\sin 4\alpha}{32}\right)\right]_0^{\pi/2} = \frac{7N^2\mu_0 a^4\pi}{8l^3}$$

so that

$$M_{tot} = -\frac{N^2 4\mu_0 a^2}{l}\left(-\frac{\pi}{4} + \frac{2a}{3l} + O\left(\frac{a}{l}\right)^2 + \cdots\right) = \frac{\mu_0 N^2 \pi a^2}{l}\left(1 - \frac{8a}{3\pi l} + \frac{7}{8}\frac{a^2}{l^2} + O\left(\frac{a}{l}\right)^3 + \cdots\right)$$

where the two first terms are obtained from formula (4.80).
It is seen that if a/l is approximately 10 % there is about 10 % contribution in second order and about 1 % in third order.

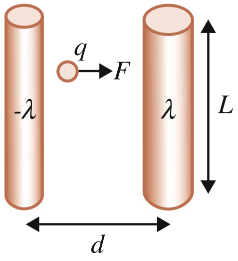
D.5 Conductors and Resistive Effects

5.1 Capacitance is defined as

$$C = \frac{Q}{\Delta\Phi}$$

where $Q = \lambda L$ and L is the common length of the cylinders. The voltage is obtained from the concept of work

$$W = U_e = q\Delta\Phi$$



To determine capacitance the work required to bring a point charge against the electric force from one to the other cylinder has to be calculated, see figure. Since the cylinders are conductors all excess charge will be collected on its surface, equivalent to a charged shell. This exercise is therefore equivalent to Exercise (4.4) with the sole difference that the radii of the two cylinders are different. Thus the work becomes

$$W = \frac{q\lambda}{2\pi\varepsilon_0}\int_a^{d-b} \frac{1}{r} + \frac{1}{d-r} dr = \frac{q\lambda}{2\pi\varepsilon_0}[\ln r - \ln|d-r|]_a^{d-b}$$
$$= \frac{q\lambda}{2\pi\varepsilon_0}[\ln(d-b) - \ln b - \ln a + \ln(d-a)]$$
$$= \frac{q\lambda}{2\pi\varepsilon_0}\ln\frac{(d-b)(d-a)}{ba}$$

and voltage

$$\Delta\Phi = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{(d-b)(d-a)}{ba}$$

and the capacitance per meter

$$C/L = \frac{Q/L}{\Delta\Phi} = \frac{2\pi\varepsilon_0}{\ln\frac{(d-b)(d-a)}{ba}}$$

5.2 This exercise is equivalent to Exercise (4.3) with the sole difference that the outer conductor has a thickness. Let λ denote charge per meter. The work to bring a point charge q from the inner to the outer conductor becomes

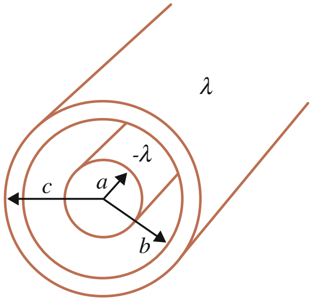
$$W = \frac{q\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{q\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} = q\Delta\Phi$$

The voltage is

$$\Delta\Phi = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

and the capacitance per meter

$$C/L = \frac{Q/L}{\Delta\Phi} = \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}}$$



5.3 Charge is collected on the surface of the spheres. The work needed to bring a point charge from one to the other object has to be determined. According to Sect. 4.1.5.2 there is no force on a charge inside a sphere so that work becomes

$$W = \frac{qQ}{4\pi\varepsilon_0}\int_a^{d-b}\left(\frac{1}{r^2}+\frac{1}{(d-r)^2}\right)dr = \frac{qQ}{4\pi\varepsilon_0}\left[-\frac{1}{r}+\frac{1}{d-r}\right]_a^{d-b}$$
$$= \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{d-b}+\frac{1}{b}-\frac{1}{d-a}\right) = q\Delta\Phi$$

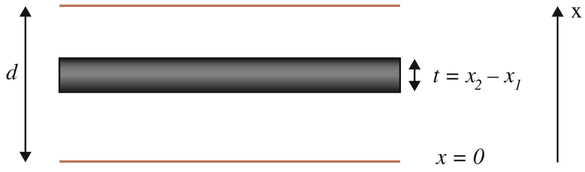
and capacitance

$$C = \frac{Q}{\Delta\Phi} = 4\pi\varepsilon_0\left(\frac{1}{a}-\frac{1}{d-b}+\frac{1}{b}-\frac{1}{d-a}\right)^{-1}$$

5.4 a. The effective distance between the plates is halved. Since the charge on the plates is unchanged, voltage is halved.

b. The effective area is halved, but the charge on this effective area is half of the original so that the surface charge density is unchanged. Since the distance between the plates is unchanged, the voltage is unchanged.

*5.5



a. The capacitance $C = Q/\Delta\Phi$ where the voltage $\Delta\Phi$ is defined from the work W needed to bring a point charge from one to the other plate:

$$W = \frac{qQ}{A\varepsilon_0}(d-t) \Rightarrow \Delta\Phi = \frac{W}{q} = \frac{Q}{A\varepsilon_0}(d-t) \Rightarrow C = \frac{A\varepsilon_0}{(d-t)}$$

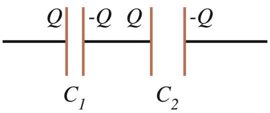
b. The upper and lower coordinates of the metal plate are denoted by x_2 and x_1 respectively, so that

$$C_1 = \frac{\varepsilon_0 A}{x_1},\ C_2 = \frac{\varepsilon_0 A}{(d-x_2)}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \varepsilon_0 A \frac{\frac{1}{x_1(d-x_2)}}{\frac{1}{x_1} + \frac{1}{(d-x_2)}} = \varepsilon_0 A \frac{1}{d + (x_1 - x_2)} = \frac{\varepsilon_0 A}{(d-t)}$$

c. Consider a series coupling of two charged capacitors. The charge is distributed equally as in the figure. The partial voltages are summed to total voltage:

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} \Rightarrow \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C = \frac{C_1 C_2}{C_1 + C_2}$$



*5.6 a. Because of the low speed.

b. The terminal speed is achieved when gravity balances the air resistance:

$$mg = +Gv + Bv^2$$

$$Bv^2 + Gv - mg = 0$$

$$v^2 + \frac{G}{B}v - \frac{mg}{B} = 0$$

$$\left(v + \frac{G}{2B}\right)^2 = \frac{mg}{B} + \left(\frac{G}{2B}\right)^2$$

$$v = \pm\sqrt{\frac{mg}{B} + \left(\frac{G}{2B}\right)^2} - \frac{G}{2B}$$

so that heavier objects fall quicker than lighter.

d. Since at that high altitude the air resistance is lower due to the lower density.

5.7 a. For a homogeneous conductor $J = \frac{I}{S}$ and the conductivity $\sigma = \frac{l}{RS}$. Using Joule's law (5.15), the heat power becomes

$$\frac{P_{fr}}{V} = \frac{J^2}{\sigma} = \frac{I^2/S^2}{l/RS} = \frac{I^2 R}{V} \Rightarrow P_{fr} = I^2 R$$

b. Ohm's law (5.5):

$$\Delta\Phi = RI = \frac{l}{\sigma S} JS \Rightarrow J = \sigma \frac{\Delta\Phi}{l} \Rightarrow \bar{J} = -\sigma\nabla\Phi$$

The minus sign in the last formula reflects the fact that $\nabla\Phi$ is oppositely directed to the current direction.

5.8 These three properties of the metal originate from the practically free conduction electrons. These are responsible for making the metal an equally good heat conductor as current carrier.

In addition, due to the conduction electrons the metal is shiny and, if well-polished, works as a mirror. To understand the mirror, one has to take into account the vision sense in the discussion. When something is registered by the vision, it basically means that an electric interaction has caused an action potential in the eye. Usually, the eye interacts with the electric dipole vibrations of the objects, e.g. a flower. Including the mirror in the dynamics a three objects dynamics arise. Firstly, the dipole vibrations is picked up by the mirror whose conduction electrons manage to exactly reproduce the charge vibration pattern of the flower. When the eye is directed towards the mirror, it interacts with the vibrating conduction electrons and registers an image identical to the real object.

The mirror is certainly a miracle and reveals in this way some of the basic mysteries of nature.

A mirror image may also be produced by water and glass surfaces, although these do not contain any conduction electrons. In these cases it is their mobile electric dipoles that mimic the charge vibration of the object.

5.9 The continuity equation

$$\nabla \cdot \bar{J} + \frac{d\rho}{dt} = 0$$

may be written

$$nq\nabla \cdot \bar{v} + n\frac{dq}{dt} = 0$$

Consider the problem in one dimension

$$q\frac{dv}{dt}\frac{dt}{dx} + \frac{dq}{dt} = 0$$

so that

$$q\frac{dv}{dt} + \frac{dx}{dt}\frac{dq}{dt} = 0$$

where $dx/dt = v$. In vector form:

$$\frac{d}{dt}(q\bar{v}) = 0$$

5.10 a. The flux density T, i.e. flux per surface, becomes:

$$\bar{T} = \bar{v}\rho$$

where ρ is mass density and v is velocity of a mass element. Without any feed or drain, i.e. no sources or losses, the flux is conserved leading to a continuity equation:

$$\nabla \cdot \bar{T} + \frac{d\rho}{dt} = 0$$

b. Consider an element of the flux with mass m. Equivalent to Exercise (5.9), we obtain

$$\frac{d}{dt}(m\bar{v}) = 0$$

i.e. Newton's force law.

5.11 The bulb interacts with the mirror whose conduction electrons are put in vibration. After the bulb has ceased the electron vibrations in the mirror continue a while given by the relaxation time. Table D.1 provides necessary input data.

a. According to formula (5.10) this time is

$$t_r = \frac{m}{q^2 n}\sigma$$

where

$$n = N/V = \frac{N}{M/\rho} = \frac{N_V N_A}{M_{mol}/\rho}$$

and N_V is the number of valence electrons.

Table D.1

Electron mass	$m = 9.1 \times 10^{-31}\,\text{kg}$
Electron charge	$q = 1.6 \times 10^{-19}\,\text{C}$
Number of valence electrons	$N_V = 1$
Atomic number	$Z = 41$
Density	$\rho = 7830\,\text{kg/m}^3$
Avogadro's number	$N_A = 6.0 \times 10^{23}$

Molar mass M_{mol} = atomic number Z in grams.

$$n = \frac{6.0 \times 10^{23}}{41 \times 10^{-3}/7.83 \times 10^{3}} = 1.15 \times 10^{29} m^{-3}$$

Resistivity

$$\rho_r = \frac{m}{q^2 n t_R} = \frac{9.1 \times 10^{-31}}{(1.6 \times 10^{-19})^2 1.15 \times 10^{29} \times 0.13} \Omega m = 2.4 \times 10^{-21} \Omega m$$

b. This low resistivity appears for superconductors, a state at an extremely low temperature. In the rewarded experiment the temperature was as low as 0.8 K.

5.12 The force is twice that of formula (4.44b)

$$\bar{F} = q\bar{v} \times \mu_0 K \hat{z}$$

where z is along the coil axis. The force is accordingly centripetal pointing toward the coil axis:

$$\frac{mv^2}{r} = qv\mu_0\,\text{K}$$

The velocity is obtained from the high voltage of the electron tube:

$$q\Delta\Phi = \frac{mv^2}{2}$$

Putting these two equations together the charge to mass ratio becomes:

$$\frac{q}{m} = \frac{2\Delta\Phi}{(\mu_0 r K)^2}$$

5.13 a. The equations of motion in the two cases are:

$$-mg + Gv = m\frac{dv}{dt} \quad \Rightarrow \quad v_f = \frac{mg}{G}$$
$$-mg - Gv + q\frac{\Delta\Phi}{d} = m\frac{dv}{dt} \quad \Rightarrow \quad v_r = \frac{q\frac{\Delta\Phi}{d} - mg}{G}$$

where formula (4.22) has been used in the second case and d is the distance between the plates.

b. Summing the two velocities

$$v_f + v_r = \frac{q\Delta\Phi}{dG} \quad \Rightarrow \quad q = (v_f + v_r)\frac{dG}{\Delta\Phi} = (v_f + v_r)\frac{d6\pi\eta a}{\Delta\Phi}$$

The radius a is obtained from v_f, the known density of the oil ρ and the viscosity of air η:

$$v_f = \frac{mg}{G} = \frac{\frac{4}{3}\pi a^3\rho g}{6\pi\eta a} = \frac{2\rho g}{9\eta}a^2 \quad \Rightarrow a = \sqrt{\frac{9\eta v_f}{2\rho g}}$$

The charge becomes

$$q = (v_f + v_r)\frac{d6\pi\eta}{\Delta\Phi}\sqrt{\frac{9\eta v_f}{2\rho g}}$$

5.14 a. Formula (3.1):

$$U_e = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = -\frac{(1.6\times 10^{-19})^2}{4\pi \times 8.9\times 10^{-12}\cdot 0.5\times 10^{-10}}J = -4.6\times 10^{-18}\text{J}$$

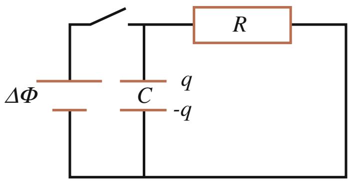
The energy is negative since the two charges have opposite signs.

b. The centripetal force is electric:

$$\frac{m_e v^2}{r} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \quad \Rightarrow \quad v = \sqrt{\frac{q_1 q_2}{4\pi\varepsilon_0 r m_e}}$$
$$= \sqrt{\frac{(1.6\times 10^{-19})^2}{4\pi \times 8.9\times 10^{-12}\times 0.5\times 10^{-10}\times 9.1\times 10^{-31}}}\ \text{m/s}$$
$$= 5.1\times 10^5\ \text{m/s}$$

D.6 Electric Circuits

6.1 Let the capacitor in the figure be charged to the battery voltage. Then disconnect the circuit with the switch so that the capacitor is discharged through the resistor.



The potential law at discharge becomes

$$\frac{q}{C} - R(-\frac{dq}{dt}) = 0 \Rightarrow q = Ae^{-\frac{t}{RC}}$$

where q is the charge on the capacitor. Since the current is positive but q decreases there is a minus sign in front of dq/dt. The current through the resistor becomes

$$i = -\frac{dq}{dt} = \frac{A}{RC}e^{-\frac{t}{RC}}$$

The constant A is determined by the initial condition

$$i(t = 0) = \frac{\Delta\Phi}{R} \Rightarrow A = C\Delta\Phi$$

so that

$$i(t) = \frac{\Delta\Phi}{R}e^{-\frac{t}{RC}}$$

To measure this current, utilize an AC squared formed voltage source and connect an oscilloscope over the resistor. Alternatively, use a large capacitor with capacitance of about 1 F and measure current directly using an ordinary ampere meter.

6.2 When voltage switches to zero the current decreases in the circuit. The coil responds by generating an induction current in the same direction, clockwise. Applying the potential law clockwise gives

$$L\left(-\frac{di}{dt}\right) - Ri = 0$$

There is a minus sign in front of the time derivative since current decreases in the circuit but the induction voltage is positive clockwise. The solution becomes

$$i = i_0e^{-tR/L}$$

where i_0 is determined from the initial condition

$$i_0 = i(t = 0) = \frac{\Delta\Phi}{R}$$

so that

$$i = \frac{\Delta\Phi}{R} e^{-tR/L}$$

6.3 a. The voltage over both inductors becomes

$$\Delta\Phi = \Delta\Phi_1 + \Delta\Phi_2 = L_1\frac{dI}{dt} + M_{12}\frac{dI}{dt} + L_2\frac{dI}{dt} + M_{12}\frac{dI}{dt} = L\frac{dI}{dt}$$
$$\Rightarrow L = L_1 + L_2 + 2M_{12}$$

b. The total inductance is defined from

$$\Delta\Phi = L\frac{d}{dt}(I_1 + I_2)$$

Further condition is

$$\begin{bmatrix} \Delta\Phi = L_1\frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt} \\ \Delta\Phi = L_2\frac{dI_2}{dt} + M_{21}\frac{dI_1}{dt} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta\Phi = L_1\frac{dI_1}{dt} + M_{12}\left(\frac{\Delta\Phi - M_{21}\frac{dI_1}{dt}}{L_2}\right) \\ \Delta\Phi = L_2\frac{dI_2}{dt} + M_{21}\left(\frac{\Delta\Phi - M_{12}\frac{dI_2}{dt}}{L_1}\right) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta\Phi(L_2 - M_{12}) = L_2L_1\frac{dI_1}{dt} - M_{12}M_{21}\frac{dI_1}{dt} \\ \Delta\Phi(L_1 - M_{12}) = L_2L_1\frac{dI_2}{dt} - M_{12}M_{21}\frac{dI_2}{dt} \end{bmatrix}$$

Summing these two equations gives

$$\Delta\Phi = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}\frac{d}{dt}(I_1 + I_2)$$

where $M = M_{12} = M_{21}$
The total inductance then becomes

$$L = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

c. The magnetic energy for a system consisting of two inductors is

$$U_m = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

For constant current the magnetic energy of the system is positive and also a maximized extreme value.
Denoting the ratio $I_1/I_2 = r$ energy becomes

$$U_m = \frac{1}{2}L_1 r^2 I_2^2 + \frac{1}{2}L_2 I_2^2 + M_{12} r I_2^2 \geq 0$$

Using

$$dU_m/dr = 0 \Rightarrow L_1 r + M_{12} = 0 \Rightarrow r = -M_{12}/L_1$$

we obtain

$$\frac{1}{2}\frac{M_{12}^2}{L_1} + \frac{1}{2}L_2 - \frac{M_{12}^2}{L_1} \geq 0 \Rightarrow \sqrt{L_1 L_2} \geq |M_{12}|$$

which is equivalent to what should be shown.

