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Control and Filtering for Semi-Markovian Jump Systems

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To my parents and Yewen

—Fanbiao Li

To Fengmei, Lisa and Michael

—Peng Shi

To Jingyan and Zhixin

—Ligang Wu

Preface

In practice, a large class of physical systems has variable structures subject to random changes. These may result from abrupt phenomena such as component and interconnection failures, parameters shifting, tracking, and the time required to measure some of the variables at different stages. Systems with this character may be modeled as hybrid ones; that is, to the continuous state variable, a discrete random variable called the mode, or regime, is appended. The mode describes the random jumps of the system parameters and the occurrence of discontinuities. Such a system model is useful particularly since it allows the decision maker to cope adequately with the discrete events that disrupt and/or change the normal operation of a system significantly, by using the knowledge of their occurrence and the statistical information on the rate at which these events take place. Markovian jump systems (MJS), with its powerful modeling capability in application areas such as the aerospace industry, industrial processes, biomedical industry, and socioeconomics, have proved to be of vital importance as a typical class of hybrid dynamical system. However, MJS have many limitations in applications, since the jump time in MJS is subject to exponential distribution or geometric distribution in continuous- and discrete-time domains, respectively. So, the results obtained for the MJS are intrinsically conservative due to constant transition rates. Compared with the MJS, semi-Markovian jump systems (S-MJS) are characterized by a fixed matrix of transition probabilities and a matrix of sojourn time probability density functions. Due to their relaxed conditions on the probability distributions, S-MJS have much broader applications than the conventional MJS. Thus, this area of research is significant because of both its theoretical and practical values.

This book aims to present up-to-date research developments and novel methodologies on S-MJS. The content of this book can be divided into three parts: Part I is focused on stability analysis and control of the considered S-MJS, Part II puts the emphasis on fault detection and filtering of S-MJS, while Part III summarizes the results of the book. These methodologies provide a framework for stability and performance analysis, robust controller design, robust filter design, and fault detection for the considered systems. The main contents of Part I include the following: Chapter 2 is concerned with stochastic stability of S-MJS with mode-dependent delays; Chapter 3 studies the constrained regulation problem of singular S-MJS; Chapter 4 addresses the state estimation and sliding mode control problems of S-MJS with mismatched uncertainties; and Chap. 5 investigates the quantized dynamic output feedback control of nonlinear S-MJS. The main contents of Part II include the following: Chapter 6 is concerned with the neural network-based passive filter design for delayed neutral-type S-MJS; Chapter 7 studies event-triggered fault detection filtering problem for sojourn information-dependent S-MJS; Chapter 8 addresses the fault detection filtering for S-MJS via T-S fuzzy approach; Chapter 9 investigates the fault detection problem for underactuated manipulators modeled by MJS; and Chap. 10 summarizes the results of the book and discusses some future works.

This book is a research monograph whose intended audience is graduate and postgraduate students as well as researchers. Prerequisite to reading this book is elementary knowledge on mathematics, matrix theory, probability, optimization techniques, and control system theory.

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Contents

Notations and Acronyms

■	End of proof
♦	End of remark
$\triangleq$	Is defined as
$\in$	Belongs to
$\forall$	For all
$\sum$	Sum
$\mathbb{C}$	Field of complex numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{R}^n$	Space of n-dimensional real vectors
$\mathbb{R}^{n\times m}$	Space of $n \times m$ real matrices
$\mathbb{Z}^+$	Field of positive integral numbers
$\mathbb{C}_{n,d}$	Set of $\mathbb{R}^n$-valued continuous functions on $[d, 0]$
$\mathbf{E}\{\cdot\}$	Mathematical expectation operator
$(\Omega, \mathscr{F}, \Pr)$	Ω represents the sample space, $\mathscr{F}$ is the σ-algebra of subsets of the sample space, and Pr is the probability measure on $\mathscr{F}$
lim	Limit
max	Maximum
min	Minimum
sup	Supremum
inf	Infimum
rank $(\cdot)$	Rank of a matrix
det $(\cdot)$	Determinant of a matrix
trace $(\cdot)$	Trace of a matrix
deg $(\cdot)$	Degree of a polynomial
$\lambda_i(\cdot)$	ith eigenvalue of a matrix
$\lambda_{\min}(\cdot)$	Minimum eigenvalue of a matrix
$\lambda_{\max}(\cdot)$	Maximum eigenvalue of a matrix
$\rho(\cdot)$	Spectral radius of a matrix
I	Identity matrix
I_n	$n \times n$ identity matrix

0	Zero matrix
$0_{n\times m}$	Zero matrix of dimension $n \times m$
X^T	Transpose of matrix X
X^{-1}	Inverse of matrix X
X^+	Moore–Penrose inverse of matrix X
$X^\perp$	Full row rank matrix satisfying $X^\perp X = 0$ and $X^\perp X^{\perp T} > 0$
diag	Block diagonal matrix with blocks $\{X_1, \ldots, X_m\}$
$sym(A)$	$A + A^T$
$X > (<)0$	X is real symmetric positive (negative) definite
$X \geq (\leq)0$	X is real symmetric positive (negative) semi-definite
$\mathcal{L}_2\{[0,\infty),[0,\infty)\}$	Space of square summable sequences on $\{[0,\infty),[0,\infty)\}$ (continuous case)
$\ell_2\{[0,\infty),[0,\infty)\}$	Space of square summable sequences $\{[0,\infty),[0,\infty)\}$ (discrete case)
$\lvert\cdot\rvert$	Euclidean vector norm
$\lVert\cdot\rVert$	Euclidean matrix norm (spectral norm)
$\lVert\cdot\rVert_2$	$\mathcal{L}_2$ - norm : $\sqrt{\int_0^\infty \lvert\cdot\rvert^2 dt}$ (continuous case) ℓ_2 - norm : $\sqrt{\sum_0^\infty \lvert\cdot\rvert^2}$ (discrete case)
$\lVert\cdot\rVert_{E_2}$	$\mathbf{E}\{\lVert\cdot\rVert_2\}$
$\lVert \mathrm{T} \rVert_\infty$	H_∞ norm of transfer function $\mathbf{T}$: $\sup_{\omega\in[0,\infty)} \lVert \mathbf{T}(j\omega) \rVert$ (continuous state) $\sup_{\omega\in[0,2\pi)} \lVert \mathbf{T}(e^{j\omega}) \rVert$ (discrete state)
$*$	Symmetric terms in a symmetric matrix
MJS	Markovian jump systems
S-MJS	Semi-Markovian jump systems
CCL	Cone complementary linearization
LMI	Linear matrix inequality
SMC	Sliding mode control

Matrices, if their dimensions are not explicitly stated, are assumed to be compatible with algebraic operations.

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Chapter 1
Introduction

Abstract This chapter introduced the research background and significance of MJS, as well as the current research status of MJS, so as to provide a basis of reference for further research of S-MJS. The main differences between S-MJS and MJS have been provided, followed by a description of the advantages of the S-MJS and its broad application prospects. Additionally, we have mentioned several problems that are yet to be solved, methods that require refinements and the main research contents of this dissertation.

In order to control the behavior of a system, it is necessary to capture the salient system features in a mathematical model. Dynamic systems are intrinsically difficult due to their system complexities, the challenge of measuring various parameters and also the uncertain and/or time-varying parameters. The development of systematic methods for efficient and reliable design of such complex control systems is a key issue in control technology and industrial information. It is currently of high interest to control engineers, computer scientists and mathematicians in research institutions as well as in many industrial sectors. MJS are a special class of parameter-switching systems, and they are modeled by a set of linear or nonlinear systems with the transitions between the models determined by a Markov chain taking values in a finite set [1]. Some of the earliest works with these features include [2–6].

Applications of MJS can be found in many real world applications, such as economic systems [7–9], flight systems [10], power systems [11–13], communication systems [14] and networked control systems [15, 16]. In the following, we will give a brief exposition of some selected topics regarding applications of MJS.

1.1 Analysis and Control of MJS

MJS can also be considered as special case of switched hybrid systems with the switching signals governed by a Markovian chain. From a mathematical point of view, MJS can be regarded as a special class of stochastic systems with system matrices changing randomly at discrete-time points governed by a Markov process

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and remaining time-invariant between random jumps. Over the past decades, a great amount of attention has been paid to MJS, due their wide applications in practical systems. In recent years, to ease the practical application of MJS, considerable efforts have been made, and a lot of progresses have been made on topics such as: (1) modeling of MJS; (2) stability and performance analysis; (3) control and filtering; (4) fault detection and fault tolerance; (5) identification via networks; and so on.

1.1.1 Stability Analysis of MJS

It is common knowledge that the stability of a dynamical system is one of the primary concerns in the design and synthesis of a control system. The study of stability of jump linear systems has attracted the attention of many researchers. The stability analysis and stabilization problems for MJS have been addressed in [17–29]. To mention a few, Cao and Lam investigated the stochastic stabilizability for discrete-time jump linear systems with time delay in [19]; de Souza studied the robust stability and stabilization problems for uncertain discrete-time MJS in [20]; Gao et al. considered the stabilization problem for two-dimensional (2-D) MJS in [21]; Sun et al. discussed the robust exponential stabilization for MJS with mode-dependent input delay in [26]; Boukas and Yang proposed an exponential stabilizability condition for MJS in [17]; while Wang et al. solved the stabilization problem for bilinear uncertain time delay MJS [27]; and some other results on MJS can be found, for example [30–37], and the references therein.

1.1.1.1 Stability of Linear MJS

Consider the following continuous- and discrete-time MJS with state vectors $x(t) \in \mathbb{R}^n$ and $x(k) \in \mathbb{R}^n$, respectively.

$$\dot{x}(t) = A(\eta_t)x(t), \tag{1.1}$$

$$x(k+1) = A(\gamma_k)x(k). \tag{1.2}$$

For continuous-time systems, $\{\eta_t, t \geqslant 0\}$ is a time homogeneous Markov process with right continuous trajectories and taking values in a finite set $\mathcal{M} = \{1, 2, \ldots, M\}$ with stationary transition rates

$$\Pr(\eta_{t+h} = j|\eta_t = i) = \begin{cases} \pi_{ij}h + o(h), & i \neq j, \\ 1 + \pi_{ij}h + o(h), & i = j, \end{cases} \tag{1.3}$$

where $h > 0$, $\lim_{h\to 0} o(h)/h = 0$ and $\pi_{ij} \geqslant 0$ is the transition rate from mode i at time t to mode j at time $t+h$, and $\pi_{ii} = -\sum_{j=1, j\neq i}^{M} \pi_{ij}$. Furthermore, the Markov process transition rate matrix Π is defined by:

$$\Pi \triangleq \begin{bmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1M} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{M1} & \pi_{M2} & \dots & \pi_{MM} \end{bmatrix}.$$

For discrete-time systems, $\{\gamma_k, k \in \mathbb{Z}^+\}$ is a time homogeneous Markov chain taking values in a finite set $\mathcal{M} = \{1, 2, \dots, M\}$ with stationary transition probabilities

$$\lambda_{ij} = \Pr(\gamma_{k+1} = j \,|\gamma_k = i), \tag{1.4}$$

where $\lambda_{ij} \geqslant 0$ is the transition probability from mode i at time k to mode j at time $k+1$ and $\sum_{j=1}^{M} \lambda_{ij} = 1$. Likewise, the transition probability matrix Λ is defined by:

$$\Lambda \triangleq \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1M} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{M1} & \lambda_{M2} & \dots & \lambda_{MM} \end{bmatrix}.$$

Much works have been done for the stability analysis of MJS (1.1) and (1.2). Stability properties of systems described by multiple models switching according to Markov processes/chains can be analyzed via the notion of stochastic stability introduced in [22]. When parameters of Markov process/chain describing the transition between different models are not completely known, it is important to know how much uncertainty can be tolerated for the system to be stochastically stable. In [36], the problem of almost sure instability was studied of the random harmonic oscillator.

Definition 1.1 [38, 39] For system (1.1) with transition probability satisfied (1.3), the equilibrium point is

(i) *asymptotically mean square stable*, if for any initial state x_0 and initial distribution η_0

$$\lim_{t \to +\infty} \mathbf{E}\left\{\|x(t, x_0, \eta_0)\|^2\right\} = 0.$$

(ii) *exponentially mean square stable*, if for every initial state x_0 and initial distribution η_0, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\mathbf{E}\left\{\|x(t, x_0, \eta_0)\|^2\right\} < \alpha\|x_0\|^2\exp(-\beta t), \quad \forall t > 0.$$

(iii) *stochastically stable*, if for every initial state x_0 and initial distribution η_0,

$$\mathbf{E}\left\{\int_0^{+\infty} \|x(t, x_0, \eta_0)\|^2 \mathrm{d}t\right\} < +\infty.$$

(iv) *almost surely (asymptotically) stable*, if for any initial state x_0 and initial distribution η_0

$$\Pr\left\{\lim_{t\to+\infty}\mathbf{E}\left\{\|x(t,x_0,\eta_0)\|\right\}=0\right\}=1.$$

Remark 1.2 By analyzing the stochastic properties of the transition matrix for jump linear systems, it has been shown in [36] that the second moment stability concepts, namely, mean square stability, stochastic stability and exponential mean square stability are all equivalent, and any one of them implies almost sure stability.

Remark 1.3 The moment stability implies almost sure sample stability was observed earlier in [40], for a special class of randomly switched systems. A general result for systems in the form (1.1) with $\{\eta_t, t \geqslant 0\}$ an arbitrary stationary random process satisfying certain separability and boundness conditions was obtained by [39]. In particular, for (1.1), the results in [39] give the fact that (ii) or (iii) implies (iv).

Remark 1.4 For discrete-time MJS (1.2) with a homogeneous Markov chain $\{\gamma_k, k \in \mathbb{Z}^+\}$ satisfying (1.4), the definitions of stochastically stabile, mean square stable and exponentially mean square stable are given in [41, 42]. Also, the work in [41] established the equivalence of these three stability concepts. Moreover, the almost sure stability for systems (1.2) and (1.4) are given in [41]. In particular, stochastically stability, mean square stability and exponentially mean square stability are each sufficient but not necessary for almost sure stability.

The following theorems on the stochastic stability of systems (1.1) and (1.2) are recalled based on Lyapunov method. The proofs can be found in the cited references.

Theorem 1.5 [18] *System* (1.1) *is stochastically stable if, and only if, there exists a set of matrices P_i, $i \in \mathcal{M}$ satisfying*

$$A_i^T P_i + P_i A_i + \mathcal{P}_i < 0, \tag{1.5}$$

where $\mathcal{P}_i \triangleq \sum_{j\in\mathcal{M}} \pi_{ij} P_j$ and $A_i \triangleq A(\eta_t)$.

Theorem 1.6 [43] *System* (1.2) *is stochastically stable if, and only if, there exists a set of matrices P_i, $i \in \mathcal{M}$ satisfying*

$$A_i^T \mathcal{P}_i A_i - P_i < 0, \tag{1.6}$$

where $\mathcal{P}_i \triangleq \sum_{j\in\mathcal{M}} \lambda_{ij} P_j$.

Remark 1.7 It should be emphasized that Theorems 1.5 and 1.6 play an important role in determining the stability of MJS, and are given in the form of strict linear matrix inequalities that allow solutions to be readily obtained via available optimization techniques.

1.1.1.2 Stability of MJS with Incomplete Transition Descriptions

In previous subsections, all the transition rates or transition probabilities in the corresponding Markov jumping process, as a crucial factor, are assumed to be completely accessible. In practice, it is difficult to obtain the exact value of the switching probabilities. For instance, in the soft landing process of a reentry body [44], the probability of opening the parachute is determined by the altitude as well as its rate of change. Another example refers to internet based networked control systems, where the packets dropouts and channel delays can be modeled by Markov chains [45], but the delay or packet loss is distinct at different periods, which leads to the resulting transition probability matrix changing throughout the running time. Similar phenomenon also arises in other systems, such as electronic circuits, mental health analysis, and manpower systems. To overcome the above issues, MJS with uncertain transition probabilities have been studied in [28, 46, 47], in which robust approaches were adopted to cope with some compact sets with polytopic-type or norm-bounded structure in the transition probability matrix.

In [48–50], the transition rates or transition probabilities of the jump are considered to be partially accessed, i.e., some elements in matrix Π or Λ are unknown. For notational clarity, denote $l = l_K^i + l_{UK}^i$ for any $i \in \mathcal{M}$ with

$$l_K^i \triangleq \{j : \pi_{ij} (\text{or } \lambda_{ij}) \text{ is known}\}, \tag{1.7}$$

$$l_{UK}^i \triangleq \{j : \pi_{ij} (\text{or } \lambda_{ij}) \text{ is unknown}\}. \tag{1.8}$$

Also, we denote $\pi_K^i \triangleq \sum_{j \in l_K^i} \pi_{ij}$ and $\lambda_K^i \triangleq \sum_{j \in l_K^i} \lambda_{ij}$, respectively.

Remark 1.8 The accessibility of the jumping process $\{\eta_t, t \geqslant 0\}$ (or $\{\gamma_k, k \in \mathbb{Z}^+\}$) in literature is commonly assumed to be completely accessible ($l_{UK} = \emptyset, l_K = l$) or completely inaccessible ($l_{UK} = l, l_K = \emptyset$). Moreover, the transition rates or probabilities with polytopic or norm-bounded uncertainties require the knowledge of bounds or structure of uncertainties, which can still be viewed as accessible. Therefore, the transition rates or probabilities matrix considered in [48–51] is a more natural assumption to MJS, thus have more practical potentials.

The following theorems present sufficient conditions on the stochastic stability of the considered system with partially known transition probabilities (1.7) and (1.8), respectively.

Theorem 1.9 [50] *Consider system* (1.1) *with partially known transition rates* (1.7). *The corresponding system is stochastically stable if there exist matrix $P_i, i \in \mathcal{M}$, such that*

$$\begin{aligned}
&(1+\pi^i_K)(A_i^T P_i + P_i A_i) + \mathcal{P}^i_K < 0,\\
&A_i^T P_i + P_i A_i + P_j \geqslant 0, \quad \forall j \in l^i_{UK},\ j = i,\\
&A_i^T P_i + P_i A_i + P_j \leqslant 0, \quad \forall j \in l^i_{UK},\ j \neq i,
\end{aligned}$$

where $\mathcal{P}^i_K \triangleq \sum_{j\in l^i_K} \pi_{ij} P_j$.

Theorem 1.10 [50] *Consider system* (1.2) *with partially known transition probabilities* (1.8). *The corresponding system is stochastically stable if there exist matrix* $P_i, i \in \mathcal{M}$, *such that*

$$\begin{aligned}
&A_i^T \mathcal{P}^i_K A_i - \lambda^i_K P_i < 0,\\
&A_i^T P_j A_i - P_i < 0, \quad \forall j \in l^i_{UK},
\end{aligned}$$

where $\mathcal{P}^i_K \triangleq \sum_{j\in l^i_K} \lambda_{ij} P_j$.

In [50], the stability and stabilization problems of a class of continuous-time and discrete-time MJS with partially known transition probabilities are investigated. The system under consideration is more general, which covers the systems with completely known and completely unknown transition probabilities as two special cases-the latter is hereby the switched linear systems under arbitrary switching. Moreover, in contrast with the uncertain transition probabilities studied recently, the concept of partially known transition probabilities proposed in [50] does not require any knowledge of the unknown elements. The sufficient conditions for stochastic stability and stabilization of the underlying systems are derived via linear matrix inequality formulation, and the relation between the stability criteria currently obtained for the usual MJS and switched linear systems under arbitrary switching, are exposed by the proposed class of hybrid systems.

1.1.2 Control and Filtering of MJS

In this section, we will review some recent advancements on control and filter design for MJS. Till date, a great number of fundamental concepts and results on MJS have been developed, for instance, optimal control [2, 4, 8, 9, 52], dissipative control [53], observer design [54–56]. Other results related to discrete-time MJS have also been reported in [19, 25, 29, 57, 58].

Definition 1.11 [22] Consider the following MJS:

$$\dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)u(t), \tag{1.9}$$

$$x(k+1) = A(\gamma_k)x(k) + B(\gamma_k)u(t). \tag{1.10}$$

If there exists a feedback control

$$u(t) = K(\eta_t)x(t), \text{ (respectively, } u(k) = K(\gamma_k)x(k)), \tag{1.11}$$

such that the resulting closed-loop control system is stable, where $K(\eta_t)$ (respectively $K(\gamma_k)$) is the controller gain to be determined, respectively. Then the control system (1.9) (respectively (1.10)) is said to be stochastically stabilizable in the corresponding sense. If the resulting closed-loop system is absolutely stable, then (1.9) (respectively (1.10)) is absolutely stabilizable.

For the stabilization problem of MJS, readers may refer to [20, 21, 25–27, 29, 59, 60]. Design of control systems that can handle model uncertainties has been one of the most challenging problems and received considerable attention from academics, scientists and engineers in the past decades. There are two major issues in robust controller design. The first is concerned with the robust stability of the uncertain closed-loop system (see for example, [20] and the references therein), and another is robust performance. On the other hand, convex analysis has shown to be a powerful tool to derive numerical algorithms for control problems. For state feedback MJS case, convex analysis had been previously considered in [61].

In order to ensure the performance of MJS, the researchers have proposed linear quadratic control theory [22], H_2 control theory [62, 63], H_∞ control theory [19, 57, 64, 65], H_∞ filtering theory [66–68], and so on by defining accordingly performance index. Since its introduction in 1980s, the so-called H_∞ optimal control has been one of the most attractive and dominated research topics in the past 30 years [69].

Consider the following MJS in probability space $(\Omega, \mathscr{F}, \Pr)$:

$$\begin{aligned} \dot{x}(t) &= A(\eta_t)x(t) + B(\eta_t)u(t) + G(\eta_t)w(t), \\ z(t) &= C(\eta_t)x(t) + D(\eta_t)u(t), \end{aligned} \tag{1.12}$$

where $\{\eta_t, t \geqslant 0\}$ are finite Markov processes satisfying (1.3); $x(t)$ is the system state; $u(t)$ is control input satisfied (1.11); $w(t)$ is the disturbance input which belongs to $\mathcal{L}_2[0, \infty)$; and $z(t)$ is the controlled output which belongs to $\mathcal{L}_2[0, \infty)$.

Definition 1.12 [64] Consider system (1.12) with $\{\eta_t, t \geqslant 0\}$ are satisfied (1.3). We are concerned with designing a state feedback controller (1.11), such that, for all nonzero $w(t) \in \mathcal{L}_2[0, \infty)$

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2, \tag{1.13}$$

where $\gamma > 0$ is a prescribed level of disturbance attenuation to be achieved and

$$\|z(t)\|_{E_2} = \mathbf{E}\left\{\int_0^T z^T(t)z(t)\mathrm{d}t\right\}^{1/2}.$$

When (1.13) is satisfied, the system (1.12) with controller (1.11) is said to have H_∞ performance (1.13) over the horizon $[0, T]$.

Both the cases of continuous-time and discrete-time dynamical linear and nonlinear systems have been intensively studied. H_∞ control for MJS were investigated in [19, 21, 57, 64, 65, 70, 71], while robust H_∞ control for MJS with unknown nonlinearities was studied in [64]. H_∞ control was designed in [57] for discrete-time MJS with bounded transition probabilities; the robust H_∞ control problem was considered in [19] for uncertain MJS with time delay; and the delay-dependent H_∞ control problem were discussed in [65, 71] for singular MJS with time-varying delays.

Remark 1.13 Note that H_∞ performance analysis plays a vital role in controller design of MJS. Apart from this approach, other methods for performance analysis have also yielded important results. For instance, H_2 performance [63], L_1 gain performance [72]. In addition, the $\mathcal{L}_2$–$\mathcal{L}_\infty$ performance, which is also referred to as the energy-to-peak performance [73] and H_2 extended performance, is an important index and has received considerable attention.

As is well known, filtering technique has been playing an important role in a variety of application areas including signal processing, target tracking, and image processing [74, 75]. Up to now, many important developments on the filtering problem have been made for MJS. The designed filters can be classified into two types: mode-dependent filters [46, 66, 67, 76–79] and mode-independent filters [68, 75, 80–82].

Consider the following MJS in probability space $(\Omega, \mathscr{F}, \Pr)$:

$$\begin{aligned}\dot{x}(t) &= A(\eta_t)x(t) + B(\eta_t)w(t),\\ y(t) &= C(\eta_t)x(t) + D(\eta_t)w(t),\\ z(t) &= L(\eta_t)x(t),\end{aligned} \tag{1.14}$$

where $\{\eta_t, t \geqslant 0\}$ is a finite Markov process satisfying (1.3); $x(t) \in \mathbb{R}^n$ is the state; $w(t) \in \mathbb{R}^m$ is the noise signal (including process and measurement noises), which is assumed to be an arbitrary signal in $\mathcal{L}_2[0, \infty)$; $y(t) \in \mathbb{R}^l$ is the measurement; and $z(t) \in \mathbb{R}^s$ is the signal to be estimated.

The filtering problem to be addressed is to obtain an estimate $\hat{z}(t)$ of $z(t)$ via a causal mode-dependent linear filter which provides a uniformly small estimation error, $\tilde{z}(t) \triangleq z(t) - \hat{z}(t)$, for all $w(t) \in \mathcal{L}_2[0, \infty)$. Attention is focused on the design of a linear time-invariant, asymptotically stable, filter with state space-realization

$$\begin{aligned}\dot{\hat{x}}(t) &= A_f(\eta_t)\hat{x}(t) + B_f(\eta_t)y(t),\\ z(t) &= C_f(\eta_t)\hat{x}(t),\end{aligned} \tag{1.15}$$

where the matrices $A_f(\eta_t) \in \mathbb{R}^{n\times n}$, $B_f(\eta_t) \in \mathbb{R}^{n\times l}$, and $C_f(\eta_t) \in \mathbb{R}^{s\times n}$ are to be designed.

It follows from (1.14)–(1.15) that the dynamics of the estimation error $\tilde{z}(t)$ can be described by the following state-space model:

$$\begin{aligned}\dot{\xi}(t) &= \tilde{A}(\eta_t)\xi(t) + \tilde{B}(\eta_t)w(t),\\ \tilde{z}(t) &= \tilde{C}(\eta_t)\xi(t),\end{aligned} \tag{1.16}$$

where

$$\tilde{A}(\eta_t) \triangleq \begin{bmatrix} A(\eta_t) & 0 \\ B_f(\eta_t)C(\eta_t) & A_f(\eta_t) \end{bmatrix}, \quad \xi(t) \triangleq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$
$$\tilde{B}(\eta_t) \triangleq \begin{bmatrix} B(\eta_t) \\ B_f(\eta_t)D(\eta_t) \end{bmatrix}, \quad \tilde{C}(\eta_t) \triangleq \begin{bmatrix} L(\eta_t) & -C_f(\eta_t) \end{bmatrix}.$$

Then, the robust H_∞ filtering problem addressed is formulated as follows: given MJS (1.14), determine a filter system (1.15) such that the filtering error system (1.16) is stochastically stability (exponential mean-square stability), and satisfies a prescribed H_∞ performance index. The problem of H_∞ filter for MJS was investigated in [46], where a method for designing a mode-independent filter was proposed. The results reported in [46] further improved in [83]. The problem of mode-independent H_∞ filtering was discussed in [84] for singular MJS, and the full-order and reduced-order filters were designed in a unified framework. In [85], the problem of mode-independent H_∞ filter was addressed for discrete-time MJS, and a design procedure has been proposed. It is noted that the mode-independent filter is very useful when the system mode information is completely unaccessible. However, it should be pointed out that the mode-independent filters cannot deal with the complex asynchronous phenomenon between filter modes and system modes, all the available modes information are neglected, which inevitably leads to conservatism to some extent. On the other hand, Kalman filtering for continuous-time uncertain MJS was considered in [76]; the H_∞ filter problem for continuous- and discrete-time MJS were studied in [66, 67], respectively. In the mean time, Wu et al. extended the H_∞ filtering problem to 2-D MJS; and the quantized H_∞ filtering for Markovian jump linear parameter varying systems was studied with intermittent measurements [82].

In addition of filtering design, the fault detection problem for MJS was investigated in [86–92]. Specifically, Meskin and Khorasani considered the fault detection and isolation problems in [86] for discrete-time MJS with application to a network of multi-agent systems with imperfect communication channels; while Nader et al. proposed a geometric approach to fault detection and isolation for continuous-time MJS in [87]. The work in [88] studied the problem of generalized H_2 fault detection for two-dimensional MJS; while the work in [89] developed fault detection filter design for singular MJS with intermittent measurements. The problems of robust fault detection were addressed in [90] and [91] respectively.

Apart from the above-mentioned synthesis problems for MJS, the model reduction problem for such systems has also been investigated, see for example, [93, 94]. In [93], the model reduction problem was considered for discrete-time MJS; and Zhang et al. considered H_∞ model reduction for both continuous- and discrete-time MJS [94].

On another research front line, sliding mode control (SMC) has received noticeable attention since it has various attractive features such as fast response, good transient performance, order reduction and so on. In particular, SMC laws are robust with respect to the so-called matched uncertainty, see for example, [72, 74, 75, 95–103]. However, due to the system is switching stochastically between different subsystems, the dynamics of the jump systems can not stay on each sliding surface of subsystems forever, therefore, it can not be determined whether the closed-loop system is stochastically stable or not. SMC of MJS with actuator nonlinearities was considered in [98]; Ma and Boukas proposed a singular system approach to robust SMC for uncertain MJS [100]; SMC problem for singular MJS was solved in [75]; the problems of state estimation and SMC of singular MJS were discussed in [74]; and also SMC design with bounded l_2 gain performance for singular time-delay MJS in [72]. Recently, Shi et al. designed the SMC of MJS [102]. Motivated by the work in [102], Wu et al. considered the SMC problem for singular MJS where the main difficulties come from the sliding surface function design and the stochastic admissibility analysis for the resulting sliding mode dynamics [74].

1.1.2.1 Networked-Based Markovian Control Systems

Networked control systems (NCS) are a type of distributed control systems, where the information of control system components is exchanged via communication networks. The introduction of networks also presents some constraints such as time delays and packet dropouts which bring difficulties for analysis and design of NCS. Nowadays, various methodologies have been proposed for modeling, stability analysis, and controller design for NCS in the presence of network-induced time delays and packet dropouts. The Markov chain, a discrete-time stochastic process with the Markov property, can be effectively used to model the network-induced delays in NCS. In [104, 105], the time delays in NCS were modeled by Markov chains, and further an linear quadratic Gaussian optimal controller design method was proposed. Xiao et al. developed two types of controller design methods for NCS modeled as finite dimensional [106], discrete-time jump linear systems: One is the state feedback controller that only depends on delays from sensor to controller (S-C delays), and is called the one-mode-dependent controller; the other is the output feedback controller that does not depend on either the S-C delays or the C-A delays (delays from controller to actuator), and called the mode-independent controller.

Zhang et al. extended the idea of [106] and used two Markov chains to model the delays in both feedback and forward channels [107]. It is assumed that at each sampling instant, the current S-C delay (τ_{sc}^k) and previous C-A delay (τ_{ca}^k) can be obtained by the time-stamping technique. However, practically the previous C-A delay is not always available because the information about C-A delays needs to be transmitted through the S-C communication link before reaching the controller. More precisely, the discretized controlled plant is considered as

$$x(k+1) = Ax(k) + Bu(k), \tag{1.17}$$
$$x(k) = \phi(k),\ k \in [-\bar{\tau} - \bar{d}, -\bar{\tau} - \bar{d} + 1, \ldots, 0]. \tag{1.18}$$

The control input can be obtained as

$$u(k) = K\left(\tau_{sc}^{k}, \tau_{ca}^{k-1}\right) x\left(k - \tau_{sc}^{k} - \tau_{ca}^{k}\right),$$

where the delays τ_{ca}^{k} and τ_{ca}^{k} are subject to Markov chain with

$$\hat{\lambda}_{ij} = \Pr\left(\tau_{sc}^{k+1} = j | \tau_{sc}^{k} = i\right),$$
$$\check{\lambda}_{rs} = \Pr\left(\tau_{ca}^{k+1} = s | \tau_{ca}^{k} = r\right),$$

where $\hat{\lambda}_{ij} \geqslant 0$, $\check{\lambda}_{rs} \geqslant 0$, $\sum_{j=0}^{\bar{r}} \hat{\lambda}_{ij} = 1$, and $\sum_{s=0}^{\bar{d}} \check{\lambda}_{rs} = 1$, $\forall i, j, r, s \in \mathcal{M}$.

Remark 1.14 Note that in [107], the designed controller gain ($K(\tau_{sc}^{k}, \tau_{ca}^{k-1})$) depends on the current feedback channel delay and the previous forward channel delay. Yet, the property of NCS listed above enables a set of control commands to be sent from the controller site, by which the control command can be selected at the smart actuator according to the current forward channel delay. In addition to the stochastic description of delay variations, nondeterministic descriptions can be considered within the framework, i.e., assuming the transitions probabilities are completely unknown and the variations of the time delays are state dependent or time dependent.

The work for network-induced delays issue is classified whether the methodology is dependent on the delay information online or not. Similar thought is also applicable to the context of packet losses problem in NCS.

Consider system (1.17). Let $\mathcal{J} \triangleq \{i_1, i_2, \ldots\}$, and a subsequence of $\{1, 2, 3, \ldots\}$ denote the sequence of time points of successful data transmissions from the sampler to the zero-order hold, and $s \triangleq \max_{i_k \in \mathcal{J}}(i_{k+1} - i_k)$ be the maximum packet-loss upper bound. Then the following concept and mathematical models are introduced to capture the nature of packet losses.

Definition 1.15 [108] A packet-loss process is defined as

$$\{\eta(i_k) \triangleq i_{k+1} - i_k : i_k \in \mathcal{L}\}, \tag{1.19}$$

which takes values in the finite state space $\mathcal{M}$.

Definition 1.16 [108] Packet-loss process (1.19) is said to be Markovian if it is a discrete-time homogeneous Markov chain on a complete probability space $(\Omega, \mathscr{F}, \Pr)$, and takes values in $\mathcal{M}$ with known transition probabilities matrix $\Lambda \triangleq (\lambda_{ij}) \in \mathbb{R}^{M\times M}$, where

$$\lambda_{ij} = \Pr(\eta(i_{k+1}) = j \,|\eta(i_k) = i), \tag{1.20}$$

for any $i, j \in \mathcal{M}$, and $\sum_{j=1}^{M} \lambda_{ij} = 1$.

From the viewpoint of the zero-order hold, the control input is

$$u(l) \triangleq u(i_k) = Kx(i_k), \tag{1.21}$$

for $i_k \leqslant l \leqslant i_{k+1} - 1$. The initial inputs are set to zeros: $u(l) = 0, 0 \leqslant l \leqslant i_1 - 1$. Hence the closed-loop system becomes

$$x(l+1) = Ax(l) + BKx(i_k), \tag{1.22}$$

for $i_k \leqslant l \leqslant i_{k+1} - 1, i_k \in \mathcal{M}$. The objective of analysis and design of NCS with packet losses is to construct controller (1.21) such that NCS (1.22) is stable.

Next, sufficient conditions for stochastic stability of the closed-loop NCS are obtained via Markovian theories and the packet-loss-dependent Lyapunov function approach.

Theorem 1.17 [108] *The closed-loop system* (1.22) *with a Markovian packet-loss process defined as in* (1.20) *is stochastically stable, if there exist positive symmetry matrices $P_i > 0, i \in \mathcal{M}$, such that*

$$(A+BK)^T\left(\sum_{j=1}^{M}\lambda_{ij}P_j\right)(A+BK) - P_i < 0, \ \forall i \in \mathcal{M}.$$

One framework for the analysis and design of NCS with packet losses is the offline framework, where the controller is designed despite any situation of real packet losses [108–110]. However, there exist the effects of the current packet cases (dropped and received successfully) on the future packet cases. A Markov process is able to represent such effect [108, 111]. In [108], the time interval between packet successful transmissions was used as a state in a Markov chain. The relations among the amount of consecutive packet dropouts are employed to establish the transition probabilities matrix. In [112], a Markov process was used to describe the quantity of packet dropouts between the current time instant and the latest successful transmission.

1.1.2.2 Control and Filtering for MJS with Incomplete Transition Probabilities

The ideal knowledge on the transition probabilities are definitely expected to simplify the system analysis and design. However, the likelihood of obtaining such available knowledge is actually questionable, and the cost is probably expensive. Therefore, rather than having a large complexity to measure or estimate all the transition probabilities, it is significant and necessary, from control perspectives, to further study more general jump systems with partially unknown transition probabilities.

The uncertainties in transition probabilities in [28] and [81] were represented by norm-bounded or polytopic description. Then, robust control and filtering methods are utilized to deal with the uncertainties presumed in the transition probabilities. Considering a more realistic situation that some parts of the elements in the desired transition probabilities matrix are hard to obtain, Zhang studied the stability, stabilization, and H_∞ filtering problems for MJS with partially known transition probabilities in [48–50]. The significance of this hypothesis lies in that rather than having a large complexity to measure all the transition probabilities, it is more meaningful to directly study MJS with partially unknown transition probabilities. The transient and steady performance for MJS with partially known transition probabilities were considered in [113, 114] in time domain.

On the other hand, there is a general lack of online sensors in many fields, such as pharmacy industry, fermentation process, and petrochemical industry. Therefore, state estimation problem is an important research issue in control discipline. Meanwhile, considering the practical applications that transition probabilities are generally determined by physical experiments or numerical simulations that lead to the transition probabilities with stochastic features, H_∞ filtering problem for MJS, viewed from the stochastic standpoint, was studied in [114]. It assumes that the exact value of transition probabilities is unknown, but the distribution can be approximated by Gaussian process. To obtain the expectation of unknown transition probabilities from Gaussian probability density function (PDF), a discretization method is developed. On this basis, a H_∞ filter was designed such that the worst-case induced l_2 gain from process noise to estimation error is minimized. Different from the existing results in literatures, a Gaussian PDF is utilized to characterize the relative likelihood for unknown transition probabilities to occur at a given constant. With the Gaussian distribution of transition probabilities, we can obtain the expectation of unknown transition probabilities. Moreover, the considered systems are more general than the systems with completely known and partially known transition probabilities, which can be viewed as two special cases of the ones tackled here.

1.2 Analysis and Control of S-MJS

1.2.1 Literature Review of S-MJS

MJS, although important in theory and useful to describe many practical systems, have many limitations in applications, since sojourn-time in MJS is subject to exponential distribution or geometric distribution in continuous- and discrete-time domains, respectively. If this assumption is not satisfied, the transition rates or transition probabilities will be time-varying in the time domain. S-MJS are characterized by a fixed matrix of transition probabilities and a matrix of sojourn-time probability density functions. Due to their relaxed conditions on the probability distributions, S-MJS have much broader applications than the conventional MJS. Indeed, most of the modeling, analysis, and design results for MJS would be special cases of S-MJS. Thus, this area of research is significant not only in theory, but also in practice.

Till now the developed theories on S-MJS are far from maturity yet. In [115, 116], probability distributions of sojourn-time on Markovian processes, from an exponential distribution to a Weibull one, were discussed. Therefore, the transition rate is time-varying instead of constant. It has been moved one step further towards the numerically solvable conditions by making use of the upper and lower bounds of the transition rate. However, the main proposition in [116] to partition the sojourn-time into M sections in each working mode is relatively conservative and practically infeasible. Stochastic stability of S-MJS was studied in [117–119]. Zhang et al. [120] proposed the concept of σ-mean square stability (σ-MSS) for S-MJS, where σ is capable of characterizing the degree of approximation error of σ-MSS to MSS, that is, some S-MJS are not MSS but σ-MSS. The semi-Markov kernel was used to derive the stability and stabilization criteria so that the information of probability density functions (PDF) of sojourn-time can be explicitly included and the PDF are dependent on both current and next system mode.

Definition 1.18 [115, 116] The evolution of the semi-Markov process $\{\eta_t, t \geqslant 0\}$ is governed by the following probability transitions:

$$\Pr(\eta_{t+h} = j \,|\eta_t = i) = \begin{cases} \pi_{ij}(h)h + o(h), & i \neq j, \\ 1 + \pi_{ij}(h)h + o(h), & i = j, \end{cases} \tag{1.23}$$

where $h > 0$ is the sojourn-time from mode i to mode j, $\lim_{h\to 0} o(h)/h = 0$ and $\pi_{ij} \geqslant 0$ is the transition rate from mode i at time t to mode j at time $t + h$, and $\pi_{ii}(h) = -\sum_{j=1, j\neq i}^{M} \pi_{ij}(h)$.

Remark 1.19 The transition rate (1.23) in Definition 1.18 is different from (1.3) since it is dependent on the sojourn-time h. ♦

The application of semi-Markov process in fault-tolerant control systems was discussed in [121], and it was shown that when a practical system does not satisfy the so-called memoryless restriction, the widely used Markov switching scheme would not be applicable. A typical transition rate in the bathtub shape in the reliability analysis was reported in [122]. Also, the work in [123] considered the control problem of singularly perturbed MJS and S-MJS, as well as a particular control problem in accelerator physics known as the bunch-train cavity interaction (BTCI). It is shown that the BTCI is in fact a physical model example of a linear S-MJS.

1.2.2 Mathematical Descriptions and Basic Concepts

In what follows, we recall some definitions and existing results, which will be used in the development of our main results of the book.

Definition 1.20 [124] A probability distribution $\mathcal{F}(\cdot)$ on $[0, \infty)$ is said to be a phase-type (PH) distribution with representation $(\mathbf{a}, T)$ if it is the distribution of the time until absorption in a finite-state Markov processes on the states $\{1, 2, \dots, m+1\}$ with generator $\mathcal{Q} = \begin{bmatrix} T & T_0 \\ 0_{1\times m} & 1 \end{bmatrix}$, where the matrix $T = (T_{ij})_{m\times m}$ satisfies $T_{ij} \geqslant 0$, $Te \leqslant e$, and e denotes an appropriately dimensioned column vector with all components equal to one. The initial PH distribution is $(\mathbf{a}, a_{m+1})$ where $\mathbf{a} = (a_1, a_2, \dots, a_m)$ and $\mathbf{a}e + a_{m+1} = 1$. The states $\{1, 2, \dots, m\}$ are transient and the state $m+1$ is absorbing.

Assumption 1 Assume that the absorbing state is reached with probability one for a finite time.

Definition 1.21 [124] Let $\mathcal{E}$ be a finite or countable set. A stochastic process $\bar{\eta}_t$ on the state space $\mathcal{E}$ is called a denumerable PH semi-Markov process, if the following conditions hold.

- (i) The sample paths of $(\bar{\eta}_t, t < +\infty)$ are right-continuous functions and have left-hand limits with probability one;
- (ii) Denote the nth jump point of the process $\bar{\eta}_t$ by τ_n, where $\tau_0 = 0 < \tau_1 < \tau_2 < \dots < \tau_n < \dots$, and $\tau_n, n = 1, 2, \dots$, are Markovian moments of the process $\bar{\eta}_t$;
- (iii) $\mathcal{F}_{ij}(t) \triangleq \Pr(\tau_{n+1} - \tau_n \leqslant t \,|\, \bar{r}_{\tau_n} = i, \bar{r}_{\tau_{n+1}} = j) = \mathcal{F}_i(t), i, j \in \mathcal{E}, t \geqslant 0$ do not depend on j and n; and
- (iv) $\mathcal{F}_i(t), i \in \mathcal{E}$ is a PH distribution.

Remark 1.22 The definition of PH semi-Markov chain, readers may refer to [117, 118, 125] for more details. It is worth noting that the PH distribution is a generalization of the exponential distribution while still preserving much of its analytic tractability, and has been used in a wide range of stochastic modeling applications in areas as diverse as reliability theory, queueing theory and biostatistics. Furthermore, the family of PH distribution is dense in all the families of distributions on $[0, +\infty)$. So, for every probability distribution on $[0, +\infty)$, we may choose a PH distribution to approximate the original distribution in any accuracy.

Lemma 1.23 [124] *Let Γ be the time until absorption in the state $m+1$ defined by:*

$$\Gamma = \inf\{t \geqslant 0 |\, \bar{\eta}_t = m+1\}.$$

Then, the probability distribution $\mathcal{F}(\cdot)$ of the time until absorption in the state $m+1$, corresponding to the initial probability vector $(\mathbf{a}, a_{m+1})$, is given by $\mathcal{F}(t) = 1 - \mathbf{a}\exp(Tt)e$, $t \geqslant 0$.

Proof From the definition of Γ, we have, for all $t \geqslant 0$,

$$\Gamma \leqslant t \Leftrightarrow \bar{\eta}_t = m+1.$$

It follows, according to the form of $P_{i,j}(t) = \Pr(\bar{\eta}_t = j |\, \bar{r}_0 = i)$, that:

$$\begin{aligned}\Pr(\Gamma \leqslant t) &= \Pr(\bar{\eta}_t = m+1)\\ &= \sum_{i=1}^{m+1} a_i P_{i,m+1}(t)\\ &= \mathbf{a}\big(e - \exp(Tt)e\big) + a_{m+1}\\ &= \mathbf{a}e + a_{m+1} - \mathbf{a}\exp(Tt)e\\ &= 1 - \mathbf{a}\exp(Tt)e,\end{aligned}$$

which completes the proof. □

Let $\mathbf{a}^{(i)} \triangleq \left(a_1^{(i)}, a_2^{(i)}, \ldots, a_{m^{(i)}}^{(i)}\right)$ and $T^{(i)} \triangleq \left(T_{vs}^{(i)},\ v, s \in \mathcal{E}^{(i)}\right)$. Then $(\mathbf{a}^{(i)}, T^{(i)})$, $i \in \mathcal{E}$, denotes the $m^{(i)}$ order representation of $\mathcal{F}_i(t)$, and $\mathcal{E}^{(i)}$ be the corresponding set of all transient states set. It is not difficult to show that the number of the elements in $\mathcal{E}^{(i)}$ is $m^{(i)}$.

Also, let

$$\begin{aligned}p_{ij} &\triangleq \Pr\Big(\bar{r}_{n+1} = j |\, \bar{r}_n = i,\ i, j \in \mathcal{E}\Big),\\ P &\triangleq (p_{ij},\ i, j \in \mathcal{E}),\ (\mathbf{a}, T) \triangleq \{(\mathbf{a}^{(i)}, T^{(i)}),\ i \in \mathcal{E}\}.\end{aligned}$$

Obviously, the probability distribution of $\bar{\eta}_t$ can be determined only by $\{P, (\mathbf{a}, T)\}$. For every n, $\tau_n \leqslant t \leqslant \tau_{n+1}$, define

$$J(t) \triangleq \text{the phase of } \mathcal{F}_{\bar{\eta}_t}(\cdot) \text{ at time } t - \tau_n.$$

For any $i \in \mathcal{E}$, define

$$T_j^{(i,0)} \triangleq -\sum_{s=1}^{m^{(i)}} T_{js}^{(i)},\ j = 1, 2, \ldots, m^{(i)},$$
$$G \triangleq \left\{(i, s^{(i)})|\, i \in \mathcal{E},\ s^{(i)} = 1, 2, \ldots, m^{(i)}\right\}. \tag{1.24}$$

Lemma 1.24 [117] *$Z(t) = (\bar{\eta}_t, J(t))$ is a Markov chain with state space G (G is finite, if and only if $\mathcal{E}$ is finite). The infinitesimal generator of $Z(t)$, given by $Q = (q_{\mu\varsigma}, \mu, \varsigma \in G)$, is determined only by the pair of $(\bar{\eta}_t, J(t))$ given by $\{P, (\mathbf{a}, T)\}$ as follows:*

$$\begin{cases} q_{(i,k^{(i)})(i,k^{(i)})} = T^{(i)}_{k^{(i)}k^{(i)}}, & (i, k^{(i)}) \in G, \\ q_{(i,k^{(i)})(i,\bar{k}^{(i)})} = T^{(i)}_{k^{(i)}\bar{k}^{(i)}}, & k^{(i)} \neq \bar{k}^{(i)},\ (i, k^{(i)}) \in G,\ (i, \bar{k}^{(i)}) \in G, \\ q_{(i,k^{(i)})(j,k^{(j)})} = p_{ij} T^{(i,0)}_{k^{(i)}} a^{(j)}_{k^{(j)}}, & i \neq j, (i, k^{(i)}) \in G,\ (j, k^{(j)}) \in G. \end{cases}$$

By (1.24), we obtain that G has $N = \sum_{i \in \mathcal{E}} m^{(i)}$ elements, then the state space of $Z(t)$ has N elements. For convenience, we denote the summation $\sum_{r=1}^{i-1} m^{(r)} + s$ by $\psi(i, s)$ for $1 \leqslant s \leqslant m^{(i)}$. Hence $\psi(i, s) = \sum\limits_{r=1}^{i-1} m^{(r)} + s,\ i \in \mathcal{E},\ 1 \leqslant s \leqslant m^{(i)}$. Moreover, define $r_t \triangleq \psi(\bar{\eta}_t, J(t)) = \psi(Z(t))$ and $\lambda_{\psi(i,s)\psi(i',s')} \triangleq q_{\psi(i,s)\psi(i',s')}$.

Therefore, by Lemma 1.24 and employing the same techniques as those used in [117], r_t is an associated Markov process of $\bar{\eta}_t$ with the state space $\mathcal{N} \triangleq \{1, 2, \ldots, N\}$ and the infinitesimal generator is $\Pi = (\pi_{\alpha\beta})$, $1 \leqslant \alpha, \beta \leqslant N$, such that

$$\Pr(r_{t+h} = \beta|\, r_t = \alpha) = \Pr(\psi(Z(t) + h) = \beta|\, \psi(Z(t)) = \alpha)$$
$$= \begin{cases} \pi_{\alpha\beta} h + o(h), & \alpha \neq \beta, \\ 1 + \pi_{\alpha\beta} h + o(h), & \alpha = \beta, \end{cases}$$

where $\pi_{\alpha\beta}$ is the transition rate from mode α at time t to mode β at time $t + h$ when $\alpha \neq \beta$ and $\pi_{\alpha\beta} = -\sum_{\beta=1,\beta\neq\alpha}^{N} \pi_{\alpha\beta}$; also we have $h > 0$ and $\lim\limits_{h\to 0} o(h)/h = 0$.

In the following, by a supplementary variable technique and a novel transformation, a finite PH semi-Markov process has been transformed into a finite Markov chain, which is called its associated Markov chain. Consequently, a class of PH S-MJS can be equivalently expressed as its associated Markovian system.

Consider a class of stochastic differential equations in the probability space $(\Omega, \mathscr{F}, \Pr)$ for $t > 0$

$$dx(t) = \hat{f}(x(t), t, \hat{r}_t)dt + \hat{g}(x(t), t, \hat{r}_t)d\omega(t),$$
$$x(0) = x_0, \tag{1.25}$$

where $\omega(t)$ be an m-dimensional Brownian motion defined on the probability space, and we assume that the semi-Markov chain $\{\hat{r}_t, t \geqslant 0\}$ is independent of the Brownian motion $\omega(t)$. The initial state $x_0 \in \mathbb{R}^n$ is a fixed constant vector. $f(\cdot) : \mathbb{R}^n \times \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \times \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}^{n\times m}$.

By the following theorem, a PH S-MJS can be equivalently expressed as its associated MJS. For the proof, the reader may refer to [117, 118, 125].

Theorem 1.25 [117] *System* (1.25) *is equivalent to the following system for* $t > 0$

$$\begin{aligned} \mathrm{d}x(t) &= f(x(t), t, r_t)\mathrm{d}t + g(x(t), t, r_t)\mathrm{d}\omega(t), \\ x(0) &= x_0, \end{aligned}$$

where $\{r_t, t \geqslant 0\}$ *is the associated Markov chain of PH semi-Markovian chain* $\{\hat{r}_t, t \geqslant 0\}$.

Remark 1.26 It is worth to mention that the advantages of Theorem 1.25 is that when we study stochastic stability and control problems, we can replace MJS with S-MJS, and achieve some similar results. Furthermore, more importantly, almost all the nice results obtained so far on MJS, for example, [59, 117, 126–128] can be extended S-MJS.

Next, we recall the following lemmas, which will be used to develop our main results.

Lemma 1.27 *Given any scalar* ε *and square matrix* $Q \in \mathbb{R}^{n\times n}$, *the following inequality*

$$\varepsilon\left(Q + Q^T\right) \leqslant \varepsilon^2 T + QT^{-1}Q^T$$

holds for any symmetric positive definite matrix $T \in \mathbb{R}^{n\times n}$.

Lemma 1.28 [129] *For any real matrices* Y, H *and* E *with appropriate dimensions, and* $F(t)$ *satisfying* $F^T(t)F(t) \leqslant I$, *the following inequality holds:*

$$Y + HF(t)E + E^T F^T(t)H^T < 0,$$

if and only if there exists a scalar $\varepsilon > 0$ *such that*

$$Y + \varepsilon HH^T + \varepsilon^{-1}E^T E < 0.$$

Lemma 1.29 [130] *Let* $W = W^T \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times m}$ *and* $V \in \mathbb{R}^{k\times n}$ *be given matrices, and suppose that* $\operatorname{rank}(U) < n$ *and* $\operatorname{rank}(V) < n$. *Consider the problem of finding some matrix* $\mathcal{G}$ *satisfying*

$$W + U\mathcal{G}V + (U\mathcal{G}V)^T < 0. \tag{1.26}$$

Then, the inequality (1.26) *is solvable for* $\mathscr{G}$ *if and only if*

$$U^{\perp}W(U^{\perp})^T < 0 \; and \; V^{T\perp}W(V^{T\perp})^T < 0. \tag{1.27}$$

Furthermore, if (1.27) *holds, then all the solutions of* $\mathscr{G}$ *are given by*

$$\mathscr{G} = U_R^+ \Psi V_L^+ + \Phi - U_R^+ U_R \Phi V_L V_L^+$$

with

$$\begin{aligned}
\Psi &\triangleq -\Pi^{-1} U_L^T \Lambda V_R^T \left(V_R \Lambda V_R^T\right)^{-1} + \Pi^{-1} \Xi^{1/2} L \left(V_R \Lambda V_R^T\right)^{-1/2}, \\
\Lambda &\triangleq \left(U_L \Pi^{-1} U_L^T - W\right)^{-1} > 0, \\
\Xi &\triangleq \Pi - U_L^T \left(\Lambda - \Lambda V_R^T \left(V_R \Lambda V_R^T\right)^{-1} V_R \Lambda\right) U_L > 0
\end{aligned}$$

where Φ, Π *and* L *are any appropriately dimensioned matrices satisfying* $\Pi > 0$ *and* $\|L\| < 1$.

Note that, to make each chapter in the book self-contained, some other definitions or lemmas will be recalled when needed.

1.3 Outline of the Book

The general layout of presentation of this book is divided into three parts. Part I: stability and control of some classes of S-MJS, Part II: fault detection and filtering of S-MJS, and Part III: summary of the book. The organization structure of this book is shown in Fig. 1.1, and the main contents of this book are shown in Fig. 1.2.

Chapter 1 introduced the research background and significance of MJS, as well as the current research status of MJS, so as to provide a basis of reference for further research of S-MJS. Secondly, the main differences between S-MJS and MJS have been provided, followed by a description of the advantages of the S-MJS and its broad application prospects. Additionally, we have mentioned several problems that are yet to be solved, methods that require refinements and the main research contents of this dissertation.

Part I focuses on the stability analysis and control design for some classes of S-MJS. Part I which begins with Chap. 2 consists of four chapters as follows.

Chapter 2 is concerned with the stochastic stability for a class of S-MJS with time-variant delays. By the Lyapunov function approach, together with a piecewise analysis method, sufficient conditions are proposed to guarantee the stochastic stability of the S-MJS. As more time-delay information is used, our results are

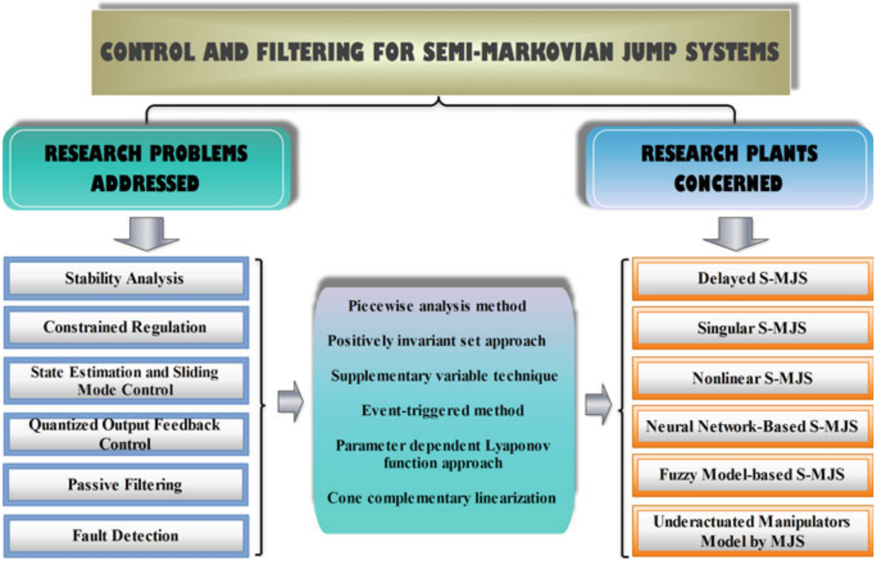

Fig. 1.1 The organization structure of the book

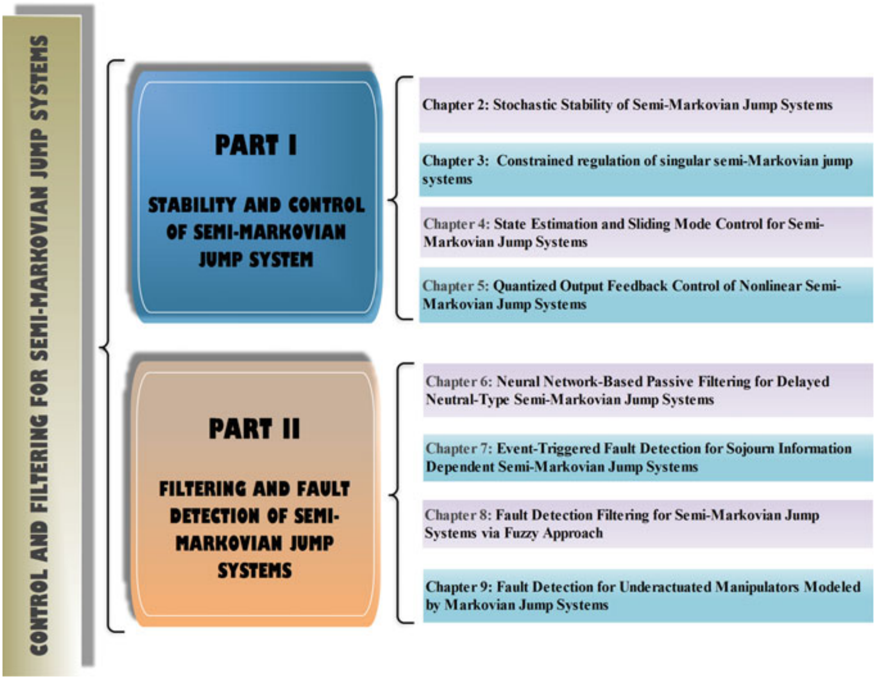

Fig. 1.2 The main contents of the book

much less conservative than some existing ones in literature. Finally, two examples are given to show the effectiveness and advantages of the proposed techniques.

Chapter 3 studies the constrained regulation problem for a class of singular S-MJS. Using a supplementary variable technique and a plant transformation presented in this Chapter, a finite PH semi-Markov process has been transformed into a finite Markov chain, which is called its associated Markov chain. Then, the S-MJS of this kind has been transformed into its associated Markovian system. Motivated by recent develops in positively invariant set, necessary and sufficient conditions for the existence of full rank solutions for a class of nonlinear equations are derived, and a new algorithm that provides a solution to the constrained regulation problem is presented.

Chapter 4 investigates the state estimation and sliding mode control problems for S-MJS with mismatched uncertainties. A sliding surface is then constructed and a sliding mode controller is synthesized to ensure that the S-MJS satisfying the reaching condition. Further, an observer-based sliding mode control problem is investigated. Sufficient conditions are established for the solvability of the desired observer. It is shown that the proposed SMC law based on the estimated states can guarantee that the sliding modes within both the state estimation space and the estimation error space are attained simultaneously.

Chapter 5 considers the quantized output feedback control problem for a class of S-MJS with repeated scalar nonlinearities. A sufficient condition for the S-MJS is developed. This condition guarantees that the corresponding closed-loop systems are stochastically stable and have a prescribed H_∞ performance. The existence conditions for full- and reduced-order dynamic output feedback controllers are proposed, and the cone complementarity linearization procedure is employed to cast the controller design problem into a sequential minimization one, which can be solved efficiently with existing optimization techniques. Finally, an application to cognitive radio systems demonstrates the efficiency of the new design method developed.

Part II studies the fault detection and filtering for S-MJS. Part II which begins with Chap. 6 consists of five chapters as follows.

Chapter 6 is concerned with the problem of exponential passive filtering for a class of stochastic neutral-type neural networks with both semi-Markovian jump parameters and mixed time delays. Our aim is to estimate the state by a Luenberger-type observer such that the filter error dynamics are exponentially mean-square stable with an expected decay rate and an attenuation level. Sufficient conditions for existence of passive filters are obtained and a cone complementarity linearization procedure is employed to transform a nonconvex feasibility problem into a sequential minimization problem, which can be readily solved by existing optimization techniques.

Chapter 7 considers the fault detection problem for a class of S-MJS with known sojourn probability. An event-driven control strategy is developed to reduce the frequency of transmission, and sufficient conditions for sojourn probability dependent jumped systems are presented. A fault detection filter is designed such that the corresponding filtering error system is stochastically stable and has a prescribed performance. In addition, a fault detection filter design algorithm is employed such that the existence conditions for the designed filer are provided.

Chapter 8 investigates the problem of fault detection filtering for S-MJS by a Takagi-Sugeno fuzzy approach. Attention is focused on the construction of a fault detection filter such that the estimation error converges to zero in the mean square and meets a prescribed system performance. The designed fuzzy model-based fault detection filter can guarantee the sensitivity of the residual signal to faults and the robustness of the external disturbances. By using the cone complementarity linearization algorithm, the existence conditions for the design of fault detection filters are provided, and the error between the residual signal and the fault signal can be made within a desired region.

Chapter 9 is concerned with the fault detection filtering problem for underactuated manipulators based on the Markovian jump model. The purpose is to design a fault detection filter such that the filter error system is stochastically stable and the prescribed probability constraint performance can be guaranteed. The existence conditions for a fault detection filter are proposed through the stochastic analysis technique, and a new fault detection filter algorithm is employed to design the desired filter gains. In addition, the cone complementarity linearization procedure is employed to cast the filter design into a sequential minimization problem.

Chapter 10 summarizes the results of the book and then proposes some related topics for the future research work.

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Part I
Stability and Control

Chapter 2
Stochastic Stability of Semi-Markovian Jump Systems

Abstract This chapter is concerned with the stochastic stability for a class of S-MJS with time-variant delays. By the Lyapunov function approach, together with a piecewise analysis method, sufficient conditions are proposed to guarantee the stochastic stability of the S-MJS. As more time-delay information is used, our results are much less conservative than some existing ones in literature. Finally, two examples are given to show the effectiveness and advantages of the proposed techniques.

2.1 Introduction

The presence of time delay is quite common in practical dynamical systems, which is frequently the main cause of instability and poor performance of the systems. As an extension of the delay-partitioning method, a piecewise analysis method (PAM) was considered in the stability analysis of the time-delay systems. The idea of PAM has initially appeared in [1], and then by this approach, some stability criteria for linear time-varying delay systems with less conservativeness were presented.

In this chapter, the stochastic stability problem has investigated for S-MJS. Specifically, the concepts of S-MJS, and mode-dependent time delays are introduced together for the stochastically stable problem in order to reflect a more realistic environment. By Lyapunov function approach, together with PAM, conditions are proposed to ensure the stochastic stability of the underlying S-MJS with time-delays. Finally, numerical examples are given to illustrate the effectiveness of the proposed control scheme.

2.2 System Description and Preliminaries

Consider a semi-Markovian jump time-delay system described by:

$$\begin{cases} \dot{x}(t) = A(\eta_t)x(t) + A_\tau(\eta_t)x(t - \tau_{\eta_t}(t)), \\ x(t) = \phi(t), \quad r(0) = r_0, \quad t \in [-\tau_M, 0], \end{cases} \tag{2.1}$$

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where $\{\eta_t, t \geqslant 0\}$ is a continuous-time Markov process on the probability space which has been defined in Definition 1.18 of Chap. 1, and $x(t) \in \mathbb{R}^n$ is the system state, $A(\eta_t)$ and $A_\tau(\eta_t)$ are matrix functions of the random jumping process $\{\eta_t, t \geqslant 0\}$. For notation simplicity, when the system operates in the ith mode, $A(\eta_t)$ and $A_\tau(\eta_t)$ are denoted by $A(i)$ and $A_\tau(i)$, respectively. $\phi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau_M, 0]$, and r_0 is the initial conditions of the continuous state and the mode. $\tau_{\eta_t}(t)$ is the mode-dependent time-varying delay (denoted by $\tau_i(t)$ for simplicity), and it satisfies

$$0 \leqslant \underline{\tau}_i \leqslant \tau_i(t) \leqslant \bar{\tau}_i, \quad \dot{\tau}_i(t) \leqslant \mu < 1, \tag{2.2}$$

where $\underline{\tau}_i$, $\bar{\tau}_i$ and μ are known scalars.

Remark 2.1 As mentioned in [2], we relax the probability distribution of sojourn-time from exponential distribution to Weibull distribution, so the transition rate in S-MJS will be time-varying instead of constant in MJS. In practice, the transition rate $\pi_{ij}(h)$ is general bounded by $\underline{\pi}_{ij} \leqslant \pi_{ij}(h) \leqslant \bar{\pi}_{ij}$, with $\underline{\pi}_{ij}$ and $\bar{\pi}_{ij}$ are real constant scalars. In this case, $\pi_{ij}(h)$ can always be described by $\pi_{ij}(h) = \pi_{ij} + \Delta\pi_{ij}$, where $\pi_{ij} = \frac{1}{2}(\bar{\pi}_{ij} + \underline{\pi}_{ij})$ and $|\Delta\pi_{ij}| \leqslant \kappa_{ij}$ with $\kappa_{ij} = \frac{1}{2}(\bar{\pi}_{ij} - \underline{\pi}_{ij})$.♦

Parallel to full MJS, We give the following stochastic stability definition for system (2.1).

Definition 2.2 System (2.1) is said to be stochastically stable if, for any finite $\phi(t)$ defined on $[-\tau_M, 0]$, and $r_0 \in \mathcal{M}$, the following condition is satisfied,

$$\lim_{t\to\infty} \mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s | (\phi, r_0)\right\} \leqslant \infty. \tag{2.3}$$

Remark 2.3 The above definition is consistent with that of stochastic stability of delayed S-MJS. It should be pointed out that Definition 2.2 has similar form with the stochastic stability definition in [3, 4], however, the MJS and the S-MJS are governed by different stochastic processes. The stochastic stability of S-MJS in Definition 2.2 mean that the trajectory of the state $x(t)$ along the semi-Markovian process from the initial state (ϕ, r_0), in the mean-square sense, asymptotically to the origin.♦

2.3 Main Results

In this section, we will use the PAM combined with the Lyapunov–Krasovskii function technique to analyze the stochastic stability for system (2.1). To this end, firstly, let $\tau_m = \min\{\underline{\tau}_i, i \in \mathcal{M}\}$ and $\tau_M = \max\{\bar{\tau}_i, i \in \mathcal{M}\}$, then, the variation interval $[\tau_m, \tau_M]$ of time-delay can be divided into l parts with equal length. So $[\tau_m, \tau_M] = \bigcup_{q=0}^{l-1}[\tau_q, \tau_{q+1}]$, where $\tau_q = \tau_m + q(\tau_M - \tau_m)/l, q = 0, 1, 2, \ldots, l$, and $\delta = \tau_q - \tau_{q-1}$. For a given l, a Lyapunov functional $V(\cdot)$ is constructed, and then by checking the variation of derivative of $V(\cdot)$ for the case $\tau_i(t) \in [\tau_q, \tau_{q+1}]$, $q = 0, 1, 2, \ldots, l-1$, thus some delay-dependent conditions with less conservativeness will be obtained. Our first result in this chapter is as follows.

