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POLYTROPES
Applications in Astrophysics and Related Fields

G.P. HOREDT

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POLYTROPES

Applications in Astrophysics and Related Fields

by

G.P. HOREDT

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1 POLYTROPIC AND ADIABATIC PROCESSES

1.1 Basic Concepts

Provided that no chemical reactions are occurring, the thermodynamic state of a system of matter can be described completely by three quantities: The pressure P, the absolute temperature T, and the volume V or the rest mass density ϱ. For any system of matter there exists a thermodynamic function F or G connecting the three state variables

$$F(P, V, T) = 0 \qquad \text{or} \qquad G(P, \varrho, T) = 0. \tag{1.1.1}$$

By thermodynamic equilibrium we understand the state of a thermodynamic system of matter that has come to a perfectly steady state, being in mechanical (hydrostatic) equilibrium and at uniform temperature. Often, in larger systems of matter, thermodynamic equilibrium is satisfied only roughly or even not at all. However, for most cases of interest large systems can be divided into smaller ones, where thermodynamic equilibrium holds, at least to a good approximation. In this case we are speaking about local thermodynamic equilibrium.

Thermodynamic changes of the state variables can be reversible processes or irreversible ones. Reversible thermodynamic processes are changes of the thermodynamic system when the system can be carried through exactly the same thermodynamic equilibrium states in reverse order as in original order, without supplying energy from outside to the system. When the thermodynamic system cannot be brought back to its original state without supply of additional energy from outside, we are speaking about irreversible processes.

A change of the state variables of a thermodynamic system that changes the state variables only infinitesimally is called an infinitesimal process of the thermodynamic system. Infinitesimal processes of a thermodynamic system can be carried out reversibly or irreversibly. A quasistatic process of a thermodynamic system is an infinitesimal process which changes the state variables in such a way that the system remains always very close to its state of thermodynamic equilibrium. Quasistatic processes are always reversible thermodynamic processes, the notions of quasistatic process or reversible process being synonymous. Infinitesimal quasistatic processes are a special class of infinitesimal processes and are always reversible processes, whereas infinitesimal processes may be reversible or irreversible processes.

Thermodynamic processes in a system of matter that are taking place without exchange of thermal energy with the outside of the system are called adiabatic processes. Perfect thermal insulation of a system implies always adiabatic thermodynamic processes in the system.

Considering only thermal and mechanical infinitesimal processes in a system of matter, and neglecting mass motions, the first law of thermodynamics (law of energy conservation) can be written as

$$dQ = dU + P\ dV. \tag{1.1.2}$$

dQ denotes the infinitesimal quantity of heat energy added to the system which is used to increase the system's internal energy U by the amount dU and to effect the mechanical work $P\ dV$, where dV is the infinitesimal change of the system's volume V, and P the hydrostatic pressure force. The volume of the thermodynamic system can be expressed by its average rest mass density ϱ as $V = m/\varrho$, where m is the total constant rest mass of the system. In the nonrelativistic case, the rest mass will simply be termed mass. Eq. (1.1.2) can be put into the form

$$dQ = dU - P\ d\varrho/\varrho^2, \qquad (m = 1;\ dV = d(1/\varrho) = -d\varrho/\varrho^2), \tag{1.1.3}$$

if the rest mass of the system is taken equal to the unit of rest mass $m = 1$. In this case the volume is equal to the specific volume $V = 1/\varrho$. "Specific" quantities will always be referred to the unit of rest mass.

Generalizations of Eq. (1.1.2) are given by Cox and Giuli (1968) when the mixed components of the pressure tensor are nonzero ($P_{ik} \neq 0$ if $i \neq k$).

The second law of thermodynamics may be stated as follows (e.g. Cox and Giuli 1968, Landau and Lifshitz 1971, Gerthsen et al. 1977): For each state of a thermodynamic system there exists a function of the state variables $S = S(P, \varrho, T)$, called entropy. The change of entropy dS during thermodynamic processes of the system is constrained by the equation

$$dS \geq dQ/T, \tag{1.1.4}$$

where dQ is the infinitesimal amount of heat added to the system, and T the temperature of the system. For reversible processes $dS = dQ/T$, for irreversible ones $dS > dQ/T$. For reversible processes dS is an exact differential.

An immediate corollary of the second law follows for adiabatically enclosed thermodynamic systems, that means when $dQ = 0,\ Q = \mathrm{const}$. Adiabatic reversible processes have $dS = dQ/T = 0,\ (Q, S = \mathrm{const}$ – isentropic processes), and adiabatic irreversible processes have $dS > dQ/T = 0$, i.e. $dS > 0,\ dQ = 0,\ (Q = \mathrm{const};\ S \neq \mathrm{const})$. The entropy S of an enclosed thermodynamic system, excepting for completely negligible fluctuations, can never decrease.

In principle, polytropic processes, to be defined in the next section, could be reversible or irreversible processes. Generally we are dealing only with reversible (quasistatic) processes of systems in thermodynamic equilibrium, without considering chemical or nuclear reactions. However, if shock waves occur for instance, the physical processes become irreversible in the polytropic gas during the shock (cf. Secs. 6.3.4, 6.4.1, Landau and Lifshitz 1959, §82).

Generally, we make a clear distinction between polytropic (adiabatic) processes (changes) occurring at a fixed location (e.g. local oscillations), and the spatial, overall distribution of matter obeying the polytropic equation of state (1.3.29) or (2.1.6). This chapter is devoted exclusively to local polytropic (adiabatic) changes occurring at a fixed space point.

1.2 Polytropic and Adiabatic Processes in a Perfect Gas

In order to better understand the definitions of the polytropic exponents and polytropic indices, it seems worthwhile to pursue the historical development, and to define them at first for thermodynamic systems obeying the equation of state of a perfect gas (Emden 1907, Milne 1930a, Chandrasekhar 1939, Eddington 1959). We distinguish between the terms ideal gas and perfect gas. Ideal gases are called systems of noninteracting particles. In practice, this amounts to systems where the energy of particle interactions is negligibly small as compared to the kinetic energy of the particles (Landau and Lifshitz 1971). In other words, when we are studying the motion of ideal gases we can always neglect their viscosity; energy dissipation during motion is negligibly small. Perfect gases are called ideal gases obeying the Maxwell-Boltzmann statistics (perfect gas = nondegenerate, ideal gas). Completely degenerate gases in our terminology are ideal gases too, but not perfect gases.

Let us suppose that we change during a reversible (quasistatic) process the amount of heat energy of a thermodynamic system by dQ, thus changing its temperature by dT. We define the specific heat as the differential quotient

$$c = dQ/dT. \tag{1.2.1}$$

In other words, the specific heat of a thermodynamic system is the amount of heat that must be added to a thermodynamic system in order to change its temperature by one degree Kelvin ($dT = \pm 1$). In this book "specific heat" means the specific heat per unit rest mass. It is clear that the heat quantity dQ of a system can be changed in infinitely different ways. It is therefore reasonable to define the specific heat at constant α, where α is a specified function of the state variables:

$$c_\alpha = (dQ/dT)_\alpha. \tag{1.2.2}$$

The most commonly used specific heats are those at constant pressure and constant volume (density). Thus: $\alpha = P$ or $\alpha = V$.

We define a polytropic change or a polytropic process as a reversible (quasistatic) change of a thermodynamic system of unit rest mass, occurring in such a way that the derivative dQ/dT, the "specific heat", varies in a prescribed manner during the reversible change:

$$c = dQ/dT = \text{defined function}, \qquad (m = 1). \tag{1.2.3}$$

A very important special case occurs when

$$c = dQ/dT = \text{const.} \tag{1.2.4}$$

Although some statements and derivations are also valid with the more general definition (1.2.3), it is generally assumed throughout this book that c = const. The definitions (1.2.3) or (1.2.4) yield a second equation besides the equation of state (1.1.1); from this set of two equations we can eliminate one of the three state variables P, ϱ, T, and the polytropic thermodynamic system has therefore a single independent state variable. Obviously, the definition of polytropic processes involves some arbitrariness. We have adopted the most restrictive definition by defining polytropic processes as reversible (quasistatic) changes of a thermodynamic system. Most equations of this chapter are valid only with this definition. It is clear that, more generally, polytropic processes could also be irreversible processes. Quite generally, polytropes could be defined as systems of matter where pressure and density are connected by a law of the form (1.3.25); however, such a kind of definition entirely forgets the thermodynamic base of the notion of polytrope.

If the specific heat c of the polytropic change is zero, the polytropic process becomes an adiabatic one ($dQ = 0$). Also, if the specific heat of the polytrope is $c = \pm\infty$, the polytropic change becomes an isothermal one ($dT = 0$). Thus, polytropic processes are intermediate between isothermal and adiabatic processes. When the specific heat of the polytropic process c is equal to the specific heat at constant pressure c_P, the polytropic process is an isobaric process (P = const), and when c equals the specific heat at constant volume c_V, the polytropic process is an isometric one. Polytropic processes were first considered by G. Zeuner in 1887 (cf. Chandrasekhar 1939).

For a perfect gas, the pressure is given by the equation

$$P = \mathcal{R}\varrho T/\mu = \mathcal{R}mT/\mu V = k\varrho T/\mu H = kn_dT. \tag{1.2.5}$$

μ is the mean molecular weight of the gas (the rest mass of free particles per mole [see Eq. (1.7.1)], and the atomic mass unit $H = 1.66055 \times 10^{-24}$ g is defined as the sixteenth part of a neutral oxygen atom, or as the twelfth part of a neutral carbon atom: $H = m(^{16}\mathrm{O})/16 = m(^{12}\mathrm{C})/12$. The mean molecular weight μ can be expressed in g/mole or in atomic mass units H (see Sec. 1.7). One mole is the rest mass of gas measured in grams, which is numerically equal to the mean mass of its individual particles measured in atomic mass units H. In Eq. (1.2.5) $k = 1.38062 \times 10^{-16}$ erg K^{-1} denotes the Boltzmann constant, equal to the number of ergs per degree Kelvin, $\mathcal{R} = k/H = 8.3143 \times 10^7$ erg K^{-1} mole^{-1} the perfect gas constant, m the rest mass of gas inside volume V, and n_d the number density of free particles, equal to the number of free particles per unit volume. The number of free particles per mole N_A (Avogadro's number) is numerically equal to the inverse of the atomic mass unit: $N_A = 1/H = 6.02217 \times 10^{23}$ mole^{-1} (see App. A).

It should be noted that the perfect gas law (1.2.5) is valid for thermodynamic systems composed of nondegenerate and noninteracting particles, both in the nonrelativistic limit (when the total relativistic energy of particles is nearly equal to their rest energy), and in the relativistic limit (when the total relativistic energy of the particles is not nearly equal to their rest energy), (Cox and Giuli 1968). For relativistic particles the density from Eq. (1.2.5) has the same meaning as in the nonrelativistic case: ϱ is always equal to the rest mass density, to be defined below.

It is well known from the kinematics of special relativity that the total relativistic kinetic energy of translational particle motion $E_r^{(kin)}$ is related to the particle mass by (e.g. Landau and Lifschitz 1987)

$$E_r^{(kin)} = m_rc^2 = mc^2/(1 - v^2/c^2)^{1/2}, \tag{1.2.6}$$

where $c = 2.997925 \times 10^{10}$ cm s^{-1} is the velocity of light [not to be confused with the polytropic specific heat from Eq. (1.2.3)], and v the velocity between particle and observer. m_r denotes the relativistic mass of the particle, and m the rest mass, when $v = 0$.

We use throughout the supplementary notation kin, if we wish to stress that only translational kinetic energy is considered. This index is omitted in the general case, when besides of purely translational kinetic motion, there are also taken into account the energy of particle interactions, of external force and radiation fields, etc. From Eq. (1.2.6) follows

$$m_r = m/(1 - v^2/c^2)^{1/2}. \tag{1.2.7}$$

If $v = 0$, the relativistic kinetic energy $E_r^{(kin)}$ of the particle is equal to its rest energy E :

$$E = mc^2; \qquad E_r^{(kin)} = E/(1 - v^2/c^2)^{1/2}. \tag{1.2.8}$$

Eqs. (1.2.6)-(1.2.8) are meaningful only for particles having nonzero rest mass $m \neq 0$ and moving with velocities $v < c$. Particles with zero rest mass ($m = 0$) are always moving with the velocity of light $v = c$. Obviously, in relativistic kinematics the kinetic energy $E^{(kin)}$, ($E^{(kin)} \approx mv^2/2$ if $v \ll c$) is just the difference between total relativistic kinetic energy $E_r^{(kin)} = m_r c^2$ and rest energy $E = mc^2$:

$$E^{(kin)} = E_r^{(kin)} - E = mc^2[1/(1 - v^2/c^2)^{1/2} - 1]. \tag{1.2.9}$$

In the general case, the total relativistic energy E_r of a thermodynamic system contains besides its rest energy E, the energy of translational kinetic motion $E^{(kin)}$, the energy of particle interactions (excepting gravitational interactions) and of external force fields, radiation energy, etc. (e.g. Zeldovich and Novikov 1971, Landau and Lifschitz 1987). In this general case, Eq. (1.2.9) writes

$$E^{(int)} = E_r - E, \tag{1.2.10}$$

where $E^{(int)}$ contains all forms of energy, excepting the rest energy of the system and gravitational interactions. The equation of momentum is via Eq. (1.2.7) equal to

$$\vec{p} = m_r \vec{v} = m\vec{v}/(1 - v^2/c^2)^{1/2} = E_r^{(kin)}\ \vec{v}/c^2. \tag{1.2.11}$$

A useful relationship between the energy of translational kinetic motion and momentum can readily be verified by direct substitution:

$$E_r^{(kin)} = (p^2c^2 + m^2c^4)^{1/2}. \tag{1.2.12}$$

The kinetic translational energy is then obtained as [cf. Eq. (1.2.9)]

$$E^{(kin)} = E_r^{(kin)} - E = (p^2c^2 + m^2c^4)^{1/2} - mc^2 = mc^2\{[1 + (p/mc)^2]^{1/2} - 1\}. \tag{1.2.13}$$

The above equations are equally valid for microscopic particles and macroscopic bodies. The relativistic volume V_r occupied by a macroscopic body changes with its velocity v relative to an observer as (e.g. Landau and Lifschitz 1987)

$$V_r = V(1 - v^2/c^2)^{1/2}, \tag{1.2.14}$$

where the proper volume or volume of rest V is obtained when $v = 0$. The total relativistic translational kinetic energy $E_r^{(kin)}$ contained in the unit of proper volume of a thermodynamic system is called relativistic energy density of translational kinetic motion $\varepsilon_r^{(kin)}$. Eq. (1.2.6) writes for the unit of proper volume ($E_r^{(kin)}, E^{(kin)}, E \rightarrow \varepsilon_r^{(kin)}, \varepsilon^{(kin)}, \varepsilon;\ m_r, m \rightarrow \varrho_r, \varrho$) :

$$\varepsilon_r^{(kin)} = \varepsilon^{(kin)} + \varepsilon = \varrho_r c^2 = \varrho c^2/(1 - v^2/c^2)^{1/2}. \tag{1.2.15}$$

$\varepsilon = \varrho c^2$ is the rest energy density, and $\varepsilon^{(kin)}$ the energy density of translational kinetic particle motion [cf. Eqs. (2.6.114), (2.6.115)]. The relativistic density $\varrho_r = \varepsilon_r^{(kin)}/c^2$ is equal to the relativistic mass in the unit of proper volume, and the rest mass density $\varrho = \varepsilon/c^2$ is equal to the rest mass contained in the unit of proper volume. As noted above, in the general case, the relativistic energy density ε_r contains besides the rest energy density ε and the energy density $\varepsilon^{(kin)}$ of translational kinetic particle motion, also the energy density of particle interactions (other than gravitational interactions), of external forces and radiation fields, etc. Thus, quite generally

$$\varepsilon_r = \varrho_r c^2 = \varepsilon + \varepsilon^{(int)} = \varrho c^2 + \varepsilon^{(int)}, \tag{1.2.16}$$

where we simply call internal energy density $\varepsilon^{(int)}$ the difference between the total relativistic energy density ε_r and the rest energy density $\varepsilon = \varrho c^2$ of the system.

The specific relativistic kinetic energy $E_r^{(kin)}$ (relativistic energy per unit rest mass) is found from Eq. (1.2.6):

$$E_r^{(kin)} = c^2/(1 - v^2/c^2)^{1/2}, \qquad (m = 1). \tag{1.2.17}$$

Inserting into Eq. (1.2.15), we obtain an important relationship between relativistic mass density ϱ_r, rest mass density ϱ, and specific relativistic kinetic energy $E_r^{(kin)}$ (e.g. Zeldovich and Novikov 1971, p. 186):

$$\varrho_r = \varrho E_r^{(kin)}/c^2, \qquad (m = 1). \tag{1.2.18}$$

After this brief excursion into special relativity, we return to the main subject of this section. The specific internal energy U, $(m = 1)$ of a perfect gas is a function only of the absolute temperature $U = U(T)$, [cf. Eq. (1.7.61)]. According to Eqs. (1.1.2), (1.2.2) the specific heat at constant volume for a perfect gas will be

$$c_V = (dQ/dT)_V = (\partial U/\partial T)_V = dU/dT, \qquad (m = 1). \tag{1.2.19}$$

The specific heat at constant pressure c_P can be determined by differentiating the equation of state (1.2.5) when $m = \varrho V = 1:$

$$P\ dV + V\ dP = \mathcal{R}\ dT/\mu. \tag{1.2.20}$$

Introducing into Eq. (1.1.2), we get

$$dQ = dU + \mathcal{R}\ dT/\mu - V\ dP. \tag{1.2.21}$$

Thus

$$c_P = (dQ/dT)_P = dU/dT + \mathcal{R}/\mu = c_V + \mathcal{R}/\mu. \tag{1.2.22}$$

To obtain the equation of a polytrope of specific heat $c = \mathrm{const}$, we substitute Eqs. (1.2.4) and (1.2.19) into Eq. (1.1.2):

$$dQ = c\ dT = c_V\ dT + P\ dV = c_V\ dT - \mathcal{R}T\ d\varrho/\mu\varrho. \tag{1.2.23}$$

Or, because $\mathcal{R}/\mu = c_P - c_V:$

$$(c_V - c)\ dT/T = (c_P - c_V)\ d\varrho/\varrho, \qquad (m = 1). \tag{1.2.24}$$

Provided that $c, c_V, c_P = \mathrm{const}$, the integration of Eq. (1.2.24) yields for the equation of a polytrope of specific heat $c:$

$$T\varrho^{(c_P - c_V)/(c - c_V)} = \mathrm{const}. \tag{1.2.25}$$

By convention, we define the polytropic index of a perfect gas n as (cf. Chandrasekhar 1939)

$$n = (c_V - c)/(c_P - c_V). \tag{1.2.26}$$

If $n = \mathrm{const}$, we obtain from Eqs. (1.2.25) and (1.2.26), with the aid of the equation of state (1.2.5), three equivalent forms for the equation governing a polytropic process:

$$P\varrho^{-1-1/n} \propto PV^{1+1/n} = \mathrm{const}, \tag{1.2.27}$$

$$PT^{-1-n} = \mathrm{const}, \tag{1.2.28}$$

$$T\varrho^{-1/n} \propto TV^{1/n} = \mathrm{const}. \tag{1.2.29}$$

From Eqs. (1.2.27) and (1.2.28) it is obvious that when $n = -1$, then $P = \mathrm{const}$ (isobaric polytropic process). From Eq. (1.2.29) we observe that $T = \mathrm{const}$ if $n = \pm\infty$ (isothermal polytropic process). If we rewrite Eq. (1.2.29) as $T^{-n}\varrho \propto T^{-n}/V = \mathrm{const}$, we observe that if $n = 0$, we have $\varrho \propto 1/V = \mathrm{const}:$ There occurs an isopycnic ($\varrho = \mathrm{const}$) or an isometric ($V = \mathrm{const}$) polytropic process.

In analogy to the adiabatic exponent $\gamma = c_P/c_V$ for a perfect gas, we can define the polytropic *exponent* of a perfect gas by

$$\gamma' = (c_P - c)/(c_V - c), \tag{1.2.30}$$

and taking into account Eq. (1.2.26), the polytropic *index* of a perfect gas is defined by

$$n = 1/(\gamma' - 1); \quad \gamma' = 1 + 1/n. \tag{1.2.31}$$

Only for an adiabatic (isentropic) perfect gas ($c = dQ/dT = 0$) we have $\gamma' = \gamma = c_P/c_V$ and $n = 1/(\gamma - 1)$. Eqs. (1.2.27)-(1.2.29) can be rewritten as

$$P\varrho^{-\gamma} \propto PV^{\gamma} = \mathrm{const}; \quad PT^{\gamma/(1-\gamma)} = \mathrm{const}; \quad T\varrho^{1-\gamma} \propto TV^{\gamma-1} = \mathrm{const},$$
$$(\gamma = \gamma' = 1 + 1/n;\ S = \mathrm{const}). \tag{1.2.32}$$

It should be stressed that these equations are valid for adiabatic and reversible, i.e. isentropic processes, as emphasized by Eq. (1.2.39). Since dissipative processes are absent by definition in a perfect gas, the adiabatic perfect gas is always isentropic.

Using the logarithmic derivative of Eq. (1.2.5), $dP/P = d\varrho/\varrho + dT/T$, we can rewrite Eq. (1.2.24) under the equivalent forms $(c_V - c)\ dP/P = (c_P - c)\ d\varrho/\varrho$ and $(c_P - c_V)\ dP/P = (c_P - c)\ dT/T$. These equations, together with Eq. (1.2.24), can be used to define the polytropic exponent from Eq. (1.2.30) in the following three ways:

$$\gamma' = d\ln P/d\ln\varrho = -d\ln P/d\ln V; \quad \gamma'/(\gamma' - 1) = d\ln P/d\ln T;$$
$$\gamma' - 1 = d\ln T/d\ln\varrho = -d\ln T/d\ln V. \tag{1.2.33}$$

Using the definitions (1.2.31) and (1.2.33), the polytropic index n for a perfect gas can be defined by one of the three equations

$$1 + 1/n = d\ln P/d\ln\varrho = -d\ln P/d\ln V; \quad 1 + n = d\ln P/d\ln T;$$
$$1/n = d\ln T/d\ln\varrho = -d\ln T/d\ln V. \tag{1.2.34}$$

From Eq. (1.2.27) we obtain the polytropic equation of state for a perfect gas as

$$P = K\varrho^{1+1/n}, \qquad (K, n = \mathrm{const}). \tag{1.2.35}$$

If $n = 0$, it would seem that $P = \infty$, but this seeming singularity can be removed at once, if we write Eq. (1.2.35) under the form $P = L^{1/n}\varrho^{1+1/n}$, or $P^n = L\varrho^{n+1}$, $(K = L^{1/n} = \mathrm{const})$. If $n = 0$, we observe that $\varrho = 1/L = \mathrm{const}$, irrespective of the value of P [cf. Eq. (1.2.27)]. It should be stressed that whenever the factor $K\varrho^{1/n}$ occurs, it can be replaced by P/ϱ according to Eq. (1.2.35), and the apparent singularity for $n = 0$ vanishes. Equating Eq. (1.2.35) with the equation of state (1.2.5), we find the equation defining the polytropic constant K for a perfect gas:

$$K = \mathcal{R}T/\mu\varrho^{1/n} \quad \text{if} \quad n \neq 0; \qquad K\varrho^{1/n} = \mathcal{R}T/\mu = P/\varrho \quad \text{if} \quad n = 0. \tag{1.2.36}$$

The so-called polytropic temperature T_p – a notion mainly of historical interest – can be defined from Eq. (1.2.29):

$$T_p = T\varrho^{-1/n} = K\varrho T/P \quad \text{and} \quad K = \mathcal{R}T_p/\mu. \tag{1.2.37}$$

Thus, the polytropic temperature is the temperature along a given polytrope for which the density is just equal to unity. The polytropic temperature characterizes a one-parametric family of polytropes in the (T, ϱ)-plane. For an isothermal process ($n = \pm\infty$), we have from Eq. (1.2.37) $T = T_p$, i.e. the polytropic temperature of an isothermal process agrees just with the real temperature.

If the polytropic processes in the perfect gas are reversible, the entropy is given via Eq. (1.1.4) by the total differential

$$dS = dQ/T = c_V\ dT/T - \mathcal{R}\ d\varrho/\mu\varrho = c_V\ dT/T - (c_P - c_V)\ d\varrho/\varrho$$
$$= c_V\ dP/P - c_P\ d\varrho/\varrho = c_P\ dT/T - (c_P - c_V)\ dP/P, \tag{1.2.38}$$

where we have used Eq. (1.2.23) and the logarithmic derivative of the equation of state (1.2.5). Integration yields

$$P/P_0 = (\varrho/\varrho_0)^{\gamma}\exp[(S - S_0)/c_V] = (T/T_0)^{\gamma/(\gamma-1)}\exp[(S - S_0)/(c_V - c_P)];$$
$$\varrho/\varrho_0 = (T/T_0)^{1/(\gamma-1)}\exp[(S - S_0)/(c_V - c_P)], \tag{1.2.39}$$

the zero indexed quantities denoting some initial state. Thus, the adiabatic relationships (1.2.32) of a perfect gas hold for isentropic processes $S = S_0 =$ const. Likewise, Eq. (1.2.24) can be written as $dT/T = (\gamma' - 1)\ d\varrho/\varrho$. Eq. (1.2.38) becomes

$$dS = c_V(\gamma' - \gamma)\ d\varrho/\varrho = c_V(\gamma' - \gamma)\ dT/(\gamma' - 1)T, \tag{1.2.40}$$

or if $\gamma', \gamma =$ const :

$$S = c_V(\gamma' - \gamma)\ln\varrho + \text{const} = [c_V(\gamma' - \gamma)/(\gamma' - 1)]\ln T + \text{const}. \tag{1.2.41}$$

For an adiabatic reversible process we have by Eq. (1.2.30): $\gamma' = \gamma,\ (c = 0)$. In this case the entropy S remains always constant (isentropic process).

1.3 Polytropic Processes for a General Equation of State

When the equation of state is of the general form (1.1.1), the polytropic exponent is no longer the same among the three equations (1.2.33). However, we can define analogously to Eqs. (1.2.33) the three polytropic exponents (Chandrasekhar 1939, Cox and Giuli 1968)

$$\Gamma_1' = d\ln P/d\ln\varrho = -d\ln P/d\ln V; \quad \Gamma_2'/(\Gamma_2' - 1) = d\ln P/d\ln T;$$
$$\Gamma_3' - 1 = d\ln T/d\ln\varrho = -d\ln T/d\ln V. \tag{1.3.1}$$

Below, we determine for a general equation of state

$$P = P(\varrho, T), \tag{1.3.2}$$

the equations for the polytropic exponents (1.3.1) as a function of the specific heats c_P, c_V, and of the partial derivatives

$$\chi_\varrho = (\partial\ln P/\partial\ln\varrho)_T = -(\partial\ln P/\partial\ln V)_T; \quad \chi_T = (\partial\ln P/\partial\ln T)_\varrho = (\partial\ln P/\partial\ln T)_V. \tag{1.3.3}$$

Dividing in Eq. (1.3.1) the first equation by the third one, we obtain the useful identity

$$\Gamma_1'/(\Gamma_3' - 1) = \Gamma_2'/(\Gamma_2' - 1), \tag{1.3.4}$$

which shows that only two of the three polytropic exponents are independent variables. Differentiating logarithmically the equation of state (1.3.2), we obtain

$$d\ln P = \chi_T\ d\ln T + \chi_\varrho\ d\ln\varrho \quad \text{or} \quad \Gamma_1' = \chi_T(\Gamma_3' - 1) + \chi_\varrho. \tag{1.3.5}$$

Thus, when the partial derivatives χ_ϱ and χ_T are computed from the equation of state (1.3.2) and if one of the polytropic exponents (1.3.1) is known, the other two exponents can be computed according to Eqs. (1.3.4) and (1.3.5). Regarding the specific internal energy U as a function of V and T, we can rewrite Eq. (1.1.2) as follows:

$$dQ = [(\partial U/\partial V)_T + P]\ dV + (\partial U/\partial T)_V\ dT. \tag{1.3.6}$$

Since polytropic changes are by definition reversible processes, the second law of thermodynamics can be written under the form

$$dS = dQ/T = (1/T)[(\partial U/\partial V)_T + P]\ dV + (1/T)(\partial U/\partial T)_V\ dT. \tag{1.3.7}$$

The entropy is an exact differential, so we have

$$\big\{\partial[(1/T)(\partial U/\partial V)_T + P/T]/\partial T\big\}_V = \big\{\partial[(1/T)(\partial U/\partial T)_V]/\partial V\big\}_T, \tag{1.3.8}$$

or by performing the derivation:

$$(\partial U/\partial V)_T = T(\partial P/\partial T)_V - P. \tag{1.3.9}$$

For the unit of rest mass we have $(\partial U/\partial V)_T = -\varrho^2(\partial U/\partial \varrho)_T$, and Eq. (1.3.9) takes the logarithmic form

$$(\partial \ln U/\partial \ln \varrho)_T = -(P/\varrho U)[(\partial \ln P/\partial \ln T)_\varrho - 1] = -(P/\varrho U)(\chi_T - 1). \tag{1.3.10}$$

Regarding the specific internal energy U as a function of ϱ and T, the first law of thermodynamics (1.1.3) becomes

$$dQ = [(\partial U/\partial \varrho)_T - P/\varrho^2]\ d\varrho + (\partial U/\partial T)_\varrho\ dT = [(\partial U/\partial \varrho)_T - P/\varrho^2]\ d\varrho + c_V\ dT, \tag{1.3.11}$$

since from the definition of the specific heat at constant volume we obtain, by using Eqs. (1.1.2) or (1.1.3):

$$c_V = (dQ/dT)_V = (dQ/dT)_\varrho = (\partial U/\partial T)_V = (\partial U/\partial T)_\varrho. \tag{1.3.12}$$

Equating dQ from Eqs. (1.3.11) and (1.2.3), we find

$$(c - c_V)\ dT/T = [\varrho(\partial U/\partial \varrho)_T - P/\varrho]\ d\varrho/\varrho T, \tag{1.3.13}$$

and finally the desired equation

$$\Gamma_3' - 1 = d\ln T/d\ln \varrho = [U(\partial \ln U/\partial \ln \varrho)_T - P/\varrho]\big/(c - c_V)T = P\chi_T\big/(c_V - c)\varrho T, \tag{1.3.14}$$

via Eq. (1.3.10). The above equation can be brought into a form equivalent to Eq. (1.2.24) by including also the value of $c_P - c_V$. To this end, we recall the definition for the specific heat at constant pressure, using the first law of thermodynamics (1.1.3):

$$c_P = (dQ/dT)_P = (\partial U/\partial T)_P - (P/\varrho^2)(\partial \varrho/\partial T)_P. \tag{1.3.15}$$

If $P = \text{const}$, we have $dP = (\partial P/\partial \varrho)_T\ d\varrho + (\partial P/\partial T)_\varrho\ dT = 0$, and dividing this equation by dT, we get for the second derivative on the right-hand side of Eq. (1.3.15):

$$(\partial \varrho/\partial T)_P = -(\partial P/\partial T)_\varrho\big/(\partial P/\partial \varrho)_T = -(\varrho/T)\ \chi_T/\chi_\varrho. \tag{1.3.16}$$

To obtain also an expression for the first derivative $(\partial U/\partial T)_P$ from Eq. (1.3.15), we differentiate the internal energy of the thermodynamic system in two ways, by taking $U = U(\varrho, T)$ and $U = U[P(\varrho, T), T]$:

$$\begin{aligned} dU &= (\partial U/\partial \varrho)_T\ d\varrho + (\partial U/\partial T)_\varrho\ dT \qquad \text{and} \\ dU &= (\partial U/\partial P)_T\ [(\partial P/\partial \varrho)_T\ d\varrho + (\partial P/\partial T)_\varrho\ dT] + (\partial U/\partial T)_P\ dT. \end{aligned} \tag{1.3.17}$$

We find, by equating the terms near $d\varrho$ and dT, respectively:

$$(\partial U/\partial \varrho)_T = (\partial U/\partial P)_T\ (\partial P/\partial \varrho)_T; \quad (\partial U/\partial T)_\varrho = (\partial U/\partial P)_T\ (\partial P/\partial T)_\varrho + (\partial U/\partial T)_P. \tag{1.3.18}$$

Using successively Eqs. (1.3.18), (1.3.16), (1.3.12), (1.3.18), (1.3.3), and (1.3.10), we get from Eq. (1.3.15) the important result

$$\begin{aligned} c_P &= (\partial U/\partial T)_\varrho - (\partial U/\partial P)_T\ (\partial P/\partial T)_\varrho + (P/\varrho T)\ \chi_T/\chi_\varrho \\ &= c_V - (\partial U/\partial \varrho)_T\ (\partial \varrho/\partial P)_T\ (\partial P/\partial T)_\varrho + (P/\varrho T)\ \chi_T/\chi_\varrho \\ &= c_V - (U/T)(\partial \ln U/\partial \ln \varrho)_T\ \chi_T/\chi_\varrho + (P/\varrho T)\ \chi_T/\chi_\varrho = c_V + (P/\varrho T)\ \chi_T^2/\chi_\varrho, \end{aligned} \tag{1.3.19}$$

or

$$\chi_T^2 = (c_P - c_V)\varrho T\chi_\varrho/P. \tag{1.3.20}$$

Eq. (1.3.14) can be written via Eq. (1.3.20) under the final form

$$\Gamma_3' - 1 = d\ln T/d\ln \varrho = (c_P - c_V)\chi_\varrho\big/(c_V - c)\chi_T. \tag{1.3.21}$$

The two other gammas can be readily found with the aid of Eqs. (1.3.4), (1.3.5):

$$\Gamma_1' = d\ln P/d\ln \varrho = (c_P - c)\chi_\varrho/(c_V - c); \quad \Gamma_2'/(\Gamma_2' - 1) = d\ln P/d\ln T = (c_P - c)\chi_T/(c_P - c_V). \tag{1.3.22}$$

If $c = dQ/dT = 0$, we obtain from Eqs. (1.3.21), (1.3.22) the *adiabatic* exponents for the general equation of state $P = P(\varrho, T)$, calculated at constant entropy, since the considered processes are reversible and adiabatic $dS = dQ/T = 0$:

$$\Gamma_1 = (\partial \ln P/\partial \ln \varrho)_S = c_P\chi_\varrho/c_V; \quad \Gamma_2/(\Gamma_2 - 1) = (\partial \ln P/\partial \ln T)_S = c_P\chi_T/(c_P - c_V);$$
$$\Gamma_3 - 1 = (\partial \ln T/\partial \ln \varrho)_S = (c_P - c_V)\chi_\varrho/c_V\chi_T. \tag{1.3.23}$$

In terms of the ratio of specific heats

$$\gamma = c_P/c_V = \Gamma_1/\chi_\varrho = 1/[1 - \chi_T(\Gamma_2 - 1)/\Gamma_2] = 1 + \chi_T(\Gamma_3 - 1)/\chi_\varrho. \tag{1.3.24}$$

The three adiabatic exponents are equal to the ratio of specific heats ($\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma$) if and only if $\chi_\varrho = \chi_T = 1$, as can be seen from Eqs. (1.3.1), (1.3.24). When $\chi_\varrho = \chi_T = 1$, the equation of state must be of the form of the perfect gas equation (1.2.5): $P \propto \varrho T$. The three adiabatic exponents can be equal among each other even if $\chi_\varrho, \chi_T \neq 1$, but in this case $\Gamma_k \neq \gamma$, $(k = 1, 2, 3)$. Such a situation occurs for instance for black body radiation, for a completely degenerate gas in the nonrelativistic and extreme relativistic limit, and for electron-positron pairs (Secs. 1.4–1.6).

Analogously to Eqs. (1.2.33), (1.2.34) we define the polytropic index n for the general equation of state by

$$1 + 1/n = \Gamma_1' = d\ln P/d\ln \varrho = (c_P - c)\chi_\varrho/(c_V - c). \tag{1.3.25}$$

The polytropic index can also be defined by (cf. Cox and Giuli 1968)

$$1 + n' = \Gamma_2'/(\Gamma_2' - 1) = d\ln P/d\ln T \quad \text{or} \quad 1/n'' = \Gamma_3 - 1 = d\ln T/d\ln \varrho. \tag{1.3.26}$$

From Eqs. (1.3.25), (1.3.26) we find relationships analogous to Eq. (1.2.31):

$$n = 1/(\Gamma_1' - 1); \quad n' = 1/(\Gamma_2' - 1); \quad n'' = 1/(\Gamma_3' - 1). \tag{1.3.27}$$

Generally, the polytropic indices n, n', n'' are different for the general equation of state (1.3.2). When the polytropic indices are functions of a radial distance they are called *effective* polytropic indices.

In the particular case, when all polytropic exponents are constant, we can integrate Eq. (1.3.1), to obtain a system equivalent to Eq. (1.2.32):

$$P\varrho^{-\Gamma_1'} \propto PV^{\Gamma_1'} = \text{const}; \quad PT^{\Gamma_2'/(1-\Gamma_2')} = \text{const}; \quad T\varrho^{1-\Gamma_3'} \propto TV^{\Gamma_3'-1} = \text{const},$$
$$(\Gamma_1', \Gamma_2', \Gamma_3' = \text{const}). \tag{1.3.28}$$

From the first of these equations, together with the definition (1.3.25), we obtain the general equation of state for a polytropic change in a form similar to Eqs. (1.2.27), (1.2.35):

$$P\varrho^{-1-1/n} \propto PV^{1+1/n} = \text{const} \quad \text{or} \quad P = K\varrho^{1+1/n}, \qquad (K, n = \text{const}). \tag{1.3.29}$$

K will be referred to as the polytropic constant. If $n = -1$ and $K, \varrho \neq 0$, we have $P = K = \text{const}$, but generally this case will be excluded [cf. discussion subsequent to Eq. (2.1.8)]. If $n = 0$, we get $\varrho = \text{const}$, as already shown subsequently to Eq. (1.2.35), by writing Eq. (1.3.29) under the form $\varrho^{1+n} = P^n/L$, $(K = L^{1/n})$. And finally, if $n = \pm\infty$, we get $P = K\varrho$. The equation of state (1.3.29) is valid for polytropic processes obeying the general equation of state $P = P(\varrho, T)$, but only in the somewhat particular case $\Gamma_1', n = \text{const}$. The more general definition (1.3.25) allows for a variable polytropic index n, but this would introduce far too many degrees of freedom; generally, the polytropic index n will be considered constant when used in connection with an equation of state of the form (1.3.29).

In the two last sections we have discussed polytropic processes for a perfect gas and for a general equation of state. A very important class of polytropic changes are reversible adiabatic (isentropic) processes $dS = dQ/T = 0$. In this case there is $\Gamma_k' = \Gamma_k$, $(k = 1, 2, 3)$ via Eqs. (1.3.22), (1.3.23). The corresponding polytropic indices from Eqs. (1.3.25), (1.3.26)

$$n = 1/(\Gamma_1 - 1); \quad n' = 1/(\Gamma_2 - 1); \quad n'' = 1/(\Gamma_3 - 1), \qquad (S = \text{const}), \tag{1.3.30}$$

generally vary between narrow limits for the simple thermodynamic systems considered in this chapter. In the next three sections we will consider adiabatic processes in three other simple thermodynamic systems, namely black body radiation, electron-positron pairs, and completely degenerate electron or neutron gas.

1.4 Adiabatic Processes in a Mixture of Black Body Radiation and Perfect Gas

At first let us consider an adiabatic enclosure of proper volume V containing black body radiation without matter. The radiation pressure P and the internal energy U of black body radiation contained inside volume V are given by (e.g. Chandrasekhar 1939, Menzel et al. 1963)

$$P = aT^4/3; \quad U = \varepsilon_r V = aT^4 V. \tag{1.4.1}$$

$a = 7.5647 \times 10^{-15}$ erg cm^{-3} K^{-4} denotes the radiation pressure constant (Stefan constant). The specific heat at constant volume of the black body radiation contained in volume V is

$$c_V = (\partial U/\partial T)_V = 4aT^3 V. \tag{1.4.2}$$

Photons have no rest mass in virtue of Eq. (1.2.7), and we infer from Eq. (1.2.16) that $\varepsilon_r = \varrho_r c^2 = \varepsilon^{(int)} = U/V = aT^4$. The specific heat at constant pressure can easily be found from Eq. (1.3.20)

$$c_P = c_V + (P/\varrho T)\ \chi_T^2/\chi_\varrho = \infty, \tag{1.4.3}$$

because

$$\chi_\varrho = -(\partial \ln P/\partial \ln V)_T = 0; \quad \chi_T = (\partial \ln P/\partial \ln T)_V = 4. \tag{1.4.4}$$

For the adiabatic exponents of black body radiation we find from Eq. (1.3.14), when $c = dQ/dT = 0$ and $\varrho \to \varrho_r = \varepsilon_r/c^2 = U/Vc^2 = 1/V,\ (U/c^2 = m_r = 1):$

$$\Gamma_3 - 1 = P\chi_T/c_V \varrho T \to P\chi_T/c_V \varrho_r T = PV\chi_T/c_V T = 1/3. \tag{1.4.5}$$

Eqs. (1.3.5) and (1.3.4) yield, respectively

$$\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\varrho = 4/3, \tag{1.4.6}$$

$$\Gamma_2/(\Gamma_2 - 1) = \Gamma_1/(\Gamma_3 - 1) = 4. \tag{1.4.7}$$

Therefore, in the case of black body radiation

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3 = \text{const}, \tag{1.4.8}$$

and the equations for the adiabates are [cf. Eq. (1.3.28)]

$$PV^4/3 = \text{const}; \quad TV^{1/3} = \text{const}; \quad PT^{-4} = \text{const}. \tag{1.4.9}$$

Adiabatic processes in an enclosure filled with black body radiation behave like a perfect gas with adiabatic exponent $\Gamma_k = 4/3,\ (k = 1, 2, 3)$. However, the ratio of specific heats $\gamma = c_P/c_V$ for black body radiation is not equal to Γ_k, as in the perfect gas case. Instead, we have from Eq. (1.3.24)

$$\gamma = c_P/c_V = \Gamma_1/\chi_\varrho = \infty, \qquad (T \neq 0). \tag{1.4.10}$$

The equations for the adiabatic exponents and the specific heats are more complicated in the case of a mixture of black body radiation and perfect gas; this case has ample application for polytropic stars. The pressure can be split into two parts

$$P = P_g + P_r = \mathcal{R}\varrho T/\mu + aT^4/3, \tag{1.4.11}$$

where P_g means the gas pressure (1.2.5), and P_r the radiation pressure (1.4.1). We find by integration of Eq. (1.2.19) the specific internal energy of a perfect gas

$$U_g = c_{Vg} T = \mathcal{R}T/\mu(\gamma_g - 1), \qquad (m = 1;\ c_{Vg} = \text{const}), \tag{1.4.12}$$

where the subscript g denotes values relative to the gas component ($\gamma_g = c_{Pg}/c_{Vg};\ c_{Pg} = c_{Vg} + \mathcal{R}/\mu$). The internal energy per unit mass of the mixture (specific internal energy) will be the sum of the gas and radiation parts from Eqs. (1.4.12) and (1.4.1):

$$U = \mathcal{R}T/\mu(\gamma_g - 1) + aT^4/\varrho, \qquad (m = 1). \tag{1.4.13}$$

ϱ is the rest mass density of the gas ($\varrho = 1/V$), the relativistic mass density $\varrho_r = aT^4/c^2$ of black body radiation being neglected; this approximation can be made for most cases of practical interest. Let us denote by β the ratio between gas pressure and total pressure

$$\beta = P_g/P. \tag{1.4.14}$$

Then

$$\chi_\varrho = (\varrho/P)(\partial P \partial \varrho)_T = \mathcal{R}\varrho T/\mu P = P_g/P = \beta, \tag{1.4.15}$$

and

$$\chi_T = (T/P)(\partial P/\partial T)_\varrho = (4aT^4/3 + \mathcal{R}\varrho T/\mu)/P = 4P_r/P + P_g/P = 4 - 3\beta. \tag{1.4.16}$$

For the specific heat at constant volume of the mixture, we get

$$\begin{aligned} c_V &= (\partial U/\partial T)_\varrho = \mathcal{R}/\mu(\gamma_g - 1) + 4aT^3/\varrho = [\mathcal{R}/\mu(\gamma_g - 1)][12P_r(\gamma_g - 1)/P_g + 1] \\ &= [\mathcal{R}/\mu\beta(\gamma_g - 1)][12(1-\beta)(\gamma_g - 1) + \beta], \end{aligned} \tag{1.4.17}$$

and for the specific heat at constant pressure

$$\begin{aligned} c_P &= c_V + (P/\varrho T)(\chi_T^2/\chi\varrho) = [\mathcal{R}/\mu(\gamma_g - 1)][12(1-\beta)(\gamma_g - 1) + \beta]/\beta \\ &+ \mathcal{R}(4-3\beta)^2/\mu\beta^2 = [\mathcal{R}/\mu\beta^2(\gamma_g - 1)]\{(\gamma_g - 1)[12\beta(1-\beta) + (4-3\beta)^2] + \beta^2\}. \end{aligned} \tag{1.4.18}$$

For the adiabatic exponents we obtain (Eqs. (1.4.5)-(1.4.7), Chandrasekhar 1939, Menzel et al. 1963)

$$\Gamma_3 - 1 = P\chi_T/c_V\varrho T = (4-3\beta)(\gamma_g - 1)/[12(1-\beta)(\gamma_g - 1) + \beta], \tag{1.4.19}$$

$$\Gamma_1 = \chi_\varrho + \chi_T(\Gamma_3 - 1) = \beta + (4-3\beta)^2(\gamma_g - 1)/[12(1-\beta)(\gamma_g - 1) + \beta], \tag{1.4.20}$$

$$\Gamma_2/(\Gamma_2 - 1) = \Gamma_1/(\Gamma_3 - 1) = \left[12\beta(1-\beta)(\gamma_g - 1) + \beta^2 + (4-3\beta)^2(\gamma_g - 1)\right]/(4-3\beta)(\gamma_g - 1), \tag{1.4.21}$$

and for the ratio of specific heats

$$\gamma = c_P/c_V = \Gamma_1/\beta = 1 + (4-3\beta^2)(\gamma_g - 1)/[12\beta(1-\beta)(\gamma_g - 1) + \beta^2]. \tag{1.4.22}$$

If $\beta \to 0$, Eqs. (1.4.19)-(1.4.22) become identical to those for black body radiation from Eqs. (1.4.8), (1.4.10), and if $\beta \to 1$ we recover the perfect gas case $\gamma_g = \gamma = \Gamma_k$. When ionization processes are included, the adiabatic indices become considerably more complicated (e.g. Cox and Giuli 1968).

1.5 Adiabatic Processes in a Mixture of Electron-Positron Pairs and Black Body Radiation

The photons from a black body radiation field having energies in excess of the rest mass energies of the electron and positron $2m_ec^2$, ($m_e = 9.10956 \times 10^{-28}$ g – rest mass of the electron or positron) can split to form electron-positron pairs ($e^{\pm}$-pairs). The equilibrium concentration of electron-positron pairs can be large at very high temperatures ($> 10^9$ K), and in the virtual absence of matter. In fact, some matter must be present, in order to assure black body radiation and $e^{\pm}$-pair production. No ionization electrons are assumed to be present, so the total number density $n_{de\pm}$ of $e^{\pm}$-pairs is just twice the electron number density n_{de} :

$$n_{de\pm} = 2n_{de}. \tag{1.5.1}$$

Since the $e^{\pm}$-pairs are "dissociated" or "ionized" photons, they form an intrinsic part of the radiation field, and therefore radiation pressure must be included when calculating the equation of state of the mixture composed of $e^{\pm}$-pairs and radiation. We neglect the pressure of nuclei, which generally obey the perfect gas law (cf. Sec. 1.7). The total pressure of the mixture is split into two parts, the radiation pressure P_r and the $e^{\pm}$-pair pressure $P_e^{\pm}$:

$$P = P_{e\pm} + P_r, \qquad (P_r = aT^4/3). \tag{1.5.2}$$

The maximum of spectral emissivity of black body radiation occurs according to Wien's law at a photon energy of (e.g. Gerthsen et al. 1977)

$$e_{ph}^{(m)} = 2.82kT = 3.89 \times 10^{-16}T. \tag{1.5.3}$$

In the nonrelativistic limit, the approximate mean photon energy from Wien's law ($\approx kT$) is much smaller than the sum of the rest energies of electrons and positrons $2m_ec^2$; most photons cannot split into $e^{\pm}$-pairs, and the mixture behaves almost like pure black body radiation, discussed in the previous section. The pressure of electron-positron pairs obeys in the nonrelativistic limit nearly the perfect gas law (e.g. Cox and Giuli 1968):

$$P_{e\pm} \approx n_{de\pm}kT = 2n_{de}kT. \tag{1.5.4}$$

Since $kT \ll m_ec^2$, very few $e^{\pm}$-pairs exist, and $P_{e\pm} \ll P_r$:

$$P \approx P_r = aT^4/3. \tag{1.5.5}$$

In the extreme relativistic limit, at very high temperatures, we have $kT \gg m_ec^2$, and the pressure exerted by the $e^{\pm}$-pairs is (e.g. Cox and Giuli 1968)

$$P_{e\pm} \approx 1.05n_{de\pm}kT = 7P_r/4 = 7aT^4/12. \tag{1.5.6}$$

The pressure exerted by the electron-positron pairs is just 7/11 parts of the total pressure, since $P_{e\pm}/P = P_{e\pm}/(P_{e\pm} + P_r) = 7/11$. In virtue of Eq. (1.5.6), the equation of state of the $e^{\pm}$-pairs is similar in the extreme relativistic limit to that of black body radiation, and we get for the total pressure of the mixture

$$P \approx 11P_r/4 = 11aT^4/12. \tag{1.5.7}$$

Since the pressure of the mixture in the nonrelativistic and extreme relativistic limit obeys nearly the same form as for simple black body radiation, the equations for the adiabatic exponents are nearly the same as for black body radiation ($\chi_\varrho = 0$) :

$$\Gamma_1 \approx \Gamma_2 \approx \Gamma_3 \approx 4/3; \quad \gamma = \infty. \tag{1.5.8}$$

In the partially relativistic case when $kT/m_ec^2 \approx O(1)$, the equation of state of the $e^{\pm}$–photon mixture is complicated and will not be quoted. Because $\chi_\varrho = 0$, there is always $\gamma = \infty$; numerical values of $\Gamma_k,\ (k = 1, 2, 3)$, as well as additional effects of ions are briefly discussed in Sec. 1.7.

1.6 Adiabatic Processes in a Completely Degenerate Electron or Neutron Gas

Obviously, a gas composed merely of electrons would be unstable; therefore the charge of the electrons should be compensated by an equivalent number of ionic charges. Fortunately, the influence of the ions on the pressure of the degenerate electron gas can be safely neglected as long as the rest mass density of the mixture is $\varrho < 10^8$ g cm^{-3}, and the temperature is sufficiently small [cf. Eq. (1.7.28)]. However, it will be shown that the ions strongly influence the values of the adiabatic exponents Γ_2, Γ_3 [Eqs. (1.6.14), (1.6.16)]. The density of the mixture will be mainly given by the density of the ions. Complete degeneracy implies zero temperature of the degenerate electron gas, so all subsequent equations are strictly valid only if $T = 0$. When the gas is completely degenerate, the influence of electron-positron pairs is completely negligible [cf. Eq. (1.5.5)].

At first we consider only the equations for the electron component of the plasma mixture [Eqs. (1.6.1)-(1.6.11)]. The pressure of the completely degenerate electron gas in the partially relativistic regime is given by (e.g. Cox and Giuli 1968, Landau and Lifschitz 1971, Zeldovich and Novikov 1971)

$$P = (\pi m_e^4 c^5/3h^3)\{x(x^2+1)^{1/2}(2x^2-3) + 3\ln[x + (x^2+1)^{1/2}]\} = Af(x);$$
$$A = \pi m_e^4 c^5/3h^3; \quad \text{arcsinh}\, x = \ln[x + (x^2+1)^{1/2}]. \tag{1.6.1}$$

The parameter x is defined through the so-called Fermi momentum p_F of the electrons

$$x = p_F/m_e c = (3h^3 n_{de}/8\pi)^{1/3}/m_e c. \tag{1.6.2}$$

$h = 6.62620 \times 10^{-27}$ erg s is the Planck constant, c the velocity of light, $m_e = 9.10956 \times 10^{-28}$ g the electron rest mass, and n_{de} the number of free ionization electrons per cm^3. The number density of free ionization electrons is connected to the mean molecular weight per free ionization electron μ_e, and to the rest mass density ϱ of the gas by [cf. Eqs. (1.7.18)-(1.7.23)]

$$n_{de} = N_A \varrho/\mu_e. \tag{1.6.3}$$

$N_A = 6.02217 \times 10^{23}$ mole^{-1} denotes the Avogadro number. Inserting Eq. (1.6.3) into Eq. (1.6.2), we obtain the parametric representation for the rest mass density of the completely degenerate electron gas:

$$\varrho = 8\pi m_e^3 c^3 \mu_e x^3/3h^3 N_A = Bx^3; \quad B = 8\pi m_e^3 c^3 \mu_e/3h^3 N_A. \tag{1.6.4}$$

Since the electrons are assumed to be noninteracting, their internal energy density $\varepsilon^{(int)}$ is equal to their kinetic energy density $\varepsilon^{(kin)}$ from Eq. (1.2.15), (cf. Sec. 1.7, Schatzman 1958, Cox and Giuli 1968):

$$\varepsilon^{(kin)} = (\pi m_e^4 c^5/3h^3)\{8x^3[(x^2+1)^{1/2} - 1] - x(x^2+1)^{1/2}(2x^2-3) - 3\ln[x + (x^2+1)^{1/2}]\}. \tag{1.6.5}$$

In the nonrelativistic limit ($x \ll 1$) we have $P \propto 8x^5/5$, and therefore the pressure is given for constant μ_e by a polytrope of index $n = 1.5$, $(P \propto \varrho^{5/3})$:

$$P = (3/\pi)^{2/3} h^2 n_{de}^{5/3}/20m_e = (3/\pi)^{2/3} h^2 (N_A\varrho/\mu_e)^{5/3}/20m_e = 1.004 \times 10^{13} (\varrho/\mu_e)^{5/3} \text{ [dyne cm}^{-2}]. \tag{1.6.6}$$

In the extreme relativistic limit ($x \gg 1$) there is $P \propto 2x^4$, and the pressure is given for constant μ_e by a polytrope of index $n = 3$, $(P \propto \varrho^{4/3})$:

$$P = (3/\pi)^{1/3} hc\, n_{de}^{4/3}/8 = (3/\pi)^{1/3} hc (N_A\varrho/\mu_e)^{4/3}/8 = 1.244 \times 10^{15} (\varrho/\mu_e)^{4/3} \text{ [dyne cm}^{-2}]. \tag{1.6.7}$$

In the extreme relativistic limit, when $\varrho \ll \varrho_r$, $\varepsilon^{(kin)} \approx \varepsilon_r^{(kin)} = \varrho_r c^2$, we get from Eqs. (1.6.1) and (1.6.5): $P \approx \varepsilon^{(kin)}/3 \approx \varepsilon_r^{(kin)}/3 = \varrho_r c^2/3$ [cf. Eqs. (1.7.37)-(1.7.39)].

When the degenerate gas is composed of neutrons instead of electrons, we have to replace in the preceding formulas the rest mass of the electron m_e and the mean molecular weight per free ionization electron μ_e by the rest mass of the neutron $m_n = 1.67482 \times 10^{-24}$ g and by the mean molecular weight of the neutrons μ_n, ($\mu_n \approx 1$), respectively (cf. Sec. 1.7).

The expression for the adiabatic exponent Γ_1 is given by Eq. (1.3.5) with $\chi_T = 0$ and $\Gamma_1' = \Gamma_1,\ \Gamma_3' = \Gamma_3$. Via Eqs. (1.6.1) and (1.6.4) we get

$$\Gamma_1 = (\partial \ln P/\partial \ln \varrho)_S = \chi_T(\Gamma_3 - 1) + \chi_\varrho = \chi_\varrho = d\ln f(x)/d\ln x^3 = 8x^5/3(x^2+1)^{1/2} f(x). \quad (1.6.8)$$

The equation for Γ_3 is obtained in a considerably more involved way, since the temperature effect on the equation of state has to be considered before turning to the limit $T \to 0$. It can be shown that (e.g. Schatzman 1958, Cox and Giuli 1968)

$$\Gamma_3 - 1 = (x^2+2)/3(x^2+1). \quad (1.6.9)$$

Using the identity (1.3.4) we obtain from Eqs. (1.6.8), (1.6.9):

$$(\Gamma_2 - 1)/\Gamma_2 = (\Gamma_3 - 1)/\Gamma_1 = (x^2+2)\ f(x)/8x^5(x^2+1)^{1/2}. \quad (1.6.10)$$

The equation (1.3.24) for the ratio of specific heats is simply

$$\gamma = c_P/c_V = \Gamma_1/\chi_\varrho = 1, \quad (1.6.11)$$

since according to Eq. (1.6.8) $\Gamma_1 = \chi_\varrho$. Numerical values of the gammas are quoted in Table 1.7.1.

The previous equations are strictly valid only for the completely degenerate electron component of a gas, neglecting the influence of the nondegenerate ion component. It has already been noted that the adiabatic exponents Γ_2, Γ_3 of a plasma composed of degenerate electrons and nondegenerate ions are strongly affected by the ions, because of the temperature dependence of their equation of state. Note however, that under certain conditions prevailing in white dwarfs the ions may arrange into a crystalline lattice, their properties resembling those of an ordinary solid (Debye solid), rather than those of a perfect gas (cf. Sec. 1.7).

Let us calculate the adiabatic exponents in a plasma mixture consisting of completely degenerate electrons and of a nondegenerate ionic component, obeying the equation of state of a perfect gas. From Eq. (1.3.14) we have for an adiabatic change

$$\Gamma_3 - 1 = P\chi_T/c_V\varrho T, \qquad (c = dQ/dT = 0;\ \Gamma_3' = \Gamma_3). \quad (1.6.12)$$

Taking into account that $\chi_T = (T/P)(\partial P/\partial T)_\varrho$ and $c_V = (\partial U/\partial T)_\varrho$, the equation (1.6.12) becomes

$$\Gamma_3 - 1 = (1/\varrho)(\partial P/\partial U)_\varrho = (1/\varrho)(\partial P/\partial \varepsilon^{(kin)})_\varrho\ (\partial \varepsilon^{(kin)}/\partial U)_\varrho = (\partial P/\partial \varepsilon^{(kin)})_\varrho, \quad (1.6.13)$$

where $\varepsilon^{(kin)}$ is exactly equal to the energy density of internal energy $\varepsilon^{(int)}$ (internal energy per unit volume), because only kinetic translational motions of noninteracting particles occur in the considered system (cf. Eq. (1.7.59) if $f = 3$). If we denote by U the specific internal energy of the gas (internal energy per unit rest mass), we can write $\varepsilon^{(int)} = \varepsilon^{(kin)} = \varrho U$, and therefore $(\partial \varepsilon^{(kin)}/\partial U)_\varrho = \varrho$. We split pressure and energy density of the plasma into the components of the completely degenerate electron gas and of the nondegenerate ion plasma ($P = P_i + P_e;\ \varepsilon^{(kin)} = \varepsilon_i^{(kin)} + \varepsilon_e^{(kin)}$), and obtain via Eq. (1.6.13)

$$\Gamma_3 - 1 = (\partial P/\partial \varepsilon^{(kin)})_\varrho = [(\partial P_i/\partial T)_\varrho + (\partial P_e/\partial T)_\varrho]/[(\partial \varepsilon_i^{(kin)}/\partial T)_\varrho + (\partial \varepsilon_e^{(kin)}/\partial T)_\varrho]$$
$$\approx (\partial P_i/\partial T)_\varrho/(\partial \varepsilon_i^{(kin)}/\partial T)_\varrho = (\partial P_i/\partial \varepsilon_i^{(kin)})_\varrho = \Gamma_{3i} - 1, \quad (1.6.14)$$

because for the degenerate electron component we have $(\partial P_e/\partial T)_\varrho \to 0$ and $(\partial \varepsilon_e/\partial T)_\varrho \to 0$, as $T \to 0$. Noting that $\chi_\varrho \to \chi_{\varrho e}$ (contribution of ionic component to the pressure is negligible), and $\chi_T \to 0$ if $T \to 0$, we get according to Eq. (1.3.5):

$$\Gamma_1 = \chi_\varrho + (\Gamma_3 - 1)\chi_T \approx \chi_{\varrho e} = \Gamma_{1e}. \quad (1.6.15)$$

Hence

$$(\Gamma_2 - 1)/\Gamma_2 = (\Gamma_3 - 1)/\Gamma_1 \approx (\Gamma_{3i} - 1)/\Gamma_{1e}. \quad (1.6.16)$$

The ratio of specific heats is also in this case equal to its value (1.6.11) for the completely degenerate electron gas:

$$\gamma = \Gamma_1/\chi_\varrho \approx \Gamma_{1e}/\chi_{\varrho e} = 1. \quad (1.6.17)$$

1.7 Numerical Survey of Equations of State, Adiabatic Exponents, and Polytropic Indices

In the equations of state quoted in the previous sections there appears frequently the mean molecular weight of the gas μ and the mean molecular weight per free ionization electron μ_e. We briefly derive the relevant equations. The mean molecular weight is defined as the mean rest mass per mole of free particles (e.g. Chandrasekhar 1939, Cox and Giuli 1968):

$$\mu = \varrho N_A/n_d = \varrho/Hn_d. \tag{1.7.1}$$

ϱ denotes the rest mass density of free particles, n_d the number density of free particles, N_A Avogadro's number, and $H = 1/N_A$ the atomic mass unit. The quantitiy ϱ/n_d is just the mean rest mass of a free particle, which yields the mean molecular weight μ, when multiplied by the number N_A of free particles per mole. The definition (1.7.1) is independent of relativistic effects, since only the rest mass density ϱ is considered. From the definition (1.7.1) it is obvious that μ is also equal to the average rest mass of free particles measured in atomic mass units H :

$$\mu = m/H, \tag{1.7.2}$$

where $m = \varrho/n_d$ means the average rest mass per free particle. Therefore, the mean molecular weight can be expressed in g mole^{-1} or in atomic mass units H. If we denote by m_k the rest mass of a free particle of type k, and by n_{dk} its number density, we have obviously

$$\varrho = \sum_k m_k n_{dk}, \tag{1.7.3}$$

where summation extends over all types of particles present in the gas. Also

$$n_d = \sum_k n_{dk}, \tag{1.7.4}$$

and

$$N_A = 1/H. \tag{1.7.5}$$

Inserting Eqs. (1.7.3)-(1.7.5) into the definition (1.7.1), we obtain

$$\mu = \sum_k m_k n_{dk} \Big/ H \sum_k n_{dk} = \sum_k A_k n_{dk} \Big/ \sum_k n_{dk}, \tag{1.7.6}$$

where $A_k = m_k/H$ is the mass of a free particle of type k measured in atomic mass units. Eq. (1.7.6) can be transformed further by inserting the atomic mass fraction x_k (relative mass abundance or shortly abundance) of the particle of type k, which is equal to the mass measured in grams of particles of type k present in one gram of gas. Therefore

$$x_k \varrho = n_{dk} m_k \quad \text{or} \quad n_{dk} = \varrho x_k/m_k = \varrho x_k/HA_k. \tag{1.7.7}$$

Since obviously

$$\sum_k x_k = 1, \tag{1.7.8}$$

we get by substitution of Eqs. (1.7.7), (1.7.8) into Eq. (1.7.6):

$$\mu = 1 \Big/ \sum_k (x_k/A_k). \tag{1.7.9}$$

Over most ranges in the temperature-density diagram from Fig. 1.7.1 the matter is partially or completely ionized. The total number density n_d can be written as

$$n_d = n_{de} + \sum_i n_{di}, \tag{1.7.10}$$

where n_{de} is the number density of free ionization electrons (no electrons from electron-positron pairs are considered), and n_{di} the number density of all particles of type i, excepting electrons; we have changed the summation index in order to distinguish summation over all particles (index k) from summation over all particles other than electrons (index i). If each particle of type i contributes on the average ν_i electrons, the number density of electrons is via Eq. (1.7.7):

$$n_{de} = \sum_i \nu_i n_{di} = (\varrho/H) \sum_i \nu_i x_i/A_i. \tag{1.7.11}$$

Eq. (1.7.10) becomes, by using again Eq. (1.7.7) for $\sum_i n_{di}$:

$$n_d = (\varrho/H) \sum_i (1+\nu_i) x_i/A_i = (\varrho/H) \sum_i s_i x_i. \tag{1.7.12}$$

$s_i = (1+\nu_i)/A_i$ is the total number of free particles per atomic mass unit H, contributed by particles of type i. Introducing Eq. (1.7.12) into Eq. (1.7.1), we get

$$\mu = 1\Big/\sum_i s_i x_i. \tag{1.7.13}$$

$\sum_i x_i$ is slightly less than 1, since the summation is not extended over the electrons; the electron rest masses are neglected. Possible occurrence of negative ions is neglected too. s_i from Eq. (1.7.13) changes between $1/A_i$ (no ionization) and $(1+Z_i)/A_i$ (complete ionization), where Z_i denotes the atomic charge number. Therefore, μ is contained between the limiting values for completely ionized and nonionized material:

$$1\Big/\sum_i [x_i(1+Z_i)/A_i] \leq \mu \leq 1\Big/\sum_i (x_i/A_i). \tag{1.7.14}$$

For nonionized matter we have $Z_i = 0$, and

$$1/\mu = \sum_i (x_i/A_i) = \langle 1/A \rangle, \tag{1.7.15}$$

where $\langle 1/A \rangle$, as defined by Eq. (1.7.15), is the average reciprocal mass of free particles with respect to mass abundance, measured in atomic mass units H. For matter composed of single elements we have for instance $\mu = 12$ for C.

In the case of completely ionized material we obtain for hydrogen $(1+Z_H)/A_H = 2/1.008 \approx 2$, for helium $(1+Z_{He})/A_{He} = 3/4.004 \approx 3/4$, and for all elements heavier than helium $(1+Z_i)/A_i \approx 1/2$. Thus, in the case of complete ionization Eq. (1.7.14) becomes

$$\mu \approx 1\big/\big[2x_H + 3x_{He}/4 + (1 - x_H - x_{He})/2\big] = 2/(1 + 3x_H + x_{He}/2), \tag{1.7.16}$$

where we have used $\sum_i x_i \approx 1$, and have denoted the mass fraction of H and He by x_H and x_{He}, respectively. The mean molecular weight of completely ionized matter is contained between the narrow limits

$$1/2 \leq \mu \leq 2. \tag{1.7.17}$$

The lower limit obtains if $x_H = 1,\ x_{He} = 0$, and the upper one if $x_H = x_{He} = 0,\ (0 \leq x_i \leq 1)$.

For degenerate matter it is appropriate to use the mean molecular weight per free ionization electron μ_e, instead of the mean molecular weight μ. In analogy to Eq. (1.7.1), we define the mean molecular weight per free ionization electron as

$$\mu_e = \varrho/Hn_{de}. \tag{1.7.18}$$

This equation is also valid for a gas composed only of electrons. For a mixture of electrons and nuclei Eq. (1.7.18) can be transformed further. The number density of free ionization electrons n_{de} (the contribution to n_{de} from electron-positron pairs is not considered, though they may be present in this context) can be obtained from Eq. (1.7.10) by using Eqs. (1.7.7), (1.7.12):

$$n_{de} = n_d - \sum_i n_{di} = (\varrho/H) \sum_i s_i x_i - (\varrho/H) \sum_i x_i/A_i = (\varrho/H) \sum_i [(x_i/A_i)(A_i s_i - 1)]. \tag{1.7.19}$$

For complete ionization there is $s_i = (1 + Z_i)/A_i$, and therefore

$$n_{de} = (\varrho/H) \sum_i x_i Z_i/A_i. \tag{1.7.20}$$

We insert Eq. (1.7.20) into Eq. (1.7.18):

$$1/\mu_e = \sum_i x_i Z_i/A_i. \tag{1.7.21}$$

For all elements heavier than hydrogen we have $Z_i/A_i \approx 1/2$. Therefore

$$1/\mu_e = x_H + (1 - x_H)/2 = (1 + x_H)/2, \tag{1.7.22}$$

and

$$1 \le \mu_e \le 2. \tag{1.7.23}$$

The lower limit applies if $x_H = 1$, the upper one if $x_H = 0$.

In the case of completely ionized matter, the mean molecular weight μ is related to the mean molecular weight per free ionization electron μ_e by

$$1/\mu = \sum_i [x_i(1 + Z_i)/A_i] = \sum_i x_i/A_i + \sum_i x_i Z_i/A_i = \langle 1/A \rangle + 1/\mu_e = 1/\mu_i + 1/\mu_e, \tag{1.7.24}$$

where we have used Eqs. (1.7.14), (1.7.15), (1.7.21). $\mu_i = 1/\langle 1/A \rangle$ can be considered to represent the mean molecular weight per ion.

Summarizing, for completely ionized matter μ and μ_e are contained within the narrow intervals $[1/2, 2]$ and $[1, 2]$, respectively. For nonionized matter μ is approximately equal to the mean mass of free particles measured in atomic mass units H.

We now turn to a brief discussion of the applicability of the simple equations of state outlined in previous sections. We start with the lowest densities occurring in interstellar clouds ($\varrho \approx 10^{-24}$ g cm^{-3}), where Eq. (1.4.11) for a mixture of perfect gas and radiation is applicable:

$$P = \mathcal{R}\varrho T/\mu + aT^4/3. \tag{1.7.25}$$

Radiation pressure is comparable to gas pressure if

$$\varrho \approx a\mu T^3/3\mathcal{R}. \tag{1.7.26}$$

This equation yields for $\mu = 1$ (as for neutral hydrogen) the delimitation line between regions 1 and 2 in Fig. 1.7.1. In region 1 the equation of state is given by Eq. (1.7.25), or even simpler by $P = aT^4/3$. In domain 2 the most appropriate equation is the perfect gas law

$$P = \mathcal{R}\varrho T/\mu. \tag{1.7.27}$$

When the density of matter increases, the volume of matter available for an atom becomes less than the atomic dimensions, and the gas becomes a highly compressed electron-nucleon plasma. The Maxwell-Boltzmann statistics of perfect gases is no longer applicable, and the electrons become degenerate, obeying the Fermi-Dirac statistics.

The pressure of the degenerate gas at densities 10^8 g cm^{-3} is due mainly to the degenerate electrons, the influence of the nondegenerate ions being negligible (e.g. Cox and Giuli 1968, Landau and Lifschitz 1971). A rough delimitation between the nondegenerate perfect gas and the degenerate electron gas is given by equating the perfect gas pressure from Eq. (1.7.27) to the pressure of the nonrelativistic completely degenerate electron gas from Eq. (1.6.6), (Schwarzschild 1958):

$$\mathcal{R}\varrho T/\mu \approx 1.004 \times 10^{13}(\varrho/\mu_e)^{5/3}. \tag{1.7.28}$$

This equation yields with $\mu = \mu_e = 1$ (pure hydrogen) the delimitation line between domains 2 and 3 from Fig. 1.7.1. The delimitation line between the nonrelativistic degenerate electron gas and the extreme

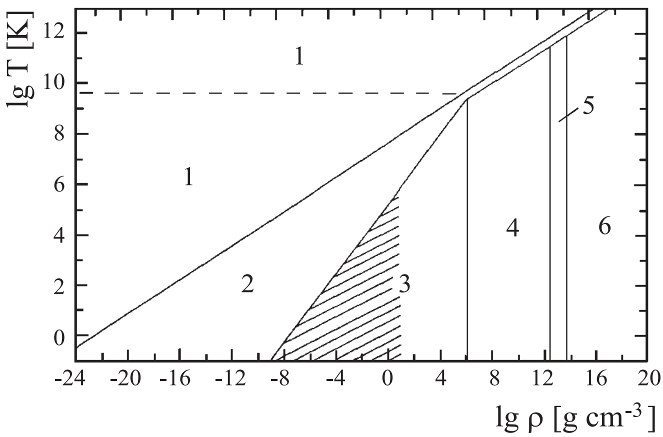

Fig. 1.7.1 Temperature – rest mass density diagram showing the principal domains where the equations of state discussed in this chapter are roughly valid ($\mu = \mu_e = \mu_n = 1$). Domains 1 and 2 constitute the nondegenerate region, domains 3-5 are degenerate regions. Domain 1 is the radiation pressure dominated region with the equation of state (1.7.25) below the broken line (1.7.30), and with Eqs. (1.5.5), (1.5.7), (1.7.31) above the broken line, when the influence of electron-positron pairs becomes important. Domain 2 is the perfect gas region with Eq. (1.7.27). Domain 3 is the nonrelativistic completely degenerate electron gas with Eq. (1.6.6). Domain 4 is the extreme relativistic degenerate electron gas with Eq. (1.6.7). Domain 5 is the nonrelativistic degenerate neutron gas with Eq. (1.7.34). Domain 6 is the nuclear interaction domain with Eqs. (1.7.35), (1.7.43). The hatched area in domain 3 is the region of planetary pressures, densities, and temperatures (planetary subregion), where Eq. (1.7.44) may be appropriate. The delimitation lines are obtained from left to right by Eqs. (1.7.30), (1.7.26), (1.7.28), (1.7.29), (1.7.36), (1.7.33), and (1.7.35), respectively.

relativistic degenerate electron gas (domains 3 and 4) can simply be found by equating the corresponding pressures from Eqs. (1.6.6) and (1.6.7), ($\mu_e = 1$) :

$$1.004 \times 10^{13}(\varrho/\mu_e)^{5/3} \approx 1.244 \times 10^{15}(\varrho/\mu_e)^{4/3} \quad \text{or} \quad \varrho \approx 1.90 \times 10^6 \text{ g cm}^{-3}. \tag{1.7.29}$$

Above the broken line in the radiation pressure domain 1, the influence of electron-positron pairs becomes important, when the mean photon energy $\approx 2.82kT$ [see Wien's law from Eq. (1.5.3)] is comparable to the minimum energy $2m_ec^2$ required to produce an electron-positron pair. Thus

$$2.82kT \approx 2m_ec^2 \quad \text{or} \quad T \approx 4.21 \times 10^9 \text{ K}. \tag{1.7.30}$$

For instance, in the extreme relativistic limit, the pressure above the broken line should be calculated by [cf. Eq. (1.5.7)]

$$P = \mathcal{R}\varrho T/\mu + 11aT^4/12, \qquad (kT \gg m_ec^2). \tag{1.7.31}$$

When the gas is highly degenerate (regions 3-5), electron-positron pairs are never important (Cox and Giuli 1968).

The broken line in Fig. 1.7.1 constitutes also an approximate delimitation between nonrelativistic and extreme relativistic nondegenerate electrons, since in this case the kinetic electron energy $3kT/w$, $[1 \leq w \leq 2$, Eq. (1.7.55)] is of the same order of magnitude as the electron rest energy m_ec^2. We find, similarly to Eq. (1.7.30)

$$kT \approx m_ec^2 \quad \text{or} \quad T \approx 5.93 \times 10^9 \text{ K}. \tag{1.7.32}$$

Protons become relativistic only at temperatures of order $T \approx m_pc^2/k = 1.09 \times 10^{13}$ K, where $m_p = 1.67261 \times 10^{-24}$ g denotes the proton rest mass. The relativistic nondegenerate domains are not shown separately, because the equation of state is invariant with respect to the rest mass density ϱ (cf. Sec. 1.2).

The equation of state in domains 5 and 6 at ultrahigh densities is uncertain. Similar to the degenerate electron gas, the temperature influence to the equation of state is generally modest, and the equation of state can be represented approximately by a pressure-density power law, resembling the form of the general polytropic equation of state (1.3.29), (cf. Zeldovich and Novikov 1971, Schaeffer et al. 1983, Shapiro and Teukolsky 1983). At densities above about 10^8 g cm^{-3} the neutronization of matter becomes possible through the so-called inverse β-process, i.e. capture of electrons by a nucleus and transformation of the protons inside the nucleus into neutrons. At still higher densities the neutron-rich nuclei decay with emission of a free neutron, and finally all the matter becomes a degenerate neutron gas. The above mentioned processes are important at densities between about $10^8 - 10^{12}$ g cm^{-3}, depending on nuclear composition. Zeldovich and Novikov (1971, p. 182) find that at a rest mass density

$$\varrho \approx 1.5 \times 10^{12} \ \mathrm{g\ cm^{-3}}, \tag{1.7.33}$$

the pressure 6.6×10^{29} dyne cm^{-2} of relativistically degenerate electrons is equal to the pressure (1.7.34) $5.461 \times 10^9[(\varrho - 4 \times 10^{11})/\mu_n]^{5/3}$ exerted by a nonrelativistic degenerate neutron gas of rest mass density $(\varrho - 4 \times 10^{11})$ g cm^{-3}, when there is equilibrium between neutron-rich heavy nuclei of density 4×10^{11} g cm^{-3}, degenerate neutrons of density $(\varrho - 4\times 10^{11})$ g cm^{-3}, and relativistically degenerate electrons. The approximate density value from Eq. (1.7.33) is taken somewhat deliberately as the delimitation line between domain 4, where the pressure is assumed to be caused mainly by the completely degenerate relativistic electrons and domain 5, where the pressure may be due mainly to the nonrelativistic degenerate neutrons (cf. Zeldovich and Novikov 1971, p. 184). The pressure of the nonrelativistic degenerate neutron gas is given analogously to Eq. (1.6.6) by

$$P = (3/\pi)^{2/3} h^2 n_{dn}^{5/3}/20m_n = 5.461 \times 10^9 (\varrho/\mu_n)^{5/3} \ [\mathrm{dyne\ cm^{-2}}], \tag{1.7.34}$$

where m_n denotes the rest mass of the neutron, n_{dn} the number density of neutrons, and $\mu_n \approx 1$ the mean molecular weight of the neutrons from Eq. (1.7.1).

At still higher densities, of the order of nuclear densities $\varrho \approx 2 \times 10^{14}$ g cm^{-3} and greater, interactions between neutrons becomes increasingly important. The neutrons can no longer be treated as separate particles – rather they form a gigantic nucleus. At these densities transformation of neutrons into other elementary particles may occur. A relativistically degenerate neutron gas cannot come to existence, because its equation of state would be pertinent only at densities $\varrho \approx 1.2 \times 10^{16}$ g cm^{-3}, as can be seen at once by equating the pressure (1.7.34) of the nonrelativistic degenerate neutrons to the pressure $1.244 \times 10^{15} (\varrho/\mu_n)^{4/3}$ [dyne cm^{-2}] of relativistically degenerate neutrons from Eq. (1.6.7), $(\mu_e \to \mu_n \approx 1)$.

As an example of an equation of state of the form (1.7.43), which may be valid at densities $\varrho \gtrsim 10^{13}$ g cm^{-3}, we quote from Zeldovich and Novikov (1971, p. 200) the relationships $P = \varrho^2 c^2/5 \times 10^{15}$, $\varrho_r = \varrho + \varrho^2/5 \times 10^{15}$. We have $\varrho_r \approx \varrho$ if $\varrho \lesssim 10^{13}$ g cm^{-3}, and $\varrho_r = \varrho^2/5 \times 10^{15}$, $(P = \varrho_r c^2)$ if $\varrho \gg 5 \times 10^{15}$ g cm^{-3}. An approximate delimitation line between domain 5 (nonrelativistically degenerate neutron gas) and the nuclear interaction domain 6 can be obtained by equating the previous equation of state to Eq. (1.7.34):

$$P = \varrho^2 c^2/5 \times 10^{15} = 5.461 \times 10^9 (\varrho/\mu_n)^{5/3} \ [\mathrm{dyne\ cm^{-2}}] \quad \text{or} \quad \varrho \approx 2.8 \times 10^{13} \ \mathrm{g\ cm^{-3}}. \tag{1.7.35}$$

The delimitation of domain 4 towards high temperatures is approximately found by equating the perfect gas law to the pressure of the relativistically degenerate electron gas $(\mu = \mu_e = 1)$:

$$\mathcal{R}\varrho T/\mu = 1.244 \times 10^{15} (\varrho/\mu_e)^{4/3}. \tag{1.7.36}$$

This equation yields a line with the same slope $(\varrho \propto T^3)$ as the delimitation line between the two nondegenerate domains 1 and 2, and opens in domain 2 a small window towards the region of ultrahigh temperatures and densities, where the simple perfect gas law is valid.

Note, that we base our discussion of the equation of state on oversimplified forms, which often agree only very roughly with more realistic equations of state. For instance, our adiabatic exponents in domains 4 and 5 should be $\Gamma_1 = 4/3$ and $5/3$ by Eqs. (1.6.6) and (1.7.34), respectively, whereas they may change between 0 and 2 for various proposed equations of state (e.g. Shapiro and Teukolsky 1983, Fig. 2.3).

The border of domains 5 and 6 towards high temperatures has deliberately been taken equal to the upper delimitation line of domain 4.

For the study of relativistic polytropes it will be important to discuss the form of the equation of state in the extreme relativistic limit. Recall that degenerate electrons start to become relativistic already at 1.90×10^6 g cm^{-3} [cf. Eq. (1.7.29)]. In the nondegenerate Maxwell-Boltzmann regime, electrons become relativistic at about 5.93×10^9 K [cf. Eq. (1.7.32)], and protons at about 1.09×10^{13} K.

For systems of noninteracting particles, the pressure is given by (e.g. Landau and Lifschitz 1987)

$$P = \varrho v^2/3(1 - v^2/c^2)^{1/2}, \tag{1.7.37}$$

where v denotes the mean velocity of microscopic kinetic translational motion of particles. Combining the above equation with Eq. (1.2.15) for the relativistic kinetic energy density of translational motion $\varepsilon_r^{(kin)}$, we obtain

$$P = \varepsilon_r^{(kin)} v^2/3c^2 = \varrho_r v^2/3, \tag{1.7.38}$$

since for systems of noninteracting particles the pressure is equal to the transfer of kinetic momentum of the particles across a surface.

Because $v \leq c$, Eq. (1.7.38) shows that for systems of noninteracting particles there is always

$$P \leq \varepsilon_r^{(kin)}/3. \tag{1.7.39}$$

In the extreme relativistic case $v = c$ we have according to Eq. (1.7.38) $P = \varepsilon_r^{(kin)}/3 = \varrho_r c^2/3$, and the adiabatic velocity of sound is

$$a^2 = (\partial P/\partial \varrho_r)_S = c^2/3; \quad a = c/3^{1/2}, \qquad (dQ = 0;\ S = \mathrm{const}). \tag{1.7.40}$$

On the one side, this result is reasonable since the particles are moving nearly with the velocity of light - in all directions however. On the other side, it would be difficult to conceive that relativistic considerations would lead to anything else than to the condition $a \leq c$, and not $a \leq c/3^{1/3}$ (e.g. Zeldovich and Novikov 1971). To obtain an equation of state compatible with relativistic principles, we generalize Eq. (1.7.38) for systems of interacting particles, by including in the pressure P also contributions from particle interactions, force and radiation fields, but exclusive of gravitational fields:

$$P = \beta \varepsilon_r v^2/c^2 = \beta \varrho_r v^2, \qquad (\varepsilon_r^{(kin)} \rightarrow \varepsilon_r). \tag{1.7.41}$$

We have $\beta = 1/3$, if only translational kinetic particle motions are present. In the extreme relativistic limit Eq. (1.7.41) becomes $P = \beta \varepsilon_r = \beta c^2 \varrho_r$, and the velocity of sound is

$$a^2 = (\partial P/\partial \varrho_r)_S = \beta c^2. \tag{1.7.42}$$

The condition $a \leq c$ leads to $\beta \leq 1$; thus, we get in the extreme relativistic limit the equation of state

$$P = \beta \varepsilon_r = \beta c^2 \varrho_r \leq \varepsilon_r = c^2 \varrho_r, \qquad (0 \leq \beta \leq 1). \tag{1.7.43}$$

This equation of state is also assumed by Eqs. (4.1.84), (4.1.86) in the extreme relativistic limit: $P = (\gamma - 1)c^2 \varrho_r = c^2 \varrho_r/n$. Such an equation of state is employed in cosmology for the study of initial isotropic singularities (e.g. Anguige and Tod 1999).

For systems of noninteracting particles, and even for electromagnetically interacting particles, Eqs. (1.7.37)-(1.7.40) remain valid. As outlined in Eq. (1.2.16), the total relativistic energy density of matter ε_r is composed of the rest energy density $\varepsilon = \varrho c^2$, and of the internal energy density $\varepsilon^{(int)}$. The internal energy density $\varepsilon^{(int)}$ itself is composed of the sum of the energies of microscopic particle motions, particle interactions (other than gravitational), external forces and radiation fields, etc. The internal energy density $\varepsilon^{(int)}$ for systems of noninteracting particles is generally composed of the energy density of (i) kinetic translational motion $\varepsilon^{(kin)}$, (ii) rotation and vibration energies in molecules, (iii) potential energy of external forces and radiation fields. The sum of the energies of translation, vibration, and rotation of the particles is called internal energy U. When monoatomic or completely ionized perfect gases, degenerate electrons (neutrons) or photons are considered, the particles possess only translational kinetic energy density $\varepsilon^{(kin)}$, which is just equal to the internal energy density $\varepsilon^{(int)}$. At nonrelativistic energies ($v \ll c$), in the case of molecular perfect gases, additional degrees of freedom of the molecules (rotation and vibration of molecules) lead to an internal energy density $\varepsilon^{(int)}$ that could be considerably larger than the translational kinetic energy density $\varepsilon^{(kin)}$, as will be obvious from Eqs. (1.7.58) and (1.7.59).

It should be noted that the temperature-density diagram from Fig. 1.7.1 is only a crude estimate. The quoted simple equations of state are very rough approximations near most delimitation lines. Also, in the hatched area of domain 3 (terrestrial and planetary gases, liquids, and solids) where interactions between atoms, molecules, ions, and electrons (Van der Waals forces, Coulomb interactions, etc.) are not negligible, the equation of state deviates considerably from that of a simple degenerate electron plasma or from the perfect gas law (e.g. Landau and Lifshitz 1971, Hubbard 1978, Robnik and Kundt 1983). However, as pointed out for instance by Slattery (1977), matter inside the giant planets obeys approximately a "perturbed" polytropic equation of state

$$P = K\varrho^{1+1/n}\exp\Big(\sum_{k=0}^{s} X_k\varrho^k\Big), \qquad [X_k = X_k(T)], \tag{1.7.44}$$

where the exponential correction factor is generally of order unity $\sum_{k=0}^{s} X_k\varrho^k \ll 1$, and the polytropic index is $n \approx 1$ (Table 6.1.2, Öpik 1962, Hubbard 1978).

We now turn to the numerical evaluation of the adiabatic exponents and of the corresponding polytropic indices for adiabatic changes occurring in the simple thermodynamic systems discussed in the previous sections. Recall that the adiabatic exponents and the isentropic polytropic indices are given via Eqs. (1.3.25), (1.3.26), (1.3.30) by

$$\begin{aligned}&1 + 1/n = \Gamma_1 = (\partial\ln P/\partial\ln\varrho)_S; \quad 1 + n' = \Gamma_2/(\Gamma_2 - 1) = (\partial\ln P/\partial\ln T)_S;\\ &1/n'' = \Gamma_3 - 1 = (\partial\ln T/\partial\ln\varrho)_S, \qquad (dQ = 0;\ S = \mathrm{const}).\end{aligned} \tag{1.7.45}$$

At first we discuss an important relationship between pressure P and kinetic energy density $\varepsilon^{(kin)}$ for noninteracting particle systems. To this end we insert into Eqs. (1.7.37), (1.7.38) the equation (1.2.15) for the relativistic mass density ϱ_r, and the absolute value of the momentum p from Eq. (1.2.11), by observing that the rest mass density can be written as $\varrho = n_d m$, (m = mass of a single particle):

$$P = \varepsilon_r^{(kin)}v^2/3c^2 = \varrho_r v^2/3 = n_d m v^2/3(1 - v^2/c^2)^{1/2} = n_d p v/3. \tag{1.7.46}$$

In virtue of Eq. (1.2.15) the kinetic energy density of translational motion is

$$\varepsilon^{(kin)} = \varepsilon_r^{(kin)} - \varepsilon = n_d e^{(kin)}, \tag{1.7.47}$$

where $e^{(kin)}$ is the translational kinetic energy per particle. Equating Eqs. (1.2.6) and (1.2.12), we get

$$e_r^{(kin)} = mc^2/(1 - v^2/c^2)^{1/2} = (p^2c^2 + m^2c^4)^{1/2}. \tag{1.7.48}$$

The last equality from Eq. (1.7.48) yields

$$v = pc/(p^2 + m^2c^2)^{1/2}. \tag{1.7.49}$$

On the other hand, we obtain from Eqs. (1.2.9) and (1.7.48) for the kinetic energy of a particle

$$e^{(kin)} = e_r^{(kin)} - e = (p^2c^2 + m^2c^4)^{1/2} - mc^2 = c\big[(p^2 + m^2c^2)^{1/2} - mc\big], \tag{1.7.50}$$

where $e = mc^2$ denotes the rest energy of a particle. We eliminate for instance, the factor c between Eqs. (1.7.49) and (1.7.50):

$$v = pe^{(kin)}/\big[p^2 + m^2c^2 - mc(p^2 + m^2c^2)^{1/2}\big]. \tag{1.7.51}$$

The result is introduced into Eq. (1.7.46):

$$P = n_d p^2 e^{(kin)}/3\big[p^2 + m^2c^2 - mc(p^2 + m^2c^2)^{1/2}\big] = \varepsilon^{(kin)}p^2/3\big[p^2 + m^2c^2 - mc(p^2 + m^2c^2)^{1/2}\big]. \tag{1.7.52}$$

Inserting in virtue of Eq. (1.2.11) $p^2 = m^2v^2/(1 - v^2/c^2)$, we obtain further:

$$\begin{aligned}&P = \varepsilon^{(kin)}v^2/3c^2[1 - (1 - v^2/c^2)^{1/2}] = \varepsilon^{(kin)}[1 + (1 - v^2/c^2)^{1/2}]/3 = w\varepsilon^{(kin)}/3;\\ &w = 1 + (1 - v^2/c^2)^{1/2}.\end{aligned} \tag{1.7.53}$$

In the nonrelativistic limit $v \ll c$ the factor w is 2, and in the extreme relativistic limit $v \approx c$ this factor becomes 1. Therefore

$$1/3 \le P/\varepsilon^{(kin)} \le 2/3. \tag{1.7.54}$$

Eqs. (1.7.46)-(1.7.54) apply to any system of noninteracting particles, whether degenerate or nondegenerate.

(i) Perfect Gases. Equating the perfect gas pressure (1.2.5) to the pressure (1.7.53), we get

$$P = kn_dT = w\varepsilon^{(kin)}/3. \tag{1.7.55}$$

From Eq. (1.7.55) we observe that the kinetic energy density of a perfect gas is $\varepsilon^{(kin)} = 3kn_dT/w$, and therefore the energy associated with a single particle equals $e^{(kin)} = 3kT/w$. For a perfect gas there is valid the principle of equipartition of energy between various degrees of freedom of particle motion. Since a single particle has only three spatial degrees of freedom, the energy per degree of freedom of a particle in a perfect gas is $e^{(kin)}/3 = kT/w$. We rewrite Eq. (1.7.55), using Eqs. (1.2.5), (1.2.22):

$$P = w\varepsilon^{(kin)}/3 = \mathcal{R}\varrho T/\mu = (c_P - c_V)\varrho T = (\gamma - 1)c_V\varrho T. \tag{1.7.56}$$

From the definition of the specific heat at constant volume $c_V = (\partial U/\partial T)_V$, we find for a perfect gas with constant specific heats

$$c_V = U/T, \tag{1.7.57}$$

where U denotes the specific internal energy, ($U =$ internal energy per unit rest mass). Combining Eqs. (1.7.56) and (1.7.57), we find ($\varrho U = \varepsilon^{(int)}$) :

$$P = (\gamma - 1)\varrho U = (\gamma - 1)\varepsilon^{(int)} = w\varepsilon^{(kin)}/3 \quad \text{or} \quad \varepsilon^{(int)} = w\varepsilon^{(kin)}/3(\gamma - 1). \tag{1.7.58}$$

As the energy per degree of freedom and per particle is kT/w, the internal energy density of particles having f degrees of freedom is

$$\varepsilon^{(int)} = fkn_dT/w = w\varepsilon^{(kin)}/3(\gamma - 1) = kn_dT/(\gamma - 1) \quad \text{or} \quad \gamma = (f + w)/f. \tag{1.7.59}$$

For the polytropic indices we get (cf. Sec. 1.2)

$$n = n' = n'' = 1/(\gamma - 1) = 1/(\Gamma_k - 1) = f/w, \quad (f \ge 3;\ 1 \le w \le 2;\ \gamma = \Gamma_k;\ k = 1,2,3). \tag{1.7.60}$$

For completely ionized or monoatomic gases (e.g. He, Ar, Hg, etc.) we have $f = 3$, $\gamma = 1 + w/3$, $(4/3 \le \gamma \le 5/3)$ and $n = 3/w$, $(1.5 \le n \le 3)$. Diatomic or multiatomic molecules exist only at nonrelativistic particle velocities $v \ll c$, and therefore the energy associated with each degree of freedom in molecules is $kT/2$, $(w = 2)$. Generally, a molecule composed of j atoms can be considered to possess roughly $3j$ degrees of freedom, stemming from the 3 degrees of freedom of translational kinetic motion of each of the j atoms (Landau and Lifschitz 1971): $\gamma = (3j + 2)/3j$. If $j \gg 1$, then $\gamma \to 1$ and $n \to \infty$. Similarly, a gas undergoing ionization behaves as having a large number of degrees of freedom: $f \gg 1$, and consequently $\gamma \approx 1$. In ionization (dissociation) zones all gammas are approximately equal, and can attain in the midstages of ionization values as low as $\gamma \approx \Gamma_1 \approx \Gamma_2 \approx \Gamma_3 \approx 1.1 - 1.25$ (Cox and Giuli 1968).

We also note the equations for the specific heats at constant volume and constant pressure for a perfect gas with constant specific heats. From Eq. (1.7.59) we get for the specific internal energy

$$U = \varepsilon^{(int)}/\varrho = fkn_dT/w\varrho = fkT/w\mu H, \tag{1.7.61}$$

and from Eq. (1.7.57)

$$c_V = U/T = fkn_d/w\varrho = fk/w\mu H = fkN_A/w\mu = f\mathcal{R}/w\mu;$$
$$c_P = c_V + \mathcal{R}/\mu = (\mathcal{R}/\mu)(f/w + 1), \qquad (\mathcal{R} = kN_A = k/H;\ 1 \le w \le 2). \tag{1.7.62}$$

For crystalline solids, as well as for the nondegenerate ion lattice in a degenerate electron gas, the specific heats at constant pressure and constant volume are nearly equal ($\gamma \approx 1$), and are given by $c_P \approx c_V \approx 3\mathcal{R}/\mu$ (Dulong-Petit value) when $T \gg T_D = 3.4 \times 10^3(Z/A)\varrho^{1/2}$, and $c_P \approx c_V \approx 12\pi^4\mathcal{R}T^3/5\mu T_D^3$

Table 1.7.1 Intervals of variation of the adiabatic exponents $\Gamma_k,\ (k=1,2,3)$, of the ratio of specific heats $\gamma = c_P/c_V$, and of the corresponding isentropic polytropic indices n, n', n'' from Eqs. (1.3.30), (1.7.45) for the simple thermodynamic systems discussed in the text.

System	Γ_1	Γ_2	Γ_3	γ	n	n'	n''
Perfect gas		$\Gamma_1=\Gamma_2=\Gamma_3=\gamma$				$n=n'=n''$	
– nonrelativistic	$[1,5/3]$	$[1,5/3]$	$[1,5/3]$	$[1,5/3]$	$[1.5,\infty)$	$[1.5,\infty)$	$[1.5,\infty)$
– extreme relativistic	$[1,4/3]$	$[1,4/3]$	$[1,4/3]$	$[1,4/3]$	$[3,\infty)$	$[3,\infty)$	$[3,\infty)$
Black body radiation	4/3	4/3	4/3	∞	3	3	3
Mixture of black body radiation and electron-positron pairs	$[1.22,4/3]$	$[1.28,4/3]$	$[1.27,4/3]$	∞	$[3,4.5]$	$[3,3.6]$	$[3,3.7]$
Completely degenerate electron or neutron gas	$[4/3,5/3]$	$[4/3,5/3]$	$[4/3,5/3]$	1	$[1.5,3]$	$[1.5,3]$	$[1.5,3]$

Table 1.7.2 Values of the adiabatic exponents $\Gamma_k,\ (k=1,2,3)$ and of the polytropic indices n, n', n'' for a mixture of a completely degenerate electron gas and a nonrelativistic, nondegenerate ion plasma according to Eqs. (1.6.12)-(1.6.17), $(\gamma = 1)$. The relativity parameter x is given by Eq. (1.6.2), (Schatzman 1958).

x	Γ_1	Γ_2	Γ_3	n	n'	n''
0	5/3	5/3	5/3	1.50	1.50	1.50
0.5	1.6168	1.7017	5/3	1.62	1.43	1.50
1	1.5333	1.7693	5/3	1.88	1.30	1.50
1.5	1.4701	1.8298	5/3	2.13	1.21	1.50
2	1.4299	1.8735	5/3	2.33	1.14	1.50
4	1.3669	1.9521	5/3	2.73	1.05	1.50
10	1.3397	1.9905	5/3	2.94	1.01	1.50
∞	4/3	2.0000	5/3	3.00	1.00	1.50

if $T \ll T_D$. The symbol T_D denotes the Debye temperature, Z the atomic charge number, and A the atomic weight expressed in atomic mass units H (e.g. Cox and Giuli 1968, Landau and Lifschitz 1971).

Matter in planetary interiors, including crystalline solids, obeys roughly the perturbed polytropic equation of state (1.7.44). For instance, inside Jupiter we have $\gamma \approx 1,\ \Gamma_1 \approx 2,\ \Gamma_3 \approx 1.64,\ \Gamma_2 = \Gamma_1/(1+\Gamma_1-\Gamma_3) \approx 1.47,\ (n\approx 1,\ n' \approx 2.13,\ n'' \approx 1.56)$, (Hubbard 1978).

(ii) Mixture of Black Body Radiation and Perfect Gas. For pure black body radiation we have shown in Sec. 1.4 that $\Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3$ and $\gamma = \infty$, and therefore $n = n' = n'' = 3$. For a mixture of black body radiation and perfect gas the problem is somewhat more complicated, but as a general rule the gammas and polytropic indices are contained between the extreme values obtained for each component of the mixture separately [cf. Eqs. (1.4.17)-(1.4.22)]: $1 \le \Gamma_k \le 5/3,\ 1.5 \le n, n', n'' \le \infty$, and $1 \le \gamma \le \infty$. Concrete values for a mixture are extensively published in the astrophysical literature (e.g. Chandrasekhar 1939, Cox and Giuli 1968).

(iii) Gas Composed of Electron-Positron Pairs and Black Body Radiation. A numerical evaluation of the gammas in the partially relativistic regime shows that they attain minimum values of

$$\Gamma_1 \approx 1.22,\ (n\approx 4.5);\quad \Gamma_2 \approx 1.28,\ (n' \approx 3.6);\quad \Gamma_3 \approx 1.27,\ (n'' \approx 3.7), \tag{1.7.63}$$

if $\alpha = kT/m_ec^2 = O(1)$. They approach 4/3 in the nonrelativistic limit $(\alpha \ll 1)$ and in the extreme relativistic limit $(\alpha \gg 1)$, (Cox and Giuli 1968). If nuclei are also present, they would tend to make $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3$ in the nonrelativistic limit.

(iv) Completely Degenerate Electron and Neutron Gas. The relevant equations for $\Gamma_k,\ (k = 1,2,3)$ and γ are Eqs. (1.6.8)-(1.6.11), (Table 1.7.1). Obviously, the adiabatic exponents are contained between the values in the nonrelativistic limit $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3,\ (x \ll 1)$, and those in the extreme relativistic limit $\Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3,\ (x \gg 1)$. Also, $1.5 \le n, n', n'' \le 3$, and $\gamma = 1$ for any x. The values of the specific heats at constant pressure and volume are nearly equal for degenerate matter, and for temperatures well below the degeneracy temperature we have for degenerate electrons $c_P \approx c_V \approx 4\pi^{8/3}k^2T/3^{2/3}h^2n_e^{2/3}$ in the nonrelativistic case, and $c_P \approx c_V \approx 2\pi^{7/3}k^2T/3^{1/2}chm_en_e^{1/3}$ in the extreme relativistic case (e.g. Landau and Lifschitz 1971). A rough approximation for the degeneracy temperature of the electron gas is Eq. (1.7.28) in the nonrelativistic case, and Eq. (1.7.36) in the extreme relativistic case, shown graphically in Fig. 1.7.1 by the delimitation lines between domain 2 on the one side, and domains 3, 4 on the other side.

(v) Mixture of Completely Degenerate Electron Plasma and Nondegenerate Nonrelativistic Ion Plasma. When appreciable amounts of ions are present in the mixture, its thermal properties are determined by the ions, whereas the pressure is due mainly to the degenerate electrons. The ratio of specific heats $\gamma = \Gamma_1/\chi_\varrho$ will remain 1, since it is determined by the pressure-density relationship dominated by the degenerate electrons. The same is true for Γ_1, $(4/3 \leq \Gamma_1 \leq 5/3;\ 1.5 \leq n \leq 3)$, but the value of Γ_3 is entirely determined by the ions, obeying the perfect gas law: $\Gamma_3 = 5/3,\ (n'' = 1.5)$. The value of Γ_2 is partly determined by the electrons, partly by the ions: $5/3 \leq \Gamma_2 \leq 2,\ (1 \leq n' \leq 1.5)$, (cf. Table 1.7.2 and Eqs. (1.6.12)-(1.6.17), Schatzman 1958)

Finally, we wish again to draw attention that the brief survey presented in this section yields several useful approximations, but does not provide exact equations of state and accurate values of the adiabatic exponents for most cases; also, it largely ignores the effects produced by chemical and nuclear reactions, superfluidity, ionization, etc.

It is obvious that for most adiabatic changes of simple thermodynamic systems the corresponding polytropic indices from Eq. (1.7.45) are contained between narrow limits $1.5 < n, n', n'' < 3$ (cf. Table 1.7.1). We briefly discuss the important generalization achieved by the introduction of a polytropic equation of state with indices in the much larger interval $-\infty \leq n \leq \infty$. For many applications the thermal properties of matter are negligible or can be ignored, and the systems are described by simple pressure-density relationships of the form $P = K\varrho^{1+1/n}$. Sometimes even the *spatial* structure of collisionless systems (e.g. stellar clusters) is described by a polytropic "pressure-density" law. The sole pertinent polytropic index for many practical problems is $n = 1/(\Gamma_1' - 1) = 1/(d\ln P/d\ln\varrho - 1)$, the two other indices n' and n'', connected with the thermal properties of matter, being ignored. Polytropic equations of state can be successfully used (preferentially with variable polytropic index) whenever equations involving the temperature are not well established or can be ignored (e.g. neutron stars, planetary interiors, solar wind, interstellar clouds with negative polytropic index, etc.).

The success encountered by the notion of polytropes (polytropic changes) becomes now more obvious: (i) It offers a concise and concrete equation of state, containing only two state variables P and ϱ, once the polytropic index n and the polytropic constant K are fixed. (ii) For certain ranges of n the polytropic equation of state is valid – at least approximately – for many thermodynamic systems occurring in the Universe: Convective zones of ordinary stars, convective stellar and planetary atmospheres, planetary interiors, interstellar matter, black and white dwarf stars, neutron stars, solar and stellar winds, mass distribution in stellar systems, black body radiation fields, electron-positron pairs, etc.

1.8 Emden's Theorem

Theorem. If two polytropes AB and CD with constant polytropic exponents $\Gamma_{11}',\ \Gamma_{21}',\ \Gamma_{31}'$ are intersected by an arbitrary polytrope with constant polytropic exponents $\Gamma_{12}',\ \Gamma_{22}',\ \Gamma_{32}',\ (\Gamma_{k1}' \neq \Gamma_{k2}',\ k = 1,2,3)$, then the ratio of the polytropic state variables P, T, ϱ, V in the two points of intersection is constant for all polytropes having polytropic exponents Γ_{k2}'.

Proof. Let AB and CD be two polytropes of specific heat c_1 and with constant polytropic exponents $\Gamma_{k1}',\ (k = 1,2,3)$. Let AD and BC be two other polytropes of specific heat c_2 and with constant polytropic exponents $\Gamma_{k2}',\ (k = 1,2,3)$, (see Fig. 1.8.1). Let the four polytropes intersect at points A, B, C, D. Let P_X, T_X, ϱ_X, and V_X be the values of the state variables at point X, $(X = A, B, C, D)$.

Polytropic changes are by definition reversible processes, and therefore the line integral

$$\oint dS = \oint dQ/T = 0, \tag{1.8.1}$$

is exactly zero, when the system goes through the closed cycle ABCDA. For a polytropic change we have by definition $dQ = c_1\ dT$ over the parts AB, CD, and $dQ = c_2\ dT$ over the parts AD, BC. Eq. (1.8.1) becomes

$$\oint dQ/T = c_1\int_A^B dT/T + c_2\int_B^C dT/T + c_1\int_C^D dT/T + c_2\int_D^A dT/T = 0, \tag{1.8.2}$$

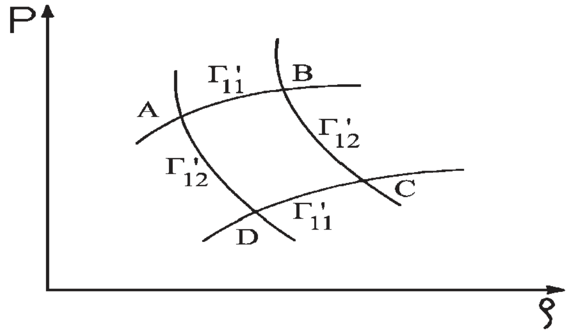

Fig. 1.8.1 Emden's theorem. The polytropes AB, CD (polytropic exponent Γ'_{11}) and the polytropes BC, AD (polytropic exponent Γ'_{12}) are shown schematically in a pressure-density diagram.

and by integrating

$$c_1 \ln(T_B/T_A) + c_2 \ln(T_C/T_B) + c_1 \ln(T_D/T_C) + c_2 \ln(T_A/T_D) = 0, \tag{1.8.3}$$

or

$$(c_1 - c_2) \ln(T_B T_D/T_A T_C) = 0, \qquad (c_1 \neq c_2), \tag{1.8.4}$$

or

$$T_A/T_D = T_B/T_C. \tag{1.8.5}$$

Thus, the theorem has been proved for the temperature T. The proof for the other state variables is readily made by observing that according to Eq. (1.3.28)

$$T_A/T_B = (\varrho_A/\varrho_B)^{\Gamma'_{31}-1} = (V_A/V_B)^{1-\Gamma'_{31}}, \tag{1.8.6}$$

and

$$T_D/T_C = (\varrho_D/\varrho_C)^{\Gamma'_{31}-1} = (V_D/V_C)^{1-\Gamma'_{31}}. \tag{1.8.7}$$

Division of Eqs. (1.8.6) and (1.8.7) yields

$$T_A T_C/T_B T_D = 1 = (\varrho_A \varrho_C/\varrho_B \varrho_D)^{\Gamma'_{31}-1} = (V_A V_C/V_B V_D)^{1-\Gamma'_{31}}, \tag{1.8.8}$$

or

$$\varrho_A/\varrho_D = \varrho_B/\varrho_C; \quad V_A/V_D = V_B/V_C. \tag{1.8.9}$$

Similarly, we derive

$$P_A/P_D = P_B/P_C. \tag{1.8.10}$$

In Chandrasekhar's (1939) formulation, Emden's theorem may be stated as follows: A polytrope AB with constant polytropic exponents $\Gamma'_{k1},\ (k = 1,2,3)$ is cut at point A by another polytrope AD belonging to another class of polytropes with constant polytropic exponents $\Gamma'_{k2},\ (k = 1,2,3)$. Along the polytrope AD we consider the point D, such that the ratio P_A/P_D (or $T_A/T_D,\ \varrho_A/\varrho_D,\ V_A/V_D$) is a certain constant. If AD is any polytrope belonging to class $\Gamma'_{k2},\ (k = 1,2,3)$, then the geometric locus of D is another polytrope belonging to the class $\Gamma'_{k1},\ (k = 1,2,3)$.

2 UNDISTORTED POLYTROPES

2.1 General Differential Equations

In an inertial frame the equation of motion of a viscous fluid under the influence of body forces and magnetic fields can be written as (e.g. Landau and Lifshitz 1959, Alfvén and Fälthammar 1963, Tohline 1982)

$$\varrho\ D\vec{v}_0/Dt = -\nabla P + \varrho\vec{F} + (1/4\pi)(\nabla\times\vec{H})\times\vec{B} + \nabla\cdot\tau. \tag{2.1.1}$$

ϱ denotes the density, $D\vec{v}_0/Dt = \partial\vec{v}_0/\partial t + (\vec{v}_0\cdot\nabla)\vec{v}_0$ the material derivative (B.23) of the velocity $\vec{v}_0$, $(v_0 \ll c)$, P the hydrostatic pressure of the fluid, $\vec{F}$ the body force (e.g. gravitation) acting on the unit of mass, $\vec{B}$ the field vector of magnetic induction in the unrationalized CGS-system (Gaussian system), and $\vec{H}$ the magnetic field intensity vector, displacement currents being neglected. The influence of dissipative forces of arbitrary origin (e.g. ordinary viscosity, turbulence, radiative viscosity, dissipation by gravitational radiation reaction) is represented by the viscous stress tensor τ, which for the moment needs no further specification (see however Secs. 3.5, 5.8.3, 5.8.4). In the case of hydrostatic equilibrium $(\vec{v}_0 = 0)$, without energy dissipation and magnetic fields, Eq. (2.1.1) simplifies considerably:

$$\nabla P = \varrho\vec{F}. \tag{2.1.2}$$

If the body force per unit mass $\vec{F}$ possesses a potential function Φ, $(\nabla\Phi = \vec{F})$, the hydrostatic equation writes as

$$\nabla P = \varrho\ \nabla\Phi. \tag{2.1.3}$$

The only body force to be considered will be gravitation, so that Φ is equal to the internal Newtonian gravitational potential, obeying Poisson's equation

$$\nabla^2\Phi = -4\pi G\varrho. \tag{2.1.4}$$

Chandrasekhar (1939) pointed out that there can be introduced instead of the polytropic equation of state (1.3.29), the more general form

$$P = K\varrho^{1+1/n} + D, \qquad (K, n, D = \text{const}), \tag{2.1.5}$$

pertinent for example to solid state matter ($\varrho \neq 0$ if $P = 0$). Most of our equations are valid with this more general form, but we will preserve for simplicity the usual equation (1.3.29) with $D = 0$:

$$P = K\varrho^{1+1/n}, \qquad (K, n = \text{const}). \tag{2.1.6}$$

(i) $\boldsymbol{n \neq -1, \pm\infty}$. Introducing Eq. (2.1.6) into the hydrostatic equation (2.1.3), we get

$$(1 + 1/n)K\varrho^{1/n}\ \nabla\varrho = \varrho\ \nabla\Phi, \tag{2.1.7}$$

and by integration

$$\Phi - \Phi_0 = (n+1)K(\varrho^{1/n} - \varrho_0^{1/n}) = (n+1)(P/\varrho - P_0/\varrho_0)$$
$$= (n+1)K^{n/(n+1)}[P^{1/(n+1)} - P_0^{1/(n+1)}]. \tag{2.1.8}$$

Φ_0, ϱ_0, and P_0 denote central values, and we may take without loss of generality $\Phi_0 = 0$.

As we have already stressed subsequently to Eqs. (1.2.35) and (1.3.29), the pressure from Eq. (2.1.6) apparently becomes infinite if $n = 0$. This singularity can be removed at once, if we rewrite Eq. (2.1.6) under the form $P = L^{1/n}\varrho^{1+1/n}$, $(K = L^{1/n})$; if $n = 0$, we have $\varrho = P^{n/(n+1)}/L^{1/(n+1)} = 1/L = \mathrm{const}$ for any $P, L \neq 0$.

We briefly discuss the special isobaric case $n = -1$. Let us suppose that $K, \varrho, \varrho_0 \neq 0$ if $n = -1$. The equation of state (2.1.6) becomes $P = K = \mathrm{const}$, and Eq. (2.1.8) would yield $\Phi - \Phi_0 = 0$. But if $\Phi = \Phi_0 = \mathrm{const}$, Poisson's equation (2.1.4) would imply $\varrho, \varrho_0 = 0$, which is contradicted by our assumption $\varrho, \varrho_0 \neq 0$ (cf. Horedt 1971, 1989). Therefore, no *hydrostatic* solution can be given in terms of physical variables for the special case $n = -1$, although this case appears as the limit of physically possible hydrostatic models having $n \approx -1$, as will be obvious from the next equations. For the reasons stated above, the *hydrostatic* case $n = -1$ will be generally excluded from our discussion (see however Fig. 2.5.1, and Sec. 6.3.1 for a polytropic flow with $n = -1$).

The case $n = \pm\infty$ is another special case; generally, this case must be treated separately. In particular, for a polytropic configuration consisting of a perfect gas, the case $n = \pm\infty$ corresponds to an isothermal configuration $T = \mathrm{const}$, according to Eq. (1.2.29).

We introduce Φ from Eq. (2.1.8) into Poisson's equation (2.1.4):

$$(n+1)K\ \nabla^2\varrho^{1/n} = -4\pi G\varrho. \tag{2.1.9}$$

If $n = 0$, this equation becomes, by replacing $K\varrho^{1/n}$ with $P/\varrho:\ \nabla^2 P = -4\pi G\varrho^2,\ (\varrho = \varrho_0 = \mathrm{const})$.

The function $\varrho^{1/n}$ under the Laplace operator takes the simplest dimensionless form, if we put $\varrho^{1/n} = \mathrm{const}\ \theta$, or

$$\varrho = \varrho_0\theta^n; \quad P = P_0\theta^{n+1}. \tag{2.1.10}$$

Eq. (2.1.9) becomes

$$[(n+1)K/4\pi G\varrho_0^{1-1/n}]\ \nabla^2\theta = -\theta^n. \tag{2.1.11}$$

In N-dimensional polar coordinates the Laplace operator $\nabla^2\theta$ is given by Eq. (C.15), and Eq. (2.1.11) takes for radial symmetry $\theta = \theta(r)$ the simple form

$$[(n+1)K/4\pi G\varrho_0^{1-1/n}r^{N-1}]\ d(r^{N-1}\ d\theta/dr)/dr = -\theta^n, \qquad (N = 1, 2, 3, ...). \tag{2.1.12}$$

In one-dimensional space r is the distance from the central symmetry-plane of a polytropic slab, in two-dimensional space r is the distance from the central symmetry-axis of a polytropic cylinder, and in N-dimensional space r is equal to the distance from the central symmetry-point of a N-dimensional sphere ($N \geq 3$). Polytropic slabs and cylinders have always infinite extension along their symmetry-plane and symmetry-axis, respectively. Values of the geometric index N differing from natural numbers ($N = 1, 2, 3, ...$), as considered for instance by Abramowicz (1983), do not seem to have any physical significance (see App. C).

If we insert the dimensionless radial coordinate $\xi = r/\mathrm{const}$ into Eq. (2.1.12), we find that with

$$r = \mathrm{const}\ \xi = [\pm(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [\pm(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi = \alpha\xi, \tag{2.1.13}$$

the equilibrium equation of a polytrope with radial symmetry obeys the well known Lane-Emden equation

$$\nabla^2\theta(\xi) = \xi^{1-N}\ d(\xi^{N-1}\ d\theta/d\xi)/d\xi = \theta'' + (N-1)\theta'/\xi = \mp\theta^n, \qquad (\theta' = d\theta/d\xi;\ \theta'' = d^2\theta/d\xi^2). \tag{2.1.14}$$

The upper sign corresponds always to values of the polytropic index $-1 < n < \infty$, the lower one to $-\infty < n < -1$. The special case $n = -1$ appears as the limiting case of two polytropic sequences having $-1 < n < \infty$ and $-\infty < n < -1$, respectively. From a formal viewpoint Eq. (2.1.14) holds also if the minus sign on its right-hand side is associated with polytropic indices $-\infty < n < -1$, and the positive sign of θ^n with $-1 < n < \infty$. But an inspection of Eq. (2.1.13) quickly shows that in this case the radial distance r becomes imaginary ($r^2 < 0$), and therefore these unphysical imaginary polytropic sequences will generally not be considered further (see however Figs. 2.4.1, 2.5.1). Likewise, as outlined subsequently to Eq. (2.1.8), the unphysical special case $n = -1$ will generally not be considered, although it represents the limiting case of physically possible hydrostatic polytropes. It should be stressed that the Lane-Emden equation (2.1.14) in terms of the variables ξ, θ is solvable for the whole range $-\infty < n < \infty$, with both signs in front of θ^n (cf. Sec. 2.5, Fig. 2.5.1).

For a sphere $N = 3$ Perov and Frolova (1994) have generalized Eq. (2.1.14) to the case when n is a function of radius [effective polytropic index from Eq. (1.3.25)].

(ii) $\boldsymbol{n = \pm\infty}$. The polytropic equation (2.1.6) becomes

$$P = K\varrho, \tag{2.1.15}$$

and we have to integrate the hydrostatic equation (2.1.7) ex novo, to obtain

$$\Phi - \Phi_0 = K \ln(\varrho/\varrho_0) = K \ln(P/P_0). \tag{2.1.16}$$

Inserting for Φ, Poisson's equation (2.1.4) becomes

$$K\ \nabla^2 \ln \varrho = -4\pi G\varrho. \tag{2.1.17}$$

If we put $\ln \varrho = \mathrm{const} - \theta$, or

$$\varrho = \varrho_0 \exp(-\theta); \quad P = P_0 \exp(-\theta), \tag{2.1.18}$$

Eq. (2.1.17) takes the simple form

$$(K/4\pi G\varrho_0)\ \nabla^2\theta = (K/4\pi G\varrho_0 r^{N-1})\ d(r^{N-1}\ d\theta/dr)/dr = \exp(-\theta), \quad (N = 1, 2, 3, ...). \tag{2.1.19}$$

The scale factor for the dimensionless radial coordinate ξ is found by inserting $r = \mathrm{const}\ \xi$ into Eq. (2.1.19):

$$r = \mathrm{const}\ \xi = (K/4\pi G\varrho_0)^{1/2}\ \xi = \alpha\xi. \tag{2.1.20}$$

Introducing Eq. (2.1.20) into Eq. (2.1.19), we obtain the Lane-Emden equation for the polytropic index $n = \pm\infty$:

$$\nabla^2\theta(\xi) = \xi^{1-N}\ d(\xi^{N-1}\ d\theta/d\xi)/d\xi = \theta'' + (N-1)\theta'/\xi = \exp(-\theta). \tag{2.1.21}$$

This equation is sometimes referred to as the *isothermal equation,* a terminology that is correct only for the particular case of a perfect gas (cf. Sec. 1.2).

We have the same equation of state for the polytropic indices $n = -\infty$ and $n = \infty$, since according to Eddington (1931) the polytropic index changes from *positive* to *negative* values through the infinity points $n = \pm\infty$; a physical condition corresponding to values of the polytropic index intermediate between $n = -n_0$ and $n = n_0$ corresponds to values $n \le -n_0$ and $n \ge n_0$, and not as it might seem at first sight to polytropic indices $-n_0 \le n \le n_0$, $(n_0 > 0)$. Therefore, Kimura and Liu (1978), and Kimura (1981a) employ another normalization procedure, introducing Φ from Eq. (2.1.8) into Poisson's equation (2.1.4):

$$K^{2n/(n+1)} P_0^{(1-n)/(1+n)}\ \nabla^2\{(n+1)[1 - (P/P_0)^{1/(n+1)}]\} = 4\pi G(P/P_0)^{n/(n+1)}, \qquad (n \ne -1, \pm\infty). \tag{2.1.22}$$

Putting

$$\varphi = (n+1)[1 - (P/P_0)^{1/(n+1)}] = (\Phi - \Phi_0)/K^{n/(n+1)} P_0^{1/(n+1)}, \qquad (n \ne -1, \pm\infty), \tag{2.1.23}$$

we find from Eq. (2.1.22)

$$(K^{2n/(n+1)} P_0^{(1-n)/(1+n)}/4\pi G)\ \nabla^2\varphi = [1 - \varphi/(n+1)]^n. \tag{2.1.24}$$

With

$$r = (K^{2n/(n+1)} P_0^{(1-n)/(1+n)}/4\pi G)^{1/2}\ \eta = (K/4\pi G\varrho_0^{1-1/n})^{1/2}\ \eta, \qquad (n \ne -1, \pm\infty), \tag{2.1.25}$$

we get the Kimura-Liu form of the Lane-Emden equation (2.1.14), that will be shown to be also valid in the limit $n = \pm\infty$:

$$\nabla^2\varphi = d^2\varphi/d\eta^2 + (N-1)(d\varphi/d\eta)/\eta = [1 - \varphi/(n+1)]^n, \qquad (n \ne -1). \tag{2.1.26}$$

Comparing Eqs. (2.1.13) and (2.1.14) with Eqs. (2.1.25) and (2.1.26), we observe that (cf. Hunter 2001, App. C)

$$\xi = \eta/[\pm(n+1)]^{1/2}, \qquad (n \neq -1, \pm\infty), \tag{2.1.27}$$

and

$$\theta = 1 - \varphi/(n+1), \qquad (n \neq -1, \pm\infty), \tag{2.1.28}$$

since $d\theta(\xi)/d\xi = [\pm(n+1)]^{1/2}\ d\theta(\eta)/d\eta = \mp[d\varphi(\eta)/d\eta]/[\pm(n+1)]^{1/2}$ and $d^2\theta(\xi)/d\xi^2 = \mp\ d^2\varphi(\eta)/d\eta^2$. To compare the Lane-Emden variables ξ, θ with the variables η, φ for the case $n = \pm\infty$, we turn in Eq. (2.1.28) to the limit $n \to \pm\infty$. We have (e.g. Smirnow 1967)

$$\lim_{n\to\pm\infty} \theta = 1; \ \lim_{n\to\pm\infty} \theta^n = \lim_{n\to\pm\infty} [1 - \varphi/(n+1)]^n = \lim_{n\to\pm\infty} (1 - \varphi/n)^n = \exp(-\varphi);$$
$$\lim_{n\to\pm\infty} [(n+1)\ d\theta/d\eta] = -d\varphi/d\eta. \tag{2.1.29}$$

Thus, by turning in Eqs. (2.1.25), (2.1.26) to the limit $n \to \pm\infty$, we get

$$r = (K/4\pi G\varrho_0)^{1/2}\eta, \qquad (n = \pm\infty), \tag{2.1.30}$$

and

$$\nabla^2\varphi = d^2\varphi/d\eta^2 + (N-1)(d\varphi/d\eta)/\eta = \exp(-\varphi), \qquad (n = \pm\infty). \tag{2.1.31}$$

Comparing Eqs. (2.1.30) and (2.1.31) with Eqs. (2.1.20) and (2.1.21), we observe at once that for the special case $n = \pm\infty$ there is identity between the two kinds of variables:

$$\xi = \eta; \quad \theta = \varphi, \qquad (n = \pm\infty). \tag{2.1.32}$$

Eq. (2.1.26) has the advantage that it offers a unified description of polytropes, including the case $n = \pm\infty$. However, because all the relevant literature is given in terms of the ξ, θ-variables, we use exclusively the Lane-Emden variables instead of a description in terms of the Kimura-Liu variables η, φ.

The Lane-Emden equations (2.1.14), (2.1.21), and (2.1.26) govern the *hydrostatic* distribution of matter in any region where the polytropic equation of state (2.1.6) is valid. The Lane-Emden equation describes the structure of a complete polytrope, if Eq. (2.1.6) is valid over the whole configuration. In this case P_0 and ϱ_0 can be taken equal to the central pressure and density of the configuration at $r = 0$. To integrate the second order Lane-Emden equation, we also need the value of $d\theta/d\xi = \theta'$ at $\xi = 0$. This value can be easily found by writing down the radially symmetric form of Poisson's equation (2.1.4), [cf. Eq. (C.15)]:

$$r^{1-N}\ d(r^{N-1}\ d\Phi/dr)/dr = -4\pi G\varrho. \tag{2.1.33}$$

Integration yields

$$d\Phi/dr = -r^{1-N}\int_0^r 4\pi G\varrho r'^{N-1}\ dr'; \qquad (d\Phi/dr)_{r=0} = 0. \tag{2.1.34}$$

Inserting into the radially symmetric form of the hydrostatic equation (2.1.3) we get [cf. Eq. (C.12)]

$$dP/dr = \varrho\ d\Phi/dr = -4\pi G\varrho r^{1-N}\int_0^r \varrho r'^{N-1}\ dr'. \tag{2.1.35}$$

If $n \neq -1, \pm\infty$, we insert P and ϱ from Eq. (2.1.10) to obtain

$$(n+1)P_0\theta^n\ d\theta/dr = -4\pi G\varrho_0\theta^n r^{1-N}\int_0^r \varrho r'^{N-1}\ dr'. \tag{2.1.36}$$

Since θ^n is not identical to zero, we can simplify and write

$$d\theta/d\xi = -[4\pi G\alpha\varrho_0/(n+1)P_0]r^{1-N}\int_0^r \varrho r'^{N-1}\ dr'. \tag{2.1.37}$$

If $r = \alpha\xi \rightarrow 0$, the density can be approximated by its value at the origin $\varrho = \varrho_0\theta^n(\xi) \approx \varrho_0\theta^n(0)$. Thus

$$d\theta/d\xi \approx -[4\pi G\alpha\varrho_0^2\theta^n(0)/(n+1)P_0]\, r^{1-N}\int_0^r r'^{N-1}\, dr' = -[4\pi G\alpha^2\varrho_0^2\theta^n(0)/(n+1)NP_0]\,\xi$$
$$= \mp\xi\theta^n(0)/N, \qquad (n \neq -1, \pm\infty;\ r = \alpha\xi \approx 0). \tag{2.1.38}$$

In the limit $\xi \rightarrow 0$ we get $(d\theta/d\xi)_{\xi=0} = 0$. Eq. (2.1.38) corresponds with Eq. (2.4.21). If $n = \pm\infty$, we find in the same manner by virtue of Eq. (2.1.18)

$$P_0 \exp(-\theta)\ d\theta/dr = 4\pi G\varrho_0 \exp(-\theta)\ r^{1-N}\int_0^r \varrho r'^{N-1}\ dr', \tag{2.1.39}$$

and

$$d\theta/d\xi = (4\pi G\alpha\varrho_0/P_0)r^{1-N}\int_0^r \varrho r'^{N-1}\ dr' \approx \{4\pi G\alpha^2\varrho_0^2 \exp[-\theta(0)]/NP_0\}\ \xi = \xi\exp[-\theta(0)]/N,$$
$$(n = \pm\infty;\ r = \alpha\xi \approx 0). \tag{2.1.40}$$

In the limit $\xi \rightarrow 0$ we have $(d\theta/d\xi)_{\xi=0} = 0$, and Eq. (2.1.40) corresponds with Eq. (2.4.36). Thus, we arrive to the following theorem (cf. Chandrasekhar 1939 if $N = 3$) :

Theorem. The derivative $d\theta/d\xi = \theta'$ must be zero at the origin $\xi = 0$ for any solution θ of the Lane-Emden equation that is finite at the origin, provided that $n \neq -1$.

If not stated explicitly otherwise, we will always assume the initial conditions to be equal to

$$\theta(0) = 1,\ \theta'(0) = 0 \quad \text{if} \quad n \neq -1, \pm\infty, \quad \text{and} \quad \theta(0) = 0,\ \theta'(0) = 0 \quad \text{if} \quad n = \pm\infty, \tag{2.1.41}$$

for the Lane-Emden equations (2.1.14) and (2.1.21) in the case of radially symmetric complete polytropes. For the initial conditions (2.1.41) P_0 and ϱ_0 are always equal to the pressure and density at the origin $\xi = 0$. Solutions of the Lane-Emden equation obeying this most important special type of initial conditions are called Lane-Emden functions. The next sections will be devoted to the study of analytical and numerical forms of Lane-Emden functions.

The spherical $N = 3$ Lane-Emden equations (2.1.14), (2.1.21) can also be written under the form (2.1.47), (2.1.48) of a Volterra type integral equation (Schaudt 2000). The general solution of the three-dimensional Poisson equation (2.1.4) at position vector $\vec{r}$ can be written via Green's first formula under the well known form (e.g. Courant and Hilbert 1962)

$$4\pi\Phi(\vec{r}) = -\int_V \nabla^2\Phi(\vec{r'})\ dV/|\vec{r}-\vec{r'}| + \int_S [\nabla\Phi(\vec{r'})\cdot\vec{\nu}(\vec{r'})]\ dS/|\vec{r}-\vec{r'}|$$
$$-\int_S \Phi(\vec{r'})\ [\nabla_{\vec{r'}}(1/|\vec{r}-\vec{r'}|)\cdot\vec{\nu}(\vec{r'})]\ dS, \qquad [\nabla^2\Phi(\vec{r'}) = -4\pi G\varrho(\vec{r'})], \tag{2.1.42}$$

where $\vec{r'}$ is the position vector of a current integration point inside volume V or on its surface S. The unit vector along the exterior normal to the surface element dS is denoted by $\vec{\nu} = \vec{\nu}(\vec{r'})$, $(|\vec{\nu}| = 1)$. The right-hand side of Poisson's equation $\nabla^2\Phi = -4\pi G\varrho$ does not appear explicitly in Eq. (2.1.42), so the general solution of the Lane-Emden equations $\nabla^2\theta(\vec{\xi}) = \mp\theta^n(\vec{\xi})$ or $\nabla^2\theta(\vec{\xi}) = \exp[-\theta(\vec{\xi})]$ can be obtained at once by replacing $\Phi, \vec{r}, \vec{r'}$ with $\theta, \vec{\xi}, \vec{\xi'}$, respectively:

$$4\pi\theta(\vec{\xi}) = -\int_V \nabla^2\theta(\vec{\xi'})\ dV/|\vec{\xi}-\vec{\xi'}| + \int_S [\nabla\theta(\vec{\xi'})\cdot\vec{\nu}(\vec{\xi'})]\ dS/|\vec{\xi}-\vec{\xi'}|$$
$$-\int_S \theta(\vec{\xi'})\ [\nabla_{\vec{\xi'}}(1/|\vec{\xi}-\vec{\xi'}|)\cdot\vec{\nu}(\vec{\xi'})]\ dS. \tag{2.1.43}$$

We particularize this equation to spherical symmetry by integrating over a sphere of radius ξ_0 [Eq.

(B.34)]. In this case we have $\nabla f(\vec{\xi'}) \cdot \vec{\nu}(\vec{\xi'}) = df(\xi')/d\xi'$, $(\vec{\xi'} = \xi'\vec{\nu}(\vec{\xi'});\ \xi' = |\vec{\xi'}|)$, and

$$4\pi\theta(\xi) = -\int_0^{\xi_0} d\xi' \int_0^{\pi} d\lambda \int_0^{2\pi} \nabla^2\theta(\xi')\ \xi'^2\ \sin\lambda\ d\varphi/(\xi^2+\xi'^2-2\xi\xi'\cos\lambda)^{1/2}$$
$$+\xi_0^2\theta'(\xi_0)\int_0^{\pi} d\lambda \int_0^{2\pi} \sin\lambda\ d\varphi/(\xi^2+\xi_0^2-2\xi\xi_0\cos\lambda)^{1/2} + \xi_0^2\theta(\xi_0)$$
$$\times\int_0^{\pi} d\lambda \int_0^{2\pi} (\xi_0-\xi\cos\lambda)\ \sin\lambda\ d\varphi/(\xi^2+\xi_0^2-2\xi\xi_0\cos\lambda)^{3/2}, \qquad (N=3;\ \xi=|\vec{\xi}|;\ \xi<\xi_0=\xi_0'). \tag{2.1.44}$$

We perform the elementary integration over the angular coordinates, taking into account that $\xi+\xi'-|\xi-\xi'|$ equals 2ξ if $\xi<\xi'$, and $2\xi'$ if $\xi>\xi'$:

$$\theta(\xi) = -(1/\xi)\int_0^{\xi} \nabla^2\theta(\xi')\ \xi'^2\ d\xi' - \int_{\xi}^{\xi_0} \nabla^2\theta(\xi')\ \xi'\ d\xi' + \xi_0\theta'(\xi_0) + \theta(\xi_0). \tag{2.1.45}$$

Since the first integral vanishes in the limit $\xi \to 0$, the initial conditions $\theta(0)=1$ if $n \neq -1, \pm\infty$, and $\theta(0)=0$ if $n=\pm\infty$ yield

$$\xi_0\theta'(\xi_0)+\theta(\xi_0) = \begin{cases} 1+\int_0^{\xi_0}\nabla^2\theta(\xi')\ \xi'\ d\xi' & \text{if} \quad n\neq -1,\pm\infty \\ \int_0^{\xi_0}\nabla^2\theta(\xi')\ \xi'\ d\xi' & \phantom{\text{if}} \quad n=\pm\infty \end{cases} \tag{2.1.46}$$

We insert this into Eq. (2.1.45), substituting also for $\nabla^2\theta$ via Eqs. (2.1.14) and (2.1.21), respectively:

$$\theta(\xi) = 1 \mp \int_0^{\xi} \xi'(1-\xi'/\xi)\ \theta^n(\xi')\ d\xi', \qquad (N=3;\ n\neq -1,\pm\infty), \tag{2.1.47}$$

$$\theta(\xi) = \int_0^{\xi} \xi'(1-\xi'/\xi)\exp[-\theta(\xi')]\ d\xi', \qquad (N=3;\ n=\pm\infty). \tag{2.1.48}$$

There are two, generally quite different physical parameters inside a polytrope: (i) The polytropic index $n=1/(\Gamma_1'-1)$ from Eq. (1.3.25) is defined with the aid of the polytropic exponent Γ_1', and determines the global overall properties of the polytropic model under consideration. (ii) The adiabatic exponent Γ_1, (generally $\Gamma_1 \neq \Gamma_1' = 1+1/n$) from Eq. (1.3.23) characterizes the local behaviour of polytropic matter when performing adiabatic oscillations, for instance [Rosseland 1964, p. 27; Zeldovich and Novikov 1971, p. 252; Shapiro and Teukolsky 1983, p. 133)]. The squared adiabatic sound velocity within a polytrope of index n is obtained from Eqs. (1.7.42), (1.3.23)

$$a^2 = (\partial P/\partial\varrho)_{S=\mathrm{const}} = \Gamma_1 P/\varrho, \qquad [\varrho_r=\varrho;\ \Gamma_1=(\varrho/P)(\partial P/\partial\varrho)_S], \tag{2.1.49}$$

rather than from Eq. (2.1.6), $(P=K\varrho^{1+1/n})$:

$$a^2 = (\partial P/\partial\varrho)_{S=\mathrm{const}} = (1+1/n)K\varrho^{1/n} = (1+1/n)P/\varrho = \Gamma_1' P/\varrho. \tag{2.1.50}$$

The adiabatic sound velocity is determined at constant entropy S, in order to stress that the underlying physical process is adiabatic and reversible (quasi-static). Eq. (2.1.50) would lead, among many other inconsistencies, to the absurd result that the sound velocity within a homogeneous Newtonian polytrope $n=0$ is infinite. The correct delimitation of the sound velocity inside a polytrope comes from the relativistic considerations of Eq. (1.7.43), where the obvious constraint $a \leq c$ leads to Eq. (4.1.68). Only for isentropic polytropes Eq. (2.1.50) subsists too, and we have by virtue of Eqs. (1.3.22), (1.3.23), (1.3.25), (1.3.30):

$$\Gamma_1 = \Gamma_1' = 1+1/n, \qquad (S,Q=\mathrm{const}). \tag{2.1.51}$$

2.2 The Homology Theorem and Transformations of the Lane-Emden Equation

2.2.1 The Homology Theorem

Theorem. If $\theta(\xi)$ is a solution of the Lane-Emden equation (2.1.14) or (2.1.21) then $A^{2/(n-1)}\theta(A\xi)$, (A = const) is also a solution of Eq. (2.1.14), and $\theta(A\xi) - \ln A^2$ is also a solution of Eq. (2.1.21).

Proof. (i) $\boldsymbol{n \neq \pm 1, \pm\infty}$**.** The case $n = 1$ has to be excluded, but it will be shown in Sec. 2.3.2 that for $n = 1$ the Lane-Emden equation can be solved explicitly in terms of Bessel functions. The proof can be most easily made by substituting directly $A^{2/(n-1)}\theta(A\xi)$ instead of $\theta(\xi)$ into the Lane-Emden equation (2.1.14). We get

$$\xi^{1-N}\ d[\xi^{N-1}A^{2/(n-1)}\ d\theta(A\xi)/d\xi]/d\xi = \mp A^{2n/(n-1)}\theta^n(A\xi). \tag{2.2.1}$$

We rearrange the exponents of A to obtain

$$(A\xi)^{1-N}\ d[(A\xi)^{N-1}\ d\theta(A\xi)/d(A\xi)]/d(A\xi) = \mp\theta^n(A\xi), \tag{2.2.2}$$

which shows that $A^{2/(n-1)}\theta(A\xi)$ indeed satisfies identically the Lane-Emden equation when $\xi \to A\xi$.

(ii) $\boldsymbol{n = \pm\infty}$**.** We substitute in Eq. (2.1.21) $\theta(A\xi) - \ln A^2$ instead of $\theta(\xi)$ to obtain

$$(A\xi)^{1-N}\ d[(A\xi)^{N-1}\ d\theta(A\xi)/d(A\xi)]/d(A\xi) = \exp[-\theta(A\xi)]. \tag{2.2.3}$$

This proves the theorem also in the special case $n = \pm\infty$.

Thus, if one solution $\theta = \theta(\xi)$ of the Lane-Emden equation is known, we can derive a whole homologous family $\{\theta(\xi)\}$ of solutions. In particular, if θ is just the Lane-Emden function defined by the initial conditions (2.1.41), then its homologous family $\{\theta(\xi)\}$ defines a whole set of solutions that are all finite at the origin $\xi = 0$. Solutions that are finite at the origin are called E-solutions and denoted by θ_E. The Lane-Emden function defined by the initial conditions from Eq. (2.1.41) is just a particular member of the set $\{\theta_E(\xi)\}$ of E-solutions. All E-solutions can be found from the Lane-Emden function through the homology transformations

$$\theta(\xi) \to A^{2/(n-1)}\theta(A\xi), \qquad (n \neq \pm 1, \pm\infty), \tag{2.2.4}$$

and

$$\theta(\xi) \to \theta(A\xi) - \ln A^2, \qquad (n = \pm\infty). \tag{2.2.5}$$

According to the theorem from the previous section any solution $\theta_E = \theta_E(\xi)$ that is finite at the origin $\xi = 0$ is an E-solution, and its derivative is zero $(d\theta_E/d\xi)_{\xi=0} = 0$. The general solution of the second order Lane-Emden equation must be characterized by two integration constants. According to the homology theorem one of the two constants must be "trivial" in the sense that it defines merely the scale factor A of the homology transformation, and we should be able to transform the second order Lane-Emden equation into a first order differential equation (Chandrasekhar 1939). Below, we present the most important transformations of the Lane-Emden equation.

2.2.2 Milne Homology Invariant Variables

Milne's homology invariant variables are introduced by the equations (Chandrasekhar 1939)

$$u = u(\xi) = \mp\xi\theta^n/\theta'; \quad v = v(\xi) = \mp\xi\theta'/\theta, \qquad (n \neq -1, \pm\infty), \tag{2.2.6}$$

and

$$u = u(\xi) = \xi\exp(-\theta)/\theta'; \quad v = v(\xi) = \xi\theta', \qquad (n = \pm\infty). \tag{2.2.7}$$

(i) $n \neq \pm 1, \pm\infty$. At first we prove that u and v are indeed homology invariant functions with respect to the transformation of the homology theorem

$$\theta_H(\xi) = A^{2/(n-1)}\theta(A\xi), \qquad (n \neq \pm 1, \pm\infty). \tag{2.2.8}$$

To this end it is convenient to introduce instead of the variable ξ from the homology transformation (2.2.8), the variable ξ/A. Eq. (2.2.8) takes the equivalent form

$$\theta_H(\xi/A) = A^{2/(n-1)}\theta(\xi). \tag{2.2.9}$$

Eq. (2.2.9) means that if $\theta(\xi/A)$ is a solution of the Lane-Emden equation

$$d[(\xi/A)^{N-1}\ d\theta(\xi/A)/d(\xi/A)]/d(\xi/A) = \mp(\xi/A)^{N-1}\theta^n(\xi/A), \tag{2.2.10}$$

then $\theta_H(\xi/A) = A^{2/(n-1)}\theta(\xi)$ is also a solution of Eq. (2.2.10); this can be easily shown by direct substitution of $\theta_H(\xi/A)$ instead of $\theta(\xi/A)$ into Eq. (2.2.10). The homologous transformation $u_H(\xi/A)$ of $u(\xi)$ is according to Eqs. (2.2.6) and (2.2.9) equal to

$$u_H(\xi/A) = \mp(\xi/A)\ \theta_H^n(\xi/A)/[d\theta_H(\xi/A)/d(\xi/A)] = \mp A^{2n/(n-1)}(\xi/A)\ \theta^n(\xi)$$
$$/[A^{2/(n-1)}\ d\theta(\xi)/d(\xi/A)] = \mp\xi\theta^n(\xi)/[d\theta(\xi)/d\xi] = u(\xi), \qquad (n \neq \pm 1, \pm\infty). \tag{2.2.11}$$

The invariance of v with respect to the homology transformation (2.2.9) is proved analogously:

$$v_H(\xi/A) = \mp(\xi/A)[d\theta_H(\xi/A)/d(\xi/A)]/\theta_H(\xi/A) = \mp A^{2/(n-1)}(\xi/A)[d\theta(\xi)/d(\xi/A)]/A^{2/(n-1)}\theta(\xi)$$
$$= \mp\xi\ [d\theta(\xi)/d\xi]/\theta(\xi) = v(\xi), \qquad (n \neq \pm 1, \pm\infty). \tag{2.2.12}$$

Since u and v are homology invariant functions, we can reduce with the aid of these functions the second order Lane-Emden equation (2.1.14) to one of the first order. We have, by using Eq. (2.1.14)

$$du/d\xi = \mp\theta^n/\theta' \mp n\xi\theta^{n-1} \pm \xi\theta^n\theta''/\theta'^2 = -\xi\theta^{2n}/\theta'^2 \mp n\xi\theta^{n-1} \mp N\theta^n/\theta'$$
$$= (\mp\theta^n/\theta')(\pm\xi\theta^n/\theta' + n\xi\theta'/\theta + N) = (u/\xi)(-u \mp nv + N), \tag{2.2.13}$$

and

$$dv/d\xi = \mp\theta'/\theta + \xi\theta''/\theta \pm \xi\theta'^2/\theta^2 = \xi\theta^{n-1} \pm \xi\theta'^2/\theta^2 \pm (N-2)\theta'/\theta$$
$$= (\mp\theta'/\theta)(\mp\xi\theta^n/\theta' - \xi\theta'/\theta - N + 2) = (v/\xi)(u \pm v - N + 2). \tag{2.2.14}$$

Dividing Eq. (2.2.14) by (2.2.13) we obtain a first order differential equation in the homology variables u and v, equivalent to the Lane-Emden equation (2.1.14):

$$dv/du = v(u \pm v - N + 2)/u(-u \mp nv + N), \qquad (N = 1, 2, 3, ...;\ n \neq -1, \pm\infty), \tag{2.2.15}$$

where the upper sign holds if $-1 < n < \infty$, and the lower one if $-\infty < n < -1$.

(ii) $n = \pm\infty$. We can analogously show that for the homologous transformation

$$\theta_H(\xi/A) = \theta(\xi) - \ln A^2, \qquad (n = \pm\infty), \tag{2.2.16}$$

u and v are homology invariant functions:

$$u_H(\xi/A) = (\xi/A)\exp[-\theta_H(\xi/A)]/[d\theta_H(\xi/A)/d(\xi/A)] = A^2(\xi/A)\exp[-\theta(\xi)]/[d\theta(\xi)/d(\xi/A)]$$
$$= \xi\exp[-\theta(\xi)]/[d\theta(\xi)/d\xi] = u(\xi), \tag{2.2.17}$$

and

$$v_H(\xi/A) = (\xi/A)[d\theta_H(\xi/A)/d(\xi/A)] = (\xi/A)\ d\theta(\xi)/d(\xi/A) = \xi\ d\theta(\xi)/d\xi = v(\xi). \tag{2.2.18}$$

The reduction to the first order of Eq. (2.1.21) proceeds as follows:

$$du/d\xi = \exp(-\theta)/\theta' - \xi\exp(-\theta) - \xi\theta''\exp(-\theta)/\theta'^2 = -\xi\exp(-2\theta)/\theta'^2 - \xi\exp(-\theta)$$
$$+N\exp(-\theta)/\theta' = [\exp(-\theta)/\theta'][-\xi\exp(-\theta)/\theta' - N\xi\theta'] = (u/\xi)(-u-v+N), \tag{2.2.19}$$

$$dv/d\xi = \theta' + \xi\theta'' = \xi\exp(-\theta) + (2-N)\theta' = \theta'[\xi\exp(-\theta)/\theta' - N + 2] = (v/\xi)(u-N+2). \tag{2.2.20}$$

Dividing the two last equations, we obtain a first order equation in the homology invariant variables that is equivalent to the Lane-Emden equation (2.1.21):

$$dv/du = v(u-N+2)/u(-u-v+N), \qquad (N = 1,2,3,...;\ n = \pm\infty). \tag{2.2.21}$$

Besides the Milne variables u and v we can form an arbitrary large number of other homology invariant functions, defining other first order differential equations, all equivalent to the usual second order Lane-Emden equations (2.1.14) or (2.1.21).

2.2.3 Emden Variables

(i) $n \neq \pm 1, \pm\infty$. It will be shown in Eq. (2.3.70) that the Lane-Emden equation (2.1.14) has a solution of the form $\theta = \mathrm{const}\ \xi^{2/(1-n)}$. Therefore, Emden (1907) makes the transformation (cf. Chandrasekhar 1939)

$$\theta(\xi) = B\xi^{2/(1-n)}\ z(\xi), \qquad (B = \mathrm{const};\ n \neq \pm 1, \pm\infty), \tag{2.2.22}$$

with the scope to reduce the order of Eq. (2.1.14). We have

$$d\theta/d\xi = B[\xi^{2/(1-n)}\ dz/d\xi + 2\xi^{(1+n)/(1-n)}z/(1-n)]; \quad d^2\theta/d\xi^2$$
$$= B[\xi^{2/(1-n)}\ d^2z/d\xi^2 + 4\xi^{(1+n)/(1-n)}\ (dz/d\xi)/(1-n) + 2(1+n)\xi^{2n/(1-n)}z/(1-n)^2]. \tag{2.2.23}$$

Eq. (2.1.14) becomes

$$\xi^2\ d^2z/d\xi^2 + [3+N+n(1-N)]\xi\ (dz/d\xi)/(1-n) + 2[N+n(2-N)]z/(1-n)^2 \pm B^{n-1}z^n = 0. \tag{2.2.24}$$

With the further change of variable

$$\xi = \exp(-t), \tag{2.2.25}$$

we find

$$d^2z/dt^2 + [2+N+n(2-N)](dz/dt)/(n-1) + 2[N+n(2-N)]z/(n-1)^2 \pm B^{n-1}z^n = 0, \tag{2.2.26}$$

where

$$dz/d\xi = -\xi^{-1}\ dz/dt; \qquad d^2z/d\xi^2 = \xi^{-2}(dz/dt + d^2z/dt^2). \tag{2.2.27}$$

The new variable

$$y = dz/dt = (dz/d\xi)\ d\xi/dt = -\xi\ dz/d\xi = -(\xi^{2/(n-1)}/B)[2\theta/(n-1) + \xi\ d\theta/d\xi]$$
$$= -2z/(n-1) - (\xi^{(n+1)/(n-1)}/B)\ d\theta/d\xi, \qquad (2.2.28)$$

transforms Eq. (2.2.26) into the first order equation

$$y\ dy/dz + [2+N+n(2-N)]y/(n-1) + 2[N+n(2-N)]z/(n-1)^2 \pm B^{n-1}z^n = 0,$$
$$(N = 1,2,3,...;\ n \neq \pm 1, \pm\infty), \qquad (2.2.29)$$

where $d^2z/dt^2 = dy/dt = (dy/dz)\ dz/dt = y\ dy/dz$.

(ii) $n = \pm\infty$. Since the Lane-Emden equation (2.1.21) admits for $n = \pm\infty$, $N \geq 3$ the singular solution $\theta = \ln[\xi^2/2(N-2)]$ from Eq. (2.3.74), we employ the transformation (Emden 1907, Chandrasekhar 1939)

$$\theta(\xi) = \ln\xi^2 - z(\xi), \qquad (n = \pm\infty). \qquad (2.2.30)$$

We have

$$d\theta/d\xi = 2/\xi - dz/d\xi; \quad d^2\theta/d\xi^2 = -2/\xi^2 - d^2z/d\xi^2. \qquad (2.2.31)$$

Eq. (2.1.21) becomes

$$\xi^2\ d^2z/d\xi^2 + (N-1)\xi\ dz/d\xi + \exp z + 2(2-N) = 0. \qquad (2.2.32)$$

With the change of variable (2.2.25) we finally find

$$d^2z/dt^2 + (2-N)\ dz/dt + \exp z + 2(2-N) = 0. \qquad (2.2.33)$$

The first order equation – equivalent to the Lane-Emden equation (2.1.21) – is

$$y\ dy/dz + (2-N)y + \exp z + 2(2-N) = 0, \qquad (N = 1,2,3,...;\ n = \pm\infty), \qquad (2.2.34)$$

where we have introduced the new variable

$$y = dz/dt = -\xi\ dz/d\xi = \xi\ d\theta/d\xi - 2. \qquad (2.2.35)$$

It remains to show that z and y are invariant with respect to the homologous transformations (2.2.9) and (2.2.16).

(i) $n \neq \pm 1, \pm\infty$. According to Eqs. (2.2.9) and (2.2.22) we have

$$z_H(\xi/A) = (\xi/A)^{2/(n-1)}\theta_H(\xi/A)/B = (\xi/A)^{2/(n-1)}A^{2/(n-1)}\theta(\xi)/B = \xi^{2/(n-1)}\theta(\xi)/B = z(\xi), \qquad (2.2.36)$$

and via Eq. (2.2.28)

$$y_H(\xi/A) = -2z_H(\xi/A)/(n-1) - [(\xi/A)^{(n+1)/(n-1)}/B]\ d\theta_H(\xi/A)/d(\xi/A)$$
$$= -2z(\xi)/(n-1) - (\xi^{(n+1)/(n-1)}/B)\ d\theta(\xi)/d\xi = y(\xi). \qquad (2.2.37)$$

(ii) $n = \pm\infty$. In virtue of Eqs. (2.2.16) and (2.2.30) we have

$$z_H(\xi/A) = \ln(\xi/A)^2 - \theta_H(\xi/A) = \ln(\xi/A)^2 - \theta(\xi) + \ln A^2 = \ln\xi^2 - \theta(\xi) = z(\xi), \qquad (2.2.38)$$

and by Eq. (2.2.35)

$$y_H(\xi/A) = (\xi/A)\ d\theta_H(\xi/A)/d(\xi/A) - 2 = (\xi/A)\ d\theta(\xi)/d(\xi/A) - 2 = \xi\ d\theta(\xi)/d\xi - 2 = y(\xi). \qquad (2.2.39)$$

This demonstrates completely the homology invariance of Emden's transformation.

The connection between Emden's variables from Eqs. (2.2.22), (2.2.28), (2.2.30), (2.2.35) and Milne's homology invariant variables from Eqs. (2.2.6), (2.2.7) follows easily by taking the arbitrary constant $B = 1$ (Chandrasekhar 1939):

$$z = \xi^{2/(n-1)}\theta = (\xi^2\theta^{n-1})^{1/(n-1)} = (uv)^{1/(n-1)}; \quad y = -2z/(n-1) - \xi^{(n+1)/(n-1)}\theta'$$
$$= -[2/(n-1)](uv)^{1/(n-1)} \pm [\xi^{n+1}(\mp\theta')^{n-1}]^{1/(n-1)} = -[2/(n-1)](uv)^{1/(n-1)} \pm (uv^n)^{1/(n-1)},$$
$$(n \neq \pm 1, \pm\infty), \qquad (2.2.40)$$

$$z = \ln\xi^2 - \theta = \ln[\xi^2\exp(-\theta)] = \ln(uv); \quad y = \xi\theta' - 2 = v - 2, \qquad (n = \pm\infty). \qquad (2.2.41)$$

2.2.4 Kelvin Variables

If $N = 2$, we put

$$x = \ln\xi, \tag{2.2.42}$$

and Eq. (2.1.14) becomes

$$\exp(-2x)\ d^2\theta(x)/dx^2 = \mp\theta^n(x), \qquad (N = 2;\ n \neq -1, \pm\infty), \tag{2.2.43}$$

since $d\theta(\xi)/d\xi = \xi^{-1}\ d\theta(x)/dx,\ d^2\theta(\xi)/d\xi^2 = -\xi^{-2}\ d\theta(x)/dx + \xi^{-2}\ d^2\theta(x)/dx^2$.

If $N = 3$, Eq. (2.1.14) turns with the transformation

$$x = 1/\xi, \tag{2.2.44}$$

into

$$x^4\ d^2\theta(x)/dx^2 = \mp\theta^n(x), \qquad (N = 3;\ n \neq -1, \pm\infty), \tag{2.2.45}$$

since $d\theta(\xi)/d\xi = -\xi^{-2}\ d\theta(x)/dx,\ d^2\theta(\xi)/d\xi^2 = 2\xi^{-3}\ d\theta(x)/dx + \xi^{-4}\ d^2\theta(x)/dx^2$.

In the particular case $n = \pm\infty$, Eq. (2.1.21) becomes in the same way

$$\exp(-2x)\ d^2\theta(x)/dx^2 = \exp[-\theta(x)], \qquad (N = 2;\ n = \pm\infty), \tag{2.2.46}$$

and

$$x^4\ d^2\theta(x)/dx^2 = \exp[-\theta(x)], \qquad (N = 3;\ n = \pm\infty). \tag{2.2.47}$$

The plane-parallel Lane-Emden equation ($N = 1$) lacks already in its original form for the first order derivative, and needs therefore no transformation of the previous kind.

2.3 Exact Analytical Solutions of the Lane-Emden Equation

If not stated explicitly otherwise, we will always restrict to solutions obeying the usual initial conditions from Eq. (2.1.41). The exact solutions can be grouped into four classes and are generally valid for any positive integer value of the geometric index N. Medvedev and Rybicki (2001) claim that the Lane-Emden equation (2.1.14) possesses in the spherical case $N = 3$ only a zero-dimensional Lie group algebra, indicating its global nonintegrability if $n \neq 0, 1, 5$.

2.3.1 Ritter's First Integral $n = 0$

Eq. (2.1.14) takes the simple form of a second order differential equation without θ-term:

$$d(\xi^{N-1}\ d\theta/d\xi) = -\xi^{N-1}\ d\xi, \qquad (n = 0;\ N = 1, 2, 3, ...). \tag{2.3.1}$$

A first elementary integration yields

$$d\theta/d\xi = -\xi/N + C\xi^{1-N}. \tag{2.3.2}$$

After a second integration we get

$$\theta = -\xi^2/2N + C\xi^{2-N}/(2-N) + D, \qquad (N \neq 2), \tag{2.3.3}$$

and if $N = 2$

$$\theta = -\xi^2/4 + C\ln\xi + D, \qquad (N = 2). \tag{2.3.4}$$

With the usual initial conditions $\theta(0) = 1,\ \theta'(0) = 0$ we obtain $C = 0,\ D = 1$, and

$$\theta = 1 - \xi^2/2N, \qquad (n = 0;\ N = 1, 2, 3, ...). \tag{2.3.5}$$

2.3.2 Ritter's Second Integral $n = 1$

In this case the Lane-Emden equation (2.1.14) becomes a somewhat more complicated, second order, linear homogeneous equation of a form similar to a Bessel differential equation:

$$\xi^2\theta'' + (N-1)\xi\theta' + \xi^2\theta = 0, \qquad (n = 1;\ N = 1, 2, 3, ...). \tag{2.3.6}$$

Quite generally, any differential equation of the form

$$x^2\ d^2y/dx^2 + (2\alpha+1)x\ dy/dx + (\alpha^2 - \beta^2\nu^2 + \beta^2\gamma^2x^{2\beta})y = 0,$$
$$(y = y(x);\ \alpha, \beta, \gamma, \nu = \mathrm{const};\ \beta, \gamma \neq 0), \tag{2.3.7}$$

can be transformed into the Bessel equation

$$\xi^2\ d^2u/d\xi^2 + \xi\ du/d\xi + (\xi^2 - \nu^2)u = 0, \tag{2.3.8}$$

with the change of variables (Smirnow 1967)

$$x = (\xi/\gamma)^{1/\beta}; \quad y(x) = x^{-\alpha}u(\xi). \tag{2.3.9}$$

The general solutions of the Bessel equation are (e.g. Spiegel 1968, Bronstein and Semendjajew 1985)

$$u(\xi) = C_1J_\nu(\xi) + C_2J_{-\nu}(\xi), \qquad (\nu \neq 0, 1, 2, 3, ...), \tag{2.3.10}$$

and

$$u(\xi) = C_1 J_\nu(\xi) + C_2 Y_\nu(\xi), \qquad (\text{all} \quad \nu). \tag{2.3.11}$$

C_1, C_2 are integration constants, $J_\nu(\xi)$ is the Bessel function of order ν, and $Y_\nu(\xi)$ the Bessel function of second kind of order ν, also called Neumann or Weber function. There is

$$J_\nu(\xi) = [\xi^\nu/2^\nu \Gamma(\nu+1)][1 - \xi^2/2(2\nu+2) + \xi^4/2 \times 4(2\nu+2)(2\nu+4) - ...]$$
$$= \sum_{k=0}^{\infty}(-1)^k (\xi/2)^{\nu+2k}/k!\ \Gamma(\nu+k+1), \tag{2.3.12}$$

and

$$\nu = 0: \ Y_0(\xi) = (2/\pi)\big[\ln(\xi/2) + \varepsilon\big] J_0(\xi) - (2/\pi)\sum_{k=0}^{\infty}(-1)^k \Xi(k)\ (\xi/2)^{2k}/(k!)^2;$$
$$\nu = 1, 2, 3, ...: \ Y_\nu(\xi) = (2/\pi)\big[\ln(\xi/2) + \varepsilon\big] J_\nu(\xi) - (1/\pi)\sum_{k=0}^{\nu-1}(\nu - k - 1)!(\xi/2)^{2k-\nu}/k!$$
$$-(1/\pi)\sum_{k=0}^{\infty}(-1)^k\big[\Xi(k) + \Xi(\nu+k)\big](\xi/2)^{2k+\nu}/k!(\nu+k)!; \quad \Xi(k) = \sum_{j=1}^{k}(1/j) \quad \text{if} \quad k = 1, 2, 3, ...;$$
$$\Xi(0) = 0; \quad \varepsilon = 0.5772... = \text{Euler's constant}. \tag{2.3.13}$$

The gamma function $\Gamma(\nu)$ is defined by Eq. (C.9) or by the recursion formula (C.11):

$$\Gamma(\nu) = \Gamma(\nu+k)/(\nu+k-1)(\nu+k-2)...(\nu+1)\nu, \qquad (k = 1, 2, 3, ...). \tag{2.3.14}$$

For our special equation (2.3.6), the transformation (2.3.9) becomes

$$x \equiv \xi; \quad y(x) \equiv \theta(\xi) = \xi^{(2-N)/2} u(\xi), \tag{2.3.15}$$

since $\alpha = \nu = (N-2)/2$, and $\beta, \gamma = 1$.

Inserting Eq. (2.3.15) into Eq. (2.3.6), we obtain the Bessel equation

$$\xi^2\ d^2u/d\xi^2 + \xi\ du/d\xi + \big\{\xi^2 - [(N-2)/2]^2\big\}u = 0. \tag{2.3.16}$$

Since N is a positive integer, we have to write down the solution of this equation only for integer and half integer values of $\nu = (N-2)/2$. If $\nu = 0, 1, 2, 3, ...$, i.e. $N = 2, 4, 6, 8, ...$, the solution of Eq. (2.3.16) is given by Eq. (2.3.11). The integration constant C_2 is zero in this case, as follows from the initial conditions (2.1.41): According to Eqs. (2.3.13) and (2.3.15) $Y_\nu(0)$ is unbounded if $\nu = 0, 1, 2, 3, ...$, while $u(0)$ must be finite if $N \geq 2$. If ν is a positive half integer $\nu = 1/2, 3/2, 5/2, ...$, i.e. if $N = 3, 5, 7, ...$, the solution of Eq. (2.3.16) is given by Eq. (2.3.10). The integration constant C_2 vanishes again, because according to Eqs. (2.3.12) and (2.3.15) $J_{-\nu}(0)$ is unbounded if $\nu = 1/2, 3/2, 5/2, ...$, and $u(0) = 0$ if $N \geq 3$. Consequently, for all values of $N \geq 2$ the solution of Eq. (2.3.16) is

$$u(\xi) = C_1 J_{(N-2)/2}(\xi), \tag{2.3.17}$$

and the solution of Eq. (2.3.6) amounts to

$$\theta = C_1 \xi^{(2-N)/2} J_{(N-2)/2}(\xi), \qquad (n = 1;\ N = 2, 3, 4, ...). \tag{2.3.18}$$

The integration constant C_1 can be determined from Eqs. (2.3.12) and (2.3.18), by observing that

$$\theta(0) = 1 = C_1/2^{(N-2)/2}\Gamma(N/2). \tag{2.3.19}$$

Thus

$$\theta = \Gamma(N/2)\ (\xi/2)^{(2-N)/2} J_{(N-2)/2}(\xi) = \sum_{k=0}^{\infty}(-1)^k (\xi/2)^{2k}\Gamma(N/2)/k!\ \Gamma(N/2+k),$$
$$(n = 1;\ N = 2, 3, 4, ...). \tag{2.3.20}$$

Via Eq. (2.3.14) and with the relationships for the Bessel functions of integer and half integer values we obtain further (Smirnow 1967, Ostriker 1964a if $N = 2$)

$$\theta = (N/2-1)!(\xi/2)^{(2-N)/2}J_{(N-2)/2}(\xi)$$
$$= (N/2-1)!(\xi/2)^{(2-N)/2}(1/\pi)\int_0^\pi \cos[(N-2)\varphi/2 - \xi\sin\varphi]\ d\varphi, \quad (n=1;\ N=2,4,6,...), \tag{2.3.21}$$

$$\theta = (N-2)(N-4)...5\times3\times1\times(\pi/2)^{1/2}\xi^{(2-N)/2}J_{(N-2)/2}(\xi)$$
$$= (N-2)(N-4)...5\times3\times1\times(-1)^{(N-3)/2}\ d^{(N-3)/2}(\sin\xi/\xi)/(\xi\ d\xi)^{(N-3)/2},$$
$$(n=1;\ N=3,5,7,...), \tag{2.3.22}$$

where we have used the familiar notation from the theory of Bessel functions

$$d^k f(\xi)/(\xi\ d\xi)^k = (1/\xi)\ d[d^{k-1}f(\xi)/(\xi\ d\xi)^{k-1}]/d\xi, \qquad (k=1,2,3,...). \tag{2.3.23}$$

If $N = 3$, the *general* solution (2.3.10) of Eq. (2.3.6) may be written under the form (2.8.83), (Chandrasekhar 1939).

The case $N = 1$, $(\nu = -1/2)$ needs special discussion. The solution of Eq. (2.3.16) is given by Eq. (2.3.10), where $\nu = (N-2)/2 = -1/2$:

$$u(\xi) = C_1J_{-1/2}(\xi) + C_2J_{1/2}(\xi), \qquad (n=1;\ N=1). \tag{2.3.24}$$

Using Eq. (2.3.15) and $J_{1/2}(\xi) = (2/\pi\xi)^{1/2}\sin\xi$, $J_{-1/2}(\xi) = (2/\pi\xi)^{1/2}\cos\xi$, we find for the solution of Eq. (2.3.6):

$$\theta = \xi^{1/2}\big[C_1J_{-1/2}(\xi) + C_2J_{1/2}(\xi)\big] = (2/\pi)^{1/2}(C_1\cos\xi + C_2\sin\xi). \tag{2.3.25}$$

With the initial conditions $\theta(0) = 1$, $\theta'(0) = 0$ we get $C_1 = (\pi/2)^{1/2} = \Gamma(1/2)/2^{1/2}$ and $C_2 = 0$ via (2.3.19). Therefore

$$\theta = \cos\xi = (\pi\xi/2)^{1/2}J_{-1/2}(\xi), \qquad (n=1;\ N=1). \tag{2.3.26}$$

Of course, this result can be obtained much easier by direct integration of Eq. (2.3.6), (cf. Harrison and Lake 1972). Thus, by virtue of Eqs. (2.3.22) and (2.3.26) – when N equals an odd integer – the solution of the Lane-Emden equation (2.1.14) for $n = 1$ can always be expressed in closed form with the aid of trigonometric functions.

2.3.3 Schuster-Emden Integral $n = (N+2)/(N-2)$

In this case Eq. (2.2.26) takes the simple form

$$d^2z/dt^2 + 2[N + n(2-N)]z/(n-1)^2 \pm B^{n-1}z^n = 0, \qquad (n \neq \pm1, \pm\infty). \tag{2.3.27}$$

This equation can be simplified further, if we take the arbitrary constant B from Emden's transformation (2.2.22) equal to

$$B^{n-1} = -2[N + n(2-N)]/(n-1)^2 = 4/(n-1)^2, \qquad [N = 2(n+1)/(n-1)]. \tag{2.3.28}$$

Eq. (2.3.27) becomes

$$d^2z/dt^2 = 4(z \mp z^n)/(n-1)^2. \tag{2.3.29}$$

After multiplication by dz/dt this equation can be integrated:

$$(dz/dt)^2/2 = 4\big[z^2/2 \mp z^{n+1}/(n+1)\big]/(n-1)^2 + C. \tag{2.3.30}$$

We turn back to the original ξ, θ-variables by using Emden's transformations (2.2.22), (2.2.25):

$$\xi = \exp(-t); \quad d\xi/dt = -\xi; \quad z = \xi^{2/(n-1)}\theta/B;$$
$$dz/dt = (dz/d\xi)\ d\xi/dt = -(\xi^{2/(n-1)}/B)\big[2\theta/(n-1) + \xi\theta'\big]$$
$$= -[(n-1)\xi/2]^{2/(n-1)}[2\theta/(n-1) + \xi\theta'], \qquad (B = [2/(n-1)]^{2/(n-1)}). \tag{2.3.31}$$

Substitution of Eq. (2.3.31) into Eq. (2.3.30) shows that with the initial conditions $\theta(0) = 1,\ \theta'(0) = 0$ we have $z = 0,\ dz/dt = 0$ if $\xi = 0$ and $n > 1$. Consequently $C = 0$ if $n > 1$. From the initial assumption $n = (N+2)/(N-2)$ follows that $N > 2$ if $n > 1$. Thus, the plane-symmetrical and the cylindrical case ($N = 1, 2$) of the Schuster-Emden integral need special discussion, and solution (2.3.36) is valid only if $N = 3, 4, 5, ...$ Then $C = 0$, and Eq. (2.3.30) becomes after some algebra via Eq. (2.3.31):

$$2\theta\theta'/(n-1) + \xi\theta'^2/2 + \xi\theta^{n+1}/(n+1) = 0, \quad (n = (N+2)/(N-2) > 1;\ N = 3, 4, 5, ...). \tag{2.3.32}$$

This equation can be written under the equivalent form

$$2d\xi/(n-1)\xi + d\theta/2\theta - d(\xi^{N-1}\theta')/(n+1)\xi^{N-1}\theta' = 0, \tag{2.3.33}$$

where we have replaced θ^n with $-\xi^{1-N}\ d(\xi^{N-1}\theta')/d\xi$ in virtue of the Lane-Emden equation (2.1.14). Integration of Eq. (2.3.33) yields

$$\xi\theta^{(n+1)/2}/\theta' = C. \tag{2.3.34}$$

To determine the new integration constant C, we use the series expansion of θ' from Eq. (2.4.21): $\theta' \approx -\xi/N,\ (\xi \approx 0;\ n > -1)$. We obtain $C = -N = -2(n+1)/(n-1)$. After a further elementary integration of Eq. (2.3.34) we find eventually

$$2\theta^{(1-n)/2}/(1-n) = (1-n)\xi^2/4(1+n) + D. \tag{2.3.35}$$

From the initial condition $\theta(0) = 1$ follows $D = 2/(1-n)$, and (Chandrasekhar 1939, Kimura and Liu 1978, Abramowicz 1983, Horedt 1986a):

$$\theta = [1 + (n-1)^2\xi^2/8(n+1)]^{2/(1-n)} = [1 + \xi^2/N(N-2)]^{(2-N)/2},$$
$$(n = (N+2)/(N-2) > 1;\ N = 3, 4, 5, ...). \tag{2.3.36}$$

The case $n = 5,\ N = 3$ is also known as the Plummer model (Eq. (2.3.90), Binney and Tremaine 1987).

Srivastava's Solution if $N = 3,\ n = 5$. Srivastava (1962) has given an additional solution in closed form, that may be applied to construct composite polytropic models (Sec. 2.8.1, Murphy 1980a, 1981). The assumption $C \neq 0$ in Eq. (2.3.30) generally involves elliptic integrals (Chandrasekhar 1939), but for the particular choice $C = 1/12$ the right-hand side of Eq. (2.3.30) can be factorized:

$$(dz/dt)^2 = -z^6/12 + z^2/4 + 1/6 = (z^2+1)^2(2-z^2)/12, \qquad (N = 3;\ n = 5). \tag{2.3.37}$$

With the substitution $z = \tan\chi$, Eq. (2.3.37) assumes the form

$$12^{1/2}\cos\chi\ d\chi/(2 - 3\sin^2\chi)^{1/2} = \pm dt. \tag{2.3.38}$$

Integration yields

$$2\arcsin[(3/2)^{1/2}\sin\chi] = \pm t + C \quad \text{or} \quad \sin\chi = (2/3)^{1/2}\sin[(\pm t + C)/2], \quad (C = \text{const}). \tag{2.3.39}$$

From Eq. (2.3.28) we obtain for the constant B

$$B = \pm 2^{-1/2}, \qquad (N = 3;\ n = 5), \tag{2.3.40}$$

and the Emden transformation (2.3.31) reads

$$\theta = \pm(2\xi)^{-1/2}z = \pm(2\xi)^{-1/2}\tan\chi = \pm(2\xi)^{-1/2}\sin\chi/(1 - \sin^2\chi)^{1/2}. \tag{2.3.41}$$

Eq. (2.3.41) transforms into Srivastava's (1962) solution by using Eqs. (2.3.31), (2.3.39):

$$\theta = \pm \sin(\ln \xi^{1/2} + D)/\xi^{1/2}\big[3 - 2\sin^2(\ln \xi^{1/2} + D)\big]^{1/2}, \qquad (D = \text{const};\ N = 3;\ n = 5). \qquad (2.3.42)$$

Because of the sine function this solution has an oscillatory behaviour, with the amplitudes growing indefinitely as $\xi \to 0$. We have $\theta \to \pm\infty$ if $\xi \to 0$, and $\theta \to 0$ if $\xi \to \infty$. The zeros of Srivastava's integral occur at $\xi = \exp[2(k\pi - D)]$, where k is zero or an integer. Eq. (2.3.42) has some practical significance only when $\theta > 0$ (Eq. (2.8.50), Murphy 1980a, 1981, 1983b, Murphy and Fiedler 1985a, b).

We now discuss the cases $N = 1, 2$, left over from the preceding derivation.

Case $N = 2$. In this case $n = (N + 2)/(N - 2) = \infty$, and Eq. (2.2.33) takes the simple form

$$d^2z/dt^2 = -\exp z, \qquad (n = \pm\infty;\ N = 2). \qquad (2.3.43)$$

Exactly as for Eq. (2.3.29) we find after multiplication with dz/dt and integration:

$$(dz/dt)^2/2 = -\exp z + C. \qquad (2.3.44)$$

With Eqs. (2.2.25) and (2.2.30) we turn back to the original (ξ, θ)-variables:

$$\xi^2(2/\xi - \theta')^2/2 = -\xi^2 \exp(-\theta) + C. \qquad (2.3.45)$$

With the initial conditions $\xi,\ \theta(0),\ \theta'(0) = 0$, we find $C = 2$. If we replace $\exp(-\theta)$ by $\xi^{-1}\ d(\xi\theta')/d\xi$ via the Lane-Emden equation (2.1.21), the preceding equation can be written under the equivalent form

$$-2d\xi/\xi + d\theta/2 + d(\xi\theta')/\xi\theta' = 0, \qquad (n = \pm\infty;\ N = 2), \qquad (2.3.46)$$

which integrates to give

$$\ln(\theta'/\xi) = -\theta/2 + C. \qquad (2.3.47)$$

From Eq. (2.4.36) we have $\theta' \approx \xi/N,\ (\xi \approx 0,\ n = \pm\infty)$, and the new integration constant becomes $C = -\ln 2$, by using the initial conditions at $\xi = 0$. A further integration of Eq. (2.3.47) yields the final result (Stodólkiewicz 1963, Ostriker 1964a):

$$\theta = 2\ln(1 + \xi^2/8), \qquad (n = \pm\infty;\ N = 2). \qquad (2.3.48)$$

Case $N = 1$. In this case we would have $n = (N + 2)/(N - 2) = -3$, but already in the original Lane-Emden equation the θ'-term is missing, as in Eqs. (2.3.27) or (2.3.43). Consequently, there is no need to turn to the Emden variables t and z. We have

$$\theta'' = \mp\theta^n, \qquad (n \neq -1, \pm\infty;\ N = 1), \qquad (2.3.49)$$

and

$$\theta'' = \exp(-\theta), \qquad (n = \pm\infty;\ N = 1). \qquad (2.3.50)$$

After multiplication with θ', Eq. (2.3.49) can be integrated in the same way as Eq. (2.3.29):

$$\theta'^2/2 = \mp\theta^{n+1}/(n + 1) + C, \qquad (C = \text{const}). \qquad (2.3.51)$$

With the initial conditions $\theta(0) = 1,\ \theta'(0) = 0$ we get

$$\theta' = \mp\big\{2(1 - \theta^{n+1})/[\pm(n + 1)]\big\}^{1/2}, \qquad (n \neq -1, \pm\infty;\ N = 1), \qquad (2.3.52)$$

where we have used the fact that $\theta' \leq 0,\ \theta \leq 1$ if $-1 < n < \infty$, and $\theta' \geq 0,\ \theta \geq 1$ if $-\infty < n < -1$ (cf. Sec. 2.7). If $\theta^{n+1} = 0$, Eq. (2.3.52) yields an important analytical expression for the value of the derivative at the finite or infinite boundary of the slab [cf. Eq. (2.4.67)]. We integrate Eq. (2.3.52) further

$$\xi = \mp\int_1^\theta [\pm(n + 1)/2(1 - \tau^{n+1})]^{1/2}\ d\tau, \qquad (n \neq -1, \pm\infty;\ N = 1). \qquad (2.3.53)$$

With the substitution

$$t = 1 - \tau^{n+1}, \tag{2.3.54}$$

Eq. (2.3.53) can be brought into the form of an incomplete beta function (Harrison and Lake 1972):

$$\xi = [\pm 2(n+1)]^{-1/2} \int_0^T t^{-1/2}(1-t)^{-n/(n+1)}\, dt, \qquad (n \neq -1, \pm\infty;\ N = 1;\ T = 1 - \theta^{n+1}). \tag{2.3.55}$$

The incomplete beta function (Smirnow 1967, Spiegel 1968)

$$B_T(p,q) = \int_0^T t^{p-1}(1-t)^{q-1}\, dt, \qquad (p, q > 0;\ 0 \leq T \leq 1), \tag{2.3.56}$$

is originally defined only for positive exponents p and q. If $p > 0$ and $q \leq 0$, the incomplete beta function becomes infinite merely for the upper limit $T = 1$ of the integration interval. Thus, we can extend the definition of the incomplete beta function (2.3.56) to

$$B_T(p,q) = \int_0^T t^{p-1}(1-t)^{q-1}\, dt, \qquad (p > 0;\ 0 \leq T \leq 1), \tag{2.3.57}$$

with the understanding that $B_T(p,q) = \infty$ if $T = 1$ and $q \leq 0$. Using the above definition we write instead of Eq. (2.3.55)

$$\xi = [\pm 2(n+1)]^{-1/2} B_T[1/2, 1/(n+1)], \qquad (n \neq -1, \pm\infty;\ N = 1;\ 0 \leq T \leq 1). \tag{2.3.58}$$

ξ from Eq. (2.3.55) or (2.3.58) becomes infinite only when $n < -1$ and $T = 1 - \theta^{n+1} = 1$, in full accordance with the general behaviour of plane-symmetric solutions (cf. Sec. 2.7).

The integrand from Eq. (2.3.55) can be brought into a simpler form (Kimura and Liu 1978)

$$\xi = \{2/[\pm(n+1)]\}^{1/2} \int_0^X (1-x^2)^{-n/(n+1)}\, dx, \qquad (n \neq -1, \pm\infty;\ N = 1;\ 0 \leq X \leq 1), \tag{2.3.59}$$

with the transformation of variables

$$x = t^{1/2} = (1 - \tau^{n+1})^{1/2}; \quad X = (1 - \theta^{n+1})^{1/2}. \tag{2.3.60}$$

From Eq. (2.3.59) it is obvious that solutions in closed form of Eq. (2.3.49) exist whenever $-n/(n+1) = k/2$, where k is an integer or zero. In this case the integrand is of the form $F[(1-x^2)^{1/2}]$ and can be integrated by standard methods, where F is a rational function of the argument $(1-x^2)^{1/2}$. For instance, when k takes values between -4 and 4, the corresponding values of n for which Eq. (2.3.49) can be solved in closed form are $n = -2, -3, \pm\infty, 1, 0, -1/3, -1/2, -3/5, -2/3$, respectively. For instance, if $n = 1$, we recover at once from Eq. (2.3.59) the solution $\theta = \cos\xi$, obtained previously in Eq. (2.3.26). If a solution $\xi = \xi(\theta)$ is available in closed form, we can find $\theta^n(\xi) = \mp d^2\theta(\xi)/d\xi^2$ with the relationships for the derivative of the inverse function:

$$d\theta(\xi)/d\xi = 1/[d\xi(\theta)/d\theta]; \quad d^2\theta(\xi)/d\xi^2 = -[d^2\xi(\theta)/d\theta^2]/[d\xi(\theta)/d\theta]^3. \tag{2.3.61}$$

A closed solution of the Lane-Emden equation (2.3.50) exists if $n = \pm\infty$. Multiplication by θ' and subsequent integration yields $(\theta(0), \theta'(0) = 0)$

$$\theta' = 2^{1/2}[1 - \exp(-\theta)]^{1/2}, \qquad (n = \pm\infty;\ N = 1). \tag{2.3.62}$$

With the substitution $x = [1 - \exp(-\theta)]^{1/2}$, we find

$$d\xi = 2^{1/2} \int_0^x dx/(1-x^2), \tag{2.3.63}$$

which can be integrated to give

$$\xi = 2^{-1/2} \ln[(1+x)/(1-x)] + C. \tag{2.3.64}$$

Since $x = 0$ if $\xi = 0$, the integration constant C turns out to be zero, and by moving back to the original θ-variable, we find (Spitzer 1942, Harrison and Lake 1972)

$$\exp(-\theta) = 4/[\exp(\xi/2^{1/2}) + \exp(-\xi/2^{1/2})]^2 = 1/\cosh^2(\xi/2^{1/2}), \quad (n = \pm\infty;\ N = 1). \tag{2.3.65}$$

Thus, the intimate connection becomes obvious between the planar case $N = 1$ if $n \neq -1$, the cylindrical case $N = 2$ if $n = \pm\infty$, and the general case $N \geq 3$ if $n = (N+2)/(N-2)$: In all these cases the Lane-Emden equation can be brought to an integrable form without first order derivatives.

2.3.4 Singular Solution

(i) $n \neq \pm 1, \pm\infty$. Let us rewrite Emden's equation (2.2.29) under the form

$$dy/\{[2+N+n(2-N)]y/(n-1)+2[N+n(2-N)]z/(n-1)^2 \pm B^{n-1}z^n\} = -dz/y,\quad (n \neq \pm 1, \pm\infty). \tag{2.3.66}$$

The singular points of a first order differential equation, written under the form

$$dy/P(y,z) = dz/Q(y,z), \tag{2.3.67}$$

are given by the solutions of the system

$$P(y,z) = 0; \quad Q(y,z) = 0. \tag{2.3.68}$$

Thus, the singular points are found from Eq. (2.3.66) to be

$$y_s = 0; \quad z_s = 0 \qquad \text{and} \qquad y_s = 0; \quad z_s = \{\mp 2[N+n(2-N)]/(n-1)^2\}^{1/(n-1)}/B. \tag{2.3.69}$$

The first singular point $y_s = z_s = 0$ is trivial in this context, but the second singular point leads to a singular solution in the ξ, θ-variables, as defined by Eq. (2.2.22):

$$\theta(\xi) = B\xi^{2/(1-n)}z_s = \{\mp 2[N+n(2-N)]/(n-1)^2\xi^2\}^{1/(n-1)}, \qquad (n \neq \pm 1, \pm\infty). \tag{2.3.70}$$

Since z_s, $(z_s \neq 0)$ must be a real number, there results the important additional constraint

$$\mp[N+n(2-N)] > 0 \quad \text{or} \quad n(N-2) \gtrless N, \qquad (n \neq \pm 1, \pm\infty). \tag{2.3.71}$$

As familiar, the upper sign holds if $-1 < n < \infty$, the lower one if $-\infty < n < -1$. No singular solution exists in the plane-symmetric case $N = 1$, since the condition (2.3.71) is not fulfilled. If $N \geq 2$, the singular solution (2.3.70) exists for polytropic indices obeying the constraint $-\infty < n < -1$. In addition, if $N \geq 3$, singular solutions exist also for polytropic indices obeying the inequality $N/(N-2) < n < \infty$.

(ii) $n = \pm\infty$. We rewrite Emden's equation (2.2.34) under the form

$$dy/[(2-N)y + \exp z + 2(2-N)] = -dz/y, \qquad (n = \pm\infty), \tag{2.3.72}$$

to obtain the singular point

$$y_s = 0; \quad z_s = \ln[2(N-2)], \qquad (n = \pm\infty;\ N \geq 3). \tag{2.3.73}$$

The singular solution in the (ξ, θ)-variables is by virtue of Eq. (2.2.30) equal to

$$\theta = \ln[\xi^2/2(N-2)]; \quad \varrho = 2\varrho_0(N-2)/\xi^2, \qquad (n = \pm\infty;\ N \geq 3). \tag{2.3.74}$$

No singular solutions exist if $n = \pm\infty$, $N = 1, 2$.

The singular solutions (2.3.70) and (2.3.74) do not obey the initial conditions (2.1.41) for the Lane-Emden functions. As will be obvious from the next section, the singular solutions of the Lane-Emden equation are important for the study of the asymptotic behaviour of solutions when $\xi \to \infty$.

So far, we have considered in this section all cases for which the Lane-Emden equations (2.1.14) and (2.1.21), or their Emden transformations (2.2.26) and (2.2.33), can be brought to simpler differential equations, integrable by standard methods. Below, we summarize the analytic solutions.

Polytropic Slabs, $N = 1$. The Lane-Emden equation

$$\theta'' = \mp\theta^n, \qquad (n \neq -1, \pm\infty), \tag{2.3.75}$$

has the solution

$$\xi = \mp\int_1^\theta [\pm(n+1)/2(1-\tau^{n+1})]^{1/2}\, d\tau, \qquad (n \neq -1, \pm\infty), \tag{2.3.76}$$

that can be integrated in closed form whenever $2n/(n+1)$ is an integer or zero. For instance, if

$$n = 0: \; \theta = 1 - \xi^2/2, \tag{2.3.77}$$

$$n = 1: \; \theta = \cos\xi = (\pi\xi/2)^{1/2} J_{-1/2}(\xi). \tag{2.3.78}$$

If $n = \pm\infty$, the Lane-Emden equation

$$\theta'' = \exp(-\theta), \qquad (n = \pm\infty), \tag{2.3.79}$$

has the solution

$$\theta = \ln[\cosh^2(\xi/2^{1/2})], \qquad (n = \pm\infty). \tag{2.3.80}$$

No singular solutions exist for polytropic slabs.

Polytropic Cylinders, $N = 2$. The Lane-Emden equation

$$\theta'' + \theta'/\xi = \mp\theta^n, \qquad (n \neq -1, \pm\infty), \tag{2.3.81}$$

has solutions in closed form if

$$n = 0: \; \theta = 1 - \xi^2/4, \tag{2.3.82}$$

$$n = 1: \; \theta = J_0(\xi) = \sum_{k=0}^{\infty} (-1)^k (\xi/2)^{2k}/(k!)^2. \tag{2.3.83}$$

There exists the singular solution

$$\theta = [4/(n-1)^2\xi^2]^{1/(n-1)}, \qquad (-\infty < n < -1). \tag{2.3.84}$$

The Lane-Emden equation

$$\theta'' + \theta/\xi = \exp(-\theta), \qquad (n = \pm\infty), \tag{2.3.85}$$

has the solution

$$\theta = 2\ln(1 + \xi^2/8), \qquad (n = \pm\infty). \tag{2.3.86}$$

Polytropic Spheres, $N = 3$. The Lane-Emden equation

$$\theta'' + 2\theta'/\xi = \mp\theta^n, \qquad (n \neq -1, \pm\infty), \tag{2.3.87}$$

has solutions in closed form if

$$n = 0: \; \theta = 1 - \xi^2/6, \tag{2.3.88}$$

$$n = 1: \; \theta = \sin\xi/\xi = (\pi/2\xi)^{1/2} J_{1/2}(\xi), \tag{2.3.89}$$

$$n = 5: \; \theta = (1 + \xi^2/3)^{-1/2}. \tag{2.3.90}$$

Srivastava's (1962) integral

$$\theta = \pm\sin(\ln\xi^{1/2} + D)/\xi^{1/2}\big[3 - 2\sin^2(\ln\xi^{1/2} + D)\big]^{1/2}, \qquad (n = 5;\; D = \mathrm{const}), \tag{2.3.91}$$

does not obey the usual initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$.

There exist the singular solutions

$$\theta = [2(3-n)/(n-1)^2\xi^2]^{1/(n-1)}, \qquad (-\infty < n < -1), \tag{2.3.92}$$

and

$$\theta = [2(n-3)/(n-1)^2\xi^2]^{1/(n-1)}, \qquad (3 < n < \infty). \tag{2.3.93}$$

The Lane-Emden equation

$$\theta'' + 2\theta'/\xi = \exp(-\theta), \qquad (n = \pm\infty), \tag{2.3.94}$$

has the singular solution

$$\theta = \ln(\xi^2/2), \qquad (n = \pm\infty). \tag{2.3.95}$$

2.4 Approximate Analytical Solutions

2.4.1 Series Expansion of Lane-Emden Functions near an Interior Point

If not stated explicitly otherwise, we will only be concerned with solutions of the Lane-Emden equations (2.1.14) or (2.1.21) satisfying the constraints

$$\xi \geq 0; \quad 0 \leq \theta(\xi) \leq \infty. \tag{2.4.1}$$

Mostly these solutions are of practical interest, because according to Eqs. (2.1.10), (2.1.13), (2.1.18), (2.1.20) we have $r \propto \xi,\ \varrho \propto \theta^n,\ P \propto \theta^{n+1}$, or $r \propto \xi,\ \varrho \propto \exp(-\theta),\ P \propto \exp(-\theta)$, the pressure P being a decreasing function as the radius r increases. Series solutions are important especially as a starting point for numerical integrations, and for the elucidation of the general topology of the Lane-Emden equation. A Taylor series expansion can be provided for intervals of ξ where the Lane-Emden function θ is continuous. Quite generally, we are seeking a power series solution of the Lane-Emden equation in the neighborhood of the initial conditions $\theta_0 = \theta(\xi_0)$ and $\theta'_0 = (d\theta/d\xi)_{\xi=\xi_0}$. The general convergence of series solutions for the Lane-Emden equation will be touched subsequently to Eq. (2.4.39).