*6.4 a.

$$\Delta\Phi = \Delta\Phi_0 e^{j\omega_s t}$$
$$P = i \cdot \Delta\Phi = \frac{\Delta\Phi_0}{Z}\sin(\omega_s t - \varphi) \cdot \Delta\Phi_0 \sin\omega_s t$$

where φ is defined in (6.23b).

b.

$$P = \frac{(\Delta\Phi_0)^2}{Z}\sin(\omega_s t - \varphi)\sin\omega_s t$$
$$= \frac{(\Delta\Phi_0)^2}{Z}(\sin\omega_s t\cos\varphi - \cos\omega_s t\sin\varphi)\sin\omega_s t$$

The average power is obtained by integrating over a time period and then dividing by a time period. The integral over the time dependent factor of the first term gives a factor 1/2 and that of the second term vanishes:

$$\langle P \rangle = \frac{(\Delta\Phi_0)^2}{Z}\frac{\cos\varphi}{2}$$

c. The phase shift $\varphi = 0$

d.

$$\tan\varphi = \frac{\omega_s L - \frac{1}{\omega_s C}}{R} = 0 \Rightarrow \omega_s = \frac{1}{\sqrt{LC}}$$

6.5 a.

$$\Delta\Phi_c = \Delta\Phi_0 e^{j\omega_s t}$$
$$i_c = \frac{\Delta\Phi_0}{Z} e^{j(\omega_s t - \varphi)}$$

b. $Z_c = Ze^{j\varphi}$ follows directly from Exercise a.

c.

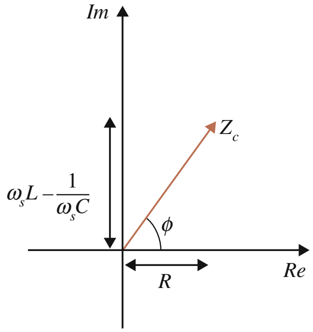
$$Z = \sqrt{\left(\frac{1}{\omega_s C} - L\omega_s\right)^2 + R^2}$$
$$\tan\varphi = \frac{\omega_s L - \frac{1}{\omega_s C}}{R}$$

Since the circuit is purely resistive for $\varphi = 0$ the projection on the real axis becomes the resistance.
The imaginary part of Z_c is therefore

$$Im\ Z_c = \omega_s L - \frac{1}{\omega_s C}$$

Thus, the complex impedance may be written

$$Z_c = R + j\left(\omega_s L - \frac{1}{\omega_s C}\right)$$



6.6 a. The initial charge on the plates is

$$Q_i = C_i \Delta\phi = \frac{\varepsilon_0 A}{d}\Delta\phi$$

Final charge on the plates is

$$Q_f = C_f \Delta\phi = \frac{\varepsilon_0 A}{d+s}\Delta\phi$$

The amount of charge that has passed through the resistor is $\Delta Q = Q_f - Q_i$

b. The movable plate is affected by the longitudinal sound wave making the distance between the plates to change.

6.7 Denote the displacement by x, the mass by m, the friction coefficient by λ and the spring constant by k. With the x axis directed upwards, the equation of motion becomes (the buoyancy force of Archimedes is neglected)

$$m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} - kx - F_g$$

where F_g is the gravitational force. This may be written

$$-F_g - \lambda\frac{dx}{dt} - kx - m\frac{d^2x}{dt^2} = 0$$

Setting $i = dq/dt$, Eq. (6.8) is

$$\Delta\Phi - R\frac{dq}{dt} - \frac{q}{C} - L\frac{d^2q}{dt^2} = 0$$

Identification gives the following interpretation:

$$\begin{aligned} x &\Leftrightarrow q \\ F_g &\Leftrightarrow -\Delta\Phi \\ \lambda &\Leftrightarrow R \\ k &\Leftrightarrow 1/C \\ m &\Leftrightarrow L \end{aligned}$$

Note the equivalence between inertial mass and inductance.

D.7 Electric and Magnetic Dipoles

7.2 a. The torque is given by formula (7.5)

$$\bar{\tau} = -\bar{p} \times \nabla\Phi$$

and the electric potential from the point charge is

$$\Phi = \frac{q}{4\pi\varepsilon_0 r}$$

The torque becomes

$$\bar{\tau} = \frac{q}{4\pi\varepsilon_0 r^2}\bar{p} \times \hat{r}$$

7.3 a. The force on the dipole is given by formula (7.10)

$$\bar{F} = \frac{q}{4\pi\varepsilon_0}\left(-3\frac{\bar{p}\cdot\hat{r}}{r^3}\hat{r} + \frac{\bar{p}}{r^3}\right)$$

where $\bar{r}$ is directed towards the dipole. The first term vanishes and the second term becomes

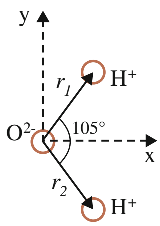
$$\bar{F} = \frac{q}{4\pi\varepsilon_0}\frac{p\hat{z}}{x^3} = \left(\frac{1.6\times 10^{-19}}{4\pi \times 8.85\times 10^{-12}}\frac{6.2\times 10^{-30}}{(10^{-6})^3}\right)\hat{z}N$$
$$= 8.9\times 10^{-21}\hat{z}N$$

The torque is (see Exercise 7.2)

$$\bar{\tau} = \frac{q}{4\pi\varepsilon_0 r^2}\bar{p}\times\hat{r} = \frac{qp}{4\pi\varepsilon_0 x^2}(\hat{z}\times -\hat{x}) = -\frac{qp}{4\pi\varepsilon_0 x^2}\hat{y}$$

b. In the discrete case, the dipole moment is

$$\bar{p} = \sum_i \bar{r}_i q_i$$



The coordinate system is chosen with origin in the oxygen atom

$$\bar{p} = \sum_i \bar{r}_i q_i = q(\bar{r}_1 + \bar{r}_2) = q(x_1\hat{x} + y_1\hat{y} + x_2\hat{x} + y_2\hat{y}) = 2qx\hat{x}$$

since $x_1 = x_2 = x$ and $y_1 = -y_2$. The coordinates become

$$x = \frac{p}{2q} \Rightarrow y_1 = x\tan 52.5^\circ = \frac{p}{2q}\tan 52.5^\circ = 2.5\times 10^{-11}\text{ m}$$

The distance between the hydrogen ions is $2y_1$.

However, this calculation should be considered just as an estimate since molecules are described by quantum mechanics, see Exercise (7.19). The real distance is three times larger than calculated here.

7.4 In the electric case, the dipole moment is

$$\bar{p} = \int\limits_{V'} \rho(\bar{r}')\bar{r}' dV'$$

A coordinate transformation gives

$$\bar{p} = \int\limits_{V'} \rho(\bar{r}')(\bar{r}' + \bar{a}) dV' = \int\limits_{V'} \rho(\bar{r}')\bar{r}' dV' + \bar{a} \int\limits_{V'} \rho(\bar{r}') dV' = \int\limits_{V'} \rho(\bar{r}')\bar{r}' dV'$$

In the magnetic case the dipole moment is

$$\bar{m} = \frac{1}{2} \int\limits_{V'} (\bar{r}' \times \bar{J}(\bar{r}')) dV' \to \frac{1}{2} \int\limits_{V'} ((\bar{r}' + \bar{a}) \times \bar{J}(\bar{r}')) dV'$$

$$= \frac{1}{2} \int\limits_{V'} (\bar{r}' \times \bar{J}(\bar{r}')) dV' + \frac{1}{2} \int\limits_{V'} (\bar{a} \times \bar{J}(\bar{r}')) dV'$$

The second term

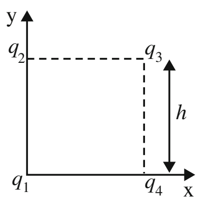
$$\int\limits_{V'} (\bar{a} \times \bar{J}(\bar{r}')) dV' = \bar{a} \times \int\limits_{V'} \bar{J}(r') dV' = 0$$

for closed currents.

7.5 a. Dipole moment is given by formula (7.22)

$$\bar{p} = \sum_i \bar{r}_i q_i = q_1 \bar{r}_1 + q_2 \bar{r}_2 + q_3 \bar{r}_3 + q_4 \bar{r}_4$$

$$= qh\hat{y} + q(h\hat{x} + h\hat{y}) - qh\hat{x} = 2qh\hat{y}$$



b. For coordinate origin in the center of the system

$$\bar{p} = \sum_i \bar{r}_i q_i = q_1 \bar{r}_1 + q_2 \bar{r}_2 + q_3 \bar{r}_3 + q_4 \bar{r}_4$$

$$= -q\left(-\frac{h}{2}\hat{x} - \frac{h}{2}\hat{y}\right) + q\left(-\frac{h}{2}\hat{x} + \frac{h}{2}\hat{y}\right)$$
$$+ q\left(\frac{h}{2}\hat{x} + \frac{h}{2}\hat{y}\right) - q\left(\frac{h}{2}\hat{x} - \frac{h}{2}\hat{y}\right) = 2qh\hat{y}$$

c. The quadrupole moment in this two-dimensional example has four elements:

$$Q_{xx} = \sum_n (3x^2 - (x^2 + y^2 + z^2))q_n$$
$$Q_{yy} = \sum_n (3y^2 - (x^2 + y^2 + z^2))q_n$$
$$Q_{xy} = \sum_n (3xyq_n) = Q_{yx}$$

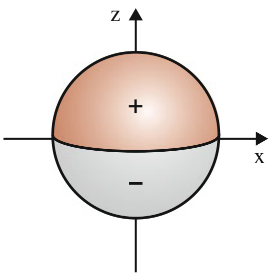
With a coordinate system as in task a the elements are

$$Q_{xx} = 2h^2 e$$
$$Q_{yy} = -4h^2 e$$
$$Q_{xy} = -3h^2 e$$

where e is the electron charge.

With a coordinate system as in task b all elements vanish. Thus, the quadrupole moment is not coordinate independent.

7.6 The dipole moment in the continuous case is given by formula (7.23).



Denote the upper half by u and the lower half by l.

$$\bar{p} = \int_V \rho(\bar{r})\bar{r}dV = \int_u \rho\bar{r}dV + \int_l -\rho\bar{r}dV$$
$$\int_u \bar{r}dV = \int\int\int r\hat{r}dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^3 \sin\theta(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})drd\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^3 \sin\theta\cos\theta\hat{z}drd\theta d\phi\hat{z} = \frac{a^4}{4}2\pi\left[-\frac{1}{4}\cos 2\theta\right]_0^{\pi/2}\hat{z} = \frac{\pi a^4}{4}\hat{z}$$

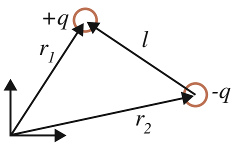
$$\int_l \bar{r}dV = \int\int\int r\hat{r}dV$$

$$= \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{a} r^3 \sin\theta(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})drd\theta d\phi$$

$$= \frac{a^4}{4}2\pi\left[-\frac{1}{4}\cos 2\theta\right]_{\pi/2}^{\pi}\hat{z} = -\frac{\pi a^4}{4}\hat{z}$$

$$\Rightarrow \bar{p} = \rho\frac{\pi a^4}{2}\hat{z} = \frac{3q}{2\pi a^3}\frac{\pi a^4}{2}\hat{z} = \frac{3qa}{4}\hat{z}$$

7.7 An idealized dipole consists of two oppositely charged point charges, as in the figure.



Formula (7.22) gives

$$\bar{p} = \sum_i \bar{r}_i q_i = q_1\bar{r}_1 + q_2\bar{r}_2 = q(\bar{r}_1 - \bar{r}_2) = q\bar{l}$$

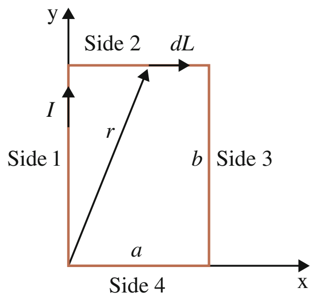
which is formula (7.1).

7.8 The magnetic dipole moment, (7.39), is

$$\bar{m} = \frac{1}{2}\int_V (\bar{r} \times \bar{J})dV$$

For a circuit with a homogeneous current distribution the current density is replaced by current according to the formula $Id\bar{L} = \bar{J}dV$, see Exercise (2.24). The dipole moment becomes

$$\bar{m} = \frac{1}{2}I\int_{Loop} \bar{r} \times d\bar{L}$$



If the coordinate system is chosen according to the figure, the contributions from sides 1 and 4 vanish since $\bar{r}//d\bar{L}$

$$\bar{m} = \frac{1}{2}I\int_{Loop}\bar{r}\times d\bar{L} = \frac{I}{2}\left(\int_{2} r dx\sin\theta_2 + \int_{3} -r dy\sin\theta_3\right)(-\hat{z})$$

$$= \frac{I}{2}\left(\int_0^a r dx\frac{b}{r} - \int_b^0 r dy\frac{a}{r}\right)(-\hat{z}) = -Iab\hat{z}$$

where θ_2 (θ_3) is the angle between r and dL for side 2 (3).
For a circuit with fixed surface normal, its dipole moment is generally its current times its surface, see formula (7.31).

7.9 *The torque* may be determined in two ways:

1. Formula (7.32) gives

$$\tau = \left|\frac{dU}{d\theta}\right|$$

   where the energy is given by (7.36)

$$U = \mu_0\frac{I_1}{2Y}I_2A\hat{n}_2\cdot\hat{y} = \mu_0\frac{I_1}{2Y}I_2A\cos\theta$$

   for this specific case where θ is the angle between the y axis and the surface normal of the loop.
   The direction of the torque is obtained by considering the magnetic forces on the sides of the loop. Since parallel currents attract, the torque will act such as the surface normal becomes parallel to the y axis, i.e. the torque is directed along negative z:

$$\bar{\tau} = -\mu_0\frac{I_1}{2Y}I_2A\sin\theta\hat{z}$$

2. Alternatively, formula (7.30) may be used:

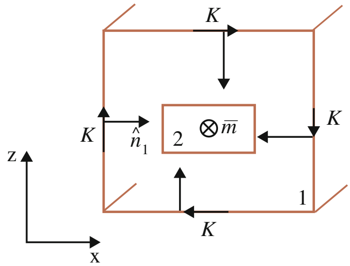
$$\bar{\tau} = \frac{\mu_0}{2}\bar{m} \times (\bar{K} \times \hat{n}) = -\frac{\mu_0}{2} m K \sin\theta \hat{z}$$

The force on the loop is given by formula (7.28):

$$d\bar{F} = I d\bar{L} \times \hat{y}\frac{\mu_0 K}{2}$$

so that the forces on parallel sides are equal in magnitude but oppositely directed so that the total force vanishes. Alternatively, the force may be obtained by taking the gradient of the energy determined above. The same result is obtained since the energy is independent of position, depending only on orientation.

7.10 The problem with the system discussed in the previous exercise and in Sect. 7.2.2 is that the large plate is not realizable. A similar system made up of a closed circuit has to replace the large plate. One possibility is to put together four small plates with a square cross section and with the loop placed inside, illustrated in the figure.



a. The torque is given by (7.30):

$$\bar{\tau} = \frac{\mu_0}{2}\bar{m} \times (\bar{K} \times \hat{n})$$

b. The induction voltage is

$$\varepsilon_2 = \frac{d}{dt}(I_1 M_{12})$$

where according to formula (7.36)

$$I_1 M_{12} = \frac{\mu_0 A K \cos\theta}{2}$$

where A is the surface of the loop. Since θ is the angle between the y axis and the surface normal of the loop we obtain

$$I_1 M_{12} = \frac{\mu_0 \bar{A} \cdot (\bar{K} \times \hat{n})}{2}$$

Since $\bar{K} \times \hat{n}$ is equal for all plates the contribution to torque and induction voltage is equal from all plates. In this way, a motor and a generator respectively may be realized.
Note that when the loop is oriented as in the figure no torque appears.
The square cross section may be generalized to any number of sides, including an infinite number corresponding to a coil with a circular cross section. Figure 2.13 shows an example of a generator or motor construction.