Theorem 2.4 *For given scalars $\bar{\tau}_i$, $\underline{\tau}_i$ and μ, system (2.1) is stochastically stable if there exist a positive integer l, and matrices $P(i) > 0$, $Q(i) > 0$, $W > 0$, $S_q > 0$ and $R_q > 0$ ($i \in \mathcal{M}$; $q = 1, 2, \ldots, l+1$), such that the following inequalities hold:*

$$\begin{bmatrix} \Gamma(i) + \Phi_k & \mathscr{A}(i)M \\ * & -M \end{bmatrix} < 0, \quad k = 0, 1, \ldots, l-1, \tag{2.4}$$

$$\sum_{j=1}^{N} \bar{\pi}_{ij} Q_j - W \leqslant 0, \tag{2.5}$$

where

$$\Gamma(i) \triangleq \mathscr{A}^T(i)P(i)\mathbb{I}_1 + \mathbb{I}_1^T P(i)\mathscr{A}(i) - \sum_{q=0}^{l} \mathbb{I}_{3i}^T S_{q+1}\mathbb{I}_{3i} - (1-\mu)\mathbb{I}_2^T Q(i)\mathbb{I}_2$$
$$+ \mathbb{I}_1^T\Bigg(Q(i) + \tau_M W + \sum_{q=0}^{l} S_{q+1} + \sum_{j=1}^{N} \pi_{ij}(h)P(j)\Bigg)\mathbb{I}_1 - \mathbb{I}_4^T R_1 \mathbb{I}_4,$$

$$\mathscr{A}(i) \triangleq \begin{bmatrix} A(i) & A_\tau(i) & 0_{m\times(l+1)m} \end{bmatrix}, \quad M \triangleq \tau_m^2 R_1 + \sum_{q=1}^{l} \delta R_{q+1},$$

$$\Phi_k \triangleq -\frac{1}{\delta} \sum_{\substack{q=1 \\ q\neq k+1}}^{l} \mathbb{I}_{5i}^T R_{q+1}\mathbb{I}_{5i} - \frac{1}{\delta}\mathbb{I}_{6k}^T R_{k+2}\mathbb{I}_{6k} - \frac{1}{\delta}\mathbb{I}_{7k}^T R_{k+2}\mathbb{I}_{7k}, \ k = 0, \ldots, l-1,$$

$$\mathbb{I}_1 \triangleq \begin{bmatrix} I_m & 0_{m\times(l+2)m} \end{bmatrix}, \quad \mathbb{I}_2 = \begin{bmatrix} 0_m & I_m & 0_{m\times(l+1)m} \end{bmatrix},$$
$$\mathbb{I}_{3q} \triangleq \begin{bmatrix} 0_{m\times(q+2)m} & I_m & 0_{m\times(l-q)m} \end{bmatrix}, \quad q = 0, 1, 2, \ldots, l,$$
$$\mathbb{I}_4 \triangleq \begin{bmatrix} I_m & 0_m & -I_m & 0_{m\times lm} \end{bmatrix},$$
$$\mathbb{I}_{5q} \triangleq \begin{bmatrix} 0_{m\times(q+1)m} & I_m & -I_m & 0_{m\times(l-q)m} \end{bmatrix}, \quad q = 1, 2, \ldots, l,$$
$$\mathbb{I}_{6k} \triangleq \begin{bmatrix} 0_m & I_m & 0_{m\times km} & -I_m & 0_{m\times(l-k)m} \end{bmatrix}, \quad k = 0, 1, \ldots, l-1,$$
$$\mathbb{I}_{7k} \triangleq \begin{bmatrix} 0_m & I_m & 0_{m\times(k+1)m} & -I_m & 0_{m\times(l-k-1)m} \end{bmatrix}, \quad k = 0, 1, \ldots, l-1.$$

Proof Let $\mathbb{C}[-\tau_M, 0]$ be the space of continuous functions. Define a process $x(t) \in \mathbb{C}[-\tau_M, 0]$, $t \geqslant 0$ by $x(t) = x(t+\theta), \theta \in [-\tau_M, 0]$. Then $\{(x(t), r_t), t \geqslant 0\}$ is a semi-Markovian process with initial state $(\phi(t), r_0)$.

Consider the following Lyapunov–Krasovskii function:

$$V\big(x(t), \eta_t\big) \triangleq \sum_{i=1}^{6} V_i\big(x(t), \eta_t\big), \tag{2.6}$$

where

$$
\begin{aligned}
V_1(x(t), \eta_t) &\triangleq x^T(t)P(\eta_t)x(t), \\
V_2\big(x(t), \eta_t\big) &\triangleq \int_{-\tau_i(t)}^{0} x^T(t+\theta)Q(\eta_t)x(t+\theta)\mathrm{d}\theta, \\
V_3\big(x(t), t\big) &\triangleq \int_{-\tau_i(t)}^{0}\int_{t+\theta}^{t} x^T(s)Wx(s)\mathrm{d}s\mathrm{d}\theta, \\
V_4\big(x(t), t\big) &\triangleq \sum_{q=0}^{l}\int_{t-\tau_q}^{t} x^T(s)S_{q+1}x(s)\mathrm{d}s, \\
V_5\big(x(t), t\big) &\triangleq \tau_m\int_{t-\tau_m}^{t}\int_{s}^{t} \dot{x}^T(\nu)R_1\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}s, \\
V_6\big(x(t), t\big) &\triangleq \sum_{q=1}^{l}\int_{t-\tau_q}^{t-\tau_{q-1}}\int_{s}^{t} \dot{x}^T(\nu)R_{q+1}\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}s,
\end{aligned}
$$

with $P(i) > 0$, $Q(i) > 0$, $W > 0$, $S_q > 0$ and $R_q > 0$, $i \in \mathcal{M}$; $q = 0, 1, \ldots, l$, are real matrices to be determined.

The infinitesimal generator $\mathcal{L}$ can be considered as a derivative of the Lyapunov function $V(x(t), \eta_t)$ along the trajectory of the semi-Markovian process $\{\eta_t, t > 0\}$ at the point $\{x(t), \eta_t\}$ at time t. The MJS and the S-MJS are governed by different stochastic processes, so the infinitesimal generator of the Lyapunov function for the S-MJS is essentially different from the one for general MJS. Firstly, we need to derive the infinitesimal generator $\mathcal{L}$. According to the definition of $\mathcal{L}$, we have

$$
\mathcal{L}V(x(t), \eta_t) \triangleq \lim_{\Delta\to 0}\frac{\mathbf{E}\Big\{V(x(t+\Delta), \eta_{t+\Delta})|x(t), \eta_t\Big\} - V(x(t), \eta_t)}{\Delta}, \tag{2.7}
$$

where Δ is a small positive number. Applying the law of total probability and conditional expectation with $\eta_t = i$, we have

$$
\begin{aligned}
\mathcal{L}V_1(x(t), \eta_t) = &\lim_{\Delta\to 0}\frac{1}{\Delta}\Bigg[\sum_{j=1, j\neq i}^{N}\Pr(\eta_{t+\Delta} = j|\eta_t = i)\; x^T(t+\Delta)P(j)x(t+\Delta) \\
&+\Pr(\eta_{t+\Delta} = i|\eta_t = i)x^T(t+\Delta)P(i)x(t+\Delta) - x^T(t)P(i)x(t)\Bigg] \\
= &\lim_{\Delta\to 0}\frac{1}{\Delta}\Bigg[\sum_{j=1, j\neq i}^{N}\frac{q_{ij}(G_i(h+\Delta) - G_i(h))}{1 - G_i(h)}\; x^T(t+\Delta)P(j)x(t+\Delta) \\
&+\frac{1 - G_i(h+\Delta)}{1 - G_i(h)}x^T(t+\Delta)P(i)x(t+\Delta) - x^T(t)P(i)x(t)\Bigg],
\end{aligned} \tag{2.8}
$$

where h is the time elapsed when the system stays at mode i from the last jump; $G_i(t)$ is the cumulative distribution function of the sojourn-time when the system remains in mode i, and q_{ij} is the probability intensity of the system jump from mode i to mode j. Given that Δ is small, the first order approximation of $x(t+\Delta)$ is

$$\begin{aligned} x(t+\Delta) &= x(t) + \dot{x}(t)\Delta + o(\Delta) \\ &= [A(i)\Delta + I]x(t) + A_\tau(i)\Delta x(t-\tau_i(t)) + o(\Delta). \end{aligned}$$

Then, (2.8) becomes

$$\begin{aligned} \mathcal{L}V_1(x(t),\eta_t) = \lim_{\Delta\to 0}\frac{1}{\Delta}\Bigg\{ &\sum_{j=1,j\neq i}^{N} \frac{q_{ij}(G_i(h+\Delta)-G_i(h))}{1-G_i(h)}\xi_1^T(t)\begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix} \\ &\times P(j)\begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix}^T \xi_1(t) + \frac{1-G_i(h+\Delta)}{1-G_i(h)}\xi_1^T(t)\begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix} \\ &\times P(i)\begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix}^T \xi_1(t) - x^T(t)P(i)x(t)\Bigg\}. \end{aligned}$$

Using the condition of $\lim_{\Delta\to 0}\frac{G_i(h+\Delta)-G_i(h)}{1-G_i(h)} = 0$, we have

$$\mathcal{L}V_1(x(t),\eta_t) = \xi_1^T(t)\begin{bmatrix} \Lambda(i) & \lim_{\Delta\to 0}\frac{1-G_i(h+\Delta)}{1-G_i(h)}P(i)A(i) \\ * & 0 \end{bmatrix}\xi_1(t)$$

where $\xi_1(t) \triangleq \begin{bmatrix} x^T(t) & x^T(t-\tau_i(t)) \end{bmatrix}^T$, and

$$\begin{aligned} \Lambda(i) \triangleq A^T(i)P(i) + P(i)A(i) + \lim_{\Delta\to 0}\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\Delta)-G_i(h))}{\Delta(1-G_i(h))}P(j) \\ + \lim_{\Delta\to 0}\frac{G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))}P(i). \end{aligned}$$

By the same techniques used in [5], we have

$$\lim_{\Delta\to 0}\frac{1-G_i(h+\Delta)}{1-G_i(h)} = 1 \quad \text{and} \quad \lim_{\Delta\to 0}\frac{G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))} = \pi_i(h),$$

where $\pi_i(h)$ is the transition rate of the system jumping from mode i.

Define

$$\pi_{ij}(h) \triangleq q_{ij}\pi_i(h), i \neq j \ \text{ and } \ \pi_{ii}(h) \triangleq -\sum_{j=1,j\neq i}^{N}\pi_{ij}(h),$$

then we obtain

$$\begin{aligned}\mathcal{L}V_1(x(t),\eta_t) &= x^T(t)\Bigg(\sum_{j=1}^{N}P(j)\pi_{ij}(h)\Bigg)x(t)+2x^T(t)P(i)\\&\quad\times\Big[A(i)x(t)+A_\tau(i)x(t-\tau_i(t))\Big]\\&=2\xi^T(t)\mathscr{A}^T(i)P(i)\mathbb{I}_1\xi(t)\\&\quad+\xi^T(t)\mathbb{I}_1^T\Bigg(\sum_{j=1}^{N}\pi_{ij}(h)P(j)\Bigg)\mathbb{I}_1\xi(t),\end{aligned}\tag{2.9}$$

where $\mathscr{A}(i)$ and $\mathbb{I}_1$ are defined as in Theorem 2.4, and

$$\xi(t)\triangleq\left[x^T(t)\ x^T(t-\tau_i(t))\ x^T(t-\tau_m)\ \ldots\ x^T(t-\tau_{l-1})\ x^T(t-\tau_M)\right]^T.$$

On the other hand,

$$\begin{aligned}&\mathbf{E}\{V_2(x(t+\Delta),\eta_{t+\Delta})|x(t),\eta_t\}\\&\quad=\mathbf{E}\Bigg\{\int_{-\tau_i(t+\Delta)}^{0}x^T(t+\Delta+\theta)Q(\eta_{t+\Delta})x(t+\Delta+\theta)\mathrm{d}\theta|x(t),\eta_t\Bigg\}\\&\quad=\mathbf{E}\Bigg\{\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta|x(t),\eta_t\Bigg\}\\&\qquad+\mathbf{E}\Bigg\{\int_{-\tau_i(t+\Delta)}^{0}x^T(t+\Delta+\theta)Q(\eta_{t+\Delta})x(t+\Delta+\theta)\mathrm{d}\theta\\&\qquad-\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta|x(t),\eta_t\Bigg\}\\&\quad=\sum_{j=1,j\neq i}^{N}\Pr(\eta_{t+\Delta}=j|\eta_t=i)\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta\\&\qquad+\Pr(\eta_{t+\Delta}=i|\eta_t=i)\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(i)x(t+\theta)\mathrm{d}\theta\\&\qquad+\mathbf{E}\Bigg\{\int_{\Delta-\tau_i(t+\Delta)}^{\Delta}x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta\\&\qquad-\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta|x(t),\eta_t\Bigg\}\\&\quad=\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\Delta)-G_i(h))}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta\\&\qquad+\frac{1-G_i(h+\Delta)}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(i)x(t+\theta)\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{E}\left\{\int_0^{\Delta} x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta\right.\\
&\left.-\int_{-\tau_i(t)}^{\Delta-(\tau_i(t)+\dot{\tau}\Delta+o(\Delta))} x^T(t+\theta)Q(\eta_{t+\Delta})x(t+\theta)\mathrm{d}\theta|x(t),\eta_t\right\}\\
=&\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\Delta)-G_i(h))}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta\\
&+\frac{1-G_i(h+\Delta)}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(i)x(t+\theta)\mathrm{d}\theta+\Delta x^T(t)Q(i)x(t)\\
&-\Delta(1-\dot{\tau}_i(t))x^T(t-\tau_i(t))Q(i)x(t-\tau_i(t))+o(\Delta).
\end{aligned}
$$

The above, together with (2.7), yields

$$
\begin{aligned}
\mathcal{L}V_2(x(t),\eta_t)=&\lim_{\Delta\to 0}\frac{1}{\Delta}\Big\{\mathbf{E}\big\{V_2(x(t+\Delta),\eta_{t+\Delta})|x(t),\eta_t\big\}-V_2(x(t),\eta_t)\Big\}\\
=&\lim_{\Delta\to 0}\frac{1}{\Delta}\Bigg\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\Delta)-G_i(h))}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta\\
&+\frac{1-G_i(h+\Delta)}{1-G_i(h)}\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(i)x(t+\theta)\mathrm{d}\theta+\Delta x^T(t)Q(i)x(t)\\
&+o(\Delta)-V_2(x(t),\eta_t)-\Delta(1-\dot{\tau}_i(t))x^T(t-\tau_i(t))Q(i)x(t-\tau_i(t))\Bigg\}.
\end{aligned}
$$

Then, similar to the development of (2.9) it follows

$$
\begin{aligned}
\mathcal{L}V_2\big(x(t),\eta_t\big)=&\sum_{j=1}^{N}\pi_{ij}(h)\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta\\
&-(1-\dot{\tau}_i(t))x^T(t-\tau_i(t))Q(i)x(t-\tau_i(t))+x^T(t)Q(i)x(t)\\
\leqslant&\ \xi^T(t)\mathbb{I}_1^TQ(i)\mathbb{I}_1\xi(t)-(1-\mu)\xi^T(t)\mathbb{I}_2^TQ(i)\mathbb{I}_2\xi(t)\\
&+\sum_{j=1}^{N}\pi_{ij}(h)\int_{-\tau_i(t)}^{0}x^T(t+\theta)Q(j)x(t+\theta)\mathrm{d}\theta. \qquad (2.10)
\end{aligned}
$$

Moreover, from (2.7), it can be shown that

$$
\begin{aligned}
\mathcal{L}\Bigg(\sum_{i=3}^{6}V_i\big(x(t),\eta_t\big)\Bigg)=&\ \xi^T(t)\mathscr{A}^T(i)M\mathscr{A}(i)\xi(t)-\int_{t-\tau_i(t)}^{t}x^T(\theta)Wx(\theta)\mathrm{d}\theta\\
&+\xi^T(t)\mathbb{I}_1^T\Big(\tau_MW+\sum_{q=0}^{l}S_{i+1}\Big)\mathbb{I}_1\xi(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{q=0}^{l} \xi^T(t)\mathbb{I}_{3i}^T S_{q+1}\mathbb{I}_{3i}\xi(t) - \tau_m \int_{t-\tau_m}^{t} \dot{x}^T(s)R_1\dot{x}(s)\mathrm{d}s \\
&- \sum_{q=1}^{l} \int_{t-\tau_q}^{t-\tau_{q-1}} \dot{x}^T(s)R_{q+1}\dot{x}(s)\mathrm{d}s. \qquad (2.11)
\end{aligned}
$$

It follows by Jensen's inequality that

$$
-\tau_m \int_{t-\tau_m}^{t} \dot{x}^T(s)R_1\dot{x}(s)\mathrm{d}s \leqslant -\xi^T(t)\mathbb{I}_4^T R_1\mathbb{I}_4\xi(t), \qquad (2.12)
$$

$$
-\sum_{\substack{q=1\\ q\neq k+1}}^{l} \int_{t-\tau_q}^{t-\tau_{q-1}} \dot{x}^T(s)R_{q+1}\dot{x}(s)\mathrm{d}s \leqslant -\frac{1}{\delta}\sum_{\substack{q=1\\ q\neq k+1}}^{l} \xi^T(t)\mathbb{I}_{5i}^T R_{q+1}\mathbb{I}_{5i}\xi(t). \qquad (2.13)
$$

Next, by checking $\tau_i(t) \in [\tau_k, \tau_{k+1}]$ for some positive integer k with $0 \leqslant k \leqslant l-1$, one has

$$
\begin{aligned}
&-\int_{t-\tau_{k+1}}^{t-\tau_k} \dot{x}^T(s)R_{k+2}\dot{x}(s)\mathrm{d}s \\
&= -\int_{t-\tau_i(t)}^{t-\tau_k} \dot{x}^T(s)R_{k+2}\dot{x}(s)\mathrm{d}s - \int_{t-\tau_{k+1}}^{t-\tau_i(t)} \dot{x}^T(s)R_{k+2}\dot{x}(s)\mathrm{d}s \\
&\leqslant -\frac{1}{\delta}\xi^T(t)\mathbb{I}_{6k}^T R_{k+2}\mathbb{I}_{6k}\xi(t) - \frac{1}{\delta}\xi^T(t)\mathbb{I}_{7k}^T R_{k+2}\mathbb{I}_{7k}\xi(t). \qquad (2.14)
\end{aligned}
$$

By combining (2.9)–(2.14), we have

$$
\begin{aligned}
\mathcal{L}V(x(t),\eta_t) = {}& \xi^T(t)\Big(\Gamma(i) + \Phi_k + \mathscr{A}^T(i)M\mathscr{A}(i)\Big)\xi(t) \\
& - \int_{t-\tau_i(t)}^{t} x^T(\theta)Wx(\theta)\mathrm{d}\theta \\
& + \sum_{j=1}^{N} \pi_{ij}(h)\int_{t-\tau_i(t)}^{t} x^T(\theta)Q(j)x(\theta)\mathrm{d}\theta,
\end{aligned}
$$

where $\xi(t)$, $\Gamma(i)$, Φ_k, and $\mathscr{A}(i)$ are defined as before. From (2.4)–(2.5) and the Schur complement, it can be seen that $\mathcal{L}V(x(t),\eta_t) < 0$ for any $i \in \mathcal{M}$.

Next, we set

$$
\Theta_i \triangleq \Gamma(i) + \Phi_k + \mathscr{A}^T(i)M\mathscr{A}(i),
$$

and

$$\lambda_1 \triangleq \min_{i\in\mathcal{M}}\{\lambda_{\min}(-\Theta_i)\},$$

then for any $t \geqslant \tau$ we have

$$\mathcal{L}V(x(t), \eta_t) \leqslant -\lambda_1\|\xi(t)\|^2 \leqslant -\lambda_1\|x(t)\|^2.$$

By Dynkin's formula, we have

$$\mathbf{E}\{V(x(t), \eta_t)|\phi, r_0\} \leqslant \mathbf{E}\{V(\phi, r_0)\} - \lambda_1\mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s\right\}. \tag{2.15}$$

So, from (2.6), it follows that for any $t \geqslant 0$

$$\mathbf{E}\{V(x(t), \eta_t)\} \geqslant \lambda_2\mathbf{E}\{\|x(t)\|^2\}, \tag{2.16}$$

where $\lambda_2 = \min_{i\in\mathcal{M}}\{\lambda_{\min}(-P(i))\} > 0$. From (2.15) and (2.16), we have

$$\mathbf{E}\left\{\|x(t)\|^2\right\} \leqslant \varepsilon_2 V(x(t), \eta_t) - \varepsilon_1\mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s\right\},$$

with $\varepsilon_1 = \lambda_1\lambda_2^{-1}$ and $\varepsilon_2 = \lambda_2^{-1}$. By Gronwell–Bellman lemma, one has

$$\mathbf{E}\left\{\|x(t)\|^2\right\} \leqslant \varepsilon_2 V(x(t), \eta_t)e^{-\varepsilon_1 t}, \tag{2.17}$$

then

$$\mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s|\phi, r_0\right\} \leqslant -\varepsilon_1^{-1}\varepsilon_2 V\left(x(t), \eta_t\right)\left(e^{-\varepsilon_1 t} - 1\right).$$

Let $t \to \infty$, we have

$$\lim_{t\to\infty}\mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s|\phi, r_0\right\} \leqslant \varepsilon_1^{-1}\varepsilon_2 V(x(t), \eta_t).$$

Then there always exists a scalar $c > 0$, such that

$$\lim_{t\to\infty}\mathbf{E}\left\{\int_0^t \|x(s)\|^2 \mathrm{d}s|\phi, r_0\right\} \leqslant c \sup_{-\tau\leqslant s\leqslant 0}\|\phi(s)\|^2,$$

which is the desired result, by Definition 2.2, and the proof is completed. ■

Remark 2.5 Based on the idea of PAM and the piecewise Lyapunov function technique, a new delay-dependent sufficient condition is proposed in Theorem 2.4 for the stochastic stability of S-MJS (2.1). Due to the instrumental idea of PAM, the proposed stability condition is much less conservative than the existing results. Moreover, the conservatism reduction becomes more obvious with the partitioning getting thinner (i.e., l becoming bigger). In addition, from the proof of Theorem 2.4, we can see that the conditions to guarantee $\mathcal{L}V(x(t), \eta_t) < 0$ are derived by checking the delay variation in the sub-intervals. So, the PAM for stability analysis in this chapter, actually, is quite different from the delay partitioning method used in [6, 7]. Compared with the technique of dividing the lower bounded of the time-delay into m parts with equal length in [6, 7], in our work, the variation interval of time-delay has divided into l parts with equal length. So the PAM can be seen as an extension of the delay-partitioning method, which may further reduce the conservativeness (see Example 2.7).♦

Due to the time-varying term $\pi_{ij}(h)$, it is difficult to solve inequalities (2.4), since they contain infinite number of inequalities, which limits the use of Theorem 2.4 in practice. To overcome this shortcoming, we present our next result which reduces infinite number of inequalities in Theorem 2.4 to finitely many ones, which will be simple to solve.

Theorem 2.6 *For given scalars $\bar{\tau}_i$, $\underline{\tau}_i$ and μ, system (2.1) is stochastically stable if there exists a positive integer l, and matrices $P(i) > 0$, $Q(i) > 0$, $Q > 0$, $S_q > 0$, $R_q > 0$, and T_{ij}, $(i, j \in \mathcal{M}; q = 1, 2, \ldots, l+1)$, such that (2.5) and the following inequalities hold:*

$$\begin{bmatrix} \tilde{\Gamma}(i) + \Phi_k & \mathscr{A}(i)M & \mathcal{M}_i \\ * & -M & 0 \\ * & * & -\Lambda_i \end{bmatrix} < 0, \quad k = 0, 1, \ldots, l-1, \tag{2.18}$$

where

$$\begin{aligned} \tilde{\Gamma}(i) &\triangleq \mathbb{I}_1^T \left(Q(i) + \tau_M W + \sum_{i=0}^{l} S_{i+1} + \sum_{j=1}^{N} \pi_{ij} P(j) + \sum_{j=1, j\neq i}^{N} \frac{\kappa_{ij}^2}{4} T_{ij} \right) \mathbb{I}_1 \\ &\quad + \mathscr{A}^T(i) P(i) \mathbb{I}_1 + \mathbb{I}_1^T P(i) \mathscr{A}(i) - \sum_{i=0}^{l} \mathbb{I}_{3i}^T S_{i+1} \mathbb{I}_{3i} \\ &\quad - \mathbb{I}_4^T R_1 \mathbb{I}_4 - (1-\mu) \mathbb{I}_2^T Q(i) \mathbb{I}_2, \\ \mathcal{M}_i &\triangleq \begin{bmatrix} P_i - P_1 & \ldots & P_i - P_{i-1} & P_i - P_{i+1} & \ldots & P_i - P_s \end{bmatrix}, \\ \Lambda_i &\triangleq \mathrm{diag}\{T_{i1}, \cdots, T_{i(i-1)}, T_{i(i+1)}, \cdots, T_{is}\}, \end{aligned}$$

and Φ_k, $\mathscr{A}(i)$, $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_{3i}$, and $\mathbb{I}_4$ are defined as in Theorem 2.4.

Proof According to Theorem 2.4, system (2.1) is stochastically stable if the following inequalities hold for all $i \in \mathcal{M}$,

$$\begin{bmatrix} \Omega_i + \mathbb{I}_1^T \left(\sum_{j=1}^{N} (\pi_{ij} + \Delta\pi_{ij}) P_j \right) \mathbb{I}_1 & \mathscr{A}(i) M \\ * & -M \end{bmatrix} < 0, \tag{2.19}$$

where Ω_i is defined as below,

$$\Omega_i \triangleq \mathscr{A}^T(i) P(i) \mathbb{I}_1 + \mathbb{I}_1^T P(i) \mathscr{A}(i) + \mathbb{I}_1^T \left(Q(i) + \tau_M W + \sum_{i=0}^{l} S_{i+1} \right) \mathbb{I}_1$$
$$- \sum_{i=0}^{l} \mathbb{I}_{3i}^T S_{i+1} \mathbb{I}_{3i} - \mathbb{I}_4^T R_1 \mathbb{I}_4 - (1-\mu) \mathbb{I}_2^T Q(i) \mathbb{I}_2 + \Phi_k,$$

while (2.19) can be further rewritten as

$$\begin{bmatrix} \Psi(i) & \mathscr{A}(i) M \\ * & -M \end{bmatrix} < 0.$$

where

$$\Psi(i) \triangleq \Omega_i + \mathbb{I}_1^T \left\{ \sum_{j=1}^{N} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{N} \left[\frac{1}{2} \Delta\pi_{ij} \big(P(j) - P(i)\big) \right.\right.$$
$$\left.\left. + \frac{1}{2} \Delta\pi_{ij} \big(P(j) - P(i)\big) \right] \right\} \mathbb{I}_1.$$

From Lemma 1.27, the above inequality holds for all $|\Delta\pi_{ij}| \leqslant \kappa_{ij}$, if there exist matrices $T_{ij} (i, j \in \mathcal{M})$, such that

$$\begin{bmatrix} \hat{\Psi}(i) & \mathscr{A}(i) M \\ * & -M \end{bmatrix} < 0, \tag{2.20}$$

with

$$\hat{\Psi}(i) \triangleq \Omega_i + \mathbb{I}_1^T \left\{ \sum_{j=1}^{N} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{N} \left[\frac{\kappa_{ij}^2}{4} T_{ij} \right.\right.$$
$$\left.\left. + (P(j) - P(i)) T_{ij}^{-1} (P(j) - P(i)) \right] \right\} \mathbb{I}_1.$$

In view of Schur complement lemma, it can be seen that (2.20) is equivalent to inequality (2.18). Then, by this fact together with Theorem 2.4, we can conclude that system (2.1) is stochastically stable. The proof is completed. ■

2.4 Illustrative Example

In this section, two examples will be presented to demonstrate the effectiveness and superiority of the methods developed previously.

Example 2.7 Consider S-MJS in (2.1) with two modes and the following parameters:

$$A(1) = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_\tau(1) = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -1.0 & 0.5 \\ 0.0 & -1.0 \end{bmatrix}, \quad A_\tau(2) = \begin{bmatrix} -1.0 & 0.0 \\ -0.1 & -1.0 \end{bmatrix}.$$

In this example, we choose $\tau_m = 1$, $\mu = 0.5$, $\pi_{11} = -0.1$ and $\pi_{22} = -0.8$. The maximum allowed τ_M, obtained by the criteria in [6, 8–10] and Theorem 2.4 in this chapter, is shown in Table 2.1.

From Table I, it is obvious that, when the delay partitioning number l becomes larger, the conservatism of the result is further reduced, while the computational cost increases. This is reasonable since l is related to the decision variables. Thus, the larger l indicates that the solution can be searched in a wider space and a lager maximum allowable delay τ_M can be obtained. Table I shows that for no partitioning ($l = 1$), the new criterion given in Theorem 2.4 has the same conservatism with the previous result in [10] and [6] ($m = 1$). Moreover, for $l > 1$, the conservatism reduction by the PAM proved to be quite effective. Moreover, we can see that even with the case $l = 3$ our results still outperform those in [6] with $m = 5$.

Table 2.1 Maximum upper delay bound τ_M with $\tau_m = 1$ and $\mu = 0.5$

Different results	Maximum allowed τ_M
Theorem 3.1 of [8]	0.2237
Theorem 1 of [9]	1.4701
Theorem 1 of [10]	1.6602
Theorem 1 of [6] ($m = 1$)	1.6600
Theorem 1 of [6] ($m = 2$)	1.6902
Theorem 1 of [6] ($m = 3$)	1.7318
Theorem 1 of [6] ($m = 4$)	1.7456
Theorem 1 of [6] ($m = 5$)	1.7527
Theorem 1 ($l = 1$)	1.6602
Theorem 1 ($l = 2$)	1.7344
Theorem 1 ($l = 3$)	1.8023
Theorem 1 ($l = 4$)	1.8072

Table 2.2 Maximum upper delay bound τ_M with different μ

Theorem 2.6	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 0.9$
$l = 3$	$\tau_M = 5.405$	$\tau_M = 4.001$	$\tau_M = 2.068$	$\tau_M = 1.437$
$l = 4$	$\tau_M = 5.476$	$\tau_M = 4.072$	$\tau_M = 2.102$	$\tau_M = 1.500$
$l = 5$	$\tau_M = 5.512$	$\tau_M = 4.101$	$\tau_M = 2.125$	$\tau_M = 1.542$
$l = 6$	$\tau_M = 5.541$	$\tau_M = 4.112$	$\tau_M = 2.161$	$\tau_M = 1.568$

Example 2.8 Consider S-MJS in (2.1) with two modes and the following parameters:

$$A(1) = \begin{bmatrix} -1.0 & 0.4 \\ 0.0 & -0.8 \end{bmatrix}, \quad A_\tau(1) = \begin{bmatrix} -0.6 & 0.0 \\ -1.0 & -1.0 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -1.1 & 1.5 \\ 0.0 & -1.2 \end{bmatrix}, \quad A_\tau(2) = \begin{bmatrix} -1.1 & 0.0 \\ -0.1 & -0.9 \end{bmatrix}.$$

In this example, the transition rates in the model are $\pi_{11}(h) \in (-2.2, -1.8)$, $\pi_{12}(h) \in (1.8, 2.2)$, $\pi_{21}(h) \in (2.6, 3.4)$ and $\pi_{22}(h) \in (-3.4, -2.6)$, so we can always set $\pi_{11} = -2$, $\pi_{12} = 2$, $\pi_{21} = -3$, $\pi_{22} = 3$ and $\kappa_{1j} = 0.2$, $\kappa_{2j} = 0.4$ for the stochastic stability analysis. Our purpose is to find the allowable maximum time delay τ_M such that the delayed S-MJS is stochastic stable. The obtained allowable maximum delays by testing the conditions in Theorem 2.6 for various partitioning number l are shown in Table 2.2. It can be seen from Theorem 2.6 that the maximum allowable delay is becoming larger as the partitioning becomes thinner. Moreover, it should be pointed out that although conservatism is reduced as the partitioning becomes thinner, and there is no significant improvement after $l = 5$.

When $\tau_m = 1$, $\mu = 0.5$ and $l = 3$, we can obtain feasible solutions to the inequalities in Theorem 2.6 as follows:

$$P(1) = \begin{bmatrix} 0.65 & -1.8 \\ -0.28 & 1.37 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 0.49 & -1.23 \\ -1.13 & 1.29 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 1.52 & -0.25 \\ 1.15 & 0.07 \end{bmatrix},$$
$$Q(2) = \begin{bmatrix} 0.42 & -0.25 \\ -0.15 & 1.06 \end{bmatrix}, \quad W = \begin{bmatrix} 0.02 & -0.02 \\ -0.02 & 0.02 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0.27 & -1.34 \\ -0.74 & 1.35 \end{bmatrix},$$
$$T_{21} = \begin{bmatrix} 1.88 & -0.42 \\ 1.32 & 0.23 \end{bmatrix}, \quad S_1 = \begin{bmatrix} -1.48 & 0.16 \\ 0.36 & -0.46 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1.24 & 0 \\ 0 & 1.24 \end{bmatrix},$$
$$S_3 = \begin{bmatrix} 0.20 & -0.02 \\ -0.02 & 0.60 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0.50 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.35 & 0.04 \\ 0.04 & 0.18 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 0.29 & 0.05 \\ 0.05 & 0.47 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.46 & 0.05 \\ 0.05 & 0.17 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0.53 & 0.02 \\ 0.02 & 0.33 \end{bmatrix},$$

and we can obtain the maximum allowed time-delay $\tau_M = 2.18$. When setting the initial condition as $\phi(0) = 0$, $r_0 = 1$, the simulation result is shown in Figs. 2.1 and 2.2, which illustrates that the S-MJS with time-varying transition rates is stable.

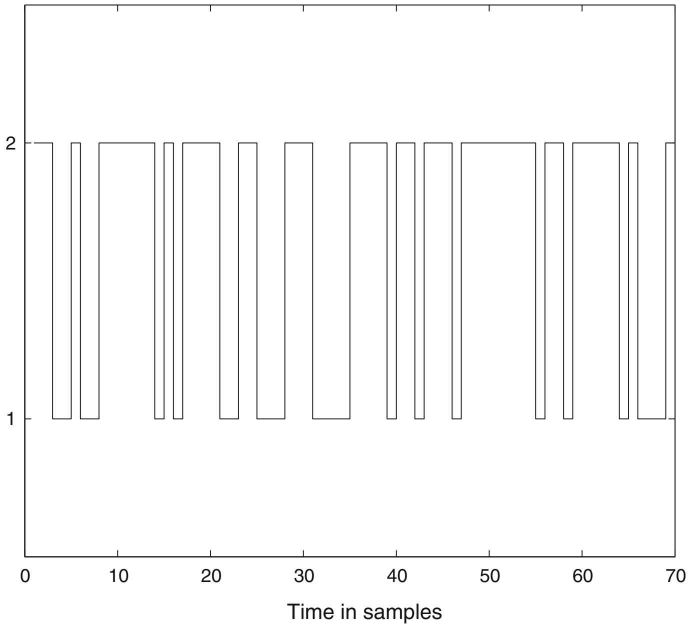

Fig. 2.1 Switched signal of the system in Example 2.8

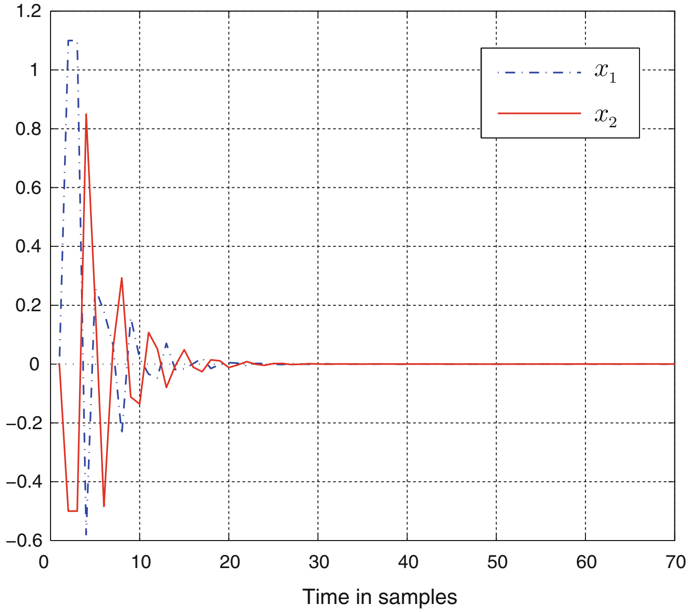

Fig. 2.2 States of the system in Example 2.8

2.5 Conclusion

In this chapter, the problem of stochastic stability for a class of linear S-MJS with time-delays has been addressed. Inspired by piecewise analysis method, a new stochastic stability condition has been established for the S-MJS, which is less conservative than some existing results. Moreover, by linear matrix inequality approach, a criterion has been presented to reduced to infinite number of inequalities to finitely many ones which are practically solvable. Two numerical examples have been given to demonstrate the potential of the new design method.

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Chapter 3
Constrained Regulation of Singular Semi-Markovian Jump Systems

Abstract This chapter studies the constrained regulation problem for a class of singular S-MJS. Using a supplementary variable technique and a plant transformation presented in Chap. 1, a finite PH semi-Markov process has been transformed into a finite Markov chain, which is called its associated Markov chain. Then, the S-MJS of this kind has been transformed into its associated Markovian system. Motivated by recent develops in positively invariant set, necessary and sufficient conditions for the existence of full rank solutions for a class of nonlinear equations are derived, and a new algorithm that provides a solution to the constrained regulation problem is presented.

3.1 Introduction

As one of the basic problems in control theory, regulation for systems subject to control constraints has long been an active and challenging area. In the past two decades, many efforts have been made to develop numerical tractable solutions to the problem, especially through the positively invariant set technique [1, 2]. However, this approach has a major disadvantage that there does not always exist a control strategy such that a given subset of the state space is a positively invariant set for the controlled system [3–5]. Meanwhile, singular systems have attracted much research interest due to the fact that singular systems can better describe the behavior of some physical systems than state-space ones [6, 7].

In this chapter, we will investigate the regulation problem for singular S-MJS subject to constrained control input. By the supplementary variable technique and the novel transformation in Chap. 1, a phase-type semi-Markov process has been transformed into a finite Markov chain which is called its associated Markov chain. Accordingly, a semi-Markovian system has been first transformed into its associated Markovian system. In light of the positively invariant set approach, we deal with the resolution of a class of nonlinear equations, which plays a fundamental rule in the design of controllers of linear systems with constrained control. Consequently, necessary and sufficient conditions are established for the existence of full rank solutions for nonlinear equations. We also give a heuristic algorithm for the constrained regulation problem.

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3.2 System Description and Preliminaries

Consider a semi-Markovian jump singular system described by

$$E\dot{x}(t) = \bar{A}(\bar{\eta}_t)x(t) + \bar{B}(\bar{\eta}_t)u(t), \quad x(t_0) = x_0, \tag{3.1}$$

where $\{\bar{\eta}_t, t \geqslant 0\}$ is a continuous-time PH semi-Markov process on the probability space which has been defined in Definition 1.21 of Chap. 1, and $x(t) \in \mathbb{R}^n$ represents the state vector; $u(t) \in \mathbb{R}^m$ is the control input; E is constant matrix; $\bar{A}(\bar{\eta}_t)$ and $\bar{B}(\bar{\eta}_t)$ are matrix functions of the PH semi-Markov process $\{\bar{\eta}_t\}$. Denote $\operatorname{rank}(E) = r \leqslant n$.

It follows from Lemma 1.24 of Chap.1, and employing the same techniques as those used in [8]. We can construct the following associated Markovian system equivalent to (3.1).

$$E\dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)u(t), \quad x(t_0) = x_0, \tag{3.2}$$

where t_0 and x_0 are the initial time and state, respectively. For notational simplicity, when the system operates in the αth mode $A(\eta_t)$ and $B(\eta_t)$ are denoted as $A(\alpha)$ and $B(\alpha)$, respectively.

For the system (3.2), we consider the following control law:

$$u(t) = F(\alpha)x(t), \tag{3.3}$$

where $F(\alpha)$ is the gain matrix to be designed for each mode α. Then, the closed-loop system is given by:

$$E\dot{x}(t) = \big[A(\alpha) + B(\alpha)F(\alpha)\big]x(t). \tag{3.4}$$

As it generally occurs in practical situations, the set of admissible controls $S\big(F(\alpha), \overline{u}(\alpha), \underline{u}(\alpha)\big)$ is an asymmetric polyhedral set defined as

$$S\big(F(\alpha), \overline{u}(\alpha), \underline{u}(\alpha)\big) \triangleq \big\{x(t) | -\underline{u}(\alpha) \leq F(\alpha)x(t) \leq \overline{u}(\alpha)\big\}, \tag{3.5}$$

where $\overline{u}(\alpha) > 0$ and $\underline{u}(\alpha) > 0, \forall \alpha \in \mathcal{M}$, represent the constraint bound on the system input, respectively. Let $\mathbb{S}$ be the common set of all the modes, $\mathbb{S} \triangleq \bigcap_{\alpha \in \mathcal{M}} S\big(F(\alpha), \overline{u}(\alpha), \underline{u}(\alpha)\big)$.

Definition 3.1 [4] A set $\mathbb{S} \in \mathbb{R}^n$ is said to be stochastically positively invariant with respect to system (3.4) if, for all $x_0 \in \Omega$, and the initial mode $r_0 \in \mathcal{M}$, $\mathbf{E}\{x(k)\}$ lies in the set $\mathbb{S}$.

Definition 3.2 [6] For $E, A(\alpha) \in \mathbb{R}^{n \times n}$, we have the following results:

(i) The pair $(E, A(\alpha))$ is said to be regular if the characteristic polynomial $\det(sE - A(\alpha))$ is not identically zero for each mode $\alpha \in \mathcal{M}$;

(ii) The pair $(E, A(\alpha))$ is said to be impulse free, if $\deg(\det(sE - A(\alpha))) = \text{rank}(E)$ for each mode $\alpha \in \mathcal{M}$;
(iii) The pair $(E, A(\alpha))$ is said to be admissible, if it is regular, impulse-free and stable; and
(iv) System (3.2) with $u(t) = 0$ is said to be admissible if the pair $(E, A(\alpha))$ is admissible.

Definition 3.3 [6] System (3.2) is said to be stabilizable if there exists a controller (3.3) with $F(\alpha)$ satisfies (3.5), such that the closed-loop system (3.4) is regular, impulse-free and stable.

Lemma 3.4 *Let* $\mathcal{A} \in \mathbb{R}^{n\times n}, \mathcal{B} \in \mathbb{R}^{m\times m}, \mathcal{C} \in \mathbb{R}^{m\times n}, \mathcal{H} = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{C} & \mathcal{B} \end{bmatrix}, \sigma(\mathcal{A}) \cap \sigma(\mathcal{B}) = \emptyset$. *Then* $\mathcal{H}$ *has* $n+m$ *linearly independent eigenvectors if and only if* $\mathcal{A}$ *has* n *linearly independent eigenvectors and* $\mathcal{B}$ *has* m *linearly independent eigenvectors.*

Proof Necessity. Let

$$\lambda_i \in \sigma(\mathcal{A}), \quad i = 1, 2, \ldots, n,$$

$$\lambda_j \in \sigma(\mathcal{B}), \quad j = n+1, n+2, \ldots, n+m,$$

then

$$\lambda_k \in \sigma(\mathcal{H}), \quad k = 1, 2, \ldots, n+m.$$

Since $\mathcal{H}$ has $n+m$ linearly independent eigenvectors, we can assume that

$$\begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{C} & \mathcal{B} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \lambda_k \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad k = 1, 2, \ldots, n+m,$$

where $\begin{bmatrix} x_k \\ y_k \end{bmatrix}$, $k = 1, 2, \ldots, n+m$ are linearly independent, and hence

$$\begin{aligned} \mathcal{A}x_k &= \lambda_k x_k, \quad k = 1, 2, \ldots, n+m, \\ \mathcal{C}x_k + \mathcal{B}y_k &= \lambda_k y_k, \quad k = 1, 2, \ldots, n+m. \end{aligned}$$

By $\sigma(\mathcal{A}) \cap \sigma(\mathcal{B}) = \emptyset$, one obtains

$$x_j = 0, \quad j = n+1, \ldots, n+m.$$

Therefore, $x_1, \ldots, x_n$ and $y_{n+1}, \ldots, y_{n+m}$ are linearly independent, respectively, and satisfying

$$\begin{aligned} \mathcal{A}x_i &= \lambda_i x_i, \quad i = 1, \ldots, n, \\ \mathcal{B}y_j &= \lambda_j y_j, \quad j = n+1, \ldots, n+m. \end{aligned}$$

Sufficiency. Since $\mathcal{A}$ has n linearly independent eigenvectors and $\mathcal{B}$ has m linearly independent eigenvectors, one can assume that

$$\begin{aligned} \mathcal{A}\xi_i &= \lambda_i \xi_i, \quad i = 1, \ldots, n, \\ \mathcal{B}\eta_j &= \lambda_j \eta_j, \quad j = n+1, \ldots, n+m, \end{aligned}$$

where $\xi_1, \ldots, \xi_n$ and $\eta_{n+1}, \ldots, \eta_{n+m}$ are linearly independent, respectively. Set

$$\begin{aligned} \gamma_i &= \begin{bmatrix} \xi_i \\ (\lambda_i I - \mathcal{B})^{-1}\mathcal{C}\xi_i \end{bmatrix}, \quad i = 1, \ldots, n, \\ \gamma_j &= \begin{bmatrix} 0 \\ \eta_j \end{bmatrix}, \quad j = n+1, \ldots, n+m. \end{aligned}$$

Then $\gamma_1, \ldots, \gamma_{n+m}$ are linearly independent and satisfy that

$$\mathcal{H}\gamma_k = \lambda_k \gamma_k, \quad k = 1, 2, \ldots, n+m.$$

This completes the proof. ■

We now formulate the constrained regulation problem as follows.
Constrained Regulation Problem: *For given positive vectors* $\overline{u}(\alpha) > 0$ *and* $\underline{u}(\alpha) > 0, \forall \alpha \in \mathcal{M}$, *design a state-feedback control law* (3.3) *such that the closed-loop system* (3.4) *is admissible and* $\mathbb{S}$ *is a positively invariant set.*

Since system (3.4) is admissible if and only if system (3.2) is stabilizable and impulse controllable [6], the following assumption is a necessary condition under which the constrained regulation problem has at least one solution.

Assumption 3.1 System (3.2) is stabilizable and impulse controllable.

Remark 3.5 The constrained regulation problem has been studied in [1] when the controlled system is assumed completely controllable, in which the imposed constraints are stronger than Assumption 3.1. ♦

For any $\alpha \in \mathcal{M}$, let P and Q be nonsingular matrices, such that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.6}$$

Correspondingly, let

$$PA(\alpha)Q \triangleq \begin{bmatrix} A_{11}(\alpha) & A_{12}(\alpha) \\ A_{21}(\alpha) & A_{22}(\alpha) \end{bmatrix}, \quad PB(\alpha) \triangleq \begin{bmatrix} B_1(\alpha) \\ B_2(\alpha) \end{bmatrix}, \tag{3.7}$$

$$F(\alpha)Q \triangleq \begin{bmatrix} F_1(\alpha) & F_2(\alpha) \end{bmatrix}. \tag{3.8}$$

Then, the closed-loop system (3.4) can be written as:

$$\dot{x}_1(t) = [A_{11}(\alpha) + B_1(\alpha)F_1(\alpha)]x_1(t) + [A_{12}(\alpha) + B_1(\alpha)F_2(\alpha)]x_2(t), \tag{3.9a}$$
$$0 = [A_{21}(\alpha) + B_2(\alpha)F_1(\alpha)]x_1(t) + [A_{22}(\alpha) + B_2(\alpha)F_2(\alpha)]x_2(t), \tag{3.9b}$$

where $A_{11}(\alpha)$, $A_{12}(\alpha)$, $A_{21}(\alpha)$, $A_{22}(\alpha)$, $B_1(\alpha)$, $B_2(\alpha)$, $F_1(\alpha)$ and $F_2(\alpha)$ are of appropriate dimensions.

To guarantee that the linear system (3.9) is regular and impulse free, the matrix $F_2(\alpha)$ is required to satisfy that $A_{22}(\alpha) + B_2(\alpha)F_2(\alpha)$ is nonsingular. The existence of such $F_2(\alpha)$ is guaranteed by Assumption 3.1.

By using (3.9b), $x_2(t)$ can be determined uniquely by

$$x_2(t) = \big(D_1(\alpha) + D_2(\alpha)F_1(\alpha)\big)x_1(t),$$

where

$$D_1(\alpha) \triangleq -(A_{22}(\alpha) + B_2(\alpha)F_2(\alpha))^{-1}A_{21}(\alpha), \tag{3.10a}$$
$$D_2(\alpha) \triangleq -\big(A_{22}(\alpha) + B_2(\alpha)F_2(\alpha)\big)^{-1}B_2(\alpha). \tag{3.10b}$$

Substituting (3.10) into (3.9a) and (3.5), we have

$$\dot{x}_1(t) = \big(\mathbb{A}(\alpha) + \mathbb{B}(\alpha)F_1(\alpha)\big)x_1(t), \tag{3.11}$$

and

$$\bar{S}(F(\alpha), \overline{u}(\alpha), \underline{u}(\alpha)) = \Bigg\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^n : -\underline{u}(\alpha) \le \mathbb{F}(\alpha)x_1(t) \le \overline{u}(\alpha) \text{and} \quad x_2(t) = \mathbb{C}(\alpha)x_1(t) \Bigg\},$$

where

$$\mathbb{A}(\alpha) \triangleq A_{11}(\alpha) + (A_{12}(\alpha) + B_1(\alpha)F_2(\alpha))D_1(\alpha), \tag{3.12a}$$
$$\mathbb{B}(\alpha) \triangleq B_1(\alpha) + (A_{12}(\alpha) + B_1(\alpha)F_2(\alpha))D_2(\alpha), \tag{3.12b}$$
$$\mathbb{C}(\alpha) \triangleq D_1(\alpha) + D_2(\alpha)F_1(\alpha), \tag{3.12c}$$
$$\mathbb{F}(\alpha) \triangleq (I + F_2(\alpha)D_2(\alpha))F_1(\alpha) + F_2(\alpha)D_1(\alpha), \tag{3.12d}$$

and $\bar{\mathbb{S}}$ be the common set of all the modes, $\bar{\mathbb{S}} \triangleq \bigcap_{\alpha\in\mathcal{M}} \bar{S}\big(F(\alpha), \overline{u}(\alpha), \underline{u}(\alpha)\big)$.

Remark 3.6 By [1, 2, 9], the constrained regulation problem can be solved if there exist matrices $F(\alpha) = [F_1(\alpha)\ F_2(\alpha)]$ such that the following conditions are satisfied:

(i) $\operatorname{rank}(A_{22}(\alpha) + B_2(\alpha)F_2(\alpha)) = n - r$; and
(ii) $\mathbb{F}(\alpha)(\mathbb{A}(\alpha) + \mathbb{B}(\alpha)F_1(\alpha)) = H(\alpha)\mathbb{F}(\alpha)$.

The existence of $F_2(\alpha)$ satisfying (i) can be guaranteed by Assumption 3.1. However, it is generally difficult to solve the nonlinear matrix equation in (ii). In this subsection we will solve the constrained regulation problem with the following assumption. ♦

Assumption 3.2 $\det(I + F_2(\alpha)D_2(\alpha)) \neq 0,\ \forall\alpha \in \mathcal{M}$.

Substituting (3.12d) into (ii) of Remark 3.6, it gives the following nonlinear matrix equation

$$\begin{aligned}&\Omega_1 + \Omega_2 F_1(\alpha) + F_1(\alpha)\Omega_3 + \Omega_4 F_1(\alpha)\Omega_3 + F_1(\alpha)\Omega_5 F_1(\alpha)\\ &\quad +\Omega_4 F_1(\alpha)\Omega_5 F_1(\alpha) = 0,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\Omega_1 &\triangleq F_2(\alpha)D_1(\alpha)\mathbb{A}(\alpha) - HF_2(\alpha)D_1(\alpha),\\ \Omega_2 &\triangleq F_2(\alpha)D_1(\alpha)\mathbb{B}(\alpha) - H(I + F_2(\alpha)D_2(\alpha)),\\ \Omega_3 &\triangleq \mathbb{A}(\alpha),\ \Omega_4 \triangleq F_2(\alpha)D_2(\alpha),\ \Omega_5 \triangleq \mathbb{B}(\alpha).\end{aligned}$$

By using Assumption 3.2, (3.13) is equivalent to

$$\mathbb{F}(\alpha)\left[\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathbb{F}(\alpha)\right] = H(\alpha)\mathbb{F}(\alpha), \tag{3.14}$$

where

$$\mathcal{A}(\alpha) \triangleq \mathbb{A}(\alpha) - \mathbb{B}(\alpha)\left[I + F_2(\alpha)D_2(\alpha)\right]^{-1} F_2(\alpha)D_1(\alpha), \tag{3.15}$$

$$\mathcal{B}(\alpha) \triangleq \mathbb{B}(\alpha)\left[I + F_2(\alpha)D_2(\alpha)\right]^{-1}. \tag{3.16}$$

Remark 3.7 From the discussions and analysis above, we know that the matrix parameters in the nonlinear matrix Eq. (3.14) are different from the parameters in [2]. In addition, necessary and sufficient conditions have been established based on Assumption 3.2 for the existence of full real rank solutions of matrix equation (3.16). So the works in this chapter are indeed different from [1] and [2]. ♦

These results allow us to establish the exponentially stability conditions for system (3.11) by the following lemma.

Lemma 3.8 [10] *If there exist a matrix $H(\alpha) \triangleq [h_{ij}(\alpha)]$ and a vector $\rho(\alpha)$, $\forall\alpha, \beta \in \mathcal{M}$, such that*

(i) $G(\alpha)\left[\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathbb{F}(\alpha)\right] = H(\alpha)\mathbb{F}(\alpha)$;
(ii) $H(\alpha)U(\alpha) \leqslant 0$; *and*
(iii) $\mathscr{A}(\alpha)\rho(\alpha) + \left[\sum_{\beta=1}^{s} \pi_{\alpha\beta} \max\limits_{1\leqslant i\leqslant n} \frac{\rho_i(\alpha)}{\rho_i(\beta)}\right]\rho(\alpha) < 0,$

where

$$G(\alpha) \triangleq \sum_{\beta=1}^{s} r_{\alpha\beta} F(\beta), \quad H(\alpha) \triangleq \begin{bmatrix} H_1(\alpha) & H_2(\alpha) \\ H_2(\alpha) & H_1(\alpha) \end{bmatrix},$$

$$U(\alpha) \triangleq \begin{bmatrix} \bar{u}(\alpha) \\ \underline{u}(\alpha) \end{bmatrix}, \quad \mathscr{A}(\alpha) \triangleq \begin{cases} |\mathbb{A}(\alpha) + \mathbb{B}(\alpha) F_1(\beta)|, & \alpha \neq \beta, \\ \mathbb{A}(\alpha) + \mathbb{B}(\alpha) F_1(\alpha), & \alpha = \beta, \end{cases}$$

$$H_1(\alpha) \triangleq \begin{cases} h_{ii}(\alpha), & i = j, \\ \sup\{h_{ij}(\alpha), 0\}, & i \neq j, \end{cases} \quad H_2(\alpha) \triangleq \begin{cases} 0, & i = j, \\ \sup\{-h_{ij}(\alpha), 0\}, & i \neq j. \end{cases}$$

Then, the close-loop system (3.11) *is exponentially stable.*

Remark 3.9 In Lemma 3.8, conditions (i) and (ii) allow the closed-loop system (3.11) to behave as a linear system inside the set $\bar{\mathbb{S}}$ that has the positive invariance property in the presence of input constraints. In addition, it follows from Assumption 1.1 and reference [10] that (iii) presents a sufficient condition of exponentially stability for the closed-loop system (3.11). ♦

Notice that the design of the controller, we are looking for the solutions of the Eq. (3.14). In the following section, we will show how to compute the matrix $H(\alpha)$ that satisfies (3.14).

3.3 Main Results

The full rank solutions to the nonlinear matrix Eq. (3.14) are derived in this section. Necessary and sufficient conditions under which the equations have full rank solutions are proposed, and an approach for constructing a full rank solution is given.

3.3.1 Full-Column Rank Solutions

The following theorem gives a necessary and sufficient condition under which the nonlinear matrix Eq. (3.14) have full-column rank solutions, and shows the construction of one full-column rank solution when such solutions exist.

Theorem 3.10 *Let* $\mathcal{A}(\alpha) \in \mathbb{R}^{r\times r}$, $\mathcal{B}(\alpha) \in \mathbb{R}^{r\times m}$, $H(\alpha) \in \mathbb{R}^{m\times m}$, $\lambda_1^{(\alpha)}, \ldots,$ *and* $\lambda_r^{(\alpha)} \in \mathbb{C}^- \backslash \sigma(A(\alpha))$. *Then the following two statements are equivalent when* $r \leqslant m$.

(i) *There exist linearly independent vectors* $\theta_1^{(\alpha)}, \ldots, \theta_r^{(\alpha)}$ *such that*

$$H(\alpha)\theta_i^{(\alpha)} = \lambda_i^{(\alpha)}\theta_i^{(\alpha)}, \quad \forall \alpha \in \mathcal{M}, i = 1, \ldots, r,$$

and that vectors $\eta_i^{(\alpha)}$ *are linearly independent, where*

$$\eta_i^{(\alpha)} \triangleq \left(\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha)\right)^{-1} \mathcal{B}(\alpha)\theta_i^{(\alpha)}, \quad \forall \alpha \in \mathcal{M}, \quad i = 1, \ldots, r. \tag{3.17}$$

(ii) *The nonlinear matrix Eq.* (3.14) *has full-column rank solutions* $X(\alpha)$ *such that*

$$\lambda_i^{(\alpha)} \in \sigma\left(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)X(\alpha)\right), \quad i = 1, 2, \ldots, r, \tag{3.18}$$

and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)X(\alpha)$ *has* r *linearly independent eigenvectors.*

Furthermore, if (i) *holds, then*

$$\mathcal{Y}(\alpha) \triangleq \left[\, \theta_1^{(\alpha)} \ldots \theta_r^{(\alpha)} \,\right] \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1}, \quad \forall \alpha \in \mathcal{M},$$

are full-column rank solutions to (3.14) *such that*

$$\lambda_i^{(\alpha)} \in \sigma\left(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)\right), \quad i = 1, 2, \ldots, r, \tag{3.19}$$

and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ *has* r *linearly independent eigenvectors.*

Proof Statement (i) $\Rightarrow$ Statement (ii). Let

$$\mathcal{X}(\alpha) = \left[\, \theta_1^{(\alpha)} \ldots \theta_r^{(\alpha)} \,\right] \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1}.$$

Obviously, $\mathcal{X}(\alpha)$ is of full-column rank. From (3.17), we have

$$\begin{aligned}
&\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha) \\
&\quad = \left(\mathcal{A}(\alpha)\left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right] + \mathcal{B}(\alpha)\left[\, \theta_1^{(\alpha)} \ldots \theta_r^{(\alpha)} \,\right]\right)\left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1} \\
&\quad = \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right] \mathrm{diag}\left\{\lambda_1, \ldots, \lambda_r\right\} \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1}.
\end{aligned}$$

Thus $\lambda_i^{(\alpha)} \in \sigma\big(\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)\big)$ and $A(\alpha)+B(\alpha)\mathcal{X}(\alpha)$ has r linearly independent eigenvectors. Furthermore, it follows that

$$\begin{aligned}
&\mathcal{X}(\alpha)\big(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha)\big) \\
&\quad = \mathcal{X}(\alpha)\left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right] \mathrm{diag}\left\{\lambda_1, \ \ldots, \ \lambda_r\right\} \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1} \\
&\quad = \left[\, \theta_1^{(\alpha)} \ldots \theta_r^{(\alpha)} \,\right] \mathrm{diag}\left\{\lambda_1, \ \ldots, \ \lambda_r\right\} \left[\, \eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)} \,\right]^{-1} \\
&\quad = H(\alpha)\mathcal{X}(\alpha).
\end{aligned}$$

Statement (ii) $\Rightarrow$ Statement (i). For any $\alpha \in \mathcal{M}$, assume that $\mathcal{X}(\alpha)$ is a full-column rank solution to (3.14) such that

$$\lambda_i^{(\alpha)} \in \sigma\big(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha)\big), \quad i = 1, 2, \ldots, r,$$

and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha)$ has r linearly independent eigenvectors $\varepsilon_1^{(\alpha)}, \ldots, \varepsilon_r^{(\alpha)}$. Without loss of generality, let

$$(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha))\varepsilon_i^{(\alpha)} \triangleq \lambda_i^{(\alpha)}\varepsilon_i^{(\alpha)}, \quad i = 1, \ldots, r \tag{3.20}$$

and

$$\mathcal{X}(\alpha) \triangleq \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathcal{B}(\alpha) \triangleq \begin{bmatrix} \mathcal{B}_1^T(\alpha) \\ \mathcal{B}_2^T(\alpha) \end{bmatrix}^T, \quad H(\alpha) \triangleq \begin{bmatrix} H_{11}(\alpha) & H_{12}(\alpha) \\ H_{21}(\alpha) & H_{22}(\alpha) \end{bmatrix},$$

where $\mathcal{B}_1(\alpha), H_{11}(\alpha) \in \mathbb{R}^{r\times r}$. Since $\mathcal{X}(\alpha)$ is a full-column rank solution to (3.14), we have

$$\begin{bmatrix} I \\ 0 \end{bmatrix}\mathcal{A}(\alpha) + \begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} \mathcal{B}_1(\alpha) & \mathcal{B}_2(\alpha) \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} H_{11}(\alpha) & H_{12}(\alpha) \\ H_{21}(\alpha) & H_{22}(\alpha) \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix},$$

thus $\mathcal{A}(\alpha) + \mathcal{B}_1(\alpha) = H_{11}(\alpha)$ and $H_{21}(\alpha) = 0$. So $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha) = H_{11}(\alpha)$, this together with (3.20), implies that

$$H_{11}(\alpha)\varepsilon_i^{(\alpha)} = \Big(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha)\Big)\varepsilon_i^{(\alpha)} = \lambda_i^{(\alpha)}\varepsilon_i^{(\alpha)}. \tag{3.21}$$

Set $\theta_i^{(\alpha)} \triangleq \begin{bmatrix} \varepsilon_i^{(\alpha)} \\ 0 \end{bmatrix}$, $i = 1, \ldots, r$. Clearly, $\theta_i^{(\alpha)}$, $i = 1, \ldots, r$ are linearly independent. According to $H_{21}(\alpha) = 0$ and (3.21), we have $H(\alpha)\theta_i^{(\alpha)} = \lambda_i^{(\alpha)}\theta_i^{(\alpha)}$, $i = 1, \ldots, r$. On the other hand, one obtains from (3.20) that

$$\begin{aligned}
&\Big(\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha)\Big)^{-1}\mathcal{B}(\alpha)\theta_i^{(\alpha)} \\
&= (\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha))^{-1}\begin{bmatrix} \mathcal{B}_1(\alpha) & \mathcal{B}_2(\alpha) \end{bmatrix}\begin{bmatrix} \varepsilon_i^{(\alpha)} \\ 0 \end{bmatrix} \\
&= \big(\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha)\big)^{-1}\mathcal{B}(\alpha)\mathcal{X}(\alpha)\varepsilon_i^{(\alpha)} \\
&= \varepsilon_i^{(\alpha)}, \ i = 1, \ 2, \ \ldots, \ r.
\end{aligned}$$

Hence, the vectors $\eta_i^{(\alpha)}$, $i = 1, \ldots, r$ defined by (3.17) are linearly independent.

Furthermore, when statement (i) is satisfied, we can conclude from the proof of statement (i) $\Rightarrow$ statement (ii) that for any $\alpha \in \mathcal{M}$, $\mathcal{Y}(\alpha)$ is a full-column rank solution to (3.14) such that (3.19) is satisfied and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ has r linearly independent eigenvectors. ■

In order to guarantee that the full-column rank solutions obtained from Theorem 3.10 are real, we offer the following corollary.

Corollary 3.11 *Let* $\mathcal{A}(\alpha) \in \mathbb{R}^{r\times r}, \mathcal{B}(\alpha) \in \mathbb{R}^{r\times m}, H(\alpha) \in \mathbb{R}^{m\times m}$*, and the group of complex numbers* $\lambda_1^{(\alpha)}, \ldots, \lambda_r^{(\alpha)} \in \mathbb{C}^-\backslash\sigma(\mathcal{A}(\alpha))$ *be symmetric, i.e., for every* $1 \leq i \leq r$*, these exists* $1 \leq j \leq r$ *such that* $\lambda_j^{(\alpha)}$ *is the conjugate of* $\lambda_i^{(\alpha)}$*. Then the following two statements are equivalent for* $r \leq m$*.*

(i) *There exist linearly independent vectors* $\theta_1^{(\alpha)}, \ldots, \theta_r^{(\alpha)}$ *such that* $\theta_i^{(\alpha)} = \bar{\theta}_j^{(\alpha)}$ *if and only if* $\lambda_i^{(\alpha)} = \bar{\lambda}_j^{(\alpha)}$, $H(\alpha)\theta_i^{(\alpha)} = \lambda_i^{(\alpha)}\theta_i^{(\alpha)}, i = 1, \ldots, r$*, and vectors* $\eta_i^{(\alpha)}$ *are linearly independent, where*

$$\eta_i^{(\alpha)} \triangleq \left[\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha)\right]^{-1} \mathcal{B}(\alpha)\theta_i^{(\alpha)}, \quad i = 1, \ldots, r. \tag{3.22}$$

(ii) *The nonlinear matrix Eq. (3.14) has full-column rank real solutions* $X(\alpha)$ *such that (3.18) is satisfied and* $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)X(\alpha)$ *has* r *linearly independent eigenvectors* $\varepsilon_1^{(\alpha)}, \ldots, \varepsilon_r^{(\alpha)}$ *subject to* $\varepsilon_i^{(\alpha)} = \bar{\varepsilon}_j^{(\alpha)}$ *if and only if* $\lambda_i = \bar{\lambda}_j^{(\alpha)}$*.*

Furthermore, if (i) *holds, then*

$$\mathcal{Y}(\alpha) \triangleq \left[\, \theta_1^{(\alpha)} \, \ldots \, \theta_r^{(\alpha)} \,\right] \left[\, \eta_1^{(\alpha)} \, \ldots \, \eta_r^{(\alpha)} \,\right]^{-1} \tag{3.23}$$

is a full-column rank real solution to (3.14) such that (3.19) is satisfied and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ *has* r *linearly independent eigenvectors.*

Proof By the proof of Theorem 3.10, it suffices to show that the full-column rank solution $\mathcal{Y}(\alpha)$ defined by (3.23) is real. By using (3.22) and $\theta_i^{(\alpha)} = \bar{\theta}_j^{(\alpha)}, \ i, \ j = 1, \ 2, \ \ldots, r$ whenever $\lambda_i^{(\alpha)} = \bar{\lambda}_j^{(\alpha)}$, we obtain that $\eta_i^{(\alpha)} = \bar{\eta}_j^{(\alpha)}, \ i, \ j = 1, \ 2, \ \ldots, r$ whenever $\lambda_i^{(\alpha)} = \bar{\lambda}_j^{(\alpha)}$. Hence (3.23) can be rewritten as

$$\mathcal{Y}(\alpha)\left[\, \mathcal{H}_1(\alpha) \; \overline{\mathcal{H}}_1(\alpha) \; \mathcal{H}_3(\alpha) \,\right] = \left[\, \mathcal{H}_2(\alpha) \; \overline{\mathcal{H}}_2(\alpha) \; \mathcal{H}_4(\alpha) \,\right],$$

where $\mathcal{H}_1(\alpha)$ and $\mathcal{H}_2(\alpha)$ are complex matrices, and $\mathcal{H}_3(\alpha)$ and $\mathcal{H}_4(\alpha)$ are real matrices. This implies that

$$\begin{aligned} &\mathcal{Y}(\alpha)\mathcal{H}_1(\alpha) = \mathcal{H}_2(\alpha), \ \mathcal{Y}(\alpha)\overline{\mathcal{H}}_1(\alpha) = \overline{\mathcal{H}}_2(\alpha), \\ &\mathcal{Y}(\alpha)\mathcal{H}_3(\alpha) = \mathcal{H}_4(\alpha), \end{aligned}$$

and hence

$$\overline{\mathcal{Y}}(\alpha)\left[\, \mathcal{H}_1(\alpha) \; \overline{\mathcal{H}}_1(\alpha) \; \mathcal{H}_3(\alpha) \,\right] = \mathcal{Y}(\alpha)\left[\, \mathcal{H}_1(\alpha) \; \overline{\mathcal{H}}_1(\alpha) \; \mathcal{H}_3(\alpha) \,\right],$$

which implies that $\overline{\mathcal{Y}}(\alpha) = \mathcal{Y}(\alpha)$, i.e., $\mathcal{Y}(\alpha)$ is real. ■

3.3.2 Full-Row Rank Solutions

The following theorem gives a necessary and sufficient condition under which the nonlinear matrix Eq. (3.14) have full-row rank solutions, and gives the construction of one full-row rank solution when such solutions exist.

Theorem 3.12 *Let* $\mathcal{A}(\alpha) \in \mathbb{R}^{r\times r}$, $\mathcal{B}(\alpha) \in \mathbb{R}^{r\times m}$, $H(\alpha) \in \mathbb{R}^{m\times m}$, $\lambda_1^{(\alpha)}, \ldots,$ *and* $\lambda_r^{(\alpha)} \in \mathbb{C}^-$. *If* $\sigma(H(\alpha)) \cap \sigma(\mathcal{A}(\alpha)) = \emptyset$. *Then the following two statements are equivalent for* $r > m$.