(i) $n \neq -1, \pm\infty$. If $\xi_0 \neq \infty$, we are seeking a solution of the Lane-Emden equation under the form (Mohan and Al-Bayaty 1980)

$$\theta = \sum_{k=0}^{\infty} a_k (\xi - \xi_0)^k. \tag{2.4.2}$$

Raising a power series to a real power yields another power series

$$\theta^n = \Big(\sum_{i=0}^{\infty} a_i x^i\Big)^n = \sum_{k=0}^{\infty} c_k x^k, \qquad (x = \xi - \xi_0), \tag{2.4.3}$$

with the coefficients (e.g. Gradshteyn and Ryzhik 1965, Seidov 1979)

$$c_0 = a_0^n; \quad c_k = (1/ka_0)\sum_{i=1}^{k}(-k + i + in)a_i c_{k-i}, \qquad (k \geq 1;\ a_0 \neq 0). \tag{2.4.4}$$

We insert Eq. (2.4.2) into the Lane-Emden equation (2.1.14) written under the from

$$(\xi - \xi_0)\theta'' + \xi_0\theta'' + (N-1)\theta' \pm (\xi - \xi_0)\theta^n \pm \xi_0\theta^n = 0, \qquad (n \neq -1, \pm\infty), \tag{2.4.5}$$

to obtain

$$\sum_{k=0}^{\infty}\Big[(k+2)(k+1)a_{k+2}(\xi - \xi_0)^{k+1} + (k+2)(k+1)a_{k+2}\xi_0(\xi - \xi_0)^k$$
$$+(N-1)(k+1)a_{k+1}(\xi - \xi_0)^k \pm c_k(\xi - \xi_0)^{k+1} \pm c_k\xi_0(\xi - \xi_0)^k\Big] = 0. \tag{2.4.6}$$

Equating equal powers of $\xi - \xi_0$, we get the identities, $(\xi_0, \theta_0 \neq 0)$

$$a_2 = [\mp c_0\xi_0 - (N-1)a_1]/2\xi_0, \qquad (k = 0), \tag{2.4.7}$$

$$a_{k+2} = \big[\mp (c_k\xi_0 + c_{k-1}) - (k+1)(N+k-1)a_{k+1}\big]/(k+1)(k+2)\xi_0, \qquad (k \geq 1), \tag{2.4.8}$$

with the c_k's given by Eq. (2.4.4). To determine the coefficients a_k completely, we need a_0 and a_1, which are obtained from Eq. (2.4.2) with the initial conditions θ_0 and θ'_0 :

$$a_0 = \theta_0; \quad a_1 = \theta'_0; \quad c_0 = a_0^n = \theta_0^n. \tag{2.4.9}$$

To get the series expansion near the origin $\xi_0 = 0$, it is advisable to start ex novo in this important particular case. The series

$$\theta = \sum_{k=0}^{\infty} a_k \xi^k, \qquad (\xi_0 = 0), \tag{2.4.10}$$

is inserted into Eq. (2.1.14):

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2}\xi^{k+1} + (N-1)(k+1)a_{k+1}\xi^k \pm c_k\xi^{k+1}\right] = 0. \tag{2.4.11}$$

Equating equal powers, we find

$$c_k = \mp(k+2)(k+N)a_{k+2}, \qquad (k \geq 0). \tag{2.4.12}$$

This equation is introduced into Eq. (2.4.4) to obtain the recurrence formula

$$a_{k+2} = \left[1/k(k+2)(k+N)a_0\right] \sum_{i=1}^{k}(-k+i+in)(k-i+2)(k-i+N)a_i a_{k-i+2}, \qquad (k \geq 1). \tag{2.4.13}$$

For the first two coefficients of the series (2.4.10) we get with the usual initial conditions (2.1.41)

$$a_0 = \theta_0 = \theta(0) = 1; \quad a_1 = \theta_0' = (d\theta/d\xi)_{\xi=0} = 0, \tag{2.4.14}$$

and from Eqs. (2.4.4), (2.4.12), if $k = 0$:

$$a_2 = \mp c_0/2N = \mp a_0^n/2N = \mp 1/2N. \tag{2.4.15}$$

Eqs. (2.4.13)-(2.4.15) completely determine the coefficients of the expansion of Lane-Emden functions near $\xi = 0$. Since $a_1 = 0$, we observe from Eq. (2.4.13) that all coefficients with odd indices are zero: Let $k = 2p-1$, (p – natural number), then either a_i or a_{2p-i+1} will be odd indexed and consequently zero. Hence, we can write instead of Eq. (2.4.10) the equivalent series

$$\theta = \sum_{k=0}^{\infty} b_{2k}\xi^{2k}, \tag{2.4.16}$$

which yields the coefficients (Seidov and Kuzakhmedov 1977, Seidov et al. 1979)

$$b_0 = 1; \quad b_2 = \mp 1/2N; \quad b_{2k+2} = [1/k(k+1)(2k+N)b_0]$$
$$\times \sum_{i=1}^{k}(-k+i+in)(k-i+1)(2k-2i+N)b_{2i}b_{2k-2i+2}, \qquad (k \geq 1). \tag{2.4.17}$$

These coefficients can be obtained simply from Eq. (2.4.13) with the transformation $k \to 2k,\ i \to 2i$.

In the particular case $n = 2$, Seidov (1979) obtains a much simpler recurrence formula than Eq. (2.4.13), by observing that from the equation

$$\theta^2 = \left(\sum_{i=0}^{\infty} b_{2i}\xi^{2i}\right)^2 = \sum_{k=0}^{\infty} c_{2k}\xi^{2k}, \tag{2.4.18}$$

we obtain

$$c_{2k} = \sum_{i=0}^{k} b_{2i}b_{2k-2i}, \qquad (n = 2), \tag{2.4.19}$$

and after substituting into Eq. (2.1.14), and equating equal powers of ξ :

$$b_{2k+2} = \mp\left[1/(2k+2)(2k+N)b_0\right] \sum_{i=0}^{k} b_{2i}b_{2k-2i}, \qquad (n = 2;\ k \geq 0). \tag{2.4.20}$$

Below, we write down the first six terms of the series (2.4.16), providing a good approximation to the Lane-Emden functions as long as $\xi \ll 1$ (cf. Seidov et al. 1979):

$$\theta \approx 1 \mp (1/2N)\xi^2 + [n/2^3N(N+2)]\xi^4 \mp \{[2n^2(N+1) - n(N+2)]/2^4 \times 3N^2(N+2)(N+4)\}\xi^6$$
$$+\{[n^3(6N^2+20N+8) + n^2(-7N^2-32N-24) + n(2N^2+12N+16)]$$
$$/2^7 \times 3N^3(N+2)(N+4)(N+6)\}\xi^8 \mp \{[n^4(24N^4+220N^3+612N^2+536N+96)$$
$$+n^3(-46N^4-488N^3-1632N^2-1920N-576) + n^2(29N^4+354N^3+1428N^2+2200N+1056)$$
$$+n(-6N^4-84N^3-408N^2-816N-576)]/2^8 \times 3 \times 5N^4(N+2)^2(N+4)(N+6)(N+8)\}\xi^{10}$$
$$+..., \qquad (n \neq -1, \pm\infty;\ N = 1,2,3,...). \tag{2.4.21}$$

For the plane-symmetrical, cylindrical, and spherical polytropes we get if $\xi \ll 1$:

$$N = 1: \quad \theta \approx 1 \mp (1/2!)\xi^2 + (n/4!)\xi^4 \mp [n(4n-3)/6!]\xi^6 + [n(34n^2-63n+30)/8!]\xi^8$$
$$\mp[n(496n^3-1554n^2+1689n-630)/10!]\xi^{10} + ... \tag{2.4.22}$$

$$N = 2: \quad \theta \approx 1 \mp [1/(2^1 \times 1!)^2]\xi^2 + [n/(2^2 \times 2!)^2]\xi^4 \mp [n(3n-2)/(2^3 \times 3!)^2]\xi^6$$
$$+[n(18n^2-29n+12)/(2^4 \times 4!)^2]\xi^8 \mp [n(180n^3-487n^2+452n-144)/(2^5 \times 5!)^2]\xi^{10} + ... \tag{2.4.23}$$

$$N = 3: \quad \theta \approx 1 \mp (1/3!)\xi^2 + (n/5!)\xi^4 \mp [n(8n-5)/3 \times 7!]\xi^6 + [n(122n^2-183n+70)/9 \times 9!]\xi^8$$
$$\mp[n(5032n^3-12642n^2+10805n-3150)/45 \times 11!]\xi^{10} + ... \tag{2.4.24}$$

We have noted the erroneous expansions of Abramowicz [1983, Eq. (7)], of Ibáñez and Sigalotti (1984, Eq. (14), $N = 1$), and the error occurring in the equation written by Ostriker (1964a), and Viala and Horedt (1974b) for the last coefficient in Eq. (2.4.23), where the numerator should read $n(180n^3 - 487n^2 + 452n - 144)$ instead of $n(180n^3 - 505n^2 + 470n - 144)$.

The spherical Lane-Emden functions of most practical interest can also be calculated with a series expansion of the form

$$\theta \approx \sum_{i=0}^{I} a_i[\xi/(\xi + 2^{5-n})]^i, \qquad (I = 14;\ a_i = \mathrm{const};\ N = 3;\ 0 < n < 5), \tag{2.4.25}$$

to a precision better than 10^{-6}, where 2^{5-n} is an empirically found number, and the coefficients a_i have been tabulated by Service (1977) for the polytropic indices $n = 1.5, 2, 2.5, 3, 3.25, 3.5, 4, 4.5$.

Other approximations to polytropic spheres near the origin have been proposed by Fowler and Hoyle [1964, Eq. (C.79)]

$$\theta \approx [1 + (n/120 - 1/72)\xi^4]\exp(-\xi^2/6) \approx \exp(-\xi^2/6), \qquad (\xi \approx 0;\ N = 3;\ -\infty < n < \infty), \tag{2.4.26}$$

and by Beech (1987)

$$\theta \approx [1 + (15/108 - n/20)\xi^4]/\cosh(\xi/3^{1/2}), \qquad (\xi \approx 0;\ N = 3;\ -\infty < n < \infty). \tag{2.4.27}$$

(ii) $n = \pm\infty$. The series (2.4.2) still holds. To obtain a useful expression of $\exp(-\theta)$ occurring on the right-hand side of Eq. (2.1.21), we employ the auxiliary function

$$\chi(\xi) = 1 - \theta(\xi)/n = 1 - \sum_{i=0}^{\infty} a_i(\xi - \xi_0)^i/n. \tag{2.4.28}$$

In virtue of Eq. (2.4.4) we have

$$\chi^n = (1 - \theta/n)^n = \sum_{k=0}^{\infty} c_k(\xi - \xi_0)^k = \left[1 - \sum_{i=0}^{\infty} a_i(\xi - \xi_0)^i/n\right]^n, \tag{2.4.29}$$

with

$$c_0 = (1 - a_0/n)^n; \quad c_k = \big[1/k(1 - a_0/n)\big] \sum_{i=1}^{k} (-k + i + in)(-a_i/n)c_{k-i}, \qquad (k \geq 1). \tag{2.4.30}$$

Turning in Eq. (2.4.30) to the limit $n \to \pm\infty$, we get

$$c_0 = \exp(-a_0); \quad c_k = -(1/k) \sum_{i=1}^{k} i a_i c_{k-i}, \tag{2.4.31}$$

and Eq. (2.4.29) becomes

$$\lim_{n\to\pm\infty} \chi^n = \exp(-\theta) = \sum_{k=0}^{\infty} c_k(\xi - \xi_0)^k = \exp(-a_0) - \sum_{k=1}^{\infty} \left\{ \big[(\xi - \xi_0)^k/k\big] \sum_{i=1}^{k} i a_i c_{k-i} \right\}. \tag{2.4.32}$$

The coefficients a_k are obtained in the same way as in Eqs. (2.4.7)-(2.4.9):

$$a_0 = \theta_0; \quad a_1 = \theta_0'; \quad c_0 = \exp(-\theta_0); \quad a_2 = [c_0\xi_0 - (N-1)a_1]/2\xi_0;$$
$$a_{k+2} = [c_k\xi_0 + c_{k-1} - (k+1)(N+k-1)a_{k+1}]\big/(k+1)(k+2)\xi_0, \qquad (k \geq 1;\ \xi_0 \neq 0). \tag{2.4.33}$$

If $\xi_0 = 0$, we find in the same manner as in Eqs. (2.4.12)-(2.4.15):

$$c_k = (k+2)(k+N)a_{k+2}, \qquad (n = \pm\infty;\ k \geq 0), \tag{2.4.34}$$

and

$$a_0 = \theta(0) = \theta_0 = 0; \quad a_1 = \theta'(0) = \theta_0' = 0; \quad c_0 = \exp(-a_0) = 1; \quad a_2 = c_0/2N = 1/2N;$$
$$a_{k+2} = -[1/k(k+2)(k+N)] \sum_{i=1}^{k} i(k-i+2)(k-i+N)a_i a_{k-i+2}, \qquad (k \geq 1). \tag{2.4.35}$$

According to Eq. (2.4.35) the first five terms of the expansion near $\xi = 0$ are

$$\theta \approx (1/2N)\xi^2 - [1/2^3N(N+2)]\xi^4 + [(N+1)/2^3 \times 3N^2(N+2)(N+4)]\xi^6$$
$$-[(3N^2 + 10N + 4)/2^6 \times 3N^3(N+2)(N+4)(N+6)]\xi^8 + [(6N^4 + 55N^3 + 153N^2 + 134N + 24)$$
$$/2^6 \times 3 \times 5N^4(N+2)^2(N+4)(N+6)(N+8)]\xi^{10} - ..., \qquad (n = \pm\infty). \tag{2.4.36}$$

For the plane-symmetrical, cylindrical, and spherical case we obtain

$$N = 1: \ \theta \approx \xi^2/2! - \xi^4/4! + 4\xi^6/6! - 34\xi^8/8! + 496\xi^{10}/10! - ... \tag{2.4.37}$$

$$N = 2: \ \theta \approx \xi^2/(2^1 \times 1!)^2 - \xi^4/(2^2 \times 2!)^2 + 3\xi^6/(2^3 \times 3!)^2 - 18\xi^8/(2^4 \times 4!)^2$$
$$+180\xi^{10}/(2^5 \times 5!)^2 - ... \tag{2.4.38}$$

$$N = 3: \ \theta \approx \xi^2/3! - \xi^4/5! + 8\xi^6/3 \times 7! - 122\xi^8/9 \times 9! + 5032\xi^{10}/45 \times 11! - ... \tag{2.4.39}$$

Formally, these expansions can be obtained from Eqs. (2.4.21)-(2.4.24) by preserving in the coefficients of $\xi^2, \xi^4, ...\xi^{10}$ only the factor near the highest power of n.

In the spherical case the series expansions (2.4.16) of the analytical solutions $1 - \xi^2/6,\ (n = 0)$ and $\xi^{-1}\sin\xi,\ (n = 1)$ converge for any value of ξ (radius of convergence $= \infty$), while the series expansion of the Schuster-Emden integral $(1+\xi^2/3)^{-1/2},\ (n = 5)$ converges only if $\xi^2/3 \leq 1$ or $\xi \leq 3^{1/2}$. Therefore, we may expect that the radius of convergence of the series (2.4.16) decreases from ∞ to $3^{1/2}$ as n grows from 1 to 5, $(N = 3)$. This conjecture has been confirmed by the numerical work of Seidov and Kuzakhmedov (1977), and Seidov (1979) with the aid of the quotient criterion of D'Alembert: $\lim_{k\to\infty} |b_{2k}\xi^2/b_{2k-2}| < 1$ (Smirnow 1967). Over twenty years later Roxburgh and Stockman (1999, Table 1) reached the same conclusion. The radius of convergence extends up to the boundary of a polytropic sphere with index $0 \leq n \leq 5$, if the expansion variable ξ is replaced by a power $(-\xi^2\theta')^{2/3}$ of the dimensionless mass.

Hunter (2001) has clarified the analytical reasons for this strange behaviour of the convergency radius: It is due to a singularity of the form [cf. Eq. (2.4.42)]

$$\theta(\xi) \propto (\xi^2 - \xi_s^2)^{2/(1-n)} \to \infty, \qquad (\xi^2 \to \xi_s^2 < 0;\ \xi^2 > \xi_s^2;\ n > 1), \tag{2.4.40}$$

at the pure imaginary values $\xi = \pm i(-\xi_s^2)^{1/2}$, $(\xi^2 = \xi_s^2 < 0)$ of the radial coordinate. The series (2.4.16) ceases to converge before the surface of the polytropic sphere is reached if $n > 1.9121$, and it converges up to the surface (convergency radius $= \xi_1$) if $1 < n < 1.9121$. With the change of variable $\zeta = \xi^2$ the Lane-Emden equation (2.3.87) becomes

$$4\zeta\ d^2\theta/d\zeta^2 + 6\ d\theta/d\zeta = -\theta^n(\zeta), \qquad (n \neq -1, \pm\infty). \tag{2.4.41}$$

Close to a singularity of the form (2.4.40) there is $d^2\theta/d\zeta^2 \gg d\theta/d\zeta$, and integration of Eq. (2.4.41) yields with this approximation:

$$(d\theta/d\zeta)^2 \approx \theta^{n+1}/2(n+1)(-\zeta_s) + \mathrm{const} \approx \theta^{n+1}/2(n+1)(-\zeta_s) \quad \text{and}$$
$$\theta \approx [(n-1)^2/8(n+1)(-\zeta_s)]^{1/(1-n)}(\zeta - \zeta_s)^{2/(1-n)}$$
$$= [(n-1)^2(-\zeta_s)/8(n+1)]^{1/(1-n)}(1 - \zeta/\zeta_s)^{2/(1-n)}, \qquad (\zeta \approx \zeta_s < 0;\ \zeta > \zeta_s;\ n > 1). \tag{2.4.42}$$

The isothermal equation (2.3.94) turns into

$$4\zeta\ d^2\theta/d\zeta^2 + 6\ d\theta/d\zeta = \exp[-\theta(\zeta)], \qquad (n = \pm\infty), \tag{2.4.43}$$

which approximates to $4\zeta_s\ d^2\theta/d\zeta^2 \approx \exp(-\theta)$ near the singularity. This integrates to

$$2\zeta_s\ (d\theta/d\zeta)^2 \approx -\exp(-\theta) + \mathrm{const} \approx -\exp(-\theta) \quad \text{and} \quad \theta \approx \ln[(\zeta - \zeta_s)^2/8(-\zeta_s)],$$
$$(\zeta \approx \zeta_s < 0;\ n = \pm\infty). \tag{2.4.44}$$

The precise value of the movable singularity ζ_s depends on the initial conditions, and can be found by numerical integration of Eqs. (2.4.41) and (2.4.43) up to a point where the asymptotic relationships (2.4.42) and (2.4.44) are well established. For instance, Hunter (2001) obtains $\zeta_s = -\infty, -40.92, -15.72, -6.63, -3.00, -2.35, -10.72$ if n =1, 1.5, 2, 3, 5, 6, $\pm\infty$, respectively.

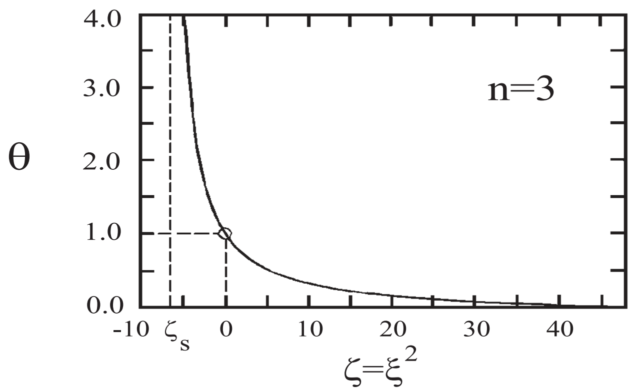

Fig. 2.4.1 The $n = 3$ polytrope for real and imaginary values of the radial coordinate $\xi = \pm\zeta^{1/2}$, with the singularity $\theta(\zeta_s) = \infty$ at $\zeta_s = \xi_s^2 = -6.63$ (Hunter 2001).

In the new variable Eq. (2.4.16) turns into

$$\theta(\zeta) = \sum_{k=0}^{\infty} \beta_k \zeta^k, \qquad (\beta_k = \text{const}). \tag{2.4.45}$$

If $\theta(\zeta)$ takes the form (2.4.42) at the singularity ζ_s which is closest to the origin $\zeta = 0$, then by Darboux's theorem the expansion coefficients β_k take for large k the form (Hunter 2001)

$$\beta_k \approx [(n-1)^2(-\zeta_s)/8(n+1)]^{1/(1-n)} k^{(n-3)/(1-n)}/\Gamma[2/(n-1)]\ \zeta_s^k, \qquad (k \to \infty), \tag{2.4.46}$$

with Γ denoting the gamma function (C.11). Application of d'Alembert's ratio criterion to the series (2.4.45) with the coefficients (2.4.46) requires for convergence that

$$\lim_{k\to\infty} |\beta_{k+1}\zeta/\beta_k| = |\zeta/\zeta_s| \lim_{k\to\infty} [(k+1)/k]^{(n-3)/(1-n)} \approx |\zeta/\zeta_s| < 1 \quad \text{or} \quad |\zeta_s| > |\zeta|. \tag{2.4.47}$$

And this rough evaluation shows that the series (2.4.16) and (2.4.45) converge up to the boundary $\theta(\zeta_1) = \theta(\xi_1^2) = 0$ only if $|\zeta_s| > \zeta_1$. If $n > 1.9121$, we have $|\zeta_s| < \zeta_1 = \xi_1^2$, and the two series become divergent beyond $\xi = |\xi_s| = |\zeta_s^{1/2}|$, as results from the radii of convergence determined by Seidov and Kuzakhmedov (1977), Seidov (1979), Roxburgh and Stockman (1999), and Hunter (2001): $|\xi_s| = |\zeta_s^{1/2}| = 4.19, 3.96, 3.06, 2.57, 2.26, 2.03, 3^{1/2}, 1.53$ if $n = 1.9, 2, 2.5, 3, 3.5, 4, 5, 6$, respectively. If $1 < n < 1.9121$, the radius of convergence is equal to the surface coordinate ξ_1. This is due to the fact that the surface coordinate ζ_1 is now the Darboux singularity ζ_s closest to the origin; the surface singularity of $\theta(\zeta)$ at ζ_1 results similarly to Eq. (2.4.61) from terms containing $(\zeta_1 - \zeta)^n$, which generally become complex if $\zeta > \zeta_1,\ (\zeta_1 > 0)$.

With an Euler transformation (Hunter 2001)

$$w = \zeta/(\zeta + \lambda) \quad \text{or} \quad \zeta = \lambda w/(1-w), \qquad (\lambda > 0), \tag{2.4.48}$$

the series (2.4.45) turns into

$$\theta(w) = \sum_{k=1}^{\infty} \gamma_k w^k, \qquad (\gamma_k = \text{const}), \tag{2.4.49}$$

converging even if $5 < n < \infty$; the surface at infinity $\zeta_1 = \infty,\ (n \geq 5)$ is mapped to $w_1 = 1$, and the singular point to $w_s = \zeta_s/(\zeta_s + \lambda) < -1$ if $|\zeta_s| < \lambda < |2\zeta_s|,\ (\zeta_s < 0)$, implying $w_1 < |w_s|$ for convergence.

For the isothermal sphere ($n = \pm\infty$) the series (2.4.45) and (2.4.49) cannot converge at $\zeta_1 = \infty,\ w_1 = 1$, because θ becomes infinite. Analytic approximations to the isothermal sphere are quoted in Eqs. (2.4.106), (2.4.127), (2.4.128).

2.4.2 Expansion of Lane-Emden Functions near the Finite Boundary

As will be obvious from the discussion of the topology of the Lane-Emden equation in Sec. 2.7, a finite boundary exists for Lane-Emden functions only in the cases $-1 < n < \infty$ if $N = 1, 2$, and $-1 < n < 5$ if $N = 3$. The discussion of this section is meaningful only if $n > 0$. We can take in a first approximation $\theta^n = 0$ on the right-hand side of the Lane-Emden equation (2.1.14), and obtain near the boundary ξ_1 :

$$\theta'' + (N-1)\theta'/\xi = 0, \qquad (n > 0;\ \xi \approx \xi_1). \tag{2.4.50}$$

Integration of Eq. (2.4.50) yields

$$\theta' = C\xi^{1-N}. \tag{2.4.51}$$

A second integration gives

$$\theta = C\xi^{2-N}/(2-N) + D, \qquad (N \neq 2), \tag{2.4.52}$$

where C and D are integration constants. If $N = 2$, we obtain instead of Eq. (2.4.52)

$$\theta = C \ln \xi + D, \qquad (N = 2). \tag{2.4.53}$$

With the conditions $\theta_1 = \theta(\xi_1) = 0$ and $\theta_1' = (d\theta/d\xi)_{\xi=\xi_1}$ at the boundary $\xi = \xi_1$, Eqs. (2.4.52) and (2.4.53) yield for the first order approximation of the Lane-Emden function near the finite boundary

$$\theta = [\theta_1'/(2-N)\xi_1^{1-N}](\xi^{2-N} - \xi_1^{2-N}), \qquad (N \neq 2), \tag{2.4.54}$$

and

$$\theta = \xi_1 \theta_1' \ln(\xi/\xi_1), \qquad (N = 2). \tag{2.4.55}$$

Expanding in series these two equations, we get for the geometric indices of principal interest (Chandrasekhar 1939, Linnell 1975a, b, Horedt 1983)

$$N = 1: \ \theta = \xi_1(-\theta_1')[(\xi_1 - \xi)/\xi_1], \tag{2.4.56}$$

$$N = 2: \ \theta = \xi_1(-\theta_1') \ln(\xi_1/\xi) = \xi_1(-\theta_1') \sum_{i=1}^{\infty} (1/i)[(\xi_1 - \xi)/\xi_1]^i, \tag{2.4.57}$$

$$N = 3: \ \theta = \xi_1(-\theta_1')(\xi_1 - \xi)/\xi = \xi_1(-\theta_1') \sum_{i=1}^{\infty} [(\xi_1 - \xi)/\xi_1]^i, \tag{2.4.58}$$

where we have taken into account that the derivative θ_1' is negative for the considered values of n (cf. Sec. 2.7).

To obtain the second order approximation, we may insert Eqs. (2.4.56)-(2.4.58) into the right-hand side of Eq. (2.1.14):

$$\theta'' + (N-1)\theta'/\xi = -\theta^n \approx -\xi_1^n(-\theta_1')^n \eta^n, \qquad (0 < n < \infty), \tag{2.4.59}$$

where

$$\eta = \begin{cases} (\xi_1 - \xi)/\xi_1 & \text{if} \quad N = 1, \ \text{slab} \\ \ln(\xi_1/\xi) & \quad\quad N = 2, \ \text{cylinder} \\ (\xi_1 - \xi)/\xi & \quad\quad N = 3, \ \text{sphere} \end{cases} \tag{2.4.60}$$

Eq. (2.4.59) suggests to seek the general solution near the boundary under the form of the doubly infinite series

$$\theta = \xi_1(-\theta_1')\eta + \sum_{i,j=1}^{\infty} c_{ij}\eta^{in+j} = \xi_1(-\theta_1')\eta + \sum_{j=1}^{\infty} C_j \eta^{n+j} + \sum_{j=1}^{\infty} D_j \eta^{2n+j} + \sum_{j=1}^{\infty} E_j \eta^{3n+j} + ...$$
$$= \xi_1(-\theta_1')\eta + \delta, \qquad (0 < n < \infty; \ c_{ij}, C_j, D_j, E_j = \text{const}). \tag{2.4.61}$$

The leading term $\xi_1(-\theta_1')\eta$ is provided by the first approximation (2.4.56)-(2.4.58). It should be stressed that the series (2.4.61) generally holds only inside the boundary where $\eta > 0$ and $\xi < \xi_1$, but for n equal to a rational fraction p/q, with q being an odd integer, it can also be used outside the boundary, where $\eta < 0$ and $\xi > \xi_1$ (e.g. Bellman 1953). Higher order terms become increasingly complicated, and therefore the expansion (2.4.61) is generally useful only when $|\eta| \ll 1$.

We expand the right-hand side of Eq. (2.4.59) into a Taylor series:

$$\theta^n = \left[\xi_1(-\theta_1')\eta + \delta\right]^n = \xi_1^n(-\theta_1')^n\eta^n + \sum_{j=1}^{\infty}\left[n(n-1)...(n-j+1)\xi_1^{n-j}(-\theta_1')^{n-j}\eta^{n-j}\delta^j/j!\right]$$
$$= \xi_1^n(-\theta_1')^n\eta^n + n\xi_1^{n-1}(-\theta_1')^{n-1}\sum_{j=1}^{\infty}(C_j\eta^{2n+j-1} + D_j\eta^{3n+j-1} + E_j\eta^{4n+j-1} + ...)$$
$$+\left[n(n-1)/2!\right]\xi_1^{n-2}(-\theta_1')^{n-2}\sum_{j,k=1}^{\infty}\left[C_jC_k\eta^{3n+j+k-2} + 2C_jD_k\eta^{4n+j+k-2}\right.$$
$$\left.+(D_jD_k + 2C_jE_k)\eta^{5n+j+k-2} + ...\right] + ... \tag{2.4.62}$$

We insert Eqs. (2.4.61) and (2.4.62) into Eq. (2.4.59); due to our choice for the variable η, the coefficient of the first order derivative $d\theta/d\eta$ cancels, and we obtain

$$(n+1)nC_1\eta^{n-1} + (n+2)(n+1)C_2\eta^n + ... + (n+i)(n+i-1)C_i\eta^{n+i-2} + ...$$
$$+(2n+1)2nD_1\eta^{2n-1} + (2n+2)(2n+1)D_2\eta^{2n} + ... + (2n+i)(2n+i-1)D_i\eta^{2n+i-2} + ...$$
$$+(3n+1)3nE_1\eta^{3n-1} + (3n+2)(3n+1)E_2\eta^{3n} + ... + (3n+i)(3n+i-1)E_i\eta^{3n+i-2} + ...$$
$$= -\Xi(\eta)\Big\{\xi_1^n(-\theta_1')^n\eta^n + n\xi_1^{n-1}(-\theta_1')^{n-1}\sum_{j=1}^{\infty}(C_j\eta^{2n+j-1} + D_j\eta^{3n+j-1} + E_j\eta^{4n+j-1} + ...)$$
$$+\big[n(n-1)/2!\big]\xi_1^{n-2}(-\theta_1')^{n-2}\sum_{j,k=1}^{\infty}\big[C_jC_k\eta^{3n+j+k-2} + 2C_jD_k\eta^{4n+j+k-2}$$
$$+(D_jD_k + 2C_jE_k)\eta^{5n+j+k-2} + ...\big] + ...\Big\}. \qquad (2.4.63)$$

The function $\Xi(\eta) = 1/(d\eta/d\xi)^2$ appears due to the change of variable from ξ to η. In virtue of Eq. (2.4.60) we have

$$N = 1: \ \Xi(\eta) = \xi_1^2, \qquad (2.4.64)$$

$$N = 2: \ \Xi(\eta) = \xi^2 = \xi_1^2\exp(-2\eta) = \xi_1^2\sum_{\ell=0}^{\infty}(-2)^\ell\eta^\ell/\ell!, \qquad (2.4.65)$$

$$N = 3: \ \Xi(\eta) = \xi^4/\xi_1^2 = \xi_1^2(1+\eta)^{-4} = \xi_1^2\left[1 + \sum_{\ell=1}^{\infty}4\times5\times6...(\ell+2)(\ell+3)(-1)^\ell\eta^\ell/\ell!\right]$$
$$= \xi_1^2\sum_{\ell=0}^{\infty}(\ell+1)(\ell+2)(\ell+3)(-1)^\ell\eta^\ell/3!. \qquad (2.4.66)$$

In principle, all coefficients C_j, D_j, E_j can be determined by equating in Eq. (2.4.63) equal powers of η (Horedt 1987a; if $N = 3$ see also Sadler and Miller 1932, Linnell 1975a, c).

2.4.3 Asymptotic Expansions of Lane-Emden Functions

If $\theta^{n+1} \neq 0$ or $\exp(-\theta) \neq 0$ for any finite ξ, the polytropes have no finite boundary, and the pressure $P \propto \theta^{n+1}$ or $P \propto \exp(-\theta)$ tends asymptotically to zero when $\xi \to \infty$. As will be obvious from Sec. 2.6.8, this is the case for Lane-Emden functions with index $-\infty < n < -1$ and $n = \pm\infty$ if $N = 1, 2, 3$, and also for spherical polytropes ($N = 3$) with index $5 \leq n < \infty$.

The plane-symmetric case can be discussed at once. From Eq. (2.3.52) we get near the finite or infinite boundary, where $\theta^{n+1} \approx 0$:

$$\theta' = \mp\{2/[\pm(n+1)]\}^{1/2}, \qquad (N = 1;\ n \neq -1, \pm\infty). \qquad (2.4.67)$$

By integration we find the behaviour of the Lane-Emden functions near the boundary [cf. Eq. (2.4.56)]:

$$\theta = \mp\{2/[\pm(n+1)]^{1/2}\}\xi + C, \qquad (C = \mathrm{const};\ N = 1;\ n \neq -1, \pm\infty). \qquad (2.4.68)$$

On the boundary, where $P \propto \theta^{n+1} \to 0$, we have $\theta \to \infty$ if $-\infty < n < -1$. From Eq. (2.4.68) it appears that in this case $\xi \to \infty$, and the asymptotic behaviour is given by

$$\theta = [-2/(n+1)]^{1/2}\xi + C \approx [-2/(n+1)]^{1/2}\xi, \qquad (\xi \to \infty;\ N = 1;\ -\infty < n < -1). \qquad (2.4.69)$$

To obtain the form of θ if $\xi \to \infty$ and $N \geq 2$, we study the asymptotic behaviour of Lane-Emden functions near the singular solution (2.3.70), that exists if $-\infty < n < -1$, $(N = 2)$, and $-\infty < n < -1,\ 3 < n < \infty$, $(N = 3)$. Taking the arbitrary constant $B = 1$, Eq. (2.3.66) becomes

$$y\ dy/dz + [2 + N + n(2 - N)]y/(n - 1) + 2[N + n(2 - N)]z/(n - 1)^2 \pm z^n = 0, \quad (n \neq \pm 1, \pm\infty), \tag{2.4.70}$$

with the singular solution from Eq. (2.3.69)

$$y_s = 0; \quad z_s = \{\mp 2[N + n(2 - N)]/(n - 1)^2\}^{1/(n-1)}, \tag{2.4.71}$$

if $\mp[N + n(2 - N)] > 0$. We write

$$z = z_s + z_1, \qquad (z_s \gg z_1), \tag{2.4.72}$$

and insert into Eq. (2.4.70):

$$\begin{aligned} &y\ dy/dz_1 + [2 + N + n(2 - N)]y/(n - 1) \pm (n - 1)z_1 z_s^{n-1} \\ &= y\ dy/dz_1 + [2 + N + n(2 - N)]y/(n - 1) - 2[N + n(2 - N)]z_1/(n - 1) = 0, \end{aligned} \tag{2.4.73}$$

where $(z_s + z_1)^n \approx z_s^n + nz_1 z_s^{n-1}$ and $dz = dz_1$. Since $y = dz/dt = dz_1/dt$ and $y\ dy/dz_1 = (dz_1/dt)\ dy/dz_1 = dy/dt = dz_1^2/dt^2$ via Eq. (2.2.28), the equation (2.4.73) transforms into the homogeneous second order equation with constant coefficients

$$d^2z_1/dt^2 + [2 + N + n(2 - N)](dz_1/dt)/(n - 1) - 2[N + n(2 - N)]z_1/(n - 1) = 0, \tag{2.4.74}$$

that is integrable by standard methods. Its characteristic equation is

$$q^2 + [2 + N + n(2 - N)]q/(n - 1) - 2[N + n(2 - N)]/(n - 1) = 0, \tag{2.4.75}$$

with the roots

$$q_{1,2} = \{-2 - N + n(N - 2) \pm [n^2(N^2 - 12N + 20) - 2n(N^2 - 8N + 4) + (N - 2)^2]^{1/2}\}/2(n - 1). \tag{2.4.76}$$

The solution of Eq. (2.4.74) should be discussed as a function of the sign of the second order expression under the square root from Eq. (2.4.76), outside and inside its two roots

$$n_{1,2} = [N^2 - 8N + 4 \pm 8(N - 1)^{1/2}]/(N - 2)(N - 10), \qquad (N \geq 3). \tag{2.4.77}$$

The discussion for general N is possible, but we restrict ourselves only to the two cases of practical interest $N = 2, 3$. For cylindrical polytropes Eq. (2.4.76) becomes

$$q_{1,2} = (-2 \pm 2n^{1/2})/(n - 1), \qquad (N = 2;\ -\infty < n < -1), \tag{2.4.78}$$

and the solution of Eq. (2.4.74) is

$$\begin{aligned} &z_1 = C_1 \exp[-2t/(n - 1)] \cos[2(-n)^{1/2}t/(n - 1) + C_2], \\ &(C_1, C_2 = \mathrm{const};\ N = 2;\ -\infty < n < -1). \end{aligned} \tag{2.4.79}$$

The general form of the solution of Eq. (2.4.70) in the vicinity of the singular solution is

$$\begin{aligned} &z = z_s + z_1 = [2/(n - 1)]^{2/(n-1)} + C_1 \exp[-2t/(n - 1)] \cos[2(-n)^{1/2}t/(n - 1) + C_2], \\ &(N = 2;\ -\infty < n < -1). \end{aligned} \tag{2.4.80}$$

We write also down the derivative of Eq. (2.4.80)

$$\begin{aligned} &y = dz/dt = C_1 \exp[-2t/(n - 1)]\{[-2/(n - 1)] \cos[2(-n)^{1/2}t/(n - 1) + C_2] \\ &-[2(-n)^{1/2}/(n - 1)] \sin[2(-n)^{1/2}t/(n - 1) + C_2]\}, \end{aligned} \tag{2.4.81}$$

and observe that if $t \to -\infty$, the solutions (2.4.80) and (2.4.81) tend toward the singular solutions z_s and y_s. With the familiar transformations from Eqs. (2.2.22) and (2.2.25)

$$\xi = \exp(-t); \quad \theta = \xi^{2/(1-n)} z, \qquad (B = 1), \tag{2.4.82}$$

Eq. (2.4.80) becomes (Viala and Horedt 1974b)

$$\theta \approx [(1-n)^2/4]^{1/(1-n)} \xi^{2/(1-n)} \{1 + c_1 \xi^{-2/(1-n)} \cos[2(-n)^{1/2} \ln \xi/(1-n) + c_2]\},$$
$$(c_1, c_2 = \mathrm{const};\ N = 2;\ \xi \to \infty \text{ if } -\infty < n < -1), \tag{2.4.83}$$

where $t \to -\infty$ corresponds to $\xi \to \infty$. We observe that θ oscillates round the singular solution $[(1-n)^2/4]^{1/(1-n)} \xi^{2/(1-n)}$ with decreasing amplitude as $\xi \to \infty$. Also $\theta \to \infty$, $P \propto \theta^{n+1} \to 0$, and $\varrho \propto \theta^n \to 0$ if $\xi \to \infty$.

The discussion for the spherical case is somewhat more complicated. The singular solution exists if $-\infty < n < -1$ and $3 < n < \infty$. The roots of the characteristic equation (2.4.75) are

$$q_{1,2} = [-5 + n \pm (-7n^2 + 22n + 1)^{1/2}]/2(n-1), \quad (N = 3;\ -\infty < n < -1;\ 3 < n < \infty). \tag{2.4.84}$$

These roots are real if $(11 - 8 \times 2^{1/2})/7 \le n \le (11 + 8 \times 2^{1/2})/7$, or equivalently if $-0.04482... \le n \le 3.18767...$, and complex outside this interval. We discuss at first the case of principal interest, when $-\infty < n < -1$ and $3.18767... < n < \infty$:

$$z_1 = C_1 \exp[(n-5)t/2(n-1)] \cos[(7n^2 - 22n - 1)^{1/2} t/2(n-1) + C_2],$$
$$(C_1, C_2 = \mathrm{const};\ N = 3;\ -\infty < n < -1;\ 3.18767... < n < \infty), \tag{2.4.85}$$

$$z = [\pm 2(n-3)/(n-1)^2]^{1/(n-1)} + z_1, \tag{2.4.86}$$

$$y = dz/dt = C_1 \exp[(n-5)t/2(n-1)]\{[(n-5)/2(n-1)]$$
$$\times \cos[(7n^2 - 22n - 1)^{1/2} t/2(n-1) + C_2] - [(7n^2 - 22n - 1)^{1/2}/2(n-1)]$$
$$\times \sin[(7n^2 - 22n - 1)^{1/2} t/2(n-1) + C_2]\}. \tag{2.4.87}$$

If $-\infty < n < -1$ and $5 < n < \infty$, we observe that y and z tend towards the singular solution y_s and z_s if $t \to -\infty$, i.e. $\xi \to \infty$. If $3.18767... < n < 5$, the singular solution y_s, z_s is approached as $t \to \infty$, i.e. $\xi \to 0$. Turning to the ξ, θ-variables, Eq. (2.4.86) becomes (Chandrasekhar 1939, Viala and Horedt 1974b):

$$\theta \approx [\pm(1-n)^2/2(n-3)]^{1/(1-n)} \xi^{2/(1-n)} \{1 + c_1 \xi^{(n-5)/2(1-n)}$$
$$\times \cos[(7n^2 - 22n - 1)^{1/2} \ln \xi/2(1-n) + c_2]\}, \qquad (c_1, c_2 = \mathrm{const};\ N = 3;$$
$$\xi \to \infty \text{ if } -\infty < n < -1 \text{ and } 5 < n < \infty;\ \xi \to 0 \text{ if } 3.18767... < n < 5). \tag{2.4.88}$$

If $-\infty < n < -1$, we observe that $\theta \to \infty$ if $\xi \to \infty$. When $5 < n < \infty$, we have $\theta \to 0$ if $\xi \to \infty$. Pressure ($P \propto \theta^{n+1}$) and density ($\varrho \propto \theta^n$) tend to zero in both cases. On the other side, when $3.18767... < n < 5$, we have $\theta \to \infty$ if $\xi \to 0$; in this case Eq. (2.4.88) does not obey the usual initial condition $\theta(0) = 1$. The singular solution $[\pm(1-n)^2/2(n-3)]^{1/(1-n)} \xi^{2/(1-n)}$ is always approached in an oscillatory manner with decreasing amplitude.

In the particular case $n = 5$, Eq. (2.4.88) leads to

$$\theta \approx (2\xi)^{-1/2}[1 + c_1 \cos(\ln \xi - c_2)], \qquad (c_1 \approx 0;\ c_1, c_2 = \mathrm{const};\ N = 3;\ n = 5;\ \text{any } \xi). \tag{2.4.89}$$

It is seen that θ oscillates for any ξ with constant amplitude c_1 round the singular solution $(2\xi)^{-1/2}$, where c_1 obeys the important constraint $c_1 \approx 0$.

We now turn to the discussion of the second case when $3 < n < 3.18767...$ In this case the roots $q_{1,2}$ are real, and we write down at once the analogue of Eq. (2.4.86):

$$z = [2(n-3)/(n-1)^2]^{1/(n-1)} + C_1 \exp\{[-5 + n + (-7n^2 + 22n + 1)^{1/2}]t/2(n-1)\}$$
$$+ C_2 \exp\{[-5 + n - (-7n^2 + 22n + 1)^{1/2}]t/2(n-1)\},$$
$$(C_1, C_2 = \mathrm{const};\ N = 3;\ 3 < n < 3.18767...). \tag{2.4.90}$$

We observe that the singular solution y_s, z_s is approached if $t \to \infty$, $(\xi \to 0)$. We rewrite Eq. (2.4.90) in the (ξ, θ)-variables:

$$\theta \approx [(1-n)^2/2(n-3)]^{1/(1-n)} \xi^{2/(1-n)} \big\{1 + c_1 \xi^{[n-5+(-7n^2+22n+1)^{1/2}]/2(1-n)}$$
$$+ c_2 \xi^{[n-5-(-7n^2+22n+1)^{1/2}]/2(1-n)}\big\}, \quad (c_1, c_2 = \text{const};\ \xi \to 0;\ N=3;\ 3 < n < 3.18767...). \quad (2.4.91)$$

In the particular case $n = (11 + 8 \times 2^{1/2})/7 = 3.18767...$ the characteristic equation has a real double root, and we write down directly the result

$$\theta \approx [(1-n)^2/2(n-3)]^{1/(1-n)} \xi^{2/(1-n)} [1 + \xi^{(n-5)/2(1-n)} (c_1 \ln \xi + c_2)],$$
$$(c_1, c_2 = \text{const};\ \xi \to 0;\ N=3;\ n = 3.18767...). \quad (2.4.92)$$

From Eqs. (2.4.91), (2.4.92) it appears that $\theta \to \infty$ if $\xi \to 0$.

Summarizing the discussion of the spherical case, we observe that for all values $3 < n \leq 5$ we have $\theta \to \infty$ if $\xi \to 0$, so equations (2.4.88)-(2.4.92) represent in this case merely an expansion near the origin, that is of interest for the general topology of solutions (Sec. 2.7). On the other hand, Eq. (2.4.88) yields the asymptotic expansion in the limit $\xi \to \infty$, if $-\infty < n < -1$ and $5 < n < \infty$. Because of the presence of the trigonometric term in Eq. (2.4.88), θ approaches the singular solution in an oscillatory manner.

If $n = \pm\infty$, we present a general discussion in N-dimensional space. The solution for the plane-symmetrical polytrope is given by Eq. (2.3.65):

$$\theta = \ln[\cosh^2(\xi/2)^{1/2}] \approx 2^{1/2}\xi, \qquad (\xi \to \infty;\ N=1;\ n = \pm\infty). \quad (2.4.93)$$

In the cylindrical case Eq. (2.3.48) yields

$$\theta = 2\ln(1 + \xi^2/8) \approx \ln(\xi^4/64), \qquad (\xi \to \infty;\ N=2;\ n = \pm\infty). \quad (2.4.94)$$

If $N \geq 3$, we write Eq. (2.3.72) under the form

$$y\ dy/dz + (2-N)y + \exp z + 2(2-N) = 0, \qquad (N \geq 3;\ n = \pm\infty), \quad (2.4.95)$$

having the singular solution (2.3.73):

$$y_s = 0; \quad z_s = \ln[2(N-2)], \qquad (N \geq 3;\ n = \pm\infty). \quad (2.4.96)$$

We substitute analogously to Eq. (2.4.72):

$$z = z_s + z_1 \quad \text{and} \quad \exp z \approx \exp z_s + z_1 \exp z_s, \qquad (z_s \gg z_1). \quad (2.4.97)$$

After a short algebra Eq. (2.4.95) becomes similar to Eq. (2.4.74):

$$d^2 z_1/dt^2 + (2-N)\ dz_1/dt + 2(N-2)z_1 = 0, \qquad (N \geq 3;\ n = \pm\infty). \quad (2.4.98)$$

The characteristic equation of Eq. (2.4.98) is

$$q^2 + (2-N)q + 2(N-2) = 0, \quad (2.4.99)$$

with the roots

$$q_{1,2} = [N - 2 \pm (N^2 - 12N + 20)^{1/2}]/2. \quad (2.4.100)$$

The roots are imaginary if $2 < N < 10$, and real outside this interval, where N has to be a positive integer. The solution for imaginary q_1, q_2 is

$$z_1 = C_1 \exp[(N-2)t/2] \cos\left[(-N^2 + 12N - 20)^{1/2} t/2 + C_2\right],$$
$$(C_1, C_2 = \text{const};\ n = \pm\infty;\ 3 \leq N \leq 9), \quad (2.4.101)$$

and

$$z = \ln[2(N-2)] + z_1. \quad (2.4.102)$$

Performing the derivative $y = dz/dt$, we observe that the singular points y_s, z_s are approached as $t \to -\infty$. With the transformations (2.2.25), (2.2.30), we find from Eqs. (2.4.101), (2.4.102):

$$\theta \approx \ln[\xi^2/2(N-2)] - C_1\xi^{(2-N)/2}\cos\{[(-N^2+12N-20)^{1/2}/2]\ln\xi - C_2\},$$
$$(\xi \to \infty;\ n = \pm\infty;\ 3 \le N \le 9). \qquad (2.4.103)$$

In the case of practical interest $N = 3$, Eq. (2.4.103) becomes (Chandrasekhar 1939)

$$\theta \approx \ln(\xi^2/2) - C_1\xi^{-1/2}\cos[(7^{1/2}/2)\ln\xi - C_2], \qquad (\xi \to \infty;\ n = \pm\infty;\ N = 3), \qquad (2.4.104)$$

and

$$\exp(-\theta) = \varrho/\varrho_0 \approx (2/\xi^2)\exp\{C_1\xi^{-1/2}\cos[(7^{1/2}/2)\ln\xi - C_2]\}$$
$$\approx (2/\xi^2)\{1 + C_1\xi^{-1/2}\cos[(7^{1/2}/2)\ln\xi - C_2]\}, \qquad (\xi \to \infty;\ n = \pm\infty;\ N = 3). \qquad (2.4.105)$$

Obviously, if $\xi \to \infty$ and $3 \le N \le 9$, the solution of the Lane-Emden equation (2.1.21) approaches the singular solution $\ln[\xi^2/2(N-2)]$ in an oscillatory manner, due to the presence of the trigonometric term in Eq. (2.4.103). An elementary density approximation for the isothermal sphere is furnished by Eq. (2.4.105), [Horedt 1976, see also Eqs. (2.4.127), (2.4.128)]:

$$\exp(-\theta) = \varrho/\varrho_0 = \begin{cases} 1 & \text{if} \quad 0 \le \xi \le 2^{1/2} \\ 2/\xi^2 & \quad\quad \xi \ge 2^{1/2} \end{cases} \qquad (n = \pm\infty;\ N = 3). \qquad (2.4.106)$$

Eq. (2.4.99) has the real double root $q_{1,2} = 4$ if $N = 10$, and the solutions are obtained in the same way as for $3 \le N \le 9$; the singular solution is approached if $t \to -\infty$ or $\xi \to \infty$:

$$z_1 = (C_1 t + C_2)\exp(4t), \qquad (2.4.107)$$

$$\theta = \ln(\xi^2/16) + \xi^{-4}(C_1\ln\xi - C_2), \qquad (\xi \to \infty;\ n = \pm\infty;\ N = 10). \qquad (2.4.108)$$

Eq. (2.4.99) has distinct real roots if $N \ge 11$, and the singular solution is approached if $t \to -\infty$ or $\xi \to \infty$. The solutions are $(N - 2 > (N^2 - 12N + 20)^{1/2}$ if $N \ge 11)$:

$$z_1 = C_1\exp\{[N - 2 + (N^2 - 12N + 20)^{1/2}]t/2\} + C_2\exp\{[N - 2 - (N^2 - 12N + 20)^{1/2}]t/2\}, \qquad (2.4.109)$$

$$\theta = \ln[\xi^2/2(N-2)] - C_1\xi^{\{[-N+2-(N^2-12N+20)^{1/2}]/2\}} - C_2\xi^{\{[-N+2+(N^2-12N+20)^{1/2}]/2\}},$$
$$(\xi \to \infty;\ n = \pm\infty;\ N \ge 11). \qquad (2.4.110)$$

Thus, when $n = \pm\infty$, Eqs. (2.4.103), (2.4.108), (2.4.110) represent the asymptotic behaviour of Lane-Emden functions $\theta \to \infty$ and $\exp(-\theta) \to 0$ if $\xi \to \infty$. Only if $3 \le N \le 9$, the singular solution is approached in an oscillatory manner, due to the absence of trigonometric terms if $N = 1, 2$, and $N \ge 10$. This completes the discussion of the behaviour of Lane-Emden functions if $\xi \to \infty$.

2.4.4 Padé Approximants for Lane-Emden Functions

When the Lane-Emden functions are approximated with polynomials of the form

$$\theta \approx \theta_P = \sum_{i=0}^{I} b_i(\xi - \xi_0)^{2i} \Big/ \sum_{j=0}^{J} c_j(\xi - \xi_0)^{2j}, \qquad (b_i, c_j = \text{const}), \qquad (2.4.111)$$

i.e. by so-called Padé approximants (Pascual 1977), we can obtain sometimes significantly better approximations than with simple power series of the form (2.4.2). Near the origin $\xi = 0$ the Padé approximant (2.4.111) can be expanded into the Maclaurin series

$$\theta \approx \theta_P = \sum_{k=0}^{\infty} \left[d^k\theta_P \Big/ d(\xi^2)^k \right]_{\xi=0} \xi^{2k}/k!, \tag{2.4.112}$$

and equated to the power series (2.4.16)

$$\theta = \sum_{k=0}^{\infty} a_{2k}\xi^{2k}, \qquad (a_{2k} = \mathrm{const}), \tag{2.4.113}$$

where the coefficients a_{2k} are given by Eq. (2.4.21), and we have taken into account that all odd indexed coefficients are zero. Equating the coefficients of equal powers in Eqs. (2.4.112) and (2.4.113), we get

$$a_{2k} = \left[d^k\theta_P \Big/ d(\xi^2)^k \right]_{\xi=0} \Big/ k!. \tag{2.4.114}$$

We now apply this technique to the most simple Padé approximants of the form

$$\theta \approx \theta_P = (1 + A\xi^2)/(1 + B\xi^2), \tag{2.4.115}$$

and

$$\theta \approx \theta_P = (1 + C\xi^2 + D\xi^4)/(1 + E\xi^2 + F\xi^4), \tag{2.4.116}$$

satisfying already the usual initial conditions $\theta(0) = 1,\ \theta'(0) = 0$. Equating according to Eq. (2.4.114) the coefficients of the series (2.4.113) with the coefficients of the series (2.4.115) and (2.4.116), respectively, we obtain after some algebra (cf. Pascual 1977, Seidov et al. 1979)

$$a_2 = A - B; \quad a_4 = B(B - A), \tag{2.4.117}$$

and

$$\begin{aligned} &a_2 = C - E; \quad a_4 = E^2 + D - CE - F; \quad a_6 = 2EF - CF - DE + CE^2 - E^3;\\ &a_8 = -3E^2F + 2CEF + F^2 - DF + DE^2 - CE^3 + E^4. \end{aligned} \tag{2.4.118}$$

Inserting the value of a_k from Eq. (2.4.21), we find if $n \neq -1, \pm\infty$:

$$A = (a_2^2 - a_4)/a_2 = \pm[(nN - 2N - 4)/4N(N + 2)]; \quad B = \pm n/4(N + 2), \tag{2.4.119}$$

and

$$\begin{aligned} &C = a_2 + E = \pm\big[n^2(2N^3 + 4N^2 + 24N + 16) - n(7N^3 + 42N^2 + 80N + 114)\\ &+6N^3 + 56N^2 + 152N + 128\big]/2^2N(N + 6)[n(N^2 + 8) - 2(N + 2)^2];\\ &D = a_4 + a_2E + F = \big[n^3(2N^5 + 8N^4 + 64N^3 + 112N^2) - n^2(11N^5 + 82N^4 + 400N^3\\ &+1040N^2 + 768N + 768) + n(20N^5 + 216N^4 + 1008N^3 + 2656N^2 + 3840N + 2304)\\ &-(12N^5 + 168N^4 + 912N^3 + 2400N^2 + 3072N + 1536)\big]\\ &/2^4 \times 3N^2(N + 2)(N + 4)(N + 6)[n(N^2 + 8) - 2(N + 2)^2];\\ &E = (a_2a_8 - a_4a_6)/(a_4^2 - a_2a_6) = \pm\big[n^2(2N^3 + 4N^2 + 24N + 16) - n(5N^3 + 30N^2 + 64N + 48)\\ &+(2N^3 + 16N^2 + 40N + 32)\big]/2^2N(N + 6)[n(N^2 + 8) - 2(N + 2)^2];\\ &F = (a_6^2 - a_4a_8)/(a_4^2 - a_2a_6) = \big[n^3(2N^3 + 4N^2 + 56N) - n^2(5N^3 + 36N^2 + 136N + 96)\\ &+n(2N^3 + 20N^2 + 80N + 96)\big]/2^4 \times 3N(N + 4)(N + 6)[n(N^2 + 8) - 2(N + 2)^2]. \end{aligned} \tag{2.4.120}$$

If $N = 3$, we recover from Eqs. (2.4.119) and (2.4.120) Pascual's (1977) result

$$\theta_P = [60 \pm (3n - 10)\xi^2]/(60 \pm 3n\xi^2), \tag{2.4.121}$$

Table 2.4.1 Comparison between the first zeros of Lane-Emden functions obtained by exact numerical integration, by the Padé approximants from Eqs. (2.4.116), (2.4.126), and by the empirical relationships from Eq. (2.4.130), (Horedt 1987a). If $N=1,\ n \gtrsim 10$ and $N=2,\ n \gtrsim 5$ and $N=3,\ n \gtrsim 3.5$, no real zeros of Eq. (2.4.116) occur. $a+b$ means $a \times 10^b$.

$N=1$			$N=2$		
n	Numerical	Eq. (2.4.116)	n	Numerical	Eq. (2.4.116)
−0.9	1.2691	1.2823	−0.9	1.7178	1.7412
0	1.4142	1.4142	0	2.0000	2.0000
1	1.5708	1.5708	1	2.4048	2.4055
3	1.8541	1.8484	3	3.5739	3.5582
5	2.1033	2.1475	5	5.4276	9.6207
10	2.6284	3.8470			

$N=3$					
n	Numerical	Eq. (2.4.116)	Eq. (2.4.126)	Eq. (2.4.130)	
−0.9	2.0504	2.0825	–	2.0365	2.0484
0	2.4495	2.4495	2.4495	2.4533	2.4600
0.5	2.7527	2.7454	2.7086	2.7620	2.7658
1	3.1416	3.1457	3.0603	3.1534	3.1546
1.5	3.6538	3.6868	3.5997	3.6645	3.6639
2	4.3529	4.4092	4.3067	4.3584	4.3577
2.5	5.3553	5.4437	5.4752	5.3507	5.3530
3	6.8968	6.9212	7.3305	6.8775	6.8880
3.5	9.5358	1.3672+1	1.0319+1	9.5058	9.5293
4	1.4972+1	–	1.5838+1	1.5000+1	1.5006+1
4.5	3.1836+1	–	3.1455+1	3.2715+1	3.2094+1
4.9	1.7143+2	–	1.5381+2	2.0003+2	1.7304+2
5	∞	–	∞	∞	∞

$$\theta_P = \big[45360(17n-50) \pm 420(178n^2-951n+1250)\xi^2 + (1290n^3-10849n^2+29100n$$
$$-24500)\xi^4\big]\big/\big[45360(17n-50) \pm 420(178n^2-645n+350)\xi^2 + 15n(86n^2-321n+190)\xi^4\big]. \tag{2.4.122}$$

To obtain an even better approximation for some values of n, $(N=3;\ -1 < n \leq 5)$, Pascual (1977) introduces the auxiliary independent variable

$$\zeta = 6[(1+\xi^2/3)^{1/2}-1] \quad \text{or} \quad \xi^2 = \zeta^2/12+\zeta, \tag{2.4.123}$$

resembling the structure of the Schuster-Emden integral (2.3.90). Inserting Eq. (2.4.123) into the series (2.4.24), the first five terms become

$$\theta = 1-(1/3!)\zeta + [(3n-5)/3\times 5!]\zeta^2 - [n(8n-26)/3\times 7!]\zeta^3$$
$$+[n(122n^2-615n+529)/9\times 9!]\zeta^4 - ..., \qquad (N=3;\ -1<n\leq 5). \tag{2.4.124}$$

Then, using the same procedure as before, Pascual (1977) obtains instead of Eqs. (2.4.121), (2.4.122):

$$\theta_P = [60+3(n-5)\zeta]/[60+(3n-5)\zeta], \qquad (N=3;\ -1<n\leq 5), \tag{2.4.125}$$

$$\theta_P = \big[45360(17n+35) + 420(178n^2-376n-630)\zeta + 3(n-5)(430n^2-1393n+1470)\zeta^2\big]$$
$$\big/[45360(17n+35) + 420n(178n-61)\zeta + 5n(258n^2-919n+3703)\zeta^2\big], \qquad (N=3;\ -1<n\leq 5). \tag{2.4.126}$$

A Padé approximant

$$\exp(-\theta_P) \approx 50/(10+\xi^2) - 48/(12+\xi^2) = (1+\xi^2/60)/(1+11\xi^2/60+\xi^4/120),$$
$$(N=3;\ n=\pm\infty), \tag{2.4.127}$$

for the isothermal sphere has been proposed by Natarajan and Lynden-Bell (1997). This approximation can be improved with the Padé approximant (Hunter 2001, Table 2)

$$\exp(-\theta_P) \approx \sum_{i=1}^{4} A_i/(B_i + \xi^2); \quad \sum_{i=1}^{4} A_i = 2, \qquad (N = 3;\ n = \pm\infty;\ A_i, B_i = \text{const}). \tag{2.4.128}$$

And a modified Padé approximant

$$\theta_P \approx (1 + A\xi^2)/(1 + B\xi^2)^S, \qquad (N = 3;\ 1 \le n \le 5;\ A, B, S = \text{const}), \tag{2.4.129}$$

has been employed by Roxburgh and Stockman (1999). More cumbersome approximations have been worked out by Liu (1996).

The first zero ξ_1 of the Lane-Emden equation results simply by equating the numerator of the corresponding Padé approximant to zero. Empirical relationships for the determination of the first zero of Lane-Emden functions have been quoted for the spherical case by Pascual (1977) and Buchdahl (1978), respectively:

$$\xi_1 = 15(5 - n)^{-9/8} \quad \text{and} \quad \xi_1 = 12.3(1 - 0.128n)/(5 - n)(1 - 0.150n), \qquad (N = 3;\ -1 < n \le 5). \tag{2.4.130}$$

The relative error of these two relationships is generally less than 1%. Table 2.4.1 shows a comparison between the exact numerical values ξ_1 from Tables 2.4.3, 2.5.2 and those from Eqs. (2.4.116), (2.4.126), (2.4.130). It should be stressed that Padé approximants yield for some values of n and N erroneous, or even imaginary zeros, as compared to exact numerical integrations (cf. Table 2.4.1).

2.4.5 Analytical Solutions Close to an Exact Solution

Let us suppose that an exact analytical solution θ_0 of the Lane-Emden equation of index n_0 is known. A method due to Seidov and Kuzakhmedov (1978) enables us to find out solutions θ of the Lane-Emden equations for polytropic indices n differing slightly from n_0. Let us write

$$n = n_0 + \varepsilon; \quad \theta(\xi) = \theta_0(\xi) + \varepsilon\theta_*(\xi) = \theta_0(\xi) + (n - n_0)\theta_*(\xi), \qquad (\varepsilon = n - n_0 \ll 1), \tag{2.4.131}$$

where $\theta_0, n_0, \varepsilon$ are known, and θ_* has to be determined. We have

$$\theta^n = \theta_0^{n_0} + \varepsilon(d\theta^n/dn)_{n=n_0} = \theta_0^{n_0} + \varepsilon\theta_0^{n_0}(\ln\theta_0 + n_0\theta_*/\theta_0), \tag{2.4.132}$$

because $d\theta^n/dn = d[\exp(n\ln\theta)]/dn = \theta^n[\ln\theta + (n/\theta)\ d\theta/dn]$. Inserting Eqs. (2.4.131), (2.4.132) into the Lane-Emden equation (2.1.14)

$$\theta'' + (N - 1)\theta'/\xi = \mp\theta^n, \qquad (n \ne -1, \pm\infty), \tag{2.4.133}$$

and equating equal powers of ε, we find

$$\theta_0'' + (N - 1)\theta_0'/\xi = \mp\theta_0^{n_0}, \tag{2.4.134}$$

and

$$\theta_*'' + (N - 1)\theta_*'/\xi = \mp\theta_0^{n_0}\ln\theta_0 \mp n_0\theta_0^{n_0-1}\theta_*, \tag{2.4.135}$$

with the initial conditions $\theta_0(0) = 1$ and $\theta_0'(0),\ \theta_*(0),\ \theta_*'(0) = 0$. By assumption Eq. (2.4.134) is satisfied identically.

Eq. (2.4.135) has explicitly been solved by Seidov and Kuzakhmedov (1978) for the cases $n_0 = 0$ and $n_0 = 5,\ (N = 3)$. If $n_0 = 1,\ N = 3$, the relevant integrals cannot be solved in terms of elementary functions. If $n_0 = 0,\ N = 3$, we have via Eq. (2.3.88) $\theta_0 = 1 - \xi^2/6$, and Eq. (2.4.135) becomes

$$\theta_*'' + 2\theta_*'/\xi = -\ln\theta_0, \qquad (n_0 = 0;\ N = 3). \tag{2.4.136}$$

This linear, inhomogeneous second order equation can be solved by standard methods (e.g. Bronstein and Semendjajew 1985). If a particular integral $y_1 = y_1(x)$ of the homogeneous equation

$$d^2y/dx^2 + \varphi(x)\ dy/dx + \psi(x)\ y = 0, \tag{2.4.137}$$

is known, we can find another linearly independent particular integral with the substitution $z = z(x) = d(y/y_1)/dx$. In this case Eq. (2.4.137) becomes

$$dz/dx + [\varphi + 2(dy_1/dx)/y_1]\ z = 0, \tag{2.4.138}$$

which can be solved for z by separation of variables. Then, the second particular integral is given by

$$y_2(x) = y_1(x) \int z(x)\ dx, \tag{2.4.139}$$

and the general solution of Eq. (2.4.137) is

$$y(x) = C_1y_1(x) + C_2y_2(x), \qquad (C_1, C_2 = \text{const}). \tag{2.4.140}$$

The general integral of the inhomogeneous equation

$$d^2y/dx^2 + \varphi(x)\ dy/dx + \psi(x)\ y = \omega(x), \tag{2.4.141}$$

is then found by the method of variation of constants, writing instead of Eq. (2.4.140)

$$y(x) = z_1(x)\ y_1(x) + z_2(x)\ y_2(x), \tag{2.4.142}$$

with the additional constraint

$$y_1(x)\ dz_1(x)/dx + y_2(x)\ dz_2(x)/dx = 0. \tag{2.4.143}$$

Substitution of Eq. (2.4.143) into Eq. (2.4.141) yields

$$(dy_1/dx)\ dz_1/dx + (dy_2/dx)\ dz_2/dx = \omega. \tag{2.4.144}$$

The system of equations (2.4.143) and (2.4.144) can now be solved for dz_1/dx and dz_2/dx, yielding for the general solution of Eq. (2.4.141)

$$y = -y_1 \int (\omega y_2/W)\ dx + y_2 \int (\omega y_1/W)\ dx + C_1y_1 + C_2y_2, \qquad (C_1, C_2 = \text{const}), \tag{2.4.145}$$

where W denotes the Wronski determinant

$$W = y_1\ dy_2/dx - y_2\ dy_1/dx. \tag{2.4.146}$$

We now apply to Eq. (2.4.136) the method briefly outlined in Eqs. (2.4.137)-(2.4.146). The general solution of the homogeneous equation $\theta_*'' + 2\theta_*'/\xi = 0$ is found by direct integration to be

$$\theta_* = C_1/\xi + C_2, \tag{2.4.147}$$

and after performing the other outlined integrations, θ_* becomes eventually:

$$\theta_* = (3 - \xi^2/6)\ln(1 - \xi^2/6) + (2 \times 6^{1/2}/\xi)\ln[(6^{1/2} + \xi)/(6^{1/2} - \xi)] + 5\xi^2/18 - 4,$$
$$(n_0 = 0;\ N = 3;\ \theta_*(0) = \theta_*'(0) = 0;\ \xi \le 6^{1/2}). \tag{2.4.148}$$

The Lane-Emden functions for polytropic indices differing slightly from zero obey therefore the form

$$\theta = (1 - \xi^2/6) + n\theta_*, \qquad (n \approx 0;\ N = 3;\ \xi \le 6^{1/2}), \tag{2.4.149}$$

where θ_* is provided by Eq. (2.4.148). We already know from Eq. (2.1.13) that ξ is proportional to the radius of the polytrope; from Secs 2.6.3, 2.6.4 it will be obvious that $-\xi^2\theta'$ is proportional to the mass of the polytrope, whereas the ratio between central density ϱ_0 and mean density ϱ_m is given by $\varrho_0/\varrho_m = -\xi/3\theta'$, $(N = 3)$. We now determine the boundary values of these quantities analytically for

Table 2.4.2 Comparison between exact numerical boundary values from Tables 2.4.3 and 2.5.2 (first columns), and the corresponding analytical boundary figures (second columns), as they result from Eqs. (2.4.152)-(2.4.156) and (2.4.166)-(2.4.169). $a + b$ means $a \times 10^b$.

n	ξ_1		$-\xi_1^2\theta_1'$		$-\xi_1/3\theta_1'$	
			$n_0 = 0$			
-0.5	2.2086	2.1805	7.4215	6.4213	0.4839	0.3598
0.5	2.7527	2.7185	3.7887	3.3766	1.8351	1.6402
			$n_0 = 1$			
0.5	2.7527	2.6990	3.7887	3.6562	1.8351	1.3604
1.5	3.6538	3.5842	2.7141	2.6270	5.9907	5.2194
			$n_0 = 5$			
4.5	3.1836+1	3.5285+1	1.7378	1.6599	6.1895+3	8.4546+3
4.75	6.6387+1	7.0570+1	1.7243	1.6960	5.6562+4	6.7636+4
4.8	8.3813+1	8.8213+1	1.7234	1.7032	1.1387+5	1.3210+5
4.85	1.1296+2	1.1762+2	1.7235	1.7104	2.7873+5	3.1313+5
4.9	1.7143+2	1.7643+2	1.7246	1.7176	9.7381+5	1.0568+6
4.95	3.4740+2	3.5285+2	1.7272	1.7248	8.0916+6	8.4546+6
4.99	1.7582+3	1.7643+3	1.7308	1.7306	1.0467+9	1.0568+9
5	∞	∞	1.7321	1.7321	∞	∞

polytropes having $n \approx 0$. Strictly speaking, we can determine the boundary values $\xi_1, \theta'(\xi_1)$ only for polytropes with indices $n \leq 0$, because the boundary of polytropes having $n > 0$ is attained for ξ_1-values larger than $6^{1/2}$ (see Table 2.5.2), and the previous equations are valid only if $\xi \leq 6^{1/2}$. However, it will be obvious from the subsequent equations that the diverging logarithmic terms $\ln(1 - \xi/6^{1/2})$ in Eq. (2.4.148) cancel out to the first order in n if $\xi \approx 6^{1/2}$, and the following results are applicable in this first approximation even if $n > 0$, $(n \approx 0)$. To obtain the boundary value ξ_1 for polytropes having $n \approx 0$, we put

$$\xi_1 = 6^{1/2}(1 + \delta), \qquad (\delta \approx 0), \tag{2.4.150}$$

and insert into Eqs. (2.4.148), (2.4.149). Expansion up to the first order in δ yields

$$\delta \approx n(2\ln 2 - 7/6), \qquad (n \approx 0), \tag{2.4.151}$$

and

$$\xi_1 \approx 6^{1/2}[1 + n(2\ln 2 - 7/6)] = 6^{1/2}(1 + 0.21963n), \qquad (n \approx 0;\ N = 3). \tag{2.4.152}$$

To obtain $\xi_1^2\theta_1'$, we derive Eq. (2.4.148) and insert for ξ_1 :

$$(\theta_*')_{\xi=\xi_1} \approx 6^{-1/2}(16/3 - 4\ln 2) = 1.04542. \tag{2.4.153}$$

With $\theta' = \theta_0' + n\theta_*'$ we find eventually

$$(-\xi^2\theta')_{\xi=\xi_1} \approx 2 \times 6^{1/2}[1 + n(8\ln 2 - 37/6)] = 2 \times 6^{1/2}(1 - 0.62149n), \quad (n \approx 0;\ N = 3). \tag{2.4.154}$$

With Eqs. (2.4.152), (2.4.153) we get for the density ratio (2.6.27):

$$\varrho_0/\varrho_m = (-\xi/3\theta')_{\xi=\xi_1} \approx 1 + n(8/3 - 2\ln 2) = 1 + 1.28037n, \qquad (n \approx 0;\ N = 3). \tag{2.4.155}$$

If $n_0 = 1,\ N = 3$, Seidov and Kuzakhmedov (1978) find

$$\xi_1 \approx \pi[1 + 0.28179(n - 1)]; \quad (-\xi^2\theta')_{\xi=\xi_1} \approx \pi[1 - 0.32762(n - 1)];$$
$$\varrho_0/\varrho_m \approx (\pi^2/3)[1 + 1.17299(n - 1)], \qquad (n \approx 1). \tag{2.4.156}$$

If $n_0 = 5,\ N = 3$, we have $\theta_0 = (1 + \xi^2/3)^{-1/2}$ via Eq. (2.3.90), and Eq. (2.4.135) reads

$$\theta_*'' + 2\theta_*'/\xi + 5\theta_*/(1 + \xi^2/3)^2 = \left[1/2(1 + \xi^2/3)^{5/2}\right] \ln(1 + \xi^2/3), \qquad (n \approx 5). \tag{2.4.157}$$

Table 2.4.3 The Seidov-Kuzakhmedov minimum for the dimensionless mass $-\xi_1^2\theta_1'$ of polytropic spheres at $n \approx 4.823$ (Horedt 1986b). $a + b$ means $a \times 10^b$.

n	ξ_1	θ_1'	$-\xi_1^2\theta_1'$	$-\xi_1/3\theta_1'$
4.75	66.38709554	−3.912321−4	1.724256470	5.656241+4
4.80	83.81283880	−2.453446−4	1.723445575	1.138709+5
4.81	88.40745274	−2.204976−4	1.723381986	1.336484+5
4.82	93.51603675	−1.970617−4	1.723354059	1.581840+5
4.821	94.05848618	−1.947952−4	1.723353281	1.609527+5
4.822	94.60706735	−1.925427−4	1.723352876	1.637854+5
4.823	95.16188436	−1.903041−4	1.723352843	1.666839+5
4.824	95.72304369	−1.880794−4	1.723353186	1.696500+5
4.825	96.29065425	−1.858687−4	1.723353906	1.726858+5
4.826	96.86482744	−1.836718−4	1.723355004	1.757933+5
4.827	97.44567726	−1.814889−4	1.723356482	1.789746+5
4.828	98.03332036	−1.793198−4	1.723358340	1.822319+5
4.829	98.62787611	−1.771645−4	1.723360582	1.855674+5
4.83	99.22946669	−1.750232−4	1.723363209	1.889835+5
4.84	105.6613548	−1.543677−4	1.723410981	2.281594+5
4.85	112.9556313	−1.350813−4	1.723499081	2.787349+5
4.90	171.4334501	−5.868160−5	1.724618596	9.738058+5
4.95	347.4003713	−1.431107−5	1.727160845	8.091645+6
4.99	1758.189155	−5.598955−7	1.730765298	1.046736+9
5.00	∞	0	1.732050808	∞

With the transformation

$$\xi = 3^{1/2}\tan\alpha, \tag{2.4.158}$$

Eq. (2.4.157) becomes

$$d^2\theta_*/d\alpha^2 + (2/\tan\alpha)\ d\theta_*/d\alpha + 15\theta_* = -3\cos\alpha\ln(\cos\alpha), \tag{2.4.159}$$

where $d\theta_*/d\xi = [1/3^{1/2}(1+\xi^2/3)]\ d\theta_*/d\alpha$.

With some skill we get a particular integral of the homogeneous equation

$$d^2\theta_*/d\alpha^2 + (2/\tan\alpha)\ d\theta_*/d\alpha + 15\theta_* = 0, \tag{2.4.160}$$

under the form $\cos 3\alpha + \cos\alpha \propto \sin 4\alpha/\sin\alpha$. Using the trigonometric relationships for multiple angles, the general solution of Eq. (2.4.160) is found by the method outlined in Eqs. (2.4.137)-(2.4.146):

$$\theta_* = C_1\sin 4\alpha/\sin\alpha + C_2\cos 4\alpha/\sin\alpha. \tag{2.4.161}$$

The general solution of the inhomogeneous equation (2.4.159) is obtained via Eq. (2.4.145), ($W = -4/\sin^2\alpha$) :

$$\theta_* = (3/8\sin\alpha)\Big[-\sin 4\alpha\int\cos 4\alpha\sin 2\alpha\ln(\cos\alpha)\ d\alpha + \cos 4\alpha\int\sin 4\alpha\sin 2\alpha\ln(\cos\alpha)\ d\alpha\Big]$$
$$+C_1\sin 4\alpha/\sin\alpha + C_2\cos 4\alpha/\sin\alpha, \qquad (C_1, C_2 = \text{const}). \tag{2.4.162}$$

This integral can be solved in terms of elementary functions, and after a lengthy calculation we finally get

$$\theta_*(\alpha) = (1/48\sin\alpha)[\sin 2\alpha - (5/4)\sin 4\alpha + 3\alpha\cos 4\alpha - 3(2\sin 2\alpha + \sin 4\alpha)\ln(\cos\alpha)];$$
$$\alpha = \arctan(\xi/3^{1/2}), \tag{2.4.163}$$

where the initial condition $\theta_*(0) = 0$ determines the integration constants as $C_1 = -1/24,\ C_2 = 0$. Medvedev and Rybicki [2001, Eq. (12)] quote Eq. (2.4.163) in terms of ξ.

Thus, the Lane-Emden functions with indices differing slightly from $n = 5$, write as

$$\theta(\alpha) = \cos\alpha + (n-5)\ \theta_*(\alpha);\ \theta_0 = (1+\xi^2/3)^{-1/2} = \cos\alpha, \quad (\alpha = \arctan(\xi/3^{1/2});\ n \approx 5;\ N = 3). \tag{2.4.164}$$

And these functions exhibit a multiple-core structure (plateaus with $\theta \approx$ const) if $n > 5$ (Medvedev and Rybicki 2001).

The case $n_0 = 5$ does not present the problems encountered for $n_0 = 0$ if $\xi > 6^{1/2}$, because all polytropes having $n \geq 5$, $(N = 3)$ extend to infinity ($\xi_1 = \infty$ if $\alpha = \pi/2$), (cf. Sec. 2.7). To obtain the boundary value ξ_1 of polytropes with $n \approx 5$, $(n \leq 5)$, we insert

$$\alpha = \pi/2 - \delta, \qquad (\delta > 0;\ \delta \approx 0), \tag{2.4.165}$$

into Eq. (2.4.164), getting $\alpha \approx \pi/2 + \pi(n-5)/32$ if $\theta(\alpha) \approx 0$. Eventually, we find with the aid of Eq. (2.4.158)

$$\xi_1 \approx 32 \times 3^{1/2}/\pi(5-n) = 17.64252/(5-n), \qquad (n \approx 5;\ n \leq 5;\ N = 3). \tag{2.4.166}$$

If $n > 5$, Eq. (2.4.166) would provide *negative* radii, whereas in fact all polytropes having $n \geq 5$, $(N = 3)$ extend to infinity.

The derivative $[d\theta_*(\alpha)/d\alpha]_{\alpha\approx\pi/2} \approx -1/12$ of Eq. (2.4.163) enables us to obtain

$$[d\theta_*(\xi)/d\xi]_{\xi=\xi_1} \approx [d\theta_*(\alpha)/d\alpha]_{\alpha\approx\pi/2}\ (d\alpha/d\xi)_{\xi=\xi_1} \approx -3^{1/2}/12\xi_1^2, \qquad (\xi_1 \gg 1), \tag{2.4.167}$$

and finally

$$(-\xi^2\theta')_{\xi=\xi_1} \approx 3^{1/2}[1 + (n-5)/12], \qquad (n \approx 5;\ n \leq 5;\ N = 3). \tag{2.4.168}$$

Eq. (2.4.168) shows that $(-\xi^2\theta')_{\xi=\xi_1}$ would decrease when the polytropic index becomes somewhat less than 5, a fact fully confirmed by the numerical integrations of Seidov and Kuzakhmedov (1978). While $(-\xi^2\theta')_{\xi=\xi_1}$ decreases from 17.79380 to $3^{1/2} = 1.732051$ if n increases from -0.9 to 5, $(N = 3$, see Table 2.5.2), it appears that its minimum is not attained for $n = 5$, but for $n \approx 4.823$ at a value of about 1.72335284 (cf. Table 2.4.3). The density ratio is obtained with the values already derived in Eqs. (2.4.166), (2.4.168):

$$\varrho_0/\varrho_m = (-\xi/3\theta')_{\xi=\xi_1} \approx [32/\pi(5-n)]^3 = 1056.82/(5-n)^3, \qquad (n \approx 5;\ n \leq 5;\ N = 3). \tag{2.4.169}$$

2.4.6 Approximate Solutions by the Method of Multiple Scales

The method of multiple scales involves an extension of the variables into a space of higher dimension. This calls for the introduction of new variables, and as a result of this process, ordinary differential equations are converted into a set of partial differential equations. These are solved approximately, and the solutions are required to coincide along certain lines – the "trajectories" – with the solutions of the original differential equation (Ramnath 1971). The following discussion is pertinent only for the spherical case $N = 3$, when $n \neq -1, \pm\infty$.

At first we transform the original Lane-Emden equation (2.1.14) in a way similar to Kelvin's transformation from Sec. 2.2.4. Putting

$$\theta = \chi/\xi, \tag{2.4.170}$$

Eq. (2.1.14) becomes

$$d^2\chi/d\xi^2 \pm \xi^{1-n}\chi^n = 0, \qquad (\chi = \chi(\xi);\ N = 3;\ n \neq -1, \pm\infty). \tag{2.4.171}$$

Replacing the power ξ^{1-n} by ξ^σ we can write this equation in the more general form

$$d^2\chi/d\xi^2 \pm \xi^\sigma\chi^n = 0. \tag{2.4.172}$$

To apply the multiple scale technique, it is useful to parameterize Eq. (2.4.172) further, by taking

$$\xi = \varepsilon^{1/(\sigma+2)}t, \qquad (0 < \varepsilon \ll 1;\ \sigma \neq -2). \tag{2.4.173}$$

Eq. (2.4.172) becomes

$$d^2\chi/dt^2 \pm \varepsilon t^\sigma \chi^n = 0, \qquad [\chi = \chi(t)]. \tag{2.4.174}$$

In the particular case $\sigma = -2$, we transform the function $\chi(\xi)$ by taking

$$\chi(\xi) = \varepsilon^{1/(n-1)} z(\xi), \qquad (0 < \varepsilon \ll 1;\ \sigma = -2;\ n \neq 1), \tag{2.4.175}$$

and Eq. (2.4.172) assumes the same form as Eq. (2.4.174):

$$d^2z/d\xi^2 \pm \varepsilon \xi^{-2} z^2 = 0, \qquad (\sigma = -2;\ n \neq 1). \tag{2.4.176}$$

Thus, the Lane-Emden equation (2.1.14) can be transformed into an equation of the form (2.4.174) or (2.4.176), provided that $n \neq \pm 1, \pm\infty,\ (N = 3)$. The extension of the variable t into bidimensional space is now effected with the transformation

$$x = t; \quad y = \varepsilon K(t) = \varepsilon K(x); \quad F(x,y) = \chi(t), \tag{2.4.177}$$

where $K(t)$ is a scale function that has to be determined. The original function $\chi(t)$ turns in (x,y)-space into the function $F(x,y)$. With the extension of variable from Eq. (2.4.177) we can write instead of Eq. (2.4.174)

$$d^2\chi(t)/dt^2 \pm \varepsilon t^\sigma \chi^n(t) = \partial^2 F/\partial x^2 + \varepsilon\big[2\ (\partial^2 F/\partial x \partial y)\ dK/dx + (\partial F/\partial y)\ d^2K/dx^2 \pm x^\sigma F^n\big]$$
$$+\varepsilon^2 (dK/dx)^2\ \partial^2 F/\partial y^2 = 0. \tag{2.4.178}$$

Equating equal powers of ε, we arrive at the system of partial differential equations

$$\partial^2 F/\partial x^2 = 0, \tag{2.4.179}$$

$$(d^2K/dx^2)\ \partial F/\partial y + 2\ (dK/dx)\ \partial^2 F/\partial x \partial y = \mp x^\sigma F^n, \tag{2.4.180}$$

$$(dK/dx)^2\ \partial^2 F/\partial y^2 = 0. \tag{2.4.181}$$

From the integration of Eq. (2.4.179) we find at once

$$F = F(x,y) = A(y)\ x + B(y). \tag{2.4.182}$$

The two terms of this sum are linearly independent with respect to x, and we endeavour to obtain a solution of the system (2.4.179)-(2.4.181) by substituting separately $A(y)\ x$ and $B(y)$ into Eq. (2.4.180).

(i) $\boldsymbol{F(x,y) = A(y)\ x}$**.** Separation of variables in Eq. (2.4.180) yields

$$A^{-n}\ dA(y)/dy = \mp x^{\sigma+n}/\big[x\ d^2K(x)/dx^2 + 2\ dK(x)/dx\big] = C_1', \qquad (C_1' = \text{const}). \tag{2.4.183}$$

The left-hand side of Eq. (2.4.183) is a function of y, and the right-hand side a function of x alone. The two sides can only be equated, if they are equal to the same constant C_1'. Thus, we have to solve separately the simple equations

$$dA/dy = C_1' A^n, \tag{2.4.184}$$

and

$$d^2K/dx^2 + (2/x)\ dK/dx = \mp x^{\sigma+n-1}/C_1'. \tag{2.4.185}$$

The integration of Eq. (2.4.184) is elementary

$$A = [C_1'(1-n)y + C_2']^{1/(1-n)}, \qquad (C_1', C_2' = \text{const}), \tag{2.4.186}$$

while Eq. (2.4.185) can be integrated with the standard method for linear, nonhomogeneous second order differential equations, as outlined in Eqs. (2.4.137)-(2.4.146):

$$K = \mp x^{\sigma+n+1}/C_1'(\sigma+n+1)(\sigma+n+2) + C_3'/x + C_4', \quad (\sigma + n \neq -1, -2;\ C_1', C_3', C_4' = \text{const}). \tag{2.4.187}$$

(ii) $F(x,y)=B(y)$. Instead of Eq. (2.4.183) we now have

$$B^{-n}\ dB/dy = \mp x^\sigma/(d^2K/dx^2) = D_1', \qquad (D_1' = \mathrm{const}), \tag{2.4.188}$$

with the integrals

$$B = [D_1'(1-n)y + D_2']^{1/(1-n)}, \qquad (D_1', D_2' = \mathrm{const}), \tag{2.4.189}$$

and

$$K = \mp x^{\sigma+2}/D_1'(\sigma+1)(\sigma+2) + D_3'x + D_4', \qquad (\sigma \neq -1,-2;\ D_1', D_3', D_4' = \mathrm{const}). \tag{2.4.190}$$

We turn back to the original variables by applying the restrictions (2.4.177) to Eq. (2.4.182):

$$\begin{aligned}\chi(t) &= F(x,y) = F[t,\varepsilon K(t)] = A[\varepsilon K(t)]t + B[\varepsilon K(t)]\\ &= \big[\mp\varepsilon(1-n)t^{\sigma+n+1}/(\sigma+n+1)(\sigma+n+2) + C_1/t + C_2\big]^{1/(1-n)}t\\ &+\big[\mp\varepsilon(1-n)t^{\sigma+2}/(\sigma+1)(\sigma+2) + D_1t + D_2\big]^{1/(1-n)},\\ &(n\neq\pm1,\pm\infty;\ \sigma+n,\ \sigma\neq-1,-2;\ C_1,C_2,D_1,D_2=\mathrm{const}).\end{aligned} \tag{2.4.191}$$

We have considered Eqs. (2.4.186), (2.4.187), (2.4.189), (2.4.190), and have matched the primed integration constants with the new ones. The previous equation is converted into the original variables via Eqs. (2.4.170), (2.4.173):

$$\begin{aligned}\theta(\xi) &= \chi(\xi)/\xi = \big[\mp(1-n)\xi^{\sigma+n+1}/(\sigma+n+1)(\sigma+n+2) + c_1/\xi + c_2\big]^{1/(1-n)}\\ &+\xi^{-1}\big[\mp(1-n)\xi^{\sigma+2}/(\sigma+1)(\sigma+2) + d_1\xi + d_2\big]^{1/(1-n)},\\ &(n\neq\pm1,\pm\infty;\ \sigma+n,\ \sigma\neq-1,-2;\ c_1,c_2,d_1,d_2=\mathrm{const}).\end{aligned} \tag{2.4.192}$$

The parameter ε cancels out from the first term within the brackets, and is matched with the new integration constants for the other terms. Comparing Eq. (2.4.171) with Eq. (2.4.172) we see at once that the case of practical interest occurs if $\sigma = 1-n$. Eq. (2.4.192) becomes

$$\begin{aligned}\theta(\xi) &= \big[\mp(1-n)\xi^2/6 + c_1/\xi + c_2\big]^{1/(1-n)}\\ &+\xi^{-1}\big[\mp(1-n)\xi^{3-n}/(n-2)(n-3) + d_1\xi + d_2\big]^{1/(1-n)}, \qquad (n\neq\pm1,2,3,\pm\infty).\end{aligned} \tag{2.4.193}$$

To get the expansion near the origin $\xi=0$, we ignore the last bracket in Eq. (2.4.193) and take $c_1=0,\ c_2=1$, in order to satisfy the initial condition $\theta(0)=1$. Eq. (2.4.193) becomes

$$\begin{aligned}\theta(\xi) &= [1\mp(1-n)\xi^2/6]^{1/(1-n)} \approx 1\mp\xi^2/3! + (5/3)n\xi^4/5! \mp (7/3)(10n^2-5n)\xi^6/3\times7! + ...,\\ &(\xi\approx0).\end{aligned} \tag{2.4.194}$$

Only the first two terms coincide with the exact expansion (2.4.24). Excepting for the constant factor, the behaviour of the singular solutions (2.3.92) and (2.3.93) can be simulated, if we choose $d_1=d_2=0$, and ignore the first bracket in Eq. (2.4.193):

$$\theta(\xi) = [\pm(n-2)(n-3)/(n-1)\xi^2]^{1/(n-1)}, \quad (-\infty<n<-1;\ 1<n<2;\ 3<n<\infty). \tag{2.4.195}$$

Although the method of multiple scales, as devised by Ramnath (1971), appears to be the most sophisticated approximation to the Lane-Emden functions, it provides the poorest results.

2.5 Exact Numerical Solutions

In Table 2.5.1 we present seven digit numerical solutions of the Lane-Emden equations (2.1.14) and (2.1.21) for the plane-parallel, cylindrical, and spherical case ($N = 1, 2, 3$) for polytropic indices $n = -10, -5, -2, -1.01, -0.9, -0.5$, 0, 0.5, 1, 2, 5, 20, $\pm\infty$, supplemented by n =1.5, 2.5, 3, 3.5, 4, 4.5, 4.99, and 6 for the spherical case.

Tabulations of Lane=Emden functions have been effected by Emden (1907) if $0 \le n \le 6$, $(N = 3)$, Sadler and Miller (1932) if $1 \le n \le 5$, $(N = 3)$, Chandrasekhar and Wares (1949) if $n = \pm\infty$, $(N = 3)$, Ostriker (1965) if $1 \le n \le 20$, $(N = 2)$, Tascione (1972) if $0.5 \le n \le 25$, $(N = 1)$, Viala and Horedt (1974b) if $-10 \le n < 0$, $(N = 1, 2, 3)$, and Horedt (1986b) if $-\infty \le n \le \infty$, $(N = 1, 2, 3)$.

Table 2.5.1 Numerical values of Lane-Emden functions. Finite boundary values are also shown in Table 2.4.3 if $N = 3$, $4.75 \le n \le 5$, and in Table 2.5.2. $aE + b$ means $a \times 10^b$.

ξ	θ	θ'	θ^{n+1}
		Polytropic Slabs ($N = 1$)	
		$n = -10$	
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	1.004959E+00	9.836838E−02	9.564565E−01
5.000E−01	1.105577E+00	3.635537E−01	4.052290E−01
1.000E+00	1.315256E+00	4.509497E−01	8.489983E−02
5.000E+00	3.193682E+00	4.713977E−01	2.893180E−05
1.000E+01	5.550699E+00	4.714045E−01	1.999269E−07
1.000E+02	4.797711E+01	4.714045E−01	7.424952E−16
5.000E+02	2.365389E+02	4.714045E−01	4.313965E−22
1.000E+03	4.722412E+02	4.714045E−01	8.561017E−25
∞	∞	4.714045E−01	0.000000E+00
		$n = -5$	
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	1.004979E+00	9.917613E−02	9.803282E−01
5.000E−01	1.114013E+00	4.187533E−01	6.492914E−01
1.000E+00	1.376785E+00	6.007019E−01	2.783144E−01
5.000E+00	4.136961E+00	7.058987E−01	3.414086E−03
5.000E+01	3.595441E+01	7.071066E−01	5.983995E−07
1.000E+02	7.130975E+01	7.071068E−01	3.867269E−08
5.000E+02	3.541525E+02	7.071068E−01	6.356806E−11
1.000E+03	7.077059E+02	7.071068E−01	3.986473E−12
∞	∞	7.071068E−01	0.000000E+00
		$n = -2$	
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	1.004992E+00	9.966849E−02	9.950331E−01
5.000E−01	1.120219E+00	4.632873E−01	8.926824E−01
1.000E+00	1.437714E+00	7.803223E−01	6.955485E−01
5.000E+00	6.045074E+00	1.291957E+00	1.654240E−01
1.000E+01	1.270838E+01	1.357433E+00	7.868823E−02
5.000E+01	6.841028E+01	1.403839E+00	1.461769E−02
1.000E+02	1.387645E+02	1.409109E+00	7.206453E−03
5.000E+02	7.036360E+02	1.413208E+00	1.421189E−03
1.000E+03	1.410395E+03	1.413712E+00	7.090213E−04
∞	∞	1.414214E+00	0.000000E+00
		$n = -1.01$	
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	1.004996E+00	9.983226E−02	9.999502E−01
5.000E−01	1.122513E+00	4.806305E−01	9.988450E−01

ξ	θ	θ'	θ^{n+1}
1.000E+00	1.465450E+00	8.734220E−01	9.961857E−01
5.000E+00	7.907860E+00	2.023178E+00	9.795338E−01
1.000E+01	1.913896E+01	2.411882E+00	9.709141E−01
5.000E+01	1.334280E+02	3.090552E+00	9.522424E−01
1.000E+02	2.945844E+02	3.324743E+00	9.447304E−01
5.000E+02	1.744939E+03	3.792806E+00	9.280731E−01
1.000E+03	3.691542E+03	3.971276E+00	9.211448E−01
∞	∞	1.414214E+01	0.000000E+00
	$n = -0.9$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.949962E−01	−1.001505E−01	9.994985E−01
5.000E−01	8.725164E−01	−5.204771E−01	9.864552E−01
1.000E+00	4.500706E−01	−1.238801E+00	9.232686E−01
1.200E+00	1.501963E−01	−1.858464E+00	8.273055E−01
1.260E+00	2.457023E−02	−2.488755E+00	6.903048E−01
1.26907230E+00	0.000000E+00	−4.472136E+00	0.000000E+00
	$n = -0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.949979E−01	−1.000835E−01	9.974958E−01
5.000E−01	8.736400E−01	−5.111277E−01	9.346871E−01
1.000E+00	4.744121E−01	−1.115749E+00	6.887758E−01
1.200E+00	2.191387E−01	−1.458599E+00	4.681226E−01
1.300E+00	6.083209E−02	−1.735924E+00	2.466416E−01
1.330E+00	6.484284E−03	−1.917785E+00	8.052505E−02
1.33333333E+00	0.000000E+00	−2.000000E+00	0.000000E+00
	$n = 0$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950000E−01	−1.000000E−01	9.950000E−01
5.000E−01	8.750000E−01	−5.000000E−01	8.750000E−01
1.000E+00	5.000000E−01	−1.000000E+00	5.000000E−01
1.300E+00	1.550000E−01	−1.300000E+00	1.550000E−01
1.400E+00	2.000000E−02	−1.400000E+00	2.000000E−02
1.410E+00	5.950000E−03	−1.410000E+00	5.950000E−03
1.41421356E+00	0.000000E+00	−1.414214E+00	0.000000E+00
	$n = 0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950021E−01	−9.991662E−02	9.925125E−01
5.000E−01	8.763133E−01	−4.894474E−01	8.203309E−01
1.000E+00	5.216345E−01	−9.115944E−01	3.767467E−01
1.400E+00	1.073903E−01	−1.134200E+00	3.519232E−02
1.450E+00	5.030978E−02	−1.148167E+00	1.128440E−02
1.490E+00	4.235670E−03	−1.154541E+00	2.756660E−04
1.49366840E+00	0.000000E+00	−1.154701E+00	0.000000E+00
	$n = 1$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950042E−01	−9.983342E−02	9.900333E−01
5.000E−01	8.775826E−01	−4.794255E−01	7.701512E−01
1.000E+00	5.403023E−01	−8.414710E−01	2.919266E−01
1.500E+00	7.073720E−02	−9.974950E−01	5.003752E−03
1.560E+00	1.079612E−02	−9.999417E−01	1.165561E−04
1.570E+00	7.963267E−04	−9.999997E−01	6.341362E−07
1.57079633E+00	0.000000E+00	−1.000000E+00	0.000000E+00
	$n = 2$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950083E−01	−9.966750E−02	9.850996E−01

ξ	θ	θ'	θ^{n+1}
5.000E−01	8.799988E−01	−4.608187E−01	6.814692E−01
1.000E+00	5.711855E−01	−7.365003E−01	1.863509E−01
1.500E+00	1.773134E−01	−8.142175E−01	5.574739E−03
1.700E+00	1.413791E−02	−8.164954E−01	2.825894E−06
1.710E+00	5.972952E−03	−8.164965E−01	2.130919E−07
1.71731534E+00	0.000000E+00	−8.164966E−01	0.000000E+00
	$n = 5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950207E−01	−9.917369E−02	9.704937E−01
5.000E−01	8.864138E−01	−4.142920E−01	4.850864E−01
1.000E+00	6.339554E−01	−5.582962E−01	6.491610E−02
1.500E+00	3.482555E−01	−5.768351E−01	1.783972E−03
2.000E+00	5.962479E−02	−5.773503E−01	4.493254E−08
2.100E+00	1.889759E−03	−5.773503E−01	4.554472E−17
2.10327316E+00	0.000000E+00	−5.773503E−01	0.000000E+00
	$n = 20$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	9.950813E−01	−9.679027E−02	9.016323E−01
5.000E−01	9.073945E−01	−2.878603E−01	1.299329E−01
1.000E+00	7.558668E−01	−3.081742E−01	2.801240E−03
2.000E+00	4.473082E−01	−3.086067E−01	4.599863E−08
3.000E+00	1.387015E−01	−3.086067E−01	9.631791E−19
3.400E+00	1.525886E−02	−3.086067E−01	7.144386E−39
3.440E+00	2.914594E−03	−3.086067E−01	5.703465E−54
3.44944436E+00	0.000000E+00	−3.086067E−01	0.000000E+00

ξ	θ	θ'	$\exp(-\theta)$
	$n = \pm\infty$		
0.000E+00	0.000000E+00	0.000000E+00	1.000000E+00
1.000E−01	4.995839E−03	9.983367E−02	9.950166E−01
5.000E−01	1.224795E−01	4.801582E−01	8.847241E−01
1.000E+00	4.631626E−01	8.610572E−01	6.292903E−01
5.000E+00	5.686471E+00	1.411813E+00	3.391539E−03
1.000E+01	1.275584E+01	1.414212E+00	2.885412E−06
5.000E+01	6.932438E+01	1.414214E+00	7.812727E−31
1.000E+02	1.400351E+02	1.414214E+00	1.525968E−61
5.000E+02	7.057205E+02	1.414214E+00	0.000000E+00
1.000E+03	1.412827E+03	1.414214E+00	0.000000E+00
∞	∞	1.414214E+00	0.000000E+00

ξ	θ	θ'	θ^{n+1}	$\xi\theta'$
		Polytropic Cylinders ($N = 2$)		
		$n = -10$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.002485E+00	4.938322E−02	9.779147E−01	4.938322E−03
5.000E−01	1.054468E+00	1.912979E−01	6.204418E−01	9.564896E−02
1.000E+00	1.163980E+00	2.289803E−01	2.549712E−01	2.289803E−01
5.000E+00	1.767163E+00	9.586834E−02	5.949908E−03	4.793417E−01
1.000E+01	2.116863E+00	5.263780E−02	1.171563E−03	5.263780E−01
5.000E+01	3.023880E+00	1.193259E−02	4.730634E−05	5.966293E−01
1.000E+02	3.446982E+00	6.245449E−03	1.455624E−05	6.245449E−01
5.000E+02	4.515506E+00	1.421744E−03	1.281274E−06	7.108722E−01
1.000E+03	5.027160E+00	7.684533E−04	4.876358E−07	7.684533E−01
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.002492E+00	4.968970E−02	9.900929E−01	4.968970E−03

ξ	θ	θ'	θ^{n+1}	$\xi\theta'$
5.000E−01	1.058124E+00	2.167741E−01	7.977266E−01	1.083871E−01
1.000E+00	1.195872E+00	3.151563E−01	4.889470E−01	3.151563E−01
5.000E+00	2.301819E+00	2.085767E−01	3.562175E−02	1.042884E+00
1.000E+01	3.112968E+00	1.292417E−01	1.064882E−02	1.292417E+00
5.000E+01	5.647209E+00	3.763304E−02	9.832512E−04	1.881652E+00
1.000E+02	7.065343E+00	2.225516E−02	4.012980E−04	2.225516E+00
5.000E+02	1.159867E+01	7.148806E−03	5.525436E−05	3.574403E+00
1.000E+03	1.439466E+01	4.542467E−03	2.329133E−05	4.542467E+00
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -2$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.002497E+00	4.987542E−02	9.975093E−01	4.987542E−03
5.000E−01	1.060648E+00	2.355674E−01	9.428195E−01	1.177837E−01
1.000E+00	1.224220E+00	4.055534E−01	8.168465E−01	4.055534E−01
5.000E+00	3.413257E+00	5.428001E−01	2.929753E−01	2.714001E+00
1.000E+01	5.863892E+00	4.444409E−01	1.705352E−01	4.444409E+00
5.000E+01	1.818926E+01	2.417748E−01	5.497750E−02	1.208874E+01
1.000E+02	2.870520E+01	1.878095E−01	3.483690E−02	1.878095E+01
5.000E+02	8.253684E+01	1.091720E−01	1.211580E−02	5.458598E+01
1.000E+03	1.307404E+02	8.701058E−02	7.648744E−03	8.701058E+01
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -1.01$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.002498E+00	4.993701E−02	9.999750E−01	4.993701E−03
5.000E−01	1.061547E+00	2.424989E−01	9.994029E−01	1.212494E−01
1.000E+00	1.236086E+00	4.475223E−01	9.978827E−01	4.475223E−01
5.000E+00	4.448128E+00	9.589581E−01	9.851860E−01	4.794791E+00
1.000E+01	9.429328E+00	1.013153E+00	9.778116E−01	1.013153E+01
5.000E+01	4.955395E+01	9.906663E−01	9.617213E−01	4.953331E+01
1.000E+02	9.875060E+01	9.794248E−01	9.551127E−01	9.794248E+01
5.000E+02	4.873803E+02	9.684580E−01	9.399859E−01	4.842290E+02
1.000E+03	9.708283E+02	9.656582E−01	9.335306E−01	9.656582E+02
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -0.9$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.974986E−01	−5.005636E−02	9.997496E−01	−5.005636E−03
5.000E−01	9.365911E−01	−2.573970E−01	9.934706E−01	−1.286985E−01
1.000E+00	7.336948E−01	−5.707421E−01	9.695083E−01	−5.707421E−01
1.500E+00	3.289180E−01	−1.137038E+00	8.947645E−01	−1.705557E+00
1.700E+00	4.082342E−02	−2.033763E+00	7.262580E−01	−3.457397E+00
1.716E+00	5.080142E−03	−2.602816E+00	5.896409E−01	−4.466432E+00
1.71782384E+00	0.000000E+00	−4.307079E+00	0.000000E+00	−7.398803E+00
		$n = -0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.974992E−01	−5.003130E−02	9.987488E−01	−5.003130E−03
5.000E−01	9.369994E−01	−2.540556E−01	9.679873E−01	−1.270278E−01
1.000E+00	7.413013E−01	−5.368733E−01	8.609886E−01	−5.368733E−01
1.500E+00	3.848069E−01	−9.184839E−01	6.203280E−01	−1.377726E+00
1.800E+00	5.083987E−02	−1.403693E+00	2.254770E−01	−2.526648E+00
1.830E+00	6.632797E−03	−1.573105E+00	8.144199E−02	−2.878783E+00
1.83413266E+00	0.000000E+00	−1.669972E+00	0.000000E+00	−3.062950E+00
		$n = 0$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975000E−01	−5.000000E−02	9.975000E−01	−5.000000E−03
5.000E−01	9.375000E−01	−2.500000E−01	9.375000E−01	−1.250000E−01

ξ	θ	θ'	θ^{n+1}	$\xi\theta'$
1.000E+00	7.500000E−01	−5.000000E−01	7.500000E−01	−5.000000E−01
1.500E+00	4.375000E−01	−7.500000E−01	4.375000E−01	−1.125000E+00
1.900E+00	9.750000E−02	−9.500000E−01	9.750000E−02	−1.805000E+00
2.00000000E+00	0.000000E+00	−1.000000E+00	0.000000E+00	−2.000000E+00
		$n = 0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975008E−01	−4.996874E−02	9.962535E−01	−4.996874E−03
5.000E−01	9.379900E−01	−2.460730E−01	9.084422E−01	−1.230365E−01
1.000E+00	7.579284E−01	−4.680380E−01	6.598456E−01	−4.680380E−01
1.500E+00	4.785050E−01	−6.383440E−01	3.310014E−01	−9.575159E−01
2.000E+00	1.347795E−01	−7.156315E−01	4.948068E−02	−1.431263E+00
2.150E+00	2.784930E−02	−7.059669E−01	4.647521E−03	−1.517829E+00
2.180E+00	6.752620E−03	−7.001004E−01	5.548920E−04	−1.526219E+00
2.18966219E+00	0.000000E+00	−6.975389E−01	0.000000E+00	−1.527374E+00
		$n = 1$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975016E−01	−4.993753E−02	9.950094E−01	−4.993753E−03
5.000E−01	9.384698E−01	−2.422685E−01	8.807256E−01	−1.211342E−01
1.000E+00	7.651977E−01	−4.400506E−01	5.855275E−01	−4.400506E−01
1.500E+00	5.118277E−01	−5.579365E−01	2.619676E−01	−8.369048E−01
2.000E+00	2.238908E−01	−5.767248E−01	5.012708E−02	−1.153450E+00
2.400E+00	2.507683E−03	−5.201853E−01	6.288476E−06	−1.248445E+00
2.40482556E+00	0.000000E+00	−5.191475E−01	0.000000E+00	−1.248459E+00
		$n = 2$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975031E−01	−4.987521E−02	9.925281E−01	−4.987521E−03
5.000E−01	9.394002E−01	−2.350047E−01	8.289951E−01	−1.175023E−01
1.000E+00	7.780992E−01	−3.933192E−01	4.710911E−01	−3.933192E−01
2.000E+00	3.426523E−01	−4.255694E−01	4.023100E−02	−8.511388E−01
2.900E+00	6.778285E−03	−3.190870E−01	3.114293E−07	−9.253523E−01
2.920E+00	4.184451E−04	−3.169018E−01	7.326820E−11	−9.253533E−01
2.92132072E+00	0.000000E+00	−3.167585E−01	0.000000E+00	−9.253533E−01
		$n = 5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975078E−01	−4.968918E−02	9.851396E−01	−4.968918E−03
5.000E−01	9.419802E−01	−2.156290E−01	6.986357E−01	−1.078145E−01
1.000E+00	8.076454E−01	−2.992337E−01	2.775391E−01	−2.992337E−01
4.000E+00	1.624658E−01	−1.330609E−01	1.838954E−05	−5.322436E−01
5.000E+00	4.368123E−02	−1.064689E−01	6.946548E−09	−5.323444E−01
5.400E+00	2.711449E−03	−9.858230E−02	3.973825E−16	−5.323444E−01
5.420E+00	7.434453E−04	−9.821853E−02	1.688473E−19	−5.323444E−01
5.42757459E+00	0.000000E+00	−9.808145E−02	0.000000E+00	−5.323444E−01
		$n = 20$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.975308E−01	−4.877951E−02	9.494065E−01	−4.877951E−03
5.000E−01	9.515602E−01	−1.528401E−01	3.525019E−01	−7.642005E−02
1.000E+00	8.762500E−01	−1.382735E−01	6.240053E−02	−1.382735E−01
5.000E+00	6.091291E−01	−3.505830E−02	3.012286E−05	−1.752915E−01
1.000E+01	4.874841E−01	−1.755776E−02	2.800078E−07	−1.755776E−01
5.000E+01	2.048883E−01	−3.511754E−03	3.482259E−15	−1.755877E−01
1.000E+02	8.318021E−02	−1.755877E−03	2.091320E−23	−1.755877E−01
1.500E+02	1.198554E−02	−1.170585E−03	4.485451E−41	−1.755877E−01
1.600E+02	6.533662E−04	−1.097423E−03	1.313075E−67	−1.755877E−01
1.60596473E+02	0.000000E+00	−1.093347E−03	0.000000E+00	−1.755877E−01

ξ	θ	θ'	$\exp(-\theta)$	$\xi\theta'$
		$n = \pm\infty$		
0.000E+00	0.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	2.498439E−03	4.993758E−02	9.975047E−01	4.993758E−03
5.000E−01	6.154332E−02	2.424242E−01	9.403122E−01	1.212121E−01
1.000E+00	2.355661E−01	4.444444E−01	7.901235E−01	4.444444E−01
5.000E+00	2.834132E+00	6.060606E−01	5.876951E−02	3.030303E+00
1.000E+01	5.205379E+00	3.703704E−01	5.486968E−03	3.703704E+00
5.000E+01	1.149560E+01	7.974482E−02	1.017478E−05	3.987241E+00
1.000E+02	1.426340E+01	3.996803E−02	6.389772E−07	3.996803E+00
5.000E+02	2.069961E+01	7.999744E−03	1.023934E−09	3.999872E+00
1.000E+03	2.347215E+01	3.999968E−03	6.399898E−11	3.999968E+00
∞	∞	0.000000E+00	0.000000E+00	4.000000E+00

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
		Polytropic Spheres ($N = 3$)		
		$n = -10$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.001658E+00	3.300334E−02	9.851975E−01	3.300334E−04
5.000E−01	1.037197E+00	1.333955E−01	7.198627E−01	3.334887E−02
1.000E+00	1.115647E+00	1.670811E−01	3.734730E−01	1.670811E−01
5.000E+00	1.540827E+00	6.274943E−02	2.042750E−02	1.568736E+00
1.000E+01	1.761992E+00	3.249452E−02	6.108914E−03	3.249452E+00
5.000E+01	2.342434E+00	8.300094E−03	4.709708E−04	2.075023E+01
1.000E+02	2.652366E+00	4.806458E−03	1.539213E−04	4.806458E+01
5.000E+02	3.559869E+00	1.301230E−03	1.089165E−05	3.253074E+02
1.000E+03	4.039407E+00	7.348356E−04	3.492538E−06	7.348356E+02
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.001663E+00	3.316755E−02	9.933775E−01	3.316755E−04
5.000E−01	1.039273E+00	1.482851E−01	8.571975E−01	3.707128E−02
1.000E+00	1.135601E+00	2.243041E−01	6.013083E−01	2.243041E−01
5.000E+00	1.928973E+00	1.447372E−01	7.222633E−02	3.618429E+00
1.000E+01	2.478760E+00	8.620524E−02	2.648878E−02	8.620524E+00
5.000E+01	4.225452E+00	2.765166E−02	3.136947E−03	6.912914E+01
1.000E+02	5.308688E+00	1.759050E−02	1.259074E−03	1.759050E+02
5.000E+02	9.083411E+00	6.071909E−03	1.468940E−04	1.517977E+03
1.000E+03	1.144890E+01	3.819149E−03	5.820287E−05	3.819149E+03
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -2$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.001665E+00	3.326683E−02	9.983378E−01	3.326683E−04
5.000E−01	1.040666E+00	1.588211E−01	9.609228E−01	3.970529E−02
1.000E+00	1.152316E+00	2.798032E−01	8.678178E−01	2.798032E−01
5.000E+00	2.712219E+00	3.859601E−01	3.687019E−01	9.649001E+00
1.000E+01	4.435735E+00	3.101642E−01	2.254418E−01	3.101642E+01
5.000E+01	1.313928E+01	1.742879E−01	7.610765E−02	4.357198E+02
1.000E+02	2.081351E+01	1.383030E−01	4.804571E−02	1.383030E+03
5.000E+02	6.081090E+01	8.111227E−02	1.644442E−02	2.027807E+04
1.000E+03	9.654558E+01	6.437779E−02	1.035780E−02	6.437779E+04
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -1.01$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.001666E+00	3.329972E−02	9.999834E−01	3.329972E−04
5.000E−01	1.041154E+00	1.626152E−01	9.995968E−01	4.065379E−02
1.000E+00	1.159022E+00	3.041212E−01	9.985253E−01	3.041212E−01

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
5.000E+00	3.409439E+00	6.751255E−01	9.878094E−01	1.687814E+01
1.000E+01	6.898565E+00	7.067390E−01	9.808722E−01	7.067390E+01
5.000E+01	3.489717E+01	6.947161E−01	9.650995E−01	1.736790E+03
1.000E+02	6.952768E+01	6.912177E−01	9.584698E−01	6.912177E+03
5.000E+02	3.446515E+02	6.858592E−01	9.432487E−01	1.714648E+05
1.000E+03	6.869356E+02	6.835353E−01	9.367654E−01	6.835353E+05
∞	∞	0.000000E+00	0.000000E+00	∞
		$n = -0.9$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983326E−01	−3.336338E−02	9.998331E−01	−3.336338E−04
5.000E−01	9.578529E−01	−1.705588E−01	9.957031E−01	−4.263971E−02
1.000E+00	8.249992E−01	−3.685921E−01	9.809466E−01	−3.685921E−01
2.000E+00	9.450454E−02	−1.572580E+00	7.898512E−01	−6.290319E+00
2.040E+00	2.396453E−02	−2.061745E+00	6.885839E−01	−8.580159E+00
2.050E+00	1.175205E−03	−2.780906E+00	5.093442E−01	−1.168676E+01
2.05040073E+00	0.000000E+00	−4.232444E+00	0.000000E+00	−1.779380E+01
		$n = -0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983329E−01	−3.335002E−02	9.991661E−01	−3.335002E−04
5.000E−01	9.580681E−01	−1.688077E−01	9.788096E−01	−4.220193E−02
1.000E+00	8.288357E−01	−3.520616E−01	9.104041E−01	−3.520616E−01
2.000E+00	2.320758E−01	−9.394628E−01	4.817425E−01	−3.757851E+00
2.208E+00	8.800112E−04	−1.482764E+00	2.966498E−02	−7.228864E+00
2.20858842E+00	0.000000E+00	−1.521468E+00	0.000000E+00	−7.421510E+00
		$n = 0$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983333E−01	−3.333333E−02	9.983333E−01	−3.333333E−04
5.000E−01	9.583333E−01	−1.666667E−01	9.583333E−01	−4.166667E−02
1.000E+00	8.333333E−01	−3.333333E−01	8.333333E−01	−3.333333E−01
2.000E+00	3.333333E−01	−6.666667E−01	3.333333E−01	−2.666667E+00
2.400E+00	4.000000E−02	−8.000000E−01	4.000000E−02	−4.608000E+00
2.44948974E+00	0.000000E+00	−8.164966E−01	0.000000E+00	−4.898979E+00
		$n = 0.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983338E−01	−3.331666E−02	9.975017E−01	−3.331666E−04
5.000E−01	9.585943E−01	−1.645770E−01	9.385388E−01	−4.114426E−02
1.000E+00	8.375345E−01	−3.164564E−01	7.664857E−01	−3.164564E−01
2.000E+00	4.025795E−01	−5.249758E−01	2.554333E−01	−2.099903E+00
2.700E+00	2.674118E−02	−5.138861E−01	4.372913E−03	−3.746230E+00
2.750E+00	1.350271E−03	−5.009125E−01	4.961712E−05	−3.788151E+00
2.75269805E+00	0.000000E+00	−4.999971E−01	0.000000E+00	−3.788651E+00
		$n = 1$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983342E−01	−3.330001E−02	9.966711E−01	−3.330001E−04
5.000E−01	9.588511E−01	−1.625370E−01	9.193954E−01	−4.063426E−02
1.000E+00	8.414710E−01	−3.011687E−01	7.080734E−01	−3.011687E−01
2.000E+00	4.546487E−01	−4.353978E−01	2.067055E−01	−1.741591E+00
3.000E+00	4.704000E−02	−3.456775E−01	2.212762E−03	−3.111097E+00
3.140E+00	5.072143E−04	−3.186325E−01	2.572663E−07	−3.141589E+00
3.14159265E+00	0.000000E+00	−3.183099E−01	0.000000E+00	−3.141593E+00
		$n = 1.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983346E−01	−3.328337E−02	9.958417E−01	−3.328337E−04
5.000E−01	9.591039E−01	−1.605449E−01	9.008741E−01	−4.013622E−02
1.000E+00	8.451698E−01	−2.872555E−01	6.566892E−01	−2.872555E−01

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
3.000E+00	1.588576E−01	−2.842527E−01	1.005820E−02	−2.558275E+00
3.600E+00	1.109099E−02	−2.093927E−01	1.295467E−05	−2.713729E+00
3.650E+00	7.639242E−04	−2.037196E−01	1.612968E−08	−2.714055E+00
3.65375374E+00	0.000000E+00	−2.033013E−01	0.000000E+00	−2.714055E+00
		$n = 2$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983350E−01	−3.326675E−02	9.950133E−01	−3.326675E−04
5.000E−01	9.593527E−01	−1.585990E−01	8.829476E−01	−3.964974E−02
1.000E+00	8.486541E−01	−2.745394E−01	6.112124E−01	−2.745394E−01
3.000E+00	2.418241E−01	−2.406215E−01	1.414160E−02	−2.165593E+00
4.000E+00	4.884015E−02	−1.504097E−01	1.165014E−04	−2.406555E+00
4.300E+00	6.810943E−03	−1.303965E−01	3.159525E−07	−2.411031E+00
4.350E+00	3.660302E−04	−1.274169E−01	4.904003E−11	−2.411046E+00
4.35287460E+00	0.000000E+00	−1.272487E−01	0.000000E+00	−2.411046E+00
		$n = 2.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983354E−01	−3.325015E−02	9.941861E−01	−3.325015E−04
5.000E−01	9.595978E−01	−1.566977E−01	8.655901E−01	−3.917443E−02
1.000E+00	8.519442E−01	−2.628722E−01	5.707409E−01	−2.628722E−01
4.000E+00	1.376807E−01	−1.340534E−01	9.684029E−04	−2.144855E+00
5.000E+00	2.901919E−02	−8.747353E−02	4.162922E−06	−2.186838E+00
5.300E+00	4.259544E−03	−7.786397E−02	5.043949E−09	−2.187199E+00
5.355E+00	2.100894E−05	−7.627276E−02	4.250249E−17	−2.187200E+00
5.35527546E+00	0.000000E+00	−7.626491E−02	0.000000E+00	−2.187200E+00
		$n = 3$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983358E−01	−3.323356E−02	9.933599E−01	−3.323356E−04
5.000E−01	9.598391E−01	−1.548396E−01	8.487772E−01	−3.870989E−02
1.000E+00	8.550576E−01	−2.521293E−01	5.345415E−01	−2.521293E−01
5.000E+00	1.108198E−01	−8.012604E−02	1.508238E−04	−2.003151E+00
6.000E+00	4.373798E−02	−5.604388E−02	3.659612E−06	−2.017580E+00
6.800E+00	4.167789E−03	−4.364697E−02	3.017332E−10	−2.018236E+00
6.896E+00	3.601115E−05	−4.244020E−02	1.681697E−18	−2.018236E+00
6.89684862E+00	0.000000E+00	−4.242976E−02	0.000000E+00	−2.018236E+00
		$n = 3.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983362E−01	−3.321699E−02	9.925349E−01	−3.321699E−04
5.000E−01	9.600768E−01	−1.530231E−01	8.324857E−01	−3.825579E−02
1.000E+00	8.580096E−01	−2.422051E−01	5.020126E−01	−2.422051E−01
5.000E+00	1.786843E−01	−7.362030E−02	4.309118E−04	−1.840508E+00
9.000E+00	1.180312E−02	−2.334019E−02	2.108560E−09	−1.890555E+00
9.500E+00	7.472341E−04	−2.094800E−02	8.522264E−15	−1.890557E+00
9.530E+00	1.207723E−04	−2.081632E−02	2.338050E−18	−1.890557E+00
9.53580534E+00	0.000000E+00	−2.079098E−02	0.000000E+00	−1.890557E+00
		$n = 4$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983367E−01	−3.320043E−02	9.917109E−01	−3.320043E−04
2.000E−01	9.933862E−01	−6.561355E−02	9.673656E−01	−2.624542E−03
5.000E−01	9.603109E−01	−1.512470E−01	8.166939E−01	−3.781176E−02
1.000E+00	8.608138E−01	−2.330096E−01	4.726570E−01	−2.330096E−01
5.000E+00	2.359227E−01	−6.788810E−02	7.308848E−04	−1.697203E+00
1.000E+01	5.967274E−02	−1.796142E−02	7.566237E−07	−1.796142E+00
1.400E+01	8.330527E−03	−9.169539E−03	4.012013E−11	−1.797230E+00
1.490E+01	5.764189E−04	−8.095266E−03	6.363425E−17	−1.797230E+00
1.49715463E+01	0.000000E+00	−8.018079E−03	0.000000E+00	−1.797230E+00

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
		$n = 4.5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983371E−01	−3.318389E−02	9.908881E−01	−3.318389E−04
5.000E−01	9.605416E−01	−1.495100E−01	8.013809E−01	−3.737749E−02
1.000E+00	8.634822E−01	−2.244656E−01	4.460604E−01	−2.244656E−01
5.000E+00	2.848977E−01	−6.286131E−02	1.001818E−03	−1.571533E+00
1.000E+01	1.189407E−01	−1.722137E−02	8.209531E−06	−1.722137E+00
2.000E+01	3.230429E−02	−4.344152E−03	6.323115E−09	−1.737661E+00
3.000E+01	3.341455E−03	−1.930888E−03	2.407943E−14	−1.737799E+00
3.140E+01	7.587396E−04	−1.762545E−03	6.926438E−18	−1.737799E+00
3.180E+01	6.258970E−05	−1.718483E−03	7.599165E−24	−1.737799E+00
3.18364632E+01	0.000000E+00	−1.714549E−03	0.000000E+00	−1.737799E+00
		$n = 4.99$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983375E−01	−3.316769E−02	9.900827E−01	−3.316769E−04
5.000E−01	9.607644E−01	−1.478442E−01	7.868198E−01	−3.696106E−02
1.000E+00	8.659757E−01	−2.166602E−01	4.223371E−01	−2.166602E−01
5.000E+00	3.265336E−01	−5.853396E−02	1.225823E−03	−1.463349E+00
1.000E+01	1.696936E−01	−1.658187E−02	2.430499E−05	−1.658187E+00
5.000E+01	3.361606E−02	−6.913154E−04	1.492848E−09	−1.728288E+00
1.000E+02	1.632192E−02	−1.730279E−04	1.970150E−11	−1.730279E+00
5.000E+02	2.477128E−03	−6.923051E−06	2.453300E−16	−1.730763E+00
1.000E+03	7.463631E−04	−1.730765E−06	1.857682E−19	−1.730765E+00
1.700E+03	3.369502E−05	−5.988807E−07	1.622253E−27	−1.730765E+00
1.75818915E+03	0.000000E+00	−5.598955E−07	0.000000E+00	−1.730765E+00
		$n = 5$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983375E−01	−3.316736E−02	9.900663E−01	−3.316736E−04
5.000E−01	9.607689E−01	−1.478106E−01	7.865271E−01	−3.695265E−02
1.000E+00	8.660254E−01	−2.165064E−01	4.218750E−01	−2.165064E−01
5.000E+00	3.273268E−01	−5.845122E−02	1.229956E−03	−1.461281E+00
1.000E+01	1.706640E−01	−1.656932E−02	2.470882E−05	−1.656932E+00
5.000E+01	3.462025E−02	−6.915751E−04	1.721794E−09	−1.728938E+00
1.000E+02	1.731791E−02	−1.731272E−04	2.697571E−11	−1.731272E+00
5.000E+02	3.464081E−03	−6.928079E−06	1.727938E−15	−1.732020E+00
1.000E+03	1.732048E−03	−1.732043E−06	2.699976E−17	−1.732043E+00
∞	0.000000E+00	0.000000E+00	0.000000E+00	−1.732051E+00
		$n = 6$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983383E−01	−3.313435E−02	9.884260E−01	−3.313435E−04
5.000E−01	9.612138E−01	−1.445204E−01	7.581235E−01	−3.613010E−02
1.000E+00	8.707732E−01	−2.021179E−01	3.796082E−01	−2.021179E−01
5.000E+00	3.973243E−01	−5.113662E−02	1.563206E−03	−1.278416E+00
1.000E+01	2.568119E−01	−1.537875E−02	7.367255E−05	−1.537875E+00
5.000E+01	1.215139E−01	−7.912354E−04	3.911844E−07	−1.978088E+00
1.000E+02	1.000997E−01	−2.421930E−04	1.007003E−07	−2.421930E+00
5.000E+02	6.833864E−02	−4.109422E−05	6.960885E−09	−1.027355E+01
1.000E+03	5.313600E−02	−2.270810E−05	1.195974E−09	−2.270810E+01
∞	0.000000E+00	0.000000E+00	0.000000E+00	∞
		$n = 20$		
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	9.983498E−01	−3.267875E−02	9.659117E−01	−3.267875E−04
5.000E−01	9.663460E−01	−1.098095E−01	4.872885E−01	−2.745239E−02
1.000E+00	9.108381E−01	−1.029657E−01	1.406905E−01	−1.029657E−01
5.000E+00	7.376408E−01	−1.785363E−02	1.677812E−03	−4.463409E−01

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
1.000E+01	6.828324E−01	−7.015413E−03	3.315862E−04	−7.015413E−01
5.000E+01	5.858471E−01	−1.128450E−03	1.328830E−05	−2.821125E+00
1.000E+02	5.465871E−01	−5.680053E−04	3.096327E−06	−5.680053E+00
5.000E+02	4.589993E−01	−1.001212E−04	7.907753E−08	−2.503031E+01
1.000E+03	4.259806E−01	−4.526416E−05	1.648873E−08	−4.526416E+01
∞	0.000000E+00	0.000000E+00	0.000000E+00	∞

ξ	θ	θ'	$\exp(-\theta)$	$\xi^2\theta'$
		$n = \pm\infty$		
0.000E+00	0.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E−01	1.665834E−03	3.330003E−02	9.983356E−01	3.330003E−04
5.000E−01	4.115396E−02	1.625969E−01	9.596814E−01	4.064923E−02
1.000E+00	1.588277E−01	3.029014E−01	8.531434E−01	3.029014E−01
5.000E+00	2.044092E+00	4.479695E−01	1.294977E−01	1.119924E+01
1.000E+01	3.736560E+00	2.510611E−01	2.383596E−02	2.510611E+01
5.000E+01	7.302273E+00	3.836249E−02	6.740049E−04	9.590622E+01
1.000E+02	8.596060E+00	1.842785E−02	1.848326E−04	1.842785E+02
5.000E+02	1.168443E+01	4.030609E−03	8.423977E−06	1.007652E+03
1.000E+03	1.309606E+01	2.047800E−03	2.053310E−06	2.047800E+03
∞	∞	0.000000E+00	0.000000E+00	∞

Table 2.5.2 Boundary values of Lane-Emden functions (see also Table 2.4.3 if $N = 3,\ 4.75 \leq n \leq 5$). $aE+b$ means $a \times 10^b$.

Polytropic Slabs ($N = 1$)

n	ξ_1	θ'_1	$-\xi_1/\theta'_1$
−0.9	1.26907230	−4.472136	2.837732E−1
−0.8	1.28498902	−3.162278	4.063492E−1
−0.5	1.33333333	−2.000000	6.666667E−1
−0.2	1.38194567	−1.581139	8.740192E−1
0	1.41421356	−1.414214	1.000000
0.5	1.49366840	−1.154701	1.293555
1	1.57079633	−1.000000	1.570796
1.5	1.64534085	−8.944272E−1	1.839547
2	1.71731534	−8.164966E−1	2.103273
3	1.85407468	−7.071068E−1	2.622058
4	1.98232217	−6.324555E−1	3.134327
5	2.10327316	−5.773503E−1	3.642976
6	2.21794979	−5.345225E−1	4.149404
10	2.62843161	−4.264014E−1	6.164219
20	3.44944436	−3.086067E−1	1.117748E+1

Polytropic Cylinders ($N = 2$)

n	ξ_1	θ'_1	$-\xi_1\theta'_1$	$-\xi_1/2\theta'_1$
−0.9	1.71782384	−4.307079	7.398803	1.994187E−1
−0.8	1.74556088	−2.935681	5.124411	2.973008E−1
−0.5	1.83413266	−1.669972	3.062950	5.491508E−1
−0.2	1.93087184	−1.193383	2.304271	8.089905E−1
0	2.00000000	−1.000000	2.000000	1.000000
0.5	2.18966219	−6.975389E−1	1.527374	1.569563
1	2.40482556	−5.191475E−1	1.248459	2.316129
1.5	2.64777677	−4.007569E−1	1.061115	3.303470
2	2.92132072	−3.167585E−1	9.253533E−1	4.611274
3	3.57390098	−2.070908E−1	7.401221E−1	8.628825
4	4.39526586	−1.407522E−1	6.186436E−1	1.561348E+1
5	5.42757459	−9.808145E−2	5.323444E−1	2.766871E+1
6	6.72452797	−6.954616E−2	4.676651E−1	4.834579E+1
10	1.62227407E+1	−1.947837E−2	3.159925E−1	4.164296E+2
20	1.60596473E+2	−1.093347E−3	1.755877E−1	7.344259E+4

Polytropic Spheres ($N = 3$)

n	ξ_1	θ_1'	$-\xi_1^2\theta_1'$	$-\xi_1/3\theta_1'$
−0.9	2.05040073	−4.232444	1.779380E+1	1.614828E−1
−0.8	2.08744257	−2.833273	1.234575E+1	2.455867E−1
−0.5	2.20858842	−1.521468	7.421510	4.838724E−1
−0.2	2.34663985	−1.020325	5.618645	7.666312E−1
0	2.44948974	−8.164966E−1	4.898979	1.000000
0.5	2.75269805	−4.999971E−1	3.788651	1.835143
1	3.14159265	−3.183099E−1	3.141593	3.289868
1.5	3.65375374	−2.033013E−1	2.714054	5.990705
2	4.35287460	−1.272487E−1	2.411046	1.140254E+1
2.5	5.35527546	−7.626491E−2	2.187200	2.340646E+1
3	6.89684862	−4.242976E−2	2.018236	5.418248E+1
3.5	9.53580534	−2.079098E−2	1.890557	1.528837E+2
4	1.49715463E+1	−8.018079E−3	1.797230	6.224079E+2
4.5	3.18364632E+1	−1.714549E−3	1.737799	6.189473E+3
4.99	1.75818915E+3	−5.598955E−7	1.730765	1.046736E+3
5	∞	0.000000	1.732051	∞

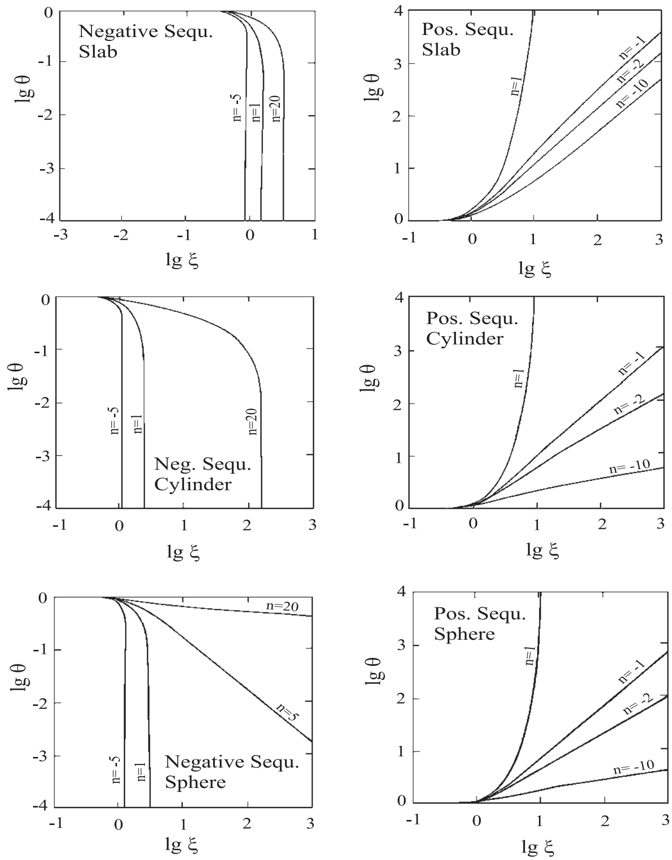

Fig. 2.5.1 Logarithmic plot of Lane-Emden variables for the negative and positive sequence of polytropic slabs, cylinders, and spheres obeying the initial conditions $\theta(0) = 1,\ \theta'(0) = 0$ (Horedt 1986a).

In Table 2.5.2 finite boundary values of polytropic slabs, cylinders, and spheres are summarized if $-1 < n < \infty$, $(N = 1, 2)$, and $-1 < n < 5$, $(N = 3)$. Comparing Eqs. (2.3.52) and (2.4.67) if $\theta = 10^{-80}$ and $n = -0.99$, we get $\theta' = -12.9731313$ and -14.1421356, respectively; this shows the inability of purely numerical iterations to reproduce even the second digit of θ' at the finite boundary ξ_1 if $n \approx -1$, $(n > -1;\ N = 1)$. If the polytropic index departs from $n = -1$ to $n = -0.9$, the values of θ' from Eqs. (2.3.52) and (2.4.67) for $\theta = 10^{-80}$ become -4.47213593 and -4.47213595, showing coincidence within the first eight digits. So, we have excluded from the seven digit tabulations the polytropic indices $-1 < n < -0.9$, since in this case the boundary value of the derivative θ' may be incorrect, as shown before for the case $N = 1$, $-1 < n < -0.9$.

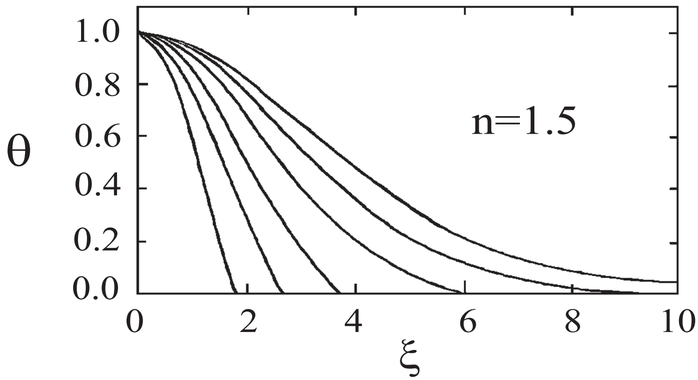

Fig. 2.5.2 The Lane-Emden functions of polytropic index $n = 1.5$ are shown from left to right in a space with dimension $N = 1, 2, 3, 5, 7, 10$, respectively (Abramowicz 1983).

If $-\infty < n < -1$, $n = \pm\infty$, $(N = 1, 2, 3)$ and $5 \leq n < \infty$, $(N = 3)$, the values for $\xi = \infty$ in Table 2.5.1 result from analytical solutions $(n = \pm\infty,\ N = 1, 2;\ n = 5,\ N = 3)$, asymptotic expansions, and from the topology of Lane-Emden functions (Secs. 2.3, 2.4, 2.7, Chandrasekhar 1939, Ostriker 1964a, Harrison and Lake 1972, Viala and Horedt 1974a, b, Kimura 1981a, Horedt 1986a, 1987a, b).

It has already been pointed out in Sec. 2.1, that from a formal mathematical viewpoint the Lane-Emden equation (2.1.14) can also be solved for $n = -1$ (Fig. 2.5.1), and for the imaginary sequence with imaginary radius $(r^2 < 0)$, when the plus sign in the equation for the radius (2.1.13) is associated with values of the polytropic index $-\infty < n < -1$, and the minus sign with $-1 < n < \infty$. If the right-hand side of Eq. (2.1.14) is equal to $-\theta^n$, the solutions form the negative sequence, while for $+\theta^n$ they belong to the positive sequence (Figs. 2.4.1, 2.5.1). Physically meaningful solutions occur for the negative sequence if $-1 < n < \infty$, and for the positive sequence when $-\infty < n < -1$. The solutions belong to the unphysical imaginary sequence if $-\infty < n \leq -1$ in the negative sequence, and $-1 \leq n < \infty$ in the positive sequence. The unphysical imaginary sequence holds for the isothermal equation (2.1.21), if we take $-\exp(-\theta)$ on its right-hand side $(r^2 < 0)$.

2.6 Physical Characteristics of Undistorted Polytropes

2.6.1 Radius

The radial distance can be written down at once according to Eqs. (2.1.13) and (2.1.20):

$$r = \alpha\xi = [\pm(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [\pm(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi, \qquad (n \neq -1, \pm\infty), \tag{2.6.1}$$

$$r = \alpha\xi = (K/4\pi G\varrho_0)^{1/2}\xi = (P_0/4\pi G\varrho_0^2)^{1/2}\xi, \qquad (n = \pm\infty). \tag{2.6.2}$$

r is the radial distance from the symmetry plane of a polytropic slab ($N = 1$), from the symmetry axis of a polytropic cylinder ($N = 2$), and from the centre of a N-dimensional sphere ($N \geq 3$). For a finite boundary, the radius of the polytrope is given by $r = r_1 = \alpha\xi_1,\ (\theta(\xi_1) = 0)$.

2.6.2 Density, Pressure, and Temperature

Density and pressure at coordinate distance ξ are given by Eqs. (2.1.10) and (2.1.18):

$$\varrho = \varrho_0\theta^n; \quad P = K\varrho^{1+1/n} = P_0\theta^{n+1}, \qquad (n \neq \pm\infty), \tag{2.6.3}$$

$$\varrho = \varrho_0 \exp(-\theta); \quad P = K\varrho = P_0 \exp(-\theta), \qquad (n = \pm\infty). \tag{2.6.4}$$

If the polytrope obeys the initial conditions (2.1.41) for the Lane-Emden variables ξ, θ, then P_0 and ϱ_0 are just equal to the central pressure and density at radial distance $r = 0$.

For a perfect gas we get, by equating Eq. (1.2.5) to Eqs. (2.6.3) and (2.6.4), respectively:

$$P = \mathcal{R}\varrho T/\mu = \mathcal{R}\varrho_0\theta^n T/\mu = P_0\theta^{n+1} = \mathcal{R}\varrho_0 T_0\theta^{n+1}/\mu = K\varrho_0^{1+1/n}\theta^{n+1},$$
$$(n \neq \pm\infty;\ K = \mathcal{R}T_0\varrho_0^{-1/n}/\mu = P_0/\varrho_0), \tag{2.6.5}$$

$$P = \mathcal{R}\varrho T/\mu = \mathcal{R}\varrho_0 \exp(-\theta)T/\mu = P_0 \exp(-\theta) = \mathcal{R}\varrho_0 T_0 \exp(-\theta)/\mu = K\varrho_0 \exp(-\theta),$$
$$(n = \pm\infty;\ K = \mathcal{R}T_0/\mu = P_0/\varrho_0). \tag{2.6.6}$$

We divide Eq. (2.6.5) by $\mathcal{R}\varrho_0\theta^n/\mu$, and Eq. (2.6.6) by $\mathcal{R}\varrho_0 \exp(-\theta)/\mu$, to obtain the temperature of a perfect gas as a function of the Lane-Emden variables:

$$T = T_0\theta = (K\mu\varrho_0^{1/n}/\mathcal{R})\theta = (\mu P_0/\mathcal{R}\varrho_0)\theta, \qquad (n \neq \pm\infty), \tag{2.6.7}$$

$$T = T_0 = K\mu/\mathcal{R} = (\mu P_0/\mathcal{R}\varrho_0) = \mathrm{const}, \qquad (n = \pm\infty). \tag{2.6.8}$$

Fig. 2.6.1 visualizes Eqs. (2.6.7) and (2.6.18) for a spherical polytrope of index $n = 3$.

When the initial conditions (2.1.41) are fulfilled, $T_0 = K\mu\varrho_0^{1/n}/\mathcal{R} = \mu P_0/\mathcal{R}\varrho_0$ is just the central temperature. If $n = \pm\infty$, the polytrope obeying the perfect gas law is isothermal: $T = T_0 = \mathrm{const}$. In the particular case $n = 0$, ($\varrho = \varrho_0 = \mathrm{const}$) the factor $K\varrho_0^{1/n}$ should be replaced by P_0/ϱ_0; quite generally, $K\varrho^{1/n}$ can always be replaced by P/ϱ.

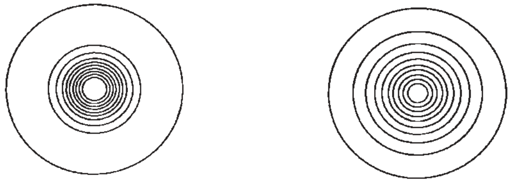

Fig. 2.6.1 Polytropic sphere of perfect gas with index $n = 3$ divided into shells corresponding to ten equal steps of temperature $\propto \theta$ (on the left), and of mass $\propto -\xi^2\theta'$ (on the right), (Eddington 1959).

2.6.3 Volume, Surface, and Mass

The volume of a polytrope inside radial distance r can be found from Eqs. (C.5), (C.10):

$$V = V(r) = \int_V dV = (2r^N/N)\{[\Gamma(1/2)]^N/\Gamma(N/2)\}, \qquad (N = 1, 2, 3, ...), \tag{2.6.9}$$

where Γ denotes the gamma function defined through Eqs. (C.9), (C.11). In the cases of practical interest $N = 1, 2, 3$ the volume (2.6.20) follows by direct integration of the volume elements

$$dV = \begin{cases} 2\ dr & \\ 2\pi r\ dr & \text{if} \\ 4\pi r^2\ dr & \end{cases} \quad \begin{matrix} N = 1, \text{ slab} \\ N = 2, \text{ cylinder} \\ N = 3, \text{ sphere} \end{matrix} \tag{2.6.10}$$

taking into account that the plane-parallel polytrope extends symmetrically above and below the symmetry plane.

The surface of a N-dimensional polytrope is simply

$$S = dV/dr = NV/r = 2r^{N-1}\{[\Gamma(1/2)]^N/\Gamma(N/2)\}. \tag{2.6.11}$$

The mass inside radial coordinate r is by virtue of Eq. (C.10) equal to

$$M = M(r) = \{2[\Gamma(1/2)]^N/\Gamma(N/2)\}\int_0^r \varrho r'^{N-1}\ dr' = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}$$
$$\times \int_0^\xi \xi'^{N-1}\theta^n\ d\xi' = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}(\mp d\theta/d\xi), \qquad (n \neq -1, \pm\infty;\ N = 1, 2, 3, ...), \tag{2.6.12}$$

where we have used Eq. (2.1.14), and α is given by Eq. (2.6.1). If $n = \pm\infty$, we find with Eqs. (2.1.21), (2.6.2):

$$M = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\int_0^\xi \xi'^{N-1}\exp(-\theta)\ d\xi' = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}\ d\theta/d\xi,$$
$$(n = \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.13}$$

In the cases of practical interest ($N = 1, 2, 3$) we can calculate the mass also by direct integration over the mass element $\varrho\ dV$.

$N = 1:$

$$M = 2\int_0^r \varrho\ dr' = 2\varrho_0\alpha\int_0^\xi \theta^n\ d\xi' = 2\varrho_0\alpha(\mp d\theta/d\xi), \qquad (n \neq -1, \pm\infty), \tag{2.6.14}$$

$$M = 2\int_0^r \varrho\ dr' = 2\varrho_0\alpha \int_0^\xi \exp(-\theta)\ d\xi' = 2\varrho_0\alpha\ d\theta/d\xi, \qquad (n = \pm\infty). \tag{2.6.15}$$

$N = 2:$

$$M = 2\pi \int_0^r \varrho r'\ dr' = 2\pi\varrho_0\alpha^2 \int_0^\xi \xi'\theta^n\ d\xi' = 2\pi\varrho_0\alpha^2\xi(\mp d\theta/d\xi), \qquad (n \neq -1, \pm\infty), \tag{2.6.16}$$

$$M = 2\pi \int_0^r \varrho r'\ dr' = 2\pi\varrho_0\alpha^2 \int_0^\xi \xi' \exp(-\theta)\ d\xi' = 2\pi\varrho_0\alpha^2\xi\ d\theta/d\xi, \qquad (n = \pm\infty). \tag{2.6.17}$$

$N = 3:$

$$M = 4\pi \int_0^r \varrho r'^2\ dr' = 4\pi\varrho_0\alpha^3 \int_0^\xi \xi'^2\theta^n\ d\xi' = 4\pi\varrho_0\alpha^3\xi^2(\mp d\theta/d\xi), \qquad (n \neq -1, \pm\infty), \tag{2.6.18}$$

$$M = 4\pi \int_0^r \varrho r'^2\ dr' = 4\pi\varrho_0\alpha^3 \int_0^\xi \xi'^2 \exp(-\theta)\ d\xi' = 4\pi\varrho_0\alpha^3\xi^2\ d\theta/d\xi, \qquad (n = \pm\infty). \tag{2.6.19}$$

With the volume elements (2.6.10) we obtain

$$V = \begin{cases} 2r = 2\alpha\xi & \text{if} \quad N = 1,\ \text{slab} \\ \pi r^2 = \pi\alpha^2\xi^2 & \phantom{\text{if}} \quad N = 2,\ \text{cylinder} \\ 4\pi r^3/3 = 4\pi\alpha^3\xi^3/3 & \phantom{\text{if}} \quad N = 3,\ \text{sphere} \end{cases} \tag{2.6.20}$$

For a finite boundary of the polytrope, total mass and total volume are obtained by putting $\xi = \xi_1$ and $\theta' = \theta'_1$. For polytropic slabs ($N = 1$) mass and volume are considered per unit surface of the symmetry plane, extending up to radial distance r above and below the symmetry plane. For infinitely long cylinders ($N = 2$) mass and volume are taken per unit length of the cylinder's symmetry axis. Clearly, the above equations can be written down without difficulty in the case when the radial coordinate changes between r_1 and r_2, rather than between 0 and r.

We can eliminate the central density ϱ_0 between Eqs. (2.6.1) and (2.6.12), obtaining the mass-radius relationship (Chandrasekhar 1939 if $N = 3$)

$$Mr^{[N(1-n)+2n]/(n-1)} = \{2[\Gamma(1/2)]^N/\Gamma(N/2)\}\big[\pm(n+1)K/4\pi G\big]^{n/(n-1)}\xi^{(n+1)/(n-1)}(\mp d\theta/d\xi),$$
$$(n \neq -1, \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.21}$$

If $n = \pm\infty$, we get by elimination of ϱ_0 between Eqs. (2.6.2) and (2.6.13):

$$Mr^{2-N} = \{2[\Gamma(1/2)]^N/\Gamma(N/2)\}(K/4\pi G)\xi(d\theta/d\xi), \qquad (n = \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.22}$$

Eq. (2.6.21) becomes in the spherical case $N = 3$ equal to $M_1 r_1^{(3-n)/(n-1)} = \text{const}$, provided that total mass M_1 and total radius r_1 are finite ($-1 < n < 5$; see Sec. 2.6.8). If the polytrope remains in hydrostatic equilibrium after a mass change $\Delta M_1 \ll M_1$, the corresponding radius variation is $\Delta r_1 = (r_1/M_1)[(n-1)/(n-3)]\ \Delta M_1$. If $-1 < n < 1$ and $3 < n < 5$, the polytrope expands with increasing mass. If $1 < n < 3$, the polytrope shrinks with mass increase. If $n = 1$, the radius remains invariant for any mass change ΔM_1, and if $n = 3$ the mass remains constant when the radius changes by Δr_1 (Paczyński 1965, Heisler and Alcock 1986).

When the initial conditions (2.1.41) are fulfilled, there subsist the equations

$$P_0 = K\varrho_0^{1+1/n}, \qquad (n \neq -1, \pm\infty), \tag{2.6.23}$$

$$P_0 = K\varrho_0, \qquad (n = \pm\infty). \tag{2.6.24}$$

Substituting for ϱ_0 and K from Eqs. (2.6.1), (2.6.21), we find

$$P_0 = \{4\pi G/[\pm(n+1)]\}\{\Gamma(N/2)/2[\Gamma(1/2)]^N\}^2 M^2 r^{2[N(1-n)+n-1]/(n-1)}/(d\theta/d\xi)^2,$$
$$(n \neq -1, \pm\infty;\ N = 1, 2, 3, ...), \tag{2.6.25}$$

and from Eqs. (2.6.2), (2.6.22)

$$P_0 = 4\pi G\{\Gamma(N/2)/2[\Gamma(1/2)]^N\}^2 M^2 r^{2(1-N)}/(d\theta/d\xi)^2, \qquad (n = \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.26}$$

2.6.4 Mean Density

The mean density inside radial coordinate r is found if we divide Eqs. (2.6.12) and (2.6.13) by Eq. (2.6.9):

$$\varrho_m = M/V = N\varrho_0(\mp d\theta/d\xi)/\xi, \qquad (n \neq -1, \pm\infty;\ N = 1, 2, 3, ...), \tag{2.6.27}$$

$$\varrho_m = N\varrho_0(d\theta/d\xi)/\xi, \qquad (n = \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.28}$$

2.6.5 Gravitational Acceleration and Gravitational Potential

Integrating the radially symmetric form of Poisson's equation (2.1.33) between the origin $r = 0$ and an arbitrary point r inside the boundary $r = r_1$ of the polytrope, we get for the derivative of the internal potential

$$F(r) = d\Phi/dr - (d\Phi/dr)_{r=0} = d\Phi/dr = -4\pi G r^{1-N}\int_0^r \varrho r'^{N-1}\,dr' = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}$$
$$\times M r^{1-N}, \qquad (r \leq r_1;\ (d\Phi/dr)_{r=0} = 0;\ M = M(r);\ N = 1, 2, 3, ...), \tag{2.6.29}$$

where we have used Eq. (C.10). The derivative of the gravitational potential function $\Phi = \Phi(r)$ is just the gravitational acceleration $F(r)$, or equivalently the gravitational force $\vec{F}(r)$ acting on the unit of mass located inside the polytrope at distance r, $(r \leq r_1)$ from the origin of coordinates. For the integration constant we can take without loss of generality $(d\Phi/dr)_{r=0} = 0$. Eq. (2.6.29) becomes for the cases of practical interest equal to

$$F(r) = \begin{cases} -2\pi GM & \\ -2GM/r & \text{if} \\ -GM/r^2 & \end{cases} \quad \begin{matrix} N = 1, \text{ slab} \\ N = 2, \text{ cylinder} \\ N = 3, \text{ sphere} \end{matrix} \tag{2.6.30}$$

For polytropic slabs F is the gravitational acceleration per unit surface in the symmetry plane, and for polytropic cylinders F is the gravitational acceleration per unit length of the cylinder. We integrate Eq. (2.6.29) again and find

$$\Phi - \Phi_0 = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}\int_0^r M(r')\, r'^{1-N}\,dr', \qquad (r \leq r_1), \tag{2.6.31}$$

where Φ_0 is the value of Φ at the origin $r = 0$.

Another equation for the internal gravitational potential in terms of Lane-Emden variables is given by Eqs. (2.1.8) and (2.1.16), respectively [cf. Eq. (6.1.200)]:

$$\Phi - \Phi_0 = (n+1)K(\varrho^{1/n} - \varrho_0^{1/n}) = (n+1)K^{n/(n+1)}[P^{1/(n+1)} - P_0^{1/(n+1)}]$$
$$= (n+1)K\varrho_0^{1/n}(\theta - 1) = [(n+1)P_0/\varrho_0](\theta - 1), \qquad (n \neq -1, \pm\infty), \tag{2.6.32}$$

$$\Phi - \Phi_0 = K\ln(P/P_0) = -K\theta, \qquad (n = \pm\infty). \tag{2.6.33}$$

Poisson's equation (2.1.33) writes for the external gravitational potential Φ_e as

$$\nabla^2\Phi_e = r^{1-N}\,d(r^{N-1}\,d\Phi_e/dr)/dr = 0, \qquad (r \geq r_1;\ \varrho = 0). \tag{2.6.34}$$

It is clear that only polytropes with a finite boundary possess an external gravitational potential. As will be shown in Sec. 2.6.8, this is the case for polytropes having indices $-1 < n < \infty$ if $N = 1, 2$, and $-1 < n < 5$ if $N = 3$.

Integrating the Laplace equation (2.6.34) twice, we get

$$\Phi_e = C_1 r^{2-N}/(2-N) + C_2 = c_1 \xi^{2-N}/(2-N) + c_2,$$
$$(r = \alpha\xi \geq r_1;\ N \neq 2;\ C_1, C_2, c_1, c_2 = \text{const}), \tag{2.6.35}$$

$$\Phi_e = C_1 \ln r + C_2 = c_1 \ln \xi + c_2, \qquad (r = \alpha\xi \geq r_1;\ N = 2;\ C_1, C_2, c_1, c_2 = \text{const}). \tag{2.6.36}$$

The integration constant C_2 can be expressed by setting the value of the external potential at the finite boundary equal to Φ_{e1} :

$$C_2 = \Phi_{e1} - C_1 r_1^{2-N}/(2-N), \qquad (N \neq 2), \tag{2.6.37}$$

$$C_2 = \Phi_{e1} - C_1 \ln r_1, \qquad (N = 2). \tag{2.6.38}$$

The value of C_1 can be determined by equating the boundary value $d\Phi/dr$ from Eq. (2.6.29) to the derivative $d\Phi_e/dr$ of Eqs. (2.6.35) and (2.6.36), respectively:

$$C_1 = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M_1, \qquad [M_1 = M(r_1)]. \tag{2.6.39}$$

Inserting Eqs. (2.6.37)-(2.6.39) into Eqs. (2.6.35) and (2.6.36) we obtain eventually

$$\Phi_e - \Phi_{e1} = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M_1(r^{2-N} - r_1^{2-N})/(2-N), \quad (N \neq 2;\ r \geq r_1), \tag{2.6.40}$$

$$\Phi_e - \Phi_{e1} = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M_1(\ln r - \ln r_1), \quad (N = 2;\ r \geq r_1). \tag{2.6.41}$$

The external potential per unit surface of the slab ($N = 1$) becomes infinite as $r \to \infty$, and the same is true for the external potential per unit length of a cylinder ($N = 2$). If $N \geq 3$, Eq. (2.6.35) shows that $\Phi_e \to C_2$ if $r \to \infty$; in this case the external potential is generally normalized by setting $C_2 = 0$. Inserting C_1 from Eq. (2.6.39), and $C_2 = 0$ into Eq. (2.6.35), we get with this particular normalization:

$$\Phi_e = 2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M_1 r^{2-N}/(N-2), \qquad (r \geq r_1;\ N \geq 3). \tag{2.6.42}$$

In the spherical case $N = 3$, Eq. (2.6.42) becomes simply

$$\Phi_e = GM_1/r, \qquad (r \geq r_1;\ N = 3). \tag{2.6.43}$$

Another equation for the external gravitational potential in terms of Lane-Emden variables can be obtained by matching the value (2.6.32) of the internal potential at the *finite* boundary $\Phi(\xi_1)$ to the boundary value $\Phi_e(\xi_1)$ of the external potential from Eqs. (2.6.35), (2.6.36), together with the corresponding derivatives $(d\Phi/d\xi)_{\xi=\xi_1}$ and $(d\Phi_e/d\xi)_{\xi=\xi_1}$. After some algebra, we find

$$c_1 = (n+1)K\varrho_0^{1/n}\xi_1^{N-1}\theta_1', \tag{2.6.44}$$

$$c_2 = \Phi_0 - (n+1)K\varrho_0^{1/n}[\xi_1\theta_1'/(2-N) + 1], \qquad (N \neq 2), \tag{2.6.45}$$

$$c_2 = \Phi_0 - (n+1)K\varrho_0^{1/n}(\xi_1\theta_1' \ln \xi_1 + 1), \qquad (N = 2). \tag{2.6.46}$$

Inserting Eqs. (2.6.44)-(2.6.46) into Eqs. (2.6.35), (2.6.36), we get, provided that ξ_1 is finite:

$$\Phi_e = \Phi_0 + (n+1)K\varrho_0^{1/n}\big[\xi_1^{N-1}\theta_1'\xi^{2-N}/(2-N) - \xi_1\theta_1'/(2-N) - 1\big],$$
$$(\xi \geq \xi_1;\ n \neq -1, \pm\infty;\ N \neq 2), \tag{2.6.47}$$

$$\Phi_e = \Phi_0 + (n+1)K\varrho_0^{1/n}[\xi_1\theta_1' \ln(\xi/\xi_1) - 1], \qquad (\xi \geq \xi_1;\ n \neq -1, \pm\infty;\ N = 2). \tag{2.6.48}$$

If $n = 0$, the factor $K\varrho_0^{1/n}$ should be replaced by P_0/ϱ_0. From Eqs. (2.6.40) and (2.6.41) we derive the gravitational acceleration acting outside the finite boundary of the polytrope (cf. Eq. (2.6.29) if $r \leq r_1$) :

$$F(r) = d\Phi_e/dr = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M_1 r^{1-N}, \quad (r \geq r_1;\ M_1 = M(r_1);\ N = 1, 2, 3, ...). \tag{2.6.49}$$

2.6.6 The Virial Theorem

Let us first derive the virial theorem in three-dimensional Euclidian space by taking the product between Eq. (2.1.1) and $x_j\ dV$ in the viscosity-free case $\tau = 0$ (Chandrasekhar 1981, Chap. XIII). Integrating over the volume V contained inside mass M, we find ($\vec{r} = \vec{r}(x_1, x_2, x_3);\ dV = dx_1dx_2dx_3;\ dM = \varrho\ dV;\ \vec{F} = \nabla\Phi$)

$$\int_M x_j(Dv_k/Dt)\ dM = -\int_V x_j(\partial P/\partial x_k)\ dV + \int_M x_j(\partial\Phi/\partial x_k)\ dM$$
$$+(p/4\pi)\int_V x_j[\partial(H_kH_\ell)/\partial x_\ell - (1/2)\ \partial H^2/\partial x_k]\ dV, \tag{2.6.50}$$

where in the sequel, summation from 1 to 3 over repeated indices is to be understood. The magnetic force field term from Eq. (2.1.1) has been transformed in virtue of the vectorial identity

$$\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}), \tag{2.6.51}$$

yielding

$$(1/4\pi)(\nabla\times\vec{H})\times\vec{B} = (p/4\pi)(\nabla\times\vec{H})\times\vec{H} = (p/4\pi)[(\vec{H}\cdot\nabla)\vec{H} - (1/2)\ \nabla H^2]. \tag{2.6.52}$$

Excepting for ferromagnetic substances, the field vector of magnetic induction $\vec{B}$ is related to the intensity of the magnetic field $\vec{H}$ by the relationship (see Sec. 3.10.1, Sommerfeld 1961, Gerthsen et al. 1977)

$$\vec{B} = p\vec{H}, \qquad (p = \mathrm{const}), \tag{2.6.53}$$

where p denotes the magnetic permeability, assumed constant in the following. To obtain the last term from Eq. (2.6.50), we have transformed Eq. (2.6.52) further

$$(\vec{H}\cdot\nabla)H_k = H_\ell\ \partial H_k/\partial x_\ell = \partial(H_\ell H_k)/\partial x_\ell, \tag{2.6.54}$$

taking into account the Maxwell equation (3.10.1):

$$\nabla\cdot\vec{B} = p\ \nabla\cdot\vec{H} = p\ \partial H_\ell/\partial x_\ell = 0. \tag{2.6.55}$$

The left-hand side of Eq. (2.6.50) can be written as follows ($v_j = dx_j/dt$) :

$$\int_V \varrho x_j(Dv_k/Dt)\ dV = \int_V \varrho[D(x_jv_k)/Dt - v_jv_k]\ dV = d\Big[\int_V \varrho x_jv_k\ dV\Big]\Big/dt - 2E_{jk}, \tag{2.6.56}$$

where

$$E_{jk} = \int_V \varrho v_jv_k\ dV/2, \tag{2.6.57}$$

is the kinetic energy tensor. Its trace

$$E_{kin} = \mathrm{Tr}\ E_{jk} = E_{11} + E_{22} + E_{33} = \int_V \varrho(v_1^2 + v_2^2 + v_3^2)\ dV/2 = \int_V \varrho|\vec{v}|^2\ dV/2, \tag{2.6.58}$$

is just the kinetic energy of internal macroscopic mass motions. The material derivative $D\ /Dt = \partial\ /\partial t + v_k\ \partial\ /\partial x_k$ in Eq. (2.6.56) can be taken in front of the integral due to the fact that for any derivable function $f(x_j, t)$ there subsists the equation (e.g. Chandrasekhar 1969, Tassoul 1978)

$$\int_V \varrho(x_j,t)\ [Df(x_j,t)/Dt]\ dV = \int_M [Df(x_j,t)/Dt]\ dM = d\Big[\int_M f(x_j,t)\ dM\Big]\Big/dt$$
$$= d\Big[\int_V \varrho(x_j,t)\ f(x_j,t)\ dV\Big]\Big/dt, \tag{2.6.59}$$

where we made use of mass conservation $dM/dt = 0$, the comoving volume being generally a time-dependent quantity $V = V(t)$. Introducing the Kronecker symbol $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$, the pressure term from Eq. (2.6.50) can be written as

$$\int_V x_j(\partial P/\partial x_k)\ dV = \int_V \left[\partial(x_j P)/\partial x_k - \delta_{jk} P\right]\ dV = \int_S P x_j\ dS_k - \delta_{jk}\int_V P\ dV, \tag{2.6.60}$$

by using the one-component form of the Gauss theorem (e.g. Bronstein and Semendjajew 1985):

$$\int_V (\partial f/\partial x_k)\ dV = \int_S f\ dS_k, \qquad (f = f(x_1, x_2, x_3);\ k = 1, 2, 3). \tag{2.6.61}$$

dS_k is the projection perpendicular to the x_k-axis of the surface element dS. The internal gravitational potential $\Phi = \Phi(\vec{r})$ in a point $\vec{r}$ due to the presence of matter in volume V is given by

$$\Phi = \Phi(\vec{r}) = G\int_M dM'/|\vec{r} - \vec{r'}| = G\int_V \varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|. \tag{2.6.62}$$

A tensor generalization of the gravitational potential (2.6.62) is provided by the gravitational potential tensor

$$\Phi_{jk} = \Phi_{jk}(\vec{r}) = G\int_V \varrho(\vec{r'})\ (x_j - x'_j)(x_k - x'_k)\ dV'/|\vec{r} - \vec{r'}|^3, \qquad (r^2 = x_1^2 + x_2^2 + x_3^2), \tag{2.6.63}$$

and its trace is just the gravitational potential (2.6.62):

$$\Phi = \mathrm{Tr}\ \Phi_{jk} = \Phi_{11} + \Phi_{22} + \Phi_{33} = G\int_V \varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|. \tag{2.6.64}$$

The trace of the second integral on the right-hand side of Eq. (2.6.50) – the virial – can be transformed as follows (e.g. Chandrasekhar 1969, 1981):

$$\begin{aligned}
&\int_M x_k(\partial\Phi/\partial x_k)\ dM = G\int_M x_k\left[\partial\left(\int_M dM'/|\vec{r} - \vec{r'}|\right)\Big/\partial x_k\right]\ dM \\
&= -G\int_M\int_M [x_k(x_k - x'_k)/|\vec{r} - \vec{r'}|^3]\ dM\ dM' \\
&= -(G/2)\int_M\int_M [(x_k - x'_k)(x_k - x'_k)/|\vec{r} - \vec{r'}|^3]\ dM\ dM' \\
&= -(G/2)\int_M\int_M dM\ dM'/|\vec{r} - \vec{r'}| = -(1/2)\int_M \Phi\ dM = W, \qquad (r^2 = x_k x_k = x_k^2;\ {r'}^2 = {x'_k}^2).
\end{aligned} \tag{2.6.65}$$

W is just the gravitational potential energy of the mass inside volume V, as will be obvious from the following. Let us adopt for the moment the discrete particle representation, and evaluate the gravitational potential energy W'_{jk} of the mass element dM_j due to the presence of all other particles dM_k, $(dM_j \neq dM_k)$ of the system. By virtue of Eq. (2.6.62) we have

$$W'_{jk} = -\Phi(\vec{r}_j)\ dM_j = -G\ dM_j \sum_k dM_k/|\vec{r}_j - \vec{r}_k|, \qquad (W'_{jj} = 0). \tag{2.6.66}$$

$\vec{r}_k$ is the position vector of the mass element dM_k. We have introduced in Eq. (2.6.66) the minus sign, because gravitation is an attractive force.

The gravitational potential energy of the whole mass M is obtained by summing up W'_{jk} over all distinct pairs of particles inside volume V :

$$W = \sum_{j<k} W'_{jk} = (1/2)\sum_{j,k} W'_{jk} = -(G/2)\sum_{j,k} dM_j\ dM_k/|\vec{r}_j - \vec{r}_k|. \tag{2.6.67}$$

If we write Eq. (2.6.67) in integral form, we have

$$W = (-G/2)\int_M\int_M dM\ dM'/|\vec{r} - \vec{r'}| = -(1/2)\int_M \Phi\ dM, \tag{2.6.68}$$

which is identical to Eq. (2.6.65).

In the spherically symmetric case we may evaluate W directly, by calculating the work done by the mass $M(r)$ contained inside radius r to bring the additional amount of matter dM from infinity up to this radius. Since the attractive force between the mass $M(r)$ and $dM(r')$ is $\vec{F} = -GM(r)\ dM(r')\ (\vec{r'}/r'^3)$, the elementary work done is (e.g. Chandrasekhar 1939, Cox and Giuli 1968)

$$dW = \int_r^\infty \vec{F} \cdot d\vec{r'} = -GM(r)\ dM \int_r^\infty \vec{r'} \cdot d\vec{r'}/r'^3 = -GM(r)\ dM \int_r^\infty dr'/r'^2 = -GM(r)\ dM/r. \tag{2.6.69}$$

The whole gravitational potential energy of the sphere is obtained by summing up dW over the mass of the sphere:

$$W = \int_M dW = -G \int_M M(r)\ dM/r, \qquad (N=3). \tag{2.6.70}$$

The tensor generalization of Eq. (2.6.65) can be effected at once by replacing Φ with Φ_{jk} (e.g. Chandrasekhar 1969)

$$W_{jk} = -(1/2) \int_V \varrho \Phi_{jk}\ dV = \int_V \varrho x_j (\partial \Phi / \partial x_k)\ dV, \tag{2.6.71}$$

where Tr $W_{jk} = W$.

The magnetic term from Eq. (2.6.50) can be transformed analogously to the pressure term:

$$\int_V x_j [\partial(H_k H_\ell)/\partial x_\ell - (1/2)\ \partial H^2/\partial x_k]\ dV = \int_V [\partial(x_j H_k H_\ell)/\partial x_\ell - H_j H_k]\ dV$$
$$-(1/2) \int_V [\partial(x_j H^2)/\partial x_k - \delta_{jk} H^2]\ dV = \int_S x_j H_k H_\ell\ dS_\ell - \int_V H_j H_k\ dV - (1/2) \int_S x_j H^2\ dS_k$$
$$+(\delta_{jk}/2) \int_V H^2\ dV. \tag{2.6.72}$$

Inserting Eqs. (2.6.56), (2.6.60), (2.6.71), (2.6.72) into Eq. (2.6.50), we obtain the virial theorem under the form (e.g. Chandrasekhar 1981, Chap. XIII)

$$d\Big(\int_V \varrho x_j v_k\ dV\Big)\Big/dt = 2E_{jk} + \delta_{jk} \int_V P\ dV + W_{jk} + \delta_{jk} \int_V (pH^2/8\pi)\ dV$$
$$-(p/4\pi) \int_V H_j H_k\ dV - \int_S x_j (P + pH^2/8\pi)\ dS_k + (p/4\pi) \int_S x_j H_k H_\ell\ dS_\ell. \tag{2.6.73}$$

We now introduce the second order moments of density distribution – the moment of inertia tensor

$$I_{jk} = \int_V \varrho x_j x_k\ dV; \quad I_{jk} = I_{kj}. \tag{2.6.74}$$

The contraction of this symmetric tensor is just the scalar moment of inertia [cf. Eq. (6.1.179)]:

$$I = \mathrm{Tr}\ I_{jk} = I_{11} + I_{22} + I_{33} = \int_V \varrho r^2\ dV = \int_M r^2\ dM. \tag{2.6.75}$$

While the pressure P generally vanishes on the boundary of the configuration, the magnetic field generally extends far beyond the boundary of the object. The surface integrals over the magnetic field in Eq. (2.6.73) will vanish when the surface and the corresponding volume of integration are extended over the whole space, since the magnetic field of any isolated configuration must decrease at least as rapidly as the dipole field from Eq. (3.10.25): $H \propto r^{-3}$. But in this case of vanishing surface integrals, the right-hand side of Eq. (2.6.73) contains only symmetric tensors, so the left-hand side must be symmetric too, and can be expressed in terms of the moment of inertia tensor (2.6.74):

$$d\Big(\int_V \varrho x_k v_j\ dV\Big)\Big/dt = d\Big(\int_V \varrho x_j v_k\ dV\Big)\Big/dt = d\Big[\int_V \varrho x_j (dx_k/dt)\ dV\Big]\Big/dt$$
$$-\int_V \varrho\big[d(x_j\ dx_k/dt)/dt\big]\ dV = (1/2) \int_V \varrho\big[d(x_j\ dx_k/dt + x_k\ dx_j/dt)/dt\big]\ dV$$
$$= (1/2) \int_V \varrho[d^2(x_j x_k)/dt^2]\ dV = (1/2)\ d^2\Big(\int_V \varrho x_j x_k\ dV\Big)\Big/dt^2 = (1/2)\ d^2 I_{jk}/dt^2, \tag{2.6.76}$$

by taking into account Eq. (2.6.59). We also introduce the magnetic energy tensor

$$H_{jk} = (p/8\pi) \int_V H_j H_k \; dV. \tag{2.6.77}$$

Its trace is just the magnetic energy of the configuration (e.g. Roberts 1967):

$$U_m = \mathrm{Tr}\; H_{jk} = H_{kk} = H_{11} + H_{22} + H_{33} = (p/8\pi) \int_V H^2 \; dV. \tag{2.6.78}$$

Thus, Eq. (2.6.73) takes the equivalent form

$$(1/2)\; d^2 I_{jk}/dt^2 = 2E_{jk} + \delta_{jk} \int_V P \; dV + W_{jk} + \delta_{jk} U_m - 2H_{jk}$$
$$- \int_S x_j (P + pH^2/8\pi) \; dS_k + (p/4\pi) \int_S x_j H_k H_\ell \; dS_\ell. \tag{2.6.79}$$

Contracting the indices in Eq. (2.6.79) we obtain the well known scalar form of the virial theorem

$$(1/2)\; d^2 I/dt^2 = 2E_{kin} + 3 \int_V P \; dV + W + U_m - \int_S x_j [\delta_{jk} P + (p/4\pi)(\delta_{jk} H^2/2 - H_j H_k)] \; dS_k, \tag{2.6.80}$$

by making $j = k$, and changing the summation index ℓ into k in the last term of Eq. (2.6.79).

The pressure tensor, including the contribution of magnetic fields, can be defined as

$$P_{jk} = \delta_{jk} P + (p/4\pi)(\delta_{jk} H^2/2 - H_j H_k), \tag{2.6.81}$$

and the total mean pressure P_{tot} is just equal to (Cox and Giuli 1968)

$$P_{tot} = (1/3)\; \mathrm{Tr}\; P_{jk} = (P_{11} + P_{22} + P_{33})/3 = (3P + 3pH^2/8\pi - pH^2/4\pi)/3$$
$$= P + pH^2/24\pi = P + P_m. \tag{2.6.82}$$

The term $P_m = pH^2/24\pi$ may be regarded as a mean magnetic pressure. The mean magnetic pressure P_m is composed of a hydrostatic magnetic pressure $pH^2/8\pi$, acting uniformly in all directions, and of a magnetic tension $pH^2/4\pi$, acting along the lines of the magnetic field intensity $\vec{H}$ (e.g. Alfvén and Fälthammar 1963, Tassoul 1978, Chandrasekhar 1981). Eq. (2.6.80) can also be written in terms of the energy of the system, by observing that according to Eqs. (1.7.53), (1.7.54) the pressure assumes in the extreme relativistic limit for noninteracting particles the value

$$P = \varepsilon^{(kin)}/3, \qquad (w = 1), \tag{2.6.83}$$

where $\varepsilon^{(kin)}$ is the energy density of kinetic translational motions. Eq. (2.6.83) also applies to radiation and magnetic fields, so we can write [cf. Eqs. (1.4.1) and (2.6.82)]:

$$P_r = \varepsilon^{(rad)}/3 = aT^4/3; \quad \varepsilon^{(rad)} = aT^4, \tag{2.6.84}$$

$$P_m = \varepsilon^{(m)}/3 = pH^2/24\pi; \quad \varepsilon^{(m)} = pH^2/8\pi. \tag{2.6.85}$$

a is Stefan's constant, and $\varepsilon^{(rad)}, \varepsilon^{(m)}$ the energy density of radiation and of the magnetic field, respectively.

In the case of noninteracting particles obeying Newtonian mechanics – in absence of radiation and magnetic fields – the mean total pressure is obtained from Eqs. (1.7.53)-(1.7.54):

$$P_{tot} = 2\varepsilon^{(kin)}/3, \qquad (w = 2). \tag{2.6.86}$$

In a system of interacting particles without radiation and magnetic fields, the pressure arises from kinetic transfer of momentum between particles and from particle interactions, such as intermolecular forces, that are dominant in liquids and solids. Denoting by P' the contribution from particle interactions, Eq. (2.6.86) extends to

$$P_{tot} = P' + 2\varepsilon^{(kin)}/3. \tag{2.6.87}$$

Generally, the total mean pressure can be split into four terms

$$P_{tot} = 2\varepsilon^{(kin)}/3 + P' + P_r + P_m, \tag{2.6.88}$$

with contributions arising respectively from Newtonian kinetic transfer of momentum, particle interactions, radiation, and magnetic fields. Thus, Eq. (2.6.80) can be written in the alternative form

$$\begin{aligned}(1/2)\ d^2I/dt^2 &= 2E_{kin} + W + 3\int_V (2\varepsilon^{(kin)}/3 + P' + P_r + P_m)\ dV - \int_S x_k P_{jk}\ dS_j\\ &= 2E_{kin} + W + \int_V [2\varepsilon^{(kin)} + 3P' + \varepsilon^{(rad)} + \varepsilon^{(m)}]\ dV - \int_S x_k P_{jk}\ dS_j\\ &= 2E_{kin} + 2E_{th} + W + 3\int_V P'\ dV + U_{rad} + U_m - \int_S x_k P_{jk}\ dS_j,\end{aligned} \tag{2.6.89}$$

where

$$E_{th} = \int_V \varepsilon^{(kin)}\ dV; \quad U_{rad} = \int_V \varepsilon^{(rad)}\ dV; \quad U_m = \int_V \varepsilon^{(m)}\ dV, \tag{2.6.90}$$

denotes the thermal, radiative, and magnetic energy of the system, respectively. The sum of macroscopic E_{kin} and microscopic E_{th} translational particle motions of the system is the total kinetic energy of the system.

As shown by Eq. (1.7.58), the pressure is related to the internal energy density of a perfect, relativistic or nonrelativistic gas by

$$P = (\gamma - 1)\varepsilon^{(int)}. \tag{2.6.91}$$

We generalize this equation also to other systems, with the understanding that Γ is equal to the ratio of specific heats $\gamma = c_P/c_V$ only for a perfect gas. Thus, including also *radiation pressure and particle interactions,* we may write

$$P = (\Gamma - 1)\varepsilon^{(int)}, \qquad (\Gamma > 1;\ P_{tot} = P + P_m;\ P = 2\varepsilon^{(kin)}/3 + P' + P_r). \tag{2.6.92}$$

Actually, in a perfect gas-radiation mixture without $e^{\pm}$-pairs the value of Γ differs at most by several percent from the first adiabatic index Γ_1 shown in Eq. (1.4.20), (see Fowler 1966, Table A1, if $\gamma_g = 5/3$). For comparison, we obtain by inserting for the pressure from Eq. (1.4.11), and for the internal energy density $\varepsilon^{(int)} = \mathcal{R}\varrho T/\mu(\gamma_g - 1) + aT^4$ from Eq. (1.4.13):

$$\Gamma = 1 + P/\varepsilon^{(int)} = [4(\gamma_g - 1) - \beta(3\gamma_g - 4)]/[3(\gamma_g - 1) - \beta(3\gamma_g - 4)]. \tag{2.6.93}$$

Returning now to Eq. (2.6.89), we get with the aid of Eq. (2.6.92)

$$(1/2)\ d^2I/dt^2 = 2E_{kin} + W + 3(\Gamma - 1)U + U_m - \int_S x_k P_{jk}\ dS_j, \tag{2.6.94}$$

where

$$U = \int_V \varepsilon^{(int)}\ dV = [1/(\Gamma - 1)] \int_V P\ dV, \tag{2.6.95}$$

is the internal energy of the system including radiation energy U_{rad}, but excepting the magnetic energy U_m. If the pressure P_{jk} on the boundary is isotropic and constant ($P_{jk} = P_S =$ const), we can transform the last term from Eq. (2.6.94) as follows:

$$\int_S x_k P_{jk}\ dS_j = P_S \int_S x_k\ dS_k = P_S \int_S \vec{r} \cdot d\vec{S} = P_S \int_V (\nabla \cdot \vec{r})\ dV = 3P_S V, \qquad (\nabla \cdot \vec{r} = 3). \tag{2.6.96}$$

If the system is in hydrostatic or quasihydrostatic equilibrium, we have $I =$ const, $d^2I/dt^2 = 0$, $E_{kin} = 0$. If the surface pressure P_S is zero too, Eq. (2.6.94) takes the simple form (Cox and Giuli 1968)

$$W + 3(\Gamma - 1)U + U_m = 0 \quad \text{or} \quad U = -(W + U_m)/3(\Gamma - 1), \tag{2.6.97}$$

and the total energy of the system becomes equal to

$$E = W + U + U_m = (-3\Gamma + 4)U = (3\Gamma - 4)(W + U_m)/3(\Gamma - 1). \tag{2.6.98}$$

The condition of dynamical stability is $E < 0$, or (Chandrasekhar and Fermi 1953)

$$(3\Gamma - 4)(W + U_m) < 0, \qquad (W < 0), \tag{2.6.99}$$

since $\Gamma > 1$ via Eq. (2.6.92). Thus, even if $\Gamma > 4/3$, a sufficiently strong magnetic field can induce dynamical instability, making $W + U_m > 0$. Provided that $\Gamma > 4/3$, the condition of dynamical stability is $U_m < -W$.

We may particularize the system further, by taking the magnetic field equal to zero: $U_m = 0$. The condition of dynamical stability (2.6.99) yields $3\Gamma - 4 > 0$, $(W < 0)$, i.e. the system is stable if $\Gamma > 4/3$. Eqs. (2.6.97) and (2.6.98) become

$$U = -W/3(\Gamma - 1) \quad \text{and} \quad E = (3\Gamma - 4)W/3(\Gamma - 1). \tag{2.6.100}$$

Inserting for the gravitational energy of a sphere from Eq. (2.6.137), we obtain $E = -(3\Gamma - 4)GM_1^2/(5 - n)(\Gamma - 1)r_1^2$, $(-1 < n < 5)$, showing that $E = 0$ when matter is dispersed to infinity, and $E < 0$ if the stable sphere ($\Gamma > 4/3$) has contracted to radius r_1 (see also Sec. 5.12).

Differentiating Eq. (2.6.100), we get (Chandrasekhar 1939)

$$\Delta U = -\Delta W/3(\Gamma - 1) \quad \text{and} \quad \Delta E = (3\Gamma - 4)\ \Delta W/3(\Gamma - 1), \qquad (\Gamma > 1). \tag{2.6.101}$$

Below, we briefly discuss the consequences of a slow quasihydrostatic contraction of the system. The only sources of energy of the system are assumed to be internal and gravitational energy. If the system contracts, we have $\Delta W < 0$ (cf. Eq. (2.6.69) for the spherically symmetric case). If $\Gamma > 4/3$, the total energy of the system decreases by the amount ΔE from Eq. (2.6.101). In other words, only the fraction $1/3(\Gamma - 1)$ of the gravitational energy change is used to increase the internal energy of the system, while the fraction $1 - 1/3(\Gamma - 1) = (3\Gamma - 4)/3(\Gamma - 1)$ leaves the system, being radiated away. If $\Gamma = 4/3$, the energies E and ΔE are zero, and the total energy of the system is conserved; the system passes from one equilibrium state to the other without energy change. All the gravitational energy released goes into internal energy: $\Delta U = -\Delta W$. If $1 < \Gamma < 4/3$, we observe from Eq. (2.6.101) that $\Delta E > 0$ if $\Delta W < 0$, i.e. a contraction of the system due to the change $\Delta W < 0$ would cause the total energy of the star to increase, which is clearly impossible for a system without nuclear energy sources. If $1 < \Gamma < 4/3$, we have $\Delta U > -\Delta W$, and the increase in internal energy would be larger than the energy ΔW, $(\Delta W < 0)$ supplied by gravitational contraction. If $1 < \Gamma < 4/3$, the system cannot be in a state of quasihydrostatic equilibrium (cf. Sec. 5.3.1). The above discussion applies to systems in quasihydrostatic equilibrium, when $d^2I/dt^2 = 0$.

For a three-dimensional space with the distribution of matter having cylindrical and planar symmetry, we present below some particular forms of the virial theorem, including magnetic fields.

For cylindrical symmetry we introduce (ℓ, φ, z)-coordinates, ℓ and φ being polar coordinates, and z directed along the axis of the cylinder. The pressure is a function of the radial coordinate $P = P(\ell)$, while the velocity is directed along the radial direction $\vec{v} = \vec{v}(v_\ell)$. The magnetic field is assumed to extend along the z-axis: $\vec{H} = \vec{H}(H_z)$, where $H_z = H_z(\ell) = H$. The sole nonzero component of Eq. (2.6.50) is along the ℓ-coordinate, and reads per unit length

$$\int_M \ell\ (Dv_\ell/Dt)\ dM = -\int_M 2GM'\ dM' - \int_V \ell\ [d(P + pH^2/8\pi)/d\ell]\ dV, \tag{2.6.102}$$

where we have taken into account that $F = \partial\Phi/\partial\ell = -2GM/\ell$ by virtue of Eq. (2.6.30). We transform Eq. (2.6.102) analogously to the three-dimensional case:

$$\begin{aligned}&(1/2)\ d^2\Big(\int_M \ell^2\ dM\Big)\Big/dt^2 - \int_M v_\ell^2\ dM\\ &= -GM^2 + 2\int_V (P + pH^2/8\pi)\ dV - \int_S (P + pH^2/8\pi)\ell\ dS,\end{aligned} \tag{2.6.103}$$

where $dM = \varrho\ dV = 2\pi\varrho\ell\ d\ell,\ dS = 2\pi\ d\ell$. The integral over V in Eq. (2.6.102) has been transformed via the divergence formula (B.46), $[\vec{r} = \vec{r}(\ell, 0, 0)]$:

$$\nabla\cdot[(P+pH^2/8\pi)\ \vec{r}] = (1/\ell)\ d[\ell^2(P+pH^2/8\pi)]/d\ell = \ell\ d(P+pH^2/8\pi)/d\ell + 2(P+pH^2/8\pi). \tag{2.6.104}$$

Eq. (2.6.103) can be written in a form analogous to Eqs. (2.6.89), (2.6.94) in virtue of Eqs. (2.6.82)-(2.6.88):

$$(1/2)\ d^2I/dt^2 = 2E_{kin} + 4E_{th}/3 + 2\int_V P'\ dV + 2U_{rad}/3 + 2U_m - GM^2 - 2P_S V. \tag{2.6.105}$$

P_S denotes the constant isotropic value of $P + pH^2/8\pi$ on the cylinder's surface. Inserting further Eq. (2.6.92) into Eq. (2.6.103), we obtain another form of the virial theorem for our particular choice of cylindrical symmetry (Chandrasekhar and Fermi 1953):

$$d^2I/dt^2 = 2E_{kin} + 2(\Gamma-1)U + 2U_m - GM^2 - 2P_S V. \tag{2.6.106}$$

In the plane-parallel case we assume $P = P(z),\ \vec{v} = \vec{v}(v_z),\ \vec{H} = \vec{H}(H_\ell),\ H_\ell = H_\ell(z) = H$, and obtain for the surface unit, analogously to Eq. (2.6.102):

$$\int_M z\ (Dv_z/Dt)\ dM = -\int_M 2\pi zGM'\ dM' - \int_V z\ [d(P+pH^2/8\pi)/dz]\ dV. \tag{2.6.107}$$

The equations for a slab are analogous to Eqs. (2.6.103)-(2.6.106), and are written down consecutively:

$$(1/2)\ d^2\left(\int_M z^2\ dM\right)\Big/dt^2 - \int_M v_z^2\ dM = -\int_M 2\pi zGM'\ dM'$$
$$+\int_V (P+pH^2/8\pi)\ dV - \int_S (P+pH^2/8\pi)z\ dS, \tag{2.6.108}$$

where $dM = \varrho\ dV = \varrho\ dz,\ dS = 1,\ \vec{r} = \vec{r}(0,0,z)$, and

$$\nabla\cdot[(P+pH^2/8\pi)\vec{r}] = d[z(P+pH^2/8\pi)]/dz = d(P+pH^2/8\pi)/dz + P + pH^2/8\pi, \tag{2.6.109}$$

$$(1/2)\ d^2I/dt^2 = 2E_{kin} + 2E_{th}/3 + \int_V P'\ dV + U_{rad}/3 + U_m - \int_M 2\pi zGM'\ dM' - P_S V, \tag{2.6.110}$$

$$(1/2)\ d^2I/dt^2 = 2E_{kin} + (\Gamma-1)U + U_m - \int_M 2\pi zGM'\ dM' - P_S V. \tag{2.6.111}$$

So far, we have discussed the virial theorem under the assumption that the microscopic and macroscopic velocities are nonrelativistic. If these velocities are relativistic, with Newtonian gravitation still valid, the problem becomes slightly more complicated (Cox and Giuli 1968). In Eq. (2.1.1) we have to replace the force per unit mass $d\vec{v}/dt = d^2\vec{r}/dt^2$ by the relativistically correct term $d\vec{p}/dt$, where $\vec{p}$ is the relativistic momentum per unit rest mass given by Eq. (1.2.11). Thus, we can write

$$\vec{r}\cdot d\vec{p}/dt = \vec{r}\cdot d[\vec{v}/(1-v^2/c^2)^{1/2}]/dt = d[\vec{r}\cdot\vec{v}/(1-v^2/c^2)^{1/2}]/dt - v^2/(1-v^2/c^2)^{1/2}$$
$$= d\{[1/2(1-v^2/c^2)^{1/2}]\ dr^2/dt\}/dt - v^2/(1-v^2/c^2)^{1/2}$$
$$= d^2[r^2/2(1-v^2/c^2)^{1/2}]/dt^2 - d[(r^2/2)\ d(1-v^2/c^2)^{-1/2}/dt]/dt - v^2/(1-v^2/c^2)^{1/2}. \tag{2.6.112}$$

We define the relativistic moment of inertia for macroscopic motion as

$$I = \int_M [r^2/(1-v^2/c^2)^{1/2}]\ dM. \tag{2.6.113}$$

The last term in Eq. (2.6.112) can be expressed through the kinetic energy density $\varepsilon^{(kin)}$ from Eq. (1.2.15). We have

$$\varepsilon^{(kin)} = \varepsilon_r^{(kin)} - \varepsilon = \varrho c^2[1/(1-v^2/c^2)^{1/2} - 1] = \varrho c^2[1-(1-v^2/c^2)^{1/2}]/(1-v^2/c^2)^{1/2}$$
$$= \varrho v^2/\{(1-v^2/c^2)^{1/2}[1+(1-v^2/c^2)^{1/2}]\}, \tag{2.6.114}$$

and

$$v^2/(1-v^2/c^2)^{1/2} = \varepsilon^{(kin)}[1+(1-v^2/c^2)^{1/2}]/\varrho. \tag{2.6.115}$$

We integrate Eq. (2.6.112) over the rest mass M, and use Eqs. (2.6.113), (2.6.115):

$$\int_M (\vec{r}\cdot d\vec{p}/dt)\, dM = (1/2)\, d^2I/dt^2 - d\left\{\int_M [(r^2/2)\, d(1-v^2/c^2)^{-1/2}/dt]\, dM\right\}\Big/ dt$$
$$-[1+(1-v^2/c^2)^{1/2}]\int_V \varepsilon^{(kin)}\, dV. \tag{2.6.116}$$

The integral of $\varepsilon^{(kin)}\, dV$ is just the kinetic energy E_{kin} of relativistic macroscopic mass motions, and Eq. (2.6.89) writes in the relativistic case

$$(1/2)\, d^2I/dt^2 - d\left\{\int_M [(r^2/2)\, d(1-v^2/c^2)^{-1/2}/dt]\, dM\right\}\Big/ dt$$
$$= [1+(1-v^2/c^2)^{1/2}]E_{kin} + W + 3\int_V \left[w\varepsilon^{(kin)}/3 + P' + P_r + P_m\right] dV - \int_S x_k P_{jk}\, dS_j, \tag{2.6.117}$$

where we have replaced via Eq. (1.7.53) the pressure arising from kinetic transfer of momentum. The factor w, $(1\le w\le 2)$ from Eq. (1.7.53) has the same form as the factor $1+(1-v^2/c^2)^{1/2}$ near E_{kin}, but the relevant velocity is the microscopic velocity of translational particle motion, instead of the velocity of macroscopic mass motions.

To obtain the virial theorem in N-dimensional space, we write

$$(1/2)\, d^2I/dt^2 - d\left\{\int_M [(r^2/2)\, d(1-v^2/c^2)^{-1/2}/dt]\, dM\right\}\Big/ dt$$
$$= [1+(1-v^2/c^2)^{1/2}]E_{kin} - \int_V \vec{r}\cdot\nabla P\, dV + \int_M \vec{r}\cdot\nabla\Phi\, dM, \tag{2.6.118}$$

combining Eqs. (2.6.50) and (2.6.117) in the nonmagnetic case.

In a N-dimensional space with radial symmetry we have in the nonmagnetic case $P = P(r) = w\varepsilon^{(kin)}/3 + P' + P_r$, $\Phi = \Phi(r)$ via Eq. (2.6.88), the pressure being isotropic. r denotes the radial distance from the origin if $N\ge 3$, from the symmetry axis if $N=2$, and from the symmetry plane if $N=1$. The pressure integral can be transformed according to

$$\int_V \vec{r}\cdot\nabla P\, dV = \int_V r\,(dP/dr)\, dV = \int_V \nabla\cdot(P\vec{r})\, dV - N\int_V P\, dV$$
$$= \int_S P\,\vec{r}\cdot d\vec{S} - N\int_V P\, dV = \int_S Pr\, dS - N\int_V P\, dV, \tag{2.6.119}$$

where we have used Eq. (C.13): $\nabla\cdot(P\vec{r}) = (1/r^{N-1})\, d(r^N P)/dr = NP + r\, dP/dr$.

Thus, the radially symmetric form of the nonmagnetic virial theorem in N-dimensional space becomes

$$(1/2)\, d^2I/dt^2 - d\left\{\int_M [(r^2/2)\, d(1-v^2/c^2)^{-1/2}/dt]\, dM\right\}\Big/ dt$$
$$= [1+(1-v^2/c^2)^{1/2}]E_{kin} + N\int_V P\, dV + \int_M r\,(d\Phi/dr)\, dM - \int_S Pr\, dS. \tag{2.6.120}$$

If the pressure on the surface P_S is constant, and $\vec{H}=0$, $v\ll c$, Eq. (2.6.120) is analogous to Eq. (2.6.80) if $N=3$, to Eq. (2.6.105) if $N=2$, and to Eq. (2.6.110) if $N=1$.