7.11 a. There is no difference since the angle between the y axis and the dipole moment of the loop varies equivalently in the two cases.

b. The induction voltage is (see Exercise 7.10)

$$\varepsilon_2 = \frac{d}{dt}(I_1 M_{12})$$

and

$$I_1 M_{12} = \frac{\mu_0 \bar{A} \cdot (\bar{K} \times \hat{n})}{2} = \frac{\mu_0 K \bar{A} \cdot \hat{y}}{2} = 0$$

since the surface normal in this case is perpendicular to the y axis. The time derivative vanishes and no voltage is induced.

7.12 Using the notation as in Exercises (7.8–7.11), the energy for an interaction between a current loop and a large current-carrying plate is

$$U_p = \frac{\mu_0 K}{2} \bar{m} \cdot \hat{y}$$

The interaction energy between a current loop and a current-carrying ideal coil, with the loop placed inside the coil, is given by formula (4.89 in Exercise 4.13)

$$U_c = \mu_0 n I_c \hat{z} \cdot \bar{m}$$

where n is the density of turns for the coil, I_c the current in its conductor and z its axis. The total current of the coil is $N I_c$ so that its linear current density is $K_c = N I_c / l$, where l is the length of the coil (along its axis) and N is the number of turns. The energy becomes

$$U_c = \mu_0 K_c \hat{z} \cdot \bar{m}$$

so in the latter case the energy is twice that of the former case if current density is equal. Therefore, the same energy appears if two large plates interact with the loop.

7.13 In this exercise mass is denoted by μ and angular momentum by L (not to be confused with self inductance).

a. The general definition of magnetic dipole moment is

$$\bar{m} = \frac{1}{2}\int_V (\bar{r} \times \bar{J})dV$$

For a discrete system each charge is treated individually. The volume containing one charge is denoted by dV. From formula (2.5)

$$\bar{J}dV = q\bar{v}$$

The integral for the dipole moment may then be replaced by the sum

$$\bar{m} = \frac{1}{2}\sum_i \bar{r}_i \times q_i\bar{v}_i$$

b. The angular momentum for a particle is defined as

$$\bar{L} = \bar{r} \times \mu_i\bar{v}$$

which gives the desired formula directly.

7.14 a. Start from formula (7.25):

$$U_m = -\frac{\mu_0}{4\pi d^3}[\bar{m}_1 \cdot \bar{m}_2 - 3(\bar{m}_1 \cdot \hat{d})(\bar{m}_2 \cdot \hat{d})]$$

Magnetic moment

$$\bar{m} = \frac{1}{2}\int_V (\bar{r} \times \bar{J})dV$$
$$\bar{J} = \rho\bar{v}$$
$$\bar{v} = \bar{\omega} \times \bar{r} = \omega r\hat{z} \times \hat{r} = \omega r(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \times \hat{r} = \omega r\sin\theta\hat{\phi}$$
$$\bar{r} \times \bar{J} = \rho\omega r^2\sin\theta\hat{r} \times \hat{\phi} = -\rho\omega r^2\sin\theta\hat{\theta}$$
$$\bar{m} = -\tfrac{1}{2}\rho\omega\int_0^a\int_0^\pi\int_0^{2\pi} r^4\sin^2\theta(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z})d\phi d\theta dr$$
$$= \tfrac{2}{15}\rho\omega a^5\hat{z}$$

where a is the radius of the sphere and r, θ, ϕ are spherical coordinates. The energy becomes

$$U_m = -\frac{\mu_0}{4\pi(x_2 - x_1)^3}\left(\frac{2}{15}\rho_1\omega_1 a_1^5\frac{2}{15}\rho_2\omega_2 a_2^5\right)$$

The magnetic force on object 2 becomes

$$\bar{F}_{m2} = \frac{d}{dx_2} U_m \hat{x} = \frac{3\mu_0}{225\pi d^4} \rho_1 \omega_1 a_1^5 \rho_2 \omega_2 a_2^5 \hat{x}$$

b. The total force is

$$\bar{F}_{tot2} = \bar{F}_{el2} + \bar{F}_{m2}$$

where the electric force is

$$\bar{F}_{el2} = \frac{1}{4\pi \varepsilon_0 d^2} \rho_1 \rho_2 \frac{4}{3}\pi a_1^3 \frac{4}{3}\pi a_2^3 \hat{x}$$

c. If the rotation is oppositely directed for the two objects, the magnetic force becomes attractive. The force action ceases when the magnetic and the electric force balance each other. Put $a_1 = a_2 = a$.

$$\bar{F}_{el2} = \bar{F}_{m2} \Rightarrow$$
$$\frac{3\mu_0}{225\pi d^4} \rho^2 \omega^2 a^{10} = \frac{16}{36\pi \varepsilon_0 d^2} \rho^2 \pi^2 a^6$$
$$\omega^2 = \frac{100}{3\mu_0 \varepsilon_0} \frac{d^2 \pi^2}{a^4}$$
$$\omega = \frac{10\pi}{\sqrt{3}} \frac{d}{a^2} c$$

The smallest possible distance between the objects is $2a$ and at this distance a speed greater than light speed is needed in order to reach balance. This speed is not possible to reach, see Chap. 9.

d. The magnetic force for charges in parallel motion is given by formula (2.17):

$$\bar{F}_{m2} = -\frac{\mu_0 q v q v \hat{x}}{4\pi d^2}$$

The two magnetic forces are both attractive while the electric force is repulsive. It follows that at balance the translational speed v is given by

$$v^2 = c^2 - \frac{3}{100 d^2 \pi^2} a^4 \omega^2$$

As an example let the distance $d = 2a$ and denote the speed at the equator of the sphere $v_{\max}$ so that $v_{\max} = \omega a$:

$$v^2 = c^2 - \frac{3}{\pi^2 400} v_{\max}^2$$

Putting the maximal rotational speed to c, we note that the rotation has very little affect on the translational speed v at balance. Hence, the self rotation contributes far less to the magnetic force than the translational motion.

Nevertheless, the self rotation makes it possible to reach balance theoretically since then the speed need not be exactly c. This is interesting philosophically since at this speed the interactions cease and time stands still. See Chap. 9.

e. As is further discussed in Chap. 9, magnetism is a motional consequence arising since interactions take time. As such, similar effects ought to appear in case of gravitation, which is also included in the theory of general relativity (so-called 'Lense-Thirring frame dragging'). Hence, a rotating vehicle should, if the result from Exercise d is taken over to gravitation, reduce the gravitational force. However, the effect is much too small to neutralize the force.

*7.15 a. Start from the energy (7.25)

$$\begin{aligned} U_m &= -\frac{\mu_0}{4\pi R^3}[\bar{m}_1 \cdot \bar{m}_2 - 3(\bar{m}_1 \cdot \hat{R})(\bar{m}_2 \cdot \hat{R})] \\ &= -\frac{\mu_0}{4\pi R^3}[m_1 m_2 \cos(\theta_1 - \theta_2) - 3m_1 m_2 \cos\theta_1 \cos\theta_2] \end{aligned}$$

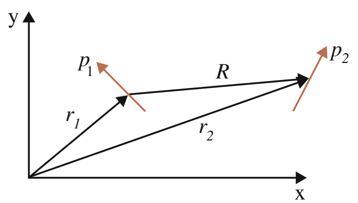
which should be maximized w.r.t. θ_2:

$$\begin{aligned} &-m_1 m_2 \sin(\theta_1 - \theta_2) - 3m_1 m_2 \cos\theta_1 \sin\theta_2 = 0 \\ &-m_1 m_2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2) - 3m_1 m_2 \cos\theta_1 \sin\theta_2 = 0 \\ &-\tan\theta_1 \cot\theta_2 - 2 = 0 \\ &\tan\theta_2 = -\tfrac{1}{2}\tan\theta_1 \end{aligned}$$

*7.16 a. *Force* is given by the gradient of energy

$$U = \frac{k}{R^3}[\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})]$$

where $k = \frac{1}{4\pi\varepsilon_0}$ in the electric case and $k = -\frac{\mu_0}{4\pi}$ in the magnetic case and p is electric *or* magnetic dipole moment, see figure.



Force on object 2 becomes

$$\bar{F}_{1\to 2} = \pm k \nabla_2 \left(\frac{1}{R^3} [\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})] \right)$$

where the plus sign is valid for magnetic interaction and the minus sign for electric.

$$\nabla_2 \left(\frac{1}{R^3} [\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})] \right) = \nabla_2 \left(\frac{1}{R^3} \bar{p}_1 \cdot \bar{p}_2 - \frac{3}{R^5} (\bar{p}_1 \cdot \bar{R})(\bar{p}_2 \cdot \bar{R}) \right)$$
$$= -3 \frac{1}{R^4} (\bar{p}_1 \cdot \bar{p}_2) \hat{R} + \frac{15}{R^6} (\bar{p}_1 \cdot \bar{R})(\bar{p}_2 \cdot \bar{R}) \hat{R} - \frac{3}{R^5} \nabla_2 (\bar{p}_1 \cdot \bar{R})(\bar{p}_2 \cdot \bar{R})$$

The third term contains

$$\nabla_2 (\bar{p}_1 \cdot \bar{R})(\bar{p}_2 \cdot \bar{R}) = (\bar{p}_2 \cdot \bar{R}) \nabla_2 (\bar{p}_1 \cdot \bar{R}) + (\bar{p}_1 \cdot \bar{R}) \nabla_2 (\bar{p}_2 \cdot \bar{R})$$

The gradient of the scalar product is

$$\nabla_2 (\bar{p}_2 \cdot \bar{R}) = \bar{p}_2$$

so that

$$\nabla_2 (\bar{p}_1 \cdot \bar{R})(\bar{p}_2 \cdot \bar{R}) = (\bar{p}_2 \cdot \bar{R}) \bar{p}_1 + (\bar{p}_1 \cdot \bar{R}) \bar{p}_2$$

and

$$\nabla_2 \left(\frac{1}{R^3} [\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})] \right)$$
$$= -3 \frac{1}{R^4} ((\bar{p}_1 \cdot \bar{p}_2) \hat{R} - 5(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{\hat{R}}) \hat{R} + (\bar{p}_2 \cdot \hat{R}) \bar{p}_1 + (\bar{p}_1 \cdot \hat{R}) \bar{p}_2)$$

The force becomes

$$\bar{F}_{1\to 2} = \mp 3k \frac{1}{R^4} ((\bar{p}_1 \cdot \bar{p}_2 - 5(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R})) \hat{R} + (\bar{p}_2 \cdot \hat{R}) \bar{p}_1 + (\bar{p}_1 \cdot \hat{R}) \bar{p}_2)$$

where the minus sign is valid for the magnetic force and the plus sign for the electric.
The torque may be determined through the derivative of the energy w.r.t. to the turning angle. Alternatively, it is obtained as follows:
The torque in the electric case is given by formula (7.5):

$$\bar{\tau} = -\bar{p} \times \nabla \Phi$$

with the potential given by formula (7.11):

$$\Phi_{dipol} = \frac{\bar{p} \cdot \hat{r}}{4\pi \varepsilon_0 r^2}$$

The torque on dipole 2 becomes

$$\bar{\tau} = -\bar{p}_2 \times \nabla\left(\frac{\bar{p}_1 \cdot \hat{R}}{4\pi\varepsilon_0 R^2}\right)$$

where the distance vector is directed towards dipole 2. The gradient may be taken from formula (7.13) so that using formulas (7.14) and (7.15) we obtain

$$\bar{\tau} = -\frac{1}{4\pi\varepsilon_0 R^3}[\bar{p}_2 \times \bar{p}_1 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R})]$$

The magnetic torque becomes

$$\bar{\tau} = \frac{\mu_0}{4\pi R^3}[\bar{p}_2 \times \bar{p}_1 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R})]$$

b. *The torque* on dipole 2 is

$$\bar{\tau} = -\frac{1}{4\pi\varepsilon_0 R^3}[\bar{p}_2 \times \bar{p}_1 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R})]$$

In this example

$$\bar{p}_1 = p\hat{z}, \qquad \bar{p}_2 = \frac{p}{\sqrt{2}}(\hat{x} + \hat{z}) \quad \hat{R} = \hat{x}$$

The second term vanishes since $\bar{p}_1 \cdot \hat{R} = 0$.

$$\bar{p}_2 \times \bar{p}_1 = \left(\frac{p}{\sqrt{2}}, 0, \frac{p}{\sqrt{2}}\right) \times (0, 0, p) = -(0, \frac{p^2}{\sqrt{2}}, 0)$$

which gives the torque

$$\bar{\tau} = \frac{1}{4\pi\varepsilon_0 x^3}\frac{p^2}{\sqrt{2}}\hat{y}$$

The force on dipole 2 is

$$\bar{F}_{1\to 2} = \frac{3}{4\pi\varepsilon_0}\frac{1}{R^4}((\bar{p}_1\cdot\bar{p}_2 - 5(\bar{p}_1\cdot\hat{R})(\bar{p}_2\cdot\hat{R}))\hat{R} + (\bar{p}_2\cdot\hat{R})\bar{p}_1 + (\bar{p}_1\cdot\hat{R})\bar{p}_2)$$

where p_1 is perpendicular to R so that the second and the fourth terms vanish.

$$\bar{F}_{1\to 2} = \frac{3}{4\pi\varepsilon_0}\frac{1}{R^4}((\bar{p}_1 \cdot \bar{p}_2)\hat{R} + (\bar{p}_2 \cdot \hat{R})\bar{p}_1)$$

$$= \frac{3p^2}{4\pi\varepsilon_0 x^4}(\cos 45^\circ \hat{x} + \cos 45^\circ \hat{z}) = \frac{3p^2}{4\sqrt{2}\pi\varepsilon_0 x^4}(\hat{x} + \hat{z})$$

7.17 a. Magnetic energy is given by formula (7.25):

$$U_m = -\frac{\mu_0}{4\pi R^3}[\bar{m}_1 \cdot \bar{m}_2 - 3(\bar{m}_1 \cdot \hat{R})(\bar{m}_2 \cdot \hat{R})]$$

The dipole moment for the loop is

$$\bar{m}_1 = IA\hat{z}$$

where the z axis is vertically upwards. Using the result in Exercise (7.14) the dipole moment for the rotating sphere is

$$\bar{m}_2 = \pm\frac{2}{15}\rho\omega a^5\hat{z}$$

For parallel dipoles the energy becomes

$$U = \frac{\mu_0}{2\pi R^3}m_1 m_2 = \frac{\mu_0}{15\pi R^3}IA\rho\omega a^5$$

and for anti-parallel dipoles

$$U = -\frac{\mu_0}{15\pi R^3}IA\rho\omega a^5$$

Stable equilibrium appears for the largest magnetic energy, i.e. for parallel dipoles.
In quantum mechanical spin resonance the two states alternate where the latter case, i.e. anti-parallel dipoles, is a so-called excited state.

b. The dipole moments are

$$\bar{m}_1 = I_1 A_1 \hat{n}_1, \quad \bar{m}_2 = I_2 A_2 \hat{n}_2$$

Parallel:

$$U = \frac{\mu_0}{2\pi R^3}m_1 m_2 = \frac{\mu_0}{2\pi R^3}I_1 A_1 I_2 A_2$$

Anti-parallel:

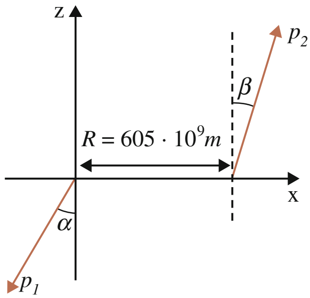
$$U = -\frac{\mu_0}{2\pi R^3}m_1 m_2$$

Perpendicular:

$$U = 0$$

since one dipole moment is perpendicular to the distance vector and the dipoles are perpendicular to each other.

7.18 a. Let the dipoles be placed on the x axis. The figure shows orientation and location of the planets' dipole moments with $\alpha = 35.0°$ and $\beta = 12.7°$.