(i) *There exist linearly independent vectors* $\theta_1^{(\alpha)}, \ldots, \theta_m^{(\alpha)}$ *and vectors* $\eta_{m+1}^{(\alpha)}, \ldots, \eta_r^{(\alpha)}$ *such that*

$$H(\alpha)\theta_i^{(\alpha)} = \lambda_i^{(\alpha)}\theta_i^{(\alpha)}, \quad i = 1, \ldots, m, \tag{3.24}$$

$$\mathcal{A}(\alpha)\eta_j^{(\alpha)} = \lambda_j^{(\alpha)}\eta_j^{(\alpha)}, \quad j = m+1, \ \ldots, r, \tag{3.25}$$

and the vectors $\eta_i^{(\alpha)}$, $i = 1, \ \ldots, r$ *are linearly independent, where*

$$\eta_i^{(\alpha)} \triangleq \left[\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha)\right]^{-1} \mathcal{B}(\alpha)\theta_i^{(\alpha)}, \quad i = 1, \ldots, m. \tag{3.26}$$

(ii) *The nonlinear matrix Eq.* (3.14) *has full-row rank solutions* $X(\alpha)$ *such that* (3.18) *is satisfied and* $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)X(\alpha)$ *has* r *linearly independent eigenvectors.*

Furthermore, if (i) *holds, then*

$$\mathcal{Y}(\alpha) \triangleq \left[\,\theta_1^{(\alpha)} \ldots \theta_m^{(\alpha)}\ 0 \ldots 0\,\right]\left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right]^{-1}$$

are full-row rank solutions to (3.14) *such that* (3.19) *is satisfied and* $\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ *has* r *linearly independent eigenvectors.*

Proof Statement (i) $\Rightarrow$ Statement (ii). Let

$$\mathcal{X}(\alpha) = \left[\,\theta_1^{(\alpha)} \ldots \theta_m^{(\alpha)}\ 0 \ldots 0\,\right]\left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right]^{-1}.$$

Obviously, $\mathcal{X}(\alpha)$ is of full-row rank. Using (3.25) and (3.26), we have

$$\begin{aligned}
&\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{X}(\alpha)\\
&= \left(\mathcal{A}(\alpha)\left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right] + \mathcal{B}(\alpha)\left[\,\theta_1^{(\alpha)} \ldots \theta_m^{(\alpha)}\ 0 \ldots 0\,\right]\right)\left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right]^{-1}\\
&= \left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right]\mathrm{diag}\left\{\lambda_1^{(\alpha)}, \ \ldots, \ \lambda_r^{(\alpha)}\right\}\left[\,\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\,\right]^{-1}.
\end{aligned}$$

Thus, $\lambda_i^{(\alpha)} \in \sigma(\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha))$, and $\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)$ has r linearly independent eigenvectors. Furthermore, it follows that

$$\begin{aligned}&\mathcal{X}(\alpha)\big(\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)\big)\\ &= \mathcal{X}(\alpha)\left[\eta_1^{(\alpha)}\ldots\eta_r^{(\alpha)}\right]\mathrm{diag}\{\lambda_1^{(\alpha)},\ \ldots,\ \lambda_r^{(\alpha)}\}\left[\eta_1^{(\alpha)}\ldots\eta_r^{(\alpha)}\right]^{-1}\\ &= \left[\theta_1^{(\alpha)}\ldots\theta_n^{(\alpha)}\ 0\ldots0\right]\mathrm{diag}\{\lambda_1^{(\alpha)},\ \ldots,\ \lambda_r^{(\alpha)}\}\left[\eta_1^{(\alpha)}\ldots\eta_r^{(\alpha)}\right]^{-1}\\ &= H(\alpha)\mathcal{X}(\alpha).\end{aligned}$$

Statement (ii) $\Rightarrow$ Statement (i). Assume that $\mathcal{X}(\alpha)$ is a full-row rank solution to (3.14) such that

$$\lambda_i \in \sigma\big(\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)\big),\quad i=1,2,\ldots,r,$$

and for any $\alpha \in \mathcal{M}$, $\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)$ has r linearly independent eigenvectors. Without loss of generality, let

$$\mathcal{A}(\alpha) \triangleq \begin{bmatrix}\mathcal{A}_{11}(\alpha) & \mathcal{A}_{12}(\alpha)\\ \mathcal{A}_{21}(\alpha) & \mathcal{A}_{22}(\alpha)\end{bmatrix},\quad \mathcal{B}(\alpha) \triangleq \begin{bmatrix}\mathcal{B}_1(\alpha)\\ \mathcal{B}_2(\alpha)\end{bmatrix},\quad \mathcal{X}(\alpha) \triangleq \begin{bmatrix} I\\ 0\end{bmatrix}^T,$$

where $\mathcal{A}_{11}(\alpha), \mathcal{B}_1(\alpha) \in \mathbb{R}^{m\times m}$. Since $\mathcal{X}(\alpha)$ is a full-row rank solution to (3.14), we have

$$\big[\mathcal{A}_{11}(\alpha)\ \mathcal{A}_{12}(\alpha)\big]+\big[\mathcal{B}_1(\alpha)\ 0\big]=H(\alpha)\big[I\ \ 0\big],$$

and thus $\mathcal{A}_{11}(\alpha)+\mathcal{B}_1(\alpha)=H(\alpha)$ and $\mathcal{A}_{12}(\alpha)=0$. Hence

$$\mathcal{A}(\alpha)+\mathcal{B}(\alpha)\mathcal{X}(\alpha)=\begin{bmatrix}H(\alpha) & 0\\ \mathcal{A}_{21}(\alpha)+\mathcal{B}_2(\alpha) & \mathcal{A}_{22}(\alpha)\end{bmatrix}.$$

According to $\sigma(H(\alpha))\cap\sigma(A(\alpha))=\emptyset$ and Lemma 3.4, one can show that

$$\begin{aligned}H(\alpha)\theta_i^{(\alpha)} &= \lambda_i^{(\alpha)}\theta_i^{(\alpha)},\quad i=1,\ldots,\ m,\\ \mathcal{A}_{22}(\alpha)\xi_j^{(\alpha)} &= \lambda_j^{(\alpha)}\xi_j^{(\alpha)},\quad j=m+1,\ldots,\ r,\end{aligned}$$

where $\theta_1^{(\alpha)},\ \ldots,\ \theta_m^{(\alpha)}$ and $\xi_{m+1}^{(\alpha)},\ \ldots,\ \xi_r^{(\alpha)}$ are linearly independent, respectively. Set

$$\eta_i^{(\alpha)} \triangleq \left[\lambda_i^{(\alpha)}I-\mathcal{A}(\alpha)\right]^{-1}\mathcal{B}(\alpha)\theta_i^{(\alpha)},\ i=1,\ldots,\ m, \tag{3.27}$$

$$\eta_j^{(\alpha)} \triangleq \begin{bmatrix}0\\ \xi_j^{(\alpha)}\end{bmatrix},\quad j=m+1,\ \ldots,\ r. \tag{3.28}$$

Then, $\mathcal{A}(\alpha)\eta_j^{(\alpha)}=\lambda_j^{(\alpha)}\eta_j^{(\alpha)}$, $j=m+1,\ldots,r$, which implies that (3.25) holds.

From (3.27), $\mathcal{A}_{12}(\alpha) = 0$ and $\mathcal{A}_{11}(\alpha) + \mathcal{B}_1(\alpha) = H(\alpha)$, we have

$$\eta_i^{(\alpha)} = \begin{bmatrix} \left[\lambda_i^{(\alpha)} I - \mathcal{A}_{11}(\alpha)\right]^{-1} \mathcal{B}_1(\alpha)\theta_i^{(\alpha)} \\ * \end{bmatrix} = \begin{bmatrix} \theta_i^{(\alpha)} \\ * \end{bmatrix}.$$

Hence the group of vectors $\eta_i^{(\alpha)},\ i = 1, \ldots,\ r$ are linearly independent.

Furthermore, when statement (i) holds, we can conclude from the proof of statement (i) $\Rightarrow$ statement (ii) that $\mathcal{Y}(\alpha)$ is a full-row rank solution to (3.14) such that (3.19) is satisfied and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ has r linearly independent eigenvectors. ■

In order to ensure that the full-row rank solutions to (3.14) constructed in Theorem 3.12 are real, we gives the following corollary.

Corollary 3.13 *Let $\mathcal{A}(\alpha) \in \mathbb{R}^{r\times r}$, $\mathcal{B}(\alpha) \in \mathbb{R}^{r\times m}$, $H(\alpha) \in \mathbb{R}^{m\times m}$, and two groups of complex numbers $\lambda_1^{(\alpha)}, \ldots, \lambda_m^{(\alpha)} \in \mathbb{C}^-$ and $\lambda_{m+1}^{(\alpha)}, \ldots, \lambda_r^{(\alpha)} \in \mathbb{C}^-$ be symmetric. If $\sigma(H(\alpha)) \cap \sigma(\mathcal{A}(\alpha)) = \emptyset$, then the following two statements are equivalent for $r > m$.*

(i) *There exist a group of linearly independent vectors $\theta_1^{(\alpha)}, \ldots, \theta_m^{(\alpha)}$ and a group of vectors $\eta_{m+1}^{(\alpha)}, \ldots, \eta_r^{(\alpha)}$ such that $\theta_i^{(\alpha)} = \bar{\theta}_j^{(\alpha)},\ i,\ j = 1,\ 2,\ \ldots, m$ whenever $\lambda_i^{(\alpha)} = \bar{\lambda}_j^{(\alpha)}$, $\eta_l^{(\alpha)} = \bar{\eta}_k^{(\alpha)}$ if and only if $\lambda_l^{(\alpha)} = \bar{\lambda}_k^{(\alpha)}$, $l, k = m + 1, \ldots, r$,*

$$\begin{aligned} H(\alpha)\theta_i^{(\alpha)} &= \lambda_i^{(\alpha)}\theta_i^{(\alpha)}, \quad i = 1, \ldots, m, \\ \mathcal{A}(\alpha)\eta_j^{(\alpha)} &= \lambda_j^{(\alpha)}\eta_j^{(\alpha)}, \ j = m + 1, \quad \ldots, r, \end{aligned}$$

and the group of vectors $\eta_1^{(\alpha)}, \ldots, \eta_r^{(\alpha)}$ are linearly independent, where

$$\eta_i^{(\alpha)} = (\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha))^{-1}\mathcal{B}(\alpha)\theta_i^{(\alpha)}, \quad i = 1, \ldots m. \tag{3.29}$$

(ii) *The nonlinear matrix Eq. (3.14) has at least one full-row rank real solution $X(\alpha)$ such that (3.18) is satisfied and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)X(\alpha)$ has r linearly independent eigenvectors $\varepsilon_1^{(\alpha)},\ \ldots,\ \varepsilon_r^{(\alpha)}$ subject to $\varepsilon_i^{(\alpha)} = \bar{\varepsilon}_j^{(\alpha)}$ if and only if $\lambda_i^{(\alpha)} = \bar{\lambda}_j^{(\alpha)}$.*

Furthermore, if (i) *holds, then*

$$\mathcal{Y}(\alpha) \triangleq \left[\theta_1^{(\alpha)} \ldots \theta_m^{(\alpha)}\ 0 \ldots 0\right]\left[\eta_1^{(\alpha)} \ldots \eta_r^{(\alpha)}\right]^{-1} \tag{3.30}$$

are full-row rank real solutions to (3.14) such that (3.19) is satisfied and $\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{Y}(\alpha)$ has r linearly independent eigenvectors.

Proof It can be carried out along the same line as that in the proof of Corollary 3.11. ■

3.3.3 A Heuristic Algorithm for Constrained Regulation Problem

If there exists $H(\alpha)$, $\alpha \in \mathcal{M}$, such that (ii) and (iii) in Lemma 3.8, and Corollaries 3.11 and 3.13 are satisfied, then

$$\mathbb{F}(\alpha) = \begin{cases} \big[\theta_1 \ldots\ \theta_r\big]\big[\eta_1 \ldots\ \eta_r\big]^{-1}, & \text{if } r \leq m, \\ \big[\theta_1 \ldots\ \theta_m\ 0_{m+1}\ \ldots 0_r\big]\big[\eta_1 \ldots\ \eta_r\big]^{-1}, & \text{if } r > m, \end{cases} \tag{3.31}$$

is a full rank solution to (3.14) such that $\sigma\big(\mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathbb{F}(\alpha)\big) \subseteq \mathbb{C}^-$. Furthermore, we can obtain $F_1(\alpha)$ and $F(\alpha)$ from (3.12d) and (3.8), respectively.

By the above discussion, we present a new algorithm to solve the regulation problem for systems (3.2) with constrained control input, as follows.

Heuristic algorithm

Step 1. Find $P(\alpha)$ and $Q(\alpha)$ satisfied (3.6).
Step 2. Compute $\mathcal{A}_{11}(\alpha)$, $\mathcal{A}_{12}(\alpha)$, $\mathcal{A}_{21}(\alpha)$, $\mathcal{A}_{22}(\alpha)$, $\mathcal{B}_1(\alpha)$ and $\mathcal{B}_2(\alpha)$ by (3.7).
Step 3. Choose $\mathcal{F}_2(\alpha)$ such that: $\mathcal{A}_{22}(\alpha) + \mathcal{B}_2(\alpha)\mathcal{F}_2(\alpha)$ and $I - \mathcal{F}_2(\alpha)\big(\mathcal{A}_{22}(\alpha) + \mathcal{B}_2(\alpha)\mathcal{F}_2(\alpha)\big)^{-1} B_2(\alpha)$ are nonsingular.
Step 4. Compute $D_1(\alpha),\ D_2(\alpha),\ \mathbb{A}(\alpha),\ \mathbb{B}(\alpha),\ \mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ by (3.10b), (3.12b) and (3.16).
Step 5. **If** $r > m$ and NUM$\{\mathcal{A}(\alpha)\} < r - m$
break, else
Compute characteristic roots $\lambda_{m+1}^{(\alpha)}, \ldots, \lambda_r^{(\alpha)}$ of $\mathcal{A}(\alpha)$ and corresponding eigenvectors $\eta_{m+1}, \ldots, \eta_r$.
End
Step 6. Set $p = \min\{r,\ m\}$ and choose matrices $H(\alpha)$, vector $\rho(\alpha)$, scalars $\lambda_i^{(\alpha)}$ and linearly independent vectors $\theta_i^{(\alpha)}$ such that ii) and iii) in Lemma 4 and the following conditions are satisfied:
(1) $H(\alpha)\theta_i^{(\alpha)} = \lambda_i^{(\alpha)}\theta_i^{(\alpha)},\ i = 1,\ \ldots,\ p$;
(2) $\sigma(\mathcal{A}(\alpha)) \cap \lambda_i^{(\alpha)} = \emptyset$;
(3) $\eta_i^{(\alpha)} \triangleq (\lambda_i^{(\alpha)} I - \mathcal{A}(\alpha))^{-1}\mathcal{B}(\alpha)\theta_i^{(\alpha)}$ are linearly independent;
Step 7. Compute $\mathbb{F}(\alpha),\ F_1(\alpha)$ and $F(\alpha)$ by (3.8) and (3.31), and then output $F(\alpha)$.

Remark 3.14 Since Corollaries 3.11 and 3.13 offer necessary and sufficient conditions, Step 6 can be easily satisfied when the nonlinear matrix Eq. (3.14) has full rank solutions. ♦

3.4 Simulation Results

In this section, we demonstrate the effectiveness of proposed constrained regulation control scheme by a DC motor driving a load [6]. Let $u(t), i(t)$ and $\omega(t)$ denote respectively the voltage of the armature, the current in the armature and the speed of the shaft at time t. If we neglect the inductance of the DC motor, then the DC motor equation can be expressed as:

$$\begin{cases} u(t) &= Ri(t) + K_w\omega(t), \\ J\dot{\omega}(t) &= K_t i(t) - b\omega(t), \end{cases}$$

where R, K_w, K_t represent respectively the electric resistor of the armature, the electromotive force constant, the torque constant, J and b are defined by $J \triangleq J_m + J_c/n^2$ and $b \triangleq b_m + b_c/n^2$ with J_m and J_c being the moments of inertia of the rotor and the load, b_m and b_c being the damping ratios of the motor and the load, and n the gear ratio of the motor and the load. We set $x_1(t) \triangleq i(t)$, $x_2(t) \triangleq \omega(t)$ and $y(t) \triangleq x_2(t)$. Then, the state-space version of the system is described by: $E\dot{x}(t) = Ax(k) + Bu(k)$ where

$$E \triangleq \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \quad A \triangleq \begin{bmatrix} R & K_w \\ K_t & -b \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now we assume that the load changes randomly and abruptly which we can model by the changes of the inertia J. Consider system (3.1) with the following system matrices:

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} -1.2 & 0.3 \\ 0.0 & 0.6 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 1.2 \\ 0.1 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -0.7 & 0.5 \\ 0.2 & 0.4 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 1.1 \\ 0.2 \end{bmatrix}.$$

The analysis of a PH S-MJS, by Lemma 1.24, is reduced to the analysis of its associated MJS. In particular, two parts can be identified in the second model for the sojourn time which is a random variable exponentially distributed with parameter λ_2 in Part 1 and λ_3 in Part 2. This view suggests that the process Ą $\bar{\eta}_t$ must stay at the first part for some time upon entering Model 2, before making its way to the second, and finally returns to Model 1 again. Therefore

$$a^{(1)} = (a_1^{(1)}) = 1, \quad a^{(2)} = \left(a_1^{(2)}, a_2^{(2)}\right) = (1, 0),$$
$$T^{(1)} = (T_{11}^{(1)}) = (-\lambda_1),$$
$$T^{(2)} = \begin{bmatrix} T_{11}^{(2)} & T_{12}^{(2)} \\ T_{21}^{(2)} & T_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{bmatrix}.$$

It is easy to see that the state space of $Z(t) = (\bar{\eta}_t, J(t))$ is $G = \big((1, 1), (2, 1), (2, 2)\big)$. We enumerate the elements of G as $\varphi\big((1, 1)\big) = 1$, $\varphi\big((2, 1)\big) = 2$, and $\varphi\big((2, 2)\big) = 3$. Hence, the infinitesimal generator of $\varphi(Z(t))$ is

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ -\lambda_3 & 0 & \lambda_3 \end{bmatrix}.$$

Now, let $\eta_t = \varphi(Z(t))$. Then, η_t is the associated Markov chain of $\bar{\eta}_t$ with state space $\{1, 2, 3\}$. The infinitesimal generator of η_t is given by Q. By choosing $H(1) = \text{diag}\{-1, -2\}$, $H(2) = \text{diag}\{-1, -3\}$, we can obtain the state-feedback controller gain matrices for Mode 1 and Mode 2 as

$$F(1) = \begin{bmatrix} -0.424 & -0.237 \\ -0.665 & 0.452 \end{bmatrix}, \quad F(2) = \begin{bmatrix} 0.342 & -0.367 \\ -0.832 & 0.205 \end{bmatrix}.$$

For each mode, we set the constraint control as: $-0.8 \leqslant u(t) \leqslant 0.8$, and choose the following different initial states $x(0) = [-1.6\ 1.1]^T$, $x(0) = [-1.6\ -0.63]^T$ and $x(0) = [1.1\ -0.63]^T$, respectively.

The simulation result is given in Fig. 3.1. It shows that $\bar{\mathbb{S}}$ is a positively invariant set of system (3.11), and that all state trajectories intersect at the origin of coordinates, which presents the stability of the system (3.11). To further illustrate the effectiveness of the proposed technique, we perform Monte Carlo simulation to capture realistic

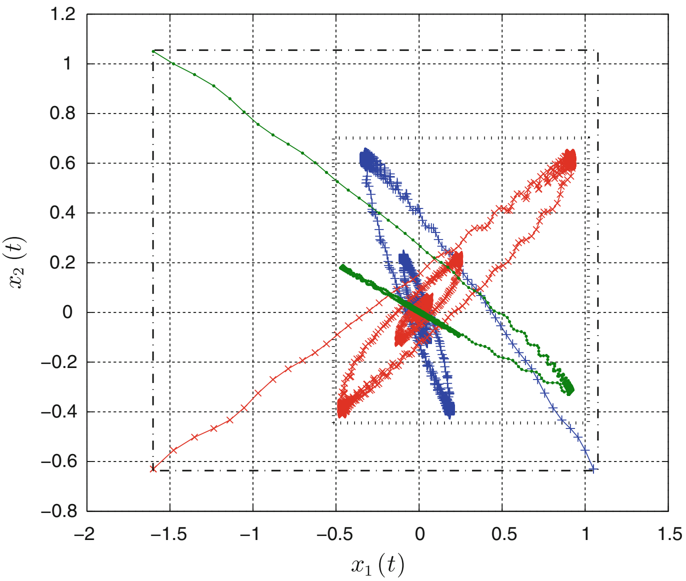

Fig. 3.1 State responses of the system inside the positive invariance set for different initial states

random behavior in real-life applications. In Fig. 3.1, the state responses of the closed-loop system for 50 runs are shown. The semi-Markov processes are unique in each run. It thus confirms that the system is stochastically stabilized for every run of simulation.

3.5 Conclusion

In this chapter, the regulation problem has been addressed for singular S-MJS with constrained control. Necessary and sufficient conditions under which the nonlinear matrix equation have full rank solutions have been derived. Based on the proposed resolution, we develop an algorithm for constructing a state-feedback control law guaranteeing that the resultant closed-loop system is admissible with a positively invariant set. A DC motor load changing simulation example is given to illustrate the effectiveness of the proposed design schemes (Fig. 3.2).

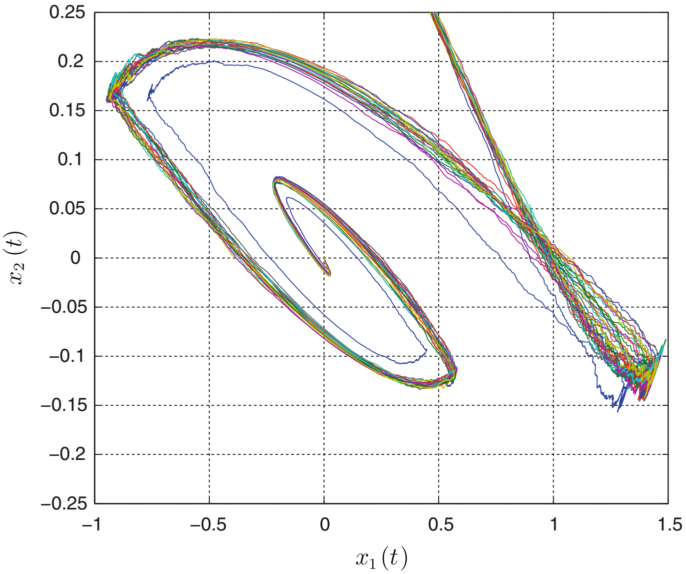

Fig. 3.2 State responses of the system by Monte Carlo simulation

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Chapter 4
State Estimation and Sliding Mode Control for Semi-Markovian Jump Systems

Abstract This chapter investigates the state estimation and sliding mode control problems for S-MJS with mismatched uncertainties. A sliding surface is then constructed and a sliding mode controller is synthesized to ensure that the S-MJS satisfying the reaching condition. Further, an observer-based sliding mode control problem is investigated. Sufficient conditions are established for the solvability of the desired observer. It is shown that the proposed SMC law based on the estimated states can guarantee that the sliding modes within both the state estimation space and the estimation error space are attained simultaneously.

4.1 Introduction

SMC is an effective control approach due to its excellent advantage of strong robustness against model uncertainties, parameter variations, and external disturbances. It is worthwhile to mention that the SMC strategy has been successfully applied to a variety of practical systems such as robot manipulators, aircraft navigation and control, and power systems stabilizers. Furthermore, the system states are not always available. Thus, sliding mode observer technique has been developed to deal with the state estimation problems for linear or nonlinear uncertain systems.

In this chapter, we will investigate the state estimation and sliding mode control problems for S-MJS. This chapter addresses two open problems: (1) how to design the appropriate sliding surface function to adjust the effect of the jumping phenomenon in the plant; and (2) how to perform the reachability analysis for the resulting sliding mode dynamics. Thus, sliding surface function design and reachability analysis of the resulting sliding mode dynamics are the main issues to be addressed in this chapter.

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4.2 System Description and Preliminaries

Consider the following S-MJS:

$$
\begin{aligned}
\dot{x}(t) &= \big[\hat{A}(\bar{\eta}_t) + \varDelta\hat{A}(\bar{\eta}_t, t)\big]x(t) + \hat{B}(\bar{\eta}_t)\big[u(t) + \varphi(t)\big],\\
y(t) &= \hat{C}(\bar{\eta}_t)x(t),
\end{aligned} \tag{4.1}
$$

where $\{\bar{\eta}_t, t \geqslant 0\}$ is a continuous-time PH semi-Markov process on the probability space which has been defined in Definition 1.21 of Chap. 1, and $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the system output, and $\varphi(t) \in \mathbb{R}^p$ is uncertainty disturbance. $\hat{A}(\bar{\eta}_t)$, $\hat{B}(\bar{\eta}_t)$ and $\hat{C}(\bar{\eta}_t)$ are matrix functions of the random process $\{\bar{\eta}_t, t \geqslant 0\}$; and $\varDelta\hat{A}(\bar{\eta}_t, t)$ is system uncertainty.

It follows from Lemma 1.24 in Chap. 1, and employing the same techniques as those used in [1]. We can construct the following associated MJS which are equivalent to (4.1):

$$
\begin{aligned}
\dot{x}(t) &= \big[A(\eta_t) + \varDelta A(\eta_t, t)\big]x(t) + B(\eta_t)\big[u(t) + \varphi(t)\big],\\
y(t) &= C(\eta_t)x(t),
\end{aligned} \tag{4.2}
$$

where $\{\eta_t, t \geqslant 0\}$ is the associated Markov chain of PH semi-Markov chain $\{\bar{\eta}_t, t \geqslant 0\}$. For notional simplicity, when the system operates in the ith mode, $A(\eta_t)$, $\varDelta A(\eta_t, t)$, $B(\eta_t)$ and $C(\eta_t)$ are denoted by $A(i)$, $\varDelta A(i, t)$, $B(i)$ and $C(i)$, respectively.

4.3 Main Results

4.3.1 Sliding Mode Control

This section presents the design results of the sliding surface and reaching motion controller. Firstly, we will analyze the sliding mode dynamics. Since $B(\eta_t)$ is of full column rank by assumption, there exists the following singular value decomposition:

$$
B(\eta_t) = \begin{bmatrix} U_1(\eta_t) & U_2(\eta_t) \end{bmatrix} \begin{bmatrix} \varSigma(\eta_t) \\ 0_{(n-p)\times p} \end{bmatrix} V^T(\eta_t),
$$

where $U_1(\eta_t) \in \mathbb{R}^{n\times p}$ and $U_2(\eta_t) \in \mathbb{R}^{n\times(n-p)}$ are unitary matrices, $\varSigma(\eta_t) \in \mathbb{R}^{p\times p}$ is a diagonal positive-definite matrix, and $V(\eta_t) \in \mathbb{R}^{p\times p}$ is a unitary matrix. Let $T(\eta_t) \triangleq \begin{bmatrix} U_2(\eta_t) \; U_1(\eta_t) \end{bmatrix}^T$. For each possible value $\eta_t = i, i \in \mathcal{M}$, we denote $T(i) \triangleq T(\eta_t = i), i \in \mathcal{M}$. Then, by the state transformation $z(t) \triangleq T(i)x(t)$, system (4.2) has the regular form given by

$$\dot{z}(t) = \big[\bar{A}(i) + \Delta\bar{A}(i,t)\big]z(t) + \begin{bmatrix} 0_{(n-p)\times p} \\ B_2(i) \end{bmatrix}\big[u(t) + \varphi(t)\big], \tag{4.3}$$

where

$$\bar{A}(i) \triangleq T(i)A(i)T^{-1}(i), \quad B_2(i) \triangleq \Sigma(i)V^T(i),$$
$$\Delta\bar{A}(i,t) \triangleq T(i)\Delta A(i,t)T^{-1}(i).$$

Let $z(t) \triangleq \big[\, z_1^T(t)\ z_2^T(t) \,\big]^T$, where $z_1(t) \in \mathbb{R}^{n-p}$ and $z_2(t) \in \mathbb{R}^p$, then system (4.3) can be transformed into

$$\begin{cases} \dot{z}_1(t) = \big[\bar{A}_{11}(i) + \Delta\bar{A}_{11}(i,t)\big]z_1(t) + \big[\bar{A}_{12}(i) + \Delta\bar{A}_{12}(i,t)\big]z_2(t), \\ \dot{z}_2(t) = \big[\bar{A}_{21}(i) + \Delta\bar{A}_{21}(i,t)\big]z_1(t) + \big[\bar{A}_{22}(i) + \Delta\bar{A}_{22}(i,t)\big]z_2(t) \\ \qquad\quad + B_2(i)\big[u(t) + \varphi(t)\big], \quad i \in \mathcal{M}, \end{cases} \tag{4.4}$$

where

$$\bar{A}_{11}(i) \triangleq U_2^T(i)A(i)U_2(i), \quad \bar{A}_{12}(i) \triangleq U_2^T(i)A(i)U_1(i),$$
$$\bar{A}_{21}(i) \triangleq U_1^T(i)A(i)U_2(i), \quad \bar{A}_{22}(i) \triangleq U_1^T(i)A(i)U_1(i).$$

The mismatched time-varying uncertainties $\Delta\bar{A}_{uv}(i,t)$, $u, v \in \{1, 2\}$ are assumed to carry the following structure

$$\begin{bmatrix} \Delta\bar{A}_{11}(i,t) & \Delta\bar{A}_{12}(i,t) \\ \Delta\bar{A}_{21}(i,t) & \Delta\bar{A}_{22}(i,t) \end{bmatrix} = \begin{bmatrix} E_1(i) \\ E_2(i) \end{bmatrix} \Delta F(i,t) \begin{bmatrix} H_1(i) & H_2(i) \end{bmatrix},$$

where $E_1(i)$, $E_2(i)$, $H_1(i)$ and $H_2(i)$, $i \in \mathcal{M}$ are known real-constant matrices, and $\Delta F(i,t)$ is the unknown time-varying matrix function satisfying

$$\Delta F^T(i,t)\Delta F(i,t) \leqslant I, \quad i \in \mathcal{M}.$$

It is obvious that (4.4) represents the sliding mode dynamics of system (4.3), and hence the corresponding sliding surface can be chosen as follows for each $i \in \mathcal{M}$,

$$s(t,i) = \big[\, -\mathbb{C}(i)\ I \,\big]z(t) = -\mathbb{C}(i)z_1(t) + z_2(t) = 0, \tag{4.5}$$

where $\mathbb{C}(i)$, $i \in \mathcal{M}$ are the parameters to be designed. When the system trajectories reach onto the sliding surface $s(t,i) = 0$, that is, $z_2(t) = \mathbb{C}(i)z_1(t)$, the sliding mode dynamics is attained. Substituting $z_2(t) = \mathbb{C}(i)z_1(t)$ into the first equation of system (4.4) gives the sliding mode dynamics:

$$\dot{z}_1(t) = \mathscr{A}(i)z_1(t), \quad i \in \mathcal{M}, \tag{4.6}$$

where

$$\mathscr{A}(i) \triangleq \bar{A}_{11}(i) + E_1(i)\Delta F_i H_1(i) + \big(\bar{A}_{12}(i) + E_1(i)\Delta F_i H_2(i)\big)\mathbb{C}(i).$$

Let us recall the definition of the exponential stable of system (4.6).

Definition 4.1 System (4.6) is said to be exponentially stable, if for any initial condition $z_1(0) \in \mathbb{R}^n$, and $r_0 \in \mathcal{M}$, there exist constants α_1 and α_2, such that

$$\mathbf{E}\left\{\|z_1(0, r_0)\|^2\right\} \leqslant \alpha_1 \|z_1(0)\|^2 \exp(-\alpha_2 t).$$

Let $C^2(\mathbb{R}^2 \times \mathcal{M}; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(z_1, i)$ on $\mathbb{R}^n \times \mathcal{M}$ which are continuously twice differentiable in z_1. For $V \in C^2(\mathbb{R}^2 \times \mathcal{M}; \mathbb{R}_+)$, define an infinitesimal operator $\mathcal{L}V(z_1, i)$ as in Mao99, Mao02. Then, we have the following lemma.

Lemma 4.2 [2, 3] *If there exists a function $V \in C^2(\mathbb{R}^2 \times \mathcal{M}; \mathbb{R}_+)$ and positive constants c_1, c_2 and c_3 such that*

$$c_1\|z_1\|^2 \leqslant V(z_1, i) \leqslant c_2\|z_1\|^2,$$

and $\mathcal{L}V(z_1, i) \leqslant -c_3\|z_1\|^2$, for all $(z_1, i) \in \mathbb{R}^n \times \mathcal{M}$, then system (4.6) is exponentially stable.

In the following, we consider the problem of sliding mode surface design.

Theorem 4.3 *Associated MJS (4.6) is exponentially stable, if there exist matrices $Q(i) > 0$ and general matrices $M(i), i \in \mathcal{M}$ such that the following inequalities hold,*

$$\begin{bmatrix} \Theta_{11}(i) & \Theta_{12}(i) & \Theta_{13}(i) \\ * & \Theta_{22}(i) & 0 \\ * & * & \Theta_{33}(i) \end{bmatrix} < 0, \tag{4.7}$$

where

$$\begin{aligned}
\Theta_{11}(i) &\triangleq \lambda_{ii}Q(i) + \bar{A}_{11}(i)Q(i) + Q(i)\bar{A}_{11}^T(i) + \bar{A}_{12}(i)M(i) + M^T(i)\bar{A}_{12}^T(i),\\
\Theta_{12}(i) &\triangleq \left[\, E_1(i)\ \epsilon_1 Q_1^T(i)H_1(i)\ E_2(i)\ \epsilon_2 M^T(i)H_1(i) \,\right],\\
\Theta_{22}(i) &\triangleq \mathrm{diag}\{-\epsilon_1 I, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_2 I\},\\
\Theta_{13}(i) &\triangleq \left[\, \sqrt{\lambda_{i1}}Q(i) \ldots \sqrt{\lambda_{iN}}Q(i) \,\right],\\
\Theta_{33}(i) &\triangleq \mathrm{diag}\{-Q(1), \ldots, -Q(N)\}.
\end{aligned}$$

Moreover, the sliding surface of system (4.6) is

$$s(t, i) = -M(i)Q^{-1}(i)z_1(t) + z_2(t), i \in \mathcal{M}. \tag{4.8}$$

Proof To analyze the stability of the sliding motion (4.6), we choose the following Lyapunov function:

$$V\big(z_1(t), i\big) \triangleq z_1^T(t)P(i)z_1(t),$$

where $P(i) \triangleq P(\eta_t = i) > 0, i \in \mathcal{M}$ are real matrices to be determined.

The infinitesimal generator $\mathcal{L}$ can be considered as a derivative of the Lyapunov function $V(z_1(t), i)$ along the trajectories of the associated Markov process $\{\eta_t, t \geqslant 0\}$. Then

$$\mathcal{L}V(z_1(t), i) = z_1^T(t)\Phi(t)z_1(t),$$

where

$$\begin{aligned}\Phi(t) \triangleq & \sum_{j=1}^{N} P(j)\lambda_{ij} + 2P(i)[\bar{A}_{11}(i) + \bar{A}_{12}(i)\mathbb{C}(i)] \\ & + 2P(i)[E_1(i)\Delta F_i H_1(i) + E_1(i)\Delta F_i H_2(i)\mathbb{C}(i)].\end{aligned}$$

Obviously, if $\Phi(t) < 0$, then $\mathcal{L}V(z_1(t), i) < 0, \forall i \in \mathcal{M}$. Next, define $Q(i) = P^{-1}(i)$, then pre- and post-multiplying $\Phi(t)$ by $Q(i)$. Using $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$ and from Schur complement, it gives that $\Phi(t) < 0$, which is equivalent to (4.7). Thus, one obtains $\mathcal{L}V(z_1(t), i) < 0$ for all $z_1(t) \neq 0$ and $i \in \mathcal{M}$. Then, $\mathcal{L}V(z_1(t), i) \leqslant -\lambda_1\|z_1(t)\|^2$, where $\lambda_1 \triangleq \min_{i\in\mathcal{M}}\{-\lambda_{\min}(\Phi(t))\} > 0$.

This result implies that the inequalities depend only on the global constant λ_1. By Lemma 4.2 and employing the same techniques as in [3], inequalities (4.7) ensure that system (4.6) is exponentially stable. ■

Next, we synthesize a SMC law to drive the system trajectories onto the predefined sliding surface $s(t, i) = 0$ in (4.8) in a finite time.

Theorem 4.4 *Suppose* (4.7) *has solutions* $M(i)$ *and* $Q(i), i \in \mathcal{M}$, *and the linear sliding surface is given by* (4.8). *Then, the following controller* (4.9) *makes the sliding surface* $s(t, i) = 0, i \in \mathcal{M}$, *stable and globally attractive in a finite time*

$$u(t) = -B_2^{-1}(i)\Big[\mathcal{M}_i\big(\bar{A}(i) + \Delta\bar{A}(i, t)\big)z(t) + \big(\varsigma(i) + \varrho(i)\big)\text{sgn}(s(t, i))\Big], \tag{4.9}$$

where

$$\mathcal{M}_i \triangleq \big[-M(i)Q^{-1}(i)\ \ I\big], \quad \varsigma(i) > 0, \quad i \in \mathcal{M},$$

are constants and $\varrho(i) \triangleq \max_{i\in\mathcal{M}}(\|B_2(i)\varphi(t)\|)$.

Proof We will complete the proof by showing that the control law (4.9) can not only make the system exponentially stable but also globally attractive in a finite time. For each $i \in \mathcal{M}$, from the sliding surface

$$s(t,i)=\big[-\mathbb{C}(i)\ I\big]z(t)\triangleq \mathcal{M}_i z(t),$$

select the following Lyapunov function:

$$V(i,s(t,i))\triangleq \frac{1}{2}s^T(t,i)s(t,i),\quad i\in\mathcal{M}.$$

According to (4.3) and (4.5), for each $i\in\mathcal{M}$, we obtain

$$\dot{s}(t,i)=\mathcal{M}_i\big[\bar{A}(i)+\Delta\bar{A}(i,t)\big]z(t)+\mathcal{M}_i\begin{bmatrix}0_{(n-p)\times p}\\ B_2(i)\end{bmatrix}\big[u(t)+\varphi(t)\big]. \tag{4.10}$$

Substituting (4.9) into (4.10) yields

$$\dot{s}(t,i)=-(\varsigma(i)+\varrho(i))\mathrm{sgn}(s(t,i))+B_2(i)\varphi(t),\quad i\in\mathcal{M}.$$

Thus, taking the derivative of $V(i,s(t,i))$ and considering $|s(t,i)|\geqslant\|s(t,i)\|$, we obtain that

$$\begin{aligned}\dot{V}(i,s(t,i))&=s^T(t,i)\dot{s}(t,i)\\&=-s^T(t,i)(\varsigma(i)+\varrho(i))\mathrm{sgn}(s(t,i))+s^T(t,i)B_2(i)\varphi(t)\\&\leqslant-\varsigma(i)\|s(t,i)\|\\&=-\varsigma(i)\sqrt{2V(i,s(t,i))}.\end{aligned}$$

Then, by applying Assumption 1.1 we obtain that the state trajectories of the dynamics (4.6) arrive in the same subsystem within a finite time. Furthermore, the state trajectories will reach the sliding path in a finite time and will remain within it. This completes the proof. ■

4.3.2 Observer-Based Sliding Mode Control

In this section, we will utilize a state observer to generate the estimate of unmeasured state components, and then synthesize a sliding mode control law based on the state estimates.

We design the following observer for systems (4.2):

$$\dot{\hat{x}}(t)=A(i)\hat{x}(t)+B(i)u(t)+L(i)\big[y(t)-C(i)\hat{x}(t)\big], \tag{4.11}$$

where $\hat{x}(t)\in\mathbb{R}^n$ represents the estimate of $x(t)$, and $L(i)\in\mathbb{R}^{n\times q}$, $i\in\mathcal{M}$, are the observer gains to be designed.

Let $\delta(t) \triangleq x(t) - \hat{x}(t)$ denote the estimation error. Consider (4.2) and (4.11), the estimate error dynamics can be obtained as

$$\dot{\delta}(t) = \Big(A(i) - L(i)C(i) + \Delta A(i,t)\Big)\delta(t) + \Delta A(i,t)\hat{x}(t) + B(i)\varphi(t), \quad (4.12)$$

where the matched uncertainty disturbance $\varphi(t)$ is unknown but bounded as $\|\varphi(t)\| \leqslant \rho\|\delta(t)\|$, where ρ is a known scalar. Define the following integral sliding mode surface function:

$$\tilde{s}(t,i) = G(i)\hat{x}(t) - \int_0^t G(i)\Big(A(i) + B(i)K(i)\Big)\hat{x}(v)\mathrm{d}v, \quad (4.13)$$

where $G(i)$ and $K(i)$, $i \in \mathcal{M}$ are coefficient matrices. In addition, $K(i)$ is chosen such that $A(i) + B(i)K(i)$ is Hurwitz, and $G(i)$ is designed such that $G(i)B(i)$ is non-singular.

It follows from (4.13) that

$$\begin{aligned}\dot{\tilde{s}}(t,i) &= G(i)\dot{\hat{x}}(t) - G(i)\Big(A(i) + B(i)K(i)\Big)\hat{x}(t)\\ &= G(i)B(i)u(t) + G(i)L(i)C(i)\delta(t) - G(i)B(i)K(i)\hat{x}(t). \quad (4.14)\end{aligned}$$

Let $\dot{\tilde{s}}(t,i) = 0$ for each $i \in \mathcal{M}$, we obtain the following equivalent control law:

$$u_{eq}(t) = K(i)\hat{x}(t) - \big(G(i)B(i)\big)^{-1}G(i)L(i)C(i)\delta(t). \quad (4.15)$$

Substituting (4.15) into (4.11), one can obtain the following sliding mode dynamics:

$$\begin{aligned}\dot{\hat{x}}(t) = &\Big(A(i) + B(i)K(i)\Big)\hat{x}(t)\\ &+ \big[I - B(i)\big(G(i)B(i)\big)^{-1}G(i)\big]L(i)C(i)\delta(t). \quad (4.16)\end{aligned}$$

In the following theorem, a sufficient condition for the stability analysis is given for the overall closed-loop system composed of the estimation error dynamics (4.12) and the sliding mode dynamics (4.16).

Theorem 4.5 *Consider the associated MJS (4.2). Its unmeasured states are estimated by the observer (4.11). The sliding surface functions in the state estimation space and in the state estimation error space are chosen as (4.13). If there exist matrices $X(i) > 0$ and $\mathcal{L}(i)$ such that for all $i \in \mathcal{M}$,*

$$\begin{bmatrix} \Omega_1(i) & \Omega_2(i) \\ * & \Omega_3(i) \end{bmatrix} < 0, \quad (4.17)$$

where

$$\begin{aligned}
\Omega_1(i) &\triangleq \begin{bmatrix} \Omega_{11}(i) & \mathcal{L}(i)C(i) & X(i)B(i) \\ * & \Omega_{22}(i) & 0 \\ * & * & -B^T(i)X(i)B(i) \end{bmatrix}, \\
\Omega_2(i) &\triangleq \begin{bmatrix} 0 & 0 & 0 \\ C^T(i)\mathcal{L}^T(i) & X(i)E(i) & X(i)B(i) \\ 0 & 0 & 0 \end{bmatrix}, \\
\Omega_3(i) &\triangleq \mathrm{diag}\{-X(i), -\varepsilon_1 I, -\varepsilon_2 I\}, \\
\Omega_{11}(i) &\triangleq X(i)\Big(A(i) + B(i)K(i)\Big) + \varepsilon_1 H^T(i)H(i) \\
&\quad + \Big(A(i) + B(i)K(i)\Big)^T X(i) + \sum_{j=1}^{N} X(j)\lambda_{ij}, \\
\Omega_{22}(i) &\triangleq X(i)A(i) + A^T(i)X(i) - \mathcal{L}(i)C(i) \\
&\quad - C^T(i)\mathcal{L}^T(i) + \varepsilon_2\rho^2 I + \sum_{j=1}^{N} X(j)\lambda_{ij},
\end{aligned}$$

then the overall closed-loop systems composed of (4.12) *and* (4.16) *are mean-square exponentially stable. Moreover, the observer gain is given by*

$$L(i) = X^{-1}(i)\mathcal{L}(i), \quad i \in \mathcal{M}. \tag{4.18}$$

Proof Select the following Lyapunov function:

$$\tilde{V}(\hat{x}, \delta, i) \triangleq \hat{x}^T(t)X(i)\hat{x}(t) + \delta^T(t)X(i)\delta(t). \tag{4.19}$$

Along the solution of systems (4.12) and (4.16), we have

$$\begin{aligned}
\mathcal{L}(\tilde{V}(\hat{x}, \delta, i)) = {}& 2\hat{x}^T(t)X(i)\Big[\Big(A(i) + B(i)K(i)\Big)\hat{x}(t) \\
&+ \Big(I - B(i)\big(G(i)B(i)\big)^{-1}G(i)\Big)L(i)C(i)\delta(t)\Big] \\
&+ 2\delta^T(t)X(i)\Big[\Big(A(i) - L(i)C(i) \\
&+ \Delta A(i,t)\Big)\delta(t) + \Delta A(i,t)\hat{x}(t) + B(i)\varphi(t)\Big] \\
&+ \hat{x}^T(t)\left(\sum_{j=1}^{N} X(j)\lambda_{ij}\right)\hat{x}(t) + \delta^T(t)\left(\sum_{j=1}^{N} X(j)\lambda_{ij}\right)\delta(t).
\end{aligned} \tag{4.20}$$

From $G(i) = B^T(i)X(i)$, for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we obtain

$$\begin{aligned}
&-2\hat{x}^T(t)X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}G(i)L(i)C(i)\delta(t)\\
&\leqslant \hat{x}^T(t)X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}B^T(i)X(i)\hat{x}(t)\\
&\quad+\delta^T(t)C^T(i)L^T(i)X(i)L(i)C(i)\delta(t),
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
-2\delta^T(t)X(i)\varDelta A(i,t)\hat{x}(t) \leqslant\ &\varepsilon_1^{-1}\delta^T(t)X(i)E(i)E^T(i)X(i)\delta(t)\\
&+\varepsilon_1\hat{x}^T(t)H^T(i)H(i)\hat{x}(t),
\end{aligned} \tag{4.22}$$

and

$$-2\delta^T(t)X(i)B(i)\varphi(t) \leqslant \varepsilon_2^{-1}\delta^T(t)X(i)B(i)B^T(i)X(i)\delta(t) + \varepsilon_2\rho^2\delta^T(t)\delta(t). \tag{4.23}$$

Substituting (4.21)–(4.23) into (4.20), we have

$$\mathcal{L}(\tilde{V}(\hat{x},\delta,i)) \leqslant \xi^T(t)\varTheta(i)\xi(t), \tag{4.24}$$

where

$$\begin{aligned}
\varTheta(i) &\triangleq \begin{bmatrix} \varTheta_{11}(i) & \mathcal{L}(i)C(i) \\ * & \varTheta_{22}(i) \end{bmatrix}, \quad \xi(t) \triangleq \begin{bmatrix} \hat{x}^T(t) & \delta^T(t) \end{bmatrix}^T,\\
\varTheta_{11}(i) &\triangleq X(i)\Big(A(i)+B(i)K(i)\Big) + \sum_{j=1}^{N} X(j)\lambda_{ij}\\
&\quad + X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}B^T(i)X(i)\\
&\quad + \Big(A(i)+B(i)K(i)\Big)^T X(i) + \varepsilon_1 H^T(i)H(i),\\
\varTheta_{22}(i) &\triangleq X(i)A(i) + A^T(i)X(i) - \mathcal{L}(i)C(i)\\
&\quad + C^T(i)L^T(i)X(i)L(i)C(i) - C^T(i)\mathcal{L}^T(i)\\
&\quad + \varepsilon_1^{-1}X(i)E(i)E^T(i)X(i) + \varepsilon_2\rho^2 I\\
&\quad + \varepsilon_2^{-1}X(i)B(i)B^T(i)X(i) + \sum_{j=1}^{N} X(j)\lambda_{ij}.
\end{aligned}$$

Then by Schur complement, it follows that (4.17) implies $\varTheta(i) < 0$. Thus,

$$\mathcal{L}(\tilde{V}(\hat{x},\delta,i)) < 0, \quad i \in \mathcal{M}.$$

Employing the same techniques as in Theorem 4.3 and [3], we know that the overall system composed of the estimation error dynamics (4.12) and the sliding mode dynamics in the state estimation space (4.16) is mean-square exponentially stable. This completes the proof. ■

In the following, we synthesize a SMC law, by which the sliding motion can be driven onto the pre-specified sliding surface $\tilde{s}(t,i)=0$ in a finite time and then are maintained there for all subsequent time.

Theorem 4.6 *The trajectories of systems* (4.11) *can be driven onto the sliding surface* $\tilde{s}(t,i)=0$ *in a finite time by the following observer-based SMC*

$$u(t) = -\varsigma\tilde{s}(t,i) + K(i)\hat{x}(t) - \chi(i,t)\mathrm{sgn}(\tilde{s}(t,i)), \tag{4.25}$$

where $\varsigma > 0$ *is a small constant, and* $\chi(i,t)$ *is given by*

$$\begin{aligned}\chi(i,t) = \max_{i\in\mathcal{M}} \Big\{ &\|B^T(i)X(i)L(i)\|\ \|y(t)\| \\ &+ \|B^T(i)X(i)C(i)\|\ \|\hat{x}(t)\|\Big\}.\end{aligned} \tag{4.26}$$

Proof Select the following Lyapunov function:

$$\mathcal{V}(\tilde{s}(t,i),i) = \frac{1}{2}\tilde{s}^T(t,i)\left[B^T(i)X(i)B(i)\right]^{-1}\tilde{s}(t,i).$$

From $\|\tilde{s}(t,i)\| \leqslant |\tilde{s}(t,i)|$ and $\tilde{s}^T(t,i)\mathrm{sgn}(\tilde{s}(t,i)) \leqslant |\tilde{s}(t,i)|$, for any $i \in \mathcal{M}$, we obtain

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{s}(t,i),i) &= \tilde{s}^T(t,i)\left(B^T(i)X(i)B(i)\right)^{-1}\dot{\tilde{s}}(t,i) \\ &= \tilde{s}^T(t,i)\left(B^T(i)X(i)B(i)\right)^{-1}B^T(i)X(i)\Big[B(i)u(t) \\ &\quad + L(i)(y(t) - C(i)\hat{x}(t)) - B(i)K(i)\hat{x}(t)\Big] \\ &= \tilde{s}^T(t,i)(B^T(i)X(i)B(i))^{-1}B^T(i)X(i)\Big[-\varsigma B(i)\tilde{s}(t,i) \\ &\quad - B(i)\chi(i,t)\mathrm{sgn}(\tilde{s}(t,i)) + L(i)(y(t) - C(i)\hat{x}(t))\Big] \\ &\leqslant -\varsigma\|\tilde{s}(t,i)\| - \|\tilde{s}(t,i)\|\ \|(B^T(i)X(i)B(i))^{-1}\| \\ &\quad \times \|B^T(i)X(i)\|\ \|B(i)\chi(i,t)\mathrm{sgn}(\tilde{s}(t,i))\| \\ &\quad + \|\tilde{s}(t,i)\|\ \|(B^T(i)X(i)B(i))^{-1}\| \\ &\quad \times \|B^T(i)X(i)\|\ \|L(i)(y(t) - C(i)\hat{x}(t))\| \\ &\leqslant -\varsigma\|\tilde{s}(t,i)\| \leqslant -\vartheta_i\mathcal{V}^{\frac{1}{2}}(t),\end{aligned}$$

where $\vartheta_i \triangleq \varsigma\sqrt{\frac{2}{\lambda_{\min}(B^T(i)X(i)B(i))}} > 0, i \in \mathcal{M}$.

Therefore, by applying Assumption 1.1 we can conclude that the state trajectories of the observer dynamics (4.11) can be driven onto the sliding surface $\tilde{s}(t, i) = 0$ by the observer-based SMC (4.25) in a finite time. This completes the proof. ■

Remark 4.7 As we know, the major drawback of SMC is that it is discontinuous across sliding surfaces. The discontinuity leads to control chattering in practice, and involves high frequency dynamics. How to reduce chattering will be a research topic in future studies. ♦

4.4 Illustrative Example

In this section, we present two examples to show the effectiveness of the control schemes proposed in this chapter.

Example 4.8 (SMC problem) Consider the S-MJS in (4.1) with two operating modes and the following parameters:

$$
\begin{aligned}
&A(1) = \begin{bmatrix} -2.4 & 0 \\ 0 & -1.9 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -1.2 & 0 \\ -1 & -1.2 \end{bmatrix}, \quad \epsilon_1 = 0.3,\\
&A(2) = \begin{bmatrix} -0.7 & 0.4 \\ 0 & -1.1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} -1.3 & 0 \\ -0.2 & -1.2 \end{bmatrix}, \quad \epsilon_2 = 0.1,\\
&E(1) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad H(1) = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},\\
&H(2) = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \quad \Delta F(1, t) = \Delta F(2, t) = 0.1\sin(t),\\
&\varphi(t) = 0.5\exp(-t)\sqrt{x_1^2 + x_2^2}.
\end{aligned}
$$

Let $\bar{\eta}_t$ be a PH semi-Markov process taking values in $\{1, 2\}$. The sojourn time in the first state is a random variable distributed according to a negative exponential distribution with parameter λ_1. The sojourn time in the second state is a random variable distributed according to a two-order Erlang distribution. From Sect. 4.2 of this chapter, we know that if we want to investigate a PH S-MJS, we can investigate its associated MJS. The key is to look for the associated Markov chain and its infinitesimal generator, and define the proper function.

In fact, the sojourn time in the second part can be divided into two parts. The sojourn time in the first (respectively second) subdivision is a random variable that is negative exponentially distributed with parameter λ_2, (respectively λ_3). More specifically, if the process $\bar{\eta}_t$ enters state 2, it must stay at the first subdivision for some time, then enter the second subdivision, and finally returns to state 1 again. We know that $p_{12} = p_{21} = 1$. Obviously,

$$\begin{aligned}
\mathbf{a}^{(1)} &= (a_1^{(1)}) = 1, \quad \mathbf{a}^{(2)} = (a_1^{(2)}, a_2^{(2)}) = (1, 0),\\
T^{(1)} &= (T_{11}^{(1)}) = (-\lambda_1),\\
T^{(2)} &= \begin{bmatrix} T_{11}^{(2)} & T_{12}^{(2)} \\ T_{21}^{(2)} & T_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{bmatrix}.
\end{aligned}$$

It is easy to see the state space of $Z(t) = (\bar{\eta}_t, J(t))$ is $G = \big((1,1), (2,1), (2,2)\big)$. We number the elements of G as $\varphi\big((1,1)\big) = 1$, $\varphi\big((2,1)\big) = 2$, and $\varphi\big((2,2)\big) = 3$. Hence, the infinitesimal generator of $\varphi(Z(t))$ is

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ -\lambda_3 & 0 & \lambda_3 \end{bmatrix}.$$

Now, let $\eta_t = \varphi(Z(t))$. It is obvious that η_t is the associated Markov chain of $\bar{\eta}_t$ with state space $\{1, 2, 3\}$. The infinitesimal generator of η_t is given by Q.

Therefore, our aim is to design a SMC such that the closed-loop system with associated Markov chain η_t is stable. To check the stability of (4.2), we solve (4.7) in Theorem 4.3, and obtain

$$\mathbb{C}(1) = -0.6480, \quad \mathbb{C}(2) = 1.1734.$$

Thus, the sliding surface function in (4.5) can be computed as

$$s(t,i) = \begin{cases} s(t,1) = \begin{bmatrix} 0.6480 & 1 \end{bmatrix} z(t), & i = 1, \\ s(t,2) = \begin{bmatrix} -1.1734 & 1 \end{bmatrix} z(t), & i = 2. \end{cases}$$

Now, we will design the SMC of (4.9) in Theorem 4.4. By computation, we have

$$\begin{aligned}
&\mathcal{M}_1 = \begin{bmatrix} -0.6480 & 1 \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} 1.1734 & 1 \end{bmatrix},\\
&\varrho(1) = 0.6548, \quad \varrho(2) = 1.2538,
\end{aligned}$$

and set $\varsigma(1) = \varsigma(2) = 0.15$. Thus, the SMC in (4.26) can be computed as

$$u(t) = \begin{cases} \begin{bmatrix} -0.6480 & 1 \end{bmatrix} z(t) + 0.704\mathrm{sgn}(s(t,1)), & i = 1, \\ \begin{bmatrix} 1.1734 & 1 \end{bmatrix} z(t) + 1.513\mathrm{sgn}(s(t,2)), & i = 2. \end{cases} \tag{4.27}$$

To prevent the control signals from chattering, we replace $\mathrm{sgn}(s(t,i))$ with $\frac{s(t,i)}{0.1+\|s(t,i)\|}$, $i = \{1, 2\}$. For a given initial condition of $x(0) = [-2 \; -1]^T$, the simulation results are given in Figs. 4.1, 4.2, 4.3 and 4.4. A switching signal is displayed in Fig. 4.1; here, '1' and '2' correspond to the first and second modes, respectively. Figure 4.2 shows the state response of the closed-loop system with control input (4.27). The SMC input and sliding function are shown in Figs. 4.3 and 4.4, respectively.

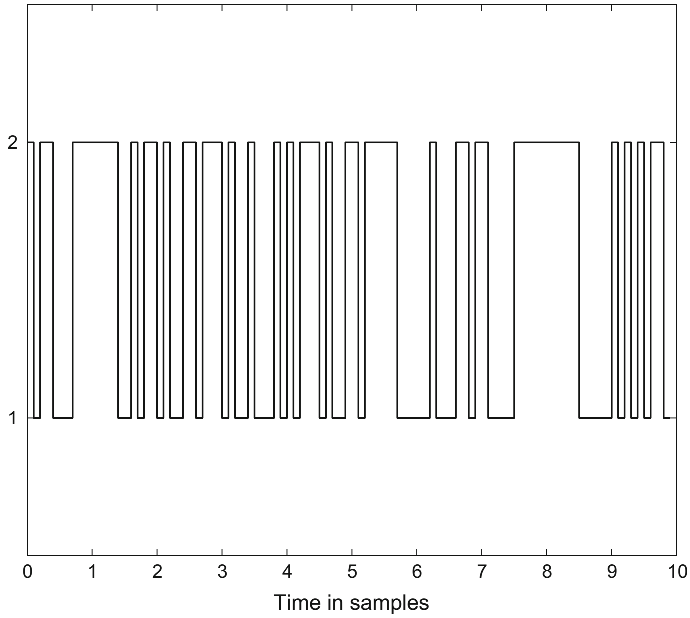

Fig. 4.1 Switching signal with two modes

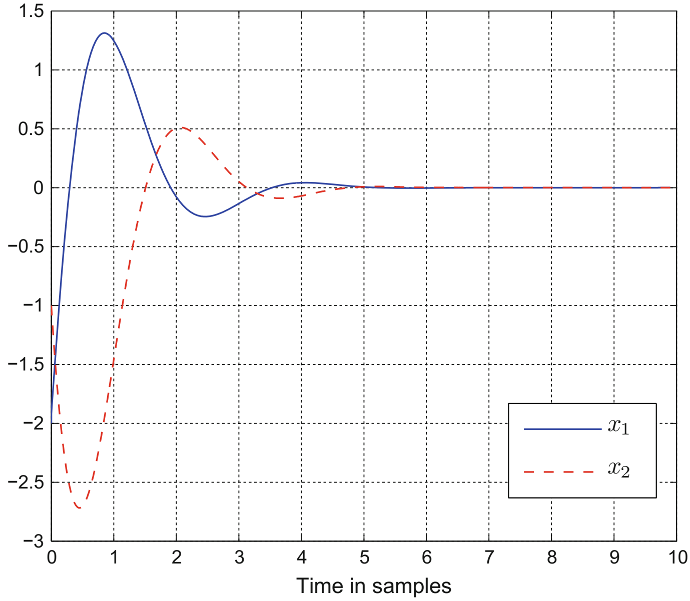

Fig. 4.2 State response of the closed-loop system

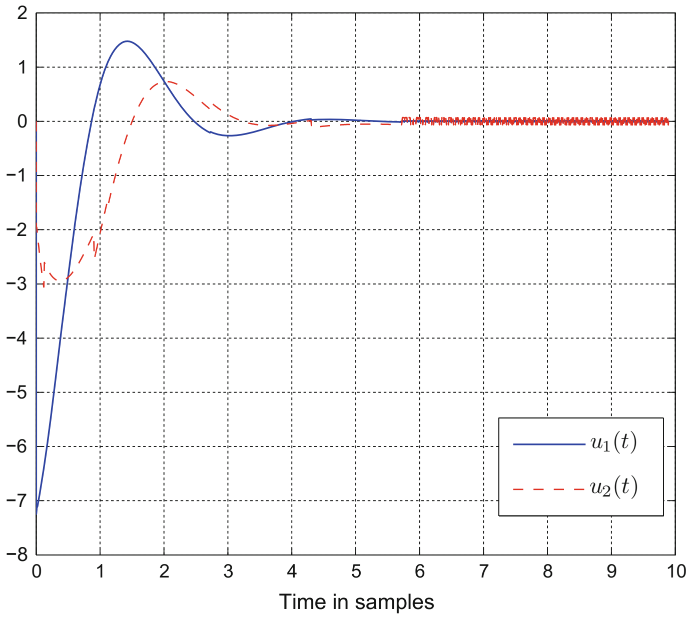

Fig. 4.3 Control input $u(t)$

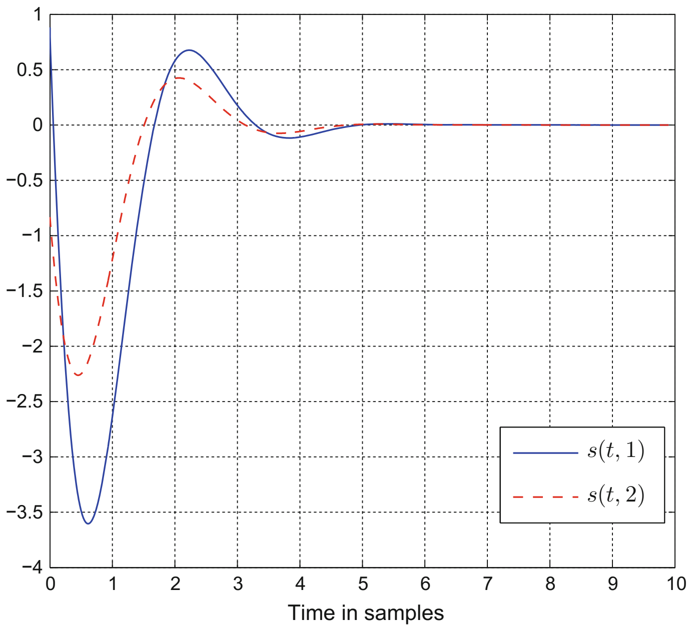

Fig. 4.4 Sliding surface function $s(t, i)$

Example 4.9 (Observer-based SMC problem) Consider the S-MJS in (4.1) with two operating modes and the following parameters:

$$
\begin{aligned}
A(1) &= \begin{bmatrix} -1.5 & -1.1 \\ 1.0 & -1.2 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.2 \\ 1.1 \end{bmatrix}, \quad E(1) = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \\
A(2) &= \begin{bmatrix} -1.5 & 0.1 \\ 2.0 & -1.5 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.2 \\ 1.0 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \\
C(1) &= \begin{bmatrix} -2.5 & -2.0 \\ -2.0 & -1.1 \end{bmatrix}, \quad C(2) = \begin{bmatrix} -2.5 & -2.0 \\ -2.0 & -1.2 \end{bmatrix}, \quad \rho = 0.1, \\
H(1) &= \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \quad \Delta F(1,t) = \Delta F(2,t) = 0.2\sin(t), \\
H(2) &= \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad \varphi(t) = \exp(-t)\sqrt{x_1^2 + x_2^2},
\end{aligned}
$$

and $\bar{\eta}_t$ is chosen as in Example 1. In this example, we consider the sliding mode observer design when some system states are not available. According to Sect. 4.4, we first design a sliding mode observer in the form of (4.11) to estimate the system states, and synthesize an observer-based SMC as in (4.25). We select matrices $K(1)$ and $K(2)$ as follows:

$$
K(1) = K(2) = \begin{bmatrix} -0.4 & -0.6 \end{bmatrix}.
$$

Solving the conditions (4.17) and (4.18) in Theorem 4.5, we have

$$
L(1) = \begin{bmatrix} -1.230 & 0.472 \\ -0.368 & 0.626 \end{bmatrix}, \quad L(2) = \begin{bmatrix} 0.233 & -0.324 \\ -0.122 & -0.061 \end{bmatrix}.
$$

According to (4.13)–(4.14), we have

$$
\tilde{s}(t,i) = \begin{cases} \tilde{s}(t,1) = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix} \hat{x}(t) + \eta_1(t), & i = 1, \\ \tilde{s}(t,2) = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix} \hat{x}(t) + \eta_2(t), & i = 2, \end{cases}
$$

with

$$
\dot{\eta}_1(t) = \begin{bmatrix} -0.07 & -0.46 \end{bmatrix} \hat{x}(t), \quad \dot{\eta}_2(t) = \begin{bmatrix} 0.28 & -0.34 \end{bmatrix} \hat{x}(t).
$$

The state observer-based SMC is designed in (4.25) with $\varsigma = 0.1$, then the sliding mode controller designed in (4.25) can be obtained as

$$
u(t) = \begin{cases} u_1(t) = -0.2\tilde{s}(t,1) + \begin{bmatrix} -0.2 & -0.27 \end{bmatrix} \hat{x}(t) \\ \qquad\qquad -\chi(i,t)\mathrm{sgn}(\tilde{s}(t,1)), i = 1, \\ u_2(t) = -0.3\tilde{s}(t,2) + \begin{bmatrix} 0.3 & -0.36 \end{bmatrix} \hat{x}(t) \\ \qquad\qquad -\chi(i,t)\mathrm{sgn}(\tilde{s}(t,2)), i = 2, \end{cases} \tag{4.28}
$$

with $\chi(i,t) = 0.0136\|y(t)\| + 0.1132\|\hat{x}(t)\|$.

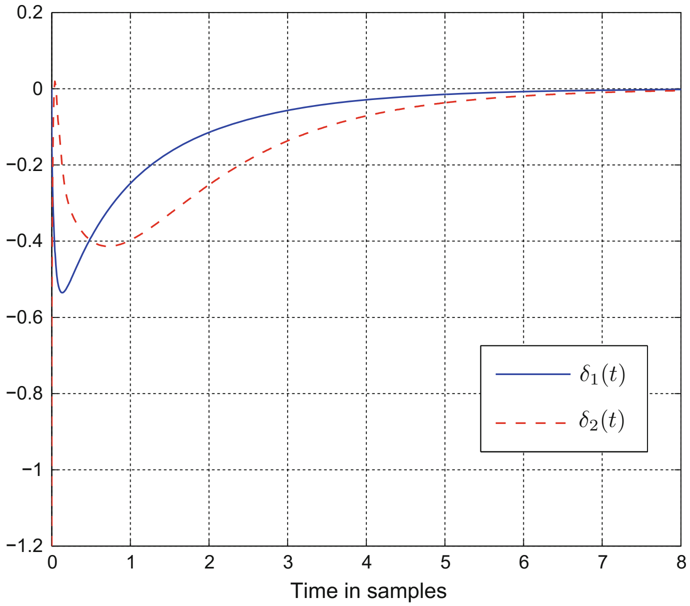

Fig. 4.5 State estimation error $\delta(t)$

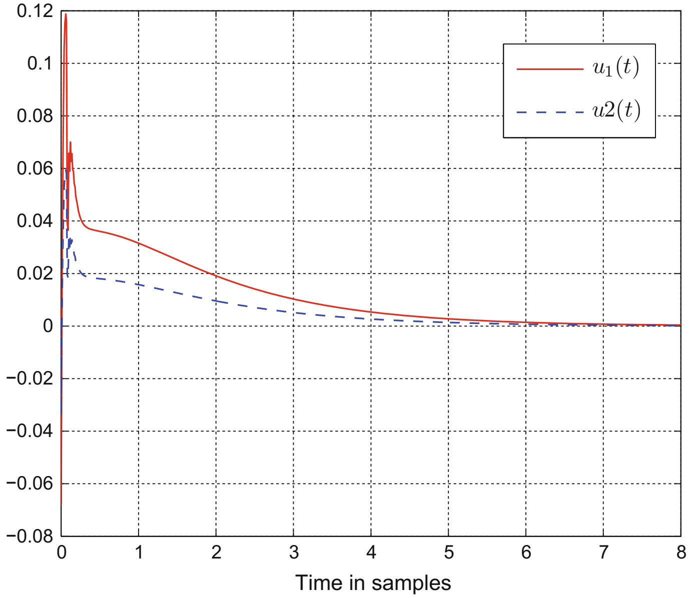

Fig. 4.6 Control input $u(t)$

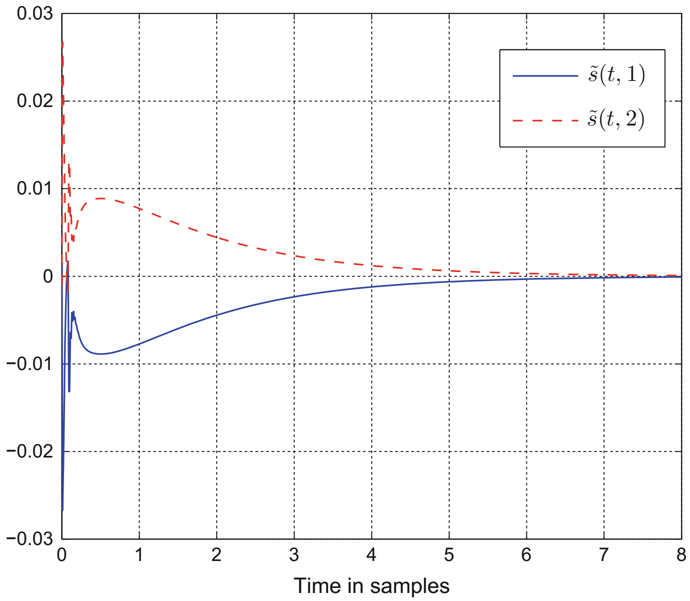

Fig. 4.7 Sliding surface function $\tilde{s}(t, i)$

To avoid the control signals from chattering, we replace $\text{sgn}(\tilde{s}(t,i))$ with $\frac{\tilde{s}(t,i)}{0.1+\|\tilde{s}(t,i)\|}$, $i = \{1, 2\}$. For a given initial condition of $x(0) = [-1.2 - 0.2]^T$, and with $\hat{x}(0) = 0$, the results from the simulation are given in Figs. 4.5, 4.6 and 4.7. Figure 4.5 shows the state response of the error system with control input (4.28). The control variables and the sliding surface functions are shown in Figs. 4.6 and 4.7, respectively.