For radial symmetry the volume V is bounded by an inner and outer surface S_0 and S_1, corresponding to the radial coordinates $r=r_0$ and r_1, $(r_1\ge r_0)$, respectively. The pressures P_{S0} and P_{S1} acting on the boundary surfaces are constant because of radial symmetry. The volume V occupied by the system is just the difference of the volumes V_1 and V_0, bounded by the radial surfaces S_1 and $S_0:\ V = V_1 - V_0$. By virtue of Eqs. (2.6.119), (C.13) we can write

$$\int_S Pr\, dS = \int_V \nabla\cdot(P\vec{r})\, dV = \int_{V_1}\nabla\cdot(P\vec{r})\, dV - \int_{V_0}\nabla\cdot(P\vec{r})\, dV = \int_{S_1} P\,\vec{r}\cdot d\vec{S} - \int_{S_0} P\,\vec{r}\cdot d\vec{S}$$
$$= P_{S1}\int_{S_1}\vec{r}\cdot d\vec{S} - P_{S0}\int_{S_0}\vec{r}\cdot d\vec{S} = P_{S1}\int_{V_1}\nabla\cdot\vec{r}\, dV - P_{S0}\int_{V_0}\nabla\cdot\vec{r}\, dV = N(P_{S1}V_1 - P_{S0}V_0). \tag{2.6.121}$$

Inserting Eq. (2.6.121) into Eq. (2.6.120), and using Eq. (2.6.95), we write down the virial theorem in a form suitable for polytropic applications:

$$(1/2)\ d^2I/dt^2 - d\Big\{\int_M [(r^2/2)\ d(1-v^2/c^2)^{-1/2}/dt]\ dM\Big\}\Big/ dt$$
$$= [1+(1-v^2/c^2)^{1/2}]E_{kin} + N(\Gamma-1)U + \int_M r\ (d\Phi/dr)\ dM - N(P_{S1}V_1 - P_{S0}V_0). \qquad (2.6.122)$$

2.6.7 Gravitational Potential Energy and Internal Energy

The gravitational energy per unit surface is infinite in the plane-symmetrical case ($N=1$), as well as the gravitational energy per unit length of a cylinder ($N=2$). This can be shown at once, if we evaluate the change of gravitational energy according to Eqs. (2.6.30) and (2.6.69):

$$dW = \int_r^\infty \vec{F}\cdot d\vec{r'} = -2\pi GM(r)\ dM\int_r^\infty dr' = -2\pi GM(r)\ dM\ r'\Big|_r^\infty = \infty, \quad (N=1), \qquad (2.6.123)$$

$$dW = \int_r^\infty \vec{F}\cdot d\vec{r'} = -2GM(r)\ dM\int_r^\infty dr'/r' = -2GM(r)\ dM\ln r'\Big|_r^\infty = \infty, \quad (N=2). \quad (2.6.124)$$

Only if $N\geq 3$, we have

$$dW = \int_r^\infty \vec{F}\cdot d\vec{r'} = -2\pi G\big\{\Gamma(N/2)/[\Gamma(1/2)]^N\big\}M(r)\ r^{2-N}\ dM/(N-2), \qquad (2.6.125)$$

where we have inserted Eq. (2.6.49). The gravitational energy of the whole mass M becomes

$$W = -[2\pi G/(N-2)]\big\{\Gamma(N/2)/[\Gamma(1/2)]^N\big\}\int_M r^{2-N}M(r)\ dM, \qquad (N\geq 3). \qquad (2.6.126)$$

If $N=3$, Eq. (2.6.126) turns into the well known result (e.g. Cox and Giuli 1968)

$$W = -G\int_M M(r)\ dM/r, \qquad (N=3). \qquad (2.6.127)$$

The gravitational potential energy from Eqs. (2.6.123), (2.6.124) is infinite if $N=1,2$, but in formal analogy to the spherical case $N=3$, we may call the integral from Eq. (2.6.65)

$$W = -(1/2)\int_M \Phi\ dM = \int_M x_k\ (\partial\Phi/\partial x_k)\ dM = \int_M \vec{r}\cdot\nabla\Phi\ dM, \qquad (2.6.128)$$

the gravitational energy of a N-dimensional radially symmetric polytrope. We insert r, dM, and $\nabla\Phi$ from Eqs. (2.6.1), (2.6.12), (2.6.32):

$$W = 2\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}(n+1)K\varrho_0^{1+1/n}\alpha^N\int_{\xi_0}^{\xi_1}\xi^N\theta^n\theta'\ d\xi, \qquad (n\neq -1,\pm\infty), \qquad (2.6.129)$$

where we consider – quite generally – the mass contained between the arbitrary radial coordinates ξ_0 and ξ_1 of a polytrope. Integrating by parts, we get in virtue of Eq. (2.1.14):

$$\int_{\xi_0}^{\xi_1}\xi^N\theta^n\theta'\ d\xi = \xi^N\theta^{n+1}/(n+1)\Big|_{\xi_0}^{\xi_1} - [N/(n+1)]\int_{\xi_0}^{\xi_1}\xi^{N-1}\theta^{n+1}\ d\xi$$
$$= \xi^N\theta^{n+1}/(n+1)\Big|_{\xi_0}^{\xi_1} \pm [N/(n+1)]\int_{\xi_0}^{\xi_1}\theta\ d(\xi^{N-1}\theta') = \xi^N\theta^{n+1}/(n+1)\Big|_{\xi_0}^{\xi_1}$$
$$\pm[N/(n+1)]\xi^{N-1}\theta\theta'\Big|_{\xi_0}^{\xi_1} \mp [N/(n+1)]\int_{\xi_0}^{\xi_1}\xi^{N-1}\theta'^2\ d\xi. \qquad (2.6.130)$$

The integral (2.6.129) can also be transformed in another way:

$$\int_{\xi_0}^{\xi_1}\xi^N\theta^n\theta'\ d\xi=\mp\int_{\xi_0}^{\xi_1}\xi\theta'\ d(\xi^{N-1}\theta')=\mp\int_{\xi_0}^{\xi_1}\xi^{N-1}\theta'\ d(\xi^{N-1}\theta')/\xi^{N-2}$$
$$=\mp\xi^N\theta'^2/2\Big|_{\xi_0}^{\xi_1}\mp[(N-2)/2]\int_{\xi_0}^{\xi_1}\xi^{N-1}\theta'^2\ d\xi. \tag{2.6.131}$$

Eliminating $\int_{\xi_0}^{\xi_1}\xi^{N-1}\theta'^2\ d\xi$ between Eqs. (2.6.130) and (2.6.131), we find

$$\int_{\xi_0}^{\xi_1}\xi^N\theta^n\theta'\ d\xi=\big\{(N-2)/[(N-2)(n+1)-2N]\big\}\Big[\xi^N\theta^{n+1}\pm N\xi^{N-1}\theta\theta'\pm N\xi^N\theta'^2/(N-2)\Big]\Big|_{\xi_0}^{\xi_1}. \tag{2.6.132}$$

Thus, Eq. (2.6.129) can finally be written as (Viala and Horedt 1974a if N=3, Kimura and Liu 1978)

$$W=8\pi G\alpha^{N+2}\varrho_0^2\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}\big\{(N-2)/[(N-2)(n+1)-2N]\big\}$$
$$\times\Big[\pm\,\xi^N\theta^{n+1}+N\xi^{N-1}\theta\theta'+N\xi^N\theta'^2/(N-2)\Big]\Big|_{\xi_0}^{\xi_1},\qquad (n\neq-1,\pm\infty;\ N=1,2,3,...), \tag{2.6.133}$$

where we have inserted for K via Eq. (2.6.1). At the finite boundary we have $\theta,\theta^{n+1}=0$, and the gravitational energy of a complete polytrope with finite boundary ξ_1 becomes

$$W_1=8\pi G\alpha^{N+2}\varrho_0^2\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}\big\{N/[(N-2)(n+1)-2N]\big\}\xi_1^N\theta_1'^2$$
$$=2\pi G\big\{[\Gamma(N/2)/[\Gamma(1/2)]^N\big\}\big\{N/[(N-2)(n+1)-2N]\big\}M_1^2/r_1^{N-2},\qquad (\xi_0=0), \tag{2.6.134}$$

by using Eqs. (2.6.1), (2.6.12), and indexing with 1 the boundary values of a complete polytrope.

For the cases of practical interest we have, by anticipating the results from the next subsection on the finiteness and infiniteness of mass and radius:

$$N=1:\ W_1=-2\pi GM_1^2r_1/(3+n),\qquad(-1<n<\infty), \tag{2.6.135}$$

$$N=2:\ W_1=-GM_1^2,\qquad(-1<n<\infty), \tag{2.6.136}$$

$$N=3:\ W_1=-3GM_1^2/(5-n)r_1,\qquad(-1<n<5). \tag{2.6.137}$$

Eqs. (2.6.135) and (2.6.136) seem to imply that the gravitational energy per unit surface of a slab, and the gravitational energy per unit length of a cylinder is finite – opposite to our earlier findings from Eqs. (2.6.123), (2.6.124). However, it should be kept in mind that Eqs. (2.6.135), (2.6.136) have been derived from a *formal* analogy with the finite sphere. Therefore, our earlier statements remain entirely valid. The same holds also in the isothermal case ($n=\pm\infty$) shown by Eqs. (2.6.153), (2.6.157).

If $(N-2)(n+1)-2N=0$ or $n=(N+2)/(N-2)$, the gravitational energy from Eq. (2.6.134) would seem at first sight to become infinite (Chandrasekhar 1939, Cox and Giuli 1968). However, since $n=(N+2)/(N-2)$ is just the value of the polytropic index corresponding to the Schuster-Emden integral (2.3.36) if $N\geq3$, the value of r_1 is infinite, and the denominator from Eq. (2.6.134) takes the undefined form $0\cdot\infty$. A more careful evaluation of Eq. (2.6.134) shows that the gravitational energy is finite if $n=(N+2)/(N-2)$, as first pointed out by Buchdahl (1978) in the spherical case $N=3$. It will be obvious from Eq. (2.6.193) that the mass M_1 of a polytrope with index $n=(N+2)/(N-2)$ tends to a finite limit if $N\geq3$, so we have to evaluate in Eq. (2.6.134) only the limit of

$$1/\big[(N-2)(n+1)-2N\big]r_1^{N-2}=\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}^2[(n+1)K\varrho_0^{1+1/n}\alpha^N/\pi GNM_1^2]$$
$$\times\int_0^\infty\xi^N\theta^n\theta'\ d\xi=[\alpha^{2-N}/N(\xi_1^{N-1}\theta_1')^2]\int_0^\infty\xi^N\theta^n\theta'\ d\xi,\qquad[N\geq3;\ n=(N+2)/(N-2)], \tag{2.6.138}$$

where we have equated Eqs. (2.6.129) and (2.6.134), and inserted for M_1 according to Eq. (2.6.12).

We evaluate the integral from Eq. (2.6.138) explicitly in the spherical case $N = 3$, and prove its finiteness in the general case $N \geq 3$ (Horedt 1987c). If $N = 3$, the Schuster-Emden integral (2.3.36) becomes $\theta = (1 + \xi^2/3)^{-1/2}$, and (Buchdahl 1978)

$$\int_0^\infty \xi^3\theta^n\theta' \, d\xi = -(1/3)\int_0^\infty \xi^4 \, d\xi/(1+\xi^2/3)^4 = \Big[\xi^4/6(1+\xi^2/3)^4 + 3\xi/8(1+\xi^2/3)^2$$
$$-9\xi/16(3+\xi^2) - (3^{3/2}/16)\arctan(\xi/3^{1/2})\Big]\Big|_0^\infty = -3^{3/2}\pi/32, \qquad (N = 3;\ n = 5). \tag{2.6.139}$$

We insert Eq. (2.6.139) into Eq. (2.6.138), taking into account that by virtue of Eq. (2.6.145) $\lim_{\xi\to\infty}(\xi^2\theta') = -3^{1/2}$:

$$1/\big[(N-2)(n+1) - 2N\big]r_1^{N-2} = 1/(n-5)r_1 = -\pi/32 \times 3^{1/2}\alpha, \qquad (N = 3;\ n = 5). \tag{2.6.140}$$

Substituting Eq. (2.6.140) into Eq. (2.6.137), we find

$$W = -3^{1/2}\pi G M_1^2/32\alpha = -3^{2/3}\pi^{4/3}G\varrho_0^{1/3}M_1^{5/3}/2^{13/3}, \qquad (N = 3;\ n = 5), \tag{2.6.141}$$

since via Eq. (2.6.18): $\alpha = (M_1/2^2 \times 3^{1/2}\pi\varrho_0)^{1/3}$.

In the general case $N \geq 3$, we have according to Eq. (2.3.36) $\theta = [1 + \xi^2/N(N-2)]^{(2-N)/2}$ and

$$\xi^N\theta^n\theta' = -\xi^{N+1}/N[1 + \xi^2/N(N-2)]^{N+1}, \tag{2.6.142}$$

$$\lim_{\xi\to\infty}(\xi^N\theta^n\theta') = -N^N(N-2)^{N+1}/\xi^{N+1} = 0, \qquad [N \geq 3;\ n = (N+2)/(N-2)]. \tag{2.6.143}$$

Thus, $\xi^N\theta^n\theta'$ is finite in the interval $[0, \infty)$, and because $\xi^N\theta^n\theta'$ changes as ξ^{-N-1} if $\xi \to \infty$, we infer that its integral from 0 to ∞ is finite. Because

$$\xi^{N-1}\theta' = -\xi^N/N[1 + \xi^2/N(N-2)]^{N/2}, \tag{2.6.144}$$

$$\lim_{\xi\to\infty}(\xi^{N-1}\theta') = -[N(N-2)]^{N/2}/N, \qquad [N \geq 3;\ n = (N+2)/(N-2)], \tag{2.6.145}$$

we conclude that Eq. (2.6.138) is finite: The gravitational potential energy W_1 from Eq. (2.6.134) is finite if $N \geq 3$ and $n = (N+2)/(N-2)$.

If $n > (N+2)/(N-2)$, the gravitational energy from Eq. (2.6.134) is positive, and consequently these polytropes extend to infinity if $N \geq 3$.

If $n = \pm\infty$, the gravitational potential energy (2.6.128) becomes

$$W = -2\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}K\varrho_0\alpha^N\int_{\xi_0}^{\xi_1}\xi^N\exp(-\theta)\theta' \, d\xi, \tag{2.6.146}$$

where we have inserted for $r, dM, \nabla\Phi$ according to Eqs. (2.6.2), (2.6.13), (2.6.33), respectively. The evaluation of the integral (2.6.146) is straightforward by using Eq. (2.1.21):

$$\int_{\xi_0}^{\xi_1}\xi^N\exp(-\theta)\theta' \, d\xi = -\xi^N\exp(-\theta)\Big|_{\xi_0}^{\xi_1} + N\int_{\xi_0}^{\xi_1}\exp(-\theta)\xi^{N-1} \, d\xi$$
$$= \big[-\xi^N\exp(-\theta) + N\xi^{N-1}\theta'\big]\Big|_{\xi_0}^{\xi_1}. \tag{2.6.147}$$

Eq. (2.6.146) becomes eventually $(K = 4\pi G\varrho_0\alpha^2)$:

$$W = 8\pi G\big\{[\Gamma(1/2)]^N/\Gamma(N/2)\big\}\varrho_0^2\alpha^{N+2}\big[\xi^N\exp(-\theta) - N\xi^{N-1}\theta'\big]\Big|_{\xi_0}^{\xi_1}, \quad (n = \pm\infty;\ N = 1, 2, 3, ...). \tag{2.6.148}$$

If $N \geq 3$, the asymptotic solution for the Lane-Emden function is given by Eqs. (2.4.103), (2.4.108), (2.4.110). In a first approximation:

$$\theta \approx \ln[\xi^2/2(N-2)], \qquad (\xi \to \infty;\ n = \pm\infty;\ N \geq 3). \tag{2.6.149}$$

Thus, $\exp(-\theta) \approx 2(N-2)/\xi^2$ and $\theta' \approx 2/\xi$ if $\xi \to \infty$, and we observe by inserting $\xi \to \infty$ into Eq. (2.6.148) that the gravitational energy of a complete N-dimensional polytrope is infinite if $n = \pm\infty$ and $N \geq 3$. However, in the plane and cylindrical case $N = 1, 2$, what we have called gravitational energy, tends to a finite limit if $r, \xi \to \infty$, as will be shown subsequently.

If $N = 1$, we have $\exp(-\theta) = 1/\cosh^2(\xi/2^{1/2})$ in virtue of Eq. (2.3.65), and

$$\theta' = 2^{1/2}\tanh(\xi/2^{1/2}); \quad \lim_{\xi\to\infty} \theta' = 2^{1/2}, \qquad (N = 1;\ n = \pm\infty). \tag{2.6.150}$$

Thus, Eq. (2.6.148) reads

$$W_1 = -2^{3/2} K\varrho_0\alpha = -2^{1/2}K^{3/2}\varrho_0^{1/2}/\pi^{1/2}G^{1/2}, \qquad (N = 1;\ n = \pm\infty), \tag{2.6.151}$$

if we insert $\alpha^2 = K/4\pi G\varrho_0$ from Eq. (2.6.2). The mass of the complete slab (2.6.15) turns out to be

$$M_1 = 2\varrho_0\alpha\theta_1' = 2^{3/2}\varrho_0\alpha = (2K\varrho_0/\pi G)^{1/2}, \qquad (N = 1;\ n = \pm\infty). \tag{2.6.152}$$

We eliminate the polytropic constant K between Eqs. (2.6.151), (2.6.152), and find

$$W_1 = -KM_1 = -\pi G M_1^3/2\varrho_0, \qquad (N = 1;\ n = \pm\infty). \tag{2.6.153}$$

In the cylindrical case Eq. (2.3.48) yields $\exp(-\theta) = 1/(1 + \xi^2/8)^2$, and

$$\theta' = \xi/2(1 + \xi^2/8); \quad \lim_{\xi\to\infty} \theta' = 4/\xi, \qquad (N = 2;\ n = \pm\infty). \tag{2.6.154}$$

Eq. (2.6.148) becomes

$$W_1 = -16\pi K\varrho_0\alpha^2 = -4K^2/G, \qquad (N = 2;\ n = \pm\infty). \tag{2.6.155}$$

From Eq. (2.6.17) we have

$$M_1 = 2\pi\varrho_0\alpha^2\xi_1\theta_1' = 8\pi\varrho_0\alpha^2 = 2K/G, \qquad (N = 2;\ n = \pm\infty), \tag{2.6.156}$$

and

$$W_1 = -2KM_1 = -GM_1^2, \qquad (N = 2;\ n = \pm\infty). \tag{2.6.157}$$

The internal energy of a polytrope U is given by Eq. (2.6.95), and can be evaluated in a similar manner as the gravitational energy (2.6.129):

$$\begin{aligned} U &= [1/(\Gamma-1)]\int_V P\ dV = [K/(\Gamma-1)]\int_V \varrho^{1+1/n}\ dV \\ &= [2K\varrho_0^{1+1/n}\alpha^N/(\Gamma-1)]\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\int_{\xi_0}^{\xi_1} \xi^{N-1}\theta^{n+1}\ d\xi, \qquad (n \neq -1, \pm\infty), \end{aligned} \tag{2.6.158}$$

where we have inserted for P and dV via Eqs. (2.6.3), (2.6.9). We have

$$\begin{aligned} &\int_{\xi_0}^{\xi_1} \xi^{N-1}\theta^{n+1}\ d\xi = \xi^N\theta^{n+1}/N\Big|_{\xi_0}^{\xi_1} - [(n+1)/N]\int_{\xi_0}^{\xi_1} \xi^N\theta^n\theta'\ d\xi \\ &= \{1/[(N-2)(n+1) - 2N]\}\Big[-2\xi^N\theta^{n+1} \mp (N-2)(n+1)\xi^{N-1}\theta\theta' \mp (n+1)\xi^N\theta'^2\Big]\Big|_{\xi_0}^{\xi_1}, \end{aligned} \tag{2.6.159}$$

via Eq. (2.6.132). Thus

$$\begin{aligned} U &= \{2K\varrho_0^{1+1/n}\alpha^N/(\Gamma-1)[(N-2)(n+1) - 2N]\}\{[\Gamma(1/2)]^N/\Gamma(N/2)\} \\ &\times\Big[-2\xi^N\theta^{n+1} \mp (N-2)(n+1)\xi^{N-1}\theta\theta' \mp (n+1)\xi^N\theta'^2\Big]\Big|_{\xi_0}^{\xi_1}, \qquad (n \neq -1, \pm\infty;\ N = 1, 2, 3, ...). \end{aligned} \tag{2.6.160}$$

If $n = \pm\infty$, we get in a similar manner

$$U = [1/(\Gamma-1)]\int_V P\ dV = [2K\varrho_0\alpha^N/(\Gamma-1)]\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\int_{\xi_0}^{\xi_1}\xi^{N-1}\exp(-\theta)\ d\xi$$
$$= [2K\varrho_0\alpha^N/(\Gamma-1)]\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\ \xi^{N-1}\theta'\Big|_{\xi_0}^{\xi_1}, \qquad (n=\pm\infty;\ N=1,2,3,...), \qquad (2.6.161)$$

where we have used Eq. (2.1.21).

Eqs. (2.6.160) and (2.6.161) can be brought into the form of the virial theorem from Eq. (2.6.122):

$$U = [2K\varrho_0^{1+1/n}\alpha^N/(\Gamma-1)]\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\left\{\xi^N\theta^{n+1}/N\Big|_{\xi_0}^{\xi_1} - [(n+1)/N]\int_{\xi_0}^{\xi_1}\xi^N\theta^n\theta'\ d\xi\right\}$$
$$= [K\varrho_0^{1+1/n}V\theta^{n+1}/(\Gamma-1)]\Big|_{\xi_0}^{\xi_1} - W/N(\Gamma-1) = [1/(\Gamma-1)](V_1P_1 - V_0P_0) - W/N(\Gamma-1),$$
$$(n \neq -1, \pm\infty;\ N=1,2,3,...), \qquad (2.6.162)$$

$$U = [2K\varrho_0\alpha^N/(\Gamma-1)]\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\left[\xi^N\exp(-\theta)/N\Big|_{\xi_0}^{\xi_1} + (1/N)\int_{\xi_0}^{\xi_1}\xi^N\exp(-\theta)\theta'\ d\xi\right]$$
$$= [K\varrho_0V\exp(-\theta)/(\Gamma-1)]\Big|_{\xi_0}^{\xi_1} - W/N(\Gamma-1) = [1/(\Gamma-1)](V_1P_1 - V_0P_0) - W/N(\Gamma-1),$$
$$(n=\pm\infty;\ N=1,2,3,...), \qquad (2.6.163)$$

by virtue of Eqs. (2.6.129) and (2.6.146), respectively.

Thus, for a radially symmetric polytrope in hydrostatic equilibrium, the virial theorem (2.6.122) can be written as ($I=$ const; $\vec{v} = d\vec{r}/dt = 0;\ \vec{H}=0$)

$$N(\Gamma-1)U + W - N(V_1P_1 - V_0P_0) = 0, \qquad (n\neq -1;\ N=1,2,3,...). \qquad (2.6.164)$$

The last term vanishes for a complete polytrope with finite boundary, since in this case $V_0, P_1 = 0$ and $P_0, V_1 =$ finite :

$$N(\Gamma-1)U_1 + W_1 = 0, \qquad (n\neq -1). \qquad (2.6.165)$$

In the cases of practical interest we obtain for complete polytropes with finite boundary

$$N=1: \quad U_1 = 2\pi GM_1^2r_1/(3+n)(\Gamma-1), \qquad (-1<n<\infty), \qquad (2.6.166)$$

$$N=2: \quad U_1 = GM_1^2/2(\Gamma-1), \qquad (-1<n<\infty), \qquad (2.6.167)$$

$$N=3: \quad U_1 = GM_1^2/(5-n)(\Gamma-1)r_1, \qquad (-1<n<5), \qquad (2.6.168)$$

by writing $U_1 = -W_1/N(\Gamma-1)$, and inserting from Eqs. (2.6.135)-(2.6.137).

Even for polytropes with infinite extension the product V_1P_1 vanishes sometimes when $r,\xi\to\infty$. As an example we quote the polytropes obeying the Schuster-Emden integral $n=(N+2)/(N-2),\ (N\geq 3)$, and the polytropes having $n=\pm\infty,\ (N=1,2)$.

If $n=(N+2)/(N-2)$, we have

$$VP \propto \xi^N\theta^{n+1} = \xi^N[1+\xi^2/N(N-2)]^{-N}; \quad \lim_{\xi\to\infty}(VP)\propto \xi^{-N} = 0. \qquad (2.6.169)$$

Since W has been shown to be finite in the case $n=(N+2)/(N-2),\ (N\geq 3)$, it turns out that U is finite too. In particular, we obtain at once from Eq. (2.6.141) in the spherical case:

$$U_1 = -W_1/3(\Gamma-1) = \pi GM_1^2/32\times 3^{1/2}(\Gamma-1)\alpha = \pi^{4/3}G\varrho_0^{1/3}M_1^{5/3}/2^{13/3}\times 3^{1/3}(\Gamma-1),$$
$$(n=5;\ N=3). \qquad (2.6.170)$$

If $N = 1$ and $n = \pm\infty$, we get

$$VP \propto \xi \exp(-\theta) = \xi/\cosh^2(\xi/2^{1/2}); \quad \lim_{\xi\to\infty} (VP) \propto \xi/\exp(2^{1/2}\xi) = 0. \tag{2.6.171}$$

If $N = 2$ and $n = \pm\infty$, we have

$$VP \propto \xi^2 \exp(-\theta) = \xi^2/(1+\xi^2/8)^2; \quad \lim_{\xi\to\infty} (VP) \propto \xi^{-2} = 0. \tag{2.6.172}$$

Thus, by using Eqs. (2.6.153) and (2.6.157):

$$U_1 = -W_1/(\Gamma-1) = \pi G M_1^3/2(\Gamma-1)\varrho_0, \qquad (N=1;\ n=\pm\infty), \tag{2.6.173}$$

$$U_1 = -W_1/2(\Gamma-1) = G M_1^2/2(\Gamma-1), \qquad (N=2;\ n=\pm\infty). \tag{2.6.174}$$

For a perfect gas the factor $\Gamma - 1$ can be replaced according to Eqs. (1.2.22), (1.7.56) by

$$\gamma - 1 = \mathcal{R}/\mu c_V. \tag{2.6.175}$$

Inserting the perfect gas law (1.2.5), the internal energy (2.6.158) can be written in this important particular case as

$$U = [1/(\gamma-1)] \int_V P\ dV = (\mu c_V/\mathcal{R}) \int_V P\ dV = c_V \int_V \varrho T\ dV = c_V \int_M T\ dM. \tag{2.6.176}$$

2.6.8 Behaviour of Physical Characteristics as the Radius Increases

We summarize the principal physical characteristics of complete polytropes as they result from the behaviour of the fundamental Lane-Emden function $\theta(\xi)$ via Eqs. (2.1.14), (2.1.21), (2.1.41), (cf. Emden 1907, Chandrasekhar 1939, Ostriker 1964a, Viala and Horedt 1974a, b, Horedt 1986b). As required by hydrostatic equilibrium, the pressure P ($\propto \theta^{n+1}$ if $n \neq -1, \pm\infty$, and $\propto \exp(-\theta)$ if $n = \pm\infty$) must decrease from its central value up to zero at the finite or infinite boundary of the complete polytrope. The following behaviour of the Lane-Emden function results from this constraint: θ decreases from its central value up to zero at the finite or infinite boundary if $-1 < n < \infty$, and increases from the central value up to ∞ if $-\infty < n < -1$ and $n = \pm\infty$. The Lane-Emden function θ is proportional to the temperature T in the case of a perfect gas if $n \neq -1, \pm\infty$ [Eq. (2.6.7)]. The density ϱ ($\propto \theta^n$ if $n \neq -1, \pm\infty$, and $\propto \exp(-\theta)$ if $n = \pm\infty$) decreases from the central value up to zero at the boundary if $-\infty < n < -1,\ 0 < n < \infty$, and $n = \pm\infty$. The density increases from the central value up to ∞ if $-1 < n < 0$. If $n = 0$, the density remains constant. With the exception of the origin, the mass inside radial coordinate ξ has to be a strictly positive quantity, in order to be physically meaningful. θ' has to be nonpositive if $-1 < n < \infty$, and nonnegative if $-\infty < n < -1$, since according to Eq. (2.6.12) $M \propto \mp\xi^{N-1}\theta'$. Eq. (2.6.13) shows that $M \propto \xi^{N-1}\theta'$ and θ' are always nonnegative if $n = \pm\infty$.

The main problem to be discussed in this subsection remains the finiteness or infiniteness of mass and radius. In the plane-symmetric case $N = 1$ the result can be found at once by rewriting Eq. (2.4.68):

$$\theta = \mp\{2/[\pm(n+1)]\}^{1/2}\xi + C = \mp\{2/[\pm(n+1)]\}^{1/2}(\xi - \xi_2) + \theta_2, \qquad (N=1;\ n \neq -1, \pm\infty). \tag{2.6.177}$$

$\theta_2 = \theta(\xi_2)$ is a value of the Lane-Emden function sufficiently close to the finite or infinite boundary of the polytropic slab: $\theta^{n+1}(\xi_2) \approx 0$. If $-1 < n < \infty$, we have $\theta_2 \approx 0$. Eq. (2.6.177) becomes near the boundary

$$\xi - \xi_2 \approx 0, \qquad (N=1;\ -1 < n < \infty), \tag{2.6.178}$$

since $\theta \to \theta_1 = 0$ as the boundary is approached. This shows that $\xi \to \xi_1 \approx \xi_2$ is finite as the boundary ξ_1 is approached. On the other hand, if $-\infty < n < -1$, the Lane-Emden function θ is increasing, and $\theta \to \theta_1 = \infty$ as the boundary is approached: $\theta_1 \gg \theta_2 \gg 1$. Eq. (2.6.177) becomes near the boundary

$$\theta \approx [-2/(n+1)^{1/2}](\xi - \xi_2), \qquad (N=1;\ -\infty < n < -1), \tag{2.6.179}$$

or

$$\xi - \xi_2 \to \infty, \qquad (N = 1;\ -\infty < n < -1). \tag{2.6.180}$$

Since ξ_2 is finite, we have $\xi \to \xi_1 \to \infty$ if the boundary ξ_1 is approached.

If $n = \pm\infty$, $(N = 1)$, the exact analytical solution (2.3.65) shows that no finite zero of the Lane-Emden function exists: $\theta \to \infty$ if $\xi \to \infty$.

Thus, polytropic slabs have finite extension if $-1 < n < \infty$. Their radius is infinite if $-\infty < n < -1$ and $n = \pm\infty$.

By virtue of Eqs. (2.4.67), (2.6.150) the boundary value of the derivative θ' is always finite for the polytropic slab:

$$\theta_1' \to \mp\{2/[\pm(n+1)]\}^{1/2}, \qquad (N = 1;\ n \neq -1, \pm\infty), \tag{2.6.181}$$

$$\theta_1' \to 2^{1/2}, \qquad (N = 1;\ n = \pm\infty). \tag{2.6.182}$$

Inserting the boundary values from Eqs. (2.6.181) and (2.6.182) into Eqs. (2.6.14) and (2.6.15), respectively, we get

$$M_1 = \{8/[\pm(n+1)]\}^{1/2}\varrho_0\alpha = [2K\varrho_0^{1+1/n}/\pi G]^{1/2} = (2P_0/\pi G)^{1/2}, \qquad (N = 1;\ n \neq -1, \pm\infty), \tag{2.6.183}$$

$$M_1 = 8^{1/2}\varrho_0\alpha = (2K\varrho_0/\pi G)^{1/2} = (2P_0/\pi G)^{1/2}, \qquad (N = 1;\ n = \pm\infty). \tag{2.6.184}$$

Thus, the total mass per unit surface of a polytropic slab is finite and depends only on the pressure P_0 in the symmetry plane. For a polytrope composed of a perfect gas we may insert $P_0 = \mathcal{R}\varrho_0 T_0/\mu$.

In the cylindrical case $N = 2$ there exists the asymptotic solution (2.4.83)

$$\theta \approx [(1-n)\xi/2]^{2/(1-n)}, \qquad (\xi \to \infty;\ N = 2;\ -\infty < n < -1), \tag{2.6.185}$$

showing that if the radius r or ξ becomes infinite, θ becomes infinite too. Pressure $\propto \theta^{n+1}$ and density $\propto \theta^n$ tend to zero if $\xi \to \infty$ and $-\infty < n < -1$. The total mass is infinite if $\xi \to \infty$, since

$$M_1 \propto \lim_{\xi\to\infty}(\xi\theta') \approx [(1-n)/2]^{(1+n)/(1-n)} \lim_{\xi\to\infty} \xi^{2/(1-n)} = \infty, \qquad (N = 2;\ -\infty < n < -1). \tag{2.6.186}$$

If $n = \pm\infty$, we have [cf. Eq. (2.4.94)]

$$\theta \approx \ln(\xi^4/64), \qquad (\xi \to \infty;\ N = 2;\ n = \pm\infty), \tag{2.6.187}$$

showing that $\theta \to \infty$ and $\exp(-\theta) \to 0$ if $\xi \to \infty$. The mass per unit length remains finite, since [cf. Eq. (2.6.156)]

$$M_1 = 2\pi\varrho_0\alpha^2 \lim_{\xi\to\infty}(\xi\theta') = 8\pi\varrho_0\alpha^2 = 2K/G = 2P_0/G\varrho_0, \qquad (\xi \to \infty;\ N = 2;\ n = \pm\infty). \tag{2.6.188}$$

For a perfect gas the total mass per unit length of an isothermal cylinder is

$$M_1 = 2\mathcal{R}T_0/G\mu, \qquad (K = \mathcal{R}T_0/\mu). \tag{2.6.189}$$

In the spherical case $N = 3$ there exists the asymptotic solution (2.4.88)

$$\theta \approx [\pm(1-n)^2\xi^2/2(n-3)]^{1/(1-n)}, \qquad (\xi \to \infty;\ N = 3;\ -\infty < n < -1 \quad \text{and} \quad 5 < n < \infty), \tag{2.6.190}$$

showing that when the radius r or ξ becomes infinite, θ becomes infinite if $-\infty < n < -1$, and zero if $5 < n < \infty$. Pressure and density tend to zero. The total mass becomes infinite if $\xi \to \infty$, since

$$M_1 \propto \lim_{\xi\to\infty}(\mp\xi^2\theta') \approx [\mp 2/(1-n)][\pm(1-n)^2/2(n-3)]^{1/(1-n)} \lim_{\xi\to\infty} \xi^{(3-n)/(1-n)} = \infty,$$
$$(N = 3;\ -\infty < n < -1 \quad \text{and} \quad 5 < n < \infty). \tag{2.6.191}$$

If $n = (N+2)/(N-2),\ N \geq 3$, the Schuster-Emden integral (2.3.36) subsists

$$\theta = [1 + \xi^2/N(N-2)]^{(2-N)/2}, \tag{2.6.192}$$

showing that $\theta, \theta^n, \theta^{n+1} \to 0$ if $\xi \to \infty,\ (N \geq 3)$. The total mass of the Schuster-Emden polytropes is finite, because $\xi^{N-1}\theta'$ is finite via Eqs. (2.6.144), (2.6.145). The mass equation (2.6.12) reads

$$M_1 = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}N^{(N-2)/2}(N-2)^{N/2}, \quad (\xi \to \infty;\ n = (N+2)/(N-2);\ N \geq 3). \tag{2.6.193}$$

In the spherical case Eq. (2.6.193) turns into

$$M_1 = 4 \times 3^{1/2}\pi\varrho_0\alpha^3, \qquad (\xi \to \infty;\ n = 5;\ N = 3). \tag{2.6.194}$$

If $n = \pm\infty$ and $N \geq 3$, the asymptotic solution (2.6.149) exists: $\theta \to \infty$ and $\exp(-\theta) \to 0$ if $\xi \to \infty$. The total mass of these polytropes is always infinite, since

$$M_1 \propto \lim_{\xi\to\infty} (\xi^{N-1}\theta') = \lim_{\xi\to\infty} (2\xi^{N-2}) = \infty, \qquad (n = \pm\infty;\ N \geq 3). \tag{2.6.195}$$

Total mass M_1 ($\propto \mp\xi^{N-1}\theta'$) and radius r_1 ($\propto \xi$) behave as follows:

(i) Slabs, $N = 1$. Complete polytropic slabs have infinite radius if $-\infty < n < -1$ and $n = \pm\infty$. They have finite extension if $-1 < n < \infty$. Their mass per unit surface remains always finite.

(ii) Cylinders, $N = 2$. Complete polytropic cylinders have infinite radius if $-\infty < n < -1$ and $n = \pm\infty$. If $-1 < n < \infty$, their extension is finite. They have infinite mass per unit length if $-\infty < n < -1$. Their mass per unit length is finite if $-1 < n < \infty$ and $n = \pm\infty$.

(iii) Spheres, $N = 3$. Complete polytropic spheres have infinite radius if $-\infty < n < -1,\ 5 \leq n < \infty$, and $n = \pm\infty$. They have finite extension if $-1 < n < 5$. Their mass is infinite if $-\infty < n < -1,\ 5 < n < \infty$, and $n = \pm\infty$. Their mass is finite if $-1 < n \leq 5$.

2.6.9 Uniform Contraction or Expansion of N-dimensional Polytropes

A contraction or expansion of a material system is said to be uniform if the distance from the origin to any point is altered in the same way as the distance from the origin to the boundary of the system. Let R_1 be the finite boundary radius of a radially symmetric configuration, and R_2 its radius after contraction or expansion. Let also r_1 be the radial distance of an arbitrary point inside the configuration, and r_2 the distance of the same point after contraction or expansion. Then, according to the definition of uniform contraction or expansion, we have

$$R_2/R_1 = r_2/r_1 = y = \mathrm{const.} \tag{2.6.196}$$

The hydrostatic equation (2.1.35) becomes with Eq. (2.6.12):

$$dP/dr = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M(r)\ \varrho r^{1-N}. \tag{2.6.197}$$

In the cases of practical interest the hydrostatic equation (2.6.197) writes [cf. Eq. (2.6.30)]:

$$N = 1: \quad dP/dr = -2\pi GM(r)\ \varrho, \tag{2.6.198}$$

$$N = 2: \quad dP/dr = -2GM(r)\ \varrho/r, \tag{2.6.199}$$

$$N = 3: \quad dP/dr = -GM(r)\ \varrho/r^2. \tag{2.6.200}$$

Conservation of mass inside the radial distance $r_i,\ (i = 1, 2)$ yields

$$M(r_1) = M(r_2) \quad \text{and} \quad dM(r_1) = dM(r_2). \tag{2.6.201}$$

We insert

$$dM(r) = 2\{[\Gamma(1/2)]^N/\Gamma(N/2)\}\varrho r^{N-1}\ dr, \tag{2.6.202}$$

into Eq. (2.6.201), to find

$$\varrho_2/\varrho_1 = r_1^{N-1}\ dr_1/r_2^{N-1}\ dr_2 = y^{-N} = (R_1/R_2)^N, \qquad [dr_2 = y\ dr_1;\ \varrho_i = \varrho(r_i)]. \tag{2.6.203}$$

The hydrostatic equation (2.6.197) becomes at the point $r = r_2$

$$dP_2 = -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}M(r_2)\ \varrho_2 r_2^{1-N}\ dr_2$$
$$= -2\pi G\{\Gamma(N/2)/[\Gamma(1/2)]^N\}y^{2-2N}M(r_1)\ \varrho_1 r_1^{1-N}\ dr_1 = y^{2-2N}\ dP_1, \qquad [P_i = P(r_i)], \tag{2.6.204}$$

and by integration

$$P_2/P_1 = y^{2-2N} = (R_1/R_2)^{2N-2}. \tag{2.6.205}$$

If the pressure obeys the perfect gas law (1.2.5), we have additionally

$$P_2 = \mathcal{R}\varrho_2 T_2/\mu = \mathcal{R}y^{-N}\varrho_1 T_2/\mu = y^{2-2N}P_1 = y^{2-2N}\mathcal{R}\varrho_1 T_1/\mu, \qquad [T_i = T(r_i)], \tag{2.6.206}$$

and

$$T_2/T_1 = y^{2-N} = (R_1/R_2)^{N-2}. \tag{2.6.207}$$

Thus, we have proved the following

Theorem. Pressure and density in the same arbitrary point of a uniformly contracting or expanding configuration change as the $(2-2N)$-th and as the $(-N)$-th power of the ratio between the final and the initial radius of the configuration.

Eliminating y between Eqs. (2.6.203) and (2.6.205), we get

$$P_2/P_1 = (\varrho_2/\varrho_1)^{(2N-2)/N} \quad \text{or} \quad P = \text{const}\ \varrho^{(2N-2)/N}. \tag{2.6.208}$$

If the configuration is a polytrope, we can equate the polytropic pressure $P = \text{const}\ \varrho^{1+1/n}$ to the pressure from Eq. (2.6.208), and the equality of the exponents of ϱ yields the values of the polytropic index of a uniformly contracting or expanding polytrope:

$$n = N/(N-2) = \text{const}. \tag{2.6.209}$$

In the cases of practical interest $N = 1, 2, 3$ we obtain from Eq. (2.6.209) $n = -1, \pm\infty$, and 3, respectively. Thus, no uniform contraction or expansion of a polytropic slab $(N = 1)$ is possible, since the polytropic index $n = -1$ leads to unphysical solutions (Sec. 2.1). Uniformly contracting or expanding polytropic cylinders are of index $n = \pm\infty$ (isothermality for a perfect gas), while uniformly contracting or expanding polytropic spheres are of polytropic index $n = 3$.

From the constancy of the polytropic index in Eq. (2.6.209) we can derive another important property: A uniformly contracting or expanding N-dimensional polytrope has the polytropic index $n = N/(N-2)$, $(N \geq 2)$, and the configuration resulting after uniform contraction or expansion is another polytrope having the same polytropic index.

When the uniformly contracting or expanding polytrope is composed of perfect gas, we have from Eq. (1.2.31)

$$\gamma' = 1 + 1/n = 1 + (N-2)/N, \tag{2.6.210}$$

where $\gamma' = (c_P - c)/(c_V - c)$ is the polytropic exponent that is generally different from the adiabatic exponent $\gamma = c_P/c_V$.

The change of entropy connected with the uniform contraction or expansion of a thermodynamic system is by virtue of Eqs. (1.2.40) and (2.6.210) equal to

$$dS = c_V(\gamma' - \gamma)\ d\varrho/\varrho = c_V(2N - 2 - N\gamma)\ d\varrho/N\varrho = -c_V(2N - 2 - N\gamma)\ dr/r, \qquad (N \geq 2). \tag{2.6.211}$$

The results of this subsection have been worked out for the spherical case ($N = 3$) mostly by Lane, Ritter, and Rudzki (cf. Chandrasekhar 1939).

Using a so-called pseudopolytropic density distribution law

$$\varrho = \varrho_0(1 - r/r_1)^c, \qquad (c = \mathrm{const}), \tag{2.6.212}$$

Guseinov and Kasumov (1972) obtain explicit analytic expressions for the mean density, mass, moment of inertia, and gravitational energy of undistorted spheres, although the authors claim that their results are appropriate to distorted rotating stars. If $c = 1.4$ and 4.9 for instance, the density distribution (2.6.212) is fairly close to a polytrope of index $n = 1.5$ and 3, respectively.

2.7 Topology of the Lane-Emden Equation

The second order Lane-Emden equations (2.1.14) and (2.1.21) seem to be not suitable for a direct topological study. Therefore, the topological behaviour of the Lane-Emden equations is generally discussed with the aid of the transformed first order differential equations (2.2.15), (2.2.21) or (2.2.29), (2.2.34) in terms of Milne's homology invariant variables u, v, or in terms of the Emden variables y, z.

In terms of the Emden variables y and z the topology of the Lane-Emden equation has been presented by Chandrasekhar (1939), with main contributions due to Emden (1907), Hopf (1931), Fowler (1930, 1931). Chandrasekhar's (1939) study was limited to the spherical case $N = 3$ and to polytropic indices $1 < n < \infty$ and $n = \pm\infty$; the treatment for other polytropic indices has not yet been undertaken, because of their reduced practical importance and due to mathematical difficulties occurring with Emden's variables.

A topological study of the asymptotic behaviour of second order equations, similar to the transformed Lane-Emden equations (2.2.43) and (2.2.45), has been presented by Bellman (1953, Chap. 7, $n > 1$), based mainly on Fowler's (1930, 1931) work.

The discussion of the topology in terms of Milne's homology variables seems to be more straightforward. Without using Kimura's (1981a) supplementary transformation of ξ and θ from Eqs. (2.1.27), (2.1.28), we present below the topology of the Lane-Emden equations (2.1.14), (2.1.21) in terms of Milne's homology variables u, v (Horedt 1987b).

Depending on the values of the polytropic index we distinguish two different forms of the homology invariant variables and of the related first order differential equations [cf. Eqs. (2.2.6), (2.2.7), (2.2.13)-(2.2.15) and (2.2.19)-(2.2.21)]:

(i) $n \neq -1, \pm\infty$:

$$u = \mp\xi\theta^n/\theta'; \quad v = \mp\xi\theta'/\theta, \tag{2.7.1}$$

$$du/d\ln\xi = u(-u \mp nv + N); \quad dv/d\ln\xi = v(u \pm v - N + 2), \tag{2.7.2}$$

$$dv/du = v(u \pm v - N + 2)/u(-u \mp nv + N). \tag{2.7.3}$$

(ii) $n = \pm\infty$:

$$u = \xi\exp(-\theta)/\theta'; \quad v = \xi\theta', \tag{2.7.4}$$

$$du/d\ln\xi = u(-u - v + N); \quad dv/d\ln\xi = v(u - N + 2), \tag{2.7.5}$$

$$dv/du = v(u - N + 2)/u(-u - v + N). \tag{2.7.6}$$

It is easily seen that in terms of the physical variables P, M, and r, the homology invariant variables can be written as

$$u = d\ln M/d\ln r = \mp\xi\theta^n/\theta'; \quad v = [\mp 1/(n+1)]\ d\ln P/d\ln r = \mp\xi\theta'/\theta, \quad (n \neq -1, \pm\infty), \tag{2.7.7}$$

and

$$u = d\ln M/d\ln r = \xi\exp(-\theta)/\theta'; \quad v = -d\ln P/d\ln r = \xi\theta', \qquad (n = \pm\infty), \tag{2.7.8}$$

where we have inserted Eqs. (2.6.1)-(2.6.4), (2.6.12), (2.6.13).

As shown in Sec. 2.6.8, the Milne variables u and v both are nonnegative for physically meaningful solutions, and the discussion of the topology of the Lane-Emden equation can be reduced to the first positive (u, v)-quadrant. As follows from Cauchy's theorem concerning the existence and uniqueness of solutions (e.g. Smirnow 1967, Bronstein and Semendjajew 1985), there passes one and only one integral curve (solution curve) through a finite and nonsingular point in the (u, v)-plane. Thus, the integral curves

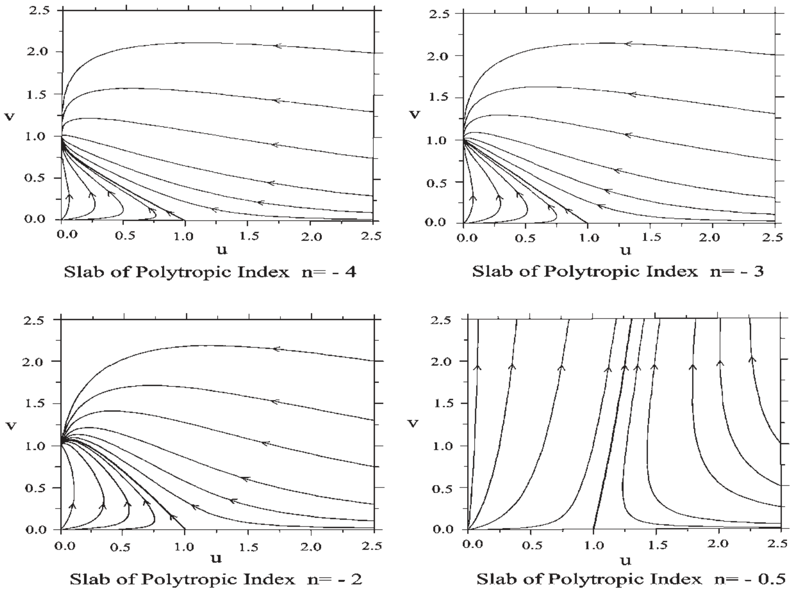

Fig. 2.7.1 Integral curves in the (u, v)-plane for polytropic slabs with index $n = -4, -3, -2, -0.5$. Arrows mark the sense of increasing ξ. E-solutions are designated by thick curves (Horedt 1987b).

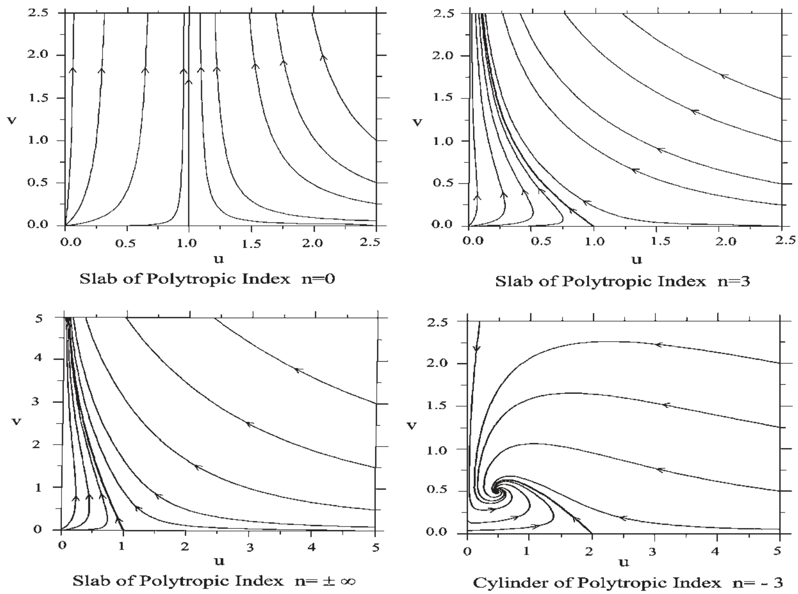

Fig. 2.7.2 Same as Fig. 2.7.1 for polytropic slabs of index $n = 0, 3, \pm\infty$, and for cylinders of index $n = -3$.

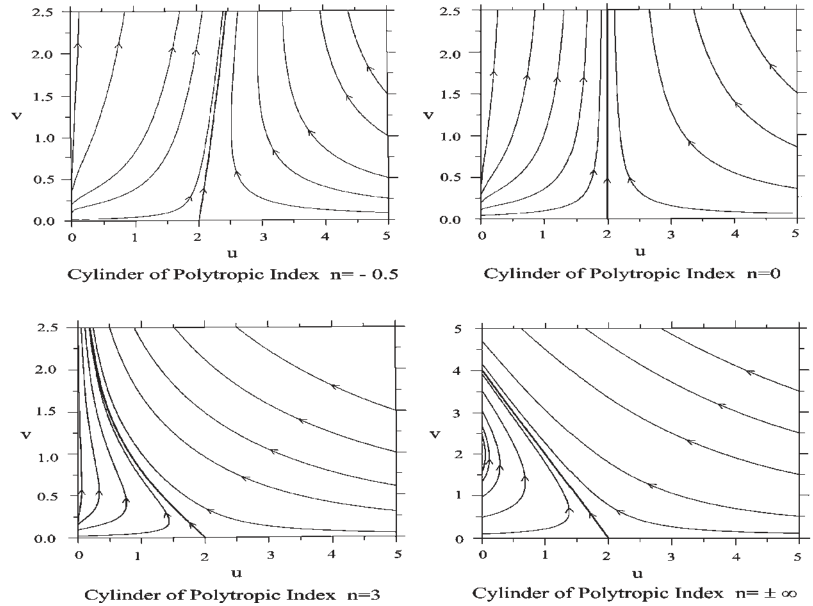

Fig. 2.7.3 Same as Fig. 2.7.1 for polytropic cylinders of index $n = -0.5, 0, 3, \pm\infty$.

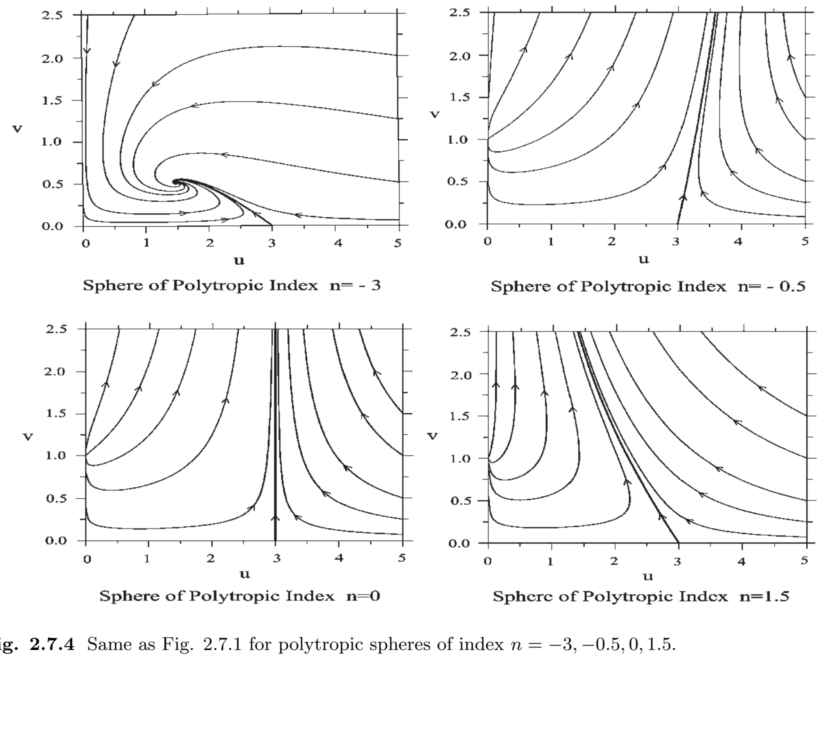

Fig. 2.7.4 Same as Fig. 2.7.1 for polytropic spheres of index $n = -3, -0.5, 0, 1.5$.

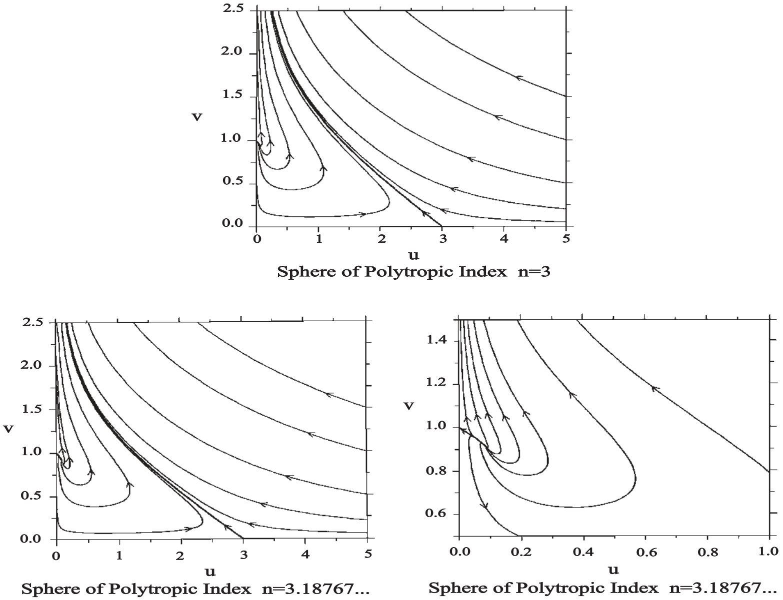

Fig. 2.7.5 Same as Fig. 2.7.1 for polytropic spheres of index $n = 3$ and $n = (11 + 8 \times 2^{1/2})/7 = 3.18767...$

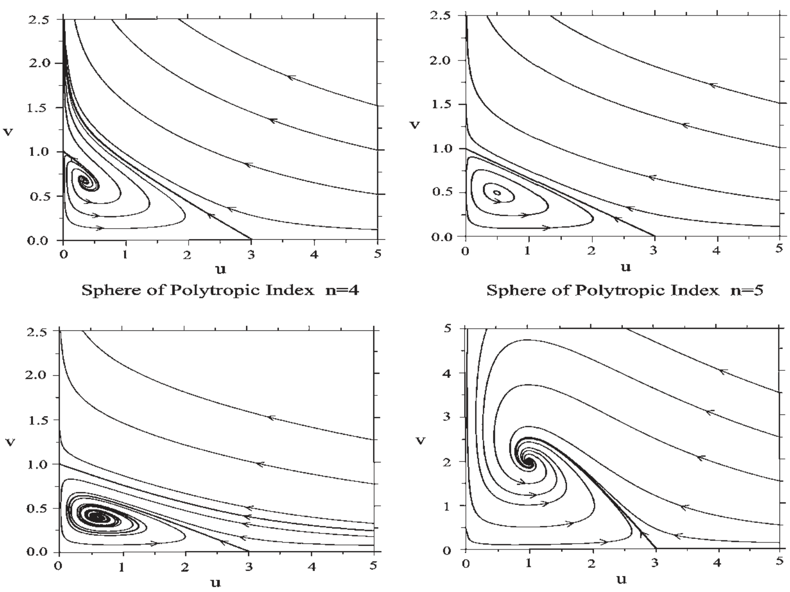

Fig. 2.7.6 Same as Fig. 2.7.1 for polytropic spheres of index $n = 4, 5, 6, \pm\infty$.

of the Lane-Emden equation are not cutting or touching among themselves, excepting possibly at the singular or infinity points.

The integral curves of Eqs. (2.7.3) or (2.7.6) are either bounded by two limiting points, or closed curves, or merely singular points. Since the limiting points of the integral curves can be either finite singular points or points at infinity, we are lead to the investigation of the integral curves in the vicinity of the finite singular points and in the vicinity of the asymptotic points.

We will be principally concerned with the cases of practical interest $N = 1, 2, 3$, although many of the subsequent equations are also valid for general N.

2.7.1 Polytropic Indices $n \neq -1, \pm\infty$

The singular points of the first order differential equation (2.7.3) are obtained if its numerator and denominator vanish simultaneously, and Eq. (2.7.3) takes the undetermined form $dv/du = 0/0$. Thus, the singular points of Eq. (2.7.3) are just the solutions of the system

$$v(u \pm v - N + 2) = 0; \quad u(-u \mp nv + N) = 0. \tag{2.7.9}$$

We denote the four solutions by $O_s(0,0)$, $U_s(N,0)$, $V_s[0, \pm(N-2)]$, $G_s\{[n(N-2)-N]/(n-1), \pm 2/(n-1)\}$. The singular point O_s is located in the origin, U_s is on the u-axis, and V_s on the v-axis. The singular point G_s is just the intersection of the loci where the integral curves of Eq. (2.7.3) have horizontal tangents ($dv/du = 0$ or $u \pm v - N + 2 = 0$) with the loci where the integral curves have vertical tangents ($du/dv = 0$ or $-u \mp nv + N = 0$), (Chandrasekhar 1939).

The infinity points of Eq. (2.7.3) can be of the form $U_\infty(\infty, v_\infty)$, $V_\infty(u_\infty, \infty)$, and $G_\infty(\infty, \infty)$, where $u_\infty, v_\infty = \text{const} \neq \infty$.

We discuss the form of the integral curves near these seven special points separately. Generally, the precise form of the integral curves at intermediate points can be found only numerically (Figs. 2.7.1-2.7.6).

(i) $O_s(0,0)$. Eq. (2.7.2) becomes in the vicinity of the origin

$$\xi\ du/d\xi = Nu; \quad \xi\ dv/d\xi = (-N+2)v, \qquad (N \neq 2;\ u, v \approx 0). \tag{2.7.10}$$

The integrals of Eq. (2.7.10) are

$$u = A\xi^N; \quad v = B\xi^{2-N}, \qquad (N \neq 2;\ \xi \approx 0;\ A, B = \text{const};\ 0 < A, B < \infty), \tag{2.7.11}$$

showing that for polytropic slabs ($N = 1$) the singular point O_s is a node and an initial point ($\xi \to 0$), (Figs. 2.7.1, 2.7.2). If $N \geq 3$, the singular point O_s cannot be approached by the integral curves (cf. Figs. 2.7.4-2.7.6). If $N = 2$, Eq. (2.7.2) turns into

$$u = A\xi^2; \quad \xi\ dv/d\xi = uv \pm v^2 = A\xi^2 v \pm v^2, \qquad (N = 2;\ \xi \approx 0). \tag{2.7.12}$$

The equation for v is a Bernoulli equation that can be integrated by standard methods (e.g. Smirnow 1967, Bronstein and Semendjajew 1985). With the substitution $v = 1/w$, Eq. (2.7.12) reads

$$dw/d\xi = -A\xi w \mp 1/\xi. \tag{2.7.13}$$

We apply the method of variation of constants to the solution

$$w = C\exp(-A\xi^2/2), \qquad (C = \text{const}), \tag{2.7.14}$$

of the homogeneous equation

$$dw/d\xi = -A\xi w, \tag{2.7.15}$$

to obtain

$$C = C(\xi) = \mp \int \exp(A\xi^2/2)\ d\xi/\xi \approx \mp\ln(B\xi), \qquad (\xi \approx 0;\ B = \text{const};\ 0 < B < \infty). \tag{2.7.16}$$

Thus, $1/v = \mp \ln(B\xi)$, $(\xi \approx 0)$, and we observe at once that the condition $v \geq 0$ is satisfied only with the minus sign:

$$v = -1/\ln(B\xi) = 1/\ln(1/B\xi), \qquad (N = 2;\ n > -1;\ \xi \approx 0). \tag{2.7.17}$$

To obtain the behaviour of θ', θ, and $\mp\xi^{N-1}\theta' \propto M$, we write the Lane-Emden equation (2.1.14) under the form (Kimura 1981a)

$$\theta'' = (1-N)\theta'/\xi \mp \theta^n = \theta'(1-N+u)/\xi \approx \theta'(1-N+A\xi^N)/\xi. \tag{2.7.18}$$

Integration yields

$$|\theta'| = C\xi^{1-N}\exp(A\xi^N/N), \qquad (\xi \approx 0;\ 0 < A, C < \infty). \tag{2.7.19}$$

If $N = 1$, we get $\theta' \to \mp C = \text{const}$, and if $N = 2,\ n > -1$, we obtain $\theta' \to -C/\xi \to -\infty$ as $\xi \to 0$. The function θ can be determined from

$$\theta = \mp\xi\theta'/v \approx C\exp(A\xi^N/N)/B \approx C/B = \text{const}, \qquad (N = 1;\ \xi \approx 0;\ 0 < A, B, C < \infty), \tag{2.7.20}$$

$$\theta = -\xi\theta'/v \approx C\exp(A\xi^N/N)\ln(1/B\xi) \to \infty, \qquad (N = 2;\ n > -1;\ \xi \approx 0;\ 0 < A, B, C < \infty). \tag{2.7.21}$$

Using Eq. (2.7.19) we get for the dimensionless mass

$$\mp\xi^{N-1}\theta' \approx C\exp(A\xi^N/N) \to C = \text{const}, \qquad (N = 1;\ n > -1 \text{ if } N = 2;\ \xi \approx 0). \tag{2.7.22}$$

Eliminating ξ in Eq. (2.7.11), we find

$$u/v = A/B = \text{const}, \qquad (N = 1;\ \xi \approx 0). \tag{2.7.23}$$

Elimination of ξ between Eqs. (2.7.12) and (2.7.17) yields

$$u\exp(2/v) = A/B^2 = \text{const}, \qquad (N = 2;\ n > -1;\ \xi \approx 0). \tag{2.7.24}$$

(ii) $\boldsymbol{U_s(N,0)}$. The shape of the integral curves in the vicinity of this singular point can be found along the lines developed by Chandrasekhar (1939) for the case $N = 3$. We find from Eq. (2.7.2)

$$\xi\ dv/d\xi = 2v, \qquad (u \approx N;\ v \approx 0), \tag{2.7.25}$$

and by integration

$$v = B\xi^2, \qquad (\xi \approx 0;\ 0 < B < \infty). \tag{2.7.26}$$

The first equation (2.7.2) becomes

$$\xi\ du/d\xi = -u^2 + u(\mp nB\xi^2 + N), \tag{2.7.27}$$

which can be integrated with the substitution $u = 1/w$ in the same way as Eq. (2.7.12):

$$1/u \approx (1 \pm nB\xi^2/2)\big[1/N \mp nB\xi^2/2(N+2) + A\xi^{-N}\big],$$
$$(\xi \approx 0;\ A, B = \text{const};\ -\infty < A < \infty;\ 0 < B < \infty), \tag{2.7.28}$$

or

$$u \approx N(N+2)/\big[N + 2 \pm nB\xi^2 + AN(N+2)/\xi^N\big]. \tag{2.7.29}$$

And with the help of Eq. (2.7.26):

$$v^{N/2}[(N+2)(u-N) \pm nNv] = -AB^{N/2}N^2(N+2) = \text{const}, \qquad (N = 1, 2, 3, ...;\ u \approx N;\ v \approx 0). \tag{2.7.30}$$

We observe from Eq. (2.7.28) that u approaches the singular point $U_s(N,0)$ only if $A=0$. Only a single integral curve passes through U_s. We obtain its parametric representation by putting $A=0$ in Eq. (2.7.29):

$$u = N \mp nNB\xi^2/(N+2); \quad v = B\xi^2, \qquad (\xi \approx 0;\ A=0;\ 0<B<\infty). \tag{2.7.31}$$

If $A \neq 0$, Eq. (2.7.30) shows that the integral curves in the vicinity of U_s are generalized hyperbolas, tangent to the asymptotes $v=0$ and $(N+2)(u-N)\pm nNv=0$. As seen from Eq. (2.7.30), the generalized hyperbolas are located on the left-hand side ($A>0$), and on the right-hand side ($A<0$) of the single curve passing through U_s if $A=0$. If $A\neq 0$, the integral curves avoid U_s; this singular point is a saddle point (Smirnow 1967). U_s is an initial point ($\xi\to 0$) for the single integral curve passing through this point. If $A=0$, we obtain for the Lane-Emden variable

$$\theta''/\theta' = (1-N+u)/\xi = 1/\xi; \quad \ln|\theta'| = \ln(C\xi); \quad \theta' = \mp C\xi, \qquad (\xi\approx 0;\ u\approx N;\ 0<C<\infty), \tag{2.7.32}$$

$$\theta = \mp\xi\theta'/v \approx C/B = \text{const}, \qquad (\xi\approx 0). \tag{2.7.33}$$

Thus, $\theta=$ const, $\theta'\to 0$, and $\mp\xi^{N-1}\theta' = C\xi^N \to 0$ if $\xi\to 0$ and $A=0$. The fact that $\theta=$ const and $\theta'=0$ at the origin $\xi=0$, reminds us that these initial conditions are just meet by the set $\{\theta_E(\xi)\}$ of E-solutions of which the Lane-Emden function is a particular member: All members of this set are finite at the origin with the derivative equal to zero. Since the set $\{\theta_E(\xi)\}$ forms a homology family, and since u, v are homology invariant variables, all members of this set are represented by the single integral curve – the E-curve – shown by the thick line starting at $U_s(N,0)$ in Figs. 2.7.1-2.7.6 (cf. Sec. 2.2.1). It will be therefore sufficient to consider only the behaviour of the Lane-Emden function obeying the initial conditions (2.1.41). For this particular member of the set $\{\theta_E(\xi)\}$ we have via Eqs. (2.1.41), (2.7.1)

$$\theta = C/B = 1 \quad \text{and} \quad \theta^{n+1} = {\theta'}^2 u/v = NC^2/B = 1, \qquad (\xi\approx 0;\ A=0). \tag{2.7.34}$$

This shows that $B=C=1/N$. Therefore $\theta' = \mp\xi/N,\ \theta'' = \mp 1/N$, and the Lane-Emden function is approximately equal to

$$\theta(\xi) = 1 + \theta'(0)\,\xi + \theta''(0)\,\xi^2/2 = 1 \mp \xi^2/2N, \qquad (\xi\approx 0), \tag{2.7.35}$$

which are just the first two terms of the expansion (2.4.21). The homology invariant Milne variables u, v corresponding to the homology family $\{\theta_E(\xi)\}$ – to the E-solutions – are obtained from Eq. (2.7.31):

$$u = N \mp n\xi^2/(N+2);\ v = \xi^2/N, \qquad (\xi\approx 0;\ N=1,2,3,...). \tag{2.7.36}$$

It follows from Eq. (2.7.30) that if $A=0$, the E-curve near the singular point is represented by the asymptote $(N+2)(u-N)\pm nNv=0$. Differentiating this equation, we get the slope of the E-curve near U_s :

$$dv/du = \mp(N+2)/nN, \qquad (u\approx N;\ v\approx 0). \tag{2.7.37}$$

(iii) $V_s[0,\pm(N-2)]$. This singular point exists in the positive quadrant if $N=1,\ (n<-1),\ N=2$, and $N\geq 3,\ (n>-1)$. The case $N=2$ has already been discussed, because it amounts to $V_s\equiv O_s$. If $N=1,3$, the singular point has the same coordinates $V_s=V_s(0,1)$. Eq. (2.7.2) reads

$$\xi\ du/d\xi = u(-u\mp nv+N) \approx u[-n(N-2)+N], \qquad [n\neq N/(N-2)], \tag{2.7.38}$$

and after integration

$$u \approx A\xi^{n(2-N)+N}, \qquad (A=\text{const};\ 0<A<\infty). \tag{2.7.39}$$

We have excluded the polytropic index $n=N/(N-2)$, because in this case u from Eq. (2.7.38) cannot be neglected with respect to $\mp nv+N$. For polytropic slabs Eq. (2.7.39) turns into

$$u = A\xi^{n+1}, \qquad (N=1;\ n<-1;\ \xi\to\infty), \tag{2.7.40}$$

showing that V_s is a final point ($\xi \to \infty$). The restriction $n \neq N/(N-2)$ is not pertinent for polytropic slabs, since it amounts to $n \neq -1$. For polytropic spheres Eq. (2.7.39) reads

$$u = A\xi^{3-n}, \qquad (N=3;\ \xi \to 0 \text{ if } -1 < n < 3;\ \xi \to \infty \text{ if } n > 3), \tag{2.7.41}$$

showing that V_s is an initial point $\xi \to 0$ if $-1 < n < 3$, and a final point $\xi \to \infty$ if $n > 3$. For polytropic spheres the omitted polytropic index $n = N/(N-2)$ leads to $n = 3$, needing special discussion [Eqs. (2.7.60)-(2.7.68)]. From Eqs. (2.7.2) and (2.7.39) we obtain

$$\xi\ dv/d\xi = v(u \pm v - N + 2) \approx \pm v^2 + v(A\xi^{n(2-N)+N} - N + 2), \tag{2.7.42}$$

where the upper sign holds in the case $N = 3,\ (n > -1)$, and the lower one if $N = 1,\ (n < -1)$. The Bernoulli equation (2.7.42) can be integrated with the substitution $w = 1/v$ in the same way as Eq. (2.7.12):

$$w = 1/v = C(\xi)\xi^{N-2}\exp\big\{-A\xi^{n(2-N)+N}/[n(2-N)+N]\big\}, \tag{2.7.43}$$

$$dC(\xi)/d\xi = \mp\xi^{1-N}\exp\big\{A\xi^{n(2-N)+N}/[n(2-N)+N]\big\}$$
$$\approx \mp\xi^{1-N}\big\{1 + A\xi^{n(2-N)+N}/[n(2-N)+N]\big\}. \tag{2.7.44}$$

The integrals of this equation are

$$C \approx \mp\xi^{2-N}/(2-N) \mp A\xi^{n(2-N)+2}/[n(2-N)+N][n(2-N)+2] + B,$$
$$(n \neq N/(N-2);\ n \neq 2/(N-2);\ N = 1,3;\ -\infty < B < \infty), \tag{2.7.45}$$

$$C \approx \mp\xi^{2-N}/(2-N) \mp A\ln\xi/[n(2-N)+N] + B,$$
$$(n \neq N/(N-2);\ n = 2/(N-2);\ N = 1,3;\ -\infty < B < \infty). \tag{2.7.46}$$

Inserting into Eq. (2.7.43), and restricting to first order terms, we get

$$v = \pm(N-2)/\big\{1 - A\xi^{n(2-N)+N}/[n(2-N)+2] + B\xi^{N-2}\big\},$$
$$(\xi \to \infty \text{ if } N = 1,\ n < -1 \text{ and } N = 3,\ n > 3;\ \xi \to 0 \text{ if } N = 3,\ -1 < n < 3;$$
$$n \neq N/(N-2);\ n \neq 2/(N-2);\ 0 < A < \infty;\ -\infty < B < \infty), \tag{2.7.47}$$

$$v = \pm(N-2)/\big\{1 + A(2-N)\xi^{N-2}\ln\xi/[n(2-N)+N] + B\xi^{N-2}\big\},$$
$$(\xi \to \infty \text{ if } N = 1,\ n = -2;\ \xi \to 0 \text{ if } N = 3,\ n = 2). \tag{2.7.48}$$

Inserting for ξ from Eq. (2.7.39), we find

$$u^{(2-N)/[n(2-N)+N]}\{u/[n(2-N)+2] - v \pm (N-2)\} \approx A^{(2-N)/[n(2-N)+N]}B = \text{const},$$
$$[N = 1,3;\ n \neq N/(N-2);\ n \neq 2/(N-2)], \tag{2.7.49}$$

$$u^{-1}(\pm u\ln u - v + 1) \approx B/A = \text{const}, \qquad (n = -2 \text{ if } N = 1;\ n = 2 \text{ if } N = 3). \tag{2.7.50}$$

For polytropic slabs Eqs. (2.7.47)-(2.7.49) become

$$v = 1/\big[1 - A\xi^{n+1}/(n+2) + B/\xi\big], \qquad (N = 1;\ n < -1;\ n \neq -2;\ \xi \to \infty), \tag{2.7.51}$$

$$v = 1/\big[1 - (A/\xi)\ln\xi + B/\xi\big], \qquad (N = 1;\ n = -2;\ \xi \to \infty), \tag{2.7.52}$$

$$u^{1/(n+1)}[u/(n+2) - v + 1] \approx A^{1/(n+1)}B, \qquad (N = 1;\ n < -1;\ n \neq -2;\ \xi \to \infty). \tag{2.7.53}$$

For polytropic spheres Eqs. (2.7.47)-(2.7.49) take the form

$$v = 1/\big[1 - A\xi^{3-n}/(2-n) + B\xi\big], \qquad (N=3;\ n \neq 2;\ \xi \to 0 \text{ if } -1 < n < 3;\ \xi \to \infty \text{ if } n > 3), \tag{2.7.54}$$

$$v = 1/\big(1 - A\xi \ln\xi + B\xi\big), \qquad (N=3;\ n=2;\ \xi \to 0), \tag{2.7.55}$$

$$u^{1/(n-3)}[u/(2-n) - v + 1] \approx A^{1/(n-3)}B, \quad (N=3;\ n \neq 2;\ \xi \to 0 \text{ if } -1 < n < 3;\ \xi \to \infty \text{ if } n > 3). \tag{2.7.56}$$

From Eqs. (2.7.50)-(2.7.56) it is seen that the singular point V_s is a node if $N=1,\ n<-1,\ (\xi\to\infty)$, and if $N=3,\ -1<n<3,\ (\xi\to 0)$. The point V_s is a saddle point, similar to U_s, if $N=3,\ n>3$, as it is obvious from Eq. (2.7.56). In this case the integral curves in the vicinity of V_s are generalized hyperbolas, tangent to the asymptotes $u=0$ and $u/(2-n)-v+1=0$. If $B=0$ and $N=3,\ n>3$, a single integral curve passes through V_s, having the initial slope $dv/du = 1/(2-n)$, called D-curve (Chandrasekhar 1939). Eq. (2.7.56) shows that if $B<0$ and $B>0$, the generalized hyperbolas are located above and below the D-curve, respectively. As seen from Figs. 2.7.5, 2.7.6, the D-curve connects V_s with the singular point G_s if $3<n<5,\ N=3$, and tends to U_∞ if $n>5,\ N=3$. If $n=5,\ N=3$, the D-curve coincides with the E-curve, connecting V_s with U_s [see Eq. (2.7.74)].

The behaviour of $\theta', \theta, \mp\xi^{N-1}\theta'$ is found in a similar manner as for U_s. We have

$$\theta''/\theta' = (1-N+u)/\xi \approx [1-N+A\xi^{n(2-N)+N}]/\xi, \tag{2.7.57}$$

$$\theta' \approx \mp C\xi^{1-N}\exp\big\{A\xi^{n(2-N)+N}/[n(2-N)+N]\big\} \approx \mp C\xi^{1-N}, \quad (n \neq N/(N-2);\ C = \text{const};\ 0<C<\infty). \tag{2.7.58}$$

If $N=1,\ (n<-1)$, we have $\theta' \approx C = \text{const},\ (\xi\to\infty)$. For polytropic spheres $(N=3)$ we obtain from Eq. (2.7.58) $\theta' \approx -C/\xi^2$, observing that $\theta' \to -\infty$ if $-1<n<3,\ (\xi\to 0)$, and $\theta' \to 0$ if $n>3,\ (\xi\to\infty)$. The Lane-Emden variable θ becomes in virtue of Eq. (2.7.58)

$$\theta = \mp\xi\theta'/v \approx \pm C\xi^{2-N}/(N-2) = C\xi^{2-N}, \qquad (v\approx 1;\ \pm(N-2)=1). \tag{2.7.59}$$

If $N=1,\ (n<-1)$, we get $\theta = C\xi \to \infty$ as $\xi\to\infty$. If $N=3$, there is $\theta \approx C/\xi$. We have $\theta\to\infty$ if $-1<n<3,\ (\xi\to 0)$, and $\theta \to 0$ if $n>3,\ (\xi\to\infty)$. The dimensionless mass is $\mp\xi^{N-1}\theta' \approx C = \text{const}$ in the considered cases.

We now turn to the particular case $N=3,\ n=3$, left over from the preceding discussion. In this case the singular points V_s and G_s coincide. Eq. (2.7.2) reads

$$\xi\ du/d\xi = u(-u-3v+3); \quad \xi\ dv/d\xi = v(u+v-1). \tag{2.7.60}$$

Inserting $v = 1+v_1,\ (v_1\approx 0)$, we obtain

$$\xi\ du/d\xi = u(-u-3v_1); \quad \xi\ dv_1/d\xi = (1+v_1)(u+v_1) \approx u+v_1, \qquad (N=3;\ n=3;\ u,v_1\approx 0). \tag{2.7.61}$$

As the singular point V_s or G_s is approached, three possibilities exist concerning the behaviour of u/v_1 :

$$\lim_{u,v_1\to 0}(u/v_1) = 0, \infty, \text{ or } c, \qquad (c = \text{const};\ c \neq 0, \infty). \tag{2.7.62}$$

It is seen at once, by inserting $u \ll v_1$ and $u \gg v_1$ into Eq. (2.7.61), that these assumptions lead to solutions contradicting the requirements $u, v_1 \approx 0$. Thus $\lim_{u,v_1\to 0}(u/v_1) = \text{const}$, and $u \approx cv_1$ if $u, v_1 \approx 0$. Eq. (2.7.61) becomes

$$\xi\ dv_1/d\xi \approx -(3+c)v_1^2; \quad \xi\ dv_1/d\xi \approx (1+c)v_1 + O(v_1^2), \qquad (u\approx cv_1 \text{ if } u,v_1\approx 0). \tag{2.7.63}$$

Since $\xi\ dv_1/d\xi$ cannot be at the same time of first and second order, the first order term $(1+c)v_1$ must vanish identically, leading to $c = -1$ and $\xi\ dv_1/d\xi \approx -2v_1^2$. Integration yields

$$v_1 = v - 1 = -u = 1/\ln(1/A\xi^2) = -1/\ln(A\xi^2), \qquad (\xi \approx 0;\ 0 < A < \infty). \tag{2.7.64}$$

Thus, if $N = 3$ and $n = 3$, the singular point V_s or G_s is an initial point ($\xi \to 0$) and a node. The functions θ', θ, and $-\xi^2\theta'$ are obtained from

$$\theta''/\theta' = (1 - N + u)/\xi = -2/\xi - 1/[\xi \ln(1/A\xi^2)], \qquad (\xi \approx 0). \tag{2.7.65}$$

Taking into account that θ' is always negative, we find by integration

$$\theta' \approx -1/\{\xi^2[\ln(1/A\xi^2)]^{1/2}\}, \qquad (\xi \approx 0;\ N = 3;\ n = 3). \tag{2.7.66}$$

Combining Eqs. (2.7.64) and (2.7.66), we get (Chandrasekhar 1939)

$$\theta = -\xi\theta'/v = 1/\{\xi\ [\ln(1/A\xi^2)]^{1/2}\}, \qquad (\xi \approx 0;\ N = 3;\ n = 3), \tag{2.7.67}$$

and

$$-\xi^2\theta' = [\ln(1/A\xi^2)]^{-1/2}. \tag{2.7.68}$$

Thus, $\theta' \to -\infty,\ \theta \to \infty,\ -\xi^2\theta' \to 0$ as the initial point $\xi = 0$ is approached.

We now turn to the connection between the singular points $U_s(N, 0)$ and $V_s(0, 1)$. For polytropic slabs the E-curve near the final point $\xi \to \infty$ is given by the Lane-Emden functions (2.4.67), (2.4.68), and we have

$$\theta' = 2/[-(n+1)]^{1/2};\quad \theta = \{2/[-(n+1)]^{1/2}\}\xi;\quad u = \{2/[-(n+1)]^{1/2}\}^{n-1}\xi^{n+1};\quad v = 1,$$
$$(\xi \to \infty;\ N = 1;\ n < -1). \tag{2.7.69}$$

Eq. (2.7.69) meets just the conditions near the singular point $V_s(0, 1)$. So, we conclude that the E-curves of polytropic slabs start at the singular point U_s, ($\xi \to 0$) and end at V_s, ($\xi \to \infty$), provided that $n < -1$ (see Fig. 2.7.1).

The tangent to the E-curve $(N + 2)(u - N) \pm nNv = 0$ from Eq. (2.7.30) intersects the v-axis at $v = \pm(N + 2)/n$. This intersection point coincides with the singular point V_s if $\pm(N + 2)/n = \pm(N - 2)$ or $n = (N + 2)/(N - 2)$, and in this case the E-curve is a straight line connecting the two singular points U_s and V_s. If $N = 1$, the straight-line solution is obtained when $n = (N + 2)/(N - 2) = -3$. In this case Eq. (2.3.53) can be integrated to give the Lane-Emden function

$$\theta = (1 + \xi^2)^{1/2}, \qquad (N = 1;\ n = -3). \tag{2.7.70}$$

Insertion of Eq. (2.7.70) into Eq. (2.7.1) yields

$$u = 1/(1 + \xi^2);\quad v = \xi^2/(1 + \xi^2), \tag{2.7.71}$$

with the straight-line solution

$$u + v = 1, \qquad (N = 1;\ n = -3). \tag{2.7.72}$$

If $N = 2$, the straight-line solution is obtained for $n = \pm\infty$, and will be discussed in Sec. 2.7.2. If $N \geq 3$, the polytropic index corresponding to the generalized Schuster-Emden integral (2.3.36) is just equal to $n = (N + 2)/(N - 2)$. Indeed, inserting Eq. (2.3.36) and its derivative into Eq. (2.7.1), we find

$$u = N/[1 + \xi^2/N(N - 2)];\quad v = \xi^2/N[1 + \xi^2/N(N - 2)], \tag{2.7.73}$$

which can be combined into the straight-line solution

$$u + Nv/(N - 2) = N, \qquad (n = (N + 2)/(N - 2);\ N \geq 3). \tag{2.7.74}$$

We conclude that the generalized Schuster-Emden integral (2.3.36) is represented in the (u, v)-plane by a straight line connecting the singular points $U_s(N, 0)$ and $V_s(0, N - 2)$. In the spherical case $N = 3$ the above results are obtained if $n = 5$.

Moreover, if $n=(N+2)/(N-2),\ (N=1,2,3,...)$, all integral curves in the (u,v)-plane can be found analytically. Substitution of Eq. (2.3.31) into Eq. (2.3.30) yields

$$4\xi^{(n+3)/(n-1)}\theta\theta'/(n-1)+\xi^{2(n+1)/(n-1)}\theta'^2\pm 2\xi^{2(n+1)/(n-1)}\theta^{n+1}/(n+1)=C,$$
$$(n=(N+2)/(N-2);\ N=1,3,4,5,...). \tag{2.7.75}$$

Inserting further Eq. (2.7.1) into Eq. (2.7.75), and combining the terms, we eventually obtain the analytical representation of the integral curves (cf. Chandrasekhar 1939 if $N=3,\ n=5$) :

$$\pm u+(n+1)v/2=\pm 2(n+1)/(n-1)+C\big(uv^{(n+1)/2}\big)^{2/(1-n)},$$
$$(C=\mathrm{const};\ n=(N+2)/(N-2);\ N=1,3,4,5,...), \tag{2.7.76}$$

or

$$\pm u+Nv/(N-2)=\pm N+C\big(uv^{N/(N-2)}\big)^{(2-N)/2},\quad (N=2(n+1)/(n-1);\ N=1,3,4,5,...). \tag{2.7.77}$$

The upper sign holds if $N\geq 3,\ (n>1)$, and the lower sign if $N=1,\ (n=-3)$. If $C=0$, we recover the straight-line E-solutions from Eqs. (2.7.72) and (2.7.74), respectively.

(iv) $G_s\{[n(N-2)-N]/(n-1),\pm 2/(n-1)\}$. This singular point exists in the positive quadrant if $n<-1,\ (N=2)$, and $-1>n\geq 3,\ (N=3)$. The singular point G_s is closely connected to the existence of the singular solution (2.3.70), (Chandrasekhar 1939 if $N=3$). From Eq. (2.7.1) we get

$$uv=\xi^2\theta^{n-1}, \tag{2.7.78}$$

and inserting for u and v the coordinates u_G and v_G of G_s, we find

$$\mp[2/(n-1)^2][N+n(2-N)]=\xi^2\theta^{n-1},\qquad (\mp[N+n(2-N)]>0), \tag{2.7.79}$$

identical to the singular solution (2.3.70). The coordinates of G_s can be written under the form

$$u_G=N-2-2/(n-1);\quad v_G=\pm 2/(n-1). \tag{2.7.80}$$

Eliminating n between the coordinates we obtain the geometric locus of G_s :

$$u_G\pm v_G=N-2,\qquad (n(N-2)\gtrless N). \tag{2.7.81}$$

Because the singular point G_s corresponds to the singular solution (2.3.70), and since we have already found in Sec. 2.4.3 the expansion of θ near the singular solution, it is most straightforward to use these expansions and calculate directly the homology invariant variables through Eq. (2.7.1).

From Eqs. (2.4.83), (2.4.88), (2.4.89), (2.4.91), (2.4.92) and from their derivatives the following behaviour results as the singular point G_s is approached: $\theta\propto\xi^{2/(1-n)}$, $\theta'\propto[2/(1-n)]\xi^{(1+n)/(1-n)}$, $\xi\theta'\propto[2/(1-n)]\xi^{2/(1-n)}$, $\mp\xi^2\theta'\propto\mp[2/(1-n)]\xi^{(3-n)/(1-n)}$. There is $\theta\to\infty$ if $\xi\to\infty,\ (n<-1,\ N=2,3)$ and $\xi\to 0,\ (3<n\leq 5,\ N=3)$. We have $\theta\to 0$ if $\xi\to\infty,\ (n\geq 5,\ N=3)$. For the derivative we get $\theta'\to 0$ if $\xi\to\infty,\ (n<-1,\ N=2,3$ and $n\geq 5,\ N=3)$. And $\theta'\to-\infty$ if $\xi\to 0,\ (3<n\leq 5,\ N=3)$. The behaviour of the dimensionless mass is given by $\mp\xi^{N-1}\theta'\to\infty$ if $\xi\to\infty,\ (n<-1,\ N=2,3$ and $n\geq 5,\ N=3)$. And $-\xi^2\theta'\to 0$ if $\xi\to 0,\ (3<n\leq 5,\ N=3)$. The singular point G_s is an initial point $\xi\to 0$ if $3<n\leq 5,\ N=3$, and a final point $\xi\to\infty$ if $n<-1,\ N=2,3$ and $n\geq 5,\ N=3$. If $n=5,\ N=3$, the singular point G_s is at the same time an initial and a final point, as will be discussed subsequently to Eq. (2.7.84).

Inserting Eq. (2.4.83) and its derivative into Eq. (2.7.1), we get for cylindrical polytropes

$$u=[2/(1-n)]\Big[1+c_1\xi^{-2/(1-n)}\big\{n\cos\big[2(-n)^{1/2}\ln\xi/(1-n)+c_2\big]$$
$$+(-n)^{1/2}\sin\big[2(-n)^{1/2}\ln\xi/(n-1)+c_2\big]\big\}\Big];$$
$$v=[2/(1-n)]\Big[1-c_1\xi^{-2/(1-n)}\big\{\cos\big[2(-n)^{1/2}\ln\xi/(n-1)+c_2\big]$$
$$+(-n)^{1/2}\sin\big[2(-n)^{1/2}\ln\xi/(n-1)+c_2\big]\big\}\Big],\qquad (N=2;\ n<-1;\ \xi\to\infty;\ c_1,c_2=\mathrm{const}). \tag{2.7.82}$$

For cylindrical polytropes with indices $n < -1$, G_s is a focus or spiral point (Smirnow 1967). u and v oscillate with decreasing amplitude round G_s if $\xi \to \infty$.
For spherical polytropes we find with Eq. (2.4.88)

$$u = [(n-3)/(n-1)]\Big[1 + c_1\xi^{(n-5)/2(1-n)}\big\{[(1+3n)/4]\cos\big[(7n^2-22n-1)^{1/2}\ln\xi/2(1-n)+c_2\big]$$
$$+[(7n^2-22n-1)^{1/2}/4]\sin\big[(7n^2-22n-1)^{1/2}\ln\xi/2(1-n)+c_2\big]\big\}\Big];$$
$$v = \pm[2/(n-1)]\Big[1 + c_1\xi^{(n-5)/2(1-n)}\big\{[(n-5)/4]\cos\big[(7n^2-22n-1)^{1/2}\ln\xi/2(1-n)+c_2\big]$$
$$-[(7n^2-22n-1)^{1/2}/4]\sin\big[(7n^2-22n-1)^{1/2}\ln\xi/2(1-n)+c_2\big]\big\}\Big],$$
$$(N=3;\ \xi\to\infty \text{ if } -1>n\geq 5;\ \xi\to 0 \text{ if } 3.18767...<n\leq 5). \tag{2.7.83}$$

If $n \neq 5$, the behaviour of u and v is the same as for cylindrical polytropes. The Milne variables oscillate with decreasing amplitude round the singular point: G_s is a focus. G_s is a final point $\xi \to \infty$ if $-1 > n > 5$, and an initial point $\xi \to 0$ if $3.18767... < n < 5$.
Eq. (2.7.83) simplifies in the particular case $n = 5$ to

$$u = (1/2)\big\{1 + 2c_1\big[2\cos(\ln\xi - c_2) - \sin(\ln\xi - c_2)\big]\big\};\quad v = (1/2)[1 + 2c_1\sin(\ln\xi - c_2)],$$
$$(c_1 \approx 0;\ N=3;\ n=5;\ \text{any } \xi). \tag{2.7.84}$$

In this particular case the singular point is surrounded by closed integral curves, degenerating into G_s if $c_1 = 0$ (see Fig. 2.7.6). The singular point is a vortex (a centre), (e.g. Smirnow 1967).
If $3 < n < (11 + 8\times 2^{1/2})/7 = 3.18767...$, Eq. (2.4.91) yields

$$u = [(n-3)/(n-1)]\Big[1 + c_1\big\{\big[3n+1-(-7n^2+22n+1)^{1/2}\big]/4\big\}\xi^{[n-5+(-7n^2+22n+1)^{1/2}]/2(1-n)}$$
$$+c_2\big\{\big[3n+1+(-7n^2+22n+1)^{1/2}\big]/4\big\}\xi^{[n-5-(-7n^2+22n+1)^{1/2}]/2(1-n)}\Big];$$
$$v = [2/(n-1)]\Big[1 + c_1\big\{\big[n-5+(-7n^2+22n+1)^{1/2}\big]/4\big\}\xi^{[n-5+(-7n^2+22n+1)^{1/2}]/2(1-n)}$$
$$+c_2\big\{\big[n-5-(-7n^2+22n+1)^{1/2}\big]/4\big\}\xi^{[n-5-(-7n^2+22n+1)^{1/2}]/2(1-n)}\Big],$$
$$(N=3;\ \xi\to 0 \text{ if } 3<n<3.18767...). \tag{2.7.85}$$

Eq. (2.4.92) gives in the particular case $n = (11 + 8\times 2^{1/2})/7 = 3.18767...$:

$$u = [(n-3)/(n-1)]\Big[1 + \xi^{(n-5)/2(1-n)}\big\{\big[(3n+1)/4\big](c_1\ln\xi + c_2) + c_1(n-1)/2\big\}\Big];$$
$$v = [2/(n-1)]\Big[1 + \xi^{(n-5)/2(1-n)}\big\{[(n-5)/4](c_1\ln\xi + c_2) + c_1(1-n)/2\big\}\Big],$$
$$(N=3;\ \xi\to 0 \text{ if } n=3.18767...). \tag{2.7.86}$$

Thus, if $N = 3,\ 3 < n \leq 3.18767...$, the singular point G_s is a node and an initial point. The same conclusion has already been reached for the particular case $N = 3,\ n = 3$, discussed in connection with the singular point V_s [cf. Eqs. (2.7.60)-(2.7.68)].
(v) $\boldsymbol{U_\infty(\infty, v_\infty),\ v_\infty = \text{const} \neq \infty}$. Eq. (2.7.2) becomes

$$\xi\ du/d\xi = -u^2;\quad \xi\ dv/d\xi = uv,\qquad (u\to\infty). \tag{2.7.87}$$

We integrate the first equation, insert the result into the second one, and integrate again:

$$u = 1/\ln(A\xi);\quad v = B\ln(A\xi),$$
$$(N = 1,2,3,...;\ A, B = \text{const};\ 0 < A, B < \infty;\ \xi\to 1/A;\ \xi\geq 1/A). \tag{2.7.88}$$

It is seen that $v \to 0$ if $\xi \to 1/A$, and since $\xi \geq 1/A$, we infer that the infinity point U_∞ is an initial point located on the u-axis. To obtain θ', θ, and $\mp\xi^{N-1}\theta'$, we employ Eq. (2.7.18) and integrate (Kimura 1981a):

$$\theta'' = \theta'(1-N+u)/\xi \approx \theta' u/\xi = \mp\theta^n;\quad \theta'^2 = \mp 2\theta^{n+1}/(n+1) + \text{const},\qquad (u\to\infty). \tag{2.7.89}$$

We insert $\theta^{n+1} = \theta'^2 u/v$ and Eq. (2.7.88) into Eq. (2.7.89):

$$\theta'^2 = \text{const}/\{1 \pm [2/(n+1)](u/v)\}; \quad \theta' = \mp C[\pm B(n+1)/2]^{1/2} \ln(A\xi),$$
$$(\xi \to 1/A;\ \xi \geq 1/A;\ C = \text{const};\ 0 < A, B, C < \infty). \tag{2.7.90}$$

θ and $\mp\xi^{N-1}\theta'$ are obtained from

$$\theta = \mp\xi\theta'/v = C[\pm(n+1)/2B]^{1/2}\xi, \tag{2.7.91}$$

$$\mp\xi^{N-1}\theta' = C[\pm B(n+1)/2]^{1/2}\xi^{N-1}\ln(A\xi). \tag{2.7.92}$$

Thus, near U_∞ we have $\theta' \to 0,\ \theta \to \text{const},\ \mp\xi^{N-1}\theta' \to 0,\ uv \to B = \text{const}$ if $\xi \to 1/A,\ (\xi \geq 1/A)$.
(vi) $V_\infty(u_\infty, \infty)$; $u_\infty = \text{const} \neq \infty$. Eq. (2.7.2) yields

$$\xi\, dv/d\xi = \pm v^2; \quad v = \mp 1/\ln(B\xi), \qquad (B = \text{const};\ 0 < B < \infty;\ v \to \infty \text{ if } \xi \to 1/B). \tag{2.7.93}$$

If $n \neq 0$, we obtain from Eq. (2.7.2)

$$\xi\, du/d\xi = \mp nuv, \qquad (v \to \infty), \tag{2.7.94}$$

which can be integrated by inserting for v from Eq. (2.7.93):

$$u = A\big[|\ln(B\xi)|\big]^n, \qquad (A = \text{const};\ 0 < A < \infty;\ \xi \to 1/B). \tag{2.7.95}$$

If $n < 0$, we get $u \to \infty$, contradicting the assumption $u \neq \infty$. Thus, we must have $n > 0$. Eqs. (2.7.93) and (2.7.95) write

$$u = A[-\ln(B\xi)]^n = A[\ln(1/B\xi)]^n; \quad v = -1/\ln(B\xi) = 1/\ln(1/B\xi),$$
$$(n > 0;\ N = 1, 2, 3, ...;\ 0 < A, B < \infty;\ \xi \to 1/B;\ \xi \leq 1/B). \tag{2.7.96}$$

If $n > 0$, the infinity point is a final point located on the v-axis: $u_\infty = 0$. Eq. (2.7.18) reads

$$\theta''/\theta' = (1 - N + u)/\xi \approx (1 - N)/\xi + A[\ln(1/B\xi)]^n/\xi, \tag{2.7.97}$$

and

$$\theta' \approx -C\xi^{1-N}, \qquad (\xi \to 1/B;\ \xi \leq 1/B;\ C = \text{const};\ 0 < B, C < \infty). \tag{2.7.98}$$

We also have

$$\theta = -\xi\theta'/v = C\xi^{2-N}\ln(1/B\xi); \quad -\xi^{N-1}\theta' = C, \qquad (n > 0;\ N = 1, 2, 3, ...;\ \xi \to 1/B;\ \xi \leq 1/B). \tag{2.7.99}$$

Thus, in the vicinity of V_∞ there is $\theta' \to -CB^{N-1} = \text{const},\ \theta \to 0,\ -\xi^{N-1}\theta' \to C = \text{const}$, and $uv^n \to A = \text{const}$.

In the particular case $n = 0$ all integral curves can be found analytically from the integrals of the Lane-Emden equation (2.3.3), (2.3.4):

$$u = -\xi/\theta' = 1/(1/N - C/\xi^N);$$
$$N \neq 2: \qquad v = -\xi\theta'/\theta = (1/N - C/\xi^N)/\big[-1/2N + C\xi^{-N}/(2-N) + D/\xi^2\big];$$
$$N = 2: \qquad v = (1/N - C/\xi^N)/\big[-1/2N + (C/\xi^2)\ln\xi + D/\xi^2\big],$$
$$(n = 0;\ C, D = \text{const};\ -\infty < C, D < \infty). \tag{2.7.100}$$

C, D, and ξ have to be chosen in such a way that $u, v \geq 0$. The infinity point V_∞ is approached if ξ tends to a finite positive zero ξ_1 of θ. If $n = 0$, the integral curves have asymptotes parallel to the v-axis, with the u_∞-coordinate equal to $u_\infty = 1/(1/N - C/\xi_1^N) \neq 0$. Since θ is a decreasing function of ξ, the infinity point is a final point. The E-solution is obtained if $C = 0$ [cf. Eq. (2.3.5)], and is equal to the straight line $u = N$ (see Figs. 2.7.2-2.7.4). Eq. (2.7.100) shows that the integral curves are located on the left of the E-curve if $C < 0$, and on its right if $C > 0$. Eqs. (2.3.2)-(2.3.4) give θ', θ, and

$-\xi^{N-1}\theta' = \xi^N/N - C$. If $\xi \to \xi_1$, $(\xi \le \xi_1;\ 0 < \xi_1 < \infty)$, we observe at once that $\theta \to 0$, $\theta' \to$ const, and $-\xi^{N-1}\theta' \to$ const.

(vii) $\boldsymbol{G_\infty(\infty,\infty)}$**.** Instead of studying the behaviour of the Milne variables near $u, v \to \infty$, it is easier to study the behaviour of the integral curves of Eq. (2.7.3) near $x, y \to 0$, by making the substitutions $u = 1/x$ and $v = 1/y$. Eq. (2.7.2) becomes up to the first order

$$\xi\ dx/d\xi = \pm nx/y + 1; \quad \xi\ dy/d\xi = -y/x \mp 1, \qquad (x, y \approx 0). \tag{2.7.101}$$

As G_∞ is approached, we have $du, dv \ge 0$, and $dx = -x^2\ du \le 0,\ dy = -y^2\ dv \le 0,\ (x, y \ge 0)$. Therefore $dy/dx \ge 0$, and Eq. (2.7.3) writes as

$$dy/dx = (y/x)(\pm x + y)/(\mp nx - y) \ge 0. \tag{2.7.102}$$

This condition cannot be fulfilled if $0 \le n < \infty$. We are left with the polytropic indices $-1 < n < 0$ and $-\infty < n < -1$. Inserting Eq. (2.7.2) into $\xi\ d(u/v)/d\xi = (\xi/v)\ du/d\xi - (\xi u/v^2)\ dv/d\xi$, we find the exact equation

$$\xi\ d(u/v)/d\xi = (u/v)[-2u \mp (1+n)v + 2N - 2]. \tag{2.7.103}$$

We show that $u/v \to 0$ if $u, v \to \infty$ and $-1 < n < 0$, by introducing the new variable $z = y/x = u/v$ into Eqs. (2.7.101), (2.7.103). These equations become up to the first order

$$\xi\ dx/d\xi = (\pm n + z)/z; \quad \xi\ dy/d\xi = -z \mp 1; \quad \xi\ dz/d\xi = [-2z - \mp(n+1)]/x,$$
$$(-1 < n < 0 \text{ and } -\infty < n < -1;\ x, y \approx 0), \tag{2.7.104}$$

and by division

$$dz/dx = -z[2z \pm (n+1)]/x(z \pm n). \tag{2.7.105}$$

Integration yields

$$1/x = cz^{n/(n+1)}[2z \pm (n+1)]^{(1-n)/2(n+1)}, \qquad (c = \text{const} > 0). \tag{2.7.106}$$

Turning back to the original (x, y)-variables, we get

$$2(xy)^{(1+n)/(1-n)} \pm (n+1)(xy^n)^{2/(1-n)} = c^{2(n+1)/(n-1)},$$
$$(-1 < n < 0 \text{ and } -\infty < n < -1;\ x, y \approx 0). \tag{2.7.107}$$

If $x, y \to 0$, the first term of the above equation becomes infinite if $-\infty < n < -1$, and cannot be equal to the finite constant $c^{2(n+1)/(n-1)}$; the interval $-\infty < n < -1$ has to be discarded. If $-1 < n < 0$, $(x, y \to 0)$, the first term tends to zero, and Eq. (2.7.107) reduces to $xy^n \to$ const, or $uv^n \to$ const. Thus (Kimura 1981a)

$$u/v = uv^n/v^{n+1} \to 0, \qquad (u, v \to \infty;\ -1 < n < 0). \tag{2.7.108}$$

Eq. (2.7.2) transforms with this important delimitation into

$$\xi\ du/d\xi = uv(-u/v - n + N/v) \approx -nuv; \quad (\xi/v^2)\ dv/d\xi = u/v + 1 - (N-2)/v \approx 1,$$
$$(-1 < n < 0;\ u, v \to \infty). \tag{2.7.109}$$

The integration of Eq. (2.7.109) is effected in the same way as in Eqs. (2.7.93), (2.7.94):

$$u = A[\ln(1/B\xi)]^n; \quad v = 1/\ln(1/B\xi),$$
$$(-1 < n < 0;\ N = 1, 2, 3, ...;\ A, B = \text{const};\ 0 < A, B < \infty;\ \xi \to 1/B;\ \xi \le 1/B). \tag{2.7.110}$$

The infinity point $G_\infty(\infty,\infty)$ exists only if $-1 < n < 0$, and is a final point. Via Eqs. (2.7.90), (2.7.108) we find

$$\theta' = -C/\{1 + [2/(n+1)](u/v)\}^{1/2} \approx -C,$$
$$(N = 1, 2, 3, ...;\ -1 < n < 0;\ C = \text{const};\ 0 < C < \infty;\ u, v \to \infty). \tag{2.7.111}$$

We have also

$$\theta = -\xi\theta'/v = C\xi \ln(1/B\xi), \qquad (\xi \to 1/B), \tag{2.7.112}$$

and

$$-\xi^{N-1}\theta' = C\xi^{N-1}, \qquad (\xi \to 1/B). \tag{2.7.113}$$

Thus, if $-1 < n < 0$ and $\xi \to 1/B$, we observe that $\theta' \to -C,\ \theta \to 0,\ -\xi^{N-1}\theta' \to B^{1-N}C$, and $uv^n = A = \text{const}$.

The three infinity points $U_\infty, V_\infty, G_\infty$ are closely connected to the finite zeros and the finite extremes of the Lane-Emden variable θ. Recall that to a whole homology family $\{\theta(\xi)\}$ only a single integral curve corresponds in the (u, v)-plane (cf. Sec. 2.2). If the Lane-Emden variable has a finite zero ξ_1, we can easily show that along the corresponding integral curve we must have $u \to u_\infty,\ v \to \infty$ if $n \geq 0$, and $u, v \to \infty$ if $-1 < n < 0$, i.e. the infinity points V_∞ and G_∞ are approached if $\theta \to 0$. Zeros of the Lane-Emden variable can only occur if $n > -1$, because when $n < -1$ we have $\theta' > 0$, and θ is an increasing function of ξ; the infinity points V_∞, G_∞ are not approached in this case by the integral curves (see Figs. 2.7.1-2.7.6). We find, by inserting into Eq. (2.7.2) the Taylor expansion of θ near the finite zero ξ_1 :

$$u = -\xi\theta^n/\theta' \approx \xi_1(-\theta_1')^{n-1}(\xi_1 - \xi)^n; \quad v = -\xi\theta'/\theta \approx \xi_1/(\xi_1 - \xi), \qquad (n > -1;\ \xi \approx \xi_1;\ \theta(\xi_1) = 0). \tag{2.7.114}$$

If $\xi \to \xi_1$, we observe that $u \to \infty$ if $-1 < n < 0$, and $u \to u_\infty = -\xi_1/\theta_1' = \text{const}$ if $n = 0$, and $u \to 0$ if $n > 0$. From Eq. (2.7.114) follows that near the finite zero of θ we have (cf. Chandrasekhar 1939 if $N = 3,\ n > 1$)

$$uv^n \approx \xi_1^{n+1}(-\theta_1')^{n-1} = \text{const}, \qquad (\xi \approx \xi_1). \tag{2.7.115}$$

From Figs. 2.7.1-2.7.6 it appears that in the cases of practical interest $N = 1, 2, 3$ all integral curves can be classified into five groups ($D,\ E,\ F,\ M$, and O-curves), according to the scheme outlined by Chandrasekhar (1939). If $N = 1,\ (n \neq -1)$, and $N = 2,\ (n > -1)$, and $N = 3,\ (-1 < n \leq 5)$, all integral curves located on the left of the E-curve are called M-curves [excepting for the D-curve that occurs if $N = 3,\ (3 < n < 5)$]. The curves located on the right of the E-curve are called F-curves. As we have outlined in the discussion of the singular point V_s, a special curve exists – the D-curve – joining the singular point V_s with U_s, G_s, or U_∞. If $N = 1,\ n < -1$ and $N = 3,\ n = 5$, the D-curve joins V_s with U_s, and is identical to the E-curve. If $N = 3,\ 3 < n < 5$, the D-curve joins V_s with G_s, and if $N = 3,\ n > 5$, the D-curve joins V_s with U_∞. If $N = 2, 3,\ (n < -1)$ and $N = 3,\ (n > 5)$, a fifth class of curves exists, called O-curves joining the infinity point U_∞ with G_s. If $N = 3,\ n > 5$, all integral curves below the D-curve are O-curves (excepting for the E-curve), and the integral curves above the D-curve are F-curves. If $N = 2, 3,\ (n < -1)$, there exist only O-curves and the E-curve; no delimitation curve occurs, and the E-curve is the sole curve cutting the u-axis at the finite singular point U_s. The E-curve is the principal delimitation curve between M and F-curves if $N = 1,\ (n \neq -1),\ N = 2,\ (n > -1)$, and $N = 3,\ (-1 < n \leq 5)$. If $N = 3,\ (n > 5)$, the D-curve takes over the role of a principal delimitation curve between O and F-curves. As θ tends to its finite zero, the corresponding $E,\ F$, and M-curves tend to V_∞ or G_∞.

Another classification of the integral curves in the (u, v)-plane can be given according to the various initial and boundary conditions that may occur (Kimura 1981a). The arrows in Figs. 2.7.1-2.7.6 represent the sense of increasing ξ. Depending on the values of $\xi,\ \theta^{n+1}$, and $\mp\xi^{N-1}\theta'$ at the initial point, we distinguish five different classes of integral curves in the (u, v)-plane, where $0 < \text{const} < \infty$ and $N = 1, 2, 3$:

(i) $\xi = 0,\ \theta^{n+1} = \text{const},\ \mp\xi^{N-1}\theta' = 0$. These integral curves are represented by the E-curves starting at U_s, called normal solutions by Kimura (1981a). They occur if $n \neq -1,\ (N = 1, 2, 3, ...)$, and have finite pressure ($P \propto \theta^{n+1}$) and zero mass ($M \propto \mp\xi^{N-1}\theta'$) at the origin $\xi = 0$.

(ii) $\xi = 0,\ \theta^{n+1} = \text{const},\ \mp\xi^{N-1}\theta' = \text{const}$. These integral curves are represented by the M-curves starting at $O_s,\ (N = 1)$, called loaded type solutions, having finite pressure and mass at the origin $\xi = 0$. They occur if $N = 1,\ (n \neq -1)$.

(iii) $\xi = 0,\ \theta^{n+1} = \infty,\ -\xi^{N-1}\theta' = \text{const}$. These integral curves are represented by the M-curves starting at O_s if $N = 2,\ (n > -1)$, and at V_s if $N = 3,\ (-1 < n < 3)$. These polytropes are termed loaded singular type solutions, with infinite pressure and finite mass at the origin $\xi = 0$.

(iv) $\xi = 0,\ \theta^{n+1} = \infty,\ -\xi^2\theta' = 0$. These integral curves are represented by the D-solution and M-solutions starting at G_s if $N = 3,\ (3 \leq n < 5)$, and by the M-solutions surrounding G_s if $N = 3,\ (n = 5)$. These polytropes are called singular type solutions with infinite pressure and zero mass at the origin $\xi = 0$.

(v) $\xi = \mathrm{const},\ \theta^{n+1} = \mathrm{const},\ \mp\xi^{N-1}\theta' = 0$. All these integral curves start at U_∞, and are represented by the F-solutions occurring if $N = 1,\ (n \neq -1),\ N = 2,\ (n > -1)$, and $N = 3,\ (-1 < n \leq 5)$, as well as by the O-solutions occurring if $N = 2, 3,\ (n < -1)$ and $N = 3,\ (n > 5)$. The D-solution occurring if $N = 3,\ (n > 5)$ is also represented by this kind of initial conditions. The pressure is finite and the mass zero at nonzero radial distance; this class of polytropes may be called vacant core type solutions.

The pressure $P \propto \theta^{n+1}$ at the boundary has always to vanish, as results from the form of θ near the final points $G_s, G_\infty, V_s, V_\infty$, as well as from the requirement of hydrostatic equilibrium. Depending on the boundary values of $\xi,\ \theta^n, \mp\xi^{N-1}\theta'$ we may distinguish five classes of final points (Kimura 1981a):

(i) $\xi = \mathrm{const},\ \theta^n = 0,\ -\xi^{N-1}\theta' = \mathrm{const}$. These integral curves approach the infinity point $V_\infty(0, \infty)$, and are represented by the E and M-curves if $N = 1, 2,\ (n > 0)$ and $N = 3,\ (0 < n < 5)$, as well as by the F-curves occurring if $N = 1, 2, 3,\ (n > 0)$. The density ($\varrho \propto \theta^n$) becomes zero as the final finite value of ξ is attained.

(ii) $\xi = \mathrm{const},\ \theta^n = 1,\ -\xi^{N-1}\theta' = \mathrm{const}$. To this class there belongs only the polytrope $n = 0$ with the $E,\ F$, and M-curves ($N = 1, 2, 3, ...$). The density ($\varrho \propto \theta^n = 1$) is constant, and the integral curves approach the infinity point $V_\infty(u_\infty, \infty)$ as the boundary of the polytrope is attained: $\theta = 0$ for finite ξ.

(iii) $\xi = \mathrm{const},\ \theta^n = \infty,\ -\xi^{N-1}\theta' = \mathrm{const}$. These integral curves approach the infinity point G_∞ and are represented by the $E,\ F$, and M-curves if $-1 < n < 0,\ (N = 1, 2, 3, ...)$. These polytropes have a finite zero of θ as the boundary of the polytrope is attained. The density ($\varrho \propto \theta^n$) becomes infinite and the mass ($M \propto -\xi^{N-1}\theta'$) remains finite at the boundary.

(iv) $\xi = \infty,\ \theta^n = 0,\ \mp\xi^{N-1}\theta' = \mathrm{const}$. The integral curves approach V_s as the final infinite value of ξ is attained. The density of these polytropes becomes zero and the mass remains finite as the boundary is approached. They are represented by the $E,\ F$, and M-curves of polytropic slabs $N = 1,\ (n < -1)$, as well as by the D-curve if $N = 3,\ (n > 3)$; the D-curve coincides with the E-curve if $N = 3,\ n = 5$.

(v) $\xi = \infty,\ \theta^n = 0,\ \mp\xi^{N-1}\theta' = \infty$. The integral curves approach G_s as the final infinite value of ξ is attained. The density tends to zero and the mass of these polytropes becomes infinite as the infinite boundary is attained. These boundary conditions are represented by the O-solutions of polytropes having $N = 2, 3,\ (n < -1)$ and $N = 3,\ (5 < n < \infty)$, as well as by the closed M-curves surrounding G_s if $N = 3,\ n = 5$.

This completes our topological study of polytropes having $n \neq -1, \pm\infty$, and we now turn to the special case $n = \pm\infty$, left over from the preceding discussion.

2.7.2 Polytropic Index $n = \pm\infty$

The singular points of Eq. (2.7.6) are given by the solutions of the system

$$v(u - N + 2) = 0; \quad u(-u - v + N) = 0, \tag{2.7.116}$$

yielding the four singular points $O_s(0, 0),\ U_s(N, 0),\ V_s(0, v_s),\ G_s(N - 2, 2)$. The points O_s and U_s are possible for any N, while V_s appears if $N = 2$, and G_s if $N \geq 2$.

The infinity points of Eq. (2.7.6) can be of the form $U_\infty(\infty, v_\infty),\ V_\infty(u_\infty, \infty)$, and $G_\infty(\infty, \infty)$, where $u_\infty, v_\infty = \mathrm{const} \neq \infty$.

(i) $\boldsymbol{O_s(0, 0)}$. Eq. (2.7.5) becomes

$$\xi\ du/d\xi = Nu; \quad \xi\ dv/d\xi = v(2 - N), \qquad (N \neq 2;\ u, v \approx 0), \tag{2.7.117}$$

with the solutions

$$u = A\xi^N; \quad v = B\xi^{2-N}, \qquad (N \neq 2;\ \xi \approx 0;\ A, B = \mathrm{const};\ 0 < A, B < \infty), \tag{2.7.118}$$

showing that this finite singular point exists only if $N = 1$. If $N = 2$, the solution of Eq. (2.7.5) is

$$u = A\xi^2; \quad v = B\exp(A\xi^2/2) \approx B, \qquad (N = 2;\ \xi \approx 0;\ A, B = \mathrm{const};\ 0 < A, B < \infty), \tag{2.7.119}$$

and if $B \approx 0$ there exist integral curves arbitrary close to the vicinity of O_s (cf. Eq. (2.7.136) and Fig. 2.7.3). According to Eq. (2.7.4) we have

$$\theta' = v/\xi \approx B; \quad \theta = \ln(\xi/u\theta') \approx \ln(1/AB); \quad \xi^{N-1}\theta' = \theta' \approx B, \qquad (N = 1;\ \xi \approx 0). \qquad (2.7.120)$$

$\theta', \theta, \xi^{N-1}\theta'$ are nonzero and finite as the singular point O_s is approached. If $N = 1$, the point O_s is a node and an initial point.

(ii) $U_s(N, 0)$. Eq. (2.7.5) becomes

$$\xi\ du/d\xi = u(-u - v + N); \quad \xi\ dv/d\xi = 2v, \qquad (u \approx N;\ v \approx 0). \qquad (2.7.121)$$

Integration of the second equation yields

$$v = B\xi^2, \qquad (\xi \approx 0;\ B = \text{const};\ 0 < B < \infty). \qquad (2.7.122)$$

The first equation (2.7.121) reads

$$\xi\ du/d\xi = -u^2 + u(N - B\xi^2), \qquad (2.7.123)$$

and can be solved in the same manner as Eq. (2.7.27):

$$u \approx N(N+2)/[N + 2 + B\xi^2 + AN(N+2)/\xi^N],$$
$$(\xi \approx 0;\ A, B = \text{const};\ -\infty < A < \infty;\ 0 < B < \infty). \qquad (2.7.124)$$

Inserting Eq. (2.7.122) into Eq. (2.7.124), we obtain [cf. Eq. (2.7.30)]

$$v^{N/2}[(N+2)(u - N) + Nv] = -AB^{N/2}N^2(N+2) = \text{const}, \qquad (u \approx N;\ v \approx 0). \qquad (2.7.125)$$

The E-curve results if $A = 0$. When $A \neq 0$, the integral curves are generalized hyperbolas, tangent to the u-axis and to the straight line $(N+2)(u - N) + Nv = 0$. The singular point U_s is a saddle point. U_s is an initial point for the single integral curve passing through U_s. If $A = 0$, we get for θ', θ, and $\xi^{N-1}\theta'$:

$$\theta' = v/\xi \approx B\xi; \quad \theta = \ln(\xi/u\theta') \approx \ln(1/NB); \quad \xi^{N-1}\theta' \approx B\xi^N, \qquad (A = 0;\ \xi \approx 0). \qquad (2.7.126)$$

Thus, $\theta',\ \xi^{N-1}\theta' \to 0,\ \theta \to$ const if $\xi \to 0$ and $A = 0$. From the initial conditions (2.1.41) we have $\theta(0) = 0$, and therefore $B = 1/N$. If $A = 0$, the Milne variables become by Eqs. (2.7.122) and (2.7.124)

$$u = N - \xi^2/(N+2); \quad v = \xi^2/N, \qquad (\xi \approx 0), \qquad (2.7.127)$$

and [see Eq. (2.4.36)]

$$\theta = \theta'(0)\ \xi + \theta''(0)\ \xi^2/2 = \xi^2/2N, \qquad (\xi \approx 0;\ \theta' = \xi/N). \qquad (2.7.128)$$

(iii) $V_s(0, v_s)$. This singular point exists only if $N = 2$, and amounts to the whole positive v-axis ($v_s =$ const; $0 < v_s < \infty$). It has been pointed out in connection with Eq. (2.7.76) that all integral curves can be represented analytically if $n = (N+2)/(N-2)$, i.e. $n = \pm\infty$ if $N = 2$. We rewrite therefore Eq. (2.3.45)

$$\xi \exp(-\theta)/\theta' + \xi\theta'/2 + 2/\xi\theta' = 2 + C/\xi\theta', \qquad (2.7.129)$$

and insert from Eq. (2.7.4):

$$u + v/2 + 2/v = 2 + C/v \quad \text{or} \quad u + (v-2)^2/2v = C/v, \qquad (N = 2;\ C = \text{const};\ 0 \leq C < \infty). \qquad (2.7.130)$$

The straight-line E-solution $2u + v = 4$ is obtained if $C = 2$, and cuts the v-axis at $v_s = 4$. The integral curves (2.7.130) intersect the v-axis at

$$v = v_s = 2 \pm (2C)^{1/2}, \qquad (N = 2;\ u = 0;\ 0 \leq C < \infty), \qquad (2.7.131)$$

and we observe that the integral curves intersect symmetrically with respect to the singular point $G_s(0, 2)$ as long as $0 \leq C \leq 2$. If $C > 2$, we have only a single intersection with the positive v-axis (Fig. 2.7.3).

The integral curves located on the left of the E-curve have $0 \leq C < 2$, and those located on the right have $C > 2$. Eqs. (2.7.130) and (2.7.131) show that the singular point $G_s(0,2)$ is approached if $C \to 0$. In the vicinity of the singular point $G_s(0,2)$ Eq. (2.7.130) takes the form

$$u = -(v-2)^2/4 + C/2, \qquad (u \approx 0;\ v \approx 2;\ C \approx 0), \tag{2.7.132}$$

and it is seen that the integral curves are parabolas with the symmetry axes on the line $v = 2$, and the vertices located at $u = C/2$, (Fig. 2.7.3).

Exact expressions can be found for the Milne variables in the whole (u,v)-plane by inserting Eq. (2.7.130) into Eq. (2.7.5):

$$dv/d\ln\xi = uv = [2C - (v-2)^2]/2, \qquad (N = 2). \tag{2.7.133}$$

An elementary integration yields

$$v = 2 + (2C)^{1/2}[(B\xi)^{(2C)^{1/2}} - 1]/[(B\xi)^{(2C)^{1/2}} + 1], \quad (B, C = \mathrm{const};\ 0 < B < \infty;\ 0 \leq C < \infty). \tag{2.7.134}$$

u is obtained by inserting into Eq. (2.7.130):

$$u = 4C(B\xi)^{(2C)^{1/2}} \Big/ \Big[[(B\xi)^{(2C)^{1/2}} + 1]\big\{2[(B\xi)^{(2C)^{1/2}} + 1] + (2C)^{1/2}[(B\xi)^{(2C)^{1/2}} - 1]\big\}\Big]. \tag{2.7.135}$$

The v-axis is approached if $\xi \to 0,\ (0 \leq C < 2)$, and if $\xi \to \infty,\ (0 \leq C < \infty)$. The v-axis is cutted at $v_s = 2 - (2C)^{1/2}$ if $\xi \to 0$, and at $v_s = 2 + (2C)^{1/2}$ if $\xi \to \infty$. Eq. (2.7.135) becomes near the v-axis

$$u = 4C\big/\big\{[2 \pm (2C)^{1/2}]\ (B\xi)^{\pm(2C)^{1/2}}\big\}, \qquad (u \approx 0), \tag{2.7.136}$$

where the upper sign holds if $\xi \to \infty,\ (0 \leq C < \infty)$, and the lower one if $\xi \to 0,\ (0 \leq C < 2)$. The behaviour of θ', θ, and $\xi\theta'$ near the v-axis is as follows:

$$\theta' = v/\xi \approx [2 \pm (2C)^{1/2}]/\xi; \quad \theta = \ln(\xi/u\theta') \approx \ln[B^{\pm(2C)^{1/2}} \xi^{2\pm(2C)^{1/2}}/4C];$$
$$\xi\theta' \approx 2 \pm (2C)^{1/2}, \qquad (u \approx 0). \tag{2.7.137}$$

$\theta' \to 0,\ \theta \to \infty$ if $\xi \to \infty,\ (0 \leq C < \infty)$, and $\theta' \to \infty,\ \theta \to -\infty$ if $\xi \to 0,\ (0 \leq C < 2)$. The product $\xi\theta'$ is always constant. V_s is an initial point if $0 < v_s \leq 2$, and a final point if $v_s \geq 2$.

(iv) $\boldsymbol{G_s(N-2, 2)}$. This singular point exists in the positive quadrant if $N \geq 2$. For polytropic cylinders ($N = 2$) the shape of the integral curves in the vicinity of G_s is given by Eqs. (2.7.132), (2.7.134), (2.7.135) if $C \approx 0$, and has already been discussed in connection with the singular point V_s : The integral curves in the vicinity of G_s are parabolas, and G_s appears like a semivortex, the parabolas degenerating into G_s if $C = 0$.

For polytropic spheres ($N = 3$) the Milne variables in the vicinity of G_s can be most easily found with the aid of the asymptotic expansions (2.4.104), (2.4.105):

$$u = 1 + (C_1\xi^{-1/2}/4)\big\{3\cos[(7^{1/2}/2)\ln\xi - C_2] - 7^{1/2}\sin[(7^{1/2}/2)\ln\xi - C_2]\big\};$$
$$v = 2 + (C_1\xi^{-1/2}/2)\big\{\cos[(7^{1/2}/2)\ln\xi - C_2] + 7^{1/2}\sin[(7^{1/2}/2)\ln\xi - C_2]\big\},$$
$$(N = 3;\ \xi \to \infty;\ C_1, C_2 = \mathrm{const}). \tag{2.7.138}$$

u and v oscillate with decreasing amplitude round the singular point if $\xi \to \infty$. The point $G_s(1,2)$ is a focus and a final point ($\xi \to \infty$). From Eq. (2.4.104) follows that $\theta' \approx 2/\xi,\ \theta \approx \ln(\xi^2/2),\ \xi^2\theta' \approx 2\xi$ if $\xi \to \infty$.

(v) $\boldsymbol{U_\infty(\infty, v_\infty),\ v_\infty = \mathrm{const} \neq \infty}$. Eq. (2.7.5) becomes [cf. Eq. (2.7.87)]

$$\xi\ du/d\xi = -u^2; \quad \xi\ dv/d\xi = uv, \qquad (u \to \infty), \tag{2.7.139}$$

and by integration

$$u = 1/\ln(A\xi); \quad v = B\ln(A\xi), \qquad (N = 1, 2, 3, ...;\ 0 < A, B < \infty;\ \xi \to 1/A;\ \xi \geq 1/A). \tag{2.7.140}$$

Thus, U_∞ is an initial point ($\xi \geq 1/A$), and of the form $U_\infty(\infty, 0)$, i.e. $v_\infty = 0$. We have

$$\theta' = v/\xi \approx B\ln(A\xi)/\xi \approx AB\ln(A\xi) \to 0; \quad \theta = \ln(\xi/u\theta') \approx \ln(\xi^2/B) \to -\ln(A^2B) = \text{const};$$
$$\xi^{N-1}\theta' \approx B\xi^{N-2}\ln(A\xi) \approx A^{2-N}B\ln(A\xi) \to 0; \quad uv = B = \text{const}, \qquad (N = 1, 2, 3, ...;\ \xi \to 1/A). \tag{2.7.141}$$

(vi) $V_\infty(u_\infty, \infty)$, $u_\infty = \text{const} \neq \infty$. Eq. (2.7.6) reads

$$dv/du = -1 + (N-2)/u, \qquad (v \to \infty), \tag{2.7.142}$$

which can be integrated, to give

$$v = \ln u^{N-2} - u + C, \qquad (C = \text{const};\ -\infty < C < \infty). \tag{2.7.143}$$

We observe that V_∞ exists if $N = 1$. We have $v \to \infty$ if $u \to 0$, and Eq. (2.7.5) turns into

$$\xi\ du/d\xi = -uv; \quad \xi\ dv/d\xi = v, \qquad (u \to 0;\ v \to \infty), \tag{2.7.144}$$

with the solutions

$$u = \exp(-B\xi + A); \quad v = B\xi, \qquad (N = 1;\ \xi \to \infty;\ A, B = \text{const};\ -\infty < A < \infty;\ 0 < B < \infty). \tag{2.7.145}$$

For polytropic slabs $V_\infty(0, \infty)$ is a final point:

$$\theta' = v/\xi \approx B = \text{const}; \quad \theta = \ln(\xi/u\theta') \approx \ln(\xi/B) + B\xi - A \approx B\xi \to \infty;$$
$$\xi^{N-1}\theta' = \theta' \approx B = \text{const}, \qquad (N = 1;\ \xi \to \infty). \tag{2.7.146}$$

(vii) $G_\infty(\infty, \infty)$. Eq. (2.7.6) takes the form

$$du/dv = -u/v - 1, \qquad (u, v \to \infty). \tag{2.7.147}$$

This equation can be integrated by the method of variation of constants

$$u = -v/2 - C/v, \qquad (u, v \to \infty;\ C = \text{const}), \tag{2.7.148}$$

showing that no positive infinity points exist in the first quadrant. The singular point G_∞ does not exist for the polytropic index $n = \pm\infty$, $(N = 1, 2, 3, ...)$.

Excepting for the D-curves, all other classes of integral curves also occur if $n = \pm\infty$. If $N = 1, 2$, the solution curves are M, E, and F-curves (Figs. 2.7.2, 2.7.3), cutting the v-axis if $N = 2$. M-curves are located on the left of the E-curve, and F-curves on its right. If $N = 3$, there occur only O-curves and the familiar E-curve (Fig. 2.7.6 and Chandrasekhar 1939).

According to Kimura's (1981a) classification scheme we can distinguish four classes of initial points ($N = 1, 2, 3;\ 0 < \text{const} < \infty$) :

(i) $\xi = 0$, $\exp(-\theta) = \text{const}$, $\xi^{N-1}\theta' = 0$. These integral curves are represented by E-curves starting at $U_s(N, 0)$. They have finite pressure and density $[P, \varrho \propto \exp(-\theta)]$, and zero mass ($M \propto \xi^{N-1}\theta'$) at the origin $\xi = 0$.

(ii) $\xi = 0$, $\exp(-\theta) = \text{const}$, $\xi^{N-1}\theta' = \text{const}$. These integral curves are represented by the M-curves starting at O_s if $N = 1$, having finite pressure and mass at the origin $\xi = 0$.

(iii) $\xi = 0$, $\exp(-\theta) = \infty$, $\xi^{N-1}\theta' = \text{const}$. These polytropes have infinite pressure and density, but finite mass at the origin $\xi = 0$. They are represented by the curves starting on the v-axis with $0 < v_s < 2$ if $N = 2$.

(iv) $\xi = \text{const}$, $\exp(-\theta) = \text{const}$, $\xi^{N-1}\theta' = 0$. These integral curves are represented by the F-solutions occurring if $N = 1, 2$, and by the O-solutions occurring for polytropic spheres ($N = 3$), having finite pressure and zero mass at a nonzero initial radial distance.

The final points can be classified into two groups:

(i) $\xi = \infty$, $\xi^{N-1}\theta' = \text{const}$. These integral curves are represented by the M, E, and F-curves if $N = 1, 2$.

(ii) $\xi = \infty$, $\xi^{N-1}\theta' = \infty$. These are the E and O-curves if $N = 3$.

The pressure $P \propto \exp(-\theta)$ tends always to zero.

The polytropes described in this section and differing from the familiar E-solution may have importance in constructing models of massive interstellar clouds with a hot ionized core of negligible mass (pertinent initial conditions are from (v) if $n \neq -1, \pm\infty$ and from (iv) if $n = \pm\infty$), or models of clouds or stellar and globular systems having a compact mass concentration at its centre [pertinent initial conditions are from (ii)]. The very importance of non-E-solutions is their role for constructing composite polytropes, providing more realistic models of stars, clusters, and clouds than simple polytropes do. Composite polytropes will be described subsequently.

2.8 Composite and Other Spherical Polytropes

2.8.1 Composite Polytropes

Since stars often have a core surrounded by an envelope, it was tempting to generalize polytropes with a single index to polytropic models having two or even more polytropic indices. We confine ourselves to two-zone polytropes, and denote by n_c, n_e the polytropic index of the core and of the envelope, respectively. The total mass M_{e1} and the total radius r_{e1} of the polytrope are specified, as well as the ratio $q = r_{ci}/r_{e1} = r_{ei}/r_{e1}$ between core radius $r_{ci} = r_{ei}$ and total radius. Generally, the core solution is chosen to be the Lane-Emden function of index n_c, satisfying the Lane-Emden equation (2.1.14)

$$\theta_c'' + (N-1)\theta_c'/\xi_c = \mp\theta_c^{n_c}, \qquad (n_c \neq -1, \pm\infty;\ \theta_c' = d\theta_c(\xi_c)/d\xi_c), \tag{2.8.1}$$

with the initial conditions $\theta_c(0) = 1,\ \theta_c'(0) = 0$. To be concise, we omit for the moment the special case $n_c = \pm\infty$ (see however Eqs. (2.8.15)-(2.8.32), and Sec. 2.8.2 for the so-called isothermal spheres). The Lane-Emden equation writes in the envelope

$$\theta_e'' + (N-1)\theta_e'/\xi_e = \mp\theta_e^{n_e}, \qquad (n_e \neq -1, \pm\infty;\ \theta_e' = d\theta_e(\xi_e)/d\xi_e), \tag{2.8.2}$$

where we have introduced the dimensionless polytropic envelope variables ξ_e and θ_e, in order to distinguish them from the core variables ξ_c and θ_c. The initial conditions of Eq. (2.8.2) need no longer to be the usual conditions of the Lane-Emden function θ_c from Eq. (2.8.1); generally, the initial envelope conditions $\theta_{ei} = \theta_e(\xi_{ei}),\ \theta_{ei}' = \theta_e'(\xi_{ei})$ of Eq. (2.8.2) are given at the core-envelope interface for a certain value of $\xi_e = \xi_{ei} \neq 0$.

The physical variables radius, density, pressure, and mass of the core are given by (cf. Secs. 2.6.1-2.6.3)

$$r_c = [\pm(n_c+1)K_c/4\pi G\varrho_{c0}^{1-1/n_c}]^{1/2}\xi_c; \quad \varrho_c = \varrho_{c0}\theta_c^{n_c}; \quad P_c = K_c\varrho_{c0}^{1+1/n_c}\theta_c^{n_c+1};$$
$$M_c = \big\{2\varrho_{c0}[\pm(n_c+1)K_c/4\pi G\varrho_{c0}^{1-1/n_c}]^{N/2}\Gamma^N(1/2)/\Gamma(N/2)\big\}\xi_c^{N-1}(\mp\theta_c'), \tag{2.8.3}$$

and for the envelope

$$r_e = [\pm(n_e+1)K_e/4\pi G\varrho_{e0}^{1-1/n_e}]^{1/2}\xi_e; \quad \varrho_e = \varrho_{e0}\theta_e^{n_e}; \quad P_e = K_e\varrho_{e0}^{1+1/n_e}\theta_e^{n_e+1};$$
$$M_e = \big\{2\varrho_{e0}[\pm(n_e+1)K_e/4\pi G\varrho_{e0}^{1-1/n_e}]^{N/2}\Gamma^N(1/2)/\Gamma(N/2)\big\}\xi_e^{N-1}(\mp\theta_e'), \qquad [(\xi_e^{N-1}\theta_e')_{\xi_e=0} = 0]. \tag{2.8.4}$$

M_e is the whole mass inside radius r_e, including also the core mass. Strictly speaking the physical parameters for the core are defined only between the origin $r_c = 0,\ (\xi_c = 0)$ and the core-envelope interface at radius $r_{ci},\ (\xi_c = \xi_{ci})$. Similarly, the envelope variables are defined between the interface at $r_e = r_{ei} = r_{ci} > 0,\ (\xi_e = \xi_{ei})$ and the boundary of the polytrope at $r_e = r_{e1},\ (\xi_e = \xi_{e1})$. In order to get simple equations for the mass M_e, we can formally extend the integration up to the centre where $r_e, \xi_e = 0$. We have assumed $(\xi_e^{N-1}\theta_e')_{\xi_e=0} = 0$, so that the simple equation (2.8.4) for M_e results, although θ_e may even have a singularity at the origin. ϱ_{c0} is just the central density of the composite polytrope, while ϱ_{e0} is generally adjusted in order to select a particular composite model (Chandrasekhar 1939). The prescribed total mass and total radius of the composite polytrope is

$$M_{e1} = \big\{2\varrho_{e0}[\pm(n_e+1)K_e/4\pi G\varrho_{e0}^{1-1/n_e}]^{N/2}\Gamma^N(1/2)/\Gamma(N/2)\big\}\xi_{e1}^{N-1}(\mp\theta_{e1}'), \tag{2.8.5}$$

$$r_{e1} = [\pm(n_e+1)K_e/4\pi G\varrho_{e0}^{1-1/n_e}]^{1/2}\xi_{e1}. \tag{2.8.6}$$

The physical quantities at the core-envelope interface must be equal in order to satisfy continuity. The Lane-Emden variables at the interface are indexed with i, and the equations of fit at the interface are obtained at once by equating Eqs. (2.8.3) and (2.8.4), respectively:

$$[\pm(n_c+1)K_c/\varrho_{c0}^{1-1/n_c}]^{1/2}\xi_{ci} = [\pm(n_e+1)K_e/\varrho_{e0}^{1-1/n_e}]^{1/2}\xi_{ei}, \tag{2.8.7}$$

$$\varrho_{c0}\theta_{ci}^{n_c} = \varrho_{e0}\theta_{ei}^{n_e}, \tag{2.8.8}$$

$$K_c\varrho_{c0}^{1+1/n_c}\theta_{ci}^{n_c+1} = K_e\varrho_{e0}^{1+1/n_e}\theta_{ei}^{n_e+1}, \tag{2.8.9}$$

$$\varrho_{c0}[\pm(n_c+1)K_c/\varrho_{c0}^{1-1/n_c}]^{N/2}\xi_{ci}^{N-1}(\mp\theta'_{ci}) = \varrho_{e0}[\pm(n_e+1)K_e/\varrho_{e0}^{1-1/n_e}]^{N/2}\xi_{ei}^{N-1}(\mp\theta'_{ei}). \tag{2.8.10}$$

We raise Eq. (2.8.7) to the N-th power, multiply the result with Eq. (2.8.8), and finally divide by Eq. (2.8.10). We just get equality of the homology variable u as a fitting condition at the interface:

$$\xi_{ci}\theta_{ci}^{n_c}/(\mp\theta'_{ci}) = \xi_{ei}\theta_{ei}^{n_e}/(\mp\theta'_{ei}) \quad \text{or} \quad u(n_c,\xi_{ci}) = u(n_e,\xi_{ei}). \tag{2.8.11}$$

Now, we divide Eq. (2.8.9) by Eq. (2.8.8), to obtain

$$K_c\varrho_{c0}^{1/n_c}\theta_{ci} = K_e\varrho_{e0}^{1/n_e}\theta_{ei}. \tag{2.8.12}$$

As already noted several times, factors of the form $K\varrho^{1/n}$ may be replaced by P/ϱ, thus avoiding the apparent singularity of $K\varrho^{1/n}$ if $n=0$. We raise Eq. (2.8.7) to the $(N-2)$-th power, and multiply by Eq. (2.8.12). Then we divide Eq. (2.8.10) by the obtained product, and get a relationship for the other homology variable v at the interface:

$$(n_c+1)\xi_{ci}(\mp\theta'_{ci})/\theta_{ci} = (n_e+1)\xi_{ei}(\mp\theta'_{ei})/\theta_{ei} \quad \text{or} \quad (n_c+1)\ v(n_c,\xi_{ci}) = (n_e+1)\ v(n_e,\xi_{ei}). \tag{2.8.13}$$

A distinct chemical composition of core and envelope has been considered by Beech and Mitalas (1990) in composite polytropes composed of perfect gas $P = \mathcal{RT}\varrho/\mu$. From the continuity of pressure and temperature at the core-envelope interface results $\varrho_c/\mu_c = \varrho_e/\mu_e$, or instead of Eq. (2.8.8) $\varrho_{c0}\theta_{ci}^{n_c}/\mu_c = \varrho_{e0}\theta_{ei}^{n_e}/\mu_e$, where μ_c, μ_e denotes the mean molecular weight in the core and envelope, respectively. In this case the left- and right-hand sides of Eqs. (2.8.11) and (2.8.13) have to be divided by μ_c and μ_e, respectively.

The solution for the core of the composite polytrope in the $[u,(n+1)v]$-plane is just given by the E-curve, corresponding to the Lane-Emden function for the core, and extending up to the radial coordinate $\xi_c = \xi_{ci}$. The solution curves for the envelope in the $[u,(n+1)v]$-plane depend on a single integration constant, forming a one-parametric grid. Some or all curves of the grid cut the E-curve of the core at certain points, corresponding to particular interface solutions of the composite polytropic model. The intersection point of the $[u_e,(n_e+1)v_e]$-curves with the E-curve is selected in such a way that the corresponding value of $\xi_e = \xi_{ei}$ at the interface obeys just the prescribed ratio $r_{ci}/r_{e1} = r_{ei}/r_{e1} = \xi_{ei}/\xi_{e1} = q$. Note, that the value of q along a solution curve in the $[u,(n+1)v]$-plane is the same for all members of a homology family, since the homology constant A, $(\xi \rightarrow A\xi)$ cancels out from the ratio q. Once ξ_{ei} is determined, the corresponding values θ_{ei} and θ'_{ei} at the interface $\xi = \xi_{ei}$ are also known, as well as $\theta_{e1} = 0$ and θ'_{e1} at the surface $\xi_e = \xi_{e1}$ of the envelope, i.e. at the surface of the composite polytrope. Actually, ξ_{ei} is known merely aside the homology constant ϱ_{e0}. To obtain ϱ_{e0}, we determine at first the polytropic constant K_e from the prescribed mass-radius relationship [cf. Eqs. (2.6.21), (2.8.5) and (2.8.6)]:

$$K_e = \{4\pi G/[\pm(n_e+1)]\}\{2[\Gamma(1/2)]^N/\Gamma(N/2)\}^{(1-n_e)/n_e}$$
$$\times M_{e1}^{(n_e-1)/n_e} r_{e1}^{[N(1-n_e)+2n_e]/n_e} \xi_{e1}^{-(n_e+1)/n_e} (\mp\theta'_{e1})^{(1-n_e)/n_e}. \tag{2.8.14}$$

ϱ_{e0} can now be readily determined from Eq. (2.8.6) with the aid of the prescribed radius. Then, the core parameters result at once, since the intersection point between the integral curves of the envelope and the E-curve of the core already yields the value of ξ_{ci} for the Lane-Emden function θ_c of the core. Hence θ_{ci} and θ'_{ci} are known too, as well as the values of the physical quantities $r_{ci} = r_{ei},\ \varrho_{ci} = \varrho_{ei},\ P_{ci} = P_{ei}$, and $M_{ci} = M_{ei}$ at the core-envelope interface. So, ϱ_{c0} and K_c can be found at once from Eq. (2.8.3), by using for instance the values of radius and density at the interface (Chandrasekhar 1939). The whole structure of the composite polytrope is now completely determined. The outlined method (cf. Milne 1930b, 1932, Cowling 1931, Russell 1931) is mainly of historical interest, due to the computerization of science.

In the present context we shall confine ourselves to the presentation of two composite models, namely the isothermal core surrounded by a polytropic envelope (Henrich and Chandrasekhar 1941, Yabushita 1975), and the composite models devised by Murphy (1982, 1983a) from the superposition of the analytic solutions for polytropes of index $n_c = 0, n_e = 1$, and $n_c = 1, n_e = 5$.

Yabushita (1975) constructs a Newtonian analogue to the model of a neutron star. For the neutrons in the core an equation of state of the form (1.7.41) may be appropriate, connecting core pressure P_c with relativistic density ϱ_{cr} :

$$P_c = \beta v^2 \varrho_{cr}. \tag{2.8.15}$$

In the Newtonian approximation the relativistic mass density ϱ_{cr} is equal to the rest mass density ϱ_c, and the pressure-density relationship in the core becomes

$$P_c = \beta v^2 \varrho_c = K_c \varrho_c, \qquad (K_c = \beta v^2 = \mathrm{const}), \tag{2.8.16}$$

which is just the equation of state of a polytrope with index $n_c = \pm\infty$ (isothermal in the case of a perfect gas). The neutrons in the core of a neutron star may be surrounded by an envelope consisting of a nonrelativistic, completely degenerate electron gas, obeying the equation of state (1.6.6) of a polytrope with index $n_e = 1.5$:

$$P_e = K_e \varrho_e^{5/3}. \tag{2.8.17}$$

The Lane-Emden equations for the core and the envelope of this composite polytrope are therefore

$$\theta_c'' + 2\theta_c'/\xi_c = \exp(-\theta_c), \qquad (n_c = \pm\infty), \tag{2.8.18}$$

and

$$\theta_e'' + 2\theta_e'/\xi_e = -\theta_e^{3/2}, \qquad (n_e = 3/2). \tag{2.8.19}$$

Yabushita (1975) fixes one of the homology constants by adopting the same central density ϱ_{c0} in both, the core and the envelope: $\varrho_{c0} = \varrho_{e0}$. Therefore, radius, density, pressure, and mass are written in Lane-Emden variables as

$$r_c = (K_c/4\pi G\varrho_{c0})^{1/2}\xi_c; \quad \varrho_c = \varrho_{c0}\exp(-\theta_c); \quad P_c = K_c\varrho_{c0}\exp(-\theta_c);$$
$$M_c = 4\pi(K_c/4\pi G)^{3/2}\varrho_{c0}^{-1/2}\xi_c^2\theta_c', \tag{2.8.20}$$

$$r_e = (5K_e/8\pi G)^{1/2}\varrho_{c0}^{-1/6}\xi_e; \quad \varrho_e = \varrho_{c0}\theta_e^{3/2}; \quad P_e = K_e\varrho_{c0}^{5/3}\theta_e^{5/2};$$
$$M_e = -4\pi(5K_e/8\pi G)^{3/2}\varrho_{c0}^{1/2}\xi_e^2\theta_e'. \tag{2.8.21}$$

Continuity of physical quantities at the interface yields

$$(K_c/\varrho_{c0})^{1/2}\xi_{ci} = (5K_e/2)^{1/2}\varrho_{c0}^{-1/6}\xi_{ei}, \tag{2.8.22}$$

$$\exp(-\theta_{ci}) = \theta_{ei}^{3/2}, \tag{2.8.23}$$

$$K_c\exp(-\theta_{ci}) = K_e\varrho_{c0}^{2/3}\theta_{ei}^{5/2}, \tag{2.8.24}$$

$$K_c^{3/2}\xi_{ci}^2\theta_{ci}' = -(5K_e/2)^{3/2}\varrho_{c0}\xi_{ei}^2\theta_{ei}'. \tag{2.8.25}$$

Elimination of θ_{ei} between Eqs. (2.8.23) and (2.8.24) gives

$$\varrho_{c0} = (K_c/K_e)^{3/2}\exp\theta_{ci}. \tag{2.8.26}$$

When this is inserted into Eq. (2.8.22), one finds that

$$\xi_{ei} = (2/5)^{1/2}\xi_{ci}\exp(-\theta_{ci}/3), \tag{2.8.27}$$

while Eq. (2.8.25) furnishes the corresponding derivative

$$\theta'_{ei} = -(2/5)^{1/2}\theta'_{ci}\exp(-\theta_{ci}/3). \qquad (2.8.28)$$

The tabulated Lane-Emden functions θ_c, θ'_c are known for any ξ_c, and we have to choose merely an arbitrary $\xi_c = \xi_{ci}$ as the value of the radial coordinate at the core-envelope interface, in order to fix the values of θ_{ci} and θ'_{ci}. Since K_c and K_e are assumed to take prescribed values, we can easily find out from the four equations (2.8.23), (2.8.26)-(2.8.28) the values of $\xi_{ei}, \theta_{ei}, \theta'_{ei}, \varrho_{c0}$ with the aid of the known quantities $K_c, K_e, \xi_{ci}, \theta_{ci}, \theta'_{ci}$. Now we can integrate the Lane-Emden equation (2.8.19) for a spherical polytrope of index $n_e = 1.5$ with the initial conditions $\theta_{ei} = \theta(\xi_{ei})$ and $\theta'_{ei} = \theta'_e(\xi_{ei})$ up to its first zero $\theta_{e1} = \theta_e(\xi_{e1}) = 0$. Total mass and total radius of this composite polytrope are obtained at once from Eq. (2.8.21), by inserting for ϱ_{c0} via Eq. (2.8.26):

$$M_{e1} = -4\pi(5/8\pi G)^{3/2}(K_cK_e)^{3/4}\xi_{e1}^2\theta'_{e1}\exp(\theta_{ci}/2); \quad r_{e1} = (5/8\pi G)^{1/2}(K_e^3/K_c)^{1/4}\xi_{e1}\exp(-\theta_{ci}/6). \qquad (2.8.29)$$

Core mass M_{ci}, core radius r_{ci}, and core density at the interface ϱ_{ci} result from Eq. (2.8.20):

$$M_{ci} = 4\pi(1/4\pi G)^{3/2}(K_cK_e)^{3/4}\xi_{ci}^2\theta'_{ci}\exp(-\theta_{ci}/2);$$
$$r_{ci} = (1/4\pi G)^{1/2}(K_e^3/K_c)^{1/4}\xi_{ci}\exp(-\theta_{ci}/2); \quad \varrho_{ci} = (K_c/K_e)^{3/2}. \qquad (2.8.30)$$

The core mass approaches a finite limiting value even if the radial coordinate at the interface ξ_{ci} grows indefinitely. This can be shown by using the asymptotic expansion of θ_c from Eqs. (2.4.104), (2.4.105):

$$\theta_c \approx \ln(\xi_c^2/2); \quad \theta'_c \approx 2/\xi_c; \quad \exp(-\theta_c/2) \approx 2^{1/2}/\xi_c \quad \text{if} \quad \xi_c \to \infty. \qquad (2.8.31)$$

Eq. (2.8.30) already shows that the interface density $\varrho_{ci} = \varrho_{ei}$ is independent of ξ_{ci}, and if we insert Eq. (2.8.31), we get at once

$$\lim_{\xi_{ci}\to\infty} M_{ci} = (2/\pi)^{1/2}G^{-3/2}(K_cK_e)^{3/4}; \quad \lim_{\xi_{ci}\to\infty} r_{ci} = (1/2\pi G)^{1/2}(K_e^3/K_c)^{1/4}. \qquad (2.8.32)$$

As outlined by Eqs. (2.7.99), (2.7.114), and shown graphically in Fig. 2.7.4, all integral curves in the (u,v)-plane approach the infinity point $V_\infty(0,\infty)$ if $n_e = 1.5$, that means there exists a finite zero ξ_{e1}. Moreover, Eq. (2.7.99) shows that $\xi_{e1}^2\theta'_{e1}$ is finite, so we conclude that the total mass of the composite polytrope given by Eq. (2.8.29) is finite too. From the tables of the isothermal function (Table 2.5.1 and Horedt 1986b) it is easily seen that the factor $\xi_{ci}^2\theta'_{ci}\exp(-\theta_{ci}/2)$ from the core mass in Eq. (2.8.30) obeys a maximum value of ≈ 4.19 if $\xi_{ci} \approx 6.5$; if $\xi_{ci} \to \infty$, this value drops to $2^{3/2} \approx 2.83$. Thus, the isothermal core of the considered composite model exhibits a similar behaviour as the general relativistic hydrostatic equilibrium model of a large cold mass, possessing a maximum mass of about 0.6-2.7 $M_\odot$ (Oppenheimer-Volkoff limit from Fig. 5.12.1). The scale of Yabushita's (1975) Fig. 1 seems inappropriate: His abscissa $\exp(\theta_{ci})$ takes values < 1, and his plotted ordinate is generally several times smaller than $\xi_{ci}^2\theta'_{ci}\exp(-\theta_{ci}/2)$.

Henrich and Chandrasekhar (1941) have determined the maximum mass that can be contained in an isothermal core surrounded by a $n_e = 3$ envelope. Further studies on this subject have been effected by Schönberg and Chandrasekhar (1942) for the case of a changing molecular weight, by Gabriel and Ledoux (1967) for the secular stability of models with isothermal cores, and by Beech (1988b). The physical parameters of the E-solution for the isothermal core ($n_c = \pm\infty$) are given by Eq. (2.8.20), while radius, density, pressure, and mass in the envelope are given by [cf. Eq. (2.8.4)]

$$r_e = (K_e/\pi G)^{1/2}\varrho_{e0}^{-1/3}\xi_e; \quad \varrho_e = \varrho_{e0}\theta_e^3; \quad P_e = K_e\varrho_{e0}^{4/3}\theta_e^4;$$
$$M_e = -4\pi(K_e/\pi G)^{3/2}\xi_e^2\theta'_e, \qquad (n_e = 3). \qquad (2.8.33)$$

At the interface the two sets of formulas should be identical:

$$K_c^{1/2}\varrho_{c0}^{-1/2}\xi_{ci}/2 = K_e^{1/2}\varrho_{e0}^{-1/3}\xi_{ei}; \quad \varrho_{c0}\exp(-\theta_{ci}) = \varrho_{e0}\theta_{ei}^3; \quad K_c\varrho_{c0}\exp(-\theta_{ci}) = K_e\varrho_{e0}^{4/3}\theta_{ei}^4;$$
$$K_c^{3/2}\varrho_{c0}^{-1/2}\xi_{ci}^2\theta'_{ci}/8 = -K_e^{3/2}\xi_{ei}^2\theta'_{ei}. \qquad (2.8.34)$$

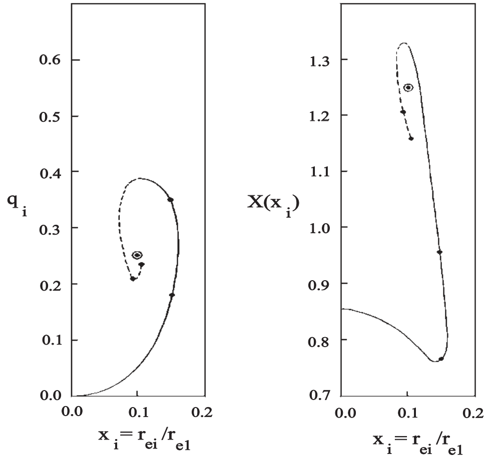

Fig. 2.8.1 Plot of fractional core mass q_i and of total radius $r_{e1} \propto X(x_i)$ versus fractional core radius $x_i = r_{ei}/r_{e1}$. The four dots on each curve represent the four solutions of Henrich and Chandrasekhar (1941), and the dot circle is the approximate location of the spiraling point.

These four equations of fit can be reduced in a similar way as Eqs. (2.8.7)-(2.8.13) to two equations involving only the homology invariant variables (2.2.6) and (2.2.7):

$$u_{ci} = \xi_{ci} \exp(-\theta_{ci})/\theta'_{ci} = u_{ei} = -\xi_{ei}\theta_{ei}^3/\theta'_{ei}; \quad v_{ci} = \xi_{ci}\theta'_{ci} = 4v_{ei} = -4\xi_{ei}\theta'_{ei}/\theta_{ei}. \tag{2.8.35}$$

The fraction of the radius occupied by the core is

$$x_i = r_{ci}/r_{e1} = r_{ei}/r_{e1} = \xi_{ei}/\xi_{e1}, \tag{2.8.36}$$

and the core fraction q_i of the total mass M_{e1} is also readily obtained:

$$q_i = M_{ci}/M_{e1} = M_{ei}/M_{e1} = M(\xi_{ei})/M(\xi_{e1}) = \xi_{ei}^2\theta'_{ei}/\xi_{e1}^2\theta'_{e1}. \tag{2.8.37}$$

Henrich and Chandrasekhar (1941) have obtained four solutions of the equations of fit (2.8.35), using M-solutions with $-\xi_{e1}^2\theta'_{e1} = 1.5$ and 1.9 (Fig. 2.8.1). To determine the $r_{e1}(x_i)$ relation, we eliminate ϱ_{e0} from the relationships

$$\varrho_{e0}^{-1/3} = K_e\theta_{ei}/K_c; \quad r_{e1} = (K_e/\pi G)^{1/2}\varrho_{e0}^{-1/3}\xi_{e1}, \tag{2.8.38}$$

to obtain

$$r_{e1} = K_e^{3/2}\xi_{e1}\theta_{ei}/K_c(\pi G)^{1/2} = -(GM_{e1}/4K_c)(u_{ei}v_{ei})^{1/2}/\xi_{e1}^2\theta'_{e1}x_i = GM_{e1}X(x_i)/K_c, \tag{2.8.39}$$

where

$$X(x_i) = -(u_{ei}v_{ei})^{1/2}/4\xi_{e1}^2\theta'_{e1}x_i. \tag{2.8.40}$$

From Fig. 2.8.1 results that the fractional core mass q_i increases if the fractional core radius x_i grows up to its maximum value $x_{i,max} \approx 0.17$. The fractional core mass continues to increase up to $q_i \approx 0.39$, when $x_i \approx 0.1$. Thus, an upper limit exists to the mass fraction that could be contained in an isothermal core – the Chandrasekhar-Schönberg limit $q_i \lesssim 0.39$ if $n_c = \pm\infty$ [Cox and Giuli 1968, Eq. (26.25)]. Both curves in Fig. 2.8.1 exhibit a spiraling round the dot circle, similarly to the asymptotic form (2.4.104) of the isothermal solution. The total radius of the composite polytrope $r_{e1} \propto X(x_i)$ first decreases when the core radius increases. If $x_i \approx 0.13$, the total radius passes through a minimum. As x_i decreases again, the total radius r_{e1} increases rapidly, reaches a maximum, and begins spiraling round the dot circle.

Table 2.8.1 Numerical values for the first four roots ξ_{im} of Eq. (2.8.43) for Murphy's (1982, 1983a) composite polytropic models with $n_c = 0,\ n_e = 1$. There are tabulated the transformation constants A_c, A_e, the central value $\theta_c(0)$ of the Lane-Emden function, and the interface values $\xi_{im},\ \theta_{ci} = \theta_{ei} = 0.5$. The surface value (2.8.48) of ξ is $\xi_{1m} = (m+1)\pi$ if $\theta_e = 0$. And $a + b$ means $a \times 10^b$.

Symbol	$m = 1$	$m = 2$	$m = 3$	$m = 4$
A_c	4.07−1	2.65−1	1.97−1	1.57−1
A_e	5.80	1.40+1	2.56+1	4.04+1
$\theta_c(0)$	6.04	1.43+1	2.58+1	4.06+1
$\theta_{ci} = \theta_{ei}$	5.00−1	5.00−1	5.00−1	5.00−1
ξ_{im}	5.76	9.10	1.23+1	1.55+1
ξ_{1m}	6.28	9.42	1.26+1	1.57+1

Similar results have been reached by Eggleton et al. (1998) if $n_c \geq 5$ and $n_e \leq 5$, specifically if $n_c = 5,\ n_e = 1$, with a mean molecular weight jump of $\mu_c/\mu_e \geq 3$ at the interface.

Another composite model that can be solved exclusively by analytical means has been devised by Murphy (1982, 1983a), consisting of the Lane-Emden type functions for a constant density core ($n_c = 0$), surrounded by an envelope of polytropic index $n_e = 1$.

In virtue of Eqs. (2.2.4), (2.3.88) the core solution is

$$\theta_c = A_c^{2/(n_c-1)}\theta(A_c\xi) = (1 - A_c^2\xi^2/6)/A_c^2 = A_c^{-2} - \xi^2/6, \qquad (n_c = 0;\ A_c = \mathrm{const}), \tag{2.8.41}$$

and the envelope solution is [cf. Eq. (2.3.89)]

$$\theta_e = A_e\theta(\xi) = A_e \sin\xi/\xi, \qquad (n_e = 1;\ A_e = \mathrm{const}). \tag{2.8.42}$$

Murphy (1982, 1983a) postulates the same radial coordinate ξ for core and envelope: $\xi \equiv \xi_c \equiv \xi_e$. The factor A_c is the homology constant from Eq. (2.2.4), and A_e is a linear scaling constant applicable to the linear and homogeneous Lane-Emden equation (2.3.6) if $n_e = 1$. The structure of the composite polytrope is completely determined if the two constants A_c and A_e are fixed by the continuity of the physical variables at the core-envelope interface. Eqs. (2.8.11) and (2.8.13) must be satisfied at the interface $\xi = \xi_i$:

$$u_{ci} = 3 = u_{ei} = \xi_i^2/(1 - \xi_i \cot\xi_i), \qquad (\xi_i > \pi), \tag{2.8.43}$$

$$v_{ci} = 2A_c\xi_i^2/(6 - A_c^2\xi_i^2) = 2v_{ei} = 2(1 - \xi_i\cot\xi_i), \qquad (\xi_i > \pi). \tag{2.8.44}$$

The transcendental equation (2.8.43) yields the successive roots $\xi_{im},\ (m = 1, 2, 3, ...)$, the first four being shown in Table 2.8.1. Eq. (2.8.44) yields for successive roots ξ_{im} the value of the homology constant equal to

$$A_c = \left[6(\tan\xi_{im} - \xi_{im})/(2\tan\xi_{im} - \xi_{im})\right]^{1/2}/\xi_{im}, \qquad (m = 1, 2, 3, ...). \tag{2.8.45}$$

The scaling factor A_e applied to the envelope solution θ_e can be established with the aid of Eq. (2.8.43), by equating θ_c and θ_e at the interface ξ_{im} :

$$A_e = \xi_{im}^3/6(1 - \xi_{im}\cot\xi_{im})\sin\xi_{im} = \xi_{im}/2\sin\xi_{im}. \tag{2.8.46}$$

From Eqs. (2.8.42), (2.8.46) follows that the interface occurs at $\theta_{ci} = \theta_{ei} = 0.5$ for all values of ξ_{im}. Fig. 2.8.2 represents in the $[u, (n+1)v]$-plane the core solution E_0 (Lane-Emden function), together with the first four envelope solutions, which are of F-type (see Sec. 2.7.1). Thus, the analytical solution for this particular composite polytrope is given by

$$\theta = \theta_c = \xi_{im}^2(2\tan\xi_{im} - \xi_{im})/6(\tan\xi_{im} - \xi_{im}) - \xi^2/6 \quad \text{if} \quad 0 \leq \xi \leq \xi_{im},\ m = 1, 2, 3, ..., \tag{2.8.47}$$

$$\theta = \theta_e = \xi_{im}\sin\xi/2\xi\sin\xi_{im} \quad \text{if} \quad \xi_{im} \leq \xi \leq (m+1)\pi,\ m = 1, 2, 3, ..., \tag{2.8.48}$$

where Eq. (2.8.47) holds for the core, and Eq. (2.8.48) for the envelope.

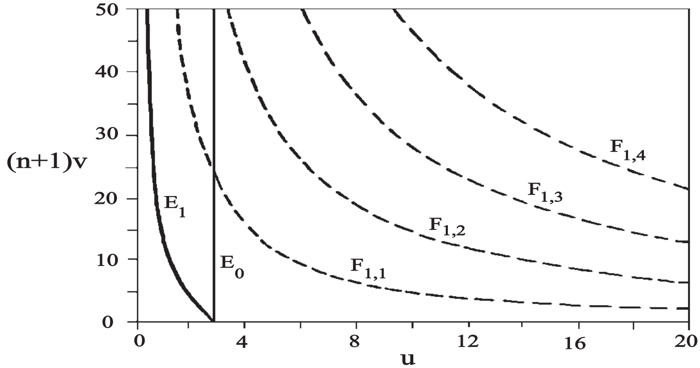

Fig. 2.8.2 Representation of Murphy's (1982) analytical composite $n_c = 0,\ n_e = 1$ polytrope in the $[u, (n+1)v]$-plane. There are shown the E-curves for the core [E_0 corresponding to Eq. (2.8.47)] and for the envelope (E_1 corresponding to Eq. (2.8.48) if $m = 0$), together with the four F_1-curves for the envelope from Eq. (2.8.48) if $m = 1, 2, 3, 4$.

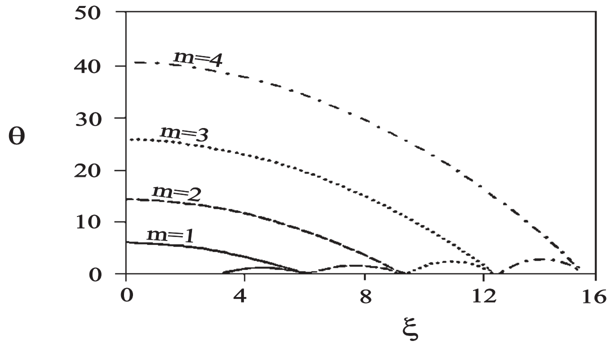

Fig. 2.8.3 Lane-Emden variables for Murphy's (1982) analytical composite $n_c = 0,\ n_e = 1$ polytrope. Above $\theta = 0.5$ the solution is given by the curves from Eq. (2.8.47) labeled $m = 1, 2, 3, 4$, respectively; these correspond to the E_0-curve in Fig. 2.8.2. Below the value $\theta = \theta_{ci} = \theta_{ei} = 0.5$, the solution is given by the small portions from Eq. (2.8.48) of the sinusoidal F_1-curves on the right of the intersection points ξ_{im}, which are not well represented at the scale of the figure, and correspond to the curves labeled $F_{1,1}, F_{1,2}, F_{1,3}, F_{1,4}$ in Fig. 2.8.2.

Murphy (1983b), and Murphy and Fiedler (1985a) have devised another composite model, consisting of a core constructed on a $n_c = 1$ polytrope [cf. Eq. (2.8.42)]

$$\theta_c = A \sin\xi/\xi, \qquad (n_c = 1;\ \theta_c(0) = A = \text{const}), \tag{2.8.49}$$

and of an envelope obeying Srivastava's (1962) singular solution from Eq. (2.3.42), written under the equivalent form

$$\theta_e = \sin[\ln(B\xi)^{1/2}]/\xi^{1/2}\{2 + \cos[\ln(B\xi)]\}^{1/2}, \qquad (n_e = 5;\ B = \text{const}). \tag{2.8.50}$$

A discontinuity is allowed for the derivatives of the Lane-Emden functions at the interface $\xi = \xi_i$, and only the values of θ are required to coincide:

$$\theta_c(\xi_i) = \theta_e(\xi_i), \tag{2.8.51}$$

Table 2.8.2 Characteristic values for some selected composite $n_c = 1,\ n_e = 5$ polytropes (Murphy and Fiedler 1985a). $1 - q_i$ means the fractional mass in the envelope, and $a + b$ is $a \times 10^b$.

Model	ξ_i	ξ_{e1}	$\theta_c(0) = A$	$\theta_c(\xi_i) = \theta_e(\xi_i)$	$1 - q_i$
1	2.50	7.90	9.32−1	2.22−1	3.0−3
2	2.63	5.60	7.37−1	1.38−1	1.8−4
3	2.70	4.85	6.64−1	1.04−1	5.0−5
4	2.88	3.87	5.60−1	5.04−2	7.0−7
5	3.07	3.29	5.01−1	1.15−2	1.3−10

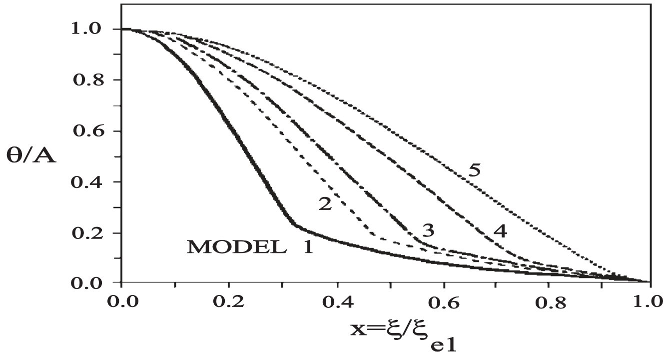

Fig. 2.8.4 Normalized Lane-Emden functions $\theta_c(\xi)/A,\ (0 \le \xi \le \xi_i)$ and $\theta_e(\xi)/A,\ (\xi_i \le \xi \le \xi_{e1})$ as a function of dimensionless radius $x = \xi/\xi_{e1} = r/r_{e1}$ for the composite $n_c = 1,\ n_e = 5$ polytropes from Table 2.8.2 (Murphy and Fiedler 1985a).

or

$$A = \xi_i^{1/2} \sin[\ln(B\xi_i)^{1/2}]/\{2 + \cos[\ln(B\xi_i)]\}^{1/2} \sin \xi_i. \tag{2.8.52}$$

The radial oscillations of this model are touched in Sec. 5.3.7 (Murphy and Fiedler 1985b).

A three-zone composite polytropic model of the Sun has been proposed by Hendry (1993), consisting of a convective envelope ($n_e = 1.5$), and a two-zone radiative core with polytropic indices $n_c = 3.79$ and 20, respectively. A composite model of main sequence stars with masses $0.4 - 0.8\ M_\odot$ has been considered by Beech (1988a). It consists of a $n_c = 3$ core with the approximate solution $\theta_c \approx 1/\cosh(\xi_c/3^{1/3})$ from Eq. (2.4.27), and a $n_e = 1$ envelope with the general exact solution $\theta_e = A \sin(\xi_e + B)/\xi_e,\ (A, B = \mathrm{const})$ from Eq. (2.8.83).

And Rappaport et al. (1983) assume composite $n_c = 3,\ n_e = 1.5$ polytropes to study the evolution of compact binary stars. Gyration factors of such stars have been calculated by Ruciński (1988), [cf. Eq. (6.1.179)].

Composite models of prestellar cores in interstellar clouds ($n_c = \pm\infty,\ n_e < -1$) have been studied by Curry and McKee (2000), adopting a density discontinuity at the interface: $\varrho_{ci} \neq \varrho_{ei}$.

2.8.2 Two-component Polytropes

The equation of state for the s species of a perfect gas is (cf. Eq. (1.2.5), Taff et al. 1975)

$$n_d = \sum_{j=1}^{s} n_{dj}; \quad \varrho = \sum_{j=1}^{s} \varrho_j = H \sum_{j=1}^{s} \mu_j n_{dj}; \quad \varrho_j = H\mu_j n_{dj};$$

$$P = \sum_{j=1}^{s} P_j = \sum_{j=1}^{s} K_j \varrho_j = H \sum_{j=1}^{s} K_j \mu_j n_{dj}, \qquad (n = \pm\infty). \tag{2.8.53}$$

The number density, the density, the molecular weight, and the polytropic constant of species j is denoted by n_{dj}, ϱ_j, μ_j, and K_j, respectively. For a perfect gas $K_j = \mathcal{R}T_0/\mu_j = kT_0/H\mu_j$, where T_0 is the isothermal temperature and k the Boltzmann constant. Instead of the mean molecular weight μ_j, we may use the mass $m_j = H\mu_j$ of individual particles. The hydrostatic equation (2.1.3) becomes

$$\nabla P = \sum_{j=1}^{s} K_j \ \nabla \varrho_j = \varrho \ \nabla\Phi = \nabla\Phi \sum_{j=1}^{s} \varrho_j. \tag{2.8.54}$$

It is seen that this equation can be split into s equations in the variables ϱ_j, as results from Dalton's law, with the pressures P_j changing independently, but with the gravitational potential Φ arising from all species. Hence, the solution of Eq. (2.8.54) is the sum

$$\varrho = \sum_{j=1}^{s} \varrho_j = \sum_{j=1}^{s} \varrho_{j0} \exp(\Phi/K_j), \tag{2.8.55}$$

of the individual solutions

$$\varrho_j = \varrho_{j0} \exp(\Phi/K_j), \qquad (j = 1, 2, 3, ...s), \tag{2.8.56}$$

of the s equations $K_j \ \nabla\varrho_j = \varrho_j \ \nabla\Phi$. The central value of ϱ_j is denoted by ϱ_{j0}, and we have taken without loss of generality $\Phi = \Phi_0 = 0$ at the centre. The spherically symmetric form of Poisson's equation (2.1.4) is

$$\nabla^2\Phi = r^{-2}d(r^2 \ d\Phi/dr)/dr = -4\pi G\varrho = -4\pi G \sum_{j=1}^{s} \varrho_j = -4\pi G \sum_{j=1}^{s} \varrho_{j0} \exp(\Phi/K_j). \tag{2.8.57}$$

Φ may be defined for the multi-component gas by either one of Eqs. (2.8.56). For instance, if $j = 1$

$$\Phi = K_1 \ln(\varrho_1/\varrho_{10}). \tag{2.8.58}$$

The analogue of the Lane-Emden variables is introduced by [cf. Eqs. (2.6.2), (2.6.33)]

$$\xi = (4\pi G\varrho_{10}/K_1)^{1/2} r = r/\alpha; \quad \theta = -\Phi/K_1 = \ln(\varrho_{10}/\varrho_1), \tag{2.8.59}$$

where θ may be regarded as a dimensionless gravitational potential. Poisson's equation (2.8.57) turns with this substitution into the analogue of the Lane-Emden equation for a multi-component spherical polytrope having index $n = \pm\infty$:

$$\xi^{-2} \ d(\xi^2 \ d\theta/d\xi)/d\xi = \sum_{j=1}^{s} (\varrho_{j0}/\varrho_{10}) \exp(-K_1\theta/K_j). \tag{2.8.60}$$

For a two-component gas Eq. (2.8.60) simplifies with the notations $\kappa = K_1/K_2, \ \kappa\lambda = \varrho_{20}/\varrho_{10}$:

$$\xi^{-2} \ d(\xi^2 \ d\theta/d\xi)/d\xi = \exp(-\theta) + \kappa\lambda \exp(-\kappa\theta), \qquad (n = \pm\infty). \tag{2.8.61}$$

The initial conditions $\theta(0) = 0, \ \theta'(0) = 0$ are the same as in the one-component case. For a perfect gas $\kappa = K_1/K_2 = \mu_2/\mu_1 = m_2/m_1$ is equal to the ratio of particle masses. There is also $\kappa\lambda = \varrho_{20}/\varrho_{10} = \mu_2 n_{d20}/\mu_1 n_{d10}$, while $\lambda = n_{d20}/n_{d10}$ is equal to the ratio of the number densities at the centre $r = 0$. Pressure P, total density ϱ, and the densities ϱ_1, ϱ_2 of the individual components are given by

$$\begin{aligned} &P = K_1\varrho_1 + K_2\varrho_2 = K_1\varrho_{10}[\exp(-\theta) + \lambda \exp(-\kappa\theta)]; \\ &\varrho = \varrho_1 + \varrho_2 = \varrho_{10} \exp(-\theta) + \varrho_{20} \exp(-\kappa\theta) = \varrho_{10}[\exp(-\theta) + \kappa\lambda \exp(-\kappa\theta)]. \end{aligned} \tag{2.8.62}$$

The mass of the two-component polytrope within the radii r_1 and r_2 is

$$\begin{aligned} M &= 4\pi \int_{r_1}^{r_2} \varrho r^2 \ dr = M_1 + M_2 = 4\pi \int_{r_1}^{r_2} (\varrho_1 + \varrho_2) r^2 \ dr \\ &= 4\pi\varrho_{10}\alpha^3 \int_{\xi_1}^{\xi_2} \xi^2 [\exp(-\theta) + \kappa\lambda \exp(-\kappa\theta)] \ d\xi = 4\pi\varrho_{10}\alpha^3\xi^2\theta' \Big|_{\xi_1}^{\xi_2}, \end{aligned} \tag{2.8.63}$$

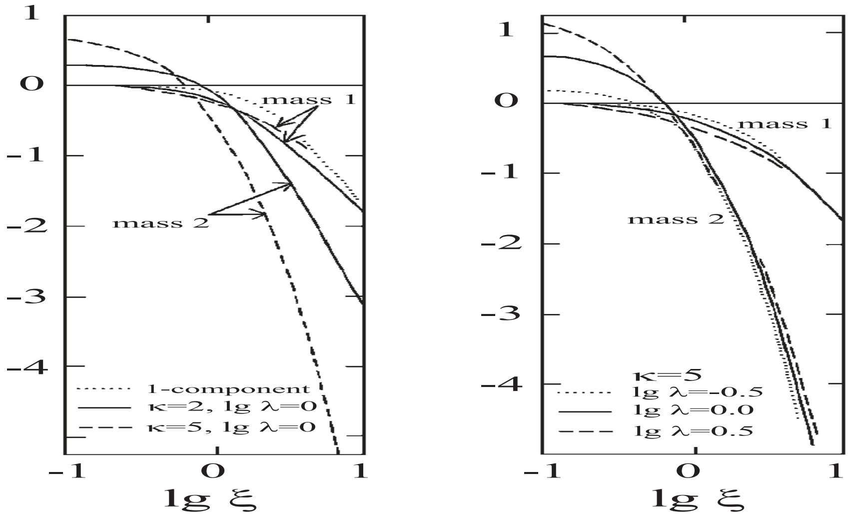

Fig. 2.8.5 Logarithmic run of the dimensionless density $\exp(-\theta)$ for mass 1, and $\kappa\lambda\exp(-\kappa\theta)$ for mass 2 as a function of dimensionless radius ξ. The one-component model corresponds to $\lambda = 0$ (Taff et al. 1975).

where M_1 and M_2 denote the contributions to the total mass from the two individual components. The gravitational energy is by virtue of Eq. (2.6.127) equal to

$$W = -G\int_M M(r)\ dM/r = -16\pi\varrho_{10}^2\alpha^5\int_{\xi_1}^{\xi_2}\xi^3\theta'[\exp(-\theta)+\kappa\lambda\exp(-\kappa\theta)]\ d\xi, \tag{2.8.64}$$

and seems not suitable for further analytical evaluations. Taff et al. (1975) find near the origin

$$\theta \approx (1+\kappa\lambda)\xi^2/3! - (1+\kappa\lambda)(1+\kappa^2\lambda)\xi^4/5! + (1+\kappa\lambda)[(1+\kappa^2\lambda)^2 \\ +5(1+\kappa\lambda)(1+\kappa^3\lambda)/3]\xi^6/7! - ..., \qquad (\xi\approx 0;\ n=\pm\infty), \tag{2.8.65}$$

where the one-component result (2.4.39) is recovered if $\lambda = 0$. To obtain the asymptotic form of the solution of Eq. (2.8.61), we observe that if $\kappa > 1$, the second exponential decreases more rapidly than the first one, so the solution should approach for $\xi \gg 1$ the form of the one-component solution for the lighter particles. Physically, this is caused by the fact that in hydrostatic equilibrium the heavier particles are concentrated towards the centre. We apply therefore the procedure for the one-component gas, inserting Emden's transformation (2.2.25), (2.2.30) into Eq. (2.8.61), and taking into account Eqs. (2.2.27), (2.2.31):

$$d^2z/dt^2 - dz/dt - 2 + \exp z + \kappa\lambda\exp[2(\kappa-1)t+\kappa z] = 0. \tag{2.8.66}$$

Eq. (2.8.66) becomes near the singular solution $z_s = \ln 2$ equal to [see Eqs. (2.4.95)-(2.4.105)]

$$dz_1^2/dt^2 - dz_1/dt + 2z_1 = -2^\kappa\kappa\lambda\exp[2(\kappa-1)t], \tag{2.8.67}$$

where $z = \ln 2 + z_1$, $(z_1 \ll z)$, and we have restricted to first order terms. The solution of this inhomogeneous equation with constant coefficients is (cf. Eq. (2.4.101) for the one-component case)

$$z_1 = C_1\exp(t/2)\cos(7^{1/2}t/2 + C_2) - [2^{\kappa-1}\kappa\lambda/(2\kappa^2-5\kappa+4)]\exp[2(\kappa-1)t], \qquad (t\to-\infty). \tag{2.8.68}$$

Turning back to the Lane-Emden type variable, we find

$$\theta = \ln\xi^2 - z = \ln(\xi^2/2) - z_1 = \ln(\xi^2/2) - C_1\xi^{-1/2}\cos[(7^{1/2}/2)\ln\xi - C_2] \\ +[2^{\kappa-1}\kappa\lambda/(2\kappa^2-5\kappa+4)]\xi^{2(1-\kappa)}, \qquad (\xi\to\infty;\ n=\pm\infty). \tag{2.8.69}$$

An interesting property of this two-component model occurs if the ratio of particle masses is sufficiently large ($\kappa > 1.5$). In this case the heavier particles are concentrated so much towards the centre that their mass M_2 is finite, though the total mass M of the two-component isothermal sphere is infinite by virtue of Eqs. (2.8.63), (2.8.69), as in the case of the one-component sphere. We have

$$M_2 = 4\pi \int_0^{r_2} \varrho_2 r^2 \, dr = 4\pi\alpha^3 \varrho_{10}\kappa\lambda \int_0^{\xi_2} \exp(-\kappa\theta)\xi^2 \, d\xi. \tag{2.8.70}$$

The part of M_2 contained between ξ_1 and ξ_2, $(\xi_1, \xi_2 \gg 1)$ is equal to

$$\Delta M_2 = 4\pi\alpha^3\varrho_{10}\kappa\lambda \int_{\xi_1}^{\xi_2} \exp(-\kappa\theta)\xi^2 \, d\xi \approx 4\pi\alpha^3\varrho_{10}\kappa\lambda \int_{\xi_1}^{\xi_2} \exp\{\kappa[z_1 - \ln(\xi^2/2)]\}\xi^2 \, d\xi$$
$$= 2^{\kappa+2}\pi\alpha^3\varrho_{10}\kappa\lambda \int_{\xi_1}^{\xi_2} \exp(\kappa z_1)\xi^{2-2\kappa} \, d\xi. \tag{2.8.71}$$

If $\xi_2 \to \infty$, this integral converges if $2 - 2\kappa < -1$ or $\kappa > 3/2$. Thus, ΔM_2 and consequently M_2 approach a finite limit if $\kappa > 3/2$.

We use Eqs. (2.8.59), (2.8.62), (2.8.63) to get the Milne variables from Eq. (2.7.8):

$$u = (r/M)\, dM/dr = \xi[\exp(-\theta) + \kappa\lambda\exp(-\kappa\theta)]/\theta';$$
$$v = -(r/P)\, dP/dr = \xi\theta'[\exp(-\theta) + \kappa\lambda\exp(-\kappa\theta)]/[\exp(-\theta) + \lambda\exp(-\kappa\theta)]. \tag{2.8.72}$$

These equations for the two-component sphere lack the homology invariance found for the one-component sphere. The related differential equations for u and v are obtained via Eq. (2.8.61):

$$d\ln u/d\ln\xi = -u - vw + 3; \quad d\ln v/d\ln\xi = u + v(1-w) - 1, \tag{2.8.73}$$

where

$$w = [\exp(-\theta) + \kappa^2\lambda\exp(-\kappa\theta)][\exp(-\theta) + \lambda\exp(-\kappa\theta)]/[\exp(-\theta) + \kappa\lambda\exp(-\kappa\theta)]^2. \tag{2.8.74}$$

Using the expansion (2.8.65) of the E-solution near the origin $\xi \approx 0$, we get

$$u \approx 3 - \xi^2(1 + \kappa^2\lambda)/5; \quad v \approx \xi^2(1 + \kappa\lambda)^2/3(1 + \lambda), \tag{2.8.75}$$

similarly to Eq. (2.7.127) in the one-component case. The E-solution of the two-component isothermal sphere joins the two singular points $U_s(3,0)$, $(\xi \to 0)$ and $G_s(1,2)$, $(\xi \to \infty;\ w \to 1;\ \kappa > 1)$ of the differential equation obtained from Eq. (2.8.73):

$$dv/du = v[u + v(1-w) - 1]/u(-u - vw + 3). \tag{2.8.76}$$

Opposite to the one-dimensional case, the E-solution exhibits self-intersections (see Taff et al. 1975, Fig. 4).

Two-component spheres with $n = \pm\infty$ provide idealized systems of two-component spherical star clusters for instance, whereby the macroscopic physics can be studied in principle with arbitrary high precision, complementing thus numerical stellar dynamics calculations (see also Eqs. (6.1.201)-(6.1.249) and Sec. 6.2.4 for further applications with $n \neq \pm\infty$).

2.8.3 Loaded Polytropes

Such polytropes have been envisaged by Huntley and Saslaw (1975) to study the distribution of stars in galactic nuclei, the massive "load" in the centre being an object with a certain constant mass M_0 (e.g. a black hole or supermassive star). Eq. (2.1.2) reads

$$dP/dr = \varrho\vec{F}, \tag{2.8.77}$$

where $\vec{F}$ is the radial gravitational force that acts on the unit of mass. The force $\vec{F}$ is composed of the gravitational force $-GM_0/r^2$ of the central load plus the force $-GM(r)/r^2$ arising from the mass

$$M(r) = \int_{r_0}^{r} 4\pi\varrho r'^2\, dr', \tag{2.8.78}$$

contained between r_0 and r. The radius r_0 is the inner border of a polytropic distribution of matter. If $M_0, r_0 = 0$, the loaded polytrope becomes an ordinary polytrope. We insert for $\vec{F}$ into Eq. (2.8.77):

$$dP/dr = -(G\varrho/r^2)[M_0 + M(r)]. \tag{2.8.79}$$

Expressing r, ϱ, P, and M in the dimensionless polytropic variables from Eqs. (2.6.1), (2.6.3), (2.6.18), we can write Eq. (2.8.79) under the form

$$\xi^2\, d\theta/d\xi = \mp\beta \mp \int_{\xi_0}^{\xi} \theta^n \xi'^2\, d\xi', \qquad (\xi \geq \xi_0;\ n \neq -1, \pm\infty), \tag{2.8.80}$$

where $\beta = M_0/4\pi\varrho_0\alpha^3$, $\xi_0 = r_0/\alpha$, and α is given by Eq. (2.6.1). Huntley and Saslaw (1975) set $\theta(\xi_0) = 1$, while Eq. (2.8.80) yields $\theta'(\xi_0) = \mp\beta/\xi_0^2$ as the second boundary condition.

If $n = \pm\infty$ (isothermal sphere in the case of a perfect gas), we get similarly to Eq. (2.8.80)

$$\xi^2\, d\theta/d\xi = \beta + \int_{\xi_0}^{\xi} \exp(-\theta)\xi'^2\, d\xi', \qquad (\xi \geq \xi_0;\ n = \pm\infty), \tag{2.8.81}$$

where α is now given by Eq. (2.6.2), and r, ϱ, P, M from Eq. (2.8.79) are substituted via Eqs. (2.6.2), (2.6.4), (2.6.19). The boundary conditions at $\xi = \xi_0$ are equal to $\theta(\xi_0) = 0,\ \theta'(\xi_0) = \beta/\xi_0^2$.

Loaded polytropes satisfy the familiar Lane-Emden equations (2.1.14) and (2.1.21), as can be seen at once by derivation of Eqs. (2.8.80) and (2.8.81), respectively. Note however, that the initial conditions are different from Eq. (2.1.41). Simple analytic solutions can be found for the polytropic indices $n = 0, 1$, while the general Schuster-Emden integral $n = 5$ generally involves elliptic integrals if $C \neq 0$ in Eq. (2.3.30). Ritter's first integral is given for arbitrary boundary conditions by Eq. (2.3.3):

$$\theta = -\xi^2/6 + C_1/\xi + C_2, \qquad (n = 0;\ N = 3;\ C_1, C_2 = \text{const}). \tag{2.8.82}$$

And in the case $n = 1$ we get from Eqs. (2.3.10), (2.3.15), (2.3.18) the general solution (Chandrasekhar 1939, Smirnow 1967)

$$\theta = C_1\xi^{-1/2}J_{1/2}(\xi) + C_2\xi^{-1/2}J_{-1/2}(\xi) = C_1'\sin\xi/\xi + C_2'\cos\xi/\xi = A\sin(\xi + B)/\xi,$$
$$(n = 1;\ N = 3;\ A, B, C_1, C_2, C_1', C_2' = \text{const}). \tag{2.8.83}$$

With the initial conditions $\theta(\xi_0) = 1,\ \theta'(\xi_0) = -\beta/\xi_0^2$ these equations become the analytic solutions for loaded polytropes in the particular cases $n = 0, 1$:

$$\theta = 1 - (\xi^2 - \xi_0^2)/6 + (\beta - \xi_0^3/3)(1/\xi - 1/\xi_0), \qquad (n = 0;\ \xi \geq \xi_0), \tag{2.8.84}$$

$$\theta = (\xi_0\sin\xi_0 - \beta\cos\xi_0/\xi_0 + \cos\xi_0)\sin\xi/\xi + (\xi_0\cos\xi_0 + \beta\sin\xi_0/\xi_0 - \sin\xi_0)\cos\xi/\xi,$$
$$(n = 1;\ \xi \geq \xi_0). \tag{2.8.85}$$

Since the virial theorem quoted by Huntley and Saslaw [1975, Eq. (29)] cannot be recovered from the equations shown in Secs. 2.6.6, 2.6.7, their conclusion that stable loaded polytropes exist only if $n > 2$ appears inadequate.

Loaded polytropes appear as a more elementary kind of composite polytropes, the core of undefined structure being determined merely by its mass M_0 and radius r_0, while the envelope is equivalent to that of a composite polytrope.

2.8.4 Boltzmann Factor Polytropes

Callebaut et al. (1982) argue that the simple polytropic law should be improved by addition of a so-called Boltzmann factor. The density distribution of a perfect isothermal gas due to the potential Φ of an arbitrary force field is given by the Boltzmann law (e.g. Eq. (2.8.56), Feynman et al. 1965)

$$\varrho = \varrho_0 \exp[\mu(\Phi - \Phi_0)/\mathcal{R}T], \qquad (T = \mathrm{const};\ n = \pm\infty), \tag{2.8.86}$$

where Φ_0 is the force potential when the density ϱ takes some fixed initial value ϱ_0. If we insert from Eq. (1.4.11) the perfect gas law with radiation pressure included

$$P = P_g + P_r = \mathcal{R}\varrho T/\mu + aT^4/3, \tag{2.8.87}$$

into the hydrostatic equation (2.1.3) for an isothermal sphere

$$\nabla P/\varrho = \nabla\Phi, \qquad (T = \mathrm{const}), \tag{2.8.88}$$

we recover by integration just Eq. (2.8.86), where Φ now means the gravitational potential of a sphere. This is one of the reasons that Callebaut et al. (1982) argue for a density law being a combination of the exact isothermal solution (2.8.86) and the familiar polytropic law from Eq. (1.2.29):

$$\varrho = \varrho_0 (T/T_0)^n \exp[\mu(\Phi - \Phi_0)/\mathcal{R}T]. \tag{2.8.89}$$

We substitute this proposed equation of state into the equation of state (2.8.87), and calculate the expression

$$\nabla P/\varrho = [(n+1)\mathcal{R}/\mu - (\Phi - \Phi_0)/T + 4aT^3/3\varrho]\ \nabla T + \nabla\Phi. \tag{2.8.90}$$

This equation is compared with the general equation of hydrostatic equilibrium (2.1.3) $\nabla P/\varrho = \nabla\Phi$. The first term on the right-hand side of Eq. (2.8.90) must be zero, or equivalently, either one of the two subsequent equations must be fulfilled:

$$\nabla T = 0, \tag{2.8.91}$$

$$(n+1)\mathcal{R}/\mu - (\Phi - \Phi_0)/T + 4aT^3/3\varrho = 0. \tag{2.8.92}$$

Eq. (2.8.91) is automatically fulfilled for the isothermal sphere ($T = \mathrm{const};\ n = \pm\infty$), while Eq. (2.8.92) has to be zero for all nonisothermal perfect gas spheres. The density in the latter case is obtained by inserting Eq. (2.8.92) into Eq. (2.8.89):

$$\varrho = \varrho_0 (T/T_0)^n \exp(n + 1 + 4a\mu T^3/3\mathcal{R}\varrho), \qquad (n \neq \pm\infty). \tag{2.8.93}$$

Callebaut et al. (1982) introduce two additional parameters

$$\sigma = P_g/4P_r = 3\mathcal{R}\varrho/4a\mu T^3; \quad \tau^{n-3} = 4a\mu T_0^n/3\mathcal{R}\varrho_0 \exp(n+1), \tag{2.8.94}$$

and Eq. (2.8.93) becomes

$$\sigma \exp(-1/\sigma) = (T/\tau)^{n-3}. \tag{2.8.95}$$

For a general equation of state the polytropic index n' has been defined through Eq. (1.3.26):

$$1 + n' = d\ln P/d\ln T. \tag{2.8.96}$$

Analogously, Callebaut et al. (1982) get the polytropic index for a polytropic equation of state with a Boltzmann factor

$$1 + n'_B = d\ln P/d\ln T = (d\ln P/d\sigma)\big/(d\ln T/d\sigma) = 1 + 3(1 + 5\sigma + 5n\sigma^2/3)\big/(1 + 5\sigma + 4\sigma^2), \tag{2.8.97}$$

where the physical variables can be expressed with the aid of Eqs. (2.8.87), (2.8.94), (2.8.95) as

$$T = \tau\sigma^{1/(n-3)}\exp[-1/(n-3)\sigma], \tag{2.8.98}$$

$$\varrho = (4a\mu\tau^3/3\mathcal{R})\sigma^{n/(n-3)}\exp[-3/(n-3)\sigma], \tag{2.8.99}$$

$$P = [a\tau^4(1+4\sigma)\sigma^{4/(n-3)}/3]\exp[-4/(n-3)\sigma]. \tag{2.8.100}$$

From Eq. (2.8.97) it appears that the polytropic index n'_B of Boltzmann factor polytropes is contained approximately between 3, $(\sigma = 0)$ and the polytropic index n. If $\sigma \gg 1$, we have $n'_B = 5n/4 \approx n$. For a standard solar-type numerical model of a zero-age main sequence star Callebaut et al. (1982) find that n'_B changes between 2.4 and 3.5, whereas n' would change between 1.6 and 4.5 in the temperature range $2\times10^6 - 2\times10^7$ K, yielding thus a better fit for Boltzmann factor models approximated by $n'_B = \mathrm{const.}$

Note, that the Boltzmann factor from Eq. (2.8.89) proposed by Callebaut et al. (1982) applies to spherical polytropes $(N = 3,\ 0 \le n < 5)$, being introduced in a somewhat heuristic manner, whereas the polytropic equation of state (Chap. 1) arises from self-consistent physical principles.