The magnetic force was determined in Exercise (7.16):

$$\bar{F}_{1\to 2} = \frac{3\mu_0}{4\pi}\frac{1}{R^4}((\bar{p}_1 \cdot \bar{p}_2 - 5(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \cdot \hat{R}))\hat{R} + (\bar{p}_2 \cdot \hat{R})\bar{p}_1 + (\bar{p}_1 \cdot \hat{R})\bar{p}_2)$$

The angle between the dipole moments is $\phi = \alpha - \beta + 180°$ and the magnetic force becomes

$$\begin{aligned}
\bar{F}_{1\to 2} &= \frac{3\mu_0}{4\pi}\frac{1}{R^4}((p_1 p_2 \cos(\alpha + 180° - \beta) - 5p_1(\cos(90° + \alpha))p_2(\cos(90° - \beta)))\hat{R} \\
&\quad + p_2(\cos(90° - \beta))\bar{p}_1 + p_1 \cos(90° + \alpha)\bar{p}_2) \\
&= \frac{3\mu_0}{4\pi}\frac{p_1 p_2}{R^4}[(\cos 202.3° - 5\cos 122.5° \cos 77.3°)\hat{R} + \cos 77.3° \hat{p}_1 + \cos 122.5° \hat{p}_2] \\
&= \frac{3\mu_0}{4\pi}\frac{p_1 p_2}{R^4}[(\cos 202.3° - 5\cos 122.5° \cos 77.3°)\hat{x} \\
&\quad + \cos 77.3°(-\hat{x}\cos 35° - \hat{z}\sin 35°) + \cos 122.5°(\hat{x}\sin 12.7° + \hat{z}\cos 12.7°)] \\
&= 3 \times 10^{-7}\frac{13.2 \times 10^{49}}{605^4 \times 10^{36}}(-0.335\hat{x} - 0.180\hat{x} - 0.126\hat{z} - 0.118\hat{x} - 0.524\hat{z}) \\
&= -2.96 \times 10^{-4}(0.633\hat{x} + 0.650\hat{z})N
\end{aligned}$$

The gravitational force is

$$F_{gr} = G\frac{M_1 M_2}{R^2} = 6.67 \times 10^{-11}\frac{5.97 \times 1.90 \times 10^{51}}{605^2 \times 10^{18}}N = 2.07 \times 10^{18}N$$

i.e. 22 orders of magnitude larger than the magnetic force.

b. The earth is approximately a magnetic dipole, as such originating from a spherical or approximately a circular current distribution, whose source presumably is the inner part of the planet consisting of a significant amount of iron. Due to an external magnetic influence, presumably from the sun, a displacement of the conduction electrons in the iron occurs, caused by the rotation of the earth. This rotation also causes the displaced charges to move in circles and thereby give rise to geomagnetism. This model of the geomagnetism is the conventional one and is called the dynamo model.

c. Nobody knows for certain but a well-educated guess could be that Venus doesn't contain metal. A special condition with Venus is that it rotates in the opposite direction relative to other planets, indicating a unique origin.

d. Magnetic torque on object 2 is

$$\bar{\tau} = \frac{\mu_0}{4\pi R^3}[\bar{p}_2 \times \bar{p}_1 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R})]$$

Let the magnetic cell of the crane be object 2. For perpendicular moments and with distance vector R in parallel to the geomagnetic moment, we obtain

$$\tau = -\frac{\mu_0}{2\pi R^3} p_2 p_1 = -2 \times 10^{-7} \times 10^{-13} \times 8.8 \times 10^{22}/6.39 \times 10^6 = 2.8 \times 10^{-4}\,\text{Nm}$$

directed perpendicular to the moment of the crane's cell.

D.8 Material Properties

8.1 a. Energy is given by

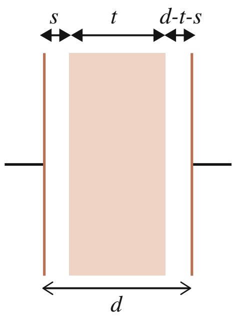
$$U_e = \frac{1}{2}C(\Delta\Phi)^2$$

Using index 0 for quantities without material and index m with material and relating to formulas (8.1) and (8.2) we obtain

$$\frac{U_{e0}}{U_{em}} = \frac{C_0(\Delta\Phi_0)^2}{C_m(\Delta\Phi_m)^2} = \frac{C_0(\Delta\Phi_0)^2}{\kappa_e C_0(\Delta\Phi_0/\kappa_e)^2} = \kappa_e$$

b. Since the material acts so that the voltmeter effectively sees less charge, the charge will increase to maintain the voltage. Accordingly, the voltage is unchanged and the capacitance increases.

8.2 a. Placing the material at an arbitrary position gives three series coupled capacitors:
C(s), C(t) and $C(d - t - s)$.



The capacitance over the length $s + t$ becomes (compare Exercise 5.5)

$$C(s+t) = \varepsilon_0 A \frac{\kappa_e}{\kappa_e s + t}$$

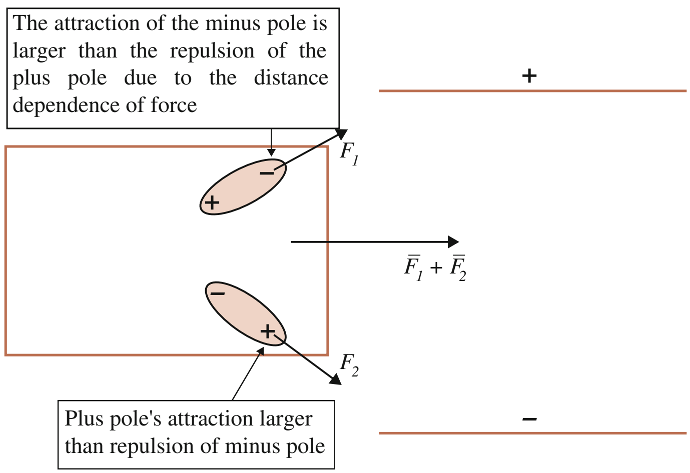
Total capacitance becomes

$$C(d) = \frac{C(t+s)C(d-t-s)}{C(t+s)+C(d-t-s)} = \frac{\varepsilon_0 A \frac{\kappa_e}{\kappa_e s+t} \varepsilon_0 A \frac{1}{d-s-t}}{\varepsilon_0 A \frac{\kappa_e}{\kappa_e s+t} + \varepsilon_0 A \frac{1}{d-s-t}}$$
$$= \frac{\varepsilon_0 A \frac{\kappa_e}{\kappa_e s+t} \frac{1}{d-s-t}}{\frac{\kappa_e(d-s-t)+\kappa_e s+t}{(\kappa_e s+t)(d-s-t)}} = \varepsilon_0 A \frac{\kappa_e}{\kappa_e(d-t)+t}$$

independent of position s.

b. Yes, it follows directly from Exercise (5.5).

8.3



8.4 a. Total interaction energy is (see Sect. 7.2)

$$U = \frac{1}{2}\int_V \bar{P} \cdot \nabla \Phi dV$$

corrected with a factor 1/2 since the polarisation is induced, see Sect. 8.2.2. The polarisation is given by formula (8.26) and the potential by (8.25). With the left end of the cylinder at $z = b$ and the right end at $z = c$ the energy becomes

$$U = \frac{1}{2}\varepsilon_0\left(\frac{1}{\kappa_e} - 1\right)\left(\frac{\lambda}{2\pi\varepsilon_0}\right)^2 \int_b^c \int_0^a \int_0^{2\pi} \frac{1}{(d-z)^2}\rho d\phi d\rho dz$$

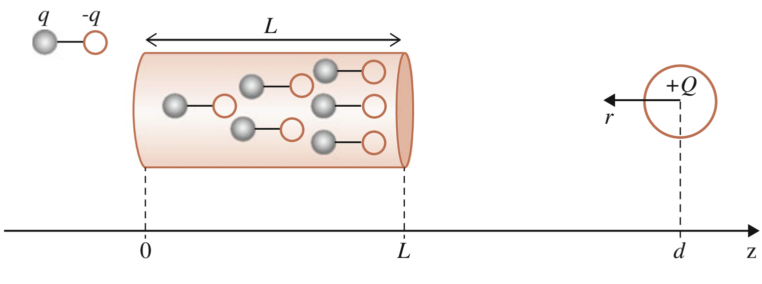
$$= \frac{1}{2}\varepsilon_0\left(\frac{1}{\kappa_e} - 1\right)\left(\frac{\lambda}{2\pi\varepsilon_0}\right)^2 \pi a^2\left(\frac{1}{d-b-L} - \frac{1}{d-b}\right)$$

where a is the radius of the cylinder and $L = c - b$ is its length. The force on the cylinder becomes

$$\bar{F} = -\frac{d}{db}U\hat{z} = -\frac{1}{2}\varepsilon_0\left(\frac{1}{\kappa_e} - 1\right)\left(\frac{\lambda}{2\pi\varepsilon_0}\right)^2 \pi a^2\left(\frac{1}{(d-b-L)^2} - \frac{1}{(d-b)^2}\right)\hat{z}$$

which is an attractive force

8.5 a.



b. A coordinate axis pointing radially out from the sphere is introduced, see figure. Using formulas (4.17) and (3.4), the potential outside the sphere along the axis of the cylinder may then be written

$$\Phi_f = \frac{Q}{4\pi\varepsilon_0 r}$$

so that

$$\nabla\Phi_f = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0(d-z)^2}\hat{z}$$

to the left of the sphere. The polarisation direction becomes approximately parallel to the cylinder axis so that formula (8.20) may be used:

$$\bar{P} = \varepsilon_0 \nabla \Phi_f \left(\frac{1}{\kappa_e} - 1 \right) = \varepsilon_0 \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi \varepsilon_0 (d-z)^2} \hat{z}$$
$$= \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2 \left(1 - \frac{z}{d}\right)^2} \hat{z} \approx \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2} (1 + 2\frac{z}{d}) \hat{z}$$

Total dipole moment becomes

$$\bar{p} = \int_V \bar{P} dV = \int_V \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2} (1 + 2\frac{z}{d}) \hat{z} dV = \pi a^2 \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2} [z + \frac{2}{d}\frac{z^2}{2}]_0^L \hat{z}$$
$$= \pi a^2 \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2} (L + \frac{L^2}{d}) \hat{z}$$

c. The surface charge density on the plates is

$$\sigma_b = \bar{P} \cdot \hat{n}$$
$$\sigma_b(z = L) = \left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2} (1 + 2\frac{L}{d})$$
$$\sigma_b(z = 0) = -\left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{4\pi d^2}$$

and on the mantle

$$\sigma_b(\rho = a) = 0$$

Note that the surface normal is negative at $z = 0$.
The volume density of charge becomes

$$\rho_b = -\nabla \cdot \bar{P} = -\frac{dP_z}{dz} = -\left(\frac{1}{\kappa_e} - 1 \right) \frac{Q}{2\pi d^3}$$

The total amount of charge vanishes.

d. Since the interaction takes place at a long distance, the cylinder may be regarded as a dipole and the sphere as a point charge. Treating the cylinder as a point-like dipole placed at $z = 0$ and using formula (7.7), the energy becomes

$$U = \frac{1}{2} \bar{p} \cdot \nabla \Phi = -\frac{1}{2} \frac{Q}{4\pi \varepsilon_0 d^2} \hat{r} \cdot \bar{p} = \frac{1}{2} \frac{Q}{4\pi \varepsilon_0 d^2} \hat{z} \cdot \bar{p}$$

corrected by a factor $1/2$ as in Exercise (8.4). Using formula (7.10), the force on the dipole (the cylinder) becomes

$$\bar{F} = \frac{1}{2}\frac{Q}{4\pi\varepsilon_0}(-3\frac{\bar{p}\cdot\hat{z}}{d^3}\hat{z} + \frac{\bar{p}}{d^3}) = \frac{1}{2}\frac{Q\pi a^2}{4\pi\varepsilon_0 d^3}\left(-2\left(\frac{1}{\kappa_e}-1\right)\frac{Q}{4\pi d^2}(L+\frac{L^2}{d})\right)\hat{z}$$
$$= -\frac{1}{2}\frac{Q^2a^2L}{8\pi\varepsilon_0 d^5}\left(\frac{1}{\kappa_e}-1\right)\left(1+\frac{L}{d}\right)\hat{z}$$

which is a positive force, as expected.

8.6 a. The polarisation is given by formula (8.33)

$$\bar{P} = -3\varepsilon_0\frac{\kappa_e-1}{\kappa_e+2}\nabla\Phi_f = 3\sigma\frac{\kappa_e-1}{\kappa_e+2}\hat{z}$$

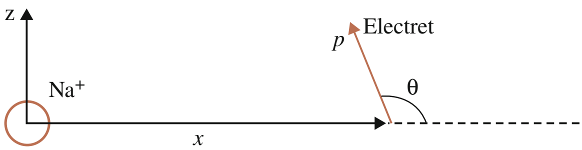
where $\hat{z}$ is directed from the plus plate to the minus plate. The dipole moment becomes

$$\bar{p} = \int_V \bar{P}dV = \int_V 3\sigma\frac{\kappa_e-1}{\kappa_e+2}dV\hat{z} = 4\pi a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}\hat{z}$$

b. There is no net charge inside the volume since $\rho_b = -\nabla\cdot\bar{P}$ and the polarisation is homogeneous. On the surface of the sphere the charge density is

$$\sigma_b = \bar{P}\cdot\hat{n} = 3\sigma\frac{\kappa_e-1}{\kappa_e+2}\hat{z}\cdot\hat{n}$$

c.



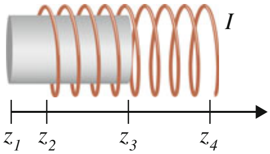
The objects are considered as a point charge and a dipole. The force on the point charge, i.e. the sodium ion, is given by formula (7.10) with opposite direction, see Exercise (7.2).

$$\bar{F} = -\frac{q}{4\pi\varepsilon_0}\left(-3\frac{\bar{p}\cdot\hat{x}}{x^3}\hat{x} + \frac{\bar{p}}{x^3}\right)$$
$$= -\frac{q}{4\pi\varepsilon_0 x^3}\left(-3\left(4\pi a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}\right)(\hat{p}\cdot\hat{x})\hat{x} + 4\pi a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}\hat{p}\right)$$
$$= -\frac{q}{\varepsilon_0 x^3}a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}(-3(\hat{p}\cdot\hat{x})\hat{x}+\hat{p}) = \frac{q}{\varepsilon_0 x^3}a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}(3\cos\theta\hat{x}-\hat{p})$$

Using formula (7.5) and Exercise (7.2), the torque becomes

$$\bar{\tau} = \frac{q}{4\pi\varepsilon_0 x^2}\bar{p}\times\hat{x} = \frac{q}{\varepsilon_0 x^2}a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}\hat{p}\times\hat{x} = \frac{q}{\varepsilon_0 x^2}a^3\sigma\frac{\kappa_e-1}{\kappa_e+2}p\sin\theta\hat{y}$$

8.7



Introducing four free coordinates as in the figure, formula (8.40) becomes

$$U_m = \frac{1}{2}I^2[\kappa_m\mu_0 n^2 S(z_3 - z_2) + \mu_0 n^2 S(z_4 - z_3)]$$
$$= \frac{1}{2}I^2[\kappa_m\mu_0 n^2 S(z_3 - z_2) + \mu_0 n^2 S(d + z_2 - z_3)]$$

where d is the length of the coil. Denoting the coil as object 2 and the material as object 1 the force on the coil becomes

$$\bar{F}_{1\to 2} = \nabla_2 U_m = \frac{d}{dz_2}U_m\hat{z} = -\frac{1}{2}I^2(\kappa_m - 1)\mu_0 n^2 S\hat{z}$$

and the force on the material

$$\bar{F}_{2\to 1} = \nabla_3 U_m = \frac{d}{dz_3}U_m\hat{z} = \frac{1}{2}I^2(\kappa_m - 1)\mu_0 n^2 S\hat{z}$$

8.8 Formula (8.45) is

$$\frac{1}{2}(\bar{r}\times\bar{J}) = \bar{M}(\bar{r})$$
$$\nabla_r \times (\bar{r}\times\bar{J}) = (\bar{J}\cdot\nabla)\bar{r} - (\bar{r}\cdot\nabla)\bar{J} + \bar{r}(\nabla\cdot\bar{J}) - \bar{J}(\nabla\cdot\bar{r}) = (\bar{J}\cdot\nabla)\bar{r} - \underbrace{(\bar{r}\cdot\nabla)\bar{J}}_{=0} - 3\bar{J}$$

Express in terms of coordinates

$$(\bar{J}\cdot\nabla)\bar{r} = \left(J_x\frac{d}{dx} + J_y\frac{d}{dy} + J_z\frac{d}{dz}\right)(x, y, z) = (J_x, J_y, J_z) = \bar{J}$$

The second term $-(\bar{r}\cdot\nabla)\bar{J} = 0$ since the position vector $\bar{r} = 0$. Thus

$$\frac{1}{2}\nabla_r \times (\bar{r}\times\bar{J}) = \frac{1}{2}(-2\bar{J})$$

so that

$$\bar{J} = -\nabla\times\bar{M}$$

8.9 The energy is given by formula (8.66), modified for actual conditions

$$U_{int} = \frac{\mu_0}{2\pi^2(x_2 - x_1)^6}(\kappa_m - 1)\left(NI_f S_f\right)^2 V$$

The force on the material is

$$\begin{aligned}\bar{F}_{2\to 1} &= \frac{d}{dx_1}U_{int}\hat{x} = \frac{6\mu_0}{2\pi^2(x_2 - x_1)^7}(\kappa_m - 1)\left(NI_f S_f\right)^2 V\hat{x} \\ &= (\kappa_m - 1)\frac{12\times 10^{-7}}{\pi 0.2^7}10^6 \times 9 \times 10^{-4}\pi \cdot 0.25 \times 10^{-4} \times 0.03N\hat{x} \\ &= (\kappa_m - 1)6.3 \times 10^{-5}N\hat{x}\end{aligned}$$

8.10 Horizontal force may be measured by means of a torsional balance as in Exercise (2.5). Sources of error concern mainly the calibration of the wire, i.e. the determination of the torsional constant.
Using balances of different kinds, the force may be determined vertically by comparing with the object's weight. The calibration is also in this case the main error source. Using a spring balance the calibration aims at determining the spring constant. Other types of balances are calibrated using e.g. a reference weight which in turn has to be determined somehow.