4.5 Conclusion

In this chapter, the state estimation and sliding mode control problems have been addressed for S-MJS with mismatched uncertainties. Sufficient conditions for the existence of sliding mode dynamics have been established, and an explicit parametrization for the desired sliding surface has also been given. Then, the sliding mode controller for reaching motion has been synthesized. Moreover, an observer-based sliding mode controller has been synthesized to guarantee the reachability of the system's trajectories to the predefined integral-type sliding surface. Finally, two numerical examples have been provided to illustrate the effectiveness of the proposed design schemes.

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Chapter 5
Quantized Output Feedback Control of Nonlinear Semi-Markovian Jump Systems

Abstract This chapter considers the quantized output feedback control problem for a class of S-MJS with repeated scalar nonlinearities. A sufficient condition for the S-MJS is developed. This condition guarantees that the corresponding closed-loop systems are stochastically stable and have a prescribed H_∞ performance. The existence conditions for full- and reduced-order dynamic output feedback controllers are proposed, and the cone complementarity linearization procedure is employed to cast the controller design problem into a sequential minimization one, which can be solved efficiently with existing optimization techniques. Finally, an application to cognitive radio systems demonstrates the efficiency of the new design method developed.

5.1 Introduction

During the past decades, with the growing interest in networked control systems, the problem of stabilizing plants over saturating quantized measurements has attracted increasing attention from the research community. In the network environment, the system outputs are always required to be quantized before transmission. In other words, real-valued signals are mapped into piecewise-constant signals taking values in finite sets, which are employed when the observation and control signals are sent via constrained communication channels. More importantly, new quantization techniques are needed for the sensor measurements and control commands that are sent over networks. Some efforts have been made toward this line [1, 2]. In fact, modern control theory has been the focus of significant research subjects in networks, remote control technology, and communication.

In this chapter, we will consider the quantized H_∞ dynamic output feedback controller (DOFC) design problem for S-MJS with repeated scalar nonlinearities. The main contributions of this paper can be summarized as follows: (1) Introducing a novel nonlinear model with a semi-Markov process, which is described by a discrete-time state equation involving a repeated scalar nonlinearity that typically appears in recurrent neural networks and hybrid systems with finite discrete operation modes; (2) Based on the mode-dependent positive definite diagonally dominant Lyapunov

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function approach, the designed dynamic output feedback controller can guarantee the corresponding closed-loop systems are stochastically stable and have a prescribed H_∞ performance; (3) Establishing a sufficient condition for existence of admissible controllers in terms of matrix equalities, and a cone complementarity linearization (CCL) procedure is employed to transform a nonconvex feasibility problem into a sequential minimization problem, which can be readily solved by existing optimization techniques; and (4) Designing full and reduced-order DOFCs to handle the case of unmeasured states.

5.2 Problem Formulation and Preliminaries

Consider the following nonlinear S-MJS:

$$\begin{cases} x(k+1) = \bar{A}(\bar{\gamma}_k)g\left(x(k)\right) + \bar{B}(\bar{\gamma}_k)u(k) + \bar{F}(\bar{\gamma}_k)\omega(k), \\ \quad y(k) = \bar{C}(\bar{\gamma}_k)g\left(x(k)\right) + \bar{D}(\bar{\gamma}_k)\omega(k), \\ \quad z(k) = \bar{E}(\bar{\gamma}_k)g\left(x(k)\right), \end{cases} \tag{5.1}$$

where $\{\bar{\gamma}_k, k \in \mathbb{Z}^+\}$ is a discrete-time phase-type (PH) semi-Markov process on the probability space which has been defined in [3], and $x(k) \in \mathbb{R}^n$ represents the state vector; $y(k) \in \mathbb{R}^p$ is the measured output; $z(k) \in \mathbb{R}^q$ is the controlled output; $u(k) \in \mathbb{R}^m$ is the control input; $\omega(k) \in \mathbb{R}^l$ is exogenous disturbance input which belongs to $\ell_2[0, \infty)$; $\bar{A}(\bar{\gamma}_k)$, $\bar{B}(\bar{\gamma}_k)$, $\bar{C}(\bar{\gamma}_k)$, $\bar{D}(\bar{\gamma}_k)$, $\bar{E}(\bar{\gamma}_k)$ and $\bar{F}(\bar{\gamma}_k)$ are matrix functions of PH semi-Markov process $\{\bar{\gamma}_k, k \in \mathbb{Z}^+\}$.

Employing the same techniques as those used in Lemma 1.24 and [4]. We can construct the an associated Markov process γ_k of $\bar{\gamma}_k$ with the state space $\mathcal{M} = \{1, 2, \ldots, N\}$ and the infinitesimal generator $\Lambda = (\lambda_{ij})$, $1 \leqslant i, j \leqslant M$, such that

$$\begin{aligned} \lambda_{ij} &= \Pr\left(r_{k+h} = j \,|\, \gamma_k = i\right) \\ &= \Pr\left(\psi(Z(k)+h) = j \,|\, \psi(Z(k)) = i\right), \end{aligned}$$

where λ_{ij} is the transition rate from mode i at time k to mode j at time $k+h$ when $i \neq j$ and $\sum_{j=1}^{N} \lambda_{ij} = 1$ for every $i \in \mathcal{M}$.

Then, we can construct the associated MJS which is equivalent to (5.1), as follows:

$$\begin{cases} x(k+1) = A_i g\left(x(k)\right) + B_i u(k) + F_i \omega(k), \\ \quad y(k) = C_i g\left(x(k)\right) + D_i \omega(k), \\ \quad z(k) = E_i g\left(x(k)\right), \end{cases}$$

where matrices $A_i \triangleq A(\gamma_k = i)$, $B_i \triangleq B(\gamma_k = i)$, $C_i \triangleq C(\gamma_k = i)$, $D_i \triangleq D(\gamma_k = i)$, $E_i \triangleq E(\gamma_k = i)$ and $F_i \triangleq F(\gamma_k = i)$ are known real constant matrices of appropriate dimensions. The function $g(\cdot)$ is a nonlinear function satisfying the following assumption as in [5].

Assumption 5.1 The nonlinear function $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

$$\forall x, y \in \mathbb{R}, \quad |g(x) + g(y)| \leqslant |x + y| \,. \tag{5.2}$$

In the sequel, for the vector

$$x(k) = \left[x_1(k) \; x_2(k) \; \ldots \; x_n(k) \right]^T ,$$

we denote

$$g(x) \triangleq \left[g\left(x_1(k)\right) \; g\left(x_2(k)\right) \; \ldots \; g\left(x_n(k)\right) \right]^T .$$

Before entering the controller, the signal $y(k)$ is quantized by the quantizer $q_i(\cdot) \triangleq q_{\gamma_k}(\cdot)$ described by

$$q_i(\cdot) \triangleq \left[q_i^{(1)}(\cdot) \; q_i^{(2)}(\cdot) \; \ldots \; q_i^{(p)}(\cdot) \right]^T , \quad i \in \mathcal{M}, \tag{5.3}$$

where $q_i^{(j)}(\cdot)$ is assumed to be symmetric, that is,

$$q_i^{(j)}(y_j(k)) = -q_i^{(j)}(-y_j(k)), \; j = 1, \ldots, p.$$

For any $i \in \mathcal{M}$, the sets of quantized levels are described by

$$\begin{aligned} \Upsilon_j \triangleq & \left\{ \pm\eta_l^{(i,j)} | \, \eta_l^{(i,j)} = (\rho^{(i,j)})^l \cdot \eta_{(0)}^{(i,j)}, \; l = \pm 1, \pm 2, \ldots \right\} \\ & \cup \left\{ \pm\eta_{(0)}^{(i,j)} \right\} \cup \{0\}, \quad 0 < \rho^{(i,j)} < 1, \; \eta_{(0)}^{(i,j)} > 0, \end{aligned}$$

with $\rho^{(i,j)}$ represents the ith quantizer density of the sub-quantizer $q_i^{(j)}(\cdot)$, and $\eta_l^{(i,j)}$ denotes the initial values for sub-quantizer $q_i(\cdot)$. The associated quantizer $q_i^{(j)}(\cdot)$ is defined as follows:

$$q_i^{(j)}(y_j(k)) \triangleq \begin{cases} \eta_l^{(i,j)}, & \text{if } \frac{\eta_l^{(i,j)}}{1+\delta^{(i,j)}} < y_j(k) \leqslant \frac{\eta_l^{(i,j)}}{1-\delta^{(i,j)}}, \\ 0, & \text{if } y_j(k) = 0, \\ -q_i^{(j)}(-y_j(k)), & \text{if } y_j(k) < 0, \end{cases} \tag{5.4}$$

where $\delta^{(i,j)} = (1 - \rho^{(i,j)})/(1 + \rho^{(i,j)})$.

Define Δ_i for all $i \in \mathcal{M}$ as $\Delta_i \triangleq \text{diag}\{\delta^{(i,1)}, \ldots, \delta^{(i,p)}\}$. It is obvious that $0 < \Delta_i < I_p$. From (5.4), it is not difficult to verify that the logarithmic quantizer can be characterized as follows

$$(1 - \delta^{(i,j)})y_j^2(k) \leqslant q_j^{(i,j)}(y_j(k)) \cdot y_j(k) \leqslant (1 + \delta^{(i,j)})y_j^2(k). \tag{5.5}$$

Note that (5.5) is equivalent to

$$[q_i(y(k)) - (I_p - \Delta_i)y(k)]^T[q_i(y(k)) - (I_p + \Delta_i)y(k)] \leqslant 0.$$

Thus, $q_i(\cdot)$ can be decomposed as follows:

$$q_i(y(k)) = (I_p - \Delta_i) \cdot y(k) + q_i^s(y(k)),$$

where $q_i^s(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ satisfies $q_i^s(\cdot) = 0$, and

$$(q_i^s(y(k)))^T \cdot [q_i^s(y(k)) - 2 \cdot \Delta_i \cdot y(k)] \leqslant 0. \tag{5.6}$$

For convenience, let

$$\mathcal{K}_{1i} \triangleq I_p - \Delta_i, \quad \mathcal{K}_{2i} \triangleq I_p + \Delta_i, \quad \mathcal{K}_i \triangleq \mathcal{K}_{2i} - \mathcal{K}_{1i} = 2\Delta_i.$$

So, (5.6) can be written as

$$(q_i^s(y(k)))^T \cdot [q_i^s(y(k)) - \mathcal{K}_i \cdot y(k)] \leqslant 0. \tag{5.7}$$

In the following, it is assumed that the output data are quantized before being transmitted to another node in the network. Thus, we get the following associated MJS.

$$\begin{cases} x(k+1) = A_i g(x(k)) + B_i u(k) + F_i \omega(k), \\ y_q(k) = q_i\Big[C_i g(x(k)) + D_i \omega(k)\Big], \\ z(k) = E_i g(x(k)). \end{cases} \tag{5.8}$$

For system (5.8), we are interested in designing a nonlinear DOFC of the following form:

$$\begin{cases} \hat{x}(k+1) = \hat{A}_i g\left(\hat{x}(k)\right) + \hat{B}_i y_q(k), \\ u(k) = \hat{C}_i g\left(\hat{x}(k)\right) + \hat{D}_i y_q(k), \end{cases} \tag{5.9}$$

where $\hat{x}(k) \in \mathbb{R}^s$ is the state vector of the DOFC. The matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ are the controller parameters to be designed.

Augmenting system (5.8) to include the states of system (5.9), the closed-loop system is governed by

$$\begin{cases} \xi(k+1) = \tilde{A}_i g(\xi(k)) + \tilde{B}_i \omega(k) + \tilde{D}_i q_i^{(s)}(y(k)), \\ z(k) = \tilde{C}_i g(\xi(k)), \end{cases} \tag{5.10}$$

where $\xi(k) \triangleq \begin{bmatrix} x^T(k) & \hat{x}^T(k) \end{bmatrix}^T$ and

$$\begin{aligned} \tilde{A}_i &\triangleq \begin{bmatrix} A_i + B_i \hat{D}_i K_{1i} C_i & B_i \hat{C}_i \\ \hat{B}_i K_{1i} C_i & \hat{A}_i \end{bmatrix}, \quad \tilde{C}_i \triangleq \begin{bmatrix} E_i & 0 \end{bmatrix}, \\ \tilde{B}_i &\triangleq \begin{bmatrix} F_i + B_i \hat{D}_i K_{1i} D_i \\ \hat{B}_i K_{1i} D_i \end{bmatrix}, \quad \tilde{D}_i \triangleq \begin{bmatrix} B_i \hat{D}_i \\ \hat{B}_i \end{bmatrix}. \end{aligned} \tag{5.11}$$

Definition 5.1 [6] The closed-loop system (5.10) with $\omega(k) = 0$ is said to be stochastically stable if the following condition hold for any initial condition $\xi_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{M}$.

$$\lim_{T\to\infty} \mathbf{E} \left\{ \sum_{k=0}^{T} \xi^T(k)\xi(k) \big| (\xi_0, r_0) \right\} \leqslant M(\xi_0, r_0).$$

Definition 5.2 [7] For a given scalar $\gamma > 0$, system (5.10) is said to be stochastically stable with an H_∞ performance γ, if it is stochastically stable with $\omega(t) = 0$, and under zero initial condition, the following condition holds for all nonzero $\omega(t) \in \ell_2[0, \infty)$,

$$\mathbf{E} \left\{ \sum_{k=0}^{\infty} z^T(k) z(k) \right\} < \gamma^2 \sum_{k=0}^{\infty} \omega^T(k)\omega(k).$$

Definition 5.3 [5] A square matrix $P \triangleq \begin{bmatrix} p_{ij} \end{bmatrix} \in \mathbb{R}^{n\times n}$ is said to be positive diagonally dominant if $P > 0$ (positive definite) and (row) diagonally dominant, i.e.,

$$\forall i, \quad |p_{ii}| \geqslant \sum_{j\neq i} |p_{ij}|.$$

Lemma 5.4 [5] *Suppose a matrix $P \geqslant 0$ is diagonally dominant, then for all nonlinear functions $g(\cdot)$ satisfying (5.2), it holds that*

$$\forall x \in \mathbb{R}^n, \quad g^T(x) P g(x) \leqslant x^T P x.$$

Lemma 5.5 [5] *A matrix P is positive diagonally dominant if and only if $P > 0$ and there exists a symmetric matrix R such that*

$$\begin{aligned} \forall i \neq j, \ r_{ij} \geqslant 0, \ p_{ij} + r_{ij} \geqslant 0 \\ \forall i, \ p_{ii} \geqslant \sum_{j\neq i} \left(p_{ij} + 2r_{ij} \right), \end{aligned}$$

which involves only $n(n-1)/2$ variables r_{ij} in addition to p_{ij} and n^2 inequalities in addition to $P > 0$.

In this paper, the quantized H_∞ DOFC design problem to be solved can be expressed as follows.

Quantized H_∞ DOFC design problem: Given a PH S-MJS (5.8) with repeated scalar nonlinearities, develop a mode-dependent quantized DOFC (5.9) such that for all admissible $\omega(k) \in \ell_2[0,\infty)$, the closed-loop system (5.10) is stochastically stable with an H_∞ disturbance attenuation level γ.

5.3 Main Results

We first investigate the stochastic stability with an H_∞ disturbance attenuation level γ of the closed-loop system (5.10).

Theorem 5.6 *Consider the associated nonlinear MJS (5.8), and suppose that the controller gains $(\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$ of system (5.9) are given. Then, the closed-loop system (5.10) is stochastically stable with an H_∞ disturbance attenuation level γ, if there exist a set of positive diagonally dominant matrices P_i such that*

$$\begin{bmatrix} -\tilde{\mathscr{P}}_i^{-1} & 0 & \tilde{\mathscr{A}}_i & \tilde{\mathscr{D}}_i & \tilde{\mathscr{B}}_i \\ * & -I & \tilde{C}_i & 0 & 0 \\ * & * & -P_i & \Phi_i^T \mathcal{K}_i & 0 \\ * & * & * & -2I & \mathcal{K}_i D_i \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall\, i \in \mathcal{M}, \tag{5.12}$$

where

$$\tilde{\mathscr{P}}_i \triangleq \mathrm{diag}\{\lambda_{i1}P_1, \lambda_{i2}P_2, \ldots, \lambda_{iN}P_N\}, \quad \Phi_i \triangleq [C_i \quad 0],$$
$$\tilde{\mathscr{A}}_i \triangleq \begin{bmatrix} \tilde{A}_i^T & \tilde{A}_i^T & \ldots & \tilde{A}_i^T \end{bmatrix}^T, \quad \tilde{\mathscr{B}}_i \triangleq \begin{bmatrix} \tilde{B}_i^T & \tilde{B}_i^T & \ldots & \tilde{B}_i^T \end{bmatrix}^T,$$
$$\tilde{\mathscr{D}}_i \triangleq \begin{bmatrix} \tilde{D}_i^T & \tilde{D}_i^T & \ldots & \tilde{D}_i^T \end{bmatrix}^T.$$

Proof By Schur complement, (5.12) is equivalent to

$$\begin{bmatrix} -P_i + \tilde{C}_i^T\tilde{C}_i & \Phi_i^T\mathcal{K}_i & 0 \\ * & -2I & \mathcal{K}_i D_i \\ * & * & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \tilde{\mathscr{A}}_i^T \\ \tilde{\mathscr{D}}_i^T \\ \tilde{\mathscr{B}}_i^T \end{bmatrix} \tilde{\mathscr{P}}_i \begin{bmatrix} \tilde{\mathscr{A}}_i^T \\ \tilde{\mathscr{D}}_i^T \\ \tilde{\mathscr{B}}_i^T \end{bmatrix}^T < 0. \tag{5.13}$$

Let $\tilde{P}_i \triangleq \sum_{j=1}^{M} \lambda_{ij} P_j$, then (5.13) yields

$$\Psi_i \triangleq \begin{bmatrix} \Psi_{1i} & \tilde{A}_i^T\tilde{P}_i\tilde{D}_i + \Phi_i^T\mathcal{K}_i & \tilde{A}_i^T\tilde{P}_i\tilde{B}_i \\ * & \tilde{D}_i^T\tilde{P}_i\tilde{D}_i - 2I & \tilde{D}_i^T\tilde{P}_i\tilde{B}_i + \mathcal{K}_i D_i \\ * & * & -\gamma^2 I + \tilde{B}_i^T\tilde{P}_i\tilde{B}_i \end{bmatrix} < 0, \tag{5.14}$$

where $\Psi_{1i} \triangleq \tilde{A}_i^T\tilde{P}_i\tilde{A}_i - P_i + \tilde{C}_i^T\tilde{C}_i$.

First, we demonstrate the stability of the closed-loop system (5.10) with $\omega(k) = 0$. The inequality (5.14) implies that

$$\Theta_i \triangleq \begin{bmatrix} \tilde{A}_i^T \tilde{P}_i \tilde{A}_i - P_i & \tilde{A}_i^T \tilde{P}_i \tilde{D}_i + \Phi_i^T \mathcal{K}_i \\ * & \tilde{D}_i^T \tilde{P}_i \tilde{D}_i - 2I \end{bmatrix} < 0.$$

Therefore, considering (5.10) with $\omega(k) = 0$, we obtain

$$\begin{aligned} &\mathbf{E}\{V_{r_{k+1}}(\xi(k+1), k+1)|\,(\xi(k), \gamma_k = i)\} - V_{\gamma_k}(\xi(k), k) \\ &\quad \leqslant \tilde{\xi}^T(k)\Theta_i\tilde{\xi}(k) \\ &\quad \leqslant -\lambda_{\min}(\Theta_i)\,\tilde{\xi}^T(k)\tilde{\xi}(k) \\ &\quad \leqslant -\beta\xi^T(k)\xi(k), \end{aligned}$$

where

$$\beta \triangleq \inf_{i\in\mathcal{S}} \{\lambda_{\min}(\Theta_i)\} \text{ and } \tilde{\xi}(k) \triangleq \begin{bmatrix} \xi(k) \\ q_i^{(s)}(y(k)) \end{bmatrix}.$$

Then, for any $T > 0$,

$$\begin{aligned} &\mathbf{E}\Big\{V_{r_{T+1}}\big(\xi(T+1), T+1\big)|\,(\xi(T), r_T)\Big\} - V_{r_0}(\xi(0), 0) \\ &\quad \leqslant -\beta\sum_{k=0}^{T}\mathbf{E}\big\{\xi^T(k)\xi(k)\big\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{k=0}^{T}\mathbf{E}\big\{\xi^T(k)\xi(k)\big\} &\leqslant \frac{1}{\beta}\Big(\mathbf{E}\big\{V_{r_0}(\xi(0), 0)\big\} \\ &\qquad -\mathbf{E}\big\{V_{r_{T+1}}\big(\xi(T+1), T+1\big)|(\xi(T), r_T)\big\}\Big) \\ &\leqslant \frac{1}{\beta}\mathbf{E}\big\{V_{r_0}(\xi(0), 0)\big\}. \end{aligned}$$

Hence, it follows that

$$\lim_{T\to\infty}\mathbf{E}\left\{\sum_{k=0}^{T}\xi^T(k)\xi(k)\Big|\,(\xi_0, r_0)\right\} \leqslant M(\xi_0, r_0),$$

where $M(\xi_0, r_0)$ is a positive number. Note that by applying Assumption 1.1 we obtain that the state trajectories of the closed-loop system arrives in the same subsystem within a finite time. Thus, the closed-loop system (5.10) is stochastically stable in the sense of Definition 5.1.

Next, we show that the H_∞ performance in the sense of Definition 5.2 is assured under zero initial condition. Choose a stochastic global Lyapunov function candidate as:

$$V_{\gamma_k}(\xi(k), k) \triangleq \xi^T(k) P(r_k) \xi(k),$$

where $P_i \triangleq P(\gamma_k = i)$, for $i \in \mathcal{M}$, and P_i are positive diagonally dominant matrices to be determined.

Also, consider the following index:

$$\mathcal{J}_T^\gamma \triangleq \sum_{k=0}^{T} \mathbf{E}\left\{z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k)\right\}.$$

Therefore, under zero initial condition, that is, $V_{r_0}(\xi(0), 0) = 0$ for initial mode r_0, we obtain

$$\begin{aligned}
\mathcal{J}_T^\gamma = & \sum_{k=0}^{T} \mathbf{E}\Big\{z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k) - V_{\gamma_k}(\xi(k), k)\Big\} \\
& + \sum_{k=0}^{T} \mathbf{E}\Big\{V_{r_{k+1}}(\xi(k+1), k+1) |\, (\xi(k), \gamma_k = i)\Big\} \\
& - \mathbf{E}\Big\{V_{r_{T+1}}(\xi(T+1), T+1)\Big\}.
\end{aligned} \tag{5.15}$$

On the other hand, for $\gamma_k = i$ and $r_{k+1} = j$, we have

$$\begin{aligned}
& \mathbf{E}\big\{V_{r_{k+1}}(\xi(k+1), k+1)|(\xi(k), \gamma_k = i)\big\} - V_{\gamma_k}(\xi(k), k) \\
& = \sum_{j=1}^{M} \Pr\{r_{k+1} = j | \gamma_k = i\} \xi^T(k+1) P_j \\
& \quad \times \xi(k+1) - \xi^T(k) P_i \xi(k) \\
& = \xi^T(k+1) \tilde{P}_i \xi(k+1) - \xi^T(k) P_i \xi(k),
\end{aligned} \tag{5.16}$$

where $\tilde{P}_i$ is defined in (5.14). By Lemma 5.4, we obtain

$$g^T(\xi(k)) P_i g(\xi(k)) \leqslant \xi^T(k) P_i \xi(k). \tag{5.17}$$

Considering (5.7) and combining with (5.15)–(5.17) yields the following inequalities:

$$\mathcal{J}_T^\gamma \leqslant \mathbf{E}\left\{\sum_{k=0}^{T} \begin{bmatrix} g(\xi(k)) \\ q_i^{(s)}(y(k)) \\ \omega(k) \end{bmatrix}^T \Psi_i \begin{bmatrix} g(\xi(k)) \\ q_i^{(s)}(y(k)) \\ \omega(k) \end{bmatrix}\right\},$$

where Ψ_i is defined in (5.14). By $\Psi_i < 0$ in (5.14), we have $\mathcal{J}_T^{\gamma} < 0$ for all nonzero $\omega(k) \in \ell_2[0, \infty)$. The proof is completed. ∎

We now shift our design focus to the full-order and reduced-order DOFC in (5.9). A sufficient condition for the existence of such a DOFC for an nonlinear associated MJS (5.8) is presented as follows.

Theorem 5.7 *Given a constant $\gamma > 0$, the closed-loop system (5.10) is stochastically stable with an H_∞ disturbance attenuation level γ, if there exist matrices $0 < P_i \triangleq [p_{\alpha\beta}]_i \in \mathbb{R}^{(n+s)\times(n+s)}$, $R_i = R_i^T \triangleq [r_{\alpha\beta}]_i \in \mathbb{R}^{(n+s)\times(n+s)}$, and $\mathcal{P}_i > 0$, $\alpha, \beta \in \{1, 2, \ldots, (n+s)\}$, such that for all $i \in \mathcal{M}$, the following inequalities are satisfied*

$$\left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|cccc} -\tilde{\mathcal{P}}_i & 0 & \bar{\mathscr{A}}_i 0 & & \bar{\mathscr{B}}_i \\ \hline * & -I & \bar{C}_i & 0 & 0 \\ * & * & -P_i & \Phi_i^T \mathcal{K}_i & 0 \\ * & * & * & -2I & \mathcal{K}_i D_i \\ * & * & * & * & -\gamma^2 I \end{array}\right] \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right]^T < 0, \tag{5.18}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & N_i^{\perp} \end{array}\right] \left[\begin{array}{cc|ccc} -\tilde{\mathcal{P}}_i & 0 & \bar{\mathscr{A}}_i 0 & & \bar{\mathscr{B}}_i \\ * & -I & \bar{C}_i & 0 & 0 \\ \hline * & * & -P_i & \Phi_i^T \mathcal{K}_i & 0 \\ * & * & * & -2I & \mathcal{K}_i D_i \\ * & * & * & * & -\gamma^2 I \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & N_i^{\perp} \end{array}\right]^T < 0, \tag{5.19}$$

$$p_{\alpha\alpha i} - \sum_{\beta \neq \alpha} \left(p_{\alpha\beta i} + 2r_{\alpha\beta i}\right) \geqslant 0, \ \forall \alpha, \tag{5.20}$$

$$r_{\alpha\beta i} \geqslant 0, \ \forall \alpha \neq \beta, \tag{5.21}$$

$$p_{\alpha\beta i} + r_{\alpha\beta i} \geqslant 0, \ \forall \alpha \neq \beta, \tag{5.22}$$

$$P_i \mathcal{P}_i = I. \tag{5.23}$$

Moreover, if the aforementioned conditions hold, then the system matrices of DOFC (5.9) are given by
$\mathcal{G}_i \triangleq \begin{bmatrix} \hat{D}_i & \hat{C}_i \\ \hat{B}_i & \hat{A}_i \end{bmatrix}$, *where*

$$\begin{aligned}
\mathcal{G}_i &\triangleq -\Pi_i^{-1} U_i^T \Lambda_i V_i^T (V_i \Lambda_i V_i^T)^{-1} + \Pi_i^{-1} \Xi_i^{1/2} L_i (V_i \Lambda_i V_i^T)^{-1/2}, \\
\Lambda_i &\triangleq (U_i \Pi_i^{-1} U_i^T - \Psi_i)^{-1} > 0, \\
\Xi_i &\triangleq \Pi_i - U_i^T (\Lambda_i - \Lambda_i V_i^T (V_i \Lambda_i V_i^T)^{-1} V_i \Lambda_i) U_i > 0.
\end{aligned}$$

In addition, Π_i and L_i are any appropriate matrices satisfying $\Pi_i > 0$, $\|L_i\| < 1$, and

$$
\begin{aligned}
\Psi_i &\triangleq \begin{bmatrix} -\tilde{\mathscr{P}}_i^{-1} & 0 & \bar{\mathscr{A}}_i & 0 & \bar{\mathscr{B}}_i \\ * & -I & \bar{C}_i & 0 & 0 \\ * & * & -P_i & \Phi_i^T \mathcal{K}_i & 0 \\ * & * & * & -2I & \mathcal{K}_i D_i \\ * & * & * & * & -\gamma^2 I \end{bmatrix}, \quad \mathcal{I} \triangleq \begin{bmatrix} I_p \\ 0_{s\times p} \end{bmatrix}, \\
U_i &\triangleq \begin{bmatrix} \mathscr{X}_i \\ 0_{q\times(m+s)} \\ 0_{(n+s)\times(m+s)} \\ 0_{p\times(m+s)} \\ 0_{l\times(m+s)} \end{bmatrix}, \quad Y_i \triangleq \begin{bmatrix} \mathcal{K}_{1i} C_i & 0_{p\times s} \\ 0_{s\times n} & I_s \end{bmatrix}, \\
V_i &\triangleq \begin{bmatrix} 0_{(p+s)\times(n+s)} & 0_{(p+s)\times q} & Y_i & \mathcal{I} & Z_i \end{bmatrix}, \quad M_i^{\perp} \triangleq \mathbb{I}_N^T \otimes X_i^{\perp}, \\
N_i &\triangleq \begin{bmatrix} \mathcal{K}_{1i} C_i & 0_{p\times s} & I_p & \mathcal{K}_{1i} D_i \\ 0_{s\times n} & I_s & 0_{s\times p} & 0_{s\times l} \end{bmatrix}^T, \quad X_i \triangleq \begin{bmatrix} B_i & 0_{n\times s} \\ 0_{s\times m} & I_s \end{bmatrix}, \\
\tilde{\mathcal{P}}_i &\triangleq \operatorname{diag}\{\pi_{i1}^{-1}\mathcal{P}_1, \pi_{i2}^{-1}\mathcal{P}_2, \ldots, \pi_{iN}^{-1}\mathcal{P}_N\}, \quad \bar{\mathscr{A}}_i \triangleq \mathbb{I}_N \otimes \bar{A}_i, \\
Z_i &\triangleq \begin{bmatrix} \mathcal{K}_{1i} D_i \\ 0_{s\times l} \end{bmatrix}, \quad \mathscr{X}_i \triangleq \mathbb{I}_N \otimes X_i^{\perp}, \ \bar{\mathscr{B}}_i \triangleq \mathbb{I}_N \otimes \bar{B}_i,
\end{aligned} \tag{5.24}
$$

Proof We can rewrite $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$ and $\tilde{D}_i$ in (5.11) as follows:

$$
\begin{aligned}
&\tilde{A}_i = \bar{A}_i + X_i \mathcal{G}_i Y_i, \quad \tilde{B}_i = \bar{B}_i + X_i \mathcal{G}_i Z_i, \\
&\tilde{C}_i = \bar{C}_i, \quad \tilde{D}_i = X_i \mathcal{G}_i \mathcal{I},
\end{aligned} \tag{5.25}
$$

where

$$
\bar{A}_i \triangleq \begin{bmatrix} A_i & 0_{n\times s} \\ 0_{s\times n} & 0_{s\times s} \end{bmatrix}, \quad \bar{B}_i \triangleq \begin{bmatrix} F_i \\ 0_{s\times l} \end{bmatrix}, \quad \bar{C}_i \triangleq \tilde{C}_i, \tag{5.26}
$$

and X_i, $\mathcal{I}$, Y_i and Z_i are defined in (5.24). Using (5.25), (5.12) can be rewritten as

$$
\Psi_i + U_i \mathcal{G}_i V_i + (U_i \mathcal{G}_i V_i)^T < 0, \tag{5.27}
$$

where Ψ_i, U_i and V_i are defined in (5.24).

Next, we assign

$$
U_i^{\perp} = \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right], \quad V_i^{T\perp} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & N_i^{\perp} \end{array}\right].
$$

It follows from Lemma 1.29 that (5.27) is solvable for $\mathcal{G}_i$ if and only if the (5.18), (5.19) and (5.23) are satisfied. In addition, from (5.20)–(5.22), we have

$$p_{\alpha\alpha i} \geqslant \sum_{\beta\neq\alpha}(p_{\alpha\beta i} + 2r_{\alpha\beta i})$$
$$= \sum_{\beta\neq\alpha} |p_{\alpha\beta i} + r_{\alpha\beta i}| + |-r_{\alpha\beta i}| \geqslant \sum_{\beta\neq\alpha} |p_{\alpha\beta i}|,$$

which implies, in view of Lemma 5.5, that the positive definite matrices P_i are diagonally dominant. On the other hand, if conditions (5.18)–(5.23) are satisfied, the parameters of an H_∞ output feedback controller corresponding to a feasible solution can be obtained using the results in [8]. This completes the proof. ■

Remark 5.8 The result in Theorem 5.7, in fact, includes the reduced-order dynamic output feedback control design. In (5.9), the reduced-order output feedback controller is designed when $s < n$. ♦

Remark 5.9 It is worth noting that the convex optimization algorithm cannot be used to find a minimum γ, since the conditions are no longer LMIs due to the matrix equation (5.23). However, we can solve this problem by using the CCL algorithm proposed in [9]. The core idea of the CCL algorithm is that if the following inequality

$$\begin{bmatrix} P_i & I \\ I & \mathcal{P}_i \end{bmatrix} \geqslant 0, \quad \forall\, i \in \mathcal{M} \tag{5.28}$$

are solvable for $P_i > 0$ and $\mathcal{P}_i > 0$, $\forall\, i \in \mathcal{M}$, then $\mathrm{tr}\left(\sum_i P_i\mathcal{P}_i\right) \geqslant n$, moreover, $\mathrm{tr}\left(\sum_i P_i\mathcal{P}_i\right) = n$ if and only if $P_i\mathcal{P}_i = I$. ♦

From the above discussion, we can solve the nonconvex feasibility problem by formulating it into a sequential optimization problem.

The quantized H_∞ DOFC design problem:

$$\begin{aligned} &\min \quad \mathrm{tr}\left(\sum\nolimits_i P_i\mathcal{P}_i\right), \\ &\text{subject to} \quad (5.18)\text{–}(5.22) \text{ and } (5.28). \end{aligned}$$

If there exists solutions that min $\mathrm{tr}\left(\sum_i P_i\mathcal{P}_i\right)$ subject to (5.18)–(5.22), and $\mathrm{tr}\left(\sum_i P_i\mathcal{P}_i\right) = n$, then the inequalities in Theorem 5.7 are solvable.

Therefore, we propose the following algorithm to solve the above problem.

Step 1. Find a feasible set $\left(P_i^{(0)}, \mathcal{P}_i^{(0)}, R_i^{(0)}\right)$ satisfying (5.18)–(5.22) and (5.28). Set $\kappa = 0$.

Step 2. Solve the following optimization problem:

$$\begin{aligned} &\min \quad \mathrm{tr}\left(\sum\nolimits_i \left(P_i^{(\kappa)}\mathcal{P}_i + P_i\mathcal{P}_i^{(\kappa)}\right)\right), \\ &\text{subject to} \quad (5.18)\text{–}(5.22) \text{ and } (5.28). \end{aligned}$$

and denote the optimum value as f^*.

Step 3. Substitute the obtained matrix variables $(P_i, \mathcal{P}_i, R_i)$ into (5.18)–(5.19). If (5.18)–(5.19) are satisfied with

$$\left|f^* - 2N(n+s)\right| < \delta$$

for a sufficiently small scalar $\delta > 0$, then output the feasible solutions $(P_i, \mathcal{P}_i, R_i)$ and EXIT.

Step 4. If $\kappa > \mathbb{N}$, where $\mathbb{N}$ is the maximum allowed iteration number, then EXIT.

Step 5. Set $\kappa = \kappa + 1$, $\left(P_i^{(\kappa)}, \mathcal{P}_i^{(\kappa)}, R_i^{(\kappa)}\right) = (P_i, \mathcal{P}_i, R_i)$, and go to *Step 2*.

Remark 5.10 Note that in the above algorithm, an iteration method has been employed to solve the minimization problem instead of the original nonconvex feasibility problem addressed in (5.23). In order to solve the minimization problem, the stopping criterion $|f^* - 2N(n+s)|$ should be checked since it can be numerically difficult to obtain the optimal solutions to meet the condition that $\mathrm{tr}\left(\sum_i P_i \mathcal{P}_i\right) = n$. ♦

5.4 Illustrative Example

In this section, we demonstrate the effectiveness of proposed quantized output feedback control scheme by a PH semi-Markov model over cognitive radio networks (Fig. 5.1). It is observed in [10] that cognitive radio systems hold promise in the design of large-scale systems due to huge needs of bandwidth during interaction and communication between subsystems. Each channel has two states (busy and idle) and the number of times the channel stays in each state are independent and identically distributed random variables following certain probability distribution functions that having possible connection to both states to be switched. A semi-Markov process has been employed in [10] for the cognitive radio structure to represent the switch between idle and busy states.

In the following, we consider a nonlinear S-MJS (5.8) with two modes over cognitive radio links:

Mode 1.

$$A_1 = \begin{bmatrix} -2.0 & 1.0 \\ -4.0 & -3.0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.4 \\ 0.7 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1.1 & -0.4 \end{bmatrix},$$
$$D_1 = 3.6, \quad E_1 = \begin{bmatrix} 3.5 & 1.1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1.5 \\ 0.4 \end{bmatrix},$$
$$K_{11} = 0.1, \quad K_{21} = 0.4.$$

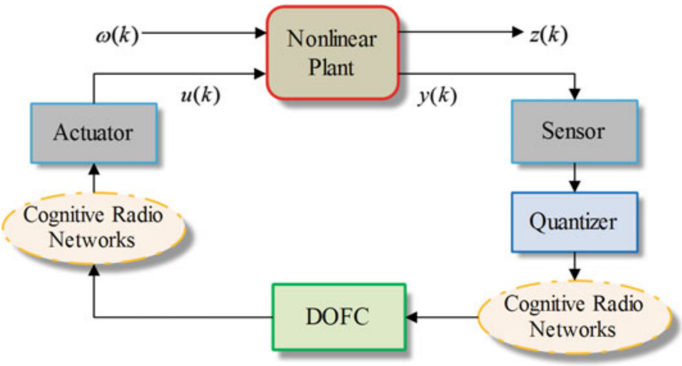

Fig. 5.1 Structure of a quantized closed-loop system with cognitive radio networks

Mode 2.

$$A_2 = \begin{bmatrix} -3.1 & 5.1 \\ 4.2 & -3.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.1 \\ 0.2 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0.3 \; 0.45 \end{bmatrix},$$
$$D_2 = 4.2, \quad E_2 = \begin{bmatrix} 0.3 \; 1.0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.5 \\ 1.3 \end{bmatrix},$$
$$K_{12} = 0.1, \quad K_{22} = 0.3.$$

Assume that at each step the sensor in cognitive radio infrastructure scans only one channel. This assumption avoids the use of costly and complicated multichannel sensors. The sensor first picks a channel to scan, then transmits the signal through it if the channel is idle, or stops transmission to avoid collision otherwise. In addition, we assume that the switch between the modes is governed by a semi-Markov process taking values in $\{1, 2\}$. In these two models, the sojourn times are the random variables distributed according to a negative exponential distribution with parameter λ_1 and according to a 2-order Erlang distribution, respectively. The analysis of a PH S-MJS, by Lemma 1, is reduced to the analysis of its associated MJS. The main idea is to search for the associated Markov chain and its infinitesimal generator and define the proper representation.

In particular, two parts can be identified in the second model for the sojourn time which is a random variable exponentially distributed with parameter λ_2 in part 1 and λ_3 in part 2. This view suggests that the process $\bar{\gamma}_k$ must stay at the first part for some time upon entering model 2, before making its way to the second, and finally return to model 1 again. We know that $p_{12} = p_{21} = 1$. Therefore,

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{a}^{(1)} = (a_1^{(1)}) = 1,$$
$$\mathbf{a}^{(2)} = \left(a_1^{(2)}, a_2^{(2)}\right) = (1, 0), \quad T^{(1)} = \left(T_{11}^{(1)}\right) = (-\lambda_1),$$

$$T^{(2)} = \begin{bmatrix} T_{11}^{(2)} & T_{12}^{(2)} \\ T_{21}^{(2)} & T_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{bmatrix}.$$

It is easy to see that the state space of $Z(k) = (\bar{\gamma}_k, J(k))$ is $G = \big((1, 1), (2, 1), (2, 2)\big)$. We enumerate the elements of G as $\varphi\big((1, 1)\big) = 1$, $\varphi\big((2, 1)\big) = 2$, and $\varphi\big((2, 2)\big) = 3$. Hence, the infinitesimal generator of $\varphi(Z(k))$ is

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ -\lambda_3 & 0 & \lambda_3 \end{bmatrix}.$$

Now, let $\gamma_k = \varphi(Z(k))$. Then, γ_k is the associated Markov chain of $\bar{\gamma}_k$ with state space $\{1, 2, 3\}$. The infinitesimal generator of γ_k is given by Q.

Consider the reduced-order DOFC design, that is, $s = 1 < n$. As shown in Fig. 5.1, before entering the controller via the CR network, the signal $y(k)$ is quantized by the mode-dependent logarithmic quantizer (5.3). The quantizer density is chosen as $\rho^{(1,1)} = 0.6667$, $\rho^{(1,2)} = 0.7391$, and $\eta_0^{(1,1)} = \eta_0^{(1,2)} = 0.001$. It can be calculated that $\delta^{(1,1)} = 0.4$ and $\delta^{(1,2)} = 0.5$. Solving the quantized H_∞ DOFC problem by means of the designed algorithm of quantized H_∞ DOFC implies that the minimum γ is equal to $\gamma^\star = 2.304$, and the corresponding reduced-order DOFC parameters are designed as

$$\begin{aligned} &\hat{A}_1 = -2.214, \quad \hat{B}_1 = 2.250, \quad \hat{C}_1 = -1.317, \quad \hat{D}_1 = -1.347, \\ &\hat{A}_2 = -8.257, \quad \hat{B}_2 = 0.155, \quad \hat{C}_2 = -1.156, \quad \hat{D}_2 = 2.231. \end{aligned}$$

Now, we illustrate the effectiveness of the quantized full-order DOFC designed in (5.9) through simulations. The repeated scalar nonlinearity in (5.8) is chosen as $g(x(k)) = \sin(x(k))$, which satisfies (5.2). Let the initial conditions be $x(0) = \begin{bmatrix} 1.2 & -0.4 \end{bmatrix}^T$ and $\hat{x}(0) = 0$. The disturbance input $\omega(k)$ is assigned as $\omega(k) = 0.2e^{-2k}$.

The simulation results are shown in Figs. 5.2, 5.3, 5.4, 5.5 and 5.6. Figure 5.2 displays a switching signal; here, '1' and '2' correspond to the first and second mode, respectively. Under this mode sequence, the trajectories of $y(k)$ and quantized measurements $q(y(k))$ are shown in Fig. 5.3. It can be seen that the mode-dependent quantizer is adjusted according to the mode jumping sequences. The trajectory of $q_i^s(y(k))$ is demonstrated in Fig. 5.4. To further illustrate the effectiveness of the proposed technique, we perform the Monte Carlo simulation. In Figs. 5.5 and 5.6, the state responses of the closed-loop system and the full-order DOFC for 50 runs are shown, respectively. The semi-Markov processes are unique in each run. It can be seen that the system is stochastically stabilized for every run of simulation.

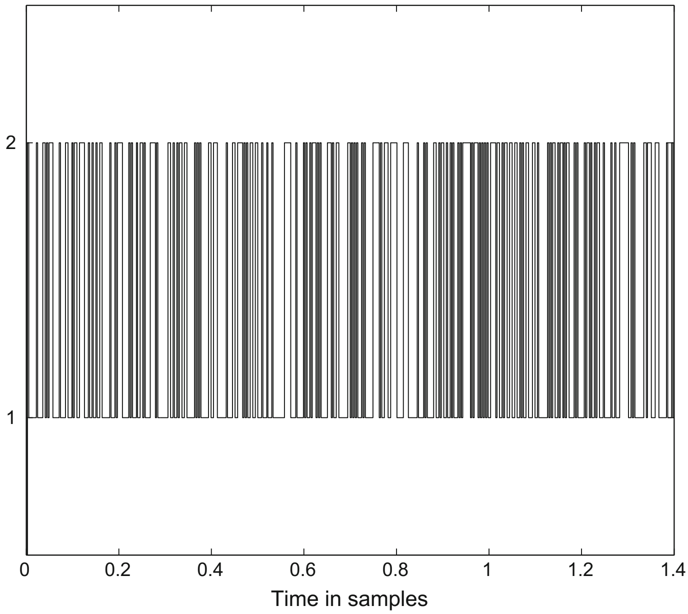

Fig. 5.2 Switching signal

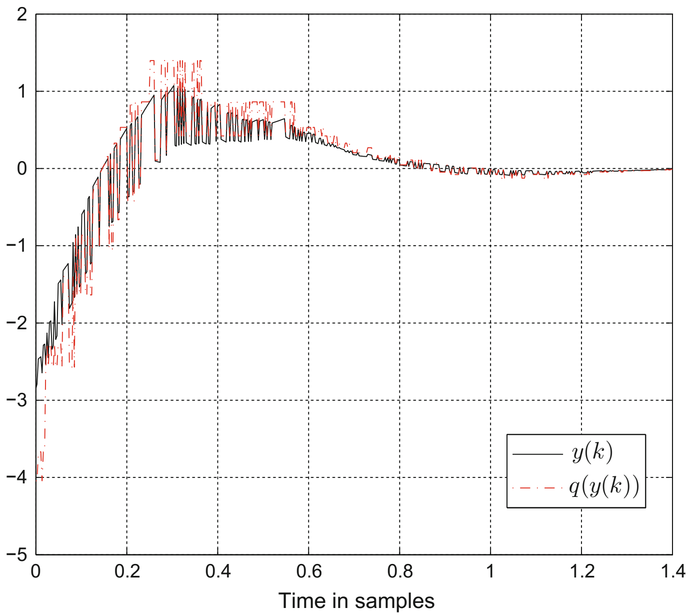

Fig. 5.3 The trajectories of output $y(k)$ and quantized output $q(y(k))$

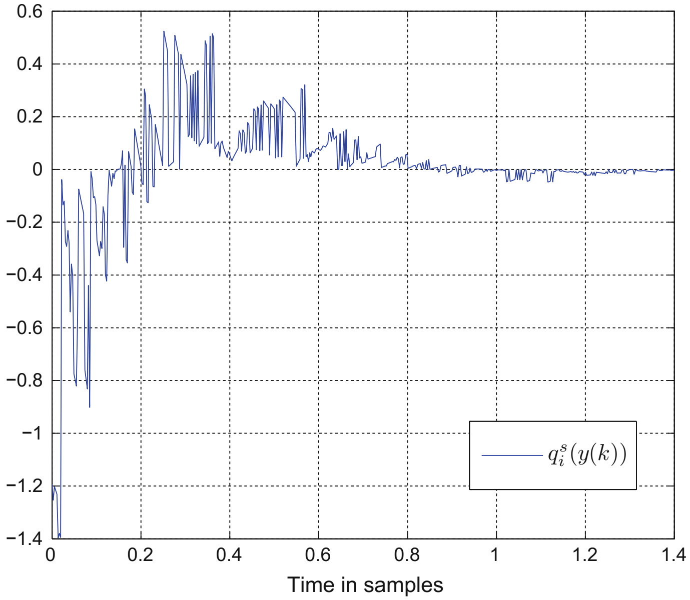

Fig. 5.4 The trajectory of output $q_i^s(y(k))$

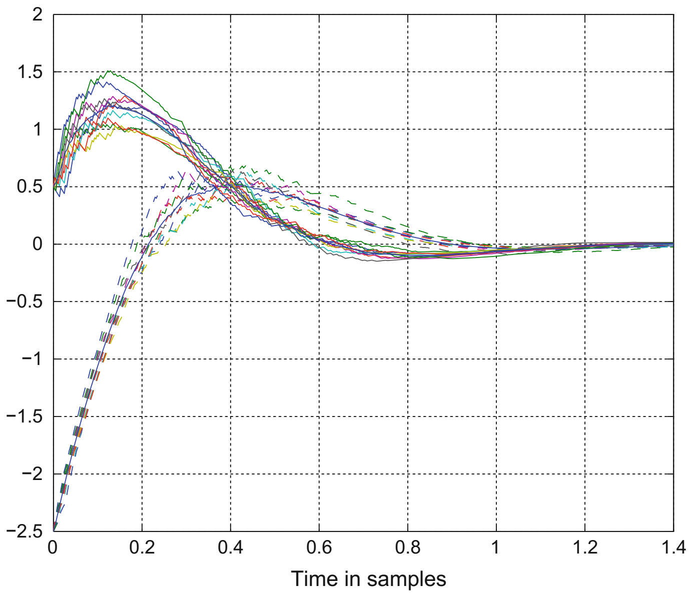

Fig. 5.5 States responses of the closed-loop system with full-order DOFC

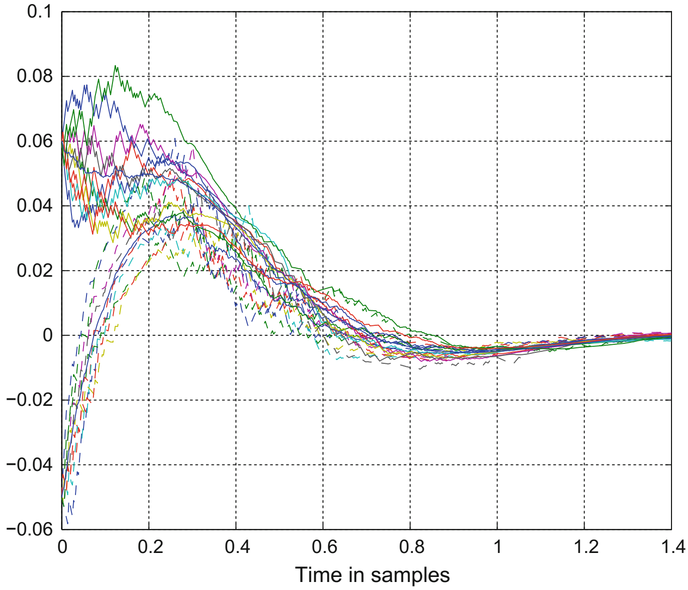

Fig. 5.6 States responses of the full-order dynamic output feedback controller

5.5 Conclusion

In this chapter, the problem of quantized output feedback control has been addressed for a class of S-MJS with repeated scalar nonlinearities. A mode-dependent logarithmic quantizer has been employed to quantize the measured output signal. Then, using the positive definite diagonally dominant Lyapunov function technique, a sufficient condition has been proposed to ensure the stochastic stability with an H_∞ performance for the closed-loop system. Furthermore, a sufficient condition has been decoupled into a convex optimization problem, which can be efficiently handled using standard numerical software. The corresponding mode-dependent quantized controller has been successfully designed for nonlinear S-MJS. Based on cognitive radio communication networks, an example has been provided to illustrate the applicability of the proposed techniques.

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Part II
Filtering and Fault Detection

Chapter 6
Passive Filtering for Delayed Neutral-Type Semi-Markovian Jump Systems

Abstract This chapter is concerned with the problem of exponential passive filtering for a class of stochastic neutral-type neural networks with both semi-Markovian jump parameters and mixed time delays. Our aim is to estimate the state by a Luenberger-type observer such that the filter error dynamics are exponentially mean-square stable with an expected decay rate and an attenuation level. Sufficient conditions for existence of passive filters are obtained and a cone complementarity linearization procedure is employed to transform a nonconvex feasibility problem into a sequential minimization problem, which can be readily solved by existing optimization techniques.

6.1 Introduction

Neural networks (NNs) have been successfully applied to various areas such as economic load dispatch, signal processing, pattern recognition, automatic control and combinatorial optimization. Note also that many neural networks may experience in their structure and parameters abrupt changes caused by some phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. In this situation, there exist finite modes in the neural networks, and the modes may switch (or jump) from one to another at different times. In practical reality, delays seldom remain the same but are subject to variations in time, and the problem of achieving exponential stability for delayed neural networks has been investigated by many researchers [1, 2]. Meanwhile, studies continue in modeling more realistic but complex delay dynamical behaviors by incorporating delay differential equations of neutral types. This development stems from many system behaviors found in applications such as the metal strip transmission lines used in VLSI circuits, combustion systems, and controlled constrained manipulators [3, 4].

In this chapter, we focus on the exponentially passive filter design problem for neural-type neural networks (NTNN) with semi-Markovian jump parameters and mixed time delays. Based on passivity theory, sufficient conditions are given to ensure that the filtering error dynamics are strictly exponentially passive with performance

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level γ. Comparing with existing works in [5, 6], the differential of the stability analysis is considered in this chapter. The method proposed in [5, 6] only guarantees asymptotical stability of the resulting system, but the method proposed in this chapter can ensure that the system is exponentially stable. This guarantees that the resulting system will attain a fast and satisfactory response. By contrast of the models for neutral-type time-delay neural networks used in [5–7], our work takes into account the semi-Markovian jump parameters and mixed time-varying delays. To reduce the computational burden of the cone complementarity linearization procedure, we convert the corresponding passive filter design into a convex optimization problem so that its solution can be efficiently found.

6.2 Problem Formulation and Preliminaries

Let $\{\eta_t, t \geqslant 0\}$ be a semi-Markov chain taking values in state-space $\mathcal{M} = \{1, 2, \ldots, M\}$. As mentioned in Chap. 2, we relax the probability distribution of sojourn time from exponential distribution to Weibull distribution, so the transition rate in S-MJS will be time varying instead of constant in MJS.

In this chapter, it is assumed that the transition probabilities $\Pi = [\pi_{ij}(h)]_{M\times M}$ belongs to a polytope P_Π, with vertices $\{\Pi^{(r)}, r = 1, 2, \ldots, S\}$ as follows:

$$P_\Pi \triangleq \left\{ \Pi | \Pi = \sum_{r=1}^{S} a_r \Pi^{(r)}, \ \sum_{r=1}^{S} a_r = 1, \ a_r \geqslant 0 \right\}.$$

For notation clarity, for any $i \in \mathcal{M}$, we denote $\mathcal{M} \triangleq \mathcal{M}^i_{uc} \cup \mathcal{M}^i_{uk}$, where

$$\begin{cases} \mathcal{M}^i_{uc} \triangleq \left\{ j | \ \pi_{ij}(h) \text{ is uncertain} \right\}, \\ \mathcal{M}^i_{uk} \triangleq \left\{ j | \ \pi_{ij}(h) \text{ is unknown} \right\}. \end{cases} \tag{6.1}$$

Also, define $\pi^i_{uc} \triangleq \sum_{j\in\mathcal{M}^i_{uc}} \pi^{(r)}_{ij}, \forall r = 1, 2, \ldots, S$.

Consider the following uncertain NTNN with time-varying discrete and distributed delays:

$$\begin{cases} \dot{x}(t) = -\mathbf{A}(\eta_t)x(t) + \mathbf{B}(\eta_t)g(x(t)) \\ \qquad\quad +\mathbf{C}(\eta_t)g(x(t-\tau_{1,\eta_t})) + \mathbf{D}(\eta_t)\dot{x}(t-\tau_{2,\eta_t}) \\ \qquad\quad +\mathbf{E}(\eta_t)\int_{t-\tau_{3,\eta_t}}^{t} g(x(s))\mathrm{d}s + \mathbf{C}_d(\eta_t)\omega(t), \\ y(t) = C_1(\eta_t)x(t) + C_g(\eta_t)g(x(t)) + D_d(\eta_t)\omega(t), \\ z(t) = D_1(\eta_t)x(t) + E_d(\eta_t)\omega(t), \\ x(t) = \phi(t), \ t \in [-\tau, 0], \end{cases} \tag{6.2}$$

where $x(t) = \left[\, x_1(t)\ x_2(t)\ \ldots\ x_n(t) \,\right]^T \in \mathbb{R}^n$ is the state vector associated with n neurons, with $x_i(t)$ being the state of the ith neuron; $\omega(t) \in \mathbb{R}^l$ is exogenous disturbance input belonging to $\mathcal{L}_2[0, \infty)$; $z(t) \in \mathbb{R}^p$ is the controlled output, and $y(t) \in \mathbb{R}^q$ is the measurable output of the neural networks. The continuous vector-valued function $\phi(t)$ denotes the initial data.

The variables τ_{1,η_t}, τ_{2,η_t} and τ_{3,η_t} are time-varying transmission delays, where τ_{1,η_t} and τ_{2,η_t} denote the mode-dependent discrete-time delays; and τ_{3,η_t} characterizes the mode-dependent upper bound of the distributed time delay. For presentation convenience, we denote:

$$
\begin{aligned}
&\bar{\tau}_1 \triangleq \max\{\tau_{1,i}, i \in \mathcal{M}\}, \quad \underline{\tau}_1 \triangleq \min\{\tau_{1,i}, i \in \mathcal{M}\},\\
&\bar{\tau}_2 \triangleq \max\{\tau_{2,i}, i \in \mathcal{M}\}, \quad \underline{\tau}_2 \triangleq \min\{\tau_{2,i}, i \in \mathcal{M}\},\\
&\bar{\tau}_3 \triangleq \max\{\tau_{3,i}, i \in \mathcal{M}\}, \quad \tau \triangleq \max\{\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\},\\
&\tilde{\tau}_1 \triangleq \max\{\dot{\tau}_{1,i}, i \in \mathcal{M}\}, \quad \tilde{\tau}_2 \triangleq \max\{\dot{\tau}_{2,i}, i \in \mathcal{M}\},\\
&\delta_{\tau_1} \triangleq \bar{\tau}_1 - \underline{\tau}_1, \quad \delta_{\tau_2} \triangleq \bar{\tau}_2 - \underline{\tau}_2.
\end{aligned}
$$

Remark 6.1 A neutral-type system is considered in system (6.2) since it contains derivatives in delayed states. Unlike retarded systems, linear neutral systems may be destabilized by small changes of the delay [8]. To guarantee robustness of the results with respect to small changes of delay, we assume that all eigenvalues of the matrix $\mathbf{D}(\eta_t)$ are inside the unit circle, which guarantees that the difference operator $\mathbb{T} : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$, defined by $\mathbb{T}x(t) = x(t) - \mathbf{D}(\eta_t)x(t - \tau_{2,\eta_t})$, is delay-independently stable with respect to all τ_{2,η_t}. ♦

The functions $g(x(t)) = [g_1(x_1(t))\ g_2(x_2(t))\ \ldots\ g_n(x_n(t))]^T$, $g_i(s)(i = 1, \ldots, n)$ denote the neuron activation functions that satisfy the following assumption [9].

Assumption 6.1 The neuron activation functions $g_j(s)(j = 1, \ldots, n)$ satisfy:

$$
g_j(0) = 0 \quad \text{and} \quad l_j^- \leqslant \frac{g_j(\zeta_1) - g_j(\zeta_2)}{\zeta_1 - \zeta_2} \leqslant l_j^+, \tag{6.3}
$$

where $\forall \zeta_1, \zeta_2 \in \mathbb{R}, \zeta_1 \neq \zeta_2$, l_j^- and l_j^+ are some constants.

In system (6.2), $\mathbf{A}(\eta_t)$, $\mathbf{B}(\eta_t)$, $\mathbf{C}(\eta_t)$, $\mathbf{D}(\eta_t)$, $\mathbf{E}(\eta_t)$, $\mathbf{C}_d(\eta_t)$, $C_1(\eta_t)$, $D_1(\eta_t)$, $C_g(\eta_t)$, $D_d(\eta_t)$ and $E_d(\eta_t)$ are connection weight matrices with appropriate dimensions, with the system parameter uncertainties taking the form

$$
\begin{aligned}
&\left[\, \mathbf{A}(\eta_t)\ \mathbf{B}(\eta_t)\ \mathbf{C}(\eta_t)\ \mathbf{D}(\eta_t)\ \mathbf{E}(\eta_t)\ \mathbf{C}_d(\eta_t) \,\right]\\
&\quad = \left[\, A(\eta_t)\ B(\eta_t)\ C(\eta_t)\ D(\eta_t)\ E(\eta_t)\ C_d(\eta_t) \,\right]\\
&\qquad + M(\eta_t)F(t)\left[L_a(\eta_t)\ L_b(\eta_t)\ L_c(\eta_t)\ L_d(\eta_t)\ L_e(\eta_t)\ L_{c_d}(\eta_t) \right],
\end{aligned}
$$

where $A(\eta_t), B(\eta_t), C(\eta_t), D(\eta_t), E(\eta_t), C_d(\eta_t), M(\eta_t), L_a(\eta_t), L_b(\eta_t), L_c(\eta_t), L_d(\eta_t)$, $L_e(\eta_t)$ and $L_{c_d}(\eta_t)$ are known real constant matrices with appropriate dimensions, and $F(t)$ is the system uncertainty matrix such that

$$F^T(t)F(t) \leqslant I, \ \forall t \geqslant 0. \tag{6.4}$$

For system (6.2), we consider the following full-order filter.

$$\begin{cases} \dot{\hat{x}}(t) = -A(\eta_t)\hat{x}(t) + B(\eta_t)g(\hat{x}(t)) + C(\eta_t)g(\hat{x}(t-\tau_{1,\eta_t})) \\ \qquad +D(\eta_t)\dot{\hat{x}}(t-\tau_{2,\eta_t}) + E(\eta_t)\int_{t-\tau_{3,\eta_t}}^{t} g(\hat{x}(s))\mathrm{d}s \\ \qquad +H(\eta_t)(y(t)-\hat{y}(t)), \\ \hat{y}(t) = C_1(\eta_t)\hat{x}(t) + C_g(\eta_t)g(\hat{x}(t)), \\ \hat{z}(t) = D_1(\eta_t)\hat{x}(t), \\ \hat{x}(t) = 0, \ t \in [-\tau, 0], \end{cases} \tag{6.5}$$

where $\hat{x}(t)$, $\hat{y}(t)$ and $\hat{z}(t)$ are the estimations of $x(t)$, $y(t)$, and $z(t)$, respectively and $H(\eta_t)$ are the filter gain matrices to be determined.

For each possible value $\eta_t = i \in \mathcal{M}$, the filtering error dynamics can be obtained from (6.2) and (6.5) as follows:

$$\begin{cases} \dot{e}(t) = \hat{A}(i)e(t) + \hat{B}(i)g(e(t)) + \hat{C}(i)g(e(t-\tau_{1,i})) \\ \qquad +\hat{D}(i)\dot{e}(t-\tau_{2,i}) + \hat{E}(i)\int_{t-\tau_{3,i}}^{t} g(e(s))\mathrm{d}s + \hat{C}_d(i)\omega(t), \\ \delta(t) = \hat{D}_1(i)e(t) + E_d(i)\omega(t), \\ e(t) = \hat{\phi}(t), \ t \in [-\tau, 0], \end{cases} \tag{6.6}$$

where

$$\begin{aligned} e(t) &\triangleq \begin{bmatrix} x^T(t) & \hat{x}^T(t) \end{bmatrix}^T, \quad \hat{D}_1(i) \triangleq \begin{bmatrix} D_1(i) & -D_1(i) \end{bmatrix}, \\ \hat{A}(i) &\triangleq \begin{bmatrix} -A(i) & 0 \\ H(i)C_1(i) & -A(i)-H(i)C_1(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_a(i), \\ \hat{B}(i) &\triangleq \begin{bmatrix} B(i) & 0 \\ H(i)C_g(i) & B(i)-H(i)C_g(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_b(i), \\ \hat{C}(i) &\triangleq \begin{bmatrix} C(i) & 0 \\ 0 & C(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_c(i), \\ \hat{C}_d(i) &\triangleq \begin{bmatrix} C_d(i) \\ H(i)D_d(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_{c_d}(i), \quad \hat{M}(i) \triangleq \begin{bmatrix} M(i) & 0 \\ 0 & M(i) \end{bmatrix}, \\ \hat{D}(i) &\triangleq \begin{bmatrix} D(i) & 0 \\ 0 & D(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_d(i), \\ \hat{E}(t) &\triangleq \begin{bmatrix} E(i) & 0 \\ 0 & E(i) \end{bmatrix} + \hat{M}(i)\hat{F}(t)\bar{l}_e(i), \quad \hat{\phi}(t) \triangleq \begin{bmatrix} \phi(t) \\ 0 \end{bmatrix}, \end{aligned}$$

$$\hat{F}(t) \triangleq \begin{bmatrix} F(t) & 0 \\ 0 & F(t) \end{bmatrix}, \quad \bar{l}_a(i) \triangleq \begin{bmatrix} -L_a(i) & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{l}_s(i) \triangleq \begin{bmatrix} L_s(i) & 0 \\ 0 & 0 \end{bmatrix} \ (s = b, c, d, e), \quad \bar{l}_{c_d}(i) \triangleq \begin{bmatrix} L_{c_d}(i) \\ 0 \end{bmatrix},$$

and $\delta(t) \triangleq z(t) - \hat{z}(t)$ is the output error.

Before ending this section, let us recall the following definitions and lemmas, which will be used in the next section.

Definition 6.2 The filtering error dynamics (6.6) with $\omega(t) = 0$ is said to be exponentially mean-square stable, if for all finite initial functions $\phi(t)$ on $(-\infty, 0]$, all initial conditions $x_0 \in \mathbb{R}^n$, and $r_0 \in \mathcal{M}$, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\mathbf{E}\{\|e(t)\|_2 \mid (x_0, r_0)\} \leqslant \beta e^{-\alpha t} \|\phi\|_{\mathcal{C}^1}, \quad \forall t \geqslant 0,$$

where $\|\phi\|_{\mathcal{C}^1} \triangleq \sup_{s \in [-\kappa, 0]} \{\|\phi(s)\|_2, \|\dot{\phi}(s)\|_2\}$.

Clearly, if system (6.6) is exponentially stable, we can achieve $x(t) \to \hat{x}(t)$. Then system (6.2) can be observed by the full-order filter (6.5), which is a Luenberger-type observer.

Definition 6.3 Given a scalar $\gamma > 0$, (6.6) is said to be strictly exponentially passive with performance level $\gamma > 0$, if it is exponentially mean-square stable and under zero initial conditions, the following relation holds

$$\int_0^T \mathbf{E}\left\{\omega^T(t)\delta(t) - \gamma \omega^T(t)\omega(t)\right\} \mathrm{d}t \geqslant 0. \tag{6.7}$$

Passive filter design problem: The objective of this chapter is to design a passive filter to estimate the neuron states from the available network output, and derive sufficient conditions under which the augmented filtering system (6.6) is strictly exponentially passive with desired disturbance attenuation $\gamma > 0$. Furthermore, the desired filter that meets the specified performance criterion by employing a convex optimization algorithm.

6.3 Main Results

6.3.1 Passive Filtering Analysis

In this section, our aim is to design the parameters of the filter in (6.5) such that the filtering error dynamics (6.6) are exponentially mean-square stable subject to meeting desired disturbance attenuation $\gamma > 0$.