2.8.5 The Emden-Fowler Equation

The Emden-Fowler equation (2.8.105) is of a more general type than the spherical Lane-Emden equation (2.3.87). It results from the spherical stationary distribution function over phase space (phase density function), (Eddington 1916, Hénon 1973):

$$f(\vec r,\vec v) = \begin{cases} C'(H_1 - H)^{n-3/2}J^{2m} & \text{if } H < H_1 \\ 0 & \phantom{\text{if }} H \ge H_1 \end{cases}$$

$$(n > 1/2;\ m > -1;\ H_1, C' = \mathrm{const}). \tag{2.8.101}$$

The energy constant H from the energy integral is [cf. Eq. (6.1.188)]

$$H = v^2/2 - \Phi(\vec r) = (v_r^2 + v_t^2)/2 - \Phi(\vec r) = \mathrm{const}, \tag{2.8.102}$$

where $\Phi(\vec r)$ is the gravitational potential at radius vector $\vec r$, and v_r, v_t are the radial and transversal components of the velocity $\vec v$. The squared angular momentum per unit mass is denoted by $J^2 = r^2v^2\sin^2\lambda = r^2v_t^2$, where λ is the angle between $\vec r$ and $\vec v$. The choice (2.8.101) for the distribution function results from the fact that in systems with spherical symmetry f is merely a function of two integrals of motion – the energy and the angular momentum: $f = f(H, J^2)$, (Ogorodnikov 1965).

The spatial density distribution of particles is obtained by integration of the distribution function over velocity space with "radial" coordinate v, polar angle λ, and azimuth coordinate φ (Batt and Pfaffelmoser 1988):

$$\varrho(\vec r) = \int_0^\infty dv\int_0^\pi d\lambda\int_0^{2\pi} f\ v^2\sin\lambda\ d\varphi = 4\pi C' r^{2m}\int_0^{v_1} dv\int_0^{\pi/2}(H_1 + \Phi - v^2/2)^{n-3/2}v^{2m+2}$$
$$\times\sin^{2m+1}\lambda\ d\lambda = 2^{3/2-n}\pi C' r^{2m}v_1^{2n+2m}\int_0^1 x^{m+1/2}(1-x)^{n-3/2}\ dx\int_0^1 y^m(1-y)^{-1/2}\ dy$$
$$= 2^{m+3/2}\pi C'\ B(m+3/2, n-1/2)\ B(m+1, 1/2)\ r^{2m}(H_1+\Phi)^{n+m} = Cr^{2m}(H_1+\Phi)^{n+m},$$
$$(n > 1/2;\ m > -1;\ \Phi > -H_1;\ v_1 = (2H_1 + 2\Phi)^{1/2};\ x = v^2/v_1^2;\ y = \sin^2\lambda). \tag{2.8.103}$$

For the limiting exponents of the beta function (2.3.56), viz. $n = 1/2$ and $m = -1$, the distribution function (2.8.101) is expressed with the aid of the Dirac function (5.10.99): $f = C'J^{2m}\ \delta_D(H_1 - H)$ if $n = 1/2$, and $f = C'(H_1 - H)^{n-3/2}\ \delta_D(J^2)$ if $m = -1$ (Eqs. (6.1.189)-(6.1.198), Hénon 1973, Barnes et al. 1986).

The equivalents of the Lane-Emden variables are introduced by

$$r = [(H_1+\Phi_0)^{1-n-m}/4\pi GC]^{1/(2m+2)}\xi = \beta\xi; \quad \theta(\xi) = [H_1+\Phi(r)]/(H_1+\Phi_0). \tag{2.8.104}$$

The potential at the centre $r = 0$ is $\Phi(0) = \Phi_0$, while $\Phi(r_1) = \Phi_1 = -H_1$ denotes the surface potential at $r = r_1$ where $\theta(\xi_1) = 0$. Poisson's equation (2.1.4) turns in radial spherical coordinates via Eqs. (2.8.103), (C.15) into the Emden-Fowler equation

$$d(\xi^2\ d\theta/d\xi)/d\xi = -\xi^{2m+2}\theta^{n+m}, \qquad (0.5 < n < \infty;\ m > -1;\ \varrho \propto \xi^{2m}\theta^{n+m}). \tag{2.8.105}$$

This is equal to the Lane-Emden equation (2.3.87) if $m = 0$. The initial conditions at the centre are $\theta(0) = 1$ and $\theta'(0) = 0$, as results from the series expansion near the origin, which may be found in a similar way as in Sec. 2.4.1:

$$\theta \approx 1 - \xi^{2m+2}/(2m+2)(2m+3), \qquad (\xi \approx 0). \tag{2.8.106}$$

The mass inside radius r follows from a first integration of Poisson's equation [cf. Eq. (2.1.34)], where $(d\Phi/dr)_{r=0} = [(H_1+\Phi_0)/\beta](d\theta/d\xi)_{\xi=0} = 0$ via Eq. (2.8.104):

$$r^2\ d\Phi/dr = -4\pi G\int_0^r \varrho r'^2\ dr' = -GM \quad \text{or} \quad M = -[\beta(H_1+\Phi_0)/G]\xi^2\ d\theta/d\xi. \tag{2.8.107}$$

The total gravitational (potential) energy (2.6.128) becomes via Eqs. (2.8.103)-(2.8.105) after integrations by parts equal to

$$W_1 = \int_{M_1} \vec{r}\cdot\nabla\Phi\ dM = G^{-1}\beta(H_1+\Phi_0)^2\int_0^{\xi_1}\xi^{2m+3}\theta^{n+m}\theta'\ d\xi$$
$$= -[G^{-1}\beta(H_1+\Phi_0)^2(2m+3)/(n+m+1)]\int_0^{\xi_1}\xi^{2m+2}\theta^{n+m+1}\ d\xi = [G^{-1}\beta(H_1+\Phi_0)^2(2m+3)$$
$$/(n+m+1)]\int_0^{\xi_1}\theta\ d(\xi^2\theta') = -[G^{-1}\beta(H_1+\Phi_0)^2(2m+3)/(n+m+1)]\int_0^{\xi_1}\xi^2\theta'^2\ d\xi. \tag{2.8.108}$$

The integral from this equation can also be evaluated similarly to Eq. (2.6.131):

$$\int_0^{\xi_1}\xi^{2m+3}\theta^{n+m}\theta'\ d\xi = -\int_0^{\xi_1}\xi\theta'\ d(\xi^2\theta') = -\int_0^{\xi_1}\xi^2\theta'\ d(\xi^2\theta')/\xi = -\xi_1^3{\theta_1'}^2/2 - (1/2)\int_0^{\xi_1}\xi^2\theta'^2\ d\xi. \tag{2.8.109}$$

Comparing Eqs. (2.8.108) and (2.8.109) we get $\int_0^{\xi_1}\xi^2\theta'^2\ d\xi = (n+m+1)\xi_1^3{\theta_1'}^2/(3m-n+5)$, and the total gravitational energy (2.8.108) writes as

$$W_1 = -G^{-1}\beta(H_1+\Phi_0)^2(2m+3)\xi_1^3{\theta_1'}^2/(3m-n+5) = -(2m+3)GM_1^2/(3m-n+5)r_1. \tag{2.8.110}$$

This reduces to Eq. (2.6.137) if $m = 0$. The homology invariant transformation (Broek and Verhulst 1982)

$$u = -\xi^{2m+1}\theta^{n+m}/\theta'; \quad v = -\xi\theta'/\theta, \tag{2.8.111}$$

reduces Eq. (2.8.105) to the first order system (cf. Eqs. (2.7.1)-(2.7.3) if $m = 0,\ N = 3;\ 0.5 < n < \infty$) :

$$du/d\ln\xi = u[-u-(n+m)v+2m+3]; \quad dv/d\ln\xi = v(u+v-1). \tag{2.8.112}$$

The Emden-Fowler equation has applications to the study of collisionless, stationary, spherical stellar and galactic systems (Sec. 6.1.9, Hénon 1973, Barnes et al. 1986, Kandrup et al. 1994). Such systems appear to be essentially stable, as shown among others by Wolansky (1999), Guo (1999), Guo and Rein (2001).

3 DISTORTED POLYTROPES

3.1 Introduction

While it seems possible to present a fairly complete unified theory of undistorted polytropes, as attempted in the previous chapter, the theory of distorted polytropes is much more extended and sophisticated, so that I present merely a brief overview of the theories that seem to me most interesting and important. Basically, the methods proposed to study the hydrostatic equilibrium of a distorted self-gravitating mass can be divided into two major groups (Blinnikov 1975): (i) Analytic or semianalytic methods using a small parameter connected with the distortion of the polytrope. (ii) More or less accurate numerical methods.

Lyapunov and later Carleman (see Jardetzky 1958, p. 13) have demonstrated that a sphere is a unique solution to the problem of hydrostatic equilibrium for a fluid mass at rest in tridimensional space. The problem complicates enormously if the sphere is rotating rigidly or differentially in space round an axis, and/or if it is distorted magnetically or tidally. Even for the simplest case of a uniformly rotating fluid body with constant density not all possible solutions have been found (Zharkov and Trubitsyn 1978, p. 222). The sphere becomes an oblate figure, and we have no a priori knowledge of its stratification, boundary shape, planes of symmetry, transfer of angular momentum in differentially rotating bodies, etc. For a general equation of state the isobaric (constant pressure) surfaces and the isopycnic (constant density) surfaces of a distorted configuration are generally inclined, and cut themselves at a certain angle; in this case the material system is called a barocline (lit. "inclined over the pressure"). If the equation of state is of the special form

$$P = P(\varrho), \tag{3.1.1}$$

the system is called a barotrope (lit. "behaving as the pressure"). Obviously, in a barotrope the surfaces of equal pressure and density coincide, since $\varrho = \text{const}$ implies $P = \text{const}$ in Eq. (3.1.1), and vice versa. Of course, the polytropic equation of state (2.1.5) is a special case of a barotrope.

We neglect at first the influence of tides, viscosity, internal motions, magnetic fields, and confine ourselves to configurations rotating with angular velocity $\vec{\Omega}$ round a fixed axis directed along the z-direction. In cylindrical coordinates ℓ, φ, z – as seen in an inertial frame of reference with the origin in the centre of mass of the configuration – the equation of motion (2.1.1) of the rotating fluid writes (cf. Eqs. (B.43)-(B.51); $v_\ell, v_z = 0;\ v_\varphi = \Omega\ell$) :

$$\begin{cases} \partial P/\partial\ell - \varrho\ \partial\Phi/\partial\ell = \varrho\Omega^2\ell \\ (1/\ell)\ \partial P/\partial\varphi - (\varrho/\ell)\ \partial\Phi/\partial\varphi = -\varrho\ \partial(\Omega\ell)/\partial t - \varrho\Omega\ \partial(\Omega\ell)/\partial\varphi. \\ \partial P/\partial z - \varrho\ \partial\Phi/\partial z = 0 \end{cases} \tag{3.1.2}$$

If the motion is also stationary (independent of time), and the density of each mass element remains constant along its path, the rotation is said to be permanent (cf. Tassoul 1978). In the case of permanent rotation Eq. (3.1.2) simplifies further. The equation of continuity

$$\partial\varrho/\partial t + \nabla\cdot(\varrho\vec{v}) = \partial\varrho/\partial t + \vec{v}\cdot\nabla\varrho + \varrho(\nabla\cdot\vec{v}) = D\varrho/Dt + \varrho(\nabla\cdot\vec{v}) = 0, \tag{3.1.3}$$

reduces to

$$\varrho(\nabla\cdot\vec{v}) = 0, \tag{3.1.4}$$

because $D\varrho/Dt = 0$ in virtue of the assumption of permanent rotation. Since $v_\ell, v_z = 0$, Eq. (3.1.4) amounts to

$$(\varrho/\ell)\ \partial(\Omega\ell)/\partial\varphi = 0 \quad \text{or} \quad \Omega = \Omega(\ell, z). \tag{3.1.5}$$

Since the configuration is in a state of permanent rotation, i.e. stationary, we have $\partial\varrho/\partial t = 0$, and according to Eqs. (3.1.3) and (3.1.4)

$$\vec{v} \cdot \nabla\varrho = 0. \tag{3.1.6}$$

Because the gradient $\nabla\varrho$ is directed along the exterior normal to the equidensity (isopycnic) surface, the velocity $\vec{v}$ is perpendicular to $\nabla\varrho$, i.e. $\vec{v}$ is in the tangent plane to the equidensity surface. The sole nonzero component of $\vec{v}$ is $v_\varphi = \Omega\ell$, so Eq. (3.1.6) becomes

$$\Omega\ \partial\varrho/\partial\varphi = 0, \tag{3.1.7}$$

and since $\Omega \neq 0$, we get $\varrho = \varrho(\ell, z)$. Because the density ϱ of a mass element is invariant, and the velocities are independent of φ (as seen from the inertial frame), the gravitational potential Φ is also independent of $\varphi:\ \Phi = \Phi(\ell, z)$. The second equation (3.1.2) becomes for the state of permanent rotation equal to $\partial P/\partial\varphi = \varrho\ \partial\Phi/\partial\varphi = 0$, and therefore $P = P(\ell, z)$. Thus, in permanent rotation the physical parameters of the rotating configuration possess axial symmetry. As will be shown in Sec. 3.8.1, it is possible to build steady distorted three-axial polytropes for which axial symmetry ceases when the polytropic index is $0 \leq n < 0.808$ (Jeans 1919, James 1964). However, these equilibrium configurations are time independent only in a frame rotating together with the polytrope, rather than in an inertial frame, so the basic assumption of permanent rotation does not apply to these triaxial polytropes. Thus, for the case of permanent rotation Eq. (3.1.2) simplifies to

$$\partial P/\partial\ell = \varrho\ \partial\Phi/\partial\ell + \varrho\Omega^2(\ell, z)\ \ell; \quad \partial P/\partial z = \varrho\ \partial\Phi/\partial z. \tag{3.1.8}$$

For permanent rotation the condition on Ω can be further strengthened, provided that isobaric and isopycnic surfaces coincide. If we eliminate the gravitational potential between the equations (3.1.8), we obtain

$$\partial(\Omega^2\ell)/\partial z = [\partial(1/\varrho)/\partial z]\ \partial P/\partial\ell - [\partial(1/\varrho)/\partial\ell]\ \partial P/\partial z, \tag{3.1.9}$$

or, since $\vec{v} = \vec{v}(0, \Omega\ell, 0)$:

$$\partial(\Omega^2\ell)/\partial z = 2\Omega\ell\ \partial\Omega/\partial z = \nabla(1/\varrho) \times \nabla P = -(1/\varrho^2)\ \nabla\varrho \times \nabla P. \tag{3.1.10}$$

The isobaric and isopycnic surfaces determined by ∇P and $\nabla\varrho$ coincide (their vector product will be zero), if and only if

$$\partial\Omega/\partial z = 0 \quad \text{or} \quad \Omega = \Omega(\ell). \tag{3.1.11}$$

This condition, namely that the angular velocity is constant over cylinders centered about the rotation axis, is automatically satisfied for barotropes in permanent rotation (including of course polytropes).

For solid-body rotation of a polytrope it will be often useful to write down the equilibrium equation in a system with the origin in the centre of mass of the configuration, rotating rigidly with the polytrope at angular velocity $\vec{\Omega} = \vec{\Omega}(t)$. In a coordinate system moving with respect to an inertial frame, the equation of motion (2.1.1) turns into (e.g. Landau and Lifshitz 1960)

$$\varrho\ D\vec{v}/Dt = -\varrho\ d\vec{v}_{tr}/dt - \varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - \varrho\ (d\vec{\Omega}/dt) \times \vec{r} - 2\varrho\ \vec{\Omega} \times \vec{v} - \nabla P + \varrho\vec{F}$$
$$+(1/4\pi)(\nabla \times \vec{H}) \times \vec{B} + \nabla \cdot \tau. \tag{3.1.12}$$

A mass element of density ϱ has the radius vector $\vec{r}$ and the velocity $\vec{v}$ with respect to the moving frame, while $\vec{v}_{tr}$ denotes the translational velocity, and $\vec{\Omega} = \vec{\Omega}(t)$ the instantaneous angular velocity vector of the moving frame with respect to the inertial one. The nabla operator now acts with respect to the moving frame. Eq. (3.1.12) can easily be deduced from Eq. (2.1.1) with the aid of the transformation formula for the acceleration in two frames:

$$D\vec{v}_0/Dt = D\vec{v}/Dt + d\vec{v}_{tr}/dt + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + (d\vec{\Omega}/dt) \times \vec{r} + 2\vec{\Omega} \times \vec{v}. \tag{3.1.13}$$

$D\vec{v}_0/Dt$ is the acceleration of the mass element with respect to the inertial frame. If we insert $D\vec{v}_0/Dt$ from Eq. (3.1.13) into Eq. (2.1.1), we recover just Eq. (3.1.12). In a Cartesian (x_1, x_2, x_3)-frame, rotating uniformly with the polytrope round the x_3-axis, the equation of hydrostatic equilibrium of a polytropic fluid – without energy dissipation and magnetic fields – can be written down at once from Eq. (3.1.12), $(\vec{v}, \vec{v}_{tr}, \vec{B}, \vec{H}, \tau = 0;\ \vec{\Omega} = \vec{\Omega}(0,0,\Omega) = \mathrm{const};\ \vec{F} = \nabla\Phi)$:

$$\nabla P = \varrho\ \nabla\Phi - \varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}), \tag{3.1.14}$$

or explicitly

$$\begin{cases} \partial P/\partial x_1 = \varrho\ \partial\Phi/\partial x_1 + \varrho\Omega^2 x_1 \\ \partial P/\partial x_2 = \varrho\ \partial\Phi/\partial x_2 + \varrho\Omega^2 x_2, \qquad (\Omega = \mathrm{const}). \\ \partial P/\partial x_3 = \varrho\ \partial\Phi/\partial x_3 \end{cases} \tag{3.1.15}$$

Eq. (3.1.14) can also be written in right-handed spherical (r, λ, φ)-coordinates, where r is the radial distance from the origin located in the centre of mass of the polytrope, λ the angle between rotation axis and radius vector (= polar angle or colatitude), and φ the azimuth angle. The Cartesian components $(\Omega^2 x_1, \Omega^2 x_2, 0)$ of the vectorial product $-\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ from Eq. (3.1.15) turn with the orthogonal transformation matrix (3.8.135) into the spherical components $(r\Omega^2 \sin^2\lambda, r\Omega^2 \sin\lambda\cos\lambda, 0)$. Eq. (3.1.14) becomes, by taking into account the expression (B.36) of the gradient in spherical coordinates:

$$\begin{cases} \partial P/\partial r = \varrho\ \partial\Phi/\partial r + \varrho r\Omega^2 \sin^2\lambda \\ \partial P/\partial\lambda = \varrho\ \partial\Phi/\partial\lambda + \varrho r^2\Omega^2 \sin\lambda\cos\lambda \\ \partial P/\partial\varphi = \varrho\ \partial\Phi/\partial\varphi \end{cases} \tag{3.1.16}$$

For axisymmetric bodies the φ-component vanishes, and with the notation $\mu = \cos\lambda$ we find

$$\partial P/\partial r = \varrho\ \partial\Phi/\partial r + \varrho r\Omega^2(1-\mu^2); \quad \partial P/\partial\mu = \varrho\ \partial\Phi/\partial\mu - \varrho r^2\Omega^2\mu. \tag{3.1.17}$$

For axial symmetry Poisson's equation (2.1.4) writes [cf. Eq. (B39)]

$$\partial(r^2\ \partial\Phi/\partial r)/\partial r + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu = -4\pi G\varrho r^2. \tag{3.1.18}$$

Inserting for Φ from Eq. (3.1.17), we deduce the fundamental equation of the rotational problem with axisymmetric rotation:

$$\partial[(r^2/\varrho)\ \partial P/\partial r]/\partial r + \partial\{[(1-\mu^2)/\varrho]\ \partial P/\partial\mu\}/\partial\mu = -4\pi G\varrho r^2 + 2\Omega^2 r^2. \tag{3.1.19}$$

We now briefly touch the question of an equatorial plane of symmetry in rotating configurations – a completely nontrivial problem, originally discussed by Lichtenstein, Wavre, and Dive. When the angular velocity in cylindrical coordinates does not depend on z, $(\partial\Omega/\partial z = 0)$, there exists always a plane of symmetry perpendicular to the Oz-axis of rotation. This condition is automatically fulfilled by barotropes in a state of permanent rotation (in particular by polytropes). For differentially rotating baroclines, when $\vec{\Omega} = \vec{\Omega}(\ell, z)$, the problem is more involved. Such configurations possess an equatorial plane of symmetry if the angular velocity Ω is a single valued function of ϱ and ℓ (e.g. Tassoul 1978).

A further interesting question concerning rotating barotropic configurations, raised first by Hamy, concerns the representation of isobaric surfaces by a set of concentric (homothetic) ellipsoids. It can be shown that the sole solutions are uniformly rotating ellipsoids of constant density throughout the configuration, with the pressure being a constant over concentric ellipsoids. Thus, a centrally condensed body cannot be rigorously represented by a set a concentric ellipsoids. However, the stratification in rotating barotropes can always be *approximated* by ellipsoidal surfaces (Secs. 3.7, 3.8.5, 5.7.4). The above result does not apply to genuine baroclines, where many models can be constructed having concentric ellipsoidal stratification, although realistic stellar models cannot be described by these simple means (Tassoul 1978).

In passing we also note an interesting inequality due to Poincaré. If we introduce in Eq. (3.1.8) the effective gravity

$$\vec{g} = \vec{g}(g_\ell, g_\varphi, g_z) = \vec{g}(\partial\Phi/\partial\ell + \Omega^2\ell,\ 0,\ \partial\Phi/\partial z), \tag{3.1.20}$$

we can write Eq. (3.1.8) under the form

$$(1/\varrho)\ \nabla P = \vec{g}. \tag{3.1.21}$$

By virtue of Eq. (3.1.8), the effective gravity can be derived from a total potential Φ_{tot}, $(\partial^2\Phi_{tot}/\partial\ell\partial z = \partial^2\Phi_{tot}/\partial z\partial\ell)$ if and only if Ω does not depend on z. In this case we have

$$\vec{g} = \nabla\Phi_{tot} = \nabla\Big[\Phi + \int \Omega^2(\ell)\ \ell\ d\ell\Big]. \tag{3.1.22}$$

Such an equation is automatically fulfilled for barotropes rotating differentially or uniformly [see Eqs. (3.1.10), (3.1.11)]. Eq. (3.1.22) writes

$$\Phi_{tot} = \Phi + \Omega^2\ell^2/2, \qquad (\Omega = \mathrm{const}), \tag{3.1.23}$$

for uniform rotation. Summation of the components of Eq. (3.1.21) yields

$$\begin{aligned}&(1/\varrho)\ [(\partial P/\partial\ell)\ d\ell + (\partial P/\partial z)\ dz] = (1/\varrho)\ dP = g_\ell\ d\ell + g_z\ dz\\ &= (\partial\Phi_{tot}/\partial\ell)\ d\ell + (\partial\Phi_{tot}/\partial z)\ dz = d\Phi_{tot}.\end{aligned} \tag{3.1.24}$$

By definition, on a level surface we have $\Phi_{tot} = \mathrm{const}$, so $d\Phi_{tot} = 0$, and Eq. (3.1.24) shows that at the same time $dP = 0$, or $P = \mathrm{const}$. Thus, the level surfaces coincide with the isobaric surfaces, when a potential Φ_{tot} of the effective gravity exists. Since $d\Phi_{tot}/dP = 1/\varrho$, the density is constant too over the level surface $\Phi_{tot} = \mathrm{const}$; isobaric, isopycnic, and level surfaces all coincide, and the vectors $\nabla P,\ \nabla\varrho,\ \vec{g} = \nabla\Phi_{tot}$ are all normal to the level surface $\Phi_{tot} = \mathrm{const}$.

We apply the Laplacian operator to Eq. (3.1.23), and obtain by virtue of Poisson's equation (2.1.4)

$$\nabla^2\Phi_{tot} = -4\pi G\varrho + 2\Omega^2, \qquad (\nabla^2(\ell^2) = (1/\ell)\ d(\ell\ d\ell^2/d\ell)/d\ell = 4). \tag{3.1.25}$$

We integrate this equation over the entire volume V, to find

$$\int_V \nabla^2\Phi_{tot}\ dV = \int_S \nabla\Phi_{tot}\cdot d\vec{S} = \int_S (\nabla\Phi_{tot}\cdot\vec{n})\ dS = \int_S (\partial\Phi_{tot}/\partial n)\ dS = -4\pi GM + 2\Omega^2 V, \tag{3.1.26}$$

where M is the mass inside volume V. The surface of the configuration is a level surface $\Phi_{tot} = \mathrm{const}$, with the effective gravity vector $\vec{g} = \nabla\Phi_{tot}$ pointing throughout towards the direction of the inner normal. Since $\vec{n}$ is by definition the outer normal to the surface S, and $\vec{g} = \nabla\Phi_{tot}$ is directed along the inner normal, we conclude that the surface integral in Eq. (3.1.26) is strictly negative, i.e.

$$\int_S (\nabla\Phi_{tot}\cdot\vec{n})\ dS = \int_S (\vec{g}\cdot\vec{n})\ dS = -\int_S |\vec{g}|\ dS < 0 \quad \text{or} \quad \Omega^2 < 2\pi G\varrho_m, \tag{3.1.27}$$

where $\varrho_m = M/V$ is the average density of the configuration. Quilghini improved Poincaré's inequality (3.1.27) to (Tassoul 1978)

$$\Omega^2 < \pi G\varrho_m. \tag{3.1.28}$$

This limit is still well above the limiting angular velocity of uniformly rotating, homogeneous biaxial ellipsoids (Maclaurin ellipsoids of polytropic index $n = 0$, Sec. 3.2): $\Omega^2 \approx 0.45\pi G\varrho_m$. Biaxial revolution ellipsoids are sometimes also called spheroids, although a spheroid means a distorted sphere, so revolution ellipsoids are merely a particular case of spheroids (Zharkov and Trubitsyn 1978).

For a differentially rotating barotrope we get from Eq. (3.1.22)

$$\Phi_{tot} = \Phi + \int \Omega^2(\ell)\ \ell\ d\ell + \mathrm{const}, \tag{3.1.29}$$

or

$$\nabla^2\Phi_{tot} = -4\pi G\varrho + 2\Omega^2 + \ell\ d\Omega^2/d\ell, \tag{3.1.30}$$

and, as derived by Wilczynski (Tassoul 1978)

$$\int_V \nabla^2\Phi_{tot}\ dV = \int_S (\nabla\Phi_{tot}\cdot\vec{n})\ dS = -4\pi GM + \int_V (2\Omega^2 + \ell\ d\Omega^2/d\ell)\ dV < 0, \tag{3.1.31}$$

or

$$(1/V)\int_V [\Omega^2 + (\ell/2)\ d\Omega^2/d\ell)]\ dV < 2\pi G\varrho_m. \tag{3.1.32}$$

For a configuration in equilibrium with zero surface pressure, the virial theorem (2.6.80) simplifies to $(d^2I/dt^2, \vec{H}, P_{jk} = 0)$

$$2E_{kin} + W + 3\int_V P\ dV = 0, \tag{3.1.33}$$

where in the case of permanent rotation E_{kin} reduces to

$$E_{kin} = (1/2)\int_M v_\varphi^2\ dM = (1/2)\int_V \varrho\Omega^2\ell^2\ dV. \tag{3.1.34}$$

A useful parameter to characterize rotating configurations is [cf. Eq. (2.6.68)]

$$\tau = E_{kin}/|W| = E_{kin}/(-W), \qquad \Big(W = -(1/2)\int_M \Phi\ dM\Big). \tag{3.1.35}$$

From Eq. (3.1.33) follows

$$0 \le \tau \le 0.5, \tag{3.1.36}$$

because the pressure integral is always a nonnegative quantity, and therefore $2E_{kin} \le -W$, $(E_{kin} \ge 0,\ W < 0)$.

We now turn to the important problem of representing the gravitational potential of a distorted configuration. Any function $f(\lambda, \varphi)$ satisfying certain general conditions of continuity (e.g. Smirnow 1967, Vol. 3) can be expanded into a convergent series of spherical harmonics (spherical functions) $Y_j(\lambda, \varphi)$:

$$f(\lambda, \varphi) = \sum_{j=0}^{\infty} Y_j(\lambda, \varphi). \tag{3.1.37}$$

A surface harmonic Y_j of order j can be taken as a combination of Legendre polynomials $P_j(\cos\lambda)$ and associated Legendre polynomials $P_j^k(\cos\lambda)$ with $\exp(ik\varphi) = \cos k\varphi + i\sin k\varphi$:

$$Y_j(\lambda, \varphi) = \sum_{k=0}^{j}(a_{jk}\cos k\varphi + b_{jk}\sin k\varphi)\ P_j^k(\cos\lambda), \qquad (a_{jk}, b_{jk} = \mathrm{const}). \tag{3.1.38}$$

We also recall the definitions of P_j and P_j^k (e.g. Hobson 1931, p. 99, Abramowitz and Stegun 1965)

$$P_j(\mu) = P_j^0(\mu) = (1/2^j j!)\ d^j(\mu^2-1)^j/d\mu^j; \quad P_j^k(\mu) = [(1-\mu^2)^{k/2}/2^j j!]\ d^{j+k}(\mu^2-1)^j/d\mu^{j+k};$$
$$P_j^{-k}(\mu) = (j-k)!\ P_j^k(\mu)/(j+k)!; \quad P_j^k(\mu) \equiv 0 \quad \text{if} \quad |k| > j, \qquad \big(\mu = \cos\lambda;\ j = 0, 1, 2, 3, ...;$$
$$k = -j, -j+1, ...j-1, j\big), \tag{3.1.39}$$

together with the differential equations satisfied by the Legendre polynomials P_j and by the associated Legendre polynomials P_j^k of order j (e.g. Spiegel 1968):

$$d[(1-\mu^2)\ dP_j/d\mu]/d\mu + j(j+1)P_j = 0, \tag{3.1.40}$$

$$d[(1-\mu^2)\ dP_j^k/d\mu]/d\mu + [j(j+1) - k^2/(1-\mu^2)]P_j^k = 0. \tag{3.1.41}$$

The generating function of Legendre polynomials is

$$(1 - 2r\cos\gamma + r^2)^{-1/2} = \sum_{j=0}^{\infty} r^j P_j(\cos\gamma). \tag{3.1.42}$$

This infinite series converges if $|r| < 1$. An important property of Legendre and associated Legendre polynomials is the so-called summation theorem of Legendre polynomials. If γ is the angle between two radius vectors $\vec{r}$ and $\vec{r'}$ on a sphere of unit radius, having the spherical coordinates (λ, φ) and (λ', φ'), respectively, then the cosine theorem yields at once

$$\cos\gamma = \cos\lambda\cos\lambda' + \sin\lambda\sin\lambda'\cos(\varphi - \varphi'), \tag{3.1.43}$$

and the summation theorem of Legendre polynomials writes (e.g. Smirnow 1967)

$$P_j(\cos\gamma) = \sum_{k=0}^{j}[2(j-k)!/\delta_k(j+k)!]\ P_j^k(\cos\lambda)\ P_j^k(\cos\lambda')\cos[k(\varphi-\varphi')],$$
$$(\delta_k = 2 \text{ if } k = 0 \quad \text{and} \quad \delta_k = 1 \text{ if } k > 0). \tag{3.1.44}$$

Quite generally, a level surface Φ_{tot} = const of a distorted sphere can be represented in spherical coordinates as a sum of Legendre polynomials [cf. Eqs. (3.1.37), (3.1.38)]:

$$r = r(\lambda, \varphi) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}(a_{jk}\cos k\varphi + b_{jk}\sin k\varphi)\ P_j^k(\cos\lambda). \tag{3.1.45}$$

If the level surfaces are surfaces of revolution (as for a configuration in permanent rotation), r is independent of the azimuth angle φ, $(k = 0)$, and must be an even function of the colatitude λ; all odd indexed Legendre polynomials $P_{2j+1}(\cos\lambda)$ are odd functions of $\cos\lambda$, and must vanish in order to assure symmetry with respect to the equatorial plane. Thus, in this important particular case the equation of a level surface (3.1.45) simplifies to

$$r = r(\lambda) = \sum_{j=0}^{\infty} a_{2j}P_{2j}(\cos\lambda), \qquad (a_{2j} = \text{const}). \tag{3.1.46}$$

As already written down by Eq. (2.6.62), the internal or external potential produced by a gravitating mass M at an arbitrary space point of radius vector $\vec{r}(r, \lambda, \varphi)$ is

$$\Phi = \Phi(\vec{r}) = G\int_V \varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|. \tag{3.1.47}$$

V is the volume of the mass M in question, $\varrho(\vec{r'}) = \varrho(r', \lambda', \varphi')$ the density at radius vector $\vec{r'} = \vec{r'}(r', \lambda', \varphi')$, $dV' = {r'}^2\sin\lambda'\ dr'\ d\lambda'\ d\varphi'$ the volume element, and $|\vec{r} - \vec{r'}|$ the distance between the vectors $\vec{r}$ and $\vec{r'}$:

$$|\vec{r} - \vec{r'}| = (r^2 + {r'}^2 - 2rr'\cos\gamma)^{1/2}. \tag{3.1.48}$$

Using the generating function of Legendre polynomials (3.1.42) we have

$$1/|\vec{r} - \vec{r'}| = (1/r)[1 - 2(r'/r)\cos\gamma + (r'/r)^2]^{-1/2} = (1/r)\sum_{j=0}^{\infty}(r'/r)^j P_j(\cos\gamma) \quad \text{if} \quad r > r', \tag{3.1.49}$$

and

$$1/|\vec{r} - \vec{r'}| = (1/r')[1 - 2(r/r')\cos\gamma + (r/r')^2]^{-1/2} = (1/r')\sum_{j=0}^{\infty}(r/r')^j P_j(\cos\gamma)$$
$$= (1/r)\sum_{j=0}^{\infty}(r'/r)^{-j-1}P_j(\cos\gamma) \quad \text{if} \quad r < r'. \tag{3.1.50}$$

Inserting Eqs. (3.1.49), (3.1.50) into Eq. (3.1.47), we get

$$\Phi(\vec{r}) = (G/r)\int_V \sum_{j=0}^{\infty}\varrho(\vec{r'})\ (r'/r)^h P_j(\cos\gamma)\ dV', \tag{3.1.51}$$

where

$$h = \begin{cases} j & \text{if} \quad r > r' \\ -j-1 & \quad\quad r < r' \end{cases} \tag{3.1.52}$$

We substitute $P_j(\cos\gamma)$ via the summation theorem (3.1.44), and finally obtain the expansion into Legendre polynomials of the internal or external gravitational potential (Zharkov and Trubitsyn 1978):

$$\Phi = (G/r)\int_V \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\varrho(\vec{r'})\ (r'/r)^h[2(j-k)!/\delta_k(j+k)!]P_j^k(\cos\lambda)\ P_j^k(\cos\lambda')\cos[k(\varphi-\varphi')]\right\} dV'$$

$$= (G/r)\sum_{j=0}^{\infty}\left[P_j(\cos\lambda)\int_V \varrho(\vec{r'})\ (r'/r)^h P_j(\cos\lambda')\ dV'\right]$$

$$+(G/r)\sum_{j=1}^{\infty}\sum_{k=1}^{j}\left\{P_j^k(\cos\lambda)\cos k\varphi\int_V [2(j-k)!/(j+k)!]\ \varrho(\vec{r'})\ (r'/r)^h P_j^k(\cos\lambda')\cos k\varphi'\ dV'\right.$$

$$\left.+P_j^k(\cos\lambda)\sin k\varphi\int_V [2(j-k)!/(j+k)!]\ \varrho(\vec{r'})\ (r'/r)^h P_j^k(\cos\lambda')\sin k\varphi'\ dV'\right\}. \tag{3.1.53}$$

We may also represent the equation of a level surface (3.1.45) under the form (cf. Zharkov and Trubitsyn 1978)

$$r = s(1+\zeta), \tag{3.1.54}$$

where $\zeta = \zeta(s,\lambda,\varphi)$ is a (generally small) unknown function, and s the average radius of the level surface, i.e. the radius of a sphere with volume $4\pi s^3/3$, equal to the volume inside the level surface. If we also represent the level surfaces corresponding to the radius vector r' by an equation $r' = s'(1+\zeta')$ similar to Eq.(3.1.54), and denote by z the absolute value of the extreme of ζ and ζ' over a level surface ($|\zeta|, |\zeta'| \leq z < 1$), we observe that the limits of convergence of the two series from Eqs. (3.1.49), (3.1.50) are given respectively by (e.g. Jardetzky 1958, p. 18)

$$s' < s(1-z)/(1+z) \quad \text{and} \quad s' > s(1+z)/(1-z), \tag{3.1.55}$$

since $r > r'$ means that even the minimum $s(1-z)$ of r must be larger than the maximum $s'(1+z)$ of r', and $r < r'$ means that even the maximum $s(1+z)$ of r must be smaller than the minimum $s'(1-z)$ of r'. Thus, there is a layer of width

$$s(1-z)/(1+z) < s' < s(1+z)/(1-z), \tag{3.1.56}$$

where the Legendre expansions (3.1.49), (3.1.50) diverge. If $z \ll 1$, the diverging layer has a maximum width between $s(1-2z)$ and $s(1+2z)$. As shown by Zharkov and Trubitsyn (1978), the subsequent expansions of the gravitational potential can also be derived without the partially diverging series (3.1.49), (3.1.50), by using the average radius s of a level surface as a separation between two expansions of the form (3.1.49) and (3.1.50), rather than the sphere $r = r'$. Laplace's use of the partially divergent Legendre series (3.1.51) is quite valid due to the precise cancellation of divergent terms.

We can proceed further to simplify Eq. (3.1.53) for the case of an axially symmetric body. In this case the potential Φ is independent of the azimuth φ, $(k = 0)$. As outlined subsequently to Eq. (3.1.19), an equatorial symmetry plane exists under fairly general conditions, and in this case Φ must be an even function of the colatitude λ. The same reasoning holds for the density $\varrho(\vec{r'}) = \varrho(r',\lambda')$. Because all odd indexed Legendre polynomials $P_{2j+1}(\cos\lambda')$ are odd functions of $\cos\lambda'$, they cancel out by integration over the volume V of the configuration, so that only even indices will enter in the potential of a rotationally distorted polytrope in this important particular case:

$$\Phi = (G/r)\left[\sum_{j=0}^{\infty}P_{2j}(\cos\lambda)\int_V \varrho(r',\lambda')\ (r'/r)^h P_{2j}(\cos\lambda')\ dV'\right]$$

$$= (G/r)\sum_{j=0}^{\infty}\left(D_{2j}r^{-2j} + D'_{2j}r^{2j+1}\right)P_{2j}(\cos\lambda); \quad D_{2j} = \int_{r>r'}\varrho(r',\lambda')\ r'^{2j}P_{2j}(\cos\lambda')\ dV';$$

$$D'_{2j} = \int_{r<r'}\varrho(r',\lambda')\ r'^{-2j-1}P_{2j}(\cos\lambda')\ dV'. \tag{3.1.57}$$

The external potential of a mass simplifies with respect to Eq. (3.1.53), because in this case $r > r'$ throughout, and $h = j$:

$$\Phi_e = (G/r)\int_V \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\varrho(\vec{r'})\ (r'/r)^j[2(j-k)!/\delta_k(j+k)!]P_j^k(\cos\lambda)\ P_j^k(\cos\lambda')\cos[k(\varphi-\varphi')]\right\}dV'$$
$$= (G/r)\sum_{j=0}^{\infty}\left[P_j(\cos\lambda)\int_V \varrho(\vec{r'})\ (r'/r)^j P_j(\cos\lambda')\ dV'\right]$$
$$+(G/r)\sum_{j=1}^{\infty}\sum_{k=1}^{j}\left\{P_j^k(\cos\lambda)\cos k\varphi\int_V 2[(j-k)!/(j+k)!]\ \varrho(\vec{r'})\ (r'/r)^j P_j^k(\cos\lambda')\cos k\varphi'\ dV'\right.$$
$$\left.+P_j^k(\cos\lambda)\sin k\varphi\int_V 2[(j-k)!/(j+k)!]\ \varrho(\vec{r'})\ (r'/r)^j P_j^k(\cos\lambda')\sin k\varphi'\ dV'\right\}$$
$$= (GM/r)\left[1-\sum_{j=1}^{\infty}(a_1/r)^j J_j P_j(\cos\lambda)+\sum_{j=1}^{\infty}\sum_{k=1}^{j}(a_1/r)^j P_j^k(\cos\lambda)\ (C_{jk}\cos k\varphi+S_{jk}\sin k\varphi)\right];$$
$$J_j = -(1/M)\int_V \varrho(\vec{r'})\ (r'/a_1)^j P_j(\cos\lambda')\ dV' = -D_j/Ma_1^j;$$
$$C_{jk} = (1/M)[2(j-k)!/(j+k)!]\int_V \varrho(\vec{r'})\ (r'/a_1)^j P_j^k(\cos\lambda')\cos k\varphi'\ dV';$$
$$S_{jk} = (1/M)[2(j-k)!/(j+k)!]\int_V \varrho(\vec{r'})\ (r'/a_1)^j P_j^k(\cos\lambda')\sin k\varphi'\ dV',\quad (J_j, C_{jk}, S_{jk} = \mathrm{const}). \tag{3.1.58}$$

By convention, a_1 is generally equal to the maximum radius of the mass M. The mass of the configuration is obtained if $j = 0$:

$$M = P_0(\cos\lambda)\int_V \varrho(\vec{r'})\ P_0(\cos\lambda')\ dV' = \int_V \varrho(\vec{r'})\ dV' = D_0,\qquad (P_0(\cos\lambda) = 1). \tag{3.1.59}$$

For axial symmetry $C_{jk}, S_{jk} = 0$. If the figure possesses an equatorial symmetry plane, we have further $J_{2j+1} = 0$. Choosing the origin of coordinates in the centre of mass of the configuration, we are able to eliminate the coefficients J_1, C_{11}, S_{11} in the expansion of the external potential. According to Eq. (3.1.58) we have

$$J_1 = -D_1/Ma_1 = -(1/Ma_1)\int_V \varrho r'\cos\lambda'\ dV' = -(1/Ma_1)\int_M x_3'\ dM' = -x_{3c}/a_1;$$
$$C_{11} = (1/Ma_1)\int_V \varrho r'\sin\lambda'\cos\varphi'\ dV' = (1/Ma_1)\int_M x_1'\ dM' = x_{1c}/a_1;$$
$$S_{11} = (1/Ma_1)\int_V \varrho r'\sin\lambda'\sin\varphi'\ dV' = (1/Ma_1)\int_M x_2'\ dM' = x_{2c}/a_1, \tag{3.1.60}$$

where the mass element is $dM' = \varrho\ dV'$, and x_{1c}, x_{2c}, x_{3c} are the coordinates of the centre of mass. The second order coefficients $J_2, C_{21}, C_{22}, S_{21}, S_{22}$ can be expressed with the aid of the moments of inertia

$$A = \int_V ({x_2'}^2 + {x_3'}^2)\varrho\ dV';\quad B = \int_V ({x_3'}^2 + {x_1'}^2)\varrho\ dV';\quad C = \int_V ({x_1'}^2 + {x_2'}^2)\varrho\ dV';$$
$$D = \int_V x_2'x_3'\varrho\ dV';\quad E = \int_V x_3'x_1'\varrho\ dV';\quad F = \int_V x_1'x_2'\varrho\ dV'. \tag{3.1.61}$$

The products $r^2P_2^k,\ (k = 0, 1, 2)$ occurring in the coefficients $J_2, C_{21}, ...,$ can be transformed as follows:

$$r^2P_2^0 = r^2(3\cos^2\lambda - 1)/2 = x_3^2 - (x_1^2 + x_2^2)/2 = (x_2^2 + x_3^2)/2 + (x_3^2 + x_1^2)/2 - (x_1^2 + x_2^2);$$
$$P_2^1 = 3\sin\lambda\cos\lambda;\quad r^2P_2^2\cos 2\varphi = 3r^2\sin^2\lambda(1 - 2\sin^2\varphi) = 3r^2 - 3x_3^2 - 6x_2^2$$
$$= 3(x_1^2 + x_3^2) - 3(x_2^2 + x_3^2). \tag{3.1.62}$$

The coefficients from Eq. (3.1.58) are expressed in virtue of Eqs. (3.1.61), (3.1.62) under the form (e.g. Zharkov and Trubitsyn 1978)

$$J_2 = -D_2/Ma_1^2 = -(1/Ma_1^2)[(A+B)/2 - C]; \quad C_{21} = E/Ma_1^2;$$
$$C_{22} = (B-A)/4Ma_1^2; \quad S_{21} = D/Ma_1^2; \quad S_{22} = F/2Ma_1^2. \tag{3.1.63}$$

If the coordinate axes are chosen along the principal axes of inertia of the body, the centrifugal moments D, E, F vanish. If an equatorial symmetry plane exists, all uneven multipole moments $D_{2j+1} = -a_1^{2j+1}MJ_{2j+1}$ vanish too. The zeroth multipole moment D_0 of the configuration is simply equal to its mass M [cf. Eq. (3.1.59)], the dipole moment D_1 is zero if the origin is taken in the mass centre [cf. Eq. (3.1.60)], and the quadrupole moment $D_2 = -a_1^2MJ_2$ is expressible through the principal moments of inertia.

Another important second order expansion of the external potential is MacCullagh's formula. We start with Eq. (3.1.47), and use the expansion (3.1.49), confining to second order terms:

$$\Phi_e(\vec{r}) = G\int_V \varrho(\vec{r'})\, dV'/|\vec{r}-\vec{r'}| = (G/r)\int_M \left[\sum_{j=0}^{\infty}(r'/r)^j P_j(\cos\gamma)\right] dM'$$
$$\approx (G/r)\int_M dM' + (G/r^2)\int_M r'\cos\gamma\, dM' + (G/r^3)\int_M r'^2\, dM' - (3G/2r^3)\int_M r'^2\sin^2\gamma\, dM',$$
$$(P_2(\cos\gamma) = 1 - 3\sin^2\gamma/2). \tag{3.1.64}$$

The first integral on the right-hand side is the potential of a point mass having mass M, the second integral vanishes if the centre of mass is chosen as the origin of coordinates, the third integral can be written via Eq. (3.1.61) as $G(A+B+C)/2r^3$, and the fourth integral is the moment of inertia $I_{\vec{r}}$ about the radius vector $\vec{r}$, so that MacCullagh's formula becomes (e.g. Stacey 1969)

$$\Phi_e(\vec{r}) \approx GM/r + G(A+B+C-3I_{\vec{r}})/2r^3. \tag{3.1.65}$$

The last integral in Eq. (3.1.64) can be transformed by observing that $r'\sin\gamma$ is equal to the absolute value of the vectorial product $\vec{r'}\times(\vec{r}/|\vec{r}|)$, where the components of $\vec{r}/|\vec{r}|$ are the direction cosines ℓ, m, n of the vector $\vec{r}$:

$$I_{\vec{r}} = \int_M r'^2\sin^2\gamma\, dM' = \int_M [\vec{r'}\times(\vec{r}/|\vec{r}|)]^2\, dM' = \int_M [x_1'^2(m^2+n^2) + x_2'^2(\ell^2+n^2) + x_3'^2(\ell^2+m^2)$$
$$-2x_1'x_2'\ell m - 2x_1'x_3'\ell n - 2x_2'x_3'mn]\, dM' = A\ell^2 + Bm^2 + Cn^2 - 2F\ell m - 2E\ell n - 2Dmn. \tag{3.1.66}$$

If the coordinate axes are chosen as the principal axes which diagonalize the moment of inertia tensor, then D, E, F vanish. If the principal axis Oz is the rotation axis, then

$$n^2 = \cos^2\lambda = 1 - \ell^2 - m^2. \tag{3.1.67}$$

For axial symmetry we have $A = B$, and combining this with Eqs. (3.1.66), (3.1.67), we get from Eq. (3.1.65)

$$\Phi_e(r,\lambda) \approx GM/r + G(A-C)(3\cos^2\lambda - 1)/2r^3 = (GM/r)[1 - (a_1/r)^2 J_2 P_2(\cos\lambda)], \quad (C_{22} = 0). \tag{3.1.68}$$

In hydrostatic equilibrium the surface of the configuration is a level surface, so the total potential Φ_{tot} from Eq. (3.1.23) must be a constant (on the surface $r_1 = r_1(\lambda)$ the internal potential Φ must be equal to the external potential Φ_e) :

$$\Phi_{tot}(r_1,\lambda) = \Phi_e(r_1,\lambda) + \Omega^2 r_1^2\sin^2\lambda/2 = \text{const}. \tag{3.1.69}$$

We rewrite Eq. (3.1.69) for the equatorial and polar radius a_1 and a_3, respectively, ($\lambda = 0, \pi/2$, and $a_1 = a_2$) :

$$GM/a_1 + GMJ_2/2a_1 + \Omega^2 a_1^2/2 = GM/a_3 - GMa_1^2J_2/a_3^3. \tag{3.1.70}$$

Then, to the first order ($J_2 \ll 1$), the oblateness f becomes for an axisymmetric configuration rotating at constant angular velocity Ω, $(a_1 \approx a_3)$ equal to

$$f = (a_1 - a_3)/a_1 = 3J_2/2 + \Omega^2 a_1^3/2GM. \tag{3.1.71}$$

The first order equation of the rotationally distorted surface is obtained by equating in Eq. (3.1.69) the total potential Φ_{tot} in an arbitrary surface point to its value at the equator a_1, for instance:

$$r_1 = a_1(1 - f\cos^2\lambda), \qquad (f \ll 1). \tag{3.1.72}$$

Eq. (3.1.14) possesses an important prime integral (e.g. Jeans 1919, Chandrasekhar 1961), that will be used for the exact calculation of the gravitational energy of a rotating configuration. If P is polytropic, Eq. (3.1.15) can be written as

$$(1 + 1/n)K\varrho^{1/n-1}\nabla\varrho = \nabla[\Phi + \Omega^2(x_1^2 + x_2^2)/2], \tag{3.1.73}$$

or integrating

$$(n+1)P/\varrho = \Phi + \Omega^2(x_1^2 + x_2^2)/2 + \mathrm{const} = \Phi + \Phi_f + \mathrm{const} = \Phi + \Phi_f - \Phi_p, \qquad (\Omega = \mathrm{const}), \tag{3.1.74}$$

where $\Phi_f = \Omega^2(x_1^2 + x_2^2)/2$ is called the centrifugal potential, and the integration constant is given by the gravitational potential at the pole of the configuration Φ_p, where $P = 0$ and $x_1, x_2 = 0$. In spherical coordinates Eq. (3.1.74) becomes equal to

$$(n+1)P/\varrho = \Phi + \Omega^2 r^2 \sin^2\lambda/2 - \Phi_p = \Phi + |\vec{\Omega} \times \vec{r}|^2/2 - \Phi_p. \tag{3.1.75}$$

Integrating this prime integral over the volume of the configuration, we find

$$(n+1)\int_V P\ dV = \int_V \varrho\Phi\ dV + \Omega^2 I_\Omega/2 - \Phi_p M. \tag{3.1.76}$$

The moment of inertia about the rotation axis is [cf. Eq. (3.1.66)]:

$$I_\Omega = \int_V \varrho(x_1^2 + x_2^2)\ dV = \int_V \varrho r^2 \sin^2\lambda\ dV. \tag{3.1.77}$$

We introduce the gravitational energy W from Eq. (2.6.68) into Eq. (3.1.76) to obtain:

$$(n+1)\int_V P\ dV = -2W + \Omega^2 I_\Omega/2 - \Phi_p\ M. \tag{3.1.78}$$

We can write the equation of motion of an inviscid, nonmagnetic, uniformly rotating fluid in a rotating Cartesian (x_1, x_2, x_3)-frame under the form [cf. Sec. 2.6.6 and Eq. (3.1.12)]

$$\varrho\ Dv_k/Dt = -\partial P/\partial x_k + \varrho\ \partial(\Phi + |\vec{\Omega} \times \vec{r}|^2/2)/\partial x_k + 2\varrho\varepsilon_{k\ell m} v_\ell \Omega_m,$$
$$(k = 1, 2, 3;\ r^2 = x_1^2 + x_2^2 + x_3^2), \tag{3.1.79}$$

where summation occurs over the repeated indices ℓ, m, and $|\vec{\Omega} \times \vec{r}|^2/2$ is the centrifugal potential $[\nabla|\vec{\Omega} \times \vec{r}|^2/2 = -\vec{\Omega} \times (\vec{\Omega} \times \vec{r})]$. The last term represents just the components of the Coriolis force $2\varrho\vec{v} \times \vec{\Omega}$ acting on the unit of volume. To shorten the notations, we have introduced the alternating symbol

$$\varepsilon_{k\ell m} = (-1)^{I(k,\ell,m)}, \tag{3.1.80}$$

where $\varepsilon_{k\ell m} = \pm 1$, depending on the number of inversions $I(k, \ell, m)$ occurring in the permutation of the three distinct elements

$$\begin{bmatrix} 1 & 2 & 3 \\ k & \ell & m \end{bmatrix} \tag{3.1.81}$$

In the same way as effected subsequently to Eq. (2.6.50), we transform the vectorial equation (3.1.79) into a tensorial form, by multiplying with $x_j\ dV$ and integrating over the volume of the configuration. Instead of the vectorial product from Eq. (3.1.79) we can write

$$(1/2)\int_V \varrho x_j\ (\partial|\vec{\Omega}\times\vec{r}|^2/\partial x_k)\ dV = \Omega^2\int_V \varrho x_j x_k\ dV - \Omega_k\Omega_\ell\int_V \varrho x_j x_\ell\ dV = \Omega^2 I_{jk} - \Omega_k\Omega_\ell I_{j\ell},$$
$$[\vec{\Omega} = \vec{\Omega}(\Omega_1,\Omega_2,\Omega_3)], \tag{3.1.82}$$

where I_{jk} are the second order moments of density distribution from Eq. (2.6.74).

Thus, Eq. (3.1.79) takes eventually the equivalent form [cf. Eqs. (2.6.56)-(2.6.79)]

$$d\left(\int_V \varrho x_j v_k\ dV\right)\Big/dt = 2E_{jk} + W_{jk} + \Omega^2 I_{jk} - \Omega_k\Omega_\ell I_{j\ell} + \delta_{jk}\int_V P\ dV + 2\varepsilon_{k\ell m}\int_V \varrho x_j v_\ell\Omega_m\ dV. \tag{3.1.83}$$

If the x_3-axis is chosen along the rotation axis $\vec{\Omega} = \vec{\Omega}(0,0,\Omega)$, and if rotation is stationary without internal motions ($v_k = 0$), Eq. (3.1.83) simplifies to (Chandrasekhar 1969)

$$W_{jk} + \Omega^2(I_{jk} - \delta_{3k}I_{j3}) = -\delta_{jk}\int_V P\ dV. \tag{3.1.84}$$

Contracting this tensorial equation we get (Chandrasekhar 1961)

$$W + \Omega^2 I_\Omega = -3\int_V P\ dV, \qquad \left(I_\Omega = \int_V (x_1^2 + x_2^2)\varrho\ dV = I_{11} + I_{22}\right). \tag{3.1.85}$$

The pressure integral can be eliminated between Eqs. (3.1.78) and (3.1.85), to give

$$W = [-3\Phi_p M + (n+5/2)\Omega^2 I_\Omega]/(5-n), \tag{3.1.86}$$

which represents a generalization of Eq. (2.6.137), ($\Omega = 0;\ M \to M_1;\ \Phi_p \to GM_1/r_1$) to the case of uniformly rotating, distorted spheres. Anand and Kushwaha (1962a) have slightly generalized Eq. (3.1.86) for a particular toroidal axisymmetric magnetic field ($H_\ell, H_z = 0,\ H_\varphi \propto \varrho\ell$, cf. Sec. 3.10.3).

We write out explicitly Eq. (3.1.84) and get, by using Eq. (3.1.78):

$$W_{11} + \Omega^2 I_{11} = W_{22} + \Omega^2 I_{22} = W_{33} = -\int_V P\ dV = [2(W_{11} + W_{22} + W_{33})$$
$$-\Omega^2(I_{11} + I_{22})/2 + \Phi_p M]/(n+1); \quad W_{12} + \Omega^2 I_{12} = W_{21} + \Omega^2 I_{21} = 0; \quad W_{13} + \Omega^2 I_{13} = 0;$$
$$W_{31} = 0; \quad W_{23} + \Omega^2 I_{23} = 0; \quad W_{32} = 0. \tag{3.1.87}$$

Since $W_{jk} = W_{kj}$ and $I_{jk} = I_{kj}$, there results from the last four equations that $W_{13}, W_{23} = 0$ and $I_{13}, I_{23} = 0$. For rotation about the x_3-axis the components $W_{12} = W_{21}$ and $I_{12} = I_{21}$ can be arranged to vanish, so the tensors W_{jk} and I_{jk} assume a diagonal form. However, it is not required that $W_{11} = W_{22}$ and $I_{11} = I_{22}$, as for axisymmetric bodies (Chandrasekhar 1969). The first set of equations from Eq. (3.1.87) can be solved to give (Chandrasekhar 1961)

$$W_{11} = [-\Phi_p M + (n-5/2)\Omega^2 I_{11} + 5\Omega^2 I_{22}/2]/(5-n);$$
$$W_{22} = [-\Phi_p M + 5\Omega^2 I_{11}/2 + (n-5/2)\Omega^2 I_{22}]/(5-n);$$
$$W_{33} = [-\Phi_p M + 5\Omega^2(I_{11} + I_{22})/2]/(5-n), \tag{3.1.88}$$

and their sum is just equal to Eq. (3.1.86).

We now turn to the brief discussion of the combined effects of rotational and tidal distortions, termed by Chandrasekhar (1933c) "the double star problem". To this end we introduce a rotating Cartesian (x_1, x_2, x_3)-frame with the origin in the primary mass M (the body on which we are interested), rotating about another body of mass M' (the secondary, which is the origin of tidal effects on the primary). The distance between the primaries D remains constant (circular orbit). The relative dispositions of the bodies remain unchanged, so that their spin angular velocity is the same as the constant orbital angular velocity $\vec{\Omega}(0,0,\Omega)$ round their centre of mass C (Fig. 3.1.1). The secondary M' is located on the Mx_1-axis, and the centre of mass C has the constant coordinates $(M'D/(M+M'), 0, 0)$.

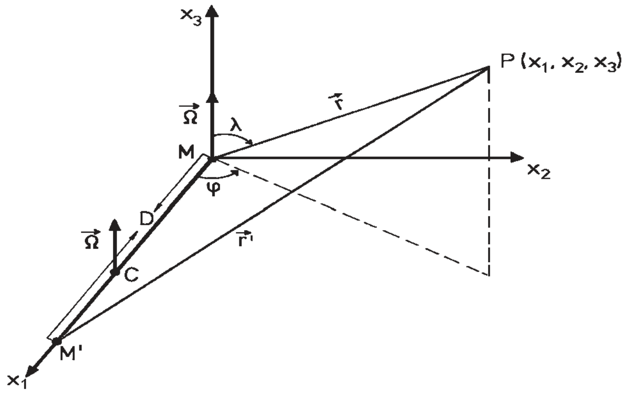

Fig. 3.1.1 Geometry of the double star problem.

Let us write down at first the equations of motion in a rotating system with the origin in the centre of mass C. They can be deduced at once from Eq. (3.1.79), where we have to add to the internal potential $\Phi(\vec{r})$ of M the external potential $\Phi'_e(\vec{r'})$ of M' (the tidal potential), and take into account that $\vec{\Omega} = \vec{\Omega}(0, 0, \Omega)$:

$$\varrho\ Dv_k/Dt = -\partial P/\partial x_k + \varrho\ \partial\big[\Phi + \Phi'_e + \Omega^2(x_1^2 + x_2^2)/2\big]/\partial x_k + 2\varrho\Omega\varepsilon_{k\ell 3}v_\ell, \qquad (k = 1, 2, 3). \tag{3.1.89}$$

We turn to the system rotating round M by making simply a translation from C to M along Mx_1, i.e. $x_1 \to x_1 - M'D/(M + M')$, so that Eq. (3.1.89) becomes eventually (e.g. Chandrasekhar 1969):

$$\varrho\ Dv_k/Dt = -\partial P/\partial x_k + \varrho\ \partial\big\{\Phi + \Phi'_e + (\Omega^2/2)\big[\big(x_1 - M'D/(M + M')\big)^2 + x_2^2\big]\big\}/\partial x_k$$
$$+2\varrho\Omega\varepsilon_{k\ell 3}v_\ell. \tag{3.1.90}$$

In a first approximation MacCullagh's formula (3.1.65) is simply $\Phi'_e(\vec{r'}) \approx GM'/r'$, where $\vec{r'}$ is the distance between a point $P(x_1, x_2, x_3)$ inside the primary and $M'(D, 0, 0)$: $r' = [(x_1 - D)^2 + x_2^2 + x_3^2]^{1/2}$. Since $|x_1|, |x_2|, |x_3| < D$, we obtain up to the second order:

$$\Phi'_e(\vec{r'}) \approx GM'/r' = GM'/D[1 - 2x_1/D + (x_1^2 + x_2^2 + x_3^2)/D^2]^{1/2}$$
$$\approx (GM'/D)[1 + x_1/D + (2x_1^2 - x_2^2 - x_3^2)/2D^2]. \tag{3.1.91}$$

The equations of motion (3.1.90) read, by dropping constant terms:

$$\varrho\ Dv_k/Dt = -\partial P/\partial x_k + \varrho\ \partial\big[\Phi + GM'x_1/D^2 - \Omega^2 M'Dx_1/(M + M')$$
$$+(GM'/2D^3)(2x_1^2 - x_2^2 - x_3^2) + \Omega^2(x_1^2 + x_2^2)/2\big]/\partial x_k + 2\varrho\Omega\varepsilon_{k\ell 3}v_\ell. \tag{3.1.92}$$

The angular velocity of revolution Ω of two tidally interacting bodies differs from its Keplerian value $G(M + M')/D^3$, (point mass approximation) by a factor ε :

$$\Omega^2 = G(M + M')(1 + \varepsilon)/D^3. \tag{3.1.93}$$

ε is of order $(r_m/D)^5$, where r_m is a mean radius of the components (Martin 1970, App. I. E). According to Jeans (1919, p. 258; see also Eqs. (3.7.93)-(3.7.97), and Lai et al. 1993, Figs. 10, 17) ε takes a maximum value of about 0.22 for two ellipsoids of equal mass and constant density (congruent Darwin ellipsoids) in contact, and will be neglected for simplicity. Inserting from Eq. (3.1.93) with $\varepsilon = 0$ into Eq. (3.1.92), we obtain the equation of motion for the so-called Roche problem:

$$\varrho\ Dv_k/Dt = -\partial P/\partial x_k + \varrho\ \partial\big[\Phi + (GM'/2D^3)(2x_1^2 - x_2^2 - x_3^2) + \Omega^2(x_1^2 + x_2^2)/2\big]/\partial x_k$$
$$+2\varrho\Omega\varepsilon_{k\ell 3}v_\ell, \qquad (\Omega^2 = G(M + M')/D^3). \tag{3.1.94}$$

$P = \text{const}$ on a level surface, and the total potential must be constant too. In hydrostatic equilibrium we have $v_k = 0$, and

$$\Phi(\vec{r}) + (GM'/2D^3)(2x_1^2 - x_2^2 - x_3^2) + \Omega^2(x_1^2 + x_2^2)/2 = \text{const.} \tag{3.1.95}$$

On the surface, the internal potential Φ must be continuous with the external potential of M [cf. Eqs. (3.1.58)-(3.1.63)]. Because of symmetry reasons the principal axes coincide just with the coordinate axes:

$$\Phi(\vec{r}_1) = \Phi_e(\vec{r}_1) \approx (GM/r_1)[1 - J_2(a_1/r_1)^2(3\cos^2\lambda - 1)/2 + 3C_{22}(a_1/r_1)^2\cos 2\varphi \sin^2\lambda]$$
$$= (GM/r_1)\big\{1 - J_2(a_1^2/r_1^4)[x_3^2 - (x_1^2 + x_2^2)/2] + 3C_{22}(a_1^2/r_1^4)(x_1^2 - x_2^2)\big\}. \tag{3.1.96}$$

We insert this into Eq. (3.1.95), and obtain along the principal axes of inertia $\vec{r}_1(a_1, 0, 0)$, $\vec{r}_1(0, a_2, 0)$, $\vec{r}_1(0, 0, a_3)$:

$$(GM/a_1)[1 + J_2/2 + 3C_{22}] + GM'a_1^2/D^3 + \Omega^2 a_1^2/2 = (GM/a_2)[1 + (a_1^2/a_2^2)J_2/2 - 3C_{22}(a_1^2/a_2^2)]$$
$$-GM'a_2^2/2D^3 + \Omega^2 a_2^2/2 = (GM/a_3)[1 - (a_1^2/a_3^2)J_2] - GM'a_3^2/2D^3 = \text{const.} \tag{3.1.97}$$

Taking $a_1 \approx a_2 \approx a_3$, we obtain for the first order distortions along the principal moments of inertia $(J_2, C_{22}, \Omega, a_1/D \ll 1)$:

$$(a_1 - a_2)/a_1 \approx 6C_{22} + 3M'a_1^3/2MD^3;$$
$$(a_1 - a_3)/a_1 \approx 3J_2/2 + 3C_{22} + 3M'a_1^3/2MD^3 + \Omega^2 a_1^3/2GM;$$
$$(a_2 - a_3)/a_1 \approx 3J_2/2 - 3C_{22} + \Omega^2 a_1^3/2GM. \tag{3.1.98}$$

Eq. (3.1.95) becomes on the surface of the configuration equal to

$$(GM/r_1)\big[1 - J_2(a_1/r_1)^2(3\cos^2\lambda - 1)/2 + 3C_{22}(a_1/r_1)^2\cos 2\varphi \sin^2\lambda\big]$$
$$+(GM'/2D^3)r_1^2(3\cos^2\varphi \sin^2\lambda - 1) + (\Omega^2 r_1^2/2)\sin^2\lambda = \text{const.} \tag{3.1.99}$$

Equating this, for instance, to the first value from Eq. (3.1.97), we obtain a first order representation of the surface of M :

$$r_1 = a_1\big[1 - (3J_2/2 + \Omega^2 a_1^3/2GM)\cos^2\lambda + (6C_{22} + 3a_1^3 M'/2MD^3)\cos^2\varphi \sin^2\lambda$$
$$-3C_{22}(\sin^2\lambda + 1) - 3a_1^3 M'/2MD^3\big] = a_1 + (a_3 - a_2)\cos^2\lambda + (a_1 - a_2)(\cos^2\varphi \sin^2\lambda - 1)$$
$$= a_1 + (a_3 - a_2)x_3^2/a_1^2 + (x_1^2 - a_1^2)(a_1 - a_2)/a_1^2. \tag{3.1.100}$$

In the case of axial symmetry ($a_1 = a_2$) this equation is identical to Eq. (3.1.72).

A particular, and somewhat artificial case of Roche's problem is the pure tidal problem considered by Jeans (1919), when rotation is absent ($\Omega = 0$). Eq. (3.1.92) becomes

$$\varrho\, Dv_k/Dt = -\partial P/\partial x_k + \varrho\, \partial\big[\Phi + GM'x_1/D^2 + (GM'/2D^3)(2x_1^2 - x_2^2 - x_3^2)\big]/\partial x_k. \tag{3.1.101}$$

The term $\partial(GM'x_1/D^2)/\partial x_1 = GM'/D^2$ is just the gravitational acceleration due to M', acting on the mass M as a whole. This term can be dropped if we consider the equation of motion in a frame that moves with acceleration GM'/D^2 with respect to the original position of M. In this way the centre of mass of M would always remain at the origin of the new frame, and the equations of motion of the tidally distorted mass become in the uniformly accelerated frame (Sec. 3.3, Jeans 1919, Chandrasekhar 1969):

$$\varrho\, Dv_k/Dt = -\partial P/\partial x_k + \varrho\, \partial\big[\Phi + (GM'/2D^3)(2x_1^2 - x_2^2 - x_3^2)\big]/\partial x_k. \tag{3.1.102}$$

3.2 Chandrasekhar's First Order Theory of Rotationally Distorted Spheres

The pioneering work on the analytical theory of uniformly rotating polytropes is due to Chandrasekhar (1933a, d), based on previous investigations by Milne and von Zeipel (Kopal 1983). It is clear that the rotational problem can be posed only for a finite mass of the undistorted spherical polytrope, i.e. we must have throughout $-1 < n \leq 5$, $(N = 3)$. Moreover, Chandrasekhar's theory can be applied only if $0 \leq n \leq 5$, because the density from Eq. (3.2.1) has a singularity at the boundary if $-1 < n < 0$, together with the derivatives of the associated Emden-Chandrasekhar functions ψ_j [cf. Eqs. (3.2.93), (3.2.95)]. The limiting cases $n = 0$ and $n = 5$ will be discussed after presenting the solution for $0 < n < 5$.

The dimensionless ξ, Θ-variables are introduced in the same manner as for the undistorted problem in Eqs. (2.1.10), (2.1.13):

$$r = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi = \alpha\xi; \quad \varrho = \varrho_0\Theta^n;$$
$$P = K\varrho_0^{1+1/n}\Theta^{n+1} = P_0\Theta^{n+1}. \tag{3.2.1}$$

We assume the distorted sphere to be axisymmetric, so the pertinent equilibrium equation in the corotating frame is Eq. (3.1.17). After some algebra, Eq. (3.1.19) writes $[\Theta = \Theta(\xi, \mu)]$:

$$\partial(\xi^2\ \partial\Theta/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Theta/\partial\mu]/\partial\mu = (-\Theta^n + \beta)\xi^2, \qquad (\mu = \cos\lambda;\ 0 \leq n \leq 5), \tag{3.2.2}$$

where

$$\beta = \Omega^2/2\pi G\varrho_0, \qquad (\Omega = \mathrm{const};\ \beta \ll 1). \tag{3.2.3}$$

Obviously, in the nonrotating case we have $\beta = 0,\ \Theta = \Theta(\xi) = \theta(\xi)$, and Eq. (3.2.2) turns into the well known Lane-Emden equation (2.1.14) if $N = 3$. For sufficiently slow rotation there is $\beta \ll 1$, and we can seek a solution of the rotationally distorted problem (3.2.2) in terms of a small deviation of order β from the solution θ of the undistorted problem, i.e. we can assume that

$$\Theta(\xi, \mu) = \theta(\xi) + \beta\Psi(\xi, \mu). \tag{3.2.4}$$

Since θ is required to be the Lane-Emden function, we have $\theta(0) = 1$. As the origin is approached, the radial component of the hydrostatic equation (3.1.16) becomes identical to the radially symmetric hydrostatic equation (2.1.35):

$$\partial P/\partial r = \varrho\ \partial\Phi/\partial r, \qquad (r \to 0). \tag{3.2.5}$$

Thus, in the limit $r = 0$ the initial conditions for the rotating and nonrotating case must coincide:

$$\Theta(0, \mu) = \theta(0) = 1; \quad \Psi(0, \mu) = 0; \quad [\partial\Theta(\xi, \mu)/\partial\xi]_{\xi=0} = \theta'(0) = 0; \quad [\partial\Psi(\xi, \mu)/\partial\xi]_{\xi=0} = 0. \tag{3.2.6}$$

With the aid of Eq. (3.2.4) the basic equation (3.2.2) reduces to

$$\partial(\xi^2\ \partial\Psi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Psi/\partial\mu]/\partial\mu = (-n\theta^{n-1}\Psi + 1)\xi^2, \tag{3.2.7}$$

if we take into account that θ satisfies the Lane-Emden equation (2.1.14). Θ^n has been expanded in a Taylor series with respect to the small quantity $\beta\Psi:\ \Theta^n = (\theta + \beta\Psi)^n \approx \theta^n + \beta n\theta^{n-1}\Psi$. Near the boundary we have $\theta \to 0,\ \Theta \approx \beta\Psi$, and $\beta\Psi$ becomes the leading term; therefore Smith (1975, 1976) claims that near the boundary Chandrasekhar's approach from Eq. (3.2.4) leads to a singular perturbation problem. However, since near the boundary $\Theta \approx \beta\Psi$ is a small first order quantity, all involved errors remain small, and no break-down of the theory occurs. In this respect an interesting point has been made by Hubbard et al. (1975): Near the boundary, the spherically symmetric hydrostatic equation becomes for a polytropic equation of state equal to [cf. Eqs. (2.1.35), (2.6.29), (2.6.30)]

$$dP/dr = -GM_1\varrho/r_1^2 \quad \text{or} \quad d\varrho/dr = -GM_1\varrho^{1-1/n}/K(1+1/n)r_1^2, \qquad (r_1, M_1 = \mathrm{const}), \tag{3.2.8}$$

and if $n \neq 1$, various derivatives of ϱ become unbounded as ϱ goes smoothly to zero when the boundary r_1 of the polytrope of mass M_1 is approached. Therefore, the density at the boundary cannot be calculated by a Taylor expansion. However, since ϱ is generally analytic within a small boundary shell, the involved errors are negligible if an infinitesimal boundary shell is omitted, and all involved integrals will converge to their correct values (cf. Zharkov and Trubitsyn 1978, §38).

In view of Eqs. (3.1.37), (3.1.46) Chandrasekhar (1933a) writes for Ψ the equation

$$\Psi(\xi,\mu) = \psi_0(\xi) + \sum_{j=1}^{\infty} A_j\psi_j(\xi)\ P_j(\mu), \qquad (A_j = \mathrm{const};\ \mu = \cos\lambda), \tag{3.2.9}$$

where odd indexed Legendre polynomials vanish. If this equation is substituted into Eq. (3.2.7), we obtain the fundamental equations of the associated Emden-Chandrasekhar functions ψ_j, by equating the coefficients of $P_j(\mu)$:

$$d(\xi^2\ d\psi_0/d\xi)/d\xi = \xi^2(-n\theta^{n-1}\psi_0 + 1), \qquad (j = 0;\ \psi_0(0), \psi_0'(0) = 0), \tag{3.2.10}$$

$$d(\xi^2\ d\psi_j/d\xi)/d\xi = [j(j+1) - n\xi^2\theta^{n-1}]\psi_j, \qquad (j = 1,2,3,...;\ \psi_j(0), \psi_j'(0) = 0), \tag{3.2.11}$$

where we have also used Eq. (3.1.40). To determine the unknown constants A_j from Eq. (3.2.9), we must evaluate the internal potential from Poisson's equation (3.1.18), since the fundamental equation (3.1.19) contains no explicit reference to the potential. Eq. (3.1.18) writes $(\Theta^n \approx \theta^n + \beta n\theta^{n-1}\Psi)$:

$$\partial(\xi^2\ \partial\Phi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu = -(n+1)K\varrho_0^{1/n}\xi^2\left[\theta^n + \beta n\theta^{n-1}\left(\psi_0 + \sum_{j=1}^{\infty} A_j\psi_jP_j\right)\right]. \tag{3.2.12}$$

If $n = 0$, the factor $K\varrho_0^{1/n}$ has to be replaced in virtue of Eq. (3.2.1) by P_0/ϱ_0.

Chandrasekhar (1933a) employs for the internal potential Φ an expression equivalent to Eq. (3.1.57), emphasizing the internal potential of the nonrotating spherical polytrope U_0, and the small distortion parameter β. We also preserve the vanishing odd indices:

$$\Phi = U_0(\xi) + \beta\sum_{j=0}^{\infty} V_j(\xi)\ P_j(\mu). \tag{3.2.13}$$

If Eq. (3.2.13) is substituted into Eq. (3.2.12), we find by equating the coefficients of $P_j(\mu)$:

$$d(\xi^2\ dU_0/d\xi)/d\xi = -(n+1)K\varrho_0^{1/n}\xi^2\theta^n, \tag{3.2.14}$$

$$d(\xi^2\ dV_0/d\xi)/d\xi = -n(n+1)K\varrho_0^{1/n}\xi^2\theta^{n-1}\psi_0, \tag{3.2.15}$$

$$d(\xi^2\ dV_j/d\xi)/d\xi - j(j+1)V_j = -n(n+1)K\varrho_0^{1/n}A_j\xi^2\theta^{n-1}\psi_j, \qquad (j \geq 1), \tag{3.2.16}$$

where we have substituted for the derivatives of P_j via Eq. (3.1.40). The above equations are Euler type equations, and we solve at first the homogeneous equations, adding then a particular solution of the inhomogeneous equation, in order to get the general solution (e.g. Linnell 1981a). The homogeneous part of Eq. (3.2.14) is

$$d(\xi^2\ dU_0/d\xi)/d\xi = \xi^2\ d^2U_0/d\xi^2 + 2\xi\ dU_0/d\xi = 0, \tag{3.2.17}$$

with the general solution

$$U_0 = a_1' + a_2'/\xi, \qquad (a_1', a_2' = \mathrm{const}). \tag{3.2.18}$$

A particular solution of the inhomogeneous equation is just equal to the gravitational potential of the undistorted polytropic sphere from Eq. (2.6.32):

$$U_0 = (n+1)K\varrho_0^{1/n}\theta + a_3', \qquad (a_3' = \mathrm{const}). \tag{3.2.19}$$

The general solution of Eq. (3.2.14) is the sum of Eqs. (3.2.18) and (3.2.19):

$$U_0 = (n+1)K\varrho_0^{1/n}(\theta + c_0 + c_1/\xi), \qquad (c_0, c_1 = \mathrm{const}), \tag{3.2.20}$$

where $c_1 = 0$, in order to avoid a singularity at the origin. The general solution of the homogeneous part of Eq. (3.2.15) is analogous:

$$V_0 = b_1'/\xi + b_2', \qquad (b_1', b_2' = \mathrm{const}). \tag{3.2.21}$$

A particular solution of the inhomogeneous equation (3.2.15) can be found by inserting into Eq. (3.2.15) for ψ_0 from Eq. (3.2.10):

$$d(\xi^2\ dV_0/d\xi)/d\xi = (n+1)K\varrho_0^{1/n}[d(\xi^2\ d\psi_0/d\xi)/d\xi - \xi^2] = (n+1)K\varrho_0^{1/n}\ d[\xi^2\ d(\psi_0 - \xi^2/6)/d\xi]/d\xi. \tag{3.2.22}$$

A particular solution is therefore

$$V_0 = (n+1)K\varrho_0^{1/n}(\psi_0 - \xi^2/6) + b_3', \qquad (b_3' = \mathrm{const}). \tag{3.2.23}$$

The general solution of Eq. (3.2.15) is

$$V_0 = (n+1)K\varrho_0^{1/n}(\psi_0 - \xi^2/6 + c_{10} + c_{11}/\xi), \qquad (c_{10}, c_{11} = \mathrm{const}), \tag{3.2.24}$$

where again $c_{11} = 0$.

The homogeneous part of Eq. (3.2.16) has the general solution (e.g. Smirnow 1967, Linnell 1981a)

$$V_j = B_j'\xi^j + C_j'\xi^{-j-1}, \qquad (j \geq 1;\ B_j', C_j' = \mathrm{const}). \tag{3.2.25}$$

A particular solution of the inhomogeneous equation (3.2.16) can be found by inserting into Eq. (3.2.16) for ψ_j from Eq. (3.2.11):

$$d(\xi^2\ dV_j/d\xi)/d\xi - j(j+1)V_j = -(n+1)K\varrho_0^{1/n}A_j[d(\xi^2\ d\psi_j/d\xi)/d\xi - j(j+1)\psi_j], \qquad (j \geq 1). \tag{3.2.26}$$

It is easily observed that this equation has a particular integral of the form

$$V_j = (n+1)K\varrho_0^{1/n}A_j\psi_j, \tag{3.2.27}$$

and the general solution of Eq. (3.2.16) is

$$V_j = (n+1)K\varrho_0^{1/n}(A_j\psi_j + B_j\xi^j + C_j\xi^{-j-1}), \qquad (j \geq 1;\ A_j, B_j, C_j = \mathrm{const}), \tag{3.2.28}$$

where again $C_j = 0$. Thus, by inserting Eqs. (3.2.20), (3.2.24), (3.2.28) into Eq. (3.2.13), the internal potential becomes

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\theta(\xi) + c_0 + \beta\left[c_{10} + \psi_0(\xi) - \xi^2/6 + \sum_{j=1}^{\infty}\big(A_j\psi_j(\xi) + B_j\xi^j\big)\ P_j(\mu)\right]\right\}. \tag{3.2.29}$$

The first equation (3.1.17) can be written in the dimensionless variables from Eq. (3.2.1) as

$$(n+1)K\varrho_0^{1/n}\ \partial\Theta/\partial\xi = \partial\Phi/\partial\xi + \beta(n+1)K\varrho_0^{1/n}\xi[1 - P_2(\mu)]/3, \qquad \big(P_2(\mu) = 3\mu^2/2 - 1/2\big). \tag{3.2.30}$$

The constants B_j are determined by inserting the derivative $\partial\Phi/\partial\xi$ of Eq. (3.2.29) into Eq. (3.2.30). $\partial\Theta/\partial\xi$ and $\beta(n+1)K\varrho_0^{1/n}\xi/3$ cancel out, and we get by equating the corresponding coefficients of $P_j(\mu)$:

$$B_j = 0 \quad \text{if} \quad j \neq 2 \qquad \text{and} \qquad 2B_2\xi - \xi/3 = 0 \quad \text{or} \quad B_2 = 1/6. \tag{3.2.31}$$

Thus, the internal potential of the rotating polytrope is up to the first order in β equal to

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\theta(\xi) + c_0 + \beta\left[c_{10} + \psi_0(\xi) - \xi^2/6 + \xi^2 P_2(\mu)/6 + \sum_{j=1}^{\infty} A_j\psi_j(\xi)\ P_j(\mu)\right]\right\}. \tag{3.2.32}$$

The external potential Φ_e of axially symmetric configurations in hydrostatic equilibrium is independent of φ, and Eq. (3.1.58) reads, by emphasizing the expansion parameter β :

$$\Phi_e = k_0/\xi + \beta\sum_{j=0}^{\infty} k_{1j}\xi^{-j-1}P_j(\mu), \qquad (k_0, k_{1j} = \mathrm{const}). \tag{3.2.33}$$

The odd indexed, vanishing coefficients $k_{1,2j+1}$ have been included for simplicity in the sum.

If the parameter β of Chandrasekhar's theory becomes zero, all solutions turn into those for the spherical polytrope, and if the same natural requirement is imposed for the boundary Ξ_1 of the rotationally distorted polytrope, then Eq. (3.1.46) can be written as (Chandrasekhar and Lebovitz 1962d)

$$\Xi_1 = \Xi_1(\lambda) = \xi_1 + \beta\sum_{j=0}^{\infty} q_jP_j(\cos\lambda) = \xi_1 + \beta\sum_{j=0}^{\infty} q_jP_j(\mu), \qquad (q_j = \mathrm{const};\ q_{2j+1} = 0). \tag{3.2.34}$$

ξ_1 is the radial coordinate on the surface of the undistorted spherical polytrope, i.e. the first zero of the Lane-Emden function $\theta(\xi)$. Note, that Ξ_1 is a radial coordinate depending on λ, or equivalently on μ. On the surface, we have

$$\Theta = \Theta(\Xi_1,\mu) = \theta(\Xi_1) + \beta\left[\psi_0(\Xi_1) + \sum_{j=1}^{\infty} A_j\psi(\Xi_1)\ P_j(\mu)\right] = 0. \tag{3.2.35}$$

Since β is small, we have $\Xi_1 \approx \xi_1$, and we can expand all functions of Ξ_1 in the vicinity of ξ_1 :

$$\Xi_1 - \xi_1 = \beta\sum_{j=0}^{\infty} q_jP_j(\mu); \quad \theta(\Xi_1) \approx \theta(\xi_1) + (\Xi_1 - \xi_1)\ \theta'(\xi_1) = \theta(\xi_1) + \beta\theta'(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu);$$

$$\psi_k(\Xi_1) \approx \psi_k(\xi_1) + (\Xi_1 - \xi_1)\ \psi_k'(\xi_1) = \psi_k(\xi_1) + \beta\psi_k'(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu);$$

$$\theta'(\Xi_1) \approx \theta'(\xi_1) + \beta\theta''(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu), \qquad (k = 0, 1, 2, 3, ...). \tag{3.2.36}$$

Eq. (3.2.36) is inserted into Eq. (3.2.35), and Θ becomes up to the first order in β equal to

$$\Theta(\Xi_1,\mu) = \theta(\xi_1) + \beta\left[\theta'(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu) + \psi_0(\xi_1) + \sum_{j=1}^{\infty} A_j\psi_j(\xi_1)\ P_j(\mu)\right] = 0. \tag{3.2.37}$$

The coefficients of $P_j(\mu)$ must be zero, and therefore

$$q_0 = -\psi_0(\xi_1)/\theta'(\xi_1); \quad q_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1), \qquad (j \geq 1). \tag{3.2.38}$$

The coefficients A_j from the expansion of Θ are still undetermined, and can be found if continuity of the internal and external gravitational potential is implemented at the boundary of the polytrope, avoiding a slight inconsistency of Chandrasekhar's (1933a) original presentation (cf. Chandrasekhar and Lebovitz 1962d, Linnell 1977a, 1981a). Eqs. (3.2.32), (3.2.33), and their derivatives with respect to ξ become on the boundary $[\Theta(\Xi_1,\mu) = 0;\ \theta''(\xi_1) = -2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]$:

$$\Phi(\Xi_1,\mu) = (n+1)K\varrho_0^{1/n}\big\{c_0 + \beta\big[c_{10} - \xi_1^2/6 + \xi_1^2P_2(\mu)/6\big]\big\}, \tag{3.2.39}$$

$$\Phi_e(\Xi_1,\mu) = k_0/\xi_1 + \beta\sum_{j=0}^{\infty}(-k_0q_j/\xi_1^2 + k_{1j}\xi_1^{-j-1})\ P_j(\mu), \tag{3.2.40}$$

$$(\partial\Phi/\partial\xi)_{\xi=\Xi_1} = (n+1)K\varrho_0^{1/n}\Big\{\theta'(\xi_1) + \beta\Big[[-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]\sum_{j=0}^{\infty} q_j P_j(\mu) + \psi_0'(\xi_1)$$
$$-\xi_1[1-P_2(\mu)]/3 + \sum_{j=1}^{\infty} A_j\psi_j'(\xi_1)\ P_j(\mu)\Big]\Big\}, \tag{3.2.41}$$

$$(\partial\Phi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + \beta\sum_{j=0}^{\infty}\left[2k_0q_j/\xi_1^3 - (j+1)k_{1j}\xi_1^{-j-2}\right]\ P_j(\mu). \tag{3.2.42}$$

If $0 < n < 5$, we put $\theta^n(\xi_1) = 0$ in Eq. (3.2.41), and require equality of Eqs. (3.2.39), (3.2.41) with Eqs. (3.2.40), (3.2.42), respectively:

$$c_0 = -\xi_1\theta'(\xi_1); \quad c_{10} = \xi_1^2/2 - \psi_0(\xi_1) - \xi_1\psi_0'(\xi_1); \quad k_0 = -(n+1)K\varrho_0^{1/n}\xi_1^2\theta'(\xi_1);$$
$$k_{10} = (n+1)K\varrho_0^{1/n}\xi_1^2[\xi_1/3 - \psi_0'(\xi_1)]; \quad k_{12} = (n+1)K\varrho_0^{1/n}\xi_1^5[\xi_1\psi_2'(\xi_1) - 2\psi_2(\xi_1)]$$
$$/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]; \quad k_{1j} = 0 \text{ if } j \neq 0,2; \quad A_2 = -5\xi_1^2/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)];$$
$$A_j = 0 \text{ if } j \neq 2; \quad q_j = 0 \text{ if } j \neq 0,2, \qquad (0 < n < 5). \tag{3.2.43}$$

The constants $A_j, k_{1j},\ (j > 2)$ are zero because they turn out to be solutions of the homogeneous system $(n+1)K\varrho_0^{1/n}A_j\psi_j(\xi_1) - k_{1j}\xi_1^{-j-1} = 0$ and $(n+1)K\varrho_0^{1/n}A_j\psi_j'(\xi_1) + (j+1)k_{1j}\xi_1^{-j-2} = 0$. They could be nonzero if the determinant of this system vanishes: $D_j(\xi_1) = \xi_1^{-j-2}[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)] = 0$. If we insert for the constants $\psi_j(\xi_1), \psi_j'(\xi_1)$ a simple Taylor expansion near the boundary [e.g. $\psi_j(\xi_1) \approx \psi_j(\xi) - (\xi - \xi_1)\psi_j'(\xi_1)$], the condition $D_j(\xi_1) = 0$ writes approximately $(j+1)\psi_j(\xi) + \xi_1\psi_j'(\xi) \approx (\xi - \xi_1)[(j+1)\psi_j'(\xi_1) + \xi_1\psi_j''(\xi_1)]$, which is contradicted by the boundary expansions from Eqs. (3.2.95), (3.2.96). Thus A_j, k_{1j} vanish if $j > 2$ (cf. Kovetz 1968, Eqs. (28)-(30) for another proof).

With Eqs. (3.2.4), (3.2.9), (3.2.32)-(3.2.35), (3.2.38), (3.2.43) we obtain if $0 < n < 5$:

$$\Theta = \Theta(\xi,\mu) = \theta(\xi) + \beta[\psi_0(\xi) + A_2\psi_2(\xi)\ P_2(\mu)], \tag{3.2.44}$$

$$\Xi_1 = \Xi_1(\mu) = \xi_1 - \beta[\psi_0(\xi_1) + A_2\psi_2(\xi_1)\ P_2(\mu)]/\theta'(\xi_1), \tag{3.2.45}$$

$$\Phi = (n+1)K\varrho_0^{1/n}\big\{\theta(\xi) + c_0 + \beta\big[c_{10} + \psi_0(\xi) - \xi^2/6 + \xi^2P_2(\mu)/6 + A_2\psi_2(\xi)\ P_2(\mu)\big]\big\}, \tag{3.2.46}$$

$$\Phi_e = k_0/\xi + \beta[k_{10}/\xi + k_{12}P_2(\mu)/\xi^3]. \tag{3.2.47}$$

In the particular case $n = 1$ the rotational problem can be solved more exactly, since we have $\Theta^n = \Theta$ in Eq. (3.2.2), and Eqs. (3.2.10), (3.2.11) admit analytical solutions (cf. Eqs. (3.2.74), (3.2.78); Kopal 1937, 1939, Papoyan et al. 1967, Blinnikov 1972, Hubbard 1974, Kozenko 1975, Cunningham 1977, Caimmi 1980b, Williams 1988). We insert the attempt (Kopal 1939)

$$\Theta(\xi,\mu) = \chi(\xi)\ \Pi(\mu) + \beta, \tag{3.2.48}$$

into the fundamental equation (3.2.2), to obtain

$$\nabla^2\Theta = \Pi(\mu)\ \nabla^2\chi(\xi) + \chi(\xi)\ \nabla^2\Pi(\mu) = -\Theta + \beta = -\chi(\xi)\ \Pi(\mu), \qquad (n=1). \tag{3.2.49}$$

We divide by $\chi(\xi)\ \Pi(\mu)$ and use Eq. (B.39):

$$\xi^2[\nabla\chi^2(\xi)/\chi(\xi) + 1] = -\xi^2\ \nabla^2\Pi(\mu)/\Pi(\mu) = \text{const.} \tag{3.2.50}$$

We take the arbitrary constant equal to $j(j+1)$, (j – nonnegative integer). Then, Eq. (3.2.50) can be split into the system of equations

$$d(\xi^2\ d\chi_j/d\xi)/d\xi + [\xi^2 - j(j+1)]\chi_j = 0; \quad d[(1-\mu^2)\ d\Pi_j/d\mu]/d\mu + j(j+1)\Pi_j = 0,$$
$$(n = 1;\ j = 0,1,2,3,...). \tag{3.2.51}$$

The first set of equations is identical to the Lane-Emden equation (2.1.14) if $j = 0$, and to Eq. (3.2.11) for the associated Emden-Chandrasekhar functions if $j = 1, 2, 3, ...$ The solutions are given by Eqs. (2.3.89), (3.2.74), and (3.2.78):

$$\chi_0 = \theta = 1 - \psi_0 = \sin\xi/\xi; \quad \chi_j = \psi_j = (-1)^j (2j+1)!!\ \xi^j\ d^j(\sin\xi/\xi)/(\xi\ d\xi)^j,$$
$$(n = 1;\ j = 2, 3, 4, ...), \tag{3.2.52}$$

where the notation (2.3.23) is assumed, and $(2j+1)!! = 1 \times 3 \times 5 \times ...(2j-1)(2j+1)$. The second set of equations (3.2.51) is identical to the Legendre equation (3.1.40). Thus, the solution of Eq. (3.2.2) in the case $n = 1$ is equal to

$$\Theta(\xi, \mu) = \sum_{j=0}^{\infty} B_j \chi_j(\xi)\ P_j(\mu) + \beta, \qquad (B_j = \mathrm{const}). \tag{3.2.53}$$

The initial conditions $\Theta(0, \mu) = \theta(0) = \chi_0(0) = 1$ yield, no matter what $\beta:\ B_0 = 1 - \beta$, so that Eq. (3.2.53) becomes more familiarly

$$\Theta(\xi, \mu) = \theta(\xi) + \beta\psi_0(\xi) + \sum_{j=1}^{\infty} B_{2j}\psi_{2j}(\xi)\ P_{2j}(\mu), \qquad (n = 1), \tag{3.2.54}$$

where $\psi_0 = 1 - \theta$, and the constants B_{2j} have to be determined from the boundary conditions (cf. Papoyan et al. 1967).

An incomplete form of Eq. (3.2.2) without μ-dependence has been employed by Sharma and Yadav (1992, 1993) for the study of $n = 1$ polytropes with the aid of Padé approximants (see Sec. 2.4.4).

The physical characteristics of rotationally distorted spherical polytropes can be obtained with Eqs. (3.2.44), (3.2.45). The surface of the polytrope is represented by Eq. (3.2.45), showing an expansion of the polytrope as a whole by an amount $-\beta\psi_0(\xi_1)/\theta'(\xi_1)$, $(\theta' < 0)$, as compared to the nonrotating spherically symmetric polytrope. Superimposed on this there is a distortion, and the corresponding oblateness (flattening, ellipticity) is up to the first order in β equal to [cf. (Eq. (3.1.98)]

$$f = (a_1 - a_3)/a_1 = -5\beta\xi_1\psi_2(\xi_1)/4\theta'(\xi_1)\ [3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)], \qquad (0 < n < 5), \tag{3.2.55}$$

where

$$a_1 = \alpha\{\xi_1 - \beta\psi_0(\xi_1)/\theta'(\xi_1) - 5\beta\xi_1^2\psi_2(\xi_1)/12\theta'(\xi_1)\ [3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\}, \qquad (P_2(0) = -1/2), \tag{3.2.56}$$

is the equatorial radius, and

$$a_3 = \alpha\{\xi_1 - \beta\psi_0(\xi_1)/\theta'(\xi_1) + 5\beta\xi_1^2\psi_2(\xi_1)/6\theta'(\xi_1)\ [3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\}, \qquad (P_2(\pm 1) = 1), \tag{3.2.57}$$

the polar radius. For ellipsoidal figures of equilibrium the eccentricity $e = (a_1^2 - a_3^2)^{1/2}/a_1$ is generally used to characterize the rotational distortion. The relevant deformations will be discussed more closely in Sec. 3.4, in connection with the more general double star problem. The total mass of the configuration is

$$M = 2\pi \int_0^\pi d\lambda \int_0^{r_1(\lambda)} \varrho(r,\lambda)\ r^2 \sin\lambda\ dr = 2\pi \int_{-1}^1 d\mu \int_0^{r_1(\mu)} \varrho(r,\mu)\ r^2\ dr$$
$$= 2\pi\varrho_0\alpha^3 \int_{-1}^1 d\mu \int_0^{\Xi_1(\mu)} \Theta^n\xi^2\ d\xi \approx 2\pi\varrho_0\alpha^3 \int_{-1}^1 d\mu \int_0^{\Xi_1(\mu)} [\theta^n + \beta n\theta^{n-1}(\psi_0 + A_2\psi_2P_2)]\xi^2\ d\xi$$
$$\approx 2\pi\varrho_0\alpha^3 \Big\{ \int_{-1}^1 d\mu \int_{\xi_1}^{\Xi_1(\mu)} [\theta^n + \beta n\theta^{n-1}(\psi_0 + A_2\psi_2P_2)]\xi^2\ d\xi$$
$$+ \int_0^{\xi_1} d\xi \int_{-1}^1 [\theta^n + \beta n\theta^{n-1}(\psi_0 + A_2\psi_2P_2)]\xi^2\ d\mu \Big\}$$
$$\approx 2\pi\varrho_0\alpha^3 \int_{-1}^1 \xi_1^2[\Xi_1(\mu) - \xi_1]\{\theta^n(\xi_1) + \beta n\theta^{n-1}(\xi_1)[\psi_0(\xi_1) + A_2\psi_2(\xi_1)\ P_2(\mu)]\}\ d\mu$$
$$+4\pi\varrho_0\alpha^3 \int_0^{\xi_1} [\theta^n(\xi) + \beta n\theta^{n-1}(\xi)\ \psi_0(\xi)]\xi^2\ d\xi \approx 4\pi\varrho_0\alpha^3 \int_0^{\xi_1} \{-d(\xi^2\theta')/d\xi$$
$$+\beta[-d(\xi^2\psi_0')/d\xi + \xi^2]\}\ d\xi = 4\pi\varrho_0\alpha^3\{-\xi_1^2\theta'(\xi_1) + \beta[-\xi_1^2\psi_0'(\xi_1) + \xi_1^3/3]\}$$
$$= -4\pi[(n+1)K/4\pi G\varrho_0^{(n-3)/3n}]^{3/2}\xi_1^2\theta'(\xi_1)\{1 + \beta[\psi_0'(\xi_1) - \xi_1/3]/\theta'(\xi_1)\}, \qquad (0 < n < 5), \quad (3.2.58)$$

where we have taken into account that $\Xi_1(\mu) - \xi_1 = O(\beta)$, $\theta^n(\xi_1) = 0$, and that the integral over $P_2(\mu) = (3\mu^2 - 1)/2$ is zero. The mass of the nonrotating configuration from Eq. (2.6.18) is recovered if $\beta = 0$. As compared to the nonrotating mass m, the total equilibrium mass M of the rotating configuration is larger for the same central density ϱ_0 (cf. Table 3.2.1), because centrifugal force lowers gravity:

$$M = m\{1 + \beta[\psi_0'(\xi_1) - \xi_1/3]/\theta'(\xi_1)\}. \qquad (3.2.59)$$

The calculation of the total volume of the rotating polytrope proceeds in the same way as for the mass:

$$V = 2\pi \int_0^\pi d\lambda \int_0^{r_1(\lambda)} r^2 \sin\lambda\ dr = 2\pi\alpha^3 \int_{-1}^1 d\mu \int_0^{\Xi_1(\mu)} \xi^2\ d\xi = 2\pi\alpha^3 \Big[\int_{-1}^1 d\mu \int_0^{\xi_1} \xi^2\ d\xi$$
$$+ \int_{-1}^1 d\mu \int_{\xi_1}^{\Xi_1(\mu)} \xi^2\ d\xi \Big] = 4\pi\alpha^3\xi_1^3/3 + 2\pi\alpha^3\xi_1^2 \int_{-1}^1 d\mu \int_{\xi_1}^{\Xi_1(\mu)} d\xi = 4\pi\alpha^3\xi_1^3/3$$
$$+2\pi\alpha^3\xi_1^2 \int_{-1}^1 [\Xi_1(\mu) - \xi_1]\ d\mu = (4\pi\xi_1^3/3)\big[(n+1)K/4\pi G\varrho_0^{(n-1)/n}\big]^{3/2}\big[1 - 3\beta\psi_0(\xi_1)/\xi_1\theta'(\xi_1)\big], \qquad (3.2.60)$$

where we have used Eq. (3.2.45).

The mean density of the rotationally distorted polytrope is simply

$$\varrho_m = M/V = -[3\varrho_0\theta'(\xi_1)/\xi_1]\{1 + \beta[\psi_0'(\xi_1) - \xi_1/3]/\theta'(\xi_1)\}/[1 - 3\beta\psi_0(\xi_1)/\xi_1\theta'(\xi_1)]$$
$$\approx -[3\varrho_0\theta'(\xi_1)/\xi_1]\{1 + \beta[3\psi_0(\xi_1) + \xi_1\psi_0'(\xi_1) - \xi_1^2/3]/\xi_1\theta'(\xi_1)\}, \qquad (0 < n < 5), \qquad (3.2.61)$$

and turns into Eq. (2.6.27) for the undistorted sphere if $\beta = 0$. We now discuss the limiting cases $n = 0$ and 5.

(i) $\boldsymbol{n = 0}$. At first we briefly recall some well known results concerning the equilibrium of uniformly rotating, homogeneous configurations, equivalent to the polytrope of index $n = 0$ (e.g. Lyttleton 1953, Jardetzky 1958, Chandrasekhar 1969, Tassoul 1978). These studies have been initiated primarily by Maclaurin, Jacobi, Dedekind, Riemann, Poincaré, etc. In general, there exist even in this simplest case several possible equilibrium figures, and it appears that not all possible equilibrium solutions of Eq. (3.1.14) have been found. Likewise, polytropes with $n \neq 0$ can also have in general several possible solutions for each $\vec{\Omega}$ (Zharkov and Trubitsyn 1978).

If the equilibrium figure is constrained to be a uniformly rotating ellipsoid of constant density, without internal mass motions, the relevant equilibrium figures are the biaxial Maclaurin ellipsoids and the triaxial Jacobi ellipsoids with the three principal axes a_1, a_2, a_3. The rotation parameter $\beta = \Omega^2/2\pi G\varrho$ of Maclaurin spheroids ($a_1 = a_2$) is zero if $e = (1 - a_3^2/a_1^2)^{1/2} = 0$, ($a_1 = a_2 = a_3$; $\tau = E_{kin}/|W| = 0$), and also if $e = 1$, $[a_1 = a_2 = \infty;\ a_3 = 0;\ \tau = 0.5$; cf. Eqs. (5.10.217)-(5.10.223)]. The maximum value of the rotation parameter $\beta \approx 0.2247$ occurs at about $e = 0.9299$. At $e = 0.81267$, $(\tau = 0.1375)$ the triaxial Jacobi ellipsoids branch off from the Maclaurin sequence. At this bifurcation point there occurs their maximum of $\beta = 0.18711$, $(a_1 = a_2)$, which decreases to zero if $a_1 = \infty$, $a_2 = a_3 = 0$, and $\tau = 0.5$. Congruent to the Jacobi ellipsoids are the triaxial Dedekind ellipsoids, which unlike the Jacobi ellipsoids are stationary as seen in an inertial frame of reference, due to internal motions which prevail in the Dedekind ellipsoids, maintaining their ellipsoidal form. These motions can be characterized by the vorticity $\nabla\times\vec{v}$, where $\vec{v}$ is the velocity vector of internal motions, as seen in a frame rotating with angular velocity $\vec{\Omega}$ with respect to an inertial frame. Riemann generalized the results of Maclaurin, Jacobi, and Dedekind, showing that ellipsoidal figures of equilibrium are possible only if: (i) Internal motions are absent $(\vec{v} = 0)$, leading to the Maclaurin-Jacobi ellipsoids. (ii) $\vec{\Omega}$ and $\nabla\times\vec{v}$ are both directed along one of the three axes of the ellipsoid (in particular, if $\nabla\times\vec{v} = 0$, there results again the sequence of Maclaurin-Jacobi ellipsoids, and if $\vec{\Omega} = 0$, triaxial Dedekind ellipsoids arise). (iii) $\vec{\Omega}$ and $\nabla\times\vec{v}$ are both located in one of the three principal planes of the ellipsoid.

A spherical $n = 0$ model with internal motions vanishing on the surface (hidden angular momentum) has been envisaged by Pekeris (1980).

Lyapunov developed a method of successive approximations to find near-ellipsoidal figures of equilibrium. And finally, in 1895 Poincaré discovered the existence of his famous "pear-shaped" (more exactly ovoidal, egg-like) figures, that branch off from the Jacobi ellipsoids in a similar way as the Jacobi ellipsoids branch off from the Maclaurin sequence (see also Secs. 3.8.4, 5.8.2, 6.1.8). Unfortunately, the pear-shaped configurations are secularly unstable (e.g. Jeans 1919, Lyttleton 1953), i.e. if a slight disturbance is applied to these configurations, the presence of internal frictional forces will destroy them. More recently, Hachisu and Eriguchi (1984a, cf. also Secs. 3.8.4, 3.8.7) have computed an impressing number of new equilibrium sequences if $n = 0$. Their stability has been investigated in a series of papers by Christodoulou et al. (1995a, b, see also Secs. 5.8.2, 6.1.8, 6.4.3).

The above mentioned investigations have been concerned mainly with homogeneous, ellipsoidal or nearly ellipsoidal configurations, so they were mentioned in this place. Other methods for investigating the equilibrium structure of rotating fluid masses (e.g. the methods of Clairaut-Laplace, Wavre, Lichtenstein etc.) can easily be particularized to the case $n = 0$ (e.g. Jardetzky 1958, Tassoul 1978). Compressible Maclaurin, Jacobi, and Riemann-S ellipsoids with $n \neq 0$ have been constructed by Lai et al. (1993, cf. Sec. 5.7.4).

If $n = 0$, the coefficients from Eq. (3.2.43) can be determined by replacing $K\varrho_0^{1/n}$ with P_0/ϱ_0 via Eq. (3.2.1); also, we have to replace in Eq. (3.2.41) the otherwise vanishing value $\theta^n(\xi_1) = 0$ by $\theta^0(\xi_1) = 1$, taking into account that $\theta^0(\xi) = (1-\xi^2/6)^0$ is discontinuous only if $\xi = \xi_1 = 6^{1/2}$:

$$\begin{aligned}
&c_0 = -\xi_1\theta'(\xi_1);\quad c_{10} = \xi_1^2/2 - \psi_0(\xi_1) - \xi_1\psi_0'(\xi_1) - \xi_1\psi_0(\xi_1)/\theta'(\xi_1);\\
&k_0 = -(P_0/\varrho_0)\xi_1^2\theta'(\xi_1);\quad k_{10} = (P_0/\varrho_0)\xi_1^2[\xi_1/3 - \psi_0'(\xi_1) - \psi_0(\xi_1)/\theta'(\xi_1)];\\
&k_{12} = (P_0/\varrho_0)\xi_1^5[\xi_1\psi_2'(\xi_1) - 2\psi_2(\xi_1) + \xi_1\psi_2(\xi_1)/\theta'(\xi_1)]/6[\xi_1\psi_2'(\xi_1) + 3\psi_2(\xi_1) + \xi_1\psi_2(\xi_1)/\theta'(\xi_1)];\\
&A_2 = -5\xi_1^2/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1) + \xi_1\psi_2(\xi_1)/\theta'(\xi_1)],\qquad (n=0).
\end{aligned}\tag{3.2.62}$$

Using Eqs. (2.3.88) and (3.2.73), we obtain $c_0 = 2$, $c_{10} = 3$, $k_0 = 2\times 6^{1/2}P_0/\varrho_0$, $k_{10} = 3\times 6^{1/2}P_0/\varrho_0$, $k_{12} = -9\times 6^{1/2}P_0/\varrho_0$, $A_2 = -5/12$ (Chandrasekhar 1933d). Hence, we can write out explicitly Eqs. (3.2.44)-(3.2.47), by inserting for ψ_0, ψ_2 from Eq. (3.2.73):

$$\begin{aligned}
&\Theta(\xi,\mu) = 1 - \xi^2/6 + (\beta\xi^2/6)[1 - 5P_2(\mu)/2];\quad \Xi_1(\mu) = 6^{1/2}\{1 + \beta[1/2 - 5P_2(\mu)/4]\};\\
&\Phi = (P_0/\varrho_0)\{3 - \xi^2/6 + \beta[3 - \xi^2P_2(\mu)/4]\};\quad \Phi_e = (P_0/\varrho_0)\{2\times 6^{1/2}/\xi + 6^{1/2}\beta[3/\xi - 9P_2(\mu)/\xi^3]\},\\
&(n=0).
\end{aligned}\tag{3.2.63}$$

The constants A_j, k_{1j}, $(j > 2)$ are zero, as can be seen by inserting the analytical values from Eq. (3.2.73) into the determinant $D_j(\xi_1) = \xi_1^{-j-2}\{(j+1)\psi_j(\xi_1) + \xi_1[\psi_j(\xi_1)/\theta'(\xi_1) + \psi_j'(\xi_1)]\} \neq 0$ of the homogeneous system defining A_j, k_{1j}.

The equatorial and polar axis, and the oblateness are given by [cf. Eqs. (3.2.55)-(3.2.57)]:

$$\begin{aligned}
&a_1 = 6^{1/2}\alpha(1 + 9\beta/8);\quad a_3 = 6^{1/2}\alpha(1 - 3\beta/4);\\
&f = (a_1 - a_3)/a_1 = 15\beta/8(1 + 9\beta/8) \approx 15\beta/8,\qquad (n=0).
\end{aligned}\tag{3.2.64}$$

This first order value for the oblateness may be compared to the exact value of the eccentricity obtained for the homogeneous Maclaurin ellipsoids. For Maclaurin spheroids the exact value is (e.g. Lyttleton 1953, Chandrasekhar 1969)

$$\beta = \Omega^2/2\pi G\varrho_0 = [(3-2e^2)/e^3](1-e^2)^{1/2}\arcsin e - 3(1/e^2-1), \tag{3.2.65}$$

which for small eccentricities becomes $\beta = 4e^2/15 + O(e^4)$, or $e^2 = 15\beta/4$. And in the small eccentricity case we also have in agreement with Eq. (3.2.64): $e^2 = (a_1^2 - a_3^2)/a_1^2 \approx 2(a_1-a_3)/a_1 = 2f = 15\beta/4$ (Chandrasekhar 1933d).

Mass, volume, and mean density are [cf. Eqs. (3.2.58)-(3.2.61)]:

$$\begin{aligned} M &= 2\pi\varrho_0\alpha^3\int_{-1}^{1} d\mu \int_0^{\Xi_1(\mu)} \Theta^n\xi^2\ d\xi = (2\pi\varrho_0\alpha^3/3)\int_{-1}^{1}\Xi_1^3(\mu)\ d\mu \\ &= 6^{3/2}(4\pi\varrho_0\alpha^3/3)(1+\beta/2)^3 \approx m(1+3\beta/2); \quad V = (4\pi\alpha^3\xi_1^3/3)(1+\beta/2)^3 \\ &\approx 8\times 6^{1/2}\pi\alpha^3(1+3\beta/2); \quad \varrho_m = \varrho_0 = \mathrm{const}, \qquad (n=0). \end{aligned} \tag{3.2.66}$$

(ii) *n* = 5. The undistorted Schuster-Emden integral $\theta = (1+\xi^2/3)^{-1/2}$ has infinite extension and finite mass [Eqs. (2.3.90), (2.6.194)], so if rotation is included, one has to be careful in performing the limiting process. As the polytropic index approaches $n=5$, the value of the Lane-Emden function θ is very nearly zero long before ξ becomes equal to its boundary value $\xi_1 = \infty$. The asymptotic solution of Eqs. (3.2.10), (3.2.11) if $n=5$ and $\xi \gg 1$ is given by Eq. (3.2.89): $\psi_0 \approx \xi^2/6,\ \psi_j \approx B_j\xi^j$. We now insert Eq. (3.2.89) into Eq. (3.2.45), to obtain the equivalent of the boundary of a distorted polytrope if $n=5$ and $\xi_1 \gg 1$, $(A_2 = -1/6B_2;\ \theta \approx 0;\ \theta' \approx -3^{1/2}/\xi^2)$:

$$\Xi_1(\mu) = \xi_1 + (\beta\xi_1^4/6\times 3^{1/2})[1-P_2(\mu)] = \xi_1 + \beta\xi_1^4\sin^2\lambda/4\times 3^{1/2}, \qquad (n=5;\ \xi_1 \gg 1;\ \beta \ll 1). \tag{3.2.67}$$

Inserting $\psi_j,\ (j>2)$ from Eq. (3.2.89) into the determinant $D_j(\xi_1),\ (\xi_1 \gg 1)$ of the homogeneous system defining A_j and k_{1j}, we observe at once that $D_j(\xi_1) \neq 0$, so A_j, k_{1j} are zero if $j>2$.

For sufficiently large ξ_1 the equatorial radius becomes $a_1 = \alpha\Xi_1(0) = \alpha\xi_1(1+\beta\xi_1^3/4\times 3^{1/2})$, and the polar radius is $a_3 = \alpha\Xi_1(1) = \alpha\xi_1$, equal to the radius of the spherically symmetric polytrope. The oblateness is $f = (a_1-a_3)/a_1 \approx \beta\xi_1^3/4\times 3^{1/2}$, [cf. Eq. (3.8.160)]. The distorted Lane-Emden function is by virtue of Eq. (3.2.44) equal to

$$\Theta(\xi,\mu) \approx 3^{1/2}/\xi + (\beta\xi^2/6)[1-P_2(\mu)] = 3^{1/2}/\xi + \beta\xi^2\sin^2\lambda/4, \qquad (n=5;\ \xi \gg 1). \tag{3.2.68}$$

Caimmi (1980b, 1987) has presented a more complete discussion of this particular case, and has also investigated rotationally distorted polytropes with polytropic indices differing slightly from $n = 0, 1, 5$ (Caimmi 1988).

It is instructive to compare the polytrope $n=5$ with the so-called Roche model, if the whole mass M of the rotating polytrope is assumed to be concentrated at the centre. Eq. (3.1.23) for the total potential of the $n=5$ polytrope becomes

$$\begin{aligned} \Phi_{tot} &= GM/R_1(\lambda) + \Omega^2\ell^2/2 = GM/\alpha\Xi_1(\lambda) + \Omega^2\alpha^2\Xi_1^2(\lambda)\sin^2\lambda/2 \\ &\approx GM/\alpha\xi_1 - (GM/\alpha)(\beta\xi_1^2\sin^2\lambda/4\times 3^{1/2}) + \beta\pi G\varrho_0\alpha^2\xi_1^2\sin^2\lambda \\ &= GM/r_1 - (GM/r_1)(\beta\xi_1^3\sin^2\lambda/4\times 3^{1/2}) + \beta\pi G\varrho_m\alpha^2\xi_1^5\sin^2\lambda/3\times 3^{1/2} \\ &= GM/r_1 = \mathrm{const}, \qquad (\xi_1 \gg 1;\ n=5), \end{aligned} \tag{3.2.69}$$

where $r_1 = \alpha\xi_1$ and $R_1(\lambda) = \alpha\Xi_1(\lambda)$ are the distances from the centre of the undistorted and of the distorted configuration, respectively. We have also used Eq. (2.6.27): $\varrho_0 = \varrho_m\xi_1^3/3^{3/2} = M\xi_1^3/4\times 3^{1/2}\pi r_1^3$. Thus, the surface determined by Eq. (3.2.69) is an equipotential, and the matter inside the undistorted radius $r_1 = \alpha\xi_1,\ (\xi_1 \gg 1)$ is just equal to the whole mass M of the configuration. This shows the first order equivalence between the point mass rotating Roche model (negligible mass of the "atmosphere") and the rotating polytrope $n=5$.

This section is concluded with the study of the associated Emden-Chandrasekhar functions from Eqs. (3.2.10), (3.2.11), (cf. Horedt 1990). So far, analytical and numerical solutions of associated

Emden-Chandrasekhar functions are spread out over a dozen of papers, and are summarized below. The function ψ_1 has no practical significance and will be largely ignored.

Since exact analytical solutions of the Lane-Emden equation (2.3.87) are known so far only for the polytropic indices $n = 0, 1, 5$, [Eqs. (2.3.88)-(2.3.90)], exact analytical expressions of ψ_j, $(j = 0, 1, 2, 3, ...)$ can be provided only for these indices.

(i) $\boldsymbol{n = 0}$. Eqs. (3.2.10) and (3.2.11) read

$$\psi_0'' + 2\psi_0'/\xi - 1 = 0; \quad \psi_j'' + 2\psi_j'/\xi - j(j+1)\psi_j/\xi^2 = 0, \qquad (n = 0;\ j = 1, 2, 3, ...), \tag{3.2.70}$$

with the solutions (Chandrasekhar 1933d, Caimmi 1980b)

$$\psi_0 = \xi^2/6 + B_0/\xi + C_0, \qquad (B_0, C_0 = \mathrm{const}), \tag{3.2.71}$$

$$\psi_j = B_j\xi^j + C_j\xi^{-j-1}, \qquad (B_j, C_j = \mathrm{const};\ j = 1, 2, 3, ...). \tag{3.2.72}$$

The initial conditions from Eq. (3.2.6) $\psi_j(0), \psi_j'(0) = 0$ yield $B_0, B_1, C_j = 0$, leaving B_j, $(j = 2, 3, 4, ...)$ undetermined. The constants B_j can be normalized to 1 [cf. Eqs. (3.2.85)-(3.2.88)], and in this case the Emden-Chandrasekhar functions of the homogeneous polytrope read

$$\psi_0 = \xi^2/6; \quad \psi_j = \xi^j, \qquad (n = 0;\ j = 2, 3, 4, ...). \tag{3.2.73}$$

(ii) $\boldsymbol{n = 1}$. This case is somewhat more complicated, and solutions have been published by Chandrasekhar (1933a) for ψ_0, by Steensholt (1935), and Chandrasekhar and Lebovitz (1962d) for ψ_2, and by Kopal (1939), Papoyan et al. (1967), and Caimmi (1980b) for $j = 0, 1, 2, 3, ...$ Chandrasekhar (1933a) makes the transformation $\psi_0 = 1 - u$, so that Eq. (3.2.10) becomes identical to the Lane-Emden equation (2.3.6) if $N = 3$, with the well known solution (2.3.89): $u = \sin\xi/\xi$. Thus

$$\psi_0 = 1 - \sin\xi/\xi = 1 - (\pi/2\xi)^{1/2}J_{1/2}(\xi), \qquad (n = 1), \tag{3.2.74}$$

satisfies the initial conditions $\psi_0(0), \psi_0'(0) = 0$. To obtain the solution for ψ_j, $(j = 1, 2, 3, ...)$, we implement the theory outlined in Eqs. (2.3.7)-(2.3.16), and transform Eq. (3.2.11) into the Bessel equation

$$\xi^2 u_j'' + \xi u_j' + [\xi^2 - (j + 1/2)^2]u_j = 0, \qquad (n = 1), \tag{3.2.75}$$

where $u_j = \xi^{1/2}\psi_j$. The general solution is

$$u_j = B_jJ_{j+1/2}(\xi) + C_jJ_{-j-1/2}(\xi) \quad \text{or} \quad \psi_j = \xi^{-1/2}[B_jJ_{j+1/2}(\xi) + C_jJ_{-j-1/2}(\xi)],$$
$$(j + 1/2 \neq 0, 1, 2, 3, ...;\ B_j, C_j = \mathrm{const}). \tag{3.2.76}$$

The expansion of the Bessel function of order ν near $\xi \approx 0$ is given by Eq. (2.3.12), and since $\psi_j(0) = 0$, we get $C_j = 0$. Eq. (3.2.76) writes near the origin as

$$\psi_j = B_j\xi^{-1/2}J_{j+1/2}(\xi) \approx B_j\xi^j/2^{j+1/2}\Gamma(j + 3/2) = B_j(2/\pi)^{1/2}\xi^j/(2j+1)!!, \qquad (\xi \approx 0). \tag{3.2.77}$$

Near the origin we have $\psi_j \approx \xi^j$, and the initial conditions $\psi_j(0), \psi_j'(0) = 0$ yield $B_1 = 0$ and $B_j = (\pi/2)^{1/2}(2j+1)!!$, $(j = 2, 3, 4, ...)$. Eq. (3.2.76) takes the form (cf. Eqs. (2.3.22), (2.3.23), Kopal 1939, Papoyan et al. 1967)

$$\psi_j = (\pi/2)^{1/2}(2j+1)!!\ \xi^{-1/2}J_{j+1/2}(\xi) = (-1)^j(2j+1)!!\ \xi^j\ d^j(\sin\xi/\xi)/(\xi\ d\xi)^j,$$
$$(n = 1;\ j = 2, 3, 4, ...). \tag{3.2.78}$$

Thus, if $n = 1$, the Emden-Chandrasekhar functions can be expressed under the form of trigonometric polynomials.

(iii) $\boldsymbol{n = 5}$. The total mass of the spherical polytrope is finite, although it extends with vanishing density up to infinity. Solutions of Eqs. (3.2.10), (3.2.11) can be provided by using the transformation of Seidov and Kuzakhmedov (1978) for the Schuster-Emden integral:

$$\theta(\xi) = (1 + \xi^2/3)^{-1/2} = \cos\alpha; \quad \xi = 3^{1/2}\tan\alpha. \tag{3.2.79}$$

Eqs. (3.2.10), (3.2.11) become

$$d^2\psi_0/d\alpha^2 + (2/\tan\alpha)\ d\psi_0/d\alpha + 15\psi_0 - 3/\cos^4\alpha = 0, \tag{3.2.80}$$

$$d^2\psi_j/d\alpha^2 + (2/\tan\alpha)\ d\psi_j/d\alpha + [15 - j(j+1)\sin^2\alpha\cos^2\alpha]\psi_j = 0. \tag{3.2.81}$$

These equations can only be solved if a particular integral is known. With some skill we find for the homogeneous part of Eq. (3.2.80) the particular integrals $\sin 4\alpha/\sin\alpha \propto \cos 3\alpha + \cos\alpha$ and $\cos 4\alpha/\sin\alpha$ (cf. Eqs. (2.4.157)-(2.4.163), Seidov and Kuzakhmedov 1978). The solution of the nonhomogeneous equation (3.2.80) can now be found by standard methods:

$$\begin{aligned}\psi_0 &= (3\sin 4\alpha/4\sin\alpha)\int(\sin\alpha\cos 4\alpha/\cos^4\alpha)\ d\alpha - (3\cos 4\alpha/4\sin\alpha)\int(\sin\alpha\sin 4\alpha/\cos^4\alpha)\ d\alpha\\ &+E_0\sin 4\alpha/\sin\alpha + F_0\cos 4\alpha/\sin\alpha = -60\cos^2\alpha + 20 + 1/2\cos^2\alpha\\ &-(15\cos 4\alpha/2\sin\alpha)\ln[\tan(\pi/4+\alpha/2)] + E_0\sin 4\alpha/\sin\alpha + F_0\cos 4\alpha/\sin\alpha\\ &= -60\cos^2\alpha + 20 + 1/2\cos^2\alpha - (15\cos 4\alpha/2\sin\alpha)\ln[(1+\sin\alpha)/\cos\alpha] + (47/4)\sin 4\alpha/\sin\alpha,\\ &(n=5;\ \alpha = \arctan(\xi/3^{1/2});\ E_0 = 47/4;\ F_0 = 0).\end{aligned} \tag{3.2.82}$$

If $\xi \approx 0$, we have $\xi \approx 3^{1/2}\alpha$, and Eq. (3.2.82) shows that $\psi_0 \approx \alpha^2/2 \approx \xi^2/6$. If $\xi \to \infty$, there is $3^{1/2}/\xi \approx \pi/2 - \alpha$, $(\alpha \approx \pi/2)$, and $\psi_0 \approx 1/2\cos^2\alpha \approx 1/2\sin^2(3^{1/2}/\xi) \approx \xi^2/6$, [cf. Eq. (3.2.89)].

To seek a particular integral of Eq. (3.2.81), we adopt the attempt $\psi_j \propto \sin^a\alpha\cos^b\alpha$, and find that solutions of this form exist merely if $j = 1$, $(a = 1;\ b = 2)$ and $j = 2$, $(a = -3;\ b = -2)$. Since ψ_1 lacks practical importance, we focus on ψ_2, and get with the standard method for the solution of a second order homogeneous equation:

$$\begin{aligned}\psi_2 &= [E_2(3\alpha/8 - \sin 4\alpha/8 + \sin 8\alpha/64) + F_2]/\sin^3\alpha\cos^2\alpha = E_2(3\alpha/8\sin^3\alpha\cos^2\alpha\\ &-3/8\sin^2\alpha\cos\alpha + 3/4\cos\alpha - \cos\alpha + 2\sin^2\alpha\cos\alpha) + F_2/\sin^3\alpha\cos^2\alpha = (15/32)\\ &\times(3\alpha - \sin 4\alpha + \sin 8\alpha/8)/\sin\alpha\sin^2 2\alpha, \qquad (n=5;\ \alpha=\arctan(\xi/3^{1/2});\ E_2 = 15/16;\ F_2 = 0).\end{aligned} \tag{3.2.83}$$

This equation is equivalent to Eq. (11d) of Caimmi (1987). The constant F_2 must be zero, and E_2 has been determined by equating the approximation near the origin $\psi_2 \approx 16E_2\alpha^2/5$ of Eq. (3.2.83) to the approximation $\psi_2 \approx \xi^2 \approx 3\tan^2\alpha \approx 3\alpha^2$, $(\alpha, \xi \approx 0)$ from Eq. (3.2.86).

Series expansions of ψ_0 and ψ_2 near the first finite zero ξ_1 of θ have been derived by Linnell (1975a, c), and for ψ_2 at an arbitrary point by Caimmi (1983). The same author (Caimmi 1988) has calculated approximate forms of ψ_2 if n differs slightly from 0, 1, and 5. Simple approximate forms of ψ_j have been published by Chandrasekhar for general n if $\xi \approx 0$ (Chandrasekhar 1933a, Eqs. (52), (53) for ψ_0, ψ_2; Chandrasekhar 1933b, Eqs. (42), (43) for ψ_3, ψ_4). While the first coefficient of the expansion of ψ_0 can be determined exactly $\psi_0 \approx \xi^2/6$, $(\xi \approx 0)$, the homogeneous equations (3.2.11) together with the initial conditions $\psi_j(0), \psi_j'(0) = 0$, $(j = 2, 3, 4, ...)$ provide only an expansion including an arbitrary constant; it is a simple exercise to show that [see Eqs. (2.4.10)-(2.4.24)]:

$$\psi_j \approx B_j\xi^j, \qquad (\xi \approx 0;\ B_j = \mathrm{const};\ j = 2, 3, 4, ...). \tag{3.2.84}$$

Following Chandrasekhar (1933a, b) we set the arbitrary constants B_j equal to 1, so that $\psi_j \approx \xi^j$, which has been used to fix the integration constants in previous equations. If $\xi \approx 0$, we get:

$$\psi_0 \approx \xi^2/6 - n\xi^4/120 + n(13n-10)\xi^6/42\times 360 - n(90n^2 - 157n + 70)\xi^8/42\times 72\times 360, \tag{3.2.85}$$

$$\psi_2 \approx \xi^2 - n\xi^4/14 + n(10n-7)\xi^6/36\times 42 - n(308n^2 - 503n + 210)\xi^8/36\times 42\times 330, \tag{3.2.86}$$

$$\psi_3 \approx \xi^3 - n\xi^5/18 + n(4n-3)\xi^7/18\times 44 - n(434n^2 - 749n + 330)\xi^9/18\times 44\times 1170, \tag{3.2.87}$$

$$\psi_4 \approx \xi^4 - n\xi^6/22 + n(14n-11)\xi^8/52\times 66 - n(1744n^2 - 3129n + 1430)\xi^{10}/52\times 66\times 1350. \tag{3.2.88}$$

Table 3.2.1 Values of associated Emden-Chandrasekhar functions $\psi_0, \psi_2, \psi_3, \psi_4$, and of their derivatives at the boundary ξ_1 of undistorted spherical polytropes for polytropic indices n between 0 and 5. The surface value of the derivative of the Lane-Emden function θ_1' is also shown in the third column (Horedt 1990). $aE+b$ means $a \times 10^b$.

n	ξ_1	θ_1'	ψ_0	ψ_0'	ψ_2
0.0	2.449490E+0	−8.164966E−1	1.000000E+0	8.164966E−1	6.000000E+0
0.5	2.752698E+0	−4.999971E−1	9.096316E−1	−1.085581E−2	4.891598E+0
1.0	3.141593E+0	−3.183099E−1	1.000000E+0	3.183099E−1	4.559453E+0
1.5	3.653754E+0	−2.033013E−1	1.294395E+0	6.407239E−1	4.787407E+0
2.0	4.352875E+0	−1.272487E−1	1.915282E+0	9.969734E−1	5.643151E+0
2.5	5.355275E+0	−7.626491E−2	3.166462E+0	1.435192E+0	7.479146E+0
3.0	6.896849E+0	−4.242976E−2	5.837811E+0	2.039084E+0	1.127721E+1
3.5	9.535805E+0	−2.079098E−2	1.239378E+1	2.997633E+0	2.006616E+1
4.0	1.497155E+1	−8.018079E−3	3.351425E+1	4.878946E+0	4.660961E+1
4.5	3.183646E+1	−1.714549E−3	1.628123E+2	1.056119E+1	1.992369E+2
5.0	∞	0	∞	∞	∞

n	ψ_2'	ψ_3	ψ_3'	ψ_4	ψ_4'
0.0	4.898979E+0	1.469694E+1	1.800000E+1	3.600000E+1	5.878775E+1
0.5	−6.505232E−1	1.475847E+1	5.232472E+0	4.306763E+1	3.440906E+1
1.0	4.206911E−1	1.737363E+1	9.795382E+0	6.115690E+1	5.902842E+1
1.5	1.161345E+0	2.336790E+1	1.490714E+1	1.001094E+2	9.678811E+1
2.0	1.757774E+0	3.557219E+1	2.185527E+1	1.877232E+2	1.637728E+2
2.5	2.338160E+0	6.161833E+1	3.290931E+1	4.094269E+2	2.998889E+2
3.0	3.040719E+0	1.248076E+2	5.335848E+1	1.085133E+3	6.253689E+2
3.5	4.106089E+0	3.160249E+2	9.891934E+1	3.843113E+3	1.609422E+3
4.0	6.189315E+0	1.176785E+3	2.355588E+2	2.268051E+4	6.057844E+3
4.5	1.250775E+1	1.088652E+4	1.025754E+3	4.501658E+5	5.655844E+4
5.0	∞	∞	∞	∞	∞

A simple asymptotic expansion of ψ_j if $n=5$ has been provided by Chandrasekhar (1933d), by ignoring in Eqs. (3.2.10), (3.2.11) the term $5\theta^4 \approx 45/\xi^4$ if $\xi \to \infty$. The asymptotic form of Eqs. (3.2.10), (3.2.11) becomes in this case equal to Eq. (3.2.70), with the solutions [cf. Eqs. (3.2.71), (3.2.72)]:

$$\psi_0 \approx \xi^2/6; \quad \psi_2 \approx B_2\xi^2 = 15\pi\xi^2/256; \quad \psi_j \approx B_j\xi^j, \quad (n=5;\ \xi\to\infty;\ B_j = \mathrm{const};\ j=1,2,3,...). \tag{3.2.89}$$

Since the value of ψ_j is already fixed by the choice $\psi_j \approx \xi^j$ near the origin, the B_j's are not arbitrary, and depend on the whole march of ψ_j. Using the exact solution of ψ_2 from Eq. (3.2.83), we find at once that $B_2 = 15\pi/256 = 0.18408$, ($\alpha \approx \pi/2$; $\cos\alpha \approx 3^{1/3}/\xi$ if $\xi \to \infty$). If $j=3,4$, we have found from numerical integrations, extended up to $\xi = 1000$, that $B_3 \approx 0.32214$ and $B_4 \approx 0.42280$, $(n=5)$.

Values of the Emden-Chandrasekhar functions have been published by Chandrasekhar (1933a, b), and Chandrasekhar and Lebovitz (1962d) for polytropic indices $n = 0, 1, 1.5, 2, 3, 3.5, 4$, by Aikawa (1968, 1971, excepting for ψ_3, ψ_3') if $n = 1.5, 2, 2.5, 3, 3.5$, by Jabbar (1984) for ψ_0, $(0 \le n \le 5$ with a step size of 0.1), by Caimmi (1985) for ψ_2, ψ_2', $(0 \le n \le 5$ with a step size of 0.25), and by Horedt (1990, see Table 3.2.1).

Below, we provide a simple first order expansion of ψ_j near the boundary, showing that if $n \approx 0$, $(n > 0)$, no exact values of $\psi_j'(\xi_1)$ can be obtained by numerical integration. The Lane-Emden function can be expanded near the finite boundary as

$$\theta(\xi) \approx \theta(\xi_1) + (\xi - \xi_1)\ \theta'(\xi_1) = (\xi_1 - \xi)[-\theta'(\xi_1)], \qquad (\theta_1' = \theta'(\xi_1) < 0;\ \xi \approx \xi_1;\ -1 < n < 5). \tag{3.2.90}$$

Eqs. (3.2.10), (3.2.11) become readily integrable if we approximate $\psi_j(\xi)$, $(j = 0, 1, 2, 3, ...)$ by its boundary value $\psi_{j1} = \psi_j(\xi_1)$:

$$\psi_0'' + 2\psi_0'/\xi + [n(-\theta_1')^{n-1}\psi_{01}(\xi_1 - \xi)^{n-1} - 1] = 0, \qquad (\xi \approx \xi_1;\ -1 < n < 5), \tag{3.2.91}$$

$$\psi_j'' + 2\psi_j'/\xi + [n(-\theta_1')^{n-1}(\xi_1 - \xi)^{n-1} - j(j+1)/\xi^2]\psi_{j1} = 0,$$
$$(\xi \approx \xi_1;\ -1 < n < 5;\ j = 1, 2, 3, ...). \qquad (3.2.92)$$

The solutions of these equations are elementary:

$$\psi_0' = (-\theta_1')^{n-1}\psi_{01}(\xi_1 - \xi)^n[1 + 2(\xi_1 - \xi)/(n+1)\xi + 2(\xi_1 - \xi)^2/(n+1)(n+2)\xi^2] + \xi/3 + B_0/\xi^2, \qquad (3.2.93)$$

$$\psi_0 = -[(-\theta_1')^{n-1}\psi_{01}(\xi_1 - \xi)^{n+1}/(n+1)][1 + 2(\xi_1 - \xi)/(n+2)\xi] + \xi^2/6 - B_0/\xi + C_0,$$
$$(B_0, C_0 = \mathrm{const}), \qquad (3.2.94)$$

$$\psi_j' = (-\theta_1')^{n-1}\psi_{j1}(\xi_1 - \xi)^n[1 + 2(\xi_1 - \xi)/(n+1)\xi + 2(\xi_1 - \xi)^2/(n+1)(n+2)\xi^2]$$
$$+ j(j+1)\psi_{j1}/\xi + B_j/\xi^2, \qquad (3.2.95)$$

$$\psi_j = -[(-\theta_1')^{n-1}\psi_{j1}(\xi_1 - \xi)^{n+1}/(n+1)][1 + 2(\xi_1 - \xi)/(n+2)\xi]$$
$$+ j(j+1)\psi_{j1}\ln\xi - B_j/\xi + C_j, \qquad (B_j, C_j = \mathrm{const}). \qquad (3.2.96)$$

It is obvious at once that the derivatives have a singularity at the boundary if $-1 < n < 0$, but since Chandrasekhar's (1933a-d) theory is only applicable if $0 \le n \le 5$, this result is merely of philosophical interest.

If $0 < n < 5$, the integration constants are $B_0 = \xi_1^2(\psi_{01}' - \xi_1/3),\ B_j = \xi_1^2\psi_{j1}' - j(j+1)\xi_1\psi_{j1}$, and if further $n \approx 0,\ (n > 0)$, Eqs. (3.2.93) and (3.2.95) write approximately

$$\psi_j' \approx \psi_{j1}' + (-\theta_1')^{n-1}\psi_{j1}(\xi_1 - \xi)^n, \qquad (n \approx 0;\ n > 0;\ j = 0, 1, 2, 3, ...). \qquad (3.2.97)$$

These equations show that it has not much sense to calculate numerically the boundary value ψ_{j1}' for polytropic indices $0 < n \lesssim 0.1$ – as done by Caimmi (1985) – since even for $n = 0.1$ we obtain if $\xi_1 - \xi = 10^{-12}:\ \psi_0' - \psi_{01}' \approx 0.080,\ \psi_2' - \psi_{21}' \approx 0.47,\ \psi_3' - \psi_{31}' \approx 1.2,\ \psi_4' - \psi_{41}' \approx 3.1$ (Horedt 1990).

Caimmi (1980b, 1983, 1985, 1987) has developed a variant of Chandrasekhar's (1933a, d) perturbation theory, by introducing instead of Eqs. (3.2.4), (3.2.9) the expansion

$$\Theta(\xi, \mu, \beta) = \theta_\beta(\xi, \beta) + \sum_{j=1}^{\infty} A_{2j}(\beta)\ \psi_{2j}(\xi)\ P_{2j}(\mu), \qquad [\theta_\beta(\xi, 0) = \theta(\xi)], \qquad (3.2.98)$$

where $\theta_\beta(\xi, \beta)$ differs from the usual Lane-Emden function $\theta(\xi)$ by terms of order β.

Bhatnagar (1940) has applied Chandrasekhar's (1933a) theory to differentially rotating polytropes.

3.3 Chandrasekhar's First Order Theory of Tidally Distorted Polytropes

This theory is based on Eqs. (3.1.101) and (3.1.102). Because the problem is symmetrical with respect to the line joining the two masses M and M', it is advisable to place in this special case the mass M' on the Mx_3-axis, rather than on the Mx_1-axis as in Fig. 3.1.1. The tide-generating external potential of the mass M' is approximated by its point mass value $\Phi'_e(\vec{r'}) \approx GM'/r'$. A second order approximation would be for instance MacCullagh's formula (3.1.65): $\Phi'_e(\vec{r'}) \approx GM'/r' + G(A' + B' + C' - 3I'_{\vec{r''}})/2r'^3$. To the order of magnitude, we have

$$|C' - I'_{\vec{r''}}| \approx |B' - I'_{\vec{r''}}| \approx |A' - I'_{\vec{r''}}| \approx \int_{M'} |x_2''^2 + x_3''^2 - (x_1''^2 + x_2''^2 + x_3''^2)\sin^2\gamma''|\, dM''$$
$$= \int_{M'} |r''^2 \cos^2\gamma'' - x_1''^2|\, dM'' \approx O[|r_1'^2 - a_1'^2|M'] \approx O[|r_1' - a_1'|r_1'M'], \tag{3.3.1}$$

where a_1' is a principal axis, and r_1' an average radius of M'. We have denoted by γ'' the angle between the radius vector $\vec{r'}$ of an exterior point to M' and the radius vector $\vec{r''} = \vec{r''}(x_1'', x_2'', x_3'')$ of an interior point (x_1'', x_2'', x_3'') of M', both measured from the mass centre of M'. Eq. (3.1.63) shows that J_2 and C_{22} for M' are of order $O[|A' - I'_{\vec{r''}}|/M'r_1'^2] \approx O[|r_1' - a_1'|/r_1']$, $(r_1' \approx a_1' \approx a_2' \approx a_3')$. Therefore, if we disregard rotation in Eq. (3.1.98), we obtain the magnitude of tidal distortion of M' due to M, by interchanging M with M' :

$$|a_1' - a_2'|,\ |a_1' - a_3'| \approx O(r_1'^4/D^3), \qquad (O(M/M') \approx 1). \tag{3.3.2}$$

Thus, the second order term in MacCullagh's formula (3.1.65) is of order (Chandrasekhar 1933b)

$$G|A' + B' + C' - 3I'_{\vec{r''}}|/2r'^3 \approx G|A' + B' + C' - 3I'_{\vec{r''}}|/2D^3 \approx O\big[G|a_1' - r_1'|r_1'M'/D^3\big]$$
$$\approx O\big[G|a_1' - a_2'|r_1'M'/D^3\big] \approx O[GM'r_1'^5/D^6], \tag{3.3.3}$$

since $r' \approx D$ if $r' \gg r_1'$. Hence, if we neglect quantities of order $O[(GM'/r_1')(r_1'/D)^6]$, the tide-generating potential of M' can be approximated by its point mass value $\Phi'_e(\vec{r'}) = GM'/r'$. The external potential Φ'_e of M' takes in the neighborhood of the primary M the form [cf. Eq. (3.1.42)]

$$\Phi'_e(\vec{r'}) = GM'/r' = (GM'/D)[1 - 2(r/D)\cos\lambda + (r/D)^2]^{-1/2} = (GM'/D)\sum_{j=0}^{\infty}(r/D)^j P_j(\cos\lambda)$$
$$= (GM'/D)\sum_{j=1}^{\infty}(r/D)^j P_j(\cos\lambda) + \mathrm{const} \approx GM'r\cos\lambda/D^2 + (GM'/D)\sum_{j=2}^{4}(r/D)^j P_j(\mu) + \mathrm{const},$$
$$(\mu = \cos\lambda = x_3/D), \tag{3.3.4}$$

where λ is the zenith angle between $\vec{r}$ and the axis Mx_3, the mass M' being located in this section on the axis Mx_3 of Fig. 3.1.1. Consistently with our scheme of approximation we have neglected quantities of order $(GM'/r_1)(r_1/D)^6$ and higher, where r_1 denotes an average radius of M. As outlined subsequently to Eq. (3.1.101), the term $GM'r\cos\lambda/D^2 = GM'x_3/D^2$ in the last equation (3.3.4) yields a uniform gravitational force $\partial(GM'x_3/D^2)/\partial x_3 = GM'/D^2$, acting on the primary M as a whole. This term is eliminated by considering the motion in a new frame, moving with acceleration GM'/D^2 (Jeans 1919), the mass M remaining always at the origin of the new frame; the adoption of this accelerated frame might not be quite rigorous, because D changes. In this new frame we are left with an external tidal potential of M' of the form

$$\Phi'_e(\vec{r'}) \approx (GM'/D)\sum_{j=2}^{4}(r/D)^j P_j(\mu) + \mathrm{const}. \tag{3.3.5}$$

The constant can be taken equal to zero. The equations of the problem are given by the hydrostatic equation (2.1.3) and by Poisson's equation (2.1.4):

$$\nabla P = \varrho\ \nabla(\Phi + \Phi_e'); \quad \nabla^2\Phi = -4\pi G\varrho; \quad \nabla^2\Phi_e' = 0. \tag{3.3.6}$$

These equations can be combined into the fundamental equation of the tidal problem:

$$\nabla[(1/\varrho)\ \nabla P] = \nabla^2(\Phi + \Phi_e') = -4\pi G\varrho. \tag{3.3.7}$$

We turn with Eq. (3.2.1) to dimensionless variables, and obtain in spherical coordinates an equation similar to Eq. (3.2.2) if $\beta = 0$:

$$\partial(\xi^2\ \partial\Theta/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Theta/\partial\mu]/\partial\mu = -\xi^2\Theta^n. \tag{3.3.8}$$

The special cases $n = 0$ and 5 are considered after presenting the subsequent solution for $0 < n < 5$. Similarly to Eq. (3.2.4) the solution is assumed under the form

$$\Theta(\xi,\mu) = \theta(\xi) + \Psi(\xi,\mu), \qquad (|\Psi| \ll \theta). \tag{3.3.9}$$

Substituting Eq. (3.3.9) into Eq. (3.3.8), and neglecting quantities in Ψ^2 and higher orders, we get the differential equation defining Ψ :

$$\partial(\xi^2\ \partial\Psi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Psi/\partial\mu]/\partial\mu = -n\xi^2\theta^{n-1}\Psi. \tag{3.3.10}$$

Analogously to Eq. (3.2.9) we assume for Ψ the following form:

$$\Psi = \sum_{j=0}^{\infty} A_j\psi_j(\xi)\ P_j(\mu), \qquad (A_j = \mathrm{const}). \tag{3.3.11}$$

Substitution of Eq. (3.3.11) into Eq. (3.3.10) yields for ψ_j the differential equation

$$d(\xi^2\ d\psi_j/d\xi)/d\xi = [j(j+1) - n\xi^2\theta^{n-1}]\psi_j, \qquad (j = 0,1,2,3,...;\ \psi_j(0), \psi_j'(0) = 0). \tag{3.3.12}$$

The internal potential of M is written analogously to Eq. (3.2.13):

$$\Phi = U_0(\xi) + \sum_{j=0}^{\infty} V_j(\xi)\ P_j(\mu). \tag{3.3.13}$$

Poisson's equation (3.1.18) reads in the (ξ,μ)-variables as

$$\partial(\xi^2\ \partial\Phi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu = -(n+1)K\varrho_0^{1/n}\xi^2\left(\theta^n + n\theta^{n-1}\sum_{j=0}^{\infty} A_j\psi_jP_j\right). \tag{3.3.14}$$

Due to the slight inconsistency in Chandrasekhar's (1933a) original presentation, mentioned subsequently to Eq. (3.2.38), we present the calculations in some detail. We substitute Eq. (3.3.13) into Eq. (3.3.14), and obtain analogously to Eqs. (3.2.14)-(3.2.16)

$$d(\xi^2\ dU_0/d\xi)/d\xi = -(n+1)K\varrho_0^{1/n}\xi^2\theta^n, \tag{3.3.15}$$

$$d(\xi^2\ dV_j/d\xi)/d\xi - j(j+1)V_j = -n(n+1)K\varrho_0^{1/n}A_j\xi^2\theta^{n-1}\psi_j, \qquad (j = 0,1,2,3,...), \tag{3.3.16}$$

with the solutions [cf. Eqs. (3.2.17)-(3.2.28)]:

$$U_0 = (n+1)K\varrho_0^{1/n}(\theta + c_0); \quad V_j = (n+1)K\varrho_0^{1/n}(A_j\psi_j + B_j\xi^j). \tag{3.3.17}$$

The internal potential of M becomes

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\theta(\xi) + c_0 + \sum_{j=0}^{\infty}[A_j\psi_j(\xi) + B_j\xi^j]\ P_j(\mu)\right\}. \tag{3.3.18}$$

The radial component of the hydrostatic equilibrium equation (3.3.6) writes

$$\partial P/\partial\xi = \varrho(\partial\Phi/\partial\xi + \partial\Phi'_e/\partial\xi) \quad \text{or} \quad (n+1)K\varrho_0^{1/n}\ \partial\Theta/\partial\xi = \partial\Phi/\partial\xi + \partial\Phi'_e/\partial\xi. \tag{3.3.19}$$

The external potential (3.3.5) of M' reads in terms of ξ as

$$\Phi'_e = (GM'/D)\sum_{j=2}^{4}(\alpha/D)^j\xi^j P_j(\mu). \tag{3.3.20}$$

We derive Eqs. (3.3.18), (3.3.20) with respect to ξ, and insert into Eq. (3.3.19). Equating the coefficients of various Legendre polynomials, we find

$$B_0 = c_{10} = \text{const};\quad B_j = -[GM'/D(n+1)K\varrho_0^{1/n}](\alpha/D)^j \text{ if } j=2,3,4;\quad B_j = 0 \text{ if } j \neq 0,2,3,4. \tag{3.3.21}$$

Hence, our expression for the internal potential of M is

$$\Phi = (n+1)K\varrho_0^{1/n}\Big\{\theta(\xi) + c_0 + \sum_{j=0}^{\infty}A_j\psi_j(\xi)\ P_j(\mu) + c_{10}P_0(\mu)$$
$$-[GM'/D(n+1)K\varrho_0^{1/n}]\sum_{j=2}^{4}(\alpha/D)^j\xi^j P_j(\mu)\Big\}. \tag{3.3.22}$$

The equation of the surface is [cf. Eq. (3.2.34)]

$$\Xi_1 = \Xi_1(\mu) = \xi_1 + \sum_{j=0}^{\infty}q_jP_j(\mu), \qquad (\Xi_1 \approx \xi_1). \tag{3.3.23}$$

On the surface we have [cf. Eq. (3.2.36)]

$$\theta(\Xi_1) \approx \theta(\xi_1) + \theta'(\xi_1)\sum_{j=0}^{\infty}q_jP_j(\mu);\quad \psi_k(\Xi_1) \approx \psi_k(\xi_1) + \psi'_k(\xi_1)\sum_{j=0}^{\infty}q_jP_j(\mu);$$
$$\theta'(\Xi_1) \approx \theta'(\xi_1) + \theta''(\xi_1)\sum_{j=0}^{\infty}q_jP_j(\mu), \tag{3.3.24}$$

and

$$\Theta(\Xi_1,\mu) = \theta(\Xi_1) + \sum_{j=0}^{\infty}A_j\psi_j(\Xi_1)\ P_j(\mu) \approx \theta(\xi_1) + \theta'(\xi_1)\sum_{j=0}^{\infty}q_jP_j(\mu) + \sum_{j=0}^{\infty}A_j\psi_j(\xi_1)\ P_j(\mu) = 0, \tag{3.3.25}$$

to the order of precision we are working. The coefficients of $P_j(\mu)$ must be zero, and therefore

$$q_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1). \tag{3.3.26}$$

Φ from Eq. (3.3.22) and its derivative $\partial\Phi/\partial\xi$ must be continuous with the external potential of M

$$\Phi_e = k_0/\xi + \sum_{j=0}^{\infty}k_{1j}\xi^{-j-1}P_j(\mu), \tag{3.3.27}$$

on the surface $\Xi_1(\mu)$, [cf. Eq. (3.2.33)]. On the boundary we have $\Theta = 0$, and we get by emphasizing the first order contribution c_{10} to the distorted potential:

$$\Phi(\Xi_1,\mu) = (n+1)K\varrho_0^{1/n}\Big\{c_0 + c_{10}P_0(\mu) - [GM'/D(n+1)K\varrho_0^{1/n}]\sum_{j=2}^{4}(\alpha/D)^j\xi_1^j P_j(\mu)\Big\}, \tag{3.3.28}$$

$$\Phi_e(\Xi_1,\mu) = k_0/\xi_1 + \sum_{j=0}^{\infty}(-k_0q_j/\xi_1^2 + k_{1j}\xi_1^{-j-1})\ P_j(\mu), \tag{3.3.29}$$

$$(\partial\Phi/\partial\xi)_{\xi=\Xi_1} = (n+1)K\varrho_0^{1/n}\Big\{\theta'(\xi_1) + [-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]\sum_{j=0}^{\infty} q_jP_j(\mu) + \sum_{j=0}^{\infty} A_j\psi_j'(\xi_1)\ P_j(\mu)$$
$$-[GM'/D(n+1)K\varrho_0^{1/n}]\sum_{j=2}^{4} j(\alpha/D)^j\xi_1^{j-1}P_j(\mu)\Big\}, \tag{3.3.30}$$

$$(\partial\Phi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + \sum_{j=0}^{\infty}[2k_0q_j/\xi_1^3 - (j+1)k_{1j}\xi_1^{-j-2}]\ P_j(\mu). \tag{3.3.31}$$

If $0 < n < 5$, we put $\theta^n(\xi_1) = 0$ in Eq. (3.3.30), and equate the corresponding coefficients from Eqs. (3.3.28), (3.3.30) on the one side, to those from Eqs. (3.3.29), (3.3.31) on the other side:

$$c_0 = -\xi_1\theta'(\xi_1);\quad c_{10} = -A_0[\psi_0(\xi_1) + \xi_1\psi_0'(\xi_1)];\quad k_0 = -(n+1)K\varrho_0^{1/n}\xi_1^2\theta'(\xi_1);$$
$$k_{10} = -(n+1)K\varrho_0^{1/n}A_0\xi_1^2\psi_0'(\xi_1);\quad k_{1j} = (GM'/D)(\alpha/D)^j\xi_1^{2j+1}[j\psi_j(\xi_1) - \xi_1\psi_j'(\xi_1)]$$
$$/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]\ \text{if}\ j = 2,3,4;\quad A_j = [GM'/D(n+1)K\varrho_0^{1/n}](\alpha/D)^j(2j+1)\xi_1^j$$
$$/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]\ \text{if}\ j = 2,3,4;\quad k_{1j}, A_j, q_j = 0\ \text{if}\ j \neq 0,2,3,4,\qquad (0 < n < 5). \tag{3.3.32}$$

The vanishing constants $A_j, k_{1j},\ (j > 4)$ are related by exactly the same equations as in the rotational problem, and the proof given subsequently to Eq. (3.2.43) subsists too.

The constant A_0 is left so far undetermined by the conditions of the problem, and Θ would be of the form $\Theta = \theta + A_0\psi_0 + A_2\psi_2 + A_3\psi_3 + A_4\psi_4$. If the distance D between the mass M and M' increases indefinitely, Eq. (3.3.32) shows that $A_2, A_3, A_4 \to 0$ if $D \to \infty$; also $\Theta \to \theta$, since the tidal effect of M' vanishes if $D \to \infty$. Thus, we are left with $A_0 = 0,\ (c_{10}, k_{10} = 0)$. In the tidal problem there is no purely radial function $\psi_0(\xi)$ as in the rotational problem; $\psi_0(\xi)$ refers to an expansion of the configuration as a whole, which is not possible in the tidal problem (Chandrasekhar 1933b).

If $0 < n < 5$, we obtain analogously to Eqs. (3.2.44)-(3.2.47), by inserting Eq. (3.3.32) into Eqs. (3.3.9), (3.3.22), (3.3.23), (3.3.27):

$$\Theta(\xi,\mu) = \theta(\xi) + [GM'/D(n+1)K\varrho_0^{1/n}]\sum_{j=2}^{4}(\alpha/D)^j(2j+1)\xi_1^j\psi_j(\xi)P_j(\mu)/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)], \tag{3.3.33}$$

$$\Xi_1(\mu) = \xi_1 - \sum_{j=2}^{4} A_j\psi_j(\xi_1)\ P_j(\mu)/\theta'(\xi_1), \tag{3.3.34}$$

$$\Phi = (n+1)K\varrho_0^{1/n}[\theta(\xi) - \xi_1\theta'(\xi_1)] + (GM'/D)\sum_{j=2}^{4}(\alpha/D)^jP_j(\mu)\big\{(2j+1)\xi_1^j\psi_j(\xi)$$
$$/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)] - \xi^j\big\}, \tag{3.3.35}$$

$$\Phi_e = k_0/\xi + \sum_{j=2}^{4} k_{1j}\xi^{-j-1}P_j(\mu). \tag{3.3.36}$$

The total mass of the configuration is analogous to Eq. (3.2.58):

$$M = 2\pi \int_0^\pi d\lambda \int_0^{r_1(\lambda)} \varrho(r,\lambda)\; r^2 \sin\lambda\; dr = 2\pi \int_{-1}^1 d\mu \int_0^{r_1(\mu)} \varrho(r,\mu)\; r^2\; dr$$
$$= 2\pi\varrho_0\alpha^3 \int_{-1}^1 d\mu \int_0^{\Xi_1(\mu)} \Theta^n\xi^2\; d\xi \approx 2\pi\varrho_0\alpha^3 \int_{-1}^1 d\mu \int_0^{\Xi_1(\mu)} \Bigg[\theta^n(\xi)$$
$$+ n\theta^{n-1}(\xi)\sum_{j=2}^4 A_j\psi_j(\xi)\; P_j(\mu)\Bigg]\xi^2\; d\xi \approx 2\pi\varrho_0\alpha^3\Bigg\{\int_{-1}^1 d\mu \int_{\xi_1}^{\Xi_1(\mu)} \Bigg[\theta^n + n\theta^{n-1}\sum_{j=2}^4 A_j\psi_jP_j\Bigg]\xi^2\; d\xi$$
$$+ \int_0^{\xi_1} d\xi \int_{-1}^1 \Bigg[\theta^n + n\theta^{n-1}\sum_{j=2}^4 A_j\psi_jP_j\Bigg]\xi^2\; d\mu\Bigg\}$$
$$\approx 2\pi\varrho_0\alpha^3 \int_{-1}^1 \xi_1^2[\Xi_1(\mu) - \xi_1]\Bigg[\theta^n(\xi_1) + n\theta^{n-1}(\xi_1)\sum_{j=2}^4 A_j\psi_j(\xi_1)\; P_j(\mu)\Bigg]\; d\mu$$
$$+ 4\pi\varrho_0\alpha^3 \int_0^{\xi_1} \theta^n\xi^2\; d\xi = 4\pi\varrho_0\alpha^3\xi_1^2[-\theta'(\xi_1)] = m, \qquad \Big(0 < n < 5;\; \theta^n(\xi_1) = 0;\; \int_{-1}^1 P_j(\mu)\; d\mu = 0\Big). \tag{3.3.37}$$

Similarly, volume and mean density obey the same relations as for the undistorted polytrope:

$$V = 4\pi\alpha^3\xi_1^3/3; \quad \varrho_m = M/V = 3\varrho_0[-\theta'(\xi_1)]/\xi_1, \tag{3.3.38}$$

the "ellipticity-terms" vanishing again up to the order of accuracy we are concerned.

Hence, we find the following theorem (Chandrasekhar 1933b): If a configuration of given mass M is tidally distorted by a mass M', it becomes distorted in such a way that its volume and mean density remain constant in the considered approximation. By virtue of Eq. (3.3.37) we can write instead of Eq. (3.3.33):

$$\Theta(\xi,\mu) = \theta(\xi) + \xi_1[-\theta'(\xi_1)](M'/M)\sum_{j=2}^4 (2j+1)(\alpha\xi_1/D)^{j+1}\psi_j(\xi)\; P_j(\mu)/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]$$
$$= \theta(\xi) + \xi_1[-\theta'(\xi_1)](M'/M)\sum_{j=2}^4 \Delta_j\delta^{j+1}[\psi_j(\xi)/\psi_j(\xi_1)]\; P_j(\mu), \qquad (0 < n < 5), \tag{3.3.39}$$

where we have used the notations

$$\delta = \alpha\xi_1/D; \quad \Delta_j = (2j+1)\psi_j(\xi_1)/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]; \quad \alpha^2 = (n+1)K/4\pi G\varrho_0^{1-1/n}. \tag{3.3.40}$$

A_j from Eq. (3.3.32) reads

$$A_j = \xi_1[-\theta'(\xi_1)](M'/M)\sum_{j=2}^4 \Delta_j\delta^{j+1}/\psi_j(\xi_1). \tag{3.3.41}$$

The boundary equation (3.3.23) writes via Eq. (3.3.26) as

$$\Xi_1(\mu) = \xi_1\Bigg[1 + (M'/M)\sum_{j=2}^4 \Delta_j\delta^{j+1}P_j(\mu)\Bigg]. \tag{3.3.42}$$

In a first approximation the equation of the tidally distorted surface is $\Xi_1(\mu) \approx \xi_1[1 + (M'/M)\Delta_2\delta^3P_2(\mu)]$, with the oblateness of the biaxial ellipsoid given by

$$f = [\Xi_1(1) - \Xi_1(0)]/\Xi_1(1) = (a_3 - a_1)/a_3 \approx 3(M'/M)\Delta_2\delta^3/2 + O(\delta^4),$$
$$\big(\Xi_1(-1) = \Xi_1(1);\; a_1 < a_3;\; a_1 = a_2\big). \tag{3.3.43}$$

If terms in δ^4 are not neglected, the tidal deformations are no longer symmetrical with respect to $\lambda = \pi/2,\ \mu = 0$ (Chandrasekhar 1933b). For instance, at the nearest point of M with respect to M', (M' located on the Mx_3-axis), the distortion is

$$[\Xi_1(1) - \xi_1]/\xi_1 = (M'\delta^3/M)(\Delta_2 + \Delta_3\delta + \Delta_4\delta^2),$$
$$\big(\mu = 1;\ P_2(\mu) = (3\mu^2 - 1)/2;\ P_3(\mu) = (5\mu^3 - 3\mu)/2;\ P_4(\mu) = (35\mu^4 - 30\mu^2 + 3)/8\big), \tag{3.3.44}$$

and on the farthest point with respect to M', $(\mu = -1)$

$$[\Xi_1(-1) - \xi_1]/\xi_1 = (M'\delta^3/M)(\Delta_2 - \Delta_3\delta + \Delta_4\delta^2). \tag{3.3.45}$$

The relevant distortions will be discussed more closely in the next section (Table 3.4.1), in connection with the so-called double star problem.

Again, we conclude this section with the discussion of the special cases $n = 0$ and 5.

(i) $\boldsymbol{n = 0}$. Eqs. (2.3.5), (3.2.73) yield $\theta = 1 - \xi^2/6,\ \psi_j = \xi^j,\ (j \geq 2)$. We put $\theta^n(\xi_1) = 1$ in Eq. (3.3.30), and replace $(n+1)K\varrho_0^{1/n}$ by P_0/ϱ_0. Eqs. (3.3.28)-(3.3.31) yield

$$c_0 = -\xi_1\theta'(\xi_1);\quad c_{10} = -A_0[\xi_1\psi_0(\xi_1)/\theta'(\xi_1) + \psi_0(\xi_1) + \xi_1\psi_0'(\xi_1)];$$
$$k_0 = -(P_0/\varrho_0)\xi_1^2\theta'(\xi_1);\quad k_{10} = -A_0(P_0/\varrho_0)[\xi_1^2\psi_0(\xi_1)/\theta_1'(\xi_1) + \xi_1^2\psi_0'(\xi_1)];$$
$$k_{1j} = 3\xi_1^{2j+1}(GM'/D)(\alpha/D)^j/2(j-1) \text{ if } j = 2,3,4;$$
$$A_j = (2j+1)(GM'\varrho_0/P_0D)(\alpha/D)^j/2(j-1) \text{ if } j = 2,3,4;\quad k_{1j}, A_j = 0 \text{ if } j \neq 0,2,3,4. \tag{3.3.46}$$

In the same way as for $0 < n < 5$, we infer that $A_0 = 0$. The proof given subsequently to Eq. (3.2.63) concerning the vanishing constants $A_j, k_{1j},\ (j > 4)$ remains valid. We rewrite at once Eqs. (3.3.33)-(3.3.36) for the special case $n = 0$:

$$\Theta(\xi,\mu) = \theta(\xi) + (GM'\varrho_0/P_0D)\sum_{j=2}^{4}(2j+1)(\alpha/D)^j\psi_j(\xi)\ P_j(\mu)/2(j-1)$$
$$= 1 - \xi^2/6 - (M'/M)\xi_1^2[-\theta'(\xi_1)]\sum_{j=2}^{4}(2j+1)(\alpha/D)^{j+1}\psi_j(\xi)\ P_j(\mu)/2(j-1)$$
$$= 1 - \xi^2/6 + 2(M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}(\xi^j/6^{j/2})\ P_j(\mu),\qquad [\delta = 6^{1/2}\alpha/D;\ \Delta_j = (2j+1)/2(j-1)], \tag{3.3.47}$$

$$\Xi_1(\mu) = \xi_1 - \sum_{j=2}^{4}A_j\psi_j(\xi_1)\ P_j(\mu)/\theta'(\xi_1) = \xi_1\left[1 + (M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}P_j(\mu)\right]$$
$$= \xi_1\big\{1 + (M'/M)\delta^3[2.5P_2(\mu) + 1.75\delta P_3(\mu) + 1.5\delta^2P_4(\mu)]\big\}, \tag{3.3.48}$$

$$\Phi = (P_0/\varrho_0)\left[3 - \xi^2/6 + (GM'\varrho_0/P_0D)\sum_{j=2}^{4}3\xi^j(\alpha/D)^jP_j(\mu)/2(j-1)\right], \tag{3.3.49}$$

$$\Phi_e = (P_0/\varrho_0)\left[2\times 6^{1/2}/\xi + (M'/M)\sum_{j=2}^{4}6^{j+2}(\alpha/D)^{j+1}\xi^{-j-1}P_j(\mu)/2(j-1)\right]. \tag{3.3.50}$$

As outlined by Eqs. (3.3.43)-(3.3.45), if only terms up to δ^3 are preserved, the oblateness becomes in virtue of Eq. (3.3.48) equal to

$$f = [\Xi_1(1) - \Xi_1(0)]/\Xi_1(1) = (a_3 - a_1)/a_3 = 3.75(M'/M)\delta^3 + O(\delta^4). \tag{3.3.51}$$

Jeans (1919) has found the ellipsoidal surface

$$M'/\pi\varrho_0D^3 = [(1-e^2)/e^3]\ln[(1+e)/(1-e)] - 6(1-e^2)/e^2(3-e^2) \approx 8e^2/45 + O(e^4), \tag{3.3.52}$$

for the tidal problem of the uniform polytrope ($n = 0$), if only the first term $GM'\alpha^2\xi^2 P_2(\mu)/D^3$ is preserved in Eq. (3.3.20). $e^2 = (a_3^2 - a_1^2)/a_3^2 \approx 2f$ denotes the square of the eccentricity of the ellipsoidal polytrope. Eq. (3.3.52) reads, if ϱ_0 is expressed by the mass M from Eq. (2.6.18):

$$f \approx e^2/2 = 3.75(M'/M)\delta^3 + O(\delta^6). \tag{3.3.53}$$

This agrees up to terms of order δ^3 with Chandrasekhar's (1933d) value (3.3.51), whereby Chandrasekhar's solution (3.3.48) is preferable in comparison to Jeans' (1919) solution (3.3.52), because it contains terms of order δ^4, δ^5, which are a priori neglected in Jeans' solution.

(ii) $n = 5$. This case is particularly simple in the tidal problem, because all Δ_j's in Eq. (3.3.40) become unity at large distances in virtue of Eq. (3.2.89): $\psi_j \approx B_j\xi^j$ if $\xi \gg 1$. The equation of the surface (3.3.42) is

$$\Xi_1(\mu) = \xi_1\left[1 + (M'/M)\sum_{j=2}^{4}\delta^{j+1}P_j(\mu)\right], \qquad (\xi_1 \gg 1;\ n = 5). \tag{3.3.54}$$

The proof given subsequently to Eq. (3.2.67) that $A_j, k_{1j} = 0$ if $j > 4$ subsists too.

The first order equivalence between the polytrope $n = 5$ and the point mass Roche model can be shown in a similar way as outlined in Eq. (3.2.69). To this end, we take in Eq. (3.1.102) $v_k = 0$ as for equilibrium, $\Phi = GM/R_1(\lambda)$ as for a point mass, and interchange the x_1-axis with the x_3-axis:

$$(1/\varrho)\ \nabla P = \nabla[GM/R_1(\lambda) + (GM'/2D^3)(2x_3^2 - x_1^2 - x_2^2)] = \nabla\Phi_{tot}. \tag{3.3.55}$$

Turning to spherical coordinates, the total gravitational potential (3.3.55) becomes

$$\begin{aligned}\Phi_{tot} &= GM/R_1(\lambda) + (GM'/2D^3)[2R_1^2(\lambda)\cos^2\lambda - R_1^2(\lambda)\sin^2\lambda]\\ &\approx GM/R_1(\lambda) + (GM'r_1^2/D^3)\ P_2(\cos\lambda) \approx GM/\alpha\Xi_1(\mu) + (GM'\alpha^2\xi_1^2/D^3)\ P_2(\mu)\\ &\approx GM/\alpha\xi_1 - (GM/\alpha\xi_1)(M'/M)\delta^3P_2(\mu) + GM'\delta^3P_2(\mu)/\alpha\xi_1 = GM/r_1 = \text{const.}\end{aligned} \tag{3.3.56}$$

We have used $\Xi_1(\mu) \approx \xi_1[1 + (M'/M)\delta^3P_2(\mu)]$ from Eq. (3.3.54), and $R_1(\mu) = \alpha\Xi_1(\mu),\ r_1 = \alpha\xi_1$. Thus, at large distances from the centre of M the surfaces (3.3.54) determined by the polytrope $n = 5$ are just the level surfaces $\Phi_{tot} = \text{const}$, generated by a point mass M.

If $n = 1$, we get with the attempt $\Theta(\xi,\mu) = \chi(\xi)\ \Pi(\mu)$ exactly in the same way as outlined by Eqs. (3.2.48)-(3.2.54), (Kopal 1939):

$$\Theta(\xi,\mu) = \sum_{j=0}^{\infty}B_j\chi_j(\xi)\ P_j(\mu) = \theta(\xi) + \sum_{j=2}^{\infty}B_j\psi_j(\xi)\ P_j(\mu), \qquad (n = 1;\ B_j = \text{const}). \tag{3.3.57}$$

3.4 Chandrasekhar's Double Star Problem

While the theory of rotating polytropes from Sec. 3.2 is readily applicable, the tidal theory from the previous section is artificial, since it ignores orbital motion of the tidally interacting components. The model adopted by Chandrasekhar (1933c) is the usual one for studying two-component systems (e.g. Kopal 1978): The orbital (x_1, x_2)-plane is perpendicular to the rotation axis directed along Mx_3 (Fig. 3.1.1), a circular orbit and complete synchronization between spin and orbital angular velocity Ω being assumed. This last assumption is suitable especially for close binary stars, where tidal forces are important. The x_1-axis is directed towards the secondary. Since the tide-generating potential (3.3.4) of M' is expanded up to order $(r_m/D)^5$, and since ε from Eq. (3.1.93) is of order $(r_m/D)^5$, (Martin 1970), the Keplerian angular velocity

$$\Omega^2 = G(M + M')/D^3, \tag{3.4.1}$$

is consistent with the adopted scheme of approximation. Further, it is clear from Eq. (3.4.1) that neglecting quantities of order $(r_m/D)^6$ is equivalent to neglect quantities of order Ω^4 – again consistent with the adopted order of approximation. r_m denotes an average radius of the components.

The polytrope $n = 3$ has already been treated by von Zeipel in 1924, but we will follow Chandrasekhar's (1933c) general presentation. The equations of equilibrium are found from Eq. (3.1.90) with $v_k = 0$, the constant term $[M'D/(M + M')]^2$ being dropped:

$$\nabla P = \varrho \,\nabla\big\{\Phi + \Phi'_e + (\Omega^2/2)[x_1^2 + x_2^2 - 2M'Dx_1/(M + M')]\big\}. \tag{3.4.2}$$

The internal potential Φ of M and the external potential Φ'_e of M' satisfy Poisson's and Laplace's equation, respectively:

$$\nabla^2\Phi = -4\pi G\varrho; \quad \nabla^2\Phi'_e = 0. \tag{3.4.3}$$

Taking the divergence of Eq. (3.4.2) and inserting Eq. (3.4.3), we deduce as in previous sections the fundamental equation of the problem:

$$\nabla[(1/\varrho)\,\nabla P] = -4\pi G\varrho + 2\Omega^2. \tag{3.4.4}$$

Changing over to spherical (ξ, λ, φ)-coordinates, and introducing the variables ξ and Θ as defined by Eq. (3.2.1), we find that Eq. (3.4.4) reduces to [cf. Eq. (B.39)]

$$\nabla^2\Theta = \big\{\partial(\xi^2\,\partial\Theta/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\,\partial\Theta/\partial\mu]/\partial\mu + (1-\mu^2)^{-1}\,\partial^2\Theta/\partial\varphi^2\big\}/\xi^2 = -\Theta^n + \beta, \tag{3.4.5}$$

where β is defined by Eq. (3.2.3). We seek a solution of Eq. (3.4.5) under the form

$$\Theta = \Theta(\xi, \mu, \varphi) = \theta(\xi) + \beta\Psi(\xi, \mu, \varphi), \tag{3.4.6}$$

and insert this approach into Eq. (3.4.5):

$$\nabla^2\Psi = -n\theta^{n-1}\Psi + 1, \qquad (\nabla^2\theta = -\theta^n). \tag{3.4.7}$$

Ψ is assumed under the form

$$\Psi = \psi_0 + \sum_{j=1}^{\infty} A_j\psi_j(\xi)\;Y_j(\mu, \varphi), \qquad (A_j = \mathrm{const}), \tag{3.4.8}$$

where $Y_j(\mu, \varphi)$ is a spherical surface harmonic of order j, satisfying the differential equation (cf. Eqs. (3.1.38)-(3.1.41), Smirnow 1967)

$$\nabla^2 Y_j = \partial[(1-\mu^2)\,\partial Y_j/\partial\mu]/\partial\mu + (1-\mu^2)^{-1}\,\partial^2 Y_j/\partial\varphi^2 = -j(j+1)Y_j, \tag{3.4.9}$$

with the eigenvalue $j(j+1)$. Eq. (3.4.8) is substituted into Eq. (3.4.7), and we obtain via Eq. (3.4.9), by equating successive orders of surface harmonics:

$$d(\xi^2\ d\psi_0/d\xi)/d\xi = \xi^2(-n\theta^{n-1}\psi_0 + 1), \tag{3.4.10}$$

$$d(\xi^2\ d\psi_j/d\xi)/d\xi = [j(j+1) - n\xi^2\theta^{n-1}]\ \psi_j, \qquad (j = 1, 2, 3, ...). \tag{3.4.11}$$

To determine the unknown constants A_j, we have to evaluate the internal potential from Eq. (3.4.3), turning to dimensionless variables [cf. Eq. (3.2.12)]:

$$\partial(\xi^2\ \partial\Phi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu + (1-\mu^2)^{-1}\ \partial^2\Phi/\partial\varphi^2$$
$$= -(n+1)K\varrho_0^{1/n}\xi^2\left[\theta^n + \beta n\theta^{n-1}\left(\psi_0 + \sum_{j=1}^{\infty} A_j\psi_j Y_j\right)\right]. \tag{3.4.12}$$

We expand the internal potential of M into spherical harmonics:

$$\Phi(\xi,\mu,\varphi) = U_0(\xi) + \beta\sum_{j=0}^{\infty} V_j(\xi)\ Y_j(\mu,\varphi). \tag{3.4.13}$$

The spherical functions Y_j are identical to those in Eq. (3.4.8), as can be shown if we insert Eq. (3.4.13) into Eq. (3.4.12), equating harmonics of the same order, and taking into account Eq. (3.4.9). The equations defining U_0 and V_j turn out to be the same as those written down in Eqs. (3.2.14)-(3.2.16), and consequently Φ is similar to Eq. (3.2.29):

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\theta(\xi) + c_0 + \beta\left[c_{10} + \psi_0(\xi) - \xi^2/6 + \sum_{j=1}^{\infty}[A_j\psi_j(\xi) + B_j\xi^j]\ Y_j(\mu,\varphi)\right]\right\}$$
$$= (n+1)K\varrho_0^{1/n}\left\{\Theta(\xi,\mu,\varphi) + c_0 + \beta\left[c_{10} - \xi^2/6 + \sum_{j=1}^{\infty} B_j\xi^j Y_j(\mu,\varphi)\right]\right\}. \tag{3.4.14}$$

The radial component of the equation of hydrostatic equilibrium (3.4.2) is

$$\partial P/\partial r = \varrho\ \partial\{\Phi + \Phi_e' + (\Omega^2/2)[r^2\sin^2\lambda - 2M'Dr\sin\lambda\cos\varphi/(M+M')]\}/\partial r. \tag{3.4.15}$$

The external potential Φ_e' of M' writes by virtue of Eqs. (3.1.42), (3.3.4) as

$$\Phi_e'(\vec{r'}) = GM'/r' = GM'/(D^2 - 2rD\cos\gamma + r^2)^{1/2} = GM'/(D^2 - 2x_1D + r^2)^{1/2}$$
$$= (GM'/D)/[1 - 2r\sin\lambda\cos\varphi/D + (r/D)^2]^{1/2} = (GM'/D)\sum_{j=0}^{\infty}(r/D)^j P_j(\sin\lambda\cos\varphi)$$
$$\approx (GM'/D)\sum_{j=1}^{4}(r/D)^j P_j(\sin\lambda\cos\varphi) + \mathrm{const}, \tag{3.4.16}$$

where $\cos\gamma = x_1/r = \sin\lambda\cos\varphi$, and γ is the angle between $\vec{r} = \vec{r}(x_1, x_2, x_3)$ and the direction of D. Eq. (3.4.15) reads in dimensionless coordinates as [cf. Eqs. (3.2.30), (3.3.19)]

$$(n+1)K\varrho_0^{1/n}\ \partial\Theta/\partial\xi = \partial(\Phi + \Phi_e')/\partial\xi + \beta(n+1)K\varrho_0^{1/n}\xi[1 - P_2(\mu)]/3 - GM'\alpha\sin\lambda\cos\varphi/D^2, \tag{3.4.17}$$

by replacing Ω^2 via Eq. (3.4.1). To determine the integration constants B_j, we insert the derivatives of Eqs. (3.4.14), (3.4.16) into Eq. (3.4.17), and equate the coefficients of harmonics of the same order. It turns out that $\left(\beta = \Omega^2/2\pi G\varrho_0 = 2G\alpha^2(M+M')/(n+1)K\varrho_0^{1/n}D^3\right)$:

$$B_2\ Y_2(\lambda,\varphi) = P_2(\cos\lambda)/6 - [M'/2(M+M')]\ P_2(\sin\lambda\cos\varphi);$$
$$B_j\ Y_j(\lambda,\varphi) = -[M'(\alpha/D)^{j-2}/2(M+M')]\ P_j(\sin\lambda\cos\varphi) \quad \text{if} \quad j = 3, 4, \tag{3.4.18}$$

and

$$\Phi(\xi,\lambda,\varphi) = (n+1)K\varrho_0^{1/n}\Big\{\Theta(\xi,\lambda,\varphi) + c_0 + \beta\Big[c_{10} - \xi^2/6 + \xi^2 P_2(\cos\lambda)/6$$
$$-\sum_{j=2}^{4} M'(\alpha/D)^{j-2}\xi^j P_j(\sin\lambda\cos\varphi)/2(M+M')\Big]\Big\}. \qquad (3.4.19)$$

The equation of the surface is given by

$$\Xi_1 = \Xi(\lambda,\varphi) = \xi_1 + \beta\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi), \qquad (q_j = \mathrm{const};\ \Xi_1 \approx \xi_1), \qquad (3.4.20)$$

and on the surface we have [cf. Eqs. (3.2.36), (3.3.24)]:

$$\theta(\Xi_1) \approx \theta(\xi_1) + \beta\theta'(\xi_1)\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi); \quad \psi_k(\Xi_1) \approx \psi_k(\xi_1) + \beta\psi_k'(\xi_1)\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi);$$
$$\theta'(\Xi_1) \approx \theta'(\xi_1) + \beta\theta''(\xi_1)\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi), \qquad (3.4.21)$$

$$\Theta(\Xi_1,\lambda,\varphi) = \theta(\Xi_1) + \beta\Big[\psi_0(\Xi_1) + \sum_{j=1}^{\infty} A_j\psi_j(\Xi_1)\ Y_j(\lambda,\varphi)\Big]$$
$$= \theta(\xi_1) + \beta\Big[\theta'(\xi_1)\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi) + \psi_0(\xi_1) + \sum_{j=1}^{\infty} A_j\psi_j(\xi_1)\ Y_j(\lambda,\varphi)\Big] = 0. \qquad (3.4.22)$$

The coefficients of $Y_j(\lambda,\varphi)$ must be zero, and therefore

$$q_0 = -\psi_0(\xi_1)/\theta'(\xi_1); \quad q_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1). \qquad (3.4.23)$$

The external potential of M can be written as [cf. Eqs. (3.1.38), (3.1.58)]

$$\Phi_e = k_0/\xi + \beta\sum_{j=0}^{\infty} k_{1j}\xi^{-j-1}Y_j(\lambda,\varphi), \qquad (k_0, k_{1j} = \mathrm{const}). \qquad (3.4.24)$$

We now calculate, as in previous sections, the values of the internal and external potential, together with their derivatives on the boundary $\Theta(\Xi_1,\lambda,\varphi) = 0:$

$$\Phi(\Xi_1,\lambda,\varphi) = (n+1)K\varrho_0^{1/n}\Big\{c_0 + \beta\Big[c_{10} - \xi_1^2/6 + \xi_1^2 P_2(\cos\lambda)/6$$
$$-\sum_{j=2}^{4} M'(\alpha/D)^{j-2}\xi_1^j P_j(\sin\lambda\cos\varphi)/2(M+M')\Big]\Big\}, \qquad (3.4.25)$$

$$\Phi_e(\Xi_1,\lambda,\varphi) = k_0/\xi_1 + \beta\sum_{j=0}^{\infty}(-k_0q_j/\xi_1^2 + k_{1j}\xi_1^{-j-1})\ Y_j(\lambda,\varphi), \qquad (3.4.26)$$

$$(\partial\Phi/\partial\xi)_{\xi=\Xi_1} = (n+1)K\varrho_0^{1/n}\Big\{\theta'(\xi_1) + \beta\Big[[-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]\sum_{j=0}^{\infty} q_j Y_j(\lambda,\varphi) + \psi_0'(\xi_1) - \xi_1$$
$$\times[1 - P_2(\cos\lambda)]/3 + \sum_{j=1}^{\infty} A_j\psi_j'(\xi_1)\ Y_j(\lambda,\varphi) - \sum_{j=2}^{4} jM'(\alpha/D)^{j-2}\xi_1^{j-1}P_j(\sin\lambda\cos\varphi)/2(M+M')\Big]\Big\},$$
$$(3.4.27)$$

$$(\partial\Phi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + \beta\sum_{j=0}^{\infty}[2k_0q_j/\xi_1^3 - (j+1)k_{1j}\xi_1^{-j-2}]\ Y_j(\lambda,\varphi). \tag{3.4.28}$$

Continuity of Φ and Φ_e, and of their derivatives on the boundary of M yields ($\theta^n(\xi_1) = 0$ if $0 < n < 5$)

$$\begin{aligned}
&c_0 = -\xi_1\theta'(\xi_1);\quad c_{10} = \xi_1^2/2 - \psi_0(\xi_1) - \xi_1\psi_0'(\xi_1);\quad k_0 = -(n+1)K\varrho_0^{1/n}\xi_1^2\theta'(\xi_1);\\
&k_{10} = (n+1)K\varrho_0^{1/n}\xi_1^2[\xi_1/3 - \psi_0'(\xi_1)];\quad k_{12}Y_2(\lambda,\varphi) = (n+1)K\varrho_0^{1/n}\xi_1^5[2\psi_2(\xi_1) - \xi_1\psi_2'(\xi_1)]\\
&\times[-P_2(\cos\lambda)/6 + M'P_2(\sin\lambda\cos\varphi)/2(M+M')]/[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)];\\
&k_{1j}Y_j(\lambda,\varphi) = (n+1)K\varrho_0^{1/n}\{[M'/2(M+M')](\alpha/D)^{j-2}\xi_1^{2j+1}[j\psi_j(\xi_1) - \xi_1\psi_j'(\xi_1)]\ P_j(\sin\lambda\cos\varphi)\\
&/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]\}\ \text{if } j = 3,4;\\
&A_2Y_2(\lambda,\varphi) = 5\xi_1^2[-P_2(\cos\lambda)/6 + M'P_2(\sin\lambda\cos\varphi)/2(M+M')]/[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)];\\
&A_jY_j(\lambda,\varphi) = [M'/2(M+M')](\alpha/D)^{j-2}(2j+1)\xi_1^jP_j(\sin\lambda\cos\varphi)/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]\\
&\text{if } j = 3,4;\quad k_{1j}, q_j = 0 \text{ if } j \neq 0,2,3,4;\quad A_j = 0 \text{ if } j \neq 2,3,4.
\end{aligned} \tag{3.4.29}$$

Again, the proof given subsequently to Eq. (3.2.43) concerning the vanishing constants A_j, k_{1j}, $(j > 4)$ remains valid. The solution of the problem from Eqs. (3.4.6), (3.4.8) is

$$\begin{aligned}
&\Theta(\xi,\lambda,\varphi) = \theta(\xi) + \beta\Big\{\psi_0(\xi) - 5\xi_1^2\psi_2(\xi)\ P_2(\cos\lambda)/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\\
&+[M'/2(M+M')]\sum_{j=2}^{4}(2j+1)(\alpha/D)^{j-2}\xi_1^j\psi_j(\xi)\ P_j(\sin\lambda\cos\varphi)\\
&/[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]\Big\},\qquad (0 < n < 5).
\end{aligned} \tag{3.4.30}$$

This equation can be written with the notations from Eq. (3.3.40) as

$$\begin{aligned}
&\Theta(\xi,\lambda,\varphi) = \theta(\xi) + \beta\{\psi_0(\xi) - 5\xi_1^2\psi_2(\xi)\ P_2(\cos\lambda)/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\}\\
&+\xi_1[-\theta'(\xi_1)](M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}[\psi_j(\xi)/\psi_j(\xi_1)]\ P_j(\sin\lambda\cos\varphi),
\end{aligned} \tag{3.4.31}$$

since by virtue of Eqs. (2.6.18), (3.4.1) we have

$$\begin{aligned}
&\beta(\alpha/D)^{j-2}\xi_1^j/2(M+M') = (\Omega^2/4\pi G\varrho_0)(\alpha/D)^{j-2}\xi_1^j/(M+M') = \alpha^{j-2}\xi_1^j/4\pi\varrho_0D^{j+1}\\
&= (\alpha\xi_1/D)^{j+1}\xi_1[-\theta'(\xi_1)]/M = \delta^{j+1}\xi_1[-\theta'(\xi_1)]/M.
\end{aligned} \tag{3.4.32}$$

On comparing the solution (3.4.31) with the corresponding solutions of the rotational and tidal problems (Eqs. (3.2.44) and (3.3.39), respectively), we notice that the difference $\Theta-\theta$ is exactly the sum of the corresponding terms in those two problems. In other words, the distortion of the double-star component M is just the sum of its separate distortions arising from simple rotation round its spin axis and from tidal influence due to the mass M' at distance D. This superposition theorem is true to the fifth order in the ratio $\delta = r_1/D$, and it is not likely to be valid for higher orders, since for sixth and higher order terms the density distribution in the secondary M' has to be taken explicitly into account (Chandrasekhar 1933c). Note, that $P_j(\cos\lambda)$ from Eq. (3.3.39) is replaced by $P_j(\sin\lambda\cos\varphi)$ in Eq. (3.4.31), because M' is now located on the Mx_1-axis ($x_1 = r\sin\lambda\cos\varphi$), rather than on the Mx_3-axis ($x_3 = r\cos\lambda$) as in Sec. 3.3.

Because mass, volume, and mean density are not affected by pure tidal distortions, the equations for these physical quantities in the double star problem agree with those of the rotational problem [Eqs. (3.2.58), (3.2.60), (3.2.61)]. The expressions for the inner and outer potential of M are omitted for brevity, but can be written down at once with the aid of Eqs. (3.4.19), (3.4.24), (3.4.29) in a similar way as Eqs. (3.2.46), (3.2.47), (3.3.35), (3.3.36). The equation of the boundary is deduced from Eqs. (3.4.20),

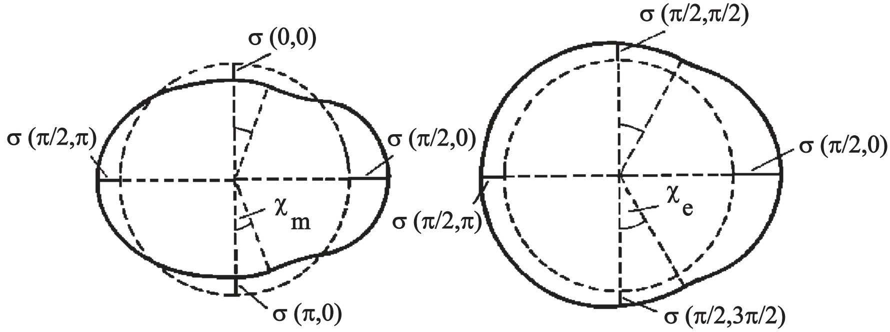

Fig. 3.4.1 Qualitative form of the deviations $\sigma(\lambda,\varphi)$ from spherical shape according to Eq. (3.4.34). On the left the meridional cross-section $\varphi = 0,\pi$, and on the right the equatorial cross-section $\lambda = \pi/2$. Shown are also the small angles χ_m and χ_e from Eqs. (3.4.73), (3.4.78), (Chandrasekhar 1933c).

(3.4.23), (3.4.29):

$$\Xi_1 = \Xi_1(\lambda,\varphi) \approx \xi_1 + \beta\big\{\psi_0(\xi_1) - 5\xi_1^2\psi_2(\xi_1)P_2(\cos\lambda)/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\big\}/[-\theta'(\xi_1)]$$
$$+\xi_1(M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}P_j(\sin\lambda\cos\varphi) = \xi_1 + 2(1+M'/M)\delta^3\psi_0(\xi_1)/\xi_1$$
$$-(1+M'/M)\Delta_2\delta^3\xi_1P_2(\cos\lambda)/3 + \xi_1(M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}P_j(\sin\lambda\cos\varphi), \tag{3.4.33}$$

where we made use of the assumption of synchronism between spin and orbital motion, expressed by Eqs. (3.4.1) and (3.4.32).