8.11 a. According to Exercise (8.9) the force between a material and an external influence is in the dipole approximation

$$F_{2\to 1} = (\kappa_m - 1)\frac{3\mu_0}{\pi^2(x_2 - x_1)^7}\left(NI_f S_f\right)^2 V$$

so that the following table can be constructed

Material	κ_m
Para	Small and greater than one
Ferro	Large and greater than one
Dia	Just below one
Super	Close to zero

b. In the particle model of a microcosm the motion of the atomic electrons are described by closed orbits around the atomic nucleus, so-called orbital motion, and rotation around its own axis, so-called spin. In addition to these motional types, the conduction electrons exhibit an almost *free* motion which is affected by the slightest influence.

c. Orbital motion and spin shall be considered as closed *and* with maintained current, equivalent to a current loop driven by a battery. When interacting with an external influence, magnetic energy should then be maximized. Therefore, the dipoles of the material align parallel to the dipole moment of the external influence and an attractive force occur. Compare formula (7.25).

The third type of motion, the free motion of conduction electrons, is not maintained but is a pure inductive effect. Basically it is caused by Weber's acceleration dependent force, formula (2.22), however mostly discussed in terms of Faraday-Henry's induction law (3.33). Thus, when an external influence with constant current approaches the material, the material responds with an oppositely directed current and the two objects repel.

d. If the resistance vanishes, the conduction electrons may move completely freely leading to maximal diamagnetism.
e. The principle of the magnet break is based on the repelling force diamagnetism gives rise to.
f. In the water molecule the dipole moments from the closed electron motions, i.e. the orbital and spin moments, are cancelled. Since the water molecule is a permanent electric dipole it follows that the electrons posses a certain freedom to move. As in metal this freedom causes diamagnetism. According to Table (9.2), the diamagnetism of water is stronger than for e.g. copper.

8.12 a. This is due to internal interactions between the dipoles of the material.
b. The material responds with a dipole moment in a different direction than that of the external influence. This effect is formalised by letting the material constants become a matrix in three dimensions:

$$\kappa = \begin{pmatrix} \kappa_{xx} & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & \kappa_{yy} & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & \kappa_{zz} \end{pmatrix}$$

so that an influence in e.g. x direction might give a response in the y or/and z direction.
c. In the electric case, the voltage over the material is measured in different directions. In the magnetic case the magnetization in different directions may be measured using e.g. a Hall sensor, see Exercise (8.14).

8.13 The torque on dipole 2 caused by dipole 1 is obtained from Exercise (7.16):

$$\begin{aligned} \bar{\tau} &= -\frac{1}{4\pi\varepsilon_0 R^3}[\bar{p}_2 \times \bar{p}_1 - 3(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R})] \\ &= \frac{3}{4\pi\varepsilon_0 R^3}(\bar{p}_1 \cdot \hat{R})(\bar{p}_2 \times \hat{R}) \end{aligned}$$

since the dipoles are parallel.
The distance vector R is directed towards the middle dipole.

1. The torque from the lower right and upper left becomes

$$\begin{aligned} \tau &= \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 45°)\sin(\theta + 45°) + \cos(-\theta + 135°)\sin(-\theta + 135°)) \\ &= \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 45°)\sin(\theta + 45°) - \cos(\theta - 135°)\sin(\theta - 135°)) \end{aligned}$$

$$= \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 45^\circ)\sin(\theta + 45^\circ) - \cos(\theta + 45^\circ)\sin(\theta + 45^\circ)) = 0$$

2. The torque from the upper right and lower left becomes

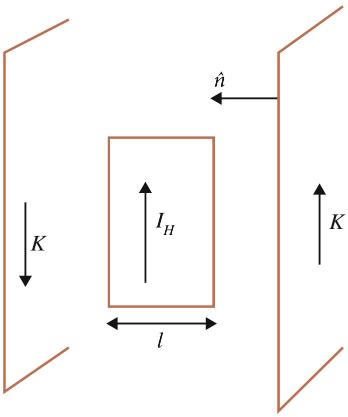
$$\tau = \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 135^\circ)\sin(\theta + 135^\circ) + \cos(-\theta + 45^\circ)\sin(-\theta + 45^\circ))$$
$$= \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 135^\circ)\sin(\theta + 135^\circ) - \cos(\theta - 45^\circ)\sin(\theta - 45^\circ))$$
$$= \frac{3p^2}{4\pi\varepsilon_0 R^3}(\cos(\theta + 135^\circ)\sin(\theta + 135^\circ) - \cos(\theta + 135^\circ)\sin(\theta + 135^\circ)) = 0$$

*8.14 The force between a straight current and two parallel large current-carrying plates is twice that of formula (4.46)

$$\bar{F} = I_H \bar{L} \times \mu_0(\bar{K} \times \hat{n})$$

where L is directed along the Hall current. The vectors are perpendicular so that

$$\bar{F} = -I_H L \mu_0 K \hat{n}$$



The magnetic force on a single electron is

$$\bar{f}_e = nevAdL\mu_0 K\hat{n} = ev\mu_0 K\hat{n}$$

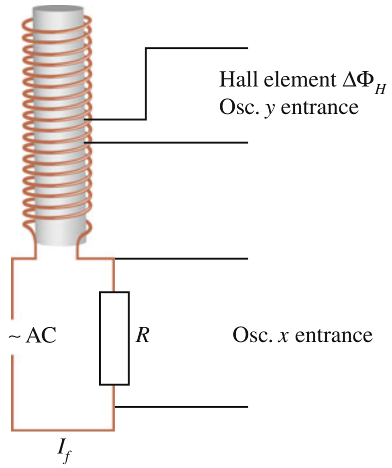
where e is electron charge with a positive sign. The force along the Hall element becomes

$$f_{eH} = ev\mu_0 K \cos\theta$$

where θ is the angle between the surface normals of the Hall element's front side and the cross section of the coil. This force is balanced by an electric force. Using formula (4.22), voltage is defined from

$$e\Delta\Phi = lev\mu_0 K\cos\theta$$
$$\Delta\Phi = lv\mu_0 K\cos\theta$$

*8.15 a.



b. For ferromagnetic materials a bound current I_b is induced in the same direction as the inducing current I_f. The Hall sensor is therefore influenced by a current corresponding to the sum of these two currents. Using the result obtained in Exercise (8.14) we obtain

$$\Delta\Phi_H = lv\mu_0 K = lv\mu_0(K_f + K_b) = lv\mu_0\left(\frac{NI_f}{l_s} + \frac{I_b}{l_m}\right)$$

l_s is the length of the coil and l_m is the length of the material.
The relation between bound and free (inducing) current is obtained from formula (8.59). For $l_s = l_m$ we obtain

$$K_b = (\kappa_m - 1)K_f$$

where the currents in formula (8.59) have been replaced by current densities.
The Hall voltage becomes

$$\Delta\Phi_H = lv\mu_0(K_f + (\kappa_m - 1)K_f) = lv\mu_0\kappa_m K_f$$

c. The slope of the curve is $lv\mu_0\kappa_m$
At the maximum value of I_f in the figure, the slope is $lv\mu_0$ because then the permeability $\kappa_m = 1$. The reason is that at this state the material

is saturated, i.e. all its dipoles are maximally aligned meaning that the material cannot respond any further.

d. Using formula (8.53), the remanent magnetization M^r is given by

$$\Delta\Phi_0 = lv\mu_0 K_b^r = lv\mu_0 M^r$$

where $\Delta\Phi_0$ is the Hall voltage at $I_f = 0$.

8.16 a. Let $C_R = \alpha C_0$, where $C_D = \kappa_e C_0$. Since the two capacitors possess equal charge we obtain

$$C_0\Delta\Phi_0 = C_R\Delta\Phi_R$$
$$\frac{\Delta\Phi_0}{\kappa_e} = \frac{C_R}{\kappa_e C_0}\Delta\Phi_R$$

so that

$$\Delta\Phi_D = \frac{\alpha}{\kappa_e}\Delta\Phi_R$$

For the Sawyer-Tower circuit the source voltage is

$$\Delta\Phi_S = \Delta\Phi_D + \Delta\Phi_R$$

Since $C_D \ll C_R$ the source voltage is approximately

$$\Delta\Phi_S \approx \Delta\Phi_D$$

so that if the oscilloscope channels are fed by $\Delta\Phi_R$ versus $\Delta\Phi_S$ it displays a curve with the slope κ_e/α.

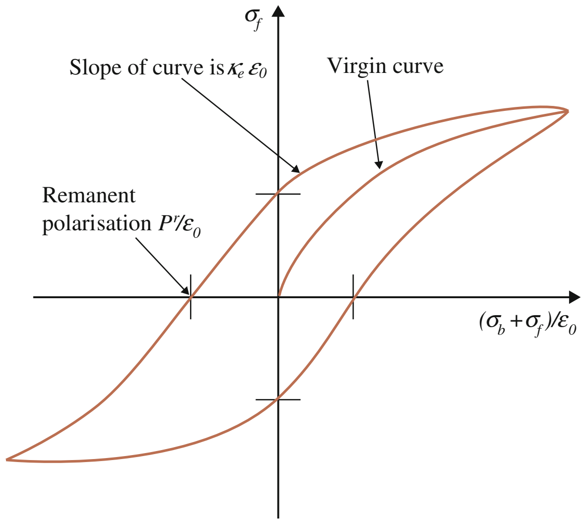
b. The axes of the hysteresis curve become, see figure

$$\sigma_f = \frac{C_R\Delta\Phi_R}{A} = \varepsilon_0\frac{C_R}{C_0}\frac{\Delta\Phi_R}{d} = \varepsilon_0\alpha\frac{\Delta\Phi_R}{d}$$

and

$$\frac{\sigma_b + \sigma_f}{\varepsilon_0} = \frac{\Delta\Phi_D}{d}$$

so that the slope of the hysteresis curve is $\varepsilon_0\kappa_e$.



The remanent polarisation is obtained when $\Delta\Phi_S = 0$. From formula (8.19) $P = \sigma_b$ so that

$$P^r = \varepsilon_0 \frac{\Delta\Phi_D(\Delta\Phi_R = 0)}{d}$$

i.e. the crossing with the horizontal axis gives P^r. However, it is customary that $\kappa_e \gg 1$ so that

$$\sigma_b = \sigma_f \left(\frac{1}{\kappa_e} - 1\right) \approx -\sigma_f$$

i.e. P^r is then also given by the crossing on the vertical axis.

c. Ferroelectric (as ferromagnetic) materials are non-linear, i.e. the dielectric constant varies with the strength of the external influence. Consider first a linear material. The interaction energy between the material and the external influence is given by

$$U_e = \frac{1}{2}\int_V \bar{P} \cdot \nabla\Phi_f dV$$

where V is the volume of the material and the factor 1/2 has been introduced since the dipoles are induced. Formula (8.24) gives

$$\bar{P} = -\varepsilon_0(\kappa_e - 1)\nabla\Phi_D$$

so that

$$U_e = -\frac{1}{2}\varepsilon_0 \int\limits_V (\kappa_e - 1)\nabla\Phi_D \cdot \nabla\Phi_f dV$$

Assuming the two gradients to be parallel we obtain

$$U_e = -\frac{1}{2}\varepsilon_0 \int\limits_V (\kappa_e - 1)\nabla\Phi_D \cdot \nabla\Phi_f dV = -\frac{1}{2}\varepsilon_0 \int\limits_V (\kappa_e - 1)\frac{\sigma_b + \sigma_f}{\varepsilon_0}\frac{\sigma_f}{\varepsilon_0} dV$$
$$= -\frac{1}{2}\varepsilon_0 \int\limits_V (\kappa_e - 1)\frac{\sigma_f}{\kappa_e\varepsilon_0}\frac{\sigma_f}{\varepsilon_0} dV = -\frac{1}{2}\varepsilon_0 \int\limits_V \frac{\sigma_f}{\varepsilon_0}\frac{\sigma_f}{\varepsilon_0} dV + \frac{1}{2}\varepsilon_0 \int\limits_V \frac{\sigma_f}{\kappa_e\varepsilon_0}\frac{\sigma_f}{\varepsilon_0} dV$$

which is thus the interaction energy. Term no. 2 corresponds to the energy in the material due to the external influence. This is now expressed as a volume density:

$$w_{mat} = \frac{1}{2}\varepsilon_0 \frac{\sigma_f}{\kappa_e\varepsilon_0}\frac{\sigma_f}{\varepsilon_0} = \frac{1}{2}\frac{\sigma_b + \sigma_f}{\varepsilon_0}\sigma_f$$

Consider next a general material, i.e. a non-linear material. The total work done on the material is obtained in the general case by summing small steps of the change of the external influence. This means that the work done over one cycle may be written

$$w_{mat} = \frac{1}{\varepsilon_0} \oint\limits_{Curve} (\sigma_b + \sigma_f) d(\sigma_f)$$

where the factor $1/2$ has been removed since it is a result of the integration process for linear materials.
The integral corresponds to the surface enclosed by the hysteresis curve. Since the curve is closed no energy has been stored in the material. The work done has been converted to heat.

8.17 a. According to Exercise (6.5) the impedance for the capacitor is

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega\kappa_e^c C_0} = \frac{1}{j\omega(\kappa_R + j\kappa_I)C_0} = -\frac{j\kappa_R + \kappa_I}{\omega(\kappa_R^2 + \kappa_I^2)C_0}$$

The resistive impedance is the real part of the impedance

$$R = Re\, Z_C = -\frac{\kappa_I}{\omega(\kappa_R^2 + \kappa_I^2)C_0}$$

b. The source is current in the isolator, due to free charges and dipole vibrations.
c. The heat power is $RI^2 = \Delta\Phi^2/R$

*8.18 a. 1. Formula (8.24) relates polarisation to potential for air as surrounding and for objects without form (large plate).

$$\bar{P} = -\varepsilon_0(\kappa_e - 1)\nabla\Phi \qquad (8.24)$$

2. Formula (8.33) is valid for polarisation of a dielectric *sphere* in air with a homogeneous influence.

$$\bar{P} = -3\varepsilon_0 \frac{\kappa_e - 1}{\kappa_e + 2} \nabla\Phi_f \qquad (8.33)$$

3. Formula (8.37) modifies (8.24) for a surrounding with arbitrary dielectric constant κ_{w}

$$\bar{P} = \varepsilon_0 \nabla\Phi_w \left(\frac{\kappa_w}{\kappa_p} - 1\right) \qquad (8.37)$$

First, formula (8.37) is reformulated to be equivalent to (8.24), i.e. the potential is given inside the material. Using formula (8.36)

$$\nabla\Phi_p = \frac{\kappa_w}{\kappa_p} \nabla\Phi_w \qquad (8.36)$$

Formula (8.37) is rewritten as

$$\bar{P} = \varepsilon_0 \frac{\kappa_p}{\kappa_w} \nabla\Phi_p \left(\frac{\kappa_w}{\kappa_p} - 1\right) = -\varepsilon_0 \left(\frac{\kappa_p}{\kappa_w} - 1\right) \nabla\Phi_p$$

Thus, this formula is obtained by making the following replacement in formula (8.24):

$$\kappa_e \rightarrow \frac{\kappa_p}{\kappa_w}$$

which is now done in formula (8.33):

$$\bar{P} = -3\varepsilon_0 \frac{\kappa_p/\kappa_w - 1}{\kappa_p/\kappa_w + 2} \nabla\Phi_w = -3\varepsilon_0 \frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w} \nabla\Phi_w$$

valid for objects with spherical form.
Note the index *w* for the electric potential indicating that the potential is given in the surrounding medium.

b. Since the polarisation P corresponds to dipole moment per volume, the dipole moment for a homogeneous sphere under homogeneous influence becomes

$$\bar{p} = \bar{P}\frac{4}{3}\pi a^3 = -4\pi\varepsilon_0 a^3 \frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\nabla\Phi_w$$

which is Clausius-Mossottis' formula.