Theorem 6.4 *For given scalars* $\underline{\tau}_1$, $\bar{\tau}_1$, $\tilde{\tau}_1$, $\underline{\tau}_2$, $\bar{\tau}_2$, $\tilde{\tau}_2$, $\bar{\tau}_3$, *and* r_3, *system* (6.6) *is strictly exponentially passive with performance level* $\gamma > 0$, *if there exist real symmetric positive-definite matrices* $\mathcal{P}(i) \triangleq \mathcal{P}(\eta_t = i) = \begin{bmatrix} P(i) & 0 \\ 0 & P(i) \end{bmatrix}$, $\mathcal{Q}(i) \triangleq \mathcal{Q}(\eta_t = i)$, $\mathcal{R}(i) \triangleq \mathcal{R}(\eta_t = i)$, $\mathcal{R}_\nu$, $\mathcal{S}_\nu$, $(\nu = 1, 2)$, $\mathcal{Q}_s$, $\mathcal{T}_s$, $(s = 1, 2, 3)$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{R}}$, *and diagonal matrices* $\mathcal{Z}_1 > 0$, $\mathcal{Z}_2 > 0$, *and* $\mathcal{H}(i)$ *and positive scalar* ε *such that*

$$\begin{bmatrix} \Sigma(i) + \Sigma_0(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} < 0, \tag{6.8}$$

$$\sum_{j=1}^{M} \pi_{ij}(h)\mathcal{Q}(j) \leqslant \tilde{\mathcal{Q}}, \tag{6.9}$$

$$\sum_{j=1}^{M} \pi_{ij}(h)\mathcal{R}(j) \leqslant \tilde{\mathcal{R}}, \tag{6.10}$$

where

$$\begin{aligned}
\Sigma(i) &\triangleq \begin{bmatrix} \Sigma_1(i) & \Psi^T(i) \\ \Psi(i) & \Phi(i) \end{bmatrix}, \quad \ell_5(i) \triangleq \begin{bmatrix} P(i)E(i) & 0 \\ 0 & P(i)E(i) \end{bmatrix}, \\
\Psi(i) &\triangleq \begin{bmatrix} \Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i) \end{bmatrix}^T, \\
\Lambda(i) &\triangleq \begin{bmatrix} \ell_1(i) & 0 & 0 & 0 & 0 & 0 & 0 & \ell_2(i) & \ell_3(i) & \ell_4(i) & \ell_5(i) & \ell_6(i) \end{bmatrix}, \\
\ell_1(i) &\triangleq \begin{bmatrix} -P(i)A(i) & 0 \\ \mathcal{H}(i)C_1(i) & -P(i)A(i) - \mathcal{H}(i)C_1(i) \end{bmatrix}, \\
\ell_2(i) &\triangleq \begin{bmatrix} P(i)D(i) & 0 \\ 0 & P(i)D(i) \end{bmatrix}, \quad \ell_6(i) \triangleq \begin{bmatrix} P(i)C_d(i) \\ \mathcal{H}(i)D_d(i) \end{bmatrix}, \\
\ell_3(i) &\triangleq \begin{bmatrix} P(i)B(i) & 0 \\ \mathcal{H}(i)C_g(i) & P(i)B(i) - \mathcal{H}(i)C_g(i) \end{bmatrix}, \\
\ell_4(i) &\triangleq \begin{bmatrix} P(i)C(i) & 0 \\ 0 & P(i)C(i) \end{bmatrix}, \quad \Phi_{22}(i) \triangleq -\mathcal{P}(i)\mathcal{Q}_3^{-1}\mathcal{P}(i), \\
\Phi(i) &\triangleq \mathrm{diag}\{\Phi_{22}(i), \Phi_{33}(i), \quad \Phi_{44}(i), \Phi_{55}(i), \Phi_{66}(i)\}, \\
\Phi_{33}(i) &\triangleq -\delta_{\tau_1}^{-1}\mathcal{P}(i)\mathcal{S}_1^{-1}\mathcal{P}(i), \quad \Phi_{44}(i) \triangleq -\bar{\tau}_1^{-1}\mathcal{P}(i)\mathcal{S}_2^{-1}\mathcal{P}(i), \\
\Phi_{55}(i) &\triangleq -\delta_{\tau_2}^{-1}\mathcal{P}(i)\mathcal{T}_1^{-1}\mathcal{P}(i), \quad \Phi_{66}(i) \triangleq -\bar{\tau}_2^{-1}\mathcal{P}(i)\mathcal{T}_2^{-1}\mathcal{P}(i), \\
\Sigma_1(i) &\triangleq \Sigma_{11}(i) + \Sigma_{12}(i) + \Sigma_{13}(i) + \Lambda^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda(i), \\
\Sigma_{11}(i) &\triangleq \mathcal{W}_1^T\Bigg(\sum_{i=1}^{2}\mathcal{R}_i + \sum_{i=1}^{2}\mathcal{Q}_i + \mathcal{R}(i) - \mathcal{L}_1\mathcal{Z}_1 + \mathcal{Q}(i) \\
&\quad + \sum_{j=1}^{M}\pi_{ij}(h)\mathcal{P}(j) - \mathcal{Z}_1^T\mathcal{L}_1^T + \bar{\tau}_1\tilde{R} + \bar{\tau}_2\tilde{Q}\Bigg)\mathcal{W}_1,
\end{aligned}$$

$$
\begin{aligned}
\Sigma_{12}(i) &\triangleq -\mathcal{W}_{12}^T \left(E_d^T(i) + E_d(i) - 2\gamma I\right) \mathcal{W}_{12} - \mathcal{W}_1^T \hat{D}_1^T(i)\mathcal{W}_{12} - \mathcal{W}_{12}^T \hat{D}_1(i)\mathcal{W}_1, \\
\Sigma_{13}(i) &\triangleq -\mathcal{W}_2^T\big((1-\tilde{\tau}_1)\mathcal{R}_3(i) + \mathcal{L}_1\mathcal{Z}_2 + \mathcal{Z}_2^T\mathcal{L}_1^T\big)\mathcal{W}_2 - \mathcal{W}_3^T\mathcal{R}_1\mathcal{W}_3 \\
&\quad -\mathcal{W}_4^T\mathcal{R}_2\mathcal{W}_4 - (1-\tilde{\tau}_2)\mathcal{W}_5^T\mathcal{Q}_3\mathcal{W}_5 - \mathcal{W}_6^T\mathcal{Q}_1\mathcal{W}_6 - \mathcal{W}_7^T\mathcal{Q}_2\mathcal{W}_7 \\
&\quad -(1-\tilde{\tau}_2)\mathcal{W}_8^T\mathcal{Q}_3\mathcal{W}_8 - \mathcal{W}_{10}^T\big(\mathcal{Z}_2 + \mathcal{Z}_2^T\big)\mathcal{W}_{10} + \mathcal{W}_1^T\mathcal{L}_2\mathcal{Z}_1\mathcal{W}_9 \\
&\quad -\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_4)^T\mathcal{S}_1(\mathcal{W}_2 - \mathcal{W}_4) + \mathcal{W}_9^T\mathcal{Z}_1^T\mathcal{L}_2^T\mathcal{W}_1 - \bar{\tau}_3^{-1}\mathcal{W}_{11}^T\mathcal{T}_3\mathcal{W}_{11} \\
&\quad +\mathcal{W}_9^T\big(\bar{\tau}_3\mathcal{T}_3 - \mathcal{Z}_1 - \mathcal{Z}_1^T\big)\mathcal{W}_9 + \mathcal{W}_{10}^T\mathcal{Z}_2^T\mathcal{L}_2^T\mathcal{W}_2 + \mathcal{W}_2^T\mathcal{L}_2\mathcal{Z}_2\mathcal{W}_{10} \\
&\quad -\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_3)^T\mathcal{S}_1(\mathcal{W}_2 - \mathcal{W}_3) - \delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_7)^T\mathcal{T}_1(\mathcal{W}_5 - \mathcal{W}_7) \\
&\quad -\delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_6)^T\mathcal{T}_1(\mathcal{W}_5 - \mathcal{W}_6), \\
\Sigma_0(i) &\triangleq \mathrm{diag}\left\{\Sigma_{01}, 0, 0, 0, 0, 0\right\}, \quad \check{T}_1(i) \triangleq \left[\, \tau_e(i)\ 0\ 0\ 0\ 0\ 0 \,\right], \\
\Sigma_{01}(i) &\triangleq -\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_4)^T\mathcal{S}_2(\mathcal{W}_2 - \mathcal{W}_4) - \delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_3)^T\mathcal{S}_2(\mathcal{W}_2 - \mathcal{W}_3) \\
&\quad -\underline{\tau}_1^{-1}(\mathcal{W}_1 - \mathcal{W}_3)^T\mathcal{S}_2(\mathcal{W}_1 - \mathcal{W}_3) - \delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_7)^T\mathcal{T}_2(\mathcal{W}_5 - \mathcal{W}_7) \\
&\quad -\underline{\tau}_2^{-1}(\mathcal{W}_1 - \mathcal{W}_6)^T\mathcal{T}_2(\mathcal{W}_1 - \mathcal{W}_6) - \delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_6)^T\mathcal{T}_2(\mathcal{W}_5 - \mathcal{W}_6), \\
\check{T}_2(i) &\triangleq \Big[\hat{M}^T\mathcal{P}(i)\mathcal{W}_1 \quad \hat{M}^T\mathcal{P}(i) \quad \hat{M}^T\mathcal{P}(i) \quad \hat{M}^T\mathcal{P}(i) \quad \hat{M}^T\mathcal{P}(i) \quad \hat{M}^T\mathcal{P}(i)\Big], \\
\tau_e(i) &\triangleq \left[\, \bar{l}_a(i)\ 0\ 0\ 0\ 0\ 0\ 0\ \bar{l}_d(i)\ \bar{l}_b(i)\ \bar{l}_c(i)\ \bar{l}_e(i)\ \bar{l}_{c_d}(i) \,\right], \\
\mathcal{W}_i &\triangleq \left[\, 0_{2n\times(i-1)2n}\ I_{2n}\ 0_{2n\times(11-i)2n} \,\right], \quad i = 1, \ldots, 10, \\
\mathcal{W}_{11} &\triangleq \left[\, 0_{2n}\ \ldots\ I_{2n}\ 0_{2n\times n} \,\right], \quad \mathcal{W}_{12} \triangleq \left[\, 0_{n\times 2n}\ \ldots\ 0_{n\times 2n}\ I_n \,\right], \\
\bar{L}_1 &\triangleq \mathrm{diag}\{l_1^- l_1^+, l_2^- l_2^+, \ldots, l_n^- l_n^+\}, \quad \mathcal{L}_1 \triangleq \mathrm{diag}\{\bar{L}_1, \bar{L}_1\}, \\
\bar{L}_2 &\triangleq \mathrm{diag}\{l_1^- + l_1^+, l_2^- + l_2^+, \ldots, l_n^- + l_n^+\}, \quad \mathcal{L}_2 \triangleq \mathrm{diag}\{\bar{L}_2, \bar{L}_2\}.
\end{aligned}
$$

Furthermore, when inequality (6.8) *is feasible, a desired filter is given by* (6.5) *with* $H(i) = P^{-1}(i)\mathcal{H}(i)$.

Proof If inequality (6.8) is feasible, then there exists a scalar $0 < \varrho < 1$, such that

$$
\begin{bmatrix} \Sigma(i) + \varrho\Sigma_0(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} < 0.
$$

By the Schur complement formula, we obtain

$$
\Sigma(i) + \varrho\Sigma_0(i) + \varepsilon\check{T}_1^T\check{T}_1 + \varepsilon^{-1}\check{T}_2^T\check{T}_2 < 0.
$$

This inequality, together with Lemma 1.28, implies that

$$
\Sigma(i) + \varrho\Sigma_0(i) + \check{T}_1^T\hat{F}^T(t)\check{T}_2 + \check{T}_2^T\hat{F}(t)\check{T}_1 < 0, \tag{6.11}
$$

for any $\hat{F}(t)$ satisfying $\hat{F}^T(t)\hat{F}(t) \leqslant I$. Clearly, Eq. (6.11) implies that

$$\begin{bmatrix} \Sigma_3(i) & \tilde{\Psi}^T(i) \\ \tilde{\Psi}(i) & \Phi(i) \end{bmatrix} < 0, \quad t \geqslant 0, \tag{6.12}$$

where

$$\begin{aligned} \tilde{\Psi}(i) &\triangleq \begin{bmatrix} \Lambda_1^T(i) & \Lambda_1^T(i) & \Lambda_1^T(i) & \Lambda_1^T(i) & \Lambda_1^T(i) \end{bmatrix}^T, \\ \Lambda_1(i) &\triangleq \Lambda(i) + \mathcal{P}(i)\hat{M}\hat{F}(t)\tau_e, \\ \Sigma_3(i) &\triangleq \Sigma_{11}(i) + \Sigma_{12}(i) + \Sigma_{13}(i) + \Lambda_1^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda_1(i) + \varrho\Sigma_{01}(i). \end{aligned}$$

Set

$$\Lambda_2(i) \triangleq \begin{bmatrix} \hat{A}(i) & 0 & 0 & 0 & 0 & 0 & 0 & \hat{D}(i) & \hat{B}(i) & \hat{C}(i) & \hat{E}(i) \end{bmatrix}.$$

Letting $H(i) = P^{-1}(i)\mathcal{H}(i)$. We have

$$\Lambda_1(i) = \mathcal{P}(i)\begin{bmatrix} \Lambda_2(i) & s\hat{C}_d(i) \end{bmatrix}. \tag{6.13}$$

Then, by Schur complement formula and (6.12), we have

$$\Sigma_3(i) + \begin{bmatrix} \Lambda_2^T(i) \\ \hat{C}_d^T(i) \end{bmatrix} \Xi_1 \begin{bmatrix} \Lambda_2(i) & \hat{C}_d(i) \end{bmatrix} < 0, \tag{6.14}$$

where

$$\Xi_1 \triangleq \mathcal{Q}_3 + \delta_{\tau_1}\mathcal{S}_1 + \bar{\tau}_1\mathcal{S}_2 + \delta_{\tau_2}\mathcal{T}_1 + \bar{\tau}_2\mathcal{T}_2.$$

Since (6.14) can be written as:

$$\begin{bmatrix} \Sigma_5(i) & \Xi(i) - \mathcal{W}_1^T\hat{D}_1^T(i) \\ \Xi^T(i) - \hat{D}_1(i)\mathcal{W}_1 & \Xi_2(i) \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Xi(i) &\triangleq \Lambda_2^T(i)\Xi_1(i)\hat{C}_d(i) + \mathcal{W}_1^T\mathcal{P}(i)\hat{C}_d(i), \\ \Xi_2(i) &\triangleq -E_d^T(i) - E_d(i) + 2\gamma I + \hat{C}_d^T(i)\Xi_1(i)\hat{C}_d(i), \\ \Sigma_5(i) &\triangleq \Sigma_{11}(i) + \Sigma_{13}(i) + \Lambda_2^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda_2(i) + \varrho\Sigma_{01}(i), \end{aligned}$$

it follows that

$$\Sigma_5(i) < 0. \tag{6.15}$$

For the augmented filtering system (6.6) to be strictly exponentially passive, it suffices to show, from Definitions 6.2 and 6.3, that (6.6) is exponentially stable when $\omega(t) = 0$ and satisfies (6.7) under the zero initial condition.

Choose a Lyapunov-Krasovskii functional as follows:

$$V(x(t), \eta_t) \triangleq \sum_{i=1}^{5} V_i(x(t), \eta_t), \tag{6.16}$$

with

$$\begin{aligned}
V_1(x(t), \eta_t) &\triangleq e^T(t)\mathcal{P}(i)e(t) + \int_{-\tau_{1,\eta_t}}^{0}\int_{t+\theta}^{t} e^T(s)\tilde{\mathcal{R}}e(s)\mathrm{d}s\mathrm{d}\theta \\
&\quad + \int_{-\tau_{2,\eta_t}}^{0}\int_{t+\theta}^{t} e^T(s)\tilde{\mathcal{Q}}e(s)\mathrm{d}s\mathrm{d}\theta, \\
V_2(x(t), \eta_t) &\triangleq \int_{t-\tau_{1,\eta_t}}^{t} e^T(s)\mathcal{R}(i)e(s)\mathrm{d}s + \int_{t-\bar{\tau}_2}^{t} e^T(s)\mathcal{Q}_2 e(s)\mathrm{d}s, \\
&\quad + \int_{t-\tau_{2,\eta_t}}^{t} \Big[e^T(s)\mathcal{Q}(i)e(s) + \dot{e}^T(s)\mathcal{Q}_3\dot{e}(s)\Big]\mathrm{d}s \\
&\quad + \int_{t-\underline{\tau}_1}^{t} e^T(s)\mathcal{R}_1 e(s)\mathrm{d}s + \int_{t-\bar{\tau}_1}^{t} e^T(s)\mathcal{R}_2 e(s)\mathrm{d}s \\
&\quad + \int_{t-\underline{\tau}_2}^{t} e^T(s)\mathcal{Q}_1 e(s)\mathrm{d}s \\
V_3(x(t), \eta_t) &\triangleq \int_{-\bar{\tau}_1}^{-\underline{\tau}_1}\int_{t+\theta}^{t} \dot{e}^T(s)\mathcal{S}_1\dot{e}(s)\mathrm{d}s\mathrm{d}\theta + \int_{-\bar{\tau}_1}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)\mathcal{S}_2\dot{e}(s)\mathrm{d}s\mathrm{d}\theta, \\
V_4(x(t), \eta_t) &\triangleq \int_{-\bar{\tau}_2}^{-\underline{\tau}_2}\int_{t+\theta}^{t} \dot{e}^T(s)\mathcal{T}_1\dot{e}(s)\mathrm{d}s\mathrm{d}\theta + \int_{-\bar{\tau}_2}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)\mathcal{T}_2\dot{e}(s)\mathrm{d}s\mathrm{d}\theta, \\
V_5(x(t), \eta_t) &\triangleq \int_{-\bar{\tau}_3}^{0}\int_{t+\theta}^{t} g^T(e(s))\mathcal{T}_3 g(e(s))\mathrm{d}s\mathrm{d}\theta.
\end{aligned}$$

Set

$$\begin{aligned}
\hat{\xi}(t) \triangleq \Big[\, & e^T(t) \quad e^T(t-\tau_{1,\eta_t}) \quad e^T(t-\underline{\tau}_1) \quad e^T(t-\bar{\tau}_1) \\
& e^T(t-\tau_{2,\eta_t}) \quad e^T(t-\underline{\tau}_2) \quad e^T(t-\bar{\tau}_2) \quad \dot{e}^T(t-\tau_{2,\eta_t}) \\
& g^T(e(t)) \quad g^T(e(t-\tau_{1,\eta_t})) \quad \left(\int_{t-\tau_{3,\eta_t}}^{t} g(e(\theta))\mathrm{d}\theta\right)^T \Big]^T.
\end{aligned}$$

Then the first equation in (6.6) can be written as:

$$\dot{e}(t) = \Lambda_2(i)\hat{\xi}(t). \tag{6.17}$$

Firstly, we need to derive the infinitesimal generator $\mathcal{L}$. According to the definition of $\mathcal{L}$ in [10], we have

$$\mathcal{L}V(x(t),\eta_t) \triangleq \lim_{\Delta\to 0}\frac{\mathbf{E}\{V(x(t+\Delta),r_{t+\Delta})|x(t),\eta_t\}-V(x(t),\eta_t)}{\Delta},$$

where Δ is a small positive number. Then, we have

$$\begin{aligned}
\mathcal{L}V_1(x(t),\eta_t) &= 2e^T(t)\mathcal{P}(i)\dot{e}(t)-\int_{t-\tau_{1,\eta_t}}^{t}e^T(s)\tilde{\mathcal{R}}e(s)\mathrm{d}s\\
&\quad +e^T(t)\left(\sum_{j=1}^{M}\pi_{ij}(h)\mathcal{P}(j)+\bar{\tau}_1\tilde{\mathcal{R}}+\bar{\tau}_2\tilde{\mathcal{Q}}\right)e(t)\\
&\quad -\int_{t-\tau_{2,\eta_t}}^{t}e^T(s)\tilde{\mathcal{Q}}e(s)\mathrm{d}s\\
&= \hat{\xi}^T(t)\Big[\mathcal{W}_1^T\mathcal{P}(i)\Lambda(i)+\Lambda^T(i)\mathcal{P}(i)\mathcal{W}_1\\
&\quad +\mathcal{W}_1^T\left(\sum_{j=1}^{M}\pi_{ij}(h)\mathcal{P}(j)+\bar{\tau}_1\tilde{\mathcal{R}}+\bar{\tau}_2\tilde{\mathcal{Q}}\right)\mathcal{W}_1\Bigg]\hat{\xi}(t)\\
&\quad -\int_{t-\tau_{1,\eta_t}}^{t}e^T(s)\tilde{\mathcal{R}}e(s)\mathrm{d}s-\int_{t-\tau_{2,\eta_t}}^{t}e^T(s)\tilde{\mathcal{Q}}e(s)\mathrm{d}s, \qquad (6.18)\\
\mathcal{L}V_2(x(t),\eta_t &\leqslant \hat{\xi}^T(t)\mathcal{W}_1^T\left(\sum_{i=1}^{2}\mathcal{Q}_i+\sum_{i=1}^{2}\mathcal{R}_i+\mathcal{Q}(i)+\mathcal{R}(i)\right)\mathcal{W}_1\hat{\xi}(t)\\
&\quad +\hat{\xi}^T(t)\Lambda_2^T(i)\mathcal{Q}_3\Lambda_2(i)\hat{\xi}(t)+\hat{\xi}^T(t)\Big(-(1-\tilde{\tau}_2)\\
&\quad \times\mathcal{W}_5^T\mathcal{Q}(i)\mathcal{W}_5-\mathcal{W}_7^T\mathcal{Q}_2\mathcal{W}_7\Big)\hat{\xi}(t)+\hat{\xi}^T(t)\big(-(1-\tilde{\tau}_2))\\
&\quad \times\mathcal{W}_8^T\mathcal{Q}_3\mathcal{W}_8-\mathcal{W}_6^T\mathcal{Q}_1\mathcal{W}_6-\mathcal{W}_3^T\mathcal{R}_1\mathcal{W}_3\Big)\hat{\xi}(t)\\
&\quad +\hat{\xi}^T(t)\big(-(1-\tilde{\tau}_1)\mathcal{W}_2^T\mathcal{R}(i)\mathcal{W}_2-\mathcal{W}_4^T\mathcal{R}_2\mathcal{W}_4\big)\hat{\xi}(t)\\
&\quad +\int_{t-\tau_{2,\eta_t}}^{t}e^T(s)\left(\sum_{j=1}^{M}\pi_{ij}(h)\mathcal{Q}(j)\right)e(s)\mathrm{d}s\\
&\quad +\int_{t-\tau_{1,\eta_t}}^{t}e^T(s)\left(\sum_{j=1}^{M}\pi_{ij}(h)\mathcal{R}(j)\right)e(s)\mathrm{d}s, \qquad (6.19)\\
\mathcal{L}V_3(x(t),\eta_t) &= \dot{e}^T(t)(\delta_{\tau_1}\mathcal{S}_1+\bar{\tau}_1\mathcal{S}_2)\dot{e}(t)-\int_{t-\bar{\tau}_1}^{t-\underline{\tau}_1}\dot{e}^T(s)\mathcal{S}_1\dot{e}(s)\mathrm{d}s\\
&\quad -\varrho\int_{t-\bar{\tau}_1}^{t}\dot{e}^T(s)\mathcal{S}_2\dot{e}(s)\mathrm{d}s-(1-\varrho)\int_{t-\bar{\tau}_1}^{t}\dot{e}^T(s)\mathcal{S}_2\dot{e}(s)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \hat{\xi}^T(t)\Lambda_2^T(i)(\delta_{\tau_1}\mathcal{S}_1 + \bar{\tau}_1\mathcal{S}_2)\Lambda_2(i)\hat{\xi}(t) \\
&\quad -\hat{\xi}^T(t)\Big(\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_4)^T\mathcal{S}_1(\mathcal{W}_2 - \mathcal{W}_4) \\
&\quad +\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_3)^T\mathcal{S}_1(\mathcal{W}_2 - \mathcal{W}_3) \\
&\quad +\varrho\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_4)^T\mathcal{S}_2(\mathcal{W}_2 - \mathcal{W}_4) \\
&\quad +\varrho\delta_{\tau_1}^{-1}(\mathcal{W}_2 - \mathcal{W}_3)^T\mathcal{S}_2(\mathcal{W}_2 - \mathcal{W}_3) \\
&\quad +\varrho\underline{\tau}_1^{-1}(\mathcal{W}_1 - \mathcal{W}_3)^T\mathcal{S}_2(\mathcal{W}_1 - \mathcal{W}_3)\Big)\hat{\xi}(t) \\
&\quad -(1-\varrho)\int_{t-\bar{\tau}_1}^{t}\dot{e}^T(s)\mathcal{S}_2\dot{e}(s)\mathrm{d}s,
\end{aligned} \tag{6.20}
$$

$$
\begin{aligned}
\mathcal{L}V_4(x(t),\eta_t) &= \dot{e}^T(t)(\delta_{\tau_2}\mathcal{T}_1 + \bar{\tau}_2\mathcal{T}_2)\dot{e}(t) - \int_{t-\bar{\tau}_2}^{t-\underline{\tau}_2}\dot{e}^T(s)\mathcal{T}_1\dot{e}(s)\mathrm{d}s \\
&\quad -\varrho\int_{t-\bar{\tau}_2}^{t}\dot{e}^T(s)\mathcal{T}_2\dot{e}(s)\mathrm{d}s - (1-\varrho)\int_{t-\bar{\tau}_2}^{t}\dot{e}^T(s)\mathcal{T}_2\dot{e}(s)\mathrm{d}s \\
&\leqslant \hat{\xi}^T(t)\Lambda_2^T(i)(\delta_{\tau_2}\mathcal{T}_1 + \bar{\tau}_2\mathcal{T}_2)\Lambda_2(i)\hat{\xi}(t) \\
&\quad -\hat{\xi}^T(t)\Big(\delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_7)^T\mathcal{T}_1(\mathcal{W}_5 - \mathcal{W}_7) \\
&\quad +\delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_6)^T\mathcal{T}_1(\mathcal{W}_5 - \mathcal{W}_6) \\
&\quad +\varrho\delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_7)^T\mathcal{T}_2(\mathcal{W}_5 - \mathcal{W}_7) \\
&\quad +\varrho\delta_{\tau_2}^{-1}(\mathcal{W}_5 - \mathcal{W}_6)^T\mathcal{T}_2(\mathcal{W}_5 - \mathcal{W}_6) \\
&\quad +\varrho\underline{\tau}_2^{-1}(\mathcal{W}_1 - \mathcal{W}_6)^T\mathcal{T}_2(\mathcal{W}_1 - \mathcal{W}_6)\Big)\hat{\xi}(t) \\
&\quad -(1-\varrho)\int_{t-\bar{\tau}_2}^{t}\dot{e}^T(s)\mathcal{T}_2\dot{e}(s)\mathrm{d}s,
\end{aligned} \tag{6.21}
$$

$$
\begin{aligned}
\mathcal{L}V_5(x(t),\eta_t) &= \bar{\tau}_3 g^T(e(t))\mathcal{T}_3 g(e(t)) - \int_{t-\bar{\tau}_3}^{t} g^T(e(t))\mathcal{T}_3 g(e(t))\mathrm{d}s \\
&\leqslant \bar{\tau}_3\hat{\xi}^T(t)\mathcal{W}_9^T\mathcal{T}_3\mathcal{W}_9\hat{\xi}(t) - \bar{\tau}_3^{-1}\hat{\xi}^T(t)\mathcal{W}_{11}^T\mathcal{T}_3\mathcal{W}_{11}\hat{\xi}(t).
\end{aligned} \tag{6.22}
$$

For the function $g(s)$, by using (6.3), we have

$$
\begin{aligned}
\left[g_i(e_i(t)) - l_i^- e_i(t)\right]\left[g_i(e_i(t)) - l_i^+ e_i(t)\right] &\leqslant 0, \\
\left[g_i(e_i(t-\tau(t))) - l_i^- e_i(t-\tau(t))\right]\left[g_i(e_i(t-\tau(t))) - l_i^+ e_i((t-\tau(t))\right] &\leqslant 0.
\end{aligned}
$$

Then, for $Z_j \triangleq \mathrm{diag}\{z_{1j}, \ldots, z_{nj}\} \geqslant 0, j = 1, 2$, we obtain

$$
\begin{aligned}
0 \leqslant & -2\sum_{i=1}^{n} z_{i1}[g_i(e_i(t)) - l_i^- e_i(t)][g_i(e_i(t)) - l_i^+ e_i(t)] \\
= & -2\hat{\xi}^T(t)\mathcal{W}_9^T Z_1 \mathcal{W}_9 \hat{\xi}(t) + 2\hat{\xi}^T(t)\mathcal{W}_1^T L_2 Z_1 \mathcal{W}_9 \hat{\xi}(t) \\
& -2\hat{\xi}^T(t)\mathcal{W}_1^T L_1 Z_1 \mathcal{W}_1 \hat{\xi}(t),
\end{aligned} \tag{6.23}
$$

$$
\begin{aligned}
0 \leqslant & -2\sum_{i=1}^{n} z_{i2}[g_i(e_i(t-\tau(t))) - l_i^- e_i(t-\tau(t))] \\
& \times [g_i(x_i(t-\tau(t))) - l_i^+ e_i(t-\tau(t))] \\
= & -2\hat{\xi}^T(t)\mathcal{W}_{10}^T Z_2 \mathcal{W}_{10} \hat{\xi}(t) + 2\hat{\xi}^T(t)\mathcal{W}_2^T L_2 Z_2 \mathcal{W}_{10} \hat{\xi}(t) \\
& -2\hat{\xi}^T(t)\mathcal{W}_2^T L_1 Z_2 \mathcal{W}_2 \hat{\xi}(t),
\end{aligned} \tag{6.24}
$$

where

$$
\begin{aligned}
L_1 &\triangleq \text{diag}\{\bar{L_1}, \bar{L_1}\}, \quad L_2 = \text{diag}\{\bar{L_2}, \bar{L_2}\}, \\
\bar{L}_1 &\triangleq \text{diag}\left\{l_1^- l_1^+, l_2^- l_2^+, \ldots, l_n^- l_n^+\right\}, \\
\bar{L}_2 &\triangleq \text{diag}\left\{l_1^- + l_1^+, l_2^- + l_2^+, \ldots, l_n^- + l_n^+\right\}.
\end{aligned}
$$

It follows from (6.18)–(6.24) that

$$
\begin{aligned}
\mathcal{L}V(x(t), \eta_t) = & \sum_{i=1}^{5} \mathcal{L}V_i(x(t), \eta_t) \\
& \leqslant \hat{\xi}^T(t)\Sigma_5(i)\hat{\xi}(t) - (1-\delta)\int_{t-\tau_2}^{t} \dot{e}^T(s)\mathcal{S}_2\dot{e}(s)\mathrm{d}s \\
& \quad -(1-\delta)\int_{t-h_2}^{t} \dot{e}^T(s)\mathcal{T}_2\dot{e}(s)\mathrm{d}s.
\end{aligned} \tag{6.25}
$$

Applying Dynkin's formula, we have

$$
\begin{aligned}
& \mathbf{E}\left\{e^{2\alpha t}V(x(t), \eta_t)\right\} - \mathbf{E}\left\{V(x_0, r_0)\right\} \\
& \quad = \mathbf{E}\int_0^t e^{2\alpha \upsilon}\big(2\alpha V(x(\upsilon), r_\upsilon) + \mathcal{L}V(x(\upsilon), r_\upsilon)\big)\mathrm{d}\upsilon,
\end{aligned} \tag{6.26}
$$

where $\alpha > 0$ is a parameter to be determined.

On the other hand, for any $t > 0$, it follows from (6.25) that there exist positive scalars κ_i $(i = 1, \ldots, 5)$ such that the following inequality holds:

$$
\int_0^t e^{2\alpha t}\Big(2\alpha V(x(\upsilon), r_\upsilon) + \mathcal{L}V(x(\upsilon), r_\upsilon)\Big)\mathrm{d}\upsilon \leqslant \Theta(\eta_t, t),
$$

where

$$\begin{aligned}\Theta(\eta_t, t) \triangleq & \left(2\alpha\lambda_{\max}(\mathcal{P}(i)) - \lambda_{\min}(-\Sigma_5)\right)\int_0^t e^{2\alpha\upsilon}\|e(\upsilon)\|^2\mathrm{d}\upsilon \\ & +\left(2\alpha\kappa_3 - (1-\varrho)\lambda_{\min}(\mathcal{S}_2)\right)\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_1}^{\upsilon}\|\dot{e}(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ & +\left(2\alpha\kappa_4 - (1-\varrho)\lambda_{\min}(\mathcal{T}_2)\right)\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_1}^{\upsilon}\|\dot{e}(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ & +2\alpha\kappa_5\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_3}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ & +2\alpha\kappa_1\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_2}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ & +2\alpha\kappa_2\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_1}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon.\end{aligned}$$

Subsequently, by exchanging integral order, we obtain

$$\begin{aligned}&\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_2}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ &\quad \leqslant \int_{-\bar{\tau}_2}^{t}\int_s^{s+\bar{\tau}_2} e^{2\alpha\upsilon}\|e(s)\|^2\mathrm{d}\upsilon\mathrm{d}s \\ &\quad \leqslant \int_{-\bar{\tau}_2}^{t}\bar{\tau}_2 e^{2\alpha(s+\bar{\tau}_2)}\|e(s)\|^2\mathrm{d}s \\ &\quad \leqslant \bar{\tau}_2 e^{2\alpha\bar{\tau}_2}\int_{-\bar{\tau}_2}^{0}\|e(t)\|^2\mathrm{d}t + \bar{\tau}_2 e^{2\alpha\bar{\tau}_2}\int_0^t e^{2\alpha s}\|e(s)\|^2\mathrm{d}s.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_1}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ &\quad \leqslant \bar{\tau}_1 e^{2\alpha\bar{\tau}_1}\int_{-\bar{\tau}_1}^{0}\|e(t)\|^2\mathrm{d}t + \bar{\tau}_1 e^{2\alpha\bar{\tau}_1}\int_0^t e^{2\alpha s}\|e(s)\|^2\mathrm{d}s, \\ &\int_0^t e^{2\alpha\upsilon}\int_{\upsilon-\bar{\tau}_3}^{\upsilon}\|e(s)\|^2\mathrm{d}s\mathrm{d}\upsilon \\ &\quad \leqslant \bar{\tau}_3 e^{2\alpha\bar{\tau}_3}\int_{-\bar{\tau}_3}^{0}\|e(t)\|^2\mathrm{d}t + \bar{\tau}_3 e^{2\alpha\bar{\tau}_3}\int_0^t e^{2\alpha s}\|e(s)\|^2\mathrm{d}s.\end{aligned}$$

Choose the appropriate $\alpha > 0$ such that

$$
\begin{aligned}
2\alpha\lambda_{\max}\big(\mathcal{P}(i)\big) - \lambda_{\min}(-\Sigma_5(i)) + 2\alpha\tau_2\kappa_2 \mathrm{e}^{2\alpha\tau_2} & \\
+2\alpha h_2\kappa_1 \mathrm{e}^{2\alpha h_2} + 2\alpha r_2\kappa_5 \mathrm{e}^{2\alpha r_2} &< 0, \\
2\alpha\kappa_4 - (1-\delta)\lambda_{\min}(\mathcal{T}_2) &< 0, \\
2\alpha\kappa_3 - (1-\delta)\lambda_{\min}(\mathcal{S}_2) &< 0.
\end{aligned}
$$

Then, the following can be derived:

$$
\begin{aligned}
&\mathbf{E}\left\{e^{2\alpha t}V(x(t),\eta_t)\right\} - \mathbf{E}\left\{V(x(0),r_0)\right\} \\
&\quad \leqslant 2\alpha\left(\kappa_2\bar{\tau}_1^2 e^{2\alpha\bar{\tau}_1} + \kappa_1\bar{\tau}_2^2 e^{2\alpha\bar{\tau}_2} + \kappa_5\bar{\tau}_3^2 e^{2\alpha\bar{\tau}_3}\right)\mathbf{E}\left\{\|\hat{\phi}\|_2^2\right\}.
\end{aligned}
$$

On the other hand, there exist a scalar $\chi > 0$ such that

$$
\mathbf{E}\left\{V(x(0),r_0)\right\} \leqslant \chi\mathbf{E}\left\{\|\hat{\phi}\|^2\right\},
$$

which yields

$$
\lambda_{\min}(\mathcal{P}(i))\mathbf{E}\left\{\|e(t)\|^2\right\} \leqslant \mathbf{E}\left\{V(x(t),\eta_t)\right\} \leqslant e^{-2\alpha t}\bar{\chi}\mathbf{E}\left\{\|\hat{\phi}\|^2\right\},
$$

where

$$
\bar{\chi} \triangleq 2\alpha\left(\kappa_2\bar{\tau}_1^2 e^{2\alpha\bar{\tau}_1} + \kappa_1\bar{\tau}_2^2 e^{2\alpha\bar{\tau}_2} + \kappa_5\bar{\tau}_3^2 e^{2\alpha\bar{\tau}_3}\right) + \chi.
$$

Thus, for the arbitrariness of $t > 0$, we obtain

$$
\mathbf{E}\left\{\|e(t)\|_2\right\} \leqslant \sqrt{\frac{\bar{\chi}}{\lambda_{\min}(\mathcal{P}(i))}}\mathrm{e}^{-\alpha t}\mathbf{E}\left\{\|\hat{\phi}\|_2\right\}. \tag{6.27}
$$

Therefore, according to Definition 6.2, system (6.6) is exponentially mean-square stable.

Next, we will show that condition (6.7) is satisfied for the filtering error dynamics under the zero initial condition. By using a similar approach to get (6.18)–(6.22), we can prove that the derivative of $V(x(t),\eta_t)$ along the trajectories of (6.6) satisfies

$$
\begin{aligned}
\mathcal{L}V(x(t),\eta_t) \leqslant \xi^T(t)\Big(&\Sigma_4(i) + \Lambda_3^T(i)(\mathcal{Q}_3 + \delta_{\tau_1}\mathcal{S}_1 \\
&+\bar{\tau}_1\mathcal{S}_2 + \delta_{\tau_2}\mathcal{T}_1 + \bar{\tau}_2\mathcal{T}_2)\Lambda_3(i)\Big)\xi(t)
\end{aligned} \tag{6.28}
$$

with

$$
\xi(t) \triangleq \begin{bmatrix} \hat{\xi}(t) \\ \omega(t) \end{bmatrix}, \quad \Lambda_3(i) \triangleq \begin{bmatrix} \Lambda_2(i) & \hat{C}_d(i) \end{bmatrix}.
$$

Introduce the following index:

$$J(T) \triangleq \int_0^T [\omega^T(t)\delta(t) - \gamma\omega^T(t)\omega(t)]\mathrm{d}t, \quad \forall T \geqslant 0.$$

Then

$$\begin{aligned} -2J(T) = \int_0^T \Big[&\mathcal{L}V(x(t),\eta_t) - \omega^T(t)\delta(t) - \delta^T(t)\omega(t) \\ &+2\gamma\omega^T(t)\omega(t)\Big]\mathrm{d}t - V(x(t),\eta_t)|_0^T. \end{aligned} \tag{6.29}$$

By combining of (6.28) and (6.29), we obtain from $V(x(0), r_0) = 0$ that

$$-2J(T) \leqslant \int_0^T \xi^T(t)\overline{\Lambda}(i)\xi(t)\mathrm{d}t - V(x(T),\eta_t),$$

where

$$\overline{\Lambda}(i) \triangleq \Sigma_3(i) + \Lambda_3^T(i)(\mathcal{Q}_3 + \delta_{\tau_1}\mathcal{S}_1 + \bar{\tau}_1\mathcal{S}_2 + \delta_{\tau_2}\mathcal{T}_1 + \bar{\tau}_2\mathcal{T}_2)\Lambda_3(i).$$

Then, applying Schur complement formula and (6.12), we have $\overline{\Lambda}(i) < 0$, and hence $J(T) > 0$, i.e., condition (6.7) holds. Thus, by Definition 6.3, it follows that the augmented filtering system (6.6) is strictly exponentially passive with performance level $\gamma > 0$. This completes the proof. ■

6.3.2 Passive Filter Design

With the time-varying term $\pi_{ij}(h)$, it becomes difficult to design the filter since Theorem 6.4 contains an infinite number of inequalities. In the following, a new theorem is presented to ensure the existence of such an exponential passive filter.

Theorem 6.5 *For given scalars* $\underline{\tau}_1$, $\bar{\tau}_1$, $\tilde{\tau}_1$, $\underline{\tau}_2$, $\bar{\tau}_2$, $\tilde{\tau}_2$, $\bar{\tau}_3$, *and* r_3, *system* (6.6) *is strictly exponentially passive with desired disturbance attenuation* $\gamma > 0$, *if there exist real symmetric positive-definite matrices* $\mathcal{P}(i) = \begin{bmatrix} P(i) & 0 \\ 0 & P(i) \end{bmatrix}$, $\mathcal{Q}(i)$, $\mathcal{R}(i)$, $\mathcal{R}_\nu$, $\mathcal{S}_\nu$, $(\nu = 1, 2)$, $\mathcal{Q}_s$, $\mathcal{T}_s$, $(s = 1, 2, 3)$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{R}}$, *and diagonal matrices* $\mathcal{Z}_1 > 0$, $\mathcal{Z}_2 > 0$, *and* $\mathcal{H}(i)$ *and positive scalar* ε *such that* (6.9), (6.10), *and the following inequalities hold,*

$$\begin{bmatrix} \Sigma^{(r)}(i) + \Sigma_0(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} < 0, \quad r = 1, 2, \ldots, M, \tag{6.30}$$

$$\Lambda^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda(i) + \mathcal{W}_1^T\mathcal{P}(j)\mathcal{W}_1 < 0, \quad j \in \mathcal{M}_{uk}^i, \quad j \neq i, \tag{6.31}$$
$$\Lambda^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda(i) + \mathcal{W}_1^T\mathcal{P}(j)\mathcal{W}_1 > 0, \quad j \in \mathcal{M}_{uk}^i, \quad j = i, \tag{6.32}$$

where

$$\Sigma^{(r)}(i) \triangleq \begin{bmatrix} \Sigma_1^{(r)}(i) & \Psi^T(i) \\ \Psi(i) & \Phi(i) \end{bmatrix},$$

$$\Sigma_1^{(r)}(i) \triangleq \Sigma_{11}^{(r)}(i) + \Sigma_{12}(i) + \Sigma_{13}(i) + \left(1 + \sum_{j\in\mathcal{M}_{uc}^i} \pi_{ij}^{(r)}\right)\left(\Lambda^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda(i)\right),$$

$$\Sigma_{11}^{(r)}(i) \triangleq \mathcal{W}_1^T\left(\sum_{i=1}^{2}\mathcal{R}_i + \sum_{i=1}^{2}\mathcal{Q}_i + \mathcal{R}(i) - \mathcal{L}_1\mathcal{Z}_1 + \mathcal{Q}(i)\right.$$
$$\left. + \sum_{j\in\mathcal{M}_{uc}^i}\pi_{ij}^{(r)}\mathcal{P}(j) - \mathcal{Z}_1^T\mathcal{L}_1^T + \bar{\tau}_1\tilde{R} + \bar{\tau}_2\tilde{Q}\right)\mathcal{W}_1.$$

Proof Since $\sum_{j=1}^{M}\pi_{ij}(h) = 0$, we can rewrite the left-hand side of (6.8) as:

$$\Upsilon(i) \triangleq \begin{bmatrix} \Upsilon_1(i) + \Sigma_0(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} + \sum_{j=1}^{N}\pi_{ij}(h)\begin{bmatrix} \Omega_r(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\Omega_r(i) \triangleq \Lambda^T(i)\mathcal{W}_1 + \mathcal{W}_1^T\Lambda(i), \quad \Upsilon_1(i) \triangleq \begin{bmatrix} \Upsilon_{11}(i) & \Psi^T(i) \\ \Psi(i) & \Phi(i) \end{bmatrix},$$
$$\Upsilon_{11}(i) \triangleq \Sigma_{11}(i) + \Sigma_{12}(i) + \Sigma_{13}(i) + \Omega_r(i),$$

where $\Sigma_0(i)$, $\check{T}_1$, $\check{T}_2$, $\Lambda(i)$, $\mathcal{W}_1$, $\Psi(i)$, $\Phi(i)$, $\Sigma_{11}(i)$, $\Sigma_{12}(i)$, and $\Sigma_{13}(i)$ are defined as in Theorem 6.4.

Thus, from (6.1), we have

$$\Upsilon(i) \triangleq \begin{bmatrix} \tilde{\Upsilon}_1(i) + \Sigma_0(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} + \sum_{j\in\mathcal{M}_{uk}^i}\pi_{ij}^{(r)}\begin{bmatrix} \Omega_r(i) + \mathcal{P}(j) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Upsilon}_{11}(i) \triangleq \Sigma_{11}^{(r)}(i) + \Sigma_{12}(i) + \Sigma_{13}(i) + \left(1 + \sum_{j\in\mathcal{M}_{uc}^i}\pi_{ij}^{(r)}\right)\Omega_r(i),$$

$$\tilde{\Upsilon}_1(i) \triangleq \begin{bmatrix} \tilde{\Upsilon}_{11}(i) & \Psi^T(i) \\ \Psi(i) & \Phi(i) \end{bmatrix}.$$

Then, for $j \in \mathcal{M}_{uk}^i$ and if $i \in \mathcal{M}_{uc}^i$, $\Upsilon(i) < 0$ can be guaranteed by (6.30), (6.31) and $\pi_{ij}(h) \geqslant 0$, $i, j \in \mathcal{M}$, $i \neq j$. On the other hand, for $j \in \mathcal{M}_{uk}^i$ and if $i \notin \mathcal{M}_{uc}^i$, $\Upsilon(i)$ can be further expressed as

$$\Upsilon(i) \triangleq \begin{bmatrix} \tilde{\Upsilon}_1(i) + \Sigma_0(i) & \varepsilon \check{T}_1^T & \check{T}_2^T \\ \varepsilon \check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2 & 0 & -\varepsilon I \end{bmatrix} + \sum_{j \in \mathcal{M}_{uk}^i, j \neq i} \pi_{ij}^{(r)} \begin{bmatrix} \Omega_r(i) + \mathcal{P}(j) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+\pi_{ii}^{(r)} \begin{bmatrix} \Omega_r(i) + \mathcal{P}(j) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\pi_{ii}(h) = -\sum_{j=1, j \neq i}^{N} \pi_{ij}(h) < 0$, then according to (6.30), (6.31) and (6.32), one can also obtain $\Upsilon(i) < 0$. This completes the proof. ■

6.3.3 Further Extensions

Note that Theorem 6.5 is derived from the augmented filtering system (6.6), which requires more computing time to design the system parameters. Next, we will offer a novel criterion for the existence of an exponentially passive filter of the form (6.5).

The nominal system of NTNN (6.2) can be described as follows:

$$\begin{cases} \dot{x}(t) = -A(i)x(t) + B(i)g(x(t)) + C(i)g(x(t - \tau_{1,\eta_t})) \\ \qquad\quad +D(i)\dot{x}(t - \tau_{2,\eta_t}) + E(i) \int_{t-\tau_{3,\eta_t}}^{t} g(x(\theta))\mathrm{d}` \\ \qquad\quad +C_d(i)\omega(t), \\ y(t) = C_1(i)x(t) + C_g(i)g(x(t)) + D_d(i)\omega(t), \\ z(t) = D_1(i)x(t) + E_d(i)\omega(t), \\ x(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{6.33}$$

For this system, Theorem 6.4 is still available by setting $\check{T}_1 = 0$ and $\check{T}_2 = 0$. Define

$$\bar{e}(t) \triangleq x(t) - \hat{x}(t).$$

Combining (6.5) with (6.33), we obtain the augmented filtering system as follows:

$$\begin{cases} \dot{\bar{e}}(t) = -(A(i) + H(i)C_1(i))\bar{e}(t) + C(i)\varphi(\bar{e}(t - \tau_{1,\eta_t})) \\ \qquad\quad +(B(i) - H(i)C_g(i))\varphi(\bar{e}(t)) + D(i)\dot{\bar{e}}(t - \tau_{2,\eta_t}) \\ \qquad\quad +(C_d(i) - H(i)D_d(i))\omega(t) + E(i) \int_{t-\tau_{3,\eta_t}}^{t} \varphi(\bar{e}(s))\mathrm{d}s, \\ \bar{\delta}(t) = D_1(i)\bar{e}(t) + E_d(i)\omega(t), \\ \bar{e}(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{6.34}$$

where $\bar{\delta}(t)$ is the output error, and

$$\varphi(\bar{e}(t)) \triangleq \left[\varphi_1(\bar{e}_1(t))\ \varphi_2(\bar{e}_2(t))\ \ldots\ \varphi_n(\bar{e}_n(t))\right)^T,$$
$$\varphi_j(\bar{e}_j(t)) \triangleq g_j(x_j(t)) - g_j(x_j(t) - \bar{e}_j(t)), \quad j = 1, \ldots, n.$$

From (6.3), we have

$$l_i^- \leqslant \frac{\varphi_j(\bar{e}_j(t))}{\bar{e}_j(t)} \leqslant l_i^+, \quad j = 1, 2, \ldots, n.$$

Theorem 6.6 *For given scalars $\underline{\tau}_1$, $\bar{\tau}_1$, $\tilde{\tau}_1$, $\underline{\tau}_2$, $\bar{\tau}_2$, $\tilde{\tau}_2$, $\bar{\tau}_3$ and r_3, system (6.34) is strictly exponentially passive, if there exist real symmetric positive-definite matrices $\bar{P}(i)$, $\bar{\mathcal{Q}}(i)$, $\bar{\mathcal{R}}(i)$, $\bar{\mathcal{R}}_\nu$, $\bar{\mathcal{S}}_\nu$, $(\nu = 1, 2)$, $\bar{\mathcal{Q}}_s$, $\bar{\mathcal{T}}_s$, $(s = 1, 2, 3)$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{R}}$, and diagonal matrices $\bar{\mathcal{Z}}_1 > 0$, $\bar{\mathcal{Z}}_2 > 0$, and $\bar{\mathcal{H}}(i)$ such that (6.9), (6.10), and the following inequality hold,*

$$\begin{cases} \begin{bmatrix} \tilde{\Lambda}^{(r)}(i) & \bar{\Psi}^T(i) \\ \bar{\Psi}(i) & -\bar{\Phi}(i) \end{bmatrix} < 0, \quad r = 1, 2, \ldots, M, \\ \tilde{\Lambda}^T(i)\tilde{\mathcal{W}}_1 + \tilde{\mathcal{W}}_1^T\tilde{\Lambda}(i) + \tilde{\mathcal{W}}_1^T\bar{P}(j)\tilde{\mathcal{W}}_1 < 0, \quad j \in \mathcal{M}_{uk}^i, \quad j \neq i, \\ \tilde{\Lambda}^T(i)\tilde{\mathcal{W}}_1 + \tilde{\mathcal{W}}_1^T\tilde{\Lambda}(i) + \tilde{\mathcal{W}}_1^T\bar{P}(j)\tilde{\mathcal{W}}_1 > 0, \quad j \in \mathcal{M}_{uk}^i, \quad j = i, \end{cases} \tag{6.35}$$

where

$$\begin{aligned}
\bar{\Psi}^T(i) &\triangleq \left[\tilde{\Lambda}^T(i)\ \tilde{\Lambda}^T(i)\ \tilde{\Lambda}^T(i)\ \tilde{\Lambda}^T(i)\ \tilde{\Lambda}^T(i)\right], \quad \bar{\Phi}_{22}(i) \triangleq -\bar{P}(i)\mathcal{Q}_3^{-1}\bar{P}(i), \\
\bar{\Phi}(i) &\triangleq \mathrm{diag}\{\bar{\Phi}_{22}(i), \bar{\Phi}_{33}(i), \bar{\Phi}_{44}(i), \bar{\Phi}_{55}(i), \bar{\Phi}_{66}(i)\}, \\
\bar{\Phi}_{33}(i) &\triangleq -\delta_{\tau_1}^{-1}\bar{P}(i)\mathcal{S}_1^{-1}\bar{P}(i), \quad \bar{\Phi}_{44}(i) \triangleq -\bar{\tau}_1^{-1}\bar{P}(i)\mathcal{S}_2^{-1}\bar{P}(i), \\
\bar{\Phi}_{55}(i) &\triangleq -\delta_{\tau_2}^{-1}\bar{P}(i)\mathcal{T}_1^{-1}\bar{P}(i), \quad \bar{\Phi}_{66}(i) \triangleq -\bar{\tau}_2^{-1}\bar{P}(i)\mathcal{T}_2^{-1}\bar{P}(i), \\
\tilde{\Lambda}^{(r)}(i) &\triangleq \tilde{\Lambda}_1^{(r)}(i) - \tilde{\mathcal{W}}_{12}^T\left(E_d^T(i) + E_d(i) - 2\gamma I\right)\tilde{\mathcal{W}}_{12} \\
&\quad -\tilde{\mathcal{W}}_1^T D_1^T(i)\tilde{\mathcal{W}}_{12} - \tilde{\mathcal{W}}_{12}^T D_1(i)\tilde{\mathcal{W}}_1, \\
\tilde{\Lambda}_1^{(r)}(i) &\triangleq \tilde{\mathcal{W}}_1^T\left(\sum_{i=1}^{2}\bar{\mathcal{R}}_i + \sum_{i=1}^{2}\bar{\mathcal{Q}}_i + \bar{\mathcal{R}}(i) - \mathcal{L}_1\mathcal{Z}_1 + \bar{\mathcal{Q}}(i)\right. \\
&\quad \left. + \sum_{j\in\mathcal{M}_{uc}^i}\pi_{ij}^{(r)}\bar{P}(j) - \mathcal{Z}_1^T\mathcal{L}_1^T + \bar{\tau}_1\tilde{R} + \bar{\tau}_2\tilde{Q}\right)\tilde{\mathcal{W}}_1 - \tilde{\mathcal{W}}_3^T\bar{\mathcal{R}}_1\tilde{\mathcal{W}}_3 \\
&\quad -\delta_{\tau_1}^{-1}(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_4)^T\mathcal{S}_1(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_4) - \delta_{\tau_1}^{-1}(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_3)^T\mathcal{S}_1(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_3) \\
&\quad -\delta_{\tau_1}^{-1}(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_4)^T\mathcal{S}_2(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_4) - \delta_{\tau_1}^{-1}(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_3)^T\mathcal{S}_2(\tilde{\mathcal{W}}_2 - \tilde{\mathcal{W}}_3) \\
&\quad -\underline{\tau}_1^{-1}(\tilde{\mathcal{W}}_1 - \tilde{\mathcal{W}}_3)^T\mathcal{S}_2(\tilde{\mathcal{W}}_1 - \tilde{\mathcal{W}}_3) - \delta_{\tau_2}^{-1}(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_7)^T\mathcal{T}_1(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_7) \\
&\quad -\delta_{\tau_2}^{-1}(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_6)^T\mathcal{T}_1(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_6) - \delta_{\tau_2}^{-1}(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_7)^T\mathcal{T}_2(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_7) \\
&\quad -\underline{\tau}_2^{-1}(\tilde{\mathcal{W}}_1 - \tilde{\mathcal{W}}_6)^T\mathcal{T}_2(\tilde{\mathcal{W}}_1 - \tilde{\mathcal{W}}_6) - \delta_{\tau_2}^{-1}(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_6)^T\mathcal{T}_2(\tilde{\mathcal{W}}_5 - \tilde{\mathcal{W}}_6)
\end{aligned}$$

$$
\begin{aligned}
&-\tilde{\mathcal{W}}_2^T\Big((1-\tilde{\tau}_1)\bar{\mathcal{R}}(i)+\mathcal{L}_1\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_2^T\mathcal{L}_1^T\Big)\tilde{\mathcal{W}}_2-\tilde{\mathcal{W}}_4^T\bar{\mathcal{R}}_2\tilde{\mathcal{W}}_4\\
&+\left(1+\sum_{j\in\mathcal{M}_{uc}^i}\pi_{ij}^{(r)}\right)\Big(\tilde{\Lambda}^T(i)\tilde{\mathcal{W}}_1+\tilde{\mathcal{W}}_1^T\tilde{\Lambda}(i)\Big)-\tilde{\mathcal{W}}_6^T\bar{\mathcal{Q}}_1\tilde{\mathcal{W}}_6\\
&-(1-\tilde{\tau}_2)\tilde{\mathcal{W}}_8^T\bar{\mathcal{Q}}_3\tilde{\mathcal{W}}_8-\tilde{\mathcal{W}}_7^T\bar{\mathcal{Q}}_2\tilde{\mathcal{W}}_7-(1-\tilde{\tau}_2)\tilde{\mathcal{W}}_5^T\bar{\mathcal{Q}}_3\tilde{\mathcal{W}}_5\\
&-\tilde{\mathcal{W}}_{10}^T(\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_2^T)\tilde{\mathcal{W}}_{10}-\bar{\tau}_3^{-1}\tilde{\mathcal{W}}_{11}^T\mathcal{T}_3\tilde{\mathcal{W}}_{11}+\tilde{\mathcal{W}}_1^T\mathcal{L}_2\bar{\mathcal{Z}}_1\tilde{\mathcal{W}}_9\\
&+\tilde{\mathcal{W}}_9^T\bar{\mathcal{Z}}_1^T\mathcal{L}_2^T\tilde{\mathcal{W}}_1+\tilde{\mathcal{W}}_2^T\mathcal{L}_2\bar{\mathcal{Z}}_2\tilde{\mathcal{W}}_{10}+\tilde{\mathcal{W}}_{10}^T\bar{\mathcal{Z}}_2^T\mathcal{L}_2^T\tilde{\mathcal{W}}_2\\
&+\tilde{\mathcal{W}}_9^T(\bar{\tau}_3\mathcal{T}_3-\mathcal{Z}_1-\mathcal{Z}_1^T)\tilde{\mathcal{W}}_9,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Lambda}(i) \triangleq \big[&-(P(i)A(i)+\bar{\mathcal{H}}(i)C_1(i))\ 0\ 0\ 0\ 0\ 0\ 0\\
&P(i)D(i)\ P(i)B(i)-\bar{\mathcal{H}}(i)C_g(i)\ P(i)C(i)\\
&P(i)E(i)\quad P(i)C_d(i)-\bar{\mathcal{H}}(i)D_d(i)\big],\\
\tilde{\mathcal{W}}_i \triangleq \big[&0_{n\times(i-1)}\quad I_n\quad 0_{n\times(12-i)}\big],\quad i=1,\ldots,12,
\end{aligned}
$$

and δ_{τ_1}, δ_{τ_2}, $\mathcal{L}_1$, *and* $\mathcal{L}_2$ *are defined as in Theorem* 6.4.

Furthermore, when inequality (6.35) *is feasible, a desired filter is given by* (6.5) *with* $H(i)=\bar{P}^{-1}(i)\bar{\mathcal{H}}(i)$.

Proof The desired result can be worked out along the same lines as those in the proofs of Theorems 6.4 and 6.5. ■

Remark 6.7 We consider system (6.2) as a parameter uncertainty system, whereas system (6.33) is a nominal system. The main results in this chapter are obtained based on various error dynamic systems, i.e., based on output errors $z(t)-\hat{z}(t)$, Theorem 6.5 is established for the error dynamics (6.6), and Theorem 6.6 is derived for the error dynamic system (6.34) based on state errors $x(t)-\hat{x}(t)$. We will compare the performance of these error dynamic systems via numerical simulations. It will reveal that the filter design method presented in Theorem 6.6 is more efficient than that of Theorem 6.5. ♦

6.3.4 Convex Optimization Algorithm for Filter Design

It is worth noting that the convex optimization algorithm cannot be used to find a minimum γ, since the nonlinearity of the existing conditions. We can solve this problem by using the cone complementarity linearization algorithm. Before discussing the algorithm we introduce the real symmetric positive-definite matrices $\tilde{\mathcal{P}}(i)$, $\mathcal{G}_s$, $\tilde{\mathcal{G}}_s$, $(s=1,\ldots,5)$, $\tilde{\mathcal{S}}_i$, $\tilde{\mathcal{T}}_i$, $(i=1,2)$ and $\tilde{\mathcal{Q}}_3$ satisfying

$$\left.\begin{array}{l}\begin{bmatrix}-\tilde{\mathcal{G}}_1 & \tilde{\mathcal{P}}(i)\\ \tilde{\mathcal{P}}(i) & -\tilde{\mathcal{Q}}_3\end{bmatrix}\leqslant 0,\quad \begin{bmatrix}-\tilde{\mathcal{G}}_2 & \tilde{\mathcal{P}}(i)\\ \tilde{\mathcal{P}}(i) & -\delta_{\tau_1}^{-1}\tilde{\mathcal{S}}_1\end{bmatrix}\leqslant 0,\quad \begin{bmatrix}-\tilde{\mathcal{G}}_3 & \tilde{\mathcal{P}}(i)\\ \tilde{\mathcal{P}}(i) & -\bar{\tau}_1^{-1}\tilde{\mathcal{S}}_2\end{bmatrix}\leqslant 0,\\ \begin{bmatrix}-\tilde{\mathcal{G}}_4 & \tilde{\mathcal{P}}(i)\\ \tilde{\mathcal{P}}(i) & -\delta_{\tau_2}^{-1}\tilde{\mathcal{T}}_1\end{bmatrix}\leqslant 0,\quad \begin{bmatrix}-\tilde{\mathcal{G}}_5 & \tilde{\mathcal{P}}(i)\\ \tilde{\mathcal{P}}(i) & -\bar{\tau}_2^{-1}\tilde{\mathcal{T}}_2\end{bmatrix}\leqslant 0,\end{array}\right\} \tag{6.36}$$

with

$$\begin{array}{l}\mathcal{P}(i)\tilde{\mathcal{P}}(i)=I,\quad \mathcal{G}_s\tilde{\mathcal{G}}_s=I(s=1,\ldots,5),\quad \mathcal{S}_1\tilde{\mathcal{S}}_1=I\\ \mathcal{S}_2\tilde{\mathcal{S}}_2=I,\quad \mathcal{Q}_3\tilde{\mathcal{Q}}_3=I,\quad \mathcal{T}_1\tilde{\mathcal{T}}_1=I,\quad \mathcal{T}_2\tilde{\mathcal{T}}_2=I.\end{array}$$

Obviously, inequality (6.8) is feasible if it satisfies (6.9), (6.10), (6.36), and

$$\begin{bmatrix}\Theta^{(r)}(i) & \varepsilon\check{T}_1^T & \check{T}_2^T\\ \varepsilon\check{T}_1 & -\varepsilon I & 0\\ \check{T}_2^T & 0 & -\varepsilon I\end{bmatrix}<0,\quad r=1,\ldots,M, \tag{6.37}$$

where

$$\begin{array}{l}\Theta^{(r)}(i)\triangleq\begin{bmatrix}\Sigma_1^{(r)}(i) & \Theta_{12}(i)\\ \Theta_{12}^T(i) & -\Theta_{22}(i)\end{bmatrix},\quad \Theta_{22}(i)\triangleq\mathrm{diag}\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4,\mathcal{G}_5\},\\ \Theta_{12}(i)\triangleq\begin{bmatrix}\Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i) & \Lambda^T(i)\end{bmatrix},\end{array}$$

where $\Sigma_1^{(r)}(i)$ and $\Lambda(i)$ are defined as in Theorem 6.5.

We propose the following algorithm to solve the passive filter design problem.

Convex optimization algorithm for filter design:

Step 1. Find a feasible set of $\mathcal{P}_0(i)$, $\tilde{\mathcal{P}}_0(i)$, $\mathcal{G}_{0s}$, $\tilde{\mathcal{G}}_{0s}$ $(s=1,\ldots,5)$, $\mathcal{S}_{01}$, $\tilde{\mathcal{S}}_{01}$, $\mathcal{S}_{02}$, $\tilde{\mathcal{S}}_{02}$, $\mathcal{Q}_{03}$, $\tilde{\mathcal{Q}}_{03}$, $\mathcal{T}_{01}$, $\tilde{\mathcal{T}}_{01}$, $\mathcal{T}_{02}$, $\tilde{\mathcal{T}}_{02}$, satisfying (6.36), (6.37), and

$$\left.\begin{array}{l}\begin{bmatrix}\mathcal{P}(i) & I\\ I & \tilde{\mathcal{P}}(i)\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}\mathcal{S}_1 & I\\ I & \tilde{\mathcal{S}}_1\end{bmatrix}\geqslant 0,\\ \begin{bmatrix}\mathcal{S}_2 & I\\ I & \tilde{\mathcal{S}}_2\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}\mathcal{Q}_3 & I\\ I & \tilde{\mathcal{Q}}_3\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}\mathcal{T}_1 & I\\ I & \tilde{\mathcal{T}}_1\end{bmatrix}\geqslant 0,\\ \begin{bmatrix}\mathcal{T}_2 & I\\ I & \tilde{\mathcal{T}}_2\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}\mathcal{G}_s & I\\ I & \tilde{\mathcal{G}}_s\end{bmatrix}\geqslant 0\quad (s=1,\ldots,5).\end{array}\right\} \tag{6.38}$$

Set $k=0$.

Step 2. Solve the following problem for the variables $\tilde{H}(i)$, $\mathcal{P}(i)$, $\tilde{\mathcal{P}}(i)$, $\mathcal{G}_s$, $\tilde{\mathcal{G}}_s$, $(s=1,\ldots,5)$, $\tilde{\mathcal{S}}_i$, $\tilde{\mathcal{T}}_i$, $(i=1,2)$ and $\mathcal{Q}_3$:

$$\min_{\text{subject to}\ (6.36)-(6.38)}\ \mathrm{tr}\ \Theta_k(i),$$

where

$$\Theta_k(i) \triangleq \mathcal{P}_k(i)\tilde{P}(i) + \mathcal{P}(i)\tilde{\mathcal{P}}_k(i) + \sum_{s=1}^{5}(\mathcal{G}_s\tilde{\mathcal{G}}_{ks} + \mathcal{G}_{ks}\tilde{\mathcal{G}}_s) + \mathcal{S}_1\tilde{\mathcal{S}}_{k1} + \mathcal{S}_{k1}\tilde{\mathcal{S}}_1 + \mathcal{S}_2\tilde{\mathcal{S}}_{k2} + \mathcal{S}_{k2}\tilde{\mathcal{S}}_2 + \mathcal{Q}_{k3}\tilde{\mathcal{Q}}_3 + \mathcal{T}_1\tilde{\mathcal{T}}_{k1} + \mathcal{T}_{k1}\tilde{\mathcal{T}}_1 + \mathcal{T}_2\tilde{\mathcal{T}}_{k2} + \mathcal{T}_{k2}\tilde{\mathcal{T}}_2 + \mathcal{Q}_3\tilde{\mathcal{Q}}_{k3}.$$

Set $\mathcal{P}_{k+1}(i) = \mathcal{P}(i)$, $\tilde{\mathcal{P}}_{k+1}(i) = \tilde{\mathcal{P}}(i)$, $\mathcal{G}_{(k+1)s} = \mathcal{G}_s$, $\tilde{\mathcal{G}}_{(k+1)s} = \tilde{\mathcal{G}}_s$, $(s = 1, \ldots, 5)$, $\mathcal{S}_{(k+1)1} = \mathcal{S}_1$, $\mathcal{S}_{(k+1)2} = \mathcal{S}_2$, $\mathcal{T}_{(k+1)1} = \mathcal{T}_1$, $\mathcal{T}_{(k+1)2} = \mathcal{T}_2$, and $\mathcal{Q}_{(k+1)3} = \mathcal{Q}_3$.

Step 3. If the following condition in (6.39) is feasible for the variables $\mathcal{Q}(i)$, $\mathcal{R}(i)$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{Q}_s$, $\mathcal{T}_s$, $(s = 1, 2, 3)$, and the matrices $\mathcal{H}(i)$ and $\mathcal{P}(i)$ obtained in Step 2, then set $H(i) = P^{-1}(i)\mathcal{H}(i)$ and stop. If (6.39) is infeasible within a specified number of iterations, then stop; otherwise, set $k = k + 1$ and go to Step 2.

$$\begin{bmatrix} \bar{\Theta}^{(r)}(i) & \varepsilon\check{T}_1^T & \check{T}_2^T \\ \varepsilon\check{T}_1 & -\varepsilon I & 0 \\ \check{T}_2^T & 0 & -\varepsilon I \end{bmatrix} < 0, \quad r = 1, \ldots, M, \tag{6.39}$$

where

$$\bar{\Theta}^{(r)}(i) \triangleq \begin{bmatrix} \Sigma_3^{(r)}(i) & \bar{\Theta}_{12}(i) \\ \bar{\Theta}_{12}^T(i) & -\bar{\Theta}_{22}(i) \end{bmatrix},$$

$$\bar{\Theta}_{12}(i) \triangleq \begin{bmatrix} \Lambda_3^T(i)\mathcal{Q}_3 & \Lambda_3^T(i)\mathcal{S}_1 & \Lambda_3^T(i)\mathcal{S}_2 & \Lambda_3^T(i)\mathcal{T}_1 & \Lambda_3^T(i)\mathcal{T}_2 \end{bmatrix},$$

$$\bar{\Theta}_{22}(i) \triangleq \text{diag}\left\{\mathcal{Q}_3, \delta_{\tau_1}^{-1}\mathcal{S}_1, \bar{\tau}_1^{-1}\mathcal{S}_2, \delta_{\tau_2}^{-1}\mathcal{T}_1, \bar{\tau}_2^{-1}\mathcal{T}_2\right\},$$

and $\Sigma_3^{(r)}(i)$ and $\Lambda_3(i)$ are defined as in Theorem 6.4.

6.4 Illustrative Example

In this section, we provide three numerical examples with simulation results to demonstrate the effectiveness of the developed method in designing the passive filter for system (6.6). In the following three examples, we assume the systems involve two jump modes and the transition probabilities matrix comprises two vertices $\Pi^{(r)}$, $(r = 1, 2)$. The first lines of $\Pi^{(r)}$, i.e., $\pi_1^{(r)}$, are given by

$$\pi_1^{(1)} \triangleq \begin{bmatrix} ? & 0.2 \end{bmatrix}, \quad \pi_1^{(2)} \triangleq \begin{bmatrix} ? & 0.5 \end{bmatrix},$$

and the second lines of $\Pi^{(r)}$, are given by

$$\pi_2^{(1)} \triangleq \begin{bmatrix} 0.4 & -0.4 \end{bmatrix}, \quad \pi_2^{(2)} \triangleq \begin{bmatrix} 0.5 & ? \end{bmatrix},$$

where "?" represents the unknown entries.