On comparing Eq. (3.4.33) with the corresponding equations of the surface (3.2.45) and (3.3.42) for the rotational and tidal problems, we again meet an example of the superposition theorem stated subsequently to Eq. (3.4.32). We see that there is an expansion of the configuration as a whole of amount $2(1+M'/M)\delta^3\psi_0(\xi_1)/\xi_1$, as in the purely rotational case, and superposed on this general expansion there are the harmonic terms $P_2(\cos\lambda)$ and $P_j(\sin\lambda\cos\varphi)$. The deviation from spherical shape (not from the undistorted sphere) is therefore given by the function

$$\sigma(\lambda,\varphi) = (\Xi_1 - \xi_1)/\xi_1 - 2(1+M'/M)\delta^3\psi_0(\xi_1)/\xi_1^2$$
$$= -(1+M'/M)\Delta_2\delta^3P_2(\cos\lambda)/3 + (M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}P_j(\sin\lambda\cos\varphi). \tag{3.4.34}$$

The deformations are symmetrical with respect to the equatorial plane, and with respect to the meridional plane $\varphi = 0,\pi$. Numerically, the deviations shown in Fig. 3.4.1 are (Chandrasekhar 1933c):

$$\sigma(\pi/2,0) = (1+7M'/M)\Delta_2\delta^3/6 + M'\Delta_3\delta^4/M + M'\Delta_4\delta^5/M;$$
$$\sigma(\pi/2,\pi) = (1+7M'/M)\Delta_2\delta^3/6 - M'\Delta_3\delta^4/M + M'\Delta_4\delta^5/M;$$
$$\sigma(\pi/2,\pi/2) = \sigma(\pi/2,3\pi/2) = (1-2M'/M)\Delta_2\delta^3/6 + 3M'\Delta_4\delta^5/8M;$$
$$\sigma(0,0) = \sigma(\pi,0) = -(1+5M'/2M)\Delta_2\delta^3/3 + 3M'\Delta_4\delta^5/8M. \tag{3.4.35}$$

If we restrict to third order terms, the equilibrium configurations are triaxial ellipsoids with the oblateness given by

$$f_1 = (a_1-a_3)/a_1 = \sigma(\pi/2,0) - \sigma(0,0) = (1+4M'/M)\Delta_2\delta^3/2;$$
$$f_2 = (a_1-a_2)/a_1 = \sigma(\pi/2,0) - \sigma(\pi/2,\pi/2) = 3M'\Delta_2\delta^3/2M;$$
$$f_3 = (a_2-a_3)/a_2 = \sigma(\pi/2,\pi/2) - \sigma(0,0) = (1+M'/M)\Delta_2\delta^3/2, \tag{3.4.36}$$

and

$$f_1 > f_3 > f_2 \quad \text{if} \quad M'/M < 0.5; \qquad f_1 > f_2 \geq f_3 \quad \text{if} \quad M'/M \geq 0.5. \tag{3.4.37}$$

It should be noted that for obtaining the actual distortions $(\Xi_1 - \xi_1)/\xi_1$ with respect to the undistorted sphere, we have to add to σ the general expansion term $2(1+M'/M)\delta^3\psi_0(\xi_1)/\xi_1^2$.

A curious fact appears in the double star problem concerning the equatorial axis a_2. A genuine contraction of this axis with respect to the undistorted sphere can arise, as seen by calculating from Eq. (3.4.34) the difference

$$a_2 - r_1 \propto \Xi_1(\pi/2,\pi/2) - \xi_1 = \xi_1\sigma(\pi/2,\pi/2) + 2(1+M'/M)\delta^3\psi_0(\xi_1)/\xi_1$$
$$= \delta^3\big[2(1+M'/M)\psi_0(\xi_1)/\xi_1 + (1-2M'/M)\Delta_2\xi_1/6 + O(\delta)\big]. \tag{3.4.38}$$

The above expression becomes negative $a_2 < r_1$ for a mass ratio

$$M'/M > [\Delta_2 + 12\psi_0(\xi_1)/\xi_1^2]/2[\Delta_2 - 6\psi_0(\xi_1)/\xi_1^2]. \tag{3.4.39}$$

Since we have $\Delta_2 = 5\psi_2(\xi_1)\big/[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]$ via Eq. (3.3.40), we observe that the condition (3.4.39) can be written as $M'/M > (a_1 - r_1)/(r_1 - a_3) > 0$, where $r_1 = \alpha\xi_1$ is the radius of the undistorted Lane-Emden sphere, and a_1, a_3 the equatorial and polar radius of the rotationally distorted polytrope from Eqs. (3.2.56), (3.2.57).

The particular cases $n = 0$ and 5 are evaluated below.

(i) $\boldsymbol{n=0}$. There is no need to go into details, since from the superposition theorem we can write down at once the equations, by adding together Eqs. (3.2.63) and (3.3.47), (Chandrasekhar 1933d):

$$\Theta(\xi,\lambda,\varphi) = \theta(\xi) + \beta\Big\{\psi_0(\xi) - 5\psi_2(\xi)\ P_2(\cos\lambda)/12 + [M'/2(M+M')]$$
$$\times\sum_{j=2}^{4}(2j+1)(\alpha/D)^{j-2}\psi_j(\xi)\ P_j(\sin\lambda\cos\varphi)/2(j-1)\Big\} = 1 - \xi^2/6$$
$$+\beta\xi^2[1 - 5P_2(\cos\lambda)/2]/6 + 2(M'/M)\sum_{j=2}^{4}(2j+1)\delta^{j+1}\psi_j(\xi)\ P_j(\sin\lambda\cos\varphi)/2(j-1)\xi_1^j = 1 - \xi^2/6$$
$$+\delta^3\xi^2(1+M'/M)[1 - 5P_2(\cos\lambda)/2]/9 + 2(M'/M)\sum_{j=2}^{4}\delta^{j+1}\Delta_j(\xi^j/6^{j/2})\ P_j(\sin\lambda\cos\varphi),$$
$$[n=0;\ \Delta_j = (2j+1)/2(j-1)], \tag{3.4.40}$$

since $\beta = \Omega^2/2\pi G\varrho_0 = 2(M+M')\delta^3/3M,\ (2\pi\varrho_0 = M/4\times 6^{1/2}\alpha^3 = 3M/2\alpha^3\xi_1^3)$. For the tidal problem we have $A_j \to \beta A_j$, and we get via Eqs. (3.2.62), (3.3.46) $(\alpha^2 = P_0/4\pi G\varrho_0^2)$:

$$A_2\ Y_2(\lambda,\varphi) = -5P_2(\cos\lambda)/12 + 5[M'/4(M+M')]\ P_2(\sin\lambda\cos\varphi);$$
$$A_j\ Y_j(\lambda,\varphi) = [M'/2(M+M')](2j+1)(\alpha/D)^{j-2}P_j(\sin\lambda\cos\varphi)/2(j-1) \quad \text{if} \quad j = 3,4;$$
$$A_j = 0 \quad \text{if} \quad j \neq 2,3,4. \tag{3.4.41}$$

The proof enunciated subsequently to Eq. (3.2.63) that $A_j, k_{1j} = 0$ if $j > 4$, subsists again. The equation of the boundary writes

$$\Xi_1 = \Xi_1(\lambda,\varphi) = \xi_1 - \beta\psi_0(\xi_1)/\theta'(\xi_1) - \beta\sum_{j=2}^{4}A_j\psi_j(\xi_1)\ Y_j(\lambda,\varphi)/\theta'(\xi_1)$$
$$= 6^{1/2} + (3\beta/6^{1/2})\Big\{1 - 5P_2(\cos\lambda)/2 + [M'/2(M+M')]\sum_{j=2}^{4}\xi_1^2\delta^{j-2}\Delta_j P_j(\sin\lambda\cos\varphi)\Big\}$$
$$= 6^{1/2} + 2(M+M')\delta^3/6^{1/2}M - 5(M+M')\delta^3P_2(\cos\lambda)/6^{1/2}M$$
$$+(6^{1/2}M'/M)\sum_{j=2}^{4}\delta^{j+1}\Delta_jP_j(\sin\lambda\cos\varphi), \qquad (n=0;\ \theta'(\xi_1) = -6^{1/2}/3). \tag{3.4.42}$$

Thus, if $n = 0$, an expansion of amount $2(M + M')\delta^3/6^{1/2}M$ occurs, and superposed on this are the harmonic terms. To the order of accuracy $O(\delta^3)$, when the equilibrium configurations are ellipsoids, the oblateness in the three principal planes is [cf. Eq. (3.4.36)]:

$$f_1 = (1 + 4M'/M)\Delta_2\delta^3/2 = 5(1 + 4M'/M)\delta^3/4; \quad f_2 = 3M'\Delta_2\delta^3/2M = 15M'\delta^3/4M;$$
$$f_3 = (1 + M'/M)\Delta_2\delta^3/2 = 5(1 + M'/M)\delta^3/4, \qquad (n = 0). \qquad (3.4.43)$$

(ii) $\boldsymbol{n = 5.}$ The equation of the boundary can easily be deduced by putting together Eqs. (3.2.67) and (3.3.54), or from Eq. (3.4.33) with $\psi_0 = \xi^2/6,\ \psi_2 = B_2\xi^2,\ \theta' = -3^{1/2}/\xi^2,\ \Delta_j = 1$ if $\xi_1 \to \infty$:

$$\Xi_1 = \xi_1 + (\beta\xi_1^4/6 \times 3^{1/2})[1 - P_2(\cos\lambda)] + \xi_1(M'/M)\sum_{j=2}^{4}\delta^{j+1}P_j(\sin\lambda\cos\varphi), \qquad (n = 5;\ \xi_1 \gg 1). \qquad (3.4.44)$$

The comment subsequent to Eq. (3.2.67) remains valid, i.e. $A_j, k_{1j} = 0$ if $j \neq 2, 3, 4$.

To the order of accuracy $O(\delta^3)$ the equilibrium configurations are ellipsoids, and the oblateness in the three principal planes is

$$f_1 = (1 + 4M'/M)\delta^3/2; \quad f_2 = 3M'\delta^3/2M; \quad f_3 = (1 + M'/M)\delta^3/2, \qquad (n = 5). \qquad (3.4.45)$$

We conclude the presentation of Chandrasekhar's pioneering work with a more quantitative discussion of his results (Chandrasekhar 1933d).

(i) Rotational Problem. So far, our treatment has been based on the comparison of distorted and undistorted polytropes having the same central density ϱ_0 and the same polytropic constant K, (cf. Chandrasekhar and Lebovitz 1962d, §IX). On the other side, if we wish to compare rotating and nonrotating polytropes having the same mass and volume, we have to compare configurations with the same *mean* density ϱ_m, and Chandrasekhar (1933d) introduces therefore the new parameter

$$\beta_m = \Omega^2/2\pi G\varrho_m = (\Omega^2/2\pi G\varrho_0)[-\xi_1/3\theta'(\xi_1)] = \beta\xi_1/3[-\theta'(\xi_1)], \qquad \big(\varrho_m = -3\varrho_0\theta'(\xi_1)/\xi_1\big). \qquad (3.4.46)$$

Note, that in our first order approximation the mean density (3.2.61) of the rotating polytrope can be replaced in the small parameter β_m by the mean density of the undistorted polytrope (2.6.27). We rewrite some of the relevant equations in terms of β_m. Eq. (3.2.59) becomes

$$M = m\{1 + \beta_m[1 - 3\psi_0'(\xi_1)/\xi_1]\}, \qquad (0 < n \leq 5), \qquad (3.4.47)$$

and Eq. (3.2.66) turns into

$$M = m(1 + 3\beta_m/2), \qquad (n = 0;\ \beta = \beta_m). \qquad (3.4.48)$$

The values of $(M - m)/m\beta_m$ are tabulated in Table 3.4.1, representing the fractional increase in total mass between a rotating and a nonrotating equilibrium configuration having the same *central* density ϱ_0. The equation of the boundary (3.2.45) writes in terms of β_m as

$$\Xi_1 = \Xi_1(\cos\lambda) = \xi_1\{1 + \beta_m[3\psi_0(\xi_1)/\xi_1^2 - \Delta_2P_2(\cos\lambda)/2]\}, \qquad (0 \leq n \leq 5), \qquad (3.4.49)$$

where Δ_2 is given by Eq. (3.3.40) if $0 < n \leq 5$, and by Eq. (3.3.47) if $n = 0$. The oblateness from Eq. (3.2.55) writes

$$f = (a_1 - a_3)/a_1 = [\Xi_1(0) - \Xi_1(1)]/\Xi_1(0) = 3\beta_m\Delta_2/4, \qquad (0 \leq n \leq 5). \qquad (3.4.50)$$

The fractional elongation at the equator and the fractional contraction at the poles is

$$[\Xi_1(0) - \xi_1]/\xi_1 = \beta_m[3\psi_0(\xi_1)/\xi_1^2 + \Delta_2/4]; \quad [\Xi_1(1) - \xi_1]/\xi_1 = \beta_m[3\psi_0(\xi_1)/\xi_1^2 - \Delta_2/2],$$
$$(0 \leq n \leq 5). \qquad (3.4.51)$$

To determine the distortion of the boundary when the *same mass* m is set rotating, we have to determine at first the corresponding change in central density ϱ_0, i.e. we have to find $\delta\varrho_0$ such that the masses of rotating and nonrotating configurations coincide:

$$m(\varrho_0, 0) = M(\varrho_0 + \delta\varrho_0, \Omega). \qquad (3.4.52)$$

We insert from Eqs. (2.6.18) and (3.2.58) to find

$$4\pi[(n+1)K\varrho_0^{(3-n)/3n}/4\pi G]^{3/2}\xi_1^2(-\theta_1') = 4\pi[(n+1)K(\varrho_0+\delta\varrho_0)^{(3-n)/3n}/4\pi G]^{3/2}\xi_1^2(-\theta_1')$$
$$\times\{1+[\Omega^2/2\pi G(\varrho_0+\delta\varrho_0)][\psi_0'(\xi_1)-\xi_1/3]/\theta_1'\}, \qquad (0<n\leq 5), \qquad (3.4.53)$$

or to the first order

$$\delta\varrho_0/\varrho_0 = [2n\beta_m/(3-n)][3\psi_0'(\xi_1)/\xi_1 - 1], \qquad (0<n\leq 5). \qquad (3.4.54)$$

Since $P = K\varrho^{1+1/n}$ and $\delta P/P = (1+1/n)\ \delta\varrho/\varrho$, the fractional change in central pressure when a polytropic sphere of constant mass is set rotating, becomes

$$\delta P_0/P_0 = [2(n+1)\beta_m/(3-n)][3\psi_0'(\xi_1)/\xi_1 - 1], \qquad (0<n\leq 5). \qquad (3.4.55)$$

Eqs. (3.4.54) and (3.4.55) break down if $n\approx 3$, since in this case the assumption $\delta\varrho_0 \ll \varrho_0,\ \delta P_0 \ll P_0$ is violated. This is due to the fact that $m \propto \varrho_0^{(3-n)/2n}$, and the mass does not depend on ϱ_0 if $n=3$, so the question of fractional change of central density has no meaning for this particular polytropic index.

If $n=0$, we have $\varrho = \mathrm{const}$, and $\delta\varrho_0/\varrho_0 = 0$. To find $\delta P_0/P_0$ we implement the condition

$$m(P_0, 0) = M(P_0+\delta P_0, \Omega), \qquad (3.4.56)$$

or with Eq. (3.2.66):

$$8\times 6^{1/2}\pi(P_0/4\pi G\varrho_0^{4/3})^{3/2} = 8\times 6^{1/2}\pi[(P_0+\delta P_0)/4\pi G\varrho_0^{4/3}]^{3/2}(1+3\beta_m/2)$$
$$= 8\times 6^{1/2}\pi(P_0/4\pi G\varrho_0^{4/3})^{3/2}(1+3\beta_m/2+3\delta P_0/2P_0), \qquad (3.4.57)$$

and

$$\delta P_0/P_0 = -\beta_m = -\beta, \qquad (n=0). \qquad (3.4.58)$$

We are now able to calculate the boundary r_{1m} when a sphere of constant mass m is set rotating:

$$r_{1m}(\mu) = [(n+1)K(\varrho_0+\delta\varrho_0)^{1/n-1}/4\pi G]^{1/2}\Xi_1(\mu)$$
$$\approx [(n+1)K/4\pi G]^{1/2}\varrho_0^{(1-n)/2n}\xi_1\{1+[(1-n)/2n]\ \delta\varrho_0/\varrho_0\}\{1+\beta_m[3\psi_0(\xi_1)/\xi_1^2 - \Delta_2P_2(\mu)/2]\}$$
$$\approx \alpha\xi_1\{1+\beta_m[(n-1)/(n-3)][3\psi_0'(\xi_1)/\xi_1 - 1] + 3\psi_0(\xi_1)/\xi_1^2 - \Delta_2P_2(\mu)/2\}, \qquad (0<n\leq 5), \qquad (3.4.59)$$

where we have used Eqs. (3.4.49) and (3.4.54). If $n=0$, we find with Eqs. (3.4.49) and (3.4.58), $[\Delta_j = (2j+1)/2(j-1)]$:

$$r_{1m}(\mu) = [(P_0+\delta P_0)/4\pi G\varrho_0^2]^{1/2}\Xi_1(\mu) \approx (P_0/4\pi G\varrho_0^2)^{1/2}(1+\delta P_0/2P_0)\xi_1$$
$$\times\{1+\beta_m[1/2 - 5P_2(\mu)/4]\} \approx \alpha\xi_1[1-5\beta_mP_2(\mu)/4], \qquad (n=0). \qquad (3.4.60)$$

The oblateness $f_m = [r_{1m}(0) - r_{1m}(1)]/r_{1m}(0) = 3\beta_m\Delta_2/4 = f$ has the same expression, whether we are comparing polytropes with equal central density or with equal mass.

The volume relation (3.2.60) between a nonrotating (V_0) and a rotating configuration with equal central density can be rewritten in terms of β_m as

$$V = (4\pi\alpha^3\xi_1^3/3)[1+9\beta_m\psi_0(\xi_1)/\xi_1^2] = V_0[1+9\beta_m\psi_0(\xi_1)/\xi_1^2], \qquad (0\leq n\leq 5;\ V_0 = 4\pi\alpha^3\xi_1^3/3). \qquad (3.4.61)$$

On the other hand, the volumes of two configurations of equal mass m behave like

$$V_m = (4\pi\alpha^3\xi_1^3/3)\{1+\beta_m[(n-1)/(n-3)][9\psi_0'(\xi_1)/\xi_1 - 3] + 9\beta_m\psi_0(\xi_1)/\xi_1^2\}, \qquad (0<n\leq 5), \qquad (3.4.62)$$

where we have simply expanded r_{1m} from Eq. (3.4.59) according to $(1+\varepsilon)^3 \approx 1+3\varepsilon$, and taken into account that the integral over the Legendre polynomial $P_2(\mu)$ is zero. If $n=0$, Eq. (3.4.60) yields simply

$$V_m = 4\pi\alpha^3\xi_1^3/3 = V_0, \qquad (n=0). \qquad (3.4.63)$$

Table 3.4.1 Numerical values of some physical parameters of distorted polytropes. The distortion coefficients Δ_j from Eqs. (3.3.40), (3.3.47), and the oblateness of rotating polytropes $f/\beta_m = f_m/\beta_m = (a_1 - a_3)/\beta_m a_1$ from Eq. (3.4.50) are shown in the upper part. The fractional elongation at the equator and the contraction at the poles of rotationally distorted polytropes (with the same central density, and the same mass, respectively) are in the middle part [Eqs. (3.4.51), (3.4.59), (3.4.60)]. The lower part shows the fractional change of mass, volume (at constant central density, and at constant mass, respectively), of central density, and central pressure for rotationally distorted polytropes [Eqs. (3.4.47), (3.4.48), (3.4.61)-(3.4.63), (3.4.54), (3.4.55), (3.4.58)]. $aE + b$ means $a \times 10^b$.

n	Δ_2	Δ_3	Δ_4	$f/\beta_m = f_m/\beta_m$
0.0	2.500000	1.750000	1.500000	1.875E+0
0.5	1.898308	1.406769	1.250125	1.424E+0
1.0	1.519818	1.212908	1.120482	1.140E+0
1.5	1.286558	1.105697	1.054786	9.649E−1
2.0	1.147877	1.048788	1.023016	8.609E−1
2.5	1.069705	1.020384	1.008683	8.023E−1
3.0	1.028886	1.007400	1.002819	7.717E−1
3.5	1.009838	1.002174	1.000732	7.574E−1
4.0	1.002390	1.000446	1.000131	7.518E−1
4.5	1.000272	1.000042	1.000011	7.502E−1
5.0	1.000000	1.000000	1.000000	7.500E−1

n	$[\Xi_1(0) - \xi_1]/\beta_m\xi_1$	$[\Xi_1(1) - \xi_1]/\beta_m\xi_1$	$[r_{1m}(0) - r_1]/\beta_m r_1$	$[r_{1m}(1) - r_1]/\beta_m r_1$
0.0	1.125E+0	−7.500E−1	6.250E−1	−1.250E+0
0.5	8.347E−1	−5.890E−1	6.323E−1	−7.914E−1
1.0	6.839E−1	−4.559E−1	6.839E−1	−4.559E−1
1.5	6.125E−1	−3.524E−1	7.705E−1	−1.944E−1
2.0	5.902E−1	−2.707E−1	9.031E−1	4.220E−2
2.5	5.987E−1	−2.036E−1	1.187E+0	3.844E−1
3.0	6.254E−1	−1.463E−1	−	−
3.5	6.614E−1	−9.603E−2	3.767E−1	−3.807E−1
4.0	6.992E−1	−5.264E−2	6.321E−1	−1.197E−1
4.5	7.320E−1	−1.823E−2	7.208E−1	−2.944E−2
5.0	7.500E−1	0.000E+0	7.500E−1	0.000E+0

n	$(M - m)/\beta_m m$	$(V - V_0)/\beta_m V_0$	$(V_m - V_0)/\beta_m V_0$	$\delta\varrho_0/\beta_m\varrho_0$	$\delta P_0/\beta_m P_0$
0.0	1.500E+0	1.500E+0	0.000E+0	0.000E+0	−1.000E+0
0.5	1.012E+0	1.080E+0	4.733E−1	−4.047E−1	−1.214E+0
1.0	6.960E−1	9.119E−1	9.119E−1	−6.960E−1	−1.392E+0
1.5	4.739E−1	8.726E−1	1.347E+0	−9.478E−1	−1.580E+0
2.0	3.129E−1	9.098E−1	1.848E+0	−1.252E+0	−1.877E+0
2.5	1.960E−1	9.937E−1	2.758E+0	−1.960E+0	−2.744E+0
3.0	1.130E−1	1.105E+0	−	−	−
3.5	5.693E−2	1.227E+0	3.727E−1	7.971E−1	1.025E+0
4.0	2.236E−2	1.346E+0	1.144E+0	1.789E−1	2.236E−1
4.5	4.802E−3	1.446E+0	1.412E+0	2.881E−2	3.522E−2
5.0	0.000E+0	1.500E+0	1.500E+0	0.000E+0	0.000E+0

We now turn to the determination of the radial component of the effective gravity g_{r1} at the boundary. Eq. (3.1.16) can be written as

$$(1/\varrho)\ \nabla P = \nabla[\Phi + \Omega^2 r^2(1-\mu^2)/2] = \nabla\{\Phi + \Omega^2 r^2[1 - P_2(\mu)]/3\}$$
$$= \nabla\{\Phi + (n+1)K\varrho_0^{1/n}\xi^2\beta[1 - P_2(\mu)]/6\} = \nabla(\Phi + \Phi_f) = \vec{g}, \qquad (3.4.64)$$

where $\Phi_f = \Omega^2 r^2(1-\mu^2)/2$ is the centrifugal potential. The effective radial surface gravity (3.1.22) is

$$g_{r1} = [\partial(\Phi + \Phi_f)/\partial r]_{r=r_1} = (1/\varrho)(\partial P/\partial r)_{r=r_1} = [(n+1)K\varrho_0^{1/n}/\alpha](\partial\Theta/\partial\xi)_{\xi=\Xi_1}$$
$$= [(n+1)K\varrho_0^{1/n}/\alpha]\big\{\theta'(\xi_1) + \beta\big[\big(-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)\big)\big(q_0 + q_2P_2(\mu)\big) + \psi_0'(\xi_1)$$
$$+A_2\psi_2'(\xi_1)\ P_2(\mu)\big]\big\} = [(n+1)K\varrho_0^{1/n}/\alpha]\big\{\theta'(\xi_1) + \beta\big[2\psi_0(\xi_1)/\xi_1 + \psi_0'(\xi_1)$$
$$+A_2P_2(\mu)\ \big(2\psi_2(\xi_1)/\xi_1 + \psi_2'(\xi_1)\big)\big]\big\} = [(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\big\{1 - \beta_m\big[6\psi_0(\xi_1)/\xi_1^2 + 3\psi_0'(\xi_1)/\xi_1$$

$$-5\big(2\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)\big)\ P_2(\mu)/2\big(3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)\big)\big]\big\}$$
$$=[(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\big\{1-\beta_m[6\psi_0(\xi_1)/\xi_1^2+3\psi_0'(\xi_1)/\xi_1+(\Delta_2-5)P_2(\mu)/2]\big\},\qquad (0<n\le 5), \tag{3.4.65}$$

where we have used Eqs. (3.2.38), (3.2.41), (3.3.40), (3.4.46), and $\theta^n(\xi_1)=0$.

If $n=0$, we have to put $\xi_1=6^{1/2},\ \theta^n(\xi_1)=1,\ -2\theta'(\xi_1)/\xi_1-\theta^n(\xi_1)=-1/3,\ \psi_0=\xi^2/6,\ \psi_2=\xi^2,\ A_2=-5/12$, and $\beta=\beta_m$ in the previous equation:

$$g_{r1}=(P_0/\alpha\varrho_0)\big\{\theta'(\xi_1)+\beta_m\big[(-2\theta'(\xi_1)/\xi_1-1)\big(q_0+q_2P_2(\mu)\big)+\psi_0'(\xi_1)+A_2\psi_2'(\xi_1)\ P_2(\mu)\big]\big\}$$
$$=(P_0/\alpha\varrho_0)\big\{\theta'(\xi_1)+\beta_m\big[-\psi_0(\xi_1)/\xi_1+\psi_0'(\xi_1)+5\big(\psi_2(\xi_1)/\xi_1-\psi_2'(\xi_1)\big)\ P_2(\mu)/12\big]\big\}$$
$$=-(6^{1/2}P_0/3\alpha\varrho_0)\{1+\beta_m[-1/2+5P_2(\mu)/4]\},\qquad (n=0). \tag{3.4.66}$$

The surface gravity of the nonrotating configuration ($\beta_m=0$) turns out to be $g_{r1}=(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha=(n+1)P_0\theta'(\xi_1)/\alpha\varrho_0$, in accordance with the derivative $d\Phi/dr=g_{r1}$ of Eq. (2.6.32). Gravity values are provided by Chandrasekhar (1933d).

(ii) Tidal Problem. The boundary in the tidal problem is given by Eqs. (3.3.42) and (3.3.48). The surface gravitational acceleration is calculated analogously to Eqs. (3.4.65)-(3.4.66). In virtue of Eqs. (3.3.20), (3.3.30), (3.3.40), (3.3.41) we find

$$g_{r1}=[\partial(\Phi+\Phi_e')/\partial r]_{r=r_1}=[(n+1)K\varrho_0^{1/n}/\alpha]\Big\{\theta'(\xi_1)+[-2\theta'(\xi_1)/\xi_1-\theta^n(\xi_1)]\sum_{j=2}^4 q_jP_j(\mu)$$
$$+\sum_{j=2}^4 A_j\psi_j'(\xi_1)\ P_j(\mu)\Big\}=[(n+1)K\varrho_0^{1/n}/\alpha]\Big\{\theta'(\xi_1)+\sum_{j=2}^4[2\psi_j(\xi_1)/\xi_1+\psi_j'(\xi_1)]A_jP_j(\mu)\Big\}$$
$$=[(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\Big\{1-(M'/M)\sum_{j=2}^4[2\psi_j(\xi_1)+\xi_1\psi_j'(\xi_1)]\Delta_j\delta^{j+1}P_j(\mu)/\psi_j(\xi_1)\Big\}$$
$$=[(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\Big\{1-(M'/M)\sum_{j=2}^4[2j+1-(j-1)\Delta_j]\delta^{j+1}P_j(\mu)\Big\},\quad (0<n\le 5). \tag{3.4.67}$$

If $n=0$, we have $\theta^n(\xi_1)=1$ in the previous equation, and after some algebra we get

$$g_{r1}=[P_0\theta'(\xi_1)/\alpha\varrho_0]\Big[1-(M'/2M)\sum_{j=2}^4(2j+1)\delta^{j+1}P_j(\mu)\Big],\qquad (n=0), \tag{3.4.68}$$

which can formally be obtained from Eq. (3.4.67) if $\Delta_j=(2j+1)/2(j-1)$ via Eq. (3.3.47).

The meridional cross-section of the tidally distorted polytrope presents a furrow, that develops at a certain small angle χ_m (Fig. 3.4.1), to be discussed subsequently within the more general context of the double star problem.

(iii) The Double Star Problem. The deviations $\sigma(\lambda,\varphi)$ from *spherical* shape [Eq. (3.4.34)] are qualitatively shown in Fig. 3.4.1, and quantitatively in Fig. 3.4.2 for a double star system having $M'=2M$ and $\delta=r_1/D=\alpha\xi_1/D=0.1$. The deviations from spherical shape in the meridional plane $\varphi=0$ can be found at once from Eq. (3.4.34):

$$\sigma(\lambda,0)=-(1+M'/M)\Delta_2\delta^3P_2(\cos\lambda)/3+(M'/M)\sum_{j=2}^4\Delta_j\delta^{j+1}P_j(\sin\lambda). \tag{3.4.69}$$

The extremes are found with $P_2(\mu)=(3\mu^2-1)/2,\ P_3(\mu)=(5\mu^3-3\mu)/2,\ P_4(\mu)=(35\mu^4-30\mu^2+3)/8:$

$$d\sigma(\lambda,0)/d\lambda=(1+M'/M)\Delta_2\delta^3\sin\lambda\cos\lambda+(M'/M)[3\Delta_2\delta^3\sin\lambda\cos\lambda$$
$$+(\Delta_3\delta^4/2)(15\sin^2\lambda\cos\lambda-3\cos\lambda)+(\Delta_4\delta^5/2)(35\sin^3\lambda\cos\lambda-15\sin\lambda\cos\lambda)]=0. \tag{3.4.70}$$

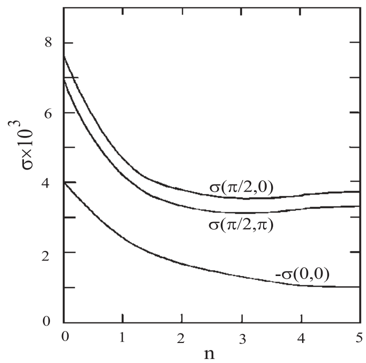

Fig. 3.4.2 Deviations from spherical shape for the component M of a double star system having $M/M' = 0.5$ and $\delta = 0.1$ (Chandrasekhar 1933d).

One extreme is given by $\cos\lambda = 0$, $(\lambda = \pi/2)$, and another extreme is found by ignoring in Eq. (3.4.70) the highest order term. We get the second order equation

$$15(M'/M)\Delta_3\delta\sin^2\lambda/2 + (1+4M'/M)\Delta_2\sin\lambda - 3(M'/M)\Delta_3\delta/2 = 0, \tag{3.4.71}$$

with one meaningful approximate solution

$$\sin\lambda = 3M'\Delta_3\delta/2\Delta_2(M+4M') + O(\delta^3). \tag{3.4.72}$$

Since $\sin\lambda \ll 1$, we have

$$\sin\lambda \approx \lambda = \chi_m = 3M'\Delta_3\delta/2\Delta_2(M+4M') + O(\delta^3) \quad \text{and} \quad \lambda = \pi - \chi_m, \qquad (0 < n \leq 5). \tag{3.4.73}$$

At the symmetric angles χ_m, $(\chi_m \ll 1)$ shown in Fig. 3.4.1, a furrow occurs in the meridional plane $\varphi = 0$, where the maximum contraction with respect to spherical shape takes place. The maximum elongation subsists at $\lambda = \pi/2$, $(\varphi = 0)$. A similar furrow develops also in the equatorial plane of M, where

$$\sigma(\pi/2,\varphi) = (1+M'/M)\Delta_2\delta^3/6 + (M'/M)\sum_{j=2}^{4}\Delta_j\delta^{j+1}P_j(\cos\varphi). \tag{3.4.74}$$

The extremes are found from

$$d\sigma(\pi/2,\varphi)/d\varphi = -(M'/M)\big[3\Delta_2\delta^3\sin\varphi\cos\varphi + (\Delta_3\delta^4/2)(15\sin\varphi\cos^2\varphi - 3\sin\varphi)$$
$$+(\Delta_4\delta^5/2)(35\sin\varphi\cos^3\varphi - 15\sin\varphi\cos\varphi)\big] = 0. \tag{3.4.75}$$

One extreme is given by $\sin\varphi = 0$, $(\varphi = 0,\pi)$, and another extreme is found by ignoring in Eq. (3.4.75) the highest order term. We get the second order equation

$$15\Delta_3\delta\cos^2\varphi/2 + 3\Delta_2\cos\varphi - 3\Delta_3\delta/2 = 0, \tag{3.4.76}$$

with one meaningful approximate solution

$$\cos\varphi = \Delta_3\delta/2\Delta_2 + O(\delta^3). \tag{3.4.77}$$

Since $\cos\varphi \ll 1$, we have

$$\varphi = \pi/2 - \chi_e \text{ and } \varphi = 3\pi/2 + \chi_e; \quad \chi_e = \Delta_3\delta/2\Delta_2, \qquad (0 < n \leq 5). \tag{3.4.78}$$

At the symmetric angles χ_e, $(\chi_e \ll 1)$ shown in Fig. 3.4.1, a furrow develops in the equatorial plane, where the maximum contraction with respect to the undistorted sphere takes place. The two maximum elongations occur at $\varphi = 0$ and $\varphi = \pi$, $(\lambda = \pi/2)$.

If $n = 0$, we insert $\Delta_j = (2j+1)/2(j-1)$ from Eq. (3.3.47) into Eqs. (3.4.73), (3.4.78):

$$\chi_m = 21M'\delta/20(M+4M') + O(\delta^3); \quad \chi_e = 7\delta/20 + O(\delta^3), \qquad (n=0). \tag{3.4.79}$$

The variation of the radial component of apparent gravity is another example of the superposition theorem enunciated subsequently to Eq. (3.4.32), since we have to add simply Eqs. (3.4.65) and (3.4.67):

$$g_{r1} = [(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\Big\{1 - \beta_m\big[6\psi_0(\xi_1)/\xi_1^2 + 3\psi_0'(\xi_1)/\xi_1 + (\Delta_2 - 5)P_2(\cos\lambda)/2\big]$$
$$-(M'/M)\sum_{j=2}^{4}[2j+1-(j-1)\Delta_j]\delta^{j+1}P_j(\sin\lambda\cos\varphi)\Big\}$$
$$= [(n+1)K\varrho_0^{1/n}\theta'(\xi_1)/\alpha]\Big\{1 - [(M+M')\delta^3/M]\big[4\psi_0(\xi_1)/\xi_1^2 + 2\psi_0'(\xi_1)/\xi_1 + (\Delta_2 - 5)P_2(\cos\lambda)/3\big]$$
$$-(M'/M)\sum_{j=2}^{4}[2j+1-(j-1)\Delta_j]\delta^{j+1}P_j(\sin\lambda\cos\varphi)\Big\}, \qquad (0 < n \le 5). \tag{3.4.80}$$

We have substituted for β_m via Eqs. (2.6.18), (3.4.1), (3.4.46):

$$\beta_m = \Omega^2\xi_1/6\pi G\varrho_0[-\theta'(\xi_1)] = (M+M')\xi_1/6\pi\varrho_0 D^3[-\theta'(\xi_1)] = 2(M+M')\delta^3/3M, \qquad (0 \le n \le 5). \tag{3.4.81}$$

If $n = 0$, we have to add Eqs. (3.4.66) and (3.4.68):

$$g_{r1} = [P_0\theta'(\xi_1)/\alpha\varrho_0]\Big\{1 + \beta_m[-1/2 + 5P_2(\cos\lambda)/4] - (M'/2M)\sum_{j=2}^{4}(2j+1)\delta^{j+1}P_j(\sin\lambda\cos\varphi)\Big\}$$
$$= -(6^{1/2}P_0/3\alpha\varrho_0)\Big\{1 + [(M+M')\delta^3/M][-1/3 + 5P_2(\cos\lambda)/6]$$
$$-(M'/2M)\sum_{j=2}^{4}(2j+1)\delta^{j+1}P_j(\sin\lambda\cos\varphi)\Big\}, \qquad (n=0). \tag{3.4.82}$$

The coefficients of the Legendre polynomials in Eq. (3.4.80) are the same as those of the function $\sigma(\lambda,\varphi)$ in Eq. (3.4.34) if we replace Δ_j by $-[2j+1-(j-1)\Delta_j]$. With this substitution, the angles at which the furrow occurs in the radial component of the gravitational acceleration are of the same form as for the external shape:

$$\chi_{gm} = 3M'(7-2\Delta_3)\delta/2(M+4M')(5-\Delta_2) + O(\delta^3);$$
$$\chi_{ge} = (7-2\Delta_3)\delta/2(5-\Delta_2) + O(\delta^3), \qquad (0 < n \le 5). \tag{3.4.83}$$

If $n = 0$, we insert $\Delta_j = (2j+1)/2(j-1)$ from Eq. (3.3.47) into Eq. (3.4.83):

$$\chi_{gm} = 21M'\delta/10(M+4M') + O(\delta^3); \quad \chi_{ge} = 7\delta/10 + O(\delta^3), \qquad (n=0). \tag{3.4.84}$$

Finally, we notice the relationship between the furrow angles in the meridional plane $(\varphi = 0)$ and in the equatorial plane $(\lambda = \pi/2)$:

$$\chi_m/\chi_e = \chi_{gm}/\chi_{ge} = 3M'/(M+4M'). \tag{3.4.85}$$

3.5 Second Order Extension of Chandrasekhar's Theory to Differentially Rotating Polytropes

Occhionero (1967a) has extended Chandrasekhar's (1933a) theory to second order, succeeded by Aikawa (1968, 1971), and Anand (1968). Geroyannis et al. (1979) have extended the theory to differential rotation, and Geroyannis and Antonakopoulos (1981a) to third order (see also Geroyannis 1984, Geroyannis and Valvi 1985, 1986a, b, c, 1987, 1988).

By virtue of Eq. (3.1.11) the angular velocity depends in a permanently rotating polytrope (special case of a barotrope) only on the radial cylindrical coordinate ℓ : $\Omega = \Omega(\ell)$. The stability condition against *axisymmetric* perturbations on surfaces of constant entropy S is provided by the Solberg-Høiland criterion (5.7.90) if $\vec{A}\cdot\nabla P \geq 0$ (e.g. Stoeckly 1965, Tassoul 1978, Sec. 7.3):

$$d[\ell^2\Omega(\ell)]/d\ell > 0. \tag{3.5.1}$$

The angular momentum per unit mass $\Omega\ell^2$ must necessarily increase outward. Eq. (3.5.1) is known as the Rayleigh criterion, originally derived for an inviscid incompressible fluid. To obtain a somewhat realistic law of differential rotation, let us consider an initially homogeneous sphere of mass M_1, density ϱ_1, and angular momentum $J_1 = 2\Omega_1 M_1 r_1^2/5$, where Ω_1 and r_1 denote the initial angular velocity and the initial radius, respectively. Suppose the sphere contracts in such a way that cylindrical surfaces remain cylindrical, conserving their mass and angular momentum. Let us denote by $M(\ell_1)$ the mass inside distance ℓ_1 from the rotation axis. The mass of the sphere outside this distance is then given by (Stoeckly 1965)

$$M_1 - M(\ell_1) = 4\pi\int_{\ell_1}^{r_1}\varrho_1 r^2\,dr\int_{\arcsin(\ell_1/r)}^{\pi/2}\sin\lambda\,d\lambda = 4\pi\varrho_1\int_{\ell_1}^{r_1} r^2(1-\ell_1^2/r^2)^{1/2}\,dr$$
$$= 4\pi\varrho_1(r_1^2-\ell_1^2)^{3/2}/3, \qquad (\sin\lambda = \ell/r). \tag{3.5.2}$$

λ is the zenith angle, ℓ the distance from the rotation axis, and r the radial distance from the centre of the sphere, the rotational distortion being neglected for the moment. Conservation of angular momentum of a cylindrical surface contracting from ℓ_1 to ℓ yields

$$\Omega_1\ell_1^2 = \Omega\ell^2. \tag{3.5.3}$$

Inserting for ℓ_1 into Eq. (3.5.2), and replacing by virtue of mass conservation $M(\ell_1)$ with $M(\ell)$, we obtain for the angular velocity Ω of the cylindrical surface at distance ℓ from the rotation axis

$$\Omega = \Omega(\ell) = (\Omega_1 r_1^2/\ell^2)\big\{1-[1-M(\ell)/M_1]^{2/3}\big\} = (5J_1/2M_1\ell^2)\big\{1-[1-M(\ell)/M_1]^{2/3}\big\}. \tag{3.5.4}$$

We turn to the dimensionless coordinates from Eq. (2.1.13):

$$r = \alpha\xi;\quad \ell = r\sin\lambda = \alpha\xi\sin\lambda = \alpha s = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}s = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}s;$$
$$s = \xi\sin\lambda, \tag{3.5.5}$$

and generalize Eq. (3.5.2) for a polytrope ($\varrho_1 \rightarrow \varrho = \varrho_0\theta^n$) :

$$M_1 - M(s_1) = 4\pi\varrho_0\alpha^3\int_{s_1}^{\xi_1}\xi^2\theta^n(1-s_1^2/\xi^2)^{1/2}\,d\xi. \tag{3.5.6}$$

Eq. (3.5.4) becomes $[M(s_1) = M(s)]$:

$$\Omega = \Omega(s) = (5J_1/2M_1\alpha^2 s^2)\left\{1-(4\pi\varrho_0\alpha^3/M_1)^{2/3}\left[\int_s^{\xi_1}\xi^2\theta^n(1-s^2/\xi^2)^{1/2}\,d\xi\right]^{2/3}\right\}. \tag{3.5.7}$$

For polytropic indices $2 \leq n \leq 3.25$ Clement (1967) has found that Eq. (3.5.7) can be fitted to within 1% by the analytical form

$$\omega = \omega(s) = \Omega(s)/\Omega(0) = \left[\sum_{i=1}^{3} a_i\exp(-b_i s^2)\right]^{1/2}, \qquad (a_i, b_i = \mathrm{const}), \tag{3.5.8}$$

where $\omega(s)$ is the dimensionless angular velocity, and $\Omega(0)$ the angular velocity on the rotation axis.

Eq. (3.5.8) has to satisfy the stability condition (3.5.1). Some numerical values for the constants a_j, b_j have been provided for instance by Clement (1967): $a_1 = 0.55,\ a_2 = 0.54,\ a_3 = -0.09,\ b_1 = 0.12,\ b_2 = 0.39,\ b_3 = 0.71$ for the polytropic index $n = 2$, and $a_1 = 0.10,\ a_2 = 0.56,\ a_3 = 0.35,\ b_1 = 0.05,\ b_2 = 0.20,\ b_3 = 0.59$ if $n = 3$. For this adopted set of constants the ratio between polar and equatorial angular velocity $1/\omega(\Xi_1) = \Omega(0)/\Omega(\Xi_1)$ is 4.09 ($n = 2$), and 10.8 ($n = 3$).

In an inertial cylindrical frame with the origin in the centre of mass of the configuration, the equations of motion of the differentially rotating, axisymmetric fluid are given by Eq. (3.1.8), where $\vec{\Omega} = \vec{\Omega}[0, 0, \Omega(\ell)]$:

$$\partial P/\partial \ell = \varrho\ \partial\Phi/\partial\ell + \varrho\Omega^2(\ell)\ \ell; \quad \partial P/\partial z = \varrho\ \partial\Phi/\partial z. \tag{3.5.9}$$

With the polytropic relationship $P = K\varrho^{1+1/n}$ we obtain at once the prime integral [cf. Eq. (3.1.22)]

$$\Phi = (n+1)P/\varrho - \int_0^\ell \Omega^2(\ell')\ \ell'\ d\ell' + \mathrm{const.} \tag{3.5.10}$$

We turn to the dimensionless coordinates defined by Eqs. (3.2.1), (3.5.5), (3.5.8), and insert $z = r\cos\lambda = \alpha\xi\cos\lambda = \alpha\zeta$. Eq. (3.5.10) becomes

$$\Phi = (n+1)K\varrho_0^{1/n}\left[\Theta(s,\zeta) - (\beta/2)\int_0^s \omega^2(s')\ s'\ ds'\right] + \mathrm{const}, \quad \big(\beta = \Omega^2(0)/2\pi G\varrho_0;\ \zeta = \xi\cos\lambda\big). \tag{3.5.11}$$

If we insert for Φ into Poisson's equation (2.1.4), we obtain in cylindrical (s, φ, ζ)-coordinates the fundamental equation

$$\nabla^2\Theta(s,\zeta) = -\Theta^n(s,\zeta) + \beta[\omega^2(s) + (s/2)\ d\omega^2(s)/ds], \tag{3.5.12}$$

where we have used Eq. (B.48). Since the undistorted polytrope has radial symmetry, spherical coordinates are more appropriate:

$$\Theta(\xi,\mu) = \theta(\xi) + \beta\sum_{j=0}^{A}\psi_j(\xi)\ P_j(\mu) + \beta^2\sum_{j=0}^{B} f_j(\xi)\ P_j(\mu) + O(\beta^3),$$
$$[\mu = \cos\lambda;\ s = \xi\sin\lambda = \xi(1-\mu^2)^{1/2}]. \tag{3.5.13}$$

A and B are integers to be determined from the boundary conditions. For the terms connected with the angular velocity Clement (1967) adopts, as for Θ, an expansion in terms of Legendre polynomials, similarly to Blinnikov [1972, Eq. (12)]:

$$\omega^2 = \sum_{i=1}^{3} a_i\exp(-b_i s^2) = \sum_{j=1}^{4}\pi_j(\xi)\ P_j(\mu); \quad (s/2)\ d\omega^2/ds = -s^2\sum_{i=1}^{3} a_i b_i\exp(-b_i s^2)$$
$$= \sum_{j=0}^{4}\chi_j(\xi)\ P_j(\mu); \quad \int_0^s \omega^2(s')\ s'\ ds'/2 = \sum_{i=1}^{3}(a_i/4b_i)[1-\exp(-b_i s^2)] = \sum_{j=0}^{4}\varphi_j(\xi)\ P_j(\mu). \tag{3.5.14}$$

Clement (1967) and Geroyannis et al. (1979) truncate the expansions at $P_4(\mu)$. The unknown radial expansion functions π_j, χ_j, and φ_j can be determined if we multiply the equations (3.5.14) consecutively by $P_k(\mu)$, making use of the orthogonality property (3.5.16) of Legendre polynomials, and taking into account that odd indexed functions are zero because of equatorial symmetry:

$$\pi_j(\xi) = (2j+1)\sum_{i=1}^{3} a_i \exp(-b_i\xi^2)\int_0^1 \exp(b_i\xi^2\mu^2)\ P_j(\mu)\ d\mu$$

$$= (2j+1)\sum_{i=1}^{3} a_i \exp(-b_i\xi^2)\sum_{k=0}^{j} p_{jk}\int_0^1 \mu^k \exp(b_i\xi^2\mu^2)\ d\mu$$

$$= (2j+1)\sum_{i=1}^{3} a_i (b_i^{1/2}\xi)^{-k-1}\exp(-b_i\xi^2)\sum_{k=0}^{j} p_{jk}\int_0^{b_i^{1/2}\xi} y^k \exp(y^2)\ dy$$

$$= (2j+1)\sum_{i=1}^{3} a_i \sum_{k=0}^{j} p_{jk}\ \psi_{ik}(\xi);\quad \pi_{2j+1}(\xi) = 0. \tag{3.5.15}$$

With δ_{jk} equal to the Kronecker delta, we have

$$\int_{-1}^{1} P_j(\mu)\ P_k(\mu)\ d\mu = 2\delta_{jk}/(2j+1);\quad P_j(\mu) = \sum_{k=0}^{j} p_{jk}\mu^k,\qquad (p_{jk} = \mathrm{const});$$

$$\psi_{ik}(\xi) = (b_i^{1/2}\xi)^{-k-1}\exp(-b_i\xi^2)\int_0^{b_i^{1/2}\xi} y^k \exp(y^2)\ dy,\qquad (y = b_i^{1/2}\xi\mu). \tag{3.5.16}$$

In the same way we get

$$\chi_j(\xi) = (2j+1)\sum_{i=1}^{3} a_i b_i\xi^2 \exp(-b_i\xi^2)\int_0^1 (\mu^2-1)\exp(b_i\xi^2\mu^2)\ P_j(\mu)\ d\mu$$

$$= (2j+1)\xi^2\sum_{i=1}^{3} a_i b_i \sum_{k=0}^{j} p_{jk}[\psi_{i,k+2}(\xi) - \psi_{ik}(\xi)];\quad \chi_{2j+1}(\xi) = 0;$$

$$\varphi_j(\xi) = (2j+1)\sum_{i=1}^{3}(a_i/4b_i)\left[\delta_{j0} - \exp(-b_i\xi^2)\int_0^1 \exp(b_i\xi^2\mu^2)\ P_j(\mu)\ d\mu\right]$$

$$= (2j+1)\sum_{i=1}^{3}(a_i/4b_i)\left[\delta_{j0} - \sum_{k=0}^{j} p_{jk}\psi_{ik}(\xi)\right];\quad \varphi_{2j+1}(\xi) = 0. \tag{3.5.17}$$

In spherical coordinates the fundamental equation (3.5.12) assumes the form

$$\nabla^2\Theta(\xi,\mu) = -\Theta^n(\xi,\mu) + \beta\sum_{j=0}^{4}[\pi_j(\xi) + \chi_j(\xi)]\ P_j(\mu). \tag{3.5.18}$$

The Laplacian of Eq. (3.5.13) is

$$\nabla^2\Theta = \nabla^2\theta + \beta\sum_{j=0}^{A}(P_j\ \nabla^2\psi_j + \psi_j\ \nabla^2 P_j) + \beta^2\sum_{j=0}^{B}(P_j\ \nabla^2 f_j + f_j\ \nabla^2 P_j)$$

$$= D_0\theta + \beta\sum_{j=0}^{A} P_j\ D_j\psi_j + \beta^2\sum_{j=0}^{B} P_j\ D_j f_j, \tag{3.5.19}$$

since according to Eqs. (B.39), (3.1.40): $\nabla^2\psi_j = d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi$; $\nabla^2 f_j = d^2 f_j/d\xi^2 + (2/\xi)\ df_j/d\xi$; $\nabla^2 P_j = (1/\xi^2)\ d[(1-\mu^2)\ dP_j/d\mu]/d\mu = -j(j+1)P_j/\xi^2$. We have also introduced the operator (Chandrasekhar and Lebovitz 1962d)

$$D_j g = d^2 g/d\xi^2 + (2/\xi)\ dg/d\xi - j(j+1)g/\xi^2. \tag{3.5.20}$$

The expansion of Θ^n up to the second power in β yields

$$\Theta^n = \theta^n + \beta n\theta^{n-1}\sum_{j=0}^{A}\psi_j P_j + \beta^2\Big\{[n(n-1)/2]\theta^{n-2}\Big(\sum_{j=0}^{A}\psi_j P_j\Big)^2 + n\theta^{n-1}\sum_{j=0}^{B} f_j P_j\Big\}$$
$$= \theta^n + \beta n\theta^{n-1}\sum_{j=0}^{A}\psi_j P_j + \beta^2\sum_{j=0}^{C}\big\{[n(n-1)/2]\theta^{n-2}S_j + n\theta^{n-1}f_j\big\}P_j, \qquad (3.5.21)$$

where we have substituted $C = \max\{2A, B\}$ and

$$\Big[\sum_{j=0}^{A}\psi_j(\xi)\ P_j(\mu)\Big]^2 = \sum_{j=0}^{2A} S_j(\xi)\ P_j(\mu). \qquad (3.5.22)$$

The comments subsequent to Eq. (3.2.7) subsist too for the expansion (3.5.21). We insert Eqs. (3.5.19), (3.5.21) into Eq. (3.5.18), and equate the coefficients of Legendre polynomials $P_j(\mu)$ having the same power of β. We obtain the set of differential equations

$$d(\xi^2\ d\theta/d\xi)/d\xi = -\xi^2\theta^n; \quad d(\xi^2\ d\psi_j/d\xi)/d\xi = [j(j+1) - n\xi^2\theta^{n-1}]\psi_j + \xi^2\pi_j + \xi^2\chi_j;$$
$$d(\xi^2\ df_j/d\xi)/d\xi = [j(j+1) - n\xi^2\theta^{n-1}]f_j - [n(n-1)/2]\xi^2\theta^{n-2}S_j, \quad (\pi_j, \chi_j = 0 \text{ if } j > 4). \quad (3.5.23)$$

The initial conditions at the origin $\xi = 0$ are analogous to Eq. (3.2.6):

$$\Theta(0,\mu) = 1;\ [\partial\Theta(\xi,\mu)/\partial\xi]_{\xi=0} = 0;\ \theta(0) = 1;\ \theta'(0) = 0;\ \psi_j(0), \psi_j'(0), f_j(0), f_j'(0) = 0. \qquad (3.5.24)$$

Geroyannis et al. (1979) assume the differentially rotating boundary under the form [cf. Eq. (3.2.34)]

$$\Xi_1(\mu) = \xi_1 + \beta\sum_{j=0}^{A} q_j P_j(\mu) + \beta^2\sum_{j=0}^{B} t_j P_j(\mu), \qquad (q_j, t_j = \text{const}). \qquad (3.5.25)$$

We have similarly to Eq. (3.2.36)

$$\theta(\Xi_1) \approx \theta(\xi_1) + (\Xi_1 - \xi_1)\ \theta'(\xi_1) + (\Xi_1 - \xi_1)^2\ \theta''(\xi_1)/2$$
$$= \theta(\xi_1) + \beta\theta'(\xi_1)\sum_{j=0}^{A} q_j P_j(\mu) + \beta^2\theta'(\xi_1)\sum_{j=0}^{B} t_j P_j(\mu) + [\beta^2\theta''(\xi_1)/2]\Big[\sum_{j=0}^{A} q_j P_j(\mu)\Big]^2$$
$$= \theta(\xi_1) + \beta\theta'(\xi_1)\sum_{j=0}^{A} q_j P_j(\mu) + \beta^2\theta'(\xi_1)\sum_{j=0}^{C}(t_j - Q_j/\xi_1)\ P_j(\mu), \qquad (0 < n < 5), \qquad (3.5.26)$$

where $\theta''(\xi_1) = -2\theta'(\xi_1)/\xi_1$ via Eq. (2.1.14), and

$$\Big[\sum_{j=0}^{A} q_j P_j(\mu)\Big]^2 = \sum_{j=0}^{2A} Q_j P_j(\mu), \qquad (Q_j = \text{const};\ 0 < n < 5). \qquad (3.5.27)$$

The products from Eq. (3.5.13) are expanded on the boundary as

$$\beta\sum_{j=0}^{A}\psi_j(\Xi_1)\ P_j(\mu) \approx \beta\sum_{j=0}^{A}\psi_j(\xi_1)\ P_j(\mu) + \beta(\Xi_1 - \xi_1)\sum_{j=0}^{A}\psi_j'(\xi_1)\ P_j(\mu) = \beta\sum_{j=0}^{A}\psi_j(\xi_1)\ P_j(\mu)$$
$$+\beta^2\sum_{k=0}^{A} q_k P_k(\mu)\Big[\sum_{j=0}^{A}\psi_j'(\xi_1)\ P_j(\mu)\Big] = \beta\sum_{j=0}^{A}\psi_j(\xi_1)\ P_j(\mu) + \beta^2\sum_{j=0}^{A}\sum_{k=0}^{2A} Q_{jk}\psi_j'(\xi_1)\ P_k(\mu), \qquad (3.5.28)$$

where

$$P_j(\mu)\sum_{k=0}^{A} q_k P_k(\mu) = \sum_{k=0}^{2A} Q_{jk} P_k(\mu), \qquad (0 \le j \le A;\ Q_{jk} = \text{const};\ Q_{jk} = 0 \text{ if } k > j + A), \qquad (3.5.29)$$

since according to Eq. (3.1.39) $P_j(\mu)$ is a polynomial of degree j. We approximate also

$$\beta^2\sum_{j=0}^{B} f_j(\Xi_1)\ P_j(\mu) \approx \beta^2\sum_{j=0}^{B} f_j(\xi_1)\ P_j(\mu). \tag{3.5.30}$$

Eqs. (3.5.26)-(3.5.30) are inserted into Eq. (3.5.13):

$$\Theta(\Xi_1,\mu) = \theta(\xi_1) + \beta\sum_{j=0}^{A}[q_j\theta'(\xi_1)+\psi_j(\xi_1)]\ P_j(\mu) + \beta^2\sum_{j=0}^{C}\Bigg[(t_j - Q_j/\xi_1)\ \theta'(\xi_1)$$
$$+\sum_{k=0}^{A} Q_{kj}\psi_k'(\xi_1) + f_j(\xi_1)\Bigg]\ P_j(\mu) = 0, \qquad (0<n<5). \tag{3.5.31}$$

The coefficients with the same β and $P_j(\mu)$ must vanish separately, and therefore

$$q_j = -\psi_j(\xi_1)/\theta'(\xi_1);\quad q_j = 0 \text{ if } j>A;\quad t_j = Q_j/\xi_1 - \sum_{k=0}^{A} Q_{kj}\psi_k'(\xi_1)/\theta'(\xi_1) - f_j(\xi_1)/\theta'(\xi_1);$$
$$t_j = 0 \quad \text{if} \quad j>B. \tag{3.5.32}$$

To simplify the notations, we express the internal potential (3.5.11) in the unit $(n+1)K\varrho_0^{1/n}$, and expand $\varphi_j(\Xi_1)$ from Eq. (3.5.14) analogously to Eqs. (3.5.28), (3.5.29). In the constant term from Eq. (3.5.11) we emphasize the contribution of β and β^2 : const $= c_0 + c_{10}\beta + c_{20}\beta^2$. Thus

$$\Phi(\Xi_1,\mu) = c_0 + \beta\Bigg[c_{10} - \sum_{j=0}^{4}\varphi_j(\xi_1)\ P_j(\mu)\Bigg] + \beta^2\Bigg[c_{20} - \sum_{j=0}^{4+A}\sum_{k=0}^{4} Q_{kj}\varphi_k'(\xi_1)\ P_j(\mu)\Bigg], \quad \big(\Theta(\Xi_1,\mu)=0\big). \tag{3.5.33}$$

To calculate the partial derivative of the internal potential (3.5.11), we must first obtain $\partial\Theta/\partial\xi$ on the boundary, by differentiating Eq. (3.5.13) and using Eqs. (3.5.26)-(3.5.30):

$$(\partial\Theta/\partial\xi)_{\xi=\Xi_1} = \theta'(\xi_1) + \beta\sum_{j=0}^{A}[-2q_j\theta'(\xi_1)/\xi_1 + \psi_j'(\xi_1)]\ P_j(\mu)$$
$$+\beta^2\sum_{j=0}^{C}\Bigg[-2t_j\theta'(\xi_1)/\xi_1 + \sum_{k=0}^{A} Q_{kj}\psi_k''(\xi_1) + 3Q_j\theta'(\xi_1)/\xi_1^2 + f_j'(\xi_1)\Bigg]\ P_j(\mu), \qquad (1<n<5), \tag{3.5.34}$$

where the derivatives on the boundary

$$\theta''(\xi_1) = -2\theta'(\xi_1)/\xi_1 \quad \text{if} \quad 0<n<5;\quad \theta'''(\xi_1) = 6\theta'(\xi_1)/\xi_1^2 \quad \text{if} \quad 1<n<5, \tag{3.5.35}$$

of the Lane-Emden equation (2.1.14) have been used. The constraint $1<n<5$ for $\theta'''(\xi_1)$ further limits the domain of applicability of the present theory, since $\theta'''(\xi_1)$ is infinite if $0<n<1$.

The derivative of Eq. (3.5.11) becomes on the boundary via Eqs. (3.5.14), (3.5.34):

$$[\partial\Phi(\xi,\mu)/\partial\xi]_{\xi=\Xi_1} = \theta'(\xi_1) + \beta\sum_{j=0}^{A}[-2q_j\theta'(\xi_1)/\xi_1 + \psi_j'(\xi_1) - \varphi_j'(\xi_1)]\ P_j(\mu)$$
$$+\beta^2\sum_{j=0}^{C}\Bigg\{-2t_j\theta'(\xi_1)/\xi_1 + \sum_{k=0}^{A} Q_{kj}[\psi_k''(\xi_1) - \varphi_k''(\xi_1)] + 3Q_j\theta'(\xi_1)/\xi_1^2 + f_j'(\xi_1)\Bigg\}\ P_j(\mu), \tag{3.5.36}$$

where it has been supposed, as will be verified later, that $A\geq 4$. We now turn to the outer potential [cf. Eq. (3.1.58)], truncating the expansions in accordance with the scheme of approximation adopted by Geroyannis et al. (1979):

$$\Phi_e(\xi,\mu) = k_0/\xi + \beta\sum_{j=0}^{A} k_{1j}P_j(\mu)/\xi^{j+1} + \beta^2\sum_{j=0}^{B} k_{2j}P_j(\mu)/\xi^{j+1}, \qquad (k_0, k_{1j}, k_{2j} = \text{const}). \tag{3.5.37}$$

We expand on the boundary via Eq. (3.5.26):

$$k_0/\Xi_1 = k_0/\xi_1 - \beta\sum_{j=0}^{A} k_0 q_j P_j(\mu)/\xi_1^2 + \beta^2\sum_{j=0}^{C}(-k_0t_j/\xi_1^2 + k_0Q_j/\xi_1^3)\ P_j(\mu) + O(\beta^3);$$

$$\beta\sum_{j=0}^{A} k_{1j}P_j(\mu)/\Xi_1^{j+1} = \beta\sum_{j=0}^{A} k_{1j}P_j(\mu)/\xi_1^{j+1} - \beta^2\sum_{j=0}^{A}\sum_{k=0}^{2A}(j+1)k_{1j}Q_{jk}P_k(\mu)/\xi_1^{j+2} + O(\beta^3);$$

$$\beta^2\sum_{j=0}^{B} k_{2j}P_j(\mu)/\Xi_1^{j+1} = \beta^2\sum_{j=0}^{B} k_{2j}P_j(\mu)/\xi_1^{j+1} + O(\beta^3). \tag{3.5.38}$$

Insertion of Eq. (3.5.38) into Eq. (3.5.37) yields

$$\Phi_e(\Xi_1,\mu) = k_0/\xi_1 + \beta\sum_{j=0}^{A}(-k_0q_j/\xi_1^2 + k_{1j}/\xi_1^{j+1})\ P_j(\mu)$$

$$+\beta^2\sum_{j=0}^{C}\left[-k_0t_j/\xi_1^2 + k_0Q_j/\xi_1^3 - \sum_{\ell=0}^{A}(\ell+1)k_{1\ell}Q_{\ell j}/\xi_1^{\ell+2} + k_{2j}/\xi_1^{j+1}\right]\ P_j(\mu). \tag{3.5.39}$$

Similarly, the derivative of Eq. (3.5.37) becomes on the boundary

$$[\partial\Phi_e(\xi,\mu)/\partial\xi]_{\xi=\Xi_1} = -k_0/\Xi_1^2 - \beta\sum_{j=0}^{A}(j+1)k_{1j}P_j(\mu)/\Xi_1^{j+2} - \beta^2\sum_{j=0}^{B}(j+1)k_{2j}P_j(\mu)/\Xi_1^{j+2}$$

$$= -k_0/\xi_1^2 + \beta\sum_{j=0}^{A}[2k_0q_j/\xi_1^3 - (j+1)k_{1j}/\xi_1^{j+2}]\ P_j(\mu)$$

$$+\beta^2\sum_{j=0}^{C}\left[2k_0t_j/\xi_1^3 - 3k_0Q_j/\xi_1^4 + \sum_{\ell=0}^{A}(\ell+1)(\ell+2)k_{1\ell}Q_{\ell j}/\xi_1^{\ell+3} - (j+1)k_{2j}/\xi_1^{j+2}\right]\ P_j(\mu). \tag{3.5.40}$$

Like in Sec. 3.2, we now equate Eqs. (3.5.33), (3.5.36) with Eqs. (3.5.39), (3.5.40), respectively. First, we remark that the zero order terms give [cf. Eq. (3.2.43)]

$$k_0 = -\xi_1^2\theta'(\xi_1);\quad c_0 = k_0/\xi_1 = -\xi_1\theta'(\xi_1),\qquad (1<n<5). \tag{3.5.41}$$

Equating the coefficients of $\beta P_0(\mu)$, we get

$$k_{10} = \xi_1^2[\varphi_0'(\xi_1) - \psi_0'(\xi_1)];\quad c_{10} = \varphi_0(\xi_1) - \psi_0(\xi_1) + \xi_1[\varphi_0'(\xi_1) - \psi_0'(\xi_1)],\qquad (1<n<5). \tag{3.5.42}$$

The coefficients of $\beta P_j(\mu)$ provide two relationships for k_{1j} :

$$k_{1j} = \xi_1^{j+1}[\psi_j(\xi_1) - \varphi_j(\xi_1)];\quad k_{1j} = \xi_1^{j+2}[\varphi_j'(\xi_1) - \psi_j'(\xi_1)]/(j+1),\qquad (j=1,2,3,...;\ 1<n<5). \tag{3.5.43}$$

These have to be identical, yielding a basic boundary condition. All $\pi_j(\xi),\ \chi_j(\xi),\ \varphi_j(\xi)$ have been assumed equal to zero if $j>4$; equatorial symmetry implies that $\pi_j(\xi),\ \chi_j(\xi),\ \varphi_j(\xi)\equiv 0$ if $j=1,3$. In this case Eq. (3.5.43) writes

$$A_j\psi_{Aj}(\xi_1) - k_{1j}\xi_1^{-j-1} = 0;\quad A_j\psi_{Aj}'(\xi_1) + (j+1)k_{1j}\xi_1^{-j-2} = 0,\qquad (j\neq 0,2,4;\ 1<n<5). \tag{3.5.44}$$

We have emphasized the constants A_j, as in Chandrasekhar's theory from Sec. 3.2, by writing $\psi_j(\xi)\equiv A_j\psi_{Aj}(\xi)$. The homogeneous system (3.5.44) has nonzero solutions if its determinant vanishes: $D_j(\xi_1) = \xi_1^{-j-2}[(j+1)\psi_{Aj}(\xi_1) + \xi_1\psi_{Aj}'(\xi_1)] = 0$, as already outlined subsequently to Eq. (3.2.43). Since the condition $D_j(\xi_1)=0$ is in conflict with the expansions (3.2.95), (3.2.96), we conclude that A_j and k_{1j} vanish identically, and therefore $\psi_j(\xi)\equiv 0$ if $j\neq 0,2,4$. And $A=4$ results from Eq. (3.5.13).

The preceding result permits to evaluate the functions S_j, Q_j, and Q_{jk}. Products of two zonal harmonics $P_j(\mu)\ P_k(\mu)$ can be decomposed into a sum of individual Legendre polynomials by the Adams-Neumann formula (e.g. Kopal 1983):

$$P_j(\mu)\ P_k(\mu) = \sum_{\ell=0}^{j} [A_{j-\ell}A_\ell A_{k-\ell}/A_{j+k-\ell}][(2j+2k-4\ell+1)/(2j+2k-2\ell+1)]\ P_{j+k-2\ell}(\mu),$$
$$(j \le k;\ A_0 = 1;\ A_\ell = 1\times 3\times 5\times ...\times(2\ell-1)/\ell!\quad \text{if}\quad \ell \ge 1). \tag{3.5.45}$$

Thus, we can write instead of the products from Eqs. (3.5.22), (3.5.27), (3.5.29):

$$(P_2)^2 = A_{10}P_0 + A_{12}P_2 + A_{14}P_4 = (1/5)P_0 + (2/7)P_2 + (18/35)P_4;\quad (P_4)^2 = A_{20}P_0 + A_{22}P_2$$
$$+A_{24}P_4 + A_{26}P_6 + A_{28}P_8 = (1/9)P_0 + (100/693)P_2 + (162/1001)P_4 + (20/99)P_6 + (490/1287)P_8;$$
$$P_2P_4 = A_{30}P_0 + A_{32}P_2 + A_{34}P_4 + A_{36}P_6 = (2/7)P_2 + (20/77)P_4 + (5/11)P_6,\quad (A_{30}=0;\ P_0=1). \tag{3.5.46}$$

Via Eqs. (3.5.22) and (3.5.46) we find after some algebra:

$$S_0(\xi) = \psi_0^2 + A_{10}\psi_2^2 + A_{20}\psi_4^2 + 2A_{30}\psi_2\psi_4;\quad S_2(\xi) = A_{12}\psi_2^2 + A_{22}\psi_4^2 + 2\psi_0\psi_2 + 2A_{32}\psi_2\psi_4;$$
$$S_4(\xi) = A_{14}\psi_2^2 + A_{24}\psi_4^2 + 2\psi_0\psi_4 + 2A_{34}\psi_2\psi_4;\quad S_6(\xi) = A_{26}\psi_4^2 + 2A_{36}\psi_2\psi_4;\quad S_8(\xi) = A_{28}\psi_4^2. \tag{3.5.47}$$

Analogously, Eqs. (3.5.27) and (3.5.46) yield

$$Q_0 = q_0^2 + A_{10}q_2^2 + A_{20}q_4^2 + 2A_{30}q_2q_4;\quad Q_2 = A_{12}q_2^2 + A_{22}q_4^2 + 2q_0q_2 + 2A_{32}q_2q_4;$$
$$Q_4 = A_{14}q_2^2 + A_{24}q_4^2 + 2q_0q_4 + 2A_{34}q_2q_4;\quad Q_6 = A_{26}q_4^2 + 2A_{36}q_2q_4;\quad Q_8 = A_{28}q_4^2. \tag{3.5.48}$$

Finally, Eqs. (3.5.29) and (3.5.46) give

$$Q_{00} = q_0;\quad Q_{02} = q_2;\quad Q_{04} = q_4;\quad Q_{20} = A_{10}q_2 + A_{30}q_4;\quad Q_{22} = q_0 + A_{12}q_2 + A_{32}q_4;$$
$$Q_{24} = A_{14}q_2 + A_{34}q_4;\quad Q_{26} = A_{36}q_4;\quad Q_{40} = A_{30}q_2 + A_{20}q_4;\quad Q_{42} = A_{32}q_2 + A_{22}q_4;$$
$$Q_{44} = q_0 + A_{34}q_2 + A_{24}q_4;\quad Q_{46} = A_{36}q_2 + A_{26}q_4;\quad Q_{48} = A_{28}q_4, \tag{3.5.49}$$

where all other S_j, Q_j, Q_{jk} are zero. Equating the second order terms $\beta^2 P_j(\mu)$ from Eqs. (3.5.36) and (3.5.40), we find

$$k_{2j} = [\xi_1^{j+2}/(j+1)]\left\{\sum_{\ell=0}^{A} Q_{\ell j}[\varphi_\ell''(\xi_1) - \psi_\ell''(\xi_1) + (\ell+1)(\ell+2)k_{1\ell}/\xi_1^{\ell+3}] - f_j'(\xi_1)\right\},$$
$$(1 < n < 5;\ j = 0,1,2,3,...), \tag{3.5.50}$$

where terms with Q_j and t_j cancel out if k_0 is inserted via Eq. (3.5.41). The two terms containing $\beta^2 P_0(\mu)$ in Eqs. (3.5.33), (3.5.39) yield

$$c_{20} = \sum_{\ell=0}^{A} Q_{\ell 0}[\varphi_\ell'(\xi_1) - (\ell+1)k_{1\ell}/\xi_1^{\ell+2}] + (k_0/\xi_1^2)(Q_0/\xi_1 - t_0) + k_{20}/\xi_1,\qquad (A=4;\ 1<n<5), \tag{3.5.51}$$

and the terms connected with $\beta^2 P_j(\mu),\ (j = 1,2,3,...)$:

$$k_{2j} = \xi_1^{j+1}\left\{\sum_{\ell=0}^{A} Q_{\ell j}[-\varphi_\ell'(\xi_1) + (\ell+1)k_{1\ell}/\xi_1^{\ell+2}] - (k_0/\xi_1^2)(Q_j/\xi_1 - t_j)\right\},\qquad (A=4;\ 1<n<5). \tag{3.5.52}$$

The two relationships (3.5.50) and (3.5.52) for $k_{2j},\ (j = 1,2,3,...)$ must be identical, yielding a basic boundary condition. If $j \ne 0,2,4,6,8$, we observe that Eqs. (3.5.50), (3.5.52) simplify to

$$k_{2j} = -\xi_1^{j+2}f_j'(\xi_1)/(j+1);\quad k_{2j} = k_0t_j\xi_1^{j-1} = \xi_1^{j+1}f_j(\xi_1),\qquad (1<n<5;\ j\ne 0,2,4,6,8), \tag{3.5.53}$$

Table 3.5.1 Boundary values of relevant functions for Chandrasekhar's second order theory of uniformly rotating polytropes (Geroyannis and Valvi 1986c). Note, that the functions ψ_j from Eq. (3.5.13) are different from those in Secs. 3.2-3.4, as they include the factors A_j from Eq. (3.2.9). For constant angular velocity we have $\pi_0 = 1,\ \varphi_0 = -\varphi_2 = \xi^2/6$. And $aE + b$ means $a \times 10^b$.

n	2	3	4	n	2	3	4
π_0	1.000E+0	1.000E+0	1.000E+0	ψ_2	−3.625E+0	−8.157E+0	−3.745E+1
φ_0	3.158E+0	7.928E+0	3.736E+1	ψ_2'	−1.129E+0	−2.199E+0	−4.973E+0
φ_0'	1.451E+0	2.300E+0	4.991E+0	ψ_2''	−6.291E−1	−3.911E−1	3.381E−1
φ_0''	3.333E−1	3.333E−1	3.333E−1	f_0	−3.625E+0	−1.728E+1	−1.485E+2
φ_2	−3.158E+0	−7.928E+0	−3.736E+1	f_0'	−4.153E+0	−6.108E+0	−1.437E+1
φ_2'	−1.451E+0	−2.300E+0	−4.991E+0	f_2	−5.748E+0	−9.260E+0	−3.027E+1
φ_2''	−3.333E−1	−3.333E−1	−3.333E−1	f_2'	3.961E+0	4.028E+0	6.065E+0
ψ_0	1.915E+0	5.838E+0	3.351E+0	f_4	1.623E+0	1.650E+0	3.226E+0
ψ_0'	9.970E−1	2.039E+0	4.879E+0	f_4'	−1.865E+0	−1.196E+0	−1.077E+0
ψ_0''	5.419E−1	4.087E−1	3.482E−1				

where we have used Eqs. (3.5.32), (3.5.41), (3.5.47)-(3.5.49). The equation (3.5.53) is similar to Eq. (3.5.43) with $\varphi_j(\xi_1) \equiv 0$, and therefore we conclude that k_{2j} and $f_j(\xi)$ vanish if $j \neq 0, 2, 4, 6, 8$. Also, from Eq. (3.5.13) results that $C = B = 2A = 8$. In the case of constant angular velocity we have $\omega \equiv 1 = \mathrm{const}$, and from Eq. (3.5.14) follows $\pi_0 = 1,\ \varphi_0 = \xi^2/6,\ \varphi_2 = -\xi^2/6,\ \big(s^2/4 = \xi^2[1 - P_2(\mu)]/6\big)$, the other functions π_j, χ_j, φ_j being equal to zero.

In this way, all relevant functions of the second order perturbation theory are completely determined.

Within the domain of real numbers the functions $\theta^n, \theta^{n-1}, \theta^{n-2}, ...$ become in most cases indefinite for negative values of θ, i.e. beyond the first zero ξ_1 of the Lane-Emden function θ. To avoid this detriment, Geroyannis (1988) adopts a complex-plane strategy, extending the functions from the expansion (3.5.13) into the complex plane if $\xi > \xi_1$. The representation of Θ if $\xi \leq \xi_1$ is

$$\Theta(\xi, \mu) = \mathrm{Re}(\theta) + \beta \sum_{j=0}^{A} \mathrm{Re}[\psi_j(\xi)]\ P_j(\mu) + \beta^2 \sum_{j=0}^{B} \mathrm{Re}[f_j(\xi)]\ P_j(\mu) + O(\beta^3), \tag{3.5.54}$$

where Re denotes the real part of the complex functions θ, ψ_j, f_j. Recall that the principal value of the n-th power of the complex function $\theta = a + ib$ is (e.g. Bronstein and Semendjajew 1985)

$$\theta^n = (a + ib)^n = (r \exp i\varphi)^n = [r(\cos\varphi + i \sin\varphi)]^n = r^n(\cos n\varphi + i \sin n\varphi);$$
$$r = (a^2 + b^2)^{1/2}; \quad \cos\varphi = a/r; \quad \sin\varphi = b/r. \tag{3.5.55}$$

Geroyannis (1988, 1990a, 1992) claims that his complex-plane strategy and multiple partition technique improve perturbation theories for uniformly and differentially rotating polytropes (cf. Sec. 3.8.7, Table 3.8.1).

Geroyannis (1990b, 1991, 1993a) has also studied viscopolytropic models with the aid of Eq. (2.1.1) when $\tau \neq 0$. A general expression for the viscous stress tensor in Cartesian coordinates is (e.g. Landau and Lifshitz 1959, Tassoul 1978)

$$\tau_{jk} = (\mu + \mu_R)[\partial v_j/\partial x_k + \partial v_k/\partial x_j - (2\delta_{jk}/3)\ \partial v_i/\partial x_i] + (\mu_B + 5\mu_R/3)\ \delta_{jk}\ \partial v_i/\partial x_i,$$
$$(i, j, k, = 1, 2, 3), \tag{3.5.56}$$

where v_1, v_2, v_3 are the velocity components, and μ, μ_B, μ_R denote the coefficients of dynamic (shear), bulk, and radiative viscosity, respectively. For axial symmetry and $v_\ell, v_z = 0,\ v_\varphi = v_\varphi(\ell, t),\ \mu = \mu_1(\ell)\ \mu_2(z),\ \mu_B, \mu_R = 0$, the φ-component of Eq. (2.1.1) becomes in cylindrical (ℓ, φ, z)-coordinates equal to (e.g. Batchelor 1967, Tassoul 1978, App. B)

$$\varrho\ Dv_\varphi/Dt = (1/\ell)\ \partial(\ell\tau_{\ell\varphi})/\partial\ell + \tau_{\ell\varphi}/\ell, \qquad (\partial P/\partial\varphi,\ \partial\Phi/\partial\varphi = 0), \tag{3.5.57}$$

where $\tau_{\ell\varphi} = \ell\mu_1(\ell)\ \mu_2(z)\ \partial(v_\varphi/\ell)/\partial\ell$. We do not pursue this subject further, as Geroyannis (1990b, 1991, 1993a) obtains numerical results under very special assumptions:

$$v_\varphi(\ell, t) = \ell\ \Omega(t)\ \omega(\ell); \quad [1/\Omega(t)]\ d\Omega(t)/dt = \mathrm{const}; \quad \mu_1(\ell) = c + \Theta^d(\ell, 0) + F(\ell),$$
$$[c, d = \mathrm{const};\ \Theta = \Theta(\ell, z)], \tag{3.5.58}$$

where $\Omega(t)$ is the angular velocity on the rotation axis, $F(\ell)$ some particular function, and $\omega(\ell) = \omega(\alpha s)$ has the form (3.5.8).

Although Smith (1975) has criticized expansions of the form (3.5.21), as leading to singular perturbation problems [cf. remarks subsequent to Eq. (3.2.7)], his method of matched asymptotic expansions bears some resemblance to the higher order perturbation theories discussed in this section. Smith (1975) takes the view that the expansions (3.5.13) and (3.5.21) become inadequate if $\theta(\xi) = O(\beta)$; from the Taylor expansion $\theta(\xi) \approx \theta'(\xi_1)(\xi-\xi_1),\ (\theta(\xi_1)=0)$ results that in this case we have $\xi_1 - \xi = O(\beta)$. In the region where $\xi_1 - \xi = O(\beta)$, we have of course also $\Theta(\xi,\mu) \approx \theta(\xi) \approx O(\beta)$, and Smith (1975) introduces the new "stretched" variables η and τ :

$$\Theta(\xi,\mu) = \beta\ \tau(\eta,\mu); \quad \xi = \xi_1 - \beta\eta. \tag{3.5.59}$$

These new variables are of order unity in the region $\xi_1-\xi = O(\beta)$, because $\Theta = O(\beta)$ and $\xi_1-\xi = O(\beta)$. As ξ approaches ξ_1 even closer, the function τ decreases, and approaches the region where $\tau = O(\beta)$, $[\Theta = O(\beta^2)]$ and $\eta_1 - \eta = O(\beta)$. In this new region Smith (1975) adopts the stretched variables ζ and ε :

$$\tau(\eta,\mu) = \beta\ \varepsilon(\zeta,\mu); \quad \eta = \eta_1 - \beta\zeta. \tag{3.5.60}$$

Smith (1975) stops at this point the introduction of new, higher order, stretched variables.

The most important detriment of higher order perturbation theories is the hopeless increase of expansion terms, the relatively modest increase of theoretical insight, and even of precision (cf. Table 3.8.1).

3.6 Double Approximation Method for Rotationally and Tidally Distorted Polytropic Spheres

A variant of the perturbation method outlined in the previous four sections is the so-called double approximation technique. In fact, this method is merely an extension of earlier work due to Jeans (1919) and Takeda (1934) on rotating nonhomogeneous stars. In the inner region, where the centrifugal force is small, Chandrasekhar's perturbation theory from the previous sections is used. In the outer zone the contribution of the mass from the outer layers is neglected, taking the gravitational force as arising solely from the matter in the slightly distorted inner zone (Tassoul 1978). The basic idea of the method arises from the fact that most of the material of a polytrope of index $1.5 \lesssim n \leq 5$ will be in the innermost region of the configuration. For example, 90% of the mass of a nonrotating polytrope of index $n = 2$ is contained within 40% of its volume. The inner region of an axisymmetric, uniformly rotating polytrope – up to the fitting radius ξ_f – is assumed to be described by Chandrasekhar's theory from Sec. 3.2:

$$\Theta(\xi,\mu) = \theta(\xi) + \beta\psi_0(\xi) + \beta\sum_{j=1}^{\infty} A_j\psi_j(\xi)\ P_j(\mu), \qquad (\xi \leq \xi_f). \tag{3.6.1}$$

The prime integral of the equation of hydrostatic equilibrium, valid over the whole polytrope is [cf. Eqs. (3.1.74), (3.5.10)]:

$$\Phi = (n+1)P/\varrho - \Omega^2\ell^2/2 + \mathrm{const.} \tag{3.6.2}$$

With Eqs. (3.2.1), (3.2.3) we turn to dimensionless coordinates:

$$\Phi = [(n+1)K\varrho_0^{1/n}]\big\{\Theta(\xi,\mu) + c_0 + \beta[c_{10} - \xi^2/6 + \xi^2 P_2(\mu)/6]\big\}, \qquad (\mathrm{const} = c_0 + \beta c_{10}). \tag{3.6.3}$$

In the inner region we insert for Θ from Eq. (3.6.1):

$$\Phi = [(n+1)K\varrho_0^{1/n}]\left\{\theta(\xi) + c_0 + \beta\left[c_{10} + \psi_0(\xi) - \xi^2/6 + \xi^2P_2(\mu)/6 + \sum_{j=1}^{\infty} A_j\psi_j(\xi)\ P_j(\mu)\right]\right\},$$
$$(\xi \leq \xi_f), \tag{3.6.4}$$

where $\ell^2 = \alpha^2\xi^2\sin^2\lambda = \alpha^2\xi^2(1-\mu^2) = 2\alpha^2\xi^2[1-P_2(\mu)]/3$. For the outer region beyond ξ_f the density is assumed so small that Poisson's equation $\nabla^2\Phi = -4\pi G\varrho$ turns into Laplace's equation $\nabla^2\Phi = 0$, that is rigorously valid only outside the boundary. Thus, we approximate Φ in the outer region with the external potential (3.2.33):

$$\Phi \approx \Phi_e = [(n+1)K\varrho_0^{1/n}]\left[k_0/\xi + \beta\sum_{j=0}^{\infty} k_{1j}\xi^{-j-1}P_j(\mu)\right], \qquad (\xi \geq \xi_f). \tag{3.6.5}$$

The solution of the problem is now completed by matching the inner and outer solution (Eqs. (3.6.4) and (3.6.5), respectively) on some suitably chosen spherical interface $\xi = \xi_f$. The fitting coordinate is determined to give maximum accuracy, by imposing that the relative error of the inner and outer solution is of the same order at ξ_f. In a rough approximation, the relative error $(\Theta - \theta)/\theta \approx (\beta\psi_0/\theta)^2$ of the inner solution (3.6.1) is equal to the square of the first order term $\beta\psi_0/\theta$, by ignoring angular dependences. The error $[M_1 - M(\xi_f)]/M_1$ of the outer solution results from the fact that the mass of the polytrope outside ξ_f is neglected, where $M(\xi_f)$ denotes the mass inside radius ξ_f, and M_1 the total mass. Thus, Monaghan and Roxburgh (1965) determine ξ_f roughly from

$$[\beta_c\psi_0(\xi_f)/\theta(\xi_f)]^2 = [M_1 - M(\xi_f)]/M_1, \tag{3.6.6}$$

where the value $\beta_c = \Omega_c^2/2\pi G\varrho_0 = (\Omega_c^2/2\pi G\varrho_m)(\varrho_m/\varrho_0) = \beta_{cm}(\varrho_m/\varrho_0) \approx 0.36(\varrho_m/\varrho_0)$ corresponds to equatorial mass loss, and is called the critical rotation parameter β_c (see Eqs. (3.4.46), (3.6.36), Table 3.8.1). On the fitting interface $\xi = \xi_f$ we impose the condition of continuity $\Phi(\xi_f,\mu) = \Phi_e(\xi_f,\mu)$ of the inner and outer potential, by equating the coefficients of the same Legendre polynomials in Eqs. (3.6.4) and (3.6.5):

$$k_0/\xi_f = \theta(\xi_f) + c_0; \quad k_{10}/\xi_f = \psi_0(\xi_f) - \xi_f^2/6 + c_{10}; \quad k_{12}/\xi_f^3 = A_2\psi_2(\xi_f) + \xi_f^2/6;$$
$$k_{1j}/\xi_f^{j+1} = A_j\psi_j(\xi_f) \text{ if } j \neq 0, 2. \tag{3.6.7}$$

Continuity of the derivatives $(\partial\Phi/\partial\xi)_{\xi=\xi_f} = (\partial\Phi_e/\partial\xi)_{\xi=\xi_f}$ of Eqs. (3.6.4) and (3.6.5) at $\xi = \xi_f$ yields

$$-k_0/\xi_f^2 = \theta'(\xi_f); \quad -k_{10}/\xi_f^2 = \psi_0'(\xi_f) - \xi_f/3; \quad -3k_{12}/\xi_f^4 = A_2\psi_2'(\xi_f) + \xi_f/3;$$
$$-(j+1)k_{1j}/\xi_f^{j+2} = A_j\psi_j'(\xi_f) \text{ if } j \neq 0, 2. \tag{3.6.8}$$

We have $k_{1j}, A_j = 0$ if $j \neq 0, 2$, as in Sec. 3.2, and

$$k_0 = -\xi_f^2\theta'(\xi_f); \quad c_0 = -\xi_f\theta'(\xi_f) - \theta(\xi_f); \quad k_{10} = -\xi_f^2\psi_0'(\xi_f) + \xi_f^3/3;$$
$$c_{10} = -\psi_0(\xi_f) - \xi_f\psi_0'(\xi_f) + \xi_f^2/2; \quad k_{12} = \xi_f^5[\xi_f\psi_2'(\xi_f) - 2\psi_2(\xi_f)]/6[3\psi_2(\xi_f) + \xi_f\psi_2'(\xi_f)];$$
$$A_2 = -5\xi_f^2/6[3\psi_2(\xi_f) + \xi_f\psi_2'(\xi_f)]. \tag{3.6.9}$$

The solution (3.6.1) for the inner region becomes

$$\Theta(\xi,\mu) = \theta(\xi) + \beta[\psi_0(\xi) + A_2\psi_2(\xi)\ P_2(\mu)], \qquad (\xi \leq \xi_f;\ 0 \leq n \leq 5). \tag{3.6.10}$$

For the outer layers Eq. (3.6.3) writes, by inserting for Φ the outer potential Φ_e from Eq. (3.6.5):

$$\Theta(\xi,\mu) = k_0/\xi - c_0 + \beta[k_{10}/\xi + \xi^2/6 - c_{10} + (k_{12}/\xi^3 - \xi^2/6)\ P_2(\mu)], \quad (\xi_f \leq \xi \leq \Xi_1;\ 0 \leq n \leq 5). \tag{3.6.11}$$

The critical configuration of a rotating polytrope occurs if the polytrope is on the verge of equatorial break-up, i.e. when the centrifugal force at the equator just balances gravity. In other words, if Φ_{tot} denotes the total potential, the effective gravity $\vec{g}(r,\mu) = \nabla\Phi_{tot}$ has to vanish at the critical equatorial radius $r = r_{ce} = r_{ce}(0)$, $[r = r(\mu)]$. The vanishing radial component of the effective gravity at the equator writes

$$g_{r_{ce}} = (\partial\Phi_{tot}/\partial r)_{r=r_{ce}} = [\partial(\Phi + \Phi_f)/\partial r]_{r=r_{ce}} = [(n+1)K\varrho_0^{1/n}/\alpha][\partial\Theta(\xi,\mu)/\partial\xi]_{\xi=\Xi_{ce}}$$
$$= [(n+1)K\varrho_0^{1/n}/\alpha][-k_0/\Xi_{ce}^2 - \beta_c(k_{10}/\Xi_{ce}^2 + 3k_{12}/\Xi_{ce}^4 - \Xi_{ce}/2)] = 0,$$
$$[\Xi_1 = \Xi_1(\mu);\ \Xi_{ce} = \Xi_1(0)], \tag{3.6.12}$$

where we have inserted for Φ from Eq. (3.6.3), for Θ from Eq. (3.6.11), and for the centrifugal potential $\Phi_f = \Omega_c^2\ell^2/2 = (n+1)K\varrho_0^{1/n}\beta_c\xi^2[1 - P_2(\mu)]/6$ from Eqs. (3.1.23), (3.4.64). On the equatorial boundary we have further

$$\Theta(\Xi_{ce}, 0) = k_0/\Xi_{ce} - c_0 + \beta_c(k_{10}/\Xi_{ce} + k_{12}/\Xi_{ce}^3 + \Xi_{ce}^2/4 - c_{10}) = 0. \tag{3.6.13}$$

The two previous equations $\partial\Theta/\partial\xi = 0$ and $\Theta = 0$ determine the critical rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0$, and the critical equatorial coordinate $\Xi_{ce} = \Xi_{ce}(0) = \Xi_1(0)$, [cf. Eqs. (3.8.158)-(3.8.164)].

The double approximation method is applicable for polytropic indices $0 \leq n \leq 5$, but with increasing mass concentration in the outer layers ($n \to 0$), its accuracy decreases sharply. The method has been extended to second order by Singh and Singh (1984), showing no drastic improvement of results.

Concerning the double star problem, Orlov (1960) has considered a binary model consisting of stars with polytropic core and mass-less (Roche type) envelope. Double approximation techniques have been applied to the double star problem nearly concomitantly by Martin (1970, second order theory), Naylor and Anand (1970), and Durney and Roxburgh (1970). In the inner region of the primary component M the fundamental function is assumed under the form (3.4.6), (3.4.8):

$$\Theta(\xi,\lambda,\varphi) = \theta(\xi) + \beta\psi_0(\xi) + \beta\sum_{j=1}^{\infty} A_j\psi_j(\xi)\ Y_j(\lambda,\varphi), \qquad (\xi \leq \xi_f). \tag{3.6.14}$$

The integral of the equation of hydrostatic equilibrium (3.4.2), valid over the whole polytrope is

$$\Phi = (n+1)P/\varrho - \Phi'_e - (\Omega^2/2)[x_1^2 + x_2^2 - 2M'Dx_1/(M+M')] + \mathrm{const} = (n+1)K\varrho^{1/n}$$
$$-(GM'/D)\sum_{j=0}^{4}(r/D)^j P_j(\sin\lambda\cos\varphi) - (\Omega^2/2)[\ell^2 - 2M'Dr\sin\lambda\cos\varphi/(M+M')] + \mathrm{const}$$
$$= (n+1)K\varrho_0^{1/n}\Theta(\xi,\lambda,\varphi) - (GM'\alpha^2/D^3)\sum_{j=1}^{4}(\alpha/D)^{j-2}\xi^j P_j(\sin\lambda\cos\varphi) - \pi\beta G\varrho_0\alpha^2\xi^2(1-\cos^2\lambda)$$
$$+GM'r\sin\lambda\cos\varphi/D^2 + \mathrm{const} = (n+1)K\varrho_0^{1/n}\Big\{\Theta(\xi,\lambda,\varphi) + c_0$$
$$+\beta\Big[c_{10} - \sum_{j=2}^{4}M'(\alpha/D)^{j-2}\xi^j P_j(\sin\lambda\cos\varphi)/2(M+M') - \xi^2/6 + \xi^2 P_2(\cos\lambda)/6\Big]\Big\}, \qquad (3.6.15)$$

where $G\alpha^2/D^3 = (n+1)\beta K\varrho_0^{1/n}/2(M+M')$, $\pi G\varrho_0\alpha^2 = (n+1)K\varrho_0^{1/n}/4$, $\mathrm{const} = c_0 + \beta c_{10}$, and we have used Eq. (3.4.16) for the external potential Φ'_e of the secondary M'.

The gravitational potential in the *inner* region is obtained by inserting for Θ from Eq. (3.6.14):

$$\Phi = (n+1)K\varrho_0^{1/n}\Big\{\theta(\xi) + c_0 + \beta\Big[c_{10} + \psi_0(\xi) + \sum_{j=1}^{\infty}A_j\psi_j(\xi)\ Y_j(\lambda,\varphi)$$
$$-\sum_{j=2}^{4}M'(\alpha/D)^{j-2}\xi^j P_j(\sin\lambda\cos\varphi)/2(M+M') - \xi^2/6 + \xi^2 P_2(\cos\lambda)/6\Big]\Big\}, \qquad (\xi \le \xi_f). \qquad (3.6.16)$$

In the *outer* region the internal potential is approximated by Eq. (3.4.24):

$$\Phi \approx \Phi_e = [(n+1)K\varrho_0^{1/n}]\Big[k_0/\xi + \beta\sum_{j=0}^{\infty}k_{1j}\xi^{-j-1}Y_j(\lambda,\varphi)\Big], \qquad (\xi \ge \xi_f). \qquad (3.6.17)$$

We now equate, as in the rotational problem, the coefficients of harmonics of the same order between Eqs. (3.6.16) and (3.6.17) at the fitting radius ξ_f :

$$k_0/\xi_f = \theta(\xi_f) + c_0; \quad k_{10}/\xi_f = \psi_0(\xi_f) - \xi_f^2/6 + c_{10};$$
$$k_{12}\xi_f^{-3}Y_2(\lambda,\varphi) = A_2\psi_2(\xi_f)\ Y_2(\lambda,\varphi) - M'\xi_f^2 P_2(\sin\lambda\cos\varphi)/2(M+M') + \xi_f^2 P_2(\cos\lambda)/6;$$
$$k_{1j}\xi_f^{-j-1}Y_j(\lambda,\varphi) = A_j\psi_j(\xi_f)\ Y_j(\lambda,\varphi) - M'(\alpha/D)^{j-2}\xi_f^j P_j(\sin\lambda\cos\varphi)/2(M+M') \text{ if } j = 3,4. \qquad (3.6.18)$$

The equations for $j \ne 0,2,3,4$ lead ultimately to $k_{1j}, A_j = 0$, as in Sec. 3.4, and are omitted for brevity. The continuity of the radial derivatives of Eqs. (3.6.16) and (3.6.17) at $\xi = \xi_f$ yields

$$-k_0/\xi_f^2 = \theta'(\xi_f); \quad -k_{10}/\xi_f^2 = \psi_0'(\xi_f) - \xi_f/3; \quad -3k_{12}\xi_f^{-4}Y_2(\lambda,\varphi) = A_2\psi_2'(\xi_f)\ Y_2(\lambda,\varphi)$$
$$-M'\xi_f P_2(\sin\lambda\cos\varphi)/(M+M') + \xi_f P_2(\cos\lambda)/3; \quad -(j+1)k_{1j}\xi_f^{-j-2}Y_j(\lambda,\varphi)$$
$$= A_j\psi_j'(\xi_f)\ Y_j(\lambda,\varphi) - jM'(\alpha/D)^{j-2}\xi_f^{j-1}P_j(\sin\lambda\cos\varphi)/2(M+M') \text{ if } j = 3,4. \qquad (3.6.19)$$

From the two previous equations we find

$$k_0 = -\xi_f^2\theta'(\xi_f); \quad c_0 = -\xi_f\theta'(\xi_f) - \theta(\xi_f); \quad k_{10} = -\xi_f^2\psi_0'(\xi_f) + \xi_f^3/3; \quad c_{10} = -\psi_0(\xi_f)$$
$$-\xi_f\psi_0'(\xi_f) + \xi_f^2/2; \quad k_{12}Y_2(\lambda,\varphi) = \xi_f^5[2\psi_2(\xi_f) - \xi_f\psi_2'(\xi_f)]\big[-P_2(\cos\lambda)/6 + M'P_2(\sin\lambda\cos\varphi)$$
$$/2(M+M')\big]/[3\psi_2(\xi_f) + \xi_f\psi_2'(\xi_f)]; \quad k_{1j}Y_j(\lambda,\varphi) = [M'/2(M+M')](\alpha/D)^{j-2}\xi_f^{2j+1}$$
$$\times[j\psi_j(\xi_f) - \xi_f\psi_j'(\xi_f)]P_j(\sin\lambda\cos\varphi)/[(j+1)\psi_j(\xi_f) + \xi_f\psi_j'(\xi_f)] \text{ if } j = 3,4; \quad A_2Y_2(\lambda,\varphi)$$
$$= 5\xi_f^2[-P_2(\cos\lambda)/6 + M'P_2(\sin\lambda\cos\varphi)/2(M+M')]/[3\psi_2(\xi_f) + \xi_f\psi_2'(\xi_f)]; \quad A_jY_j(\lambda,\varphi)$$
$$= [M'/2(M+M')](\alpha/D)^{j-2}(2j+1)\xi_f^j P_j(\sin\lambda\cos\varphi)/[(j+1)\psi_j(\xi_f) + \xi_f\psi_j'(\xi_f)] \text{ if } j = 3,4. \qquad (3.6.20)$$

Table 3.6.1 Function values at the fitting radius ξ_f according to Naylor and Anand (1970). $aE+b$ means $a\times 10^b$.

n	1.5	2	3	4	4.9
ξ_f	3.2E+0	3.6E+0	5.0E+0	8.0E+0	9.74E+1
θ	1.0455E−1	1.1525E−1	1.1082E−1	1.0450E−1	7.6465E−3
θ'	−2.5875E−1	−1.8269E−1	−8.0126E−2	−2.7957E−2	−1.8179E−4
ψ_0	1.0423E+0	1.2961E+0	2.7202E+0	8.1810E+0	1.5688E+3
ψ_0'	4.9660E−1	6.7495E−1	1.2512E+0	2.3563E+0	3.2434E+1
ψ_2	4.3134E+0	4.5360E+0	6.5108E+0	1.3843E+1	1.7738E+3
ψ_2'	1.0255E+0	1.2617E+0	1.9891E+0	3.1887E+0	3.6417E+1
ψ_3	1.7411E+1	2.2090E+1	4.9772E+1	1.8284E+2	3.0095E+5
ψ_3'	1.1811E+1	1.4617E+1	2.7318E+1	6.6699E+1	9.2691E+3
ψ_4	6.3560E+1	9.3676E+1	3.0845E+2	1.8700E+3	3.8377E+7
ψ_4'	6.7034E+1	9.2461E+1	2.3611E+2	9.2231E+2	1.5760E+6

Thus, the solution (3.6.14) for the inner region becomes [cf. Eq. (3.4.30)]:

$$\Theta(\xi,\lambda,\varphi)=\theta(\xi)+\beta\Big\{\psi_0(\xi)-5\xi_f^2\psi_2(\xi)\ P_2(\cos\lambda)/6[3\psi_2(\xi_f)+\xi_f\psi_2'(\xi_f)]+[M'/2(M+M')]$$
$$\times\sum_{j=2}^{4}(2j+1)(\alpha/D)^{j-2}\xi_f^j\psi_j(\xi)\ P_j(\sin\lambda\cos\varphi)/[(j+1)\psi_j(\xi_f)+\xi_f\psi_j'(\xi_f)]\Big\},\quad (\xi\le\xi_f;\ 0\le n\le 5). \tag{3.6.21}$$

In the outer layers Eq. (3.6.15) writes, by inserting for Φ the outer potential Φ_e from Eq. (3.6.17):

$$\Theta(\xi,\lambda,\varphi)=-\xi_f^2\theta'(\xi_f)/\xi+\xi_f\theta'(\xi_f)+\theta(\xi_f)+\beta\Big\{[-\xi_f^2\psi_0'(\xi_f)+\xi_f^3/3]/\xi-\xi_f^5[2\psi_2(\xi_f)-\xi_f\psi_2'(\xi_f)]$$
$$\times\xi^{-3}P_2(\cos\lambda)/6[3\psi_2(\xi_f)+\xi_f\psi_2'(\xi_f)]+[M'/2(M+M')]\sum_{j=2}^{4}(\alpha/D)^{j-2}\xi_f^{2j+1}[j\psi_j(\xi_f)-\xi_f\psi_2'(\xi_f)]$$
$$\times\xi^{-j-1}P_j(\sin\lambda\cos\varphi)/[(j+1)\psi_j(\xi_f)+\xi_f\psi_j'(\xi_f)]+\psi_0(\xi_f)+\xi_f\psi_0'(\xi_f)-\xi_f^2/2+\xi^2/6$$
$$-\xi^2P_2(\cos\lambda)/6+\sum_{j=2}^{4}M'(\alpha/D)^{j-2}\xi^jP_j(\sin\lambda\cos\varphi)/2(M+M')\Big\},\qquad(\xi_f\le\xi\le\Xi_1;\ 0\le n\le 5). \tag{3.6.22}$$

Function values for some selected polytropic indices n at the fitting radius ξ_f are quoted in Table 3.6.1 according to Naylor and Anand (1970); some fitting constants in the tables of Monaghan and Roxburgh (1965), and Durney and Roxburgh (1970) seem unreliable. Naylor and Anand (1970) found that the effect of changing the fitting radius ξ_f from Table 3.6.1 by 10% is minor, and as expected on general grounds, it is better to overestimate the fitting radius. With the function values given in Table 3.6.1 we can calculate all fitting constants from Eqs. (3.6.9) and (3.6.20), excepting for the ratios $q=M'/M$ and α/D. While the mass ratio q is a free constant $(0\le q\le\infty)$, the ratio α/D occurring in the double star problem is not independent (Chandrasekhar 1933c):

$$\beta=\Omega^2/2\pi G\varrho_0=[(M+M')/MD^3](M/2\pi\varrho_0)=2(1+q)(\alpha/D)^3\xi_1^2(-\theta_1')\quad\text{or}$$
$$(\alpha/D)^3=\eta^{-3}=\beta/2(1+q)\xi_1^2(-\theta_1')\ll 1,\qquad(D=\alpha\eta). \tag{3.6.23}$$

Since β from Eq. (3.6.23) is already a small first order quantity, we have replaced – within our first order approximation – the mass M by its spherical value (2.6.18). Thus, for any given value of n,q, and β, the double star problem is completely determined.

If β is sufficiently small, D from Eq. (3.6.23) must be necessarily large, but if β increases, the separation distance between the binaries D becomes smaller and smaller, attaining a critical minimum limit D_c at which mass begins to escape at the point closest to the secondary. This configuration is called the critical configuration, and corresponds to equatorial break-up in the purely rotational case $q=0$. In the sequence of equipotentials drawn about any two components of the double star system, there is one

Table 3.6.2 Values of the critical rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0$ for synchronous rotation in a double star system ($q = M'/M$) according to the tables of Singh and Singh (1983, 1984a).

n	$q = 0$	$q = 0.1$	$q = 0.5$	$q = 1$	$q = 2$	$q = 10$
1.5	0.0362	0.0236	0.0139	0.0114	0.00995	0.00909
2	0.0194	0.0126	0.00746	0.00613	0.00537	0.00491
3	0.00395	0.00262	0.00158	0.00130	0.00115	0.00105

Table 3.6.3 Dimensions of the critical primary M in units of the dimensionless critical separation distance $\eta_c = D_c/\alpha$ along the $x_1, x_2, x_3, (-x_1)$-axes, respectively (Naylor and Anand 1970, Singh and Singh 1983 if $q = M'/M = 10$). Values for the Roche model are from Kopal (1978).

			$q = 0.1$			
n	1.5	2	3	4	4.9	Roche
$\Xi_c(\pi/2, 0)/\eta_c$	0.830	0.826	0.822	0.819	0.818	0.717
$\Xi_c(\pi/2, \pi/2)/\eta_c$	0.655	0.653	0.651	0.649	0.648	0.596
$\Xi_c(0, \varphi)/\eta_c$	0.560	0.562	0.565	0.565	0.564	0.534
$\Xi_c(\pi/2, \pi)/\eta_c$	0.692	0.688	0.685	0.683	0.682	–
			$q = 1$			
$\Xi_c(\pi/2, 0)/\eta_c$	0.533	0.532	0.530	0.530	0.529	0.500
$\Xi_c(\pi/2, \pi/2)/\eta_c$	0.383	0.394	0.385	0.385	0.385	0.374
$\Xi_c(0, \varphi)/\eta_c$	0.360	0.362	0.365	0.365	0.365	0.356
$\Xi_c(\pi/2, \pi)/\eta_c$	0.426	0.425	0.424	0.423	0.423	–
			$q = 10$			
$\Xi_c(\pi/2, 0)/\eta_c$	0.284	0.285	0.286	–	–	0.282
$\Xi_c(\pi/2, \pi/2)/\eta_c$	0.195	0.197	0.198	–	–	0.197
$\Xi_c(0, \varphi)/\eta_c$	0.186	0.188	0.190	–	–	0.190
$\Xi_c(\pi/2, \pi)/\eta_c$	0.229	0.230	0.231	–	–	–

equipotential (the critical equipotential – the thick curve in Fig. 3.6.1) for which the potential lobes are just touching. Its shape depends on n, β, q, and at the point of contact the sum of all forces must vanish, so that $\vec{g}(r_c, \pi/2, 0) = \nabla\Phi_{tot} = 0,\ [r = r(\lambda, \varphi);\ r_c = r(\pi/2, 0)]$. For the present purpose the vanishing radial component of the effective gravity is of interest:

$$\begin{aligned} g_{r_c} &= (\partial\Phi_{tot}/\partial r)_{r=r_c} = \big\{\partial\big[\Phi + \Phi'_e + \Phi_f - \Omega^2 M' D r \sin\lambda\cos\varphi/(M+M')\big]/\partial r\big\}_{r=r_c} \\ &= [(n+1)K\varrho_0^{1/n}/\alpha][\partial\Theta(\xi,\lambda,\varphi)/\partial\xi]_{\xi=\Xi_c} = [(n+1)K\varrho_0^{1/n}/\alpha]\xi_f^2\theta'(\xi_f)/\Xi_c^2 + \beta[(n+1)K\varrho_0^{1/n}/\alpha] \\ &\times\Big\{[\xi_f^2\psi_0'(\xi_f) - \xi_f^3/3]/\Xi_c^2 + \xi_f^5[2\psi_2(\xi_f) - \xi_f\psi_2'(\xi_f)]\Xi_c^{-4}P_2(0)/2[3\psi_2(\xi_f) + \xi_f\psi_2'(\xi_f)] \\ &-(j+1)[M'/2(M+M')]\sum_{j=2}^{4}(\alpha/D)^{j-2}\xi_f^{2j+1}[j\psi_j(\xi_f) - \xi_f\psi_2'(\xi_f)]\Xi_c^{-j-2}P_j(1) \\ &/[(j+1)\psi_j(\xi_f) + \xi_f\psi_j'(\xi_f)] + \Xi_c/3 - \Xi_c P_2(0)/3 + j\sum_{j=2}^{4}M'(\alpha/D)^{j-2}\Xi_c^{j-1}P_j(1) \\ &/2(M+M')\Big\} = 0, \qquad [\Xi_1 = \Xi_1(\lambda,\varphi);\ \Xi_c = \Xi_1(\pi/2, 0)], \end{aligned} \tag{3.6.24}$$

where we have used Eqs. (3.4.2), (3.6.15) to replace Φ_{tot} by $(n+1)K\varrho_0^{1/n}\Theta$, inserting for Θ from Eq. (3.6.22). The previous equation together with $\Theta(\Xi_c, \pi/2, 0) = 0$ from Eq. (3.6.22) can be solved numerically, to obtain the critical values β_c and $\Xi_c = \Xi_1(\pi/2, 0)$ of β and $\Xi_1 = \Xi_1(\lambda, \varphi)$. The sole reliable values of β_c so far published seem to be those of Singh and Singh (1983, 1984a) shown in Table 3.6.2.

Configurations with $\beta > \beta_c$ lose mass from the equator – they are nonequilibrium figures and cannot be considered within our hydrostatic approach, opposite to Table III of Durney and Roxburgh (1970). The critical rotation parameters β_c of these authors agree to those of Singh and Singh (1983, 1984a) merely for $q = 1$, and are not reliable if $q < 1,\ n \leq 3$, as they do not approach the limiting value for the purely rotational case $q = 0$.

Once β_c is known, the axes of the critical configuration can be found by inserting β_c and $\eta_c = D_c/\alpha$ into $\Theta(\Xi_c, \lambda, \varphi) = 0$ from Eq. (3.6.22). Since the equations yield only the dimensionless distance η_c

between the binaries, it is useful to express the axes of the critical configuration in terms of the ratio $\Xi_c(\lambda,\varphi)/\eta_c$. Table 3.6.3 shows this ratio for the (λ,φ)-sets $(\pi/2,0),(\pi/2,\pi/2),(0,\varphi),(\pi/2,\pi)$, directed along the $x_1,x_2,x_3,(-x_1)$-axes in rectangular coordinates. Even in the second order approximation, the principal axes of the distorted mass M are located along the rectangular coordinate axes, because any angular contribution to the total gravity is at most of order $(\alpha\xi_1/D)^8$, (Martin 1970). Table 3.6.3 shows errors in the equal mass case $q=1$, when $\Xi_c(\pi/2,0)/\eta_c$ should be exactly 0.5, rather than about 0.53, which may serve as an indication of the overall errors of the method. These errors are almost completely removed in the much more complicated second order theory of Martin (1970): $\Xi_c(\pi/2,0)/\eta_c=0.503$ if $n=2$, and 0.50005 if $n=4.9$, $(q=1)$, as compared to the exact value 0.5. Miketinac and Parter (1981) have entirely confirmed Martin's (1970) results, by using a so-called semidiscrete pseudospectral method (cf. Sec. 3.8.6).

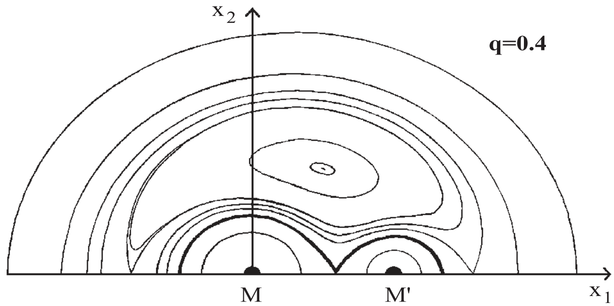

Fig. 3.6.1 Roche equipotentials in the (x_1,x_2)-plane if $q=M'/M=0.4$. The thick curve depicts the critical equipotential through the inner Lagrangian point (Kopal 1978).

As already noted [cf. Eq. (3.2.69)], the polytrope $n=5$ approximates the point mass Roche model, so the results for a polytrope with index $n\approx 5$, $(n<5)$ should be nearly equal to those of the Roche model quoted for comparison in Table 3.6.3. For this reason we briefly present some features of the Roche model, starting with the hydrostatic approach $v_k=0$ of Eq. (3.1.90), and inserting into the potential function the point mass approximations $\Phi=GM/r$ and $\Phi'_e=GM'/r'$:

$$\partial P/\partial x_k=\varrho\,\partial\big\{GM/r+GM'/r'+(\Omega^2/2)[\big(x_1-M'D/(M+M')\big)^2+x_2^2]\big\}/\partial x_k=\varrho\,\partial\Phi_{tot}/\partial x_k;$$
$$r^2=x_1^2+x_2^2+x_3^2;\ r'^2=(D-x_1)^2+x_2^2+x_3^2;\ \Omega^2=G(M+M')/D^3,\qquad (k=1,2,3). \qquad (3.6.25)$$

Subtracting from Φ_{tot} the constant term $\Omega^2M'^2D^2/2(M+M')^2=GM'^2/2D(M+M')$, we define the dimensionless potential function Λ by (Kopal 1978)

$$\Lambda=\Phi_{tot}D/GM-M'^2/2M(M+M')=1/(r/D)+q/(r'/D)$$
$$+[(1+q)/2](x_1^2+x_2^2)/D^2-qx_1/D,\qquad (q=M'/M). \qquad (3.6.26)$$

Taking the unit of length equal to the separation distance D between the binaries, Eq. (3.6.26) becomes

$$\Lambda=1/r+q(1/r'-x_1)+(1+q)(x_1^2+x_2^2)/2;\quad r^2=x_1^2+x_2^2+x_3^2;$$
$$r'^2=(1-x_1)^2+x_2^2+x_3^2,\qquad (D=1). \qquad (3.6.27)$$

Eq. (3.6.27) writes in spherical coordinates

$$\Lambda=1/r+q[1/(1-2r\sin\lambda\cos\varphi+r^2)^{1/2}-r\sin\lambda\cos\varphi]+(1+q)r^2\sin^2\lambda/2, \qquad (3.6.28)$$

or, by expanding the radical via Eq. (3.1.42):

$$(\Lambda-q)r=1+q\sum_{j=2}^{\infty}r^{j+1}P_j(\sin\lambda\cos\varphi)+(1+q)r^2[1-P_2(\cos\lambda)]/3. \qquad (3.6.29)$$

The derivatives of Eq. (3.6.27) are

$$\partial\Lambda/\partial x_1 = -x_1/r^3 + q[(1-x_1)/r'^3 - 1] + (1+q)x_1; \quad \partial\Lambda/\partial x_2 = x_2[-1/r^3 - q/r'^3 + (1+q)x_2];$$
$$\partial\Lambda/\partial x_3 = x_3(-1/r^3 - q/r'^3), \qquad (D=1). \tag{3.6.30}$$

In the contact point on the critical Roche equipotential all forces have to vanish, i.e. $\nabla\Lambda = 0$. On the x_1-axis $\partial\Lambda/\partial x_2$ and $\partial\Lambda/\partial x_3$ vanish identically, and $\partial\Lambda/\partial x_1 = 0$ yields a fifth order equation for the calculation of the critical contact point $x_{c1},\ (r = x_1;\ r' = 1 - x_1):$

$$(1+q)x_1^5 - (2+3q)x_1^4 + (1+3q)x_1^3 - x_1^2 + 2x_1 - 1 = 0. \tag{3.6.31}$$

With x_{c1} known, the value of Λ on the critical Roche equipotential surface results from Eq. (3.6.27):

$$\Lambda_c = \Lambda(x_{c1}, 0, 0) = 1/x_{c1} + q[1/(1-x_{c1}) - x_{c1}] + (1+q)x_{c1}^2/2. \tag{3.6.32}$$

With x_{c1} and Λ_c known, we put $\partial\Lambda/\partial x_2,\ \partial\Lambda/\partial x_3 = 0$, in order to calculate the maximum extension of the critical Roche lobe along the x_2 and x_3-axis, respectively, as shown in Table 3.6.3.

Simple analytical expressions for the critical mean rotation parameter β_{cm} can be found if $q \approx 0$ or $q \gg 1$. Eq. (3.6.31) can be rewritten under the equivalent form

$$(x_1 - 1)^3(x_1^2 + x_1 + 1) + q x_1^3(x_1^2 - 3x_1 + 3) = 0, \tag{3.6.33}$$

showing that $x_{c1} = 1$ is a triple root for the critical Roche equipotential if $q = 0$, the other two roots being irrelevant complex numbers. In this case (single, critically rotating mass) we find from Eq. (3.6.27): $\Lambda = \Lambda_c = 1.5,\ (r = x_1;\ x_2, x_3, q = 0)$. With this value of the critical equipotential, the critical polar axis results from Eq. (3.6.27): $x_{c3} = 2/3,\ (x_1, x_2, q = 0)$. Thus, the oblateness of the centrally concentrated, critically rotating single mass M is $f_c = (x_{c1} - x_{c3})/x_{c1} = 1/3$. With $\Lambda = \Lambda_c = 1.5$, Eq. (3.6.27) reads in cylindrical coordinates as

$$1.5 = 1/(\ell^2 + z^2)^{1/2} + \ell^2/2 \quad \text{or} \quad z = z(\ell) = (1-\ell^2)(4-\ell^2)^{1/2}/(3-\ell^2)$$
$$= 2\cos b\ (1 - 4\sin^2 b)/(3 - 4\sin^2 b), \qquad (\ell = 2\sin b;\ q = 0;\ r^2 = \ell^2 + z^2;\ \ell^2 = x_1^2 + x_2^2). \tag{3.6.34}$$

If $q = 0$, the volume inside the critical Roche surface becomes (Jeans 1919)

$$V_c = 4\pi\int_0^1 \ell\ d\ell \int_0^{z(\ell)} dz = 4\pi\int_0^1 z(\ell)\ \ell\ d\ell = -32\pi\int_0^{\pi/6}(4\cos^2 b - 3)\cos^2 b\ d(\cos b)/(4\cos^2 b - 1)$$
$$= 32\pi\{\gamma^3/3 - \gamma/2 - (1/8)\ln[(2\gamma-1)/(2\gamma+1)]\}\Big|_{3^{1/2}/2}^{1}$$
$$= (4\pi/3)\{3^{3/2} - 4 + 3\ln[3(3^{1/2}-1)/(3^{1/2}+1)]\} = 2.26662..., \qquad (q = 0;\ \gamma = \cos b). \tag{3.6.35}$$

The mean critical rotation parameter of the mass M is

$$\beta_{cm} = \Omega_c^2/2\pi G\varrho_m = 2(1+q)r_m^3/3,$$
$$\big(\Omega_c^2 = GM(1+q);\ r_m = (3V_c/4\pi)^{1/3} = (3M/4\pi\varrho_m)^{1/3};\ D = 1\big). \tag{3.6.36}$$

With Eq. (3.6.35) we obtain $\beta_{cm} = 0.36074...,\ (q = 0;\ r_m = 0.81489...)$, corresponding to equatorial break-up.

To determine β_{cm} if $q \gg 1,\ (M' \gg M)$, we observe from Eq. (3.6.33) that – as mass and radius of M approach zero – the value of x_1 at the critical contact point x_{c1} approaches zero too. Eq. (3.6.33) reduces in this case in a first approximation to $x_1 = x_{c1} = 1/(3q)^{1/3}$. The corresponding value of the critical Roche equipotential follows from Eq. (3.6.27) if $x_1 = x_{c1}$ and $x_2, x_3 = 0$. Expanding $1/r' = 1/(1-x_1)$ as $1 + x_1 + x_1^2,\ (x_1 = (3q)^{-1/3})$, we get $\Lambda = \Lambda_c = q + 3(3q)^{1/3}/2$. With this value of the potential function we are in position to determine the form of the critical Roche lobe round M if $M' \gg M$. Eq. (3.6.28) can be written as

$$r'[r^3\sin^2\lambda - 2qr^2\cos\varphi\sin\lambda/(q+1) - 2\Lambda r/(q+1) + 2/(q+1)] = -2qr/(q+1). \tag{3.6.37}$$

Since $r \approx 0$ and $O(\Lambda) = q$ if $q \gg 1$, we can neglect the first two terms in the bracket with respect to $-2\Lambda r/(q+1)$, and obtain

$$r \approx 1/(\Lambda - q), \qquad (q \gg 1). \tag{3.6.38}$$

With the previously found critical value Λ_c, this equation becomes in a first approximation simply

$$r = r_m \approx 2/3(3q)^{1/3}, \qquad (q \gg 1). \tag{3.6.39}$$

The critical mean rotation parameter (3.6.36) is equal to

$$\beta_{cm} \approx 2^4(1+q)/3^5 q \approx 2^4/3^5 = 0.065844..., \qquad (q \gg 1). \tag{3.6.40}$$

Thus, according to Kopal's (1978) Table VI. 2, the mean critical rotation parameter β_{cm} of the mass M diminishes from its value 0.36074 ($q = M'/M = 0$) to 0.065844 if $q = \infty$, taking an approximate minimum value 0.061967 if $q = 6.666$. If $q = 1$, we have $\beta_{cm} = 0.072267$.

Naylor and Anand (1970, and Table 3.6.3) have demonstrated that if $1.5 \leq n \leq 4.9$, the relative dimensions of synchronously rotating polytropic binaries do not differ substantially from those of the Roche model, which closely resembles a polytrope of index $n = 4.9$. Naylor (1972) has proved this result also for nonsynchronously rotating binaries of polytropic index $n = 3$ and 3.5. He related the spin angular velocity Ω of the two polytropic stars to their orbital angular velocity Ω_{orb} by the simple relationship

$$\Omega = (1+k)\Omega_{orb}, \qquad (-1 < k < 3), \tag{3.6.41}$$

where the constant k is restricted to the range -1 to 3, in order to maintain approximate hydrostatic equilibrium. While the relative dimensions of nonsynchronous polytropes ($n = 3, 3.5$) differ from the Roche model ($n = 4.9$) by at most 1%, they are relatively closer together in comparison to their Roche counterparts by as much as 3% ($n = 3$), and 2% ($n = 3.5$), corresponding to orbital period differences of 4.5% and 3%, respectively.

Several composite models of critically rotating and tidally distorted polytropes have been computed with the double approximation technique by Singh and Singh (1984a): They take the polytropic index of the core equal to $n = 0.5, 1.5$, and the envelope index equal to $n = 3, 4$. And Roxburgh (1974) has applied the same method to differentially rotating $n = 1.5, 3$ polytropes with meridional circulation.

The stability against mass loss of a polytrope (Heisler and Alcock 1987) filling its Roche lobe in a close binary system has been investigated by Paczyński (1965), and Hjellming and Webbink (1987).

3.7 Second Order Level Surface Theory of Rotationally Distorted Polytropes

We present the level surface theory under the form given to it by Zharkov and Trubitsyn (1978, Chap. 3), based on previous work of Clairaut, Laplace, Legendre, Darwin, Radau, Lyapunov, de Sitter, and others. The second order Zharkov-Trubitsyn theory can be extended to higher order (e.g. Zharkov and Trubitsyn 1969, 1975, 1978). The equipotential surfaces or the level surfaces of a uniformly rotating polytrope in hydrostatic equilibrium are given by [cf. Eq. (3.1.23)]

$$\Phi_{tot} = \Phi + \Phi_f = \text{const}, \tag{3.7.1}$$

where [cf. Eq. (3.4.64)]

$$\Phi_f = \Omega^2 r^2 \sin^2\lambda/2 = \Omega^2 r^2[1 - P_2(\mu)]/3, \qquad (\Omega = \text{const};\ \mu = \cos\lambda), \tag{3.7.2}$$

denotes the centrifugal potential, and the internal gravitational potential Φ is given by Eq. (3.1.57). r is the radius vector, λ the zenith angle (colatitude), Ω the angular velocity, and P_2 the Legendre polynomial of second order.

We wish to find the shape of the uniformly rotating polytrope together with its external potential Φ_e, when the density distribution $\varrho(\vec{r})$ along the radius vector $\vec{r}$ is given. In a first approximation we may write the equation of a level surface under the form of an ellipsoid of revolution

$$r^2\cos^2\lambda/a^2(1-f)^2 + r^2\sin^2\lambda/a^2 = 1, \tag{3.7.3}$$

where a is the equatorial radius of a level surface, $a(1-f)$ its polar radius, and $f = f(a)$ the oblateness of the configuration from Eq. (3.2.55). The equation (3.7.3) writes up to terms of second order

$$r(\lambda) = a[1 - f\cos^2\lambda - (3f^2/8)\sin^2 2\lambda]. \tag{3.7.4}$$

Darwin sought the equilibrium figure under the form of a spheroid that differs from an ellipsoid of revolution by the small, second order correction $g = g(a)$, [Zharkov and Trubitsyn 1969, Eq. (17)]:

$$r(\lambda) = a\big[1 - f\cos^2\lambda - (3f^2/8 + g)\sin^2 2\lambda\big] = a\big[1 - f/3 - f^2/5 - 8g/15$$
$$+(-2f/3 - f^2/7 - 8g/21)P_2(\mu) + (12f^2/35 + 32g/35)P_4(\mu)\big]. \tag{3.7.5}$$

The average radius s of a level surface (3.1.54) can be defined by the equivalence of the two volumes:

$$4\pi s^3/3 = \int_0^\pi \sin\lambda\, d\lambda \int_0^{r(\lambda)} r^2(\lambda)\, dr \int_0^{2\pi} d\varphi = (2\pi/3)\int_{-1}^{1} r^3(\mu)\, d\mu. \tag{3.7.6}$$

Inserting for r from Eq. (3.7.5), and using the expression (3.1.39) for the Legendre polynomials, we find

$$(s/a)^3 = 1 - f - 8g/5 \quad \text{or} \quad a = s(1 + f/3 + 2f^2/9 + 8g/15) + O(f^3). \tag{3.7.7}$$

With the aid of Eqs. (3.7.5), (3.7.7) we can express the radius vector r in terms of the average radius s :

$$r = r(\mu) = s[1 - 4f^2/45 + (-2f/3 - 23f^2/63 - 8g/21)P_2(\mu) + (12f^2/35 + 32g/35)P_4(\mu)]. \tag{3.7.8}$$

The essence of the axially symmetric level surface theory consists in representing the radius vector under the form (3.7.8):

$$r = r(\mu) = s\left[1 + \sum_{j=0}^{\infty} s_{2j}(s)\ P_{2j}(\mu)\right]. \tag{3.7.9}$$

This relationship is substituted into the equation for the total potential (3.7.1), so that

$$\Phi_{tot} = \Phi_{tot}(s,\mu) = \Phi(s,\mu) + \Phi_f(s,\mu) = \sum_{j=0}^{\infty} A'_{2j}(s)\ P_{2j}(\mu), \tag{3.7.10}$$

where the internal potential Φ is given by Eq. (3.1.57). On the level surface (3.7.1) we have by definition $\Phi_{tot}(s,\mu) = \text{const}$. Therefore, in terms of the new variables s and μ the total potential must be independent of μ on a level surface with the average radius $s = \text{const}$. Consequently, all coefficients $A'_{2j}(s)$ associated with angular functions have to vanish: $A'_{2j}(s) = 0$ if $j > 0$, and $\Phi_{tot}(s) = A'_0(s) = \text{const}$ on the level surface $s = \text{const}$. The conditions $A'_{2j}(s) = 0,\ (j > 0)$ provide a set of integro-differential equations for the figure functions $s_{2j}(s),\ (j > 0)$. The remaining figure function $s_0(s)$ can be determined by observing that the figure functions are coupled for instance by Eqs. (3.7.6), (3.7.9):

$$2 = \int_{-1}^{1} \left[1 + \sum_{j=0}^{\infty} s_{2j}(s)\ P_{2j}(\mu)\right]^3 d\mu. \tag{3.7.11}$$

On a level surface we also have $P, \varrho = \text{const}$, as shown by Eq. (3.1.24). Since $\Phi_{tot}(s,\mu) = \text{const}$ on a level surface characterized by $s = \text{const}$, we infer that on a level surface Φ_{tot}, P, ϱ must be independent of μ. In terms of the (s,μ)-coordinates, the physical quantities Φ_{tot}, P, ϱ depend only on $s:\ \Phi_{tot} = \Phi_{tot}(s),\ P = P(s),\ \varrho = \varrho(s)$. Thus, the equation of hydrostatic equilibrium (3.1.21) takes the form

$$(1/\varrho)\ \nabla P = \nabla\Phi_{tot} \quad \text{or} \quad (1/\varrho)\ dP/ds = d\Phi_{tot}/ds = dA'_0(s)/ds. \tag{3.7.12}$$

We now insert the internal gravitational potential Φ from Eq. (3.1.57) and the centrifugal potential from Eq. (3.7.2) into Eq. (3.7.10):

$$\Phi_{tot} = (G/r)\sum_{j=0}^{\infty}(D_{2j}r^{-2j} + D'_{2j}r^{2j+1})\ P_{2j}(\mu) + \Omega^2 r^2[1 - P_2(\mu)]/3 = \sum_{j=0}^{\infty} A'_{2j}(s)\ P_{2j}(\mu). \tag{3.7.13}$$

We convert the radius vector r into the mean radius s of the corresponding level surface. Because $\varrho = \varrho(s)$, we can integrate the relationships (3.1.57) for D_{2j}, D'_{2j} as follows:

$$D_{2j}(s) = [2\pi/(2j+3)] \int_0^s \varrho(s')\ ds' \int_{-1}^{1} P_{2j}(\mu)\ (dr^{2j+3}/ds')\ d\mu;$$
$$D'_{2j}(s) = [2\pi/(2-2j)] \int_s^{s_1} \varrho(s')\ ds' \int_{-1}^{1} P_{2j}(\mu)\ (dr^{2-2j}/ds')\ d\mu \quad \text{if} \quad j \neq 1;$$
$$D'_2(s) = 2\pi \int_s^{s_1} \varrho(s')\ ds' \int_{-1}^{1} P_2(\mu)\ (d\ln r/ds')\ d\mu, \qquad \big(dr = (dr/ds')\ ds'\big), \tag{3.7.14}$$

where s_1 denotes the outermost level surface of the planet. The powers of r can be expanded according to Eq. (3.7.9):

$$r^k = s^k + s^k \sum_{i=1}^{\infty}\left\{[k(k-1)...(k-i+1)/i!]\left[\sum_{j=0}^{\infty} s_{2j}(s)\ P_{2j}(\mu)\right]^i\right\}, \qquad (s_{2j} \ll 1). \tag{3.7.15}$$

Comparing Eqs. (3.7.8) and (3.7.9), we find

$$s_0 = -4f^2/45; \quad s_2 = -2f/3 - 23f^2/63 - 8g/21; \quad s_4 = 12f^2/35 + 32g/35, \tag{3.7.16}$$

$$s_0 = -s_2^2/5 + O(f^3). \tag{3.7.17}$$

We insert Eq. (3.7.15) into Eq. (3.7.14). After reducing the powers of Legendre polynomials to algebraic integrals via Eq. (3.1.39), and after integrating with respect to μ, we obtain

$$D_{2j}(s) = (4\pi/3) \int_0^s \varrho(s')\ d[s'^{2j+3} f_{2j}(s')]; \quad D'_{2j}(s) = (4\pi/3) \int_s^{s_1} \varrho(s')\ d[s'^{2-2j} f'_{2j}(s')],$$
$$(j = 0, 1, 2, ...), \tag{3.7.18}$$

where

$$f_0 = 1; \quad f_2 = 3s_2/5 + 12s_2^2/35; \quad f_4 = 18s_2^2/35 + s_4/3;$$
$$f_0' = 3/2 - 3s_2^2/10; \quad f_2' = 3s_2/5 - 3s_2^2/35; \quad f_4' = -9s_2^2/35 + s_4/3. \tag{3.7.19}$$

The logarithmic term in D_2' is expanded up to powers of order 2 according to Eq. (3.7.15). Zharkov and Trubitsyn (1978) introduce instead of D_{2j}, D_{2j}' the dimensionless functions S_{2j} and S_{2j}' :

$$S_{2j}(b) = 3D_{2j}/4\pi\varrho_m s^{2j+3} = b^{-2j-3}\int_0^b \delta(b')\ d(b'^{2j+3}f_{2j});$$
$$S_{2j}'(b) = 3D_{2j}'/4\pi\varrho_m s^{2-2j} = b^{2j-2}\int_b^1 \delta(b')\ d(b'^{2-2j}f_{2j}'), \tag{3.7.20}$$

where ϱ_m is the mean density of the configuration, and

$$b = s/s_1; \quad \delta(b) = \varrho(b)/\varrho_m. \tag{3.7.21}$$

The total potential (3.7.13) writes

$$\Phi_{tot} = G\big[D_0(s)/r + D_2(s)\ P_2(\mu)/r^3 + D_4(s)\ P_4(\mu)/r^5 + D_0'(s) + D_2'(s)\ P_2(\mu)r^2$$
$$+D_4'(s)\ P_4(\mu)r^4\big] + \Omega^2 r^2[1 - P_2(\mu)]/3. \tag{3.7.22}$$

Inserting for r from Eq. (3.7.15), and using Eqs. (3.7.17), (3.7.20), we eventually establish after some algebra up to the second order in f :

$$\Phi_{tot} = (4\pi G\varrho_m s^2/3)\big\{(1 + s_2^2/5 - s_2P_2 - s_4P_4 + s_2^2P_2^2)S_0 + (1 - 3s_2P_2)S_2P_2 + S_4P_4 + S_0'$$
$$+(1 + 2s_2P_2)S_2'P_2 + S_4'P_4 + (m/3)[1 + (-1 + 2s_2)P_2 - 2s_2P_2^2]\big\}$$
$$= (4\pi G\varrho_m s^2/3)(A_0 + A_2P_2 + A_4P_4). \tag{3.7.23}$$

We have introduced the important notation [cf. Eq. (3.4.46)]

$$m = 3\Omega^2/4\pi G\varrho_m = \Omega^2 r_1^3/GM_1 = 3\beta_m/2, \tag{3.7.24}$$

where M_1 denotes the total mass of the configuration, and via Eq. (3.7.13):

$$A_{2j} = 3A_{2j}'/4\pi G\varrho_m s^2. \tag{3.7.25}$$

To evaluate the total potential (3.7.23) up to the second order, we have taken into account that by virtue of Eqs. (3.7.16)-(3.7.20) S_0, S_0' are of zeroth order, S_2, S_2' of first order, and S_4, S_4' of second order in f, while $m = 3\beta_m/2$ is a small first order quantity; at the same time s_2 is a first order quantity, and s_0, s_4 are of second order in f. Using the orthogonality property (3.5.16) of Legendre polynomials, we obtain from Eq. (3.7.23) for the coefficients A_0, A_2, A_4 the following set of integral equations, where we have already shown subsequently to Eq. (3.7.10) that $\Phi_{tot} = A_0' = 4\pi G\varrho_m s^2 A_0/3$, and $A_2, A_4 = 0$ [cf. Zharkov and Trubitsyn 1978, Eqs. (28.7), (28.8), (29.4)]:

$$A_0(b) = 3\Phi_{tot}(b)/4\pi G\varrho_m s^2 = (1 + 2s_2^2/5)S_0 - 3s_2S_2/5 + S_0' + 2s_2S_2'/5 + m(1 - 2s_2/5)/3;$$
$$A_2(b) = (-s_2 + 2s_2^2/7)S_0 + (1 - 6s_2/7)S_2 + (1 + 4s_2/7)S_2' + m(-1 + 10s_2/7)/3 = 0;$$
$$A_4(b) = (-s_4 + 18s_2^2/35)S_0 - 54s_2S_2/35 + S_4 + 36s_2S_2'/35 + S_4' - 12ms_2/35 = 0. \tag{3.7.26}$$

Using Eq. (3.7.16) for the relationship between the figure functions and the distortion parameters f, g, we can easily transform Eqs. (3.7.19) and (3.7.26):

$$f_0 = 1; \quad f_2 = -2f/5 - f^2/15 - 8g/35; \quad f_4 = 12f^2/35 + 32g/105;$$
$$f_0' = 3/2 - 2f^2/15; \quad f_2' = -2f/5 - 9f^2/35 - 8g/35; \quad f_4' = 32g/105, \tag{3.7.27}$$

and

$$A_0 = (1 + 8f^2/45)S_0 + 2fS_2/5 + S_0' - 4fS_2'/15 + m(1 + 4f/15)/3;$$
$$A_2 = (2f/3 + 31f^2/63 + 8g/21)S_0 + (1 + 4f/7)S_2 + (1 - 8f/21)S_2' + m(-1 - 20f/21)/3 = 0;$$
$$A_4 = (-4f^2/35 - 32g/35)S_0 + 36fS_2/35 + S_4 - 24fS_2'/35 + S_4' + 8mf/35$$
$$= (12f^2/35 - 32g/35)S_0 + 12fS_2/7 + S_4 + S_4' = 0. \tag{3.7.28}$$

We have eliminated the rotation parameter m in the equation for A_4 by inserting its first order value from $A_2 = 0: \quad m \approx 2fS_0 + 3S_2 + 3S_2'$. The equation of hydrostatic equilibrium (3.7.12) writes in dimensionless variables as follows:

$$d\Pi/db = \delta(b)\ d(b^2A_0)/db, \qquad \big(\delta(b) = \varrho(b)/\varrho_m;\ b = s/s_1;\ \Phi_{tot} = A_0' = 4\pi G\varrho_m A_0 s^2/3;$$
$$\Pi = 3P/4\pi G\varrho_m^2 s_1^2 = s_1P/GM_1\varrho_m\big). \tag{3.7.29}$$

The second order figure equations (3.7.28) together with the hydrostatic equilibrium equation (3.7.29) and a given equation of state $\Pi = \Pi(\delta)$ can be used to determine the density distribution $\delta(b)$ and the figure functions $s_2(b), s_4(b)$ to accuracy of order f^2. Numerical results obtained with the third order level surface theory, applied to axially symmetric rotating polytropes, are listed in Table 3.8.1 (Horedt 1983).

To first order accuracy, the equation of hydrostatic equilibrium (3.7.29) can be written via Eqs. (3.7.20), (3.7.27), (3.7.28) as

$$d\Pi/db = \delta(b)\ d[b^2(S_0 + S_0' + m/3)]/db$$
$$= \delta(b)\ d\left[(3/b)\int_0^b \delta(b')\ b'^2\ db' + 3\int_b^1 \delta(b')\ b'\ db' + mb^2/3\right]\Big/ db = b\ \delta(b)\ (-S_0 + 2m/3). \tag{3.7.30}$$

If we transform back to physical variables, we obtain the well-known first order equation of hydrostatic equilibrium for a uniformly rotating body:

$$(1/\varrho)\ dP/ds = -GM(s)/s^2 + 2\Omega^2 s/3; \quad M(s) = 4\pi\int_0^s \varrho s^2\ ds. \tag{3.7.31}$$

From Eqs. (3.1.58), (3.7.20) results that for an axially symmetric configuration the external potential can be written as

$$\Phi_e = (GM_1/r)\left[1 - \sum_{j=1}^{\infty}(a_1/r)^{2j}J_{2j}P_{2j}(\mu)\right] = G\sum_{j=0}^{\infty} D_{2j}r^{-2j-1}P_{2j}(\mu)$$
$$= (GM_1/r)\sum_{j=0}^{\infty} S_{2j}(1)\ (s_1/r)^{2j}P_{2j}(\mu). \tag{3.7.32}$$

The gravitational moments from Eq. (3.1.58) are via Eq. (3.7.9) equal to

$$J_{2j} = -(s_1/a_1)^{2j}S_{2j}(1) = -\left[1 + \sum_{k=0}^{\infty} s_{2k}(s_1)\ P_{2k}(0)\right]^{-2j} S_{2j}(1). \tag{3.7.33}$$

The elegance and generality of the Zharkov-Trubitsyn theory for axially symmetric rotating figures is illustrated by the fact that we can obtain at once, by simple particularization, the well known equations of Clairaut, Darwin, and de Sitter. The equation $A_2 = 0$ from Eq. (3.7.28) yields Clairaut's integral equation, when second order terms are discarded:

$$2fS_0/3 + S_2 + S_2' - m/3 = 0. \tag{3.7.34}$$

The expressions of the coefficients S_0, S_2, S_2' are especially simple, since Clairaut's equation is of first order in f and m; so S_2, S_2' need to be evaluated only to first order, and S_0 to zeroth order, since it is multiplied by f:

$$S_0 = -b^{-3}\int_0^b \delta(b')\ db'^3; \quad S_2 = -(2b^{-5}/5)\int_0^b \delta(b')\ d(b'^5 f); \quad S_2' = -(2/5)\int_b^1 \delta(b')\ df. \tag{3.7.35}$$

The dimensionless density can be calculated from the zero order approximation of the hydrostatic equation (3.7.30):

$$d\Pi/db = -bS_0\ \delta(b). \tag{3.7.36}$$

In Clairaut's first order approximation, the relationships between radius vector, equatorial radius, and average radius of a level surface are via Eqs. (3.7.7), (3.7.8) simply

$$a/s = 1 + f/3; \quad r = s[1 - 2fP_2(\mu)/3] = a(1 - f\cos^2\lambda). \tag{3.7.37}$$

Thus, the equation of a level surface $r = r(\lambda)$ is in a first approximation identical to an ellipsoid of revolution [cf. Eq. (3.7.4)]. The first order value of the total potential Φ_{tot} on the surface of the Clairaut spheroid is by virtue of Eqs. (3.7.20), (3.7.26), (3.7.28) equal to

$$\Phi_{tot}(s_1) = GM_1A_0/s_1 = (GM_1/s_1)(S_0 + m/3) = (GM_1/s_1)\left[3\int_0^1 \delta(b')\ b'^2\ db' + m/3\right]$$
$$= (GM_1/s_1)(1 + m/3) = (GM_1/a_1)[1 + (m + f_1)/3], \qquad (a_1/s_1 = 1 + f_1/3). \tag{3.7.38}$$

Darwin's integral equation for $g(b)$ is given by $A_4 = 0$ from Eq. (3.7.28):

$$(3f^2 - 8g)S_0 + 15fS_2 + 35S_4/4 + 35S_4'/4 = 0. \tag{3.7.39}$$

All terms in this equation are of second order, so the relevant expressions (3.7.20) of S_j, S_j' are

$$S_0 = b^{-3}\int_0^b \delta(b')\ db'^3; \quad S_2 = -(2b^{-5}/5)\int_0^b \delta(b')\ d(b'^5 f);$$
$$S_4 = (12b^{-7}/35)\int_0^b \delta(b')\ d[b'^7(f^2 + 8g/9)]; \quad S_4' = (32b^2/105)\int_b^1 \delta(b')\ d(b'^{-2}g). \tag{3.7.40}$$

To get the shape of the rotationally distorted figure up to second order, we need besides Darwin's equation (3.7.39) also the Darwin-de Sitter equation, given by $A_2 = 0$ from Eq. (3.7.28):

$$(f + 31f^2/42 + 4g/7)S_0 + (3/2 + 6f/7)S_2 + (3/2 - 4f/7)S_2' + m(-1/2 - 10f/21) = 0. \tag{3.7.41}$$

The coefficients of the Darwin-de Sitter equation show that S_0 must be evaluated up to the first order, and S_2, S_2' up to the second order:

$$S_0 = b^{-3}\int_0^b \delta(b')\ db'^3; \quad S_2 = -(2b^{-5}/5)\int_0^b \delta(b')\ d[b'^5(f + f^2/6 + 4g/7)];$$
$$S_2' = -(2/5)\int_b^1 \delta(b')\ d(f + 9f^2/14 + 4g/7). \tag{3.7.42}$$

The system of equations (3.7.39) and (3.7.42) can be solved by successive approximations to get f and g, provided the first order density distribution $\delta(b)$ is known from the solution of the hydrostatic equation (3.7.30). The Darwin-de Sitter spheroid is obtained by keeping first and second order terms in the surface equations (3.7.5), (3.7.7), (3.7.9), (3.7.16).

If we insert the polytropic equation of state (3.2.1) into the hydrostatic equation (3.7.12), the level surface theory can be easily applied to polytropes (cf. Kopal 1983, p. 167). We particularize the level surface theory to polytropes of index $n = 0, 1$, and to the generalized Roche model with massless envelope, bearing some resemblance to the polytrope of index $n = 5$.

(i) $n = 0$, ($\delta(b) \equiv 1$). The figure equations (3.7.28) are solved by successive approximations. In a first approximation we solve the equation $A_2 = 0$ up to the first order to obtain $f_1 = 5m/4$ and $f = m/2 + 3f_1/5 = 5m/4 = f_1$, where f_1 is the surface value of f. This value is identical to the first order value obtained with Chandrasekhar's theory, since according to Eq. (3.2.64): $f = 15\beta/8 = 15\Omega^2/16\pi G\varrho_m = 5m/4$. The second order approximations (3.7.20) for the coefficients S_{2j}, S_{2j}' are

$$S_0 = 1; \quad S_2 = -2(f + f^2/6 + 4g/7)/5; \quad S_2' = -2[(f_1 - f) + 9(f_1^2 - f^2)/14 + 4(g_1 - g)/7]/5;$$
$$S_4 = 12(f^2 + 8g/9)/35; \quad S_4' = 32(b^2g_1 - g)/105. \tag{3.7.43}$$

f_1, g_1 denote the values of f, g on the surface. With the result $f = f_1 = 5m/4$ we enter $A_4 = 0$, and solve up to second order. The simple result is $g_1 = 0$ and $g = 0$. With this finding we can solve $A_2 = 0$ up to second order, by putting $f = (5m/4)(1 + \varepsilon)$, where ε is of order f. At first we calculate by particularization the new second order surface value f_1 of f, by putting $f(1) = f_1 = (5m/4)(1 + \varepsilon_1)$, and obtain $f_1 = (5m/4)(1 + 15m/56)$. With this boundary value we can calculate the oblateness for general $b = s/s_1$, and find after some algebra

$$f(b) = f(1) = f_1 = (5m/4)(1 + 15m/56). \tag{3.7.44}$$

(ii) $n = 1$. This case has already been discussed in connection with Chandrasekhar's first order perturbation method [Eqs. (3.2.48)-(3.2.54)]. The polytropic equation of state (3.2.1) becomes

$$P = K\varrho^2 \quad \text{or} \quad \Pi(b) = c\delta^2(b), \tag{3.7.45}$$

where $P = GM_1\varrho_m\Pi/s_1,\ \varrho(b) = \varrho_m\delta(b),\ c = K\varrho_m s_1/GM_1$. Inserting into the equation of hydrostatic equilibrium (3.7.30), we get up to the first order

$$2cb^2\ d\delta(b)/db = -3\int_0^b \delta(b')\ b'^2\ db' + 2mb^3/3 \quad \text{and} \quad M(b)/M_1 = b^3S_0 = 3\int_0^b \delta(b')\ b'^2\ db'. \tag{3.7.46}$$

We derive the first equation (3.7.46) to obtain

$$d^2\delta(b)/db^2 + (2/b)\ d\delta(b)/db + 3\delta(b)/2c - m/c = 0. \tag{3.7.47}$$

At first we solve the homogeneous equation, without the constant term $-m/c$. By particularizing Eqs. (2.3.7)-(2.3.18), we get

$$\delta(b) = Cb^{-1/2}J_{1/2}[(3/2c)^{1/2}b] = C_1\sin[(3/2c)^{1/2}b]/b = C_1\sin(C_2b)/b,$$
$$(C, C_1, C_2 = \text{const};\ C_2 = (3/2c)^{1/2}). \tag{3.7.48}$$

The solution of the nonhomogeneous equation (3.7.47) is found at once:

$$\delta(b) = C_1\sin(C_2b)/b + 2m/3. \tag{3.7.49}$$

With this result, the mass ratio from Eq. (3.7.46) becomes

$$M(b)/M_1 = (3C_1/C_2^2)[\sin(C_2b) - C_2b\cos(C_2b)] + 2mb^3/3. \tag{3.7.50}$$

The conditions $\delta(1) = 0$ and $M(1)/M_1 = 1$ determine the integration constants up to the first order, as follows:

$$C_1 = \pi/3 + m(4/3\pi - 2\pi/9); \quad C_2 = (3/2c)^{1/2} = \pi + 2m/\pi. \tag{3.7.51}$$

(iii) Generalized Roche Model with Massless Envelope. The density distribution within the configuration is given by

$$\delta = \begin{cases} \delta_c = \text{const} & \text{if} \quad 0 \le b \le b_c \\ 0 & \phantom{\text{if}} \quad b_c < b \le 1 \end{cases} \tag{3.7.52}$$

From the conservation of mass $4\pi\varrho_c r_c^3/3 = 4\pi\varrho_m r_1^3/3$ we get

$$(r_c/r_1)^3(\varrho_c/\varrho_m) = 1 \quad \text{or} \quad b_c^3\delta_c = 1, \tag{3.7.53}$$

where r_c and ϱ_c denote the radius and density of the core. We find at first the solution for the inner core region, the result being almost identical to the case $n = 0$ from Eq. (3.7.44):

$$f(b) = f_c = (5mb_c^3/4)(1 + 15mb_c^3/56) = (5m/4\delta_c)(1 + 15m/56\delta_c);$$
$$g(b) = g_c = 0 \quad \text{if} \quad 0 \le b \le b_c. \tag{3.7.54}$$

In the exterior massless region we obtain after some algebra by successive approximations in the same way as outlined for the case $n = 0$ (Zharkov and Trubitsyn 1969, 1978):

$$f(b) = (mb^3/2)[1 + (3/2)(b_c/b)^5] + (m^2b^6/8)[(b_c/b)^5 + (20/7)(b_c/b)^8 - (33/28)(b_c/b)^{10}];$$
$$g(b) = (3m^2b^6/32)[1 - 2(b_c/b)^5 + (b_c/b)^{10}], \tag{3.7.55}$$

where

$$S_0 = b^{-3}; \quad S_2 = (-2b_c^2/5b^5)(f_c + f_c^2/6 + 4g_c/7); \quad S_4 = (12b_c^4/35b^7)(f_c^2 + 8g_c/9);$$
$$S_2' = S_4' = 0 \quad \text{if} \quad b_c < b \le 1. \tag{3.7.56}$$

f_c, g_c are the values of f, g at the core boundary $b = b_c$. Because of continuity reasons, the two equations (3.7.54) and (3.7.55) are identical at the core boundary $b = b_c$. The configuration is similar to the polytrope of index $n = 5$ if $b_c \to 0$.

A method that uses an expansion of the density in Legendre polynomials, and that can be developed analytically in a manner analogous to the level surface theory, has been devised by Hubbard et al. (1975).

The strained co-ordinate method adopted by Smith (1976) is similar to the level surface theory. Smith (1976) does not present numerical results concerning critically rotating polytropes.

3.8 Numerical and Seminumerical Methods Concerning Distorted Polytropic Spheres

3.8.1 James' Calculations on Rotating Polytropes

Poisson's equation $\nabla^2\Phi = -4\pi\varrho$ writes explicitly as [cf. Eq. (B.39)]

$$\partial(r^2\ \partial\Phi/\partial r)/\partial r + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu + (1-\mu^2)^{-1}\ \partial^2\Phi/\partial\varphi^2 = -4\pi G\varrho r^2. \tag{3.8.1}$$

This equation determines the structure of the configuration, together with the hydrostatic equation (3.1.16) in the uniformly rotating frame:

$$\begin{cases} \partial P/\partial r = \varrho\ \partial\Phi/\partial r + \varrho\Omega^2 r(1-\mu^2) \\ \partial P/\partial\mu = \varrho\ \partial\Phi/\partial\mu - \varrho\Omega^2 r^2\mu \\ \partial P/\partial\varphi = \varrho\ \partial\Phi/\partial\varphi \end{cases} \tag{3.8.2}$$

The equation of hydrostatic equilibrium (3.8.2) can be written in condensed form as

$$(1/\varrho)\ \nabla P = \nabla[\Phi + \Omega^2 r^2(1-\mu^2)/2], \qquad (\Omega = \mathrm{const}). \tag{3.8.3}$$

With the substitutions from Eq. (3.2.1) this integrates at once:

$$(n+1)K\varrho_0^{1/n}\Theta = \Phi + \Omega^2 r^2(1-\mu^2)/2 + \mathrm{const.} \tag{3.8.4}$$

We take the integration constant for convenience equal to zero, and introduce the dimensionless radial coordinate ξ, as well as the dimensionless internal gravitational potential χ by

$$r = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = \alpha\xi \quad \text{and} \quad \chi = \Phi/(n+1)K\varrho_0^{1/n}. \tag{3.8.5}$$

Eq. (3.8.4) takes the dimensionless form

$$\Theta = \chi + \Omega^2\xi^2(1-\mu^2)/8\pi G\varrho_0. \tag{3.8.6}$$

The substitutions (3.2.1), (3.8.5) reduce Poisson's equation (3.8.1) to

$$\partial(\xi^2\ \partial\chi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\chi/\partial\mu]/\partial\mu + (1-\mu^2)^{-1}\partial^2\chi/\partial\varphi^2 = -\xi^2\Theta^n. \tag{3.8.7}$$

James (1964) considers at first the axially symmetric case, when Eq. (3.8.7) amounts to

$$\partial(\xi^2\ \partial\chi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\chi/\partial\mu]/\partial\mu = -\xi^2\Theta^n. \tag{3.8.8}$$

The rotating polytrope is divided into three regions: Region I near the centre of mass, where $\xi \approx 0$; region II between the point ξ_0 – where the series (3.8.9) becomes inappropriate – and the polar radius ξ_p; and finally, region III between the polar radius ξ_p and the equatorial radius ξ_e. James (1964) takes in region I the expansions of χ and Θ^n under the form [cf. Eqs. (2.4.10), (3.1.53)]:

$$\chi = \Theta - \Omega^2\xi^2(1-\mu^2)/8\pi G\varrho_0 = \sum_{j,k=0}^{\infty} A_{jk}\xi^j P_k(\mu);$$

$$\Theta^n = \sum_{j,k=0}^{\infty} B_{jk}\xi^j P_k(\mu), \qquad (A_{jk}, B_{jk} = \mathrm{const};\ \xi \approx 0). \tag{3.8.9}$$

The initial conditions at $\xi = 0$ are in virtue of Eq. (3.2.6) equal to $\Theta(0,\mu) = 1$ and $[\partial\Theta(\xi,\mu)/\partial\xi]_{\xi=0} = 0$. The potential χ is related to Θ by the simple equation (3.8.6), so we also have $\chi(0,\mu) = 1$ and $[\partial\chi(\xi,\mu)/\partial\xi]_{\xi=0} = 0$, which yields $A_{0k} = \delta_{0k}$ and $A_{1k} = 0$. Uniformly rotating polytropes have always an equatorial plane of symmetry, as mentioned subsequently to Eq. (3.1.19). Therefore $A_{jk}, B_{jk} = 0$ if k is odd, because odd indexed Legendre polynomials $P_k(\mu)$ are not symmetrical with respect to the equatorial plane $\mu = 0$ [cf. Eq. (3.1.39)]. Inserting into Eq. (3.8.9) the expansion of Θ near the origin [cf. Eqs. (2.4.16), (3.2.4), (3.2.9), (3.2.84)-(3.2.88)], we observe at once that $A_{jk} = 0$ if $j < k$. Because Θ is connected to Θ^n by an equation similar to Eq. (2.4.3), we also infer that $B_{jk} = 0$ if $j < k$. Moreover, from the mentioned expansions of Θ near the origin results that odd powers of ξ vanish, i.e. $A_{jk}, B_{jk} = 0$ if j is odd. Summarizing, $A_{jk}, B_{jk} = 0$ if j or k are odd, and if $j < k$ (James 1964, Ostriker and Mark 1968). If we insert Eq. (3.8.9) into Eq. (3.8.8), and use Eq. (3.1.40), we obtain similarly as in Chandrasekhar's method (Secs. 3.2-3.4) a relationship among the coefficients of the two series (3.8.9):

$$[j(j+1) - k(k+1)]A_{jk} = -B_{j-2,k}. \tag{3.8.10}$$

This equation determines all coefficients A_{jk} for which $j \neq k$. The determination of the coefficients B_{jk} is outlined by Eqs. (3.8.15), (3.8.16). Obviously, if $j = k$, Eq. (3.8.10) is satisfied for any value of A_{kk}, and these coefficients have to be determined in the axially symmetric case from the boundary conditions (3.8.27), (3.8.28). James (1964) fixes the termination of region I at the point $\xi = \xi_0$, where the magnitude of the term $A_{20,k}\xi_0^{20}$ in the expansion (3.8.9) is less than 10^{-10}.

In region II the solution of χ and Θ^n is extended by analytic continuation from ξ_0 up to the polar radius $\xi = \xi_p$. Let us assume that $\chi(\xi_1, \mu)$ and $[\partial\chi(\xi, \mu)/\partial\xi]_{\xi=\xi_1}$ are known at some point ξ_1 in region II. In the vicinity of ξ_1 we have

$$\xi = \xi_1 + \eta, \qquad (\eta \ll 1), \tag{3.8.11}$$

and Eq. (3.8.8) becomes

$$\eta^2\ \partial^2\chi/\partial\eta^2 + 2\eta\ \partial\chi/\partial\eta + \partial[(1-\mu^2)\ \partial\chi/\partial\mu]/\partial\mu + \eta^2\Theta^n + 2\xi_1(\eta\ \partial^2\chi/\partial\eta^2 + \partial\chi/\partial\eta + \eta\Theta^n)$$
$$+\xi_1^2(\partial^2\chi/\partial\eta^2 + \Theta^n) = 0. \tag{3.8.12}$$

Similarly to region I we substitute

$$\chi = \sum_{j,k=0}^{\infty} a_{jk}\eta^j P_k(\mu); \quad \Theta^n = \sum_{j,k=0}^{\infty} b_{jk}\eta^j P_k(\mu), \qquad (a_{jk}, b_{jk} = \mathrm{const}), \tag{3.8.13}$$

and by inserting into Eq. (3.8.12), we get the relationship

$$j(j-1)\xi_1^2 a_{jk} = -[(j-1)(j-2) - k(k+1)]a_{j-2,k} - b_{j-4,k}$$
$$-2\xi_1[(j-1)^2 a_{j-1,k} + b_{j-3,k}] - \xi_1^2 b_{j-2,k}. \tag{3.8.14}$$

Thus, we may find all coefficients a_{jk}, where a_{0k} and a_{1k} are already determined by the initial conditions $\chi(\xi_1, \mu)$ and $[\partial\chi(\xi, \mu)/\partial\xi]_{\xi=\xi_1}$. The precision of the series expansion (3.8.13) is provided by the condition $a_{10,k}\eta^{10} < 10^{-10}$.

Consider the situation when A_{jk}, a_{jk} are known. James (1964) determines the coefficients B_{jk} and b_{jk} as functions of $\beta = \Omega^2/2\pi G\varrho_0$, A_{jk}, and a_{jk}, respectively. Let us denote by D_j and d_j the coefficients of ξ^j and η^j in the expansions (3.8.9) and (3.8.13), respectively:

$$D_j(\mu) = \sum_{k=0}^{\infty} B_{jk}P_k(\mu) \quad \text{and} \quad d_j(\mu) = \sum_{k=0}^{\infty} b_{jk}P_k(\mu). \tag{3.8.15}$$

From these equations the coefficients B_{jk} and b_{jk} can be found at once, by using the orthogonality condition (3.5.16) of Legendre polynomials:

$$B_{jk} = (k+1/2)\int_{-1}^{1} D_j(\mu)\ P_k(\mu)\ d\mu \quad \text{and} \quad b_{jk}(\mu) = (k+1/2)\int_{-1}^{1} d_j(\mu)\ P_k(\mu)\ d\mu. \tag{3.8.16}$$

In the outer region III between ξ_p and ξ_e the basic functions χ and Θ^n are not analytic. $\partial^r\Theta^n/\partial\xi^r$ becomes discontinuous across the surface of the configuration, if the order of the derivative r is larger than the polytropic index n, because $\Theta = 0$ on the surface. At the same time $\partial^s\chi/\partial\xi^s$ becomes infinite across the surface if $s > n+2$ in virtue of Eq. (3.8.8). In the outer region James (1964) approximates χ and Θ^n by the series

$$\chi = \sum_{j=0}^{\infty} \chi_j(\xi)\ P_j(\mu) \quad \text{and} \quad \Theta^n = \sum_{j=0}^{\infty} \theta_j^n(\xi)\ P_j(\mu), \tag{3.8.17}$$

which are substituted into Eq. (3.8.8) to give via Eq. (3.1.40):

$$\xi^2\ d^2\chi_j/d\xi^2 + 2\xi\ d\chi_j/d\xi - j(j+1)\chi_j = -\xi^2\theta_j^n. \tag{3.8.18}$$

The two substitutions

$$\chi_j(\xi) = \xi^{-j-1}g_j(\xi) \quad \text{and} \quad dg_j/d\xi = \xi^{2j}h_j(\xi), \tag{3.8.19}$$

transform Eq. (3.8.18) into

$$dh_j/d\xi = -\xi^{-j+1}\theta_j^n(\xi). \tag{3.8.20}$$

James (1964) integrates the set of equations (3.8.19), (3.8.20), starting from the known initial values at the polar radius $\alpha\xi_p$, and ending at the equatorial radius $\alpha\xi_e$. The coefficients $\theta_j^n(\xi)$ are determined by inverting the expansion (3.8.17), analogously to Eq. (3.8.16):

$$\theta_j^n(\xi) = (j+1/2)\int_{-1}^{1} \Theta^n(\xi,\mu)\ P_j(\mu)\ d\mu. \tag{3.8.21}$$

As already mentioned subsequently to Eq. (3.8.10), the determination of the coefficients A_{kk} is effected with the aid of the boundary conditions. Quite generally, the external gravitational potential can be written via Eq. (3.1.58) under the form

$$\chi_e = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \chi_{ejk}(\xi)\ P_j^k(\mu)\ (C_{jk}\cos k\varphi + S_{jk}\sin k\varphi) = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \chi_{ejk}(\xi)\ P_j^k(\mu)\cos(k\varphi + \alpha_{jk}),$$
$$(C_{jk}, S_{jk}, \alpha_{jk} = \mathrm{const}). \tag{3.8.22}$$

Outside the configuration, the density $\varrho = \varrho_0\Theta^n$ vanishes, and Poisson's equation (3.8.7) turns into the Laplace equation $\nabla^2\chi_e = 0$. The Laplace equation eventually reduces to

$$d(\xi^2\ d\chi_{ejk}/d\xi)/d\xi - j(j+1)\chi_{ejk} = 0, \tag{3.8.23}$$

where we have inserted Eq. (3.8.22), taking into account Eq. (3.1.41) for the associated Legendre polynomials $P_j^k(\mu)$. The solution of the homogeneous Euler equation (3.8.23) is found at once:

$$\chi_{ejk} = C_1\xi^j + C_2\xi^{-j-1}, \qquad (C_1, C_2 = \mathrm{const}). \tag{3.8.24}$$

To avoid a singularity of χ_e as $\xi \to \infty$, we have to put $C_1 = 0$ and $\chi_{ejk}(\xi) = C_2\xi^{-j-1}$, which obviously satisfies outside the configuration the simple differential equation

$$d\chi_{ejk}/d\xi + (j+1)\chi_{ejk}/\xi = 0, \qquad (\xi \geq \xi_e), \tag{3.8.25}$$

in the nonaxisymmetric case. For the axisymmetric case Eqs. (3.8.22)-(3.8.25) can be particularized at once. The radial functions $\chi_{ej}(\xi)$ from the expansion of the axisymmetric external potential satisfy the same equation (3.8.25):

$$d\chi_{ej}/d\xi + (j+1)\chi_{ej}/\xi = 0, \qquad (\xi \geq \xi_e). \tag{3.8.26}$$

The external and internal potential must coincide at the equatorial boundary $\xi = \xi_e$ of the polytrope. Thus, the coefficients $\chi_j(\xi_e)$ of the internal axisymmetric potential (3.8.17) have to satisfy the same equation (3.8.26) as the coefficients $\chi_{ej}(\xi_e)$ of the external axisymmetric potential:

$$R_j = [d\chi_j/d\xi + (j+1)\chi_j/\xi]_{\xi=\xi_e} = 0. \tag{3.8.27}$$

James (1964) determines the coefficients A_{kk} from Eq. (3.8.9) in such a way as to satisfy the boundary conditions (3.8.27). The multidimensional Newton-Raphson procedure is applied to the set of equations $R_j = 0$:

$$\sum_{k=0}^{N} (\partial R_j/\partial A_{kk})\ \delta A_{kk} + R_j = 0. \tag{3.8.28}$$

This equation can easily be deduced as a generalization of the Newton-Raphson iteration process for the determination of the zeros of a function $f(x)$. The correction to the m-th iteration is

$$\delta x_m = -f(x_m)/[df(x)/dx]_{x=x_m}. \tag{3.8.29}$$

James (1964) takes $N = 10$, and because of equatorial symmetry we have $R_{2j+1} \equiv 0$. Each pair of values n and Ω defines a unique axially symmetric equilibrium configuration. The determination of the equilibrium configuration effectively amounts to the finding of the constants A_{kk} from Eq. (3.8.9).

The structure of the nonaxisymmetric configuration is assumed near the origin by the expansion [cf. Eqs. (3.1.53), (3.8.9)]:

$$\chi = \sum_{\ell,j,k=0}^{\infty} \xi^\ell (A_{\ell jk} \cos k\varphi + C_{\ell jk} \sin k\varphi)\ P_j^k(\mu); \quad \Theta^n = \sum_{\ell,j,k=0}^{\infty} \xi^\ell (B_{\ell jk} \cos k\varphi + D_{\ell jk} \sin k\varphi)\ P_j^k(\mu),$$
$$(\xi \approx 0;\ A_{\ell jk}, B_{\ell jk}, C_{\ell jk}, D_{\ell jk} = \mathrm{const}). \tag{3.8.30}$$

Similarly to the coefficients A_{jk}, B_{jk} from Eq. (3.8.9), we observe that $A_{\ell jk}, C_{\ell jk} = 0$ if $\ell < j,\ j < k$, and if $j - k$ is an odd number: If $\ell < j$, the coefficients are zero due to the expansions (2.4.16), (3.2.84)-(3.2.88), if $j < k$ the associated Legendre polynomials P_j^k vanish identically, and if $j - k$ is odd, P_j^k is not symmetrical with respect to the equatorial plane.

If we insert Eq. (3.8.30) into Eq. (3.8.7), and use Eq. (3.1.41), we obtain analogously to Eq. (3.8.10) the relationships

$$[\ell(\ell+1) - j(j+1)]A_{\ell jk} = -B_{\ell-2,jk}; \quad [\ell(\ell+1) - j(j+1)]C_{\ell jk} = -D_{\ell-2,jk}. \tag{3.8.31}$$

If $\ell = j$, the coefficients $A_{\ell\ell k}$ and $C_{\ell\ell k}$ have to be determined from the boundary conditions, as in the axisymmetric case [cf. Eqs. (3.8.27), (3.8.35)]. For the study of nonaxisymmetric rotating polytropes James (1964) restricts to the case when the nonaxisymmetric term $\delta\chi$ in the potential (3.8.32) is small:

$$\chi = \chi_a + \delta\chi, \qquad (\delta\chi \ll 1). \tag{3.8.32}$$

χ_a denotes the internal potential of the axially symmetric configuration. We write the small nonaxisymmetric part of the internal gravitational potential under the form (3.1.53):

$$\delta\chi = \sum_{j=0}^{\infty} \sum_{k=0}^{j} [\chi_{jk}^{(1)}(\xi)\ P_j^k(\mu) \cos k\varphi + \chi_{jk}^{(2)}(\xi)\ P_j^k(\mu) \sin k\varphi]. \tag{3.8.33}$$

If we insert Eqs. (3.8.32), (3.8.33) into Poisson's equation (3.8.7), we get – after using Eq. (3.1.41) – the mutually independent equations

$$d(\xi^2\ d\chi_{jk}^{(m)}/d\xi)/d\xi - j(j+1)\chi_{jk}^{(m)} = -\xi^2\theta^*\chi_{jk}^{(m)}, \qquad (m = 1, 2), \tag{3.8.34}$$

where $\Theta^n = \Theta_a^n + \theta^*\ \delta\chi$, and Θ_a^n denotes the axisymmetric contribution to the dimensionless density Θ^n. At the equatorial boundary the coefficients $\chi_{jk}^{(m)}$ of the nonaxisymmetric correction $\delta\chi$ satisfy the boundary conditions (3.8.27) for the external potential:

$$R_{jk} = [d\chi_{jk}^{(m)}/d\xi + (j+1)\chi_{jk}^{(m)}/\xi]_{\xi=\xi_e} = 0. \tag{3.8.35}$$

The necessary and sufficient condition for the existence of a nonaxisymmetric equilibrium form – adjacent to the symmetric form – is that there exists a set of coefficients $A_{\ell\ell k}$ and $C_{\ell\ell k}$ from Eq. (3.8.30), not all zero, such that $R_{jk} = 0$. As in the axisymmetric case, the coefficients $A_{\ell\ell k}$ and $C_{\ell\ell k}$ are determined by the Newton-Raphson procedure [cf. Eq. (3.8.28)]:

$$\sum_{k=0}^{N}(\partial R_{jk}/\partial A_{\ell\ell k})\ \delta A_{\ell\ell k} + R_{jk} = 0. \tag{3.8.36}$$

A lot of further computational details are mentioned in James' (1964) original paper, but we restrict ourselves merely to the presentation of his numerical results (see Table 3.8.1), found by direct numerical integration of Poisson's equation (3.8.7), subject to the boundary conditions (3.8.27) and (3.8.35).

For values of $n \leq 3$ the Newton-Raphson procedure from Eq. (3.8.27) proved satisfactory, but if $n > 3$, James' procedure failed to produce solutions because of numerical instabilities with Eq. (3.8.18) near the boundary. Surely, this fact is a major detriment of James' method. Fortunately, we know that the point mass Roche model (e.g. Kopal (1978) and end of Sec. 3.6), resembling the polytrope of index $n = 5$, has no bifurcation points as the angular velocity increases. The series of Roche models terminates without branching off, when the surface gravity at the equator of the Roche model becomes zero. As polytropes with index $n \approx 5,\ (n \leq 5)$ are very similar to the Roche model, we expect for these polytropes a similar behaviour as for the Roche model, when their angular velocity is increasing monotonically.

James (1964) has found numerically from Eqs. (3.8.30)-(3.8.36) that all polytropic sequences with $n > 0.808$ have no bifurcation points, just as the Roche model, whereas polytropes with $0 \leq n \leq 0.808$ have a bifurcation point, showing besides the axially symmetric equilibrium configurations also nonaxisymmetric equilibrium forms, just as the homogeneous polytrope $n = 0$ (cf. Secs. 3.2, 3.8.4). The limiting bifurcation form is a polytrope with index $n = 0.808$, and a critical rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0 = 0.106$ corresponding to equatorial mass loss (James 1964, Table 4, Bonazzola et al. 1996). This result is in very good agreement with the marvelous more theoretical evaluations of Jeans (1919, §182), who found $\gamma \approx 2.2,\ n = 1/(\gamma - 1) \approx 0.83$, the term associated with $P_2^2(\mu)\cos 2\varphi$ being dominant in the expansion of the gravitational potential. Some related work has also been effected by Vandervoort (1980b).

Vandervoort and Welty (1981) found that the limiting critical bifurcating polytrope occurs at $n = 0.794$ and $\beta_c = 0.116$, so Ipser and Managan (1981) conclude that somewhere in the range $0.78 < n < 0.82$ the nonaxisymmetric polytropes cease to exist. In the following we adopt James' (1964) value of $n = 0.808$ as the limiting value for the polytropic index of bifurcation.

The analysis of Jeans (1919, 1961) and James (1964) deals only with first order linear effects of nonaxisymmetry [cf. Eq. (3.8.32)], while Ipser and Managan (1981) regard their method as a generalization of Stoeckly's (1965) method (Sec. 3.8.2), and integrate the fundamental equations (3.1.75), (3.8.1) by successive approximations, demonstrating effectively the existence of polytropic nonaxisymmetric Jacobi-like sequences for polytropic indices $n = 0.5$ and 0.6. These sequences first appear at the point of bifurcation, where $\beta = \Omega^2/2\pi G\varrho_0 \approx 0.135$ and 0.125, terminating by equatorial mass loss at the critical values $\beta_c = \Omega_c^2/2\pi G\varrho_0 \approx 0.129$ and 0.123, respectively. Recall that the Jacobi sequence ($n = 0$, Sec. 3.2) starts with $\beta = 0.18711$, and terminates without mass loss at $\beta = 0$.

Applying an improved variant of Eriguchi's (1978) complex-plane strategy (Sec. 3.8.7), Hachisu and Eriguchi (1982, Fig. 3) have shown that both, the axisymmetric and the nonaxisymmetric rotating polytropes terminate with mass loss from the surface, at least for values of the polytropic index $n \geq 0.1$ (see Fig. 3.8.7 and Table 3.8.1). The homogeneous rotating polytropes ($n = 0$), i.e. the Maclaurin and Jacobi ellipsoids, do not exhibit this property [cf. discussion subsequent to Eq. (3.2.61)]. At least if $n \gtrsim 0.1$, no equilibrium configurations exist when the angular momentum is larger than a certain critical value. Equatorial mass loss can be avoided if appropriate differential rotation is introduced (cf. Figs. 3.8.1, 3.8.2).

3.8.2 Stoeckly's Method of Linearized Differences

Stoeckly (1965) implements a Henyey-type iteration scheme (as adopted for the calculation of stellar models) to obtain the solution of the linearized Poisson equation. The basic equations are given by Eqs. (3.8.1), (3.8.2) without the φ-terms. The nonuniform rotation of a polytrope of index $n = 1.5$ is approximated similarly to Eqs. (3.5.1)-(3.5.8) by

$$\Omega(\ell) = \Omega(0)\exp(-c\ell^2/r_e^2), \qquad (c = \mathrm{const}), \tag{3.8.37}$$

where $\ell = r\sin\lambda = r(1-\mu^2)^{1/2}$ is the distance from the rotation axis, and r_e the equatorial radius. In an inertial frame the integral of the hydrostatic equation is given by [cf. Eq. (3.5.10)]:

$$(n+1)P/\varrho = (n+1)K\varrho^{1/n} = \Phi - \Phi_p + \int_0^\ell \Omega^2(\ell')\ \ell'\ d\ell'. \tag{3.8.38}$$

Φ_p is the surface value of the gravitational potential at the poles, where $\ell = 0$. Stoeckly (1965) retains for convenience a continuation of this equation outside the surface, so we will omit in this subsection a definite distinction between internal and external gravitational potential, and denote it simply by Φ. The axially symmetric potential is expanded in terms of Legendre polynomials $P_j(\mu)$:

$$\Phi(r,\mu) = \sum_{j=0}^{\infty} \Phi_j(r)\ P_j(\mu). \tag{3.8.39}$$

Outside the equatorial radius r_e of the polytrope, where $\varrho = 0$, the functions $\Phi_j(r)$ are of the form [cf. Eqs. (3.1.58), (3.2.33)]

$$\Phi_j(r) \propto 1/r^{j+1}, \qquad (r \geq r_e). \tag{3.8.40}$$

At some radius $r = r_H$, completely outside the polytrope, the previous equation is equivalent to

$$(d\Phi_j/dr)_{r=r_H} + (j+1)\ \Phi_j(r_H)/r_H = 0, \qquad (r_H \geq r_e;\ j = 0,1,2,3,...). \tag{3.8.41}$$

Stoeckly (1965) argues that it would be much more convenient to choose, besides the two normalization constants $4\pi G,\ (n+1)K$, the value of Φ_p, rather than the central density ϱ_0, as the third normalization constant. Also, he employs the boundary conditions (3.8.41) instead of the common condition $\Phi_p = 0$. So, his new dimensionless variables $\varphi(x,\mu), x, \Theta^*(x,\mu)$, and ω are given by [cf. Eq. (3.2.1)]

$$\begin{aligned}&\Phi = \Phi_p\varphi; \quad r = \{[(n+1)K]^n/4\pi G\Phi_p^{n-1}\}^{1/2}x = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi;\\ &\varrho = [\Phi_p/(n+1)K]^n\Theta^{*n}(x,\mu) = \varrho_0\Theta^n(\xi,\mu); \quad \Omega = \{4\pi G\ [\Phi_p/(n+1)K]^n\}^{1/2}\omega.\end{aligned} \tag{3.8.42}$$

Poisson's equation (3.8.1) writes with these dimensionless variables

$$\partial^2\varphi/\partial x^2 + (2/x)\ \partial\varphi/\partial x + x^{-2}\ \partial[(1-\mu^2)\ \partial\varphi/\partial\mu]/\partial\mu + \Theta^{*n} = 0, \tag{3.8.43}$$

and the hydrostatic integral (3.8.38) becomes

$$\Theta^*(x,\mu) = \varphi(x,\mu) - 1 + \int_0^{x(1-\mu^2)^{1/2}} \omega^2(y)\ y\ dy, \tag{3.8.44}$$

where the centrifugal potential can be integrated for the law of differential rotation (3.8.37):

$$\begin{aligned}&\int_0^{x(1-\mu^2)^{1/2}} \omega^2(y)\ y\ dy = \omega^2(0)\int_0^{x(1-\mu^2)^{1/2}} \exp(-2by^2)\ y\ dy\\ &= [\omega^2(0)/4b]\{1-\exp[-2bx^2(1-\mu^2)]\}, \qquad (b = c/x_e^2).\end{aligned} \tag{3.8.45}$$

x_e denotes the value of x corresponding to $r = r_e$. Eqs. (3.8.43)-(3.8.45), together with the exterior boundary condition (3.8.41), written under the form

$$(d\varphi_j/dx)_{x=x_H} + (j+1)\ \varphi_j(x_H)/x_H = 0, \tag{3.8.46}$$

constitute the basic set for the determination of the equilibrium structure of the differentially rotating polytrope, where $\varphi_j = \Phi_j/\Phi_p$, and x_H is the value of x corresponding to $r = r_H$. Retaining only terms up to the tenth order, Eq. (3.8.39) writes in dimensionless coordinates as

$$\varphi(x,\mu) = \sum_{j=0}^{10} \varphi_j(x)\ P_j(\mu). \tag{3.8.47}$$

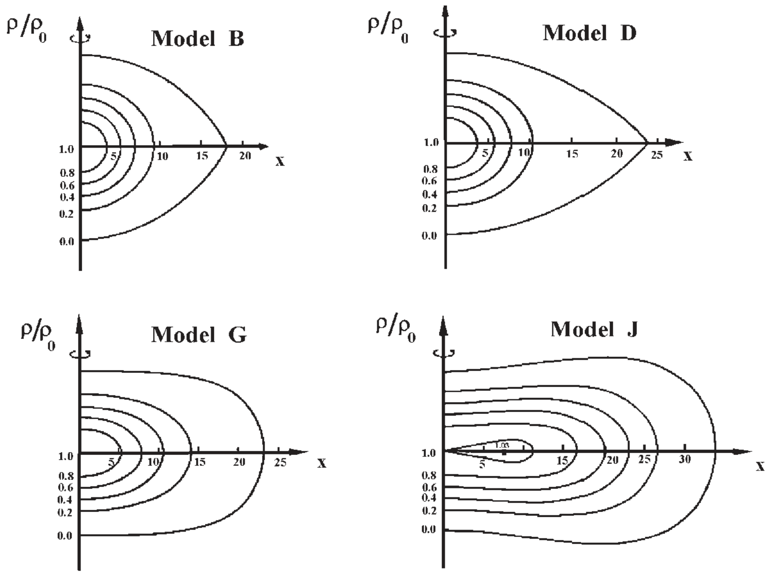

Fig. 3.8.1 Equidensity contours of selected models with low (models B, D) and high (models G, J) interior distortion for a polytropic index $n = 1.5$. The nonuniformity parameter c from Eq. (3.8.37) is zero (uniform rotation), 0.54, 0.94, and 1.05 for models B, D, G, and J, respectively (Stoeckly 1965).

The functions $\varphi_j(x)$ are obtained at once by using the relationship (3.5.16) for the integral of Legendre polynomials

$$\varphi_j(x)= [(2j+1)/2]\int_{-1}^{1} P_j(\mu)\ \varphi(x,\mu)\ d\mu \approx [(2j+1)/2]\sum_{k=1}^{6} H_k P_j(\mu_k)\ \varphi(x,\mu_k), \quad (j=0,2,...10), \tag{3.8.48}$$

where odd j's vanish because of equatorial symmetry. The eleven point Gauss-Legendre quadrature formula from Eq. (3.8.48) contains only six weight coefficients H_k, instead of eleven, because of symmetry with respect to the equatorial plane. The weighting coefficients along the six angular directions $\lambda = 90°, 74°, 59°, 43°, 27°, 12°,\ (\mu = \cos\lambda)$ are equal to $H_k = 0.273, 0.526, 0.466, 0.373, 0.251, 0.111$, respectively. The μ-derivative along the direction $\mu = \mu_k,\ (k = 1, 2, ...6)$ from Poisson's equation (3.8.43) reads

$$\partial[(1-\mu_k^2)\ \partial\varphi/\partial\mu_k]/\partial\mu_k = \partial\Big\{(1-\mu_k^2)\ \partial\Big[\sum_{j=0}^{10}\varphi_j(x)\ P_j(\mu_k)\Big]\Big/\partial\mu_k\Big\}\Big/\partial\mu_k$$

$$= -\sum_{j=0}^{10} j(j+1)\ \varphi_j(x)\ P_j(\mu_k) = \sum_{\ell=1}^{6} B_{k\ell}\ \varphi(x,\mu_\ell);$$

$$B_{k\ell} = -\sum_{j=0}^{10}[(2j+1)/2]j(j+1)H_\ell\ P_j(\mu_k)\ P_j(\mu_\ell), \qquad (k=1,2,...6), \tag{3.8.49}$$

where we have used Eqs. (3.1.40) and (3.8.48). Similarly, the external boundary condition (3.8.46), expressed in terms of $\varphi(x,\mu_\ell)$, is via Eq. (3.8.48) equal to

$$[d\varphi(x,\mu_k)/dx]_{x=x_H} + (1/x_H)\sum_{\ell=1}^{6} C_{k\ell}\ \varphi(x_H,\mu_\ell) = 0;$$

$$C_{k\ell} = \sum_{j=0}^{10}[(2j+1)/2](j+1)H_\ell\ P_j(\mu_k)\ P_j(\mu_\ell), \qquad (k=1,2,...6). \tag{3.8.50}$$

The differential operators from Eqs. (3.8.43), (3.8.50) are approximated by their simplest central differences on an equidistant grid with the nodes $x_h = h\Delta x,\ (h=0,1,2,...H)$, (Miketinac and Barton 1972, Miketinac 1984):

$$[df(x)/dx]_{x=x_h} = [f(x_{h+1}) - f(x_{h-1})]/2\Delta x; \quad [d^2f(x)/dx^2]_{x=x_h}$$
$$= [(df/dx)_{x=x_{h+1/2}} - (df/dx)_{x=x_{h-1/2}}]/\Delta x = [f(x_{h+1}) - 2f(x_h) + f(x_{h-1})]/(\Delta x)^2. \tag{3.8.51}$$

With these relationships Stoeckly (1965) casts the left-hand sides of Eqs. (3.8.43), (3.8.50) into the following difference representation:

$$B = (\Delta x)^{-2}[(1-1/h)\ \varphi(x_{h-1},\mu_k) - 2\varphi(x_h,\mu_k) + (1+1/h)\ \varphi(x_{h+1},\mu_k)]$$
$$+(h\Delta x)^{-2}\sum_{\ell=1}^{6} B_{k\ell}\ \varphi(x_h,\mu_\ell) + \Theta^{*n}(x_h,\mu_k), \tag{3.8.52}$$

$$C = [\varphi(x_{H+1},\mu_k) - \varphi(x_{H-1},\mu_k)]/2\Delta x + (1/H\Delta x)\sum_{\ell=1}^{6} C_{k\ell}\ \varphi(x_H,\mu_\ell). \tag{3.8.53}$$

The next step is the linearization of Eqs. (3.8.43), (3.8.50) in terms of the correction $\delta\varphi(x_h,\mu_k)$ added to the independent variable $\varphi(x_h,\mu_k),\ (k=1,2,...6)$ during successive iterations. The correction $\delta\Theta^{*n}$ to the dimensionless density Θ^{*n} in terms of $\delta\varphi$ reads

$$\delta\Theta^{*n} = n\Theta^{*n-1}\ \delta\Theta^* = n\Theta^{*n-1}(\partial\Theta^*/\partial\varphi)\ \delta\varphi, \tag{3.8.54}$$

where according to Eq. (3.8.44) $\partial\Theta^*/\partial\varphi$ equals 1 inside, and 0 outside the polytrope. Since Θ^* is zero on the surface, Stoeckly's (1965) method is not applicable if $n < 1$. The equations for the corrections $\delta\varphi(x_h,\mu_k)$ are by virtue of Eqs. (3.8.52)-(3.8.54) equal to

$$(\Delta x)^{-2}[(1-1/h)\ \delta\varphi(x_{h-1},\mu_k) - 2\ \delta\varphi(x_h,\mu_k) + (1+1/h)\ \delta\varphi(x_{h+1},\mu_k)]$$
$$+(h\Delta x)^{-2}\sum_{\ell=1}^{6} B_{k\ell}\ \delta\varphi(x_h,\mu_\ell) + n\Theta^{*n-1}(x_h,\mu_k)\ \delta\varphi(x_h,\mu_k) = -B,$$
$$(n\geq 1;\ h=1,2,...H;\ k=1,2,...6), \tag{3.8.55}$$

$$[\delta\varphi(x_{H+1},\mu_k) - \delta\varphi(x_{H-1},\mu_k)]/2\Delta x + (1/H\Delta x)\sum_{\ell=1}^{6} C_{k\ell}\ \delta\varphi(x_H,\mu_\ell) = -C. \tag{3.8.56}$$

The corrections $\delta\varphi$ would vanish only for the exact solution, if $B, C = 0$ according to Eqs. (3.8.43), (3.8.50). The function $\varphi(x_{H+1},\mu_k)$ can be eliminated between Eqs. (3.8.52) and (3.8.53) by putting $h = H$ in Eq. (3.8.52). An explicit condition at the centre is unnecessary, since the coefficient of $\varphi(x_0,\mu_k)$ vanishes in Eq. (3.8.52) for $h=1$, and the remaining relation between $\varphi(x_1,\mu_k)$ and $\varphi(x_2,\mu_k)$ serves as the central boundary condition.

Stoeckly (1965) has calculated about 400 models for a fixed step length of 0.0464 with the nonuniformity parameter c between 0 and 1.15, and with the unique polytropic index $n=1.5$. So, his conclusions remain isolated. Stoeckly (1965) suggests that there are two sequences of differentially rotating polytropes if $n=1.5$. One sequence terminates by mass loss at the equator (models B and D from Fig. 3.8.1) with modest interior distortion, and with a maximum value of the nonuniformity parameter $c = 0.67 \pm 0.12$. The other sequence contains models with more nonuniform rotation (up to $c=1.05$; model J from Fig. 3.8.1). These models may contain unstable configurations with pressure and density increasing outward (model J with $\varrho/\varrho_0 = 1.03$), or pass through models with a detached outer ring, which terminate or are discontinuous soon after.

3.8.3 Williams' Optimal Matching Method

Williams (1975) applies the optimal matching method to rotating polytropes, based on the work of Faulkner et al. (1968) for uniformly rotating stellar models. To some extent this method is similar to the double approximation technique presented in Sec. 3.6. In the inner region Williams (1975) represents the fundamental function Θ under the form (3.2.98):

$$\Theta(\xi,\mu,\beta) \approx \theta_0(\xi,\beta) + A_2(\beta)\ \psi_2(\xi)\ P_2(\mu). \tag{3.8.57}$$

We insert Eq. (3.8.57) into the fundamental equation of hydrostatic equilibrium (3.2.2). This equation has to be satisfied separately for the purely radial part of Θ, and for its angular part associated with $P_2(\mu)$. Thus, Eq. (3.2.2) splits after insertion into the two equations

$$d(\xi^2\ d\theta_0/d\xi)/d\xi = (-\theta_0^n + \beta)\xi^2, \tag{3.8.58}$$

$$d(\xi^2\ d\psi_2/d\xi)/d\xi = (6 - n\xi^2\theta_0^{n-1})\psi_2, \tag{3.8.59}$$

with the obvious initial conditions [cf. Eq. (3.2.6)]: $\theta_0(0) = 1;\ \theta_0'(0), \psi_2(0), \psi_2'(0) = 0.$

In the outer regions of the configuration, the internal gravitational potential is represented by its external empty-space value, similarly to Eq. (3.6.5):

$$\Phi \approx \Phi_e \approx [(n+1)K\varrho_0^{1/n}][k_0/\xi + k_2\xi^{-3}P_2(\mu)], \qquad [k_0 = k_0(\beta);\ k_2 = k_2(\beta)]. \tag{3.8.60}$$

Inserting for Φ into Eq. (3.6.3), we get the function Θ in the outer part of the configuration [cf. Eq. (3.6.11)]:

$$\Theta(\xi,\mu) \approx c + k_0/\xi + k_2\xi^{-3}P_2(\mu) + \beta\xi^2[1 - P_2(\mu)]/6, \qquad [c = c(\beta)]. \tag{3.8.61}$$

Demanding continuity of Θ and $\partial\Theta/\partial\xi$ at a certain fitting radius ξ_f, we get by equating the corresponding terms in Eqs. (3.8.57) and (3.8.61):

$$\theta_0(\xi_f) = c + k_0/\xi_f + \beta\xi_f^2/6; \quad A_2\psi_2(\xi_f) = k_2/\xi_f^3 - \beta\xi_f^2/6;$$
$$\theta_0'(\xi_f) = -k_0/\xi_f^2 + \beta\xi_f/3; \quad A_2\psi_2'(\xi_f) = -3k_2/\xi_f^4 - \beta\xi_f/3. \tag{3.8.62}$$

Williams (1975) integrates Eq. (3.8.58) along the radius $\xi^{(0)}$, corresponding to $P_2(\mu) = 0$ or $\mu = 3^{-1/2}$, so that Eq. (3.8.59) for the function ψ_2 can be neglected.