8.19 Let the index w denote the surrounding medium of the sphere which is air and water respectively. The index p denotes the material of the sphere.
To determine force calculate first energy, given by formula (7.2) corrected by a factor 1/2

$$U = \frac{1}{2}\int_v \bar{P}\cdot\nabla\Phi_w dV$$

The polarisation is taken from Exercise (8.18):

$$\bar{P} = -3\varepsilon_0\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\nabla\Phi_w$$

and Φ_w is the electric potential generated by the wire in the surrounding medium, given by formula (8.25):

$$\Phi_w = \frac{\lambda}{2\pi\varepsilon_0\kappa_w}\ln|x| + \text{constant}$$

The energy becomes

$$U = -\frac{1}{2}3\varepsilon_0\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\left(\frac{\lambda}{2\pi\varepsilon_0\kappa_w}\right)^2\int_V \frac{1}{x^2}dV$$

For a small sphere with radius a the energy is

$$U = -\frac{1}{2}3\varepsilon_0\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\left(\frac{\lambda}{2\pi\varepsilon_0\kappa_w}\right)^2\frac{1}{x^2}\frac{4}{3}\pi a^3 = -\frac{1}{2\pi\varepsilon_0}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\left(\frac{\lambda}{\kappa_w}\right)^2\frac{1}{x^2}a^3$$

so that the force on the sphere becomes

$$\bar{F} = -\nabla U = -\frac{1}{\pi\varepsilon_0}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\left(\frac{\lambda}{\kappa_w}\right)^2\frac{1}{x^3}a^3\hat{x}$$

giving a repulsive force if $\kappa_w > \kappa_p$, e.g. if the sphere is plastic and the surrounding is water.

8.20 a. The polarisation is given by

$$\bar{P}_p = -3\varepsilon_0\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\nabla\Phi_w$$

where the index p refers to 'particle', i.e. the foreign object. The source of the potential is approximated to a dipole. Using formula (7.11)

$$\Phi_w = \frac{\bar{p}_f \cdot \hat{r}}{4\pi\kappa_w\varepsilon_0 r^2}$$

where index f stands for 'fish'. The gradient of the potential is

$$\nabla\Phi_w = \frac{1}{4\pi\kappa_w\varepsilon_0} \cdot \left[\bar{p}_f \times \left(\nabla \times \frac{\hat{r}}{r^2}\right) + \left(\bar{p}_f \cdot \nabla\right)\frac{\hat{r}}{r^2}\right]$$

where the first term vanishes and the second term gives (compare formula 7.10)

$$\nabla\Phi_w = \frac{1}{4\pi\kappa_w\varepsilon_0}\left(-3\frac{\bar{p}_f \cdot \hat{r}}{r^3}\hat{r} + \frac{\bar{p}_f}{r^3}\right)$$

so that the general formula for the polarisation becomes

$$\bar{P}_p = -\frac{3}{4\pi\kappa_w}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\left(-3\frac{\bar{p}_f \cdot \hat{r}}{r^3}\hat{r} + \frac{\bar{p}_f}{r^3}\right)$$

In this specific case the distance vector is perpendicular to the dipole moment of the fish

$$\bar{P}_p = -\frac{3}{4\pi\kappa_w}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\frac{\bar{p}_f}{r^3}$$

b. The interaction energy between two dipoles is given by formula (7.16), corrected by a factor of $1/2$:

$$U = \frac{1}{2}\frac{1}{4\pi\varepsilon_0 r^3}[\bar{p}_1 \cdot \bar{p}_2 - 3(\bar{p}_1 \cdot \hat{r})(\bar{p}_2 \cdot \hat{r})]$$

Since the foreign object is considered as small its dipole moment becomes

$$\bar{p}_p = \bar{P}_p V = -\frac{a^3}{\kappa_w}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\frac{\bar{p}_f}{r^3}$$

Since both dipole moments are perpendicular to the distance vector, the energy is

$$U = -\frac{1}{2}\frac{p_f^2}{4\pi\varepsilon_0\kappa_w r^6}\frac{a^3}{\kappa_w}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}$$

taking into account that the surrounding is water.
The force on the foreign object is

$$\bar{F} = -\nabla U = \hat{r}\frac{1}{2}\frac{d}{dr}\frac{p_f^2}{4\pi\varepsilon_0 r^6}\frac{a^3}{\kappa_w^2}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w} = -\frac{1}{2}\frac{3p_f^2}{2\pi\varepsilon_0 r^7}\frac{a^3}{\kappa_w^2}\frac{\kappa_p - \kappa_w}{\kappa_p + 2\kappa_w}\hat{r}$$

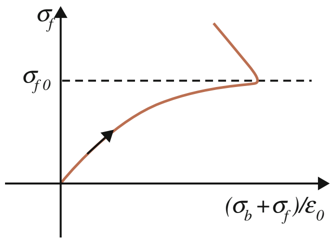
which is a repulsive force for $\kappa_w > \kappa_p$.

c. The potential on the body of the fish caused by the foreign object is given by formula (7.11)

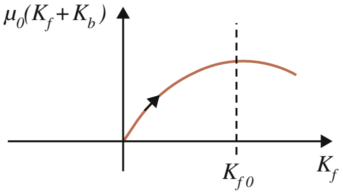
$$\Phi = \frac{\bar{p}_p \cdot \hat{r}}{4\pi\varepsilon_0 r_1^2} = \frac{p_p \cos\theta}{4\pi\varepsilon_0 (x_1^2 + y_1^2)}$$

where $\theta = 90° - \arctan\frac{5}{80}$, $x_1 = 0.05$ and $y_1 = 0.80\,\text{m}$.
A fish of this kind, e.g. the elephant nose fish, is able to sense a potential change of about $1\,\mu\text{V}$.

8.21 In the electric case the external influence appears on the vertical axis. According to Exercise (8.16), the slope of the curve is $\kappa_e\varepsilon_0$. The transition to meta-material occurs at $\sigma_f = \sigma_{f0}$. At stronger external influence, the addition of bound charge increases faster than that of free charge. Remember the two charges have opposite sign.



In the magnetic case the external influence appears on the horizontal axis. The slope of the curve is $\kappa_m\mu_0$. The transition to meta-material appears at $K_f = K_{f0}$. At stronger external influence the addition of bound current increases quicker than that of the free current. The two currents are oppositely directed.



Note that these phenomena are only hypothetical and have not been observed.

8.22 The bar magnet may be considered as a current-carrying coil. Since the bar is homogeneously magnetized currents flow only on the surface. The problem is therefore equivalent to Exercise (4.12) replacing the coil with this bar magnet.
In Exercise (4.12), the force on the loop is

$$\bar{F}_{2\to 1} = -\mu_0 I_1 I_2 \frac{N\pi a^2 b^2}{2l}\left(\frac{1}{\left[(a+b)^2 + \left(\frac{l}{2} - z_1\right)^2\right]^{3/2}} - \frac{1}{\left[(a+b)^2 + (-\frac{l}{2} - z_1)^2\right]^{3/2}}\right)\hat{z}$$

For same current directions the force will be directed towards the center of the coil.
The magnetization of the bar M is given by formula (8.53)

$$\bar{K} = \bar{M} \times \hat{n}$$

The current density $\bar{K} = NI_2/l\,\hat{\phi}$ so that magnetization $\bar{M} = NI_2/l\,\hat{z}$ since $\hat{n} = \hat{\rho}$.
The force is

$$\bar{F}_{2\to 1} = -\bar{M}\mu_0 I_1 \frac{\pi a^2 b^2}{2}\left(\frac{1}{\left[(a+b)^2 + \left(\frac{l}{2} - z_1\right)^2\right]^{3/2}} - \frac{1}{\left[(a+b)^2 + (-\frac{l}{2} - z_1)^2\right]^{3/2}}\right)$$

and magnetization becomes

$$\bar{M} = -\bar{F}_{2\to 1}\left[\mu_0 I_1 \frac{\pi a^2 b^2}{2}\left(\frac{1}{\left[(a+b)^2 + \left(\frac{l}{2} - z_1\right)^2\right]^{3/2}} - \frac{1}{\left[(a+b)^2 + (-\frac{l}{2} - z_1)^2\right]^{3/2}}\right)\right]^{-1}$$

where positive current is directed along $\hat{\phi}$.

D.10 Field Theory

10.1 Formula (8.74) gives the magnetization

$$\bar{M} = (\kappa_m - 1)\frac{m_f}{4\pi R^3}[(2\cos^2\theta_f - \sin^2\theta_f)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}]$$

Using (10.19)

$$\mu_0\bar{M} = (\kappa_m - 1)\frac{\mu_0}{4\pi}\int\limits_{V_f}\frac{\bar{J}_f \times \bar{R}}{R^3}dV_f$$

and (10.20)

$$\bar{H} = \frac{1}{4\pi} \int\limits_{V_f} \frac{\bar{J}_f \times \bar{R}}{R^3} dV_f$$

gives

$$\mu_0 \bar{M} = \mu_0(\kappa_m - 1)\bar{H}$$

so that

$$\begin{aligned}\bar{H} &= \frac{m_f}{4\pi R^3}[(2\cos^2\theta_f - \sin^2\theta_f)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}] \\ &= \frac{I_f A}{4\pi R^3}[(2\cos^2\theta_f - \sin^2\theta_f)\hat{y} + 3\sin\theta_f\cos\theta_f\hat{x}]\end{aligned}$$

where A is the loop area.

10.2 Using formula (8.89), the interaction energy between inducing system and bound charge is

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int\limits_{V_b}\int\limits_{V_f} \frac{\rho_b(\bar{r}_b)\rho_f(\bar{r}_f)}{R} dV_f dV_b$$

where the factor $1/2$ is introduced since the bound charge is induced. The distance vector is

$$\bar{R} = \bar{r}_b - \bar{r}_f$$

Using formula (8.19), i.e. $\rho_b = -\nabla \cdot \bar{P}$, we obtain

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b} \frac{-\nabla_b \cdot \bar{P}\rho_f(\bar{r}_f)}{R} dV_b dV_f$$

where the integration order has been interchanged. Integration by parts over V_b gives

$$\int\limits_{V_b} \frac{-\nabla_b \cdot \bar{P}\rho_f(\bar{r}_f)}{R} dV_b = -\left[\frac{\bar{P}\rho_f}{R}\right]^{S_b} - \int\limits_{V_b} \bar{P}\frac{\rho_f}{R^2}\hat{R} dV_b = -\int\limits_{V_b} \bar{P}\frac{\rho_f}{R^2}\hat{R} dV_b$$

where the first term in the second step vanishes since there is no free charge on the border of V_b. Using formula (10.36), the energy becomes

$$U_e = -\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int\limits_{V_f}\int\limits_{V_b} \bar{P}\frac{\rho_f}{R^2}\hat{R} dV_b dV_f = -\frac{1}{2}\bar{E} \cdot \int\limits_{V_b} \bar{P} dV_b = -\frac{1}{2}\frac{\bar{D}}{\varepsilon_0} \cdot \int\limits_{V_b} \bar{P} dV_b$$

*10.3 a. Integrate (10.59) over a volume and use the divergence theorem

$$\int_V \nabla \cdot \bar{E} dV = \frac{1}{\varepsilon_0} \int_V \rho dV \Rightarrow \oint_S \bar{E} \cdot d\bar{a} = \frac{1}{\varepsilon_0} \int_V \rho dV$$

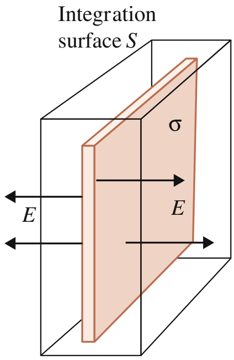
b. Integrate (10.62) over a surface and use Stokes' theorem

$$\int_S \nabla \times \bar{B} \cdot d\bar{a} = \mu_0 \int_S \bar{J} \cdot d\bar{a} \Rightarrow \oint_C \bar{B} \cdot d\bar{l} = \mu_0 \int_S \bar{J} \cdot d\bar{a}$$

c. Integrate (10.60) over a volume and use the divergence theorem

$$\int_V \nabla \cdot \bar{B} = 0 \Rightarrow \oint_S \bar{B} \cdot d\bar{a} = 0$$

*10.4 a. Large plate
By reasons of symmetry the electric field is parallel to the surface normal of the plate, see figure.

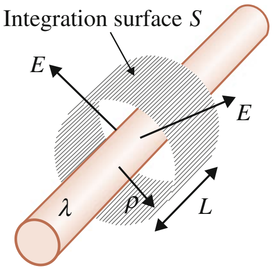


$$\oint_S \bar{E} \cdot d\bar{a} = E2S$$

where the edges of the plate has been disregarded since the plate is large. Thus,

$$E2S = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

b. Long cylinder
By reasons of symmetry the electric field is radially directed as in the figure. The cylinder plates are disregarded since the cylinder is long. We thus obtain



$$\oint_S \bar{E} \cdot d\bar{a} = ES = EL2\pi\rho$$

so that

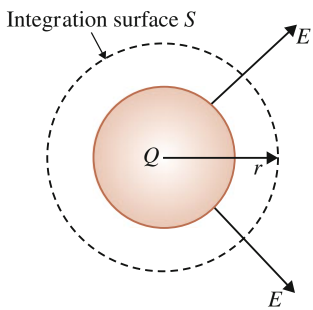
$$EL2\pi\rho = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0\rho}$$

c. Sphere
The electric field is directed radially from the sphere as in the figure. Thus

$$\oint_S \bar{E} \cdot d\bar{a} = ES = E4\pi r^2$$

so that

$$E4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



*10.5 a. Straight long current

Formula (10.2)

$$\bar{B} = \kappa_m \frac{\mu_0}{4\pi} \int_{V_f} \frac{\bar{J}_f \times \bar{R}}{R^3} dV_f = \frac{\mu_0}{4\pi} I \int_{V_f} \frac{d\bar{L} \times \bar{R}}{R^3}$$

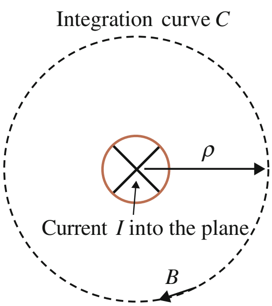
shows that the magnetic field B is perpendicular to the distance vector and current direction. Hence, it forms circles around a straight current. Apply formula (10.86):

$$\oint_C \bar{B} \cdot d\bar{l} = \mu_0 I$$

A circle is formed around the current where the magnetic field is constant:

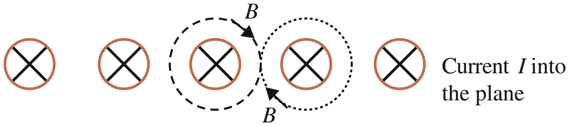
$$\oint_C \bar{B} \cdot d\bar{l} = B 2\pi\rho$$

$$B 2\pi\rho = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi\rho}$$

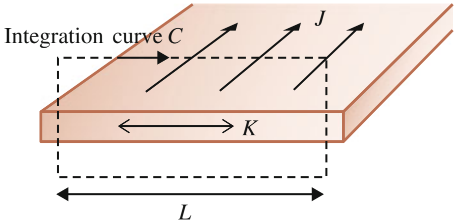


b. Large current-carrying plate

The plate is considered as consisting of infinitesimal parallel straight conductors.



Vertical components are cancelled.

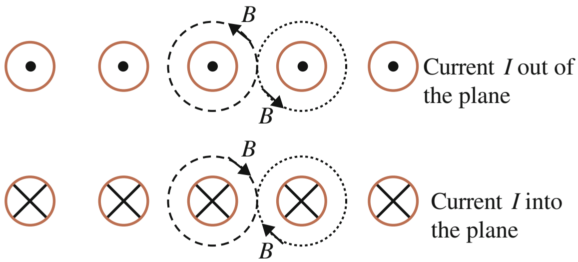


$$\oint_C \bar{B} \cdot d\bar{l} = B2L$$

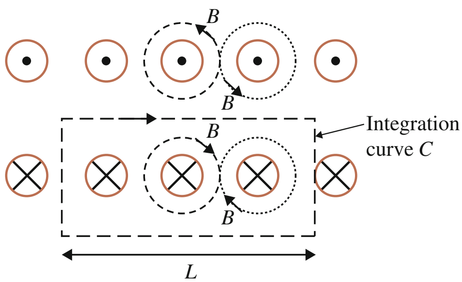
$$B2L = \mu_0 K L \Rightarrow B = \frac{\mu_0 K}{2}$$

Note that the field above the plate is oppositely directed to that below the plate.

c. Long current-carrying coil
The figure shows a cross section of a coil along its axis.