Example 6.8 In order to evaluate the performance of proposed exponential passive filter, we consider system (6.2) with two subsystems. The values for systems (6.2) parameters are:

$$\begin{aligned}
&A(1) = \begin{bmatrix} 3.00 & 0 \\ 0 & 4.00 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 1.00 & 0 \\ 0 & 2.00 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -0.22 & 0 \\ -0.43 & 0.25 \end{bmatrix}, \\
&B(2) = \begin{bmatrix} -0.33 & 0 \\ -0.14 & 0.15 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 0.23 & 0.14 \\ -0.15 & -0.18 \end{bmatrix}, \quad C(2) = \begin{bmatrix} 1.06 & 0.25 \\ -0.45 & -1.08 \end{bmatrix}, \\
&D(1) = D(2) = 0.1I, \quad E(1) = \begin{bmatrix} 0.41 & -0.58 \\ 0.29 & 0.31 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.21 & -0.18 \\ 0.29 & 0.51 \end{bmatrix}, \\
&C_d(1) = \begin{bmatrix} -0.34 & 2.21 \\ 0.31 & 0.30 \end{bmatrix}, \quad C_d(2) = \begin{bmatrix} 1.23 & 3.33 \\ 0.23 & -1.18 \end{bmatrix}, \quad C_1(1) = \begin{bmatrix} -1.20 & 0.13 \\ -1.23 & -0.34 \end{bmatrix}, \\
&C_1(2) = \begin{bmatrix} -1.31 & 0.13 \\ -2.23 & 1.54 \end{bmatrix}, \quad C_g(1) = \begin{bmatrix} -1.34 & 1.14 \\ 0.23 & 0.43 \end{bmatrix}, \quad C_g(2) = \begin{bmatrix} -1.63 & 0.13 \\ 0.83 & -0.74 \end{bmatrix}, \\
&D_d(1) = \begin{bmatrix} 0.27 & 0.84 \\ 2.01 & 1.26 \end{bmatrix}, \quad D_d(2) = \begin{bmatrix} 0.23 & 0.83 \\ 2.93 & 1.24 \end{bmatrix}, \quad E_d(1) = \begin{bmatrix} 2.92 & 2.16 \\ 0.19 & 3.23 \end{bmatrix}, \\
&E_d(2) = \begin{bmatrix} 2.30 & 2.32 \\ 0.63 & 3.16 \end{bmatrix}, \quad F(t) = \text{diag}\{0.2, 0.2\}, \quad D_1(1) = D_1(2) = 0.01I, \\
&M(1) = M(2) = I, \quad l_a(1) = \begin{bmatrix} 0 & 0 \\ 0.21 & 0.11 \end{bmatrix}, \quad l_a(2) = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.13 \end{bmatrix}, \\
&l_b(1) = \begin{bmatrix} 0 & 0 \\ 0.02 & 0.03 \end{bmatrix}, \quad l_b(2) = \begin{bmatrix} 0 & 0 \\ 0.12 & 0.23 \end{bmatrix}, \quad l_c(1) = \begin{bmatrix} 0 & 0 \\ 0.12 & 0.03 \end{bmatrix}, \\
&l_c(2) = \begin{bmatrix} 0 & 0 \\ 0.02 & 0.02 \end{bmatrix}, \quad l_e(1) = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.01 \end{bmatrix}, \quad l_e(2) = \begin{bmatrix} 0 & 0 \\ 0.11 & 0.21 \end{bmatrix}, \\
&l_{c_d}(1) = \begin{bmatrix} 0 & 0 \\ 0.15 & 0.02 \end{bmatrix}, \quad l_{c_d}(2) = \begin{bmatrix} 0 & 0 \\ 0.05 & 0.03 \end{bmatrix}, \quad l_d(1) = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.21 \end{bmatrix}, \\
&l_d(2) = \begin{bmatrix} 0 & 0 \\ 0.11 & 0.21 \end{bmatrix}, \quad \gamma = 1.04, \quad \tau_{1,\eta_t} = 0.2 + 0.1\sin(4t), \quad \eta_t \in \{1, 2\} \\
&\tau_{2,\eta_t} = 0.25 + 0.15\sin(2t), \quad \tau_{3,\eta_t} = 0.2 + 0.2\sin t, \quad g(s) = \tanh(s/2), \\
&l_1^- = 0.3, \quad l_1^+ = 0.7, \quad l_2^- = 0.2, \quad l_2^+ = 0.8.
\end{aligned}$$

Solving the matrix inequality in Theorem 6.5 by the Toolbox YALMIP of MATLAB, we obtain the desired filter gains

$$H(1) = \begin{bmatrix} 0.8500 & 0.2023 \\ -0.3209 & 0.5625 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 0.3540 & 1.0053 \\ -0.2379 & 0.8625 \end{bmatrix}.$$

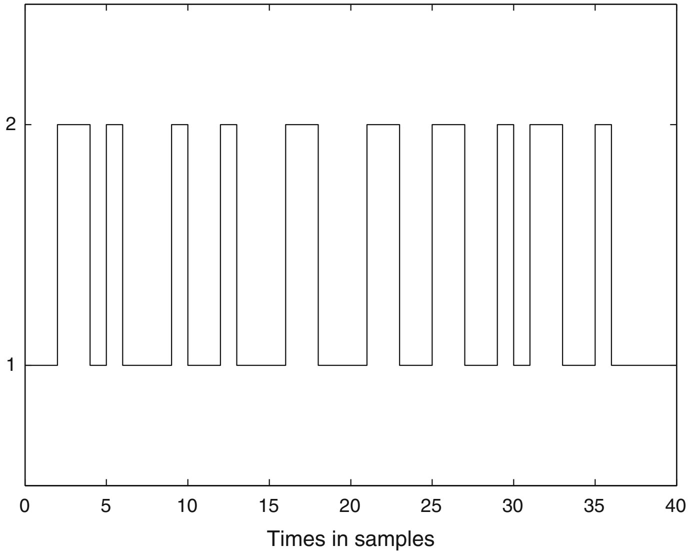

Fig. 6.1 Switching signal with two modes

We set the initial function $\phi(t) = 0$ and the disturbance input $\omega(t) = e^{-0.1t} \left[\sin(t)\ \sin(0.4t)\right]^T$, and use the switching signal of the form shown in Fig. 6.1, where '1' and '2' correspond to the first and second modes, respectively. The actual and estimated values of $x_1(t)$ and $x_2(t)$ in Figs. 6.2 and 6.3 clearly show that the estimated states can track the real states smoothly, which illustrates the effectiveness of the proposed approach in this chapter.

Example 6.9 Consider systems (6.33) with parameters as follows:

$$
\begin{aligned}
A(1) &= \begin{bmatrix} 3.00 & 0 \\ 0 & 4.00 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 2.00 & 0 \\ 0 & 1.00 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -0.12 & 0 \\ -0.27 & 0.13 \end{bmatrix}, \\
B(2) &= \begin{bmatrix} -0.34 & 0 \\ -0.41 & 0.21 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 0.11 & 0.24 \\ -0.05 & -0.38 \end{bmatrix}, \quad C(2) = \begin{bmatrix} 0.14 & 0.33 \\ -0.05 & -0.38 \end{bmatrix}, \\
D(1) &= D(2) = 0.2I, \quad E(1) = \begin{bmatrix} 0.21 & -0.15 \\ 0.59 & 0.31 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.21 & -0.32 \\ 0.29 & 0.13 \end{bmatrix}, \\
C_d(1) &= \begin{bmatrix} -1.10 & 0.23 \\ 0.23 & 1.31 \end{bmatrix}, \quad C_d(2) = \begin{bmatrix} -0.21 & 1.02 \\ 0.13 & 2.11 \end{bmatrix}, \quad C_1(1) = \begin{bmatrix} -1.02 & 0.14 \\ -1.01 & -1.04 \end{bmatrix}, \\
C_1(2) &= \begin{bmatrix} -1.04 & 0.21 \\ -1.32 & -1.03 \end{bmatrix}, \quad C_g(1) = \begin{bmatrix} -1.20 & 0.31 \\ 0.48 & -1.47 \end{bmatrix}, \quad C_g(2) = \begin{bmatrix} -1.10 & 0.31 \\ 0.82 & -0.37 \end{bmatrix}, \\
D_d(1) &= \begin{bmatrix} 0.22 & 0.84 \\ 2.01 & 1.22 \end{bmatrix}, \quad D_d(2) = \begin{bmatrix} 0.24 & 0.82 \\ 2.60 & 1.21 \end{bmatrix}, \quad E_d(1) = \begin{bmatrix} 2.02 & 0.42 \\ 0 & 0.23 \end{bmatrix}, \\
E_d(2) &= \begin{bmatrix} 2.01 & 2.63 \\ 0 & 3.01 \end{bmatrix}, D_1(1) = D_1(2) = 0.01I, \quad \gamma = 1.05,
\end{aligned}
$$

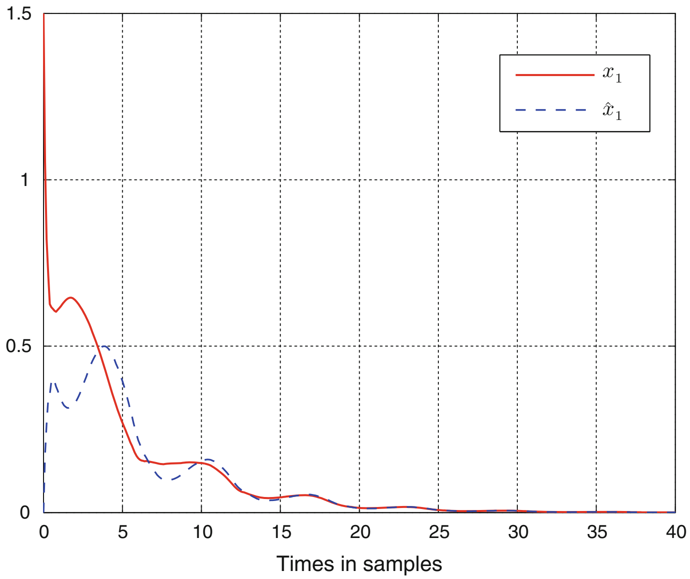

Fig. 6.2 State $x_1(t)$ and its estimation $\hat{x}_1(t)$

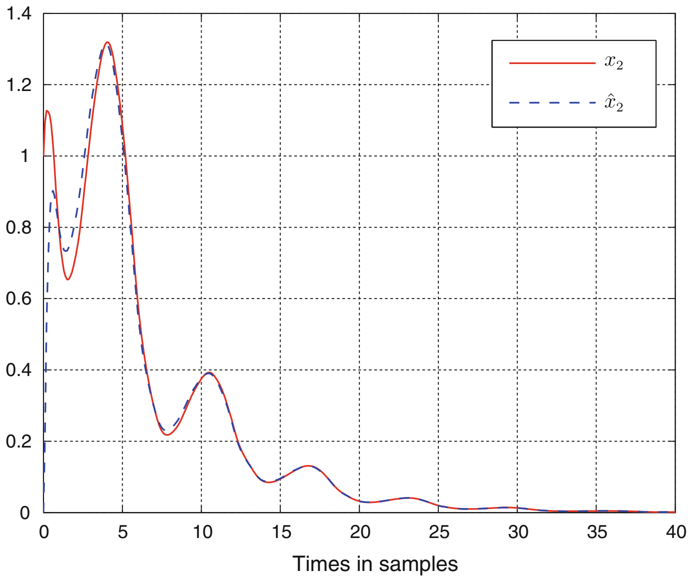

Fig. 6.3 State $x_2(t)$ and its estimation $\hat{x}_2(t)$

$\tau_{1,\eta_t} = 0.35 + 0.15\sin 4t, \quad \tau_{2,\eta_t} = 0.75 + 0.55\sin 0.4t, \quad \tau_{3,\eta_t} = 0.34 + 0.42\sin t,$
$g(s) = \tanh(s/2), l_1^- = 0.34, \quad l_1^+ = 0.56, \quad l_2^- = 0.23, \quad l_2^+ = 0.78.$

Using the Toolbox YALMIP of MATLAB to solve the conditions in Theorem 6.5, we obtain the desired filter gains

$$H(1) = \begin{bmatrix} 1.1612 & 0.0535 \\ -1.0342 & 1.0329 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 2.6218 & -0.3788 \\ -2.2147 & 1.8230 \end{bmatrix}.$$

For performance comparison, we apply Theorem 6.6 to obtain the filter gains; it gives

$$H(1) = \begin{bmatrix} 0.4342 & 0.4865 \\ -1.2662 & 0.9809 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 1.1928 & -1.9258 \\ -0.0687 & 0.0610 \end{bmatrix}.$$

Let the initial function $\phi(t) = 0$ and the disturbance input $\omega(t) = e^{-0.3t} \left[\sin(t) \ \sin(2t) \right]^T$, and use the switching signal as shown in Fig. 6.1, the state responses of the error between $x(t)$ and $\hat{x}(t)$ of these two filter designs are shown in Figs. 6.4 and 6.5.

Remark 6.10 The filter based on the design criterion given in Theorem 6.6 takes 8 s for the two states to converge to 0, whereas more than 14 s is required for the

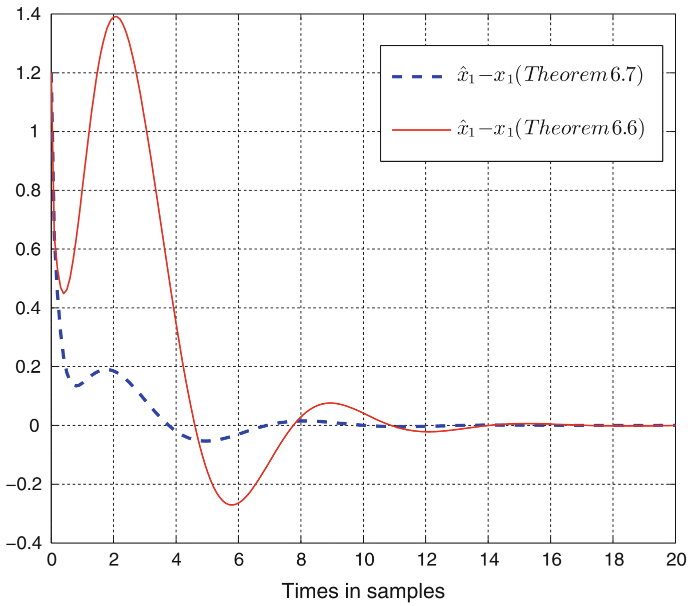

Fig. 6.4 Error of $\hat{x}_1(t) - x_1(t)$

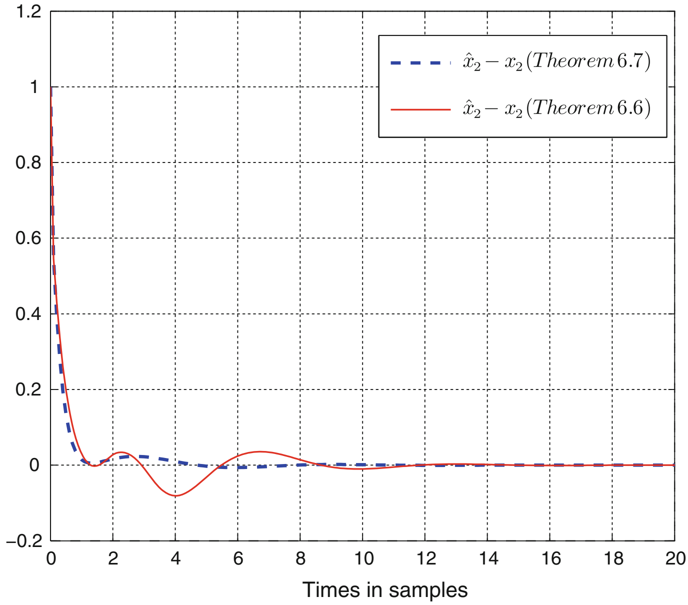

Fig. 6.5 Error of $\hat{x}_2(t) - x_2(t)$

filter designed using Theorem 6.5. It thus confirms that the filter design technique of Theorem 6.6 outperforms that of Theorem 6.5. ♦

Example 6.11 To illustrate applications of the results developed in this chapter, we employ a synthetic oscillatory network of transcriptional regulators in *Escherichia coli*, which has been used to model repressilators, and experimentally investigated in [11, 12]. We consider a synthetic oscillatory network of transcriptional regulators with three repressor protein concentrations and their corresponding mRNA concentrations. In this example, it is assumed that the switching between different modes can be governed by a semi-Markov chain. Consider the following genetic regulatory network:

$$\begin{aligned}\dot{m}(t) &= -\bar{A}(\eta_t)m(t) + \dot{m}(t-\tau_{2,t}) + \bar{B}(\eta_t)f(p(t-\tau_{1,t}))\\ &\quad+\bar{E}_d(\eta_t)\omega(t),\\ \dot{p}(t) &= -\bar{C}(\eta_t)p(t) + \dot{p}(t-\tau_{2,t}) + \bar{D}(\eta_t)m(t-\tau_{1,t}),\end{aligned} \tag{6.40}$$

where $m(t) = [m_1(t), \ldots, m_n(t)]^T$, $p(t) = [p_1(t), \ldots, p_n(t)]^T$, and $m_i(t)$ and $p_i(t)$ are concentrations of mRNA and protein of the ith node. $\bar{A}(\eta_t) = \text{diag}\{\bar{a}_1(\eta_t), \ldots, \bar{a}_n(\eta_t)\}$ with $|\bar{a}_i(\eta_t)| < 1$, $\bar{C}(\eta_t) = \text{diag}\{\bar{c}_1(\eta_t), \ldots, \bar{c}_n(\eta_t)\}$ with $|\bar{c}_i(\eta_t)| < 1$, where $\bar{a}_i(\eta_t)$ and $\bar{c}_i(\eta_t)$ are the decay rates of mRNA and protein, respectively. $\bar{D}(\eta_t) = \text{diag}\{\bar{d}_1(\eta_t), \ldots, \bar{d}_n(\eta_t)\}$. $f(p(t)) = [f_1(p_1(t)), \ldots, f_n(p_n(t))]^T$, and $f_i(p_i(t))$ represents the feedback regulation of the protein on the transcription, which is generally a nonlinear function but has a form of monotonicity with each variable.

$\bar{B} = (b_{ij}) \in \mathbb{R}^{n\times n}$ is the delayed connection weight matrix of the genetic network, which is defined as in [11]. Letting

$$
\begin{aligned}
x(t) &= \begin{bmatrix} m(t) \\ p(t) \end{bmatrix}, \quad g(x(t-\tau_{1,t})) = \begin{bmatrix} m(t-\tau_{1,t}) \\ f(p(t-\tau_{1,t})) \end{bmatrix}, \\
A(\eta_t) &= \begin{bmatrix} \bar{A}(\eta_t) & 0 \\ 0 & \bar{C}(\eta_t) \end{bmatrix}, \quad C(\eta_t) = \begin{bmatrix} 0 & \bar{B}(\eta_t) \\ \bar{D}(\eta_t) & 0 \end{bmatrix}, \\
C_d(\eta_t) &= \begin{bmatrix} \bar{E}_d(\eta_t) \\ 0 \end{bmatrix},
\end{aligned}
$$

we can transform genetic regulatory network (6.40) into neural network (6.33) with $B(\eta_t) = E(\eta_t) = 0$ and $D(\eta_t) = I$.
It is assumed that the time delay satisfying $\tau_{1,t} = 0.3 + 0.1\sin(t)$, and $\tau_{2,t} = 0.1 + 0.1\sin(t)$. In order to design a filter to estimate the states of biological model (6.40), we take the values for the parameters as follows:

$$
\begin{aligned}
&\bar{A}(1) = 0.2I, \quad \bar{A}(2) = 0.1I, \quad \bar{C}(1) = 0.09I, \quad \bar{C}(2) = 0.1I, \\
&\bar{B}(1) = -0.2\mathcal{W}, \quad \bar{B}(2) = -0.1\mathcal{W}, \bar{D}(1) = 0.08I, \\
&\mathcal{W} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{E}_d(1) = [0.1, 0.2, 0]^T, \\
&\bar{E}_d(1) = [0.2, 0, 0.1]^T, \quad \bar{D}(2) = 0.09I.
\end{aligned}
$$

Let us choose the regulation function $f(\cdot)$ as $f_m(s) = s^2/(1+s^2)$, $m = 1, 2, 3$, and the exogenous disturbance input is assumed to be $\omega(t) = 0.2\exp(0.2t)\upsilon_t$, where υ_t is a random noise which is binomially distributed over $[-0.1, 0.1]$. The parameters of the measurement output in (6.33) are selected as follows:

$$
\begin{aligned}
C_1(1) &= [0.2, 0, 0.1, 0.2, 0.3, 0.1], \\
C_1(2) &= [0.1, 0, 0.1, 0.1, 0.2, 0.1], \\
D_1(1) &= [0, 0.1, 0, 0, 0.2, 0.1], \\
D_1(2) &= [0, 0.2, 0.1, 0, 0.2, 0.2],
\end{aligned}
$$

and $C_g(i) = E_d(i) = 0$, $i = 1, 2$. The residual errors of the state vector $m(t)$ of mRNAs and the state vector $p(t)$ of protein are shown in Fig. 6.6 with the initial condition $[0.2, 0.4, 0.5]^T$. In addition, if we set $\gamma = 0.8763$, we have $\bar{\chi} = 0.8422$, and it follows from (6.27) that the estimate of state decay is given by:

$$
\mathbf{E}\{\|e(t)\|_2\} \leqslant 2.1385e^{-0.2401t}\|\phi\|_{C^1}, \quad \forall t \geqslant 0.
$$

Therefore, the filtering error dynamics (6.34) is exponentially stable in mean square with a prescribed H_∞ performance.

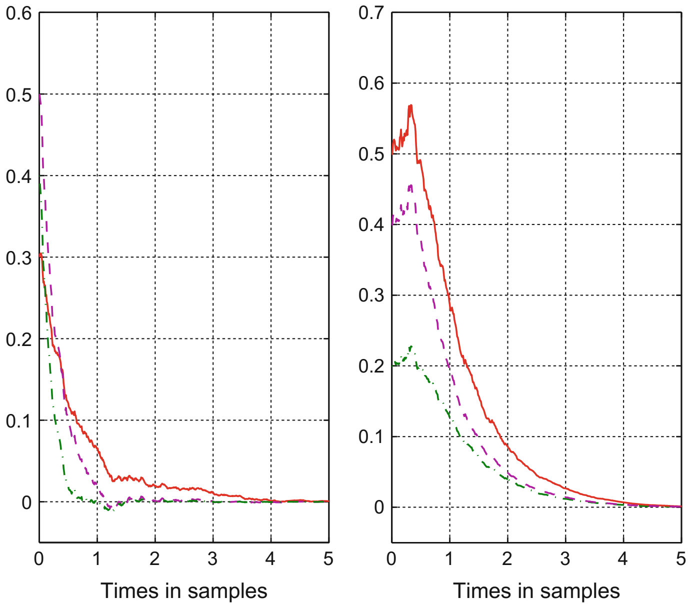

Fig. 6.6 Residual errors of $m(t)$ and $p(t)$

6.5 Conclusion

In this chapter, the exponential passive filtering problem for a class of neutral-type neural networks with semi-Markovian jump parameters, discrete and distributed delays has been investigated. The semi-Markovian switching signal has been simultaneously introduced to reflect the complex hybrid nature existing in the neural networks. Sufficient conditions for the existence of a desired filter are given to ensure that the filtering error system is strictly exponentially passive with desired disturbance attenuation. In addition, a convex optimization algorithm for the filter design has been developed. Finally, three numerical examples are given to demonstrate the effectiveness of the proposed methods.

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Chapter 7
Event-Triggered Fault Detection for Semi-Markovian Jump Systems

Abstract This chapter considers the fault detection problem for a class of S-MJS with known sojourn probability. An event-driven control strategy is developed to reduce the frequency of transmission, and sufficient conditions for sojourn probability dependent jumped systems are presented. A fault detection filter is designed such that the corresponding filtering error system is stochastically stable and has a prescribed performance. In addition, a fault detection filter design algorithm is employed such that the existence conditions for the designed filer are provided.

7.1 Introduction

In networked control systems, an important issue is how to transmit signals more effectively by utilizing the available but limited network bandwidth. To alleviate the unnecessary waste of communication/computation resources that often occurs in conventional time-triggered signal transmissions, a recently popular communication schedule called event-triggered strategy has been proposed. In this chapter, we focus on the fault detection problem for S-MJS with known sojourn probability. In light of an event-triggered technique and the cone complementarity linearization approach, a nonconvex feasibility problem has been transformed into a sequential minimization problem, and then, sufficient conditions are established for the existence of event-triggered fault detection filters. The new features of this chapter are the following three folds: (1) Based on the relative error with respect to the measurement signal, an event indicator variable is introduced and the corresponding event-triggered scheme is proposed in order to reduce the sampling or frequency of communication between the components of the systems; (2) different from the existing works, the sojourn probability is assumed to be known a prior. By using the sojourn probability information, a new kind of jump system model is introduced, which is easier to be measured than transitions probabilities in MJS; and (3) the developed fault detection filter design algorithm is recursive and is thus suitable for online applications.

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7.2 Problem Formulation

Consider the following S-MJS in the probability space $(\Omega, \mathscr{F}, \Pr)$ for $k \in \mathbb{Z}^{+}$:

$$\begin{aligned} x(k+1) &= \textstyle\sum_{\gamma_k=1}^{M}\alpha_{\gamma_k}(k)\big[A_{\gamma_k}x(k) + B_{\gamma_k}u(k) + E_{\gamma_k}\omega(k) + F_{\gamma_k}f(k)\big], \\ y(k) &= \textstyle\sum_{\gamma_k=1}^{M}\alpha_{\gamma_k}(k)\big[C_{\gamma_k}x(k) + D_{r_k}u(k) + G_{r_k}\omega(k) + H_{r_k}f(k)\big], \end{aligned} \tag{7.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $y(k) \in \mathbb{R}^p$ is the measured output; $u(k) \in \mathbb{R}^m$ is the deterministic input vector; $\omega(k) \in \mathbb{R}^q$ is exogenous disturbance input which belongs to $\ell_2[0, \infty)$; and $f(k) \in \mathbb{R}^l$ is the fault vector which is also deterministic.

It is assumed that the transition probabilities Λ belongs to a polytope P_Λ, with vertices $\Lambda^{(r)}$, $r = 1, 2, \ldots, N$ as follows:

$$P_\Lambda \triangleq \left\{ \Lambda | \Lambda = \sum_{\nu=1}^{N} \alpha_\nu(k)\Lambda^{(r)}, \sum_{\nu=1}^{N} \alpha_\nu(k) = 1, \alpha_\nu(k) \geqslant 0 \right\}$$

where $\Lambda^{(r)} \triangleq [\lambda_{ij}^{(r)}]_{M\times M}$ $(i, j \in \mathcal{M}; r = 1, \ldots, N)$ denotes the given transition probability matrices that contain unknown elements. For notation simplicity, for any $i \in \mathcal{M}$, we denote $\mathcal{M} \triangleq \mathcal{M}_{uc}^{i} \cup \mathcal{M}_{uk}^{i}$, where

$$\begin{cases} \mathcal{M}_{uc}^{i} \triangleq \left\{ j | \lambda_{ij}^{h} \text{ is uncertain} \right\}, \\ \mathcal{M}_{uk}^{i} \triangleq \left\{ j | \lambda_{ij}^{h} \text{ is unknown} \right\}. \end{cases}$$

Also, define $\lambda_{uc}^{i} \triangleq \sum_{j\in\mathcal{M}_{uc}^{i}} \lambda_{ij}^{(r)}$, $\forall r = 1, 2, \ldots, N$.

In this chapter, the probability of a jumped system staying in each subsystem is assumed to be known a prior, i.e.

$$\Pr\{\gamma_k = i | k \in \mathbb{Z}^{+}, i \in \mathcal{M}\} = \beta_i.$$

where $\beta_i \in [0, 1]$ is called "sojourn probability" of the ith subsystem. Then, the sojourn probability β_i can be obtained through the following statistical method

$$\beta_i = \lim_{n\to\infty} \frac{k_i}{n},$$

where $k_i, n \in \mathbb{Z}^{+}$, and k_i is the times of $\gamma_n = i$ in the interval $[1, n]$.

A set of stochastic variables $\alpha_i(k)$ are defined as

$$\alpha_i(k) = \begin{cases} 1, & \gamma_k = i, \\ 0, & \gamma_k \neq i, \end{cases} \quad i \in \mathcal{M},$$

and the expectation of $\alpha_i(k)$ is β_i. $\alpha_i(k)$ and β_i satisfy

$$\sum_{i=1}^{M}\alpha_i(k) = 1, \quad \sum_{i=1}^{M}\beta_i = 1.$$

The sojourn probability of the ith subsystem β_i, $i \in \mathcal{M}$ can be obtained by solving the following set of equations:

$$\begin{cases} \Lambda^{(r)}\beta = \beta, \\ \beta_1 + \beta_2 + \cdots + \beta_M = 1, \end{cases} \tag{7.2}$$

where $\beta = [\beta_1, \ldots, \beta_M]^T$.

Remark 7.1 The sojourn probability information of jump systems has been considered in this chapter. One of the motivations to adopt sojourn probabilities in jump systems is the difficulty to obtain the transition probabilities of S-MJS. Compared with the transition probabilities in S-MJS, the sojourn probability information is easier to measure: for a jump system with n subsystems, there are n^2 transition probabilities while there are only n sojourn probabilities to be known for analysis and design purpose. ♦

For the purpose of reducing data communication frequency, the event generator is constructed which uses the previously measurement output to determine whether the newly measurement output will be sent out to the controller or not. As such, we define the event generator function $\mathcal{G}(\cdot, \cdot)$ as follows:

$$\mathcal{G}(\psi(k), \varrho) \triangleq \psi^T(k)\psi(k) - \varrho y^T(k)y(k), \tag{7.3}$$

where $\psi(k) \triangleq y(k_i) - y(k)$, and $y(k_i)$ is the measurement at latest event time, $y(k)$ is the current measurement and $\varrho \in [0, 1)$.

The execution is triggered as long as the condition

$$\mathcal{G}(\psi(k), \varrho) > 0, \tag{7.4}$$

is satisfied. Therefore, the sequence of event-triggered instants $0 \leqslant k_0 \leqslant k_1 \leqslant \cdots \leqslant k_i \leqslant \cdots$ is determined iteratively by

$$k_{i+1} = \inf\{k \in \mathbb{N} | k > k_i, \quad \mathcal{G}(\psi(k), \varrho) > 0\}. \tag{7.5}$$

Accordingly, any measurement data satisfying the event condition (7.4) will be transmitted to the controller.

For the stochastic system (7.1), we are interested in designing a fault detection filter of the following form, for $\gamma_k = i$,

$$\begin{cases} x_f(k+1) = \sum_{i=1}^{M} \alpha_i(k) \left[A_{fi} x_f(k) + B_{fi} y(k_i) \right], \\ \delta(k) = \sum_{i=1}^{M} \alpha_i(k) C_{fi} x_f(k), \end{cases} \tag{7.6}$$

where $x_f(k) \in \mathbb{R}^s$ is the state of the fault detection filter; and $\delta(k) \in \mathbb{R}^l$ is the residual signal. A_{fi}, B_{fi}, and C_{fi} are the filter parameters to be designed.

To improve or enhance the performance of fault detection system, we add a weighting matrix function into the fault $f(z)$, that is, $f_w(z) = W(z) f(z)$, where $f(z)$ and $f_w(z)$ denote respectively the 'z' transforms of $f(k)$ and $f_w(k)$. Here, $W(z)$ is given *a priori*, and the choice of $W(z)$ is to impose frequency weighting on the spectrum of the fault signal for detection. One state space realization of $f_w(z) = W(z) f(z)$ can be

$$\begin{cases} x_w(k+1) = A_w x_w(k) + B_w f(k), \\ f_w(k) = C_w x_w(k), \end{cases} \tag{7.7}$$

where $x_w(k) \in \mathbb{R}^k$ is the state vector, and matrices A_w, B_w, and C_w are priorly chosen.

Denoting $e(k) \triangleq \delta(k) - f_w(k)$, and augmenting the model of (7.1) to include the states of (7.6) and (7.7), then the overall dynamics of fault detection system is governed by

$$\begin{cases} \xi(k+1) = \sum_{i=1}^{M} \alpha_i(k) \left[\tilde{A}_i \xi(k) + \tilde{B}_i \upsilon(k) + \tilde{D}_i \psi(k) \right], \\ e(k) = \sum_{i=1}^{M} \alpha_i(k) \tilde{C}_i \xi(k), \end{cases} \tag{7.8}$$

where

$$\xi(k) \triangleq \left[x^T(k) \; x_f^T(k) \; x_w^T(k) \right]^T,$$
$$\upsilon(k) \triangleq \left[u^T(k) \; \omega^T(k) \; f^T(k) \right]^T,$$

and

$$\begin{cases} \tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 & 0 \\ B_{fi} C_i & A_{fi} & 0 \\ 0 & 0 & A_w \end{bmatrix}, \quad \tilde{D}_i \triangleq \begin{bmatrix} 0 \\ B_{fi} \\ 0 \end{bmatrix}, \\ \tilde{B}_i \triangleq \begin{bmatrix} B_i & E_i & F_i \\ B_{fi} D_i & B_{fi} G_i & B_{fi} H_i \\ 0 & 0 & B_w \end{bmatrix}, \\ \tilde{C}_i \triangleq \left[0 \; C_{fi} \; -C_w \right]. \end{cases} \tag{7.9}$$

Before presenting our main results in this chapter, we introduce the following definitions.

Definition 7.2 The error system (7.8) with $\upsilon(k) = 0$ is said to be stochastically stable, if for any initial state $(\xi(0), r_0)$, the following condition holds

$$\mathbf{E}\left\{\sum_{k=0}^{\infty}\|\xi(k)\|^2|(\xi(0), r_0)\right\} < \Gamma(\xi(0), r_0),$$

where $\Gamma(\xi(0), r_0)$ is a nonnegative function of the system initial values.

Definition 7.3 For a given scalar $\gamma > 0$, the fault detection system (7.8) is said to be stochastically stable with a generalized H_2 performance level γ, if it is stochastically stable with $\upsilon(k) = 0$, and under zero initial condition, that is, $\xi(0) = 0$, $\|e(k)\|_\infty < \gamma\,\|\upsilon(k)\|_2$ for all nonzero $\upsilon(k) \in \ell_2[0, \infty)$, where $\|e(k)\|_\infty \triangleq \sup_k \sqrt{\mathbf{E}\{|e(k)|^2\}}$.

Now, the problem of event-triggered fault detection problem can be transformed into H_2 filtering problem for system (7.1): develop a filter (7.6) for a residual signal $\delta(k)$ to assure that the resulting overall fault detection system (7.8) to be stochastically stable with a disturbance attenuation γ.

After generating the residual signal, a residual evaluation value will be computed through a prescribed evaluation function. Here, we consider the following evaluation function: $\mathcal{J}(\delta)$ (where δ denotes $\delta(k)$ for simplicity) and a threshold $\mathcal{J}_{th}$

$$\mathcal{J}(\delta) \triangleq \left(\sum_{k=k_0}^{k_0+k^\star}\delta^T(k)\delta(k)\right)^{1/2},$$

$$\mathcal{J}_{th} \triangleq \sup_{0\neq\omega\in\ell_2, 0\neq u\in\ell_2, f=0} \mathcal{J}(\delta),$$

where k_0 denotes the initial evaluation time instant, and $k^\star$ stands for the evaluation time. For the detailed discussion of the threshold $\mathcal{J}_{th}$, readers are referred to [1].

Based on above, the occurrence of faults can be detected by comparing $\mathcal{J}(\delta)$ and $\mathcal{J}_{th}$ according to the following test:

$$\begin{cases} \mathcal{J}(\delta) > \mathcal{J}_{th} \Rightarrow \text{with faults} \Rightarrow \text{alarm} \\ \mathcal{J}(\delta) \leqslant \mathcal{J}_{th} \Rightarrow \text{no faults} \end{cases}$$

7.3 Main Results

7.3.1 Fault Detection Filtering Analysis

In the following, sufficient conditions will be developed to guarantee the asymptotical stability of the residual system in (7.8) with a performance described.

Theorem 7.4 *For given scalars $\gamma > 0$ and $0 < \varrho < 1$, the fault detection system (7.8) is stochastically stable with a generalized H_2 performance level γ, if there exist matrices $P_i > 0$, $i \in \mathcal{M}$ such that*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & -\Xi_{22} & 0 \\ * & * & -\Xi_{33} \end{bmatrix} < 0, \tag{7.10}$$

$$\begin{bmatrix} -P_i & \Xi_{14} \\ * & -\Xi_{44} \end{bmatrix} < 0, \tag{7.11}$$

where

$$\begin{aligned}
\Xi_{11} &\triangleq \mathrm{diag}\{-P_i, -I, -I\}, \quad \mathcal{P}_{uc}^i \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} P_j, \\
\mathcal{P}_j &\triangleq \mathcal{P}_{uc}^i + \left(1 - \lambda_{uc}^i\right) P_j, \quad \forall i, j \in \mathcal{M} \times \mathcal{M}_{uk}^i, \\
\Xi_{12} &\triangleq \left[\, \Phi_1, \ldots, \Phi_M \,\right], \quad \Xi_{13} \triangleq \left[\, \Psi_1, \ldots, \Psi_M \,\right], \\
\Phi_i &\triangleq \left[\, \sqrt{\beta_i} \mathcal{P}_j \tilde{A}_i \ \sqrt{\beta_i} \mathcal{P}_j \tilde{B}_i \ \sqrt{\beta_i} \mathcal{P}_j \tilde{D}_i \,\right]^T, \\
\Psi_i &\triangleq \left[\, \varrho \sqrt{\beta_i} \mathscr{C}_i \ \varrho \sqrt{\beta_i} \mathscr{D}_i \ 0 \,\right]^T, \\
\Xi_{22} &\triangleq \mathrm{diag}\{\underbrace{\mathcal{P}_j, \ldots, \mathcal{P}_j}_{M}\}, \quad \Xi_{33} \triangleq \mathrm{diag}\{\underbrace{\varrho I, \ldots, \varrho I}_{M}\}, \\
\Xi_{14} &\triangleq \left[\, (\tilde{C}_1)^T, \ldots, (\tilde{C}_M)^T \,\right], \quad \Xi_{44} \triangleq \mathrm{diag}\{\underbrace{\gamma^2 I, \ldots, \gamma^2 I}_{M}\}, \\
\mathscr{C}_i &\triangleq \left[\, C_i \ 0 \ 0 \,\right], \quad \mathscr{D}_i \triangleq \left[\, D_i \ G_i \ H_i \,\right],
\end{aligned}$$

and $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$, and $\tilde{D}_i$ are defined as in (7.9).

Proof First, let us consider generalized H_2 performance criterion. Choose a stochastic Lyapunov function candidate as:

$$V(\xi(k), \alpha(k), \gamma_k) \triangleq \xi^T(k) \left(P_{\gamma_k}\right) \xi(k), \tag{7.12}$$

where $P_i \triangleq P_{\gamma_k = i}$, $i \in \mathcal{M}$ are positive define matrices to be determined.

Note that

$$\mathbf{E}\{\alpha_i(k)\alpha_j(k)\} = \begin{cases} \beta_i, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then, for $\gamma_k = i$, the increment of $V(\xi(k), \alpha(k), \gamma_k)$ from k to $k+1$ satisfies

$$\begin{aligned}
&\mathbf{E}\{\Delta V(\xi(k), \alpha(k), \gamma_k)\} \\
&\quad = \mathbf{E}\Big\{V\left(\xi(k+1), \gamma_{k+1}\right) \Big| \left(\xi(k), \alpha(k), \gamma_k\right)\Big\} - V\left(\xi(k), \gamma_k\right) \\
&\quad = \mathbf{E}\Big\{\xi^T(k+1) \left(P_{(\gamma_{k+1}=j|\gamma_k=i)}\right) \xi(k+1)\Big\} - \xi^T(k) \left(P_i\right) \xi(k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{M} \beta_i \Big(\tilde{A}_i \xi(k) + \tilde{B}_i \upsilon(k) + \tilde{D}_i \psi(k) \Big)^T \Big(\sum_{j=1}^{M} \lambda_{ij}^h P_j \Big) \\
&\quad \times \Big(\tilde{A}_i \xi(k) + \tilde{B}_i \upsilon(k) + \tilde{D}_i \psi(k) \Big) - \xi^T(k) P_i \xi(k) \\
&= \sum_{i=1}^{M} \beta_i \zeta^T(k) \big(\mathscr{A}_i\big)^T \Big(\sum_{j \in \mathcal{M}} \lambda_{ij}^h P_j \Big) \mathscr{A}_i \zeta(k) - \xi^T(k) P_i \xi(k)
\end{aligned} \tag{7.13}
$$

where $\zeta(k) \triangleq \begin{bmatrix} \xi(k) \\ \upsilon(k) \\ \psi(k) \end{bmatrix}$ and $\mathscr{A}_i \triangleq \begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{D}_i \end{bmatrix}$.

To establish the generalized H_2 performance for the fault detection system (7.8), we assume zero initial condition, that is, $\xi(0) = 0$, then we have $V(\xi(k), \alpha(k), \gamma_k)|_{t=0} = 0$. Consider the index

$$
\mathcal{J} \triangleq V(\xi(k), \alpha(k), \gamma_k) - \sum_{s=0}^{k-1} \upsilon^T(s) \upsilon(s),
$$

then for any nonzero $\upsilon(k) \in \ell_2[0, \infty)$ and $t > 0$, we have

$$
\begin{aligned}
\mathcal{J} &= \mathbf{E}\Big\{ V(\xi(k), \alpha(k), \gamma_k) - V(\xi(0), \alpha(0), r_0) \Big\} - \sum_{s=0}^{k-1} \upsilon^T(s) \upsilon(s) \\
&= \sum_{s=0}^{k-1} \mathbf{E}\left\{ \Delta V(\xi(s), \alpha(s)) - \upsilon^T(s) \upsilon(s) \right\}.
\end{aligned}
$$

On the other hand, it follows from (7.4) and (7.13), that

$$
\begin{aligned}
&\mathbf{E}\big\{ \Delta V(\xi(k), \alpha(k)) - \upsilon^T(k) \upsilon(k) \big\} \\
&\quad \leqslant \mathbf{E}\Big\{ \zeta^T(k) \big(\mathscr{A}_i\big)^T \Big(\sum_{j \in \mathcal{M}} \lambda_{ij}^h P_j \Big) \mathscr{A}_i \zeta(k) + \varrho \zeta^T(k) \big(\mathscr{B}_i\big)^T \mathscr{B}_i \zeta(k) \\
&\quad + \zeta^T(k) \mathrm{diag}\{-P_i, -I, -I\} \zeta(k) \Big\},
\end{aligned} \tag{7.14}
$$

where $\mathscr{B}_i \triangleq \begin{bmatrix} \mathscr{C}_i & \mathscr{D}_i & 0 \end{bmatrix}$.

Now, we decompose the defective transition probability matrix as follows:

$$
\sum_{j \in \mathcal{M}} \lambda_{ij}^h P_j = \Big(\sum_{j \in \mathcal{M}_{uc}^i} \sum_{\nu=1}^{M} \alpha_\nu \lambda_{ij}^{(\nu)} + \sum_{j \in \mathcal{M}_{uk}^i} \lambda_{ij}^h \Big) P_j, \tag{7.15}
$$

where $\sum_{\nu=1}^{M} \alpha_\nu \lambda_{ij}^{(\nu)}$, $j \in \mathcal{M}_{uc}^i$ represents an uncertain element in the polytope uncertainty description.

Then, by using $\sum_{\nu=1}^{M}\alpha_\nu = 1$ and $\alpha_\nu \geqslant 0$, (7.15) holds if and only if

$$\begin{aligned}\sum_{j\in\mathcal{M}}\lambda_{ij}^{h}P_j &= \sum_{\nu=1}^{M}\alpha_\nu\bigg(\sum_{j\in\mathcal{M}_{uc}^{i}}\lambda_{ij}^{(\nu)} + \sum_{j\in\mathcal{M}_{uk}^{i}}\lambda_{ij}^{h}\bigg)P_j,\\ &= \bigg(\sum_{j\in\mathcal{M}_{uc}^{i}}\lambda_{ij}^{(\nu)} + \sum_{j\in\mathcal{M}_{uk}^{i}}\lambda_{ij}^{h}\bigg)P_j,\\ &= \mathcal{P}_{uc}^{i} + \left(1-\lambda_{uc}^{i}\right)\sum_{j\in\mathcal{M}_{uk}^{i}}\frac{\lambda_{ij}^{h}}{1-\lambda_{uc}^{i}}P_j. \end{aligned}\tag{7.16}$$

Since $\frac{\lambda_{ij}^{h}}{1-\lambda_{uc}^{i}} \geqslant 0$ and $\sum_{j\in\mathcal{M}_{uk}^{i}}\frac{\lambda_{ij}^{h}}{1-\lambda_{uc}^{i}} = 1$, (7.16) becomes

$$\sum_{j\in\mathcal{M}}\lambda_{ij}^{h}P_j = \sum_{j\in\mathcal{M}_{uk}^{i}}\frac{\lambda_{ij}^{h}}{1-\lambda_{uc}^{i}}\Big(\mathcal{P}_{uc}^{i} + \left(1-\lambda_{uc}^{i}\right)P_j\Big).$$

Thus, for $0 \leqslant \lambda_{ij}^{h} \leqslant 1-\lambda_{uc}^{i}$, (7.14) is equivalent to $\forall j \in \mathcal{M}_{uk}^{i}$

$$\begin{aligned}&\mathbf{E}\big\{\Delta V(\xi(k),\alpha(k)) - \upsilon^{T}(k)\upsilon(k)\big\}\\ &\quad\leqslant \sum_{i=1}^{M}\beta_i\zeta^{T}(k)\left(\mathscr{A}_i\right)^{T}\Big(\mathcal{P}_{uc}^{i} + \left(1-\lambda_{uc}^{i}\right)P_j\Big)\mathscr{A}_i\zeta(k)\\ &\qquad + \varrho\zeta^{T}(k)\left(\mathscr{B}_i\right)^{T}\mathscr{B}_i\zeta(k) + \zeta^{T}(k)\mathrm{diag}\{-P_i,-I,-I\}\zeta(k).\end{aligned}$$

Thus by (7.10), together with Schur complement, we have

$$\Delta V(\xi(k),\alpha(k)) - \upsilon^{T}(k)\upsilon(k) < 0,$$

this guarantees $\mathcal{J} < 0$, which further implies

$$V(\xi,\alpha) < \sum_{s=0}^{k-1}\upsilon^{T}(s)\upsilon(s).\tag{7.17}$$

On the other hand, by Schur complement again, (7.11) yields

$$\sum_{i=1}^{M}\beta_i(\tilde{C}_i)^{T}\tilde{C}_i - \gamma^2 P_i < 0.\tag{7.18}$$

Then, one can obtain that, for all $t > 0$,

$$\mathbf{E}\{e^T(k)e(k)\} - \gamma^2 V(\xi, \alpha) \leqslant \xi(k)\left(\sum_{i=1}^{M} \beta_i \tilde{C}_i^T \tilde{C}_i - \gamma^2 P_i\right)\xi(k).$$

Combining with (7.17)–(7.18) yields the following inequalities:

$$\begin{aligned}\mathbf{E}\{e^T(k)e(k)\} &< \gamma^2 V(\xi, \alpha) \\ &\leqslant \gamma^2 \sum_{s=0}^{k-1} \upsilon^T(s)\upsilon(s) \leqslant \gamma^2 \sum_{s=0}^{\infty} \upsilon^T(s)\upsilon(s),\end{aligned}$$

which implies that $\|e(k)\|_\infty < \gamma \|\upsilon(k)\|_2$ for all nonzero $\upsilon(k) \in \ell_2[0, \infty)$.

Next, we will show the stochastic stability of system (7.8) with $\upsilon(k) = 0$. From (7.10) we have

$$\Upsilon_i \triangleq \begin{bmatrix} (\tilde{A}_i)^T \\ (\tilde{D}_i)^T \end{bmatrix} \mathcal{P}_j \begin{bmatrix} \tilde{A}_i & \tilde{D}_i \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & I \end{bmatrix} < 0.$$

Next we calculate stochastic Lyapunov function (7.12) along the trajectories of system, we have

$$\begin{aligned}&\mathbf{E}\{V_{\gamma_{k+1}}(\xi(k+1), k+1)|(\xi(k), \gamma_k = i)\} - V_{\gamma_k}(\xi(k), k) \\ &\quad\leqslant -\lambda_{\min}(\Upsilon_i)\, \xi^T(k)\xi(k) \\ &\quad\leqslant -\beta\xi^T(k)\xi(k),\end{aligned}$$

where $\beta \triangleq \inf_{i \in \mathcal{M}} \{\lambda_{\min}(\Upsilon_i)\}$.

Hence, it follows that

$$\lim_{T\to\infty} \mathbf{E}\left\{\sum_{k=0}^{T} \xi^T(k)\xi(k) \big| (\xi_0, r_0)\right\} \leqslant \varGamma(\xi_0, r_0),$$

where $\varGamma(\xi_0, r_0)$ is a positive constant. Thus, the fault detection system $(\tilde{\varPi})$ in (7.8) with $\upsilon(k) = 0$ is stochastically stable in the sense of Definition 7.2. This completes the proof. ■

Remark 7.5 It should be emphasized that the sojourn probabilities of the subsystems appear in Theorem 7.4, which will affect the validity of Theorem 7.4. If the sojourn probabilities of the stable subsystems are big, the whole switched system is more likely to be stable (or stable with large steady margin). ♦

7.3.2 Fault Detection Filter Design

We now shift our design focus on the fault detection filter in (7.6) based on Theorem 7.4 and Lemma 1.29. To this end, the filter matrices (A_{fi}, B_{fi}, C_{fi}) should be determined to guarantee the stochastic stability of the filter error system (7.8) with a disturbance attenuation level γ. We establish a sufficient condition for the existence of such a filter through the following theorem:

Theorem 7.6 *Consider the stochastic system* (7.1) *with fault detection filter* (7.6). *For the given disturbance attenuation level* γ *and the scalar* $\varrho \in [0, 1)$, *the fault detection system* (7.8) *is stochastically stable with a generalized disturbance attenuation level, if there exist positive diagonally dominant matrices* $X_i > 0$, $Y_i > 0$, $\mathscr{X}_i > 0$, *and* $\mathscr{Y}_i > 0$, $i \in \mathcal{M}$ *satisfying*

$$\begin{bmatrix} \bar{\Xi}_{11} & \bar{\Xi}_{12} & \bar{\Xi}_{13} \\ * & \bar{\Xi}_{22} & 0 \\ * & * & \bar{\Xi}_{33} \end{bmatrix} < 0, \tag{7.19}$$

$$\left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right] \begin{bmatrix} \bar{\Xi}_{11} & \tilde{\Xi}_{12} & \bar{\Xi}_{13} \\ * & \tilde{\Xi}_{22} & 0 \\ * & * & \bar{\Xi}_{33} \end{bmatrix} \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right]^T < 0, \tag{7.20}$$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ * & -\Xi_{44} \end{bmatrix} < 0, \tag{7.21}$$

$$\mathcal{X}_j \mathscr{X}_j = I, \quad \mathcal{Y}_j \mathscr{Y}_j = I, \tag{7.22}$$

where

$$\begin{aligned}
\bar{\Xi}_{11} &\triangleq \text{diag}\{-X_i, -Y_i, -I, -I\}, \quad \bar{\Xi}_{12} \triangleq \left[\bar{\Phi}_1, \ldots, \bar{\Phi}_M\right], \\
\bar{\Phi}_i &\triangleq \begin{bmatrix} \sqrt{\beta_i} J \bar{A}_i & 0 & \sqrt{\beta_i} J \bar{B}_i & \sqrt{\beta_i} J \bar{D}_i \\ 0 & \sqrt{\beta_i} A_w & \sqrt{\beta_i} \hat{B}_w & 0 \end{bmatrix}^T, \quad i \in \mathcal{M}, \\
\bar{\Xi}_{22} &\triangleq \text{diag}\{-J \mathscr{X}_j J^T, -\mathscr{Y}_j, \ldots, -J \mathscr{X}_j J^T, -\mathscr{Y}_j\}, \\
\bar{\Xi}_{13} &\triangleq \left[\bar{\Psi}_1, \ldots, \bar{\Psi}_M\right], \quad \bar{\Psi}_i \triangleq \left[\varrho \sqrt{\beta_i} \bar{\mathscr{C}_i} \quad 0 \quad \varrho \sqrt{\beta_i} \mathscr{D}_i \quad 0\right], \\
\tilde{\Xi}_{12} &\triangleq \left[\tilde{\Phi}_1, \ldots, \tilde{\Phi}_M\right], \quad \Lambda_{11} \triangleq \text{diag}\{-J X_i J^T, -Y_i\}, \\
\tilde{\Phi}_i &\triangleq \begin{bmatrix} \sqrt{\beta_i} \bar{A}_i & 0 & \sqrt{\beta_i} \bar{B}_i & \sqrt{\beta_i} \bar{D}_i \\ 0 & \sqrt{\beta_i} A_w & \sqrt{\beta_i} \hat{B}_w & 0 \end{bmatrix}^T, \quad i \in \mathcal{M}, \\
\tilde{\Xi}_{22} &\triangleq \text{diag}\{-\mathscr{X}_j, -\mathscr{Y}_j, \ldots, -\mathscr{X}_j, -\mathscr{Y}_j\}, \\
\Lambda_{12} &\triangleq [\Theta_1, \ldots, \Theta_M], \quad \Theta_i \triangleq [0 \quad -C_w]^T, \\
\mathcal{X}_j &\triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} X_j + (1 - \lambda_{uc}^i) X_j, \forall j \in \mathcal{M}_{uk}^i, \\
\mathcal{Y}_j &\triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} Y_j + (1 - \lambda_{uc}^i) Y_j, \forall j \in \mathcal{M}_{uk}^i.
\end{aligned}$$

Moreover, if the above conditions (7.19)–(7.22) *are feasible, then the system matrices of a fault detection filter* (7.6) *are given by*

$$\begin{aligned}
\mathcal{G}_i &= -\Pi_{1i}^{-1}U_{1i}^T\Lambda_{1i}V_{1i}^T(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1} + \Pi_{1i}^{-1}\Xi_{1i}^{1/2}L_{1i}(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1/2} \\
\mathcal{K}_i &= -\Pi_{2i}^{-1}U_{2i}^T\Lambda_{2i}V_2^T(V_2\Lambda_{2i}V_2^T)^{-1} + \Pi_{2i}^{-1}\Xi_{2i}^{1/2}L_{2i}(V_2\Lambda_{2i}V_2^T)^{-1/2} \\
\Lambda_{1i} &= (U_{1i}\Pi_{1i}^{-1}U_{1i}^T - W_{1i})^{-1} > 0 \\
\Lambda_{2i} &= (U_{2i}\Pi_{2i}^{-1}U_{2i}^T - W_{2i})^{-1} > 0 \\
\Xi_{1i} &= \Pi_{1i} - U_{1i}^T(\Lambda_{1i} - \Lambda_{1i}V_{1i}^T(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1}V_{1i}\Lambda_{1i})U_{1i} > 0 \\
\Xi_{2i} &= \Pi_{2i} - U_{2i}^T(\Lambda_{2i} - \Lambda_{2i}V_2^T(V_2\Lambda_{2i}V_2^T)^{-1}V_2\Lambda_{2i})U_{2i} > 0
\end{aligned}$$

where $\mathcal{G}_i \triangleq \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix}$, $\mathcal{K}_i \triangleq C_{fi}$. *In addition,* $\Pi_{\kappa i}$ *and* $L_{\kappa i}$, $\kappa = 1, 2$ *are any appropriate matrices satisfying* $\Pi_{\kappa i} > 0$, $\|L_{\kappa i}\| < 1$ *and*

$$\begin{aligned}
\bar{A}_i &\triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 \triangleq \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \bar{B}_i \triangleq \begin{bmatrix} B_i & E_i & F_i \\ 0 & 0 & 0 \end{bmatrix}, \\
M_i &\triangleq \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ C_i & 0 & 0 & 0 & D_i & G_i & H_i \end{bmatrix}^T, \quad R_i \triangleq \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix}, \\
S_i &\triangleq \begin{bmatrix} 0 & 0 & 0 \\ D_i & G_i & H_i \end{bmatrix}, \quad E \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad V_2 \triangleq \begin{bmatrix} T & 0 & 0 \end{bmatrix}, \\
U_1 &\triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & E^T & 0 & 0 \end{bmatrix}^T, \quad W_{2i} \triangleq \begin{bmatrix} \bar{\Lambda}_{11} & \Lambda_{12} \\ * & -\Xi_{44} \end{bmatrix}, \\
W_{1i} &\triangleq \begin{bmatrix} \bar{\Xi}_{11} & \tilde{\Xi}_{12} & \bar{\Xi}_{13} \\ * & \tilde{\Xi}_{22} & 0 \\ * & * & \bar{\Xi}_{33} \end{bmatrix}, \quad V_{1i} \triangleq \begin{bmatrix} R_i & 0 & 0 & S_i & 0 & 0 & 0 \end{bmatrix}, \\
\bar{\Lambda}_{11} &\triangleq \mathrm{diag}\{-X_i, -Y_i\}, \quad T \triangleq \begin{bmatrix} 0 & I \end{bmatrix}, \quad J \triangleq \begin{bmatrix} I & 0 \end{bmatrix}.
\end{aligned} \tag{7.23}$$

Proof Set $P_i \triangleq \mathrm{diag}\,(X_i, Y_i)$, $i \in \mathcal{M}$ in (7.10), where $X_i \in \mathbb{R}^{(n+s)\times(n+s)}$ and $Y_i \in \mathbb{R}^{k\times k}$. Then, from Theorem 7.4, the fault detection system (7.8) is stochastically stable with a performance γ, if there exist matrices $X_i > 0$ and $Y_i > 0$ such that

$$\begin{bmatrix} \bar{\Xi}_{11} & \hat{\Xi}_{12} & \bar{\Xi}_{13} \\ * & \tilde{\Xi}_{22} & 0 \\ * & * & \bar{\Xi}_{33} \end{bmatrix} < 0, \tag{7.24}$$

$$\begin{bmatrix} \bar{\Lambda}_{11} & \hat{\Lambda}_{12} \\ * & -\Xi_{44} \end{bmatrix} < 0, \tag{7.25}$$

where

$$
\begin{aligned}
\hat{\Xi}_{12} &\triangleq \left[\hat{\Phi}_1, \ldots, \hat{\Phi}_M\right], \quad \hat{\Lambda}_{12} \triangleq \left[\hat{\Theta}_1, \ldots, \hat{\Theta}_M\right], \\
\hat{\Phi}_i &\triangleq \begin{bmatrix} \sqrt{\beta_i}\hat{A}_i & 0 & \sqrt{\beta_i}\hat{B}_i & \sqrt{\beta_i}\hat{D}_i \\ 0 & \sqrt{\beta_i}A_w & \sqrt{\beta_i}\hat{B}_w & 0 \end{bmatrix}^T, \\
\hat{A}_i &\triangleq \begin{bmatrix} A_i & 0 \\ B_{fi}C_i & A_{fi} \end{bmatrix}, \quad \hat{D}_i \triangleq \begin{bmatrix} 0 \\ B_{fi} \end{bmatrix}, \\
\hat{\mathscr{C}}_i &\triangleq \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad \hat{C}_i \triangleq \begin{bmatrix} 0 & C_{fi} \end{bmatrix}, \\
\hat{B}_i &\triangleq \begin{bmatrix} B_i & E_i & F_i \\ B_{fi}D_i & B_{fi}G_i & B_{fi}H_i \end{bmatrix}, \\
\hat{B}_w &\triangleq \begin{bmatrix} 0 & 0 & B_w \end{bmatrix}, \quad \hat{\Theta}_i \triangleq \begin{bmatrix} \hat{C}_i & -C_w \end{bmatrix}^T .
\end{aligned} \tag{7.26}
$$

Rewrite (7.26) in the following form:

$$
\begin{aligned}
\hat{A}_i &= \bar{A}_i + E\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} R_i, \quad \hat{C}_i = C_{fi}T, \\
\hat{B}_i &= \bar{B}_i + E\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} S_i,
\end{aligned} \tag{7.27}
$$

where $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, E, R_i, S_i and T are defined in (7.23).

Using (7.27), the inequalities (7.24) and (7.25) can be rewritten as

$$
W_{1i} + U_1\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} V_{1i} + \left(U_1\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} V_{1i}\right)^T < 0, \tag{7.28}
$$

$$
W_{2i} + U_2C_{fi}V_2 + \left(U_2C_{fi}V_2\right)^T < 0, \tag{7.29}
$$

where W_{1i}, W_{2i}, U_1, V_{1i}, U_2 and V_2 are defined in (7.23).

Next, we assign

$$
U_1^{\perp} = \mathrm{diag}\{I, I, I, I, J, I, I\}, \quad V_{1i}^{T\perp} = \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right].
$$

It follows from Lemma 1.29 that inequality (7.28) is solvable for $\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix}$ if and only if (7.19) and (7.20) are satisfied.

In addition, set

$$
U_2^{\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad V_2^{T\perp} = \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

Then, inequality (7.29) is solvable for C_{fi} if and only if (7.21) hold. This completes the proof. ■

Remark 7.7 It's worth noting that the convex optimization algorithm cannot be used to find a minimum γ, since the conditions are no longer LMIs due to the matrix equation (7.22). However, we can solve this problem by using the cone complementarity linearization algorithm proposed in [2]. ♦

From the above discussion, we can solve the nonconvex feasibility problem by formulating it into the following sequential optimization problem.
Fault Detection Filter Design Problem:

$$\begin{aligned} \min \quad & \text{trace}\left(\sum\nolimits_i (X_i \mathscr{X}_i)\right) + \text{trace}\left(\sum\nolimits_i (Y_i \mathscr{Y}_i)\right) \\ \text{subject to} \quad & (7.19) - (7.21), \text{ and } \forall i \in \mathcal{M} \\ & \begin{bmatrix} X_i & I \\ I & \mathscr{X}_i \end{bmatrix} \geqslant 0, \begin{bmatrix} Y_i & I \\ I & \mathscr{Y}_i \end{bmatrix} \geqslant 0. \end{aligned} \tag{7.30}$$

If there exists solutions such that

$$\text{trace}\left(\sum\nolimits_i (X_i \mathscr{X}_i)\right) + \text{trace}\left(\sum\nolimits_i (Y_i \mathscr{Y}_i)\right) = N\,(n + s + k)\,, \tag{7.31}$$

then the conditions in Theorem 7.6 are solvable.

Therefore, we propose the following event-triggered fault detection filter design algorithm to solve the event-triggered fault detection filter design problem.

The fault detection filter design algorithm:
Step 1. Given the disturbance attenuation level $\gamma > 0$ and the scalar $\varrho \in [0, 1)$.
Step 2. Find a feasible set $(X_i^{(0)}, Y_i^{(0)}, \mathscr{X}_i^{(0)}, \mathscr{Y}_i^{(0)})$ satisfying (7.19)–(7.21) and (7.30). Set $\kappa = 0$.
Step 3. Solve the following optimization problem: $\min \text{trace}\left(\sum_i \left(X_i^{(\kappa)} \mathscr{X}_i + X_i \mathscr{X}_i^{(\kappa)}\right)\right)$ $+\text{trace}\left(\sum_i \left(Y_i^{(\kappa)} \mathscr{Y}_i + Y_i \mathscr{Y}_i^{(\kappa)}\right)\right)$ subject to (7.19)–(7.21) and (7.30), and denote f^* to be the optimized value.
Step 4. Substitute the obtained matrix variables $(X_i, Y_i, \mathscr{X}_i, \mathscr{Y}_i)$ into (7.24)–(7.25). If there exists a sufficiently small scalar ϵ, such that $\lvert f^* - 2N(n + s + k)\rvert < \epsilon$, then output the feasible solutions $(X_i, Y_i, \mathscr{X}_i, \mathscr{Y}_i)$. Stop.
Step 5. If $\kappa > \mathbb{N}$, where $\mathbb{N}$ is the maximum number of iterations allowed, stop.
Step 6. Set $\kappa = \kappa + 1$, and $(X_i^{(\kappa)}, Y_i^{(\kappa)}, \mathscr{X}_i^{(\kappa)}, \mathscr{Y}_i^{(\kappa)}) = (X_i, Y_i, \mathscr{X}_i, \mathscr{Y}_i)$, and go to *Step 3*.

Remark 7.8 Note that, in the aforementioned algorithm, an iteration method has been employed to solve the minimization problem instead of the original nonconvex feasibility problem. In order to solve the minimization problem, the stopping criterion $|f^* - 2N(n+s+k)|$ should be checked since it can be numerically difficult to obtain the optimal solutions to meet the condition (7.31). ♦

7.4 Illustrative Example

In this section, we use a numerical example to illustrate the effectiveness of the proposed approach. Consider the stochastic system (7.1) with two subsystems.

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.5 & 0.1 \\ -0.2 & -0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, E_1 = \begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & -0.7 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, E_2 = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, \\
F_1 &= \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.5 & 0.3 \end{bmatrix}, D_1 = 0.3, \\
F_2 &= \begin{bmatrix} -0.1 \\ 0.3 \end{bmatrix}, C_2 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix}, D_2 = 0.4, \\
G_1 &= 0.8, \quad H_1 = 1.5, \quad G_2 = 0.5, \quad H_2 = 0.5.
\end{aligned}
$$

The weighting matrix $W(z)$ in $f_w(z) = W(z)f(z)$ is taken as $W(z) = 5/(z+5)$. Its state space realization is given as (7.7) with $A_w = 0.5$, $B_w = 0.5$ and $C_w = 1$. The transition probabilities matrix comprises two vertices $\Lambda^{(r)}$, $(r = 1, 2)$. The first rows of $\Lambda^{(r)}$, i.e., $\Lambda_1^{(r)}$ are given by

$$
\Lambda_1^{(1)} \triangleq \begin{bmatrix} ? & 0.7 \end{bmatrix}, \quad \Lambda_1^{(2)} \triangleq \begin{bmatrix} ? & 0.3 \end{bmatrix},
$$

and the second rows of $\Lambda^{(r)}$ are given by

$$
\Lambda_2^{(1)} \triangleq \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad \Lambda_2^{(2)} \triangleq \begin{bmatrix} 0.3 & ? \end{bmatrix},
$$

where ? represents the unknown entries.

Our purpose here is to design a full-order filter system (7.6) to generate the residual signal $\delta(t)$ such that the fault detection system (7.8) is stochastically stable with a performance index. Solving fault detection filter problem by using Algorithm of fault detection filter design, it follows that the minimized feasible γ is $\gamma^* = 3.5022$, and the corresponding full-order filter parameters are given as

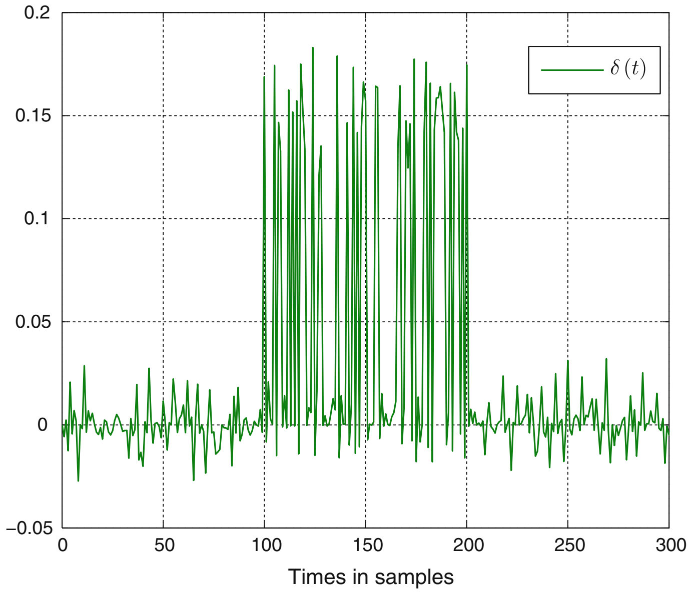

Fig. 7.1 Generated residual $\delta(t)$

$$
\begin{aligned}
A_{f1} &= \begin{bmatrix} -0.6047 & -0.4312 \\ -1.3923 & -1.2383 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} 1.4367 \\ -0.6434 \end{bmatrix}, \\
C_{f1} &= \begin{bmatrix} 2.2321 & 0.3660 \end{bmatrix}, \\
A_{f2} &= \begin{bmatrix} -3.3582 & 0.5149 \\ -1.0526 & -1.1935 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} -0.5447 \\ -0.3294 \end{bmatrix}, \\
C_{f2} &= \begin{bmatrix} -1.3638 & 0.5123 \end{bmatrix}.
\end{aligned}
$$

Figure 7.1 depicts the generated residual signal $\delta(k)$; and Fig. 7.2 presents the evaluation function of $\mathcal{J}(r)$ for both the fault case (solid line) and fault-free case (dash-dot line). With a selected threshold

$$\mathcal{J}_{th} = \sup_{\omega \neq 0, u \neq 0, f=0} \left(\sum_{k=0}^{300} \delta^T(k)\delta(k) \right)^{1/2} = 8.265,$$

the simulation results show that $\left(\sum_{k=0}^{100} \delta^T(k)\delta(k) \right)^{1/2} = 8.373 > \mathcal{J}_{th}$. Thus, the appeared fault can be detected after some time steps. The simulation result illustrates the efficiency of the design techniques proposed in this chapter.

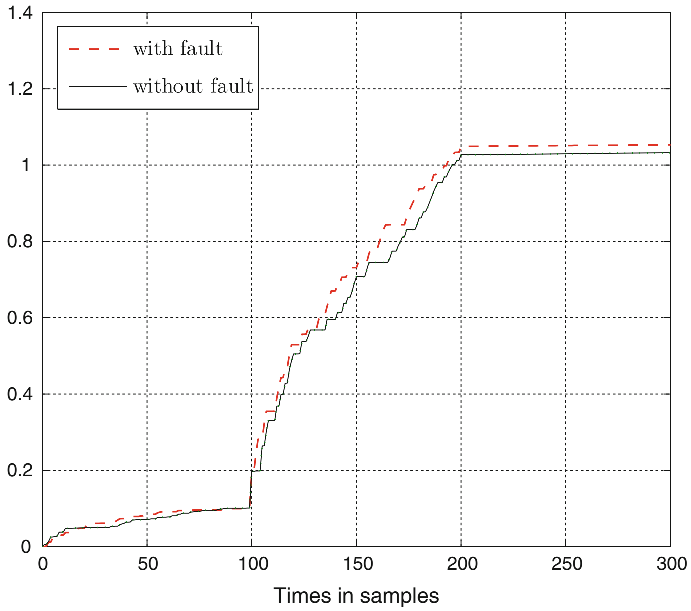

Fig. 7.2 Evaluation function of $\mathcal{J}(r)$

7.5 Conclusion

In this chapter, the problem of fault detection filtering has been addressed for a class of S-MJS with known sojourn probability. A sojourn probability dependent event-triggered filter has been designed to estimate the fault of the measured output signal. Then, a sufficient condition has been proposed to ensure the stochastic stability with an H_2 performance for the filtering error system. Furthermore, the filter has been designed, and a sufficient condition has been decoupled into a convex optimization problem, which can be efficiently handled using standard method. An example has been provided to illustrate the applicability of the proposed techniques.