The critical case of equatorial break-up will occur when the radial component of the effective gravity $\partial(\Phi + \Phi_f)/\partial r \propto \partial\Theta/\partial\xi$ is zero at the critical equatorial radius $r_{ce} = \alpha\Xi_{ce}$ [cf. Eq. (3.6.12)]. Since in the outer region Θ is given by Eq. (3.8.61), the conditions $\Theta = 0$ and $\partial\Theta/\partial\xi = 0$ yield at the critical equatorial coordinate $\xi = \Xi_{ce},\ \mu = 0$:

$$c + k_0/\Xi_{ce} - k_2/2\Xi_{ce}^3 + \beta_c\Xi_{ce}^2/4 = 0; \quad -k_0/\Xi_{ce}^2 + 3k_2/2\Xi_{ce}^4 + \beta_c\Xi_{ce}/2 = 0, \qquad (\beta = \beta_c). \tag{3.8.63}$$

Because the numerical integration proceeds along the radius $\xi^{(0)}$, $(P_2(\mu) = 0)$, we have via Eqs. (3.8.57), (3.8.61):

$$\Theta(\xi^{(0)}, 3^{-1/2}) \approx \theta_0(\xi^{(0)}), \qquad \text{(inner region)};$$
$$\Theta(\xi^{(0)}, 3^{-1/2}) = c + k_0/\xi^{(0)} + \beta_c{\xi^{(0)}}^2/6, \qquad \text{(outer region)}. \tag{3.8.64}$$

If the fitting radius ξ_f is just at the critical surface coordinate $\xi_f = \Xi_c^{(0)}$ along $P_2(\mu) = 0$, the fitting conditions (3.8.62) become with the simplified function Θ from Eq. (3.8.64) equal to

$$\Theta(\Xi_c^{(0)}, 3^{-1/2}) \approx \theta_0(\Xi_c^{(0)}) = c + k_0/\Xi_c^{(0)} + \beta_c{\Xi_c^{(0)}}^2/6 = 0, \tag{3.8.65}$$

$$(\partial\Theta/\partial\xi)_{\xi=\Xi_c^{(0)}} \approx \theta_0'(\Xi_c^{(0)}) = -k_0/{\Xi_c^{(0)}}^2 + \beta_c\Xi_c^{(0)}/3. \tag{3.8.66}$$

The boundary conditions (3.8.63) yield with the simplification $k_2 = 0$, corresponding to $P_2(\mu) = 0$:

$$c + k_0/\Xi_{ce} + \beta_c \Xi_{ce}^2/4 = 0; \quad \beta_c = 2k_0/\Xi_{ce}^3. \tag{3.8.67}$$

If Eqs. (3.8.65) and (3.8.67) are put together, we get the cubic equation

$$2(\Xi_c^{(0)}/\Xi_{ce})^3 - 9\Xi_c^{(0)}/\Xi_{ce} + 6 = 0, \tag{3.8.68}$$

which admits according to the formula for the casus irreducibilis of a cubic equation the meaningful solution

$$\Xi_c^{(0)}/\Xi_{ce} = -6^{1/2}\cos\{[2\pi - \arccos(2/3)^{1/2}]/3\} = 0.7669. \tag{3.8.69}$$

Eq. (3.8.58) can now be integrated, subject to the bursting condition (3.8.69), yielding numerical results in satisfactory agreement to those of James (1964), and Monaghan and Roxburgh (1965), (see Table 3.8.1).

3.8.4 The Self-Consistent Field Method of Ostriker and Mark

The self-consistent field method, applied first by Ostriker and Mark (1968) to polytropes, is especially designed to calculate numerically the structure of severely distorted (differentially rotating) configurations. In fact, a self-consistent field scheme has been originally devised by Hartree and Fock in connection with molecular structure calculations (cf. Tassoul 1978). The self-consistent field method adopts an integral representation, rather than the use of differential equations, because the boundary conditions are easier to handle in integral form.

Eq. (3.1.8) can be written in condensed form for differentially rotating barotropes as

$$(1/\varrho)\ \nabla P = \nabla\left[\Phi + \int_0^\ell j^2(\ell')\ d\ell'/\ell'^3\right], \tag{3.8.70}$$

where

$$j = j(\ell') = \Omega(\ell')\ \ell'^2, \tag{3.8.71}$$

is the angular momentum per unit mass. Hydrostatic equilibrium generally requires the angular momentum to be an increasing function of ℓ. Note however, that some differentially rotating rings (tori) which are *axisymmetrically* stable, satisfying the Rayleigh criterion (3.5.1), (5.10.1), or (6.4.160), are subject to violent *nonaxisymmetric* instabilities (Sec. 6.4.3, Tohline and Hachisu 1990). Differentially rotating models can be calculated without additional labour if the angular momentum distribution over the configuration is introduced as a prescribed quantity. We designate by M_1 the total mass, by J the total angular momentum of the polytropic (barotropic) configuration, and by r_1 its maximum radius, which may be larger than the equatorial radius. The fractional mass interior to a cylinder of radius ℓ is

$$m(\ell) = (2\pi/M_1)\int_0^\ell \ell'\ d\ell' \int_{-\infty}^{\infty} \varrho(\ell', z)\ dz, \tag{3.8.72}$$

where $\varrho(\ell', z)$ denotes the density of the mass element $dm = 2\pi\varrho\ell'\ d\ell'\ dz$. Further, for a polytropic equation of state $P = P_0(\varrho/\varrho_0)^{1+1/n}$ we can integrate Eq. (3.8.70) between the centre and an arbitrary point of the configuration:

$$\varrho^{1/n} = \left[\varrho_0^{1+1/n}/(n+1)P_0\right]\left[\Phi - \Phi_0 + \int_0^\ell (j^2/\ell'^3)\ d\ell'\right]. \tag{3.8.73}$$

Φ_0 denotes the central value of the interior gravitational potential (3.1.47):

$$\Phi = \Phi(\vec{r}) = G\int_V \varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|. \tag{3.8.74}$$

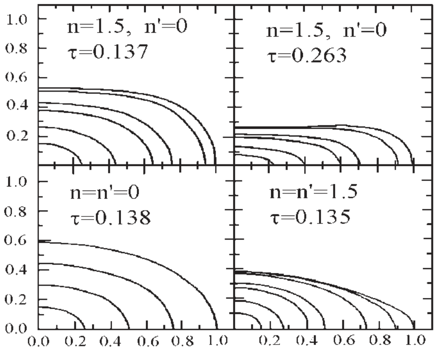

Fig. 3.8.2 Equidensity contours in the (ℓ, z)-plane of differentially rotating polytropes of index n, referenced to uniformly rotating polytropes of index n'. The ratio (3.1.35) between kinetic and gravitational energy is denoted by τ (Bodenheimer and Ostriker 1973).

In cylindrical coordinates the density can be expanded under the polynomial form

$$\varrho(\ell, z) = \sum_{i,k=0}^{\infty} a'_{ik}\ell^i z^k, \qquad (a'_{ik} = \text{const}), \tag{3.8.75}$$

and the gravitational potential under the form

$$\Phi = \sum_{\ell,m=0}^{\infty} A_{\ell m}\ell^\ell z^m, \qquad (A_{\ell m} = \text{const}). \tag{3.8.76}$$

Inserting Eqs. (3.8.75), (3.8.76) into Eq. (3.8.74), we get a tensor $T_{ik\ell m}$ such that

$$A_{\ell m} = \sum_{i,k=0}^{\infty} T_{ik\ell m}a'_{ik}. \tag{3.8.77}$$

In fact, Ostriker and Mark (1968) expand the density in spherical coordinates as

$$\varrho(x, \mu) \approx (3M_1/4\pi r_1^3)\sum_{i,k=0}^{N} a_{ik}x^{2k}P_{2i}(\mu) = (3M_1/4\pi r_1^3)\sum_{i=0}^{N}\sum_{k=i}^{N} a_{ik}x^{2k}P_{2i}(\mu), \qquad (a_{ik} = \text{const}), \tag{3.8.78}$$

where $x = r/r_1$ is the dimensionless radial coordinate, and the constants are $a_{ik} = 0$ if i, k are odd numbers or if $i > k$, as already outlined subsequently to Eq. (3.8.9). The even-order Legendre polynomials are expanded as [cf. Eq. (3.5.16)]

$$P_{2i}(\mu) = \sum_{m=0}^{i} p_{im}\mu^{2m}, \tag{3.8.79}$$

and the calculation of the "Green function" $T_{ik\ell m}$ proceeds accordingly. Inserting Eq. (3.8.78) into Eq. (3.1.57), and using Eq. (3.5.16), we obtain after some algebra:

$$\Phi(x, \mu) = 2\pi G r_1^2 \sum_{\ell=0}^{\infty} P_{2\ell}(\mu)\int_{-1}^{1} P_{2\ell}(\mu')\ d\mu'\left[\int_0^x \varrho(x', \mu')\ (x'^{2\ell+2}/x^{2\ell+1})\ dx' + \int_x^1 \varrho(x', \mu')\ (x^{2\ell}/x'^{2\ell-1})\ dx'\right] = (3GM_1/2r_1)\sum_{i=0}^{N}\sum_{k=i}^{N}[a_{ik}P_{2i}(\mu)/(k-i+1)(4i+1)] \times[x^{2i} - (4i+1)x^{2k+2}/(2k+2i+3)], \qquad (i = \ell). \tag{3.8.80}$$

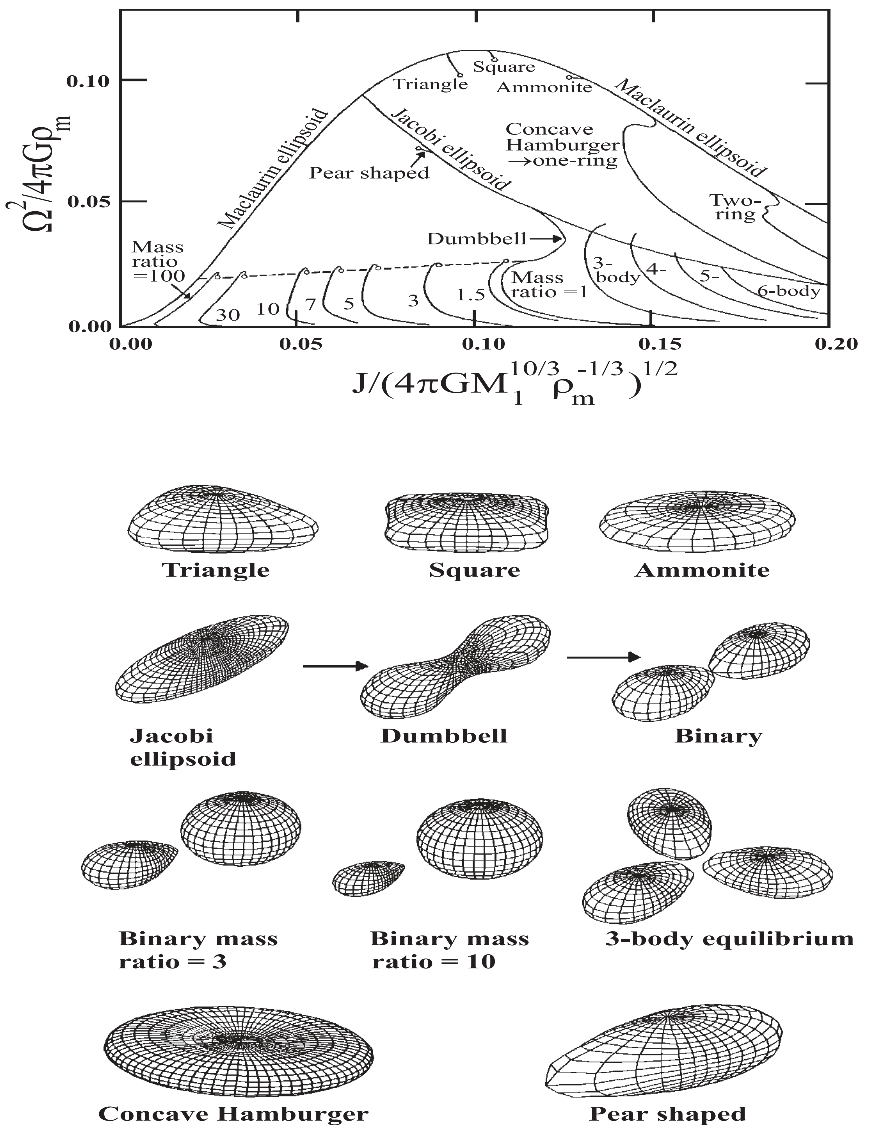

Fig. 3.8.3 Diagram of dimensionless angular momentum versus dimensionless angular velocity for hydrostatic configurations of constant density $\varrho = \varrho_m$. The small circles at the end of curves denote the termination of equilibrium sequences due to mass loss from the surface, or – in the binary case of unequal masses – due to mass flow from the secondary to the primary. The termination of the binary sequences is located on the dashed line (Hachisu and Eriguchi 1984a). Slight quantitative corrections to parts of the diagram can be found in Christodoulou et al. (1995a, b). Note, that binary mass ratio means the ratio of the more massive primary to the less massive secondary.

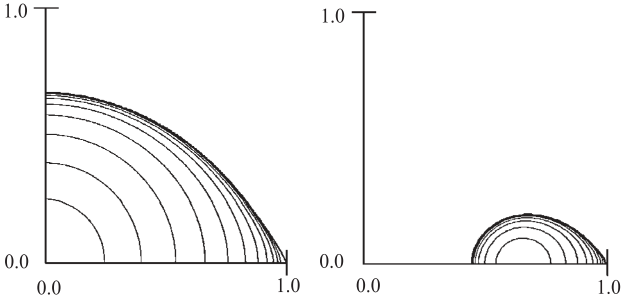

Fig. 3.8.4 Meridional density profile (density changes 10 times between successive contours) for the axially symmetric spheroid (on the left) and for the ring (on the right) with polytropic index $n = 3$ at critical rotation Ω_c (Hachisu 1986a).

Inserting Eq. (3.8.79) into Eq. (3.8.80), we observe after some algebra that the potential Φ and the density ϱ are indeed related by the four-indexed tensor $T_{ik\ell m}$, as already noted in Eq. (3.8.77). The centrifugal potential from Eq. (3.8.73) is expanded under a similar form (Ostriker and Mark 1968):

$$\int_0^\ell j^2(\ell')\,d\ell'/\ell'^3 = (J/M_1r_1)^2\int_0^a h^2(a')\,da'/a'^3 \approx (J/M_1r_1)^2\sum_{i=1}^N Q_i a^{2i},$$
$$(a' = \ell'/r_1;\ h = jM_1/J). \tag{3.8.81}$$

Thus, for any trial density distribution (3.8.78) we can find from Eqs. (3.8.80) and (3.8.81) the gravitational and centrifugal potential in analytic form. The next step in the iterative process is the determination of a new density profile via Eq. (3.8.73). Iteration is continued until self-consistency among the physical quantities of the rotating configuration is obtained. Clement (1974) employs instead of the series expansions (3.8.78)-(3.8.80) an interesting finite-difference scheme for solving Poisson's equation (3.8.1), claiming that the series expansions involve uncertain truncation errors and do not always represent radial variations with sufficient accuracy.

A simplified version of the self-consistent field method has been applied by Vandervoort and Welty (1981) to the construction of axisymmetric and nonaxisymmetric models of uniformly rotating polytropes. Blinnikov (1975) has presented another variant of the self-consistent field method with some applications to polytropes.

Fig. 3.8.2 shows equipotential (equidensity) surfaces of differentially rotating polytropes with index $n = 0, 1.5$ calculated with the self-consistent field method (Bodenheimer and Ostriker 1973). The initial angular momentum distribution is assumed to be the same as that of a uniformly rotating polytrope of index n', having such a large radius, that this reference polytrope can be considered nearly spherical. Bodenheimer and Ostriker (1973) increase the central density of the polytrope with index n, by preserving the initial angular momentum distribution. In this way, they obtain a whole sequence of contracting, axially symmetric, differentially rotating equilibrium configurations of polytropic index n. Numerical difficulties prevent Ostriker and Bodenheimer (1973) to calculate the models beyond $\tau = E_{kin}/|W| \approx 0.26$, but up to this limit their differentially rotating polytropic sequences mimic closely the behaviour of the Maclaurin-Jacobi sequence ($n, n' = 0$; Sec. 3.2), reaching points of bifurcation instead of terminating with the critical angular velocity Ω_c and mass shedding from the equator.

If one specifies a few points (usually two) of the shape of the polytrope, the rest of the configuration cannot deviate from the true solution, and converges into it, provided it exists at all (Hachisu et al. 1987, 1988). In this way there can be avoided the numerical difficulties of previous calculations by Ostriker and Bodenheimer (1973), who specified mass and angular momentum instead of fixed surface points and maximum density. With this choice of the initial conditions Hachisu (1986a, b) has extended previous numerical investigations to a lot of new equilibrium sequences, which have enlarged considerably our view about rotating and tidally distorted polytropes. In fact, most of the equilibrium figures obtained with the self-consistent field method by Hachisu (1986a, b), have already been found with similar precision by

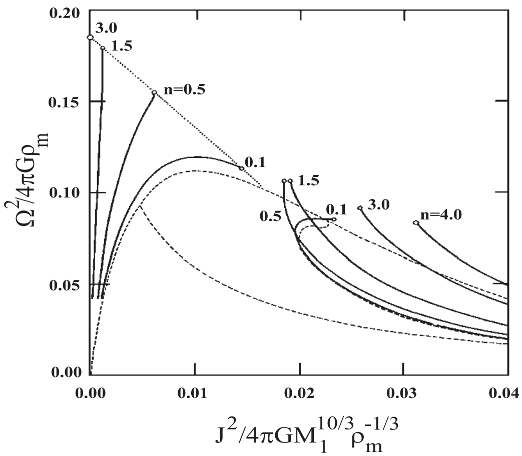

Fig. 3.8.5 Diagram of dimensionless angular momentum versus dimensionless angular velocity for axially symmetric polytropes with indices $n = 0.1, 0.5, 1.5, 3, 4$. Spheroids are on the left, polytropic rings on the right. Broken lines show the Maclaurin, Jacobi, and one-ring sequence from Fig. 3.8.3 if $n = 0$. All spheroidal sequences terminate by mass loss at critical rotation along the dotted line (Hachisu 1986a).

Eriguchi's (1978) complex-plane strategy (see Sec. 3.8.7, Fukushima et al. 1980, Hachisu and Eriguchi 1982, 1984a, b, c, Hachisu et al. 1982). We present all these results together, within the context of the more straightforward self-consistent field method.

With the polytropic pressure-density relationship $P = K\varrho^{1+1/n}$ Eq. (3.8.70) can be integrated at once:

$$H = (n+1)P/\varrho = \Phi + \int_0^\ell \Omega^2(\ell')\,\ell'\,d\ell' + \mathrm{const}, \qquad (dH = dP/\varrho;\ S = \mathrm{const}), \tag{3.8.82}$$

where H is the enthalpy or heat function per unit mass. Hachisu (1986a, b) adopts three types of rotation laws: (i) Rigid rotation $\Omega = \Omega_0 = \mathrm{const.}$ (ii) Rotation with constant velocity $v = \Omega\ell$, $[\Omega = v_0/(g+\ell^2)^{1/2}]$. (iii) Rotation with constant angular momentum $j = \Omega\ell^2$, $[\Omega = j_0/(g+\ell^2)]$, where $v_0, j_0, g = \mathrm{const.}$ The rotation laws (ii) and (iii) approach the true rotation, as denoted by name, if $g \to 0$. For these three rotation laws the integral in Eq. (3.8.82) can be solved:

$$\int_0^\ell \Omega^2(\ell')\,\ell'\,d\ell' = h_0^2\Psi + \mathrm{const}, \tag{3.8.83}$$

where

$$h_0 = \mathrm{const} = \begin{bmatrix} \Omega_0 \\ v_0 \\ j_0 \end{bmatrix} \qquad \text{and} \qquad \Psi = \begin{cases} \ell^2/2 & \text{law (i)} \\ (1/2)\ln(g+\ell^2) & \text{law (ii)} \\ -1/2(g+\ell^2) & \text{law (iii)} \end{cases} \tag{3.8.84}$$

Eq. (3.8.82) reads

$$H = \Phi + h_0^2\Psi + C, \qquad (C = \mathrm{const}). \tag{3.8.85}$$

At the surface, both pressure and density have to vanish, and this condition amounts to $H = (n+1)P/\varrho = 0$. If two boundary points A and B are fixed, Eq. (3.8.85) writes on the boundary as

$$H(A) = \Phi(A) + h_0^2\Psi(A) + C = 0, \tag{3.8.86}$$

$$H(B) = \Phi(B) + h_0^2\Psi(B) + C = 0, \tag{3.8.87}$$

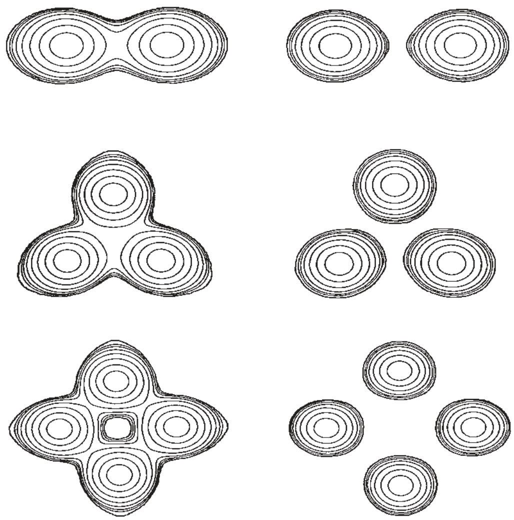

Fig. 3.8.6 Equatorial density profiles (tenfold density increase between successive contours) for binary, triple, and quadruple equilibrium configurations with polytropic index $n = 3$ and equal mass components. The left-hand side show the systems just prior to mass loss from the common outer envelope. As the distance among the components increases, the systems turn into the detached masses shown on the right (Hachisu 1986b).

or

$$h_0^2 = -[\Phi(A) - \Phi(B)]/[\Psi(A) - \Psi(B)], \tag{3.8.88}$$

$$C = -\Phi(A) - h_0^2\Psi(A) = -\Phi(B) - h_0^2\Psi(B). \tag{3.8.89}$$

According to the self-consistent field method, the gravitational potential Φ and the centrifugal potential Ψ are calculated with a guessed density approximation from Eqs. (3.8.80), (3.8.81). Then h_0^2 is obtained from Eq. (3.8.88), and after calculating C from Eq. (3.8.89), the enthalpy H can be determined. A new improved density approximation can be established via Eq. (3.8.82): $\varrho = [H/K(n+1)]^n$. The iteration continues until the absolute values of the three relative differences $\Delta H/H,\ \Delta C/C,\ \Delta h_0^2/h_0^2$ fall below a certain small number (Hachisu 1986a, b).

Fig. 3.8.3 completes the discussion from Sec. 3.2 on the equilibrium sequences of homogeneous polytropes ($n = 0$), viz. on the Maclaurin and Jacobi ellipsoids, as well as on the pear-shaped ovoids. The dimensionless units adopted by Hachisu and Eriguchi (1984a) are $\Omega^2/4\pi G\varrho_m$ [instead of $\beta = \Omega^2/2\pi G\varrho_0$ from Eq. (3.2.3)] and $J/(4\pi G)^{1/2}M_1^{5/3}\varrho_m^{-1/6}$. The dimensionless unit of the angular momentum follows from the fact that the angular momentum of a homogeneous sphere is equal to $J = 2\Omega r_1^2 M_1/5 \propto (G\varrho_m)^{1/2}(M_1/\varrho_m)^{2/3}M_1 \propto (4\pi G)^{1/2}M_1^{5/3}\varrho_m^{-1/6}$, where the radius is $r_1 = (3M_1/4\pi\varrho_m)^{1/3}$, and Ω is expressed in units of $(4\pi G\varrho_m)^{1/2}$.

It has already been shown by Chandrasekhar (1969) that a point of bifurcation must also be a neutral stability point along a certain equilibrium sequence of rotating configurations. A neutral point along an equilibrium sequence is defined as a point where a characteristic frequency, belonging to some proper normal mode of oscillation, vanishes. In other words, a nontrivial Lagrangian displacement of the configuration exists, such that its equilibrium is unaffected by the deformation due to this displacement (cf. Sec. 5.8.1). Eriguchi and Hachisu (1982) start from the neutral points along the Maclaurin and Jacobi sequence (generally with an artificial deformation of a trial solution), and obtain the new equilibrium sequences shown in Fig. 3.8.3 for the incompressible case $n = 0$. For instance, from the neutral points occurring at $\Omega^2/4\pi G\varrho_m = 0.110\ (e = 0.899),\ 0.112\ (e = 0.933),\ 0.104\ (e = 0.969),\ 0.0873\ (e = 0.985),$

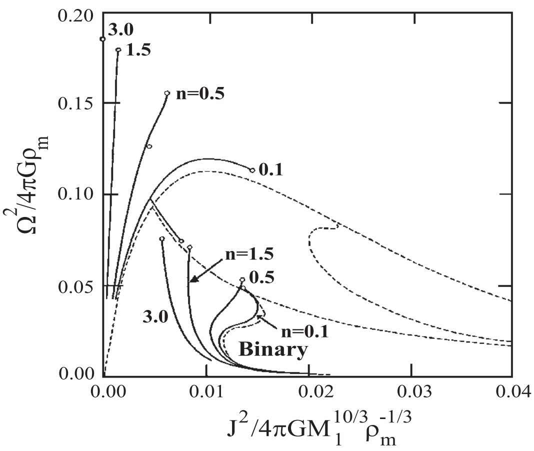

Fig. 3.8.7 Diagram of dimensionless angular momentum versus dimensionless angular velocity for polytropes with index $n = 0.1, 0.5, 1.5, 3$. The lines on the upper left are identical to the axially symmetric spheroids already shown in Fig. 3.8.5. The triaxial equilibrium models bifurcate from these polytropes if $0 < n < 0.808$ (cf. Sec. 3.8.1), terminating by equatorial mass loss too, as shown by the circles at the end of the bifurcating curves for $n = 0.1$ and 0.5. In the lower part of the diagram, polytropic binaries with equal mass components are plotted, terminating by mass loss from the outer common envelope (outer critical Roche lobe). Dashed lines show the Maclaurin sequence from which the one-ring sequence bifurcates, and the Jacobi sequence from which the dumbbell binary sequence bifurcates, as already plotted in Fig. 3.8.3 if $n = 0$ (Hachisu 1986b).

and 0.0658 ($e = 0.994$) on the Maclaurin sequence, there bifurcates the triangle, square, ammonite, one-ring (toroidal or concave hamburger), and the two-ring sequence, respectively. Recall that the Jacobi sequence branches off from the Maclaurin sequence at $\Omega^2/4\pi G\varrho_m = 0.0936$, ($e = 0.813$), (Secs. 3.2, 3.8.1).

The one-ring (toroidal) sequence evolves from the Maclaurin sequence through concave hamburger-like configurations (Fig. 3.8.3). The dumbbell sequence, bifurcating from the Jacobi-like ellipsoids, turns into the binary sequence with two equal masses. Hachisu and Eriguchi (1984c) have also computed constant density binaries with unequal mass ratios, that turn into the Maclaurin sequence when the secondary vanishes (purely rotational case). These binary sequences terminate when the smaller secondary fills its Roche lobe, and mass begins to flow from the secondary to the primary through the inner Lagrange point (e.g. Kopal 1978). Hachisu and Eriguchi (1984a) have also computed multi-body sequences with equal components shown in the lower right part of the diagram from Fig. 3.8.3, and in Figs. 3.8.6, 3.8.8.

As seen from Fig. 3.8.3, there are on the Jacobi sequence two neutral points from which new equilibrium sequences start: The famous pear-shaped sequence at $\Omega^2/4\pi G\varrho_m = 0.0710$ for the neutral mode of oscillation of the Jacobi ellipsoid belonging to the third zonal harmonic $P_3(\mu)$, (Jeans 1919), and the dumbbell sequence (which passes into the binary sequence with mass ratio $q = 1$) starting at $\Omega^2/4\pi G\varrho_m = 0.0532$ for the $P_4^2(\mu)\cos 2\varphi$ type perturbation (Chandrasekhar 1969, Eriguchi and Hachisu 1982, Eriguchi et al. 1982). Pear-shaped, triangle, square, and ammonite sequences terminate soon by mass loss from the surface.

For polytropes of index $0 < n < 5$ Hachisu (1986a, b) has found exciting new equilibrium figures that will be briefly presented in the following. The computation of equilibrium sequences is started with the two fixed boundary points A and B from Eqs. (3.8.86)-(3.8.89). Point A is fixed at $r = 1$, $\lambda = \pi/2$, while point B is located on the rotation axis, being moved for some models to the equatorial axis. For a given value of Ω there exist generally two ($n \neq 0$, Figs. 3.8.5, 3.8.7) or even four ($n = 0$, Fig. 3.8.3) axisymmetric equilibrium solutions.

As seen from Fig. 3.8.5, if $n > 0$, no connection appears between axisymmetric spheroids and polytropic rings, opposite to the constant-density case ($n = 0$), when the Maclaurin ellipsoids bifurcate into the one-ring sequence, as shown in Fig. 3.8.3. Earlier studies on homogeneous rings ($n = 0$) have

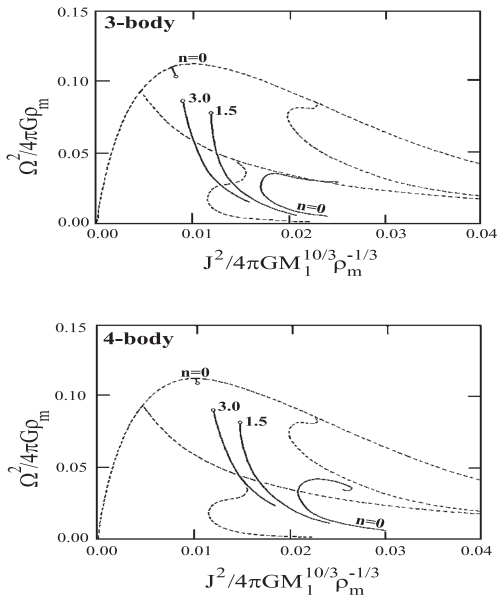

Fig. 3.8.8 Diagram of dimensionless angular momentum versus dimensionless angular velocity for three- and four-body equilibrium configurations of equal mass and polytropic index $n = 0, 1.5, 3$. The triangle and three-body sequence (upper figure), and the square and four-body sequence (lower figure) for homogeneous masses ($n = 0$) have already been depicted in Fig. 3.8.3. Dashed lines have the same meaning as in Fig. 3.8.7 (Hachisu 1986b).

been undertaken by Laplace, Maxwell, Poincaré, Kowalewsky, Dyson, and Randers, as noted by Ostriker (1964b) and Wong (1974), (see Sec. 3.9.2). If $n \neq 0$, both spheroidal and ring structures terminate at a certain critical rotation, when mass begins to be shed from the equator of these axisymmetric polytropic configurations (Fig. 3.8.5). The same situation also occurs for nonaxisymmetric polytropes with index $0 < n < 0.808$: No connection exist between Jacobi-like triaxial polytropes and polytropic binaries with equal mass components. These polytropic sequences both terminate by mass loss (Fig. 3.8.7). No dumbbell polytropes exist if $n \neq 0$. In the constant density case $n = 0$, dumbbell configurations connect smoothly Jacobi ellipsoids and homogeneous binaries, as shown in Fig. 3.8.3. No triangle and square sequences exist in the polytropic $n \neq 0$ case (Fig. 3.8.8).

3.8.5 The Ellipsoidal Method of Roberts

This method is based on a variational principle, in order to pick up the "best" solution from a set of ellipsoidal shells into which the polytrope is decomposed. Some similarities occur with the treatment of Vandervoort (1980b), Vandervoort and Welty (1981), as well as with the ellipsoidal energy variational method of Lai et al. (1993).

The total energy of the rotating polytrope is the sum of three parts

$$E = U + E_{kin} + W, \tag{3.8.90}$$

where

$$U = Kn \int_{V_1} \varrho^{1+1/n}\, dV, \qquad [n = 1/(\Gamma - 1) = 1/(\gamma - 1)], \tag{3.8.91}$$

is the internal energy of an isentropic polytrope from Eq. (2.6.158). Further

$$E_{kin} = (\Omega^2/2)\int_{V_1}\varrho\ell^2\ dV, \tag{3.8.92}$$

denotes the kinetic energy of rotation [cf. Eq. (3.1.34)], and

$$W = -(1/2)\int_{V_1}\Phi\varrho\ dV, \tag{3.8.93}$$

the gravitational energy from Eq. (2.6.68).

We express the integrals (3.8.91)-(3.8.93) in terms of the elements of ellipsoidal shells. To this end we remember at first the volume of an ellipsoid with semiaxes $a_i,\ (i = 1, 2, 3)$:

$$V_1 = \int_{-a_3}^{a_3} dx_3 \int_{-x_2'}^{x_2'} dx_2 \int_{-x_1'}^{x_1'} dx_1 = 4\pi a_1a_2a_3/3,$$
$$\left[x_2' = a_2(1 - x_3^2/a_3^2)^{1/2};\ x_1' = a_1(1 - x_2^2/a_2^2 - x_3^2/a_3^2)^{1/2}\right], \tag{3.8.94}$$

where the ellipsoidal surface is given by

$$\sum_{i=1}^{3}(x_i/a_i)^2 = 1. \tag{3.8.95}$$

For an ellipsoid of revolution we have $a_1 = a_2,\ a_3 = a_1(1-e_1^2)^{1/2}$, and the volume of the axisymmetric ellipsoid becomes $V_1 = 4\pi a_1^3(1 - e_1^2)^{1/2}/3$. We designate by a the semimajor axis of an ellipsoidal shell inside V_1, and assume the density ϱ and the eccentricity e of the ellipsoidal strata to be functions solely of $a : \varrho = \varrho(a),\ e = e(a)$. The volume inside a biaxial ellipsoid of semimajor axis a is accordingly $V = 4\pi a^3(1 - e^2)^{1/2}/3$, and the volume of an ellipsoidal shell of semimajor axis a and infinitesimal thickness da is $dV = (4\pi/3)\ d[a^3(1-e^2)^{1/2}]$. The mass of this ellipsoidal shell is simply $dM = \varrho\ dV$. The total mass of an axisymmetric configuration composed of ellipsoidal shells is therefore (Roberts 1963b)

$$M_1 = (4\pi/3)\int_{M_1}\varrho\ d[a^3(1 - e^2)^{1/2}] = (4\pi/3)\int_0^{a_1}\varrho\ \{d[a^3(1 - e^2)^{1/2}]/da\}\ da. \tag{3.8.96}$$

The internal energy (3.8.91) can be written as

$$U = (4\pi Kn/3)\int_0^{a_1}\varrho^{1+1/n}\{d[a^3(1 - e^2)^{1/2}]/da\}\ da. \tag{3.8.97}$$

Eq. (3.8.92) is transformed in a similar way, by calculating at first

$$\int_{V_1}\ell^2\ dV = \int_{-a_3}^{a_3} dx_3 \int_{-x_2'}^{x_2'} dx_2 \int_{-x_1'}^{x_1'} (x_1^2 + x_2^2)\ dx_1 = 4\pi a_1a_2a_3(a_1^2 + a_2^2)/15. \tag{3.8.98}$$

Putting $a_1 = a_2 = a$ and $a_3 = a(1 - e^2)^{1/2}$, we get

$$\ell^2\ dV = (8\pi/15)\ d[a^5(1 - e^2)^{1/2}], \tag{3.8.99}$$

and finally

$$E_{kin} = (4\pi\Omega^2/15)\int_0^{a_1}\varrho\ \{d[a^5(1 - e^2)^{1/2}]/da\}\ da. \tag{3.8.100}$$

Likewise, the angular momentum of the ellipsoidal configuration is

$$J = \Omega\int_{M_1}\varrho\ell^2\ dV = (8\pi\Omega/15)\int_0^{a_1}\varrho\ \{d[a^5(1 - e^2)^{1/2}]/da\}\ da. \tag{3.8.101}$$

On making a small change $\delta\varrho$ and δe in the density and eccentricity, the corresponding changes in mass and angular momentum are

$$\delta M_1 = (4\pi/3)\int_0^{a_1}\delta\varrho\ d[a^3(1-e^2)^{1/2}] - (4\pi/3)\int_0^{a_1}\varrho\ \delta e\ d[a^3 e/(1-e^2)^{1/2}]$$
$$= (4\pi/3)\int_0^{a_1}\delta\varrho\ d[a^3(1-e^2)^{1/2}] + (4\pi/3)\int_0^{a_1}[a^3 e/(1-e^2)^{1/2}]\ \delta e\ d\varrho, \qquad (3.8.102)$$

$$\delta J = (8\pi\ \delta\Omega/15)\int_0^{a_1}\varrho\ d[a^5(1-e^2)^{1/2}] + (8\pi\Omega/15)\int_0^{a_1}\delta\varrho\ d[a^5(1-e^2)^{1/2}]$$
$$+(8\pi\Omega/15)\int_0^{a_1}[a^5 e/(1-e^2)^{1/2}]\ \delta e\ d\varrho, \qquad (3.8.103)$$

where we have integrated by parts and taken into account that $\varrho(a_1) = 0$.

The variational principle enunciated by Roberts (1963b) requires the first order change in the total energy

$$\delta E = \delta U + \delta E_{kin} + \delta W, \qquad (3.8.104)$$

to be zero for all first order changes of $\delta\varrho$ and δe, preserving mass and angular momentum of the body, i.e. $\delta M_1,\ \delta J = 0$.

The dimensionless polytropic coordinates are introduced by [cf. Eqs. (3.2.1), (3.2.3)]

$$\varrho = \varrho_0\Theta^n(x); \quad a = \alpha x; \quad \beta = \Omega^2/2\pi G\varrho_0, \qquad (0 \le a \le a_1), \qquad (3.8.105)$$

where the radial x-coordinate is measured in the equatorial plane of the polytrope. In these new coordinates the mass (3.8.96) inside a is measured by Hurley and Roberts (1965) in units of $4\pi\varrho_0\alpha^3$, and becomes

$$M = M(x) = (1/3)\int_0^x \Theta^n\{d[x'^3(1-e^2)^{1/2}]/dx'\}\ dx'. \qquad (3.8.106)$$

This equation writes in differential from as

$$dM/dx = (\Theta^n/3)\ d[x^3(1-e^2)^{1/2}]/dx = x^2\Theta^n(1-e^2)^{1/2}[1 - xe\ (de/dx)/3(1-e^2)]$$
$$= x^2\Theta^n(1-e^2)^{1/2}[1 - e^2\eta/6(1-e^2)], \qquad (3.8.107)$$

where $\eta = (2x/e)\ de/dx$. The equations needed to solve the rotational problem are casted by Hurley and Roberts (1965) into the form

$$d\Theta/dx = [3/x^2(1-e^2)^{1/2}(3-2e^2)][\beta x^3(1-e^2)^{3/2}/3 - (1-e^2)M(x)$$
$$+e^2 D(x)\ g(e)\ (\eta+2)(5-2e^2)/20 - e^2\eta\ D(x)/4];$$
$$de/dx = [3e(3-2e^2)\ D(x)\ g(e)/x]/[-2\beta x^3(1-e^2)^{1/2} + 9D(x) + 6M(x) - 3(3-2e^2)\ D(x)\ g(e)];$$
$$dD/dx = [-(\eta+2)/x][D(x) - M(x) + x^3\Theta^n(1-e^2)^{1/2}/3];$$
$$g(e) = [15(1-e^2)^{1/2}/4e^5][(3-2e^2)\arcsin e - 3e(1-e^2)^{1/2}];$$
$$D(x) = M(x) - 5N(x)/e^2x^2; \quad N(x) = (1/15)\int_0^x \Theta^n\{d[x'^5 e^2(1-e^2)^{1/2}]/dx'\}\ dx'. \qquad (3.8.108)$$

These equations have to obey the obvious boundary values $\Theta(0) = 1;\ \Theta'(0) = 0;\ M(0) = D(0) = 0;\ \Theta(x_1) = 0$. They are also subject to the surface condition $\Delta(x_1) = 0$, where $x_1 = a_1/\alpha$ and $\Delta(x) = x^3(1-e^2)^{1/2}[\beta - 4e^2 g(e)/15]/3 - 2e^2 g(e)\ [3D(x) + 2M(x)]/15$. Numerical values are plotted in Table 3.8.1 ($x_1 = \Xi_e$), and obviously suffer from the constraint that equidensity surfaces are forced into ellipsoidal shape.

3.8.6 Miketinac's Method

This method (Miketinac 1984) is a modification of the semidiscrete pseudospectral method of Miketinac and Parter (1981) already mentioned in Sec. 3.6. The problem is formulated as a free boundary problem, and the method is obtained by combining Newton-Raphson's procedure and two different types of discretization. We insert Eq. (3.8.5) into Eq. (3.8.4) to obtain

$$\Theta = \chi + \Omega^2\xi^2(1-\mu^2)/8\pi G\varrho_0 + \mathrm{const.} \tag{3.8.109}$$

Miketinac (1984) determines the integration constant by the potential χ_p at the pole of the configuration [cf. Eq. (3.1.74)], where $\mu = 1$ and $\Theta = 0$. We get: const $= -\chi_p$. Eq. (3.8.7) can be written in condensed form as $\nabla^2\chi = -\Theta^n$. Miketinac (1984) considers a sphere of some finite radius ξ_H, that contains all equilibrium figures of a given sequence. Outside this sphere the external potential χ_e satisfies Laplace's equation $\nabla^2\chi_e = 0$. The differential equations for the internal and external potential can be written under the simplified form $\nabla^2\psi = -\eta^n$, by defining the new potential function (cf. Clement 1974): $\psi = \chi - \chi_p,\ (\eta = \Theta$ if $\Theta \geq 0)$, and $\psi = \chi_e - \chi_p,\ (\eta = 0$ if $\Theta \leq 0)$. The potential is sought under the well known form

$$\psi(\xi,\mu) = \sum_{j=0}^{\infty} \psi_j(\xi)\ P_j(\mu). \tag{3.8.110}$$

Outside the sphere $\xi = \xi_H$ the functions $\psi_j(\xi)$ satisfy the Laplace equation $\nabla^2\psi = -\eta^n = 0$, or [cf. Eqs. (3.8.23)-(3.8.26)]

$$d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi - j(j+1)\psi_j/\xi^2 = 0, \qquad (\xi \geq \xi_H), \tag{3.8.111}$$

with the solution $\psi_j = C_1\xi^j + C_2\xi^{-j-1},\ (C_1 = 0;\ C_2 = \mathrm{const})$, obeying the equation

$$d\psi_j/d\xi + (j+1)\psi_j/\xi = 0, \qquad (\xi \geq \xi_H). \tag{3.8.112}$$

Eq. (3.8.112) is taken as the boundary condition. Conservation of mass inside ξ_H writes as

$$M_1 = \varrho_0\alpha^3\int_{V_\xi}\Theta^n\ dV_\xi = -\varrho_0\alpha^3\int_{V_\xi}\nabla^2\psi\ dV_\xi = -\varrho_0\alpha^3\int_{S_\xi}\nabla\psi\cdot d\vec{S}_\xi$$
$$= -2\pi\varrho_0\alpha^3\int_{-1}^{1}(\partial\psi/\partial\xi)_{\xi=\xi_H}\ \xi_H^2\ d\mu = -4\pi\varrho_0\alpha^3\xi_H^2\ (d\psi_0/d\xi)_{\xi=\xi_H}, \tag{3.8.113}$$

since

$$\int_{-1}^{1} P_j(\mu)\ d\mu = 0 \quad \text{if} \quad j = 1,2,3,...; \qquad \int_{-1}^{1} P_0(\mu)\ d\mu = 2. \tag{3.8.114}$$

The notations V_ξ and S_ξ mean that volume and surface are measured over the dimensionless coordinate ξ, rather than over radial distance r. Combining Eqs. (3.8.112) and (3.8.113), we find the boundary condition if $j = 0:$

$$\psi_0(\xi_H) = M_1/4\pi\varrho_0\alpha^3\xi_H. \tag{3.8.115}$$

If $\Theta > 0$, the combination of Eqs. (3.8.109) and (3.8.110) yields simply

$$\eta = \psi + \Omega^2\xi^2(1-\mu^2)/8\pi G\varrho_0 \approx \sum_{j=0}^{N}\psi_j(\xi)\ P_j(\mu) + \Omega^2\xi^2[1 - P_2(\mu)]/12\pi G\varrho_0, \tag{3.8.116}$$

where Miketinac (1984) truncates the expansion (3.8.110) at a suitable cut-off N. We insert the truncated expansion (3.8.116) into $\nabla^2\psi = -\eta^n$, or equivalently into

$$\partial(\xi^2\ \partial\psi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\psi/\partial\mu]/\partial\mu = -\xi^2\eta^n. \tag{3.8.117}$$

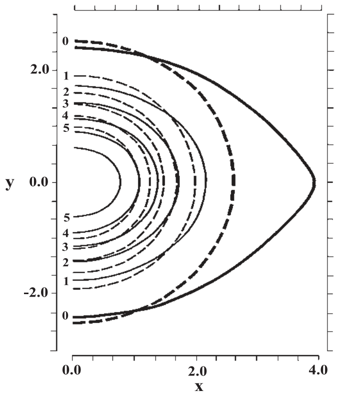

Fig. 3.8.9 Equidensity contours of the critically rotating (continuous lines) and the nonrotating (broken lines) polytrope of index $n = 1.5$ are numbered from 0 to 5 (Miketinac 1984).

Via Eq. (3.1.40) we get eventually

$$\sum_{j=0}^{N}[d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi - j(j+1)\psi_j/\xi^2]\ P_j(\mu) = -\eta^n(\xi,\mu). \tag{3.8.118}$$

Using again the orthogonality of Legendre polynomials from Eq. (3.5.16), we transform Eq. (3.8.118) into

$$d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi - j(j+1)\psi_j/\xi^2 = -[(2j+1)/2]\int_{-1}^{1}\eta^n(\xi,\mu)\ P_j(\mu)\ d\mu$$
$$\approx -\sum_{k=0}^{N}H_k\eta^n(\xi,\mu_k)\ P_j(\mu_k), \qquad (j = 0,1,2,...N), \tag{3.8.119}$$

where a Gauss $(2N+1)$-point quadrature formula has been used to discretize the integral, and μ_k, H_k are the nodes and weights, respectively. The system (3.8.119) is discretized by choosing through numerical experimentation a value $\Delta\xi$ such that $\xi_H = H\ \Delta\xi$, (H – integer). The differential operators appearing in Eq. (3.8.119) are approximated by their simplest central difference equivalents on an equidistant grid with the nodes $\xi_h = h\ \Delta\xi$, $(h = 0,1,2,...H)$, as already outlined in Eq. (3.8.51). The equation (3.8.119) reads

$$(1-1/h)\ \psi_j(\xi_{h-1}) - [2 + j(j+1)/h^2]\ \psi_j(\xi_h) + (1+1/h)\ \psi_j(\xi_{h+1})$$
$$= -(\Delta\xi)^2\sum_{k=0}^{N}H_k\eta^n(\xi_h,\mu_k)\ P_j(\mu_k), \qquad (h = 1,2,3,...H-1;\ j = 0,1,2,...N). \tag{3.8.120}$$

If $h = 1$, the coefficient of $\psi_j(\xi_0)$ vanishes, so the value of $\psi_j(\xi)$ at the singular point $\xi_0 = 0$ decouples from the system (3.8.120). If $h = H$ and $j \neq 0$, the value $\psi_j(\xi_{H+1})$ is eliminated by using the central difference equivalent of the boundary condition (3.8.112):

$$\psi_j(\xi_{H+1}) = \psi_j(\xi_{H-1}) - 2(j+1)\ \psi_j(\xi_H)/H, \qquad (j \neq 0). \tag{3.8.121}$$

If $j = 0$, the boundary condition (3.8.115) explicitly gives the potential function $\psi_0(\xi_H)$, once the mass M_1 is selected. Miketinac (1984) solves the system (3.8.115), (3.8.120), (3.8.121) of $H(N+1)$ simultaneous equations iteratively with the Newton-Raphson algorithm, the initial guess for the solution being supplied by the nonrotating model.

3.8.7 Eriguchi's Complex-Plane Strategy

Many of Hachisu's (1986a, b) results, obtained with the self-consistent field method from Sec. 3.8.4, have been previously found by the method of analytic extension of solutions into the complex plane (Courant and Hilbert 1962, p. 499; Garabedian 1986, Chap. 16). This method has been originally applied by Eriguchi (1978) to rotating polytropes. It has been improved and extended mainly by Hachisu et al. (1982), and Hachisu and Eriguchi (1982, 1984b). Another continuation into the complex plane has been proposed by Geroyannis (1988), [cf. Eq. (3.5.54)]. The basic idea of the complex-plane strategy comes from the fact that Poisson's equation [cf. Eq. (B.39)]

$$(1/r^2)\ \partial(r^2\ \partial\Phi/\partial r)/\partial r + (1/r^2\sin\lambda)\ \partial(\sin\lambda\ \partial\Phi/\partial\lambda)/\partial\lambda + (1/r^2\sin^2\lambda)\ \partial^2\Phi/\partial\varphi^2 = -4\pi G\varrho, \tag{3.8.122}$$

is an elliptic differential equation, and initial value problems (Cauchy problems) are apt to lead to numerical instabilities in the case of elliptic equations, while for the hyperbolic type this problem seems to be absent (cf. Eriguchi 1978). Introducing the complex variables

$$\lambda + i\kappa \quad \text{and} \quad \varphi + i\psi, \tag{3.8.123}$$

instead of the real spherical coordinates λ and φ, the elliptic differential equation (3.8.122) is transformed into the hyperbolic type, as already outlined in 1929 by H. Lewy (Courant and Hilbert 1962, pp. 499-507). The derivative of a complex function $f(z) = f(x + iy)$ is defined as (Spiegel 1974)

$$\begin{aligned} df(z)/dz &= \lim_{\Delta z\to 0}[f(z+\Delta z) - f(z)]/\Delta z \\ &= \lim_{(\Delta x + i\Delta y)\to 0}[f(x+iy+\Delta x+i\Delta y) - f(x+iy)]/(\Delta x + i\Delta y), \end{aligned} \tag{3.8.124}$$

provided that the limit exists independent of the way in which Δz approaches zero. If $df(z)/dz$ exists in a certain domain, $f(z)$ is said to be analytic in this region. If we let $\Delta z = \Delta x + i\Delta y$ approach zero along the path $\Delta y = 0$, and then along the path $\Delta x = 0$, Eq. (3.8.124) becomes

$$df(z)/dz = \partial f(z)/\partial x = \partial f(z)/i\partial y = -i\ \partial f(z)/\partial y, \tag{3.8.125}$$

provided that $f(z)$ is analytic in the neighborhood of z. The derivative of Eq. (3.8.125) is analogous

$$d^2 f(z)/dz^2 = \partial^2 f(z)/\partial x^2 = -i\ \partial^2 f(z)/i\partial y^2 = -\partial^2 f(z)/\partial y^2, \tag{3.8.126}$$

where we have again particularized the derivation paths along $\Delta y = 0$, and $\Delta x = 0$, respectively. Replacing x and y by λ and κ, or by φ and ψ, respectively, we get the equalities

$$\partial\Phi/\partial\lambda = -i\ \partial\Phi/\partial\kappa;\quad \partial\Phi/\partial\varphi = -i\ \partial\Phi/\partial\psi;\quad \partial^2\Phi/\partial\lambda^2 = -\partial^2\Phi/\partial\kappa^2;\quad \partial^2\Phi/\partial\varphi^2 = -\partial^2\Phi/\partial\psi^2. \tag{3.8.127}$$

Thus, the elliptic Poisson equation (3.8.122) can be replaced in the complex domain by a hyperbolic equation, since the second order derivatives $\partial^2\Phi/\partial\lambda^2$, $\partial^2\Phi/\partial\varphi^2$ change their sign in the complex domain, being equal to $-\partial^2\Phi/\partial\kappa^2$, $-\partial^2\Phi/\partial\psi^2$. After this analytic continuation into the complex plane, r, κ, ψ are treated as new independent variables, and λ, φ as parameters. Before turning to the complex domain, Eriguchi (1978) transforms the second order elliptic equation (3.8.122) into a system of first order partial differential equations, by introducing the dimensionless variables

$$\begin{aligned} &\vec F = \vec F(F_r, F_\lambda, F_\varphi) = (R_0/\Phi_0)\ \nabla\Phi;\quad F_r = (R_0/\Phi_0)\ \partial\Phi/\partial r;\quad F_\lambda = (R_0/\Phi_0 r)\ \partial\Phi/\partial\lambda; \\ &F_\varphi = (R_0/\Phi_0 r\sin\lambda)\ \partial\Phi/\partial\varphi, \end{aligned} \tag{3.8.128}$$

$$F_P = (P/P_0)^{1/(n+1)} = (\varrho/\varrho_0)^{1/n}, \tag{3.8.129}$$

$$t = \ln(r/R_0);\quad dt = dr/r, \tag{3.8.130}$$

$$R_0 = (P_0/4\pi G\varrho_0^2)^{1/2}; \quad \Phi_0 = P_0/\varrho_0. \tag{3.8.131}$$

P_0, ϱ_0 denote central values of pressure and density, respectively. As seen from Eq. (3.8.131), the constants R_0 and Φ_0 have the dimensions of length and potential, respectively.

The spherical (r, λ, φ)-coordinates adopted by Hachisu and Eriguchi (1982, Fig. 1) are introduced with respect to a Cartesian (x, y, z)-system, with its x-axis being rotated by the azimuth angle φ^* and by the polar angle λ^* with respect to a rotating Cartesian (x_0, y_0, z_0)-frame, with the z_0-axis directed along the angular velocity vector $\vec{\Omega}$. Clearly, the axes Oz_0 and Oz form the angle $\pi/2 - \lambda^*$, and the coordinates of the two Cartesian systems are connected by the transformation matrix (e.g. Bronstein and Semendjajew 1985)

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \sin\lambda^*\cos\varphi^* & -\sin\varphi^* & -\cos\lambda^*\cos\varphi^* \\ \sin\lambda^*\sin\varphi^* & \cos\varphi^* & -\cos\lambda^*\sin\varphi^* \\ \cos\lambda^* & 0 & \sin\lambda^* \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3.8.132}$$

The equation of hydrostatic equilibrium in a uniformly rotating frame is [cf. Eq. (3.1.14)]

$$\nabla P/\varrho = \nabla\Phi - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}). \tag{3.8.133}$$

In the (x_0, y_0, z_0)-system the angular velocity $\vec{\Omega}$ has the components $(0, 0, \Omega)$. As the transformation of the components from the (x_0, y_0, z_0)-frame to the (x, y, z)-frame proceeds with the inverse of the transformation matrix (3.8.132), the components of the angular velocity in the (x, y, z)-system are $(\Omega\cos\lambda^*, 0, \Omega\sin\lambda^*)$. The components of the vectorial product from Eq. (3.8.133) in the (x, y, z)-frame are accordingly $[\vec{r} = \vec{r}(r\sin\lambda\cos\varphi, r\sin\lambda\sin\varphi, r\cos\lambda)]$

$$\vec{Z} = \vec{Z}(Z_x, Z_y, Z_z) = \vec{\Omega}\times(\vec{\Omega}\times\vec{r}); \quad Z_x = \Omega^2 r\sin\lambda^*(\cos\lambda^*\cos\lambda - \sin\lambda^*\sin\lambda\cos\varphi);$$
$$Z_y = -\Omega^2 r\sin\lambda\sin\varphi; \quad Z_z = \Omega^2 r\cos\lambda^*(-\cos\lambda^*\cos\lambda + \sin\lambda^*\sin\lambda\cos\varphi). \tag{3.8.134}$$

The transformation from the Cartesian (x, y, z)-system to the right-handed spherical (r, λ, φ)-frame proceeds with the transformation matrix (e.g. Bronstein and Semendjajew 1985, p. 565)

$$\begin{bmatrix} Z_r \\ Z_\lambda \\ Z_\varphi \end{bmatrix} = \begin{bmatrix} \sin\lambda\cos\varphi & \sin\lambda\sin\varphi & \cos\lambda \\ \cos\lambda\cos\varphi & \cos\lambda\sin\varphi & -\sin\lambda \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix}. \tag{3.8.135}$$

Hachisu and Eriguchi [1982, Eq. (14)] consider only the radial component of the hydrostatic equation (3.8.133), i.e.

$$\partial P/\partial r = \varrho\ \partial\Phi/\partial r - \varrho Z_r = \varrho\ \partial\Phi/\partial r + \varrho\Omega^2 r\big[(\sin\lambda^*\sin\lambda\cos\varphi - \cos\lambda^*\cos\lambda)^2 + \sin^2\lambda\sin^2\varphi\big]. \tag{3.8.136}$$

The partial derivatives of Φ can be transformed by taking into account the continuity of the second order derivatives of the gravitational potential:

$$\partial^2\Phi/\partial r\partial\lambda = \partial^2\Phi/\partial\lambda\partial r; \quad \partial^2\Phi/\partial r\partial\varphi = \partial^2\Phi/\partial\varphi\partial r. \tag{3.8.137}$$

Inserting Eqs. (3.8.128)-(3.8.131) consecutively into Eqs. (3.8.122), (3.8.137), (3.8.136), we eventually obtain the first order system

$$\partial F_r/\partial t = -2F_r - F_\lambda\cot\lambda - \partial F_\lambda/\partial\lambda - (1/\sin\lambda)\ \partial F_\varphi/\partial\varphi - F_P^n\exp t, \tag{3.8.138}$$

$$\partial F_\lambda/\partial t = -F_\lambda + \partial F_r/\partial\lambda, \tag{3.8.139}$$

$$\partial F_\varphi/\partial t = -F_\varphi + (1/\sin\lambda)\ \partial F_r/\partial\varphi, \tag{3.8.140}$$

$$\partial F_P/\partial t = [\exp t/(n+1)]\big\{F_r + (\beta/2)\exp t[(\sin\lambda^*\sin\lambda\cos\varphi - \cos\lambda^*\cos\lambda)^2 + \sin^2\lambda\sin^2\varphi]\big\}. \tag{3.8.141}$$

If the radial coordinate $r = R_0 \exp t$ is interpreted as the "time", the elliptic differential equation (3.8.122) becomes in the complex domain a two-dimensional hyperbolic wave-type equation in the (κ, ψ)-plane. The system (3.8.138)-(3.8.140) represents a transformed wave-type equation, and can be written in the complex domain in condensed matricial form ($\lambda \to \lambda + i\kappa$; $\varphi \to \varphi + i\psi$; $\partial\,/\partial\lambda \to -i\,\partial\,/\partial\kappa$; $\partial\,/\partial\varphi = -i\,\partial\,/\partial\psi$) :

$$\partial F/\partial t = A\ \partial F/\partial\kappa + B\ \partial F/\partial\psi + C, \tag{3.8.142}$$

where

$$F = \begin{bmatrix} F_r \\ F_\lambda \\ F_\varphi \end{bmatrix}; \qquad A = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 & 0 & i/\sin(\lambda+i\kappa) \\ 0 & 0 & 0 \\ -i/\sin(\lambda+i\kappa) & 0 & 0 \end{bmatrix};$$
$$C = \begin{bmatrix} -2F_r - F_\lambda\cot(\lambda+i\kappa) - F_P^n \exp t \\ -F_\lambda \\ -F_\varphi \end{bmatrix}. \tag{3.8.143}$$

The characteristics along the κ-direction ($\psi = \text{const}$) and along the ψ-direction ($\kappa = \text{const}$) are obtained as the roots of the determinants (e.g. Chester 1971, Garabedian 1986)

$$\text{Det}\ |A - I\alpha_\kappa| = \begin{vmatrix} -\alpha_\kappa & i & 0 \\ -i & -\alpha_\kappa & 0 \\ 0 & 0 & -\alpha_\kappa \end{vmatrix} = -\alpha_\kappa^3 - i^2\alpha_\kappa = 0, \tag{3.8.144}$$

$$\text{Det}\ |B - I\alpha_\psi| = \begin{vmatrix} -\alpha_\psi & 0 & i/\sin(\lambda+i\kappa) \\ 0 & -\alpha_\psi & 0 \\ -i/\sin(\lambda+i\kappa) & 0 & -\alpha_\psi \end{vmatrix} = -\alpha_\psi^3 - i^2\alpha_\psi/\sin^2(\lambda+i\kappa) = 0, \tag{3.8.145}$$

where I denotes the identity matrix. The characteristic roots of Eqs. (3.8.144) and (3.8.145) are therefore

$$\alpha_{\kappa 1} = 0;\ \alpha_{\kappa 2} = 1;\ \alpha_{\kappa 3} = -1 \quad \text{and} \quad \alpha_{\psi 1} = 0;\ \alpha_{\psi 2} = 1/\sin(\lambda+i\kappa);\ \alpha_{\psi 3} = -1/\sin(\lambda+i\kappa). \tag{3.8.146}$$

The system (3.8.142) is totally hyperbolic if the respective eigenvalues $\alpha_{\kappa j}$ and $\alpha_{\psi j}$, $(j = 1, 2, 3)$ are real and distinct, so the roots $\alpha_{\psi 2}$ and $\alpha_{\psi 3}$ have to be real. This can be achieved for instance by putting $\lambda = \pi/2$, because in this case

$$\alpha_{\psi 2,\psi 3} = \pm 1/\sin(\pi/2 + i\kappa) = \mp 1/\cos(i\kappa) = \mp 1/\cosh\kappa, \tag{3.8.147}$$

and the elliptic Poisson equation (3.8.122) is indeed transformed into the totally hyperbolic system (3.8.142). If for certain values of λ^* and φ^* the potential (3.8.148) is known in the central region $r \approx 0$, the system (3.8.142) can be solved along the x-axis, where $\lambda = \pi/2$ and $\varphi = 0$. Thus, for the analytic continuation of Eqs. (3.8.138)-(3.8.141) into the complex domain, λ and φ are replaced throughout by $\pi/2 + i\kappa$ and $0 + i\psi$, respectively.

Let us denote by $(r, \lambda_0, \varphi_0)$ the spherical system associated with the (x_0, y_0, z_0)-coordinates. In this frame Hachisu and Eriguchi (1982, 1984b) expand the potential near the centre of the configuration in a form similar to Eq. (3.8.30):

$$\Phi/\Phi_0 = \sum_{j,k,\ell=0}^{\infty} (r/R_0)^\ell \big(A_{\ell jk}\cos k\varphi_0 + B_{\ell jk}\sin k\varphi_0\big)\ P_j^k(\cos\lambda_0), \qquad (r \approx 0;\ A_{\ell jk}, B_{\ell jk} = \text{const}). \tag{3.8.148}$$

The constants $A_{\ell jk}, B_{\ell jk}$ determine the initial values (Cauchy data) of the problem at hand, and constitute its eigenvalues. Let us denote by

$$\vec{F_0} = \vec{F_0}(F_r, F_{\lambda 0}, F_{\varphi 0}) = (R_0/\Phi_0)\ \nabla\Phi; \quad F_r = (R_0/\Phi_0)\ \partial\Phi/\partial r; \quad F_{\lambda 0} = (R_0/\Phi_0 r)\ \partial\Phi/\partial\lambda_0;$$
$$F_{\varphi 0} = (R_0/\Phi_0 r\sin\lambda_0)\ \partial\Phi/\partial\varphi_0, \tag{3.8.149}$$

the equivalent of the dimensionless gravitational force $\vec{F}$ from Eq. (3.8.128) in the $(r, \lambda_0, \varphi_0)$-frame. The respective components are connected by the transformation matrix

$$\begin{bmatrix} F_r \\ F_{\lambda 0} \\ F_{\varphi 0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \partial\lambda/\partial\lambda_0 & \sin\lambda\ \partial\varphi/\partial\lambda_0 \\ 0 & (1/\sin\lambda_0)\ \partial\lambda/\partial\varphi_0 & (\sin\lambda/\sin\lambda_0)\ \partial\varphi/\partial\varphi_0 \end{bmatrix} \begin{bmatrix} F_r \\ F_\lambda \\ F_\varphi \end{bmatrix}. \tag{3.8.150}$$

The elements of this transformation matrix are just the direction cosines among the axes of the two spherical coordinate systems $(r, \lambda_0, \varphi_0)$ and (r, λ, φ).

The metrics of two orthogonal, right-handed curvilinear coordinate systems (q_{10}, q_{20}, q_{30}) and (q_1, q_2, q_3) are given by [cf. Eq. (B.4)]

$$ds_0^2 = \sum_{j=1}^{3} h_{j0}^2\ dq_{j0}^2; \quad ds^2 = \sum_{j=1}^{3} h_j^2\ dq_j^2. \tag{3.8.151}$$

The direction cosines $(h_k/h_{j0})(\partial q_k/\partial q_{j0}),\ (j,k = 1,2,3)$ are just equal to the ratios between the line elements $h_k\ dq_k$ and $h_{j0}\ dq_{j0}$ along the axes. Thus, the components of a vector $\vec{V}_k(V_1, V_2, V_3)$ in the q_k-system transform to the vector $\vec{V}_{j0}(V_{10}, V_{20}, V_{30})$ in the q_{j0}-system via the matricial form [cf. Eq. (3.8.150)]

$$\big[V_{j0}\big] = \big[(h_k/h_{j0})\ (\partial q_k/\partial q_{j0})\big]\big[V_k\big]. \tag{3.8.152}$$

In the present case we have: $h_{10} = 1;\ h_{20} = r;\ h_{30} = r\sin\lambda_0;\ h_1 = 1;\ h_2 = r;\ h_3 = r\sin\lambda$.

The condition that no mass distributions exist outside the surface is imposed by the integral representation (3.1.53) of the gravitational potential:

$$\Phi(r, \lambda_0, \varphi_0) = G\int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\lambda'\ d\lambda' \int_0^{\infty} \varrho(r', \lambda', \varphi')\ r'^2\ dr'/|\vec{r} - \vec{r'}|$$
$$= G\sum_{j=0}^{\infty}\sum_{k=0}^{j}[f_1(t)\cos k\varphi_0 + f_2(t)\sin k\varphi_0]\ P_j^k(\cos\lambda_0), \qquad \big[\vec{r} = \vec{r}(r, \lambda_0, \varphi_0);\ \vec{r'} = \vec{r'}(r', \lambda', \varphi')\big]. \tag{3.8.153}$$

The dimensionless gravitational force can be obtained from this equation via Eq. (3.8.149): $\vec{F_0} = (R_0/\Phi_0)\ \nabla\Phi$. The dimensionless pressure (3.8.129) has a similar form as the potential, as can be seen from the prime integral of the hydrostatic equation (3.8.133) in the $(r, \lambda_0, \varphi_0)$ system [cf. Eq. (3.8.4)]:

$$(n+1)K\varrho^{1/n} = \Phi + (1/2)\Omega^2 r^2 \sin^2\lambda_0 + \mathrm{const.} \tag{3.8.154}$$

With the notations (3.8.129), (3.8.131) this equation can be written in dimensionless form as

$$(n+1)F_P = \Phi/\Phi_0 + (\beta/6)(r/R_0)^2[1 - P_2(\cos\lambda_0)] + \mathrm{const.} \tag{3.8.155}$$

Indeed, the dimensionless pressure has a similar series representation as the potential Φ from Eq. (3.8.153):

$$F_P(t, \lambda_0, \varphi_0) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}[p_1(t)\cos k\varphi_0 + p_2(t)\sin k\varphi_0]\ P_j^k(\cos\lambda_0). \tag{3.8.156}$$

As already mentioned, λ and φ are replaced by $\lambda + i\kappa = \pi/2 + i\kappa$ and $\varphi + i\psi = 0 + i\psi$, respectively. Eqs. (3.8.141) and (3.8.142) are integrated for a given value of λ^*, φ^* with a presumed set of the eigenvalues $A_{\ell jk}, B_{\ell jk}$ from Eq. (3.8.148), where the transition from the (r, λ, φ)-frame to the $(r, \lambda_0, \varphi_0)$-frame is effected through Eqs. (3.8.132), (3.8.150). An approximate value of the density distribution within the polytrope can be guessed with an approximate data set $A_{\ell jk}, B_{\ell jk}$ from Eq. (3.8.148). With this assumed density distribution the potential (3.8.153) can be found throughout the configuration, where it is sufficient to solve the equations (3.8.153) and (3.8.156) in the $(r, \lambda_0, \varphi_0)$ coordinates for a single set of values λ^*, φ^*, because of the analyticity of all functions. The potential Φ and the corresponding gravitational force $\vec{F_0}$ are in general not consistent with the force $\vec{F}$ found previously from the integration of Eq. (3.8.142), so corrected eigenvalues $A_{\ell jk}, B_{\ell jk}$ have to be sought. The criterion of consistency is conveniently expressed by the constancy of the total potential from Eq. (3.8.155) over the surface $r_s = r_s(\lambda_0, \varphi_0)$ where P, ϱ, and F_P vanish:

$$\Phi(r_s, \lambda_0, \varphi_0)/\Phi_0 + (\beta/4)(r_s \sin\lambda_0/R_0)^2 + \mathrm{const} = 0. \tag{3.8.157}$$

Hachisu and Eriguchi (1982) claim that the convergence of the series (3.8.148) is also guaranted in the region far from the real axis, i.e. even if the imaginary variables κ and ψ are large.

3.8.8 Tabulation of Numerical Results for Critically Rotating Spheroids

The parameters characterizing best the critically rotating polytropes seem to be the critical rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0$ [cf. Eqs. (3.2.3), (3.6.6)], the mean critical rotation parameter $\beta_{cm} = \Omega_c^2/2\pi G\varrho_m$ [cf. Eqs. (3.4.46), (3.6.36)], and the critical oblateness (flattening, ellipticity) $f_c = (a_1 - a_3)/a_1 = (\Xi_{ce} - \Xi_{cp})/\Xi_{ce}$, where $a_1 = \alpha\Xi_{ce}$ and $a_3 = \alpha\Xi_{cp}$ means the equatorial and polar radius of the critical surface. Ξ_{ce} and Ξ_{cp} are the values of the critical dimensionless radial coordinate at the equator and pole, respectively.

The general condition of critical rotation is simply given by the vanishing radial component of the effective gravity $g_{r_{ce}}$ at the critical equatorial radius r_{ce} [cf. Eqs. (3.1.22), (3.4.65), (3.6.12)]:

$$g_{r_{ce}} = (\partial\Phi_{tot}/\partial r)_{r=r_{ce}} = [\partial(\Phi + \Phi_f)/\partial r]_{r=r_{ce}} = [(n+1)K\varrho_0^{1/n}/\alpha][\partial\Theta(\xi,\mu,\varphi)/\partial\xi]_{\xi=\Xi_{ce}} = 0, \tag{3.8.158}$$

where $\Phi_f = \Omega_c^2\ell^2/2 = (n+1)K\varrho_0^{1/n}\xi^2\beta_c[1-P_2(\mu)]/6$ denotes the centrifugal potential from Eqs. (3.1.23), (3.4.64), or (3.6.12). We have also used the hydrostatic equation from Eq. (3.4.64) or (3.8.3).

The critical values result either in the course of less transparent numerical manipulations, or they require the knowledge of certain constants, as for Chandrasekhar's perturbation method [Eq. (3.4.65)], for the double approximation technique [Eqs. (3.6.12), (3.6.13)], or for Williams' optimal matching method [Eq. (3.8.67)]. The finding of critical values can be best illustrated with Chandrasekhar's first order theory. Equating Eq. (3.4.65) to zero, we get at once the critical rotation parameter:

$$g_{r_{ce}} = [(n+1)K\varrho_0^{1/n}/\alpha]\big\{\theta'(\xi_1) + \beta_c\big[2\psi_0(\xi_1)/\xi_1 + \psi_0'(\xi_1) + A_2P_2(0)\ \big(2\psi_2(\xi_1)/\xi_1 + \psi_2'(\xi_1)\big)\big]\big\} = 0,$$
$$\big(\theta^n(\xi_1) = 0;\ 0 < n \le 5\big). \tag{3.8.159}$$

Substituting for A_2 from Eq. (3.2.43), we obtain

$$\beta_c = -\theta'(\xi_1)/\big\{2\psi_0(\xi_1)/\xi_1 + \psi_0'(\xi_1) + 5\xi_1^2\big[2\psi_2(\xi_1)/\xi_1 + \psi_2'(\xi_1)\big]/12\big[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)\big]\big\};$$
$$\beta_{cm} = \beta_c\varrho_0/\varrho_m = -[\beta_c\xi_1/3\theta'(\xi_1)][1 - 3\beta_c\psi_0(\xi_1)/\xi_1\theta'(\xi_1)]/\{1 + \beta_c[\psi_0'(\xi_1) - \xi_1/3]/\theta'(\xi_1)\}$$
$$\approx -\beta_c\xi_1/3\theta'(\xi_1), \qquad (0 < n \le 5), \tag{3.8.160}$$

with $P_2(0) = -1/2$, and ϱ_0/ϱ_m taken from Eq. (3.2.61). The corresponding dimensionless equatorial and polar radii result via Eqs. (3.2.56), (3.2.57):

$$\Xi_{ce} = \xi_1 - \beta_c\big\{\psi_0(\xi_1)/\theta'(\xi_1) + 5\xi_1^2\psi_2(\xi_1)/12\theta'(\xi_1)[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\big\}, \tag{3.8.161}$$

$$\Xi_{cp} = \xi_1 - \beta_c\big\{\psi_0(\xi_1)/\theta'(\xi_1) - 5\xi_1^2\psi_2(\xi_1)/6\theta'(\xi_1)[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\big\}. \tag{3.8.162}$$

If $n = 0$, we get in the same manner from Eq. (3.4.66), ($\theta = 1 - \xi^2/6;\ \theta^n(\xi_1) = 1;\ -2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1) = -1/3 = \theta'(\xi_1)/\xi_1;\ \psi_0 = \xi^2/6;\ \psi_2 = \xi^2;\ A_2 = -5/12$)

$$\beta_c = \beta_{cm} = -\theta'(\xi_1)/\big\{-\psi_0(\xi_1)/\xi_1 + \psi_0'(\xi_1) + A_2P_2(0)\ [-\psi_2(\xi_1)/\xi_1 + \psi_2'(\xi_1)]\big\} = 8/9, \quad (n = 0), \tag{3.8.163}$$

and from Eqs. (3.2.45), (3.2.64)

$$\Xi_{ce} = \xi_1 - \beta_c[\psi_0(\xi_1) + A_2P_2(0)\ \psi_2(\xi_1)]/\theta'(\xi_1) = 6^{1/2}(1 + 9\beta_c/8) = 2 \times 6^{1/2};$$
$$\Xi_{cp} = \xi_1 - \beta_c[\psi_0(\xi_1) + A_2P_2(1)\ \psi_2(\xi_1)]/\theta'(\xi_1) = 6^{1/2}(1 - 3\beta_c/4) = 6^{1/2}/3, \qquad (n = 0). \tag{3.8.164}$$

Table 3.8.1 Critical values of rotationally distorted spheres: $\beta_c = \Omega_c^2/2\pi G\varrho_0$ – critical rotation parameter, $\beta_{cm} = \Omega_c^2/2\pi G\varrho_m$ – mean critical rotation parameter, Ξ_{ce} – critical dimensionless equatorial radius (maximum equatorial radius for nonaxisymmetric models), Ξ_{cp} – critical dimensionless polar radius, $f_c = (\Xi_{ce} - \Xi_{cp})/\Xi_{ce}$ – critical oblateness. ξ_1 denotes the dimensionless radius of the polytropic sphere from Table 2.5.2. The critical values for the limiting bifurcation sequence by Vandervoort and Welty (1981) are for $n = 0.794$ instead of $n = 0.808$. Values of Chandrasekhar (1933a, d) and β_{cm} of James (1964) are taken from Hurley and Roberts (1964, Table 4). Values of Stoeckly (1965) are from Martin (1970, Table I). Values of Ξ_{ce}, Ξ_{cp} by Ipser and Managan (1981) are taken from Vandervoort and Welty (1981, Table 3). $a + b$ means $a \times 10^b$.

Author	β_c	β_{cm}	Ξ_{ce}	Ξ_{cp}	f_c
		$n = 0;\ \xi_1 = 2.45$			
Caimmi (1985)	6.67−1	6.67−1	∞	2.45	1.000
Chandrasekhar (1933a, d)	8.89−1	8.89−1	4.90	0.82	0.833
		$n = 0.1$			
Hachisu et al. (1982)	1.82−1	2.09−1	–	–	0.773
Hachisu (1986a)	–	2.26−1	–	–	0.729
		Nonaxisymmetric polytrope : $n = 0.1$			
Hachisu and Eriguchi (1982)	1.25−1	1.41−1	–	–	0.677
		$n = 0.5;\ \xi_1 = 2.75$			
Caimmi (1985)	2.18−1	–	4.33	2.50	0.424
Hachisu et al. (1982)	1.53−1	3.00−1	–	–	0.565
Hachisu (1986a)	–	3.10−1	–	–	0.558
Horedt (1983)	1.44−1	2.76−1	–	–	0.440
		Nonaxisymmetric polytrope : $n = 0.5$			
Hachisu and Eriguchi (1982)	1.32−1	2.46−1	–	–	0.524
Ipser and Managan (1981)	1.29−1	–	4.97	2.24	0.549
Vandervoort and Welty (1981)	1.31−1	–	5.15	2.25	0.564
		Bifurcating polytrope : $n = 0.808$			
Caimmi (1985)	1.33−1	–	4.34	2.56	0.410
Horedt (1983)	1.22−1	3.85−1	–	–	0.503
James (1964)	1.06−1	3.30−1	4.77	2.49	0.478
Vandervoort and Welty (1981)	1.16−1	–	4.76	2.40	0.495
		$n = 1;\ \xi_1 = 3.14$			
Anand (1968)	8.31−2	–	4.24	3.21	0.242
Caimmi (1985)	1.20−1	–	4.94	2.55	0.483
Chandrasekhar (1933a, d)	1.71−1	5.75−1	4.35	2.34	0.462
Geroyannis (1988)	7.51−2	–	4.81	2.69	0.440
Horedt (1983)	9.46−2	3.93−1	–	–	0.457
James (1964)	8.37−2	3.39−1	4.83	2.69	0.442
Monaghan and Roxburgh (1965)	7.59−2	–	4.54	2.92	0.357
Vandervoort and Welty (1981)	8.89−2	–	4.78	2.65	0.447
		$n = 1.5;\ \xi_1 = 3.65$			
Anand (1968)	4.63−2	–	4.85	3.73	0.230
Caimmi (1985)	5.19−2	–	5.08	3.16	0.378
Chandrasekhar (1933a, d)	8.19−2	5.56−1	4.75	3.02	0.365
Eriguchi (1978)	4.32−2	–	–	–	–
Hachisu (1986a)	–	3.60−1	–	–	0.383
Horedt (1983)	4.80−2	4.07−1	–	–	0.384
Hurley and Roberts (1964)	4.45−2	3.55−1	4.95	3.16	0.361
James (1964)	4.36−2	3.52−1	5.36	3.30	0.385
Martin (1970)	4.16−2	–	5.30	–	–
Miketinac and Barton (1972)	4.36−2	–	5.36	–	–
Monaghan and Roxburgh (1965)	4.10−2	–	5.24	3.49	0.334
Naylor and Anand (1970)	3.75−2	–	5.37	–	–
Singh and Singh (1984)	3.71−2	–	5.32	3.48	0.347
Smith (1975)	4.37−2	–	5.33	–	–

Author	β_c	β_{cm}	Ξ_{ce}	Ξ_{cp}	f_c
Stoeckly (1965)	4.36−2	–	5.30	–	–
Williams (1975)	4.34−2	–	5.23	3.48	0.335
		$n = 2;\ \xi_1 = 4.35$			
Anand (1968)	2.30−2	–	5.62	4.38	0.221
Caimmi (1985)	2.50−2	–	5.95	3.97	0.333
Chandrasekhar (1933a, d)	3.88−2	5.34−1	5.46	3.81	0.302
Geroyannis (1984)	2.75−2	–	5.90	3.95	0.331
Geroyannis (1988)	2.11−2	–	6.25	4.06	0.351
Horedt (1983)	2.34−2	4.10−1	–	–	0.350
James (1964)	2.16−2	3.58−1	6.31	4.06	0.357
Martin (1970)	2.14−2	–	6.29	–	–
Monaghan and Roxburgh (1965)	1.99−2	–	6.33	4.15	0.344
Naylor and Anand (1970)	1.94−2	–	6.37	–	–
Occhionero (1967a)	3.51−2	–	–	–	–
Singh and Singh (1984)	1.97−2	–	6.32	4.16	0.342
Stoeckly (1965)	2.15−2	–	6.13	–	–
Williams (1975)	2.15−2	–	6.25	–	–
		$n = 2.5;\ \xi_1 = 5.36$			
Caimmi (1985)	1.15−2	–	7.41	5.04	0.320
Geroyannis (1984)	1.24−2	–	7.14	5.03	0.296
Geroyannis (1988)	9.94−3	–	7.76	5.10	0.343
Horedt (1983)	1.07−2	4.12−1	–	–	0.334
Hurley and Roberts (1964)	1.01−2	3.59−1	7.10	4.90	0.310
James (1964)	9.93−3	3.60−1	7.76	5.10	0.343
Martin (1970)	9.90−3	–	7.76	–	–
Monaghan and Roxburgh (1965)	9.31−3	–	7.72	5.18	0.329
Occhionero (1967a)	1.50−2	–	–	–	–
Williams (1975)	9.90−3	–	7.73	–	–
		$n = 3;\ \xi_1 = 6.90$			
Anand (1968)	4.43−3	–	8.81	6.84	0.224
Caimmi (1985)	4.69−3	–	9.62	6.63	0.311
Chandrasekhar (1933a, d)	7.05−3	5.31−1	8.55	6.51	0.238
Geroyannis (1984)	5.08−3	–	9.20	6.62	0.281
Geroyannis (1988)	4.13−3	–	10.2	6.67	0.345
Hachisu (1986a)	–	3.67−1	–	–	0.338
Horedt (1983)	4.36−3	4.13−1	–	–	0.327
Hurley and Roberts (1964)	4.13−3	3.59−1	9.19	6.41	0.302
James (1964)	3.93−3	–	–	6.58	–
Linnell (1977a)	4.12−3	–	10.3	6.68	0.353
Martin (1970)	4.08−3	–	10.1	–	–
Miketinac and Barton (1972)	4.07−3	–	10.1	–	–
Monaghan and Roxburgh (1965)	3.95−3	–	10.1	6.72	0.336
Naylor and Anand (1970)	3.93−3	–	10.1	–	–
Occhionero (1967a)	5.96−3	–	–	–	–
Singh and Singh (1984)	3.97−3	–	10.1	6.71	0.335
Stoeckly (1965)	4.03−3	–	9.81	–	–
Vandervoort and Welty (1981)	4.09−3	–	10.1	6.68	0.336
Williams (1975)	4.07−3	–	10.1	6.70	0.334
		$n = 3.5;\ \xi_1 = 9.54$			
Anand (1968)	1.51−3	–	12.2	9.39	0.232
Caimmi (1985)	1.58−3	–	13.4	9.31	0.307
Chandrasekhar (1933a, d)	2.37−3	5.27−1	11.8	9.20	0.221
Geroyannis (1984)	1.72−3	–	12.8	9.29	0.272
Geroyannis (1988)	1.39−3	–	14.0	9.35	0.334
Horedt (1983)	1.48−3	4.13−1	–	–	0.324

Author	β_c	β_{cm}	Ξ_{ce}	Ξ_{cp}	f_c
Hurley and Roberts (1964)	1.40−3	3.59−1	12.8	8.97	0.299
Monaghan and Roxburgh (1965)	1.25−3	–	14.1	9.56	0.322
Occhionero (1967a)	2.00−3	–	–	–	–
Williams (1975)	1.38−3	–	14.0	–	–
	$n = 4;\ \xi_1 = 14.97$				
Caimmi (1985)	3.74−4	–	21.3	14.8	0.306
Chandrasekhar (1933a, d)	5.59−4	5.23−1	18.6	14.7	0.210
Eriguchi (1978)	3.22−4	–	–	–	–
Geroyannis (1984)	4.08−4	–	20.2	14.8	0.268
Geroyannis (1988)	3.29−4	–	22.2	14.8	0.333
Hachisu (1986a)	–	3.69−1	–	–	0.333
Horedt (1983)	3.50−4	4.13−1	–	–	0.323
Hurley and Roberts (1964)	3.33−4	3.59−1	20.3	14.2	0.298
Martin (1970)	3.29−4	–	22.2	–	–
Monaghan and Roxburgh (1965)	3.27−4	–	22.3	14.8	0.334
Naylor and Anand (1970)	3.22−4	–	22.3	–	–
Williams (1975)	3.29−4	–	22.2	–	–
	$n = 4.5;\ \xi_1 = 31.84$				
Caimmi (1985)	3.66−5	–	45.6	31.7	0.305
	$n = 4.9;\ \xi_1 = 171.43$				
Eriguchi (1978)	2.00−7	–	–	–	–
Horedt (1983)	2.88−7	4.13−1	–	–	0.323
Martin (1970)	2.03−7	–	257	–	–
Naylor and Anand (1970)	2.03−7	–	257	–	–
	$n = 5;\ \xi_1 = \infty$				
Caimmi (1985)	0	–	∞	∞	0.333
Chandrasekhar (1933a, d)	0	5.00−1	∞	∞	0.200
Roche model	–	3.61−1	–	–	0.333

The perturbation theories of Chandrasekhar (1933a, d) and Caimmi (1985) are inadequate to represent the behaviour of the critically rotating constant density polytrope $n = 0$; this results by comparing their values to those obtained numerically for the similar polytrope $n = 0.1$ from Table 3.8.1. Their values should also be contrasted with the maximum rotation parameter $\beta = 0.2247$ of Maclaurin ellipsoids. The maximum value of β for Jacobi ellipsoids is 0.18711 (Lyttleton 1953, Tassoul 1978).

For the *axisymmetric* models with $0.3 \lesssim n < 5$ the critical parameters β_c and β_{cm} are just the maximum values of β and β_m, respectively. No calculations have been effected so far for polytropic indices in the interval $0 < n < 0.1$, so we exclude this range from our subsequent discussion. If $0.1 < n \lesssim 0.3$, the rotation parameters of axisymmetric models attain a certain maximum value (e.g. $\beta = 0.21,\ \beta_m = 0.24$ if $n = 0.1$), falling with increasing oblateness and increasing angular momentum towards the critical values (e.g. $\beta_c = 0.182,\ \beta_{cm} = 0.226$ if $n = 0.1$, Hachisu et al. 1982, Hachisu and Eriguchi 1982, Hachisu 1986a, b; see also Figs. 3.8.5, 3.8.7).

The *nonaxisymmetric* models with $0.1 < n < 0.808$ bifurcate at their maximum values of β, β_m, the critical values being the smallest ones (Sec. 3.8.1 and Fig. 3.8.7, Hachisu and Eriguchi 1982, Hachisu 1986b). The bifurcation of the nonaxisymmetric sequences occurs for all polytropic indices $0.1 \leq n \leq 0.808$ at nearly the same oblateness $f = 0.4$, and at nearly the same dimensionless angular momentum $J/(4\pi G M_1^{10/3} \varrho_0^{-1/3})^{1/2} \approx 0.07$ (Hachisu and Eriguchi 1982).

The values of the polytrope $n = 5$, ($\beta_c = 3^{1/2}/\xi^3;\ \beta_{cm} = 1/2;\ f_c = 1/5$ if $\xi \to \infty$) for Chandrasekhar's theory (1933a, d) are obtained from Eqs. (3.8.160)-(3.8.162) with the asymptotic values quoted in Eqs. (3.2.68) and (3.2.89): $\theta' \approx -3^{1/2}/\xi^2;\ \psi_0 \approx \xi^2/6;\ \psi_2 \approx 15\pi\xi^2/256$. As shown by Eq. (3.2.69), the polytrope $n = 5$ resembles the Roche model. For the critically rotating Roche model we have shown in Eqs. (3.6.33)-(3.6.36) that $f_c = 1/3$, and $\beta_{cm} = 0.36074$; these values are quoted for comparison in the last line of Table 3.8.1.

The ratio $\tau_c = E_{kin}/|W|$ between kinetic and gravitational energy of critically rotating polytropes diminishes continuously as the polytropic index increases from 0.1 to 5. No critical configuration exists for the nonrelativistic homogeneous ellipsoids $n = 0$: The ratio τ attains its maximum value 0.5 from Eqs. (3.1.36), (5.10.217)-(5.10.223) for Maclaurin ellipsoids degenerating into an infinitely thin disk ($e = 1;\ a_1 = a_2 = \infty;\ a_3 = 0;\ \beta = \Omega^2/2\pi G\varrho = 0$), as well as for Jacobi ellipsoids degenerating into an infinitely thin needle ($a_1 = \infty;\ a_2, a_3 = 0;\ \beta = 0;\ \tau = 0.5$). In the range $0.1 < n < 0.808$ there exist

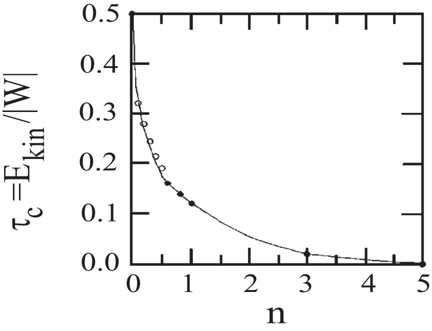

Fig. 3.8.10 Critical ratio τ_c between rotational and gravitational energy of axisymmetric polytropes as a function of polytropic index n (cf. Table 5.8.2). Boss (1986) interpolates the values of Tassoul (1978, Chap. 10, filled circles) and Hachisu et al. (1982, open circles).

critically rotating, axisymmetric and nonaxisymmetric configurations with equatorial mass shedding. τ_c is always larger for axisymmetric polytropes as compared to nonaxisymmetric ones: $\tau_c = 0.32$ versus $\tau_c = 0.16$ if $n = 0.1$, and $\tau_c = 0.19$ versus $\tau_c = 0.14$ if $n = 0.5$ (Hachisu et al. 1982, Hachisu and Eriguchi 1982). Hurley and Roberts (1964) find $\tau_c = 6.10\times10^{-2},\ 9.14\times10^{-3},\ 1.20\times10^{-3}$ if $n = 1.5, 3, 4$, whereas Hachisu (1986a) quotes in close agreement $\tau_c = 5.95\times10^{-2},\ 9.00\times10^{-3},\ 1.19\times10^{-3}$, respectively (Table 5.8.2).

Below, we list the various methods adopted by the authors from Table 3.8.1. Anand (1968) adopts the second order perturbation method of Chandrasekhar (1933a, d; Sec. 3.5). Caimmi [1985; see Eq. (3.2.98)] also uses a variant of Chandrasekhar's (1933a, d) method. The perturbation method of Chandrasekhar (1933a, d) is exposed in Secs. 3.2-3.5. Eriguchi (1978), Hachisu et al. (1982), and Hachisu and Eriguchi (1982) implement the complex-plane strategy from Sec. 3.8.7. Hachisu (1986a, b) adopts the self-consistent field method from Sec. 3.8.4. Geroyannis (1984; cf. Sec. 3.5) uses the third order Chandrasekhar (1933a, d) method. Geroyannis (1988) introduces his complex-plane strategy touched at the end of Sec. 3.5. Horedt (1983, Sec. 3.7) uses the third-order level surface theory of Zharkov and Trubitsyn (1978). Hurley and Roberts (1964) calculate with the ellipsoidal method from Sec. 3.8.5. Ipser and Managan (1981) generalize Stoeckly's (1965) method from Sec. 3.8.2. James (1964) implements his numerical method from Sec. 3.8.1. Linnell (1977a) expands on Chandrasekhar's (1933a, d) method. Martin (1970) mixes a higher order Chandrasekhar (1933a, d) method with the double approximation technique from Sec. 3.6. Miketinac and Barton (1972) use Stoeckly's (1965) method from Sec. 3.8.2. Monaghan and Roxburgh (1965, see Sec. 3.6) implement the double approximation technique. Naylor and Anand (1970) extend the double approximation technique of Monaghan and Roxburgh (1965). Occhionero (1967a, cf. Sec. 3.5) implements the second order perturbation theory of Chandrasekhar (1933a, d). Singh and Singh (1984) expand the double approximation technique of Monaghan and Roxburgh (1965). Smith [1975, cf. Eqs. (3.5.59), (3.5.60)] introduces stretched variables for his asymptotic matching procedure. Stoeckly (1965, see Sec. 3.8.2) adopts for his calculations a Henyey-type algorithm. Vandervoort and Welty (1981) calculate with a simplified version of the self-consistent field method from Sec. 3.8.4. And finally, Williams (1975, cf. Sec. 3.8.3) uses the optimal matching technique after Faulkner et al. (1968).

Table 3.8.1 shows that the differences among various methods are often large and exhibit systematic trends; these differences are caused by the different approximation procedures for critically rotating polytropes. Of course, since the critical polytropes are rotating rapidly, the most reliable results seem to be produced by the more numerical methods, as those of James (1964), Martin (1970), Hachisu and Eriguchi (1982), Geroyannis (1988). For a statistical evaluation of the results see Geroyannis (1988).

3.9 Rotating Polytropic Cylinders and Polytropic Rings

3.9.1 Rotating Polytropic Cylinders

We consider only polytropic indices $0 \le n < \infty$, because in this case the physical parameters of undistorted cylinders remain finite. The limiting case of the "isothermal" cylinder ($n = \pm\infty$) is marginally touched. The study of rotating polytropic cylinders seems to have been initiated by Robe (1968b). In an inertial cylindrical frame the equation of hydrostatic equilibrium (3.1.2) becomes for a cylinder rotating differentially with angular velocity $\vec{\Omega} = \vec{\Omega}[0, 0, \Omega(\ell)]$:

$$\begin{cases} \partial P/\partial \ell = \varrho\ \partial\Phi/\partial\ell + \varrho\Omega^2\ell \\ \partial P/\partial\varphi = \varrho\ \partial\Phi/\partial\varphi \\ \partial P/\partial z = \varrho\ \partial\Phi/\partial z \end{cases} \qquad [\Omega = \Omega(\ell)]. \tag{3.9.1}$$

Cartesian vector components $\vec{Z} = \vec{Z}(Z_x, Z_y, Z_z)$ transform into cylindrical components $\vec{Z} = \vec{Z}(Z_\ell, Z_\varphi, Z_z)$ according to the matrix (e.g. Bronstein and Semendjajew 1985, p. 564)

$$\begin{bmatrix} Z_\ell \\ Z_\varphi \\ Z_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix}, \tag{3.9.2}$$

where $x = \ell\cos\varphi,\ y = \ell\sin\varphi$. With the notations from Eq. (3.2.1)

$$\varrho = \varrho_0\Theta^n; \quad P = K\varrho^{1+1/n} = K\varrho_0^{1+1/n}\Theta^{n+1} = P_0\Theta^{n+1}, \qquad (0 \le n < \infty), \tag{3.9.3}$$

we obtain from the hydrostatic equation (3.9.1) the prime integral

$$(n+1)K\varrho^{1/n} = (n+1)K\varrho_0^{1/n}\Theta = \Phi + \int \Omega^2(\ell)\ \ell\ d\ell + \text{const}. \tag{3.9.4}$$

Poisson's equation in cylindrical coordinates is via Eq. (B.48) equal to:

$$\nabla^2\Phi = (1/\ell)\ \partial(\ell\ \partial\Phi/\partial\ell)/\partial\ell + (1/\ell^2)\ \partial^2\Phi/\partial\varphi^2 + \partial^2\Phi/\partial z^2 = -4\pi G\varrho. \tag{3.9.5}$$

With the dimensionless quantities from Eqs. (3.2.1), (3.2.3)

$$\ell = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi = \alpha\xi;$$
$$z = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\zeta = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\zeta = \alpha\zeta;$$
$$\beta = \Omega^2/2\pi G\varrho_0, \qquad (0 \le n < \infty), \tag{3.9.6}$$

Poisson's equation becomes, after inserting for the derivatives of Φ via Eq. (3.9.4), (Robe 1979, Simon et al. 1981, Veugelen 1985a):

$$(1/\xi)\ \partial(\xi\ \partial\Theta/\partial\xi)/\partial\xi + (1/\xi^2)\ \partial^2\Theta/\partial\varphi^2 + \partial^2\Theta/\partial\zeta^2 = -\Theta^n + (1/2\xi)\ \partial[\xi^2\beta(\xi)]/\partial\xi. \tag{3.9.7}$$

If density and pressure depend only on ℓ, and if the angular velocity Ω is constant, Eq. (3.9.7) writes (Robe 1968b)

$$(1/\xi)\ d(\xi\ d\Theta/d\xi)/d\xi = d^2\Theta/d\xi^2 + (1/\xi)\ d\Theta/d\xi = -\Theta^n + \beta, \qquad (\beta = \text{const};\ 0 \le n < \infty), \tag{3.9.8}$$

subject to the boundary conditions $\Theta(0) = 1$ and $(d\Theta/d\xi)_{\xi=0} = 0$.

In a similar way as for Eq. (2.4.23), Robe (1968b) finds for the expansion near the origin:

$$\Theta \approx 1 - (1-\beta)\xi^2/(2^1 \times 1!)^2 + n(1-\beta)\xi^4/(2^2 \times 2!)^2$$
$$-n(1-\beta)[n + 2(n-1)(1-\beta)]\xi^6/(2^3 \times 3!)^2 + ..., \qquad (\xi \approx 0). \tag{3.9.9}$$

Table 3.9.1 Critical values of polytropic cylinders and rings: $\beta_c = \Omega_c^2/2\pi G\varrho_0$ – critical rotation parameter, $\beta_{cm} = \Omega_c^2/2\pi G\varrho_m$ – mean critical rotation parameter, ξ_1 – dimensionless surface coordinate of nonrotating cylinder from Table 2.5.2, Ξ_{c1} – dimensionless critical surface coordinate of rotating cylinder. r_{c1}/R_1 – ratio between mean critical minor radius r_{c1} of the ring (measured in the equatorial plane) and its major radius R_1 (Fig. 3.9.2). For polytropic cylinders β_{cm} should be throughout exactly 1 [see Eq. (3.9.12)]. $a+b$ means $a\times 10^b$.

A. Cylinders (Robe 1968b)

n	β_c	β_{cm}	ξ_1	Ξ_{c1}
0	1.00	1.00	2.00	∞
1	2.87−1	1.00	2.40	3.82
3	5.47−2	1.00	3.57	5.49
6	8.21−3	9.47−1	6.72	1.08+1

B. Rings (Hachisu 1986a)

n	β_{cm}	r_{c1}/R_1
0.1	0.170	– (concave hamburger)
0.5	0.211	0.431
1.5	0.211	0.337
3.0	0.183	0.279
4.0	0.166	0.254

When the centrifugal force at the cylindrical surface is just equal to gravitation, we obtain the critically rotating cylinder by equating to zero the total gravitational acceleration at the critical surface $\ell_{c1} = \alpha\Xi_{c1}$:

$$g_{\ell c1} = \left\{\partial\left[\Phi + \int \Omega^2(\ell')\ \ell'\ d\ell'\right]\Big/\partial\ell\right\}_{\ell=\ell_{c1}} = [(n+1)K\varrho_0^{1/n}/\alpha][\partial\Theta(\xi,\varphi,\zeta)/\partial\xi]_{\xi=\Xi_{c1}} = 0. \quad (3.9.10)$$

The mass per unit length of the rotating cylinder is

$$M_1 = 2\pi\int_0^{\ell_1}\varrho\ell\ d\ell = 2\pi\varrho_0\alpha^2\int_0^{\Xi_1}\Theta^n\xi\ d\xi = 2\pi\varrho_0\alpha^2[-\xi(d\Theta/d\xi)_{\xi=\Xi_1} + \beta\Xi_1^2/2]$$
$$= \pi\varrho_m\ell_1^2 = \pi\varrho_m\alpha^2\Xi_1^2, \qquad (0\le n<\infty;\ \beta = \mathrm{const}). \quad (3.9.11)$$

ϱ_m denotes the mean density, and we have used the fundamental equation (3.9.8). For critically rotating cylinders Eq. (3.9.11) yields with the condition of critical rotation $(d\Theta/d\xi)_{\xi=\Xi_{c1}} = 0$, as outlined in Eq. (3.9.10):

$$\varrho_0\beta_c/\varrho_m = \Omega_c^2/2\pi G\varrho_m = \beta_{cm} = 1. \quad (3.9.12)$$

As results from the numerical tables of Robe (1968b), the mass of the rotating cylinder is always larger than the mass of the nonrotating one, provided the central densities are the same (cf. Eq. (3.2.58) for the spheroidal case). If we consider two cylinders of equal mass, the central density of the nonrotating cylinder is larger.

Like in the nonrotating case we can find exact analytical solutions in the particular cases $n=0$ and $n=1$. If $n=0$, we get from a simple integration of Eq. (3.9.8), $(\Theta(0)=1;\ \Theta'(0)=0;\ \varrho=\varrho_0)$:

$$\Theta = 1-(1-\beta)\xi^2/4 \qquad \text{or} \qquad P = P_0\Theta = P_0[1-\pi G\varrho_0^2\ell^2(1-\beta)/P_0] = \pi G\varrho_0^2(1-\beta)(\ell_1^2-\ell^2). \quad (3.9.13)$$

We have inserted for P and ℓ from Eqs. (3.9.3) and (3.9.6), respectively. The central pressure P_0 has been eliminated by putting in Eq. (3.9.13) $P(\ell_1) = 0$, where ℓ_1 is the surface value of ℓ : $P_0 = \pi G\varrho_0^2\ell_1^2(1-\beta)$.

For the critically rotating homogeneous cylinder ($n=0$) we have by virtue of Eq. (3.9.12): $\beta=\beta_c=\beta_{cm}=1$. Opposite to Robe's (1968b) findings the pressure within this particular cylinder stays constant, since from $\Theta = 1-(1-\beta)\xi^2/4$ we get $\Theta = P/P_0 = 1$ if $\beta\to\beta_c = 1$, at any finite distance $\ell=\alpha\xi$. Likewise, the boundary $\ell_1\propto\Xi_1 = 2/(1-\beta)^{1/2}$, $(\Theta=0)$ is attained at larger and larger distances if $\beta\to\beta_c=1$. In the limit an infinite radius is attained: $\ell_{c1}\propto\Xi_{c1}=\infty$.

Eq. (3.9.8) is linear and nonhomogeneous in the second particular case $n=1$, admitting the solution (cf. Eqs. (2.3.21), (2.3.83); Robe 1968b)

$$\Theta = (1-\beta)\ J_0(\xi) + \beta, \qquad (n=1). \quad (3.9.14)$$

The factor $1-\beta$ near the Bessel function J_0 appears in order to satisfy the expansion (3.9.9) near the origin, since $J_0 \approx 1-\xi^2/4$ [cf. Eq. (2.3.12)]. The critical value of β_c in the case $n=1$ is given by the two equations $\Theta(\Xi_{c1})=0$ and $(d\Theta/d\xi)_{\xi=\Xi_{c1}}=0$ [cf. Eq. (3.9.10)], corresponding to the first minimum of $J_0(\xi)$ at $\xi=\Xi_{c1}=3.82$, when $dJ_0/d\xi=0$ (Spiegel 1968).

Differentially rotating polytropic cylinders have been considered by Robe (1979), Simon et al. (1981), Veugelen (1985a), Schneider and Schmitz (1995). Assuming rotation laws of the form

$$\beta = a_0/(1+a_1\xi^2) \quad \text{and} \quad \beta=\sum_{k=0}^{\infty} b_{2k}\xi^{2k}, \qquad (a_0, b_0>0;\ a_1\geq 0), \tag{3.9.15}$$

there result equilibrium structures of cylinders with an angular velocity on the axis, that is larger than the critical break-up velocity of uniformly rotating cylinders from Table 3.9.1. The critical values quoted by Simon et al. (1981) in the ninth line of their Table for $n=1.5$ and $\beta_c=0.16$ are not listed in Table 3.9.1, because one obtains $\beta_{cm}\approx 0.7$ instead of 1 from Eq. (3.9.12).

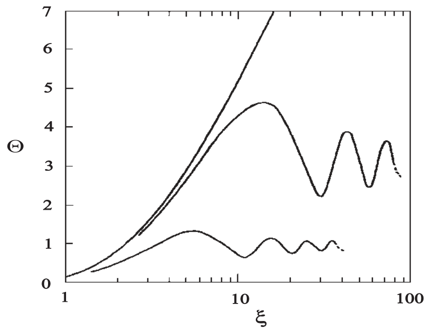

Fig. 3.9.1 The density parameter Θ of a rotating "isothermal" cylinder ($n=\pm\infty$) as a function of the dimensionless distance from the rotation axis ξ, and for rotation parameters $\beta=0$ (upper curve), $\beta=0.05$ (middle curve), and $\beta=0.4$ (lower curve). The upper curve ($\beta=0$) is identical to the analytic solution for the nonrotating cylinder from Eq. (2.3.48), (Hansen et al. 1976).

The rotating isothermal cylinders have been discussed by Hansen et al. (1976). We introduce the notations from Eqs. (2.1.15), (2.1.18), and (2.1.20):

$$P=K\varrho=K\varrho_0\exp(-\Theta)=P_0\exp(-\Theta); \quad \ell=(K/4\pi G\varrho_0)^{1/2}\xi=\alpha\xi;$$
$$z=(K/4\pi G\varrho_0)^{1/2}\zeta=\alpha\zeta, \qquad (n=\pm\infty). \tag{3.9.16}$$

The prime integral of the hydrostatic equation (3.9.1) writes as

$$K\ln\varrho = K\ln\varrho_0-\Theta=\Phi+\int\Omega^2(\ell)\ \ell\ d\ell+\text{const}, \qquad (n=\pm\infty), \tag{3.9.17}$$

and the isothermal equivalent of Eq. (3.9.7) becomes

$$(1/\xi)\ \partial(\xi\ \partial\Theta/\partial\xi)/\partial\xi+(1/\xi^2)\ \partial^2\Theta/\partial\varphi^2+\partial^2\Theta/\partial\zeta^2=\exp(-\Theta)-(1/2\xi)\ \partial[\xi^2\beta(\xi)]/\partial\xi. \tag{3.9.18}$$

If $\beta=\text{const}$ and $\Theta=\Theta(\xi)$, we get analogously to Eq. (3.9.8):

$$(1/\xi)\ d(\xi\ d\Theta/d\xi)/d\xi=\exp(-\Theta)-\beta, \qquad (\beta=\text{const};\ n=\pm\infty). \tag{3.9.19}$$

The equivalent of Eq. (3.9.11) for the case $n=\pm\infty$ writes

$$M_1=2\pi\int_0^{\ell_1}\varrho\ell\ d\ell=2\pi\varrho_0\alpha^2\int_0^{\Xi_1}\exp(-\Theta)\ \xi\ d\xi=2\pi\varrho_0\alpha^2[\xi\ (d\Theta/d\xi)_{\xi=\Xi_1}+\beta\Xi_1^2/2]$$
$$=(K/2G)[\xi\ (d\Theta/d\xi)_{\xi=\Xi_1}+\beta\Xi_1^2/2]. \tag{3.9.20}$$

The numerical integration of Eq. (3.9.19) for the cases $\beta = 0.05$ and 0.4 exhibits oscillations of Θ, and consequently of the density; for instance, if $\beta = 0.05$, the first density inversion occurs at $\xi = 14$ (Fig. 3.9.1). As it is to be expected on general grounds, these density inversions give rise to dynamically unstable density perturbations, to be touched in Sec. 5.9.2 (Hansen et al. 1976). The density inversions seem to appear because undistorted isothermal cylinders have finite mass, but infinite extension (Sec. 2.6.8). The structure of the rotating isothermal cylinders is one of concentric rings of varying density, and Hansen et al. (1976) invoke an artificial external pressure in order to prevent the configuration from expanding beyond a certain radius.

3.9.2 Polytropic Rings

Rings with infinitely large major radius R_1 resemble cylinders, and are therefore treated within this context (see also Secs. 3.8.4, 6.4.3). To be analytically tractable the rings should be slender, i.e. the minor radius r_1 of the ring (half-thickness of the ring) has to be much lower than its major radius $R_1 : r_1 \ll R_1$ (Fig. 3.9.2). Also, the rotation of ring material round the central axis Oz is assumed to be uniform: $\Omega = \mathrm{const}$.

Ostriker's (1964b) perturbation method for the study of polytropic rings is similar to Chandrasekhar's (1933a, d) first order theory of polytropic spheroids from Sec. 3.2. Let us consider a Cartesian frame $Oxyz$ in the central point of the ring, and let us denote by R_1 the distance between O and the point O' of *maximum central pressure* P_0 inside the ring (Fig. 3.9.2). A right-handed polar system (r, λ, φ) is defined, which should not be confused with a spherical coordinate system. The radial coordinate r is measured from O', and the polar angle φ with its pole in O' is in the vertical cross-section of the ring. The angle λ with its pole located in O is taken in the equatorial (x, y)-plane of the ring. The minor radius of the ring is denoted by r_1, and is equal to the radius of the circular cross-section of the ring. As will be obvious from Eqs. (3.9.84)-(3.9.89), this circular cross-section is displaced outwards by the first order amount from Eq. (3.9.87) with respect to the point of maximum pressure O'.

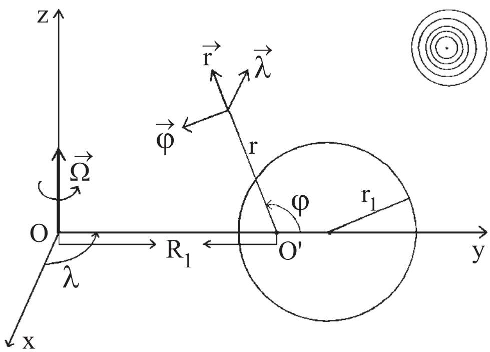

Fig. 3.9.2 Geometry of polytropic rings along a cross-section in the (y, z)-plane, $(\lambda = \pi/2)$. The origin O' of the radial r-coordinate is located in the point of maximum pressure P_0 inside the ring, and is displaced inward with respect to the circular cross-section of the ring by the amount $bf_1(\xi_1)/|\theta'(\xi_1)|$, [Eq. (3.9.87)]. Equidensity surfaces are shown on the upper right $(\varrho/\varrho_0 = 1, 0.8, 0.6, 0.4, 0.2, 0)$ for a ring with polytropic index $n = 1.5$ and $b = r_1/\xi_1 R_1 = 0.1$ (Ostriker 1964b).

From Fig. 3.9.2 we realize that the relationship between the inertial (x, y, z) coordinate system and the (r, λ, φ)-frame is

$$x = (R_1 + r\cos\varphi)\cos\lambda; \quad y = (R_1 + r\cos\varphi)\sin\lambda; \quad z = r\sin\varphi. \tag{3.9.21}$$

The line elements along the (r, λ, φ)-axes are simply $dr,\ (R_1+r\cos\varphi)\ d\lambda,\ r\ d\varphi$, and the corresponding metric is given by [cf. Eq. (B.4)]

$$ds^2 = dr^2 + (R_1 + r\cos\varphi)^2\ d\lambda^2 + r^2\ d\varphi^2. \tag{3.9.22}$$

In virtue of Eq. (B.21) the Laplace equation writes for this line element as

$$\nabla^2\Phi_e = \partial^2\Phi_e/\partial r^2 + [(R_1 + 2r\cos\varphi)/r(R_1 + r\cos\varphi)]\ \partial\Phi_e/\partial r + [1/(R_1 + r\cos\varphi)^2]\ \partial^2\Phi_e/\partial\lambda^2$$
$$+(1/r^2)\ \partial^2\Phi_e/\partial\varphi^2 - [\sin\varphi/r(R_1 + r\cos\varphi)]\ \partial\Phi_e/\partial\varphi = 0. \tag{3.9.23}$$

We introduce the dimensionless radial coordinate ξ from Eq. (3.2.1)

$$r = \alpha\xi;\quad \alpha = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2} = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\quad \text{if}\quad 0 \le n < \infty,\quad \text{and}$$
$$\alpha = (K/4\pi G\varrho_0)^{1/2} = (P_0/4\pi G\varrho_0^2)^{1/2}\quad \text{if}\quad n = \pm\infty, \tag{3.9.24}$$

and the parameter

$$b = \alpha/R_1 = r_1/\xi_1 R_1,\qquad (b \ll 1), \tag{3.9.25}$$

where $b \ll 1$, according to the initial assumption of a slender ring. Note, that in Eq. (3.9.25) we have used the fact that the minor radius of the ring r_1 is just equal to the radius of the undistorted polytropic cylinder $\alpha\xi_1$, as will be obvious from Eqs. (3.9.84)-(3.9.89).

Before proceeding to the evaluation of the fundamental function Θ inside the ring, we calculate the external potential of a thin hoop, and the external potential Φ_e of a slender ring. With the notations from Eqs. (3.9.24), (3.9.25), the Laplace equation (3.9.23) becomes up to the first order in b :

$$\nabla^2\Phi_e = \partial^2\Phi_e/\partial\xi^2 + (1/\xi)\ \partial\Phi_e/\partial\xi + (1/\xi^2)\ \partial^2\Phi_e/\partial\varphi^2 + b[\cos\varphi\ \partial\Phi_e/\partial\xi - (\sin\varphi/\xi)\ \partial\Phi_e/\partial\varphi]$$
$$+O(b^2) = 0,\qquad (b = \alpha/R_1 \ll 1). \tag{3.9.26}$$

The expansion of the potential Φ_e in terms of b is simply $\Phi_e = \Phi_{e0} + b\Phi_{e1} + ...$ We insert this expansion into Eq. (3.9.26), equating to zero equal powers of b, so that Eq. (3.9.26) splits into the following two parts:

$$\partial^2\Phi_{e0}/\partial\xi^2 + (1/\xi)\ \partial\Phi_{e0}/\partial\xi + (1/\xi^2)\ \partial^2\Phi_{e0}/\partial\varphi^2 = 0, \tag{3.9.27}$$

$$\cos\varphi\ \partial\Phi_{e0}/\partial\xi - (\sin\varphi/\xi)\ \partial\Phi_{e0}/\partial\varphi + \partial^2\Phi_{e1}/\partial\xi^2 + (1/\xi)\ \partial\Phi_{e1}/\partial\xi + (1/\xi^2)\ \partial^2\Phi_{e1}/\partial\varphi^2 = 0. \tag{3.9.28}$$

We apply Fourier's method of separation of variables to Eq. (3.9.27), seeking solutions under the form $\Phi_{e0}(\xi,\varphi) = u(\xi)\ v(\varphi)$. Eq. (3.9.27) becomes

$$[\xi^2 u''(\xi) + \xi u'(\xi)]/u(\xi) = -v''(\varphi)/v(\varphi) = j^2 = \text{const}. \tag{3.9.29}$$

The two sides of this equation must be equal to the constant j^2, as they are merely functions of a single independent variable ξ or φ, respectively. The first equation in (3.9.29) is an Euler equation with the general solution $u = A_j\xi^{-j} + B_j\xi^j$ if $j \ne 0$, and $u = A_0\ln\xi + B_0$ if $j = 0$. The second equation is an equation with constant coefficients, having the solution $v = C_j\sin j\varphi + D_j\cos j\varphi$ if $j \ne 0$, and $C_0\varphi + D_0$ if $j = 0$. Obviously, j^2 cannot be negative, because the resulting solution would be $v = C_j\exp[(-j^2)^{1/2}\varphi] + D_j\exp[-(-j^2)^{1/2}\varphi]$, which has no period of 2π, as required by the angular coordinate φ. The general solution of Eq. (3.9.27) is therefore

$$\Phi_{e0} = (A_0\ln\xi + B_0)(C_0\varphi + D_0) + \sum_{j=1}^{\infty}(A_j\xi^{-j} + B_j\xi^j)(C_j\sin j\varphi + D_j\cos j\varphi),$$
$$(A_0, B_0, C_0, D_0, A_j, B_j, C_j, D_j = \text{const}). \tag{3.9.30}$$

The external potential of the ring must be symmetrical with respect to the (x,y)-plane, i.e. $C_0, C_j = 0$. Also, the potential has to be a periodic function with a period of 2π : $\cos j\varphi = \cos j(\varphi + 2\pi)$. This implies that j is an integer, where we can restrict ourselves to positive integers, because the constants D_j are arbitrary, and $\cos(-j\varphi) = \cos j\varphi$. Thus, the pertinent form of Eq. (3.9.30) is

$$\Phi_{e0} = a_0 \ln\xi + b_0 + \sum_{j=1}^{\infty}(a_j\xi^{-j} + b_j\xi^j)\cos j\varphi, \qquad (a_0, b_0, a_j, b_j = \mathrm{const}). \tag{3.9.31}$$

We now insert Eq. (3.9.31) into Eq. (3.9.28), and get

$$a_0\xi\cos\varphi + \sum_{j=1}^{\infty} j\{-a_j\xi^{-j+1}\cos[(j+1)\varphi] + b_j\xi^{j+1}\cos[(j-1)\varphi]\}$$
$$+\xi\ \partial(\xi\ \partial\Phi_{e1}/\partial\xi)/\partial\xi + \partial^2\Phi_{e1}/\partial\varphi^2 = 0. \tag{3.9.32}$$

The general solution Φ_{e1} of this nonhomogeneous equation is the sum of the general solution of the homogeneous equation $\nabla^2\Phi_{e1} = 0$ – which is of the form (3.9.31) – and a particular integral of the nonhomogeneous equation. The form of the nonhomogeneous terms suggests to seek a particular solution under the form (Ostriker 1964b):

$$y = A_0\xi\ln\xi\cos\varphi + \sum_{j=1}^{\infty}\{A_j\xi^{-j+1}\cos[(j+1)\varphi] + B_j\xi^{j+1}\cos[(j-1)\varphi]\}, \qquad (A_0, A_j, B_j = \mathrm{const}). \tag{3.9.33}$$

If we insert this attempt into Eq. (3.9.32), we find $A_0 = -a_0/2,\ A_j = -a_j/4,\ B_j = -b_j/4$, and the general solution of Eq. (3.9.32) is

$$\Phi_{e1} = -(a_0/2)\xi\ln\xi\cos\varphi - (1/4)\sum_{j=1}^{\infty}\{a_j\xi^{-j+1}\cos[(j+1)\varphi] + b_j\xi^{j+1}\cos[(j-1)\varphi]\}$$
$$+c_0\ln\xi + d_0 + \sum_{j=1}^{\infty}(c_j\xi^{-j} + d_j\xi^j)\cos j\varphi, \qquad (c_0, d_0, c_j, d_j = \mathrm{const}). \tag{3.9.34}$$

In order to apply the previous results to rings, it will be useful to calculate, as a preparation, the external potential Φ_{eH} of a thin hoop, located along the ring's major circle of radius R_1 in the (x, y)-plane of Fig. 3.9.2. The radius vector $\vec{r'}$ along the hoop has therefore the coordinates $\vec{r'}(0, \lambda', 0)$ in the (r, λ, φ)-system. The radius vector $\vec{r}$ in the observation point of the potential can be taken – because of symmetry reasons – in the plane $\lambda = 0$, and therefore $\vec{r} = \vec{r}(r, 0, \varphi)$. The external potential of the point $(r, 0, \varphi)$ is [cf. Eq. (3.1.47)]

$$\Phi_{eH}(\vec{r}) = G\int_{V_H}\varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|, \qquad [\vec{r} = \vec{r}(r, 0, \varphi);\ \vec{r'} = \vec{r'}(0, \lambda', 0)]. \tag{3.9.35}$$

In virtue of Eq. (3.9.21) the distance $|\vec{r} - \vec{r'}|$ becomes

$$(r - r')^2 = (R_1 + r\cos\varphi - R_1\cos\lambda')^2 + R_1^2\sin^2\lambda' + r^2\sin^2\varphi = 4R_1(R_1 + r\cos\varphi)\sin^2(\lambda'/2) + r^2. \tag{3.9.36}$$

The "volume" element of the hoop is equal to $R_1\ d\lambda'$, and its density $\varrho(\vec{r'})$ is just the constant surface density Σ of the hoop. Thus

$$\Phi_{eH}(\vec{r}) = G\Sigma R_1\int_0^{2\pi} d\lambda'/[4R_1(R_1 + r\cos\varphi)\sin^2(\lambda'/2) + r^2]^{1/2}$$
$$= (4G\Sigma R_1/r)\int_0^{\pi/2} d\gamma/[1 + 4R_1(R_1 + r\cos\varphi)\sin^2\gamma/r^2]^{1/2}, \qquad (\gamma = \lambda'/2). \tag{3.9.37}$$

With the notation $s^2 = 4R_1(R_1 + r\cos\varphi)/r^2$, the integral from Eq. (3.9.37) becomes

$$\int_0^{\pi/2} d\gamma/(1 + s^2\sin^2\gamma)^{1/2} = \int_0^{\pi/2} d\gamma/[1 + s^2\cos^2(\pi/2 - \gamma)]^{1/2} = \int_0^{\pi/2} d\gamma/(1 + s^2\cos^2\gamma)^{1/2}$$
$$= (1 + s^2)^{-1/2}\int_0^{\pi/2} d\gamma/\{1 - [s^2/(1 + s^2)]\sin^2\gamma\}^{1/2} = (1 + s^2)^{-1/2}K[s/(1 + s^2)^{1/2}] = k'K(k), \tag{3.9.38}$$

which constitutes an elliptic integral $K(k)$ of the first kind with the modulus $k = s/(1+s^2)^{1/2}$, and with the complementary modulus $k' = (1-k^2)^{1/2} = 1/(1+s^2)^{1/2}$ (Abramowitz and Stegun 1965). If we assume that $|\vec{r}| = r \ll R_1$, then $s \gg 1$, and therefore $k \approx 1,\ k' \approx 0$. Tedious calculations involving the theory of theta functions lead to the following approximation for the elliptic integral if $k \approx 1$ or $k' \approx 0$ (e.g. Tölke 1966):

$$K(k) = \int_0^{\pi/2} d\gamma/(1-k^2\sin^2\gamma)^{1/2} = \int_0^1 d\nu/(1-\nu^2)^{1/2}(1-k^2\nu^2)^{1/2}$$
$$= \int_0^1 d\nu/(1-\nu^2)^{1/2}(1-\nu^2+k'^2\nu^2)^{1/2} \approx \ln(4/k') + O[(k'^2/4)\ln(4/k')], \qquad (k' \approx 0;\ \nu = \sin\gamma). \tag{3.9.39}$$

Substituting into this approximation the quantities

$$k' = 1/(1+s^2)^{1/2} \approx 1/s = (r/2R_1)/[1+(r/R_1)\cos\varphi]^{1/2} \approx (r/2R_1)[1-(r/2R_1)\cos\varphi];$$
$$\ln(4/k') \approx \ln(8R_1/r) - \ln[1-(r/2R_1)\cos\varphi] \approx \ln(8R_1/r) + (r/2R_1)\cos\varphi, \tag{3.9.40}$$

the potential (3.9.37) of the hoop becomes eventually:

$$\Phi_{eH}(r,\varphi) \approx (4G\Sigma R_1/r)k'\ln(4/k') \approx 2G\Sigma\Big\{\ln(8R_1/r) + [1-\ln(8R_1/r)](r/2R_1)\cos\varphi$$
$$+O[(r/R_1)^2\ln(8R_1/r)]\Big\}. \tag{3.9.41}$$

Noting that the mass of the hoop is $M_H = 2\pi R_1\Sigma$, and transforming to the dimensionless variables from Eqs. (3.9.24), (3.9.25), we find

$$\Phi_{eH}(\xi,\varphi) = (GM_H/\pi R_1)\{\ln(8/b) - \ln\xi + (b\xi/2)[1-\ln(8/b)+\ln\xi]\cos\varphi + O(b^2)\},$$
$$(r \ll R_1;\ b \ll 1). \tag{3.9.42}$$

To obtain the external potential of a slender ring, we match this equation with the previously obtained external potentials Φ_{e0} and Φ_{e1} from Eqs. (3.9.31) and (3.9.34), respectively. The matching is effected in a region far enough from the ring at $r \gg r_1$, where the external potential of the ring approaches that of an equivalent hoop, but still near enough, so that $r \ll R_1$, where the previous expansions are valid: $r_1 \ll r \ll R_1$ or $\xi_1 \ll \xi \ll 1/b$. Eq. (3.9.42) requires the angular dependence of Φ_{e0} to vanish, i.e. $a_j, b_j = 0,\ (j \geq 1)$ in Eq. (3.9.31). Also, c_0 and d_0 from Eq. (3.9.34) have to vanish, as the coefficient of b in Eq. (3.9.42) depends on φ. The nonvanishing term $d_1\xi\cos\varphi,\ (\xi \gg \xi_1)$ from Eq. (3.9.34) is seen to be equal to $(\xi/2)[1-\ln(8/b)]\cos\varphi$, whereas the terms $d_j\xi^j\cos j\varphi,\ (j \geq 2;\ \xi \gg \xi_1)$ must be absent, by comparing with Eq. (3.9.42). Only the terms $c_j\xi^{-j}\cos j\varphi$, that vanish if $\xi \gg \xi_1$, may appear in the final matched expansion of the external potential of a slender ring of mass $M_R,\ (M_H \to M_R)$:

$$\Phi_e(\xi,\varphi) = (GM_R/\pi R_1)\Big\{\ln(8/b) - \ln\xi + (b\xi/2)[1-\ln(8/b)+\ln\xi]\cos\varphi$$
$$+b\sum_{j=1}^{\infty} c_j\xi^{-j}\cos j\varphi + O(b^2)\Big\}, \qquad (b = \alpha/R_1 = r_1/\xi_1 R_1 \ll 1). \tag{3.9.43}$$

With the expansion coefficients thus fixed, the previous equation is applicable everywhere outside the ring. For the perturbation analysis a relationship between the angular velocity Ω of ring material and the dimensions of the ring (characterized by the parameter b) is absolutely necessary (Ostriker 1964b). Such a relationship can be obtained from the virial theorem. If magnetic energy U_m and external pressure forces P_{jk} are zero, the virial equation (2.6.94) takes the simple form

$$2E_{kin} + W + 3(\Gamma-1)U = 0, \tag{3.9.44}$$

where the moment of inertia I is constant for our hydrostatic rings. In a zero order approximation the mass of the ring M_R, the kinetic energy of rotation E_{kin}, the gravitational energy W, and the internal energy U are simply $2\pi R_1$ times their values per unit length of an infinitely long cylinder. As the velocity of rotation of ring material round O is in a first approximation $v_{rot} = \Omega R_1$, the kinetic energy of rotation is

$$E_{kin} = \Omega^2 M_R R_1^2/2. \tag{3.9.45}$$

The evaluation of the gravitational energy of the ring needs some effort, as we have to start with the original equation (2.6.68), bearing in mind our comments subsequent to Eq. (2.6.137). We get

$$W = -\pi R_1 \int_{M_1} \Phi\ dM = -2\pi^2 R_1 (n+1) K \varrho_0^{1+1/n} \alpha^2 \Big\{ [\Phi_0/(n+1)K\varrho_0^{1/n} - 1] \int_0^{\xi_1} \theta^n \xi\ d\xi$$
$$+ \int_0^{\xi_1} \theta^{n+1} \xi\ d\xi \Big\}, \qquad (dM = 2\pi\varrho r\ dr = 2\pi\varrho_0\alpha^2\xi\theta^n\ d\xi), \tag{3.9.46}$$

where we have inserted for the internal potential Φ from Eq. (2.6.32), and M_1 is the mass per unit length of the undistorted cylinder. The two integrals in Eq. (3.9.46) have already been solved in Eqs. (2.6.16) and (2.6.159), respectively. Thus

$$W = -2\pi^2 R_1 (n+1) K \varrho_0^{1+1/n} \alpha^2 \xi_1^2 \theta_1'^2 \{[\Phi_0/(n+1)K\varrho_0^{1/n} - 1]/(-\xi_1\theta_1') + (n+1)/4\}. \tag{3.9.47}$$

The value Φ_0 of the internal potential along the central circle $\xi = 0$ of the ring can be fixed in our zero order approximation by using the results obtained for the external potential of undistorted cylinders in Eq. (2.6.48):

$$\Phi_e = \Phi_0 + (n+1)K\varrho_0^{1/n}[\xi_1\theta_1' \ln(\xi/\xi_1) - 1]. \tag{3.9.48}$$

This value of the external potential per unit length of a cylinder is required to be equal to the zero order approximation of the external potential far from the ring, as given by Eq. (3.9.43):

$$(n+1)K\varrho_0^{1/n}[\Phi_0/(n+1)K\varrho_0^{1/n} - 1 + \xi_1\theta_1' \ln(\xi/\xi_1)] = (GM_R/\pi R_1)\ln(8/b\xi). \tag{3.9.49}$$

Inserting for the mass M_R of the ring $2\pi R_1$ times its value from Eq. (2.6.16), i.e. $M_R = 2\pi R_1 M_1 = -4\pi^2 R_1 \varrho_0 \alpha^2 \xi_1 \theta_1'$, we find after obvious simplifications: $[\Phi_0/(n+1)K\varrho_0^{1/n} - 1]/(-\xi_1\theta_1') = \ln(8R_1/r_1)$. The gravitational energy (3.9.47) becomes eventually, after inserting for $(n+1)K\varrho_0^{1/n} = 4\pi G\varrho_0\alpha^2$:

$$W = -(GM_R^2/2\pi R_1)[\ln(8R_1/r_1) + (n+1)/4]. \tag{3.9.50}$$

If also a central point mass M_c exists, about which the ring rotates, its contribution W_c to the gravitational energy of the ring will be via Eqs. (2.6.69), (2.6.70) equal to

$$dW_c = -GM_c\ dM_R \int_{R_1}^{\infty} dr'/r'^2 = -GM_c\ dM_R/R_1; \quad W_c = -(GM_c/R_1)\int_{M_R} dM_R$$
$$= -GM_cM_R/R_1. \tag{3.9.51}$$

dM_R denotes the fraction of ring material that is brought by the attraction of the central mass M_c from infinity up to the major radius R_1 of the ring. Adding the contribution of the central mass M_c to the gravitational energy of the ring, we obtain the gravitational energy of a ring rotating round the central point mass M_c :

$$W = -(GM_R^2/2\pi R_1)[\ln(8R_1/r_1) + 2\pi M_c/M_R + (n+1)/4]$$
$$= -2\pi^2 R_1 \alpha^2 (n+1) K\varrho_0^{1+1/n} \xi_1^2 \theta'^2(\xi_1)[\ln(8/b\xi_1) + 2\pi M_c/M_R + (n+1)/4],$$
$$[M_R = -4\pi^2 R_1 \alpha^2 \varrho_0 \xi_1 \theta'(\xi_1)]. \tag{3.9.52}$$

The internal energy of ring material (the internal energy of the point mass M_c is zero) is in the zeroth approximation $2\pi R_1$ times its value from Eq. (2.6.167):

$$U = \pi G R_1 M_1^2/(\Gamma - 1) = GM_R^2/4\pi R_1(\Gamma - 1) = \pi^2 R_1 \alpha^2 (n+1) K\varrho_0^{1+1/n} \xi_1^2 \theta'^2(\xi_1)/(\Gamma - 1),$$
$$(M_1 = M_R/2\pi R_1). \tag{3.9.53}$$

Inserting Eqs. (3.9.45), (3.9.52), (3.9.53) into Eq. (3.9.44), we find

$$\beta = \Omega^2/2\pi G\varrho_0 = (r_1/R_1)^2(\varrho_m/2\varrho_0)[\ln(8R_1/r_1) + 2\pi M_c/M_R + (n-5)/4], \tag{3.9.54}$$

which shows that the dimensionless rotation parameter β is of order (Ostriker 1964b)

$$\beta = O(b^2 \ln b,\ b^2 M_c/M_R) \ll 1, \tag{3.9.55}$$

because $O(r_1/R_1) = O(b\xi_1) \approx O(b)$.

From Eq. (3.1.89) we can deduce at once the equation of hydrostatic equilibrium of the ring

$$\nabla P = \varrho\, \nabla[\Phi + \Phi_c + \Omega^2\ell^2/2], \qquad (\ell^2 = x^2 + y^2), \tag{3.9.56}$$

where we have equated to zero the velocity components v_k of internal mass motions in the rotating ring system, and have denoted by $\Phi_c = GM_c/(x^2+y^2+z^2)^{1/2}$ the potential of the central point mass M_c. With the polytropic equation of state $P = K\varrho^{1+1/n}$ this equation becomes

$$\nabla[-(n+1)K\varrho^{1/n} + \Phi + \Phi_c + \Omega^2\ell^2/2] = 0. \tag{3.9.57}$$

Taking the divergence of Eq. (3.9.57), we get with the Poisson and Laplace equation of the gravitational potential ($\nabla^2\Phi = -4\pi G\varrho;\ \nabla^2\Phi_c = 0;\ \nabla^2\ell^2 = 4$)

$$(n+1)K\ \nabla^2\varrho^{1/n} = -4\pi G\varrho + 2\Omega^2. \tag{3.9.58}$$

Turning to the dimensionless variables from Eqs. (3.9.3), (3.9.6), (3.9.24), we obtain the hydrostatic equilibrium equation of a ring rotating about a central mass under the same form as for a rotationally distorted sphere [Eqs. (3.2.2), (3.4.5)]:

$$\nabla^2\Theta = -\Theta^n + \beta, \qquad (0 \le n < \infty). \tag{3.9.59}$$

Alternately, Eq. (3.9.57) may be integrated from the central circle $\ell = R_1$ up to the observation point (r,φ,λ) :

$$\Phi = \Phi_0 - \Phi_c + \Phi_{c0} + (n+1)K(\varrho^{1/n} - \varrho_0^{1/n}) + \Omega^2(R_1^2 - \ell^2)/2. \tag{3.9.60}$$

The zero subscript designates values along the central circle. The external potential of the central point mass is simply

$$\begin{aligned}\Phi_c &= GM_c/(x^2+y^2+z^2)^{1/2} = GM_c/(R_1^2 + r^2 + 2rR_1\cos\varphi)^{1/2}\\ &= (GM_c/R_1)[1 - r\cos\varphi/R_1 + (r/2R_1)^2(3\cos^2\varphi - 1) + ...],\end{aligned} \tag{3.9.61}$$

and Eq. (3.9.60) becomes

$$\begin{aligned}\Phi &= \Phi_0 + K(n+1)(\varrho^{1/n} - \varrho_0^{1/n}) + (r/R_1)(\Phi_{c0} - \Omega^2R_1^2)\cos\varphi - (r^2/4R_1^2)[\Phi_{c0}(1+3\cos 2\varphi)\\ &-\Omega^2R_1^2(1+\cos 2\varphi)] + ..., \qquad [\Phi_{c0} = GM_c/R_1;\ \ell^2 = x^2+y^2 = (R_1 + r\cos\varphi)^2].\end{aligned} \tag{3.9.62}$$

Turning to dimensionless coordinates, we recast:

$$\begin{aligned}\Phi &= \Phi_0 + (n+1)K\varrho_0^{1/n}\{\Theta - 1 + b\xi[\Phi_{c0}/(n+1)K\varrho_0^{1/n} - (\beta/2b^2)]\cos\varphi\\ &-(b^2\xi^2/4)[\Phi_{c0}(1+3\cos 2\varphi)/(n+1)K\varrho_0^{1/n} + (\beta/2b^2)(1+\cos 2\varphi)]\} + ...\end{aligned} \tag{3.9.63}$$

From the first order term we get the condition $\beta/b^2 \lesssim O(1)$, and by virtue of Eq. (3.9.55) this amounts to $M_c/M_R \lesssim O(1)$. Hence $M_c \lesssim M_R$, opposite to the case pertinent to Saturn's ring, for instance; the mass of our rings has to be larger or at least of the same order as the central mass M_c. Thus, the results of Laplace, Randers, and many others for rings with $M_c \gg M_R$ cannot be compared to the present findings (Ostriker 1964b).

Because the ring degenerates into an infinitely long cylinder if $b \to 0$, the zero order approximation of the polytropic variable $\Theta(\xi,\varphi)$ must be the cylindrical Lane-Emden function $\theta(\xi)$:

$$\Theta(\xi,\varphi) = \theta(\xi) + b\left[A_0f_0(\xi) + f_1(\xi)\cos\varphi + \sum_{j=2}^{\infty} A_jf_j(\xi)\cos j\varphi\right] + O(b^2), \qquad (0 \le n < \infty). \tag{3.9.64}$$

As symmetry with respect to the equatorial plane is required, sine terms are missing. We substitute the Laplace operator from Eq. (3.9.26) into the fundamental equation (3.9.59) to obtain

$$\partial^2\Theta/\partial\xi^2 + (1/\xi)\ \partial\Theta/\partial\xi + (1/\xi^2)\ \partial^2\Theta/\partial\varphi^2 + b[\cos\varphi\ \partial\Theta/\partial\xi - (\sin\varphi/\xi)\ \partial\Theta/\partial\varphi] = -\Theta^n + \beta,$$
$$[\beta \approx O(b^2)]. \qquad (3.9.65)$$

Inserting Eq. (3.9.64) into Eq. (3.9.65), and equating equal powers of $b\sin j\varphi$ and $b\cos j\varphi$, our problem splits into the following set of ordinary differential equations:

$$\theta'' + \theta'/\xi + \theta^n = 0; \quad f_1'' + f_1'/\xi + (n\theta^{n-1} - 1/\xi^2)f_1 = -\theta';$$
$$f_j'' + f_j'/\xi + (n\theta^{n-1} - j^2/\xi^2)f_j = 0, \qquad (j = 0, 2, 3, 4, ...;\ 0 \le n < \infty). \qquad (3.9.66)$$

The initial conditions are the same as in the spherical case, namely $\Theta(0,\varphi) = 1,\ (\partial\Theta/\partial\xi)_{\xi=0} = 0$. These conditions turn via Eq. (3.9.64) into the following initial conditions for θ and f_j :

$$\theta(0) = 1 \quad \text{and} \quad \theta'(0), f_j(0), f_j'(0) = 0, \qquad (0 \le n < \infty;\ j = 0, 1, 2, 3, ...). \qquad (3.9.67)$$

These initial conditions place the origin O' of the ξ or r-coordinate in the point of maximum pressure of the ring. The boundary condition $f_1(\xi_1) = 0,\ [f_1(\xi) = \xi^3/16 - \xi/4]$ adopted instead of $f_1'(0) = 0$ by Ostriker (1964b) for the particular case $n = 0$ is not consistent with the initial condition $(\partial\Theta/\partial\xi)_{\xi=0} = 0$, and amounts to a simple translation by the amount $b/2$ of the origin O' from the point of maximum pressure to the geometrical centre of the constant-density ring; we will expand more closely on this point subsequently to Eq. (3.9.95).

Closed solutions of $f_1(\xi)$ have been found by Ostriker (1964b) for the cases $n = 0$ and $n = 1$. If $n = 0$, Eq. (3.9.66) becomes

$$\xi^2 f_1'' + \xi f_1' - f_1 = \xi^3/2, \qquad (n = 0;\ \theta = 1 - \xi^2/4), \qquad (3.9.68)$$

with the elementary solution for this nonhomogeneous Euler equation $f_1 = A\xi + B/\xi + \xi^3/16$. The initial conditions $f_1(0) = f_1'(0) = 0$ yield

$$f_1 = \xi^3/16, \qquad (n = 0). \qquad (3.9.69)$$

Instead, Ostriker (1964b) obtains $f_1 = \xi^3/16 - \xi/4$ with his boundary conditions $f_1(0) = f_1(\xi_1) = 0$. If $n = 1$, Eq. (3.9.66) becomes

$$\xi^2 f_1'' + \xi f_1' + (\xi^2 - 1)f_1 = -\xi^2\ dJ_0(\xi)/d\xi = \xi^2 J_1(\xi), \qquad [n = 1;\ \theta = J_0(\xi)], \qquad (3.9.70)$$

where we have used the relationship $d[\xi^{-\nu}J_\nu(\xi)]/d\xi = -\xi^{-\nu}J_{\nu+1}(\xi)$ for the Bessel functions if $\nu = 0$ (e.g. Spiegel 1968). The general solution of the homogeneous part of Eq. (3.9.70) is $f_1(\xi) = C_1J_1(\xi) + C_2Y_1(\xi)$ via Eq. (2.3.11). Due to the boundary conditions $f_1(0) = f_1'(0) = 0$, both constants $C_1,\ C_2$ have to vanish, because $J_1 \approx \xi/2 - \xi^3/16$ if $\xi \approx 0$, and because the Neumann function Y_1 is unbounded if $\xi = 0$. We have to look for a particular solution of the nonhomogeneous equation (3.9.70). With the attempt $f_1 = A\xi J_2(\xi)$, and with the recurrence relationships $d[\xi^\nu J_\nu(\xi)]/d\xi = \xi^\nu J_{\nu-1}(\xi)$ and $J_{\nu+1}(\xi) = (2\nu/\xi)\ J_\nu(\xi) - J_{\nu-1}(\xi)$, we get $A = 1/2$, and

$$f_1(\xi) = (\xi/2)\ J_2(\xi), \qquad (n = 1). \qquad (3.9.71)$$

Analogously to Eq. (3.2.34) we expand the surface of the ring in powers of b and $\cos j\varphi$:

$$\Xi_1 = \Xi_1(\varphi) = \xi_1 + b\sum_{j=0}^{\infty} q_j \cos j\varphi, \qquad (q_j = \text{const}). \qquad (3.9.72)$$

On the surface we have

$$\Theta = \Theta(\Xi_1,\varphi) = \theta(\Xi_1) + b\left[A_0f_0(\Xi_1) + f_1(\Xi_1)\cos\varphi + \sum_{j=2}^{\infty} A_jf_j(\Xi_1)\cos j\varphi\right] = 0. \qquad (3.9.73)$$

Table 3.9.2 Values of $\xi_1, \theta'(\xi_1), f_1(\xi_1), f_1'(\xi_1), \varepsilon(\xi_1)$ according to Ostriker's (1964b, 1965) tables. ξ_1 and $\theta'(\xi_1)$ agree with corresponding values listed in Table 2.5.2. $a+b$ means $a\times 10^b$.

n	ξ_1	$\theta_1'(\xi_1)$	$f_1(\xi_1)$	$f_1'(\xi_1)$	$\varepsilon(\xi_1)$
0	2.0000+0	−1.0000+0	5.0000−1	7.5000−1	2.5000−1
1	2.4048+0	−5.1915−1	5.1915−1	4.0835−1	4.1583−1
1.5	2.6478+0	−4.0076−1	5.7636−1	4.4552−1	5.4316−1
2	2.9213+0	−3.1676−1	6.5466−1	4.6991−1	7.0748−1
3	3.5739+0	−2.0709−1	8.6837−1	4.9715−1	1.1733+0
4	4.3953+0	−1.4075−1	1.1604+0	5.0929−1	1.8757+0
5	5.4276+0	−9.8081−2	1.5427+0	5.1427−1	2.8980+0
6	6.7245+0	−6.9546−2	2.0351+0	5.1578−1	4.3515+0
7	8.3542+0	−4.9949−2	2.6645+0	5.1563−1	6.3854+0
8	1.0403+1	−3.6228−2	3.4667+0	5.1472−1	9.1987+0
10	1.6223+1	−1.9478−2	5.7865+0	5.1229−1	1.8312+1
12	2.5453+1	−1.0695−2	9.5382+0	5.1001−1	3.5037+1
16	6.3514+1	−3.3598−3	2.5416+1	5.0676−1	1.1910+2
20	1.6060+2	−1.0934−3	6.6973+1	5.0481−1	3.8142+2

Expanding all functions of Ξ_1 in the vicinity of ξ_1, we find

$$\Xi_1-\xi_1=b\sum_{j=0}^{\infty}q_j\cos j\varphi;\quad \theta(\Xi_1)=\theta(\xi_1)+(\Xi_1-\xi_1)\ \theta'(\xi_1)=b\theta'(\xi_1)\sum_{j=0}^{\infty}q_j\cos j\varphi+O(b^2);$$
$$f_j(\Xi_1)=f_j(\xi_1)+O(b),\qquad \big(\theta(\xi_1)=0\big).\tag{3.9.74}$$

Inserting into Eq. (3.9.73), we get

$$\Theta(\Xi_1,\varphi)=b\left[\theta'(\xi_1)\sum_{j=0}^{\infty}q_j\cos j\varphi+A_0f_0(\xi_1)+f_1(\xi_1)\cos\varphi+\sum_{j=2}^{\infty}A_jf_j(\xi_1)\cos j\varphi\right]+O(b^2)=0.\tag{3.9.75}$$

This expression is zero if all coefficients of equal $\cos j\varphi$ vanish simultaneously:

$$q_1=-f_1(\xi_1)/\theta'(\xi_1);\quad q_j=-A_jf_j(\xi_1)/\theta'(\xi_1),\qquad (j=0,2,3,4,...).\tag{3.9.76}$$

The internal potential from Eq. (3.9.63) takes on the ring's surface the value

$$\Phi(\Xi_1,\varphi)=\Phi_{00}-(n+1)K\varrho_0^{1/n}+b\Phi_{01}+b(n+1)K\varrho_0^{1/n}\left[\theta'(\xi_1)\sum_{j=0}^{\infty}q_j\cos j\varphi+A_0f_0(\xi_1)\right.$$
$$\left.+f_1(\xi_1)\cos\varphi+\sum_{j=2}^{\infty}A_jf_j(\xi_1)\cos j\varphi-(\beta/2b^2)\xi_1\cos\varphi\right]+O(b^2),\tag{3.9.77}$$

where we have split Φ_0 into its zeroth and first order components: $\Phi_0=\Phi_{00}+b\Phi_{01}$. We have also neglected, for the moment, the influence of the central mass ($\Phi_c=0$), and have expanded $\Theta(\Xi_1,\varphi)$ via Eq. (3.9.75). The derivative of Eq. (3.9.63) on the boundary is equal to

$$[\partial\Phi(\xi,\varphi)/\partial\xi]_{\xi=\Xi_1}=(n+1)K\varrho_0^{1/n}\theta'(\xi_1)+b(n+1)K\varrho_0^{1/n}\left\{[-\theta'(\xi_1)/\xi_1-\theta^n(\xi_1)]\sum_{j=0}^{\infty}q_j\cos j\varphi\right.$$
$$\left.+A_0f_0'(\xi_1)+f_1'(\xi_1)\cos\varphi+\sum_{j=2}^{\infty}A_jf_j'(\xi_1)\cos j\varphi-(\beta/2b^2)\cos\varphi\right\}+O(b^2).\tag{3.9.78}$$

We have used the cylindrical Lane-Emden equation $\theta''=-\theta'/\xi-\theta^n$, ($\theta^n(\xi_1)=0$ if $n>0$), and the derivative of Eq. (3.9.64):

$$[\partial\Theta(\xi,\varphi)/\partial\xi]_{\xi=\Xi_1}=\theta'(\xi_1)+b\left\{[-\theta'(\xi_1)/\xi_1-\theta^n(\xi_1)]\sum_{j=0}^{\infty}q_j\cos j\varphi\right.$$
$$\left.+A_0f_0'(\xi_1)+f_1'(\xi_1)\cos\varphi+\sum_{j=2}^{\infty}A_jf_j'(\xi_1)\cos j\varphi\right\}.\tag{3.9.79}$$

The external potential (3.9.43) on the ring's surface becomes equal to

$$\Phi_e(\Xi_1,\varphi) = (GM_R/\pi R_1)\Big\{\ln(8/b\xi_1) + b\Big[-(1/\xi_1)\sum_{j=0}^{\infty} q_j\cos j\varphi + (\xi_1/2)[1-\ln(8/b\xi_1)]\cos\varphi$$
$$+\sum_{j=1}^{\infty} c_j\xi_1^{-j}\cos j\varphi\Big]\Big\} + O(b^2), \tag{3.9.80}$$

$$[\partial\Phi_e(\xi,\varphi)/\partial\xi]_{\xi=\Xi_1} = (GM_R/\pi R_1)\Big\{-1/\xi_1 + b\Big[(1/\xi_1^2)\sum_{j=0}^{\infty} q_j\cos j\varphi + (1/2)[2-\ln(8/b\xi_1)]\cos\varphi$$
$$-\sum_{j=1}^{\infty} jc_j\xi_1^{-j-1}\cos j\varphi\Big]\Big\} + O(b^2). \tag{3.9.81}$$

The internal and external potential has to be continuous across the boundary of the ring, so we can equate equal terms on the right-hand sides of Eqs. (3.9.77), (3.9.80), and of Eqs. (3.9.78), (3.9.81), respectively:

$$\Phi_{00}/(n+1)K\varrho_0^{1/n} = 1 + [GM_R/\pi R_1(n+1)K\varrho_0^{1/n}]\ln(8/b\xi_1) = 1 - \xi_1\theta'(\xi_1)\ln(8R_1/r_1);$$
$$\Phi_{01} = 0;\quad f_1(\xi_1) - (\beta/2b^2)\xi_1 = -c_1\theta'(\xi_1) - [\xi_1^2\theta'(\xi_1)/2][1-\ln(8R_1/r_1)];$$
$$f_1'(\xi_1) - \beta/2b^2 = c_1\theta'(\xi_1)/\xi_1 - [\xi_1\theta'(\xi_1)/2][2-\ln(8R_1/r_1)];\quad A_jf_j(\xi_1) = -c_j\xi_1^{-j+1}\theta'(\xi_1);$$
$$A_jf_j'(\xi_1) = jc_j\xi_1^{-j}\theta'(\xi_1),\qquad (0<n<\infty;\ j=0,2,3,4,...). \tag{3.9.82}$$

The unknowns $\beta/2b^2, A_j, c_1, c_j$ can be found at once from these equations:

$$\beta = b^2\{\xi_1\theta'(\xi_1)[3/2-\ln(8R_1/r_1)] + f_1(\xi_1)/\xi_1 + f_1'(\xi_1)\};\quad c_1 = \xi_1^2/4 + [1/2\theta'(\xi_1)]$$
$$\times[-f_1(\xi_1) + \xi_1f_1'(\xi_1)];\quad A_j[jf_j(\xi_1) + \xi_1f_j'(\xi_1)] = 0,\qquad (0<n<\infty;\ j=0,2,3,4,...). \tag{3.9.83}$$

Since $jf_j(\xi_1) + \xi_1f_j'(\xi_1) \neq 0$ – analogously to the comment after Eq. (3.2.43) – we are left with the alternative $A_j = 0$. From Eqs. (3.9.76), (3.9.82) follows at once that $c_j, q_j = 0$ if $j = 0,2,3,4,...$

Thus, the fundamental function (3.9.64) and the surface coordinate (3.9.72) take the elementary form

$$\Theta(\xi,\varphi) = \theta(\xi) + bf_1(\xi)\cos\varphi;\quad \Xi_1(\varphi) = \xi_1 - b[f_1(\xi_1)/\theta'(\xi_1)]\cos\varphi. \tag{3.9.84}$$

We define an isobaric surface ($P = \mathrm{const}$) by an equation similar to the surface equation (3.9.72):

$$\Xi = \Xi(\xi,\varphi) = \xi + b\sum_{j=0}^{\infty} q_j(\xi)\cos j\varphi. \tag{3.9.85}$$

On an isobaric surface the fundamental function (3.9.84) is constant:

$$\Theta(\Xi,\varphi) = \theta(\Xi) + bf_1(\Xi)\cos\varphi \approx \theta(\xi) + (\Xi-\xi)\ \theta'(\xi) + bf_1(\xi)\cos\varphi$$
$$= \theta(\xi) + b\Big[\theta'(\xi)\sum_{j=0}^{\infty} q_j(\xi)\cos j\varphi + f_1(\xi)\cos\varphi\Big] = \mathrm{const}. \tag{3.9.86}$$

Equating equal powers of b and equal $\cos j\varphi$-terms, we get const $= \theta(\xi)$, $q_j = 0$, $(j = 0,2,3,4,...)$, and $q_1(\xi) = -f_1(\xi)/\theta'(\xi)$. Isobaric surfaces obey therefore the equation

$$\Xi = \Xi(\xi,\varphi) = \xi - b[f_1(\xi)/\theta'(\xi)]\cos\varphi + O(b^2). \tag{3.9.87}$$

But this is just the first order equation of a circle of radius ξ, whose centre is displaced outward from the origin of coordinates by the small amount $bf_1(\xi)/|\theta'(\xi)|$. Indeed, the equation of a circle of radius ξ in polar (Ξ,φ)-coordinates, with its centre displaced outwards from the origin O' by the amount $bf_1/|\theta'|$, can be approximated as

$$\xi = [\Xi^2 + (bf_1/\theta')^2 - (2\Xi bf_1/|\theta'|)\cos\varphi]^{1/2} \approx \Xi - (bf_1/|\theta'|)\cos\varphi = \Xi + (bf_1/\theta')\cos\varphi,\qquad (\theta'\leq 0), \tag{3.9.88}$$

which is just identical to Eq. (3.9.87). The quantity

$$\varepsilon(\xi_1) = -f_1(\xi_1)/\xi_1\theta'(\xi_1), \tag{3.9.89}$$

is a measure of the first order outward displacement of the ring's boundary circle with respect to the origin of coordinates O' in terms of its own radius ξ_1 (Table 3.9.2). As it is expected on general grounds, the pressure distortion increases with increasing polytropic index, because of the increasing compressibility of the models.

Thus, in our first order approximation, the ring preserves its circular cross-section, and its circular isobaric (equipotential) surfaces are displaced outward with respect to the origin of coordinates O' (with respect to the point of maximum pressure) by the amount $-bf_1(\xi)/\theta'(\xi)$. Within the limits of Ostriker's (1964b) first order theory the obvious condition $\xi_1 \geq -bf_1(\xi_1)/\theta'(\xi_1)$ or $b\varepsilon \leq 1$ must be fulfilled, because otherwise the point of maximum pressure would be outside the inner boundary of the ring.

(i) $\boldsymbol{n=0}$. In Eq. (3.9.78) $\theta^n(\xi_1)$ becomes 1 instead of zero. The expansion parameter b equals in this particular case

$$b = \alpha/R_1 = r_1/\xi_1 R_1 = r_1/2R_1, \qquad (n=0;\ \theta = 1-\xi^2/4;\ \xi_1 = 2). \tag{3.9.90}$$

The equivalent of Eq. (3.9.82) is now $(K\varrho_0^{1/n} = P_0/\varrho_0)$

$$\begin{aligned}
&\Phi_{00}\varrho_0/(n+1)P_0 = 1 + [GM_R\varrho_0/\pi R_1 P_0]\ln(8/b\xi_1) = 1 - \xi_1\theta'(\xi_1)\ln(8R_1/r_1);\\
&\Phi_{01} = 0; \quad f_1(\xi_1) - (\beta/2b^2)\xi_1 = -c_1\theta'(\xi_1) - [\xi_1^2\theta'(\xi_1)/2][1 - \ln(8R_1/r_1)];\\
&-q_1 + f_1'(\xi_1) - \beta/2b^2 = c_1\theta'(\xi_1)/\xi_1 - [\xi_1\theta'(\xi_1)/2][2 - \ln(8R_1/r_1)];\\
&A_j f_j(\xi_1) = -c_j\xi_1^{-j+1}\theta'(\xi_1); \quad -q_j + A_j f_j'(\xi_1) = jc_j\xi_1^{-j}\theta'(\xi_1), \qquad (n=0;\ j = 0,2,3,4,...).
\end{aligned} \tag{3.9.91}$$

From these equations, and by inserting for q_j from Eq. (3.9.76), we obtain the equivalent of Eq. (3.9.83):

$$\begin{aligned}
&\beta = b^2\{\xi_1\theta'(\xi_1)[3/2 - \ln(8R_1/r_1)] + f_1(\xi_1)/\theta'(\xi_1) + f_1(\xi_1)/\xi_1 + f_1'(\xi_1)\}\\
&= (r_1^2/2R_1^2)[\ln(8R_1/r_1) - 5/4]; \quad c_1 = \xi_1^2/4 + [1/2\theta'(\xi_1)][\xi_1 f_1(\xi_1)/\theta'(\xi_1) - f_1(\xi_1) + \xi_1 f_1'(\xi_1)] = 1;\\
&q_1 = 1/2; \quad A_j = q_j = 0, \qquad (n=0;\ j = 0,2,3,4,...).
\end{aligned} \tag{3.9.92}$$

Eqs. (3.9.84), (3.9.87), (3.9.89) are in the constant density case equal to

$$\begin{aligned}
&\Theta(\xi,\varphi) = 1 - \xi^2/4 + (r_1/32R_1)\xi^3\cos\varphi; \quad \Xi(\xi,\varphi) = \xi + (r_1/16R_1)\xi^2\cos\varphi;\\
&\Xi_1(\varphi) = 2 + (b/2)\cos\varphi; \quad \varepsilon = 1/4, \qquad (n=0).
\end{aligned} \tag{3.9.93}$$

The first order expansion of the internal and external potential from Eqs. (3.9.63), (3.9.43) becomes $[-(n+1)K\varrho_0^{1/n}\xi_1\theta'(\xi_1) = -(n+1)P_0\xi_1\theta'(\xi_1)/\varrho_0 = GM_R/\pi R_1]$:

$$\begin{aligned}
&\Phi = (GM_R/\pi R_1)\{\ln(8/b\xi_1) + (1-\xi^2/4)/2 + (b/2)[\xi^3/16 + \xi\big(5/4 - \ln(8/b\xi_1)\big)]\cos\varphi\}\\
&= (GM_R/\pi R_1)\{\ln(8R_1/r_1) + (1 - r^2/r_1^2)/2 + (r_1/2R_1)[r^3/4r_1^3 + (r/r_1)\big(5/4 - \ln(8R_1/r_1)\big)]\cos\varphi\},\\
&(n=0),
\end{aligned} \tag{3.9.94}$$

$$\begin{aligned}
&\Phi_e = (GM_R/\pi R_1)\{\ln(8/b\xi) + b[(\xi/2)\big(1 - \ln(8/b\xi)\big) + 1/\xi]\cos\varphi\}\\
&= (GM_R/\pi R_1)\{\ln(8R_1/r) + (r_1/2R_1)[(r/r_1)\big(1 - \ln(8R_1/r)\big) + r_1/2r]\cos\varphi\}, \qquad (n=0).
\end{aligned} \tag{3.9.95}$$

Coincidence to first order with the results of Poincaré, Dyson [1892, p. 91; 1893, Eq. (9)], and Ostriker [1964b, Eq. (80)] can be obtained at once by a translation of the origin of the polar coordinates by the amount $bf_1(\xi_1)/|\theta'(\xi_1)| = b/2 = r_1/4R_1$ from the point O' of maximum pressure to the geometrical centre of the circular cross-section of the homogeneous ring. With this translation the (ξ,φ)-coordinates transform into the new translated polar (ξ',φ')-coordinates as [cf. Eq. (3.9.88)]: $\xi = \xi' + (b/2)\cos\varphi,\ \varphi = \varphi' + O(b)$.

We now briefly discuss the influence of a central point mass M_c from Eqs. (3.9.51)-(3.9.63). The general form (3.9.43) of the external potential of a ring is not altered by the inclusion of the external potential Φ_c of a central point mass M_c. The internal potential, including the central mass M_c, has already been evaluated in Eq. (3.9.63). We proceed exactly as for Eqs. (3.9.77)-(3.9.83), matching Eqs. (3.9.43), (3.9.63), and their derivatives on the ring's surface. The results are the same, excepting for the rotation parameter β from Eqs. (3.9.83), (3.9.92), where the additional term $2\pi M_c/M_R$ appears:

$$\beta = b^2\{\xi_1\theta'(\xi_1)[3/2 - 2\pi M_c/M_R - \ln(8R_1/r_1)] + f_1(\xi_1)/\xi_1 + f_1'(\xi_1)\} \quad \text{if} \quad 0 < n < \infty;$$
$$\beta = b^2[2\ln(8R_1/r_1) - 5/2 + 4\pi M_c/M_R] \quad \text{if} \quad n = 0, \qquad (M_c \lesssim M_R). \tag{3.9.96}$$

The appearance of a central mass merely increases the equilibrium angular velocity of the ring, and does not alter otherwise its equilibrium structure in our first order approximation. Below, we show that the first order contributions to mass, kinetic energy of rotation, gravitational and internal energy are zero, so these first order integral properties of the ring are equal to their zero order values from Eqs. (3.9.45), (3.9.52), (3.9.53). This can be shown by observing that integrals of the required quantities are of the general first order form

$$I = \int_0^{2\pi} d\varphi \int_0^{\Xi_1(\varphi)} [A(\xi) + bB(\xi)\cos\varphi]\ d\xi + O(b^2), \tag{3.9.97}$$

since the volume element is by virtue of Eq. (3.9.22) equal to $dV = (R_1 + r\cos\varphi)r\ dr\ d\lambda\ d\varphi = \alpha^2 R_1(1 + b\xi\cos\varphi)\xi\ d\xi\ d\lambda\ d\varphi$. By $A(\xi)$ we denote zero order terms, and by $B(\xi)$ all first order terms. Eq. (3.9.97) can be transformed into

$$I = \int_0^{2\pi} d\varphi \int_0^{\xi_1 + bq_1\cos\varphi} [A(\xi) + bB(\xi)\cos\varphi]\ d\xi = \int_0^{2\pi} d\varphi \int_0^{\xi_1} [A(\xi) + bB(\xi)\cos\varphi]\ d\xi$$
$$+ \int_0^{2\pi} d\varphi \int_{\xi_1}^{\xi_1 + bq_1\cos\varphi} [A(\xi) + bB(\xi)\cos\varphi]\ d\xi \approx 2\pi \int_0^{\xi_1} A(\xi)\ d\xi$$
$$+ \int_0^{2\pi} d\varphi \int_{\xi_1}^{\xi_1 + bq_1\cos\varphi} \{A(\xi_1) + bB(\xi_1)\cos\varphi + [A'(\xi_1) + bB'(\xi_1)\cos\varphi](\xi - \xi_1)\}\ d\xi$$
$$= 2\pi \int_0^{\xi_1} A(\xi)\ d\xi + \int_0^{2\pi} bq_1 A(\xi_1)\cos\varphi\ d\varphi + O(b^2) = 2\pi \int_0^{\xi_1} A(\xi)\ d\xi + O(b^2), \tag{3.9.98}$$

where we have simply expanded $A(\xi)$ and $B(\xi)$ into a Taylor series. Since all contributions of order b to E_{kin}, W, U are zero, the expression (3.9.54) for the angular velocity, derived in the zeroth approximation, is also correct to the first order in b, and should be identical to Eq. (3.9.96). This can be shown by multiplying in Eq. (3.9.66) the differential equation for f_1 by $\xi\theta'$:

$$\xi\theta' f_1'' + \theta' f_1' + (n\theta^{n-1} - 1/\xi^2)\xi\theta' f_1 = -\xi\theta'^2. \tag{3.9.99}$$

If we differentiate the cylindrical Lane-Emden equation, we get

$$\theta''' + \theta''/\xi - \theta'/\xi^2 = -n\theta^{n-1}\theta'. \tag{3.9.100}$$

Substitution of $n\theta^{n-1}\theta'$ into Eq. (3.9.99) yields

$$\xi\theta' f_1'' + \theta' f_1' - \xi\theta''' f_1 - \theta'' f_1 = d(\xi\theta' f_1' + \theta' f_1 + \xi\theta^n f_1)/d\xi = -\xi\theta'^2. \tag{3.9.101}$$

Integration of this equation gives

$$f_1'(\xi_1) + f_1(\xi_1)/\xi_1 = -[1/\xi_1\theta'(\xi_1)] \int_0^{\xi_1} \xi\theta'^2\ d\xi, \qquad \big(0 < n < \infty;\ \theta^n(\xi_1) = 0\big). \tag{3.9.102}$$

On the other hand, the integral in the previous equation can also be evaluated via Eq. (2.6.159):

$$(n+1)\xi_1^2\theta_1'^2/4 = \int_0^{\xi_1} \xi\theta^{n+1}\ d\xi = \int_0^{\xi_1} \theta\ (\xi\theta^n)\ d\xi = -\int_0^{\xi_1} \theta\ [d(\xi\theta')/d\xi]\ d\xi = \int_0^{\xi_1} \xi\theta'^2\ d\xi, \tag{3.9.103}$$

where we have used the Lane-Emden equation, and have integrated by parts. Comparing Eqs. (3.9.102) and (3.9.103), we find

$$f_1'(\xi_1) + f_1(\xi_1)/\xi_1 = -(n+1)\xi_1\theta'(\xi_1)/4, \qquad (0 < n < \infty). \tag{3.9.104}$$

We insert into Eq. (3.9.96), and obtain just Eq. (3.9.54), as claimed previously:

$$\begin{aligned}\beta &= -b^2\xi_1\theta'(\xi_1)[\ln(8R_1/r_1) + 2\pi M_c/M_R + (n-5)/4]\\ &= -(r_1/R_1)^2[\theta'(\xi_1)/\xi_1][\ln(8R_1/r_1) + 2\pi M_c/M_R + (n-5)/4], \qquad (0 < n < \infty),\end{aligned} \tag{3.9.105}$$

where we have to substitute for $-\theta'(\xi_1)/\xi_1 = \varrho_m/2\varrho_0$ via Eq. (2.6.27).

If $n = 0$, Eq. (3.9.96) transforms at once into Eq. (3.9.54), where $\varrho_m = \varrho_0$ and $b = r_1/2R_1$ in virtue of Eq. (3.9.90).

The first order expression of the total energy E of the ring is equal to its zero order evaluation [cf. Eq. (2.6.98)]:

$$\begin{aligned}E &= E_{kin} + W + U = -(GM_R^2/4\pi R_1)[\ln(8R_1/r_1) + 2\pi M_c/M_R + (n+7)/4 - 1/(\Gamma-1)]\\ &= -J^2/2M_RR_1^2 - [(3\Gamma-4)/(\Gamma-1)](GM_R^2/4\pi R_1), \qquad (0 \leq n < \infty),\end{aligned} \tag{3.9.106}$$

where we have used Eqs. (3.9.45), (3.9.52), (3.9.53) for E_{kin}, W, U, respectively. The first order equation for the angular momentum is just equal to its zero order value $J = \Omega M_R R_1^2$. The angular velocity from Eq. (3.9.54) can be written as

$$\begin{aligned}&\Omega^2 = 2\pi G\varrho_0\beta = (GM_R/2\pi R_1^3)[\ln(8R_1/r_1) + 2\pi M_c/M_R + (n-5)/4],\\ &(M_c \lesssim M_R = 2\pi^2\varrho_m r_1^2 R_1;\ 0 \leq n < \infty),\end{aligned} \tag{3.9.107}$$

which is inserted into the equation of the angular momentum $J = \Omega M_R R_1^2$, in order to derive Eq. (3.9.106). From Eq. (3.9.106) we observe that the value $\Gamma = 4/3$ occupies a well known special place [cf. Eqs. (2.6.98)-(2.6.101)]. If $\Gamma \geq 4/3$, the total energy E is always negative, and cooling ($\Delta E < 0$) causes a decrease $\Delta R_1 < 0$ of the ring's major radius R_1. In absence of other external forces the total angular momentum of the ring remains constant, so for sufficiently large R_1 the energy E of the ring is positive if $\Gamma < 4/3$, and cooling ($\Delta E < 0$) will produce a general expansion $\Delta R_1 > 0$ of the ring (Ostriker 1964b).

(ii) $\boldsymbol{n = \pm\infty}$. If the ring is composed of a perfect gas, this special case amounts to an isothermal ring $T =$ const, as outlined in Sec 1.2. With $P = K\varrho = K\varrho_0\exp(-\Theta)$ from Eq. (3.9.16), the equation (3.9.56) of hydrostatic equilibrium of the isothermal ring writes as

$$\nabla(-K\ln\varrho + \Phi + \Phi_c + \Omega^2\ell^2/2) = 0, \qquad (n = \pm\infty). \tag{3.9.108}$$

Taking the divergence of Eq. (3.9.108), we obtain in the same way as for Eq. (3.9.58):

$$K\ \nabla^2\ln\varrho = -4\pi G\varrho + 2\Omega^2 \quad \text{or} \quad \nabla^2\Theta = \exp(-\Theta) - \beta, \qquad (n = \pm\infty). \tag{3.9.109}$$

Alternately, we may integrate Eq. (3.9.108) from the central circle at $\ell = R_1$ to the observation point of coordinates (r, φ, λ) :

$$\Phi = \Phi_0 - \Phi_c + \Phi_{c0} + K\ln(\varrho/\varrho_0) + \Omega^2(R_1^2 - \ell^2)/2, \qquad (n = \pm\infty). \tag{3.9.110}$$

The previous equation can be expanded in the same way as Eqs. (3.9.61)-(3.9.63):

$$\Phi = \Phi_0 + K[-\Theta + b\xi(\Phi_{c0}/K - \beta/2b^2)\cos\varphi] + O(b^2). \tag{3.9.111}$$

We expand the fundamental function Θ in powers of b up to the first order:

$$\Theta(\xi,\varphi) = \theta(\xi) + b\left[A_0f_0(\xi) + f_1(\xi)\cos\varphi + \sum_{j=2}^{\infty}A_jf_j(\xi)\cos j\varphi\right] + O(b^2). \tag{3.9.112}$$

We substitute the Laplace operator from Eq. (3.9.26) into the fundamental equation (3.9.109), and obtain

$$\begin{aligned}&\partial^2\Theta/\partial\xi^2 + (1/\xi)\ \partial\Theta/\partial\xi + (1/\xi^2)\ \partial^2\Theta/\partial\varphi^2 + b[\cos\varphi\ \partial\Theta/\partial\xi - (\sin\varphi/\xi)\ \partial\Theta/\partial\varphi] = \exp(-\Theta) - \beta,\\ &(n = \pm\infty).\end{aligned} \tag{3.9.113}$$

Inserting Eq. (3.9.112) into Eq. (3.9.113), and equating equal powers of b, $\sin j\varphi$, $\cos j\varphi$, our problem splits into the set of ordinary differential equations

$$\theta'' + \theta'/\xi - \exp(-\theta) = 0; \quad f_1'' + f_1'/\xi + [\exp(-\theta) - 1/\xi^2]f_1 = -\theta';$$
$$f_j'' + f_j'/\xi + [\exp(-\theta) - j^2/\xi^2]f_j = 0, \qquad (j = 0, 2, 3, 4, ...; \ n = \pm\infty), \tag{3.9.114}$$

with the initial conditions

$$\theta(0), \theta'(0), f_j(0), f_j'(0) = 0, \qquad (j = 0, 1, 2, 3, ...; \ n = \pm\infty), \tag{3.9.115}$$

resulting from the initial conditions $\Theta(0, \varphi)$, $(\partial\Theta/\partial\xi)_{\xi=0} = 0$. The fundamental function (3.9.112) is inserted into Eq. (3.9.111) to obtain

$$\Phi = \Phi_{00} - K\theta(\xi) + b\Phi_{01} + bK\Big[- A_0 f_0(\xi) - f_1(\xi)\cos\varphi + \xi(GM_c/KR_1 - \beta/2b^2)\cos\varphi$$
$$- \sum_{j=2}^{\infty} A_j f_j(\xi)\cos j\varphi\Big]. \tag{3.9.116}$$

As it is known from Eqs. (2.3.86), (2.6.17), the total mass per unit length of the isothermal cylinder is finite and equal to

$$M_1 = \lim_{\xi\to\infty} M(\xi) = \lim_{\xi\to\infty} 2\pi\varrho_0\alpha^2\xi\theta' = \lim_{\xi\to\infty} 8\pi\varrho_0\alpha^2/(1 + 8/\xi^2) = 8\pi\varrho_0\alpha^2,$$
$$[n = \pm\infty; \ \theta = \ln(1 + \xi^2/8)^2]. \tag{3.9.117}$$

Strictly speaking, isothermal rings cannot come to existence, because isothermal cylinders extend up to infinity, so some external pressure has to act on a certain cut-off boundary of a hypothetical isothermal ring in order to prevent its extension up to infinity. Isothermal rings are merely a useful approximation for polytropic rings of high polytropic index $n \gg 1$, which possess a finite boundary, because the undistorted cylinders do so. With this reservation in mind, we place the hypothetical boundary $r_1 = \alpha\xi_1$ of the isothermal ring at a sufficiently large coordinate distance $\xi_1 \gg 1$, where Eq. (3.9.117) is approximately valid. At the same time the constraint $r_1 \ll R_1$ or $\xi_1 \ll 1/b$ has to be fulfilled, so all our subsequent approximate evaluations are valid only in the interval

$$1 \ll \xi_1 \ll 1/b, \tag{3.9.118}$$

where ξ_1 is the dimensionless coordinate of a certain cut-off boundary of the ring. With this constraint the total mass of the isothermal ring is approximately

$$M_R \approx 2\pi R_1 M_1 = 16\pi^2\varrho_0\alpha^2 R_1 = 4\pi K R_1/G, \qquad (n = \pm\infty). \tag{3.9.119}$$

At the cut-off boundary ξ_1 the "internal" potential from Eq. (3.9.116) has to be equal to the "external" potential from Eq. (3.9.43). The defining equation (3.9.114) of f_0

$$f_0'' + f_0'/\xi + f_0/(1 + \xi^2/8)^2 = 0, \tag{3.9.120}$$

has no solution satisfying the boundary conditions $f_0(0) = f_0'(0) = 0$, excepting the trivial one $f_0 \equiv 0$. If $\xi \gg 1$, the defining equation (3.9.114) of f_j can be approximated as

$$f_j'' + f_j'/\xi + [1/(1 + \xi^2/8)^2 - j^2/\xi^2]f_j \approx f_j'' + f_j'/\xi - j^2 f_j/\xi^2 = 0, \qquad (\xi \gg 1; \ j = 2, 3, 4, ...). \tag{3.9.121}$$

This homogeneous Euler equation has the obvious solution $f_j = C_1\xi^j + C_2\xi^{-j}$, $(\xi \gg 1; \ C_1, C_2 =$ const), and the resulting form of the internal potential (3.9.116) can be matched with the external potential (3.9.43) at ξ_1 only if $A_j = 0$, $(j = 2, 3, 4, ...)$. Thus, we are left with the determination of f_1 from

$$f_1'' + f_1'/\xi + [1/(1 + \xi^2/8)^2 - 1/\xi^2]f_1 = -\theta'. \tag{3.9.122}$$

It can be verified that the elegant integral given by Ostriker (1964b)

$$f_1(\xi) = -\theta'(\xi)\int_0^\xi dy/y\ \theta'^2(y)\int_0^y x\ \theta'^2(x)\ dx, \tag{3.9.123}$$

satisfies Eq. (3.9.122), where the derivative of the Lane-Emden equation from Eq. (3.9.114) has to be used. With the substitutions $z = 1+\xi^2/8 = \exp(\theta/2),\ \theta' = \xi/2z$ this integral can be transformed through integration by parts into

$$f_1(z) = -[8(z-1)/z^2]^{1/2}\int_1^z (z'^2\ln z' - z'^2 + z')\ dz'/(z'-1)^2$$
$$= -[8(z-1)/z^2]^{1/2}\left\{3 - 2z + [(z^2-2z)/(z-1)]\ln z + 2\int_1^z \ln z'\ dz'/(z'-1)\right\}. \tag{3.9.124}$$

If $\xi, z \gg 1$, as in our case, the dilogarithmic function from this equation can be evaluated by observing that its integral is known for the interval $[0,1]$, (Fichtenholz 1964). Transforming with $z' = 1/t'$ to this interval, the dilogarithmic function becomes

$$\int_1^z \ln z'\ dz'/(z'-1) = \int_1^t \ln t'\ dt'/t'(1-t') = \int_1^t \ln t'\ dt'/t' + \int_1^t \ln t'\ dt'/(1-t')$$
$$= (1/2)\ln^2 t - \int_0^1 \ln t'\ dt'/(1-t') + \int_0^t \ln t'\ dt'/(1-t')$$
$$= (1/2)\ln^2 t + \pi^2/6 + \sum_{j=0}^{\infty}[t^{j+1}\ln t/(j+1) - t^{j+1}/(j+1)^2], \qquad (t \approx 0). \tag{3.9.125}$$

Transforming back to the original variable $1/t = z = 1+\xi^2/8 \approx \xi^2/8$, Eq. (3.9.124) reads eventually

$$f_1(\xi) = -\xi\ln(\xi^2/8) + 2\xi - (8/\xi)(\pi^2/3+3) + (16/\xi)\ln(\xi^2/8) - (8/\xi)\ln^2(\xi^2/8) + O[\xi^{-3}\ln^2(\xi^2/8)],$$
$$(\xi \gg 1). \tag{3.9.126}$$

The nonvanishing terms of f_1 if $\xi \to \infty$ are now inserted into Eq. (3.9.116):

$$\Phi = \Phi_{00} + 2K\ln(8/\xi^2) + b\Phi_{01} + bK\xi[GM_c/KR_1 - \beta/2b^2 + \ln(\xi^2/8) - 2]\cos\varphi + O(b^2),$$
$$(n = \pm\infty;\ \xi \gg 1;\ \Phi_0 = \Phi_{00} + b\Phi_{01}). \tag{3.9.127}$$

At the cut-off boundary ξ_1 of the ring this equation must be equal to the external potential (3.9.43). Comparing the equivalent coefficients, we get:

$$GM_R/\pi R_1 = 4K;\quad \Phi_{00} = 2K\ln(8/b^2);\quad \Phi_{01} = 0;\quad \beta = 2b^2[\ln(8/b^2) - 4 + 4\pi M_c/M_R],$$
$$(n = \pm\infty;\ M_c \lesssim M_R). \tag{3.9.128}$$

To the first order in b, the density distribution in the ring is

$$\varrho = \varrho_0\exp(-\Theta) = [\varrho_0/(1+\xi^2/8)^2][1 - bf_1(\xi)\cos\varphi], \qquad (n = \pm\infty). \tag{3.9.129}$$

From Eqs. (3.9.24), (3.9.25), (3.9.119), (3.9.128) we find for the expansion parameter

$$b = \alpha/R_1 = (K/4\pi G\varrho_0 R_1^2)^{1/2} = (M_R/16\pi^2\varrho_0 R_1^3)^{1/2}, \qquad (n = \pm\infty), \tag{3.9.130}$$

and for the angular velocity

$$\Omega^2 = 2\pi G\varrho_0\beta = (GM_R/4\pi R_1^3)[\ln(128\pi^2\varrho_0 R_1^3/M_R) - 4 + 4\pi M_c/M_R], \qquad (n = \pm\infty). \tag{3.9.131}$$

By the same type of argument as given for the polytropic indices $0 \le n < \infty$ one can show that the first order terms in b vanish for all integral quantities such as mass, kinetic energy of rotation, gravitational and internal energy (Ostriker 1964b). It seems not worthwhile to pursue this point further, due to the problematic existence of isothermal rings, associated with their infinite extension.

Finally, we turn to the critical rotation velocity of rings, i.e. to the maximum angular speed of a ring before equatorial mass loss or break up. To this end we calculate the radial component of the effective gravity on the surface $r = r_1$. Via Eqs. (3.4.65), (3.9.56) this amounts to

$$\begin{aligned}
g_{r_1} &= (\partial\Phi_{tot}/\partial r)_{r=r_1} = [\partial(\Phi + \Phi_c + \Omega^2\ell^2/2)/\partial r]_{r=r_1} = (1/\varrho)\ (\partial P/\partial r)_{r=r_1} \\
&= [(n+1)K\varrho_0^{1/n}/\alpha][\partial\Theta(\xi,\varphi)/\partial\xi]_{\xi=\Xi_1} = [(n+1)K\varrho_0^{1/n}/\alpha]\{\partial[\theta(\xi) + bf_1(\xi)\cos\varphi]/\partial\xi\}_{\xi=\Xi_1} \\
&= [(n+1)K\varrho_0^{1/n}/\alpha][\theta'(\xi_1) + (\Xi_1 - \xi_1)\ \theta''(\xi_1) + bf_1'(\xi_1)\cos\varphi] \\
&= [(n+1)K\varrho_0^{1/n}/\alpha]\{\theta'(\xi_1) + b[f_1(\xi_1)/\xi_1 + \theta^n(\xi_1)\ f_1(\xi_1)/\theta'(\xi_1) + f_1'(\xi_1)]\cos\varphi\}, \qquad (0 \le n < \infty).
\end{aligned} \tag{3.9.132}$$

For critical rotation the effective gravity at the outer equatorial boundary $r_{c1},\ (\xi = \xi_1,\ \varphi = 0)$ must be zero:

$$[\partial\Theta(\xi,\varphi)/\partial\xi]_{(\xi=\xi_1,\varphi=0)} = 0 \quad \text{or} \quad b_c = -\theta'(\xi_1)/[f_1(\xi_1)/\xi_1 + f_1'(\xi_1)] \quad \text{if} \quad \theta^n(\xi_1) = 0,\ 0 < n < \infty,$$
$$\text{and} \quad b_c = -\theta'(\xi_1)/[f_1(\xi_1)/\xi_1 + f_1(\xi_1)/\theta'(\xi_1) + f_1'(\xi_1)] \quad \text{if} \quad \theta^n(\xi_1) = 1,\ n = 0. \tag{3.9.133}$$

From this equation it is obvious that Ostriker's (1964b) first order theory is not adequate for the calculation of critically (i.e. rapidly) rotating rings, because if $n = 0$, we get $b_c = 2$, opposite to the basic requirement $b_c \ll 1$ from Eq. (3.9.25). Only if $n \gtrsim 4$, the condition $b_c \ll 1$ is fulfilled ($b_c = 0.18$ if $n = 4$). This inadequacy seems to arise mostly from the fact that in Ostriker's (1964b) theory all quantities are expanded only up to order $\beta^{1/2}$, $[b \propto \beta^{1/2}$; Eq. (3.9.83)], instead to order β, as in Chandrasekhar's (1933a-d) theory, for instance.

The brief discussion of stability problems and oscillations of ringlike structures and accretion tori will be deferred to Secs. 5.10 and 6.4.3.

3.10 Magnetopolytropes

3.10.1 Introduction

Magnetic fields in polytropes have already been touched in Sec. 2.6.6 in connection with the virial theorem. Basically, we study the behaviour of electrically conducting polytropic matter in the presence of magnetic fields, applying the magnetohydrodynamic equation (2.1.1) under simplifying assumptions. Even in Eq. (2.1.1) there are already contained certain simplifications, which can be illustrated by writing down the complete set of Maxwell's equations in the Gaussian unrationalized CGS-system of units (e.g. Sommerfeld 1961, Hughes and Gaylord 1964, Chandrasekhar 1981):

$$\nabla\cdot\vec D = 4\pi\varrho_e;\quad \nabla\cdot\vec B = 0;\quad \nabla\times\vec E = -(1/c)\ \partial\vec B/\partial t;\quad \nabla\times\vec H = (4\pi/c)\vec J + (1/c)\ \partial\vec D/\partial t. \tag{3.10.1}$$

$\vec D$ denotes the electric displacement vector, $\vec B$ the vector of magnetic induction, $\vec E$ and $\vec H$ are the intensities of the electric and magnetic field, ϱ_e is the electric charge density, $\vec J$ the electric current density (generally the density of electric currents), and c the velocity of light. Relativistic effects are neglected throughout this section, i.e. the velocity v of fluid elements with respect to a fixed laboratory frame is always much smaller than the velocity of light: $v \ll c$.

The term $\partial\vec D/\partial t$ (Maxwell's displacement current) in Eq. (3.10.1) is negligible in magnetohydrodynamics, referred to as the quasi-steady approximation. Neglect of the displacement current $\partial\vec D/\partial t$ is only justified when the time-scale τ of field variations is long as compared to the light-travel time L/c, where L is the characteristic length-scale of the system considered: $L/c\tau \ll 1$. To see this, we follow Roberts (1967) and determine the order of magnitude of pertinent quantities: $|\nabla\times\vec E| \approx E/L,\ |\partial\vec B/\partial t| \approx B/\tau$. From the third equation (3.10.1) we find

$$|\vec E|/|\vec B| = E/B \approx L/c\tau, \tag{3.10.2}$$

and from the last equation (3.10.1) we observe that

$$(1/c)|\partial\vec D/\partial t|/|\nabla\times\vec H| = (\varepsilon p/c)|\partial\vec E/\partial t|/|\nabla\times\vec B|$$
$$\approx (\varepsilon p/c)(L/\tau)(E/B) \approx (\varepsilon p)(L/c\tau)^2 \approx (L/c\tau)^2 \ll 1, \tag{3.10.3}$$

where we have used the phenomenological constitutive equations (3.10.5), and the fact that the dielectric constant ε and the magnetic permeability p of the highly conducting medium are related by

$$\varepsilon p \approx \varepsilon_0 p_0 = 1, \qquad (\varepsilon\approx\varepsilon_0;\ p\approx p_0), \tag{3.10.4}$$

where $\varepsilon_0 = 1$ and $p_0 = 1$ denote free-space values.

Thus, neglect of displacement currents $\partial\vec D/\partial t$ is justified if we are not concerned with the effects of propagation of electromagnetic waves, i.e. if the state of the system alters only slightly during the interval L/c taken by light to cross the system: $\tau \gg L/c$.

As suggested by experiment, the relationship between $\vec J$ and $\vec E$, $\vec D$ and $\vec E$, $\vec B$ and $\vec H$ in a comoving frame is linear in many cases (e.g. Sommerfeld 1961, Gerthsen et al. 1977):

$$\vec J = \sigma\vec E;\quad \vec D = \varepsilon\vec E;\quad \vec B = p\vec H, \tag{3.10.5}$$

where σ denotes the electric conductivity. Note, that particularly Ohm's law $\vec J = \sigma\vec E$ is valid only in a reference frame that is moving together with the fluid at the point concerned [cf. Eq. (3.10.10)]. The material parameters σ,ε,p will be considered throughout as constant, which holds true for most substances under very general conditions.

The body force $\vec f$ acting on the unit of volume, i.e. on the mass ϱ, is due to electrostatic and electromagnetic interactions between field and fluid material (Alfvén and Fälthammar 1963):

$$\vec f = \varrho_e\vec E + \vec J\times\vec B/c. \tag{3.10.6}$$

Again, we can show that the electrostatic force term $\varrho_e\vec{E}$ is negligible as compared to the electromagnetic contribution $\vec{J}\times\vec{B}/c$. We have

$$|\varrho_e\vec{E}|/|\vec{J}\times\vec{B}/c| = |\nabla\cdot\vec{D}||\vec{E}|/|\nabla\times\vec{H}||\vec{B}| \approx \varepsilon p(E^2/L)/(B^2/L) \approx (L/c\tau)^2 \ll 1, \qquad (3.10.7)$$

by considering Eqs. (3.10.1), (3.10.2), (3.10.4), (3.10.5). The magnetic force term $\vec{J}\times\vec{B}/c$, also called Lorentz force, can be transformed at once to the form (2.1.1) by inserting for $\vec{J}$ from Eq. (3.10.1):

$$\vec{f} = \vec{J}\times\vec{B}/c = (1/4\pi)(\nabla\times\vec{H})\times\vec{B} = (p/4\pi)(\nabla\times\vec{H})\times\vec{H}, \qquad (\partial\vec{D}/\partial t = 0;\ p = \mathrm{const}). \quad (3.10.8)$$

Because for most substances the magnetic permeability p is close to its Gaussian free-space value $p_0 = 1$ (e.g. Gerthsen et al. 1977), we often take $p = 1$. Eq. (2.1.1) obeys for a dissipationless fluid in an inertial laboratory frame the form (e.g. Chandrasekhar 1981)

$$\varrho\ D\vec{v}/Dt = \partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \varrho\ \nabla\Phi + (p/4\pi)(\nabla\times\vec{H})\times\vec{H}. \qquad (3.10.9)$$

Because the charge density ϱ_e does not enter into the equation of motion (3.10.9), the use of the first Maxwell equation $\nabla\cdot\vec{D} = 4\pi\varrho_e$ can generally be avoided.

A further simplification results from the assumption of stationarity: $\partial\vec{H}/\partial t = 0$. In order to see the conditions of stationarity, we write down Ohm's law $\vec{J} = \sigma\vec{E}$ – holding in a comoving frame – in a laboratory frame of reference when the nonrelativistic fluid has velocity $\vec{v}$, $(v \ll c)$ with respect to this system:

$$\vec{J} = \sigma[\vec{E} + (1/c)\ \vec{v}\times\vec{B}] + \varrho_e\vec{v}. \qquad (3.10.10)$$

Again, we show that the electric convection current $\varrho_e\vec{v}$ can be neglected with respect to the conduction current $\sigma[\vec{E} + (1/c)\vec{v}\times\vec{B}]$:

$$|\varrho_e\vec{v}/\vec{J}| = |\nabla\cdot\vec{D}||\vec{v}|/c|\nabla\times\vec{H}| \approx (\varepsilon p/c)(E/L)(L/\tau)/(B/L) \approx (L/c\tau)^2 \ll 1. \qquad (3.10.11)$$

Thus

$$\vec{E} = \vec{J}/\sigma - (p/c)\ \vec{v}\times\vec{H} = (c/4\pi\sigma)\ \nabla\times\vec{H} - (p/c)\ \vec{v}\times\vec{H}, \qquad (3.10.12)$$

or

$$-(c/p)\ \nabla\times\vec{E} = \partial\vec{H}/\partial t = -(c^2/4\pi\sigma p)\ \nabla\times(\nabla\times\vec{H}) + \nabla\times(\vec{v}\times\vec{H}). \qquad (3.10.13)$$

For systems with large dimensions (cosmic objects), and/or for high conductivity σ, the temporal variation of the magnetic field is by virtue of Eq. (3.10.13) equal to

$$\partial\vec{H}/\partial t = \nabla\times(\vec{v}\times\vec{H}). \qquad (3.10.14)$$

A steady state of the magnetic field $(\partial\vec{H}/\partial t = 0)$ will establish if circulation of matter proceeds along the field lines $(\vec{v}\times\vec{H} = 0)$, or if fluid motions cease at all $(v = 0)$.

Even with all the above assumptions, the general problem with arbitrary magnetic fields seems too formidable. The most reasonable additional simplification, which still preserves much of the physics of the problem, is a magnetic field that is symmetric to some axis. The problem of thermodynamic equilibrium is entirely eliminated by the assumption of a polytropic (barotropic) equation of state. However, it should be noted that the polytropic equation of state implies that the curl of the magnetic body force per unit mass $\nabla\times[(\nabla\times\vec{H})\times\vec{H}/4\pi\varrho]$ is zero [see Eq. (3.10.16)], a constraint that is generally not valid in real stars.