Referring to the figure, it is seen that for a long coil with close windings the vertical components are cancelled. Furthermore, the upper row generates a field oppositely directed to that generated from the lower row. The field outside the coil is cancelled since for close windings, the field is independent of distance. The following integration path C is introduced:



$$\oint_C \bar{B} \cdot d\bar{l} = BL$$

since only the upper side contributes to the integral. Thus

$$BL = \mu_0 NI \Rightarrow B = \frac{\mu_0 NI}{L}$$

where N is the number of windings crossing the surface enclosed by C.

*10.6 a. Magnetic hysteresis is discussed in Exercise (8.15):

$$\begin{aligned}\Delta\Phi_H &= lv\mu_0(K_f + K_b)\\ &= lv\mu_0(K_f + (\kappa_m - 1)K_f)\\ &= lv\mu_0\kappa_m K_f = lvB\end{aligned}$$

where the result from Exercise (10.5c) for an ideal coil has been used. Hence, the y axis becomes

$$\frac{\Delta\Phi_H}{lv} = \mu_0(K_f + K_b) = B$$

while for the x axis:

$$K_f = \frac{B}{\mu_0\kappa_m} = H$$

so that the slope of the curve is $\mu_0\kappa_{\mathrm{m}}$

b. Electric hysteresis is discussed in Exercise (8.16).
Electric field is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{\sigma_b + \sigma_f}{\varepsilon_0} = \frac{\sigma_f}{\kappa_e\varepsilon_0}$$

corresponding to the x axis.
Electric displacement is

$$D = \varepsilon_0\kappa_e E = \sigma_f$$

corresponding to the y axis.
The slope is $\varepsilon_0\kappa_e$

c. Using the result from Exercise (8.16c), we have obtained

$$w_{mat} = \oint_{Curve} E\,dD$$

for isotropic materials.

10.7 The electric field is directed along the surface normal of the charged plate. The surface normal is directed from medium 1 to 2.
The continuity condition is

$$(\bar{E}_2 - \bar{E}_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

which gives

$$2E = \frac{\sigma}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

10.8 a. The interaction energy between a small magnetic dipole with moment $\bar{m}_2 = m_2\hat{z}$ and a sphere homogeneously magnetized in the z direction is given by formula (8.79)

$$U_{int} = \frac{\mu_0}{2\pi z^3} m_2 \left(\frac{4}{3}\pi a^3 M\right) \quad z \geq a$$
$$U_{int} = \mu_0 \frac{2}{3} m_2 M \qquad z \leq a$$

in case the dipole is located on the z axis. According to formula (10.12), the magnetic energy is also

$$U_{int} = \bar{m}_2 \cdot \bar{B}$$

so that

$$\bar{B} = \frac{\mu_0}{2\pi z^3} \left(\frac{4}{3}\pi a^3 M\right) \hat{z} \quad z \geq a$$
$$\bar{B} = \mu_0 \frac{2}{3} M \hat{z} \qquad z \leq a$$

b. The magnetic field intensity is given by formula (10.19):

$$\bar{H} = \frac{\bar{B}}{\mu_0} - \bar{M}$$

which gives

$$\bar{H} = \frac{1}{2\pi z^3} \left(\frac{4}{3}\pi a^3 M\right) \hat{z} \quad z \geq a$$

$$\bar{H} = -\frac{1}{3} M \hat{z} \qquad z \leq a$$

since $M = 0$ outside the sphere.

c. The continuity condition for the normal component of $\bar{H}$ is derived from (10.56):

$$(\bar{B}_2 - \bar{B}_1) \cdot \hat{n} = 0 \Leftrightarrow (\bar{H}_2 + \bar{M}_2 - (\bar{H}_1 + \bar{M}_1)) \cdot \hat{n} = 0$$

so that

$$(\bar{H}_2 - \bar{H}_1) \cdot \hat{n} = -(\bar{M}_2 - \bar{M}_1) \cdot \hat{n}$$

For $z = a$ the left-hand side becomes

$$(\bar{H}_2 - \bar{H}_1) \cdot \hat{n} = \frac{2}{3}M - \left(-\frac{1}{3}M\right) = M$$

and the right-hand side becomes

$$-(\bar{M}_2 - \bar{M}_1) \cdot \hat{n} = -(0 - M) = M$$

d. Outside the sphere the source to $\bar{H}$ is the bound net current on the surface of the sphere. Inside the sphere the source is the net current flowing on the inner surface of a cavity, oppositely directed to the currents on the sphere's outer surface. See Sect. 8.2.2.3, especially Fig. 8.18.

10.9 a. The electric potential from the sphere is given by formula (8.31):

$$\begin{aligned}
\Phi_b &= \frac{Pa^2}{2\varepsilon_0}\int_{-1}^{1}\frac{tdt}{(z^2+a^2-2zat)^{1/2}} \\
&= \left[\frac{2(-2zat-2(z^2+a^2))}{3(2za)^2}(z^2+a^2-2zat)^{1/2}\right]_{-1}^{1} \\
&= \frac{2(-2za-2(z^2+a^2))}{3(2za)^2}(z^2+a^2-2za)^{1/2} \\
&\quad -\left[\frac{2(2za-2(z^2+a^2))}{3(2za)^2}(z^2+a^2+2za)^{1/2}\right] \\
&= \frac{2(-2za-2(z^2+a^2))}{3(2za)^2}((z-a)^2)^{1/2} - \left[\frac{2(2za-2(z^2+a^2))}{3(2za)^2}((z+a)^2)^{1/2}\right] \\
&= \frac{2(-2za-2(z^2+a^2))}{3(2za)^2}\,|z-a| - \left[\frac{2(2za-2(z^2+a^2))}{3(2za)^2}\,|z+a|\right]
\end{aligned}$$

which gives two cases:

1. $z \le a$

$$\begin{aligned}
\Phi_b &= \frac{Pa^2}{2\varepsilon_0}\left[\frac{2(-2za-2(z^2+a^2))}{3(2za)^2}(a-z) - \left[\frac{2(2za-2(z^2+a^2))}{3(2za)^2}(a+z)\right]\right] \\
&= \frac{Pa^2}{2\varepsilon_0}\left[\frac{2(-2za-2(z^2+a^2))}{3(2za)^2}(a-z) + \left[\frac{-2(2za-2(z^2+a^2))}{3(2za)^2}(a+z)\right]\right] \\
&= \frac{Pa^2}{2\varepsilon_0}\left[\frac{-8za^2}{3(2za)^2} + \frac{8z(z^2+a^2)}{3(2za)^2}\right] = \frac{Pa^2}{2\varepsilon_0}\frac{2}{3}\frac{z}{a^2} = \frac{Pz}{3\varepsilon_0} \\
\bar{E} &= -\nabla\Phi = -\nabla\Phi_b = -\frac{P}{3\varepsilon_0}\hat{z}
\end{aligned}$$

2. $z \geq a$

$$\begin{aligned}
\Phi_b &= \frac{Pa^2}{2\varepsilon_0}\left[\frac{2(-2za - 2(z^2+a^2))}{3(2za)^2}(z-a) - \left[\frac{2(2za - 2(z^2+a^2))}{3(2za)^2}(a+z)\right]\right] \\
&= \frac{Pa^2}{2\varepsilon_0}\left[\frac{2(-2za - 2(z^2+a^2))}{3(2za)^2}(z-a) + \left[\frac{-4za + 4(z^2+a^2)}{3(2za)^2}(z+a)\right]\right] \\
&= \frac{Pa^2}{2\varepsilon_0}\left[\frac{-8z^2a + 8a(z^2+a^2)}{12z^2a^2}\right] = \frac{Pa^2}{2\varepsilon_0}\frac{8a^3}{12z^2a^2} = \frac{Pa^3}{3\varepsilon_0 z^2} \\
\bar{E} &= -\nabla\Phi = -\nabla\Phi_b = \frac{2Pa^3}{3\varepsilon_0 z^3}\hat{z}
\end{aligned}$$

b. Formula (10.16)

$$\bar{D} = \bar{P} + \varepsilon_0 \bar{E}$$

gives

1. $z \leq a$

$$\bar{D} = \bar{P} + \varepsilon_0\left(-\frac{P}{3\varepsilon_0}\hat{z}\right) = \frac{2}{3}P\hat{z}$$

2. $z \geq a$

$$\bar{D} = \varepsilon_0\left(\frac{2Pa^3}{3\varepsilon_0 z^3}\right)\hat{z} = \frac{2Pa^3}{3z^3}\hat{z}$$

c. The boundary condition is defined by formula (10.54)

$$(\bar{E}_2 - \bar{E}_1)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0} = \frac{\sigma_b}{\varepsilon_0}$$

which gives

$$(\bar{D}_2 - \bar{P}_2 - (\bar{D}_1 - \bar{P}_1))\cdot\hat{n} = \sigma_b$$

$$(\bar{D}_2 - \bar{D}_1))\cdot\hat{n} = \sigma_b - \bar{P}_1\cdot\hat{n} = 0$$

so that at $z = a$ the normal component of D is continuous in accordance with Exercise b.

d. Using (10.59)

$$\nabla\cdot\bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0} = \frac{\rho_b + \rho_f}{\varepsilon_0}$$

and (10.16) we obtain

$$\nabla\cdot\left(\frac{\bar{D} - \bar{P}}{\varepsilon_0}\right) = \frac{\rho_b + \rho_f}{\varepsilon_0}$$
$$\nabla\cdot\bar{D} = \nabla\cdot\bar{P} + \rho_b + \rho_f = \rho_f$$

where formula (8.19), i.e. $\nabla \cdot \bar{P} = -\rho_b$, has been used.
In analogy with the magnetic case, Exercise (10.9), a D-field occurs due to a net charge on the outer surface of the sphere as well as on the inner surface of a cavity inside the sphere.

10.10 The time varying electric field is a further source to the magnetic field according to formula (10.77):

$$\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{d\bar{E}}{dt}$$

Apply the integral formula (10.96, Exercise 10.3), modified for a time varying electric field

$$\oint_C \bar{B} \cdot d\bar{l} = \mu_0 \int_S \bar{J} \cdot d\bar{a} + \int_S \mu_0 \varepsilon_0 \frac{d\bar{E}}{dt} \cdot d\bar{a}$$

Utilizing (10.16), i.e. $\bar{P} = \bar{D} - \varepsilon_0 \bar{E}$, to include dipoles, the integral formula becomes

$$\oint_C \bar{B} \cdot d\bar{l} = \mu_0 \int_S \bar{J} \cdot d\bar{a} + \int_S \mu_0 \frac{d(\bar{D} - \bar{P})}{dt} \cdot d\bar{a}$$

In this case the source is only dipole vibrations

$$\oint_C \bar{B} \cdot d\bar{l} = -\int_S \mu_0 \frac{d\bar{P}}{dt} \cdot d\bar{a}$$

As in Exercise (10.5) an integration path is formed around the plate so that

$$B2L = -\mu_0 \frac{dP}{dt} Lb$$

where L is the length of the integration path (plate edges neglected) and b is the thickness of the plate. Thus

$$B = -\frac{\mu_0 b}{2} \omega P_0 \cos \omega t$$

10.11 With the z axis along the cylinder axis and using cylindrical coordinates, the electric field between the conductors is given by formula (10.5)

$$\bar{E} = \frac{\lambda}{2\pi \varepsilon_0 \rho} \hat{\rho} = \frac{Q}{2\pi \varepsilon_0 \rho L} \hat{\rho} = \frac{C \Delta \Phi}{2\pi \varepsilon_0 \rho L} \hat{\rho} = \frac{\Delta \Phi}{\rho \ln \frac{b}{a}} \hat{\rho}$$

Magnetic field is given by (10.7)

$$\bar{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} = \frac{\mu_0 \Delta\Phi}{2\pi\rho R}\hat{\phi}$$

Poynting vector becomes

$$\bar{S} = \bar{E} \times \bar{H} = \frac{\Delta\Phi}{\rho \ln\frac{b}{a}} \frac{\Delta\Phi}{2\pi\rho R}\hat{\rho} \times \hat{\phi} = \frac{(\Delta\Phi)^2}{R2\pi \ln\frac{b}{a}} \frac{1}{\rho^2}\hat{z}$$

Outside the fields vanish.

b. Inside the conductors the electric field is in the z direction and the magnetic field in the ϕ direction. The Poynting vector is then directed along ρ and the power corresponds to heat.
c. Total power transfer through a cross sectional area A between the conductors is

$$P = \int_A \bar{S} \cdot d\bar{a} = \frac{(\Delta\Phi)^2}{R2\pi \ln\frac{b}{a}} \int_a^b \int_0^{2\pi} \frac{1}{\rho^2}\rho d\phi d\rho = \frac{(\Delta\Phi)^2}{R}$$

equivalent to the circuit theoretical result.

10.12 a. With the z axis along the cylinder axis and using cylindrical coordinates, the electric field between the plates is

$$\bar{E} = \frac{\Delta\Phi(t)}{d}\hat{z}$$

The magnetic field is obtained from the generalization of formula (10.96):

$$\oint_C \bar{B} \cdot d\bar{l} = \varepsilon_0\mu_0 \int_S \frac{d\bar{E}}{dt} \cdot d\bar{a}$$

since there is no current between the plates. Choosing an integration path with radius ρ and coaxial with z axis we obtain that

$$B2\pi\rho = \varepsilon_0\mu_0 \frac{dE}{dt}\pi\rho^2 \Rightarrow \bar{B} = \frac{\varepsilon_0\mu_0}{2}\frac{dE}{dt}\rho\hat{\phi}$$

Poynting vector becomes

$$\bar{S} = \bar{E} \times \bar{H} = \frac{\Delta\Phi(t)}{d^2}\frac{\varepsilon_0}{2}\frac{d\Delta\Phi(t)}{dt}\rho\hat{\rho} = \frac{1}{4}\varepsilon_0\rho\frac{d}{dt}E^2\hat{\rho}$$

Total power flow through the mantle of the cylinder

$$P(\rho = a) = \int_{Mantle} \bar{S} \cdot d\bar{a} = \frac{1}{4}\varepsilon_0 a \frac{d}{dt} E^2 2\pi a d \hat{\rho} \cdot \hat{\rho} = \frac{\varepsilon_0 \pi a^2 d}{2} \frac{d}{dt} E^2$$

b. Formula (10.25b) gives the field energy density in free space

$$\frac{dU}{dV} = \frac{1}{2}\varepsilon_0 E^2 \Rightarrow U = \frac{\varepsilon_0 \pi a^2 d}{2} E^2$$

whose time derivative equals the previous result.