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Chapter 8
Fuzzy Fault Detection Filtering for Semi-Markovian Jump Systems

Abstract This chapter investigates the problem of fault detection filtering for S-MJS by a Takagi-Sugeno fuzzy approach. Attention is focused on the construction of a fault detection filter such that the estimation error converges to zero in the mean square and meets a prescribed system performance. The designed fuzzy model-based fault detection filter can guarantee the sensitivity of the residual signal to faults and the robustness of the external disturbances. By using the cone complementarity linearization algorithm, the existence conditions for the design of fault detection filters are provided, and the error between the residual signal and the fault signal can be made within a desired region.

8.1 Introduction

In many engineering areas of the real world, most physical systems and processes are described by nonlinear models, which introduce serious difficulties in the control and design of system. It has been well recognized that fuzzy control is a simple and effective approach to controlling many complex nonlinear systems or even non-analytic systems. In many cases, fuzzy logic control (FLC) has been suggested as an alternative approach to conventional control techniques. However, it is also worth mentioning that the traditional FLC techniques suffer from a number of disadvantages such as the difficulty for systematic design and inconsistent performance. Takagi–Sugeno (T–S) fuzzy model is a fuzzy dynamic model, and has been applied to formulate a complex nonlinear systems into a framework. Consequently, T–S fuzzy models are less prone to the curse of dimensionality than other fuzzy models.

In this chapter, we focus on the fault detection filter design problem for S-MJS by a T–S fuzzy approach. Attention is focused on the construction of a fault detection filter such that the estimation error converges to zero in the mean square and the prescribed performance requirement can be guaranteed. Intensive stochastic analysis is carried out to obtain sufficient conditions for ensuring the stochastically stability of the desired fault detection filters, and then, the corresponding solvability conditions for the desired filter gains are established. By using the cone complementarity linearization algorithm, the existence conditions for the design of fault detection filters are provided, at the same time, the error between the residual signal and the fault signal is made as small as possible.

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8.2 Problem Formulation and Preliminaries

Consider the following T–S fuzzy system in complete probability space $(\Omega, \mathscr{F}, \Pr)$:

♦ **Plant Form:**

Rule ν: IF $\theta_1(t)$ is $M_{\nu 1}$, $\theta_2(t)$ is $M_{\nu 2}$, ... and $\theta_p(t)$ is $M_{\nu p}$, THEN

$$\begin{aligned} x(t+1) &= A_{\gamma_t}^{(\nu)} x(t) + B_{\gamma_t}^{(\nu)} u(t) + E_{\gamma_t}^{(\nu)} \omega(t) + F_{\gamma_t}^{(\nu)} f(t) \\ y(t) &= \vartheta(t) C_{\gamma_t}^{(\nu)} x(t) + D_{\gamma_t}^{(\nu)} u(t) + G_{\gamma_t}^{(\nu)} \omega(t) + H_{\gamma_t}^{(\nu)} f(t), \quad \nu = 1, \ldots, L, \end{aligned} \tag{8.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state; $y(t) \in \mathbb{R}^p$ is the measured output; $u(t) \in \mathbb{R}^m$ is the deterministic input; $\omega(t) \in \mathbb{R}^q$ is exogenous disturbance input which belongs to $\ell_2[0, \infty)$; and $f(t) \in \mathbb{R}^l$ is the fault to be detected. $A_{\gamma_t}^{(\nu)}$, $B_{\gamma_t}^{(\nu)}$, $E_{\gamma_t}^{(\nu)}$, $F_{\gamma_t}^{(\nu)}$, $C_{\gamma_t}^{(\nu)}$, $D_{\gamma_t}^{(\nu)}$, $G_{\gamma_t}^{(\nu)}$, and $H_{\gamma_t}^{(\nu)}$ are known constant matrices with appropriate dimensions. $M_{\nu p}$ are the fuzzy sets, L is the number of IF-THEN rules, and $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]$ is the premise variables vector, and it is assumed that the premise variables do not depend on the input variables $u(t)$. The fuzzy basis functions are given by

$$\alpha_\nu(\theta(t)) = \frac{\prod_{s=1}^{p} M_{\nu s}(\theta_\nu(t))}{\sum_{\nu=1}^{L} \prod_{s=1}^{p} M_{\nu s}(\theta_\nu(t))}, \tag{8.2}$$

with $M_{\nu s}(\theta_\nu(t))$ representing the grade of membership of $\theta_\nu(t)$ in $M_{\nu s}$ ($\nu = 1, 2, \ldots, L$; $s = 1, 2, \ldots, p$). Therefore, for all t we have $\alpha_\nu(\theta(t)) \geqslant 0$ and $\sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) = 1$.

In system (8.1), the stochastic variable $\vartheta(t)$ is a Bernoulli-distributed white noise sequence specified by the following distribution law:

$$\Pr(\vartheta(t) = 1) = \mathbf{E}\{\vartheta(t)\} = \bar{\vartheta},$$

where $\bar{\vartheta} \in [0, 1]$ is a known constant.

The variable $\vartheta(t)$ is introduced to represent missing measurements. We use $\vartheta(t)$ to indicate the arrival (with value 1) or loss (with value 0) of the packets. Clearly, for the stochastic variable $\vartheta(t)$, one has

$$\mathbf{E}\{\vartheta(t) - \bar{\vartheta}\} = 0, \quad \mathbf{E}\{|\vartheta(t) - \bar{\vartheta}|^2\} = \bar{\vartheta}(1 - \bar{\vartheta}).$$

Given a pair of $(x(t), u(t))$, the overall fuzzy system is inferred as

$$\begin{aligned} x(t+1) &= \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) \Big[A_{\gamma_t}^{(\nu)} x(t) + B_{\gamma_t}^{(\nu)} u(t) + E_{\gamma_t}^{(\nu)} \omega(t) + F_{\gamma_t}^{(\nu)} f(t) \Big] \\ y(t) &= \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) \Big[\vartheta(t) C_{\gamma_t}^{(\nu)} x(t) + D_{\gamma_t}^{(\nu)} u(t) + G_{\gamma_t}^{(\nu)} \omega(t) + H_{\gamma_t}^{(\nu)} f(t) \Big]. \end{aligned} \tag{8.3}$$

Moreover, system (8.3) can be written as

$$\begin{cases} x(t+1) = A_{\gamma_t}^{(\alpha)} x(t) + B_{\gamma_t}^{(\alpha)} u(t) + E_{\gamma_t}^{(\alpha)} \omega(t) + F_{\gamma_t}^{(\alpha)} f(t) \\ y(t) = \vartheta(t) C_{\gamma_t}^{(\alpha)} x(t) + D_{\gamma_t}^{(\alpha)} u(t) + G_{\gamma_t}^{(\alpha)} \omega(t) + H_{\gamma_t}^{(\alpha)} f(t), \end{cases} \tag{8.4}$$

where

$$\begin{aligned} A_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) A_{\gamma_t}^{(\nu)}, \quad & B_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) B_{\gamma_t}^{(\nu)}, \\ E_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) E_{\gamma_t}^{(\nu)}, \quad & F_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) F_{\gamma_t}^{(\nu)}, \\ C_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) C_{\gamma_t}^{(\nu)}, \quad & D_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) D_{\gamma_t}^{(\nu)}, \\ G_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) G_{\gamma_t}^{(\nu)}, \quad & H_{\gamma_t}^{(\alpha)} &\triangleq \sum_{\nu=1}^{L} \alpha_\nu(\theta(t)) H_{\gamma_t}^{(\nu)}. \end{aligned}$$

Let $\{\gamma_t, t \in \mathbb{Z}^+\}$ be a Markov chain taking values in state-space $\mathcal{M} = \{1, 2, \ldots, M\}$. The evolution of the Markov process $\{\gamma_t, t \in \mathbb{Z}^+\}$ is governed by the following probability transitions:

$$\Pr(\gamma_{t+h} = j | \gamma_t = i) = \lambda_{ij}^h, \tag{8.5}$$

where h is the sojourn time at the ith model, λ_{ij}^h is the transition rate from mode i at time t to mode j at time $t+h$ when $i \neq j$ and $\sum_{j=1}^{M} \lambda_{ij}^h = 1$.

It is assumed that the transition probabilities $\Lambda = [\lambda_{ij}^h]_{M\times M}$ belongs to a polytope P_Λ, with vertices $\Lambda^{(r)}$, $r = 1, 2, \ldots, N$ as follows:

$$P_\Lambda \triangleq \left\{ \Lambda | \, \Lambda = \sum_{\nu=1}^{N} \alpha_\nu(t) \Lambda^{(r)}, \ \sum_{\nu=1}^{N} \alpha_\nu(t) = 1, \ \alpha_\nu(t) \geqslant 0 \right\}, \tag{8.6}$$

where $\Lambda^{(r)} \triangleq [\lambda_{ij}^{(r)}]_{M\times M}$ $(i, j \in \mathcal{M}; r = 1, \ldots, N)$ denotes the given transition probability matrices that contain unknown elements. For notation clarity, for any $i \in \mathcal{M}$, we denote $\mathcal{M} \triangleq \mathcal{M}_{uc}^i \cup \mathcal{M}_{uk}^i$, where

$$\begin{cases} \mathcal{M}_{uc}^i \triangleq \left\{ j | \, \lambda_{ij}^h \text{ is uncertain} \right\}, \\ \mathcal{M}_{uk}^i \triangleq \left\{ j | \, \lambda_{ij}^h \text{ is unknown} \right\}. \end{cases} \tag{8.7}$$

Also, define $\lambda_{uc}^i \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)}, \forall r = 1, 2, \ldots, N$.

For the stochastic system (8.4), we adopt the following fault detection filter:

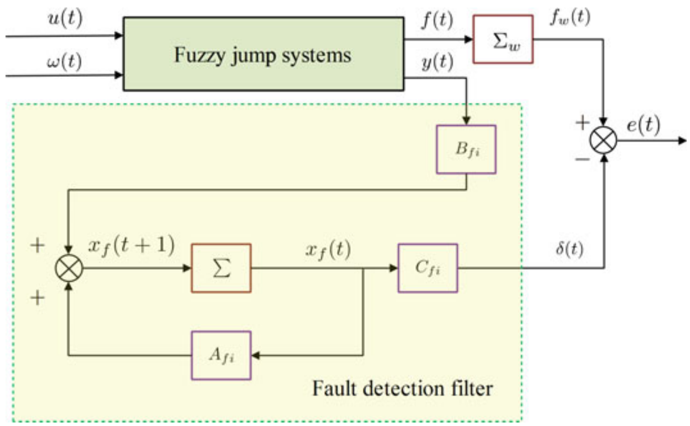

Fig. 8.1 Block diagram of the fault detection filter design

$$\begin{cases} x_f(t+1) = A_{fi}x_f(t) + B_{fi}y(t) \\ \quad\quad \delta(t) = C_{fi}x_f(t) \end{cases} \tag{8.8}$$

where $x_f(t) \in \mathbb{R}^s$ is the state vector of the fault detection filter; $\delta(t) \in \mathbb{R}^l$ is the residual signal; A_{fi}, B_{fi} and C_{fi} are the filter parameters to be designed.

To improve or enhance the performance of fault detection system, we add a weighting matrix function into the fault vector $f(z)$, that is, $f_w(z) = W(z)f(z)$, where $W(z)$ is given a priori [1]. One state space realization of $f_w(z) = W(z)f(z)$ can be

$$\begin{cases} x_w(t+1) = A_w x_w(t) + B_w f(t), \\ \quad\quad f_w(t) = C_w x_w(t) \end{cases} \tag{8.9}$$

where $x_w(t) \in \mathbb{R}^k$ is the state vector, and matrices A_w, B_w, C_w are priorly chosen. Figure 8.1 shows the block diagram of the fault detection filter design.

Denoting $e(t) \triangleq \delta(t) - f_w(t)$, and augmenting the model of (8.4) to incorporate the states of (8.8) and (8.9), then the overall dynamics of fault detection system is governed by

$$\begin{cases} \xi(t+1) = \left(\tilde{A}_i^{(\alpha)} + \tilde{\vartheta}\mathcal{A}_i^{(\alpha)}\right)\xi(t) + \tilde{B}_i^{(\alpha)}\upsilon(t) \\ \quad\quad e(t) = \tilde{C}_i^{(\alpha)}\xi(t), \end{cases} \tag{8.10}$$

where

$$\xi(t) \triangleq \begin{bmatrix} x^T(t) & x_f^T(t) & x_w^T(t) \end{bmatrix}^T, \quad \upsilon(t) \triangleq \begin{bmatrix} u^T(t) & \omega^T(t) & f^T(t) \end{bmatrix}^T,$$

and

$$\left\{\begin{array}{l} \tilde{A}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} A_i^{(\alpha)} & 0 & 0 \\ \bar{\vartheta} B_{fi} C_i^{(\alpha)} & A_{fi} & 0 \\ \hline 0 & 0 & A_w \end{array}\right], \quad \tilde{\vartheta} \triangleq \vartheta(t) - \bar{\vartheta}, \\ \mathcal{A}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} A_i^{(\alpha)} & 0 & 0 \\ B_{fi} C_i^{(\alpha)} & A_{fi} & 0 \\ \hline 0 & 0 & A_w \end{array}\right], \quad \tilde{C}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} 0 & C_{fi} & -C_w \end{array}\right], \\ \tilde{B}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} B_i^{(\alpha)} & E_i^{(\alpha)} & F_i^{(\alpha)} \\ B_{fi} D_i^{(\alpha)} & B_{fi} G_i^{(\alpha)} & B_{fi} H_i^{(\alpha)} \\ \hline 0 & 0 & B_w \end{array}\right]. \end{array}\right. \tag{8.11}$$

We introduce the following definitions, they will play key roles in deriving our main results.

Definition 8.1 [2] The error system (8.10) is said to be stochastically stable, if for any initial state $(\xi(0), r_0)$, the following condition holds

$$\mathbf{E}\left\{\sum_{k=0}^{\infty} \|\xi(t)\|^2 |(\xi(0), r_0)\right\} < \Gamma(\xi(0), r_0),$$

in case of $\upsilon(t) = 0$, where $\Gamma(\xi(0), r_0)$ is a nonnegative function of the system initial values.

Definition 8.2 [3] For given a scalar $\gamma > 0$, the fault detection system (8.10) is said to be stochastically stable with a generalized H_2 disturbance attenuation γ, if it is stochastically stable with $\upsilon(t) = 0$, and under zero initial condition, that is, $\xi(0) = 0$, $\|e(t)\|_\infty < \gamma \|\upsilon(t)\|_2$ for all nonzero $\upsilon(t) \in \ell_2[0, \infty)$, where $\|e(t)\|_\infty \triangleq \sup_t \sqrt{\mathbf{E}\{|e(t)|^2\}}$.

Therefore, the fault detection filtering problem can be expressed as the following two steps.

Step 1. For the stochastic system (8.4), develop a filter in the form of (8.8) to generate a residual signal $\delta(t)$. Meanwhile, the filter is designed to assure that the resulting overall fault detection system (8.10) to be stochastically stable with a disturbance attenuation γ.

Step 2. After generating the residual signal, a residual evaluation value will be computed through a prescribed evaluation function. One of the widely adopted approaches is to select a threshold and a residual evaluation function. We consider the following evaluation function: $\mathcal{J}(\delta)$ (where δ denotes $\delta(t)$ for simplicity) and a threshold $\mathcal{J}_{th}$ are selected as

$$\begin{aligned} \mathcal{J}(\delta) &\triangleq \left(\sum_{k=t_0}^{t_0+t^\star} \delta^T(t)\delta(t)\right)^{1/2}, \\ \mathcal{J}_{th} &\triangleq \sup_{0 \neq \omega \in \ell_2, 0 \neq u \in \ell_2, f=0} \mathcal{J}(\delta), \end{aligned} \tag{8.12}$$

where t_0 denotes the initial evaluation time, and $t^\star$ stands for the evaluation time. Then, we can detect the faults by using the following logical relationship:

$$\begin{cases} \mathcal{J}(\delta) > \mathcal{J}_{th} \Rightarrow \text{ with faults } \Rightarrow \text{ alarm,} \\ \mathcal{J}(\delta) \leqslant \mathcal{J}_{th} \Rightarrow \text{ no faults.} \end{cases}$$

Before ending this section, let us recall the following lemmas, which will be used in the next section.

8.3 Main Results

8.3.1 Fault Detection Filtering Analysis

We first investigate the stochastic stability with a disturbance attenuation level of the fault detection system (8.10).

Theorem 8.3 *Given a disturbance attenuation level $\gamma > 0$, the fault detection system (8.10) is stochastically stable with a performance disturbance γ, if there exist positive definite matrices $P_i^{(\nu)}$, $i \in \mathcal{M}$ such that for all $\nu \in \{1, 2, \ldots, L\}$,*

$$\begin{bmatrix} -P_i^{(\nu)} & 0 & \left(\tilde{A}_i^{(\nu)}\right)^T \mathcal{P}_j^{(\nu)} & \bar{\vartheta}(1-\bar{\vartheta})\left(\mathcal{A}_i^{(\nu)}\right)^T \mathcal{P}_j^{(\nu)} \\ * & -I & (\tilde{B}_i^{(\nu)})^T \mathcal{P}_j^{(\nu)} & 0 \\ * & * & -\mathcal{P}_j^{(\nu)} & 0 \\ * & * & * & -\bar{\vartheta}(1-\bar{\vartheta})\mathcal{P}_j^{(\nu)} \end{bmatrix} < 0 \tag{8.13}$$

$$\begin{bmatrix} -P_i^{(\nu)} & (\tilde{C}_i^{(\nu)})^T \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{8.14}$$

where $\begin{cases} \mathcal{P}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} P_j^{(\nu)}, & \forall j \in \mathcal{M}_{uc}^i, \\ \mathcal{P}_j^{(\nu)} \triangleq P_j^{(\nu)}, & \forall j \in \mathcal{M}_{uk}^i. \end{cases}$

Proof Choose a stochastic Lyapunov function candidate as:

$$V(\xi(t), \alpha(t), \gamma_t) \triangleq \xi^T(t)\left(\sum_{\nu=1}^{L} \alpha_\nu(t) P_{\gamma_t}^{(\nu)}\right)\xi(t) \tag{8.15}$$

where $P_i^{(\nu)}$, $i \in \mathcal{M}$ are positive definite matrices to be determined.

We first demonstrate the stochastic stability of the fault detection system (8.10) with $\upsilon(t) = 0$. From (8.15), we have

$$
\begin{aligned}
&\mathbf{E}\Big\{V\big(\xi(t+1), \alpha(t+1), r_{t+1}\big)\big|\big(\xi(t), \alpha(t), \gamma_t\big)\Big\} - V\big(\xi(t), \alpha(t), \gamma_t\big) \\
&= \mathbf{E}\Big\{\xi^T(t+1)P^{(\nu+1)}_{(r_{t+1}=j|\gamma_t=i)}\xi(t+1)\Big\} - \xi^T(t)P_i^{(l)}\xi(t) \\
&= \sum_{l=1}^{L}\sum_{m=1}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_m(t)\alpha_\nu(t+1)\Bigg[\xi^T(t)(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(m)}\xi(t) \\
&\quad + \bar{\vartheta}(1-\bar{\vartheta})\xi^T(t)(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(m)}\xi(t) - \xi^T(t)P_i^{(l)}\xi(t)\Bigg] \\
&\leqslant \sum_{l=1}^{L}\sum_{m=1}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_m(t)\alpha_\nu(t+1)\xi^T(t)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(m)} \\
&\quad + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(m)} - P_i^{(l)}\Bigg]\xi(t) \\
&= \sum_{l=1}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_\nu(t+1)\xi^T(t)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta}) \\
&\quad \times (\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} - P_i^{(l)}\Bigg]\xi(t) + \sum_{l=1}^{L}\sum_{m>l}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_m(t) \\
&\quad \times \alpha_\nu(t+1)\xi^T(t)\Bigg[(\tilde{A}_i^{(l)})^T\left(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\right)\tilde{A}_i^{(m)} - P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta}) \\
&\quad \times (\mathcal{A}_i^{(l)})^T\left(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\right)\mathcal{A}_i^{(m)} + (\tilde{A}_i^{(m)})^T\left(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\right)\tilde{A}_i^{(l)} \\
&\quad + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\left(\sum_{j=1}^{M}\lambda_{ij}^h P_j^{(\nu)}\right)\mathcal{A}_i^{(l)} - P_i^{(m)}\Bigg]\xi(t). \qquad (8.16)
\end{aligned}
$$

Now in view of $\sum_{j=1}^{M}\lambda_{ij}^h = 1$, we rewrite the right-hand side of (8.16) as $\xi^T(t)\Upsilon_i\xi(t)$, where

$$
\begin{aligned}
\Upsilon_i \triangleq &\sum_{l=1}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_\nu(t+1)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T \\
&\times\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\Bigg)P_i^{(l)}\Bigg] + \sum_{l=1}^{L}\sum_{m>l}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_m(t)
\end{aligned}
$$

$$\times \alpha_\nu(t+1)\bigg[(\tilde{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(m)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T$$
$$\times\bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(m)} - (\tilde{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(l)} - \bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\bigg)P_i^{(l)}$$
$$+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(l)} - \bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\bigg)P_i^{(m)}\bigg].$$

Therefore, considering (8.7), we obtain

$$\begin{aligned}
\Upsilon_i \triangleq{}& \sum_{l=1}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_\nu(t+1)\bigg[(\tilde{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(l)} - \bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\bigg)P_i^{(l)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(l)} + (\tilde{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(l)} \\
&-\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\bigg)P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(l)}\bigg] \\
&+\sum_{l=1}^{L}\sum_{m>l}^{L}\sum_{\nu=1}^{L}\alpha_l(t)\alpha_m(t)\alpha_\nu(t+1)\bigg[(\tilde{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(m)} \\
&-(\tilde{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(l)} - \bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\bigg)P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T \\
&\times\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(m)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(l)} \\
&-\bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\bigg)P_i^{(m)} + (\tilde{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\tilde{A}_i^{(m)} \\
&-(\mathscr{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathscr{A}_i^{(l)} - \bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\bigg)P_i^{(l)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(m)} - \bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\bigg)P_i^{(m)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\bigg)\mathcal{A}_i^{(l)}\bigg].
\end{aligned} \tag{8.17}$$

Then, since one always has $\lambda_{ij}^h \geqslant 0, \forall j \in \mathcal{M}$, when (8.13) is satisfied we have $\Upsilon_i < 0$. Obviously, if we have no knowledge on $\lambda_{ij}^h, \forall j \in \mathcal{M}_{uk}^i$ in (8.13), we have

$$(\tilde{A}_i^{(\nu)})^T \mathcal{P}_j^{(\nu)} \tilde{A}_i^{(\nu)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(\nu)})^T \mathcal{P}_j^{(\nu)} \mathcal{A}_i^{(\nu)} - P_i^{(\nu)} < 0.$$

Then, combining with (8.16) and (8.17), we have

$$\begin{aligned}
&\mathbf{E}\{V_{r_{t+1}}(\xi(t+1), t+1) | (\xi(t), \gamma_t = i)\} - V_{\gamma_t}(\xi(t), t) \\
&\quad \leqslant \xi^T(t)\Upsilon_i \xi(t) \\
&\quad \leqslant -\lambda_{\min}(\Upsilon_i)\, \xi^T(t)\xi(t) \\
&\quad \leqslant -\beta \xi^T(t)\xi(t),
\end{aligned}$$

where $\beta \triangleq \inf_{i \in \mathcal{M}} \{\lambda_{\min}(\Upsilon_i)\}$.

It follows that

$$\lim_{T\to\infty} \mathbf{E}\left\{ \sum_{k=0}^{T} \xi^T(t)\xi(t) \Big| (\xi_0, r_0) \right\} \leqslant \Gamma(\xi_0, r_0),$$

where $\Gamma(\xi_0, r_0)$ is a positive number. Thus, the fault detection system (8.10) with $\upsilon(t) = 0$ is stochastically stable in the sense of Definition 8.1.

To establish the generalized H_2 performance for the fault detection system (8.10), we assume zero initial condition, that is, $\xi(0) = 0$, then we have $V(\xi(t), \alpha(t), \gamma_t)$ $|_{t=0} = 0$. Consider the index

$$\mathcal{J} \triangleq V(\xi(t), \alpha(t), \gamma_t) - \sum_{s=0}^{t-1} \upsilon^T(s)\upsilon(s).$$

For any nonzero $\upsilon(t) \in \ell_2[0, \infty)$ and $t > 0$, we have

$$\begin{aligned}
\mathcal{J} &= \mathbf{E}\Big\{ V(\xi(t), \alpha(t), \gamma_t) - V(\xi(0), \alpha(0), r_0) \Big\} - \sum_{s=0}^{t-1} \upsilon^T(s)\upsilon(s) \\
&= \sum_{s=0}^{t-1} \mathbf{E}\left\{ \Delta V(\xi(s), \alpha(s)) - \upsilon^T(s)\upsilon(s) \right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E}\Big\{ \Delta V(\xi(s), \alpha(s)) \Big\} = \sum_{i=1}^{L} \sum_{l=1}^{L} \alpha_i(s)\alpha_l(s) \mathbf{E}\Big\{ \Big[\Big(\tilde{A}_i^{(j)} + \tilde{\vartheta} \tilde{A}_i^{(j)} \Big) \xi(s) + \tilde{B}_i^{(j)} \upsilon(s) \Big]^T \\
\mathcal{P}_j \Big[\Big(\tilde{A}_i^{(j)} + \tilde{\vartheta} \tilde{A}_i^{(j)} \Big) \xi(s) + \tilde{B}_l^{(j)} \upsilon(s) \Big] - \xi^T(s) P_i \xi(s) \Big\}
\end{aligned}$$

$$\leqslant \sum_{i=1}^{L}\sum_{l=1}^{L}\alpha_i(s)\alpha_l(s)\begin{bmatrix}\xi(s)\\ \upsilon(s)\end{bmatrix}^T \left\{\begin{bmatrix}(\tilde{A}_i^{(j)})^T\\ (\tilde{B}_i^{(j)})^T\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\tilde{A}_l^{(j)} & \tilde{B}_l^{(j)}\end{bmatrix} - \begin{bmatrix}P_i & 0\\ 0 & 0\end{bmatrix} + \bar{\vartheta}(1-\bar{\vartheta})\begin{bmatrix}(\mathcal{A}_i^{(j)})^T\\ 0\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\mathcal{A}_l^{(j)} & 0\end{bmatrix}\right\}\begin{bmatrix}\xi(s)\\ \upsilon(s)\end{bmatrix}.$$

Then,

$$\mathcal{J} \leqslant \sum_{s=0}^{t-1}\left\{\sum_{i=1}^{L}\sum_{l=1}^{L}\alpha_i(s)\alpha_l(s)\begin{bmatrix}\xi(s)\\ \upsilon(s)\end{bmatrix}^T \left(\begin{bmatrix}(\tilde{A}_i^{(j)})^T\\ (\tilde{B}_i^{(j)})^T\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\tilde{A}_l^{(j)} & \tilde{B}_l^{(j)}\end{bmatrix} - \begin{bmatrix}P_i^{(\nu)} & 0\\ 0 & I\end{bmatrix} + \bar{\vartheta}(1-\bar{\vartheta})\begin{bmatrix}(\mathcal{A}_i^{(j)})^T\\ 0\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\mathcal{A}_l^{(j)} & 0\end{bmatrix}\right)\begin{bmatrix}\xi(s)\\ \upsilon(s)\end{bmatrix}\right\}.$$

It is shown from (8.13) that

$$\begin{bmatrix}(\tilde{A}_i^{(\nu)})^T\\ (\tilde{B}_i^{(\nu)})^T\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\tilde{A}_i^{(j)} & \tilde{B}_i^{(j)}\end{bmatrix} + \bar{\vartheta}(1-\bar{\vartheta})\begin{bmatrix}(\mathcal{A}_i^{(j)})^T\\ 0\end{bmatrix}\mathcal{P}_j\begin{bmatrix}\mathcal{A}_l^{(j)} & 0\end{bmatrix} - \begin{bmatrix}P_i^{(\nu)} & 0\\ 0 & I\end{bmatrix} < 0,$$

this guarantees $\mathcal{J} < 0$, which further implies

$$V(\xi,\alpha) < \sum_{s=0}^{t-1}\upsilon^T(s)\upsilon(s). \tag{8.18}$$

Moreover, (8.14) yields

$$(\tilde{C}_i^{(j)})^T\tilde{C}_i^{(j)} - \gamma^2 P_i < 0. \tag{8.19}$$

Then, one can obtain that, for all $t > 0$,

$$\mathbf{E}\{e^T(t)e(t)\} - \gamma^2 V(\xi,\alpha) \leqslant \sum_{i=1}^{N}\sum_{l=1}^{N}\alpha_i(t)\alpha_l(t)\xi^T(t)\left(\tilde{C}_i^T\tilde{C}_l - \gamma^2 P_i\right)\xi(t).$$

Combining with (8.18)–(8.19) yields the following inequalities:

$$\mathbf{E}\{e^T(t)e(t)\} < \gamma^2 V(\xi,\alpha) \leqslant \gamma^2\sum_{s=0}^{t-1}\upsilon^T(s)\upsilon(s) \leqslant \gamma^2\sum_{s=0}^{\infty}\upsilon^T(s)\upsilon(s).$$

This implies that $\|e(t)\|_\infty < \gamma\,\|\upsilon(t)\|_2$ for all nonzero $\upsilon(t) \in \ell_2[0,\infty)$. This completes the proof. ■

8.3.2 Fault Detection Filter Design

Next, we will design the fault detection filter in (8.8) based on Theorem 8.3 and Lemma 1.29. To this end, the filter matrices (A_{fi}, B_{fi}, C_{fi}) should be determined to guarantee the stochastic stability of the filter error system (8.10) with a disturbance attenuation level γ. We establish a sufficient condition for the existence of such a filter through the following theorem:

Theorem 8.4 *Consider the stochastic system* (8.4)*. A fault detection filter* (8.8) *exists, if there exist positive definite matrices* $X_i^{(\nu)}$, $Y_i^{(\nu)}$, *and positive definition matrices* $\mathscr{X}_i^{(\nu)} > 0$, $\mathscr{Y}_i^{(\nu)} > 0$, $i \in \mathcal{M}$, *such that*

$$\left[\begin{array}{cccc|c} -X_i^{(\nu)} & 0 & 0 & (\bar{A}_i^{(\nu)})^T J^T & \\ * & -Y_i^{(\nu)} & 0 & 0 & \Psi_1 \\ * & * & -I & (\bar{B}_i^{(\nu)})^T J^T & \\ * & * & * & -J\mathscr{X}_j^{(\nu)} J^T & \\ \hline * & * & * & * & -\Psi_2 \end{array}\right] < 0 \tag{8.20}$$

$$\left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|c} -X_i^{(\nu)} & 0 & 0 & \\ * & -Y_i^{(\nu)} & 0 & \Psi_3 \\ * & * & -I & \\ \hline * & * & * & -\Psi_4 \end{array}\right] \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right]^T < 0, \tag{8.21}$$

$$\begin{bmatrix} -JX_i^{(\nu)} J^T & 0 & 0 \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{8.22}$$

$$\mathcal{X}_j^{(\nu)} \mathscr{X}_j^{(\nu)} = I,\ \mathcal{Y}_j^{(\nu)} \mathscr{Y}_j^{(\nu)} = I, \tag{8.23}$$

where

$$\Psi_1 \triangleq \begin{bmatrix} 0 & \varpi(\bar{A}_i^{(\nu)})^T J^T & 0 \\ A_w^T & 0 & \varpi A_w^T \\ \hat{B}_w^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Psi_3 \triangleq \begin{bmatrix} (\bar{A}_i^{(\nu)})^T & 0 & \varpi(\bar{A}_i^{(\nu)})^T & 0 \\ 0 & A_w^T & 0 & \varpi A_w^T \\ (\bar{B}_i^{(\nu)})^T & \hat{B}_w^T & 0 & 0 \end{bmatrix},$$

$$\Psi_2 \triangleq \text{diag}\left\{\mathscr{Y}_j^{(\nu)}, \varpi J \mathscr{X}_j^{(\nu)} J^T, \varpi \mathscr{Y}_j^{(\nu)}\right\}, \quad \varpi \triangleq \bar{\vartheta}(1-\bar{\vartheta}),$$

$$\Psi_4 \triangleq \text{diag}\left\{\mathscr{X}_j^{(\nu)}, \mathscr{Y}_j^{(\nu)}, \varpi \mathscr{X}_j^{(\nu)}, \varpi \mathscr{Y}_j^{(\nu)}\right\},$$

$$\mathcal{X}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} X_j^{(\nu)}, \forall j \in \mathcal{M}_{uc}^i, \quad \mathcal{X}_j^{(\nu)} \triangleq X_j^{(\nu)}, \forall j \in \mathcal{M}_{uk}^i,$$

$$\mathcal{Y}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} Y_j^{(\nu)}, \forall j \in \mathcal{M}_{uc}^i, \quad \mathcal{Y}_j^{(\nu)} \triangleq Y_j^{(\nu)}, \forall j \in \mathcal{M}_{uk}^i.$$

Moreover, if the above conditions (8.20)–(8.23) are feasible, then the system matrices of a fault detection filter (8.8) are given by

$$
\begin{aligned}
\mathcal{G}_i &= -\Pi_{1i}^{-1}U_{1i}^T\Lambda_{1i}V_{1i}^T(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1} + \Pi_{1i}^{-1}\Xi_{1i}^{1/2}L_{1i}(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1/2},\\
\mathcal{K}_i^{(\nu)} &= -\Pi_{2i}^{-1}U_{2i}^T\Lambda_{2i}V_{2i}^T(V_{2i}\Lambda_{2i}V_{2i}^T)^{-1} + \Pi_{2i}^{-1}\Xi_{2i}^{1/2}L_{2i}(V_{2i}\Lambda_{2i}V_{2i}^T)^{-1/2},\\
\Lambda_{1i} &= (U_{1i}\Pi_{1i}^{-1}U_{1i}^T - W_{1i})^{-1} > 0,\\
\Lambda_{2i} &= (U_{2i}\Pi_{2i}^{-1}U_{2i}^T - W_{2i})^{-1} > 0,\\
\Xi_{1i} &= \Pi_{1i} - U_{1i}^T(\Lambda_{1i} - \Lambda_{1i}V_{1i}^T(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1}V_{1i}\Lambda_{1i})U_{1i} > 0,\\
\Xi_{2i} &= \Pi_{2i} - U_{2i}^T(\Lambda_{2i} - \Lambda_{2i}V_{2i}^T(V_{2i}\Lambda_{2i}V_{2i}^T)^{-1}V_{2i}\Lambda_{2i})U_{2i} > 0,
\end{aligned}
$$

where $\mathcal{G}_i \triangleq \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix}$, $\mathcal{K}_i \triangleq C_{fi}$. *In addition,* $\Pi_{\kappa i}$ *and* $L_{\kappa i}$, $\kappa = 1, 2$ *are any appropriate matrices satisfying* $\Pi_{\kappa i} > 0$, $\|L_{\kappa i}\| < 1$ *and*

$$
\begin{aligned}
\bar{A}_i^{(\nu)} &= \begin{bmatrix} A_i^{(\nu)} & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0_{(n+s)\times l} \\ 0_{k\times l} \\ I_{l\times l} \end{bmatrix}, \quad \bar{B}_i^{(\nu)} = \begin{bmatrix} B_i^{(\nu)} & E_i^{(\nu)} & F_i^{(\nu)} \\ 0 & 0 & 0 \end{bmatrix},\\
M_i &= \begin{bmatrix} 0_{s\times n} & I_{s\times s} & 0_{s\times k} & 0_{s\times m} & 0_{s\times q} & 0_{s\times l} \\ C_i^{(\nu)} & 0_{p\times s} & 0_{p\times k} & D_i^{(\nu)} & G_i^{(\nu)} & H_i^{(\nu)} \end{bmatrix}^T, \quad R_i^{(\nu)} = \begin{bmatrix} 0_{s\times n} & I_{s\times s} \\ C_i^{(\nu)} & 0_{p\times s} \end{bmatrix},\\
E &= \begin{bmatrix} 0_{n\times s} \\ I_{s\times s} \end{bmatrix}, \quad S_i^{(\nu)} = \begin{bmatrix} 0_{s\times m} & 0_{s\times q} & 0_{s\times l} \\ D_i^{(\nu)} & G_i^{(\nu)} & H_i^{(\nu)} \end{bmatrix}, \quad V_2 = \begin{bmatrix} T & 0_{s\times k} & 0_{s\times l} \end{bmatrix},\\
W_{1ij} &= \left[\begin{array}{cccc|c} -X_i^{(\nu)} & 0 & 0 & \bar{A}_i^T & \\ * & -Y_i^{(\nu)} & 0 & 0 & \bar{\Psi}_1 \\ * & * & -I & \bar{B}_i^T & \\ * & * & * & -\mathscr{X}_j^{(\nu)} & \\ \hline * & * & * & * & -\bar{\Psi}_2 \end{array}\right], \quad U_1 = \begin{bmatrix} 0_{(n+s)\times s} \\ 0_{k\times s} \\ 0_{(m+q+l)\times s} \\ \bar{\vartheta}E \\ 0_{k\times s} \\ E \\ 0_{k\times s} \end{bmatrix},\\
\bar{\Psi}_1 &= \begin{bmatrix} 0 & \varpi\bar{A}_i^T & 0 \\ A_w^T & 0 & \varpi A_w^T \\ \hat{B}_w^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W_{2i} = \begin{bmatrix} -X_i^{(\nu)} & 0 & 0 \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix},\\
\bar{\Psi}_2 &= \mathrm{diag}\left\{\mathscr{Y}_j^{(\nu)}, \varpi\mathscr{X}_j^{(\nu)}, \varpi\mathscr{Y}_j^{(\nu)}\right\}, \quad J = \begin{bmatrix} I_{n\times n} & 0_{n\times s} \end{bmatrix},\\
V_{1i} &= \begin{bmatrix} R_i & 0_{(p+s)\times k} & S_i & 0_{(p+s)\times(n+s)} & 0_{(p+s)\times k} \end{bmatrix}, \quad T = \begin{bmatrix} 0_{s\times n} & I_{s\times s} \end{bmatrix}. \qquad (8.24)
\end{aligned}
$$

Proof Set $P_i^{(\nu)} \triangleq \text{diag}\left(X_i^{(\nu)}, Y_i^{(\nu)}\right)$, $i \in \mathcal{M}$ in (8.13), where $X_i^{(\nu)} \in \mathbb{R}^{(n+s)\times(n+s)}$ and $Y_i^{(\nu)} \in \mathbb{R}^{k\times k}$. Then, from Theorem 8.3, the fault detection system (8.10) is stochastically stable with a performance index γ, if there exist positive definite matrices $X_i^{(\nu)}$ and $Y_i^{(\nu)}$ such that

$$\begin{bmatrix} -X_i^{(\nu)} & 0 & 0 & (\hat{A}_i^{(\nu)})^T & 0 & \varpi(\hat{\mathcal{A}}_i^{(\nu)})^T & 0 \\ * & -Y_i^{(\nu)} & 0 & 0 & A_w^T & 0 & \varpi A_w^T \\ * & * & -I & (\hat{B}_i^{(\nu)})^T & \hat{B}_w^T & 0 & 0 \\ * & * & * & -\mathcal{X}_j^{(\nu)} & 0 & 0 & 0 \\ * & * & * & * & -\mathcal{Y}_j^{(\nu)} & 0 & 0 \\ * & * & * & * & * & -\varpi\mathcal{X}_j^{(\nu)} & 0 \\ * & * & * & * & * & * & -\varpi\mathcal{Y}_j^{(\nu)} \end{bmatrix} < 0, \tag{8.25}$$

$$\begin{bmatrix} -X_i^{(\nu)} & 0 & (\hat{C}_i^{(\nu)})^T \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{8.26}$$

where

$$\begin{aligned} \hat{A}_i^{(\nu)} &\triangleq \begin{bmatrix} A_i^{(\nu)} & 0 \\ \bar{\vartheta} B_{fi} C_i^{(\nu)} & A_{fi} \end{bmatrix}, \quad \hat{\mathcal{A}}_i^{(\nu)} \triangleq \begin{bmatrix} A_i^{(\nu)} & 0 \\ B_{fi} C_i^{(\nu)} & A_{fi} \end{bmatrix}, \\ \hat{B}_i^{(\nu)} &\triangleq \begin{bmatrix} B_i^{(\nu)} & E_i^{(\nu)} & F_i^{(\nu)} \\ B_{fi} D_i^{(\nu)} & B_{fi} G_i^{(\nu)} & B_{fi} H_i^{(\nu)} \end{bmatrix}, \\ \hat{B}_w &\triangleq \begin{bmatrix} 0 & 0 & B_w \end{bmatrix}, \quad \hat{C}_i^{(\nu)} \triangleq \begin{bmatrix} 0 & C_{fi} \end{bmatrix}. \end{aligned} \tag{8.27}$$

Rewrite (8.27) in the following form:

$$\begin{aligned} \hat{A}_i^{(\nu)} &= \bar{A}_i^{(\nu)} + \bar{\vartheta} E \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} R_i^{(\nu)}, \quad \hat{C}_i^{(\nu)} = C_{fi} T, \\ \hat{\mathcal{A}}_i^{(\nu)} &= \bar{A}_i^{(\nu)} + E \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} R_i^{(\nu)}, \\ \hat{B}_i^{(\nu)} &= \bar{B}_i^{(\nu)} + E \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} S_i^{(\nu)}, \end{aligned} \tag{8.28}$$

where $\bar{A}_i^{(\nu)}$, $\bar{B}_i^{(\nu)}$, $\bar{C}_i^{(\nu)}$, E, $R_i^{(\nu)}$, $S_i^{(\nu)}$ and T are defined in (8.24).

Using (8.28), the inequalities (8.25) and (8.26) can be rewritten as

$$W_{1ij} + U_1 \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} V_{1i} + \left(U_1 \begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix} V_{1i}\right)^T < 0, \tag{8.29}$$

$$W_{2i} + U_2 C_{fi} V_2 + \left(U_2 C_{fi} V_2\right)^T < 0, \tag{8.30}$$

where W_{1ij}, W_{2i}, U_1, V_{1i}, U_2 and V_2 are with same definition as in (8.24).

Next, we set

$$U_1^{\perp} = \text{diag}\{I, I, I, J, I, J, I\}, V_{1i}^{T\perp} = \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right].$$

By using Lemma 1.29, inequality (8.29) is solvable for $\begin{bmatrix} A_{fi} & B_{fi} \end{bmatrix}$ if and only if (8.20) and (8.21) are satisfied.

In addition, set

$$U_2^{\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad V_2^{T\perp} = \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then, inequality (8.30) is solvable for C_{fi} if and only if (8.22) hold. This completes the proof. ■

Remark 8.5 Due to the matrix equation (8.23) are not LMIs, it is quite difficult to find a minimum γ by using convex optimization algorithm. However, we can solve this problem utilize the cone complementarity linearization algorithm [4]. ♦

From the above discussion, we can solve the nonconvex feasibility problem by formulating it into the following sequential optimization problem.

Fault Detection Filter Design Problem:

$$\begin{aligned} \min \quad & \text{trace}\left(\sum\nolimits_i \left(X_i^{(\nu)} \mathscr{X}_i^{(\nu)}\right)\right) + \text{trace}\left(\sum\nolimits_i \left(Y_i^{(\nu)} \mathscr{Y}_i^{(\nu)}\right)\right) \\ \text{subject to} \quad & \text{(8.20)–(8.22) and } \forall i \in \{1, \ldots, S\} \\ & \begin{bmatrix} X_i^{(\nu)} & I \\ I & \mathscr{X}_i^{(\nu)} \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} Y_i^{(\nu)} & I \\ I & \mathscr{Y}_i^{(\nu)} \end{bmatrix} \geqslant 0. \end{aligned} \tag{8.31}$$

If there exists solutions such that

$$\begin{aligned} & \text{trace}\left(\sum\nolimits_i \left(X_i^{(\nu)} \mathscr{X}_i^{(\nu)}\right)\right) + \text{trace}\left(\sum\nolimits_i \left(Y_i^{(\nu)} \mathscr{Y}_i^{(\nu)}\right)\right) \\ & = N\left(n + s + k\right), \end{aligned} \tag{8.32}$$

then the conditions in Theorem 8.4 are solvable.

Therefore, we propose the following fault detection filter algorithm to design the fault detection filter.

Heuristic fault detection filter algorithm
Step 1. Given the disturbance attenuation level $\gamma > 0$.
Step 2. Find a feasible set $(X_i^{(\nu 0)}, Y_i^{(\nu 0)}, \mathscr{X}_i^{(\nu 0)}, \mathscr{Y}_i^{(\nu 0)})$ satisfying (8.20)–(8.22) and (8.31). Set $\kappa = 0$.
Step 3. Solve the following optimization problem: $\min \operatorname{trace}\left(\sum_i \left(X_i^{(\nu\kappa)}\mathscr{X}_i + X_i\mathscr{X}_i^{(\nu\kappa)}\right)\right) + \operatorname{trace}\left(\sum_i \left(Y_i^{(\nu\kappa)}\mathscr{Y}_i + Y_i\mathscr{Y}_i^{(\nu\kappa)}\right)\right)$ subject to (8.20)–(8.22) and (8.31), and denote f^* to be the optimized value.
Step 4. Substitute the obtained matrix variables $(X_i^{(\nu)}, Y_i^{(\nu)}, \mathscr{X}_i^{(\nu)}, \mathscr{Y}_i^{(\nu)})$ into (8.25)–(8.26). If there exists a sufficiently small scalar ϵ, such that $\lvert f^* - 2N(n+s+k)\rvert < \epsilon$, then output the feasible solutions $(X_i^{(\nu)}, Y_i^{(\nu)}, \mathscr{X}_i^{(\nu)}, \mathscr{Y}_i^{(\nu)})$. Stop.
Step 5. If $\kappa > \mathbb{N}$, where $\mathbb{N}$ is the maximum number of iterations allowed, stop.
Step 6. Set $\kappa = \kappa + 1$, and $(X_i^{(\nu\kappa)}, Y_i^{(\nu\kappa)}, \mathscr{X}_i^{(\nu\kappa)}, \mathscr{Y}_i^{(\nu\kappa)}) = (X_i^{(\nu)}, Y_i^{(\nu)}, \mathscr{X}_i^{(\nu)}, \mathscr{Y}_i^{(\nu)})$, and go to *Step 3*.

Remark 8.6 Since it is difficult to solve the optimal problems to meet the condition in (8.32). In above algorithm, an iteration method has been applied to solve the minimization problem, in which the termination condition $\lvert f^* - 2N(n+s+k)\rvert$ should be checked. ♦

8.4 Simulation

To illustrate the fault detection filtering scheme, an inverted pendulum system is considered in the simulation. The experimental layout picture of this inverted pendulum system is shown in Fig. 8.2.

The motion of the system can be described by [5].

$$
\begin{aligned}
&M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y + l\sin\theta) = F_\omega - F_r + u,\\
&m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y + l\sin\theta)\cdot l\cos\theta = mgl\sin\theta,
\end{aligned}
$$

where $F_r = c_r\dot{y}$ is the resultant force of the damper of the track; F_ω is the position-dependent stochastic perturbation, which is caused by the rough track. The associated parameters are as follows: M denotes the mass of the slider associated with frame; m denotes the mass of the bob on the pendulum; l denotes the length of the pendulum; g denotes the acceleration due to gravity; θ denotes the angle the pendulum makes

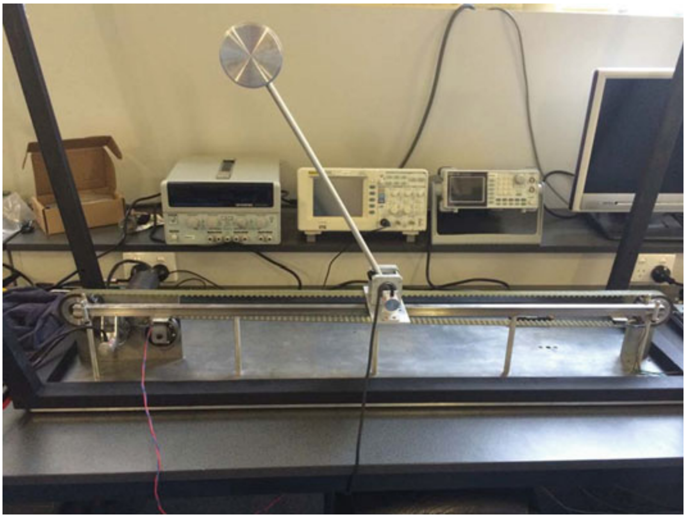

Fig. 8.2 Inverted pendulum system (from the University of Adelaide)

with vertical; y denotes the displacement of the slider; and u denotes the applied force.

By choosing of the state variables $x_1 = y$, $x_2 = \theta$, $x_3 = \dot{y}$ and $x_4 = \dot{\theta}$, the state space dynamic system is

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \frac{-mg \sin x_2}{M \cos x_2} - \frac{(c_r x_3 - F_\omega - u)}{M}, \\
\dot{x}_4 &= \frac{(M+m)g \sin x_2}{Ml \cos^2 x_2} + \frac{x_4^2 \sin x_2}{\cos x_2} + \frac{c_r x_3 - F_\omega - u}{Ml \cos x_2}.
\end{aligned}$$

To apply our filter design method, we should first describe the original nonlinear system by a T–S fuzzy model. We will obtain this model by the following approximation method.

(**i**) When x_2 is near zero, the nonlinear equations can be simplified as

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \frac{-mgx_2}{M} - \frac{c_r x_3 - F_\omega - u}{M}, \\
\dot{x}_4 &= \frac{(M+m)gx_2}{Ml} + \frac{c_r x_3 - F_\omega - u}{Ml}.
\end{aligned}$$

(**ii**) When x_2 is near ϕ $(0 < |\phi| < 90^\circ)$, the nonlinear equations can be simplified as

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \frac{-mg\beta x_2}{M\alpha} - \frac{c_r x_3 - F_\omega - u}{M}, \\
\dot{x}_4 &= \frac{(M+m)g\beta x_2}{Ml\alpha^2} + \frac{c_r x_3 - F_\omega - u}{Ml\alpha},
\end{aligned}$$

where $\alpha = \cos\phi$, and $\beta = (\sin\phi)/\phi$.

To analyze the effects of the fault and disturbance on the residual of the detection filter, we consider the occurrence of the stuck fault

$$f(t) = \begin{cases} 1, & 100 \leqslant t \leqslant 200, \\ 0, & \text{otherwise.} \end{cases}$$

The model is transformed to its discrete-time equivalence using first the local approximation in fuzzy partition spaces [6], and employing the Euler first-order approximation. It gives

Plant Rule 1: IF $x_1(t)$ is $\mathcal{M}_1$, THEN

$$\begin{cases} x(t+1) = A^{(1)}_{\gamma_t} x(t) + B^{(1)}_{\gamma_t} u(t) + E^{(1)}_{\gamma_t} \omega(t) + F^{(1)}_{\gamma_t} f(t) \\ y(t) = \vartheta(t) C^{(1)}_{\gamma_t} x(t) + D^{(1)}_{\gamma_t} u(t) + G^{(1)}_{\gamma_t} \omega(t) + H^{(1)}_{\gamma_t} f(t), \end{cases}$$

Plant Rule 2: IF $x_1(t)$ is $\mathcal{M}_2$, THEN

$$\begin{cases} x(t+1) = A^{(2)}_{\gamma_t} x(t) + B^{(2)}_{\gamma_t} u(t) + E^{(2)}_{\gamma_t} \omega(t) + F^{(2)}_{\gamma_t} f(t) \\ y(t) = \vartheta(t) C^{(2)}_{\gamma_t} x(t) + D^{(2)}_{\gamma_t} u(t) + G^{(2)}_{\gamma_t} \omega(t) + H^{(2)}_{\gamma_t} f(t), \end{cases}$$

where $\mathcal{M}_1 = 0$, $\mathcal{M}_2 = \phi$, and $\omega(t)$ is the exogenous disturbance introduced by F_ω. The system matrices are expressed as

$$A^{(1)}_{\gamma_t} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & -\frac{Tmg}{M} & 1 - \frac{Tc_r}{M} & 0 \\ 0 & \frac{T(M+m)g}{Ml} & \frac{Tc_r}{Ml} & 1 \end{bmatrix}, \quad B^{(1)}_{\gamma_t} = \begin{bmatrix} 0 \\ 0 \\ \frac{T}{M} \\ \frac{-T}{Ml} \end{bmatrix},$$

$$A^{(2)}_{\gamma_t} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & -\frac{Tmg\beta}{M\alpha} & 1 - \frac{Tc_r}{M} & 0 \\ 0 & \frac{T(M+m)g\beta}{Ml\alpha^2} & \frac{Tc_r}{Ml\alpha} & 1 \end{bmatrix}, \quad B^{(2)}_{\gamma_t} = \begin{bmatrix} 0 \\ 0 \\ \frac{T}{M} \\ \frac{-T}{Ml\alpha} \end{bmatrix},$$

$$E_{\gamma_t}^{(1)} = \begin{bmatrix} 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & \frac{Tg_\omega}{M} & 0\,0 \\ 0 & \frac{-Tg_\omega}{Ml} & 0\,0 \end{bmatrix}, \quad F_{\gamma_t}^{(1)} = F_{\gamma_t}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad H_{\gamma_t}^{(1)} = 0.3,$$

$$E_{\gamma_t}^{(2)} = \begin{bmatrix} 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & \frac{Tg_\omega}{M} & 0\,0 \\ 0 & \frac{-Tg_\omega}{Ml\alpha} & 0\,0 \end{bmatrix}, \quad D_{\gamma_t}^{(1)} = D_{\gamma_t}^{(2)} = 0.6,\, H_{\gamma_t}^{(2)} = 0.8,$$

$$C_{\gamma_t}^{(1)} = C_{\gamma_t}^{(2)} = \begin{bmatrix} 0 \; 0 \; 1 \; 1 \end{bmatrix}, \quad G_{\gamma_t}^{(1)} = G_{\gamma_t}^{(2)} = 0.4, \tag{8.33}$$

where T denotes the sampling time, g_ω the coefficient of the exogenous disturbance. It is assumed that the nonuniform sampling periods are time varying among T and $2T$, and the transition probabilities matrix comprises two vertices $\Lambda^{(r)}$, $(r = 1, 2)$. The first lines of $\Lambda^{(r)}$, i.e., $\Lambda_1^{(r)}$ are given by

$$\Lambda_1^{(1)} \triangleq \begin{bmatrix} ? \; 0.5 \end{bmatrix}, \quad \Lambda_1^{(2)} \triangleq \begin{bmatrix} ? \; 0.5 \end{bmatrix},$$

and the second lines of $\Lambda^{(r)}$ are given by

$$\Lambda_2^{(1)} \triangleq \begin{bmatrix} 0.4 \; 0.6 \end{bmatrix}, \quad \Lambda_2^{(2)} \triangleq \begin{bmatrix} 0.3 \; ? \end{bmatrix},$$

where ? represents the unknown entries.

Under the assumption that $|x_2(t)|$ should be smaller then $|\phi|$, we can represent $\mathcal{M}_1(x_2(t))$ and $\mathcal{M}_2(x_2(t))$ by the following expressions:

$$\mathcal{M}_1(x_2(t)) = 1 - \frac{|x_2(t)|}{|\phi|}, \quad \mathcal{M}_2(x_2(t)) = \frac{|x_2(t)|}{|\phi|}.$$

From (8.2), we will further obtain the following fuzzy basis functions:

$$h_1(x_2(t)) = 1 - \frac{|x_2(t)|}{|\phi|}, \quad h_2(x_2(t)) = \frac{|x_2(t)|}{|\phi|}.$$

To simulate this model, we set $g = 9.8\,\mathrm{m/s^2}$, $g_\omega = 1.1\,\mathrm{kg/(ms)}$, $m = 0.041\,\mathrm{kg}$, $l = 0.335\,\mathrm{m}$, $c_r = 5.80\,\mathrm{kg/s}$, $M = 1.298\,\mathrm{kg}$, $T = 0.026\,\mathrm{s}$, and $\theta = 15°$. Then, we have

$$A_1^{(1)} = \begin{bmatrix} 1 & 0 & 0.025 & 0 \\ 0 & 1 & 0 & 0.025 \\ 0 & -0.0091 & 0.8915 & 0 \\ 0 & 0.7817 & 0.3338 & 1 \end{bmatrix}, \quad B_1^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0.0181 \\ -0.056 \end{bmatrix},$$

$$A_2^{(1)} = \begin{bmatrix} 1 & 0 & 0.025 & 0 \\ 0 & 1 & 0 & 0.025 \\ 0 & -0.01 & 0.8915 & 0 \\ 0 & 0.9954 & 0.3855 & 1 \end{bmatrix}, \ B_2^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0.0181 \\ -0.065 \end{bmatrix},$$

$$A_1^{(2)} = \begin{bmatrix} 1 & 0 & 0.05 & 0 \\ 0 & 1 & 0 & 0.05 \\ 0 & -0.0181 & 0.783 & 0 \\ 0 & 1.5635 & 0.6676 & 1 \end{bmatrix}, \ B_1^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0363 \\ -0.112 \end{bmatrix},$$

$$A_2^{(2)} = \begin{bmatrix} 1 & 0 & 0.05 & 0 \\ 0 & 1 & 0 & 0.05 \\ 0 & -0.02 & 0.783 & 0 \\ 0 & 1.9907 & 0.7709 & 1 \end{bmatrix}, \ B_2^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0363 \\ -0.129 \end{bmatrix},$$

$$E_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.0218 & 0 & 0 \\ 0 & -0.067 & 0 & 0 \end{bmatrix}, \quad E_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.0435 & 0 & 0 \\ 0 & -0.1340 & 0 & 0 \end{bmatrix},$$

$$E_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.0218 & 0 & 0 \\ 0 & -0.0773 & 0 & 0 \end{bmatrix}, \quad E_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.0435 & 0 & 0 \\ 0 & -0.1547 & 0 & 0 \end{bmatrix},$$

and $C_\mu^{(\nu)}$, $D_\mu^{(\nu)}$, $G_\mu^{(\nu)}$, $H_\mu^{(\nu)}$ and $F_\mu^{(\nu)}$, $\mu, \nu \in \{1, 2\}$ are given as in (8.33). The two fuzzy basis functions become

$$h_1(x_2(t)) = 1 - 1.91|x_2(t)|, \quad h_2(x_2(t)) = 1.91|x_2(t)|.$$

The weighting matrix $W(z)$ in $f_w(z) = W(z)f(z)$ is taken as $W(z) = 5/(z+5)$. Its state-space realization is given as (8.9) with $A_w = 0.5$, $B_w = 0.5$ and $C_w = 1$. Suppose disturbance input $\omega(t) = 1.5e^{-t}$ and control input $u(t) = \sin(t)$. Let $\bar{\vartheta} = 0.7$. The solution to the fault detection filter problem using Algorithm FDF gives the minimized feasible $\gamma^* = 1.3272$, and the parameters of full-order filter as follows:

$$A_{f1} = \begin{bmatrix} -0.304 & 0.431 & 0.642 & 0.520 \\ -0.204 & 1.680 & 0.203 & 0.261 \\ 1.440 & -0.032 & 0.127 & 0.412 \\ -1.552 & 1.705 & 0.204 & 0.434 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} 0.246 \\ -0.442 \\ 0.340 \\ -0.143 \end{bmatrix},$$

$$C_{f1} = \begin{bmatrix} 1.432 & 0.260 & 0.310 & 0.561 \end{bmatrix},$$

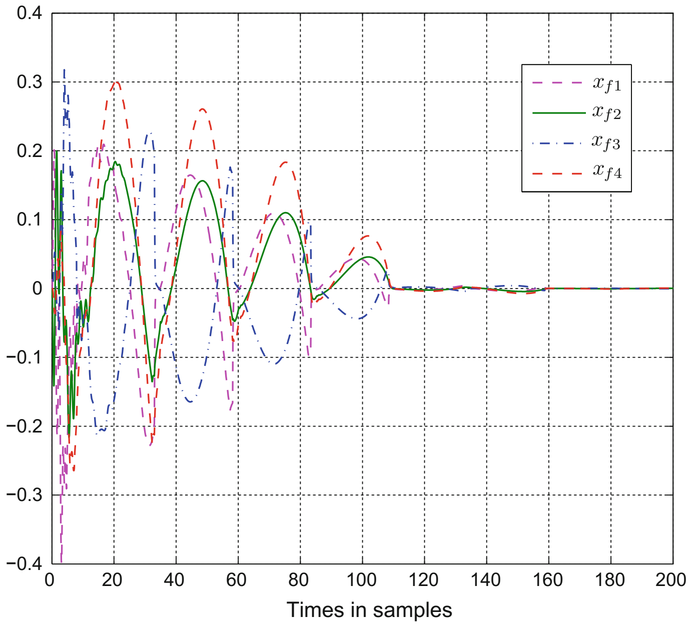

Fig. 8.3 State response of the fault detection filter

$$A_{f2} = \begin{bmatrix} 0.534 & -0.731 & 0.401 & 0.213 \\ 0.730 & 1.634 & 0.247 & 0.434 \\ 1.292 & 0.523 & -1.323 & 0.412 \\ 1.407 & 0.308 & 0.540 & 0.210 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} 0.476 \\ -0.243 \\ 1.402 \\ 0.702 \end{bmatrix},$$
$$C_{f2} = \begin{bmatrix} 1.201 & 0.330 & 1.411 & 1.652 \end{bmatrix}.$$

Set the initial function $x_f(0) = [0.2, 0, -0.3, -0.5]^T$, the state trajectories of the filter are shown in Fig. 8.3; and Fig. 8.4 shows the evolution of residual evaluation function, in which the dashed line is fault-free case, the solid line is the case with fault.

Remark 8.7 With a selected threshold

$$\mathcal{J}_{th} = \sup_{\omega \neq 0, u \neq 0, f=0} \left(\sum_{t=0}^{200} \delta^T(t)\delta(t) \right)^{1/2} = 6.2232,$$

the simulation results show that $\left(\sum_{t=0}^{100} \delta^T(t)\delta(t) \right)^{1/2} = 6.3143 > \mathcal{J}_{th}$. Thus, the appeared fault can be detected after some time steps. The simulation results thus illustrate the efficiency of the approach. ♦

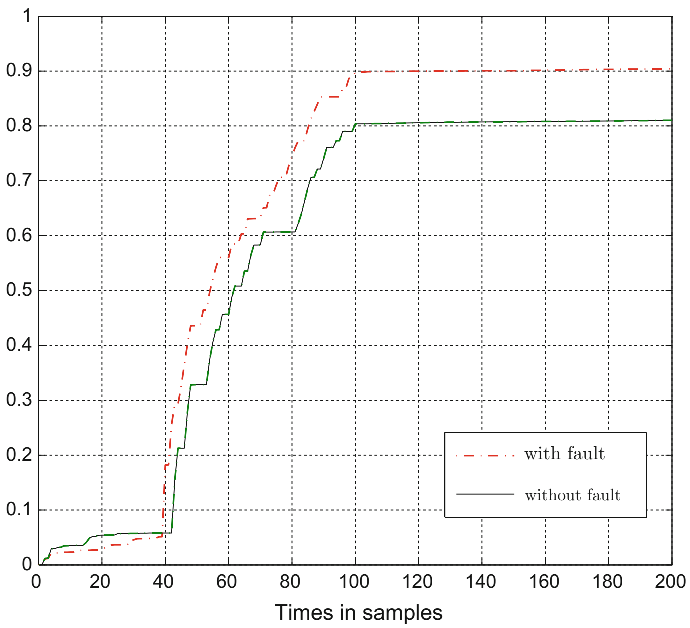

Fig. 8.4 Evaluation function of $\mathcal{J}(r)$

8.5 Conclusion

In this chapter, we have dealt with the fault detection filtering problem for a class of S-MJS by a Takagi–Sugeno fuzzy approach. A sufficient condition has been proposed to guarantee the stochastically stability for the fault detection system with a disturbance attenuation level. The corresponding fault detection filters have been successfully designed for the filtering error dynamics. In addition, a new algorithm has been given utilize the cone complementarity linearization procedure, then the fault detection filter design problem have been derived in terms of a sequential minimization problem. Finally, a simulation example is presented to illustrate the theory development.

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Chapter 9
Fault Detection for Underactuated Manipulators Modeled by MJS

Abstract This chapter is concerned with the fault detection filtering problem for underactuated manipulators based on the Markovian jump model. The purpose is to design a fault detection filter such that the filter error system is stochastically stable and the prescribed probability constraint performance can be guaranteed. The existence conditions for a fault detection filter are proposed through the stochastic analysis technique, and a new fault detection filter algorithm is employed to design the desired filter gains. In addition, the cone complementarity linearization procedure is employed to cast the filter design into a sequential minimization problem.

9.1 Introduction

Extensive research on the kinematics, dynamics and control of robots has been carried out for regular conventional manipulators, i.e., one actuator for each joint in a manipulator whose degree of freedom equals the number of actuators [1]. In contrast, an underactuated manipulator has passive joints equipped with no actuators. The passive joints, which can rotate freely, can be indirectly driven by the effect of the dynamic coupling between the active and passive joints [2]. Previous works on the modeling and control of such manipulators can be found in [3–5]. Specifically, Markov theory was used to characterize and capture the abrupt changes in the operation points of the robotic manipulator [3]. The nonlinear system is linearized around operation points, and a Markovian model was developed in regard to the changes at the operation points and the probability of a fault.

One common engineering practice is to associate a specific system performance with some desired probability of attaining that performance [6]. Motivated by this practical requirement, we adopt a probabilistic approach to solve robust H_2 performance analysis and fault detection problem. The main works of this chapter can be summarized as follows: (1) the underactuated manipulators modeled as MJSs are considered. Based on such a model, a fault detection filter will be designed to demonstrate the effectiveness of the design scheme; (2) to solve the fault detection problem for the stochastic jumping systems, the probability guaranteed H_2 index is introduced to evaluate the performance; (3) based on the stochastic analysis and Lyapunov func-

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tion techniques, sufficient conditions are proposed to guarantee the fault detection systems to be stochastically stable with a probability guaranteed performance; and (4) to combat with computation burden, via the cone complementarity linearization procedure, we convert the corresponding fault detection filter design problem into a convex optimization one such that its solutions can be found efficiently.

9.2 Problem Formulation and Preliminaries

In this section, the dynamic model of a general underactuated manipulator with single passive joint is presented. Figure 9.1 shows the model of a general 2-link underactuated manipulator with the first joint being passive and the other one being actuated. The variables related to the passive (or actuated) joint and the link attached to the joint are as follows ($j = 1, 2$): q_j is the angle of the jth link either relative to the vertical when the link is attached to the base or relative to the line described by the front link; $\dot{q}_j$ is the angular velocity of the jth link; τ_j is the torque applied to joints and g is the gravitational acceleration.

The dynamic of an actuated manipulator with n joints can be represented as:

$$\tau = M(q)\ddot{q} + b(\dot{q}, \ddot{q}), \tag{9.1}$$

where $\ddot{q} \in \mathbb{R}^n$ is the joint acceleration vector; $M(q)$ is the $n \times n$ symmetric positive definite inertia matrix; $b(\dot{q}, \ddot{q})$ is the $n \times 1$ noninertial torque vector, including the

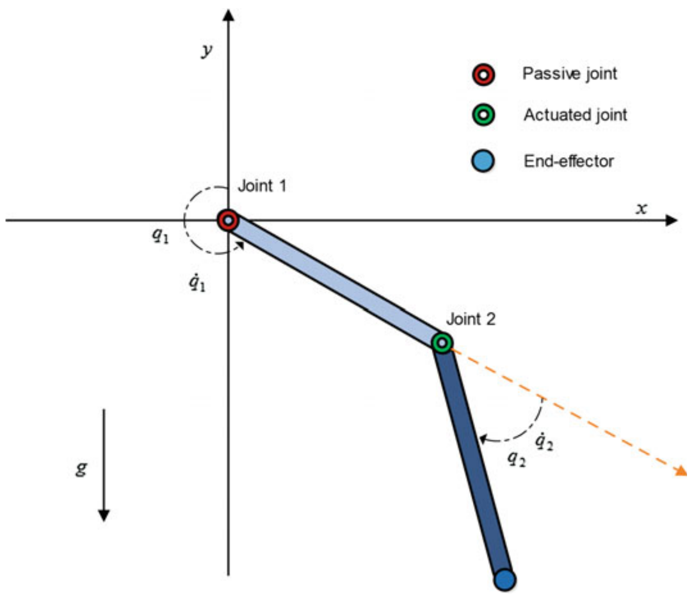

Fig. 9.1 Model of underactuated manipulators

Coriolis and centrifugal forces and the gravitational torques; τ is the $n \times 1$ applied torque vector.