With respect to an inertial frame the equation of magnetostatic equilibrium inside a magnetopolytrope writes via Eq. (3.10.9) as

$$(1/\varrho)\ \nabla P = (n+1)K\ \nabla\varrho^{1/n} = \nabla\Phi + (1/4\pi\varrho)(\nabla\times\vec{H})\times\vec{H}, \qquad (p = 1;\ \vec{v} = 0). \qquad (3.10.15)$$

Applying the curl operator to this equation, and observing that the curl of a gradient is zero, we find

$$\nabla\times\left[(\nabla\times\vec{H})\times\vec{H}/\varrho\right] = 0. \qquad (3.10.16)$$

3.10.2 Roxburgh's Theory of Spherical Magnetopolytropes

Roxburgh (1966a) assumes axial symmetry ($\partial\ /\partial\varphi = 0$) round the axis $\lambda = 0$ of a spherical (r, λ, φ)-frame: $P = P(r, \lambda),\ \Phi = \Phi(r, \lambda),\ \vec{H} = \vec{H}[H_r(r, \lambda), H_\lambda(r, \lambda), H_\varphi(r, \lambda)]$. The components of the magnetic force along the unit vectors $\vec{e}_r, \vec{e}_\lambda, \vec{e}_\varphi$ are

$$(1/4\pi)[(\nabla\times\vec{H})\times\vec{H}] = (1/4\pi)\big\{[-(H_\varphi/r)\ \partial(rH_\varphi)/\partial r - (H_\lambda/r)\ \partial(rH_\lambda)/\partial r + (H_\lambda/r)\ \partial H_r/\partial\lambda]\ \vec{e}_r + [-(H_\varphi/r\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda + (H_r/r)\ \partial(rH_\lambda)/\partial r - (H_r/r)\ \partial H_r/\partial\lambda]\ \vec{e}_\lambda + [(H_\lambda/r\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda + (H_r/r)\ \partial(rH_\varphi)/\partial r]\ \vec{e}_\varphi\big\}, \quad (3.10.17)$$

where the curl operator is calculated via Eq. (B.38). Because of the assumed axial symmetry we have $\partial P/\partial\varphi,\ \partial\Phi/\partial\varphi = 0$, so the φ-component of Eq. (3.10.15) has to vanish:

$$[(\nabla\times\vec{H})\times\vec{H}]_\varphi = (H_\lambda/r\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda + (H_r/r)\ \partial(rH_\varphi)/\partial r = 0. \quad (3.10.18)$$

This equation can be written under an equivalent form by splitting the magnetic field into a poloidal $\vec{H}_P = \vec{H}_P(H_r, H_\lambda, 0)$ and a toroidal component $\vec{H}_T = \vec{H}_T(0, 0, H_\varphi)$:

$$[(\nabla\times\vec{H})\times\vec{H}]_\varphi = [(\nabla\times\vec{H}_T)\times\vec{H}_P] = 0;$$
$$\nabla\times\vec{H}_T = (\vec{e}_r/r\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda - (\vec{e}_\lambda/r)\ \partial(rH_\varphi)/\partial r. \quad (3.10.19)$$

Another, slightly different form of Eq. (3.10.18) is

$$H_r\ \partial(rH_\varphi\sin\lambda)/\partial r + (H_\lambda/r)\ \partial(rH_\varphi\sin\lambda)/\partial\lambda = \vec{H}_P\cdot\nabla(rH_\varphi\sin\lambda) = 0. \quad (3.10.20)$$

Since the vectors $\vec{H}_P$ and $\nabla(rH_\varphi\sin\lambda)$ cannot be perpendicular throughout, the sole other possibility is that

$$\nabla(rH_\varphi\sin\lambda) = 0 \quad \text{or} \quad rH_\varphi\sin\lambda = b = \mathrm{const}(\vec{H}_P), \quad (3.10.21)$$

where $\mathrm{const}(\vec{H}_P)$ means that b is constant along the field lines of $\vec{H}_P$. From Maxwell's equations (3.10.1) we have $\nabla\cdot\vec{B} = p\ \nabla\cdot\vec{H} = 0$, or [cf. Eq. (B.37)]

$$\nabla\cdot\vec{H} = \nabla\cdot\vec{H}_P = (1/r^2)\ \partial(r^2H_r)/\partial r + (1/r\sin\lambda)\ \partial(H_\lambda\sin\lambda)/\partial\lambda = 0, \qquad (\partial H_\varphi/\partial\varphi = 0). \quad (3.10.22)$$

Analogously to fluid dynamics we can introduce the stream function S by the relationships (e.g. Batchelor 1967)

$$H_r = (1/r^2\sin\lambda)\ \partial S/\partial\lambda; \quad H_\lambda = -(1/r\sin\lambda)\ \partial S/\partial r, \quad (3.10.23)$$

which obviously satisfies the divergence-free condition (3.10.22) of the poloidal field: $\nabla\cdot\vec{H}_P = (1/r^2\sin\lambda)\ \partial^2S/\partial\lambda\partial r - (1/r^2\sin\lambda)\ \partial^2S/\partial r\partial\lambda = 0$. We introduce Eqs. (3.10.21) and (3.10.23) into Eq. (3.10.16) to obtain $\big(H_\varphi = b/r\sin\lambda;\ [(\nabla\times\vec{H})\times\vec{H}]_\varphi = 0\big)$

$$\nabla\times[(\nabla\times\vec{H})\times\vec{H}/\varrho] = (\vec{e}_\varphi/r)\big\{\partial\big[-(H_\varphi/\varrho\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda + (H_r/\varrho)\ \partial(rH_\lambda)/\partial r - (H_r/\varrho)\ \partial H_r/\partial\lambda\big]/\partial r - \partial\big[-(H_\varphi/\varrho r)\ \partial(rH_\varphi)/\partial r - (H_\lambda/\varrho r)\ \partial(rH_\lambda)/\partial r + (H_\lambda/\varrho r)\ \partial H_r/\partial\lambda\big]/\partial\lambda\big\} = (\vec{e}_\varphi/r)\big\{-\partial\big[(b/\varrho r^2\sin^2\lambda)\ \partial b/\partial\lambda + (1/\varrho r^2\sin^2\lambda)(\partial S/\partial\lambda)(\partial^2S/\partial r^2) + (1/\varrho r^4\sin\lambda)(\partial S/\partial\lambda)\ \partial(\sin^{-1}\lambda\ \partial S/\partial\lambda)/\partial\lambda\big]/\partial r + \partial\big[(b/\varrho r^2\sin^2\lambda)\ \partial b/\partial r + (1/\varrho r^2\sin^2\lambda)(\partial S/\partial r)(\partial^2S/\partial r^2) + (1/\varrho r^4\sin\lambda)(\partial S/\partial r)\ \partial(\sin^{-1}\lambda\ \partial S/\partial\lambda)/\partial\lambda\big]/\partial\lambda\big\} = 0. \quad (3.10.24)$$

In order to facilitate the solution of Eq. (3.10.15) it is usually assumed that the magnetic force is throughout small as compared to gravity, so that the equation of hydrostatic equilibrium (3.10.15) can be solved for the spherically symmetric case ($\vec{H}=0$), while the magnetic problem can be solved separately by Eq. (3.10.24). The general solution of this equation is too complicated, so Roxburgh (1966a) looks for dipole-like solutions. It is well known (e.g. Sommerfeld 1961, Alfvén and Fälthammar 1963) that a magnetic dipole field centered in the origin of coordinates has the magnetic potential $\Phi_m=\vec{a}_m\cdot\vec{r}/r^3=a_m\cos\lambda/r^2$, where $\vec{a}_m,\ (a_m=|\vec{a}_m|)$ is the magnetic moment of the dipole, directed along the axis $\lambda=0$. The intensity of the magnetic field is then given by

$$\vec{H}=-\nabla\Phi_m;\quad H_r=-\partial\Phi_m/\partial r=2a_m\cos\lambda/r^3;\quad H_\lambda=-(1/r)\ \partial\Phi_m/\partial\lambda=a_m\sin\lambda/r^3.\quad(3.10.25)$$

The magnetic stream function which yields for $f(r)\propto 1/r$ just the dipole field (3.10.25), is simply

$$S(r,\lambda)=f(r)\sin^2\lambda.\quad(3.10.26)$$

Roxburgh (1966a) assumes further that $b=CS,\ (C=\mathrm{const})$, so that the components of the magnetic field intensity are via Eqs. (3.10.21), (3.10.23) equal to

$$H_r=2f(r)\cos\lambda/r^2;\quad H_\lambda=-f'(r)\sin\lambda/r;\quad H_\varphi=Cf(r)\sin\lambda/r.\quad(3.10.27)$$

Eq. (3.10.24) then reduces to

$$-\partial\big(2C^2f^2\sin\lambda\cos\lambda/\varrho r^2+2ff''\sin\lambda\cos\lambda/\varrho r^2-4f^2\sin\lambda\cos\lambda/\varrho r^4\big)/\partial r$$
$$+\partial\big(C^2ff'\sin^2\lambda/\varrho r^2+f'f''\sin^2\lambda/\varrho r^2-2ff'\sin^2\lambda/\varrho r^4\big)/\partial\lambda=0.\quad(3.10.28)$$

As the density ϱ depends in our approximation only on r, the derivation with respect to λ can be effected at once; the derivation with respect to r is performed by splitting the factor f from the products. After simplifications we obtain eventually

$$f\sin 2\lambda\ d[(1/\varrho r^2)(f''-2f/r^2)+C^2f/\varrho r^2]/dr=0,\quad(3.10.29)$$

which can be integrated at once, by taking into account that $f\sin 2\lambda\neq 0:$

$$f''-2f/r^2+C^2f+D\varrho r^2=0,\qquad(C,D=\mathrm{const}).\quad(3.10.30)$$

To facilitate the evaluation of the magnetic field, we introduce the following transformations [cf. Eqs. (3.2.1), (3.8.5)]:

$$r=\alpha\xi=[(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi=[(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi;\quad P=P_0\theta_0^{n+1};\quad \varrho=\varrho_0\theta_0^n;$$
$$f(r)=f(\alpha\xi)=D\varrho_0\alpha^4\gamma(\xi);\quad \Phi=(n+1)K\varrho_0^{1/n}\chi,\qquad(0\le n<5),\quad(3.10.31)$$

where θ_0 denotes in this and in the next subsection the familiar Lane-Emden function. Eq. (3.10.30) reduces to

$$d^2\gamma/d\xi^2-2\gamma/\xi^2+C^2\alpha^2\gamma+\xi^2\theta_0^n=0,\qquad(0\le n<5).\quad(3.10.32)$$

We consider only polytropic indices $0\le n<5$, when all physical parameters of the undistorted spheres remain finite (Sec. 2.6.8). As will be shown subsequently, the boundary conditions on Eq. (3.10.32) are

$$\gamma(0)=0;\quad \gamma'(0)=0,\quad(3.10.33)$$

$$\gamma(\xi_1)=0;\quad \gamma'(\xi_1)=0,\qquad(C\neq 0),\quad(3.10.34)$$

where ξ_1 is the first zero of the Lane-Emden function: $\theta_0(\xi_1)=0$. The boundary conditions at the origin follow from the fact that the magnetic field (3.10.27) must remain finite at the origin, i.e. $f/r^2\propto\gamma/\xi^2$ and $f'/r\propto\gamma'/\xi$ must be finite at $\xi=0$, so γ must be at least of order ξ^2 near the origin. According to the assumptions made by Roxburgh (1966a), the magnetic field at the surface ξ_1 (where $\vec{J},\vec{D},\nabla\times\vec{H}=0$) must be continuous with an external, curl-free, axially symmetric field. Such a field has no azimuthal component: $H_\varphi=Cf(r)\sin\lambda/r=0$, or $f(r)\propto\gamma(\xi)=0$ outside the star. This implies that $H_r=2f(r)\cos\lambda/r^2=0$, and therefore also $f'\propto\gamma'=0$ outside the star, which proves the correctness of the boundary conditions (3.10.34) on the surface $\xi = \xi_1$. If $C \neq 0$, the four boundary conditions (3.10.33) and (3.10.34) define an eigenvalue problem for the function γ from Eq. (3.10.32) with the eigenvalue $C\alpha$.

The one exception from the boundary conditions (3.10.34) occurs if $C = 0$; in this case $H_\varphi = 0$ throughout the star, and the resulting purely poloidal field inside the sphere must be continuous with an external dipole field. Comparing Eqs. (3.10.25) and (3.10.27) we infer that $f = a_m/r$, in order to satisfy the magnetic field of a dipole from Eq. (3.10.25). Outside the sphere, the magnetic stream function (3.10.26) becomes $S(r,\lambda) = (a_m/r)\sin^2\lambda$, ($a_m =$ const; $C = 0$). Continuity of $S(r,\lambda)$ and $\partial S/\partial r$ on the surface r_1 of the sphere imposes the conditions

$$S(r_1,\lambda) = f(r_1)\sin^2\lambda = (a_m/r_1)\sin^2\lambda \quad \text{and} \quad (\partial S/\partial r)_{r=r_1} = f'(r_1)\sin^2\lambda = -(a_m/r_1^2)\sin^2\lambda, \quad (C=0). \tag{3.10.35}$$

Elimination of $a_m\sin^2\lambda$ between these equations yields instead of Eq. (3.10.34) the boundary condition

$$f(r_1) + r_1 f'(r_1) = 0 \quad \text{or} \quad \gamma(\xi_1) + \xi_1\gamma'(\xi_1) = 0, \qquad (C=0). \tag{3.10.36}$$

Roxburgh (1966a) discusses at first two particular cases.

(i) Purely toroidal field. In this case $C \gg 1$, so that H_φ from Eq. (3.10.27) becomes the leading component of the magnetic field intensity: $H_\varphi \gg H_r, H_\lambda$. To examine the behaviour of the solution for large C, we write $\Gamma = \alpha^2C^2\gamma$. Eq. (3.10.32) becomes

$$(d^2\Gamma/d\xi^2 - 2\Gamma/\xi^2)/C^2\alpha^2 + \Gamma = -\xi^2\theta_0^n, \tag{3.10.37}$$

which for $C \gg 1$ has the solution

$$\Gamma \approx -\xi^2\theta_0^n, \qquad (C \gg 1;\ 1 < n < 5), \tag{3.10.38}$$

provided that Γ'' remains finite. Thus, $f(\alpha\xi) \approx -D\varrho_0\alpha^2\xi^2\theta_0^n/C^2$ and

$$H_\varphi = -D\varrho_0\alpha\xi\theta_0^n\sin\lambda/C = -D\varrho r\sin\lambda/C, \qquad (H_\varphi \gg H_r, H_\lambda;\ C \gg 1). \tag{3.10.39}$$

This solution satisfies the boundary conditions (3.10.33) and (3.10.34) if

$$\Gamma(\xi_1) = 0; \quad \Gamma'(\xi_1) = 0 \quad \text{or} \quad \theta_0^n(\xi_1) = 0; \quad (d\theta_0^n/d\xi)_{\xi=\xi_1} = n\theta_0^{n-1}(\xi_1)\,\theta_0'(\xi_1) = 0. \tag{3.10.40}$$

If $n > 1$, these boundary conditions are satisfied since $\theta_0(\xi_1) = 0$. If $n \leq 1$ however, the boundary condition (3.10.40) cannot be fulfilled by the simple solution (3.10.38): Γ'' must become very large, so that the term Γ''/α^2C^2 in Eq. (3.10.37) can be of the same order as Γ. Thus, Eq. (3.10.38) is valid only if $1 < n < 5$.

The solution (3.10.39) is just one particular form of a class of solutions for the purely toroidal field in a sphere. The condition (3.10.16) writes for a purely toroidal field $[H_r, H_\lambda = 0;\ H_\varphi = H_\varphi(r,\lambda)]$:

$$\begin{aligned}&\nabla\times[(\nabla\times\vec{H})\times\vec{H}/\varrho] = (\vec{e}_\varphi/r)\big\{-\partial[(H_\varphi/\varrho\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda]/\partial r\\ &+\partial[(H_\varphi/\varrho r)\ \partial(rH_\varphi)/\partial r]/\partial\lambda\big\} = (\vec{e}_\varphi/r)\big\{-\partial[(1/2\varrho r^2\sin^2\lambda)\ \partial(r^2H_\varphi^2\sin^2\lambda)/\partial\lambda]/\partial r\\ &+\partial[(1/2\varrho r^2\sin^2\lambda)\ \partial(r^2H_\varphi^2\sin^2\lambda)/\partial r]/\partial\lambda\big\} = (\vec{e}_\varphi/2\varrho^2r^5\sin^4\lambda)\big\{[\partial(\varrho r^2\sin^2\lambda)/\partial r]\\ &\times[\partial(r^2H_\varphi^2\sin^2\lambda)/\partial\lambda] - [\partial(\varrho r^2\sin^2\lambda)/\partial\lambda][\partial(r^2H_\varphi^2\sin^2\lambda)/\partial r]\big\}\\ &= (1/2\varrho^2r^4\sin^4\lambda)[\nabla(\varrho r^2\sin^2\lambda)\times\nabla(r^2H_\varphi^2\sin^2\lambda)] = 0.\end{aligned} \tag{3.10.41}$$

This condition implies that the two surfaces $\varrho r^2\sin^2\lambda =$ const and $r^2H_\varphi^2\sin^2\lambda =$ const coincide, since the vectorial product of $\nabla(\varrho r^2\sin^2\lambda)$ and $\nabla(r^2H_\varphi^2\sin^2\lambda)$ is zero, so the normals to these two surfaces must have the same direction. Hence, the general solution of Eq. (3.10.41) is $r^2H_\varphi^2\sin^2\lambda = F(\varrho r^2\sin^2\lambda)$, (e.g. Smirnow 1967), as can be seen by direct insertion, for instance. Therefore (cf. Anand and Kushwaha 1962a)

$$H_\varphi^2 = F(\varrho r^2\sin^2\lambda)/r^2\sin^2\lambda, \tag{3.10.42}$$

where F is an arbitrary function of its argument $\varrho r^2\sin^2\lambda$. If $F = (D\varrho r^2\sin^2\lambda/C)^2$, we recover the particular solution (3.10.39).

(ii) Purely poloidal field. In this case $C = 0$, so that $H_\varphi = 0$. Eq. (3.10.32) reduces to [cf. Sinha 1968b, Eq. (11)]

$$\gamma'' - 2\gamma/\xi^2 + \xi^2\theta_0^n = 0, \qquad (0 \leq n < 5), \tag{3.10.43}$$

which can be integrated for the particular cases $n = 0$ and $n = 1$:

$$\gamma = \xi^2(1 - \xi^2/10), \qquad (n = 0;\ \theta_0 = 1 - \xi^2/6), \tag{3.10.44}$$

$$\gamma = \xi^2/3 + \xi\sin\xi - 2\sin\xi/\xi + 2\cos\xi, \qquad (n = 1;\ \theta_0 = \sin\xi/\xi). \tag{3.10.45}$$

The solutions (3.10.44), (3.10.45) of the nonhomogeneous equation (3.10.43) for the particular cases $n = 0, 1$ can also be recovered from Eq. (3.10.87). For other values of n, Eq. (3.10.43) must be integrated numerically (cf. Fig. 3.10.1).

(iii) General case. For solutions with both a toroidal and a poloidal magnetic field we must solve Eq. (3.10.32), subject to the boundary conditions (3.10.33), (3.10.34). In the neighborhood of the origin $\xi = 0$ we obtain a particular integral of Eq. (3.10.32) by a series expansion similar to Eq. (2.4.17):

$$\gamma \approx \xi^2 - \xi^4(1 + C^2\alpha^2)/10 + ..., \qquad (\theta_0 \approx 1 - \xi^2/6;\ \xi \approx 0). \tag{3.10.46}$$

The general solution near $\xi = 0$ of the homogeneous part $\gamma'' - 2\gamma/\xi^2 + C^2\alpha^2\gamma = 0$ of Eq. (3.10.32) is $A(\xi^2 - C^2\alpha^2\xi^4/10 + ...)$, $(A = \mathrm{const})$. Thus, the general solution of the nonhomogeneous equation (3.10.32) can be written near the origin under the series form

$$\gamma(\xi) = A(\xi^2 - C^2\alpha^2\xi^4/10) + \xi^2 - \xi^4(1 + C^2\alpha^2)/10 + ..., \qquad (\xi \approx 0). \tag{3.10.47}$$

In order to obtain an approximate solution of Eq. (3.10.32) in the outer layers, Roxburgh (1966a) introduces the variable

$$\eta = 1/\xi - 1/\xi_1. \tag{3.10.48}$$

The Lane-Emden equation $\theta_0''(\xi) + 2\theta_0'(\xi)/\xi + \theta_0^n(\xi) = 0$ becomes near the surface

$$d^2\theta_0/d\eta^2 = -(1/\xi_1 + \eta)^{-4}\theta_0^n(\eta) \approx 0, \qquad (\eta \approx 0), \tag{3.10.49}$$

having the approximate solution [cf. Eq. (2.4.61)]:

$$\theta_0(\eta) \approx \theta_0'(0)\ \eta + O(\eta^{n+2}), \qquad (\xi \approx \xi_1;\ \eta \approx 0). \tag{3.10.50}$$

Near the surface Eq. (3.10.32) reads in terms of the new variable η as

$$d^2\gamma/d\eta^2 + 2(d\gamma/d\eta)/(1/\xi_1 + \eta) - 2\gamma/(1/\xi_1 + \eta)^2 + C^2\alpha^2\gamma/(1/\xi_1 + \eta)^4 = -\theta_0^n(\eta)/(1/\xi_1 + \eta)^6,$$
$$(\eta \approx 0), \tag{3.10.51}$$

where the surface boundary conditions are $\gamma,\ d\gamma/d\eta = 0$ at $\eta = 0$. Near the surface we seek a series solution of this equation under the form (2.4.61): $\gamma(\eta) = \sum_{i=1}^{\infty} a_i\eta^{n+i}$, $(a_i = \mathrm{const})$. Inserting this attempt into Eq. (3.10.51), we get

$$\gamma(\eta) = -[\theta_0'^{\,n}(0)\ \xi_1^6/(n + 1)(n + 2)][\eta^{n+2} - 2\xi_1\eta^{n+3}/(n + 3) + ...],$$
$$\big(\xi \approx \xi_1;\ \eta \approx 0;\ \theta_0^n(\eta) \approx \theta_0'^{\,n}(0)\ \eta^n\big). \tag{3.10.52}$$

This surface solution was evaluated by Roxburgh (1966a) at the point $\xi_1\eta = 0.1$. In the inner region Eq. (3.10.47) was evaluated at $\xi = 0.1$ for a test value of $C\alpha$, and then extended by numerical integration up to the matching point $\xi_1\eta = 0.1$. Similarly as in Sec. 3.6, the continuity of γ and $d\gamma/d\eta$ at $\xi_1\eta = 0.1$ determines the constant A from Eq. (3.10.47) and the eigenvalue $C\alpha$ from Eq. (3.10.32). The eigenvalue $C\alpha = (2H_\varphi/H_r\xi)\cot\lambda$ results from Eq. (3.10.27) as a measure of the ratio between the toroidal and the radial component of the magnetic field. As seen from the upper part of Fig. 3.10.1, the main feature of the solution if $n \leq 1$ is the existence of nodes between $\xi = 0$ and $\xi = \xi_1$, whereas if $n > 1$, no nodes occur.

Finally, we show that the magnetic body force tends to be zero as the surface is approached. From Eqs. (3.10.31) and (3.10.52) we get $f \propto \gamma \propto \eta^{n+2}$. From Eq. (3.10.27) follows $\vec{H} \propto f, f'$, and from Eq. (3.10.50) $\varrho \propto \theta_0^n \propto \eta^n$. Thus, $(\nabla\times\vec{H})\times\vec{H} \propto ff', f'^2, ff'', f'f''$, so that the terms of lowest power in η give for the magnetic body force per unit mass $(\nabla \times \vec{H}) \times \vec{H}/4\pi\varrho \propto f'f''/\varrho \propto \gamma'\gamma''/\theta_0^n \propto \eta^{2n+1}/\eta^n = \eta^{n+1}$, which tends to zero for all $n > -1$.

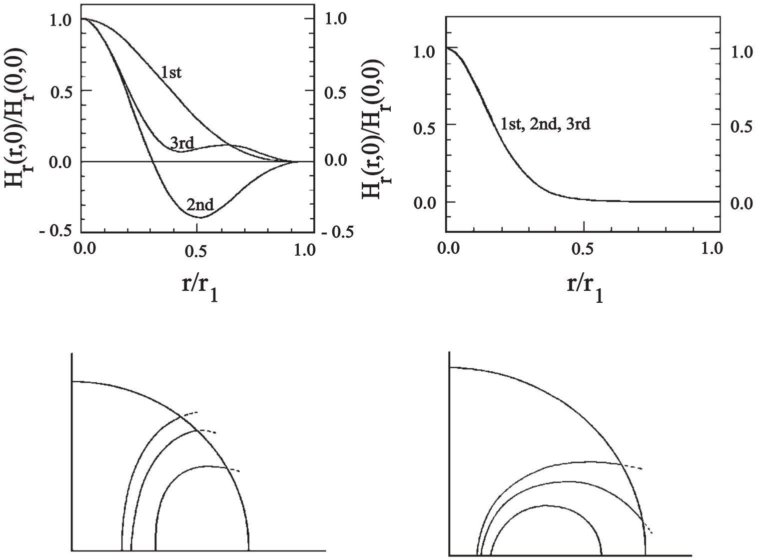

Fig. 3.10.1 Upper figures: Variation of the radial component $H_r(r,\lambda)$ of the magnetic field intensity along the symmetry axis ($\lambda = 0$) of the polytrope $n = 1$ (top left) and $n = 3$ (top right) for the first three eigenvalues $C\alpha$ from Eq. (3.10.32) in Roxburgh's (1966a) theory: $C\alpha = 2.36, 3.39, 4.40$ if $n = 1$, and $C\alpha = 6.68, 7.12, 7.62$ if $n = 3$, respectively. The ordinate axis shows the ratio $H_r(r,0)/H_r(0,0)$, and the abscissa axis the relative radius r/r_1. If $n = 3$, the three eigensolutions coincide at the scale of the figure. Lower figures: Field lines of the poloidal magnetic field in Monaghan's (1965) theory for the polytrope $n = 1$ (bottom left), and $n = 3$ (bottom right). The surface of the polytrope is depicted too.

3.10.3 Monaghan's Magnetopolytropes with Poloidal Fields

Monaghan (1965) has investigated purely poloidal fields by using Chebyshev polynomials – a most useful approximation. Analogously to Eq. (3.10.26) the magnetic stream function is sought under the form

$$S(r,\lambda) = (1-\mu^2)\sum_{j=0}^{\infty} B_{2j}(r)\ T_{2j}(\mu), \qquad (\mu = \cos\lambda). \tag{3.10.53}$$

$B_{2j}(r)$ are unknown functions of r, and $T_j(\mu)$ are Chebyshev polynomials of the first kind of order j, defined by (e.g. Abramowitz and Stegun 1965, Spiegel 1968)

$$T_j(\mu) = T(\cos\lambda) = \cos(j\arccos\mu) = \cos j\lambda = (1/2)[(2\mu)^j - (j/1)C^0_{j-2}\ (2\mu)^{j-2}$$
$$+(j/2)C^1_{j-3}\ (2\mu)^{j-4} - (j/3)C^2_{j-4}\ (2\mu)^{j-6} + ...], \qquad (j = 0, 1, 2, 3, ...), \tag{3.10.54}$$

where $C^k_j = j!/k!(j-k)!,\ (j! = 1\times 2\times 3\times ...\times (j-1)j;\ 0! = 1)$.

The components (3.10.23) of the magnetic field are written in terms of μ up to the third order:

$$H_r = -(1/r^2)\ \partial S/\partial\mu = (2/r^2)[B_0(r)\ T_1(\mu) + B_2(r)\ T_3(\mu)];$$
$$H_\mu = -[1/r(1-\mu^2)^{1/2}]\ \partial S/\partial r = -[(1-\mu^2)^{1/2}/r][B_0'(r) + B_2'(r)\ T_2(\mu)]. \tag{3.10.55}$$

Only the φ-component of $\nabla\times\vec{H}$ is pertinent for calculating the magnetic body force of the poloidal field (3.10.55):

$$(\nabla\times\vec{H})_\varphi = [\partial(rH_\mu)/\partial r + (1-\mu^2)^{1/2}\ \partial H_r/\partial\mu]\ \vec{e}_\varphi/r$$
$$= -\{B_0''(r) - 2B_0(r)/r^2 - 6B_2(r)/r^2 + T_2(\mu)\ [B_2''(r) - 12B_2(r)/r^2]\}(1-\mu^2)^{1/2}\ \vec{e}_\varphi/r$$
$$= -[F + GT_2(\mu)](1-\mu^2)^{1/2}\ \vec{e}_\varphi/r, \qquad \big(F = F(r) = B_0''(r) - 2B_0(r)/r^2 - 6B_2(r)/r^2;$$
$$G = G(r) = B_2''(r) - 12B_2(r)/r^2;\ dT_3(\mu)/d\mu = 6T_2(\mu) + 3\big). \tag{3.10.56}$$

The radial and meridional component of the magnetic force term $(\nabla\times\vec{H})\times\vec{H}$ from Eq. (3.10.15) is then obtained as

$$f_r = -H_\mu(\nabla\times\vec{H})_\varphi = -(1/2r^2)\{F(B_0' - B_2'/2) + G(B_2' - B_0')/2$$
$$+T_2[F(B_2' - B_0') + G(B_0' - 3B_2'/4)] + T_4[-FB_2'/2 + G(B_2' - B_0')/2] - T_6GB_2'/4\};$$
$$f_\mu = H_r(\nabla\times\vec{H})_\varphi = -2[(1-\mu^2)^{1/2}/r^3][T_1(FB_0 + GB_0/2 + GB_2/2) + T_3(FB_2 + GB_0/2)$$
$$+T_5GB_2/2]. \tag{3.10.57}$$

Products of Chebyshev polynomials have been transformed into higher-order Chebyshev polynomials via Eq. (3.10.54). With the dimensionless notations from Eq. (3.10.31) and with $\varrho = \varrho_0\Theta^n$, the equation of hydrostatic equilibrium (3.10.15) becomes

$$\Theta^n\ \partial\Theta/\partial\xi = \Theta^n\ \partial\chi/\partial\xi + \alpha f_\xi(\xi,\mu)/4\pi(n+1)K\varrho_0^{1+1/n} = \Theta^n\ \partial\chi/\partial\xi + \varepsilon f_\xi(\xi,\mu);$$
$$(\Theta^n/\xi)\ \partial\Theta/\partial\mu = (\Theta^n/\xi)\ \partial\chi/\partial\mu - (1-\mu^2)^{-1/2}\alpha f_\mu(\xi,\mu)/4\pi(n+1)K\varrho^{1+1/n}$$
$$= (\Theta^n/\xi)\ \partial\chi/\partial\mu - \varepsilon(1-\mu^2)^{-1/2}f_\mu(\xi,\mu). \tag{3.10.58}$$

With the transformation $r = \alpha\xi$ the functions $B_j(r), f_r(r,\mu), f_\mu(r,\mu)$ turn into $B_j(\xi), f_\xi(\xi,\mu), f_\mu(\xi,\mu)$, respectively. The parameter

$$\varepsilon = \alpha/4\pi(n+1)K\varrho_0^{1+1/n} = 1/16\pi^2G\alpha\varrho_0^2 \ll 1, \tag{3.10.59}$$

is a measure for the ratio between dimensionless magnetic and gravitational force. As this ratio is assumed small by Monaghan (1965), ε can be used as an expansion parameter:

$$\Theta(\xi,\mu) = \sum_{s=0}^{\infty}\theta_s(\xi,\mu)\ \varepsilon^s; \quad \chi(\xi,\mu) = \sum_{s=0}^{\infty}\chi_s(\xi,\mu)\ \varepsilon^s. \tag{3.10.60}$$

θ_s and χ_s are unknown functions associated with various powers of ε. We have to add Poisson's equation (3.1.18), which writes in our dimensionless notations as

$$\partial(\xi^2\ \partial\chi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\chi/\partial\mu]/\partial\mu = -\xi^2\Theta^n. \tag{3.10.61}$$

The zero order approximations ($\varepsilon = 0$) of Eqs. (3.10.58) and (3.10.61) are simply the radially symmetric equations of a sphere

$$d\theta_0/d\xi = d\chi_0/d\xi; \quad d(\xi^2\ d\chi_0/d\xi)/d\xi = -\xi^2\theta_0^n, \qquad [\theta_0 = \theta_0(\xi);\ \chi_0 = \chi_0(\xi)], \tag{3.10.62}$$

and need no further discussion. In the first approximation, terms of order ε have to be equated after insertion of Eq. (3.10.60) into Eqs. (3.10.58), (3.10.61):

$$\theta_0^n\ \partial(\theta_1 - \chi_1)/\partial\xi = f_\xi; \quad (\theta_0^n/\xi)\ \partial(\theta_1 - \chi_1)/\partial\mu = -f_\mu/(1-\mu^2)^{1/2};$$
$$\partial(\xi^2\ \partial\chi_1/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\chi_1/\partial\mu]/\partial\mu = -n\xi^2\theta_0^{n-1}\theta_1. \tag{3.10.63}$$

We now expand the coefficients $\theta_s(\xi,\mu)$ and $\chi_s(\xi,\mu)$ in terms of Chebyshev polynomials up to the sixth order, as required by Eq. (3.10.57) according to our third order approximation (3.10.55) of the magnetic field:

$$\theta_s = \sum_{t=0}^{3}\theta_{s,2t}(\xi)\ T_{2t}(\mu); \quad \chi_s = \sum_{t=0}^{3}\chi_{s,2t}(\xi)\ T_{2t}(\mu), \qquad (s = 1,2,3,...). \tag{3.10.64}$$

A simple equation for the determination of B_2 is provided by the terms associated with $T_6(\mu)$ and $T_5(\mu)$, respectively. We insert Eqs. (3.10.57), (3.10.64) into Eq. (3.10.63):

$$\theta_0^n\ d(\theta_{1,6}-\chi_{1,6})/d\xi=(B_2'/8\xi^2)(B_2''-12B_2/\xi^2);\quad 12\theta_0^n(\theta_{1,6}-\chi_{1,6})=(B_2/\xi^2)(B_2''-12B_2/\xi^2);$$
$$d(\xi^2\ d\chi_{1,6}/d\xi)/d\xi-42\chi_{1,6}=-n\xi^2\theta_0^{n-1}\theta_{1,6}. \tag{3.10.65}$$

We have taken into account that $dT_6/d\mu=12(T_5+T_3+T_1)$ and $d[(1-\mu^2)\ dT_6/d\mu]/d\mu=-42T_6+O(T_5)$. From the first two equations (3.10.65) we get at once

$$(\theta_{1,6}-\chi_{1,6})^2=C_1^2B_2^3;\quad B_2''-12B_2/\xi^2=12C_1\theta_0^nB_2^{1/2}\xi^2,\qquad (C_1=\mathrm{const}). \tag{3.10.66}$$

Once Eq. (3.10.66) has been solved, the remaining equations – arising from the coefficients of T_0,T_2,T_4 – can be solved in order to determine the remaining functions $\theta_{1,2t}(\xi)$ and $\chi_{1,2t}(\xi)$. Monaghan (1965) simplifies the problem further, by retaining only the $B_0(\xi)$-term – a valid approximation when the external field is a simple dipole field from Eq. (3.10.25). With this approximation the dimensionless form of Eq. (3.10.57) is

$$f_\xi(\xi,\mu)=-(1/2\xi^2)B_0'(\xi)\ [B_0''(\xi)-2B_0(\xi)/\xi^2][1-T_2(\mu)];$$
$$f_\mu(\xi,\mu)=-2(1-\mu^2)^{1/2}T_1(\mu)\ B_0(\xi)\ [B_0''(\xi)-2B_0(\xi)/\xi^2]/\xi^3. \tag{3.10.67}$$

It is now only necessary to expand $\theta_1(\xi,\mu)$ and $\chi_1(\xi,\mu)$ up to the second order Chebyshev polynomial $T_2(\mu)$:

$$\theta_1(\xi,\mu)=\theta_{1,0}(\xi)+\theta_{1,2}(\xi)\ T_2(\mu);\quad \chi_1(\xi,\mu)=\chi_{1,0}(\xi)+\chi_{1,2}(\xi)\ T_2(\mu). \tag{3.10.68}$$

Eq. (3.10.63) becomes

$$\theta_0^n\ d(\theta_{1,0}-\chi_{1,0})/d\xi=-B_0'(B_0''-2B_0/\xi^2)/2\xi^2;\quad \theta_0^n\ d(\theta_{1,2}-\chi_{1,2})/d\xi=B_0'(B_0''-2B_0/\xi^2)/2\xi^2;$$
$$2\theta_0^n(\theta_{1,2}-\chi_{1,2})=B_0(B_0''-2B_0/\xi^2)/\xi^2;\quad d(\xi^2\ d\chi_{1,0}/d\xi)/d\xi-2\chi_{1,2}=-n\xi^2\theta_0^{n-1}\theta_{1,0};$$
$$d(\xi^2\ d\chi_{1,2}/d\xi)/d\xi-6\chi_{1,2}=-n\xi^2\theta_0^{n-1}\theta_{1,2},\qquad \big(d[(1-\mu^2)\ dT_2(\mu)/d\mu]/d\mu=-6T_2(\mu)-2\big). \tag{3.10.69}$$

Concerning the surface boundary conditions of these equations, Monaghan (1965) refers to the unperturbed spherical surface $\xi=\xi_1$. On this spherical surface the internal gravitational potential matches in a rough approximation the Laplace solution (3.1.58) of the axially symmetric external gravitational potential, and the magnetic field matches onto the dipole field (3.10.25). For the external potential we write according to Eq. (3.1.58)

$$\chi_e=b_0'/\xi+b_2'P_2(\mu)/\xi^3=b_0'/\xi+(b_2'/\xi^3)[1/4+3T_2(\mu)/4]$$
$$=b_{00}/\xi+\varepsilon[b_{01}/\xi+b_2/4\xi^3+3b_2T_2(\mu)/4\xi^3],\qquad (b_0',b_2',b_{00},b_{01},b_2=\mathrm{const}), \tag{3.10.70}$$

where we have stressed the small parameter ε from Eq. (3.10.59) in the constants b_0',b_2' of the gravitational potential: $b_0'=b_{00}+\varepsilon b_{01},\ b_2'=\varepsilon b_2$. The internal potential is by virtue of Eqs. (3.10.60) and (3.10.68) equal to

$$\chi(\xi,\mu)=\chi_0(\xi)+\varepsilon[\chi_{1,0}(\xi)+\chi_{1,2}(\xi)\ T_2(\mu)]. \tag{3.10.71}$$

The internal and external potential, together with the corresponding derivatives with respect to ξ, must be equal on the boundary ξ_1 :

$$b_{01}/\xi_1+b_2/4\xi_1^3=\chi_{1,0};\quad 3b_2/4\xi_1^3=\chi_{1,2};\quad -b_{01}/\xi_1^2-3b_2/4\xi_1^4=\chi_{1,0}';\quad -9b_2/4\xi_1^4=\chi_{1,2}'. \tag{3.10.72}$$

Elimination of the two constants b_{01} and b_2 from these four equations yields two boundary conditions on the coefficients of the internal potential:

$$\chi_{1,0}(\xi_1)+\xi_1\chi_{1,0}'(\xi_1)=-2\chi_{1,2}(\xi_1)/3;\quad 3\chi_{1,2}(\xi_1)+\xi_1\chi_{1,2}'(\xi_1)=0. \tag{3.10.73}$$

The boundary condition for the magnetic field results from the matching on the boundary of the magnetic stream function (3.10.53) with the stream function (3.10.26) of a dipole field ($f(r) \propto 1/r$), together with their derivatives:

$$(1-\mu^2)B_0 = b\sin^2\lambda/\xi_1 = b(1-\mu^2)/\xi_1; \quad B_0' = -b/\xi_1^2, \qquad (b = \mathrm{const}). \tag{3.10.74}$$

Elimination of b from these two matching conditions yields the boundary condition consistent with Monaghan's (1965) approximation:

$$B_0(\xi_1) + \xi_1 B_1'(\xi_1) = 0. \tag{3.10.75}$$

From the second and third equation (3.10.69) follows

$$(\theta_{1,2}' - \chi_{1,2}')/(\theta_{1,2} - \chi_{1,2}) = B_0'/B_0, \tag{3.10.76}$$

and we infer by integration that

$$\theta_{1,2}(\xi) - \chi_{1,2}(\xi) = EB_0(\xi), \qquad (E = \mathrm{const}). \tag{3.10.77}$$

The third equation (3.10.69) becomes

$$B_0'' - 2B_0/\xi^2 = 2E\xi^2\theta_0^n. \tag{3.10.78}$$

With

$$B_0 = B_0(\xi) = \xi^2 Q(\xi), \tag{3.10.79}$$

Eq. (3.10.78) takes the form

$$d[\xi^4 Q'(\xi)]/d\xi = 2E\xi^4\theta_0^n. \tag{3.10.80}$$

Therefore

$$Q'(\xi)/E = c_1/\xi^4 + (2/\xi^4)\int_0^\xi \xi'^4\theta_0^n\, d\xi' = c_1/\xi^4 - 2\theta_0'(\xi) + (4/\xi^4)\int_0^\xi \xi'^3(d\theta_0/d\xi')\, d\xi'$$
$$= c_1/\xi^4 - 2\theta_0'(\xi) + 4\theta_0(\xi)/\xi - (12/\xi^4)\int_0^\xi \xi'^2\theta_0\, d\xi', \qquad (c_1 = \mathrm{const}), \tag{3.10.81}$$

where we have integrated by parts, and have used the Lane-Emden equation $d(\xi^2\, d\theta_0/d\xi)/d\xi = -\xi^2\theta_0^n$. Eq. (3.10.81) can be integrated further:

$$Q(\xi)/E = c_2 - c_1/3\xi^3 - 2\theta_0(\xi) + 4\int_0^\xi \theta_0\, d\xi'/\xi' - 12\int_0^\xi (d\xi''/\xi''^4)\int_0^{\xi''} \xi'^2\theta_0\, d\xi'$$
$$= c_2 - c_1/3\xi^3 - 2\theta_0(\xi) + (4/\xi^3)\int_0^\xi \xi'^2\theta_0\, d\xi', \qquad (c_2 = \mathrm{const}). \tag{3.10.82}$$

We have taken into account that integration by parts yields

$$\int_0^\xi (d\xi''/\xi''^4)\int_0^{\xi''} \xi'^2\theta_0\, d\xi' = -(1/3\xi^3)\int_0^\xi \xi'^2\theta_0\, d\xi' + (1/3)\int_0^\xi \theta_0\, d\xi'/\xi'. \tag{3.10.83}$$

To avoid a singularity of B_0 at the origin $\xi = 0$ we must have $c_1 = 0$. The boundary condition (3.10.75) written in terms of Q becomes

$$3Q(\xi_1) + \xi_1 Q'(\xi_1) = 0. \tag{3.10.84}$$

Substitution of Q, Q' from Eqs. (3.10.81), (3.10.82), evaluated at the surface $\xi = \xi_1$, shows that

$$c_2 = 2\xi_1\theta_0'(\xi_1)/3. \tag{3.10.85}$$

Thus

$$B_0(\xi)/E = \xi^2 Q(\xi)/E = 2\xi_1\theta_0'(\xi_1)\ \xi^2/3 - 2\xi^2\theta_0(\xi) + (4/\xi)\int_0^\xi \xi'^2\theta_0(\xi')\ d\xi';$$
$$S(\xi,\mu)/E = (1-\mu^2)\ B_0(\xi)/E. \tag{3.10.86}$$

As it is to be expected, the differential equation (3.10.78) – determining the field intensity in Monaghan's (1965) evaluation – becomes identical to the corresponding equation (3.10.43) in Roxburgh's (1966a) treatment if we put $B_0(\xi) = -2E\gamma(\xi)$. The solution of Eq. (3.10.78) for the particular cases $n = 0$ and $n = 1$ can be found at once by using Eq. (3.10.86), [cf. Eqs. (3.10.44), (3.10.45)]:

$$B_0(\xi)/E = -\xi^2(2-\xi^2/5) \quad \text{if} \quad n=0;$$
$$B_0(\xi)/E = -2\xi^2/3 - 2\xi\sin\xi + 4\sin\xi/\xi - 4\cos\xi \quad \text{if} \quad n=1. \tag{3.10.87}$$

Near the origin the expansion of the Lane-Emden function is given by Eq. (2.4.24), ($\theta_0 \approx 1-\xi^2/6$), and the stream function takes the form

$$S(\xi,\mu)/E = (1-\mu^2)\ B_0(\xi)/E = 2\xi^2(1-\mu^2)[\xi_1\theta_0'(\xi_1) - 1]/3, \qquad (\xi \approx 0), \tag{3.10.88}$$

which determines via Eq. (3.10.55) straight lines parallel to the polar axis $\mu = 1$, (see Fig. 3.10.1 bottom). This is just what one would expect, since the dipole field has this behaviour inside the current loop generating the field. Further, the radial magnetic component f_ξ from Eq. (3.10.67) is determined by $\theta_0^n B_0' \propto \varrho B_0'$, while the angular component f_μ is $\propto \varrho B_0/\xi$, as can be seen by inserting for $B_0'' - 2B_0/\xi^2$ from Eq. (3.10.78). These magnetic force components have their maximum well inside the star.

To determine in a more refined approximation the distortion of the configuration, we have to solve Eq. (3.10.69) also for $\theta_{1,0},\ \theta_{1,2},\ \chi_{1,0}$, and $\chi_{1,2}$. We insert Eq. (3.10.77) into the last equation (3.10.69):

$$d(\xi^2\ d\chi_{1,2}/d\xi)/d\xi - 6\chi_{1,2} = -n\xi^2\theta_0^{n-1}(EB_0 + \chi_{1,2}). \tag{3.10.89}$$

Substituting now Eq. (3.10.78) into the first equation (3.10.69), we get

$$d(\theta_{1,0} - \chi_{1,0})/d\xi = -E\ dB_0/d\xi, \tag{3.10.90}$$

or

$$\theta_{1,0} - \chi_{1,0} = -EB_0 + F, \qquad (F = \text{const}). \tag{3.10.91}$$

This equation is used to eliminate $\theta_{1,0}$ from the fourth equation (3.10.69):

$$d(\xi^2\ d\chi_{1,0}/d\xi)/d\xi - 2\chi_{1,2} = -n\xi^2\theta_0^{n-1}(-EB_0 + F + \chi_{1,0}). \tag{3.10.92}$$

The problem reduces to the solution of Eqs. (3.10.89), (3.10.92), subject to the boundary conditions (3.10.73) and to the initial conditions

$$\chi_{1,0}(0) = -F; \quad (d\chi_{1,0}/d\xi)_{\xi=0} = 0; \quad \chi_{1,2}(0) = 0; \quad (d\chi_{1,2}/d\xi)_{\xi=0} = 0. \tag{3.10.93}$$

These initial conditions on the internal potential follow at once from Eqs. (3.10.69), (3.10.86), (3.10.90), (3.10.91), because

$$\Theta(0,\mu),\ \theta_0(0) = 1; \quad \theta_{1,0}(0),\ \theta_{1,2}(0),\ B_0(0) = 0;$$
$$(\partial\Theta/\partial\xi)_{\xi=0},\ (d\theta_0/d\xi)_{\xi=0},\ (d\theta_{1,0}/d\xi)_{\xi=0},\ (d\theta_{1,2}/d\xi)_{\xi=0},\ (dB_0/d\xi)_{\xi=0} = 0, \tag{3.10.94}$$

where

$$\Theta(\xi,\mu) = \theta_0(\xi) + \varepsilon[\theta_{1,0}(\xi) + (2\mu^2-1)\ \theta_{1,2}(\xi)];$$
$$\chi(\xi,\mu) = \chi_0(\xi) + \varepsilon[\chi_{1,0}(\xi) + (2\mu^2-1)\ \chi_{1,2}(\xi)]. \tag{3.10.95}$$

On the boundary $\Xi_1 = \Xi_1(\mu)$ we have $\Theta(\Xi_1,\mu) = 0$ and $\theta_0(\xi_1) = 0$, or to the first order:

$$\Theta(\Xi_1,\mu) = \theta_0(\Xi_1) + \varepsilon[\theta_{1,0}(\Xi_1) + (2\mu^2-1)\ \theta_{1,2}(\Xi_1)] \approx (\Xi_1 - \xi_1)\ \theta_0'(\xi_1)$$
$$+\varepsilon[\theta_{1,0}(\xi_1) + (2\mu^2-1)\ \theta_{1,2}(\xi_1)] = 0, \qquad (\Xi_1 \approx \xi_1;\ \varepsilon \ll 1). \tag{3.10.96}$$

In a similar way as in Secs. 2.4.1, 3.2, we obtain from Eqs. (3.10.89), (3.10.92) the expansions near the origin $\big(\chi_{1,k} = a_{0k} + a_{2k}\xi^2 + a_{4k}\xi^4;\ k = 0, 2\big)$:

$$\chi_{1,0} = -F + a_{22}\xi^2/3; \quad \chi_{1,2} = a_{22}\xi^2 - (n\xi^4/14)\{2E^2[\xi_1\theta_0'(\xi_1) - 1]/3 + a_{22}\}, \qquad (\xi \approx 0). \tag{3.10.97}$$

$\chi_{1,0}$ and $\chi_{1,2}$ can be obtained with these starting sequences by numerical integration of Eqs. (3.10.89), (3.10.92). The two eigenvalues a_{22} and F are determined to satisfy the boundary conditions (3.10.73) for the potential. The problem of a distorted magnetopolytrope is solved completely by the evaluation of $\theta_{1,0}$ and $\theta_{1,2}$ from Eqs. (3.10.77) and (3.10.91). Monaghan's (1966) magnetopolytropes are oblate spheroids lying wholly ($n = 1.5, 3$), or partially ($n = 1$) inside the undistorted sphere.

3.10.4 Perturbation Methods for Magnetopolytropes

The magnetic field is weak, and its effects on the polytropic structure are calculated as a first order perturbation on a known nonmagnetic sphere, as done for instance by Van der Borght (1967) for the magnetic field (3.10.27).

(i) Toroidal field. The field assumed by Sinha (1968a) is of the form (3.10.39):

$$H_\varphi = C\varrho(r,\lambda)\ r\sin\lambda; \quad \vec{H} = \vec{H}[0, 0, H_\varphi(r,\lambda)], \qquad (C = \mathrm{const}). \tag{3.10.98}$$

The magnetostatic equation (3.10.15) writes as

$$\partial P/\partial r - \varrho\ \partial\Phi/\partial r = -(H_\varphi/4\pi r)\ \partial(rH_\varphi)/\partial r = -(C^2\varrho\sin^2\lambda/4\pi)\ \partial(\varrho r^2)/\partial r;$$
$$\partial P/\partial\lambda - \varrho\ \partial\Phi/\partial\lambda = -(H_\varphi/4\pi\sin\lambda)\ \partial(H_\varphi\sin\lambda)/\partial\lambda = -(C^2\varrho r^2/4\pi)\ \partial(\varrho\sin^2\lambda)/\partial\lambda. \tag{3.10.99}$$

The prime integral of this system becomes with the polytropic equation of state equal to

$$(n+1)K\varrho^{1/n} - \Phi + (C^2/4\pi)\varrho r^2\sin^2\lambda = \mathrm{const}. \tag{3.10.100}$$

We apply the Laplace operator ∇^2 to this equation by using the dimensionless notations

$$P = P_0\Theta^{n+1} = K\varrho_0^{1+1/n}\Theta^{n+1}; \quad \varrho = \varrho_0\Theta^n; \quad \mu = \cos\lambda; \quad h = C^2/16\pi^2 G;$$
$$r = \alpha\xi = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi. \tag{3.10.101}$$

Then, we obtain with $\nabla^2\Phi = -4\pi G\varrho_0\Theta^n$ and Eq. (B.39):

$$\partial\{\xi^2\ \partial[\Theta + h\xi^2(1-\mu^2)\Theta^n]/\partial\xi\}/\partial\xi + \partial\{(1-\mu^2)\ \partial[\Theta + h\xi^2(1-\mu^2)\Theta^n]/\partial\mu\}/\partial\mu + \xi^2\Theta^n = 0,$$
$$(0 \le n < 5). \tag{3.10.102}$$

The constant h is a measure for the strength of the toroidal field, and is assumed to be small:

$$\Theta(\xi,\mu) = \theta(\xi) + h\Psi(\xi,\mu), \qquad (h \ll 1), \tag{3.10.103}$$

where θ is the Lane-Emden function. Substituting Eq. (3.10.103) into Eq. (3.10.102), and using the Lane-Emden equation (2.3.87), we get

$$\partial(\xi^2\ \partial\Psi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Psi/\partial\mu]/\partial\mu + n\xi^2\theta^{n-1}\Psi + (1-\mu^2)\ d[\xi^2\ d(\xi^2\theta^n)/d\xi]/d\xi$$
$$+(6\mu^2 - 2)\xi^2\theta^n = 0, \qquad (\Theta^n \approx \theta^n + hn\theta^{n-1}\Psi), \tag{3.10.104}$$

with the obvious initial conditions $\Psi, \partial\Psi/\partial\xi = 0$ at $\xi = 0$. We expand Ψ in terms of the Sinha associated functions $\psi_j(\xi)$ and of Legendre polynomials $P_j(\xi)$:

$$\Psi(\xi,\mu) = \sum_{j=0}^{\infty}\psi_j(\xi)\ P_j(\mu). \tag{3.10.105}$$

We substitute into Eq. (3.10.104), equating coefficients of equal $P_j(\xi)$, and use Eq. (3.1.40) to obtain the set of Sinha's associated functions $\psi_j(\xi)$ for magnetopolytropes with purely toroidal fields:

$$d(\xi^2\ d\psi_0/d\xi)/d\xi + n\xi^2\theta^{n-1}\psi_0 + (2/3)\ d[\xi^2\ d(\xi^2\theta^n)/d\xi]/d\xi = 0;$$
$$d(\xi^2\ d\psi_2/d\xi)/d\xi + (n\xi^2\theta^{n-1} - 6)\psi_2 - (2/3)\ d[\xi^2\ d(\xi^2\theta^n)/d\xi]/d\xi + 4\xi^2\theta^n = 0;$$
$$d(\xi^2\ d\psi_j/d\xi)/d\xi + [n\xi^2\theta^{n-1} - j(j+1)]\psi_j = 0, \qquad (j \neq 0, 2). \tag{3.10.106}$$

If ψ_j is a solution of the last equation (3.10.106), so is $A_j\psi_j$, (A_j = const) too. Hence, a more appropriate expansion of Ψ from Eq. (3.10.105) is

$$\Psi(\xi,\mu) = \psi_0(\xi) + A_1\psi_1(\xi)\ P_1(\mu) + \psi_2(\xi)\ P_2(\mu) + \sum_{j=3}^{\infty} A_j\psi_j(\xi)\ P_j(\mu). \tag{3.10.107}$$

Poisson's equation writes analogously to Eq. (3.2.12):

$$\partial(\xi^2\ \partial\Phi/\partial\xi)/\partial\xi + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu$$
$$= -(n+1)K\varrho_0^{1/n}\xi^2\left[\theta^n + hn\theta^{n-1}\left(\psi_0 + A_1\psi_1P_1 + \psi_2P_2 + \sum_{j=3}^{\infty} A_j\psi_jP_j\right)\right]. \tag{3.10.108}$$

Analogously to Eq. (3.2.13) we write

$$\Phi = U_0(\xi) + h\sum_{j=0}^{\infty} V_j(\xi)\ P_j(\mu). \tag{3.10.109}$$

Equating the coefficients of P_j after substitution of Eq. (3.10.109) into Eq. (3.10.108), we recover Eqs. (3.2.14) and (3.2.16):

$$d(\xi^2\ dU_0/d\xi)/d\xi = -(n+1)K\varrho_0^{1/n}\xi^2\theta^n, \tag{3.10.110}$$

$$d(\xi^2\ dV_j/d\xi)/d\xi - j(j+1)V_j = -n(n+1)K\varrho_0^{1/n}\xi^2\theta^{n-1}A_j\psi_j, \qquad (j = 0,1,2,3,...;\ A_0, A_2 = 1). \tag{3.10.111}$$

The solution of Eq. (3.10.110) is analogous to Eq. (3.2.20):

$$U_0 = (n+1)K\varrho_0^{1/n}(\theta + c_0), \qquad (c_0 = \mathrm{const}). \tag{3.10.112}$$

Eq. (3.10.111) has the same solution as Eq. (3.2.28) if $j \neq 0, 2$:

$$V_j = (n+1)K\varrho_0^{1/n}(A_j\psi_j + B_j\xi^j), \qquad (j \neq 0, 2;\ A_j, B_j = \mathrm{const}). \tag{3.10.113}$$

Substituting for ψ_0 from the first equation (3.10.106) into Eq. (3.10.111), we get

$$d(\xi^2\ dV_0/d\xi)/d\xi = (n+1)K\varrho_0^{1/n}\ d[\xi^2\ d(\psi_0 + 2\xi^2\theta^n/3)/d\xi]/d\xi, \qquad (j = 0;\ A_0 = 1), \tag{3.10.114}$$

with the solution [cf. Eq. (3.2.24)]

$$V_0 = (n+1)K\varrho_0^{1/n}(\psi_0 + 2\xi^2\theta^n/3) + c_{10}, \qquad (c_{10} = \mathrm{const}). \tag{3.10.115}$$

If $j = 2$, $(A_2 = 1)$, we obtain analogously

$$d(\xi^2\ dV_2/d\xi)/d\xi - 6V_2 = (n+1)K\varrho_0^{1/n}\{d[\xi^2\ d(\psi_2 - 2\xi^2\theta^n/3)/d\xi]/d\xi - 6(\psi_2 - 2\xi^2\theta^n/3)\}, \tag{3.10.116}$$

with the solution [cf. Eq. (3.10.113)]

$$V_2 = (n+1)K\varrho_0^{1/n}(\psi_2 - 2\xi^2\theta^n/3 + B_2\xi^2), \qquad (B_2 = \mathrm{const}). \tag{3.10.117}$$

Substituting the values from Eqs. (3.10.112), (3.10.113), (3.10.115), (3.10.117) into Eq. (3.10.109), we get

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\Theta + c_0 + h\left[c_{10} + \sum_{j=1}^{\infty} B_j\xi^j P_j + 2\xi^2\theta^n(1-P_2)/3\right]\right\}. \tag{3.10.118}$$

The prime integral (3.10.100) writes in the dimensionless variables (3.10.101) as

$$(n+1)K\varrho_0^{1/n}\Theta - \Phi + 2h(n+1)K\varrho_0^{1/n}\xi^2\theta^n[1-P_2(\mu)]/3 = \mathrm{const.} \tag{3.10.119}$$

Inserting Eq. (3.10.118) into Eq. (3.10.119), we obtain at once $B_j = 0$. Hence

$$\Phi = (n+1)K\varrho_0^{1/n}\left\{\theta + c_0 + h\left[c_{10} + \psi_0 + A_1\psi_1 P_1 + \psi_2 P_2 + \sum_{j=3}^{\infty} A_j\psi_j P_j + 2\xi^2\theta^n(1-P_2)/3\right]\right\}. \tag{3.10.120}$$

On the surface $\Xi_1 = \Xi_1(\mu)$ we have up to the first order [cf. Eqs. (3.2.35), (3.2.37)]

$$\Theta = \Theta(\Xi_1,\mu) = \theta(\Xi_1) + h\left[\psi_0(\Xi_1) + A_1\psi_1(\Xi_1)\ P_1(\mu) + \psi_2(\Xi_1)\ P_2(\mu) + \sum_{j=3}^{\infty} A_j\psi_j(\Xi_1)\ P_j(\mu)\right] \approx$$
$$\theta(\xi_1) + h\left[\theta'(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu) + \psi_0(\xi_1) + \psi_2(\xi_1)\ P_2(\mu) + A_1\psi_1(\xi_1)\ P_1(\mu) + \sum_{j=3}^{\infty} A_j\psi_j(\xi_1)\ P_j(\mu)\right] = 0, \tag{3.10.121}$$

where the figure constants q_j occur from the equation of the boundary [cf. Eq. (3.2.34)]

$$\Xi_1 = \Xi_1(\mu) = \xi_1 + h\sum_{j=0}^{\infty} q_jP_j(\mu), \tag{3.10.122}$$

and we have analogously to Eq. (3.2.36):

$$\theta(\Xi_1) \approx \theta(\xi_1) + (\Xi_1 - \xi_1)\ \theta'(\xi_1) \approx \theta(\xi_1) + h\theta'(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu);$$
$$\theta'(\Xi_1) \approx \theta'(\xi_1) + h\theta''(\xi_1)\sum_{j=0}^{\infty} q_jP_j(\mu); \quad \psi_j(\Xi_1) \approx \psi_j(\xi_1). \tag{3.10.123}$$

Because the coefficients of $P_j(\mu)$ from Eq. (3.10.121) must be zero, the figure coefficients are equal to

$$q_0 = -\psi_0(\xi_1)/\theta'(\xi_1); \quad q_2 = -\psi_2(\xi_1)/\theta'(\xi_1); \quad q_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1), \qquad (j \neq 0,2). \tag{3.10.124}$$

The internal potential (3.10.120) and its radial derivative become on the boundary of the magnetopolytrope equal to [cf. Eqs. (3.2.39), (3.2.41)]:

$$\Phi(\Xi_1,\mu) = (n+1)K\varrho_0^{1/n}(c_0 + hc_{10}), \tag{3.10.125}$$

$$(\partial\Phi/\partial\xi)_{\xi=\Xi_1} = (n+1)K\varrho_0^{1/n}\left\{\theta'(\xi_1) + h\left[[-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]\sum_{j=0}^{\infty} q_jP_j(\mu)\right.\right.$$
$$+\psi_0' + A_1\psi_1'(\xi_1)\ P_1(\mu) + \psi_2'(\xi_1)\ P_2(\mu) + \sum_{j=3}^{\infty} A_j\psi_j'(\xi_1)\ P_j(\mu)$$
$$\left.\left. +2\xi_1\theta^{n-1}(\xi_1)\ [2\theta(\xi_1) + n\xi_1\theta'(\xi_1)][1-P_2(\mu)]/3\right]\right\}. \tag{3.10.126}$$

These equations must be equal to the corresponding equations of the external potential, which can be copied from Eqs. (3.2.40) and (3.2.42), respectively:

$$\Phi_e(\Xi_1,\mu) = k_0/\xi_1 + h\sum_{j=0}^{\infty}(-k_0q_j/\xi_1^2 + k_{1j}\xi_1^{-j-1})\ P_j(\mu), \tag{3.10.127}$$

$$(\partial\Phi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + h\sum_{j=0}^{\infty}[2k_0q_j/\xi_1^3 - (j+1)k_{1j}\xi_1^{-j-2}]\ P_j(\mu). \tag{3.10.128}$$

From Eq. (3.10.126) it is obvious that $n \geq 1$, in order to avoid a singularity of the derivative at the boundary. We will always take $n > 1$, circumventing the discussion of the insignificant particular case $n = 1$. Comparing corresponding coefficients from the last four equations, we find analogously to Eq. (3.2.43):

$$c_0 = -\xi_1\theta'(\xi_1);\quad c_{10} = -\psi_0(\xi_1) - \xi_1\psi_0'(\xi_1);\quad k_0 = -(n+1)K\varrho_0^{1/n}\xi_1^2\theta'(\xi_1);$$
$$k_{10} = -(n+1)K\varrho_0^{1/n}\xi_1^2\psi_0'(\xi_1);\quad k_{1j}, A_j = 0 \text{ if } j \neq 0,2, \qquad (n>1). \tag{3.10.129}$$

As noted subsequently to Eq. (3.2.43), the relationship $(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1) \neq 0$ is always fulfilled, so the constants $k_{1j}, A_j,\ (j \neq 0, 2)$ are zero. Two relationships for the constant k_{12} follow from Eqs. (3.10.125)-(3.10.128)

$$k_{12} = -(n+1)K\varrho_0^{1/n}\xi_1^4\psi_2'(\xi_1)/3 = (n+1)K\varrho_0^{1/n}\xi_1^3\psi_2(\xi_1), \qquad (n>1), \tag{3.10.130}$$

yielding the boundary condition

$$3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1) = 0, \qquad (n>1). \tag{3.10.131}$$

This additional constraint seems to appear because Sinha (1968a) has deliberately taken $A_2 = 1$. The fundamental function (3.10.103) for a magnetopolytrope with the toroidal field (3.10.98) is therefore

$$\Theta(\xi,\mu) = \theta(\xi) + h[\psi_0(\xi) + \psi_2(\xi)\ P_2(\mu)], \qquad (n>1), \tag{3.10.132}$$

where ψ_0 and ψ_2 are solutions of the differential equations (3.10.106), subject to the boundary condition (3.10.131), and to the initial conditions $\psi_j(0), \psi_j'(0) = 0,\ (j = 0, 2)$.

Near the centre, the following expansions of ψ_0 and ψ_2 are obtained in a similar way as for the associated Lane-Emden functions from Sec. 3.2:

$$\psi_0 \approx -2\xi^2/3;\quad \psi_2 \approx X\xi^2, \qquad (\xi \approx 0;\ \theta \approx 1 - \xi^2/6). \tag{3.10.133}$$

The unknown constant $X,\ (X \approx 1)$ has to be determined numerically, in order to satisfy the boundary condition (3.10.131), (see Table 3.10.1).

The equation of the surface is via Eqs. (3.10.122), (3.10.124), (3.10.132) equal to

$$\Xi_1 = \Xi_1(\mu) = \xi_1 - [h/\theta'(\xi_1)][\psi_0(\xi_1) + \psi_2(\xi_1)\ P_2(\mu)], \qquad (n>1). \tag{3.10.134}$$

The oblateness of the configuration is by virtue of Eq. (3.2.55) equal to

$$f = (a_1 - a_3)/a_1 = 3h\psi_2(\xi_1)/2\xi_1\theta'(\xi_1), \qquad (n>1), \tag{3.10.135}$$

where

$$a_1 = \xi_1 - [h/\theta'(\xi_1)][\psi_0(\xi_1) - \psi_2(\xi_1)/2];\quad a_3 = \xi_1 - [h/\theta'(\xi_1)][\psi_0(\xi_1) + \psi_2(\xi_1)], \qquad (n>1), \tag{3.10.136}$$

is the equatorial and polar radius, respectively. From the numerical values listed in Table 3.10.1 results that polytropes distorted by the toroidal field (3.10.98) are prolate ($a_1 < a_3;\ f < 0$), a result also confirmed by Anand (1969) and Miketinac (1973).

Table 3.10.1 Boundary values of the Sinha associated functions ψ_0, ψ_2, ψ_0' from Eq. (3.10.106), of the fitting constant X from Eq. (3.10.133), and of the oblateness f from Eq. (3.10.135), (Sinha 1968a). $aE+b$ means $a \times 10^b$.

n	ξ_1	$\psi_0(\xi_1)$	$\psi_2(\xi_1)$	$\psi_0'(\xi_1)$	X	f/h
1.5	3.654	8.823E−1	2.824E−1	3.680E−2	1.149	−0.570
2	4.353	7.746E−1	1.376E−1	−1.807E−2	1.153	−0.373
2.5	5.355	6.455E−1	6.345E−2	−3.310E−2	1.157	−0.233
3	6.897	5.111E−1	2.677E−2	−2.889E−2	1.160	−0.137
3.5	9.536	3.777E−1	1.094E−2	−1.869E−2	1.162	−0.083

The mass of the considered magnetopolytrope is similar to Eq. (3.2.58):

$$\begin{aligned} M_1 &= \int_{-1}^{1} d\mu \int_0^{r_1(\mu)} \varrho(r,\mu)\ r^2\ dr = 2\pi\varrho_0\alpha^3 \int_{-1}^{1} d\mu \int_0^{\Xi_1(\mu)} \Theta^n \xi^2\ d\xi \\ &\approx 2\pi\varrho_0\alpha^3 \int_{-1}^{1} d\mu \int_0^{\Xi_1(\mu)} [\theta^n + hn\theta^{n-1}(\psi_0+\psi_2 P_2)]\xi^2\ d\xi \\ &\approx 4\pi\varrho_0\alpha^3 \int_0^{\xi_1} \{-d(\xi^2\theta')/d\xi - h\ d(\xi^2\psi_0')/d\xi - (2h/3)\ d[\xi^2\ d(\xi^2\theta^n)/d\xi]/d\xi\}\ d\xi \\ &= -4\pi\varrho_0\alpha^3\xi_1^2\theta'(\xi_1)\ [1+h\psi_0'(\xi_1)/\theta'(\xi_1)] = m_1[1+h\psi_0'(\xi_1)/\theta'(\xi_1)], \qquad (n>1). \end{aligned} \tag{3.10.137}$$

m_1 is the mass of the undistorted polytrope, and we have inserted from Eqs. (2.3.87) and (3.10.106). Sinha (1968a) finds that ψ_0 is negative from the centre up to a fraction δ of the radius, when it becomes positive, showing that the inner layers of the magnetopolytrope are less dense, and the outer layers more dense than the undistorted polytrope having the same central density ϱ_0 [see Eq. (3.10.132)]. The fraction δ decreases with increasing value of n. The relative average expansion from Eq. (3.10.122) $(\Xi_1-\xi_1)/\xi_1 = hq_0/\xi_1 = -h\psi_0(\xi_1)/\xi_1\theta'(\xi_1)$ increases, and the oblateness $|f|$ decreases with increasing polytropic index n. The boundary value $\psi_0'(\xi_1)$ is positive up to about $n \approx 1.8$ (cf. Table 3.10.1). Thus, in virtue of Eq. (3.10.137) the mass of the considered magnetopolytrope is smaller ($n<1.8;\ \theta'(\xi_1)<0$) or larger ($n>1.8$) than the undistorted one having the same central density ϱ_0. Das and Tandon (1977b) find that the contribution of higher order nonspherical terms becomes significant for strong toroidal fields.

Miketinac (1973) uses Stoeckly's (1965) method (Sec. 3.8.2) – as improved by Miketinac and Barton (1972) – to solve numerically the partial differential equation for the structure of a polytrope in the presence of a strong toroidal field of the form (3.10.39) or (3.10.98):

$$H_\varphi(\ell,z) = C\varrho(\ell,z)\ \ell, \qquad (C=\mathrm{const}), \tag{3.10.138}$$

where (ℓ,φ,z) are cylindrical coordinates. Eq. (3.10.15) becomes

$$\begin{aligned} &(1/\varrho)\ \partial P/\partial\ell = \partial\Phi/\partial\ell - (C^2/4\pi)\ \partial(\varrho\ell^2)/\partial\ell; \\ &(1/\varrho)\ \partial P/\partial z = \partial\Phi/\partial z - (C^2/4\pi)\ \partial(\varrho\ell^2)/\partial z. \end{aligned} \tag{3.10.139}$$

With the polytropic equation of state $P = K\varrho^{1+1/n}$ this equation can be integrated at once between the surface and an arbitrary inner point of the star:

$$K(n+1)\varrho^{1/n} = \Phi - \Phi_p - (C^2/4\pi)\varrho\ell^2, \tag{3.10.140}$$

where Φ_p is the value of Φ at the pole of the configuration. With the notations

$$\begin{aligned} &r = \ell/\sin\lambda = \ell/(1-\mu^2)^{1/2} = [(n+1)^n K^n/4\pi G\Phi_p^{n-1}]^{1/2} x; \\ &\varrho(r,\mu) = [\Phi_p/(n+1)K]^n \Theta^{*n}(x,\mu); \quad \chi = \Phi/\Phi_p, \end{aligned} \tag{3.10.141}$$

from Eq. (3.8.42), we write the magnetostatic equation (3.10.140) as

$$\Theta^* = \chi - hx^2(1-\mu^2)\Theta^{*n} - 1, \qquad (0 \le n < 5;\ h = C^2/16\pi^2 G). \tag{3.10.142}$$

Miketinac (1973) integrates this equation numerically, together with Poisson's equation

$$\nabla^2\Phi = -4\pi G\varrho \quad \text{or} \quad \nabla^2\chi = -\Theta^{*n}. \tag{3.10.143}$$

Table 3.10.2 Uniformly and critically rotating polytropes with toroidal magnetic field according to Geroyannis and Sidiras (1992 and priv. comm.). $\beta_c = \Omega_c^2/2\pi G\varrho_0$ denotes the critical rotation parameter, $h = C^2/16\pi^2 G$ measures the strength of the toroidal field, Ξ_{ce} and Ξ_{cp} are the critical equatorial and polar coordinates, respectively.

n	h	β_c	Ξ_{ce}	Ξ_{cp}
	0	0.0416	5.30	3.36
1.5	0.0025	0.0407	5.24	3.38
	0.025	0.0339	4.98	3.54
	0	0.00980	7.77	5.12
2.5	0.0025	0.00969	7.80	5.14
	0.025	0.00870	8.03	5.36
	0	0.00412	10.33	6.68
3	0.0025	0.00406	10.36	6.71
	0.025	0.00357	10.61	7.02
	0	0.00246	11.90	7.82
3.25	0.0025	0.00242	11.91	7.86
	0.025	0.00215	12.35	8.21

Miketinac (1973, Tables I, II) finds $X_e/x_1 = 3.08,\ X_p/X_e = 1.18,\ M_1/m_1 = 1.72$ if $n = 1.5$ and $h = 1.96$. If $n = 3$ and $h = 1.32$, we have $X_e/x_1 = 3.00,\ X_p/X_e = 1.04,\ M_1/m_1 = 1.79$. The symbols $X_e,\ X_p$, and M_1 denote the dimensionless equatorial and polar radius, and the mass of the magnetopolytrope; x_1 and m_1 are the dimensionless radius and mass of the undistorted spherical polytrope, obtained if $h = 0$. In conclusion, Miketinac (1973) finds that mass, radius, and central density of an equilibrium polytrope – distorted by the toroidal field (3.10.138) – increase with increasing field strength. The shape of the polytropes $n = 1.5$ and $n = 3$ is prolate $(X_p/X_e > 1)$ – a finding also confirmed by Sinha (1968a) and Anand (1969, p. 265), (see Table 3.10.1).

Geroyannis and Sidiras (1992) combine Eq. (3.10.139) with Eq. (3.5.9) to study the effect of the toroidal field (3.10.138) on differentially rotating polytropes [cf. Geroyannis and Hadjopoulos 1990, Eq. (2.8)]:

$$(1/\varrho)\ \partial P/\partial\ell = \partial\Phi/\partial\ell + \Omega^2(\ell)\ \ell - (C^2/4\pi)\ \partial(\varrho\ell^2)/\partial\ell;$$
$$(1/\varrho)\ \partial P/\partial z = \partial\Phi/\partial z - (C^2/4\pi)\ \partial(\varrho\ell^2)/\partial z. \qquad (3.10.144)$$

Using the complex-plane strategy [cf. Eqs. (3.5.54), (3.5.55)] with the multiple partition technique, Geroyannis and Sidiras (1992, 1993) integrate Eq. (3.10.144) together with Poisson's and Laplace's equation for polytropic indices $n = 1, 1.5, 2, 2.5, 3, 3.25$. The critical rotation parameter β_c decreases with increasing strength of the toroidal field (Table 3.10.2). We have noted the particular decrease of the critical equatorial coordinate Ξ_{ce} if $n = 1.5$, while otherwise Ξ_{ce} and the polar critical coordinate Ξ_{cp} increase with increasing strength of the toroidal field. All critically rotating models are oblate ($\Xi_{ce} > \Xi_{cp}$, Table 3.10.2). Differentially rotating visco-magnetopolytropes have been considered by Geroyannis and Sidiras (1995).

(ii) Poloidal field. An axisymmetric poloidal field can always be written in cylindrical (ℓ, φ, z)-coordinates under the form (cf. Eq. (3.10.227); Roberts 1967, Sec. 4.6; Chandrasekhar 1981, App. III):

$$\vec{H} = \nabla \times [\ell Q(\ell, z)\ \vec{e}_\varphi] = -[\partial(\ell Q)/\partial z]\ \vec{e}_\ell + [(1/\ell)\ \partial(\ell^2 Q)/\partial\ell]\ \vec{e}_z. \qquad (3.10.145)$$

As can be verified for instance by direct insertion, Eq. (3.10.16) is satisfied in the axisymmetric case by all solutions of the equation (e.g. Woltjer 1960)

$$\nabla \times \vec{H} = b_0\vec{H} + b_1\varrho\ell\ \vec{e}_\varphi, \qquad [b_0, b_1 = \mathrm{const};\ \vec{H} = \vec{H}(\ell, z);\ \varrho = \varrho(\ell, z)]. \qquad (3.10.146)$$

Trehan and Billings (1971) take $b_0 = 0$, and insert

$$\nabla \times \vec{H} = b_1\varrho\ell\ \vec{e}_\varphi, \qquad (3.10.147)$$

together with Eq. (3.10.145) into the equation of axisymmetric magnetostatic equilibrium (3.10.15):

$$\varrho^{-1}\ \nabla P = \nabla\Phi + (b_1/4\pi)\ \nabla(\ell^2 Q). \qquad (3.10.148)$$

Taking the divergence of this equation, and turning to the (ξ,μ)-coordinates from Eq. (3.2.1), we find

$$\nabla^2\Theta = -\Theta^n + (b_1^2\alpha^2/16\pi^2 G)\ \nabla^2[\xi^2(1-\mu^2)\ Q(\xi,\mu)] = -\Theta^n + \varepsilon\ \nabla^2[\xi^2(1-\mu^2)\ Q(\xi,\mu)], \tag{3.10.149}$$

where $\ell^2 = \alpha^2\xi^2(1-\mu^2)$, and we have taken $Q(\ell,z) = Q(r,\mu) = b_1\alpha^2\varrho_0 Q(\xi,\mu)$, i.e. $Q(\xi,\mu)$ is measured in units of $b_1\alpha^2\varrho_0$. The constant

$$\varepsilon = b_1^2\alpha^2/16\pi^2 G, \tag{3.10.150}$$

should not be confused with ε from Eq. (3.10.59). Except for a numerical factor, ε is just equal to the ratio between magnetic energy (2.6.78) $U_m = p\overline{H^2}V/8\pi$ and gravitational energy (2.6.137) $|W| = 3Gm^2/(5-n)r$, where m, V, r, and $\overline{H^2} \propto b_1^2\alpha^4\varrho_0^2$ denote mass, volume, radius, and mean square magnetic field intensity.

The magnetic field is supposed to be weak, so that ε may be treated as a small perturbation parameter. Thus

$$\Theta(\xi,\mu) = \theta(\xi) + \varepsilon\Psi(\xi,\mu). \tag{3.10.151}$$

Inserting Eq. (3.10.145) into Eq. (3.10.147), we obtain after some algebra

$$\partial^2 Q/\partial r^2 + (4/r)\ \partial Q/\partial r + [(1-\mu^2)/r^2]\ \partial^2 Q/\partial\mu^2 - (4\mu/r^2)\ \partial Q/\partial\mu = -b_1\varrho. \tag{3.10.152}$$

Turning to the dimensionless coordinates from Eq. (3.2.1), and substituting into Eq. (3.10.152) for the density $\varrho_0\Theta^n$ its zero order value $\varrho_0\theta^n$, we observe that the determination of the magnetic field reduces to the solution of the equation

$$\partial^2 Q/\partial\xi^2 + (4/\xi)\ \partial Q/\partial\xi + [(1-\mu^2)/\xi^2]\ \partial^2 Q/\partial\mu^2 - (4\mu/\xi^2)\ \partial Q/\partial\mu = -\theta^n(\xi). \tag{3.10.153}$$

A solution of this equation can be sought in terms of Gegenbauer polynomials $G_j^{3/2}$ of index 3/2 (Woltjer 1960):

$$Q(\xi,\mu) = f(\xi) + \sum_{j=1}^{\infty} A_j\xi^j G_j^{3/2}(\mu), \qquad (A_j = \mathrm{const}). \tag{3.10.154}$$

Gegenbauer polynomials of order j and index k can be defined by (e.g. Sauer and Szabó 1967, p. 171)

$$G_j^k(\mu) = [(-1)^j/2^j j!]\big[\Gamma(j+2k)\ \Gamma(k+1/2)/\Gamma(2k)\ \Gamma(j+k+1/2)\big](1-\mu^2)^{1/2-k}$$
$$\times d^j(1-\mu^2)^{j+k-1/2}/d\mu^j, \qquad (j=0,1,2,...;\ k>-1/2;\ G_0^k(\mu)=1), \tag{3.10.155}$$

where the gamma function is defined via Eqs. (C.9), (C.11). The differential equation satisfied by the Gegenbauer polynomials is

$$(1-\mu^2)\ d^2G_j^k/d\mu^2 - (2k+1)\mu\ dG_j^k/d\mu + j(j+2k)G_j^k = 0. \tag{3.10.156}$$

Obviously, the Legendre polynomials from Eqs. (3.1.39)-(3.1.40) are a particular case of the Gegenbauer polynomials $P_j(\mu) = (-1)^j G_j^{1/2}(\mu)$, $(k = 1/2)$, while the Chebyshev polynomials from Eq. (3.10.54) are obtained if $k \to 0$:

$$T_j = (j/2)\lim_{k\to 0}[\Gamma(k)\ G_j^k(\mu)] = [(-1)^j 2^j j!(1-\mu^2)^{1/2}/(2j)!]\ d^j(1-\mu^2)^{j-1/2}/d\mu^j = \cos(j\arccos\mu), \tag{3.10.157}$$

by inserting $\lim_{k\to 0}[\Gamma(k)/\Gamma(2k)] = 2$ via Eq. (C.11).

We substitute Eq. (3.10.154) into Eq. (3.10.153), and equate coefficients containing Gegenbauer polynomials of the same order:

$$d^2f/d\xi^2 + (4/\xi)\ df/d\xi = -\theta^n(\xi), \qquad (j=0), \tag{3.10.158}$$

$$(1-\mu^2)\ d^2G_j^{3/2}/d\mu^2 - 4\mu\ dG_j^{3/2}/d\mu + j(j+3)G_j^{3/2} = 0, \qquad (j=1,2,3,...). \tag{3.10.159}$$

The right-hand part of Eq. (3.10.158) suggests to seek its solution under the form $f(\xi) = \theta(\xi) + g(\xi)$. On using the Lane-Emden equation (2.3.87), the equation (3.10.158) turns after integration into

$$2\theta + \xi g' + 3g = \mathrm{const}, \tag{3.10.160}$$

which can be solved at once by standard methods:

$$f = \theta + g = \theta(\xi) - (2/\xi^3)\int_0^\xi \xi'^2\theta(\xi')\ d\xi' + C_1 + C_2/\xi^3, \qquad (C_1, C_2 = \mathrm{const}). \tag{3.10.161}$$

Finiteness of the magnetic field and of its derivatives at the origin $\xi = 0$ requires $C_2 = 0$. Eq. (3.10.159) is satisfied identically via Eq. (3.10.156), so Eq. (3.10.153) has the solution

$$Q(\xi, \mu) = \theta(\xi) - (2/\xi^3)\int_0^\xi \xi'^2\theta(\xi')\ d\xi' + \sum_{j=0}^{\infty} A_j\xi^j G_j^{3/2}(\mu). \tag{3.10.162}$$

Outside the polytrope we have $\varrho = \varrho_0\theta^n = 0$, so Eq. (3.10.147) turns into $\nabla\times\vec{H} = 0$, and the function Q, characterizing the magnetic field, satisfies Eq. (3.10.153) with $\theta^n = 0$:

$$\partial^2Q/\partial\xi^2 + (4/\xi)\ \partial Q/\partial\xi + [(1-\mu^2)/\xi^2]\ \partial^2Q/\partial\mu^2 - (4\mu/\xi^2)\ \partial Q/\partial\mu = 0, \qquad (\xi \geq \xi_1). \tag{3.10.163}$$

$Q(\xi, \mu)$ must be finite at infinity, so solutions of Eq. (3.10.163) have the form

$$Q(\xi, \mu) = \sum_{j=0}^{\infty} B_j G_j^{3/2}(\mu)/\xi^{\gamma_j}, \qquad (\gamma_j, B_j = \mathrm{const};\ \xi \geq \xi_1). \tag{3.10.164}$$

Inserting this equation into Eq. (3.10.163), we observe that the differential equation (3.10.156) for the Gegenbauer polynomials with $k = 3/2$ is satisfied if $j(j+3) = \gamma_j(\gamma_j+1) - 4\gamma_j$ or if $\gamma_j = j+3$. The solution for the external magnetic field (3.10.164) becomes

$$Q(\xi, \mu) = \sum_{j=0}^{\infty} B_j G_j^{3/2}(\mu)/\xi^{j+3}, \qquad (\xi \geq \xi_1). \tag{3.10.165}$$

Since the magnetic field is regarded as a small first order perturbation, the boundary condition on the continuity of Q and $\partial Q/\partial\xi$ can be applied on the spherical surface $\xi = \xi_1$:

$$f(\xi_1) + \sum_{j=1}^{\infty} A_j\xi_1^j G_j^{3/2}(\mu) = \sum_{j=0}^{\infty} B_j G_j^{3/2}(\mu)/\xi_1^{j+3};$$
$$f'(\xi_1) + \sum_{j=1}^{\infty} jA_j\xi_1^j G_j^{3/2}(\mu) = -\sum_{j=0}^{\infty}(j+3)B_j G_j^{3/2}(\mu)/\xi_1^{j+4}. \tag{3.10.166}$$

Equating coefficients of Gegenbauer polynomials of the same order, we find the system

$$f(\xi_1) = B_0/\xi_1^3;\quad f'(\xi_1) = -3B_0/\xi_1^4;\quad A_j\xi_1^j = B_j/\xi_1^{j+3};\quad jA_j\xi_1^j = -(j+3)B_j/\xi_1^{j+4},$$
$$(j = 1, 2, 3, ...), \tag{3.10.167}$$

with the solutions

$$\xi_1 f'(\xi_1) + 3f(\xi_1) = 0;\quad B_0 = \xi_1^3 f(\xi_1);\quad A_j\xi_1^j(j\xi_1 + j + 3) = 0. \tag{3.10.168}$$

Since according to Table 2.5.2 $j\xi_1 + j + 3 \neq 0$, we infer that $A_j, B_j = 0,\ (j = 1, 2, 3, ...)$. The constant C_1 from Eq. (3.10.161) is determined at once from the first equation (3.10.168): $C_1 = -\xi_1\theta'(\xi_1)/3$. The scalar function defining the magnetic field is therefore

$$Q(\xi, \mu) = Q(\xi) = f(\xi) = -\xi_1\theta'(\xi_1)/3 + \theta(\xi) - (2/\xi^3)\int_0^\xi \xi'^2\theta(\xi')\ d\xi'. \tag{3.10.169}$$

Outside the magnetopolytrope we get via Eqs. (3.10.165), (3.10.168)

$$Q(\xi) = B_0/\xi^3 = \xi_1^3 f(\xi_1)/\xi^3, \qquad (\xi \geq \xi_1), \tag{3.10.170}$$

and the resulting external magnetic field is in virtue of Eq. (3.10.145) equal to

$$\vec{H}_e = \nabla \times (B_0 \sin\lambda\ \vec{e}_\varphi/\xi^2), \qquad (\xi \geq \xi_1), \tag{3.10.171}$$

which is just a dipole field from Eq. (3.10.25): $H_{e\xi} = 2B_0 \cos\lambda/\xi^3,\ H_{e\lambda} = B_0 \sin\lambda/\xi^3$.

The basic equation (3.10.149) for the determination of the fundamental function Θ becomes with Eqs. (3.10.169) and (B.39) equal to

$$\nabla^2\Theta = -\Theta^n + \varepsilon[\chi_0(\xi) + \chi_2(\xi)\ P_2(\mu)], \tag{3.10.172}$$

where we have denoted

$$\chi_0(\xi) = -4\xi_1\theta'(\xi_1)/3 + (4/3)\ d(\xi\theta)/d\xi + (2/3)\xi\ d^2(\xi\theta)/d\xi^2;$$

$$\chi_2(\xi) = 4\theta - (8/\xi^3)\int_0^\xi \xi'^2\theta(\xi')\ d\xi' - (4/3)\ d(\xi\theta)/d\xi - (2/3)\xi\ d^2(\xi\theta)/d\xi^2. \tag{3.10.173}$$

The perturbation function Ψ from Eq. (3.10.151) is expanded in terms of the Trehan-Billings associated functions $\psi_j(\xi)$ and of Legendre polynomials $P_j(\mu)$, [cf. Eq. (3.10.107)]:

$$\Theta(\xi,\mu) = \theta(\xi) + \varepsilon\left[\psi_0(\xi) + A_1\psi_1(\xi)\ P_1(\mu) + \psi_2(\xi)\ P_2(\mu) + \sum_{j=3}^\infty A_j\psi_j(\xi)\ P_j(\mu)\right]. \tag{3.10.174}$$

Like in Sec. 3.2 we insert Eq. (3.10.174) into Eq. (3.10.172), and equate coefficients of Legendre polynomials of the same order, obtaining the differential equations of the associated functions:

$$d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi - j(j+1)\psi_j/\xi^2 = -n\theta^{n-1}\psi_j + \chi_j, \qquad (j = 0, 2), \tag{3.10.175}$$

$$d^2\psi_j/d\xi^2 + (2/\xi)\ d\psi_j/d\xi - j(j+1)\psi_j/\xi^2 = -n\theta^{n-1}\psi_j, \qquad (j \neq 0, 2). \tag{3.10.176}$$

The initial conditions are clearly $\theta(0) = 1,\ \theta'(0) = \psi_j(0) = \psi_j'(0) = 0$. Eq. (3.10.148) can be integrated with the polytropic equation of state:

$$\Phi = K(n+1)\varrho^{1/n} - (b_1/4\pi)\ell^2 Q(r) + C = (n+1)K\varrho_0^{1/n}\Theta - (2/3)[1 - P_2(\mu)](b_1^2\alpha^4\varrho_0/4\pi)\xi^2 Q(\xi)$$
$$+C = (n+1)K\varrho_0^{1/n}\Bigg\{\theta(\xi) + c_0 + \varepsilon\Bigg[c_{10} + \psi_0(\xi) + A_1\psi_1(\xi)\ P_1(\mu) + \psi_2(\xi)\ P_2(\mu)$$
$$+\sum_{j=3}^\infty A_j\psi_j(\xi)\ P_j(\mu) - (2/3)[1 - P_2(\mu)]\xi^2 Q(\xi)\Bigg]\Bigg\}, \qquad (C = c_0 + \varepsilon c_{10};\ C, c_0, c_{10} = \mathrm{const}). \tag{3.10.177}$$

The external gravitational potential is written similarly to Eq. (3.2.33):

$$\Phi_e = k_0/\xi + \varepsilon\sum_{j=0}^\infty k_{1j}\xi^{-j-1}P_j(\mu), \qquad (k_0, k_{1j} = \mathrm{const}). \tag{3.10.178}$$

The boundary of the polytrope assumes the form

$$\Xi_1(\mu) = \xi_1 + \varepsilon\sum_{j=0}^\infty q_j P_j(\mu), \qquad (q_j = \mathrm{const}). \tag{3.10.179}$$

Eq. (3.10.174) becomes on the surface equal to [cf. Eqs. (3.2.35)-(3.2.38)]

$$\Theta(\Xi_1,\mu) = \theta(\xi_1) + \varepsilon\Bigg[\theta'(\xi_1)\sum_{j=0}^\infty q_j P_j(\mu) + \psi_0(\xi_1) + A_1\psi_1(\xi_1)\ P_1(\mu) + \psi_2(\xi_1)\ P_2(\mu)$$
$$+\sum_{j=3}^\infty A_j\psi_j(\xi_1)\ P_j(\mu)\Bigg] = 0, \tag{3.10.180}$$

and since the coefficients of $P_j(\mu)$ must be zero, we get

$$q_0 = -\psi_0(\xi_1)/\theta'(\xi_1); \quad q_2 = -\psi_2(\xi_1)/\theta'(\xi_1); \quad q_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1), \qquad (j \neq 0,2). \tag{3.10.181}$$

On the boundary, the internal and external potentials (3.10.177), (3.10.178) – together with the corresponding radial derivatives – become [cf. Eqs. (3.2.39)-(3.2.42)]:

$$\Phi(\Xi_1,\mu) = (n+1)K\varrho_0^{1/n}\{c_0 + \varepsilon[c_{10} - (2/3)(1-P_2)\xi_1^2Q(\xi_1)]\}, \tag{3.10.182}$$

$$(\partial\Phi/\partial\xi)_{\xi=\Xi_1} = (n+1)K\varrho_0^{1/n}\Bigg\{\theta'(\xi_1) + \varepsilon\Bigg[[-2\theta'(\xi_1)/\xi_1 - \theta^n(\xi_1)]\sum_{j=0}^{\infty} q_jP_j(\mu) + \psi_0'(\xi_1)$$
$$+A_1\psi_1'(\xi_1)\ P_1(\mu) + \psi_2'(\xi_1)\ P_2(\mu) + \sum_{j=3}^{\infty} A_j\psi_j'(\xi_1)\ P_j(\mu) - (2/3)[1-P_2(\mu)][2\xi_1Q(\xi_1) + \xi_1^2Q'(\xi_1)]\Bigg]\Bigg\}, \tag{3.10.183}$$

$$\Phi_e(\Xi_1,\mu) = k_0/\xi_1 + \varepsilon\sum_{j=0}^{\infty}[-k_0q_j/\xi_1^2 + k_{1j}\xi_1^{-j-1}P_j(\mu)], \tag{3.10.184}$$

$$(\partial\Phi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + \varepsilon\sum_{j=0}^{\infty}[2k_0q_j/\xi_1^3 - (j+1)k_{1j}\xi_1^{-j-2}]\ P_j(\mu). \tag{3.10.185}$$

The particular case $n=0$ is ignored by Trehan and Billings (1971), so we consider only polytropic indices $0<n<5$, when $\theta^n(\xi_1)=0$ in Eq. (3.10.183). Equating coefficients associated with Legendre polynomials of the same order in Eqs. (3.10.182), (3.10.184) and in Eqs. (3.10.183), (3.10.185), respectively, we determine the unknown constants:

$$\begin{aligned}
&c_0 = -\xi_1\theta'(\xi_1); \quad c_{10} = 2\xi_1^2Q(\xi_1) + 2\xi_1^3Q'(\xi_1)/3 - \psi_0(\xi_1) - \xi_1\psi_0'(\xi_1);\\
&k_0 = -(n+1)K\varrho_0^{1/n}\xi_1^2\theta'(\xi_1); \quad k_{10} = (n+1)K\varrho_0^{1/n}\xi_1^2\{(2\xi_1/3)[2Q(\xi_1) + \xi_1Q'(\xi_1)] - \psi_0'(\xi_1)\};\\
&k_{12} = (n+1)K\varrho_0^{1/n}\xi_1^3[2\xi_1^2Q(\xi_1)/3 + \psi_2(\xi_1)]; \quad A_j[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)] = 0;\\
&k_{1j} = (n+1)K\varrho_0^{1/n}A_j\xi_1^{j+1}\psi_j(\xi_1), \qquad (0<n<5; \quad j\neq 0,2).
\end{aligned} \tag{3.10.186}$$

As noted subsequently to Eq. (3.2.43), the relationship $(j+1)\psi_j(\xi_1)+\xi_1\psi_j'(\xi_1)\neq 0$ is always fulfilled, so the constants $k_{1j}, A_j,\ (j\neq 0,2)$ are zero. The Trehan-Billings associated function ψ_2 has to satisfy the boundary condition

$$3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1) = -(2\xi_1^2/3)[5Q(\xi_1) + \xi_1Q'(\xi_1)], \tag{3.10.187}$$

and this completes the formal solution of magnetostatic equilibrium of a polytrope with the poloidal field (3.10.145).

As an example, Trehan and Billings (1971) write down the solutions in the particular case $n=1$. The solutions for $\theta, Q, \chi_1, \chi_2$ can be found at once from Eqs. (2.3.89), (3.10.169), (3.10.173):

$$\begin{aligned}
&\theta = \sin\xi/\xi; \quad Q(\xi) = 1/3 + \sin\xi/\xi + 2\cos\xi/\xi^2 - 2\sin\xi/\xi^3; \quad \chi_1 = 4/3 + 4\cos\xi/3 - 2\xi\sin\xi/3;\\
&\chi_2 = -4\cos\xi/3 + 2\xi\sin\xi/3 + 4\sin\xi/\xi + 8\cos\xi/\xi^2 - 8\sin\xi/\xi^3, \qquad (n=1).
\end{aligned} \tag{3.10.188}$$

The homogeneous parts of Eqs. (3.10.175), (3.10.176) have the solutions [cf. Eqs. (2.3.22), (3.2.74)-(3.2.78)]:

$$\begin{aligned}
&\psi_j(\xi) = B_j\xi^{-1/2}J_{j+1/2}(\xi) = C_j\xi^j\ d^j(\sin\xi/\xi)/(\xi\ d\xi)^j,\\
&[B_j, C_j = \text{const};\ \psi_0 = C_0\sin\xi/\xi;\ \psi_2 = C_2(\sin\xi/\xi + 3\cos\xi/\xi^2 - 3\sin\xi/\xi^3)].
\end{aligned} \tag{3.10.189}$$

Table 3.10.3 Boundary values of the Trehan-Billings associated functions $\psi_0, \psi_0', \psi_2, \psi_2'$ from Eq. (3.10.175), of the functions $Q(\xi_1), Q'(\xi_1)$ from Eq. (3.10.169) characterizing the poloidal field, and of the oblateness f/ε from Eq. (3.10.195), (Trehan and Billings 1971). $a+b$ means $a \times 10^b$.

Symbol	$n=1$	$n=1.5$	$n=2$	$n=3$	$n=3.5$	$n=4$
ξ_1	3.14+0	3.65+0	4.35+0	6.90+0	9.54+0	1.50+1
$\psi_0(\xi_1)$	7.00−2	−1.80−1	−3.31−1	−5.03−1	−5.61−1	−6.15−1
$\psi_0'(\xi_1)$	−7.44−1	−4.37−1	−2.59−1	−8.42−2	−4.26−2	−1.79−2
$\psi_2(\xi_1)$	−1.57+0	−1.07+0	−7.64−1	−4.08−1	−2.98−1	−2.10−1
$\psi_2'(\xi_1)$	9.47−1	5.09−1	2.78−1	7.60−2	3.65−2	1.51−2
$Q(\xi_1)$	1.31−1	7.60−2	4.29−2	1.10−2	4.52−3	1.35−3
$Q'(\xi_1)$	−1.25−1	−6.24−2	−2.96−2	−4.80−3	−1.42−3	−2.71−4
f/ε	2.35+0	2.16+0	2.07+0	2.09+0	2.26+0	2.63+0

The finding of a particular solution of the nonhomogeneous equation (3.10.175) is a cumbersome task, and we quote subsequently merely the solution found by Trehan and Billings (1971), that can be verified by direct insertion:

$$\psi_0 = -3\sin\xi/2\xi + 4/3 + \cos\xi/6 + \xi\sin\xi/6 + \xi^2\cos\xi/9; \quad \psi_2 = (4\pi^2/3)(\sin\xi/\xi + 3\cos\xi/\xi^2$$
$$-3\sin\xi/\xi^3) - 4\cos\xi/3 - \xi\sin\xi/3 - \xi^2\cos\xi/9 + 4\sin\xi/3\xi, \qquad (n=1). \tag{3.10.190}$$

The integration constants C_0, C_2 from Eq. (3.10.189) have been determined via the initial conditions $\psi_j(0), \psi_j'(0) = 0,\ (j=0,2):\ C_0 = -3/2,\ C_2 = 4\pi^2/3.$

The mass of the magnetopolytrope with the poloidal field (3.10.145) is similar to Eq. (3.2.58):

$$M_1 = 4\pi\varrho_0\alpha^3\int_{-1}^{1} d\mu\int_0^{\Xi_1(\mu)}\xi^2\Theta^n\,d\xi \approx 2\pi\varrho_0\alpha^3\int_{-1}^{1} d\mu\int_0^{\Xi_1(\mu)}[\theta^n + \varepsilon n\theta^{n-1}(\psi_0+\psi_2P_2)]\xi^2\,d\xi$$
$$\approx 4\pi\varrho_0\alpha^3\int_0^{\xi_1}\{-d(\xi^2\theta')/d\xi + \varepsilon[-d(\xi^2\psi_0')/d\xi + \xi^2\chi_0]\}\,d\xi$$
$$= 4\pi\varrho_0\alpha^3\bigg\{-\xi_1^2\theta'(\xi_1) + \varepsilon\bigg[-\xi_1^2\psi_0'(\xi_1) + 2\xi_1^4\theta'(\xi_1)/9 + (4/3)\int_0^{\xi_1}\xi^2\theta(\xi)\,d\xi\bigg]\bigg\}$$
$$= -4\pi\varrho_0\alpha^3\xi_1^2\theta'(\xi_1)\bigg\{1 + \varepsilon\bigg[\psi_0'(\xi_1)/\theta'(\xi_1) - 2\xi_1^2/9 - [4/3\xi_1^2\theta'(\xi_1)]\int_0^{\xi_1}\xi^2\theta(\xi)\,d\xi\bigg]\bigg\}$$
$$= m_1[1+O(\varepsilon)], \qquad \big(0<n<5;\ \theta'(\xi_1)<0\big), \tag{3.10.191}$$

where m_1 denotes the mass of the undistorted sphere from Eq. (2.6.18), and we have used Eqs. (2.3.87), (3.10.173), (3.10.175), integrating by parts. The volume is similar to Eq. (3.2.60):

$$V_1 = 2\pi\alpha^3\int_{-1}^{1} d\mu\int_0^{\Xi_1(\mu)}\xi^2\,d\xi \approx 4\pi\alpha^3\xi_1^3/3 + 2\pi\alpha^3\xi_1^2\int_{-1}^{1}[\Xi_1(\mu)-\xi_1]\,d\mu$$
$$= (4\pi\alpha^3\xi_1^3/3)[1 - 3\varepsilon\psi_0(\xi_1)/\xi_1\theta'(\xi_1)], \qquad (0<n<5). \tag{3.10.192}$$

The mean density is according to Eqs. (3.10.191), (3.10.192):

$$\varrho_m = M_1/V_1 = -[3\varrho_0\theta'(\xi_1)/\xi_1]\bigg\{1 + \varepsilon\bigg[\psi_0'(\xi_1)/\theta'(\xi_1) - 2\xi_1^2/9$$
$$-[4/3\xi_1^2\theta'(\xi_1)]\int_0^{\xi_1}\xi^2\theta(\xi)\,d\xi + 3\psi_0(\xi_1)/\xi_1\theta'(\xi_1)\bigg]\bigg\}. \tag{3.10.193}$$

The boundary of the magnetopolytrope is via Eqs. (3.10.179), (3.10.181):

$$\Xi_1(\mu) = \xi_1 - [\varepsilon/\theta'(\xi_1)][\psi_0(\xi_1) + \psi_2(\xi_1)\ P_2(\mu)]. \tag{3.10.194}$$

There occurs a general expansion (contraction) of the magnetopolytrope of amount $-\varepsilon\psi_0(\xi_1)/\theta'(\xi_1)$, $(\theta'(\xi_1)<0)$, similarly to the rotating polytrope. The data of Trehan and Billings (1971), (see Table 3.10.3) suggest an expansion of the magnetopolytrope if $0<n\lesssim 1,\ (\psi_0(\xi_1)>0)$, and a contraction if

$1 \lesssim n < 5$, $(\psi_0(\xi_1) < 0)$, as compared to the undistorted sphere possessing the same central density ϱ_0. The oblateness of the configuration is similar to Eqs. (3.2.55) or (3.10.135):

$$f = (a_1 - a_3)/a_1 \approx 3\varepsilon\psi_2(\xi_1)/2\xi_1\theta'(\xi_1). \tag{3.10.195}$$

From the data of Table 3.10.3 we observe that always $\psi_2(\xi_1) < 0$, and since $\theta'(\xi_1) < 0$, we infer that $f > 0$, i.e. oblate configurations with $a_1 > a_3$.

A related approach has been elaborated by Chiam and Monaghan (1971) for $n = 1.5, 3$ polytropes.

To obtain the virial equations for the magnetopolytrope of Trehan and Billings (1971), we start with Eq. (2.6.79), by taking the velocities of internal motion $v_k = 0$, $(I_{jk}, E_{jk} = 0)$, together with the hydrostatic pressure $P = 0$ on the surface S, as required by hydrostatic equilibrium of a complete magnetopolytrope:

$$\delta_{jk}\int_V P\ dV + W_{jk} + \delta_{jk}U_m - 2H_{jk} + (1/4\pi)\int_S x_j[(-H^2/2)\ dS_k + H_kH_\ell\ dS_\ell] = 0, \qquad (p = 1). \tag{3.10.196}$$

The contraction of this equation $(j = k)$ yields the scalar virial equation of magnetostatic equilibrium [cf. Eq. (2.6.80)]:

$$3\int_V P\ dV + W + U_m + (1/4\pi)\int_S x_j(-\delta_{jk}H^2/2 + H_jH_k)\ dS_k = 0. \tag{3.10.197}$$

Eq. (3.10.177) can be written under the slightly different form

$$(n+1)P = \varrho[\Phi + (b_1/4\pi)\ Q(r)\ r^2\sin^2\lambda - C]. \tag{3.10.198}$$

On integrating over the volume of the configuration, and bearing in mind Eq. (2.6.68), we get

$$(n+1)\int_V P\ dV = -2W + (b_1/4\pi)\int_V \varrho Q(r)\ r^2\sin^2\lambda\ dV - CM. \tag{3.10.199}$$

The magnetic energy term can be transformed by observing that the integration can be carried out over the undistorted volume of the Lane-Emden sphere, since the magnetic energy is a small first order quantity:

$$(b_1/4\pi)\int_V \varrho Q(r)\ r^2\sin^2\lambda\ dV \approx (b_1/4\pi)\int_0^{2\pi} d\varphi\int_0^{\pi}\sin^3\lambda\ d\lambda\int_0^{r_1}\varrho Q(r)\ r^4\ dr$$
$$= (2b_1/3)\int_0^{r_1}\varrho Q(r)\ r^4\ dr. \tag{3.10.200}$$

r_1 is the radius of the undistorted sphere, and we have used the integral

$$\int_0^{\pi/2}\sin^{2j+1}\lambda\ d\lambda = \int_0^{\pi/2}\cos^{2j+1}\lambda\ d\lambda$$
$$= 2\times 4\times 6\times ...\times 2j/1\times 3\times 5\times ...\times(2j+1) = (2j)!!/(2j+1)!!, \qquad (j = 1, 2, 3, ...). \tag{3.10.201}$$

The integral (3.10.200) can be expressed further in terms of the magnetic energy U_m from Eq. (2.6.78). The components of the magnetic field intensity are via Eqs. (3.10.145), (B.38) equal to

$$\vec{H} = \nabla\times[rQ(r)\sin\lambda\ \vec{e}_\varphi] = 2Q\cos\lambda\ \vec{e}_r - (\sin\lambda/r)[d(r^2Q)/dr]\ \vec{e}_\lambda, \tag{3.10.202}$$

where in virtue of Eq. (3.10.169):

$$Q(r,\mu) = Q(r) = b_1\alpha^2\varrho_0 Q(\xi) = b_1\alpha^2\varrho_0\left[\theta(\xi) - (2/\xi^3)\int_0^{\xi}\xi'^2\theta(\xi')\ d\xi' - \xi_1\theta'(\xi_1)/3\right]. \tag{3.10.203}$$

With Eq. (3.10.202) the magnetic energy can be expressed as

$$U_m = \int_V H^2\ dV/8\pi = \int_V(4Q^2 + 4rQQ'\sin^2\lambda + r^2Q'^2\sin^2\lambda)\ dV/8\pi$$
$$= \int_0^{r_1}(2Q^2 + 4rQQ'/3 + r^2Q'^2/3)r^2\ dr, \qquad (Q' = dQ/dr). \tag{3.10.204}$$

Taking into account that $Q(r)$ satisfies Eq. (3.10.152) under the form

$$d^2Q/dr^2 + (4/r)\ dQ/dr = -b_1\varrho, \tag{3.10.205}$$

we can transform the integral (3.10.204) as follows:

$$U_m = \int_0^{r_1} [(2/3)\ d(r^3Q^2)/dr + (r^4/3)\ d(QQ')/dr + 4r^3QQ'/3 + b_1\varrho r^4Q/3]\ dr$$
$$= 2r_1^3Q^2(r_1)/3 + \int_0^{r_1} [(1/3)\ d(r^4QQ')/dr + b_1\varrho r^4Q/3]\ dr = -r_1^3Q^2(r_1)/3 + (b_1/3)\int_0^{r_1} \varrho r^4Q(r)\ dr. \tag{3.10.206}$$

We have also employed the boundary condition (3.10.168), written under the equivalent form

$$r_1Q'(r_1) + 3Q(r_1) = 0. \tag{3.10.207}$$

Thus, Eq. (3.10.199) becomes

$$(n+1)\int_V P\ dV = -2W + 2U_m + 2r_1^3Q^2(r_1)/3 - CM, \tag{3.10.208}$$

by using the results (3.10.200), (3.10.206). Eliminating the pressure integral between Eqs. (3.10.197) and (3.10.208), we get for the gravitational energy

$$W = [1/(5-n)]\left\{(n+7)U_m + 2r_1^3Q^2(r_1) + [(n+1)/4\pi]\int_S x_j(-\delta_{jk}H^2/2 + H_jH_k)\ dS_k - 3CM\right\}. \tag{3.10.209}$$

The surface integral over the magnetic field can be expressed in a more concise form by introducing the tensor

$$S_{jk} = (1/4\pi)\int_S x_j[(-H^2/2)\ dS_k + H_kH_\ell\ dS_\ell]. \tag{3.10.210}$$

The projection of the surface element dS perpendicular to the x_k-axis is equal to (e.g. Smirnow 1967)

$$dS_k = n_k\ dS;\quad \vec{n} = \vec{n}(n_1, n_2, n_3);\quad n_1 = \sin\lambda\cos\varphi;\quad n_2 = \sin\lambda\sin\varphi;\quad n_3 = \cos\lambda, \tag{3.10.211}$$

where $\vec{n}$ is the normal to the surface element dS. Since the magnetic field is a small perturbation, it is sufficient to integrate Eq. (3.10.210) in the first order theory of Trehan and Billings (1971) over the spherical surface of the undistorted polytrope: $dS = r^2\sin\lambda\ d\lambda\ d\varphi,\ x_j = rn_j$.

The components H_1, H_2, H_3 of the magnetic field $\vec{H}$ in Cartesian (x_1, x_2, x_3)-coordinates are found from the spherical components $H_r, H_\lambda, H_\varphi$ with the transformation matrix [cf. Eq. (3.8.135)]:

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} \sin\lambda\cos\varphi & \cos\lambda\cos\varphi & -\sin\varphi \\ \sin\lambda\sin\varphi & \cos\lambda\sin\varphi & \cos\varphi \\ \cos\lambda & -\sin\lambda & 0 \end{bmatrix} \begin{bmatrix} H_r \\ H_\lambda \\ H_\varphi \end{bmatrix}. \tag{3.10.212}$$

Thus, the Cartesian coordinates of the magnetic field from Eq. (3.10.202) are

$$H_1 = \sin\lambda\cos\lambda\cos\varphi\ [2Q - (1/r)\ d(r^2Q)/dr] = -\sin\lambda\cos\lambda\cos\varphi\ r\ dQ/dr;$$
$$H_2 = -\sin\lambda\cos\lambda\sin\varphi\ r\ dQ/dr;\quad H_3 = 2Q + \sin^2\lambda\ r\ dQ/dr. \tag{3.10.213}$$

Using Eqs. (3.10.201), (3.10.207), and inserting Eqs. (3.10.211), (3.10.213) into Eq. (3.10.210), we obtain after some algebra

$$S_{11} = 2r_1^3Q^2(r_1)/15;\quad S_{22} = S_{11};\quad S_{33} = r_1^3Q^2(r_1)/15;\quad S_{jk} = 0\quad \text{if}\quad j \neq k, \tag{3.10.214}$$

taking into account the integral

$$\int_0^{\pi/2} \sin^{2j}\lambda\ d\lambda = \int_0^{\pi/2} \cos^{2j}\lambda\ d\lambda = (\pi/2)\times 1\times 3\times 5\times ...\times(2j-1)/2\times 4\times 6\times ...\times 2j$$
$$= (\pi/2)(2j-1)!!/(2j)!!,\qquad (j = 1, 2, 3, ...). \tag{3.10.215}$$

Thus, the surface integral in Eq. (3.10.197) is just

$$(1/4\pi)\int_S x_j(-\delta_{jk}H^2/2+H_jH_k)\ dS_k=S_{11}+S_{22}+S_{33}=2S_{11}+S_{33}=r_1^3Q^2(r_1)/3, \qquad (3.10.216)$$

and Eq. (3.10.209) can be written under the equivalent form

$$W=[1/(5-n)][(n+7)U_m+(n+7)r_1^3Q^2(r_1)/3-3CM]. \qquad (3.10.217)$$

The nonvanishing components of the gravitational energy tensor W_{jk}, ($W_{jk}=0$ if $j\neq k$) can be obtained at once by eliminating the pressure integral between the nonvanishing diagonal components of Eq. (3.10.196)

$$\int_V P\ dV+W_{11}+H_{33}+S_{11}=0;\qquad \int_V P\ dV+W_{33}+2H_{11}-H_{33}+S_{33}=0,$$
$$(U_m=H_{11}+H_{22}+H_{33}=2H_{11}+H_{33};\quad W_{11}=W_{22}), \qquad (3.10.218)$$

and Eq. (3.10.208), written under the form

$$(n+1)\int_V P\ dV=-2(2W_{11}+W_{33})+2(2H_{11}+H_{33})+2(2S_{11}+S_{33})-CM. \qquad (3.10.219)$$

We get

$$W_{11}=W_{22}=[8H_{11}+(n-1)H_{33}+(n+3)S_{11}+4S_{33}-CM]/(5-n);$$
$$W_{33}=[2(n-1)H_{11}+(9-n)H_{33}+8S_{11}+(n-1)S_{33}-CM]/(5-n). \qquad (3.10.220)$$

The magnetic energy tensor (2.6.77) is obtained with Eqs. (3.10.201), (3.10.213):

$$H_{11}=(1/8\pi)\int_V H_1^2\ dV\approx(1/8\pi)\int_0^{2\pi}\cos^2\varphi\ d\varphi\int_0^{\pi}\sin^3\lambda\cos^2\lambda\ d\lambda\int_0^{r_1}r^4(dQ/dr)^2\ dr$$
$$=(1/30)\int_0^{r_1}r^4(dQ/dr)^2\ dr;\quad H_{22}=H_{11};\quad H_{33}=(1/8\pi)\int_V H_3^2\ dV$$
$$\approx(1/8\pi)\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\lambda\ d\lambda\int_0^{r_1}(2Q+\sin^2\lambda\ r\ dQ/dr)^2r^2\ dr$$
$$=2\int_0^{r_1}[Q^2+2rQ\ (dQ/dr)/3+2r^2(dQ/dr)^2/15]r^2\ dr;\quad H_{jk}=0\quad\text{if}\quad j\neq k. \qquad (3.10.221)$$

The calculation of the moment of inertia tensor (2.6.74) is straightforward:

$$I_{11}=\int_V \varrho x_1^2\ dV\approx\varrho_0\alpha^5\int_0^{2\pi}\cos^2\varphi\ d\varphi\int_0^{\pi}\sin^3\lambda\ d\lambda\int_0^{\Xi_1(\lambda)}\Theta^n(\xi,\lambda)\ \xi^4\ d\xi$$
$$=2\pi\varrho_0\alpha^5\int_0^1(1-\mu^2)\ d\mu\int_0^{\Xi_1(\mu)}\Theta^n(\xi,\mu)\ \xi^4\ d\xi$$
$$\approx 2\pi\varrho_0\alpha^5\int_0^1(1-\mu^2)\ d\mu\int_0^{\xi_1}\{\theta^n(\xi)+\varepsilon n\theta^{n-1}(\xi)\ [\psi_0(\xi)+\psi_2(\xi)\ P_2(\mu)]\}\xi^4\ d\xi+(\Xi_1-\xi_1)\xi_1^4\theta^n(\xi_1)$$
$$\approx(4\pi\varrho_0\alpha^5/3)\int_0^{\xi_1}\{\theta^n(\xi)+\varepsilon n\theta^{n-1}(\xi)\ [\psi_0(\xi)-\psi_2(\xi)/5]\}\xi^4\ d\xi;$$
$$I_{22}=I_{11};\quad I_{33}=\int_V\varrho x_3^2\ dV\approx 2\pi\varrho_0\alpha^5\int_0^{\pi}\cos^2\lambda\sin\lambda\ d\lambda\int_0^{\Xi_1(\lambda)}\Theta^n(\xi,\lambda)\ \xi^4\ d\xi$$
$$\approx(4\pi\varrho_0\alpha^5/3)\int_0^{\xi_1}\{\theta^n(\xi)+\varepsilon n\theta^{n-1}(\xi)\ [\psi_0(\xi)+2\psi_2(\xi)/5]\}\xi^4\ d\xi;\quad I_{jk}=0\quad\text{if}\quad j\neq k. \qquad (3.10.222)$$

Boundary values for the diagonal tensors from Eqs. (3.10.214) and (3.10.220)-(3.10.222) have been published by Trehan and Billings (1971) for the polytropic indices from Table 3.10.3.

Moments of inertia for rotating relativistic magnetopolytropes pervaded by a dipole field of the form (3.10.26) have been calculated by Konno (2001), as related to pulsars of polytropic index $n=0,0.5,1,1.5$.

(iii) Poloidal and toroidal field. In this more general case Trehan and Uberoi (1972) constrain the magnetopolytrope to be axisymmetric, and the magnetic field can always be written under the form (Roberts 1967, Sec. 4.6; Chandrasekhar 1981, App. III)

$$\vec{H} = \nabla \times [\ell Q(\ell, z)\ \vec{e}_\varphi] + \ell T(\ell, z)\ \vec{e}_\varphi = -(1/\ell)[\partial(\ell^2 Q)/\partial z]\ \vec{e}_\ell + \ell T\ \vec{e}_\varphi + (1/\ell)[\partial(\ell^2 Q)/\partial \ell]\ \vec{e}_z, \tag{3.10.223}$$

with Q and T equal to the poloidal and toroidal magnetic field stream functions. The φ-component of the magnetostatic equation (3.10.15) has to vanish due to axisymmetry, so it is required that

$$[(\nabla \times \vec{H}) \times \vec{H}]_\varphi = 0 \quad \text{or} \quad (\vec{e}_\varphi/\ell^2)\{[\partial(\ell^2 T)/\partial z]\ \partial(\ell^2 Q)/\partial \ell - [\partial(\ell^2 Q)/\partial z]\ \partial(\ell^2 T)/\partial \ell\} = 0, \tag{3.10.224}$$

which is fulfilled if $\ell^2 T = f(\ell^2 Q)$. For this functional form Trehan and Uberoi (1972) take the simple linear relationship

$$T = b_0 Q, \qquad (b_0 = \text{const}). \tag{3.10.225}$$

Eqs. (3.10.146), (3.10.148)-(3.10.151) for the poloidal field remain entirely valid also in the present, more general case. Inserting Eq. (3.10.223) into the φ-component of Eq. (3.10.146), we obtain instead of Eq. (3.10.152):

$$\partial^2 Q/\partial r^2 + (4/r)\ \partial Q/\partial r + [(1-\mu^2)/r^2]\ \partial^2 Q/\partial \mu^2 - (4\mu/r^2)\ \partial Q/\partial \mu + b_0^2 Q = -b_1 \varrho. \tag{3.10.226}$$

The components of the magnetic field (3.10.223) are in spherical coordinates equal to

$$\begin{aligned} &\vec{H} = \vec{H}(H_r, H_\lambda, H_\varphi) = \nabla \times [r \sin\lambda\ Q(r,\lambda)\ \vec{e}_\varphi] + r\sin\lambda\ T(r,\lambda)\ \vec{e}_\varphi \\ &= (\vec{e}_r/\sin\lambda)\ \partial(\sin^2\lambda\ Q)/\partial\lambda - (\vec{e}_\lambda \sin\lambda/r)\ \partial(r^2 Q)/\partial r + r\sin\lambda\ T\ \vec{e}_\varphi. \end{aligned} \tag{3.10.227}$$

The toroidal field intensity $H_\varphi = r\sin\lambda\ T \propto \varrho$ from Eqs. (3.10.39), (3.10.98), (3.10.227) vanishes on the surface $r_1 = r_1(\lambda)$ and in the vacuum region outside the axisymmetric polytrope, together with $Q = T/b_0$ from Eq. (3.10.225): $Q[r_1(\lambda), \lambda] = 0$. A second boundary condition $(\partial Q/\partial r)_{r=r_1(\lambda)} = 0$ results via Eq. (3.10.227) by supposing $H_\lambda = 0$ on the boundary (Trehan and Uberoi 1972).

Monaghan (1976) circumvents the draw-back of a vanishing external field $Q(r,\lambda), T(r,\lambda) = 0$, by introducing for Q mildly singular functions, allowing for a nonzero, external, poloidal magnetic field.

To find a *particular* solution of the nonhomogeneous equation (3.10.226), Woltjer (1960) expands Q into a series of Gegenbauer polynomials $G_j^{3/2}(\cos\lambda) = G_j^{3/2}(\mu)$, as in Eq. (3.10.154). Because of the orthogonality of $G_j^k(\mu)$, all angular terms ($j \neq 0$) vanish, and Eq. (3.10.226) turns into [cf. Eqs. (3.10.158), (3.10.159)]

$$d^2 Q/dr^2 + (4/r)\ dQ/dr + b_0^2 Q = -b_1 \varrho. \tag{3.10.228}$$

A particular solution of this equation can be found with the aid of the nonhomogeneous Bessel equation (e.g. Kamke 1956, p. 443):

$$Q_p(r) = -(b_0 b_1/r)\left[y_1(b_0 r)\int_0^r r'^3 j_1(b_0 r')\ \varrho(r')\ dr' + j_1(b_0 r)\int_r^{r_1} r'^3 y_1(b_0 r')\ \varrho(r')\ dr'\right]. \tag{3.10.229}$$

j_k and y_k are spherical Bessel and Neumann functions of order k, $(k = 0, \pm 1, \pm 2, ...)$, connected to the ordinary Bessel and Neumann functions from Eqs. (2.3.12) and (2.3.13) by $j_k(r) = (\pi/2r)^{1/2} J_{k+1/2}(r)$ and $y_k(r) = (\pi/2r)^{1/2} Y_{k+1/2}(r)$, respectively (e.g. Morse and Feshbach 1953, Abramowitz and Stegun 1965). Since the magnetic field is a small first order quantity, we have replaced $\varrho(r,\mu)$ by its spherical approximation $\varrho(r)$, so $r_1 = \alpha\xi_1$ is just the radius of the Lane-Emden sphere. To verify the particular solution (3.10.229) by direct insertion into Eq. (3.10.228), we have to make use of the Wronskian [see Eq. (2.4.146)]

$$W[j_k(r), y_k(r)] = j_k\ dy_k/dr - y_k\ dj_k/dr = j_{k+1}\ y_k - j_k\ y_{k+1} = 1/r^2, \tag{3.10.230}$$

together with the recurrence formula $dj_k/dr = kj_k/r - j_{k+1} = -(k+1)j_k/r + j_{k-1}$, and an analogous formula for y_k. The general solution of the homogeneous part of Eq. (3.10.226) can be expressed under the form (cf. Eqs. (3.10.154), (3.10.156); Kamke 1956, p. 440; Woltjer 1960):

$$Q_h(r,\mu) = \sum_{k=0}^{\infty} A_k\ j_{k+1}(b_0 r)\ G_k^{3/2}(\mu)/r, \qquad (A_k = \mathrm{const}). \tag{3.10.231}$$

So, the complete solution of Eq. (3.10.226) is $Q(r,\mu) = Q_h(r,\mu) + Q_p(r)$. From the boundary conditions $Q[r_1(\mu),\mu] = 0$ and $(\partial Q/\partial r)_{r=r_1(\mu)} = 0$ results at once $A_k = 0$ if $k \neq 0$, and we are left with

$$Q(r_1) \propto A_0\ j_1(b_0 r_1) - b_0 b_1\ y_1(b_0 r_1)\int_0^{r_1} r^3 j_1(b_0 r)\ \varrho(r)\ dr = 0; \quad (dQ/dr)_{r=r_1} \propto A_0\ j_2(b_0 r_1)$$
$$-b_0 b_1\ y_2(b_0 r_1)\int_0^{r_1} r^3 j_1(b_0 r)\ \varrho(r)\ dr = 0, \qquad (G_0^{3/2}(\mu) = 1). \tag{3.10.232}$$

Eq. (3.10.232) can be fulfilled only if

$$A_0 = 0 \quad \text{and} \quad \int_0^{r_1} r^3 j_1(b_0 r)\ \varrho(r)\ dr = 0, \tag{3.10.233}$$

which determines the constant b_0. Thus, the solution of Eq. (3.10.226) is given by Eq. (3.10.229), with b_0 being a root of Eq. (3.10.233).

Eqs. (3.10.226), (3.10.229), (3.10.233) become in terms of the Lane-Emden variables equal to

$$d^2Q/d\xi^2 + (4/\xi)\ dQ/d\xi + {b_0'}^2 Q = -\theta^n(\xi), \qquad (Q(r,\mu) = Q_p(r) = b_1\varrho_0\alpha^2 Q(\xi);\ b_0' = \alpha b_0), \tag{3.10.234}$$

$$Q(\xi) = -(b_0'/\xi)\left[y_1(b_0'\xi)\int_0^{\xi} \xi'^3 j_1(b_0'\xi')\ \theta^n(\xi')\ d\xi' + j_1(b_0'\xi)\int_{\xi}^{\xi_1} \xi'^3 y_1(b_0'\xi')\ \theta^n(\xi')\ d\xi'\right], \tag{3.10.235}$$

$$\int_0^{\xi_1} \xi^3 j_1(b_0'\xi)\ \theta^n(\xi)\ d\xi = 0. \tag{3.10.236}$$

The further procedure for the determination of the fundamental function Θ is quite similar to that already outlined for the purely poloidal field. The basic equation is still Eq. (3.10.149), where the magnetic term (3.10.173) now takes the form [Eqs. (B.39), (3.10.234)]

$$\nabla^2\Theta = -\Theta^n + \varepsilon\ \nabla^2[\xi^2(1-\mu^2)\ Q(\xi)] = -\Theta^n + \varepsilon\{4Q + 4\xi Q'/3$$
$$-2\xi^2({b_0'}^2 Q + \theta^n)/3 + [-4\xi Q'/3 + 2\xi^2({b_0'}^2 Q + \theta^n)/3]\ P_2(\mu)\}. \tag{3.10.237}$$

The magnetic energy of the configuration is in the spherical approximation via Eqs. (2.6.78), (3.10.204), (3.10.206), (3.10.225), (3.10.227), (3.10.228) equal to

$$U_m = \int_V H^2\ dV/8\pi = \int_0^{r_1} (2Q^2 + 4rQQ'/3 + r^2{Q'}^2/3 + b_0^2 r^2 Q^2/3) r^2\ dr$$
$$= (1/3)\int_0^{r_1} [2\ d(r^3Q^2)/dr + d(r^4QQ')/dr + b_1\varrho r^4 Q + 2b_0^2 r^4 Q^2]\ dr$$
$$= (1/3)\int_0^{r_1} (b_1\varrho + 2b_0^2 Q) r^4 Q\ dr. \tag{3.10.238}$$

The purely toroidal and poloidal component of the magnetic energy is in virtue of Eqs. (3.10.225), (3.10.227) equal to

$$U_{mT} = \int_V H_\varphi^2\ dV/8\pi = (b_0^2/3)\int_0^{r_1} r^4 Q^2\ dr; \quad U_{mP} = U_m - U_{mT}. \tag{3.10.239}$$

Structural parameters of these hydrostatic magnetopolytropes have been quoted by Trehan and Uberoi (1972), and Trehan (1984) for the polytropic indices $n = 1, 1.5, 2, 3, 3.5$. These magnetopolytropes are found to be prolate spheroids ($a_3 > a_1 = a_2$), the prolateness decreasing as the polytropic index n increases. Das and Tandon (1976) have added differential rotation to the treatment of Trehan and Uberoi (1972).

Galli (1993), and Papasotiriou and Geroyannis (2001) have calculated particular, nearly *spherical* equilibrium polytropes ($1 \leq n \leq 3.5$), resulting from counterbalancing the effects of (differential) rotation and of a poloidal field – producing oblate spheroids – with the effect of a toroidal field, generating prolate spheroids. For the same polytropes Papasotiriou and Geroyannis (2001) have calculated critically rotating configurations.

Equilibrium structures of magnetopolytropes with strong poloidal fields have been calculated by Monaghan (1968) and Miketinac (1975), indicating that the magnetic field structure is not a sensitive function of polytropic index if $1 \lesssim n \lesssim 3$. The sequences start with a sphere (no magnetic field), and terminate with a strongly distorted doughnut-shaped object, when the ratio of magnetic to gravitational energy is 0.35 (Monaghan 1968) and 0.23 (Miketinac 1975) for the polytropic index $n = 3$.

Singh et al. (1987) virtually neglect magnetic effects, their treatment being equivalent to the double approximation technique from Sec. 3.6.

A differential rotation law of the form $\Omega = c_1 + c_2 H_\varphi / r \sin\lambda,\ (c_1, c_2 = \mathrm{const})$ has been obtained by Maezawa (1979), by assuming the velocity field $\vec{v} = v_r \vec{e}_r + v_\lambda \vec{e}_\lambda + \Omega r \sin\lambda\ \vec{e}_\varphi,\ (v_\lambda \ll \Omega r)$ in a spherical magnetopolytrope of infinite conductivity.

3.10.5 Magnetic Fields in Cylindrical Polytropes

At first we consider only polytropic indices $0 \leq n < \infty$; in this case all physical parameters of undistorted cylinders remain finite (Sec. 2.6.8). In a frame rotating together with the fluid at constant angular velocity $\vec{\Omega}$, the equation of hydrostatic support of a magnetic fluid becomes [cf. Eqs. (3.1.14), (3.10.15)]

$$\nabla P = \varrho\ \nabla\Phi - \varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + (1/4\pi)(\nabla \times \vec{H}) \times \vec{H}. \tag{3.10.240}$$

In a cylindrical (ℓ, φ, z)-frame the rotational term $-\varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ has the components $(\varrho\Omega^2\ell, 0, 0)$. To get the problem tractable, Talwar and Gupta (1973) consider the prevailing current system $\vec{J}$ to be such, that the associated Lorentz force per unit volume $(\nabla \times \vec{H}) \times \vec{H}/4\pi,\ (p = 1)$ from Eq. (3.10.8) has a nonvanishing component only in the radial ℓ-direction. This requirement is fulfilled by a magnetic field of the form

$$\vec{H} = \vec{H}[0, H_\varphi(\ell), H_z(\ell)]. \tag{3.10.241}$$

Talwar and Gupta (1973) further assume that the hydrostatic magnetic pressure $H^2/8\pi,\ (p = 1)$ from Eq. (2.6.82) bears a constant ratio to the fluid pressure, so that we can write

$$H^2/8\pi = (H_\varphi^2 + H_z^2)/8\pi = aP, \qquad (a = \mathrm{const}). \tag{3.10.242}$$

This implies that fluid and magnetic pressure vanish simultaneously on the boundary of the magnetic cylinder. Calculating the magnetic force

$$(\nabla \times \vec{H}) \times \vec{H} = -[H_z\ dH_z/d\ell + (H_\varphi/\ell)\ d(\ell H_\varphi)/d\ell]\ \vec{e}_\ell = -(\nabla H^2/2 + H_\varphi^2/\ell)\ \vec{e}_\ell, \tag{3.10.243}$$

and taking the divergence of Eq. (3.10.240), we find $[\nabla^2\Phi = -4\pi G\varrho;\ \nabla \cdot (\Omega^2\ell\ \vec{e}_\ell) = 2\Omega^2;\ P = P(\ell)]:$

$$(1 + a)\ \nabla \cdot [(1/\varrho)\ \nabla P] + \nabla \cdot (H_\varphi^2\ \vec{e}_\ell/4\pi\varrho\ell) + 4\pi G\varrho - 2\Omega^2 = 0. \tag{3.10.244}$$

For a purely toroidal field ($H_z = 0$) this equation writes with the polytropic equation of state $P = K\varrho^{1+1/n}$ as

$$(1/\ell)\ d(\ell\ d\varrho^{1/n}/d\ell)/d\ell + [2a/\ell(n + 1)(1 + a)]\ d\varrho^{1/n}/d\ell$$
$$+(4\pi G\varrho - 2\Omega^2)/K(n + 1)(1 + a) = 0, \qquad [H = H_\varphi(\ell)], \tag{3.10.245}$$

and for a purely axial field ($H_\varphi = 0$) we obtain

$$(1/\ell)\ d(\ell\ d\varrho^{1/n}/d\ell)/d\ell + (4\pi G\varrho - 2\Omega^2)/K(n+1)(1+a) = 0, \qquad [H = H_z(\ell)]. \tag{3.10.246}$$

We now define dimensionless variables by

$$\ell = [(n+1)(1+a)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\eta = [(n+1)(1+a)P_0/4\pi G\varrho_0^2]^{1/2}\eta = \alpha\eta;$$
$$\varrho = \varrho_0\Theta^n; \quad P = P_0\Theta^{n+1}; \quad \beta = \Omega^2/2\pi G\varrho_0, \qquad (0 \le n < \infty). \tag{3.10.247}$$

With these dimensionless variables Eqs. (3.10.245) and (3.10.246) become, respectively

$$d^2\Theta/d\eta^2 + (1/\eta)[1 + 2a/(n+1)(1+a)]\ d\Theta/d\eta + \Theta^n - \beta = 0, \qquad (H = H_\varphi;\ 0 \le n < \infty), \tag{3.10.248}$$

$$d^2\Theta/d\eta^2 + (1/\eta)\ d\Theta/d\eta + \Theta^n - \beta = 0, \qquad (H = H_z;\ 0 \le n < \infty). \tag{3.10.249}$$

It may be noted that Eq. (3.10.249) for the purely axial field is formally the same as Eq. (3.9.8) in the field-free case $H = 0$: The structure of a rotating polytropic cylinder ($0 \le n < \infty$) with a purely axial field $H = H_z(\ell)$ is modified with respect to the rotating cylinder merely in the sense that its boundary is expanded by the factor $(1+a)^{1/2}$ from Eq. (3.10.247). Because Eq. (3.10.249) is formally a particular case ($a = 0$) of Eq. (3.10.248), we shall henceforth consider only Eq. (3.10.248). Near the origin Talwar and Gupta (1973) find the expansion [cf. Eqs. (2.4.23), (3.9.9)]

$$\Theta(\eta) \approx 1 - (1-\beta)\eta^2/(2^1 \times 1!)^2[1 + a/(n+1)(1+a)]$$
$$+n(1-\beta)\eta^4/(2^2 \times 2!)^2[1 + a/(n+1)(1+a)][2 + a/(n+1)(1+a)], \qquad (\eta \approx 0), \tag{3.10.250}$$

where the initial conditions $\Theta(0) = 1$ and $\Theta'(0) = 0$ have been used. Mass and radius of the considered magnetically distorted cylinders increase with increasing field strength. Uniform rotation of polytropic cylinders has the same effect, as shown in Table 3.9.1 and Eq. (3.9.11).

The discussion of the magnetic isothermal case $n = \pm\infty$ suffers from the detriments already pointed out in Sec. 3.9.1 in connection with the rotating isothermal cylinder: The radius of the undistorted cylinder with polytropic index $n = \pm\infty$ is infinite. Analogously to Eq. (3.9.16) we introduce (Talwar and Gupta 1973):

$$P = K\varrho = K\varrho_0 \exp(-\Theta) = P_0 \exp(-\Theta); \quad \ell = [K(1+a)/4\pi G\varrho_0]^{1/2}\eta = \alpha\eta, \qquad (n = \pm\infty). \tag{3.10.251}$$

With $P = K\varrho$ we rewrite the fundamental equation (3.10.244) as

$$K(1+a)\ \nabla^2 \ln\varrho + \nabla\cdot(H_\varphi^2\ \vec{e}_\ell/4\pi\varrho\ell) + 4\pi G\varrho - 2\Omega^2 = 0. \tag{3.10.252}$$

Turning to dimensionless variables, we observe that this equation takes the same simple form

$$d^2\Theta/d\eta^2 + (1/\xi)\ d\Theta/d\eta - \exp(-\Theta) + \beta = 0, \qquad (H = H_\varphi \text{ or } H = H_z;\ \Theta(0), \Theta'(0) = 0), \tag{3.10.253}$$

for both, the purely toroidal and the purely axial field. This equation is formally the same as Eq. (3.9.19) for the rotating isothermal cylinder, and involves the influence of the magnetic field merely through the factor $(1+a)^{1/2}$ from the definition (3.10.251) of ℓ, as in the case (3.10.249) of the purely axial field. For nonrotating isothermal cylinders ($\beta = 0$) the analytic solution $\Theta = 2\ln(1+\eta^2/8)$ of Eq. (3.10.253) is formally equal to the solution (2.3.48) of the nonmagnetic nonrotating cylinder (Stodólkiewicz 1963).

Because the topology of the magnetic field within a cylinder may be fairly complicated, it is of interest to see how the results are modified by making another assumption regarding the variation of the ambient field. Instead of Eq. (3.10.242) Karnik and Talwar (1978) assume that the Alfvén velocity is constant throughout the cylinder:

$$v_B = B/(4\pi p\varrho)^{1/2} = \text{const.} \tag{3.10.254}$$

Table 3.10.4 Values of the boundary coordinate Ξ_1 from Fig. 3.10.2, and of the corresponding dimensionless mass per unit length $-\Xi_1\Theta'(\Xi_1) - [nb/2(n+1)K\varrho_0^{1/n}][\Xi_1\Theta'(\Xi_1)/\Theta(\Xi_1)] + \beta\Xi_1^2/2$ according to Eq. (3.10.259), (Karnik and Talwar 1978).

n	$nb/2(n+1)K\varrho_0^{1/n}$	Ξ_1		Dimensionless Mass	
		$\beta = 0$	$\beta = 0.02$	$\beta = 0$	$\beta = 0.02$
1	0.00	2.40	2.44	1.25	1.28
	0.05	3.87	4.02	1.40	1.44
	0.10	5.70	6.26	1.56	1.61
1.5	0.00	2.65	2.71	1.06	1.09
	0.05	3.82	4.00	1.17	1.21
	0.10	5.20	5.71	1.28	1.32
3	0.00	3.53	3.78	0.740	0.769
	0.05	4.47	5.03	0.800	0.832
	0.10	5.48	6.86	0.860	0.896

Recall that v_B is just the velocity of magnetohydrodynamic waves propagating along the field lines in an ideal, incompressible fluid of infinite conductivity (e.g. Alfvén and Fälthammar 1963, Roberts 1967, Chap. 5). Since we are taking $p = 1$, Eq. (3.10.254) reduces to ($\vec{B} = \vec{H}$)

$$H^2/4\pi = b\varrho, \qquad (b = \text{const};\ H_\ell = 0). \tag{3.10.255}$$

Applying the divergence to the equation of hydrostatic equilibrium (3.10.240), we find with Eqs. (3.10.241), (3.10.243), (3.10.255):

$$(1/\ell)\ d\{(\ell/\varrho)\ [dP/d\ell + (b/2)\ d\varrho/d\ell] + H_\varphi^2/4\pi\varrho\}/d\ell + 4\pi G\varrho - 2\Omega^2 = 0. \tag{3.10.256}$$

It is to be noted that under the condition of a purely axial field ($H_\varphi(\ell) = 0$) or of a purely toroidal field ($H_z(\ell) = 0$) this equation reduces to

$$(1/\ell)\ d\{(\ell/\varrho)\ [dP/d\ell + (b/2)\ d\varrho/d\ell]\}/d\ell + 4\pi G\varrho - 2\Omega^2 = 0. \tag{3.10.257}$$

The analogue of Eqs. (3.10.248) and (3.10.249) is

$$(1/\xi)\ d(\xi\ d\Theta/d\xi)/d\xi + [nb/2(n+1)K\varrho_0^{1/n}\xi]\ d[(\xi/\Theta)\ d\Theta/d\xi]/d\xi + \Theta^n - \beta = 0,$$
$$(H_\varphi = 0 \text{ or } H_z = 0;\ 0 \le n < \infty), \tag{3.10.258}$$

where we now have introduced into Eq. (3.10.257) the dimensionless variables from Eqs. (3.9.3), (3.9.6). Mass per unit length and mean density of the complete magnetopolytropic cylinders are obtained at once:

$$M_1(\Xi_1) = 2\pi\varrho_0\alpha^2\int_0^{\Xi_1}\Theta^n\xi\ d\xi = 2\pi\varrho_0\alpha^2\{-\Xi_1\Theta'(\Xi_1) - [nb/2(n+1)K\varrho_0^{1/n}][\Xi_1\Theta'(\Xi_1)/\Theta(\Xi_1)]$$
$$+\beta\Xi_1^2/2\}; \qquad \varrho_m(\Xi_1) = M_1(\Xi_1)/\pi\alpha^2\Xi_1^2, \qquad (0 \le n < \infty), \tag{3.10.259}$$

implying $\Theta'(\Xi_1) = 0$ if $b \ne 0$, in order to assure a finite total mass when $\Theta(\Xi_1) = 0$ at the boundary.

Karnik and Talwar (1978) integrate Eq. (3.10.258) numerically (Fig. 3.10.2 and Table 3.10.4) for a nonrotating and a rotating cylinder ($\beta = 0.02$), Their results show that mass and radius of the cylinders increase, and the ratio ϱ_m/ϱ_0 decreases, when the strength of the considered field increases, and/or the cylinders rotate faster. If $\beta \gtrsim 0.02$, it is not possible to get hydrostatic equilibrium configurations; this fact may be connected with the occurrence of the limiting critical rotation β_c discussed in Sec. 3.9.1 (Table 3.9.1).

Sood (1980) has considered a generalization of Eq. (3.10.255) by taking

$$H_\varphi^2 = b_1\varrho; \quad dH^2/d\ell = d(H_\varphi^2 + H_z^2)/d\ell = 2b_2\varrho\ell, \qquad (b_1, b_2 = \text{const}). \tag{3.10.260}$$

The second equation can be integrated at once:

$$H_z^2 = 2b_2\int_0^\ell \varrho(\ell')\ \ell'\ d\ell' - b_1\varrho + b_3, \qquad (b_3 = \text{const}). \tag{3.10.261}$$

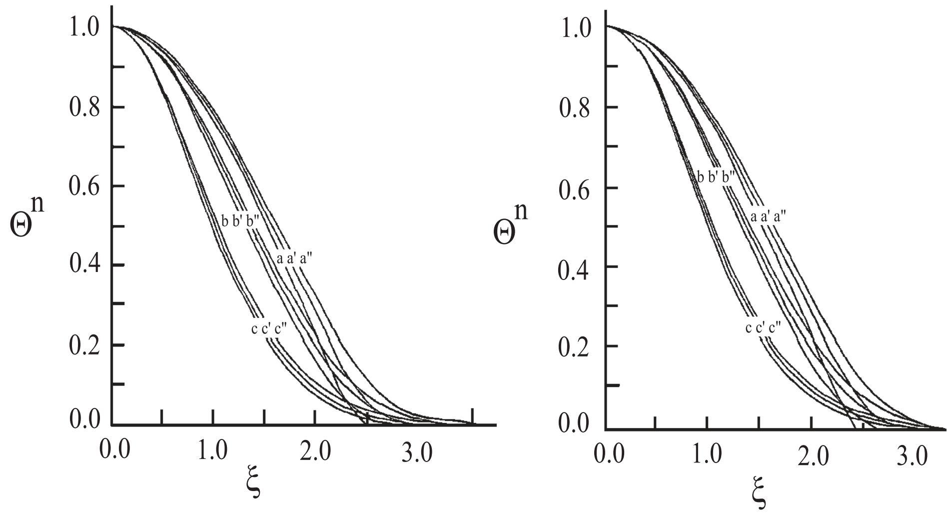

Fig. 3.10.2 Run of the dimensionless density $\Theta^n(\xi)$ for a nonrotating cylinder (on the left), and for a rotating one (on the right) with $\beta = 0.02$ according to Eq. (3.10.258). If $n = 1$, the curves labeled a, b, c correspond to $nb/2(n+1)K\varrho_0^{1/n}$ =0, 0.05, 0.1, respectively. Single- and double-primed curves are for $n = 1.5, 3$, respectively (Karnik and Talwar 1978).

Using Eqs. (3.10.243) and (3.10.260) for the evaluation of $(\nabla\times\vec{H})\times\vec{H} = -[(1/2)\, dH^2/d\ell + H_\varphi^2/\ell]\,\vec{e}_\ell = -(b_1\varrho/\ell + b_2\varrho\ell)\,\vec{e}_\ell$, and taking the divergence of Eq. (3.10.240), we find

$$[(n+1)K/\ell]\; d(\ell\; d\varrho^{1/n}/d\ell)/d\ell + 4\pi G\varrho - 2\Omega^2 + b_2/2\pi = 0. \qquad (3.10.262)$$

Obviously, the Alfvén velocity (3.10.254) is not constant for this magnetic field pattern. Sood (1980) points out some other feasible cases belonging to the category when in Eq. (3.10.240) $\nabla\cdot[(\nabla\times\vec{H})\times\vec{H}/4\pi\varrho] = \mathrm{const}$. He chooses

$$d(\ell^2 H_\varphi^2)/d\ell = 2b_1\varrho\ell^3; \quad dH_z^2/d\ell = 2b_2\varrho\ell, \qquad (3.10.263)$$

representing a toroidal field H_φ that vanishes along the axis $\ell = 0$, and assumes a constant value on the boundary of the cylinder at $\varrho = 0$. It can be verified in the same manner as for Eq. (3.10.262), that the divergence of Eq. (3.10.240) becomes

$$[(n+1)K/\ell]\; d(\ell\; d\varrho^{1/n}/d\ell)/d\ell + 4\pi G\varrho - 2\Omega^2 + (b_1+b_2)/2\pi = 0,$$
$$[(\nabla\times\vec{H})\times\vec{H} = -(b_1+b_2)\varrho\ell\;\vec{e}_\ell]. \qquad (3.10.264)$$

An alternative magnetic field pattern under this category can be obtained if we let

$$H_\varphi^2 = b_1\varrho\ell^2; \quad dH^2/d\ell = d(H_\varphi^2 + H_z^2)/d\ell = 2b_2\varrho\ell. \qquad (3.10.265)$$

The physical significance of this model can easily be seen by noting that H_φ^2 vanishes along the axis $\ell = 0$ and also on the boundary $\varrho = 0$, but has a finite maximum in between, while the axial field

$$H_z^2 = 2b_2\int_0^\ell \varrho(\ell')\;\ell'\;d\ell' - b_1\varrho\ell^2 + H_z^2(0), \qquad (3.10.266)$$

shows a more complex behaviour. With the choice (3.10.265) for the magnetic field $(\nabla\times\vec{H})\times\vec{H} = -(b_1+b_2)\varrho\ell\;\vec{e}_\ell$, we recover Eq. (3.10.264).

Outside the boundary we have a current-free region, and if $\vec{J} = 0$, the Maxwell equation (3.10.1) yields

$$\nabla \times \vec{H}_e = -(dH_{ez}/d\ell)\ \vec{e}_\varphi + (1/\ell)\ [d(\ell H_{e\varphi})/d\ell]\ \vec{e}_z = 0, \qquad (3.10.267)$$

with the elementary external solution

$$H_{e\varphi} = c_1/\ell; \quad H_{ez} = c_2, \qquad (\vec{J} = 0;\ c_1, c_2 = \mathrm{const}). \qquad (3.10.268)$$

Returning to Eq. (3.10.264), we can integrate twice $(4\pi G\varrho = -\nabla^2\Phi = -(1/\ell)\ d(\ell\ d\Phi/d\ell)/d\ell)$:

$$(n+1)K\varrho^{1/n} - \Phi - \Omega^2\ell^2/2 + \ell^2(b_1+b_2)/8\pi = \mathrm{const}. \qquad (3.10.269)$$

With the familiar dimensionless variables from Eqs. (3.9.3), (3.9.6), the equation of hydrostatic equilibrium (3.10.264) becomes for the particular magnetic fields (3.10.260), (3.10.263) or (3.10.265) equal to

$$(1/\xi)\ d[\xi\Theta'(\xi)]/d\xi + \Theta^n - \varepsilon = 0, \qquad (0 \le n < \infty;\ \Theta(0) = 1;\ \Theta'(0) = 0), \qquad (3.10.270)$$

where

$$\varepsilon = \beta - (b_1+b_2)/8\pi^2 G\varrho_0, \qquad (\beta = \Omega^2/2\pi G\varrho_0). \qquad (3.10.271)$$

In the particular case $\Omega^2 = (b_1+b_2)/4\pi,\ (\varepsilon = 0)$, Eq. (3.10.270) turns into the familiar Lane-Emden equation (2.3.81) for the undistorted cylinder: Magnetic forces exactly balance rotational distortion. Explicit expressions for the components of the magnetic field can be obtained if $\varepsilon = 0$ and $n = 1$, for instance. Via Eq. (3.10.263) we find with $\Theta(\xi) = \theta(\xi) = J_0$ from Eq. (2.3.83):

$$H_\varphi^2 = (2\alpha^2 b_1\varrho_0/\xi^2)\int_0^\xi \Theta(\xi')\ \xi'^3\ d\xi' = (2\alpha^2 b_1\varrho_0/\xi^2)\int_0^\xi J_0(\xi')\ \xi'^3\ d\xi'$$
$$= (2\alpha^2 b_1\varrho_0/\xi^2)\left[\xi^3 J_1(\xi) - 2\int_0^\xi J_1(\xi')\ \xi'^2\ d\xi'\right] = 2\alpha^2 b_1\varrho_0[\xi J_1(\xi) - 2J_2(\xi)];$$
$$H_z^2 = 2\alpha^2 b_2\varrho_0\int_0^\xi \Theta(\xi')\ \xi'\ d\xi' + H_z^2(0) = 2\alpha^2 b_2\varrho_0\xi J_1(\xi) + H_z^2(0), \qquad (\varepsilon = 0;\ n = 1;\ H_\varphi(0) = 0), \qquad (3.10.272)$$

where we have used the recurrence formula $\xi^k J_{k-1}(\xi) = d[\xi^k J_k(\xi)]/d\xi$ of Bessel functions (e.g. Smirnow 1967, Spiegel 1968). b_1 must be nonnegative $(H_\varphi^2 \ge 0)$, but b_2 may be negative, because from $H_z^2 \ge 0$ we get

$$H_z^2(0) \ge -2b_2\int_0^\ell \varrho(\ell')\ \ell'\ d\ell'. \qquad (3.10.273)$$

If b_2 is negative, H_z^2 decreases from its value $H_z^2(0)$ along the axis up to its minimum at the boundary $\xi_1 = 2.4048,\ (J_0(\xi_1) = 0)$. If b_2 is positive, H_z^2 increases from its minimum value $H_z^2(0)$ along the axis up to its maximum at the boundary ξ_1.

The corresponding external magnetic field from Eq. (3.10.268) can be determined at once by ensuring the continuity of the magnetic fields (3.10.268), (3.10.272) across the boundary:

$$H_{e\varphi} = (1/\alpha\xi)\{2\alpha^4 b_1\varrho_0[\xi_1^3 J_1(\xi_1) - 2\xi_1^2 J_2(\xi_1)]\}^{1/2};$$
$$H_{ez} = [2\alpha^2 b_2\varrho_0\xi_1 J_1(\xi_1) + H_z^2(0)]^{1/2}, \qquad (\varepsilon = 0;\ n = 1). \qquad (3.10.274)$$

From Eqs. (3.10.265), (3.10.266) we find in the same manner:

$$H_\varphi^2 = \alpha^2 b_1\varrho_0\xi^2 J_0(\xi); \quad H_z^2 = \alpha^2\varrho_0[2b_2\xi J_1(\xi) - b_1\xi^2 J_0(\xi)] + H_z^2(0),$$
$$(\varepsilon = 0;\ n = 1;\ H_\varphi(0) = 0). \qquad (3.10.275)$$

It is interesting to note that $2b_2\xi J_1 - b_1\xi^2 J_0$ is not monotonic, but has a maximum and/or minimum between 0 and $\xi_1 = 2.4048$, the "wavy" character of H_z depending on the choice of b_1 and b_2 (Sood 1980, Table III).

Table 3.10.5 Values of the boundary coordinate Ξ_1 and of the corresponding dimensionless mass per unit length from Eq. (3.10.279) for the parameter ε from Eq. (3.10.271). Note, that the values for $n = 5$ correspond to $\varepsilon = 0.01$, rather than to $\varepsilon = 0.05$, because no closed hydrostatic configurations exist if $n = 5$ and $\varepsilon \gtrsim 0.01$ (Sood 1980). If $\varepsilon = 0$, the numbers correspond to those quoted in Table 2.5.2.

	Ξ_1			$-\Xi_1\Theta'(\Xi_1) + \varepsilon\Xi_1^2/2$		
n	$\varepsilon = -0.5$	$\varepsilon = 0$	$\varepsilon = 0.05$	$\varepsilon = -0.5$	$\varepsilon = 0$	$\varepsilon = 0.05$
1	1.81	2.40	2.51	0.760	1.25	1.34
1.5	1.89	2.65	2.81	0.626	1.06	1.15
2	1.95	2.92	3.19	0.532	0.925	1.01
3	2.06	3.57	4.70	0.409	0.740	0.819
5	2.20	5.43	6.37	0.277	0.532	0.545

Analytical solutions of Eq. (3.10.270) if $\varepsilon \neq 0$ can be found in the special cases $n = 0$ and $n = 1$ with the attempt $\Theta(\xi) = C_1\theta(\xi) + C_2,\ (C_1, C_2 = \mathrm{const})$, and with the usual initial conditions $\Theta(0) = 1,\ \Theta'(0) = 0$:

$$\Theta(\xi) = (1-\varepsilon)(1-\xi^2/4) + \varepsilon = 1 - (1-\varepsilon)\xi^2/4, \qquad (n = 0;\ \theta(\xi) = 1 - \xi^2/4), \tag{3.10.276}$$

$$\Theta(\xi) = (1-\varepsilon)J_0(\xi) + \varepsilon, \qquad [n = 1;\ \theta(\xi) = J_0(\xi)]. \tag{3.10.277}$$

At the boundary $\Theta(\Xi_1) = 0$ we have

$$\Xi_1^2 = 4/(1-\varepsilon), \quad (n = 0); \qquad J_0(\Xi_1) = \varepsilon/(\varepsilon - 1), \quad (n = 1), \tag{3.10.278}$$

showing that $\varepsilon < 1$ in both cases, since $\Xi_1^2 \geq 0$ if $n = 0$, and $J_0(\Xi_1) \leq 1$ if $n = 1$. In fact, computations by Sood (1980) show that $\varepsilon \leq 0.287$ if $n = 1$.

Other relevant quantities of the magnetopolytropes obeying Eq. (3.10.270) are total mass per unit length and mean density:

$$M_1 = 2\pi \int_0^{\ell_1} \varrho(\ell)\ \ell\ d\ell = 2\pi\alpha^2\varrho_0 \int_0^{\Xi_1} \Theta^n\xi\ d\xi = 2\pi\alpha^2\varrho_0[-\Xi_1\Theta'(\Xi_1) + \varepsilon\Xi_1^2/2];$$
$$\varrho_m = M_1/\pi r_1^2 = \varrho_0[-2\Theta'(\Xi_1)/\Xi_1 + \varepsilon], \qquad (0 \leq n < \infty;\ r_1 = \alpha\Xi_1). \tag{3.10.279}$$

The numerical integration of Eq. (3.10.270) for the values of n and ε listed in Table 3.10.5 shows that both, the boundary radius $\alpha\Xi_1$ and the total mass increase as the parameter ε increases. Similarly as for the magnetic fields considered previously by Talwar and Gupta (1973), and by Karnik and Talwar (1978), it is found that ε has an upper limit for each polytropic index n, which may be interpreted as an upper limit on the angular velocity for a given magnetic field. The constant ε can take arbitrarily large negative values, but in this case the radius of the cylinder diminishes (Sood 1980, p. 223).

Eq. (3.10.271) shows that ε can be increased either by an increase in the angular velocity or by a decrease of the magnetic field strength as characterized by Eq. (3.10.263), provided that b_2 is positive $(b_1 \geq 0)$. If b_2 is negative, ε may be increased by an increase of the axial field strength H_z if the toroidal field remains unchanged. Thus, mass and radius of magnetopolytropic cylinders can increase for a decreasing magnetic field strength, opposite to the previously quoted results of Talwar and Gupta (1973), and Karnik and Talwar (1978). These diverging results can be attributed to the different geometry of the considered magnetic field patterns.

Twisted (helical) magnetic fields $\vec{H}(0, c_1\ell, c_2),\ (c_1, c_2 = \mathrm{const})$ in rotating polytropic cylinders have been considered by Karnik and Talwar (1982).

Particular analytic solutions of magnetostatic polytropic cylinders containing a free harmonic function have been found by Lerche and Low (1980) for a particular choice of the magnetic field strength $\vec{H} = \vec{H}(-\partial F/\partial y, \partial F/\partial x, G)$, with the component along the axis $H_z = G = \mathrm{const}$.

Stability and oscillations of magnetopolytropic cylinders are touched in Sec. 5.11.4.

In conclusion, I feel that results so far published on magnetopolytropes seem inconclusive to some extent, perhaps mainly due to the complexity of possible magnetic field patterns.

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4 RELATIVISTIC POLYTROPES

4.1 Undistorted Relativistic Polytropes

4.1.1 Spherical Polytropes with Einstein's General Relativity

The study of relativistic polytropes seems to have been initiated by Tooper (1964a). Generalizing Eq. (1.7.43), the pressure P and the relativistic density ϱ_r are assumed to be related by

$$P = K\varrho_r^{1+1/n}, \qquad (K, n = \mathrm{const};\ n \neq -1). \tag{4.1.1}$$

As we have already noted in Eq. (1.2.16), the relativistic mass density ϱ_r (the relativistic mass m_r per unit of proper volume) and the rest mass density ϱ (the rest mass m per unit of proper volume) are connected by the simple relationship

$$\varrho_r = \varrho + \varepsilon^{(int)}/c^2 \quad \text{or} \quad \varepsilon_r = \varepsilon + \varepsilon^{(int)}. \tag{4.1.2}$$

c denotes in relativity always the light velocity, and the internal energy density $\varepsilon^{(int)}$ is the difference between relativistic energy density $\varepsilon_r = \varrho_r c^2$ and rest energy density $\varepsilon = \varrho c^2$. The internal energy density arises from the energy of kinetic motion, particle interactions (other than gravitational), external forces, radiation, etc. For purely kinetic translational motion with relative velocity v the relationship between ϱ_r and ϱ is simply given by Eq. (1.2.15): $\varrho_r = \varrho\big/(1 - v^2/c^2)^{1/2}$. We write the metric of spacetime under the form

$$ds^2 = g_{jk}\ dx^j dx^k, \qquad (g_{jk} = g_{kj};\ j, k = 0, 1, 2, 3), \tag{4.1.3}$$

where x^0 denotes the time coordinate, and g_{jk} the metric tensor. The concordance of Eq. (4.1.3) with the metric $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ of Galilean spacetime from special relativity results if the components of the metric tensor are $g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1,\ g_{jk} = 0$ if $j \neq k$. The spacetime coordinates are considered as a contravariant four-vector, and we have adopted the common summation rule for equal covariant and contravariant indices from tensor calculus, where latin indices run from 0 to 3, and greek indices for the three spatial coordinates x^1, x^2, x^3 run from 1 to 3.

In order to facilitate the solution of the nonlinear differential equations (4.1.4) of the gravitational field, we consider only stationary (time independent) fields, when the components g_{jk} of the metric tensor do not depend on the temporal coordinate x^0. In this case x^0 is called universal time. The components $g_{\alpha 0},\ (\alpha = 1, 2, 3)$ of the metric tensor are generally nonzero in stationary gravitational fields, as for instance in the case of a uniformly rotating, axially symmetric body, when opposite directions of time ($x^0 \to -x^0$) are not equivalent, since the sign of the angular velocity changes together with time reversal (Sec. 4.2.1).

A particular case of a stationary field is the static gravitational field, when the body is at rest in a reference system where the tensor g_{jk} is independent of x^0. In this case the metric (4.1.3) is invariant everywhere with respect to time reversal. Therefore, all components $g_{\alpha 0}$ must vanish, and synchronization of clocks is possible in the whole space. In this section we deal with the static, spherically symmetric gravitational field of an undistorted spherical polytrope.

Einstein's equations in mixed components (e.g. Landau and Lifschitz 1987)

$$R_j^k - \delta_j^k R/2 = 8\pi G T_j^k/c^4, \tag{4.1.4}$$

turn with the Schwarzschild metric

$$ds^2 = \exp\nu(r)\ dt^2 - \exp\kappa(r)\ dr^2 - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2), \tag{4.1.5}$$

for the spherically symmetric static case into the set of equations

$$\exp(-\kappa)\ [(1/r)\ d\nu/dr + 1/r^2] - 1/r^2 = 8\pi GP/c^4, \tag{4.1.6}$$

$$\exp(-\kappa)\ [(1/r)\ d\kappa/dr - 1/r^2] + 1/r^2 = 8\pi G\varrho_r/c^2, \tag{4.1.7}$$

$$(1/2)\exp(-\kappa)\ [d^2\nu/dr^2 + (d\nu/dr)^2/2 + (1/r)\ (d\nu/dr - d\kappa/dr) - (d\nu/dr)(d\kappa/dr)/2] = 8\pi GP/c^4. \tag{4.1.8}$$

In Eq. (4.1.4) R_j^k are the mixed components of the curvature tensor of spacetime, δ_j^k is the Kronecker delta ($\delta_j^k = 0$ if $j \neq k,\ \delta_j^k = 1$ if $j = k$), T_j^k the energy-momentum tensor, G the gravitational constant, and R the scalar curvature:

$$R = \mathrm{Tr}\ R_j^k = R_j^j = R_0^0 + R_1^1 + R_2^2 + R_3^3 = g^{jk}R_{jk}. \tag{4.1.9}$$

R_{jk} is the covariant curvature tensor (the so-called Ricci tensor), and the contravariant metric tensor g^{jk} is connected to the covariant metric tensor g_{jk} by

$$g_{jk}g^{k\ell} = \delta_j^\ell, \qquad (g^{jk} = g^{kj}). \tag{4.1.10}$$

Comparing the metrics (4.1.3) and (4.1.5), we observe that x^0, x^1, x^2, x^3 are equal to t, r, λ, φ, where t is the timelike coordinate of the Schwarzschild metric (4.1.5): $g_{00} = \exp\nu,\ g_{11} = -\exp\kappa,\ g_{22} = -r^2,\ g_{33} = -r^2\sin^2\lambda,\ g_{jk} = 0$ if $j \neq k$. The spatial r, λ, φ-coordinates are similar to the spherical coordinates in flat space, but in curved space the radial coordinate r does not obey the property of the radius vector in Euclidian space that $2\pi r$ is the circumference of a circle, and r its radius. r has been determined in Eq. (4.1.5) by the condition that the perimeter of a circle with the centre in the origin of coordinates is just $2\pi r$.

The energy-momentum tensor of macroscopic bodies has the components

$$T_j^k = (P + \varepsilon_r)u_j u^k - P\delta_j^k, \tag{4.1.11}$$

the contravariant components of the four-velocity $u^k = dx^k/ds$ being connected to its covariant components $u_j = g_{jk}u^k$ by the simple relationship

$$u_j u^j = g_{jk}u^k u^j = g_{jk}(dx^k/ds)(dx^j/ds) = g_{jk}\ dx^j dx^k/ds^2 = 1. \tag{4.1.12}$$

The spatial components $u^\alpha = dx^\alpha/ds,\ (\alpha = 1, 2, 3)$ of the four-velocity vanish for the static sphere. Eq. (4.1.12) reduces to $u_0 u^0 = 1$, so the required components of the energy-momentum tensor are

$$T_0^0 = \varepsilon_r = \varrho_r c^2; \quad T_1^1 = T_2^2 = T_3^3 = -P; \quad T_j^k = 0 \quad \text{if} \quad j \neq k. \tag{4.1.13}$$

From the Bianchi identities results that the covariant divergence of the energy-momentum tensor vanishes

$$\nabla_k T_j^k = \partial T_j^k/\partial x^k - \Gamma_{jk}^\ell T_\ell^k + \Gamma_{\ell k}^k T_j^\ell = 0, \tag{4.1.14}$$

with the Christoffel symbols expressed through the metric tensor by

$$\Gamma_{jk}^\ell = (g^{\ell m}/2)(\partial g_{mj}/\partial x^k + \partial g_{mk}/\partial x^j - \partial g_{jk}/\partial x^m); \quad \Gamma_{jk}^\ell = \Gamma_{kj}^\ell. \tag{4.1.15}$$

The contravariant metric tensor can be obtained at once from Eq. (4.1.10):

$$g^{00} = \exp(-\nu); \quad g^{11} = -\exp(-\kappa); \quad g^{22} = -1/r^2; \quad g^{33} = -1/r^2\sin^2\lambda; \quad g^{jk} = 0 \quad \text{if} \quad j \neq k. \tag{4.1.16}$$

The nonvanishing Christoffel symbols result from Eq. (4.1.15):

$$\Gamma_{00}^1 = (1/2)\exp(\nu - \kappa)\ d\nu/dr;\ \Gamma_{10}^0 = (1/2)\ d\nu/dr;\ \Gamma_{11}^1 = (1/2)\ d\kappa/dr;\ \Gamma_{12}^2 = \Gamma_{13}^3 = 1/r;$$
$$\Gamma_{22}^1 = -r\exp(-\kappa);\ \Gamma_{23}^3 = \cot\lambda;\ \Gamma_{33}^1 = -r\sin^2\lambda\exp(-\kappa);\ \Gamma_{33}^2 = -\sin\lambda\cos\lambda. \tag{4.1.17}$$

The sole nonvanishing component of Eq. (4.1.14) is the radial one ($j = k = 1$), which reduces to the equation of hydrostatic equilibrium

$$\nabla_1 T_1^1 = \nabla_r T_r^r = dP/dr + (P/2 + \varepsilon_r/2)\ d\nu/dr = 0. \tag{4.1.18}$$

In fact, this equation follows directly from the field equations (4.1.6)-(4.1.8), by equating Eqs. (4.1.6) and (4.1.8), and using Eq. (4.1.7). So, we can drop in the sequel the complicated equation (4.1.8), and take Eqs. (4.1.1), (4.1.6), (4.1.7), (4.1.18) as a set for the determination of $\kappa, \nu, P, \varrho_r$ as functions of the radial coordinate r.

Outside the polytrope we have $\varrho, P = 0$. Adding in this case Eqs. (4.1.6) and (4.1.7), we find $d\kappa/dr = -d\nu/dr$, or

$$\kappa(r) = -\nu(r), \qquad (r \geq r_1), \tag{4.1.19}$$

where the integration constant can be set equal zero after a suitable coordinate transformation. The external Schwarzschild metric (4.1.5) is given by the well known line element (e.g. Zeldovich and Novikov 1971, Landau and Lifschitz 1987)

$$ds^2 = (1 - 2GM_{r1}/c^2r)\ dt^2 - dr^2/(1 - 2GM_{r1}/c^2r) - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2), \qquad (r \geq r_1). \tag{4.1.20}$$

r_1 is the value of the radial coordinate at the surface of the polytropic sphere. The quantity M_{r1} is the relativistic (gravitational, inertial) mass of the configuration, as determined by the motion of an external test particle. In M_{r1} the mass equivalent of all forms of energy is included [kinetic energy, interaction energy, energy of radiation, energy of gravitation and of other fields, etc.; cf. comment subsequent to Eq. (4.1.73)]. Outside the polytrope, the line elements (4.1.5) and (4.1.20) must be identical, so we get by comparison

$$\kappa = -\nu = -\ln(1 - 2GM_{r1}/c^2r), \qquad (r \geq r_1). \tag{4.1.21}$$

On the other hand, we observe from the term $(1/r)\ d\kappa/dr$ in Eq. (4.1.7) that κ must tend to zero at least as r^2 if $r \to 0$, in order to avoid a singularity; so, the initial condition is $\kappa(0) = 0$. Integrating Eq. (4.1.7) formally with this initial value, we get

$$\kappa = -\ln\left[1 - (8\pi G/c^2r)\int_0^r \varrho_r r^2\ dr\right]. \tag{4.1.22}$$

Comparing Eqs. (4.1.21) and (4.1.22) outside the polytrope ($r \geq r_1$), we obtain the important relationship

$$M_{r1} = 4\pi\int_0^{r_1} \varrho_r r^2\ dr. \tag{4.1.23}$$

Note, that $4\pi r^2\ dr$ appears in this integral, rather than the proper volume element dV, which is in the orthogonal curvilinear coordinates from Eq. (4.1.5) equal to [cf. Eq. (B.5)]

$$dV = (-g_{11}g_{22}g_{33})^{1/2}\ dx^1dx^2dx^3 = \exp(\kappa/2)\ r^2\sin\lambda\ dr\ d\lambda\ d\varphi > r^2\sin\lambda\ dr\ d\lambda\ d\varphi, \qquad (\kappa > 0). \tag{4.1.24}$$

The appearance of $r^2\ dr$ in Eq. (4.1.23) instead of $\exp(\kappa/2)\ r^2\ dr$ is related to the effect of the relativistic gravitational field upon the mass of the polytrope, i.e. to the distortion of the underlying spacetime by gravitation (Tooper 1964a, Zeldovich and Novikov 1971).

Let us define the auxiliary function

$$u = c^2r[1 - \exp(-\kappa)]/2GM_{r1}, \qquad (u(0) = 0;\ u(r_1) = 1). \tag{4.1.25}$$

Eq. (4.1.7) becomes in terms of u equal to

$$M_{r1}\ du/dr = d(M_{r1}u)/dr = 4\pi\varrho_r r^2 = dM_r/dr \quad \text{or} \quad u(r) = M_r(r)/M_{r1}. \tag{4.1.26}$$

$M_r = M_r(r) = M_{r1}u(r)$ can be interpreted in virtue of Eq. (4.1.23) as the relativistic (gravitational, inertial) mass inside coordinate radius r.

To obtain the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium for the spherically symmetric static space, we insert $d\nu/dr$ from Eq. (4.1.18), and $\exp(-\kappa)$ from Eq. (4.1.25) into Eq. (4.1.6), (e.g. Chandrasekhar 1964b):

$$dP/dr = -G(\varrho_r + P/c^2)[M_r(r) + 4\pi Pr^3/c^2]/r^2[1 - 2GM_r(r)/c^2r]. \tag{4.1.27}$$

Indeed, in the nonrelativistic limit ($c \to \infty$) we obtain just the Newtonian equation of hydrostatic equilibrium $dP/dr = -G\varrho M(r)/r^2$, ($\varrho_r, M_r \to \varrho, M$). The initial conditions of Eqs. (4.1.26) and (4.1.27) are obviously $P(0) = P_0$, $M_r(0) = 0$, $(dP/dr)_{r=0} = 0$, while the boundary conditions are $P(r_1) = 0$, $M_r(r_1) = M_{r1}$.

From the external metric (4.1.20) follows that g_{tt} becomes zero, and g_{rr} is infinite if the radial coordinate equals the gravitational radius (Schwarzschild radius)

$$r = r_g = 2GM_{r1}/c^2. \tag{4.1.28}$$

So, the configuration cannot be hydrostatic if its radial surface coordinate r_1 is smaller than the gravitational radius r_g, ($r_g = 2.96$ km for a solar mass).

We now turn to dimensionless coordinates by introducing the relativistic Lane-Emden function θ_r and the dimensionless radial coordinate ξ by the common relationships [cf. Eqs. (2.1.10), (2.1.13)]:

$$\varrho_r = \varepsilon_r/c^2 = \varrho_{r0}\theta_r^n; \quad \varepsilon_r = \varepsilon_{r0}\theta_r^n; \quad P = P_0\theta_r^{n+1} = K\varrho_{r0}^{1+1/n}\theta_r^{n+1} = K\varrho_r^{1+1/n};$$
$$r = [\pm(n+1)K/4\pi G\varrho_{r0}^{1-1/n}]^{1/2}\xi = [\pm(n+1)P_0/4\pi G\varrho_{r0}^2]^{1/2}\xi = \alpha\xi, \qquad (n \neq -1, \pm\infty). \tag{4.1.29}$$

The plus sign in the expression of α holds if $-1 < n < \infty$, and the minus sign if $-\infty < n < -1$. The central values of relativistic density and pressure are denoted by ϱ_{r0} and P_0, respectively. Eq. (4.1.18) becomes in the new variables

$$2(n+1)P_0\ d\theta_r/\varrho_{r0}c^2 + (1 + P_0\theta_r/\varrho_{r0}c^2)\ d\nu = 2(n+1)q_0\ d\theta_r + (1 + q_0\theta_r)\ d\nu = 0, \tag{4.1.30}$$

where the relativity parameter

$$q_0 = P_0/\varepsilon_{r0} = P_0/\varrho_{r0}c^2 = K\varrho_{r0}^{1/n}/c^2, \tag{4.1.31}$$

is the ratio of central pressure to central relativistic energy density. Eq. (4.1.30) can be integrated at once by separation of variables with the initial conditions $\theta_r(0) = 1$ and $\nu(0) = \nu_0$:

$$\nu = \nu_0 + \ln[(1 + q_0)/(1 + q_0\theta_r)]^{2(n+1)}. \tag{4.1.32}$$

The central value ν_0 can be specified further from the boundary condition (4.1.21). On the boundary of the relativistic polytrope we must have $P(r_1) = 0$, i.e. $\theta_r(\xi_1) = 0$ if $-1 < n < 5$ (Table 4.1.1), where ξ_1 is the boundary value of ξ. Thus, if we equate Eqs. (4.1.21) and (4.1.32) on the boundary, we find at once

$$\nu_0 = \ln[(1 - 2GM_{r1}/c^2r_1)/(1 + q_0)^{2(n+1)}]. \tag{4.1.33}$$

Eq. (4.1.32) finally becomes

$$\nu = \ln[(1 - 2GM_{r1}/c^2r_1)/(1 + q_0\theta_r)^{2(n+1)}], \qquad (-1 < n < 5). \tag{4.1.34}$$

To determine the metric component $g_{rr} = -\exp\kappa$, or equivalently the function u from Eq. (4.1.25), we substitute into Eq. (4.1.6) for $d\nu/dr$ from Eq. (4.1.30), for $\exp(-\kappa)$ from Eq. (4.1.25), and for $P_0 = q_0\varrho_{r0}c^2$ from Eq. (4.1.31):

$$[q_0(n+1)/(1 + q_0\theta_r)](1 - 2GM_{r1}u/c^2r)r\ d\theta_r/dr + GM_{r1}u/c^2r + (q_0GM_{r1}/c^2)\theta_r\ du/dr = 0. \tag{4.1.35}$$

To derive this equation we have also used Eq. (4.1.26), written under the form

$$M_{r1}\ du/dr = 4\pi\varrho_{r0}r^2\theta_r^n. \tag{4.1.36}$$

The two previous equations can be transformed into the basic form of the relativistic Lane-Emden equation [see the equivalent equations (4.1.27) and (4.1.26)]

$$\{[\pm 1 - 2q_0(n+1)\eta/\xi]/(1+q_0\theta_r)\}\xi^2\ d\theta_r/d\xi + \eta + q_0\xi\theta_r\ d\eta/d\xi = 0, \qquad (n \neq -1, \pm\infty), \quad (4.1.37)$$

$$d\eta/d\xi = \xi^2\theta_r^n, \quad (4.1.38)$$

by using $4\pi G\varrho_{r0}\alpha^2/c^2 = \pm(n+1)q_0$, and the dimensionless function

$$\eta = \eta(\xi) = M_{r1}u(r)/4\pi\varrho_{r0}\alpha^3 = M_r(r)/4\pi\varrho_{r0}\alpha^3. \quad (4.1.39)$$

Henceforth, we generally assume $-1 < n < 5$, since the dimensionless boundary radius ξ_{r1} is finite in this case (Table 4.1.1, Nilsson and Uggla 2000).

In the Newtonian limit we have $c \to \infty$, and according to Eq. (4.1.31) $q_0 \approx 0$. Eq. (4.1.37) reduces to $\xi^2\ d\theta_r/d\xi = \mp\eta$, or

$$d(\xi^2\ d\theta_r/d\xi)/d\xi = \mp d\eta/d\xi = \mp\xi^2\theta_r^n, \qquad (q_0 \approx 0), \quad (4.1.40)$$

which is identical to the familiar spherical Lane-Emden equation (2.3.87). Chen and Shao (2001), and Chen et al. (2001) introduce linear corrections to the Galilean metric in higher-order gravity, obtaining a modified Lane-Emden equation.

We have already noted in Eq. (4.1.7) that $\kappa = O(r^2)$ if $r \to 0$, so via Eq. (4.1.25) $u = O(r^3)$, and via Eq. (4.1.39) $\eta = O(\xi^3)$ if $\xi \to 0$. From Eq. (4.1.37) follows at once that also $(d\theta_r/d\xi)_{\xi=0} = 0$. Thus, the initial conditions for the system (4.1.37) and (4.1.38) are

$$\theta_r(0) = 1; \quad (d\theta_r/d\xi)_{\xi=0} = 0; \quad \eta(0) = 0. \quad (4.1.41)$$

Bludman (1973a) has pointed out that generally the relativistic Lane-Emden equations (4.1.37) and (4.1.38) do not admit a homology transformation of the form (2.2.4), i.e. if $\xi \to A\xi$ the functions θ_r and η do not transform according to the scale transformations $\theta_r \to B\theta_r(A\xi),\ \eta \to C\eta(A\xi),\ (A, B, C = \mathrm{const})$.

Near the origin we obtain from Eqs. (4.1.37), (4.1.38) for any q_0 (Sec. 2.4.1, Chu et al. 1980):

$$\theta_r \approx 1 + a_2\xi^2 + a_4\xi^4 + ...; \quad \eta \approx \xi^3/3 + na_2\xi^5/5 + [na_4 + n(n-1)a_2^2/2]\xi^7/7;$$
$$a_2 = \mp(1/3 + 4q_0/3 + q_0^2)/2; \quad a_4 = \big[n/15 + 2nq_0/9 + (16n/45 + 2/3)q_0^2$$
$$+(6n/5 + 8/3)q_0^3 + (n+2)q_0^4\big]/8, \qquad (\xi \approx 0;\ n \neq -1, \pm\infty). \quad (4.1.42)$$

If $q_0 \approx 0$, we recover $\theta_r \approx 1 \mp \xi^2/6 + n\xi^4/120$ from Eq. (2.4.24). In the post Newtonian approximation ($q_0 \ll 1$) Sharma (1981) expands θ_r up to ξ^8, calculating the relativistic second order Padé approximant $\theta_{P,r}$ (see Sec. 2.4.4 for the nonrelativistic case).

A solution in closed form of Eqs. (4.1.37), (4.1.38) can be obtained in the constant density case $n = 0$ of K. Schwarzschild [Tooper 1964a; see also Eqs. (4.1.88)-(4.1.91)]. In this special case the variables are separable, and integration with the initial conditions (4.1.41) yields

$$\theta_r = \big[(1+3q_0)(1-2q_0\xi^2/3)^{1/2} - (1+q_0)\big]/q_0\big[3(1+q_0) - (1+3q_0)(1-2q_0\xi^2/3)^{1/2}\big];$$
$$\eta = \xi^3/3, \qquad (n = 0). \quad (4.1.43)$$

In dimensional units Eq. (4.1.43) turns with Eqs. (4.1.29), (4.1.31), (4.1.39) into

$$P = \varrho_{r0}c^2\big[(3P_0 + \varrho_{r0}c^2)(1 - 8\pi G\varrho_{r0}r^2/3c^2)^{1/2} - (P_0 + \varrho_{r0}c^2)\big]/\big[3(P_0 + \varrho_{r0}c^2) - (3P_0 + \varrho_{r0}c^2)$$
$$\times(1 - 8\pi G\varrho_{r0}r^2/3c^2)^{1/2}\big]; \quad M_r = 4\pi\varrho_{r0}r^3/3, \qquad (n = 0;\ P = P_0\theta_r;\ q_0\xi^2 = 4\pi G\varrho_{r0}r^2/c^2). \quad (4.1.44)$$

The first zero of Eqs. (4.1.43) and (4.1.44) is clearly

$$\xi_1 = [6(1+2q_0)]^{1/2}/(1+3q_0);\ r_1 = [3c^2P_0(2P_0 + \varrho_{r0}c^2)]^{1/2}/(2\pi G\varrho_{r0})^{1/2}(3P_0 + \varrho_{r0}c^2), \quad (n = 0). \quad (4.1.45)$$

If $q_0 \approx 0,\ (c \to \infty)$, the three previous equations turn into their nonrelativistic counterparts: $\theta_r = 1 - \xi^2/6$, [Eq. (2.3.88)], $P = P_0 - 2\pi G\varrho_{r0}^2 r^2/3,\ \xi_1 = 6^{1/2},\ r_1 = (3P_0/2\pi G\varrho_{r0}^2)^{1/2}$.

The value r_1 of the radial coordinate at the boundary is different from the radius r_{r1} of the configuration, as measured by an external observer. This is due to the distortion of Euclidian space by the presence of gravitating matter. The true radius r_{r1} along the direction $\lambda, \varphi =$ const can be found at once by integration of the spatial line element (Eq. (5.12.95), Landau and Lifschitz 1987, §88)

$$d\ell^2 = \gamma_{\alpha\beta}\ dx^\alpha dx^\beta = (-g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00})\ dx^\alpha dx^\beta, \qquad (\alpha, \beta = 1, 2, 3), \tag{4.1.46}$$

along the radial coordinate $dx^1 = dr$, when $dx^2 = d\lambda = 0,\ dx^3 = d\varphi = 0$. For the Schwarzschild metric (4.1.5) we have $\gamma_{\alpha\beta} = -g_{\alpha\beta},\ (g_{0\alpha} = 0)$, and $\gamma_{11} = \gamma_{rr} = -g_{rr} = \exp\kappa$. The true radius of the polytropic sphere is therefore

$$r_{r1} = \int_0^{r_1} (\gamma_{rr})^{1/2}\ dr = \int_0^{r_1} \exp[\kappa(r)/2]\ dr = \int_0^{r_1} dr/(1 - 2GM_{r1}u/c^2r)^{1/2}$$
$$= \alpha \int_0^{\xi_1} d\xi/[1 - 2q_0(n+1)\eta(\xi)/\xi]^{1/2} = \alpha\xi_{r1} > r_1 = \alpha\xi_1, \qquad (-1 < n < 5). \tag{4.1.47}$$

If $n = 0$, we get in virtue of Eq. (4.1.43)

$$\xi_{r1} = (3/2q_0)^{1/2} \arcsin(2q_0\xi_1^2/3)^{1/2}, \qquad (n = 0). \tag{4.1.48}$$

The mass-radius relationship can be found from Eqs. (4.1.23), (4.1.29), (4.1.38):

$$M_{r1} = 4\pi \int_0^{r_1} \varrho_r r^2\ dr = 4\pi\varrho_{r0}\alpha^3 \int_0^{\xi_1} \theta_r^n \xi^2\ d\xi = 4\pi\varrho_{r0}\alpha^3\eta(\xi_1) = (n+1)c^2 q_0\eta(\xi_1)\ r_1/G\xi_1$$
$$= (n+1)c^2q_0\eta(\xi_1)\ r_{r1}/G\xi_{r1}, \qquad (-1 < n < 5;\ u(0), \eta(0) = 0;\ \alpha = r_1/\xi_1 = r_{r1}/\xi_{r1}). \tag{4.1.49}$$

In the particular case $n = 0$ we obtain by virtue of Eqs. (4.1.43), (4.1.45), (4.1.48):

$$M_{r1} = 2c^2r_1(q_0 + 2q_0^2)/G(1 + 3q_0)^2 = 4c^2r_{r1}(q_0 + 2q_0^2)^{3/2}/G(1 + 3q_0)^3$$
$$\times \arcsin[2(q_0 + 2q_0^2)^{1/2}/(1 + 3q_0)], \qquad (n = 0). \tag{4.1.50}$$

From Eqs. (4.1.25), (4.1.26), (4.1.39) we get at once for the mass inside the radial coordinate r :

$$M_r = M_r(r) = 4\pi \int_0^r \varrho_r r^2\ dr = M_{r1}u(r) = M_{r1}\eta(\xi)/\eta(\xi_1). \tag{4.1.51}$$

A formal definition of the mean density would be

$$\varrho_m = 4\pi \int_0^{r_1} \varrho_r r^2\ dr \Big/ 4\pi \int_0^{r_1} r^2\ dr = M_{r1} \Big/ 4\pi \int_0^{r_1} r^2\ dr = (3\varrho_{r0}/\xi_1^3) \int_0^{\xi_1} \theta_r^n \xi^2\ d\xi$$
$$= 3\varrho_{r0}\eta(\xi_1)/\xi_1^3, \qquad (-1 < n < 5), \tag{4.1.52}$$

where we have used Eq. (4.1.38). If $q_0 \approx 0$, we have pointed out that $\eta = -\xi^2\ d\theta_r/d\xi$ in Eq. (4.1.40), so Eqs. (4.1.52) and (4.1.53) reduce to the nonrelativistic equation (2.6.27). A more appropriate definition of the mean density would be (Buchdahl 1959, p. 1028):

$$\varrho_m = \int_{V_1} \varrho_r\ dV \Big/ \int_{V_1} dV = \int_0^{r_1} \varrho_r r^2 \exp(\kappa/2)\ dr \Big/ \int_0^{r_1} r^2 \exp(\kappa/2)\ dr$$
$$= \varrho_{r0} \int_0^{\xi_1} \theta_r^n \xi^2\ d\xi/[1 - 2q_0(n+1)\eta(\xi)/\xi]^{1/2} \Big/ \int_0^{\xi_1} \xi^2\ d\xi/[1 - 2q_0(n+1)\eta(\xi)/\xi]^{1/2}. \tag{4.1.53}$$

V_1 denotes the proper volume of the complete relativistic sphere, and we have used Eqs. (4.1.24) and (4.1.47).

In nonrelativistic physics the first law of thermodynamics (1.1.3) subsists for the unit of rest mass $m = \varrho V = 1$: For adiabatic reversible processes, preserving the entropy S of the system, we have $dU = P\ d\varrho/\varrho^2 = -P\ dV,\ (dS, dQ = 0)$. To obtain the equivalent of this law in relativistic physics, we consider a thought experiment, which is performed in an impenetrable vessel of proper volume $V = 1/\varrho$, where ϱ denotes the rest mass density. The rest mass $m = \varrho V = 1$ in the vessel is invariant, whereas the relativistic mass $\varrho_r V$ changes according to the work performed. The relativistic energy density $\varepsilon_r = c^2\varrho_r$ is referred to the unit of rest volume (proper volume), and if we wish to obtain the relativistic energy per unit rest mass E_r (the specific relativistic energy), we have to multiply ε_r by $V = 1/\varrho$ (e.g. Zeldovich and Novikov 1971, p. 186):

$$E_r = \varepsilon_r V = \varepsilon_r/\varrho = c^2\varrho_r/\varrho, \qquad (m = 1). \tag{4.1.54}$$

This important equation relates specific relativistic energy E_r to relativistic mass density ϱ_r and to rest mass density ϱ, where the constant c^2 is just the specific rest energy E (rest energy per rest mass unit): $c^2 = \varepsilon/\varrho = E$ if $m = 1$. Eq. (4.1.54) is a generalization of Eq. (1.2.18), where E_r was exclusively due to relativistic kinetic energy $E_r = E_r^{(kin)}$. We turn in the relationship (4.1.2) to specific quantities by division with the rest mass density ϱ :

$$E_r = \varepsilon_r/\varrho = c^2 + \varepsilon^{(int)}/\varrho = c^2 + U, \qquad (m = 1). \tag{4.1.55}$$

The specific internal energy $U = \varepsilon^{(int)}/\varrho$ is just the difference between specific relativistic energy E_r and specific rest energy c^2, including kinetic particle motions, particle interactions, force and radiation fields, but exclusive of gravitational interactions. For adiabatic reversible processes $(Q, S = \mathrm{const})$ the first law of thermodynamics (1.1.3) writes for the unit of rest mass

$$P = \varrho^2(\partial U/\partial\varrho)_S = \varrho^2(\partial E_r/\partial\varrho)_S = c^2\varrho^2[\partial(\varrho_r/\varrho)/\partial\varrho]_S, \qquad (m = 1), \tag{4.1.56}$$

replacing by virtue of Eq. (4.1.55) the derivative of the specific internal energy U with the derivative of specific relativistic energy E_r. Performing the derivation in Eq. (4.1.56), we get the relativistic first law of thermodynamics under the form (Tooper 1964a)

$$P = c^2\varrho(\partial\varrho_r/\partial\varrho)_S - \varrho_r c^2 = \varrho(\partial\varepsilon_r/\partial\varrho)_S - \varepsilon_r \quad \text{or} \quad (\partial\varepsilon_r/\partial\varrho)_S = (P + \varepsilon_r)/\varrho. \tag{4.1.57}$$

Since $m = \varrho V = 1$, we can replace $d\varrho$ by $-\varrho\, dV/V$:

$$(\partial\varepsilon_r/\partial V)_S + (P + \varepsilon_r)/V = 0, \qquad (S = \mathrm{const};\ m = 1). \tag{4.1.58}$$

Eq. (4.1.57) becomes with $\varepsilon_r = c^2\varrho_{r0}\theta_r^n$ and $P = K\varrho_{r0}^{1+1/n}\theta_r^{n+1} = q_0 c^2\varrho_{r0}\theta_r^{n+1}$ equal to

$$d\varrho/\varrho = n\ d\theta_r/\theta_r(1 + q_0\theta_r). \tag{4.1.59}$$

Integration yields

$$\varrho = \varrho_0[(1 + q_0)\theta_r/(1 + q_0\theta_r)]^n, \qquad (\theta_r(0) = 1). \tag{4.1.60}$$

ϱ_0 denotes the rest mass density at the centre, corresponding to the central density of a nonrelativistic polytropic sphere. ϱ_0 may be evaluated further, by observing that Eq. (4.1.60) becomes near the boundary

$$\varrho \approx \varrho_0(1 + q_0)^n\theta_r^n, \qquad (\theta_r \approx 0). \tag{4.1.61}$$

Relativistic effects are generally small near the boundary, so we may write

$$\varrho \approx \varrho_r = \varrho_{r0}\theta_r^n, \qquad (\theta_r \approx 0). \tag{4.1.62}$$

Comparing Eqs. (4.1.61) and (4.1.62) we realize that

$$\varrho_0 = \varrho_{r0}/(1 + q_0)^n, \tag{4.1.63}$$

and Eq. (4.1.60) reads (Tooper 1964a):

$$\varrho = \varrho_{r0}\theta_r^n/(1 + q_0\theta_r)^n = \varrho_r/(1 + q_0\theta_r)^n. \tag{4.1.64}$$

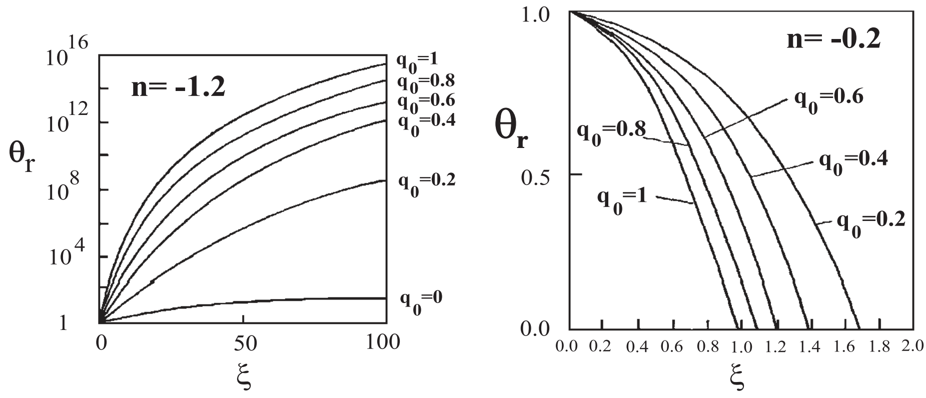

Fig. 4.1.1 Run of the relativistic Lane-Emden function θ_r if $n = -1.2$ (on the left), and $n = -0.2$ (on the right) for values of the relativity parameter q_0 listed in the figures (Chu et al. 1980).