D.11 Antenna Theory

11.2 a. *Minimum*

$$\cos\left(\frac{d\omega}{2c}\cos\theta\right) = 0$$
$$\frac{d\omega}{2c}\cos\theta = (2n+1)\frac{\pi}{2} \quad where \;\; n = 0, 1, 2, \ldots$$
$$d\cos\theta = \frac{(2n+1)}{2}\frac{c}{f}$$

Maximum
In the same manner

$$\cos\left(\frac{d\omega}{2c}\cos\theta\right) = 1$$
$$d\cos\theta = n\frac{c}{f}$$

b. For destructive interference

$$d\cos\theta = \frac{(2n+1)}{2}\frac{c}{f}$$
$$\cos\theta = \frac{(2n+1)}{2d}\frac{c}{f}$$
$$\frac{(2n+1)}{2d}\frac{c}{f} > 1$$
$$d < \frac{c}{2f}$$

11.3 At maximum force the following condition holds:

$$\cos\left(\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2}\right) = 1$$
$$\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2} = 0$$

a.

$$\frac{d\omega}{2c}\cos\theta = 0$$
$$\theta = \frac{\pi}{2}$$

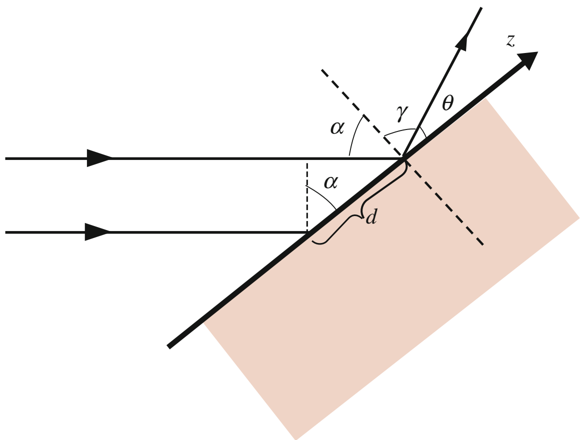
b.

$$\frac{d\omega}{2c}\cos\theta - \frac{\pi}{4} = 0$$
$$\cos\theta = +\frac{\pi}{4}\frac{2c}{d2\pi f} = 1$$
$$\theta = 0$$

c.

$$\frac{d\omega}{2c}\cos\theta + \frac{\pi}{4} = 0$$
$$\cos\theta = -\frac{\pi}{4}\frac{2c}{d2\pi f} = -1$$
$$\theta = \pi$$

11.4 Let the incoming interaction be directed horizontally and the mirror surface along the z axis as in the figure.



The mirror is modelled as two oscillators being activated with a phase shift β. Formula (11.35) is then used to find the direction of the outgoing plane wave. The phase shift β in (11.35) is

$$\beta = \omega \Delta t = \omega \frac{d \sin \alpha}{c}$$

which is introduced in (11.35):

$$\begin{aligned} F_\theta &= \frac{qIj\omega l}{4\pi \varepsilon_0 c^2} \frac{e^{-j\frac{\omega}{c}r}}{r} 2\cos\theta \cos\left(\frac{d\omega}{2c}\cos\theta - \frac{\beta}{2}\right) \\ &= \frac{qIj\omega l}{4\pi \varepsilon_0 c^2} \frac{e^{-j\frac{\omega}{c}r}}{r} 2\cos\theta \cos\left(\frac{d\omega}{2c}\cos\theta - \omega\frac{d \sin\alpha}{2c}\right) \end{aligned}$$

which has maximum when

$$\begin{aligned} &\frac{d\omega}{2c}\cos\theta - \omega\frac{d\sin\alpha}{2c} = 0 \\ &\sin\gamma = \sin\alpha \\ &\gamma = \alpha \end{aligned}$$

which is the 'reflection law'.
b. Note that the factor of formula (11.35) used here, i.e.

$$Array\ Factor = \cos\left(\frac{d\omega}{2c}\cos\theta - \frac{\beta}{2}\right)$$

is an array property and therefore independent on the particular individual antennas. The reflection law is therefore valid for any type of surface. In particular, it is well known that a water surface also obeys the reflection law although the individual antenna elements are radically different in these two cases. In case of the mirror, the conduction electrons act as antenna elements whereas in case of water it is the molecular water dipoles.

11.5 This problem is equivalent to the reflection case in Exercise (11.4). The activated oscillators at the surface of the material are now interacting on the other side of the surface. The reason for the change of direction of the lines of maximum force is that the speed of the force mediation differs in the two media. Denoting speed in the media of incoming force as c_0 and the speed in the transmitted media as c, the array factor becomes

$$Array\ factor = \cos\left(\frac{d\omega}{2c}\cos\theta - \omega\frac{d\sin\alpha}{2c_0}\right) = \cos\left(\frac{d\omega}{2c}\sin\gamma - \frac{\omega d}{2c_0}\sin\alpha\right)$$

which has maximum when

$$\frac{d\omega}{2c}\sin\gamma - \frac{\omega d}{2c_0}\sin\alpha = 0$$

so that

$$c\sin\alpha = c_0\sin\gamma$$

which is Snell's 'refraction law'.

Why is then the mediation speed different in different media? This may qualitatively be understood by understanding that materials are built up of parallel layers of surfaces consisting of arrays of dipoles. The force mediation is a matter of transmitting the dipole vibrations from layer to layer whose speed is clearly dependent on the particular material structure.

11.6 The vibrating dipole at the surface align with the force in medium 2, F_2. Along the dipole axis, there is no force mediation, see formula (11.31). If the dipole axis is oriented such that it points in the direction of the reflection, i.e. force F_2 forms an angle α with the surface normal, then the reflection vanishes. This happens when

$$\alpha + \gamma = 90^\circ$$

The refraction law, Exercise (11.5), becomes

$$c\sin\alpha = c_0\sin\gamma = c_0\sin(90^\circ - \alpha) = c_0\,\cos\alpha$$

so that the Brewster angle α_B is given by

$$\tan\alpha_B = \frac{c_0}{c}$$

11.7 The reflection appears between close layers of the atmosphere which have almost identical structure. The mediation speeds in two neighbouring layers are therefore almost the same.
When the incident angle equals the Brewster angle, one component of force will vanish. This then appears at

$$\tan\alpha_B = \frac{c_0}{c} \approx 1$$
$$\alpha_B = 45^\circ$$

Therefore, the angle β in the figure is 90°, so that the spider has maximum navigation aid directly above herself, practical of course.

11.8 From Exercise (11.7) $c\sin\alpha = c_0\sin\gamma$ which gives

$$\frac{n_2}{n_1} = \frac{c_1}{c_2}$$

Let medium 1 be a vacuum denoted by index 0 and medium 2 is arbitrary:

$$\frac{n}{n_0} = \frac{c_0}{c}$$

From Sect. 9.1, the speed of light in vacuum was obtained as

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

When electromagnetic phenomena appear in a material the formulas change by

$$\varepsilon_0 \to \varepsilon_0 \kappa_e$$
$$\mu_0 \to \mu_0 \kappa_m$$

which may be concluded from formulas (8.2) and (8.38) as well as (10.22) and (10.23). Speed of light is then in general

$$c = \frac{1}{\sqrt{\varepsilon_0 \kappa_e \mu_0 \kappa_m}}$$

so that the refractive index is

$$n = \sqrt{\kappa_e \kappa_m}$$

11.10 To obtain a plane wave to be directed to the other side of the surface normal, i.e. negative transmission angel γ, the upper oscillator in the figure of Exercise (11.5) has to start oscillate before the lower one. This means the effect of the influence has to occur before its cause.

11.11 The electric field is defined as

$$\bar{E}(\bar{r}, t) = -\nabla\Phi(\bar{r}, t) - \frac{d\bar{A}(\bar{r}, t)}{dt}$$

The vector potential is

$$\bar{A} = I\frac{\mu_0}{4\pi}\oint_{C_1}\frac{d\bar{L}}{R}$$

One moving charge is then associated with a vector potential

$$\bar{A} = I\frac{\mu_0}{4\pi}\frac{d\bar{L}}{R} = \frac{\mu_0}{4\pi R}q\bar{v}$$

Electric force on another charge Q is, neglecting the first term of (10.8),

$$\bar{F} = Q\bar{E}(\bar{r}, t) = -Q\frac{d\bar{A}(\bar{r}, t)}{dt} = -Q\frac{\mu_0}{4\pi R}q\frac{d\bar{v}}{dt}$$

which is Weber's inductive force (3.34).

11.12 The real Poynting vector is

$$\bar{S}_{real} = \bar{E}_{real} \times \bar{H}_{real}$$

where

$$\bar{E}_{real} = Re\ (\bar{E}_R + j\bar{E}_I)(\cos\omega t + j\sin\omega t) = \bar{E}_R\cos\omega t - \bar{E}_I\sin\omega t$$

and

$$\bar{H}_{real} = Re\ (\bar{H}_R + j\bar{H}_I)(\cos\omega t + j\sin\omega t) = \bar{H}_R\cos\omega t - \bar{H}_I\sin\omega t$$

Time average of function $f(t)$ over time T is

$$\langle f(t)\rangle = \int_0^T \frac{f(t)}{T}dt = \frac{1}{T}\int_0^T f(t)dt$$

The real Poynting vector becomes

$$\begin{aligned}\bar{S}_{real} = (\bar{E}_R \times \bar{H}_R)\cos^2\omega t + (\bar{E}_I \times \bar{H}_I)\sin^2\omega t \\ - [(\bar{E}_R \times \bar{H}_I) + (\bar{E}_I \times \bar{H}_R)]\sin\omega t\cos\omega t\end{aligned}$$

When taking time averages, the last two terms vanish and the first two give

$$\langle\bar{S}_{real}\rangle = \frac{1}{2}[(\bar{E}_R \times \bar{H}_R) + (\bar{E}_I \times \bar{H}_I)]$$

Consider now the following expression in phasor form

$$\begin{aligned}\bar{E} \times \bar{H}^* &= (\bar{E}_R + j\bar{E}_I)e^{j\omega t} \times (\bar{H}_R - j\bar{H}_I)e^{-j\omega t} \\ &= (\bar{E}_R \times \bar{H}_R) + (\bar{E}_I \times \bar{H}_I) \\ &\quad + j[(\bar{E}_I \times \bar{H}_R) - (\bar{E}_R \times \bar{H}_I)]\end{aligned}$$

so that

$$\langle\bar{S}_{real}\rangle = \frac{1}{2}Re(\bar{E} \times \bar{H}^*)$$

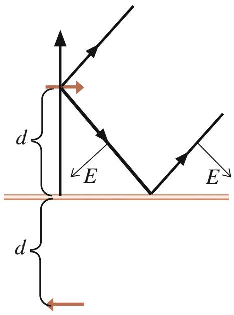
11.13 Electric field is given by (11.35)

$$E_\theta = \frac{Ij\omega l}{4\pi\varepsilon_0 c^2}\frac{e^{-j\frac{\omega}{c}r}}{r}2\cos\theta\cos\left(\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2}\right)$$

and the Poynting vector by (11.42)

$$\langle \bar{S} \rangle = \frac{1}{2}\bar{E} \times \bar{H}^* = \frac{1}{2}\bar{E} \times \frac{\bar{B}^*}{\mu_0} = \frac{|E_\theta|^2}{2\mu_0 c}\hat{r}$$
$$= \frac{1}{2\mu_0 c}\left(\frac{I_0 \omega l}{4\pi\varepsilon_0 c^2 r} 2\cos\theta\cos\left(\frac{d\omega}{2c}\cos\theta + \frac{\beta}{2}\right)\right)^2 \hat{r}$$

11.14 The antenna radiates as in the figure. The reflected power is equivalent to a power emission from an antenna placed at the real antenna's mirror image. The question is then what the relative phase is. The electric field vector is directed perpendicular to the power direction. To fit boundary condition (10.55) at the surface of the metal, the electric field vector of the reflected power has to have a horizontal component that cancels the incident horizontal field. The orientation of the mirror dipole is therefore anti-parallel to the real antenna, i.e. they oscillate in opposite phase.



The Poynting vector is obtained from Exercise (11.13):

$$\langle \bar{S} \rangle = \frac{1}{2\mu_0 c}\left(\frac{I_0 \omega l}{4\pi\varepsilon_0 c^2 r} 2\cos\theta\cos\left(\frac{2d\omega}{2c}\cos\theta + \frac{\beta}{2}\right)\right)^2 \hat{r}$$
$$= \frac{1}{2\mu_0 c}\left(\frac{I_0 \omega l}{4\pi\varepsilon_0 c^2 r} 2\cos\theta\cos\left(\frac{2d\omega}{2c}\cos\theta + \frac{\pi}{2}\right)\right)^2 \hat{r}$$
$$= \frac{1}{2\mu_0 c}\left(\frac{I_0 \omega l}{4\pi\varepsilon_0 c^2 r} 2\cos\theta\sin\left(\frac{d\omega}{c}\cos\theta\right)\right)^2 \hat{r}$$

Appendix E
General Magnetic Force Formula

In Sect. 2.2, the magnetic force formula was heuristically formulated based on measurements. In this appendix a systematic approach is presented. The objective is to find a single force formula describing both parallel and perpendicular relative motional direction. To this end, a general vector expression in each case is developed. Due to the basic property of vector formulas each of these expressions must then be valid for both cases. Thus, by requiring compatibility between the two formulas, a final concise formula will be obtained.

To summarize, it has been obtained for the two cases

$$\bar{f}_{m||} = -\frac{\mu_0 I_1 dL_1}{4\pi R^2} q_2 v_2 (a\cos\theta\,\hat{y} + \sin\theta\,\hat{x}) \tag{2.7}$$

$$\bar{f}_{m\perp} = \frac{\mu_0 I_1 dL_1}{4\pi R^2} q_2 v_2 (\sin\theta\,\hat{y} + b\cos\theta\,\hat{x}) \tag{2.10}$$

as the force acting on charge q_2.

Since the force should be written in terms of the charges, the current element is rewritten as $I_1 dL_1 = q_1 v_1$. The force must be expressed vectorially independent of coordinate system so the only usable vectors are v_1, v_2 and R. For the moment the factor $(\mu_0 q_1 q_2)/(4\pi R^2)$ is left out and formula (2.7) may be written as

$$\bar{f}_{m||} = -v_1 v_2 (a\cos\theta\,\hat{y} + \sin\theta\,\hat{x}) \tag{E.1}$$

and formula (2.10)

$$\bar{f}_{m\perp} = v_1 v_2 (\sin\theta\,\hat{y} + b\cos\theta\,\hat{x}) \tag{E.2}$$

Generally, (E.1) may be expressed, with notations as in Fig. 2.8,

$$\bar{f}_{m||} = \underbrace{A\bar{v}_2(\bar{v}_1\cdot\hat{R}) + B\bar{v}_1(\bar{v}_2\cdot\hat{R})}_{\text{y component}} + \underbrace{C\bar{v}_1\times(\bar{v}_2\times\hat{R}) + D\bar{v}_2\times(\bar{v}_1\times\hat{R})}_{\text{x component}} \tag{E.3}$$

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where the first two terms correspond to the y component and the last two to the x component.

From formula (E.2) it is similarly obtained that, utilizing the definitions of Fig. 2.9,

$$\bar{f}_{m\perp} = \underbrace{E\bar{v}_1(\bar{v}_2 \cdot \hat{R}) + F\bar{v}_2 \times (\bar{v}_1 \times \hat{R})}_{\text{y component}} + \underbrace{G\bar{v}_2(\bar{v}_1 \cdot \hat{R}) + H\bar{v}_1 \times (\bar{v}_2 \times \hat{R})}_{\text{x component}} \tag{E.4}$$

where the first two terms correspond to the y component and the last two to the x component.

The factors A-H are constants determined by the following criteria:

1. Consistency between (E.1) and (E.3) as well as between (E.2) and (E.4)
2. Compatibility between (E.3) and (E.4)

Expanding the cross products in formulas (E.3) and (E.4) gives

$$\begin{aligned}\bar{f}_{m||} = {} & A\bar{v}_2(\bar{v}_1 \cdot \hat{R}) + B\bar{v}_1(\bar{v}_2 \cdot \hat{R}) + C\bar{v}_2(\bar{v}_1 \cdot \hat{R}) - C\hat{R}(\bar{v}_1 \cdot \bar{v}_2) \\ & + D\bar{v}_1(\bar{v}_2 \cdot \hat{R}) - D\hat{R}(\bar{v}_1 \cdot \bar{v}_2)\end{aligned} \tag{E.5}$$

$$\begin{aligned}\bar{f}_{m\perp} = {} & E\bar{v}_1(\bar{v}_2 \cdot \hat{R}) + F\bar{v}_1(\bar{v}_2 \cdot \hat{R}) - F\hat{R}(\bar{v}_1 \cdot \bar{v}_2) + G\bar{v}_2(\bar{v}_1 \cdot \hat{R}) \\ & + H\bar{v}_2(\bar{v}_1 \cdot \hat{R}) - H\hat{R}(\bar{v}_1 \cdot \bar{v}_2)\end{aligned} \tag{E.6}$$

and the following conditions for the constants may be identified:

Criteria no. 1 gives:

$$\begin{aligned} A + B &= a \\ C + D &= 1 \\ E + F &= 1 \\ G + H &= b \end{aligned}$$

By comparing vectors in formulas (E.5) and (E.6) criteria no. 2 gives:

$$\begin{aligned} B + D &= E + F \\ A + C &= G + H \\ -D - C &= -F - H \end{aligned}$$

which gives $a = b$ and the following table:

$$\begin{aligned} A + B &= a \\ C + D &= 1 \\ E + F &= 1 \\ G + H &= a \\ B + D &= 1 \\ A + C &= a \\ F + H &= 1 \end{aligned} \tag{E.7}$$

These conditions are now applied to formulas (E.5) and (E.6) to give the result for general motion, including the temporarily neglected factor,

$$\bar{f}_{m2} = \frac{\mu_0 q_1 q_2}{4\pi R^2}[(-\bar{v}_1 \cdot \bar{v}_2)\hat{R} + (\bar{v}_2 \cdot \hat{R})\bar{v}_1 + a(\bar{v}_1 \cdot \hat{R})\bar{v}_2] \tag{E.8}$$

Exercises

E1. Show that the criteria (E.7) give formula (E.8).

E2. Criteria (E.7) consist of seven equations whereas there are nine unknowns. How is it then possible that only one unknown remains in formula (E.8)?

Answer: The twelve terms in (E.5) and (E.6) are not linearly independent. For example, it is necessary to know only the sum $A + C$, not each constant independently.

Index

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