Consider a manipulator with n joints, of which n_p are passive and n_a are active joints. It is known from [3] that, no more than n_a joints of the manipulator can be controlled at every instant. Based on this fact, the n_a joints being controlled is grouped in the vector $q_c \in \mathbb{R}^{n_a}$. The remaining joints are grouped in the vector $q_r \in \mathbb{R}^{n-n_a}$. Then, Eq. (9.1) can be partitioned as:

$$\begin{bmatrix} \tau_a \\ 0 \end{bmatrix} = \begin{bmatrix} M_{ac}(q) & M_{ar}(q) \\ M_{uc}(q) & M_{ur}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_c \\ \ddot{q}_r \end{bmatrix} + \begin{bmatrix} b_a(\dot{q}, \ddot{q}) \\ b_u(\dot{q}, \ddot{q}) \end{bmatrix}, \tag{9.2}$$

where the indices a and u represent the active and unlocked passive joints, respectively. Note that, the torques in the passive joints are set to zero. Isolating the vector $\ddot{q}_r$ in the second line of (9.2) and substituting in the first one:

$$\tau_a = \bar{M}(q)\ddot{q}_c + \bar{b}(q, \dot{q}), \tag{9.3}$$

where

$$\begin{aligned} \bar{M}(q) &\triangleq M_{ac}(q) - M_{ar}(q)M_{ur}^{-1}(q)M_{uc}(q), \\ \bar{b}(q, \dot{q}) &\triangleq b_a(q, \dot{q}) - M_{ar}(q)M_{ur}^{-1}(q)b_u(q, \dot{q}). \end{aligned}$$

The linearization of the manipulator around an operation point with position q_0 and velocity $\dot{q}_0$, is given by

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \tag{9.4}$$

$$z(t) = \tilde{C}x(t) + \tilde{D}u(t), \tag{9.5}$$

where

$$\begin{aligned} \tilde{A} &\triangleq \left. \begin{bmatrix} 0 & I \\ \tilde{A}_{21} & -\bar{M}^{-1}(q)\left[\frac{\partial}{\partial \dot{q}}\left(\bar{b}(q, \dot{q}) - K_D\right)\right] \end{bmatrix} \right|_{(q_0, \dot{q}_0)}, \\ \tilde{A}_{21} &\triangleq -\frac{\partial}{\partial q}\left(\bar{M}^{-1}(q)\bar{b}(q, \dot{q})\right) + \bar{M}^{-1}(q)K_p, \quad \tilde{D} \triangleq \begin{bmatrix} 0 \\ \beta I \end{bmatrix}, \\ \tilde{B} &\triangleq \left. \begin{bmatrix} 0 \\ \bar{M}^{-1}(q) \end{bmatrix} \right|_{q_0}, \quad \tilde{C} \triangleq \begin{bmatrix} \alpha I & 0 \\ 0 & 0 \end{bmatrix}, \quad x \triangleq \begin{bmatrix} q^d - q \\ \dot{q}^d - \dot{q} \end{bmatrix}. \end{aligned}$$

where q^d and $\dot{q}^d$ are the desired reference trajectory and the corresponding velocity, respectively. α and β are constants defined by the designer and are used to adjust the Markovian controllers. The discretized version of system (9.5) is described by

$$\begin{aligned} x(t+1) &= \tilde{A}x(t) + \tilde{B}u(t), \\ z(t) &= \tilde{C}x(t) + \tilde{D}u(t). \end{aligned}$$

More generally, we consider the following stochastic systems in the probability space $(\Omega, \mathscr{F}, \Pr)$ for $t > 0$:

$$(\Pi): \begin{cases} x(t+1) = A_{\gamma_t}^{(\alpha)} x(t) + B_{\gamma_t}^{(\alpha)} u(t) + E_{\gamma_t}^{(\alpha)} \omega(t) + F_{\gamma_t}^{(\alpha)} f(t), \\ \quad y(t) = \vartheta(t) C_{\gamma_t}^{(\alpha)} x(t) + D_{\gamma_t}^{(\alpha)} u(t) + G_{\gamma_t}^{(\alpha)} \omega(t) + H_{\gamma_t}^{(\alpha)} f(t), \end{cases} \tag{9.6}$$

where $\{\gamma_t, t \in \mathbb{Z}^+\}$ is a semi-Markov process on the probability space which has same definition in (8.5) and (8.6) of Chap. 9, and $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^p$ is the measured output; $u(t) \in \mathbb{R}^m$ is the deterministic input vector; $\omega(t) \in \mathbb{R}^q$ and $f(t) \in \mathbb{R}^l$ are disturbance and fault inputs, respectively. In system (Π), the stochastic variable. $\vartheta(t)$ is a Bernoulli distributed white sequences taking values on 0 or 1 with $\Pr\{\vartheta(t) = 1\} = \bar{\vartheta}$, where $\bar{\vartheta} \in [0, 1]$ is a known constant.

Clearly, for the stochastic variable $\vartheta(t)$, one has

$$\mathbf{E}\{\vartheta(t) - \bar{\vartheta}\} = 0, \quad \mathbf{E}\{|\vartheta(t) - \bar{\vartheta}|^2\} = \bar{\vartheta}(1 - \bar{\vartheta}).$$

The uncertain matrices of (9.6) belong to a convex polytope with the following form:

$$\begin{aligned} &\Phi^{\alpha} \triangleq \left(A_{\gamma_t}^{(\alpha)}, B_{\gamma_t}^{(\alpha)}, E_{\gamma_t}^{(\alpha)}, F_{\gamma_t}^{(\alpha)}, C_{\gamma_t}^{(\alpha)}, D_{\gamma_t}^{(\alpha)}, G_{\gamma_t}^{(\alpha)}, H_{\gamma_t}^{(\alpha)}\right), \\ &\Phi^{\alpha} \in \mathcal{R}, \end{aligned} \tag{9.7}$$

where $\mathcal{R}$ is a given convex-bounded polytope domain described by N vertices

$$\mathcal{R} \triangleq \left\{ \Phi^{\alpha} \middle| \; \Phi^{\alpha} = \sum_{\nu=1}^{N} \alpha_{\nu}(t) \Phi_{\gamma_t}^{\nu}, \quad \alpha(t) \in \Gamma \right\},$$

where $\Phi_{\gamma_t}^{\nu} \triangleq \left(A_{\gamma_t}^{(\nu)}, B_{\gamma_t}^{(\nu)}, C_{\gamma_t}^{(\nu)}, D_{\gamma_t}^{(\nu)}, E_{\gamma_t}^{(\nu)}, F_{\gamma_t}^{(\nu)}, G_{\gamma_t}^{(\nu)}, H_{\gamma_t}^{(\nu)}\right)$ denoting the vertices of the polytope, and $\alpha(t)$ denoting the unit simplex, that is,

$$\Gamma \triangleq \left\{ \Big(\alpha_1(t), \alpha_2(t), \ldots, \alpha_N(t) \Big) : \sum_{\nu=1}^{N} \alpha_{\nu}(t) = 1, \alpha_{\nu}(t) \geqslant 0 \right\}.$$

It is assumed that random variables $\alpha_{\nu}(t), \nu = 1, 2, \ldots, N$ are mutually independent scalars which belong to $[\beta_{\nu}, \gamma_{\nu}]$, i.e.,

$$\alpha_{\nu}(t) \in [\beta_{\nu}, \gamma_{\nu}], \quad \nu = 1, 2, \ldots, N.$$

where β_{ν} and γ_{ν} are known. Thus, the parameter vector $\alpha(t) \triangleq [\alpha_1(t), \alpha_2(t), \ldots, \alpha_N(t)]^T \in \mathbb{R}^N$ lies in an N-dimensional hyper-rectangle $\mathbb{B}$.

Note that $\mathbb{B}$ is a hyper-rectangle parameter box, but not a general polytope, it includes 2^N vertices and they are completely defined by the N pairs $\alpha_{\nu}(t) \in \{\beta_{\nu}, \gamma_{\nu}\}, \; \nu = 1, 2, \ldots, N$.

For the stochastic system (Π) in (9.6), we adopt the following fault detection filter form:

$$(\hat{\Pi}) : \quad \begin{cases} x_f(t+1) = A_{fi}^{(\alpha)} x_f(t) + B_{fi}^{(\alpha)} y(t), \\ \delta(t) = C_{fi}^{(\alpha)} x_f(t), \end{cases} \tag{9.8}$$

where $x_f(t) \in \mathbb{R}^s$ and $\delta(t) \in \mathbb{R}^l$ represent the state of the filter and the residual signal, respectively; $A_{fi}^{(\alpha)}$, $B_{fi}^{(\alpha)}$ and $C_{fi}^{(\alpha)}$ are the filter parameters to be designed with similar expressions as in (9.7).

To improve or enhance the performance of fault detection system, we add a weighting matrix function into the fault $f(z)$, that is, $f_w(z) = W(z)f(z)$, where $f(z)$ and $f_w(z)$ denote the 'z' transforms of $f(t)$ and $f_w(t)$, respectively. Here, $W(z)$ is given a priori, the choice of $W(z)$ is to impose frequency weighting on the spectrum of the fault signal for detection. One state space realization of $f_w(z) = W(z)f(z)$ can be

$$(\Pi_w) : \quad \begin{cases} x_w(t+1) = A_w x_w(t) + B_w f(t), \\ f_w(t) = C_w x_w(t), \end{cases} \tag{9.9}$$

where $x_w(t) \in \mathbb{R}^k$ is the state vector, and matrices A_w, B_w and C_w are previously chosen.

Let $e(t) \triangleq \delta(t) - f_w(t)$, the filtering error dynamics can be obtained from (Π), $(\hat{\Pi})$ and (Π_w)

$$(\tilde{\Pi}) : \quad \begin{cases} \xi(t+1) = \left(\tilde{A}_i^{(\alpha)} + \tilde{\vartheta}\mathcal{A}_i^{(\alpha)}\right)\xi(t) + \tilde{B}_i^{(\alpha)} \upsilon(t), \\ e(t) = \tilde{C}_i^{(\alpha)} \xi(t), \end{cases} \tag{9.10}$$

where

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ x_f(t) \\ x_w(t) \end{bmatrix}, \quad \upsilon(t) \triangleq \begin{bmatrix} u(t) \\ \omega(t) \\ f(t) \end{bmatrix},$$

and

$$\begin{cases} \tilde{A}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} A_i^{(\alpha)} & 0 & 0 \\ \bar{\vartheta} B_{fi}^{(\alpha)} C_i^{(\alpha)} & A_{fi}^{(\alpha)} & 0 \\ \hline 0 & 0 & A_w \end{array}\right], \quad \tilde{C}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} 0 & C_{fi}^{(\alpha)} & -C_w \end{array}\right], \\ \mathcal{A}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} A_i^{(\alpha)} & 0 & 0 \\ B_{fi}^{(\alpha)} C_i^{(\alpha)} & A_{fi}^{(\alpha)} & 0 \\ \hline 0 & 0 & A_w \end{array}\right], \quad \tilde{\vartheta} \triangleq \vartheta(t) - \bar{\vartheta}, \\ \tilde{B}_i^{(\alpha)} \triangleq \left[\begin{array}{cc|c} B_i^{(\alpha)} & E_i^{(\alpha)} & F_i^{(\alpha)} \\ B_{fi}^{(\alpha)} D_i^{(\alpha)} & B_{fi}^{(\alpha)} G_i^{(\alpha)} & B_{fi}^{(\alpha)} H_i^{(\alpha)} \\ \hline 0 & 0 & B_w \end{array}\right]. \end{cases} \tag{9.11}$$

Next, let us recall the following definitions and lemmas, which will be used in the next section.

Definition 9.1 The filtering error dynamics (9.10) is said to be stochastically stable, if for any initial state $(\xi(0), r_0)$, the following condition holds

$$\mathbf{E}\left\{\sum_{k=0}^{\infty} \|\xi(k)\|^2 |\ (\xi(0), r_0)\right\} < \Gamma(\xi(0), r_0),$$

in case of $\upsilon(t) = 0$, where $\Gamma(\xi(0), r_0)$ is a nonnegative function of the system initial values.

Definition 9.2 For given a scalar $\gamma > 0$, the filtering error dynamics $(\tilde{\Pi})$ in (9.10) is said to be stochastically stable with a generalized H_2 disturbance attenuation γ, if it is stochastically stable with $\upsilon(t) = 0$, and under zero initial condition, that is, $\xi(0) = 0$, $\|e(t)\|_\infty < \gamma \|\upsilon(t)\|_2$ for all nonzero $\upsilon(t) \in \ell_2[0, \infty)$, where $\|e(t)\|_\infty \triangleq \sup_t \sqrt{|e(t)|^2}$.

Therefore, the probability guaranteed fault detection problem to be solved can be expressed as follows.

Probability Guaranteed Fault Detection Problem: Given a probability $0 < p < 1$ and a specified disturbance attenuation level $\gamma > 0$, develop a fault detection filter $(\hat{\Pi})$ in (9.8) such that

$$\Pr\{\|e(t)\|_\infty - \gamma \|\upsilon(t)\|_2 \leqslant 0\} \geqslant p. \tag{9.12}$$

In particular, we will design filter parameters $A_{fi}^{(\alpha)}$, $B_{fi}^{(\alpha)}$ and $C_{fi}^{(\alpha)}$ in (9.8), and find a parameter-box $\mathbf{B}$ ($\mathbf{B} \in \mathbb{B}$) satisfying:

R1. The probability of $\alpha(t) \in \mathbf{B}$ is not less than p,

R2. The H_2 performance requirement can be guaranteed in the parameter-box $\mathbf{B}$, where the parameter-box $\mathbf{B}$ is generated by $\alpha_\nu(t) \in [a_\nu, b_\nu] \subseteq [\beta_\nu, \gamma_\nu]$ ($\nu = 1, 2, \ldots, N$), and the set of the 2^N vertices $V_\mathbf{B}$ of $\mathbf{B}$ is given by

$$V_\mathbf{B} \triangleq \left\{ \left[\alpha_1(t)\ \alpha_2(t)\ \ldots\ \alpha_N(t)\right]^T \middle|\ \alpha_\nu(t) \in \{a_\nu, b_\nu\}, \right. \\ \left. \nu = 1, 2, \ldots, N \right\}. \tag{9.13}$$

We will solve the probability guaranteed fault detection problem in two steps:

Step 1. Generate a Residual Signal: For the stochastic system (Π) in (9.6), develop a filter in the form of (9.8) to generate a residual signal $\delta(t)$. Meanwhile, the filter is designed to assure that the resulting overall fault detection system $(\tilde{\Pi})$ in (9.10) to be stochastically stable with a disturbance attenuation γ and a prescribed probability satisfies (9.12).

Step 2. SET UP A FAULT DETECTION MEASURE: After generating the residual signal, a residual evaluation value will be computed through a prescribed evaluation function. When the evaluation value is larger than a predefined threshold, an alarm of fault is generated. Here, we consider the following evaluation function: $\mathcal{J}(\delta)$ (where δ denotes $\delta(t)$ for simplicity) and a threshold $\mathcal{J}_{th}$ are selected as

$$\mathcal{J}(\delta) \triangleq \left(\sum_{k=t_0}^{t_0+t^\star} \delta^T(k)\delta(k) \right)^{1/2},$$
$$\mathcal{J}_{th} \triangleq \sup_{0\neq\omega\in\ell_2, 0\neq u\in\ell_2, f=0} \mathcal{J}(\delta), \tag{9.14}$$

where t_0 denotes the initial evaluation time, and $t^\star$ stands for the evaluation time. Then, we can detect the faults by using the following logical relationship:

$$\begin{cases} \mathcal{J}(\delta) > \mathcal{J}_{th} \Rightarrow & \text{with faults} \Rightarrow \quad \text{alarm} \\ \mathcal{J}(\delta) \leqslant \mathcal{J}_{th} \Rightarrow & \text{no faults} \end{cases}$$

Firstly, we will discuss the probability issue of R1. From [6], there are mutually independent $\alpha_\nu(t)$, which are uniformly distribute over $[\beta_\nu, \gamma_\nu]$, and the probability constraint of $\alpha(t) \in \mathbf{B}$ can be expressed as

$$\prod_{\nu=1}^{N} (b_\nu - a_\nu) \geqslant \bar{p}, \tag{9.15}$$

where $\bar{p} = p \prod_{\nu=1}^{N} (\gamma_\nu - \beta_\nu)$, and the endpoints a_ν, b_ν $(\nu = 1, 2, \ldots, N)$ are the parameters to be determined which are associated with the parameter-box $\mathbf{B}$ in (9.13). By using the algorithm in [6], the probability constraint in R1 can be converted into the following Lemma.

Lemma 9.3 [6] *For a given positive probability constraint p. The inequality in* (9.15) *is equivalent to*

$$\prod_{\nu=1}^{m_1} s_{1,\nu} \geqslant \sqrt{\bar{p}}, \tag{9.16}$$

where $s_{1,\nu}$ $(\nu = 1, 2, \ldots, m_1)$ are the positive scalars to be determined.

When M is even, we let $m_1 = \frac{M}{2}$ and find m_1 positive scalars $s_{1,\nu}$ such that

$$\begin{bmatrix} b_{2\nu-1} - a_{\nu-1} & s_{1,\nu} \\ * & b_{2\nu} - a_\nu \end{bmatrix} \geqslant 0, \quad \nu = 1, 2, \ldots, m_1. \tag{9.17}$$

When M is odd, we set $m_1 = \frac{M-1}{2} + 1$ and find m_1 positive scalars $s_{1,\nu}$ such that (9.17) holds for $\nu = 1, 2, \ldots, m_1 - 1$ and

$$\begin{bmatrix} b_M - a_M & s_{1,m_1} \\ * & 1 \end{bmatrix} \geqslant 0. \tag{9.18}$$

9.3 Main Results

We first investigate the stochastic stability with a probability guaranteed performance of the fault detection system $(\tilde{\Pi})$ in (9.10), and we have the following theorem.

Theorem 9.4 *For a given positive constants γ and p, the fault detection system $(\tilde{\Pi})$ in (9.10) is stochastically stable with a probability constraint disturbance γ, if there exist matrices $P_i^{(\nu)} > 0$, $i \in \mathcal{M}$ such that for all $\nu \in \mathcal{M}$,*

$$\begin{bmatrix} -P_i^{(\nu)} & 0 & \left(\tilde{A}_i^{(\nu)}\right)^T \mathcal{P}_j^{(\nu)} & \bar{\vartheta}(1-\bar{\vartheta}) \left(\mathcal{A}_i^{(\nu)}\right)^T \mathcal{P}_j^{(\nu)} \\ * & -I & (\tilde{B}_i^{(\nu)})^T \mathcal{P}_j^{(\nu)} & 0 \\ * & * & -\mathcal{P}_j^{(\nu)} & 0 \\ * & * & * & -\bar{\vartheta}(1-\bar{\vartheta})\mathcal{P}_j^{(\nu)} \end{bmatrix} < 0, \tag{9.19}$$

$$\begin{bmatrix} -P_i^{(\nu)} & (\tilde{C}_i^{(\nu)})^T \\ * & -\gamma^2 I \end{bmatrix} < 0, \tag{9.20}$$

where $\begin{cases} \mathcal{P}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} P_j^{(\nu)}, & \forall j \in \mathcal{M}_{uc}^i, \\ \mathcal{P}_j^{(\nu)} \triangleq P_j^{(\nu)}, & \forall j \in \mathcal{M}_{uk}^i. \end{cases}$

Proof Choose a stochastic Lyapunov function as follows:

$$V(\xi(t), \alpha(t), \gamma_t) \triangleq \xi^T(t) \left(\sum_{\nu=1}^{N} \alpha_\nu(t) P_{\gamma_t}^{(\nu)} \right) \xi(t) \tag{9.21}$$

where $P_i^{(\nu)}$, $i \in \mathcal{M}$ are positive diagonally dominant matrices to be determined.

First, we demonstrate stochastic stability of the fault detection system $(\tilde{\Pi})$ with $\upsilon(t) = 0$. From (9.21), we have

$$\begin{aligned} &\mathbf{E}\Big\{ V\big(\xi(t+1), \alpha(t+1), r_{t+1}\big) \big| \big(\xi(t), \alpha(t), \gamma_t\big) \Big\} - V\big(\xi(t), \alpha(t), \gamma_t\big) \\ &= \mathbf{E}\Big\{ \xi^T(t+1) P_{(r_{t+1}=j|\gamma_t=i)}^{(\nu+1)} \xi(t+1) \Big\} - \xi^T(t) P_i^{(l)} \xi(t) \\ &= \sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{\nu=1}^{N} \alpha_l(t) \alpha_m(t) \alpha_\nu(t+1) \Big[g^T(\xi(t)) (\tilde{A}_i^{(l)})^T \end{aligned}$$

$$
\begin{aligned}
&\times \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \tilde{A}_i^{(m)} g\,(\xi(t)) + \bar{\vartheta}(1-\bar{\vartheta}) g^T\,(\xi(t)) \\
&\times (\mathcal{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \mathcal{A}_i^{(m)} g\,(\xi(t)) - \xi^T(t) P_i^{(l)} \xi(t) \Bigg] \\
\leqslant &\sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{\nu=1}^{N} \alpha_l(t) \alpha_m(t) \alpha_\nu(t+1) g^T\,(\xi(t)) \\
&\times \Bigg((\tilde{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \tilde{A}_i^{(m)} + \bar{\vartheta}(1-\bar{\vartheta}) (\mathcal{A}_i^{(l)})^T \\
&\times \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \mathcal{A}_i^{(m)} - P_i^{(l)} \Bigg) g\,(\xi(t)) \\
= &\sum_{l=1}^{N} \sum_{\nu=1}^{N} \alpha_l(t) \alpha_\nu(t+1) g^T\,(\xi(t)) \Bigg[(\tilde{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \\
&\times \tilde{A}_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta}) (\mathcal{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \mathcal{A}_i^{(l)} - P_i^{(l)} \Bigg] \\
&\times g\,(\xi(t)) + \sum_{l=1}^{N} \sum_{m>l} \sum_{\nu=1}^{N} \alpha_l(t) \alpha_m(t) \alpha_\nu(t+1) \\
&\times g^T\,(\xi(t)) \Bigg[(\tilde{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \tilde{A}_i^{(m)} - P_i^{(l)} \\
&+ \bar{\vartheta}(1-\bar{\vartheta}) (\mathcal{A}_i^{(l)})^T \Bigg(\sum_{j=1}^{S} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \mathcal{A}_i^{(m)} \\
&+ (\tilde{A}_i^{(m)})^T \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \tilde{A}_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta}) (\mathcal{A}_i^{(m)})^T \\
&\times \Bigg(\sum_{j=1}^{M} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg) \mathcal{A}_i^{(l)} - P_i^{(m)} \Bigg] g\,(\xi(t))\,.
\end{aligned}
\tag{9.22}
$$

Now, due to $\sum_{j=1}^{M} \lambda_{ij}^{h} = 1$, we rewrite the right-hand side of (9.22) as

$$
\Upsilon_i \triangleq \sum_{l=1}^{N} \sum_{\nu=1}^{N} \alpha_l(t) \alpha_\nu(t+1) g^T\,(\xi(t)) \Bigg[(\tilde{A}_i^{(l)})^T \Bigg(\sum_{j \in \mathcal{M}} \lambda_{ij}^{h} P_j^{(\nu)}\Bigg)
$$

$$
\begin{aligned}
&\times\tilde{A}_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} \\
&-\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\Bigg)P_i^{(l)}\Bigg]g\left(\xi(t)\right) + \sum_{l=1}^{N}\sum_{m>l}^{N}\sum_{\nu=1}^{N}\alpha_l(t)\alpha_m(t) \\
&\times\alpha_\nu(t+1)g^T\left(\xi(t)\right)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(m)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(m)} \\
&-(\tilde{A}_i^{(m)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\Bigg)P_i^{(l)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}}\lambda_{ij}^h\Bigg)P_i^{(m)}\Bigg]g\left(\xi(t)\right).
\end{aligned}
$$

Therefore, considering (??), we have

$$
\begin{aligned}
\Upsilon_i \triangleq\ &\sum_{l=1}^{N}\sum_{\nu=1}^{N}\alpha_l(t)\alpha_\nu(t+1)g^T\left(\xi(t)\right)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(l)} \\
&-\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\Bigg)P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} \\
&+(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\Bigg)P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T \\
&\times\Bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)}\Bigg]g\left(\xi(t)\right) + \sum_{l=1}^{N}\sum_{m>l}^{N}\sum_{\nu=1}^{N}\alpha_l(t)\alpha_m(t)\alpha_\nu(t+1) \\
&\times g^T\left(\xi(t)\right)\Bigg[(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(m)} - (\tilde{A}_i^{(m)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg) \\
&\times\tilde{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\Bigg)P_i^{(l)} + \bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(m)} \\
&+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(m)})^T\Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathcal{A}_i^{(l)} - \Bigg(\sum_{j\in\mathcal{M}_{uc}^i}\lambda_{ij}^h\Bigg)P_i^{(m)} \\
&+(\tilde{A}_i^{(l)})^T\Bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\tilde{A}_i^{(m)} - (\mathscr{A}_i^{(m)})^T\Bigg(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h P_j^{(\nu)}\Bigg)\mathscr{A}_i^{(l)}
\end{aligned}
$$

$$-\Big(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\Big)P_i^{(l)}+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(l)})^T\Big(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^hP_j^{(\nu)}\Big)\mathcal{A}_i^{(m)}+\bar{\vartheta}(1-\bar{\vartheta})$$
$$\times(\mathcal{A}_i^{(m)})^T\Big(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^hP_j^{(\nu)}\Big)\mathcal{A}_i^{(l)}-\Big(\sum_{j\in\mathcal{M}_{uk}^i}\lambda_{ij}^h\Big)P_i^{(m)}\Big]g\left(\xi(t)\right). \tag{9.23}$$

Then, since one always has $\lambda_{ij}^h \geqslant 0$, $\forall j \in \mathcal{M}$, it is straightforward that $\Upsilon_i < 0$, if (9.19) holds. Obviously, if we have no knowledge on λ_{ij}^h, $\forall j \in \mathcal{M}_{uk}^i$ in (9.19), we have

$$(\tilde{A}_i^{(\nu)})^T\mathcal{P}_j^{(\nu)}\tilde{A}_i^{(\nu)}+\bar{\vartheta}(1-\bar{\vartheta})(\mathcal{A}_i^{(\nu)})^T\mathcal{P}_j^{(\nu)}\mathcal{A}_i^{(\nu)}-P_i^{(\nu)}<0.$$

Then, combining with (9.22) and (9.23), we have

$$\begin{aligned}&\mathbf{E}\{V_{r_{t+1}}(\xi(t+1),t+1)|\ (\xi(t),\gamma_t=i)\}-V_{\gamma_t}(\xi(t),t)\\&\quad\leqslant \xi^T(t)\Theta_i\xi(t)\\&\quad\leqslant -\lambda_{\min}\left(\Theta_i\right)\xi^T(t)\xi(t)\\&\quad\leqslant -\beta\xi^T(t)\xi(t),\end{aligned}$$

where

$$\beta \triangleq \inf_{i\in\mathcal{M}}\left\{\lambda_{\min}(\Theta_i)\right\}.$$

Hence, it follows that

$$\lim_{T\to\infty}\mathbf{E}\left\{\sum_{k=0}^{T}\xi^T(k)\xi(k)\big|\ (\xi_0,r_0)\right\}\leqslant \Gamma(\xi_0,r_0),$$

where $\Gamma(\xi_0, r_0)$ is a positive number. Thus, the fault detection system $(\tilde{\Pi})$ in (9.10) with $\upsilon(t) = 0$ is stochastically stable in the sense of Definition 9.1.

To establish the generalized H_2 performance for the fault detection system $(\tilde{\Pi})$ in (9.10), we assume zero initial condition, that is, $\xi(0) = 0$, then we have $V(\xi(t), \alpha(t), \gamma_t)|_{t=0} = 0$. Consider the following index:

$$\mathcal{J} \triangleq V(\xi(t),\alpha(t),\gamma_t)-\sum_{s=0}^{t-1}\upsilon^T(s)\upsilon(s),$$

then for any nonzero $\upsilon(t) \in \ell_2[0, \infty)$ and $t > 0$, we have

$$\begin{aligned}\mathcal{J} &= \mathbf{E}\Big\{V(\xi(t), \alpha(t), \gamma_t) - V(\xi(0), \alpha(0), r_0)\Big\} - \sum_{s=0}^{t-1} \upsilon^T(s)\upsilon(s) \\ &= \sum_{s=0}^{t-1} \mathbf{E}\left\{\varDelta V(\xi(s), \alpha(s)) - \upsilon^T(s)\upsilon(s)\right\},\end{aligned}$$

where

$$\begin{aligned}\mathbf{E}\Big\{\varDelta V(\xi(s), \alpha(s))\Big\} &= \sum_{i=1}^{N}\sum_{l=1}^{N} \alpha_i(s)\alpha_l(s) \\ &\times \mathbf{E}\Big\{\Big[\Big(\tilde{A}_i^{(j)} + \tilde{\vartheta}\tilde{A}_i^{(j)}\Big)g\left(\xi(s)\right) + \tilde{B}_i^{(j)}\upsilon(s)\Big]^T \mathcal{P}_j \\ &\times \Big[\Big(\tilde{A}_i^{(j)} + \tilde{\vartheta}\tilde{A}_i^{(j)}\Big)g\left(\xi(s)\right) + \tilde{B}_l^{(j)}\upsilon(s)\Big] - \xi^T(s)P_i\xi(s)\Big\} \\ &\leqslant \sum_{i=1}^{N}\sum_{l=1}^{N} \alpha_i(s)\alpha_l(s) \begin{bmatrix} g\left(\xi(s)\right) \\ \upsilon(s) \end{bmatrix}^T \left\{ \begin{bmatrix} (\tilde{A}_i^{(j)})^T \\ (\tilde{B}_i^{(j)})^T \end{bmatrix} \mathcal{P}_j \right. \\ &\times \begin{bmatrix} \tilde{A}_l^{(j)} & \tilde{B}_l^{(j)} \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix} + \bar{\vartheta}(1 - \bar{\vartheta}) \begin{bmatrix} (\mathcal{A}_i^{(j)})^T \\ 0 \end{bmatrix} \\ &\times \left. \mathcal{P}_j \begin{bmatrix} \mathcal{A}_l^{(j)} & 0 \end{bmatrix} \right\} \begin{bmatrix} g\left(\xi(s)\right) \\ \upsilon(s) \end{bmatrix}.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{J} \leqslant \sum_{s=0}^{t-1} \Bigg\{ &\sum_{i=1}^{N}\sum_{l=1}^{N} \alpha_i(s)\alpha_l(s) \begin{bmatrix} g\left(\xi(s)\right) \\ \upsilon(s) \end{bmatrix}^T \left(\begin{bmatrix} (\tilde{A}_i^{(j)})^T \\ (\tilde{B}_i^{(j)})^T \end{bmatrix} \right. \\ &\times \mathcal{P}_j \begin{bmatrix} \tilde{A}_l^{(j)} & \tilde{B}_l^{(j)} \end{bmatrix} - \begin{bmatrix} P_i^{(\nu)} & 0 \\ 0 & I \end{bmatrix} + \bar{\vartheta}(1 - \bar{\vartheta}) \begin{bmatrix} (\mathcal{A}_i^{(j)})^T \\ 0 \end{bmatrix} \\ &\times \left. \mathcal{P}_j \begin{bmatrix} \mathcal{A}_l^{(j)} & 0 \end{bmatrix} \right) \begin{bmatrix} g\left(\xi(s)\right) \\ \upsilon(s) \end{bmatrix} \Bigg\}.\end{aligned}$$

It is shown from (9.19) that

$$\begin{aligned}&\begin{bmatrix} (\tilde{A}_i^{(\nu)})^T \\ (\tilde{B}_i^{(\nu)})^T \end{bmatrix} \mathcal{P}_j \begin{bmatrix} \tilde{A}_i^{(j)} & \tilde{B}_i^{(j)} \end{bmatrix} + \bar{\vartheta}(1 - \bar{\vartheta}) \begin{bmatrix} (\mathcal{A}_i^{(j)})^T \\ 0 \end{bmatrix} \\ &\qquad \times \mathcal{P}_j \begin{bmatrix} \mathcal{A}_l^{(j)} & 0 \end{bmatrix} - \begin{bmatrix} P_i^{(\nu)} & 0 \\ 0 & I \end{bmatrix} < 0,\end{aligned}$$

this guarantees $\mathcal{J} < 0$, which further implies

$$V(\xi,\alpha) < \sum_{s=0}^{t-1} \upsilon^T(s)\upsilon(s). \tag{9.24}$$

On the other hand, by Schur complement, (9.20) yields

$$(\tilde{C}_i^{(j)})^T \tilde{C}_i^{(j)} - \gamma^2 P_i < 0. \tag{9.25}$$

Then, one can obtain that, for all $t > 0$,

$$\begin{aligned} & e^T(t)e(t) - \gamma^2 V(\xi,\alpha) \\ & \leqslant \sum_{i=1}^{N}\sum_{l=1}^{N} \alpha_i(t)\alpha_l(t) g^T(\xi(t)) \left(\tilde{C}_i^T \tilde{C}_l - \gamma^2 P_i\right) g(\xi(t)). \end{aligned}$$

Combining with (9.24)–(9.25) yields the following inequalities:

$$\begin{aligned} e^T(t)e(t) & < \gamma^2 V(\xi,\alpha) \\ & \leqslant \gamma^2 \sum_{s=0}^{t-1} \upsilon^T(s)\upsilon(s) \leqslant \gamma^2 \sum_{s=0}^{\infty} \upsilon^T(s)\upsilon(s), \end{aligned}$$

this implies that $\|e(t)\|_\infty < \gamma \|\upsilon(t)\|_2$ for all nonzero $\upsilon(t) \in \ell_2[0,\infty)$. This completes the proof. ■

We now shift our design focus to the probability guaranteed fault detection filter in (9.8) based on Theorem 9.4 and Lemma 1.29. To this end, the filter matrices $(A_{fi}^{(\alpha)}, B_{fi}^{(\alpha)}, C_{fi}^{(\alpha)})$ should be determined to guarantee the stochastic stability of the filter error system $(\tilde{\Pi})$ in (9.10) with a probability constraint disturbance attenuation level γ. We establish a sufficient condition for the existence of such a filter through the following theorem:

Theorem 9.5 *Consider the stochastic system (Π) in (9.6). A probability guaranteed fault detection filter of the form in (9.8) exists, if there exist positive definite matrices $X_i^{(\nu)}$, $Y_i^{(\nu)}$, $\mathscr{X}_i^{(\nu)}$ and $\mathscr{Y}_i^{(\nu)}$, $i \in \mathcal{M}$ such that for $i, j \in \mathcal{M}$,*

$$\left[\begin{array}{cccc|c} -X_i^{(\nu)} & 0 & 0 & (\bar{A}_i^{(\nu)})^T J^T & \\ * & -Y_i^{(\nu)} & 0 & 0 & \Psi_1 \\ * & * & -I & (\bar{B}_i^{(\nu)})^T J^T & \\ * & * & * & -J\mathscr{X}_j^{(\nu)} J^T & \\ \hline * & * & * & * & -\Psi_2 \end{array}\right] < 0, \tag{9.26}$$

$$\left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|c} -X_i^{(\nu)} & 0 & 0 & \\ * & -Y_i^{(\nu)} & 0 & \Psi_3 \\ * & * & -I & \\ \hline * & * & * & -\Psi_4 \end{array}\right] \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array}\right]^T < 0, \tag{9.27}$$

$$\begin{bmatrix} -JX_i^{(\nu)}J^T & 0 & 0 \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{9.28}$$

$$\mathcal{X}_j^{(\nu)} \mathscr{X}_j^{(\nu)} = I, \quad \mathcal{Y}_j^{(\nu)} \mathscr{Y}_j^{(\nu)} = I, \tag{9.29}$$

where

$$\Psi_1 \triangleq \begin{bmatrix} 0 & \varpi(\bar{A}_i^{(\nu)})^T J^T & 0 \\ A_w^T & 0 & \varpi A_w^T \\ \hat{B}_w^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Psi_3 \triangleq \begin{bmatrix} (\bar{A}_i^{(\nu)})^T & 0 & \varpi(\bar{A}_i^{(\nu)})^T & 0 \\ 0 & A_w^T & 0 & \varpi A_w^T \\ (\bar{B}_i^{(\nu)})^T & \hat{B}_w^T & 0 & 0 \end{bmatrix},$$

$$\Psi_2 \triangleq \operatorname{diag}\left\{\mathscr{Y}_j^{(\nu)}, \varpi J \mathscr{X}_j^{(\nu)} J^T, \varpi \mathscr{Y}_j^{(\nu)}\right\}, \quad \varpi \triangleq \overline{\vartheta}(1-\overline{\vartheta}),$$

$$\mathcal{X}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} X_j^{(\nu)}, \forall j \in \mathcal{M}_{uc}^i, \quad \mathcal{X}_j^{(\nu)} \triangleq X_j^{(\nu)}, \forall j \in \mathcal{M}_{uk}^i,$$

$$\mathcal{Y}_j^{(\nu)} \triangleq \sum_{j \in \mathcal{M}_{uc}^i} \lambda_{ij}^{(r)} Y_j^{(\nu)}, \forall j \in \mathcal{M}_{uc}^i, \quad \mathcal{Y}_j^{(\nu)} \triangleq Y_j^{(\nu)}, \forall j \in \mathcal{M}_{uk}^i,$$

$$\Psi_4 \triangleq \operatorname{diag}\left\{\mathscr{X}_j^{(\nu)}, \mathscr{Y}_j^{(\nu)}, \varpi \mathscr{X}_j^{(\nu)}, \varpi \mathscr{Y}_j^{(\nu)}\right\}.$$

Moreover, if the above conditions (9.26)–(9.29) *are feasible, then the system matrices of a fault detection filter* (9.8) *are given by*

$$\begin{cases} \mathcal{G}_i^{(\nu)} = -\Pi_{1i}^{-1} U_{1i}^T \Lambda_{1i} V_{1i}^T (V_{1i}\Lambda_{1i}V_{1i}^T)^{-1} + \Pi_{1i}^{-1} \Xi_{1i}^{1/2} L_{1i} (V_{1i}\Lambda_{1i}V_{1i}^T)^{-1/2}, \\ \mathcal{K}_i^{(\nu)} = -\Pi_{2i}^{-1} U_{2i}^T \Lambda_{2i} V_{2i}^T (V_{2i}\Lambda_{2i}V_{2i}^T)^{-1} + \Pi_{2i}^{-1} \Xi_{2i}^{1/2} L_{2i} (V_{2i}\Lambda_{2i}V_{2i}^T)^{-1/2}, \\ \Lambda_{1i} = (U_{1i}\Pi_{1i}^{-1}U_{1i}^T - W_{1i})^{-1} > 0, \\ \Lambda_{2i} = (U_{2i}\Pi_{2i}^{-1}U_{2i}^T - W_{2i})^{-1} > 0, \\ \Xi_{1i} = \Pi_{1i} - U_{1i}^T(\Lambda_{1i} - \Lambda_{1i}V_{1i}^T(V_{1i}\Lambda_{1i}V_{1i}^T)^{-1}V_{1i}\Lambda_{1i})U_{1i} > 0, \\ \Xi_{2i} = \Pi_{2i} - U_{2i}^T(\Lambda_{2i} - \Lambda_{2i}V_{2i}^T(V_{2i}\Lambda_{2i}V_{2i}^T)^{-1}V_{2i}\Lambda_{2i})U_{2i} > 0, \end{cases}$$

where $\mathcal{G}_i^{(\nu)} \triangleq \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix}$ *and* $\mathcal{K}_i^{(\nu)} \triangleq C_{fi}^{(\nu)}$. *In addition,* $\Pi_{\kappa i}$ *and* $L_{\kappa i}$, $(\kappa = 1, 2)$ *are any appropriate matrices satisfying* $\Pi_{\kappa i} > 0$, $\|L_{\kappa i}\| < 1$ *and*

$$\bar{A}_i^{(\nu)} = \begin{bmatrix} A_i^{(\nu)} & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0_{(n+s)\times l} \\ 0_{k\times l} \\ I_{l\times l} \end{bmatrix}, \quad E = \begin{bmatrix} 0_{n\times s} \\ I_{s\times s} \end{bmatrix}, \quad R_i^{(\nu)} = \begin{bmatrix} 0_{s\times n} & I_{s\times s} \\ C_i^{(\nu)} & 0_{p\times s} \end{bmatrix},$$

$$\bar{B}_i^{(\nu)} = \begin{bmatrix} B_i^{(\nu)} & E_i^{(\nu)} & F_i^{(\nu)} \\ 0 & 0 & 0 \end{bmatrix}, \quad M_i = \begin{bmatrix} 0_{s\times n} & I_{s\times s} & 0_{s\times k} & 0_{s\times m} & 0_{s\times q} & 0_{s\times l} \\ C_i^{(\nu)} & 0_{p\times s} & 0_{p\times k} & D_i^{(\nu)} & G_i^{(\nu)} & H_i^{(\nu)} \end{bmatrix}^T,$$

$$S_i^{(\nu)} = \begin{bmatrix} 0_{s\times m} & 0_{s\times q} & 0_{s\times l} \\ D_i^{(\nu)} & G_i^{(\nu)} & H_i^{(\nu)} \end{bmatrix}, \quad J = \begin{bmatrix} I_{n\times n} & 0_{n\times s} \end{bmatrix}, \quad T = \begin{bmatrix} 0_{s\times n} & I_{s\times s} \end{bmatrix},$$

$$W_{1ij} = \left[\begin{array}{cccc|c} -X_i^{(\nu)} & 0 & 0 & \bar{A}_i^T & \\ * & -Y_i^{(\nu)} & 0 & 0 & \bar{\Psi}_1 \\ * & * & -I & \bar{B}_i^T & \\ * & * & * & -\mathscr{X}_j^{(\nu)} & \\ \hline * & * & * & * & -\bar{\Psi}_2 \end{array}\right], \quad U_1 = \begin{bmatrix} 0_{(n+s)\times s} \\ 0_{k\times s} \\ 0_{(m+q+l)\times s} \\ \bar{\vartheta} E \\ 0_{k\times s} \\ E \\ 0_{k\times s} \end{bmatrix},$$

$$\bar{\Psi}_1 = \begin{bmatrix} 0 & \varpi \bar{A}_i^T & 0 \\ A_w^T & 0 & \varpi A_w^T \\ \hat{B}_w^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W_{2i} = \begin{bmatrix} -X_i^{(\nu)} & 0 & 0 \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix},$$

$$\bar{\Psi}_2 = diag\left\{\mathscr{Y}_j^{(\nu)}, \varpi \mathscr{X}_j^{(\nu)}, \varpi \mathscr{Y}_j^{(\nu)}\right\}, \quad V_2 = \begin{bmatrix} T & 0_{s\times k} & 0_{s\times l} \end{bmatrix},$$

$$V_{1i} = \begin{bmatrix} R_i & 0_{(p+s)\times k} & S_i & 0_{(p+s)\times(n+s)} & 0_{(p+s)\times k} \end{bmatrix}. \tag{9.30}$$

Proof Set $P_i^{(\nu)} \triangleq \text{diag}\left(X_i^{(\nu)}, Y_i^{(\nu)}\right)$, $i \in \mathcal{M}$ in (9.19), where $X_i^{(\nu)} \in \mathbb{R}^{(n+s)\times(n+s)}$ and $Y_i^{(\nu)} \in \mathbb{R}^{k\times k}$. Then, from Theorem 9.4, the fault detection system $(\tilde{\Pi})$ in (9.10) is stochastically stable with a probability guaranteed performance γ, if there exist positive diagonally dominant matrices $X_i^{(\nu)}$ and $Y_i^{(\nu)}$ such that

$$\begin{bmatrix} -X_i^{(\nu)} & 0 & 0 & (\hat{A}_i^{(\nu)})^T & 0 & \varpi(\hat{\mathcal{A}}_i^{(\nu)})^T & 0 \\ * & -Y_i^{(\nu)} & 0 & 0 & A_w^T & 0 & \varpi A_w^T \\ * & * & -I & (\hat{B}_i^{(\nu)})^T & \hat{B}_w^T & 0 & 0 \\ * & * & * & -\mathscr{X}_j^{(\nu)} & 0 & 0 & 0 \\ * & * & * & * & -\mathscr{Y}_j^{(\nu)} & 0 & 0 \\ * & * & * & * & * & -\varpi \mathscr{X}_j^{(\nu)} & 0 \\ * & * & * & * & * & * & -\varpi \mathscr{Y}_j^{(\nu)} \end{bmatrix} < 0, \tag{9.31}$$

$$\begin{bmatrix} -X_i^{(\nu)} & 0 & (\hat{C}_i^{(\nu)})^T \\ * & -Y_i^{(\nu)} & -C_w^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{9.32}$$

where

$$\begin{cases} \hat{A}_i^{(\nu)} \triangleq \begin{bmatrix} A_i^{(\nu)} & 0 \\ \bar{\vartheta} B_{fi}^{(\nu)} C_i^{(\nu)} & A_{fi}^{(\nu)} \end{bmatrix}, \quad \hat{\mathcal{A}}_i^{(\nu)} \triangleq \begin{bmatrix} A_i^{(\nu)} & 0 \\ B_{fi}^{(\nu)} C_i^{(\nu)} & A_{fi}^{(\nu)} \end{bmatrix}, \\ \hat{B}_i^{(\nu)} \triangleq \begin{bmatrix} B_i^{(\nu)} & E_i^{(\nu)} & F_i^{(\nu)} \\ B_{fi}^{(\nu)} D_i^{(\nu)} & B_{fi}^{(\nu)} G_i^{(\nu)} & B_{fi}^{(\nu)} H_i^{(\nu)} \end{bmatrix}, \\ \hat{B}_w \triangleq \begin{bmatrix} 0 & 0 & B_w \end{bmatrix}, \quad \hat{C}_i^{(\nu)} \triangleq \begin{bmatrix} 0 & C_{fi}^{(\nu)} \end{bmatrix}. \end{cases} \tag{9.33}$$

Rewrite (9.33) in the following form:

$$\begin{aligned} \hat{A}_i^{(\nu)} &= \bar{A}_i^{(\nu)} + \bar{\vartheta} E \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix} R_i^{(\nu)}, \quad \hat{C}_i^{(\nu)} = C_{fi}^{(\nu)} T, \\ \hat{\mathcal{A}}_i^{(\nu)} &= \bar{A}_i^{(\nu)} + E \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix} R_i^{(\nu)}, \\ \hat{B}_i^{(\nu)} &= \bar{B}_i^{(\nu)} + E \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix} S_i^{(\nu)}, \end{aligned} \tag{9.34}$$

where $\bar{A}_i^{(\nu)}$, $\bar{B}_i^{(\nu)}$, $\bar{C}_i^{(\nu)}$, E, $R_i^{(\nu)}$, $S_i^{(\nu)}$ and T are defined in (9.30).

Using (9.34), the inequalities (9.31)–(9.32) can be rewritten as

$$W_{1ij} + U_1 \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix} V_{1i} + \left(U_1 \begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix} V_{1i} \right)^T < 0, \tag{9.35}$$

$$W_{2i} + U_2 C_{fi}^{(\nu)} V_2 + \left(U_2 C_{fi}^{(\nu)} V_2 \right)^T < 0, \tag{9.36}$$

where W_{1ij}, W_{2i}, U_1, V_{1i}, U_2 and V_2 are defined in (9.30).

Next, we assign

$$U_1^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad V_{1i}^{T\perp} = \left[\begin{array}{c|c} M_i^{\perp} & 0 \\ \hline 0 & I \end{array} \right].$$

It follows from Lemma 1.29 that inequality (9.35) is solvable for $\begin{bmatrix} A_{fi}^{(\nu)} & B_{fi}^{(\nu)} \end{bmatrix}$ if and only if (9.26) and (9.27) are satisfied.

In addition, set

$$U_2^{\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad V_2^{T\perp} = \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then inequality (9.36) is solvable for C_{fi} if and only if (9.28) hold. This completes the proof. ■

Remark 9.6 Due to the matrix equation (9.29) is not linear matrix inequalities, it will bring difficult to design the minimum γ by using the convex optimization algorithm. However, we can solve this problem take advantage of the cone complementarity linearization algorithm. ♦

From the above discussion, we can transform the nonconvex feasibility problem into the following sequential optimization problem.

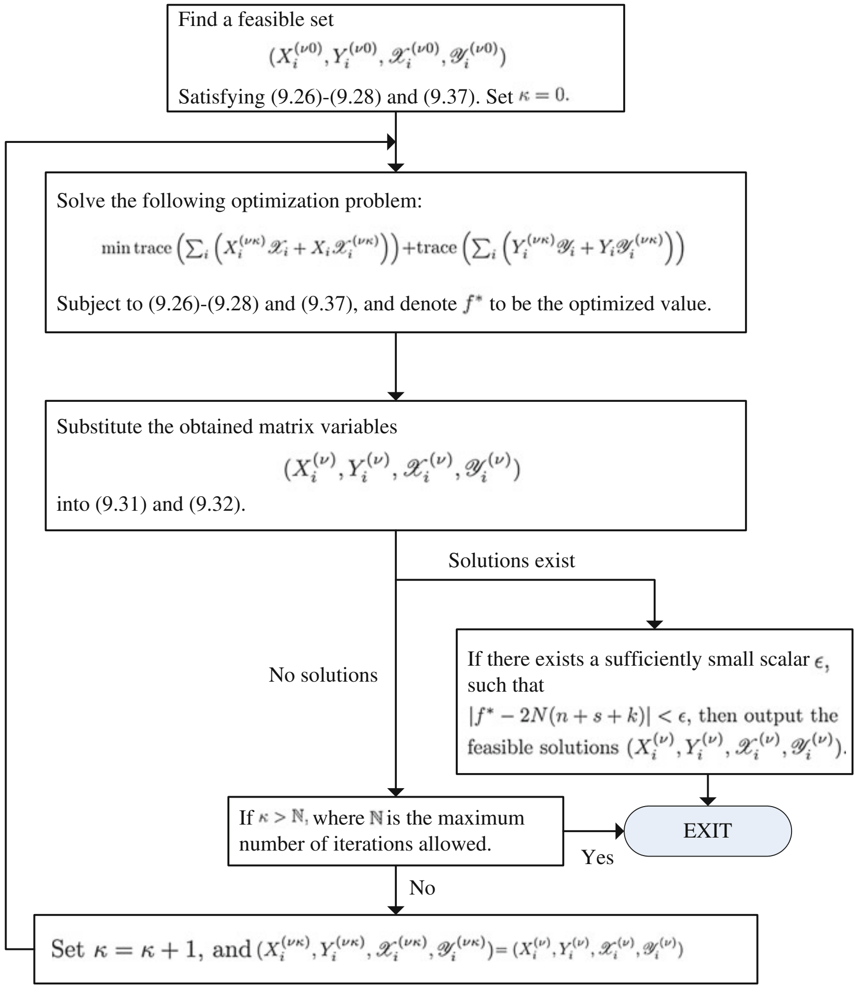

Fig. 9.2 Probability guaranteed fault detection algorithm

Probability Guaranteed Fault Detection (PGFD) Filter Design Problem:

$$\begin{aligned} \min \quad & \text{trace}\left(\sum\nolimits_i \left(X_i^{(\nu)} \mathscr{X}_i^{(\nu)}\right)\right) \\ & \quad + \text{trace}\left(\sum\nolimits_i \left(Y_i^{(\nu)} \mathscr{Y}_i^{(\nu)}\right)\right) \\ \text{subject to} \ & (9.26) - (9.28) \ \forall i \in \mathcal{M} \\ & \begin{bmatrix} X_i^{(\nu)} & I \\ I & \mathscr{X}_i^{(\nu)} \end{bmatrix} \geqslant 0, \begin{bmatrix} Y_i^{(\nu)} & I \\ I & \mathscr{Y}_i^{(\nu)} \end{bmatrix} \geqslant 0. \end{aligned} \tag{9.37}$$

If there exists solutions such that

$$\begin{aligned} & \text{trace}\left(\sum\nolimits_i \left(X_i^{(\nu)} \mathscr{X}_i^{(\nu)}\right)\right) + \text{trace}\left(\sum\nolimits_i \left(Y_i^{(\nu)} \mathscr{Y}_i^{(\nu)}\right)\right) \\ & = N\left(n + s + k\right), \end{aligned}$$

then the inequalities in Theorem 9.5 are solvable.

Therefore, we propose a novel algorithm (Fig. 9.2) to solve the probability guaranteed fault detection filtering problem.

9.4 Illustrative Example

To demonstrate the effectiveness of the proposed method, this section presents a simulation result of a 2-link underactuated manipulator being reduced to be MJS. The fault detection control system for manipulators developed in this paper utilizes the Markovian control of [3].

Unlike the work of [3], we consider a set of stochastic systems with modal transition governed by a Markovian chain. The transition probabilities matrix comprises two vertices $\Lambda^{(r)}$, $(r = 1, 2)$. The first lines of $\Lambda^{(r)}$, i.e., $\Lambda_1^{(r)}$ are given by

$$\Lambda_1^{(1)} \triangleq \begin{bmatrix} ? & 0.6 \end{bmatrix}, \quad \Lambda_1^{(2)} \triangleq \begin{bmatrix} ? & 0.4 \end{bmatrix},$$

and the second lines of $\Lambda^{(r)}$ are given by

$$\Lambda_2^{(1)} \triangleq \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad \Lambda_2^{(2)} \triangleq \begin{bmatrix} 0.3 & ? \end{bmatrix},$$

where '?' represents the unknown entries.

Consider the stochastic system in (9.6) with $\nu = 1$, and the following two subsystems:

SUBSYSTEM 1.

$$A_1 = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & -0.7 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.5 \; 0.3 \end{bmatrix}, \; D_1 = 0.3,$$
$$G_1 = 0.8, \; H_1 = 1.5,$$

SUBSYSTEM 2.

$$A_2 = \begin{bmatrix} -0.3 & -0.2 \\ 0.6 & -0.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.8 \\ -0.1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.4 \\ -0.9 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 0.8 \\ 0.3 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0.1 \; 0.5 \end{bmatrix}, \; D_2 = 0.9,$$
$$G_2 = 0.4, \; H_2 = 0.8$$

The weighting matrix $W(z)$ in $f_w(z) = W(z)f(z)$ is taken as $W(z) = 5/(z+5)$. Its state space realization is given as (9.9) with $A_w = 0.5$, $B_w = 0.5$ and $C_w = 1$. Figure 9.3 gives a switching signal, which is generated randomly, here, '1' and '2' represent respectively the first and the second subsystem.

Our purpose here is to design a full-order filter in the form of $(\hat{\Pi})$ in (9.8) to generate the residual signal $r(t)$ such that the fault detection system $(\tilde{\Pi})$ in (9.10) is stochastically stable with a probability guaranteed performance.

Next, consider the full-order case, that is $s = 2$. Solving probability guaranteed fault detection filter problem by using Algorithm PGFD, then the minimized feasible γ can be obtained $\gamma^* = 1.7532$, and the corresponding parameters of the full-order filter are:

$$A_{f1} = \begin{bmatrix} -0.6137 & -0.4202 \\ -1.2364 & -2.0083 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} 1.1327 \\ -0.2954 \end{bmatrix},$$
$$C_{f1} = \begin{bmatrix} 1.3341 \; 0.3670 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} -2.0982 & 1.1039 \\ -0.7246 & -1.0135 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} -0.8347 \\ -0.5654 \end{bmatrix},$$
$$C_{f2} = \begin{bmatrix} -1.3238 \; 0.7123 \end{bmatrix}.$$

In the following, we shall further show the effectiveness the developed results in this paper. Letting the initial condition be $x(0) = 0$. Suppose the unknown disturbance input $\omega(t) = 0.5e^{-t}$, $0 \leqslant t \leqslant 200$. It is assumed that the control input $u(t) = 0.2e^{-t}$, $0 \leqslant t \leqslant 200$; and the fault signal satisfying:

$$f(t) = \begin{cases} 2, & 50 \leqslant t \leqslant 130 \\ 0, & \text{otherwise} \end{cases}$$

Thus, Fig. 9.4 plots the weighting fault signal $f_w(t)$.

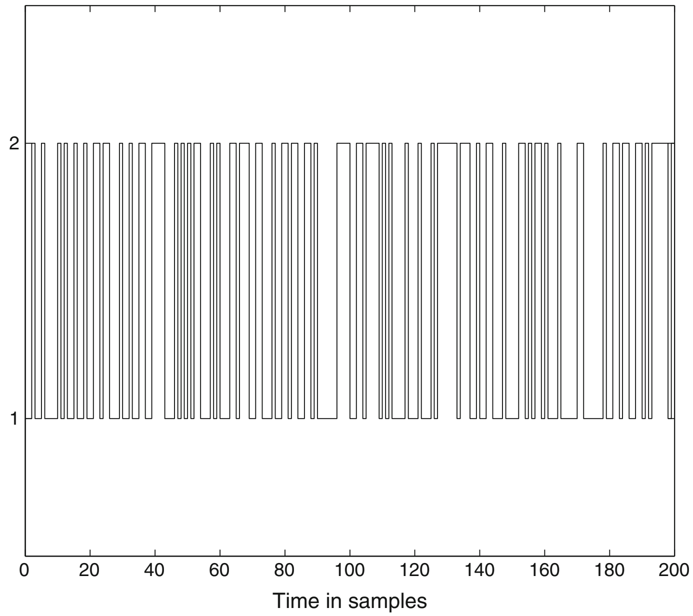

Fig. 9.3 Switching signal

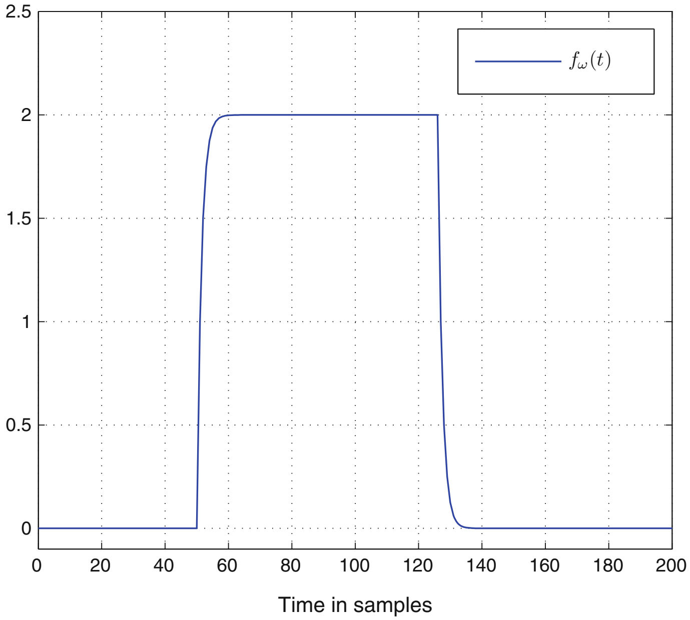

Fig. 9.4 Weighting fault signal $f_\omega(t)$

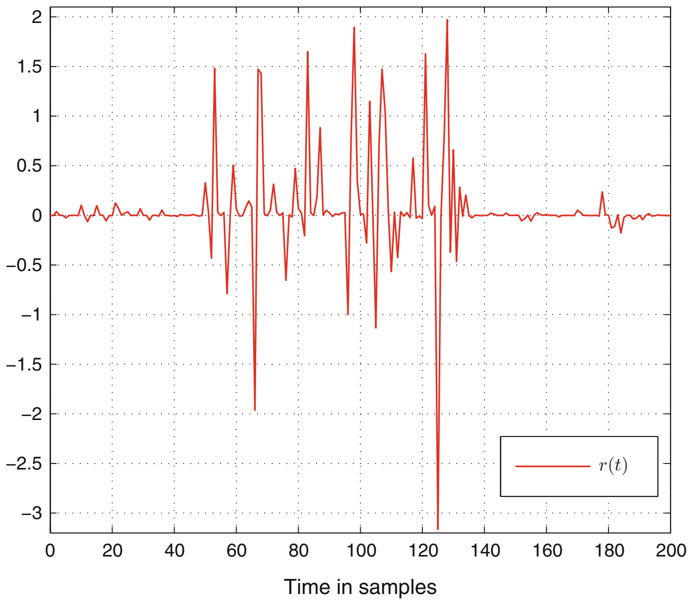

Fig. 9.5 Generated residual signal $r(t)$

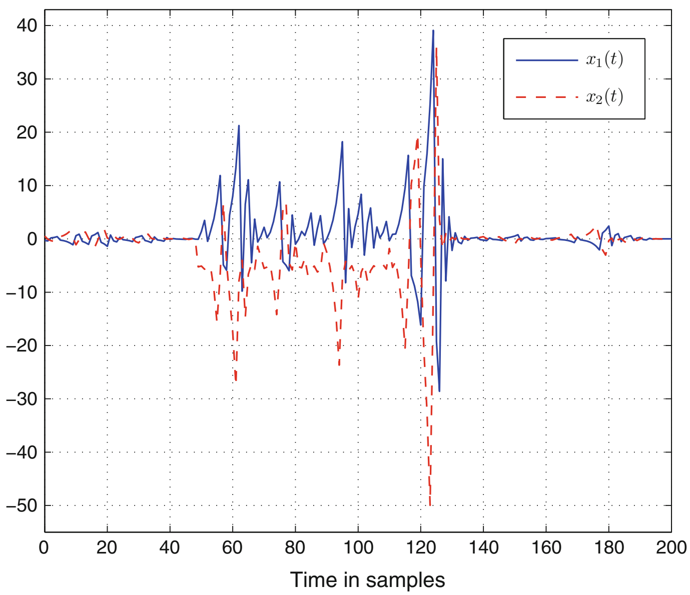

Fig. 9.6 States of the stochastic system

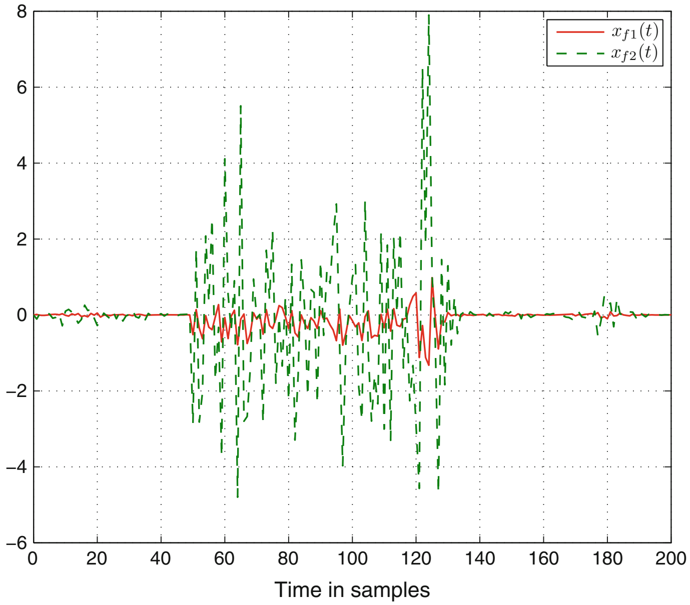

Fig. 9.7 States of the fault detection filter

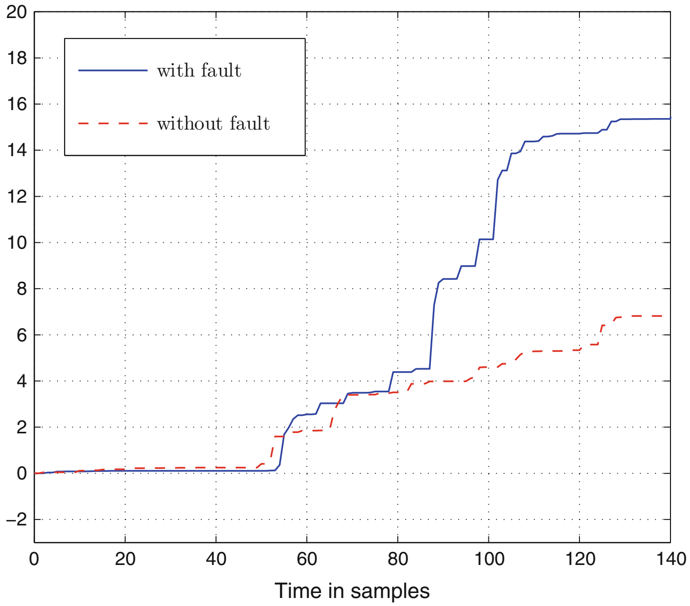

Fig. 9.8 Evaluation function of $\mathcal{J}(r)$

Consider the evaluation function and the threshold as in (9.14). The generated residual signal $r(t)$ is depicted in Figs. 9.5 and 9.6 shows the trajectories of the stochastic system, and Fig. 9.7 gives the simulation results for the trajectories of the designed fault detection filter under disturbance. Figure 9.8 shows the evaluation function of $\mathcal{J}(r)$, which clearly indicate that the residual can also detect the fault.

Next, we will set up the fault detection measure. Without loss of generality, selecting the threshold $\mathcal{J}_{th} = \sup_{\omega \neq 0, u \neq 0, f=0} \left(\sum_{t=0}^{200} r^T(t) r(t) \right)^{1/2} = 7.2961$, the simulation results show that $\left(\sum_{t=0}^{62} r^T(t) r(t) \right)^{1/2} = 7.3013 > \mathcal{J}_{th}$. Thus, the appeared fault can be detected after some time steps.

9.5 Conclusion

In this chapter, the probability guaranteed fault detection filtering problem has been investigated for an underactuated manipulator which is modeled as MJS. A sufficient condition has been proposed to guarantee the stochastically stability with a probability constraint performance for the filtering error system. The corresponding fault detection filter has been successfully designed for the MJS. In addition, a cone complementarity linearization procedure has been employed to transform a nonconvex feasibility problem into a sequential minimization problem, which can be readily solved by existing optimization techniques. Finally, a numerical example has been given to illustrate the effectiveness of the developed theoretic results.

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Part III
Summary

Chapter 10
Conclusion and Further Work

Abstract This chapter summarizes the results of the book and then proposes some related topics for the future research work.

10.1 Conclusion

The focus of the book has been placed on stability, control, filtering, and fault detection problems for some classed of S-MJS (including state-delayed systems, singular systems, neural networked systems, and T–S fuzzy systems). Specifically, several research problems have been investigated in detail.

1. New stochastic stability conditions are presented for a class of S-MJS. Specifically, the concepts of S-MJS, and mode-dependent time-delays are introduced together for the stochastically stable problem in order to reflect a more realistic environment. By Lyapunov function approach, together with piecewise analysis method, conditions are proposed to ensure the stochastic stability of the underlying S-MJS with time delays. The system under consideration not only contains time-varying delays, but also involves uncertainties in the mode transition rate matrix.
2. A new regulation methodology for the singular S-MJS with constrained control input is concerned. Motivated by recent developments in positively invariant set, necessary and sufficient conditions for the existence of full rank solutions for a class of nonlinear equations are derived, and a new algorithm that provides a solution to the constrained regulation problem is presented.
3. A new integral sliding surface has been designed and some sufficient conditions have been proposed for the stochastic stability of sliding mode dynamics in terms of strict LMI. Also, an explicit parametrization of the desired sliding surface has been given. A sliding mode controller has been synthesized to guarantee the reachability of the system state trajectories to the sliding surface. Moreover, we have further studied the observer design and observer-based SMC problems for the case that some system state components are not accessible. Sufficient conditions have also been proposed for the existence of the desired sliding mode, and the observer-based SMC law has been designed for the reaching motion.

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4. The results obtained in dynamic output feedback controller design problem for S-MJS with repeated scalar nonlinears have extended some of the results in for MJS, and the CCL procedure is employed to transform a nonconvex feasibility problem into a sequential minimization problem subject to LMIs. The desired full- and reduced-order dynamic output feedback controllers are designed in a whole framework.
5. We have utilized construction of a residual signal approach to solve the fault detection filter design problem in the case that it could be made sensitive to faults and robust to modelling errors or disturbances. Also, the weighted H_2 performance index has been employed to test the noise attenuation performance. Based on the switching-sequence dependent Lyapunov function approach and the T–S fuzzy technique, a sufficient condition, which guarantees the filtering error dynamics to be stochastically stable with a weighted H_2 error performance, has been obtained, and the developed fault detection filter design algorithm is employed that can be readily solved using optimization techniques.
6. A new structure of fault detection filter is proposed for sojourn information (sojourn time and sojourn probability) dependent S-MJS. A key restriction in the underlying S-MJS is that the sojourn time of each subsystem is subject to geometric distribution. To relax the restriction, the concept of a nonhomogeneous Markov chain, where the transition probabilities are sojourn time-dependent, has been introduced, and studies on the corresponding systems have been gradually launched.
7. A new probability guaranteed fault detection filter has been designed for an underactuated manipulator which is modeled as MJS. Some system parameters might be randomly perturbed within certain intervals. The uniform distribution has been used to characterize the statistical characteristics of the uncertain parameters. By employing the parameter-dependent Lyapunov functional approach, a novel fault detection filter has been designed and a parameter-box has been sought such that the disturbance attenuation level and the required probability are simultaneously guaranteed. Also, a computational algorithm has been proposed for the design of the robust probability-guaranteed H_2 filter.

10.2 Further Work

Related topics for the future research work are listed below:

(i) For the time-delay S-MJS, the results on stability have some conservativeness. Some recently developed methods such as delay-partitioning method, small gain based input–output method, and reciprocally convex method can be utilized to further reduce the conservativeness caused by time-delay.
(ii) The insertion of the shared communication networks in the control/filter loop, which may cause network-induced delays, packet dropouts, data disorder and so on. So, it is meaningful and important to consider the analysis and synthesis

problem with network-induced delays and data disorder phenomenon in control and filtering of S-MJS.

(iii) Chattering problem is one of the most common handicaps for applying SMC to real applications. The chattering in SMC systems is usually caused by (1) the dynamics with small time constants, which are often neglected in the ideal model; and (2) utilization of digital controllers with finite sampling rate, which causes so called 'discretization chattering'. The discontinuity leads to control chattering in practice, and involves high frequency dynamics. How to reduce chattering will be a research topic in future studies.

(iv) Another future research direction is to investigate two-dimensional (2-D) S-MJS, which consist of a family of subsystems described by 2-D dynamical systems, and a rule specifying the switching among them. Some advanced techniques (such as quadratic Lyapunov functions and piecewise Lyapunov functions) used in analyzing and designing for 1-D S-MJS can be extended to deal with 2-D S-MJS.

(v) In an industrial process, the dynamic behaviors are generally complex and nonlinear, and their real mathematical models are always difficult to obtain. How to design the filter for unknown systems using input/output data has become one main focus of research. The study of passive filter design algorithm in this book for a class of neural network systems is one solution to this challenging problem. In our future work, it will be one of our main focuses to explore the practical realization and applications of the theoretic results obtained in this book. Moreover, we will consider the robustness of a neural network against noise and variation of weights in a noisy environment.