Since in relativistic physics we distinguish two different densities (ϱ_r – relativistic density, and ϱ – rest mass density), we may define two different adiabatic exponents, depending on whether the pressure is considered as a function of ϱ_r or ϱ (cf. Eq. (1.3.23), Zeldovich and Novikov 1971, p. 186):

$$\Gamma_{r1} = (d\ln P/d\ln\varrho_r)_{S=\mathrm{const}} = (d\ln P/d\ln\varepsilon_r)_{S=\mathrm{const}};$$
$$\Gamma_1 = (d\ln P/d\ln\varrho)_{S=\mathrm{const}} = (d\ln P/d\ln\varepsilon)_{S=\mathrm{const}}. \quad (4.1.65)$$

The adiabatic velocity of sound inside a relativistic polytrope of index n is given by (cf. Eq. (2.1.49) for the Newtonian equivalent)

$$a^2 = (\partial P/\partial\varrho_r)_{S=\mathrm{const}} = \Gamma_{r1}P/\varrho_r. \quad (4.1.66)$$

Tooper (1964a) has created much confusion (e.g. Bludman 1973a, Sarkisyan and Chubaryan 1977, Glass and Harpaz 1983, Dallas and Geroyannis 1993) with his delimitation $q_0 \leq n/(n+1),\ (-\infty \leq n < -1;\ 0 \leq n \leq \infty)$ of the relativity parameter, which is valid only for isentropic polytropes ($S = \mathrm{const}$) when $P = K\varrho_r^{1+1/n} = K\varrho_r^{\Gamma_{r1}}$ and $1 + 1/n = \Gamma_{r1}$. Generally, there is no connection between Γ_{r1} and n, hence $\Gamma_{r1} \neq 1+1/n$, as outlined at the end of Sec. 2.1 (Horedt 2000a, b). Inserting in the isentropic case $\Gamma_{r1} = 1 + 1/n$ and $P = K\varrho_{r0}^{1+1/n}\theta_r^{n+1} = P_0\theta_r^{n+1},\ \varrho_r = \varrho_{r0}\theta_r^n$ from Eq. (4.1.29) into Eq. (4.1.66), we find

$$a^2 = \Gamma_{r1}P/\varrho_r = (1 + 1/n)P/\varrho_r = (1 + 1/n)P_0\theta_r/\varrho_{r0} = (n + 1)c^2q_0\theta_r/n,$$
$$(S = \mathrm{const};\ \Gamma_{r1} = 1 + 1/n;\ -\infty \leq n < -1;\ 0 \leq n \leq \infty). \quad (4.1.67)$$

At the centre we have $\theta_r = 1$, so we obtain for the central velocity of sound $a_0^2 = (n+1)c^2q_0/n$, and because $a_0 \leq c$, we get Tooper's (1964a) above mentioned delimitation: $q_0 \leq n/(n + 1),\ (S = \mathrm{const})$. As we have already remarked subsequently to Eq. (2.1.50), the generally valid delimitation of the relativity parameter results from Eq. (4.1.66):

$$a^2 = \Gamma_{r1}P/\varrho_r \leq c^2 \quad \text{or} \quad q_0 = P_0/\varrho_{r0}c^2 \leq 1/\Gamma_{r1}. \quad (4.1.68)$$

In fact, from Eq. (1.7.43) we get the important general relativistic delimitation: $q_0 = P_0/\varrho_{r0}c^2 \leq 1$.

We now turn to the calculation of the gravitational energy and of the so-called "binding energy" (Tooper 1964a, Fowler 1964). In accordance with the equivalence of mass and energy in relativity, the total relativistic energy E_{r1} of the complete spherical polytrope, including gravitational potential energy, is via Eq. (4.1.23) simply

$$E_{r1} = c^2M_{r1} = 4\pi c^2\int_0^{r_1}\varrho_r r^2\,dr = 4\pi\int_0^{r_1}\varepsilon_r r^2\,dr = 4\pi\int_0^{r_1}T_0^0 r^2\,dr. \quad (4.1.69)$$

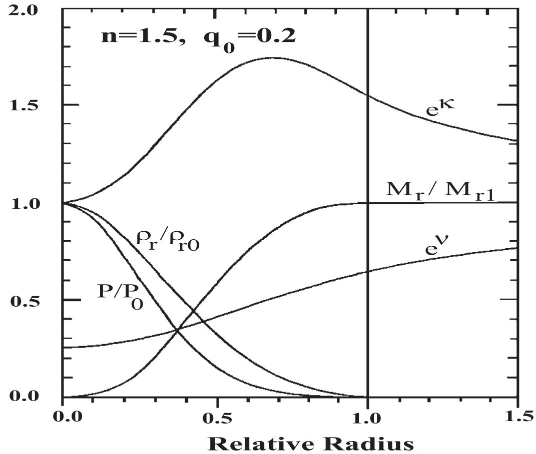

Fig. 4.1.2 Internal structure of a general relativistic polytrope with $n = 1.5,\ q_0 = 0.2$. Normalized density, pressure, mass, and the metric components $\exp\nu = g_{tt},\ \exp\kappa = -g_{rr}$ are plotted as functions of the relative radius (Tooper 1964a).

This is the energy of the relativistic polytrope as determined from the motion of an external test particle. Because of the noneuclidian nature of space induced by the gravitational field, the proper energy E_{pr} of the complete polytrope is different from E_{r1}, because E_{pr} is the integral of the relativistic energy density over all elements of proper volume dV, i.e. over the volume of rest from Eq. (4.1.24):

$$E_{pr} = \int_{V_1} \varepsilon_r\ dV = 4\pi c^2 \int_0^{r_1} \varrho_r \exp(\kappa/2)\ r^2\ dr. \tag{4.1.70}$$

In the post Newtonian approximation this equation becomes, by expanding κ from Eq. (4.1.25) in terms of $2GM_r(r)/c^2r \ll 1$:

$$\exp(\kappa/2) = [1 - 2GM_{r1}u(r)/c^2r]^{-1/2} = [1 - 2GM_r(r)/c^2r]^{-1/2} \approx 1 + GM_r(r)/c^2r. \tag{4.1.71}$$

Substitution into Eq. (4.1.70) yields

$$E_{pr} \approx 4\pi c^2 \int_0^{r_1} \varrho_r[1 + GM_r(r)/c^2r]r^2\ dr = E_{r1} + \int_0^{r_1} GM_r(r)\ dM_r/r = E_{r1} - W_r,$$
$$(q_0 \approx 0;\ dM_r = 4\pi\varrho_r r^2\ dr). \tag{4.1.72}$$

We may call W_r the relativistic gravitational energy, and generalize Eq. (2.6.127) by writing

$$W_r = -\int_{M_{r1}} GM_r(r)\ dM_r/r = E_{r1} - E_{pr} = 4\pi c^2 \int_0^{r_1} \varrho_r[1 - \exp(\kappa/2)]r^2\ dr. \tag{4.1.73}$$

Since $\exp(\kappa/2) \geq 1$ via Eq. (4.1.71), we have $E_{pr} \geq E_{r1}$, and $W_r \leq 0$. The origin of $-W_r$ is obvious: When combining the mass elements $\varrho_r\ dV$ into a relativistic sphere, we must take into account also their gravitational interaction. This gravitational interaction energy is included in E_{r1}, rather than in E_{pr}. The relativistic mass $M_{r1} = E_{r1}/c^2$ of a polytrope is not equal to the sum E_{pr}/c^2 of its individual mass elements $\varrho_r\ dV$, which already possess the assigned energy density $\varepsilon_r = \varrho_r c^2$ (Zeldovich and Novikov 1971, p. 286). The proper energy (4.1.70) writes in terms of our dimensionless variables from Eqs. (4.1.29), (4.1.31), (4.1.39), (4.1.47) as

$$E_{pr} = 4\pi c^2 \varrho_{r0}\alpha^3 \int_0^{\xi_1} \theta_r^n \xi^2\ d\xi/[1 - 2q_0(n+1)\ \eta(\xi)/\xi]^{1/2}. \tag{4.1.74}$$

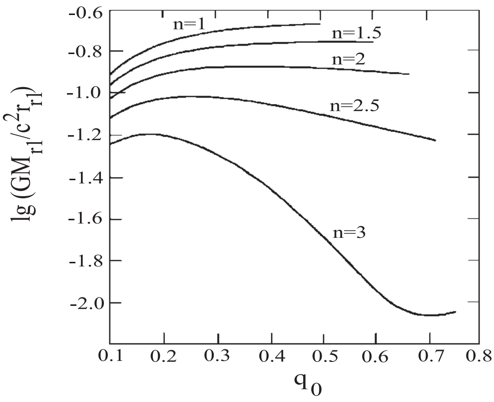

Fig. 4.1.3 Logarithmic half-ratio between the gravitational radius $2GM_{r1}/c^2$ from Eq. (4.1.28) and the geometrical radius r_{r1} according to Eq. (4.1.49): $GM_{r1}/c^2r_{r1} = q_0(n+1)\eta(\xi_1)/\xi_{r1}$ (Tooper 1964a).

We observe from Eq. (4.1.39) that $4\pi c^2\varrho_{r0}\alpha^3 = c^2M_{r1}/\eta(\xi_1)$, $(u(\xi_1) = 1)$, and since $E_{r1} = c^2M_{r1}$, we can write the relativistic gravitational energy (4.1.73) as

$$W_r = c^2M_{r1}\Big\{1 - [1/\eta(\xi_1)]\int_0^{\xi_1}\theta_r^n\xi^2\ d\xi\big/\big[1 - 2q_0(n+1)\ \eta(\xi)/\xi\big]^{1/2}\Big\}. \tag{4.1.75}$$

The rest energy of the polytrope is analogous to Eq. (4.1.70):

$$c^2M_1 = \int_{V_1}\varepsilon\ dV = 4\pi c^2\int_0^{r_1}\varrho\exp(\kappa/2)\ r^2\ dr$$
$$= [c^2M_{r1}/\eta(\xi_1)]\int_0^{\xi_1}\theta_r^n\xi^2\ d\xi/(1+q_0\theta_r)^n[1 - 2q_0(n+1)\ \eta(\xi)/\xi]^{1/2}. \tag{4.1.76}$$

M_1 is the rest mass of the polytrope inside proper volume V_1, and we have used Eqs. (4.1.39), (4.1.64). The binding energy E_b of the relativistic polytrope is defined as the difference between rest energy and relativistic energy:

$$E_b = c^2(M_1 - M_{r1}) = 4\pi c^2\int_0^{r_1}[\varrho\exp(\kappa/2) - \varrho_r]r^2\ dr. \tag{4.1.77}$$

The difference $M_1 - M_{r1}$ between rest mass and relativistic (gravitational, inertial) mass is called the mass defect. The binding energy is the energy that is restored during the formation of a dense polytrope from initially rarefied matter (Zeldovich and Novikov 1971, p. 287). It is apparent from the physics of this process that for a completely stable hydrostatic star we must have $E_b > 0$. Generally, we cannot make a definite a priori assertion with respect to the sign of E_b, since $\varrho \le \varrho_r$ and $\exp(\kappa/2) \ge 1$.

In the post Newtonian approximation we can expand Eq. (4.1.77) in terms of q_0 :

$$E_b \approx [c^2M_{r1}/\eta(\xi_1)]\int_0^{\xi_1}[1 - nq_0\theta_r + q_0(n+1)\ \eta(\xi)/\xi]\theta_r^n\xi^2\ d\xi - 4\pi c^2\int_0^{r_1}\varrho_r r^2\ dr$$
$$= -4\pi n\int_0^{r_1}Pr^2\ dr + 4\pi G\int_0^{r_1}M_r(r)\ \varrho_r r\ dr = -n\int_{V_1}P\ dV - W_r = -n(\Gamma-1)U - W_r$$
$$= nW_r/3 - W_r = (n-3)W_r/3, \qquad (q_0\approx 0;\ dV\approx 4\pi r^2\ dr), \tag{4.1.78}$$

where we have used $c^2M_{r1}/\eta(\xi_1) = 4\pi c^2\varrho_{r0}\alpha^3$ and $q_0(n+1)\eta(\xi)/\xi = GM_r(r)/c^2r$ from Eqs. (4.1.29), (4.1.31), (4.1.39), as well as Eqs. (2.6.95), (2.6.100), (4.1.72).

Table 4.1.1 Boundary values of $\xi_1, \xi_{r1}, \eta(\xi_1)$ for selected values of n and q_0 according to Tooper (1964a), Bludman (1973a), Chu et al. (1980), Dallas and Geroyannis (1993). If $n = 0$, the boundary values of $\xi_1, \xi_{r1}, \eta(\xi_1)$ are from Eqs. (4.1.45), (4.1.48), (4.1.43), respectively. If $q_0 = 0$, we have $\xi_{r1} = \xi_1$, and $\eta(\xi_1) = -\xi_1^2\theta_1'$ (cf. Table 2.5.2). $a + b$ means $a \times 10^b$.

n	q_0	ξ_1	ξ_{r1}	$\eta(\xi_1)$	n	q_0	ξ_1	ξ_{r1}	$\eta(\xi_1)$
	0.0	2.087+0	2.087+0	1.235+1		0.0	4.353+0	4.353+0	2.411+0
	0.2	1.619+0	1.650+0	6.199+0		0.2	3.398+0	4.094+0	8.403−1
−0.8	0.4	1.350+0	1.390+0	3.832+0	2	0.4	3.248+0	4.229+0	4.680−1
	0.6	1.171+0	1.212+0	2.642+0		0.6	3.399+0	4.578+0	3.201−1
	0.8	1.041+0	1.081+0	1.951+0		0.8	3.733+0	5.084+0	2.457−1
	1.0	9.411−1	9.746−1	1.462+0		0.9	3.953+0	5.387+0	2.216−1
	0.0	2.209+0	2.209+0	7.422+0		0.0	5.355+0	5.355+0	2.187+0
	0.2	1.679+0	1.756+0	3.333+0		0.2	4.721+0	5.658+0	7.606−1
−0.5	0.4	1.392+0	1.492+0	1.949+0	2.5	0.4	5.545+0	6.976+0	4.386−1
	0.6	1.206+0	1.311+0	1.302+0		0.6	7.727+0	9.633+0	3.202−1
	0.8	1.074+0	1.177+0	9.419−1		0.8	1.190+1	1.436+1	2.721−1
	1.0	9.730−1	1.073+0	7.189−1		0.9	1.485+1	1.763+1	2.619−1
	0.0	2.347+0	2.347+0	5.619+0		0.0	6.897+0	6.897+0	2.018+0
	0.2	1.753+0	1.884+0	2.353+0		0.2	7.951+0	9.262+0	7.130−1
−0.2	0.4	1.447+0	1.613+0	1.337+0	3	0.4	1.782+1	2.026+1	4.516−1
	0.6	1.255+0	1.430+0	8.804−1		0.6	9.107+1	9.583+1	4.493−1
	0.8	1.121+0	1.293+0	6.326−1		0.8	1.872+2	1.950+2	5.969−1
	1.0	1.020+0	1.190+0	4.813−1		0.9	1.870+2	1.959+2	6.375−1
	0.0	2.449+0	2.449+0	4.899+0		0.0	9.536+0	9.536+0	1.891+0
	0.2	1.811+0	1.979+0	1.981+0		0.2	2.371+1	2.582+1	7.055−1
0	0.4	1.494+0	1.706+0	1.111+0	3.5	0.4	1.382+4	1.384+4	2.735+0
	0.6	1.298+0	1.522+0	7.282−1		0.6	1.640+3	1.662+3	1.837+0
	0.8	1.162+0	1.387+0	5.226−1		0.8	1.305+3	1.331+3	1.678+0
	1.0	1.061+0	1.283+0	3.977−1		0.9	1.382+3	1.410+3	1.639+0
	0.0	2.753+0	2.753+0	3.789+0		0.0	1.497+1	1.497+1	1.797+0
	0.2	2.001+0	2.267+0	1.437+0		0.2	2.624+4	2.628+4	1.189+1
0.5	0.4	1.654+0	1.996+0	7.912−1	4	0.4	1.342+7	1.342+7	1.289+2
	0.6	1.450+0	1.818+0	5.167−1		0.6	4.945+6	4.946+6	5.974+1
	0.8	1.314+0	1.690+0	3.717−1		0.8	2.020+7	2.021+7	4.991+1
	0.9	1.260+0	1.639+0	3.232−1		0.9	2.527+10	2.527+10	1.761+3
	0.0	3.142+0	3.142+0	3.142+0		0.0	3.184+1	3.184+1	1.738+0
	0.2	2.277+0	2.659+0	1.143+0		0.2	8.426+10	8.426+10	1.395+4
1	0.4	1.913+0	2.412+0	6.249−1	4.5	0.4	6.315+18	6.315+18	4.429+7
	0.6	1.714+0	2.265+0	4.101−1		0.6	6.045+18	6.045+18	2.305+7
	0.8	1.590+0	2.168+0	2.979−1		0.8	2.139+24	2.139+24	6.038+9
	0.9	1.545+0	2.132+0	2.605−1		0.9	9.012+25	9.012+25	6.741+10
	0.0	3.654+0	3.654+0	2.714+0		0.0	1.714+2	1.714+2	1.725+0
	0.2	2.699+0	3.219+0	9.603−1		0.2	1.028+29	1.028+29	5.268+13
1.5	0.4	2.361+0	3.062+0	5.270−1	4.9	0.4	1.575+29	1.575+29	5.910+13
	0.6	2.219+0	3.018+0	3.505−1		0.6	9.808+28	9.808+28	5.027+13
	0.8	2.166+0	3.033+0	2.594−1		0.8	9.770+28	9.770+28	4.951+13
	0.9	2.158+0	3.055+0	2.291−1		0.9	9.736+28	9.736+28	4.890+13

For an adiabatic perfect gas we have $n = 1/(\gamma - 1)$ and $\Gamma = \gamma_g = \gamma = c_P/c_V$ by virtue of Eqs. (1.7.60), (2.6.93). In this case Eq. (4.1.78) becomes

$$E_b = (4 - 3\gamma)W_r/3(\gamma - 1) = -E, \qquad (q_0 \approx 0), \tag{4.1.79}$$

where E turns into the total energy (2.6.100) of a nonrelativistic sphere if $q_0 = 0$. As it was already observed in Sec. 2.6.6, the condition of stability against compression or expansion of the spherical polytrope is $E < 0$ or $E_b > 0$, which means $\gamma > 4/3$ in Eq. (4.1.79), and $n < 3$, $(W_r < 0)$ in Eq. (4.1.78). However, $n > 3$, $(\gamma < 4/3)$, i.e. a negative binding energy of the spherical relativistic polytrope is not a sufficient condition for instability against contraction or expansion (Tooper 1964a, Zeldovich and Novikov 1971, p. 293).

Eqs. (4.1.37) and (4.1.38) have been integrated with the initial conditions (4.1.41) by Tooper (1964a) for $n = 1, 1.5, 2, 2.5, 3$, the parameter q_0 increasing from 0 up to the value $n/(n+1)$ from Eq. (4.1.67).

Similar integrations have been effected by Bludman (1973a) for $0 \leq n \leq 3$ and $0 \leq q_0 \leq 0.9$. Chu et al. (1980) integrate for values of $n = -10, -4, -3, -2, -1.5, -1.2, -0.8, -0.5, -0.2$, and $0.2 \leq q_0 \leq 1$. Dallas and Geroyannis (1993) provide tabulations if $0 \leq n \leq 4.9$ and $0 \leq q_0 \leq 0.9$. Tabulations for small values of q_0 (in the so-called post Newtonian approximation $q_0 \ll 1$) have been effected by Chandrasekhar (1964b) if $n = 1, 2, 3$.

From Table 4.1.1 follows that both, the dimensionless radial coordinate ξ_1 and the dimensionless radius ξ_{r1} are always decreasing functions of q_0 if $-0.8 \leq n \leq 1.5$. If $1.5 < n \leq 3$, the values of ξ_1 and ξ_{r1} first decrease and then increase as q_0 increases, but the minima of ξ_1, ξ_{r1} do not occur at the same q_0. If $3 < n \leq 4.9$, we notice fluctuations of ξ_1, ξ_{r1} (Dallas and Geroyannis 1993).

For relativistic values of q_0 the structural features indicate a greater concentration of matter towards the centre than in the nonrelativistic case $q_0 = 0$. The normalized density $\theta_r^n = \varrho_r/\varrho_{r0}$ and pressure $\theta_r^{n+1} = P/P_0$ fall off more rapidly as a function of radius in the relativistic case, and the mass function M_r/M_{r1} rises sooner (Fig. 4.1.2).

If $n = 0,\ q_0 \to \infty$, we obtain from Eq. (4.1.50) the limiting value of the mass-radius relationship $GM_{r1}/c^2 r_{r1} \leq 2^{2/7}/27 \arcsin(2^{3/2}/3) = 0.340$, or $r_{r1} \geq 1.47 r_g$ and $r_1 \geq 9GM_{r1}/4c^2 = 9r_g/8 = 1.125 r_g$. This inequality for the radial surface coordinate holds for any hydrostatic configuration, independent of any constitutive relationship between P and ϱ_r, provided that $d\varrho_r/dr \leq 0$ (Buchdahl 1959). This inequality is further strengthened for the polytropic models discussed below. The mass-radius relationship from Eq. (4.1.49) can be written under the form

$$GM_{r1}/c^2 r_{r1} = (n+1)q_0\eta(\xi_1)/\xi_{r1} \quad \text{or} \quad r_{r1} = \xi_{r1} r_g/2(n+1)q_0\eta(\xi_1). \tag{4.1.80}$$

As seen from Fig. 4.1.3, the maximum of $GM_{r1}/c^2 r_{r1} = q_0(n+1)\eta(\xi_1)/\xi_{r1}$ for the considered values of q_0 is 0.214 if $n = 1$, and 0.0631 if $n = 3$. This means that hydrostatic configurations are possible only when $r_{r1} \geq 1.47 r_g,\ r_1 \geq 9r_g/8 = 1.125 r_g$ if $n = 0,\ r_{r1} \geq 2.34 r_g,\ r_1 \geq 1.81 r_g$ if $n = 1$, and $r_{r1} \geq 7.92 r_g,\ r_1 \geq 6.86 r_g$ if $n = 3$.

Fig. 4.1.3 also points out an important characteristic of the relativistic solutions: If $n = 2$ and 2.5, there correspond to the same pair of values M_{r1} and r_{r1} from Eq. (4.1.80) two distinct values of q_0. And if $n = 3$, when $\lg[(n+1)q_0\eta(\xi_1)/\xi_{r1}] = -2.05$, three distinct values of q_0 occur, viz. 7.62×10^{-3}, 0.665, 0.740, correlated to widely different distributions of density, pressure, and metric tensor (Tooper 1964a).

The behaviour of total mass and radius of relativistic polytropic spheres if $0 \leq n \leq \infty$ has been clarified by Nilsson and Uggla (2000), introducing the bounded relativistic Milne-like variables (4.1.82). The Newtonian Milne variables (2.2.6) can be written as (Kimura 1981a)

$$u_1 = u = d\ln M(r)/d\ln r = 4\pi\varrho r^3/M(r) = -\xi\theta^n/\theta';$$
$$v_1/(n+1) = v = -[1/(n+1)]\ d\ln P/d\ln r = G\varrho M(r)/(n+1)Pr = -\xi\theta'/(n+1)\theta. \tag{4.1.81}$$

Nilsson and Uggla (2000) take

$$U = u_1/(u_1+1) = 4\pi r^2\varrho_r/[4\pi r^2\varrho_r + M_r(r)/r];$$
$$V = v_1/(v_1+1) = [M_r(r)/r]/[M_r(r)/r + P/\varrho_r], \qquad (0 < U, V < 1), \tag{4.1.82}$$

in geometrized units $c, G = 1$.

With the third variable $y = P/(P + \varrho_r)$, the field equations (4.1.6), (4.1.7), (4.1.18) turn into a system of three regular ordinary differential equations $\{U, V, y\}$ in terms of the new independent variable $F = (1-V)(1-y) - 2yV$. The qualitative and numerical discussion of the topology in (U, V, y)-space shows that relativistic spheres have finite mass and radius if $0 \leq n \lesssim 3.339$. If $3.339 \lesssim n < 5$, a one-parameter set of regular models exist, having finite radius and mass (Table 4.1.1); a finite number of regular models possess infinite radius, and finite or infinite mass. If $n \geq 5$ all spheres have infinite radius and mass.

Other studies due to Tooper (1965, 1966b) have been effected for the case of an isentropic, relativistic perfect gas, when pressure is assumed to be connected to rest mass density according to Eqs. (1.2.31) and (1.2.32):

$$P = K\varrho^{1+1/n} = K\varrho^\gamma, \qquad (S = \text{const};\ \gamma = 1 + 1/n). \tag{4.1.83}$$

The relativistic energy density of the isentropic perfect gas is via Eqs. (1.7.56), (1.7.58), (4.1.2) equal to

$$\varepsilon_r = \varepsilon + \varepsilon^{(int)} = \varrho c^2 + w\varepsilon^{(kin)}/3(\gamma-1) = \varrho c^2 + P/(\gamma-1) = \varrho c^2 + nP. \tag{4.1.84}$$

$\varepsilon^{(int)}$ is the internal energy density, and $\varepsilon^{(kin)}$ the energy density of kinetic translational motion. A similar relationship can also be obtained from Eq. (2.6.92): $\varepsilon_r = \varepsilon + \varepsilon^{(int)} = \varrho c^2 + P/(\Gamma - 1)$. In fact, Eq. (4.1.84) is only a special case of a class of isentropic equations of state for which pressure is related to rest mass density by the power law $P = K\varrho^{1+1/n}$. In the relativistic first law of thermodynamics (4.1.57) we replace $d\varrho/\varrho$ by $[n/(n+1)]\ dP/P$ via the isentropic law (4.1.83):

$$P\ d\varepsilon_r/dP - n\varepsilon_r/(n+1) = nP/(n+1), \qquad [P = K\varrho^{1+1/n};\ S = \text{const};\ n = 1/(\gamma-1)]. \tag{4.1.85}$$

This simple inhomogeneous equation can be integrated at once by the method of variation of constants (Tooper 1965, Durgapal and Pande 1983):

$$\varepsilon_r = CP^{n/(n+1)} + nP, \qquad (C, S = \text{const}). \tag{4.1.86}$$

We recover Eq. (4.1.84) by setting $C = c^2/K^{n/(n+1)}$ and $n = 1/(\gamma - 1)$, (cf. Rosquist 1995).

Tooper (1966a) has generalized Eq. (4.1.84) further, by taking into account radiation pressure (as for massive hot stars):

$$\varepsilon_r = \varrho c^2 + P_g/(\gamma - 1) + 3P_r. \tag{4.1.87}$$

The total pressure $P = P_g + P_r$ is composed of perfect gas pressure P_g and radiation pressure $P_r = aT^4/3$ (cf. Sec. 1.4).

With the two line elements

$$ds^2 = \exp[\nu(r)]\ dt^2 - \exp[\kappa(r)]\ [dr^2 + r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2)],$$
$$\big(\exp\nu = [1 - f(r)]^2/[1 + f(r)]^2;\ \exp\kappa = [1 + f(r)]^4\big), \tag{4.1.88}$$

$$ds^2 = \exp[\nu(r)]\ dt^2 - \exp[\kappa(r)]\ dr^2 - \exp[\mu(r)]\ r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2),$$
$$(\nu = f(\mu);\ \kappa = g(\mu);\ df/d\mu + dg/d\mu = 0), \tag{4.1.89}$$

Buchdahl (1964, 1967) finds two static, spherically symmetric analytic solutions of the field equations (4.1.4), possessing respectively, the two equations of state

$$P = \varrho_r^{6/5}/[1 + 6q_0(1 - \varrho_r^{1/5})], \qquad (0 \le q_0 = P_0/c^2\varrho_{r0} \le 1), \tag{4.1.90}$$

$$(1 + k)P^{1/2} - kP = \varrho_r, \qquad (0 \le k = \text{const} \le 5/7). \tag{4.1.91}$$

In the nonrelativistic limit $(q_0, P \approx 0)$ these two equations of state become a $n = 5$ (Plummer) and $n = 1$ polytrope, respectively. Generalizations of Buchdahl's treatment have been provided by Rosquist (1999), and Beig and Karadi (2001).

Sá (1999) has considered polytropic spheres $(0 < n < \infty)$ in *three-dimensional* spacetime with the λ-coordinate in Eq. (4.1.5) equal to $\lambda \equiv \pi/2$, and with negative cosmological constant.

4.1.2 Composite Polytropes in General Relativity

Fang and Xiang (1982) have considered relativistic polytropes with a compact core, similar to the nonrelativistic loaded polytropes of Huntley and Saslaw (1975) from Sec. 2.8.3. Pandey et al. (1983) have investigated composite models composed of an isothermal core with equation of state

$$P_c = K_c\varrho_{rc}, \tag{4.1.92}$$

surrounded by a polytropic envelope with equation of state (cf. Chandrasekhar (1972) if $n = 0$, and Sec. 2.8.1 for the Newtonian case)

$$P_e = K_e\varrho_{re}^{1+1/n}. \tag{4.1.93}$$

Table 4.1.2 Parameters for isothermal core and polytropic envelope using the interface condition $H = 0$ from Eq. (4.1.97), (Pandey et al. 1983).

n	K_c	K_e	M_{rc}/M_{r1}	r_i/r_1	z_1	z_{ci}	z_{c0}
0.5	1/3	20.2	0.438	0.548	0.54	1.37	2.96
1	1/3	2.59	0.390	0.426	0.42	1.52	3.22
1.5	1/3	1.31	0.345	0.309	0.31	1.68	3.48
1	0.4	3.03	0.377	0.421	0.46	1.87	4.12
1.5	0.4	1.54	0.328	0.290	0.30	2.05	4.44
1.5	0.6	2.27	0.233	0.277	0.31	2.24	7.96
2	0.6	1.63	0.222	0.118	0.17	3.78	9.09

Choosing the units in such a way that $c, G = 1$, the Tolman-Oppenheimer-Volkoff equation (4.1.27) reads

$$dP/dr = -(\varrho_r + P)(M_r + 4\pi P r^3)/(r^2 - 2rM_r). \tag{4.1.94}$$

Defining the Bondi variables by

$$g(r) = M_r/r = (4\pi/r)\int_0^r \varrho_r r^2\, dr; \quad h(r) = 4\pi r^2 P, \tag{4.1.95}$$

the equation of relativistic hydrostatic equilibrium (4.1.94) becomes

$$(1/r)\ dr/dg = [(1-2g)\ dh/dg + (g+h)]/(2h - g^2 - 6gh - h^2),$$
$$(c, G = 1;\ \varrho_r = (1/4\pi r)\ dg/dr + g/4\pi r^2;\ dP/dr = (1/4\pi r^2)\ dh/dr - h/2\pi r^3). \tag{4.1.96}$$

The denominator

$$H = 2h - g^2 - 6gh - h^2 = 0, \tag{4.1.97}$$

defines the equation of a hyperbola, dividing the configuration into an inner core region ($H > 0$) and an outer envelope region ($H < 0$). Although the hyperbola $H = 0$ does not correspond to a physical condition, it can be taken to define the core-envelope interface $r = r_i$, where $P_{ci} = P_{ei},\ \varrho_{rci} = \varrho_{rei}$. The polytropic constants K_c, K_e from Eqs. (4.1.92), (4.1.93) are connected through the continuity of pressure at the interface by

$$K_c = K_e \varrho_{rei}^{1/n} = K_e \varrho_{rci}^{1/n}. \tag{4.1.98}$$

The velocity of sound from Eq. (4.1.66) is

$$a_c^2 = (\partial P_c/\partial \varrho_{rc})_S = K_c, \tag{4.1.99}$$

in the core, and

$$a_e^2 = (\partial P_e/\partial \varrho_{re})_S = (n+1)K_e \varrho_{re}^{1/n}/n, \tag{4.1.100}$$

in the envelope. It jumps from the constant value $K_c^{1/2}$ in the core to $[(n+1)K_e\varrho_{rei}^{1/n}/n]^{1/2} = [(n+1)K_c/n]^{1/2}$ at the inner boundary of the envelope. The redshift $z = (f - f_\infty)/f_\infty$ of photons emitted at frequency f inside the composite polytrope, and observed at large distance from the mass at frequency f_∞, is via the metrics (4.1.3), (4.1.5) equal to (e.g. Landau and Lifschitz 1987)

$$1 + z = f/f_\infty = (g_{00})^{-1/2} = \exp(-\nu/2). \tag{4.1.101}$$

At the surface of the composite polytrope we have $g_{00} = 1 - 2GM_{r1}/c^2 r_1$ from the external Schwarzschild metric (4.1.20), and

$$1 + z_1 = (1 - 2M_{r1}/r_1)^{-1/2}, \qquad (c, G = 1). \tag{4.1.102}$$

In order to calculate the gravitational redshift z_e in the envelope, we integrate the conservation law (4.1.18) with the equation of state (4.1.93):

$$1+z_e=\exp(-\nu/2)=C(K_e\varrho_{re}^{1/n}+1)^{n+1}=C(P_e/\varrho_{re}+1)^{n+1}=(1+z_1)(P_e/\varrho_{re}+1)^{n+1};$$
$$1+z_{ei}=1+z_{ci}=(1+z_1)(1+K_c)^{n+1},\qquad (C=\mathrm{const}). \tag{4.1.103}$$

Finally, the redshift in the core is calculated in the same way with the equation of state (4.1.92):

$$1+z_c=\exp(-\nu/2)=(C\varrho_{rc})^{K_c/(K_c+1)}=(1+z_{ci})(\varrho_{rc}/\varrho_{rci})^{K_c/(K_c+1)}. \tag{4.1.104}$$

Table 4.1.2 shows the ratio M_{rc}/M_{r1} between core mass M_{rc} and total mass M_{r1}, the ratio r_i/r_1 between the radial coordinates r_i, r_1 at interface and surface, as well as the redshifts at the surface, the interface, and at the centre for different values of K_c, $(K_c\leq 1)$ and n, subject to the condition $a_c, a_e \leq c$. It is seen that for a given value of K_c fractional core mass and core size both decrease as n increases. Keeping n fixed, these two quantities also decrease if K_c is increased.

4.1.3 Spherical Polytropes with Variable Gravitational Constant

These investigations are due to an Armenian group of astrophysicists, and are based on Dirac's hypothesis of a slowly changing gravitational constant G, so that the field equations depend besides the ten distinct components of the metric tensor g_{jk} also on the gravitational scalar $G=G(t,x^1,x^2,x^3)$, (Jordan 1955). From a variational principle Saakyan and Mnatsakanyan (1967) find with Newtonian gravitation for a variable G the equations

$$\varrho\ D\vec{v}/Dt+\nabla P-\varrho\ \nabla\Phi=0;\quad \nabla\cdot(\nabla\Phi/g)=-c^2\varrho/2;\quad \nabla\cdot(\nabla g/g^{3/2})$$
$$-(1/c^2)\ \partial(g^{-3/2}\ \partial g/\partial t)/\partial t=(1/2c^4\zeta)\ (\nabla\Phi)^2/g^{1/2},\qquad (g=g(t,x^1,x^2,x^3)=8\pi G/c^2). \tag{4.1.105}$$

ζ is a dimensionless parameter of variable G-theory (Saakyan and Mnatsakanyan 1968). As results from the last equation (4.1.105), the Newtonian theory with $g=$ const is recovered if $\zeta\to\pm\infty$. The first equation (4.1.105) is the hydrodynamic equation of motion (2.1.1), and the second is identical to Poisson's equation (2.1.4) if $g=8\pi G/c^2=$ const. In the static, spherically symmetric case Eq. (4.1.105) turns into

$$dP/dr=\varrho\ d\Phi/dr;\quad d[(r^2/g)\ d\Phi/dr]/dr=-\varrho r^2/2;\quad g^{1/2}\ d[(r^2/g^{3/2})\ dg/dr]/dr$$
$$=r^2(d\Phi/dr)^2/2\zeta,\qquad [c=1;\ g=g(r)=8\pi G(r)]. \tag{4.1.106}$$

Outside the polytrope we get by integration of the second equation (4.1.106):

$$d\Phi_e/dr=-g(r)\ M_1/8\pi r^2,\qquad (r\geq r_1;\ (d\Phi_e/dr)_{r\to\infty}=-G(\infty)\ M_1/r^2=-M_1/r^2;$$
$$g(\infty)=8\pi G(\infty)=8\pi;\ G(\infty)=1), \tag{4.1.107}$$

where M_1 is the total Newtonian mass of the polytrope. Substituting Eq. (4.1.107) into the last equation (4.1.106), we have $(\Phi\to\Phi_e)$:

$$d[(r^2/g^{3/2})\ dg/dr]/dr=M_1^2g^{3/2}/128\pi^2\zeta r^2=\Gamma g^{3/2}/r^2,\qquad (r\geq r_1;\ \Gamma=M_1^2/2^7\pi^2\zeta). \tag{4.1.108}$$

This equation can be integrated with the substitution

$$g^{3/2}/r^2=dy/dr, \tag{4.1.109}$$

which yields

$$g=\Gamma y^2/2+A_1y+A_2,\qquad (A_1,A_2=\mathrm{const}). \tag{4.1.110}$$

From Eq. (4.1.110) we get

$$dy/dr=\pm(dg/dr)\big/(A_1^2-2\Gamma A_2+2\Gamma g)^{1/2}, \tag{4.1.111}$$

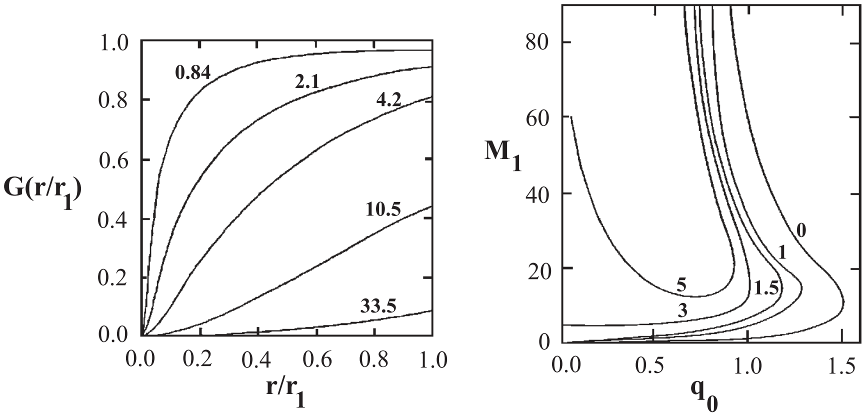

Fig. 4.1.4 Newtonian variant of variable G-theory. On the left: Variation of the dimensionless gravitational scalar within a constant density polytrope ($n = 0$; $\zeta = -30$) for values of M_1/r_1 indicated on each curve (Saakyan and Mnatsakanyan 1967). On the right: Dependence of M_1 on $q_0 = P_0/\varrho_0 = K\varrho_0^{1/n}$ for $c, K = 1$, and for the polytropic indices n indicated on each curve (Mnatsakanyan and Avakyan 1968).

and by insertion into Eq. (4.1.109)

$$dr/r^2 = \pm\, dg/g^{3/2}(A_1^2 - 2\Gamma A_2 + 2\Gamma g)^{1/2}, \tag{4.1.112}$$

or

$$g = 8\pi r^2\big/(r^2 + B_1 r + B_2), \qquad (r \geq r_1;\ B_1, B_2 = \mathrm{const}), \tag{4.1.113}$$

the two constants being connected by $B_1 = \pm(4B_2 + M_1^2/\zeta)^{1/2}$. Eliminating the potential Φ among the equations (4.1.106), and introducing the mass inside a sphere of radius r

$$M(r) = -(8\pi r^2/g\varrho)\ dP/dr, \tag{4.1.114}$$

we obtain the following basic set of equations:

$$dM(r)/dr = 4\pi\varrho r^2, \tag{4.1.115}$$

$$dP/dr = -g\varrho M(r)/8\pi r^2, \tag{4.1.116}$$

$$d^2g/dr^2 + (2/r)\ dg/dr - (3/2g)(dg/dr)^2 = \Gamma g^3 M^2(r)/M_1^2 r^4. \tag{4.1.117}$$

Solutions of the internal problem defined by Eqs. (4.1.115)-(4.1.117) must be matched to the external solution (4.1.113), which yields on the boundary $r = r_1$:

$$g(r_1) = 8\pi r_1^2\big/(r_1^2 + B_1 r_1 + B_2); \quad (dg/dr)_{r=r_1} = 16\pi r_1(B_1 r_1 + B_2)\big/(r_1^2 + B_1 r_1 + B_2)^2. \tag{4.1.118}$$

Since the value of the gravitational scalar g at the centre is unknown, it is advisable to start the numerical integration at the surface $r = r_1$ for a guessed mass $M_1 = M(r_1)$ of the polytrope with the boundary condition $P(r_1) = 0$, and $g(r_1)$, $g'(r_1)$ given by Eq. (4.1.118). By fixing r_1, and integrating from the surface $r = r_1$ to the centre $r = 0$, we will eventually find the true value M_1 for which $M(0) = 0$.

To fix the parameter ζ of the theory, Jordan (1955) demands that the change in the observed general relativistic precession effect of Mercury should not exceed the relative observational error of 2%, leading him to the value $\zeta = \pm 30$.

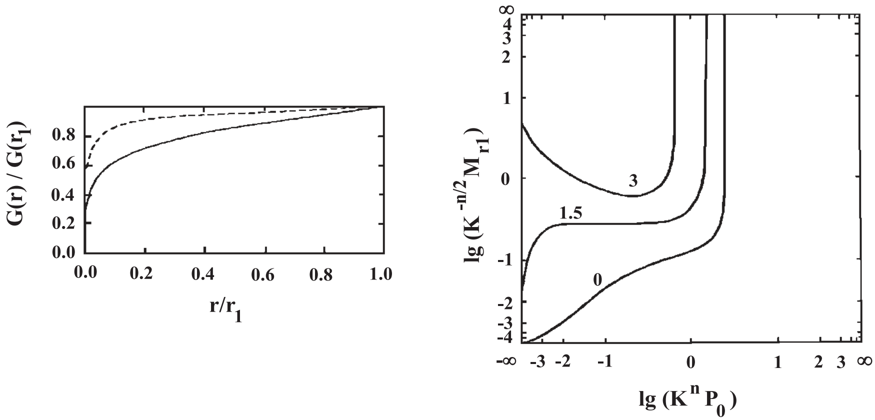

Fig. 4.1.5 Relativistic variable G-theory. On the left: Behaviour of the gravitational scalar $G(r)$ inside a polytropic sphere ($0 \leq n \leq 3$) for an intermediate value of M_{r1}/r_1 (dashed line), and if $M_{r1}/r_1 \to \infty$ (solid line). Curves for different polytropic indices coincide at the scale of the figure. On the right: Total relativistic mass M_{r1} as a function of central pressure P_0 for the polytropic indices indicated on each curve (Mnatsakanyan 1969).

Mnatsakanyan and Avakyan (1968) write Eqs. (4.1.115)-(4.1.117) under the dimensionless form

$$d\mu/dy = 4\pi M_1^2 K^{-n} q^n y^2; \quad dq/dy = -G\mu/(n+1)y^2;$$
$$d^2G/dy^2 + (2/y)\, dG/dy - (3/2G)(dG/dy)^2 = G^3\mu^2/2\zeta y^4, \tag{4.1.119}$$

by using the nonrelativistic polytropic equation of state (2.1.6) and the notations

$$q = P/\varrho = K\varrho^{1/n}; \quad y = r/M_1; \quad \mu = M(r)/M_1; \quad G(y) = g(y)/8\pi = g(r/M_1)/8\pi,$$
$$(G(\infty) = g(\infty)/8\pi = 1;\ c = 1). \tag{4.1.120}$$

For the incompressible model with polytropic index $n = 0$ the gravitational scalar $G(y) = g(y)/8\pi$ is zero at the centre $y = 0$, and tends to its Newtonian value $G = G(\infty) = 1$ if $y = r/M_1 \to \infty$ (Fig. 4.1.4 on the left). The curve representing the total mass M_1 as a function of the central value $q_0 = P_0/\varrho_0 = K\varrho_0^{1/n}$ is two-valued (Fig. 4.1.4 on the right). The lower branch is not very different from the corresponding $M_1(q_0) = M_1(\varrho_0)$-curves obtained with $G = \mathrm{const}$. The upper branch represents baryon and electron configurations. The radii of these supermassive spheres can be less than their gravitational radius r_g from Eq. (4.1.28), since in variable G-theories no Schwarzschild-type singularity exists (black holes are absent), i.e. static configurations can have radii less than r_g (cf. Jordan 1955, Saakyan and Mnatsakanyan 1968).

In the *relativistic* variable G-theory Saakyan and Mnatsakanyan (1968, 1969) find for the spherically symmetric static case the following set of equations for the determination of the four unknowns $P(r), g(r), \kappa(r), \nu(r)$, supplemented by Eqs. (4.1.1), (4.1.5), (4.1.18):

$$[(2G - r\ dG/dr)/2rG]\ d\nu/dr + (\zeta/2G^2)(dG/dr)^2 - (2/rG)\ dG/dr - (\exp\kappa - 1)/r^2$$
$$= (8\pi GP/c^4)\exp\kappa; \qquad (1/G)\ d^2G/dr^2 + [(\zeta - 4)/2G^2](dG/dr)^2 + (2/rG)\ dG/dr$$
$$+[(2G - r\ dG/dr)/2rG]\ d\kappa/dr + (\exp\kappa - 1)/r^2 = (8\pi G\varrho_r/c^2)\exp\kappa;$$
$$(1/2r)\ d\kappa/dr - (1/2r)\ d\nu/dr + (1/rG)\ dG/dr + (\exp\kappa - 1)/r^2$$
$$= 8\pi G\exp\kappa\ [\zeta P + (1-\zeta)c^2\varrho_r]/c^4(3 - 2\zeta), \qquad [G = G(r)]. \tag{4.1.121}$$

The equations of general relativity are obtained if $G = \mathrm{const}$ and $\zeta = \pm\infty$. In the limit, the equations (4.1.121) become consecutively equal to Eqs. (4.1.6), (4.1.7), and to the difference between Eqs. (4.1.7) and (4.1.6). Mnatsakanyan (1969) has integrated Eq. (4.1.121) for the polytropic indices $n = 0, 1.5, 3$.

The asymptotic, maximum admissible central values P_0 of the pressure (Fig. 4.1.5 on the right) result from a superposition of the effects associated with the curvature of space and the weakening of gravitational interaction associated with large mass concentration in variable G-theory. The maximum values of $q_0 = P_0/\varrho_{r0},\ (c = 1)$ are 2.070 ($n = 0$), 1.135 ($n = 1.5$), and 0.925 ($n = 3$), as compared to the Newtonian approximation from Fig. 4.1.4, where $q_0 = 1.51$ ($n = 0$), 1.18 ($n = 1.5$), and 1.02 ($n = 3$).

4.1.4 Polytropic Spheres in Bimetric Gravitation Theory

Sarkisyan (1980) has investigated the structure of polytropic spheres according to Rosen's (1974) bimetric gravitation theory. As suggested by name, two metrics are defined in bimetric gravitation theories: The usual Riemannian metric $ds^2 = g_{jk}\ dx^j dx^k$ from Eq. (4.1.3), which describes the true gravitational field arising from matter and other forms of energy, and a flat-space (zero curvature) metric $df^2 = f_{jk}\ dx^j dx^k$, which describes inertial forces associated with the acceleration of the reference frame. For the static, spherically symmetric case Rosen (1974) defines the two metrics

$$ds^2 = \exp[2\nu(r)]\ dt^2 - \exp[2\kappa(r)]\ dr^2 - r^2 \exp[2\chi(r)]\ (d\lambda^2 + \sin^2\lambda\ d\varphi^2);$$
$$df^2 = dt^2 - dr^2 - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2), \tag{4.1.122}$$

obtaining the field equations under the form $(c, G = 1)$:

$$\nabla^2\nu = (1/r^2)\ d(r^2\ d\nu/dr)/dr = 4\pi(3P + \varrho_r)\exp(\nu + \kappa + 2\chi), \tag{4.1.123}$$

$$(1/r^2)\ d(r^2\ d\kappa/dr)/dr + (2/r^2)\sinh[2(\chi - \kappa)] = 4\pi(P - \varrho_r)\exp(\nu + \kappa + 2\chi), \tag{4.1.124}$$

$$(1/r^2)\ d(r^2\ d\chi/dr)/dr - (1/r^2)\sinh[2(\chi - \kappa)] = 4\pi(P - \varrho_r)\exp(\nu + \kappa + 2\chi). \tag{4.1.125}$$

Subtracting Eq. (4.1.124) from Eq. (4.1.125) one finds

$$\nabla^2(\chi - \kappa) - (3/r^2)\sinh[2(\chi - \kappa)] = 0. \tag{4.1.126}$$

At large distances from the mass M_{r1}, the metric ds^2 from Eq. (4.1.122) must converge to the flat space Galilean metric of special relativity, i.e. $\nu(r), \kappa(r), \chi(r) \rightarrow 0$ as $r \rightarrow \infty$. Since near the origin $\chi - \kappa \approx Cr^2,\ (r \approx 0;\ C = \mathrm{const})$, and since $\chi - \kappa$ is a monotonically increasing (decreasing) function, the two conditions $\chi - \kappa = 0$ if $r = 0$ and $\chi - \kappa = 0$ if $r \rightarrow \infty$ can be satisfied only if $\chi - \kappa \equiv 0$ or $\chi \equiv \kappa$ (Rosen 1974). Thus, Eqs. (4.1.123)-(4.1.125) reduce to

$$(1/r^2)\ d(r^2\ d\nu/dr)/dr = 4\pi(3P + \varrho_r)\exp(\nu + 3\kappa), \tag{4.1.127}$$

$$(1/r^2)\ d(r^2\ d\kappa/dr)/dr = 4\pi(P - \varrho_r)\exp(\nu + 3\kappa). \tag{4.1.128}$$

Outside the boundary value r_1 of the radial coordinate r we have $P, \varrho_r = 0$, and the solutions of Eqs. (4.1.127), (4.1.128) become, by integrating at first between 0 and r, $(r \geq r_1)$, and then between r and ∞ :

$$\nu = -(4\pi/r)\int_0^{r_1}(3P + \varrho_r)\exp(\nu + 3\kappa)\ r'^2\ dr';$$
$$\kappa = -(4\pi/r)\int_0^{r_1}(P - \varrho_r)\exp(\nu + 3\kappa)\ r'^2\ dr', \qquad (r \geq r_1). \tag{4.1.129}$$

We have used the conditions at infinity $\nu(\infty) = 0,\ \kappa(\infty) = 0$, and the initial conditions $\nu(0) = \nu_0,\ \kappa(0) = \kappa_0,\ (d\nu/dr)_{r=0} = 0,\ (d\kappa/dr)_{r=0} = 0$, as they result from the series expansions near the origin. On the surface, ν and κ are of the form const$/r_1$, satisfying the simple boundary conditions

$$\nu(r_1) + r_1(d\nu/dr)_{r=r_1} = 0; \quad \kappa(r_1) + r_1(d\kappa/dr)_{r=r_1} = 0. \tag{4.1.130}$$

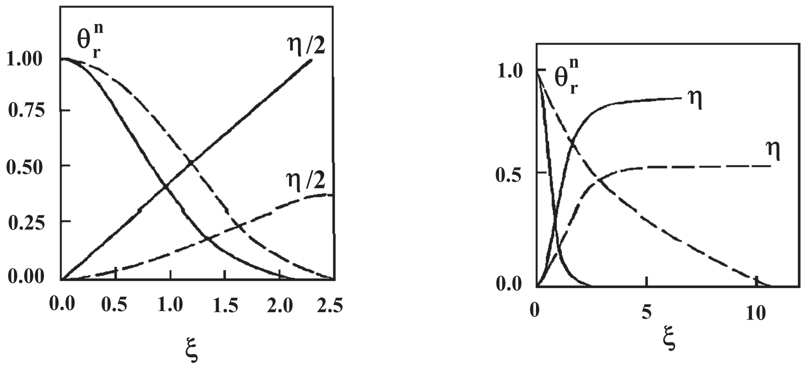

Fig. 4.1.6 Comparison of $\theta_r^n(\xi)$ and $\eta(\xi) = M_r(r)/4\pi\varrho_{r0}\alpha^3$ in bimetric theory (Eq. (4.1.150), solid curves) and in general relativity (Eq. (4.1.39), broken curves) if $n = 1.5,\ q_0 = 0.3$ (on the left), and $n = 3,\ q_0 = 0.3$ (on the right), (Sarkisyan 1980).

The sum of Eqs. (4.1.127) and (4.1.128) yields

$$(1/r^2)\ d[r^2\ d(\nu + 3\kappa)/dr]/dr = 4\pi(6P - 2\varrho_r)\exp(\nu + 3\kappa). \tag{4.1.131}$$

Making now the substitutions

$$r = Hx; \quad r^2\ d\nu/dr = Hy; \quad r^2\ d(\nu + 3\kappa)/dr = 2Hz, \tag{4.1.132}$$

and introducing the notations

$$H = \exp[-\nu(r_1)/2] = \exp[-\nu(x_1)/2]; \quad t(x) = \nu(x_1) - \nu(x);$$
$$\gamma(x) = -t(x) + 3\kappa(x) = \nu(x) + 3\kappa(x) - \nu(x_1), \qquad (x_1 = r_1/H), \tag{4.1.133}$$

the second order system (4.1.127) and (4.1.131) can be reduced to the following set of first order equations:

$$dy/dx = 4\pi H^2x^2(3P + \varrho_r)\exp(\nu + 3\kappa) = 4\pi x^2(3P + \varrho_r)\exp\gamma(x), \tag{4.1.134}$$

$$dz/dx = 4\pi H^2x^2(3P - \varrho_r)\exp(\nu + 3\kappa) = 4\pi x^2(3P - \varrho_r)\exp\gamma(x), \tag{4.1.135}$$

$$d\gamma/dx = d(\nu + 3\kappa)/dx = 2z/x^2, \tag{4.1.136}$$

$$dt/dx = -d\nu/dx = -y/x^2. \tag{4.1.137}$$

The equation of hydrostatic equilibrium (4.1.18) writes via Eqs. (4.1.132), (4.1.133) as

$$dP/dr = -(P + \varrho_r)\ d\nu/dr \quad \text{or} \quad dP/dx = (P + \varrho_r)\ dt/dx, \qquad (c, G = 1,\ \nu \to 2\nu). \tag{4.1.138}$$

Since $t(x_1) = 0$, the first boundary condition (4.1.130) is equivalent to the boundary value of Eq. (4.1.137), while the second condition (4.1.130) becomes

$$x_1\gamma(x_1) + 2z(x_1) - y(x_1) = 0, \qquad [\gamma(x_1) = 3\kappa(x_1)]. \tag{4.1.139}$$

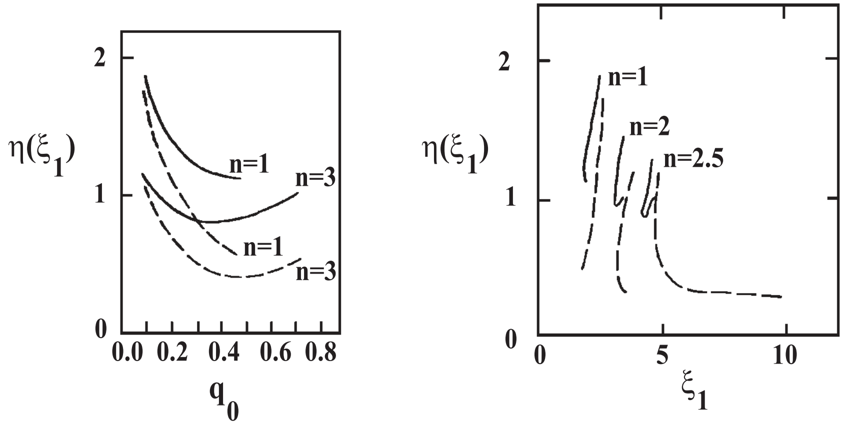

Fig. 4.1.7 Total dimensionless mass $\eta(\xi_1) = M_{r1}/4\pi\varrho_{r0}\alpha^3$, $[M_{r1} = M_r(r_{r1})]$ as a function of q_0 (on the left) and ξ_1 (on the right) according to bimetric gravitation (Eq. (4.1.150), solid lines) and general relativity (Eq. (4.1.49), dashed lines) for the polytropic indices indicated on the curves (Sarkisyan 1980).

Sarkisyan (1980) introduces dimensionless variables, similarly to Eq. (4.1.29):

$$\varrho_r = \varrho_{r0}\theta_r^n;\quad P = K\varrho_r^{1+1/n} = K\varrho_{r0}^{1+1/n}\theta_r^{n+1} = q_0\varrho_{r0}\theta_r^{n+1};\quad q_0 = P_0/\varrho_{r0} = K\varrho_{r0}^{1/n};\quad x = \alpha\zeta;$$
$$y(x) = 4\pi\varrho_{r0}\alpha^3 a(\zeta);\quad z(x) = 4\pi\varrho_{r0}\alpha^3 b(\zeta);\quad \alpha^2 = \pm(n+1)K/4\pi\varrho_{r0}^{1-1/n} = \pm(n+1)P_0/4\pi\varrho_{r0}^2,$$
$$(n \neq -1, \pm\infty;\ c, G = 1). \qquad (4.1.140)$$

The system (4.1.134)-(4.1.137) becomes in the new variables:

$$da/d\zeta = \zeta^2\theta_r^n(3q_0\theta_r + 1)\exp\gamma(\zeta), \qquad (4.1.141)$$

$$db/d\zeta = \zeta^2\theta_r^n(3q_0\theta_r - 1)\exp\gamma(\zeta), \qquad (4.1.142)$$

$$d\gamma/d\zeta = \pm 2q_0(n+1)\ b(\zeta)/\zeta^2, \qquad (4.1.143)$$

$$dt/d\zeta = \mp q_0(n+1)\ a(\zeta)/\zeta^2. \qquad (4.1.144)$$

The equation of hydrostatic support (4.1.138) transforms into

$$d\theta_r/d\zeta = \mp(q_0\theta_r + 1)\ a(\zeta)/\zeta^2, \qquad (d\nu/dr = Hy/r^2). \qquad (4.1.145)$$

Eq. (4.1.138) may be integrated between x and x_1 with the polytropic equation of state (4.1.1), to obtain

$$t(x) = (n+1)\ln(1 + q_0\theta_r) \quad \text{or} \quad t(0) = (n+1)\ln(1 + q_0), \qquad (t(x_1), \theta_r(x_1) = 0;\ \theta_r(0) = 1), \qquad (4.1.146)$$

fixing in this way the initial value of $t(0) = \nu(x_1) - \nu(0)$. The initial conditions on $a(\zeta)$ and $b(\zeta)$ are via Eqs. (4.1.132), (4.1.140) clearly $a(0), b(0) = 0$. The initial value of $\gamma(0)$ should be selected in such a way that the boundary condition (4.1.139)

$$\zeta_1\gamma(\zeta_1) \pm 2q_0(n+1)\ b(\zeta_1) \mp q_0(n+1)\ a(\zeta_1) = 0, \qquad (\zeta_1 = x_1/\alpha), \qquad (4.1.147)$$

Table 4.1.3 Boundary values of ξ_1, ξ_{r1} [Eq. (4.1.151)], and of $\eta(\xi_1) = M_{r1}/4\pi\varrho_{r0}\alpha^3$ [Eq. (4.1.150)] for selected values of n and q_0 in bimetric gravitation theory (Sarkisyan 1980).

n	q_0	ξ_1	ξ_{r1}	$\eta(\xi_1)$
1	0.2	2.018	2.820	1.446
	0.4	1.842	2.853	1.148
	0.2	2.463	3.400	1.249
1.5	0.4	2.311	3.507	1.022
	0.6	2.430	3.794	1.029
	0.2	3.116	4.213	1.110
2	0.4	2.991	4.438	0.933
	0.6	3.182	4.852	0.968
	0.2	4.246	5.369	1.006
2.5	0.4	4.164	5.876	0.868
	0.6	4.408	6.406	0.930
	0.1	6.314	7.417	1.193
3	0.2	6.365	7.973	0.940
	0.4	6.509	8.634	0.826
	0.6	6.640	9.083	0.903

should be verified. The dimensionless Lane-Emden coordinate ξ from Eq. (4.1.29) is connected to ζ by $r = Hx = \alpha H\zeta = \alpha\xi$, or

$$\xi = H\zeta. \tag{4.1.148}$$

If $\nu, \kappa \approx 0$, the metric ds^2 from Eq. (4.1.122) becomes Galilean, and if further $P \ll \varrho_r \approx \varrho$, we observe from Eqs. (4.1.127), (4.1.128) that $d(r^2\ d\nu/dr) = -d(r^2\ d\kappa/dr) \approx 4\pi\varrho r^2\ dr$, i.e. $r^2\ d\nu/dr$ becomes just equal to the Newtonian mass $M(r)$ of a sphere with radius r. Thus, in Rosen's (1974) bimetric gravitation theory we can identify

$$M_r(r) = r^2\ d\nu/dr = 4\pi \int_0^r (3P + \varrho_r) \exp(\nu + 3\kappa)\ r'^2\ dr', \tag{4.1.149}$$

with the relativistic mass inside coordinate radius r. In the dimensionless coordinates from Eqs. (4.1.132), (4.1.133), (4.1.140), (4.1.148) the previous equation becomes

$$\begin{aligned} M_r(r) &= r^2\ d\nu/dr = H\ y(x) = 4\pi\varrho_{r0}\alpha^3 H\ a(\zeta) = 4\pi H \int_0^x (3P + \varrho_r) \exp\gamma(x')\ x'^2\ dx' \\ &= 4\pi\varrho_{r0}\alpha^3 H \int_0^\zeta (3q_0\theta_r + 1) \exp\gamma(\zeta')\ \theta_r^n \zeta'^2\ d\zeta' = (4\pi\varrho_{r0}\alpha^3/H^2) \int_0^\xi (3q_0\theta_r + 1) \exp\gamma(\xi')\ \theta_r^n \xi'^2\ d\xi' \\ &= 4\pi\varrho_{r0}\alpha^3\eta(\xi); \quad \eta(\xi) = H^{-2} \int_0^\xi (3q_0\theta_r + 1) \exp\gamma(\xi')\ \theta_r^n \xi'^2\ d\xi', \end{aligned} \tag{4.1.150}$$

where we have used Eqs. (4.1.134), (4.1.141), and $x = \alpha\xi/H = \alpha\zeta$. The equivalent of Eq. (4.1.150) in general relativity is given by Eq. (4.1.39).

The radius of the configuration is similar to Eq. (4.1.47):

$$r_{r1} = \alpha\xi_{r1} = \int_0^{r_1} (\gamma_{rr})^{1/2}\ dr = \int_0^{r_1} \exp\kappa(r)\ dr = \alpha \int_0^{\xi_1} \exp\kappa(\xi)\ d\xi. \tag{4.1.151}$$

From Eq. (4.1.129) results $\nu, \kappa = \mathrm{const}/r$, $(r \geq r_1)$, and the line element (4.1.122) is well behaved for arbitrary small values of r_1, so we do not have in bimetric gravitation the possibility of black holes, as one finds in general relativity from the Schwarzschild metric (4.1.20) if $r_1 \leq r_g$.

The results of Sarkisyan (1980) indicate that θ_r drops in bimetric gravitation more rapidly than in general relativity: Bimetric configurations are more compact (Fig. 4.1.6). The dimensionless mass $\eta(\xi_1) = M_{r1}/4\pi\varrho_{r0}\alpha^3$ of bimetric polytropes exceeds the corresponding values from general relativity (Fig. 4.1.7, Tables 4.1.1, 4.1.3). The concordance between the general relativistic values from Table 4.1.1 and those quoted by Sarkisyan (1980) for Einstein's general relativity, often occurs merely in the first digit.

The bimetric solid curves in Fig. 4.1.7 for $n = 2, 2.5$ show that a single value of $\eta(\xi_1) = M_{r1}/4\pi\varrho_{r0}\alpha^3$ corresponds to different ξ_1-values, indicating that a large-volume configuration may be transformed into a more compact one.

4.1.5 Relativistic Polytropic Slabs and Cylinders

Burcev (1980) has considered a hydrostatic slab of polytropic index $n = 1$ with the metric

$$ds^2 = \exp\nu(z)\ dt^2 - \exp\kappa(z)\ (dx^2 + dy^2) - dz^2, \tag{4.1.152}$$

where the variable z is perpendicular to the (x, y)-symmetry plane and $P = K\varrho_r^2$. The external Taub metric found by Burcev (1980) possesses a singularity if $z = \pm 3D/2$, (D – thickness of the slab). The considered slab is unphysical without the existence of external masses.

Ruban (1986) has considered a metric similar to Eq. (4.1.152) for the hydrostatic homogeneous slab $\varepsilon_r = c^2\varrho_r = \mathrm{const}$, $(n = 0)$, as well as for a relativistic "isothermal" equation of state $P = \beta c^2\varrho_r$, $(0 \leq \beta \leq 1)$, [cf. Eq. (1.7.43)]. Again – unlike to the Newtonian case – the solutions are unphysical, including a "negative bare gravitational mass" in the singular plane $z = 0$: In general relativity there is no analog to the uniform Newtonian gravitational force (2.6.30) of a slab ($N = 1$).

Inside a relativistic hydrostatic cylinder Scheel et al. (1993) write the metric as

$$\begin{aligned}&ds^2 = \exp\nu(\ell)\ dt^2 - \exp[\kappa(\ell) - \mu(\ell)]\ d\ell^2 - \ell^2\exp[-\mu(\ell)]\ d\varphi^2 - \exp\mu(\ell)\ dz^2,\\ &(c, G = 1;\ \nu(0), \mu(0), \kappa(0) = 0).\end{aligned} \tag{4.1.153}$$

With the relativistic equations of state (4.1.83), (4.1.84), the field equations (4.1.4) for the static relativistic cylinder reduce to

$$\begin{aligned}&d\nu/d\ell = 4(S + 4\pi\ell^2 P\exp\nu)/\ell(1 - 8S);\quad d\mu/d\ell = [1 - (1 - 8S)^{-1/2}]/\ell;\quad dS/d\ell\\ &= 2\pi\ell(\varepsilon_r - P)\exp\nu;\quad dP/d\ell = -[(P + \varepsilon_r)/2]\ d\nu/d\ell,\qquad (S = [1 - \exp(\nu + \mu - \kappa)]/8),\end{aligned} \tag{4.1.154}$$

with the boundary conditions $S(0),\ \nu(0),\ \mu(0),\ \kappa(0),\ (dS/\ell)_{\ell=0},\ (d\nu/d\ell)_{\ell=0},\ (d\mu/d\ell)_{\ell=0},\ (d\kappa/d\ell)_{\ell=0} = 0$, and $P(\ell_1),\ \varepsilon_r(\ell_1) = 0$ at the surface coordinate $\ell = \ell_1$. Outside the cylindrical surface we have $P, \varepsilon_r = 0$, and the field equations (4.1.154) simplify after integration to

$$\begin{aligned}&\nu = \nu_1 + [4S_1/(1 - 8S_1)]\ln(\ell/\ell_1);\quad \mu = \mu_1 + [1 - (1 - 8S_1)^{-1/2}]\ln(\ell/\ell_1);\\ &S = S_1 = \mathrm{const};\quad P, \varepsilon_r = 0,\qquad (\ell \geq \ell_1;\ \nu_1, \mu_1 = \mathrm{const}).\end{aligned} \tag{4.1.155}$$

The remaining metric function reads as

$$\kappa = \nu + \mu - \ln(1 - 8S_1),\qquad (\ell \geq \ell_1), \tag{4.1.156}$$

and the external metric (4.1.153) becomes:

$$\begin{aligned}&ds^2 = \exp\nu_1\ (\ell/\ell_1)^{4S_1/(1-8S_1)}[dt^2 - d\ell^2/(1 - 8S_1)] - \ell^2\exp(-\mu_1)\ (\ell/\ell_1)^{-1+(1-8S_1)^{-1/2}}\ d\varphi^2\\ &-\exp\mu_1\ (\ell/\ell_1)^{1-(1-8S_1)^{-1/2}}\ dz^2,\qquad (\ell \geq \ell_1).\end{aligned} \tag{4.1.157}$$

In the nonrelativistic limit S_1 reduces to the Newtonian rest mass per unit length of the cylinder. The external metric (4.1.157) diverges at infinity $\ell \to \infty$ – an unrealistic feature, opposite to the spherical case. Scheel et al. (1993) exhibit the run of the metric coefficients for polytropic indices $1.1 \leq n \leq 50$.

4.2 Rotationally Distorted Relativistic Polytropes

4.2.1 Introduction

Strongly relativistic and rapidly rotating objects in hydrostatic equilibrium have been suggested to exist among others in pulsars, active galactic nuclei, and in remnants of supernova explosions (Komatsu et al. 1989a). The observed behaviour of pulsars indicates that the neutron stars, located in the centre of pulsars, are essentially uniformly rotating, relativistic objects of solar mass order, and with central densities $\varrho_{r0} \approx 5 \times 10^{15}$ g cm^{-3} (Butterworth and Ipser 1976). Relativistic and differentially rotating hydrostatic structures are expected to appear through the accretion-induced collapse of a massive white dwarf (Komatsu et al. 1989b). Self-gravitating relativistic accretion disks round compact objects (star-toroid systems) are relevant in the modelling of quasars and active galactic nuclei, or during evolution of close binary stars, when one of the components is destroyed by tidal forces (e.g. Nishida et al. 1992). Also, the collapse of rotating supermassive stars may finish prior to the onset of nuclear reactions, and there is evidence that rotation can stabilize neutron stars against collapse, even if their mass is larger than the critical one. Thus, it is not merely of theoretical interest to investigate rotating relativistic objects, and to determine how rapidly a relativistic star can rotate for a given strength of gravity (Eriguchi 1980).

Relativistic effects on rotating configurations can be considered basically in four different ways (cf. Papoyan et al. 1969): (i) Rotational and relativistic effects are taken into account as a perturbation to a nonrotating Newtonian configuration (Sec. 4.2.2, Krefetz 1967, Fahlman and Anand 1971a). (ii) Relativistic effects are taken into account as a first-order correction to Newtonian gravitation, but without limitation on angular velocity (Sec. 4.2.3, Chau 1969, Miketinac and Barton 1972). (iii) Small angular velocities, when rotation is regarded as a perturbation of a nonrotating, fully relativistic configuration (Secs. 4.2.4 and 4.2.5, Papoyan et al. 1969, Sarkisyan and Chubaryan 1977, Chubaryan et al. 1981). (iv) Numerical approaches, when practically no limitations exist on the magnitude of angular velocity and relativistic effects (Sec. 4.2.6, Butterworth and Ipser 1976, Butterworth 1976, Eriguchi 1980, Komatsu et al. 1989a, b, Nishida et al. 1992).

We limit ourselves to axially symmetric equilibrium configurations, which can be in equilibrium only if they are not radiating gravitational waves. A necessary and sufficient condition for the absence of gravitational radiation is the absence of time-dependent moments in the mass distribution (e.g. Landau and Lifschitz 1987), and this condition is guaranteed by the assumption of axially symmetric gravitational fields (Hartle and Sharp 1967, p. 318). Note, that in stationary fields (unlike to static fields), the mixed components $g_{0\alpha}$ of the metric tensor are definitely distinct from zero. We ignore the unrealistic special case of an axially symmetric field produced by a nonrotating body, which would require a very special stress field inside the mass distribution (Sedrakyan and Chubaryan 1968a).

Let us denote by Ω the angular velocity of a mass element, as measured by an observer at large distances from the configuration, i.e. as measured in asymptotically flat spacetime. First of all, we need a metric suitable for the study of stationary, axially symmetric, rotating relativistic configurations. To get some idea about the kind of metric involved, let us turn from a Galilean inertial system with metric

$$ds^2 = dt^2 - dr'^2 - r'^2(d\lambda'^2 + \sin^2\lambda'\ d\varphi'^2), \tag{4.2.1}$$

to a system rotating uniformly with angular velocity Ω. The spherical spatial coordinates are denoted by r', λ', φ', and t is the universal time coordinate in the considered stationary frame. Obviously, the spherical coordinates r, λ, φ in the rotating system are connected to r', λ', φ' by $r' = r,\ \lambda' = \lambda,\ \varphi' = \varphi + \Omega t$. Inserting this into Eq. (4.2.1), we find:

$$ds^2 = (1 - \Omega^2 r^2 \sin^2\lambda)\ dt^2 - 2\Omega r^2 \sin^2\lambda\ d\varphi\ dt - dr^2 - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2). \tag{4.2.2}$$

Since $g_{00} = 1 - \Omega^2 r^2 \sin^2\lambda > 0$, the rotating system can be used only up to a distance $r \leq 1/\Omega,\ (\sin^2\lambda \leq 1)$. In Cartesian coordinates the equivalent transformation from the inertial to the rotating frame is

$$x'^1 = x^1 \cos\Omega t - x^2 \sin\Omega t; \quad x'^2 = x^1 \sin\Omega t + x^2 \cos\Omega t; \quad x'^3 = x^3, \tag{4.2.3}$$

and the two metrics (4.2.1) and (4.2.2) become, respectively

$$ds^2 = dt^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2, \tag{4.2.4}$$

$$ds^2 = \{1 - \Omega^2[(x^1)^2 + (x^2)^2]\}\ dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + 2\Omega x^2\ dx^1\ dt - 2\Omega x^1\ dx^2\ dt. \tag{4.2.5}$$

Quite generally, a stationary metric and axial symmetry require that the metric coefficients are independent of $x^0 = t$ and $x^3 = \varphi$: $g_{jk} = g_{jk}(x^1, x^2)$. The assumption of axial symmetry also entails that the four-velocity u^j has only a temporal and φ-component: $u^j = (u^t, 0, 0, u^\varphi)$, $(u^0 = u^t,\ u^3 = u^\varphi)$. With the definition $u^j = dx^j/ds$, the angular velocity $\Omega = d\varphi/dt$ of a fluid element, as measured by an observer at infinity in an asymptotically flat spacetime, is

$$\Omega = d\varphi/dt = (d\varphi/ds)\big/(dt/ds) = u^\varphi/u^t = u^3/u^0. \tag{4.2.6}$$

Let us apply to the metric (4.1.3) the coordinate transformation $x^0 \to -x^0 = -t,\ x^3 \to -x^3 = -\varphi$ (Sedrakyan and Chubaryan 1968a). Obviously, this will not affect the sign of the angular velocity (4.2.6), and the tensor $g_{jk} = g_{jk}(x^1, x^2)$ will not undergo any change under this particular transformation. This requires that the gravitational field and its metric are invariant with respect to the simultaneous inversion of time $(t \to -t)$ and azimuth angle $(\varphi \to -\varphi)$, since each inversion alone just reverses the direction of rotation of the polytrope, i.e. it turns Ω into $-\Omega$. Therefore, all components g_{jk} connected with $dx^0 = dt$ or $dx^3 = d\varphi$ alone, must vanish: $g_{01}, g_{02}, g_{13}, g_{23} = 0$. The other nonvanishing components of the metric tensor have to remain unchanged under the coordinate transformation

$$x^1 = f_1(x'^1, x'^2); \quad x^2 = f_2(x'^1, x'^2), \tag{4.2.7}$$

i.e. a transformation of the coordinates x'^1, x'^2, x^1, x^2 among themselves. Eq. (4.2.7) yields two arbitrary conditions on the coordinates, and we choose for instance the new coordinates x^1, x^2 in such a way that $g_{12} = 0$ and $g_{11} = g_{22}$. So, the three components g_{11}, g_{12}, g_{22} can be expressed in terms of one independent function, say g_{11}. Combining all the above arguments, we see that the line element of a stationary, axially symmetric, rotating fluid depends on four distinct metric functions, and has the form (cf. Eqs. (4.2.2), (4.2.5), Hartle and Sharp 1967, Bardeen 1970)

$$ds^2 = g_{00}(x^1, x^2)\ dt^2 + g_{11}(x^1, x^2)[(dx^1)^2 + (dx^2)^2] + g_{33}(x^1, x^2)\ d\varphi^2 + 2g_{03}(x^1, x^2)\ dt\ d\varphi,$$
$$(g_{11} = g_{22}). \tag{4.2.8}$$

This Kerr type metric can also be written under the equivalent form

$$ds^2 = (g_{00} - g_{03}^2/g_{33})\ dt^2 + g_{11}[(dx^1)^2 + (dx^2)^2] + g_{33}(d\varphi + g_{03}\ dt/g_{33})^2, \tag{4.2.9}$$

which is more suitable for the calculation of the Riemann and Ricci tensors from Eqs. (4.2.19)-(4.2.24). Clearly, at spatial infinity the metrics (4.2.8), (4.2.9) have to become asymptotically Galilean: $g_{00} \to 1$; $g_{11}, g_{22}, g_{33} \to -1$; $g_{03} \to 0$. In the next section we briefly present method (i) for the study of rotating relativistic polytropes.

4.2.2 Perturbation Theory in the Post Newtonian Approximation

This subject seems to be illustrated best by particularization to the first order of Fahlman's and Anand's (1971a) second order theory. Let us write down at first the basic equations of the problem according to Chandrasekhar (1965a). The relativistic energy density from Eq. (4.1.2) can be expressed under the form

$$\varepsilon_r = \varepsilon + \varepsilon^{(int)} = \varrho c^2 + \varepsilon^{(int)} = \varrho c^2(1 + \varepsilon^{(int)}/\varrho c^2) = \varrho c^2(1 + U/c^2), \tag{4.2.10}$$

where $U = E_r - E = \varepsilon^{(int)}/\varrho$ now denotes the internal energy per unit rest mass $(m = 1)$, i.e. the specific internal energy, which is just the difference between specific relativistic energy E_r and specific rest energy E (excluding gravitational contributions). Eq. (4.1.84) is a particular case of Eq. (4.2.10).

In a first approximation to order $1/c^2$, the components of the metric tensor are (e.g. Landau and Lifschitz 1987, §106):

$$g_{00} = 1 - 2\Phi/c^2 + O(c^{-4}); \quad g_{\alpha\beta} = -(1 + 2\Phi/c^2)\,\delta_{\alpha\beta} + O(c^{-4}); \quad g_{0\alpha} = O(c^{-3}). \tag{4.2.11}$$

Φ is the Newtonian gravitational potential, satisfying Poisson's equation

$$\nabla^2\Phi = -4\pi G\varrho. \tag{4.2.12}$$

With the three-dimensional velocity $v^\alpha = dx^\alpha/d\tau$ measured by the interval of proper time $d\tau = g_{00}^{1/2}(dx^0 + g_{0\alpha}\,dx^\alpha/g_{00})/c$, we find for the contravariant four-velocity u^j in stationary fields (cf. Eqs. (5.12.94)-(5.12.98), Landau and Lifschitz 1987, §88):

$$\begin{aligned} &u^0 = [g_{00}(1 - v^2/c^2)]^{-1/2} - g_{0\alpha}v^\alpha/cg_{00}(1 - v^2/c^2)^{1/2} \approx 1 + v^2/2c^2 + \Phi/c^2; \\ &u^\alpha = v^\alpha/c(1 - v^2/c^2)^{1/2} \approx v^\alpha/c, \qquad (v^2 = v_\alpha v^\alpha). \end{aligned} \tag{4.2.13}$$

The covariant components write as

$$u_0 = g_{0\ell}u^\ell \approx 1 + v^2/2c^2 - \Phi/c^2; \quad u_\alpha = g_{\alpha\ell}u^\ell \approx -v_\alpha/c. \tag{4.2.14}$$

The covariant components of the stress-energy tensor (4.1.11) are

$$\begin{aligned} &T_{jk} = g_{j\ell}T^\ell_k = (P + \varepsilon_r)u_j u_k - Pg_{jk}; \quad T_{00} = \varrho c^2[1 + (v^2 - 2\Phi + U)/c^2] + O(c^{-2}); \\ &T_{0\alpha} = -\varrho c v_\alpha + O(c^{-1}); \quad T_{\alpha\beta} = \varrho v_\alpha v_\beta + \delta_{\alpha\beta}P + O(c^{-2}); \quad T = T^j_j = \varrho c^2 + \varrho U - 3P + O(c^{-2}). \end{aligned} \tag{4.2.15}$$

If we contract the basic field equations (4.1.4), we get $R = -8\pi GT/c^4$. Inserting this into Eq. (4.1.4), we obtain

$$R^k_j = (8\pi G/c^4)(T^k_j - \delta^k_j T/2), \tag{4.2.16}$$

or in covariant form

$$R_{jk} = (8\pi G/c^4)(T_{jk} - g_{jk}T/2). \tag{4.2.17}$$

The $(0,0)$ and $(0,\alpha)$-components of the right-hand side of the field equation (4.2.17) become with Eqs. (4.2.11) and (4.2.15):

$$T_{00} - g_{00}T/2 = \varrho(c^2/2 + v^2 - \Phi + U/2 + 3P/2\varrho) + O(c^{-2}); \quad T_{0\alpha} - g_{0\alpha}T/2 = -\varrho c v_\alpha + O(c^{-1}). \tag{4.2.18}$$

The mixed Riemann curvature tensor is given by (e.g. Landau and Lifschitz 1987)

$$R^i_{k\ell m} = \partial\Gamma^i_{km}/\partial x^\ell - \partial\Gamma^i_{k\ell}/\partial x^m + \Gamma^i_{\ell n}\Gamma^n_{km} - \Gamma^i_{mn}\Gamma^n_{k\ell}, \tag{4.2.19}$$

and the covariant curvature is

$$\begin{aligned} &R_{jk\ell m} = g_{ij}R^i_{k\ell m} = \partial(g_{ij}\Gamma^i_{km})/\partial x^\ell - \partial(g_{ij}\Gamma^i_{k\ell})/\partial x^m - \Gamma^i_{km}\,\partial g_{ij}/\partial x^\ell + \Gamma^i_{k\ell}\,\partial g_{ij}/\partial x^m \\ &+g_{ij}(\Gamma^i_{\ell n}\Gamma^n_{km} - \Gamma^i_{mn}\Gamma^n_{k\ell}) = \partial(g_{ij}\Gamma^i_{km})/\partial x^\ell - \partial(g_{ij}\Gamma^i_{k\ell})/\partial x^m + g_{np}(\Gamma^n_{k\ell}\Gamma^p_{jm} - \Gamma^n_{km}\Gamma^p_{j\ell}), \end{aligned} \tag{4.2.20}$$

where we have used the identity

$$\partial g_{jk}/\partial x^\ell = g_{kn}\Gamma^n_{j\ell} + g_{jn}\Gamma^n_{k\ell}, \tag{4.2.21}$$

resulting from Eq. (4.1.15) for the Christoffel symbols. Eq. (4.2.20) can be transformed further into

$$\begin{aligned} &R_{jk\ell m} = (1/2)\ (\partial^2 g_{jm}/\partial x^k\partial x^\ell + \partial^2 g_{k\ell}/\partial x^j\partial x^m - \partial^2 g_{j\ell}/\partial x^k\partial x^m - \partial^2 g_{km}/\partial x^j\partial x^\ell) \\ &+g_{np}(\Gamma^n_{k\ell}\Gamma^p_{jm} - \Gamma^n_{km}\Gamma^p_{j\ell}), \end{aligned} \tag{4.2.22}$$

by using the identity (e.g. Schmutzer 1968)

$$\partial(g_{ij}\Gamma^i_{km})/\partial x^\ell - \partial(g_{ij}\Gamma^i_{k\ell})/\partial x^m = (1/2)(\partial^2 g_{jm}/\partial x^k\partial x^\ell + \partial^2 g_{k\ell}/\partial x^j\partial x^m$$
$$-\partial^2 g_{j\ell}/\partial x^k\partial x^m - \partial^2 g_{km}/\partial x^j\partial x^\ell). \tag{4.2.23}$$

Contracting Eqs. (4.2.19) and (4.2.22), the Ricci tensor from Eq. (4.2.17) becomes

$$R_{km} = R^\ell_{k\ell m} = g^{j\ell}R_{jk\ell m} = \partial\Gamma^\ell_{km}/\partial x^\ell - \partial\Gamma^\ell_{k\ell}/\partial x^m + \Gamma^\ell_{\ell n}\Gamma^n_{km} - \Gamma^\ell_{mn}\Gamma^n_{k\ell}$$
$$= (g^{j\ell}/2)(\partial^2 g_{jm}/\partial x^k\partial x^\ell + \partial^2 g_{k\ell}/\partial x^j\partial x^m - \partial^2 g_{j\ell}/\partial x^k\partial x^m - \partial g_{km}/\partial x^j\partial x^\ell)$$
$$+g^{j\ell}g_{np}(\Gamma^n_{k\ell}\Gamma^p_{jm} - \Gamma^n_{km}\Gamma^p_{j\ell}). \tag{4.2.24}$$

We denote by h_{jk} the small correction tensors of the post Newtonian approximation to the Galilean metric $g^{(0)}_{00} = 1,\ g^{(0)}_{0\alpha} = 0,\ g^{(0)}_{\alpha\beta} = -\delta_{\alpha\beta}$. Eq. (4.2.11) can be written as

$$g_{00} = 1 + h_{00};\quad g_{0\alpha} = h_{0\alpha};\quad g_{\alpha\beta} = -\delta_{\alpha\beta} + h_{\alpha\beta}. \tag{4.2.25}$$

Chandrasekhar (1965a) calculates the Ricci tensor in extenso, and the components of Eq. (4.2.17) are

$$\nabla^2(h_{00}/2 - \Phi^2/c^4) - 16\pi G\varrho\Phi/c^4 = (8\pi G\varrho/c^4)(c^2/2 + v^2 - \Phi + U/2 + 3P/2\varrho), \tag{4.2.26}$$

$$\nabla^2 h_{0\alpha}/2 - (1/2c^2)\ \partial^2\Phi/\partial x^0\partial x^\alpha = -8\pi G\varrho v_\alpha/c^3, \tag{4.2.27}$$

where ∇ is the spatial nabla operator from Eq. (B.16). Using Poisson's equation (4.2.12), we can rewrite Eq. (4.2.26) under the slightly different form

$$\nabla^2(h_{00}/2 + \Phi/c^2 - \Phi^2/c^4) = (8\pi G\varrho/c^4)(v^2 + \Phi + U/2 + 3P/2\varrho). \tag{4.2.28}$$

If we define a new potential Π by means of

$$\nabla^2\Pi = -4\pi G\varrho(v^2 + \Phi + U/2 + 3P/2\varrho), \tag{4.2.29}$$

and insert into Eq. (4.2.28), this equation can be integrated at once:

$$h_{00} = -2\Phi/c^2 + 2\Phi^2/c^4 - 4\Pi/c^4 + O(c^{-6}). \tag{4.2.30}$$

Defining the further potentials

$$\nabla^2\chi = -2\Phi, \tag{4.2.31}$$

$$\nabla^2\Phi_{(\alpha)} = -4\pi G\varrho v_\alpha, \tag{4.2.32}$$

and inserting into Eq. (4.2.27), its solution is obviously

$$h_{0\alpha} = -(1/2c^2)\ \partial^2\chi/\partial x^0\partial x^\alpha + 4\Phi_{(\alpha)}/c^3, \tag{4.2.33}$$

where Chandrasekhar's (1965a) gauge condition

$$(1/2)\ \partial h^\alpha_\alpha/\partial x^0 - \partial h^\alpha_0/\partial x^\alpha = 0, \tag{4.2.34}$$

has to be verified. The spatial components of h_{jk} are via Eq. (4.2.11) equal to

$$h_{\alpha\beta} = -2\Phi\ \delta_{\alpha\beta}/c^2 + O(c^{-4}). \tag{4.2.35}$$

The identity (4.1.14) can be written under the contravariant form

$$\nabla_k T^{jk} = g^{j\ell}\nabla_k T^k_\ell = 0. \tag{4.2.36}$$

The time component $\nabla_k T^{0k} = 0$ of this equation replaces the Newtonian equation of continuity $\partial\varrho/\partial t + \nabla\cdot(\varrho\vec{v}) = 0$, and the space components $\nabla_k T^{\alpha k} = 0$ replace the Eulerian equation of Newtonian hydrodynamics $\partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} + (1/\varrho)\ \nabla P - \nabla\Phi = 0$. The explicit calculation of Eq. (4.2.36) with the aid of Eqs. (4.1.14), (4.1.15), (4.2.25), (4.2.30), (4.2.33), (4.2.35) is beyond the scope of this book, so we quote merely the post Newtonian hydrodynamic equation of stationary motion [Chandrasekhar 1965a, Eq. (68); Krefetz 1966, Eq. (4)]:

$$\varrho[1 + (v^2 + 2\Phi + U + P/\varrho)/c^2]v^\beta\ \partial v_\alpha/\partial x^\beta + \partial[(1 + 2\Phi/c^2)P]/\partial x^\alpha - \varrho\ \partial\Phi/\partial x^\alpha$$
$$+(4\varrho v^\beta/c^2)\ \partial\Phi_{(\beta)}/\partial x^\alpha + (4\varrho v^\beta/c^2)\ \partial[v_\alpha\Phi - \Phi_{(\alpha)}]/\partial x^\beta$$
$$-(2\varrho/c^2)\big[(v^2 + \Phi + U/2 + 3P/2\varrho)\ \partial\Phi/\partial x^\alpha + \partial\Pi/\partial x^\alpha\big] = 0, \qquad (v_\alpha \approx v^\alpha;\ \alpha,\beta = 1,2,3). \quad (4.2.37)$$

The contravariant velocity components in the inertial frame are for uniform rotation and axial symmetry about the x^3-axis equal to

$$\vec{v} = \vec{v}(v^1, v^2, v^3) = \vec{v}(-\Omega x^2, \Omega x^1, 0), \qquad (\Omega = \mathrm{const}). \quad (4.2.38)$$

The covariant velocity components are obtained with the tensor $\gamma_{\alpha\beta}$ of the *spatial* metric (Eqs. (5.12.94)-(5.12.96), Landau and Lifschitz 1987, §§84, 88):

$$v_\alpha = \gamma_{\alpha\beta}v^\beta = (-g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00})v^\beta \approx v^\alpha, \quad (4.2.39)$$

because to our order of approximation Eq. (4.2.11) yields $g_{\alpha\beta} = -\delta_{\alpha\beta},\ g_{0\alpha} = 0$. In virtue of Eq. (4.2.38) the velocity is obtained to this order of approximation by

$$v = (v_\alpha v^\alpha)^{1/2} = \Omega[(x^1)^2 + (x^2)^2]^{1/2} = \Omega\ell, \qquad [\ell^2 = (x^1)^2 + (x^2)^2]. \quad (4.2.40)$$

We introduce a new potential F by the substitution

$$\Phi_{(\alpha)} = v_\alpha F. \quad (4.2.41)$$

Now, $\Phi_{(\alpha)}$ satisfies Eq. (4.2.32), so F will satisfy for the *particular form* of v^α from Eq. (4.2.38) the equations $\nabla^2F + (2/x^1)\ \partial F/\partial x^1 = \nabla^2F + (2/x^2)\ \partial F/\partial x^2 = -4\pi G\varrho$, which assume in cylindrical (ℓ,φ,z)-coordinates the form

$$\nabla^2F + (2/\ell)\ \partial F/\partial\ell = -4\pi G\varrho, \qquad (\partial F/\partial x^1 = \partial F[\ell(x^1,x^2),x^3]/\partial x^1 = (\partial F/\partial\ell)\ \partial\ell/\partial x^1$$
$$= (x^1/\ell)\ \partial F/\partial\ell;\ \partial F/\partial x^2 = (\partial F/\partial\ell)\ \partial\ell/\partial x^2 = (x^2/\ell)\ \partial F/\partial\ell). \quad (4.2.42)$$

Taking into account Eqs. (4.2.38) and (4.2.40), we can transform the penultimate term of Eq. (4.2.37) as

$$v^\beta\ \partial[v_\alpha\Phi - \Phi_{(\alpha)}]/\partial x^\beta = -(1/2)(\Phi - F)\ \partial v^2/\partial x^\alpha + v_\alpha v^\beta(\partial\Phi/\partial x^\beta - \partial F/\partial x^\beta)$$
$$= -(\Phi/2)\ \partial v^2/\partial x^\alpha + \Phi_{(\beta)}\ \partial v^\beta/\partial x^\alpha, \quad (4.2.43)$$

because $(F/2)\ \partial v^2/\partial x^\alpha = F\Omega^2x^\alpha = \Phi_{(\beta)}\ \partial v^\beta/\partial x^\alpha,\ v^\beta\ \partial\Phi/\partial x^\beta = \vec{v}\cdot\nabla\Phi = 0$, and $v^\beta\ \partial F/\partial x^\beta = \vec{v}\cdot\nabla F = 0$, as can be easily verified by turning to cylindrical coordinates, and observing that $v_\ell, v_z = 0,\ \Phi = \Phi(\ell,z),\ F = F(\ell,z)$. Since $v^\beta\ \partial v_\alpha/\partial x^\beta = -\partial(\Omega^2\ell^2/2)/\partial x^\alpha$ and $\Phi_{(\beta)}\ \partial v^\beta/\partial x^\alpha + v^\beta\ \partial\Phi_{(\beta)}/\partial x^\alpha = \partial(v^\beta\Phi_{(\beta)})/\partial x^\alpha = \partial(\Omega^2\ell^2F)/\partial x^\alpha$, Eq. (4.2.37) becomes

$$-\varrho[1 + (\Omega^2\ell^2 + 2\Phi + U + P/\varrho)/c^2]\ \partial(\Omega^2\ell^2/2)/\partial x^\alpha + \partial[(1 + 2\Phi/c^2)P]/\partial x^\alpha$$
$$-\varrho\ \partial\Phi/\partial x^\alpha + (4\varrho\Omega^2/c^2)\ \partial(\ell^2F)/\partial x^\alpha - (2\varrho\Omega^2\Phi/c^2)\ \partial\ell^2/\partial x^\alpha$$
$$-(2\varrho/c^2)\big[(\Omega^2\ell^2 + \Phi + U/2 + 3P/2\varrho)\ \partial\Phi/\partial x^\alpha + \partial\Pi/\partial x^\alpha\big] = 0. \quad (4.2.44)$$

This equation can be transcribed in the slightly different form (Krefetz 1966)

$$-\varrho\ \partial(\Omega^2\ell^2/2)/\partial x^\alpha + \partial P/\partial x^\alpha - \varrho\ \partial\Phi/\partial x^\alpha + (1/c^2)\big\{-\varrho\ \partial(\Omega^2\ell^2/2)^2/\partial x^\alpha$$
$$-\varrho(2\Phi + U + P/\varrho)\big[\partial(\Omega^2\ell^2/2)/\partial x^\alpha + \partial\Phi/\partial x^\alpha\big] - 2\varrho\Omega^2\Phi\ \partial\ell^2/\partial x^\alpha - 2\varrho\Omega^2\ell^2\ \partial\Phi/\partial x^\alpha$$
$$-2P\ \partial\Phi/\partial x^\alpha + 2\ \partial(\Phi P)/\partial x^\alpha + 4\varrho\Omega^2\ \partial(\ell^2F)/\partial x^\alpha - 2\varrho\ \partial\Pi/\partial x^\alpha\big\} = 0, \quad (4.2.45)$$

or

$$[1 - (1/c^2)(U + P/\varrho)]\ \nabla P = \varrho\ \nabla\Phi_{tot}, \quad (4.2.46)$$

where the total effective potential is equal to

$$\Phi_{tot} = \Phi + \Omega^2\ell^2/2 + (1/c^2)\big[(\Omega^2\ell^2/2)^2 + 2\Omega^2\ell^2\Phi - 4\Omega^2\ell^2 F + 2\Pi\big], \tag{4.2.47}$$

and we have inserted the Newtonian equation of hydrodynamics (3.1.23) for the post Newtonian term in the bracket of Eq. (4.2.45):

$$\partial(\Omega^2\ell^2/2)/\partial x^\alpha + \partial\Phi/\partial x^\alpha = (1/\varrho)\ \partial P/\partial x^\alpha. \tag{4.2.48}$$

We have already shown in Eqs. (4.1.58), (4.1.84) that under simple isentropic conditions the specific internal energy U from Eq. (4.2.10) is

$$U = \varepsilon^{(int)}/\varrho = P/(\gamma-1)\varrho = nP/\varrho, \qquad [n = 1/(\gamma-1)]. \tag{4.2.49}$$

We substitute this equation into Eq. (4.2.46), and take its divergence in spherical coordinates:

$$\begin{aligned}&(1/r^2)\ \partial[(r^2/\varrho)\ \partial P/\partial r]/\partial r + (1/r^2)\ \partial\big\{[(1-\mu^2)/\varrho]\ \partial P/\partial\mu\big\}/\partial\mu\\ &-[(n+1)/c^2r^2]\big\{\partial[(r^2P/\varrho^2)\ \partial P/\partial r]/\partial r + \partial\big[\big(P(1-\mu^2)/\varrho^2\big)\ \partial P/\partial\mu\big]/\partial\mu\big\}\\ &= -4\pi G\varrho + 2\Omega^2 + (1/c^2)\big\{4\Omega^4r^2(1-\mu^2) + 8\Omega^2(1-\mu^2)\big[r\ \partial(\Phi-F)/\partial r - \mu\ \partial(\Phi-F)/\partial\mu\big]\\ &+8\Omega^2(\Phi-2F) - 8\pi G\varrho[\Phi + (n+3)P/2\varrho]\big\},\end{aligned} \tag{4.2.50}$$

where we have used Eqs. (4.2.12), (4.2.29). For the evaluation of the Laplacian we consider Eq. (B.39) under the form

$$\nabla^2 F(r,\mu) = \partial^2F/\partial r^2 + (2/r)\ \partial F/\partial r + [(1-\mu^2)/r^2]\ \partial^2F/\partial\mu^2 - (2\mu/r^2)\ \partial F/\partial\mu. \tag{4.2.51}$$

Eq. (4.2.42) is transcribed in spherical coordinates as

$$\partial^2F/\partial r^2 + (4/r)\ \partial F/\partial r + [(1-\mu^2)/r^2]\ \partial^2F/\partial\mu^2 - (4\mu/r^2)\ \partial F/\partial\mu = -4\pi G\varrho, \tag{4.2.52}$$

where $\nabla^2 F$ is calculated according to Eq. (4.2.51), and

$$\partial F[r(\ell,z),\ \mu(\ell,z)]/\partial\ell = (\partial F/\partial r)\ \partial r/\partial\ell + (\partial F/\partial\mu)\ \partial\mu/\partial\ell = (\ell/r)\ \partial F/\partial r - (\mu\ell/r^2)\ \partial F/\partial\mu. \tag{4.2.53}$$

The basic equations of the problem are given by Eqs. (4.2.12), (4.2.29), (4.2.46), (4.2.47), (4.2.50), (4.2.52). Unlike to Eq. (4.1.1), we now assume that the *rest mass density* ϱ obeys the isentropic polytropic relationship (4.1.83), (Fahlman and Anand 1971a):

$$P = K\varrho^{1+1/n}, \qquad (\gamma = 1+1/n). \tag{4.2.54}$$

The dimensionless polytropic variables $\xi,\ \Theta(\xi,\mu),\ \beta = \Omega^2/2\pi G\varrho_0$ are introduced via Eqs. (3.2.1), (3.2.3). With these variables the equation (4.2.46) can be integrated at once in the Newtonian limit [cf. Eqs. (3.1.74), (3.5.10), (3.8.38)]:

$$\Phi - \Phi_p = (n+1)K\varrho_0^{1/n}\Theta - \Omega^2\alpha^2\xi^2(1-\mu^2)/2, \qquad (c\to\infty). \tag{4.2.55}$$

Φ_p is the Newtonian potential at the poles of the surface, where $\mu = \pm 1,\ \Theta = 0$. The relativity parameter from Eq. (4.1.31) is now defined by

$$q_0 = P_0/\varrho_0c^2 = K\varrho_0^{1/n}/c^2. \tag{4.2.56}$$

We rewrite Eq. (4.2.50) in dimensionless form, and eliminate Φ in the post Newtonian terms with the aid of Eq. (4.2.55):

$$\begin{aligned}&(1/\xi^2)\ \partial(\xi^2\ \partial\Theta/\partial\xi)/\partial\xi + (1/\xi^2)\ \partial[(1-\mu^2)\ \partial\Theta/\partial\mu]/\partial\mu\\ &= -\Theta^n + \beta + q_0\big\{[(n+1)/2]\big[(1/\xi^2)\ \partial(\xi^2\ \partial\Theta^2/\partial\xi)/\partial\xi + (1/\xi^2)\ \partial\big((1-\mu^2)\ \partial\Theta^2/\partial\mu\big)/\partial\mu\big]\\ &-(3n+5)\Theta^{n+1} - 2(n+1)\varphi_p\Theta^n\big\} + q_0\beta(n+1)\big[4(1-\mu^2)(\xi\ \partial\Theta/\partial\xi - \mu\ \partial\Theta/\partial\mu\\ &-\xi\ \partial f/\partial\xi + \mu\ \partial f/\partial\mu) + 4\Theta - 8f + 4\varphi_p + \xi^2(1-\mu^2)\Theta^n/2\big] - 2q_0\beta^2(n+1)(1-\mu^2)\xi^2.\end{aligned} \tag{4.2.57}$$

The dimensionless potentials φ and f are connected to Φ and F by

$$\Phi = (n+1)K\varrho_0^{1/n}\varphi = (n+1)P_0\varphi/\varrho_0; \quad F = (n+1)K\varrho_0^{1/n}f = (n+1)P_0 f/\varrho_0. \tag{4.2.58}$$

In polytropic units Eqs. (4.2.52) and (4.2.29) become, respectively

$$\partial^2 f/\partial\xi^2 + (4/\xi)\ \partial f/\partial\xi + [(1-\mu^2)/\xi^2]\ \partial^2 f/\partial\mu^2 - (4\mu/\xi^2)\ \partial f/\partial\mu = -\Theta^n, \tag{4.2.59}$$

$$\partial^2\pi/\partial\xi^2 + (2/\xi)\ \partial\pi/\partial\xi + [(1-\mu^2)/\xi^2]\ \partial^2\pi/\partial\mu^2 - (2\mu/\xi^2)\ \partial\pi/\partial\mu$$
$$= -\Theta^n[\beta\xi^2(1-\mu^2)/4 + \varphi_p + (3n+5)\Theta/2(n+1)], \tag{4.2.60}$$

where

$$\Pi = [(n+1)K\varrho_0^{1/n}]^2\pi(\xi,\mu) = [(n+1)P_0/\varrho_0]^2\pi(\xi,\mu). \tag{4.2.61}$$

We now seek a solution of Eq. (4.2.57) under the form

$$\Theta(\xi,\mu) = \theta(\xi) + \beta\Psi(\xi,\mu) + q_0\Sigma(\xi,\mu), \qquad (\beta, q_0 \ll 1). \tag{4.2.62}$$

$\theta(\xi)$ is the Lane-Emden function of a spherical polytrope. Up to the first order we have

$$\Theta^n(\xi,\mu) \approx \theta^n(\xi) + \beta n\theta^{n-1}(\xi)\ \Psi(\xi,\mu) + q_0 n\theta^{n-1}(\xi)\ \Sigma(\xi,\mu). \tag{4.2.63}$$

Our comments subsequent to Eq. (3.2.7) remain entirely valid, i.e. near the boundary, where $\theta \approx 0$ and $\Theta \approx \beta\Psi + q_0\Sigma \approx 0$, no break-down of the perturbation method occurs, since all involved quantities remain small. Θ is expanded further in terms of Legendre polynomials:

$$\Theta(\xi,\mu) = \theta(\xi) + \beta\psi_0(\xi) + q_0\sigma_0(\xi) + \sum_{j=1}^{\infty}\left[\beta A_j\psi_j(\xi) + q_0 B_j\sigma_j(\xi)\right]\ P_j(\mu). \tag{4.2.64}$$

The unknown constants A_j, B_j are introduced into these expansions, because it will be obvious from Eqs. (4.2.68), (4.2.70) that ψ_j and σ_j, $(j = 1,2,3,...)$ are solutions of homogeneous equations, so $A_j\psi_j,\ B_j\sigma_j$ will also be solutions of these equations. The unknown constants A_j, B_j are determined as in Sec. 3.2 from the continuity of the gravitational potential and of its radial derivative across the boundary of the polytrope.

We insert Eq. (4.2.64) into the basic equation (4.2.57), neglecting all terms of order higher than the first in β and q_0 :

$$(1/\xi^2)\ d(\xi^2\ d\theta/d\xi)/d\xi + (\beta/\xi^2)\ d(\xi^2\ d\psi_0/d\xi)/d\xi + (q_0/\xi^2)\ d(\xi^2\ d\sigma_0/d\xi)/d\xi$$
$$+\sum_{j=1}^{\infty}\left[(\beta A_j/\xi^2)\ d(\xi^2\ d\psi_j/d\xi)/d\xi + (q_0 B_j/\xi^2)\ d(\xi^2\ d\sigma_j/d\xi)/d\xi\right]\ P_j(\mu)$$
$$-\sum_{j=1}^{\infty}[j(j+1)/\xi^2](\beta A_j\psi_j + q_0 B_j\sigma_j)\ P_j(\mu) = -\theta^n - n\theta^{n-1}(\beta\psi_0 + q_0\sigma_0)$$
$$-n\theta^{n-1}\sum_{j=1}^{\infty}(\beta A_j\psi_j + q_0 B_j\sigma_j)\ P_j(\mu) + \beta + q_0\big\{[(n+1)/2\xi^2]\ d(\xi^2\ d\theta^2/d\xi)/d\xi$$
$$-(3n+5)\theta^{n+1} - 2(n+1)\varphi_p\theta^n\big\}, \tag{4.2.65}$$

where we have used Eq. (3.1.40). Equating coefficients with the same $\beta, q_0, P_j(\mu)$, we obtain the basic set of equations:

$$(1/\xi^2)\ d(\xi^2\ d\theta/d\xi)/d\xi = -\theta^n, \tag{4.2.66}$$

$$(1/\xi^2)\ d(\xi^2\ d\psi_0/d\xi)/d\xi = -n\theta^{n-1}\psi_0 + 1, \tag{4.2.67}$$

$$(1/\xi^2)\ d(\xi^2\ d\psi_j/d\xi)/d\xi - j(j+1)\psi_j/\xi^2 = -n\theta^{n-1}\psi_j, \qquad (j = 1,2,3,...), \tag{4.2.68}$$

$$(1/\xi^2)\ d(\xi^2\ d\sigma_0/d\xi)/d\xi = -n\theta^{n-1}\sigma_0 + [(n+1)/2\xi^2]\ d(\xi^2\ d\theta^2/d\xi)/d\xi$$
$$-(3n+5)\theta^{n+1} - 2(n+1)\varphi_p\theta^n, \qquad (4.2.69)$$

$$(1/\xi^2)\ d(\xi^2\ d\sigma_j/d\xi)/d\xi - j(j+1)\sigma_j/\xi^2 = -n\theta^{n-1}\sigma_j, \qquad (j = 1, 2, 3, ...). \qquad (4.2.70)$$

The initial conditions for these five equations are obviously the same as in the nonrelativistic rotating case [cf. Eq. (3.2.6)]:

$$\Theta(0,\mu), \theta(0) = 1; \quad (\partial\Theta/\partial\xi)_{\xi=0}, (d\theta/d\xi)_{\xi=0} = 0; \quad \psi_j(0), \sigma_j(0) = 0;$$
$$(d\psi_j/d\xi)_{\xi=0}, (d\sigma_j/d\xi)_{\xi=0} = 0. \qquad (4.2.71)$$

With Eqs. (3.2.1), (4.2.49) the equation of hydrostatic equilibrium (4.2.46) turns into

$$[1 - (n+1)K\varrho_0^{1/n}\Theta/c^2]K(n+1)\varrho_0^{1/n}\ \nabla\Theta = \nabla\Phi_{tot}, \qquad (4.2.72)$$

which integrates at once (cf. Eq. (4.2.55) if $c \to \infty$):

$$\varphi = \varphi_p + \Theta - \beta\xi^2(1-\mu^2)/4 - q_0(n+1)(\Theta^2/2 + 2\pi - 2\pi_p)$$
$$+q_0\beta(n+1)\xi^2(1-\mu^2)(-\varphi + 2f) - q_0\beta^2(n+1)\xi^4(1-\mu^2)^2/16. \qquad (4.2.73)$$

π_p is the value of $\pi(\xi,\mu)$ at the surface poles of the polytrope. We observe that the dimensionless potential $\pi(\xi,\mu)$ appears only in connection with the first order relativity parameter q_0, so we need only the zero order term π_0 of its expansion

$$\pi(\xi,\mu) = \pi_0(\xi) + \sum_{j=0}^{\infty} \left[\beta a_j(\xi) + q_0 b_j(\xi)\right]\ P_j(\mu). \qquad (4.2.74)$$

Inserting into Eq. (4.2.60), and equating coefficients of equal $\beta, q_0, P_j(\mu)$, we get for the zero order function π_0 :

$$d\pi_0/d\xi^2 + (2/\xi)\ d\pi_0/d\xi = -\theta^n(\xi)\ [\varphi_{p0} + (3n+5)\ \theta(\xi)/2(n+1)], \qquad (4.2.75)$$

where we have split the surface value of the gravitational potential at the poles into its zeroth and first order parts:

$$\varphi_p = \varphi_{p0} + \beta\varphi_{p\beta} + q_0\varphi_{pq}. \qquad (4.2.76)$$

Henceforth we distinguish between internal and external gravitational potential, and expand the internal potential $\varphi = \varphi_i$ as [cf. Eq. (3.2.13)]

$$\varphi_i = U_0(\xi) + \sum_{j=0}^{\infty} \left[\beta V_j(\xi) + q_0 W_j(\xi)\right]\ P_j(\mu). \qquad (4.2.77)$$

We insert into Eq. (4.2.73), confining to first order perturbation terms $(\mu^2 = [2P_2(\mu)+1]/3)$:

$$\varphi = \varphi_i = \varphi_{p0} + \beta\varphi_{p\beta} + q_0\varphi_{pq} + \theta(\xi) + \beta\psi_0(\xi) + q_0\sigma_0(\xi) + \sum_{j=1}^{\infty} \left[\beta A_j\psi_j(\xi) + q_0 B_j\sigma_j(\xi)\right]\ P_j(\mu)$$
$$-\beta\xi^2[1 - P_2(\mu)]/6 - q_0(n+1)\left[\theta^2(\xi)/2 + 2\pi_0(\xi) - 2\pi_0(\xi_1)\right]. \qquad (4.2.78)$$

To zeroth order we have $\pi_p = \pi_0(\xi_1)$, where ξ_1 is the radius of the Lane-Emden sphere. We equate terms with equal $\beta, q_0, P_j(\mu)$ between Eqs. (4.2.77) and (4.2.78):

$$U_0(\xi) = \varphi_{p0} + \theta(\xi); \quad V_0(\xi) = \varphi_{p\beta} + \psi_0(\xi) - \xi^2/6; \quad V_2(\xi) = A_2\psi_2(\xi) + \xi^2/6;$$
$$V_j(\xi) = A_j\psi_j(\xi),\ (j \neq 0,2); \quad W_0(\xi) = \varphi_{pq} + \sigma_0(\xi) - (n+1)[\theta^2(\xi)/2 + 2\pi_0(\xi) - 2\pi_0(\xi_1)];$$
$$W_j(\xi) = B_j\sigma_j(\xi), \qquad (j = 1, 2, 3, ...). \qquad (4.2.79)$$

Regrouping the terms, the internal potential (4.2.78) becomes

$$\varphi_i = \varphi_{p0} + \theta(\xi) + \beta\Big\{\varphi_{p\beta} + \psi_0(\xi) - \xi^2[1 - P_2(\mu)]/6 + \sum_{j=1}^{\infty} A_j\psi_j(\xi)\ P_j(\mu)\Big\}$$

$$+q_0\Big\{\varphi_{pq} + \sigma_0(\xi) - (n+1)[\theta^2(\xi)/2 + 2\pi_0(\xi) - 2\pi_0(\xi_1)] + \sum_{j=1}^{\infty} B_j\sigma_j(\xi)\ P_j(\mu)\Big\}. \qquad (4.2.80)$$

The external potential $\varphi_e(\xi,\mu)$ satisfies the Laplace equation $\nabla^2\varphi_e = 0$, and its first order expansion is [cf. Eqs. (3.1.58), (3.2.33)]:

$$\varphi_e = k_0/\xi + \sum_{j=0}^{\infty}[\xi^{-j-1}(\beta k_{1j} + q_0\ell_{1j})]\ P_j(\mu), \qquad (k_0, k_{1j}, \ell_{1j} = \text{const}). \qquad (4.2.81)$$

The continuity conditions for the gravitational potential on the boundary

$$\Xi_1(\mu) = \xi_1 + \sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu), \qquad (s_j, t_j = \text{const}), \qquad (4.2.82)$$

are

$$\varphi_i(\Xi_1) = \varphi_e(\Xi_1); \quad (\partial\varphi_i/\partial\xi)_{\xi=\Xi_1} = (\partial\varphi_e/\partial\xi)_{\xi=\Xi_1}. \qquad (4.2.83)$$

On the boundary, the functions from Eq. (4.2.64) are approximated by

$$\theta(\Xi_1) \approx \theta(\xi_1) + (\Xi_1 - \xi_1)\ \theta'(\xi_1) = \theta(\xi_1) + \theta'(\xi_1)\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu);$$

$$\theta'(\Xi_1) \approx \theta'(\xi_1) + \theta''(\xi_1)\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu); \quad \psi_k(\Xi_1) \approx \psi_k(\xi_1) + \psi_k'(\xi_1)\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu);$$

$$\sigma_k(\Xi_1) \approx \sigma_k(\xi_1) + \sigma_k'(\xi_1)\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu), \qquad (k = 0, 1, 2, ...). \qquad (4.2.84)$$

Up to the first order, the surface value of $\Theta(\xi,\mu)$ becomes in virtue of Eq. (4.2.84) equal to

$$\Theta(\Xi_1,\mu) = \theta(\Xi_1) + \beta\psi_0(\Xi_1) + q_0\sigma_0(\Xi_1) + \sum_{j=1}^{\infty}\big[\beta A_j\psi_j(\Xi_1) + q_0 B_j\sigma_j(\Xi_1)\big]\ P_j(\mu) \approx \theta(\xi_1)$$

$$+\theta'(\xi_1)\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu) + \beta\psi_0(\xi_1) + q_0\sigma_0(\xi_1) + \sum_{j=1}^{\infty}[\beta A_j\psi_j(\xi_1) + q_0 B_j\sigma_j(\xi_1)]\ P_j(\mu) = 0. \qquad (4.2.85)$$

The coefficients of βP_j and $q_0 P_j$ must be zero, so the figure constants are equal to [cf. Eq. (3.2.38)]

$$s_0 = -\psi_0(\xi_1)/\theta'(\xi_1); \quad s_j = -A_j\psi_j(\xi_1)/\theta'(\xi_1);$$
$$t_0 = -\sigma_0(\xi_1)/\theta'(\xi_1); \quad t_j = -B_j\sigma_j(\xi_1)/\theta'(\xi_1), \qquad (j = 1, 2, 3, ...). \qquad (4.2.86)$$

The boundary conditions (4.2.83) read via Eqs. (4.2.80), (4.2.81), (4.2.84) as

$$\varphi_i(\Xi_1,\mu) = \varphi_{p0} + \beta[\varphi_{p\beta} - \xi_1^2/6 + \xi_1^2 P_2(\mu)/6] + q_0\varphi_{pq}, \qquad (4.2.87)$$

$$\varphi_e(\Xi_1,\mu) = k_0/\xi_1 + \sum_{j=0}^{\infty}[(-k_0/\xi_1^2)(\beta s_j + q_0 t_j) + \xi_1^{-j-1}(\beta k_{1j} + q_0\ell_{1j})]\ P_j(\mu), \qquad (4.2.88)$$

$$(\partial\varphi_i/\partial\xi)_{\xi=\Xi_1} = \theta'(\xi) - [2\theta'(\xi_1)/\xi_1]\sum_{j=0}^{\infty}(\beta s_j + q_0 t_j)\ P_j(\mu) + \beta\Big\{\psi_0'(\xi_1) - \xi_1[1-P_2(\mu)]/3$$
$$+\sum_{j=1}^{\infty} A_j\psi_j'(\xi_1)\ P_j(\mu)\Big\} + q_0\Big[\sigma_0'(\xi_1) - 2(n+1)\pi_0'(\xi_1) + \sum_{j=1}^{\infty} B_j\sigma_j'(\xi_1)\ P_j(\mu)\Big], \qquad (4.2.89)$$

$$(\partial\varphi_e/\partial\xi)_{\xi=\Xi_1} = -k_0/\xi_1^2 + \sum_{j=0}^{\infty}\big[(2k_0/\xi_1^3)(\beta s_j + q_0 t_j) - (j+1)\xi_1^{-j-2}(\beta k_{1j} + q_0\ell_{1j})\big]\ P_j(\mu), \quad (4.2.90)$$

by inserting the Lane-Emden equation (2.3.87): $\theta''(\xi_1) = -2\theta'(\xi_1)/\xi_1,\ (\theta^n(\xi_1) = 0,\ 0 < n < 5)$. Equating coefficients with the same β, q_0, P_j, we find by virtue of Eq. (4.2.83):

$$\varphi_{p0} = -\xi_1\theta'(\xi_1);\quad \varphi_{p\beta} = \xi_1^2/2 - \psi_0(\xi_1) - \xi_1\psi_0'(\xi_1);\quad \varphi_{pq} = -\sigma_0(\xi_1) - \xi_1\sigma_0'(\xi_1)$$
$$+2(n+1)\xi_1\pi_0'(\xi_1);\quad k_0 = -\xi_1^2\theta'(\xi_1);\quad k_{10} = \xi_1^2[\xi_1/3 - \psi_0'(\xi_1)];\quad \ell_{10} = \xi_1^2[2(n+1)\pi_0'(\xi_1) - \sigma_0'];$$
$$k_{12} = \xi_1^5[\xi_1\psi_2'(\xi_1) - 2\psi_2(\xi_1)]/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)];\quad A_2 = -5\xi_1^2/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]. \quad (4.2.91)$$

The remaining constants $A_j, B_j, k_{1j}, \ell_{1j}$ are solutions of the homogeneous systems

$$A_j\psi_j(\xi_1) - k_{1j}/\xi_1^{j+1} = 0;\quad A_j\psi_j'(\xi_1) + (j+1)k_{1j}/\xi_1^{j+2} = 0, \qquad (j = 1, 3, 4, ...), \qquad (4.2.92)$$

$$B_j\sigma_j(\xi_1) - \ell_{1j}/\xi_1^{j+1} = 0;\quad B_j\sigma_j'(\xi_1) + (j+1)\ell_{1j}/\xi_1^{j+2} = 0, \qquad (j = 1, 2, 3, ...). \qquad (4.2.93)$$

Since the determinants $\xi_1^{-j-2}[(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)]$ and $\xi_1^{-j-2}[(j+1)\sigma_j(\xi_1) + \xi_1\sigma_j'(\xi_1)]$ of these systems are nonzero [cf. Eq. (3.2.43)], they possess only the trivial solutions

$$A_j, k_{1j} = 0 \quad \text{if} \quad j = 1, 3, 4, ..., \quad \text{and} \quad B_j, \ell_{1j} = 0 \quad \text{if} \quad j = 1, 2, 3, ... \qquad (4.2.94)$$

Thus, the first order solution (4.2.64) for the post Newtonian rotating polytrope is

$$\Theta(\xi,\mu) = \theta(\xi) + \beta\psi_0(\xi) - 5\beta\xi_1^2\psi_2(\xi)\ P_2(\mu)/6[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)] + q_0\sigma_0(\xi). \qquad (4.2.95)$$

And the surface coordinate (4.2.82) is

$$\Xi_1(\mu) = \xi_1 - \beta\psi_0(\xi_1)/\theta'(\xi_1) + 5\beta\xi_1^2\psi_2(\xi_1)\ P_2(\mu)/6\theta'(\xi_1)[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)] - q_0\sigma_0(\xi_1)/\theta'(\xi_1). \qquad (4.2.96)$$

In the nonrelativistic case ($q_0 = 0$) all relevant equations become equal to those quoted in Sec. 3.2. The conditions for critical rotation (i.e. for mass loss from the equatorial bulge of the post Newtonian polytrope) are given as in the nonrelativistic case by [cf. Eqs. (3.6.24), (3.8.158)]

$$\Theta(\Xi_{ce}, 0) = 0;\quad [\partial\Theta(\xi,0)]_{\xi=\Xi_{ce}} = 0. \qquad (4.2.97)$$

Ξ_{ce} is the value at the equator of the surface coordinate Ξ_1 for critical rotation. The values of $\beta = \beta_c$ and Ξ_{ce} for critical rotation if $q_0 = 0.008$ are shown in the two last lines of Table 4.2.1. The agreement of Fahlman's and Anand's (1971a) values ($\beta_c = 3.76\times10^{-3},\ \Xi_{ce} = 9.90$ if $n = 3$) with those of Miketinac and Barton (1972) ($\beta_c = 4.11\times10^{-3},\ \Xi_{ce} = 9.61$ if $n = 3$) occurs only in the first digit. Tooper (1965, 1966b) has quoted a value of $q_0 \approx 0.01$, above which the post Newtonian approximation is no longer adequate.

The largest post Newtonian correction σ_0 is a purely radial effect, but for comparable values of β and q_0, the post Newtonian terms are at least as important as the rotation terms, particularly for the lower polytropic indices. An immediate application to the theory are rotating supermassive stars (Fahlman and Anand 1971b), and there is still a wide range of physical problems, including orbital perturbations of satellites, that can be investigated with the present theory. However, as stressed by Eqs. (4.2.49), (4.2.54), most equations are valid only under special isentropic conditions.

Table 4.2.1 Surface values of the post Newtonian functions $\sigma_0, \sigma_0', \pi_0, \pi_0'$ according to Fahlman and Anand (1971a). The surface values of $\psi_0, \psi_0', \psi_2, \psi_2'$ have already been quoted in Table 3.2.1. The last two lines show values of the critical rotation parameter β_c and of the critical equatorial coordinate Ξ_{ce} if $q_0 = 0.008$. The value of $\pi_0(\xi_1)$ is equal to $\Phi_{00}(\xi_1) + B_{00},\ (B_{00} = \mathrm{const})$ from Table II of Fahlman and Anand (1971a). $a + b$ means $a \times 10^b$.

n	1	2	2.5	3	3.5
$\sigma_0(\xi_1)$	5.71	2.54	1.42	4.94−1	−2.65−1
$\sigma_0'(\xi_1)$	3.56	1.49	9.01−1	5.03−1	2.45−1
$\pi_0(\xi_1)$	2.61	9.20−1	6.00−1	3.85−1	2.36−1
$\pi_0'(\xi_1)$	−7.16−1	−2.11−1	−1.12−1	−5.59−2	−2.47−2
β_c	8.25−2	2.15−2	9.56−3	3.76−3	1.18−3
Ξ_{ce}	4.43	5.96	7.53	9.90	1.42+1

4.2.3 Stoeckly's Method for Post Newtonian Rotating Polytropes

Miketinac and Barton (1972) have applied Stoeckly's (1965) method from Sec. 3.8.2 to post Newtonian polytropes with arbitrary high rotation. This theory belongs to approach (ii) from the Sec. 4.2.1. As already noted in Eq. (3.8.54), Stoeckly's (1965) method is only applicable if $1 \le n \le 5$. The relevant equations of the problem are given as before by Eqs. (4.2.12), (4.2.29), (4.2.46), (4.2.52). As in the preceding section we expand the potentials Φ and Π in series of Legendre polynomials $P_j(\mu)$, while the potential F is expanded in a series of Gegenbauer polynomials $G_j^{3/2}(\mu)$ of index 3/2, as suggested by the form of Eq. (4.2.52), [cf. Eqs. (3.10.152)-(3.10.156)]:

$$\Phi(r,\mu) = \sum_{j=0}^{\infty} \Phi_j(r)\ P_j(\mu); \quad F(r,\mu) = \sum_{j=0}^{\infty} F_j(r)\ G_j^{3/2}(\mu); \quad \Pi(r,\mu) = \sum_{j=0}^{\infty} \Pi_j(r)\ P_j(\mu). \tag{4.2.98}$$

Outside the polytrope we have $\varrho = 0$, and Eqs. (4.2.12), (4.2.52), (4.2.29) become

$$\nabla^2\Phi = 0; \quad \nabla^2 F + (2/r)\ \partial F/\partial r - (2\mu/r^2)\ \partial F/\partial\mu = 0; \quad \nabla^2\Pi = 0. \tag{4.2.99}$$

We insert the expansions (4.2.98) into Eq. (4.2.99), and equate coefficients of equal $P_j(\mu)$ and $G_j^{3/2}(\mu)$:

$$d(r^2\ d\Phi_j/dr)/dr - j(j+1)\Phi_j = 0, \tag{4.2.100}$$

$$d(r^2\ dF_j/dr)/dr + 2r\ dF_j/dr - j(j+3)F_j = 0, \tag{4.2.101}$$

$$d(r^2\ d\Pi_j/dr)/dr - j(j+1)\Pi_j = 0. \tag{4.2.102}$$

We have used Eqs. (3.1.40), (3.10.156), and because of equatorial symmetry there is $j = 0, 2, 4, ...$ Since the potentials Φ, F, Π, together with their derivatives, must remain finite as $r \to \infty$, Eqs. (4.2.100)-(4.2.102) have the solutions

$$\Phi_j, \Pi_j \propto r^{-j-1}; \quad F_j \propto r^{-j-3}, \qquad (r \to \infty). \tag{4.2.103}$$

If $j = 0$, the solution of Eqs. (4.2.100)-(4.2.102) is $\Phi_0, \Pi_0 \propto r^{-1} + \mathrm{const},\ F_0 \propto r^{-3} + \mathrm{const}$, and we normalize the potentials in such a way that const = 0. So, Eq. (4.2.103) subsists also in this case. In place of Eq. (4.2.103) we express the boundary conditions more conveniently under the equivalent form

$$(d\Phi_j/dr)_{r=r_H} + (j+1)\ \Phi_j(r_H)/r_H = 0; \quad (dF_j/dr)_{r=r_H} + (j+3)\ F_j(r_H)/r_H = 0;$$
$$(d\Pi_j/dr)_{r=r_H} + (j+1)\ \Pi_j(r_H)/r_H = 0, \tag{4.2.104}$$

where r_H is some radius completely outside the polytrope. Eq. (4.2.46) integrates at once by using Eqs. (4.2.47) and (4.2.49), $[U + P/\varrho = (n+1)P/\varrho;\ \nabla P/\varrho = (n+1)K\ \nabla\varrho^{1/n} = (n+1)\ \nabla(P/\varrho)]$:

$$(n+1)P/\varrho = \Phi + \Omega^2 r^2(1-\mu^2)/2 + (1/c^2)\big\{(n+1)^2P^2/2\varrho^2 + [\Omega^2 r^2(1-\mu^2)/2]^2$$
$$+2\Omega^2\Phi r^2(1-\mu^2) - 4\Omega^2 F r^2(1-\mu^2) + 2\Pi\big\} - \Phi_p - 2\Pi_p/c^2. \tag{4.2.105}$$

p-indexed quantities denote, as before, values at the surface poles. Similarly to Eq. (3.8.42), Miketinac and Barton (1972) introduce the dimensionless quantities

$$\Theta^*(x,\mu) = [(n+1)K/\Lambda]\varrho^{1/n}; \quad x = [4\pi G\Lambda^{n-1}/(n+1)^n K^n]^{1/2} r;$$
$$\omega = \{[(n+1)K/\Lambda]^n/4\pi G\}^{1/2}\Omega; \quad \varphi = \Phi/\Lambda; \quad f = F/\Lambda; \quad \pi^* = \Pi/\Lambda^2. \tag{4.2.106}$$

Obviously, $\varrho = \varrho_0\Theta^n(\xi,\mu) = [\Lambda/(n+1)K]^n\Theta^{*n}(x,\mu)$, and therefore $\Lambda = (n+1)K\varrho_0^{1/n}/\Theta^*(0,\mu)$, since $\Theta(0,\mu) = 1$. Eqs. (4.2.12), (4.2.52), (4.2.29), (4.2.105) become in the new variables, $(\Lambda = \Phi_p + 2\Pi_p/c^2)$:

$$\nabla\varphi = -\Theta^{*n}; \quad \nabla^2 f + (2/x)\,\partial f/\partial x - (2\mu/x^2)\,\partial f/\partial\mu = -\Theta^{*n};$$
$$\nabla\pi^* = -\Theta^{*n}[\Omega^2x^2(1-\mu^2) + \varphi + (n+3)\Theta^*/2(n+1)]; \quad \Theta^* = \varphi + \omega^2x^2(1-\mu^2)/2$$
$$+p_0\big\{\Theta^{*2}/2 + [\omega^2x^2(1-\mu^2)/2]^2 + 2\omega^2\varphi x^2(1-\mu^2) - 4\omega^2 f x^2(1-\mu^2) + 2\pi^*\big\} - 1. \tag{4.2.107}$$

To the order of approximation we are working, the quantity $p_0 = \Lambda/c^2$ is equal to

$$p_0 = \Lambda/c^2 = \Phi_p/c^2 + 2\Pi_p/c^4 \approx \Phi_p/c^2, \tag{4.2.108}$$

being connected to the relativity parameter from Eq. (4.2.56) by

$$q_0 = P_0/\varrho_0c^2 = K\varrho_0^{1/n}/c^2 = \Lambda\Theta^*(0,\mu)/c^2(n+1) = p_0\Theta^*(0,\mu)/(n+1). \tag{4.2.109}$$

In dimensionless form the boundary conditions (4.2.104) read

$$(d\varphi_j/dx)_{x=x_H} + (j+1)\ \varphi_j(x_H)/x_H = 0; \quad (df_j/dx)_{x=x_H} + (j+3)\ f_j(x_H)/x_H = 0;$$
$$(d\pi_j^*/dx)_{x=x_H} + (j+1)\ \pi_j^*(x_H)/x_H = 0, \tag{4.2.110}$$

where $\varphi_j = \Phi_j/\Lambda,\ f_j = F_j/\Lambda,\ \pi_j^* = \Pi_j/\Lambda^2$, and x_H is the value of x corresponding to $r = r_H$.

In order to solve the system (4.2.107) with the boundary conditions (4.2.110), Miketinac and Barton (1972) cast the equations, as in Sec 3.8.2, under the form of finite differences, and solve by a variant of the Gauss elimination method. The conditions for critical rotation are given by Eq. (4.2.97). The results for $q_0 = 0.008$ are $\beta_c = \Omega_c^2/2\pi G\varrho_0 = 4.48\times10^{-2}, 4.11\times10^{-3}$ and $\xi = \Xi_{ce} = 5.05, 9.61$ if $n = 1.5$ and 3, respectively, as already noted for $n = 3$ within the context of Table 4.2.1.

4.2.4 Slowly Rotating, Fully Relativistic Polytropes

The investigations from this and the next subsection belong to approach (iii) mentioned in Sec. 4.2.1, involving the first and second approximation with respect to angular velocity Ω, the tetrad formulation of general relativity, and bimetric gravitation. Sedrakyan and Chubaryan (1968a) introduce "spherical" coordinates $x^1 = r,\ x^2 = \lambda,\ x^3 = \varphi$, giving up the equality of g_{11} and g_{22} from Eq. (4.2.9). They take instead of the metric tensor g_{jk} four new unknown functions κ, μ, ν, ω, which depend on r and λ, and are more suitable for the calculation of the Ricci curvature tensor R_j^k :

$$g_{00} = \exp\nu - \omega^2\exp\mu\sin^2\lambda; \quad g_{11} = -\exp\kappa; \quad g_{22} = -\exp\nu;$$
$$g_{33} = g_{22}\sin^2\lambda = -\exp\mu\sin^2\lambda; \quad g_{03} = -\omega\exp\mu\sin^2\lambda. \tag{4.2.111}$$

In the new notations, the four-dimensional interval is written under the form

$$ds^2 = g_{00}\ dt^2 + g_{11}\ dr^2 + g_{22}\ d\lambda^2 + g_{33}\ d\varphi^2 + 2g_{03}\ dt\ d\varphi = (\exp\nu - \omega^2\exp\mu\sin^2\lambda)\ dt^2$$
$$-\exp\kappa\ dr^2 - \exp\mu\ (d\lambda^2 + \sin^2\lambda\ d\varphi^2) - 2\omega\exp\mu\sin^2\lambda\ dt\ d\varphi. \tag{4.2.112}$$

Since the metric (4.2.112) must be invariant under the transformation $t\to -t$, and because the angular velocity Ω changes sign under this transformation, it is clear that all components of the metric tensor other than ω, are even functions of Ω, and ω is an odd function of Ω. The nonzero Christoffel symbols from Eq. (4.1.15) calculated by Sedrakyan and Chubaryan (1968a) are

$$\Gamma^1_{00} = (1/2)(\partial\nu/\partial r)\exp(\nu-\kappa) - (\omega/2)(2\ \partial\omega/\partial r - \omega\ \partial\mu/\partial r)\exp(\mu-\kappa)\sin^2\lambda;$$
$$\Gamma^2_{00} = (1/2)(\partial\nu/\partial\lambda)\exp(\nu-\mu) - \omega\big[\omega\cot\lambda + \partial\omega/\partial\lambda + (\omega/2)\ \partial\mu/\partial\lambda\big]\sin^2\lambda;$$
$$\Gamma^0_{10} = (1/2)\big[\partial\nu/\partial r - \omega(\partial\omega/\partial r)\exp(\mu-\nu)\sin^2\lambda\big];\quad \Gamma^0_{20} = (1/2)\big[\partial\nu/\partial\lambda$$
$$-\omega(\partial\omega/\partial\lambda)\exp(\mu-\nu)\sin^2\lambda\big];\quad \Gamma^3_{10} = (1/2)\ (\partial\omega/\partial r + \omega\ \partial\mu/\partial r - \omega\ \partial\nu/\partial r)$$
$$+(\omega^2/2)(\partial\omega/\partial r)\exp(\mu-\nu)\sin^2\lambda;\quad \Gamma^1_{30} = -(1/2)\exp(\mu-\kappa)\ (\partial\omega/\partial r + \omega\ \partial\mu/\partial r)\sin^2\lambda;$$
$$\Gamma^2_{30} = -(1/2)(\partial\omega/\partial\lambda + \omega\ \partial\mu/\partial\lambda + 2\omega\cot\lambda)\sin^2\lambda;\quad \Gamma^0_{13} = -(1/2)(\partial\omega/\partial r)\exp(\mu-\nu)\sin^2\lambda;$$
$$\Gamma^0_{23} = -(1/2)(\partial\omega/\partial\lambda)\exp(\mu-\nu)\sin^2\lambda;\quad \Gamma^1_{11} = (1/2)\ \partial\kappa/\partial r;\quad \Gamma^2_{11} = -(1/2)(\partial\kappa/\partial\lambda)\exp(\kappa-\mu);$$
$$\Gamma^1_{12} = (1/2)\ \partial\kappa/\partial\lambda;\quad \Gamma^2_{12} = (1/2)\ \partial\mu/\partial r;\quad \Gamma^3_{13} = (1/2)\big[\partial\mu/\partial r + \omega(\partial\omega/\partial r)\exp(\mu-\nu)\sin^2\lambda\big];$$
$$\Gamma^1_{22} = -(1/2)(\partial\mu/\partial r)\exp(\mu-\kappa);\quad \Gamma^2_{22} = (1/2)\ \partial\mu/\partial\lambda;\quad \Gamma^1_{33} = -(1/2)(\partial\mu/\partial r)\exp(\mu-\kappa)\sin^2\lambda;$$
$$\Gamma^2_{33} = -(1/2)(\partial\mu/\partial\lambda + 2\cot\lambda)\sin^2\lambda. \qquad (4.2.113)$$

As already outlined in Eq. (4.2.6), the angular velocity of an axially rotating fluid element, as seen from infinity, is $\Omega = d\varphi/dt = u^3/u^0$, and $u^1, u^2 = 0$. For the case of uniform rotation Eq. (4.1.12) writes

$$u_0u^0 + u_3u^3 = g_{00}(u^0)^2 + 2g_{03}u^0u^3 + g_{33}(u^3)^2 = (u^0)^2(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2) = 1. \qquad (4.2.114)$$

The angular velocity Ω and the metric tensor g_{jk} completely specify the four-velocity (Hartle and Sharp 1967):

$$u^0 = 1/(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2)^{1/2} = 1/F;\quad u^3 = \Omega/(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2)^{1/2} = \Omega/F;$$
$$u^1, u^2 = 0, \qquad [F = (g_{00} + 2g_{03}\Omega + g_{33}\Omega^2)^{1/2}]. \qquad (4.2.115)$$

For the metric (4.2.112) the components of the four-velocity are

$$u^0 = u^t = [\exp\nu - (\omega+\Omega)^2\exp\mu\sin^2\lambda]^{-1/2};\quad u^3 = u^\varphi = \Omega u^t;\quad u^1 = u^r = 0;\quad u^2 = u^\lambda = 0. \qquad (4.2.116)$$

The nonzero components of the energy-momentum tensor (4.1.11) are

$$T^0_0 = (P+\varrho_r)(u^0)^2[\exp\nu - \omega(\omega+\Omega)\exp\mu\sin^2\lambda] - P;\quad T^1_1 = T^2_2 = -P;$$
$$T^3_3 = -(P+\varrho_r)(u^0)^2\Omega(\omega+\Omega)\exp\mu\sin^2\lambda - P;\quad T^3_0 = (P+\varrho_r)(u^0)^2\Omega$$
$$\times[\exp\nu - \omega(\omega+\Omega)\exp\mu\sin^2\lambda];\quad T^0_3 = -(P+\varrho_r)(u^0)^2(\omega+\Omega)\exp\mu\sin^2\lambda, \qquad (4.2.117)$$

where $\varepsilon_r = \varrho_r$, since we have assumed $c, G = 1$. To illustrate the complexity of the problem, we write down the $(0,0), (1,1), (2,2), (3,3), (0,3), (2,1)$-components of the Einstein equations (4.1.4), as derived by Sedrakyan and Chubaryan (1968a) with the Ricci tensor (4.2.24), the Christoffel symbols (4.2.113), and the energy-momentum tensor (4.2.117):

$$2R^0_0 - R = -\exp(-\kappa)\ [2\ \partial^2\mu/\partial r^2 + 3(\partial\mu/\partial r)^2/2 - (\partial\mu/\partial r)\ \partial\kappa/\partial r]$$
$$-\exp(-\mu)\ [\partial^2\kappa/\partial\lambda^2 + \partial^2\mu/\partial\lambda^2 + (\partial\kappa/\partial\lambda)^2/2 + (\partial\mu/\partial\lambda + \partial\kappa/\partial\lambda)\cot\lambda - 2]$$
$$-\exp(-\nu)\ \big\{\omega(\partial\omega/\partial\lambda)\ [\partial\mu/\partial\lambda + (1/2)(\partial\kappa/\partial\lambda - \partial\nu/\partial\lambda)] + (\partial\omega/\partial\lambda)^2/2 + \omega\ \partial^2\omega/\partial\lambda^2$$
$$+3\omega(\partial\omega/\partial\lambda)\cot\lambda\big\}\sin^2\lambda - \exp(\mu-\nu-\kappa)\big\{\omega(\partial\omega/\partial r)[2\ \partial\mu/\partial r$$
$$-(1/2)(\partial\nu/\partial r + \partial\kappa/\partial r)] + (\partial\omega/\partial r)^2/2 + \omega\ \partial^2\omega/\partial r^2\big\}\sin^2\lambda$$
$$= 16\pi\big\{(P+\varrho_r)[\exp\nu - \omega(\omega+\Omega)\exp\mu\sin^2\lambda]/[\exp\nu - (\omega+\Omega)^2\exp\mu\sin^2\lambda] - P\big\};$$

$$2R^1_1 - R = -\exp(-\kappa)\ [(\partial\mu/\partial r)^2/2 + (\partial\mu/\partial r)\ \partial\nu/\partial r]$$
$$-\exp(-\mu)\ [\partial^2\nu/\partial\lambda^2 + \partial^2\mu/\partial\lambda^2 + (1/2)(\partial\nu/\partial\lambda)^2 - 2 + (\partial\mu/\partial\lambda + \partial\nu/\partial\lambda)\cot\lambda]$$
$$+(1/2)\exp(-\nu)\ (\partial\omega/\partial\lambda)^2\sin^2\lambda - (1/2)\exp(\mu-\nu-\kappa)\ (\partial\omega/\partial r)^2\sin^2\lambda = -16\pi P;$$

$$2R_2^2 - R = -\exp(-\kappa)\ \big\{\partial^2\mu/\partial r^2 + \partial^2\nu/\partial r^2 + (1/2)[(\partial\nu/\partial r)^2 + (\partial\mu/\partial r)^2]$$
$$-(1/2)(\partial\kappa/\partial r)(\partial\nu/\partial r + \partial\mu/\partial r) + (1/2)(\partial\mu/\partial r)\ \partial\nu/\partial r\big\}$$
$$-\exp(-\mu)\ [(1/2)(\partial\mu/\partial\lambda)(\partial\kappa/\partial\lambda + \partial\nu/\partial\lambda) + (1/2)(\partial\nu/\partial\lambda)\ \partial\kappa/\partial\lambda + (\partial\nu/\partial\lambda + \partial\kappa/\partial\lambda)\cot\lambda]$$
$$-(1/2)\exp(-\nu)\ (\partial\omega/\partial\lambda)^2\sin^2\lambda + (1/2)\exp(\mu-\nu-\kappa)\ (\partial\omega/\partial r)^2\sin^2\lambda = -16\pi P;$$

$$2R_3^3 - R = -\exp(-\kappa)\ \big\{\partial^2\mu/\partial r^2 + \partial^2\nu/\partial r^2 + (1/2)[(\partial\nu/\partial r)^2 + (\partial\mu/\partial r)^2]$$
$$-(1/2)(\partial\kappa/\partial r)(\partial\nu/\partial r + \partial\mu/\partial r) + (1/2)(\partial\mu/\partial r)\ \partial\nu/\partial r\big\}$$
$$-\exp(-\mu)\ \big\{\partial^2\kappa/\partial\lambda^2 + \partial^2\nu/\partial\lambda^2 + (1/2)[(\partial\kappa/\partial\lambda)^2 + (\partial\nu/\partial\lambda)^2$$
$$+(\partial\kappa/\partial\lambda)\ \partial\nu/\partial\lambda - (\partial\kappa/\partial\lambda)\ \partial\mu/\partial\lambda - (\partial\mu/\partial\lambda)\ \partial\nu/\partial\lambda]\big\}$$
$$+\exp(-\nu)\ \big\{(3/2)(\partial\omega/\partial\lambda)^2 + \omega(\partial\omega/\partial\lambda)[\partial\mu/\partial\lambda + (1/2)(\partial\kappa/\partial\lambda - \partial\nu/\partial\lambda)]$$
$$+\omega\ \partial^2\omega/\partial\lambda^2 + 3\omega(\partial\omega/\partial\lambda)\cot\lambda\big\}\sin^2\lambda + \exp(\mu-\nu-\kappa)\ \big\{(3/2)(\partial\omega/\partial r)^2$$
$$+\omega(\partial\omega/\partial r)[2\ \partial\mu/\partial r - (1/2)(\partial\kappa/\partial r + \partial\nu/\partial r)] + \omega\ \partial^2\omega/\partial r^2\big\}\sin^2\lambda$$
$$= -16\pi\big\{(P+\varrho_r)\Omega(\omega+\Omega)\exp\mu\sin^2\lambda/[\exp\nu - (\omega+\Omega)^2\exp\mu\sin\lambda^2] - P\big\};$$

$$2R_0^3 = -\exp(-\kappa)\ \big\{\omega[\partial^2\nu/\partial r^2 - \partial^2\mu/\partial r^2 - (\partial\mu/\partial r)^2 + (\partial\nu/\partial r)^2/2 - (\partial\kappa/\partial r)(\partial\nu/\partial r)/2$$
$$+(\partial\kappa/\partial r)(\partial\mu/\partial r)/2 + (\partial\mu/\partial r)(\partial\nu/\partial r)/2] + (\partial\omega/\partial r)[(1/2)(\partial\kappa/\partial r + \partial\nu/\partial r) - 2\ \partial\mu/\partial r]$$
$$-\partial^2\omega/\partial r^2\big\} - \exp(-\mu)\ [-\partial^2\omega/\partial\lambda^2 + 2\omega - 3(\partial\omega/\partial\lambda)\cot\lambda + \omega(\partial^2\nu/\partial\lambda^2 - \partial^2\mu/\partial\lambda^2)$$
$$+(\omega/2)(\partial\nu/\partial\lambda)^2 - (\omega/2)(\partial\mu/\partial\lambda)(\partial\kappa/\partial\lambda + \partial\nu/\partial\lambda) + (1/2)(\partial\omega/\partial\lambda)(\partial\nu/\partial\lambda - \partial\kappa/\partial\lambda - 2\ \partial\mu/\partial\lambda)$$
$$+(\omega/2)(\partial\kappa/\partial\lambda)\ \partial\nu/\partial\lambda - \omega(\partial\kappa/\partial\lambda + \partial\mu/\partial\lambda)\cot\lambda] + \exp(-\nu)\ \big\{\omega^2\ \partial^2\omega/\partial\lambda^2 + 2\omega(\partial\omega/\partial\lambda)^2$$
$$+3\omega^2(\partial\omega/\partial\lambda)\cot\lambda + \omega^2(\partial\omega/\partial\lambda)[\partial\mu/\partial\lambda + (1/2)(\partial\kappa/\partial\lambda - \partial\nu/\partial\lambda)]\big\}\sin^2\lambda - \exp(\mu-\nu-\kappa)$$
$$\times\big[-\omega^2\ \partial^2\omega/\partial r^2 - 2\omega(\partial\omega/\partial r)^2 - 2\omega^2(\partial\omega/\partial r)\ \partial\mu/\partial r + (\omega^2/2)(\partial\omega/\partial r)(\partial\kappa/\partial r + \partial\nu/\partial r)\big]\sin^2\lambda$$
$$= 16\pi(P+\varrho_r)\Omega[\exp\nu - \omega(\omega+\Omega)\exp\mu\sin^2\lambda]/[\exp\nu - (\omega+\Omega)^2\exp\mu\sin^2\lambda];$$

$$R_2^1 = -(1/2)\exp(-\kappa)\ [(1/2)(\partial\kappa/\partial\lambda)(\partial\mu/\partial r + \partial\nu/\partial r) + (1/2)(\partial\nu/\partial\lambda)(\partial\mu/\partial r - \partial\nu/\partial r)$$
$$-\partial^2\mu/\partial r\partial\lambda - \partial^2\nu/\partial r\partial\lambda] - (1/2)\exp(\mu-\nu-\kappa)\ (\partial\omega/\partial r)(\partial\omega/\partial\lambda)\sin^2\lambda = 0, \qquad (c, G = 1). \tag{4.2.118}$$

Subsequently to Eq. (4.2.112) we have outlined that κ, ν, μ are even functions of Ω, so they remain equal to their Schwarzschild values $\kappa = \kappa(r),\ \nu = \nu(r),\ \exp\mu = r^2$ if we restrict ourselves to first order terms in Ω. Only ω will depend on the first power of Ω, and in this approximation the $(0,0), (3,3), (0,3)$-components of Eq. (4.2.118) become, respectively

$$\exp(-\kappa)\ [(1/r)\ d\kappa/dr - 1/r^2] + 1/r^2 = 8\pi\varrho_r; \quad \exp(-\kappa)\ [(1/r)\ d\nu/dr + 1/r^2] - 1/r^2 = 8\pi P, \tag{4.2.119}$$

$$\exp(-\kappa)\ \big\{\omega[d^2\nu/dr^2 + (1/2)(d\nu/dr)^2 - (1/2)(d\kappa/dr)\ d\nu/dr + (1/r)(d\kappa/dr + d\nu/dr) - 2/r^2]$$
$$+(\partial\omega/\partial r)[(1/2)(d\kappa/dr + d\nu/dr) - 4/r] - \partial^2\omega/\partial r^2\big\} + (1/r^2)[-\partial^2\omega/\partial\lambda^2 - 3(\partial\omega/\partial\lambda)\cot\lambda + 2\omega]$$
$$= -16\pi\Omega(P+\varrho_r). \tag{4.2.120}$$

Sedrakyan and Chubaryan (1968b) ignore the angular dependence of $\omega(r,\lambda)$, since the distortions remain small if $\Omega \ll 1$. Instead of the (0,3)-component from Eq. (4.2.120), Papoyan et al. (1969) use for computational reasons the $(3,0)$-component of the Einstein equations (4.1.4) in the first order approximation

$$R_3^0 = -(1/2)\exp(-\nu-\kappa)\ r^2\big\{d^2\omega/dr^2 - (d\omega/dr)[(1/2)(d\kappa/dr + d\nu/dr) - 4/r]\big\}\sin^2\lambda = 8\pi T_3^0,$$
$$(\omega = \omega(r);\ c, G = 1), \tag{4.2.121}$$

where the first order approximation of the energy-momentum tensor (4.2.117) is

$$T_3^0 \approx -(P+\varrho_r)r^2(\omega+\Omega)\exp(-\nu)\sin^2\lambda, \qquad (\exp\mu \approx r^2). \tag{4.2.122}$$

Inserting this into Eq. (4.2.121), we obtain

$$\exp(-\kappa)\ \{d^2\omega/dr^2 - (d\omega/dr)[(1/2)(d\kappa/dr + d\nu/dr) - 4/r]\}$$
$$= 16\pi(P + \varrho_r)(\omega + \Omega), \qquad [\omega = \omega(r)], \tag{4.2.123}$$

and the system to be solved now consists of Eqs. (4.2.119) and (4.2.123). We observe that the equations (4.2.119) are just equal to the Schwarzschild equations (4.1.6) and (4.1.7) for the nonrotating sphere, so we can replace the equations (4.2.119) by Eq. (4.1.26)

$$dM_r/dr = 4\pi\varrho_r r^2, \tag{4.2.124}$$

and by the Tolman-Oppenheimer-Volkoff equation (4.1.27)

$$dP/dr = -4\pi r^2(P + \varrho_r)(P + M_r/4\pi r^3)/(r - 2M_r), \qquad (c, G = 1). \tag{4.2.125}$$

Let us now introduce the relativistic rotation parameter

$$\beta_r = \Omega^2/2\pi G\varrho_{r0}, \qquad (G = 1), \tag{4.2.126}$$

and the new variable

$$Q = Q(r) = [\omega(r) + \Omega]/\beta_r^{1/2}. \tag{4.2.127}$$

Eq. (4.2.123) becomes

$$d^2Q/dr^2 + [4/r - 4\pi r^2(P + \varrho_r)/(r - 2M_r)]\ dQ/dr - 16\pi r Q(P + \varrho_r)/(r - 2M_r) = 0, \tag{4.2.128}$$

where we have also used Eq. (4.1.22), written under the form

$$\exp(-\kappa) = 1 - 2M_r/r, \qquad (c, G = 1), \tag{4.2.129}$$

together with Eq. (4.1.18):

$$d\nu/dr = -2(dP/dr)/(P + \varrho_r). \tag{4.2.130}$$

Outside the configuration Eq. (4.2.128) assumes the form

$$d^2Q/dr^2 + (4/r)\ dQ/dr = 0, \qquad (P, \varrho_r = 0), \tag{4.2.131}$$

with the elementary solution $Q = A_1 + A_2/r^3$, $(A_1, A_2 = \mathrm{const})$. From Eq. (4.2.127) results $\omega = -\Omega + \beta_r^{1/2}(A_1 + A_2/r^3)$. Since at infinite distance the metric becomes Galilean, we get from Eq. (4.2.111): $g_{03} = -\omega \exp\mu \sin^2\lambda = -\omega r^2 \sin^2\lambda = 0$, or $\omega \to 0$ if $r \to \infty$. Thus $A_1 = \Omega/\beta_r^{1/2}$, and the external solution of ω is (Papoyan et al. 1969)

$$\omega = \beta_r^{1/2} A_2/r^3, \qquad (r \geq r_1). \tag{4.2.132}$$

The polytropic variables are introduced by [cf. Eqs. (4.1.29), (4.1.31), (4.1.39)]

$$r = [(n+1)K/4\pi\varrho_{r0}^{1-1/n}]^{1/2}\xi = [(n+1)P_0/4\pi\varrho_{r0}^2]^{1/2}\xi = \alpha\xi; \quad \varrho_r = \varrho_{r0}\theta_r^n;$$
$$\eta = \eta(\xi) = M_r(r)/4\pi\varrho_{r0}\alpha^3; \quad q_0 = P_0/\varrho_{r0} = K\varrho_{r0}^{1/n}, \qquad (c, G = 1). \tag{4.2.133}$$

Eqs. (4.2.124), (4.2.125), (4.2.128) are brought into dimensionless form [cf. Eqs. (4.1.37), (4.1.38)]:

$$d\eta/d\xi = \xi^2\theta_r^n, \tag{4.2.134}$$

$$\xi^2\ d\theta_r/d\xi = -(1 + q_0\theta_r)[\eta(\xi) + q_0\xi^3\theta_r^{n+1}]/[1 - 2q_0(n+1)\eta(\xi)/\xi], \tag{4.2.135}$$

$$d^2Q/d\xi^2 + \{4/\xi - q_0(n+1)(1 + q_0\theta_r)\xi\theta_r^n/[1 - 2q_0(n+1)\eta/\xi]\}\ dQ/d\xi$$
$$-4q_0(n+1)(1 + q_0\theta_r)\theta_r^n Q/[1 - 2q_0(n+1)\eta/\xi] = 0. \tag{4.2.136}$$

Table 4.2.2 Surface coordinate ξ_1 and true dimensionless surface radius ξ_{r1} of a spherical relativistic polytrope according to Sarkisyan and Chubaryan (1977), (cf. Table 4.1.1). For critical rotation Ξ_{rce}, Ξ_{rcp} denote the true relativistic dimensionless surface radii in the equatorial and polar direction, respectively [Eq. (4.2.143)]. The relativity parameter $q_0 = P_0/\varrho_{r0}$, $(c = 1)$ is from Eq. (4.2.133), and the critical oblateness is given by $f_c = (\Xi_{rce} - \Xi_{rcp})/\Xi_{rce}$. $aE + b$ means $a \times 10^b$.

n	q_0	ξ_1	ξ_{r1}	Ξ_{rce}	Ξ_{rcp}	f_c
	0.2	2.277	2.718	2.953	2.362	2.00 E−1
1	0.4	1.915	2.461	2.778	2.172	2.18 E−1
	0.5	1.800	2.328	2.598	2.158	1.69 E−1
	0.2	2.699	3.198	3.423	2.925	1.45 E−1
1.5	0.4	2.361	3.071	3.326	2.795	1.60 E−1
	0.6	2.219	2.978	3.226	2.782	1.38 E−1
	0.2	3.398	4.140	4.318	3.836	1.12 E−1
2	0.4	3.248	4.188	4.415	3.938	1.08 E−1
	0.6	3.400	4.550	4.748	4.291	9.63 E−2
	0.2	4.724	5.688	5.854	5.381	8.08 E−2
2.5	0.4	5.520	6.950	7.220	6.732	6.76 E−2
	0.6	7.730	9.750	9.980	9.250	7.31 E−2
	0.7	9.530	11.92	12.15	11.92	1.89 E−2
	0.1	6.834	7.608	8.051	7.424	7.79 E−2
	0.2	7.951	8.980	9.381	8.620	8.11 E−2
3	0.4	18.10	20.88	21.56	20.36	5.57 E−2
	0.6	90.99	94.26	105.7	93.28	1.18 E−1
	0.7	176.9	182.1	211.0	184.3	1.27 E−1

Analysis of the behaviour of Eq. (4.2.136) near the origin shows that $Q(\xi) \approx C + O(\xi^2)$, $(C = \mathrm{const})$, (Sedrakyan and Chubaryan 1968b), so the initial conditions for the system (4.2.134)-(4.2.136) are [cf. Eq. (4.1.41)]:

$$\theta_r(0) = 1; \quad (d\theta_r/d\xi)_{\xi=0} = 0; \quad \eta(0) = 0; \quad Q(0) = C; \quad (dQ/d\xi)_{\xi=0} = 0. \tag{4.2.137}$$

The unknown constants A_2 and C from Eqs. (4.2.132), (4.2.137) can be determined in a first approximation from the continuity of Q and of its radial derivative across the sphere $\xi = \xi_1$. Taking $Q(\xi) = Cf(\xi)$, $(f(0) = 1)$, and inserting for Q from Eqs. (4.2.127), (4.2.132), we get $(Q(\xi) = A_2/\alpha^3\xi^3 + \Omega/\beta_r^{1/2}$ if $\xi \geq \xi_1)$:

$$Cf(\xi_1) = A_2/\alpha^3\xi_1^3 + \Omega/\beta_r^{1/2}; \quad Cf'(\xi_1) = -3A_2/\alpha^3\xi_1^4, \tag{4.2.138}$$

and

$$C = 3(2\pi\varrho_{r0})^{1/2}/[3f(\xi_1) + \xi_1 f'(\xi_1)]; \quad A_2 = -C\alpha^3\xi_1^4 f'(\xi_1)/3. \tag{4.2.139}$$

Sarkisyan and Chubaryan (1977) use the tetrad formulation of general relativity for the study of rotating relativistic polytropes (cf. Eqs. (4.2.151)-(4.2.171); Israel 1970, Chap. 2; Landau and Lifschitz 1987, §98), and finally arrive to the Einstein equations (4.1.4). Pressure, relativistic density, and relativistic mass are expanded up to terms of order Ω^2 :

$$P(r,\lambda) = P^{(0)}(r) + \beta_r P^{(1)}(r,\lambda); \quad \varrho_r(r,\lambda) = \varrho_r^{(0)}(r) + \beta_r \varrho_r^{(1)}(r,\lambda);$$
$$M_r(r,\lambda) = M_r^{(0)}(r) + \beta_r M_r^{(1)}(r,\lambda), \qquad (\beta_r = \Omega^2/2\pi\varrho_{r0} \ll 1). \tag{4.2.140}$$

$P^{(0)}, \varrho_r^{(0)}, M_r^{(0)}$ are the quantities corresponding to the nonrotating sphere from Sec. 4.1.1. The dimensionless surface coordinate from Eq. (4.2.82) is

$$\Xi_1(\mu) = \xi_1 + \beta_r \sum_{j=0}^{\infty} c_j P_j(\mu), \qquad (c_j = \mathrm{const}). \tag{4.2.141}$$

Similarly as in the post Newtonian approximation, Sarkisyan and Chubaryan (1977) obtain

$$\Xi_1 = \Xi_1(\mu) = \xi_1 + \beta_r[c_0 + c_2 P_2(\mu)]; \quad \Xi_e = \xi_1 + \beta_r(c_0 - c_2/2); \quad \Xi_p = \xi_1 + \beta_r(c_0 + c_2). \tag{4.2.142}$$

Ξ_e and Ξ_p are the values of the surface coordinates at the equator and at the poles, respectively. The true radius of the rotating polytrope in a certain direction $\mu =$ const is given analogously to Eq. (4.1.47) by

$$r_{r1}(\mu) = \alpha\Xi_{r1}(\mu) = \int_0^{r_1(\mu)} (\gamma_{rr})^{1/2}\ dr = \int_0^{r_1(\mu)} (-g_{rr})^{1/2}\ dr = \alpha \int_0^{\Xi_1(\mu)} \exp[\kappa(\xi)/2]\ d\xi,$$
$$(\gamma_{11} = \gamma_{rr} = -g_{rr};\ g_{01} = g_{0r} = 0). \tag{4.2.143}$$

Although the values of ξ_{r1} in the Tables 4.1.1 and 4.2.2 coincide within the first digit, the irregularities of the critical oblateness f_c (especially if $n = 2.5$ and 3) raise some doubts concerning the exactness of Ξ_{rce} and Ξ_{rcp}.

4.2.5 Slowly Rotating and Fully Relativistic Polytropes in Bimetric Gravitation

The treatment up to the first order in the angular velocity Ω by Chubaryan et al. (1981) is similar as in Sec. 4.1.4. The reference frame is essentially fixed by the particular form of the flat space metric df^2 from Eq. (4.1.122), if we take the origin of the reference frame in the centre of mass of the stationary polytrope. We have to add a mixed component to the Riemann metric ds^2 from Eq. (4.1.122), similar to g_{03} from Eq. (4.2.111):

$$ds^2 = \exp[2\nu(r)]\ dt^2 - \exp[2\kappa(r)]\ dr^2 - r^2 \exp[2\chi(r)]\ (d\lambda^2 + \sin^2\lambda\ d\varphi^2)$$
$$-2r^2\omega(r,\lambda)\ \exp[2\chi(r)] \sin^2\lambda\ d\varphi\ dt, \qquad (c, G = 1). \tag{4.2.144}$$

We have already outlined subsequently to Eq. (4.2.112) that ω is of order Ω, and therefore the components of the four-velocity from Eq. (4.2.115) are up to the first order in Ω equal to

$$u^0 \approx 1/(g_{00})^{1/2} = \exp(-\nu); \quad u^3 = \Omega u^0 \approx \Omega \exp(-\nu); \quad u^1, u^2 = 0, \qquad [g_{03} = O(\Omega)]. \tag{4.2.145}$$

Up to the first order in Ω the components of the energy-momentum tensor (4.1.11) write as [cf. Eq. (4.2.117)]

$$T_0^0 = (P + \varrho_r)(u^0)^2 \exp(2\nu) - P = \varrho_r; \quad T_1^1 = T_2^2 = T_3^3 = -P;$$
$$T_3^0 = -(P + \varrho_r)(u^0)^2(\omega + \Omega) r^2 \exp(2\chi) \sin^2\lambda = -(P + \varrho_r) r^2 (\omega + \Omega) \exp(2\chi - 2\nu) \sin^2\lambda. \tag{4.2.146}$$

The equations determining the metric functions ν and κ, $(\chi \equiv \kappa)$ in the first approximation with respect to Ω are identical to Eqs. (4.1.127), (4.1.128). For the third function ω Chubaryan et al. (1981) obtain in a way similar to Eq. (4.2.120):

$$\partial\omega^2/\partial r^2 + \big[4/r + 2(\partial\kappa/\partial r - \partial\nu/\partial r)\big]\ \partial\omega/\partial r + \big[(2/r)(\partial\kappa/\partial r - \partial\nu/\partial r)$$
$$-16\pi(\varrho_r + P)\exp(\nu + 3\kappa)\big]\ \omega + (1/r^2)\big[\partial^2\omega/\partial\lambda^2 + 3(\partial\omega/\partial\lambda)\cot\lambda\big] = 16\pi\Omega(P + \varrho_r)\exp(\nu + 3\kappa). \tag{4.2.147}$$

Rotation periods of critically rotating polytropes are found to be considerably shorter in bimetric gravitation as compared to general relativity if $1 \leq n \leq 3,\ 0.1 \leq q_0 \lesssim 0.7$. A second order approximation with respect to Ω has been devised by Grigoryan et al. (1993).

4.2.6 Rapidly Rotating and Fully Relativistic Polytropes

This case is of course the most complicated and sophisticated one. The basic stationary metric is in "cylindrical" (ℓ, φ, z)-coordinates of the general form (4.2.9), (e.g. Bardeen and Wagoner 1971)

$$ds^2 = \exp[2\nu(\ell,z)]\ dt^2 - \exp[2\kappa(\ell,z)]\ (d\ell^2 + dz^2) - \ell^2 \exp[2\gamma(\ell,z) - 2\nu(\ell,z)]\ [d\varphi - \omega(\ell,z)\ dt]^2, \tag{4.2.148}$$

and in "spherical" (r, λ, φ)-coordinates equal to (e.g. Nishida et al. 1992)

$$ds^2 = \exp[2\nu(r,\lambda)]\ dt^2 - \exp[2\kappa(r,\lambda)]\ (dr^2 + r^2\ d\lambda^2)$$
$$-r^2 \sin^2\lambda \exp[2\gamma(r,\lambda) - 2\nu(r,\lambda)]\ [d\varphi - \omega(r,\lambda)\ dt]^2. \qquad (4.2.149)$$

t denotes the universal time coordinate. The transition from Eq. (4.2.148) to Eq. (4.2.149) is effected by the simple Cartesian transformations $\ell = r\sin\lambda,\ z = r\cos\lambda,\ (d\ell^2 + dz^2 = dr^2 + r^2\ d\lambda^2)$.

The problem of fast rotation in arbitrary strong gravitational fields involves the solution of the Einstein equations (4.1.4) in a spacetime that becomes Galilean at spatial infinity (the asymptotically flat spacetime): This implies that $\nu, \kappa, \gamma, \omega$ vanish at spatial infinity. From the metric (4.2.149) it is obvious that the squared proper time interval from Eq. (5.12.94) – as measured by a static observer $dx^\alpha = 0$ – is equal to $d\tau^2 = g_{00}(dx^0 + g_{0\alpha}\ dx^\alpha/g_{00})^2/c^2 = g_{00}\ (dx^0)^2/c^2 = g_{00}\ dt^2/c^2 = [\exp(2\nu) - r^2\omega^2\sin\lambda^2\exp(2\gamma - 2\nu)]\ dt^2/c^2,\ (x^0 = t)$. This shows that g_{00} can become negative under certain circumstances – the proper time becomes imaginary, and no timelike static observers exist (cf. Landau and Lifschitz 1987, §84). It is therefore much more comfortable to use a local inertial frame of reference, tied to the so-called locally nonrotating observer (zero angular momentum observer).

In contrast to Newtonian gravitation, the inertial frames inside a relativistic rotating fluid are not at rest with respect to infinitely distant observers. Rather, the local inertial frames are dragged along by the relativistic rotating fluid (Lense-Thirring effect). This dragging of inertial frames by rotating relativistic fluids is the reason for a distinction between static observers and locally nonrotating observers (Hartle 1967, Bardeen and Wagoner 1971). The dragging of inertial frames appears in the metrics (4.2.148) and (4.2.149) as the nonvanishing of the component $g_{03} = g_{t\varphi} = \omega\ell^2\exp(2\gamma - 2\nu) = \omega r^2\sin^2\lambda\exp(2\gamma - 2\nu)$. The quantity $\omega = \omega(\ell, z) = \omega(r,\lambda)$ from the metrics (4.2.148), (4.2.149) appears as the angular velocity of the locally nonrotating observer as seen from infinity, and will be called the rate of rotation (the dragging rate) of the local inertial frame. The angular velocity $\Omega = d\varphi/dt = (d\varphi/ds)/(dt/ds) = u^3/u^0$ of the rotating mass elements is constant throughout the fluid, as seen from spatial infinity. The difference $\Omega - \omega$ is just the angular velocity of fluid elements relative to the locally nonrotating observer, as measured from infinity. The locally nonrotating observer possesses just the angular velocity $d\varphi/dt = \omega$. Therefore, the metric (4.2.149) becomes in this locus equal to $ds^2 = \exp(2\nu)\ dt^2 - \exp(2\kappa)\ (dr^2 + r^2\ d\lambda^2),\ (d\varphi - \omega\ dt = 0)$. And with this metric the interval of proper time [see Eq. (5.12.94)] of the locally nonrotating observer amounts to $d\tau = g_{00}^{1/2}(dx^0 + g_{0\alpha}\ dx^\alpha/g_{00})/c = (g_{00})^{1/2}\ dx^0/c = \exp\nu\ dt/c,\ (x^0 = t;\ g_{00} = \exp(2\nu);\ dr, d\lambda = 0)$, being smaller by the factor $\exp\nu$ with respect to the proper time interval $dt/c,\ (g_{00} = 1)$ measured at infinity. Since the angular velocities Ω and ω are inversely proportional to the proper time $d\tau$, the relative angular velocity of a fluid element, as measured by the locally nonrotating observer, is $(\Omega - \omega)\exp(-\nu)$, in comparison to its value $\Omega - \omega$ measured from infinity.

From the metric (4.2.149) we get at once the circumference of a circle, measured at the locus of the locally nonrotating observer: $2\pi r\sin\lambda\exp(\gamma - \nu) = 2\pi R_p,\ (dt, dr, d\lambda = 0,\ d\varphi = 2\pi)$. In terms of the proper circumferential radius $R_p = r\sin\lambda\exp(\gamma - \nu)$ the local circumferential velocity of fluid elements relative to the locally nonrotating observer is (Bardeen 1970, Eq. (38); Bardeen and Wagoner 1971, p. 383):

$$v = R_p(\Omega - \omega)\exp(-\nu) = r(\Omega - \omega)\sin\lambda\ \exp(\gamma - 2\nu) = \ell(\Omega - \omega)\exp(\gamma - 2\nu). \qquad (4.2.150)$$

Note, that the proper circumferential radius R_p is for strong gravitational fields quite different from the proper linear radius obtained by integration of $\exp\kappa\ dr$ [cf. Eqs. (4.1.46), (4.1.47)].

The numerical treatment of rapidly rotating and fully relativistic polytropes is based on the projection of the field equations (4.1.4) onto the tetrad of Bardeen's (1970) zero angular momentum observer. A tetrad is formed by four linearly independent four-vectors $e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}$, their covariant and contravariant components being connected by (e.g. Synge 1976, Landau and Lifschitz 1987)

$$e_{(a)j}e^j_{(b)} = \eta_{ab}, \qquad (\eta_{ab} = \eta_{ba};\ a, b, j = 0, 1, 2, 3). \qquad (4.2.151)$$

The letters $a, b, c, ...$ distinguish the four four-vectors among themselves, and are put into parentheses, if they appear together with the common tensor indices $j, k, \ell, ...$ For instance, $(e^0_{(a)}, e^1_{(a)}, e^2_{(a)}, e^3_{(a)})$ denote the four contravariant components of the four-vector $e_{(a)}$. Similarly to the metric tensor from Eq. (4.1.10), the inverse η^{ab} of the covariant symmetric matrix η_{ab} is given by

$$\eta^{ac}\eta_{bc} = \delta^a_b. \qquad (4.2.152)$$

The reciprocal $e_j^{(a)}$ of the vector $e^j_{(b)}$ of a tetrad is introduced by the orthogonality condition

$$e_j^{(a)} e^j_{(b)} = \delta^a_b, \tag{4.2.153}$$

showing that each vector $e^j_{(b)}$ is orthogonal to its reciprocal $e_j^{(a)}$ if $a \neq b$. If we multiply Eq. (4.2.153) by $e^k_{(a)}$, we get $e^k_{(a)} e_j^{(a)} e^j_{(b)} = [e^k_{(a)} e_j^{(a)}] e^j_{(b)} = e^k_{(a)} \delta^a_b = e^k_{(b)}$, where summation extends separately over the repeated indices a and j, although the labels $a, b, c, ...$ of the tetrad vectors have generally no tensorial meaning. If the summation index j takes the particular value $j = k$, we observe that the factor in the bracket must be equal to 1, while the other sums, when $j \neq k$, must be zero, in order to assure equality of both sides. Thus

$$e^k_{(a)} e_j^{(a)} = \delta^k_j. \tag{4.2.154}$$

We multiply Eq. (4.2.151) by η^{ac}, taking into account Eq. (4.2.152):

$$\eta^{ac} e_{(a)j} e^j_{(b)} = \eta_{ab}\eta^{ac} = \delta^c_b. \tag{4.2.155}$$

Comparing with Eq. (4.2.153), written under the form $e_j^{(c)} e^j_{(b)} = \delta^c_b$, we observe that

$$e_j^{(c)} = \eta^{ac} e_{(a)j}. \tag{4.2.156}$$

Multiplying Eq. (4.2.156) by η_{bc}, and taking into account Eq. (4.2.152), we also find

$$e_{(b)j} = \eta_{bc} e_j^{(c)}. \tag{4.2.157}$$

The relationship between the metric tensor g_{jk} and the matrix η_{ab} is found by remembering the equation $e_j^{(a)} = g_{j\ell} e^{(a)\ell}$ between the covariant and contravariant components of the four-vector $e^{(a)}$; if this relationship is multiplied by $e_{(a)k}$, we get $e_j^{(a)} e_{(a)k} = g_{j\ell} e^{(a)\ell} e_{(a)k} = g_{j\ell}\delta^\ell_k = g_{jk}$ via Eq. (4.2.154). Thus, taking into account Eqs. (4.2.156), (4.2.157), we have

$$g_{jk} = e_j^{(a)} e_{(a)k} = \eta_{ab} e_j^{(a)} e_k^{(b)} = \eta^{ba} e_{(b)j} e_{(a)k} = \eta^{ab} e_{(a)j} e_{(b)k}, \qquad (g_{jk} = g_{kj};\ \eta^{ab} = \eta^{ba}). \tag{4.2.158}$$

With this equation, the metric (4.1.3) of Riemannian space writes

$$ds^2 = g_{jk}\, dx^j dx^k = \eta_{ab} e_j^{(a)} e_k^{(b)}\, dx^j dx^k = \eta^{ab} e_{(a)j} e_{(b)k}\, dx^j dx^k. \tag{4.2.159}$$

The arbitrary matrix η_{ab} can be chosen in Galilean form: $\eta_{00} = 1;\ \eta_{\alpha\alpha} = -1,\ (\alpha = 1, 2, 3);\ \eta_{ab} = 0$ if $a \neq b$. With this choice the vectors of a tetrad from Eq. (4.2.151) become orthogonal

$$e^i_{(0)} e_{(0)i} = 1; \quad e^i_{(\alpha)} e_{(\alpha)i} = -1; \quad e^i_{(a)} e_{(b)i} = 0 \quad \text{if} \quad a \neq b, \tag{4.2.160}$$

the vector $e_{(0)}$ being timelike, and the vectors $e_{(\alpha)}$ spacelike. With the previous choice of η_{ab} the components of the inverse matrix η^{ab} from Eq. (4.2.152) are $\eta^{00} = 1,\ \eta^{\alpha\alpha} = -1,\ \eta^{ab} = 0$ if $a \neq b$. Eq. (4.2.159) turns into

$$ds^2 = [e_{(0)j} e_{(0)k} - e_{(1)j} e_{(1)k} - e_{(2)j} e_{(2)k} - e_{(3)j} e_{(3)k}]\, dx^j dx^k. \tag{4.2.161}$$

Bardeen (1970, App. A) chooses the nonzero components of the tetrad vectors as follows ($x^0 = t,\ x^1 = \ell,\ x^2 = z,\ x^3 = \varphi$) :

$$e_{(0)0} = -\exp\nu; \quad e_{(3)0} = -\omega\ell\exp(\gamma - \nu); \quad e_{(1)1} = e_{(2)2} = \exp\kappa; \quad e_{(3)3} = \ell\exp(\gamma - \nu). \tag{4.2.162}$$

In this way he obtains just the metric (4.2.148) with

$$g_{00} = \exp(2\nu) - \omega^2\ell^2\exp[2(\gamma - \nu)]; \quad g_{03} = \omega\ell^2\exp[2(\gamma - \nu)]; \quad g_{11} = g_{22} = -\exp(2\kappa);$$
$$g_{33} = -\ell^2\exp[2(\gamma - \nu)]; \quad g = g_{11}g_{22}(g_{00}g_{33} - g_{03}^2) = -\ell^2\exp(4\kappa + 2\gamma). \tag{4.2.163}$$

The nonvanishing contravariant components of the tetrad follow at once via Eq. (4.2.160):

$$e^0_{(0)} = -\exp(-\nu);\quad e^3_{(0)} = -\omega\exp(-\nu);\quad e^1_{(1)} = e^2_{(2)} = -\exp(-\kappa);\quad e^3_{(3)} = -(1/\ell)\exp(\nu-\gamma). \tag{4.2.164}$$

The four-velocity (4.2.115) takes a simple form by inserting for the metric tensor from Eq. (4.2.148), and for the linear velocity with respect to the zero angular momentum observer from Eq. (4.2.150):

$$u^0 = u^t = 1/(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2)^{1/2} = 1/\big[\exp(2\nu) - \omega^2\ell^2\exp(2\gamma-2\nu) + 2\Omega\omega\ell^2\exp(2\gamma-2\nu)$$
$$-\Omega^2\ell^2\exp(2\gamma-2\nu)\big]^{1/2} = 1/(1-v^2)^{1/2}\exp\nu;\quad u^1 = u^\ell = 0;\quad u^2 = u^z = 0;\quad u^3 = u^\varphi = \Omega u^0. \tag{4.2.165}$$

The tetrad components of the energy-momentum tensor are calculated from the ordinary components (4.1.11) by

$$T_{(a)(b)} = e^j_{(a)}e^k_{(b)}T_{jk} = e^j_{(a)}e^k_{(b)}[(P+\varepsilon_r)u_ju_k - g_{jk}P], \tag{4.2.166}$$

where the covariant components of the four-velocity are via Eq. (4.2.165) equal to

$$u_0 = g_{0j}u^j = (g_{00}+\Omega g_{03})u^0 = [1+\omega v^2/(\Omega-\omega)]\exp\nu/(1-v^2)^{1/2};\quad u_1 = u_2 = 0;$$
$$u_3 = g_{3j}u^j = (g_{03}+\Omega g_{33})u^0 = -v^2\exp\nu/(\Omega-\omega)(1-v^2)^{1/2}. \tag{4.2.167}$$

Finally, from Eqs. (4.2.164)-(4.2.167) we obtain for the nonvanishing tetrad components of the energy-momentum tensor (Bardeen and Wagoner 1971):

$$T_{(0)(0)} = T^{(0)}_{(0)} = T^{(0)(0)} = (Pv^2+\varepsilon_r)/(1-v^2);\quad T_{(1)(1)} = -T^{(1)}_{(1)} = T^{(1)(1)} = P;$$
$$T_{(2)(2)} = -T^{(2)}_{(2)} = T^{(2)(2)} = P;\quad T_{(3)(3)} = -T^{(3)}_{(3)} = T^{(3)(3)} = (P+\varepsilon_r v^2)/(1-v^2);$$
$$T_{(0)(3)} = -T^{(3)}_{(0)} = -T^{(0)(3)} = -(P+\varepsilon_r)v/(1-v^2);\quad T = T^{(a)}_{(a)} = \varepsilon_r - 3P. \tag{4.2.168}$$

Lowering and raising of tetrad indices proceeds with the Galilean matrices η_{ab} and η^{ab}, respectively. The Einstein equations (4.1.4) can be written under covariant form as

$$R_{jk} = 8\pi(T_{jk} - g_{jk}T/2),\qquad (c, G = 1), \tag{4.2.169}$$

where $R = -8\pi T$, $(R = R^j_j;\ T = T^j_j)$. Since

$$R_{jk} = R_{(a)(b)}e^{(a)}_j e^{(b)}_k;\quad T_{jk} = T_{(a)(b)}e^{(a)}_j e^{(b)}_k;\quad g_{jk} = \eta_{ab}e^{(a)}_j e^{(b)}_k, \tag{4.2.170}$$

the tetrad form of Einstein's equations is simply

$$R_{(a)(b)} = 8\pi[T_{(a)(b)} - \eta_{ab}T/2],\qquad (c, G = 1). \tag{4.2.171}$$

Bardeen (1970), and Bardeen and Wagoner (1971) calculate several components of the Ricci tensor $R_{(a)(b)}$, and obtain by insertion into Eq. (4.2.171) differential equations for the four unknown metric functions $\nu, \omega, \gamma, \kappa$ from Eq. (4.2.148):

$$R_{(0)(0)} = \exp(-2\kappa)\big[\exp(-\gamma)\ \nabla\cdot(\exp\gamma\ \nabla\nu) - (1/2)\ell^2\exp(2\gamma-4\nu)\ \nabla\omega\cdot\nabla\omega\big]$$
$$= 4\pi\big[(P+\varepsilon_r)(1+v^2)/(1-v^2) + 2P\big], \tag{4.2.172}$$

$$R_{(0)(3)} = (1/2\ell)\exp[2(-\gamma+\nu-\kappa)]\ \nabla\cdot[\ell^2\exp(3\gamma-4\nu)\ \nabla\omega] = -8\pi(P+\varepsilon_r)v/(1-v^2), \tag{4.2.173}$$

$$R^{(0)}_{(0)} + R^{(3)}_{(3)} = (1/\ell)\exp(-\gamma-2\kappa)\ \nabla\cdot[\ell\ \nabla(\exp\gamma)] = 16\pi P. \tag{4.2.174}$$

Since no derivatives of the metric function κ appear in these three equations, Bardeen and Wagoner (1971) consider the sum $\kappa+\nu$, and determine κ from the two equations

$$[R_{(1)(1)}-R_{(2)(2)}]\exp(2\kappa)=(1/\ell)\ \partial(\kappa+\nu)/\partial\ell+(\partial\gamma/\partial\ell)\ \partial(\kappa+\nu)/\partial\ell-(\partial\gamma/\partial z)\ \partial(\kappa+\nu)/\partial z$$
$$-(1/2\ell^2)\exp(-\gamma)\ \partial[\ell^2\ \partial(\exp\gamma)/\partial\ell]/\partial\ell+(1/2\exp\gamma)\ \partial^2(\exp\gamma)/\partial z^2-(\partial\nu/\partial\ell)^2$$
$$+(\partial\nu/\partial z)^2+(1/4)\ell^2\exp(2\gamma-4\nu)\ [(\partial\omega/\partial\ell)^2-(\partial\omega/\partial z)^2]=0;$$

$$R_{(1)(2)}\exp(2\kappa)=(1/\ell)\ \partial(\kappa+\nu)/\partial z+(\partial\gamma/\partial\ell)\ \partial(\kappa+\nu)/\partial z+(\partial\gamma/\partial z)\ \partial(\kappa+\nu)/\partial\ell$$
$$-(1/2\ell^2\exp\gamma)\ \partial[\ell^2\ \partial(\exp\gamma)/\partial z]/\partial\ell-(1/2\exp\gamma)\ \partial^2(\exp\gamma)/\partial\ell\partial z-2(\partial\nu/\partial\ell)\ \partial\nu/\partial z$$
$$+(1/2)\ell^2\exp(2\gamma-4\nu)\ (\partial\omega/\partial\ell)\ \partial\omega/\partial z=0. \qquad (4.2.175)$$

If ν,ω,γ are determined from Eqs. (4.2.172)-(4.2.174) as functions of ℓ and z, the two equations (4.2.175), in any combination, allow κ to be found by quadratures. All nabla operators act in three-dimensional Euclidian space. The transition to spherical (r,μ,φ)-coordinates in Eqs. (4.2.172)-(4.2.174) is effected at once $(\mu=\cos\lambda;\ \ell=r(1-\mu^2)^{1/2};\ z=r\mu)$:

$$\nabla\cdot(\exp\gamma\ \nabla\nu)=(1/2)r^2(1-\mu^2)\exp(3\gamma-4\nu)\ \nabla\omega\cdot\nabla\omega$$
$$+4\pi\exp(\gamma+2\kappa)\ [(P+\varepsilon_r)(1+v^2)/(1-v^2)+2P], \qquad (4.2.176)$$

$$\nabla\cdot[r^2(1-\mu^2)\exp(3\gamma-4\nu)\ \nabla\omega]=-16\pi r(1-\mu^2)^{1/2}\exp[2(\gamma-\nu+\kappa)]\ (P+\varepsilon_r)v/(1-v^2), \qquad (4.2.177)$$

$$\nabla\cdot[r(1-\mu^2)^{1/2}\ \nabla(\exp\gamma)]=16\pi r(1-\mu^2)^{1/2}\exp(\gamma+2\kappa)\ P. \qquad (4.2.178)$$

From the two equations (4.2.175) we can determine $\partial\kappa/\partial r$ and $\partial\kappa/\partial\mu$. Since the equation for $\partial\kappa/\partial r$ contains no new information, Butterworth and Ipser (1976) quote merely the equation for $\partial\kappa/\partial\mu$:

$$\partial\kappa/\partial\mu=-\partial\nu/\partial\mu-\big\{\big[r^2\ \partial^2(\exp\gamma)/\partial r^2-\partial\big((1-\mu^2)\ \partial(\exp\gamma)/\partial\mu\big)/\partial\mu-2\mu\ \partial(\exp\gamma)/\partial\mu\big]$$
$$\times[-\mu+(1-\mu^2)\ \partial\gamma/\partial\mu]/2\exp\gamma+r(\partial\gamma/\partial r)\big[\mu/2+r\mu\ \partial\gamma/\partial r+(1-\mu^2)(\partial\gamma/\partial\mu)/2\big]$$
$$+3(\partial\gamma/\partial\mu)[-\mu^2+\mu(1-\mu^2)\ \partial\gamma/\partial\mu]/2-r(1-\mu^2)[\partial^2(\exp\gamma)/\partial r\partial\mu](1+r\ \partial\gamma/\partial r)/\exp\gamma$$
$$-r^2\mu(\partial\nu/\partial r)^2-2r(1-\mu^2)(\partial\nu/\partial r)\ \partial\nu/\partial\mu+\mu(1-\mu^2)(\partial\nu/\partial\mu)^2$$
$$-2r^2(1-\mu^2)(\partial\gamma/\partial r)(\partial\nu/\partial r)\ \partial\nu/\partial\mu+(1-\mu^2)(\partial\gamma/\partial\mu)[r^2(\partial\nu/\partial r)^2-(1-\mu^2)(\partial\nu/\partial\mu)^2]$$
$$+(1-\mu^2)\exp(2\gamma-4\nu)\big[(r^4\mu/4)(\partial\omega/\partial r)^2+(r^3/2)(1-\mu^2)(\partial\omega/\partial r)\ \partial\omega/\partial\mu$$
$$-(r^2\mu/4)(1-\mu^2)(\partial\omega/\partial\mu)^2+(r^4/2)(1-\mu^2)(\partial\gamma/\partial r)(\partial\omega/\partial r)\ \partial\omega/\partial\mu$$
$$-(r^2/4)(1-\mu^2)(\partial\gamma/\partial\mu)\big(r^2(\partial\omega/\partial r)^2-(1-\mu^2)(\partial\omega/\partial\mu)^2\big)\big]\big\}$$
$$/\big\{(1-\mu^2)(1+r\ \partial\gamma/\partial r)^2+\big[\mu-(1-\mu^2)\ \partial\gamma/\partial\mu\big]^2\big\}. \qquad (4.2.179)$$

Komatsu et al. (1989a) and Nishida et al. (1992) transform Eqs. (4.2.176)-(4.2.178) further:

$$\nabla^2[(2\nu-\gamma)\exp(\gamma/2)]=\exp(\gamma/2)\big\{8\pi(P+\varepsilon_r)(1+v^2)\exp(2\kappa)/(1-v^2)+8\pi P(2\nu-\gamma)\exp(2\kappa)$$
$$+r^2(1-\mu^2)\exp(2\gamma-4\nu)\ \nabla\omega\cdot\nabla\omega+(1-\nu+\gamma/2)\big[(1/r)\ \partial\gamma/\partial r-(\mu/r^2)\ \partial\gamma/\partial\mu\big]$$
$$+(-\nu/2+\gamma/4)\ \nabla\gamma\cdot\nabla\gamma\big\}=S_\nu, \qquad (4.2.180)$$

$$\nabla^2\omega+(2/r)\ \partial\omega/\partial r-(2\mu/r^2)\ \partial\omega/\partial\mu=\nabla(4\nu-3\gamma)\cdot\nabla\omega$$
$$-16\pi(P+\varepsilon_r)(\Omega-\omega)\exp(2\kappa)/(1-v^2)=S_\omega, \qquad (4.2.181)$$

$$\nabla^2[\gamma\exp(\gamma/2)]+(1/r)\ \partial[\gamma\exp(\gamma/2)]/\partial r-(\mu/r^2)\ \partial[\gamma\exp(\gamma/2)]/\partial\mu$$
$$=\exp(\gamma/2)\ \big[16\pi P(1+\gamma/2)\exp(2\kappa)-(\gamma/4)\ \nabla\gamma\cdot\nabla\gamma\big]=S_\gamma. \qquad (4.2.182)$$

These three equations can be deduced by observing that Eqs. (4.2.176) and (4.2.178) are equivalent to

$$\nabla^2\nu + \nabla\nu\cdot\nabla\gamma = (1/2)r^2(1-\mu^2)\exp(2\gamma-4\nu)\ \nabla\omega\cdot\nabla\omega$$
$$+4\pi\exp(2\kappa)\ [(P+\varepsilon_r)(1+v^2)/(1-v^2)+2P], \tag{4.2.183}$$

and

$$\nabla^2\gamma + (1/r)\ \partial\gamma/\partial r - (\mu/r^2)\ \partial\gamma/\partial\mu + \nabla\gamma\cdot\nabla\gamma = 16\pi P\exp(2\kappa). \tag{4.2.184}$$

On the other hand, the Laplace operators from Eqs. (4.2.180), (4.2.182) can be evaluated according to Eqs. (B.36)-(B.39) as follows:

$$\nabla^2[\nu\exp(\gamma/2)] = \exp(\gamma/2)\ \big[(\nu/2)\ \nabla^2\gamma + \nabla^2\nu + (\nu/4)\ \nabla\gamma\cdot\nabla\gamma + \nabla\nu\cdot\nabla\gamma\big], \tag{4.2.185}$$

$$\nabla^2[\gamma\exp(\gamma/2)] = \exp(\gamma/2)\ \big[(1+\gamma/2)\ \nabla^2\gamma + (1+\gamma/4)\ \nabla\gamma\cdot\nabla\gamma\big]. \tag{4.2.186}$$

Inserting for $\nabla^2\gamma$ and $\nabla^2\nu$ from Eqs. (4.2.183) and (4.2.184), we indeed obtain Eqs. (4.2.180) and (4.2.182). The equation (4.2.181) for ω results by evaluating the divergence in Eq. (4.2.177), and inserting for v from Eq. (4.2.150).

The condition of asymptotic flatness ($\nu,\omega,\gamma,\kappa\to 0$ if $r\to\infty$) implies that the metric potentials ν,ω,γ have particular expansions in terms of $1/r^j$, of Legendre polynomials $P_{2k}(\mu)$, $dP_{2k+1}(\mu)/d\mu$, and of Gegenbauer polynomials $G^1_{2k}(\mu)$ of order $2k$ and index 1 [cf. Eqs. (3.10.155), (3.10.156)]. These expansions are suggested by the angular parts of the differential operators in Eqs. (4.2.176)-(4.2.178). The asymptotic forms are (Butterworth and Ipser 1976, Eriguchi 1980):

$$\nu = O(1/r);\quad \omega = O(1/r^3);\quad \gamma = O(1/r^2),\qquad (r\to\infty). \tag{4.2.187}$$

These boundary expansions result from the leading terms of Eqs. (4.2.198)-(4.2.200), by converting Eqs. (4.2.180)-(4.2.182) to the integral form with the two- and three-dimensional Green functions (e.g. Smirnow 1967):

$$F(x,y) = -(1/2\pi)\int_S f(x',y')\ \ln\big(1/|\vec r-\vec{r'}|\big)\ dx'\ dy';$$
$$F(x,y,z) = -(1/4\pi)\int_V f(x',y',z')\ dx'\ dy'\ dz'/|\vec r-\vec{r'}|. \tag{4.2.188}$$

F and f are connected to Poisson's equation by $\nabla^2 F = f$, where $|\vec r-\vec{r'}| = [(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}$. From Eq. (4.2.180) we obtain via Eq. (4.2.188):

$$(2\nu-\gamma)\exp(\gamma/2) = -(1/4\pi)\int_0^\infty dr'\int_{-1}^1 d\mu'\int_0^{2\pi} r'^2 S_\nu(r',\mu')\ d\varphi'/|\vec r-\vec{r'}|. \tag{4.2.189}$$

Eq. (4.2.182) is transformed by Komatsu et al. (1989a) to the two-dimensional form with the cylindrical coordinates $\ell = r\sin\lambda$, $z = r\cos\lambda$, the Laplace operator being calculated according to Eq. (B.48). After some algebra, Eq. (4.2.182) becomes

$$\nabla^2[\ell\gamma\exp(\gamma/2)] = \partial^2[\ell\gamma\exp(\gamma/2)]/\partial\ell^2 + \partial^2[\ell\gamma\exp(\gamma/2)]/\partial z^2 = \ell S_\gamma,\qquad (\varphi=\mathrm{const}). \tag{4.2.190}$$

We apply the two-dimensional Green function (4.2.188) to this equation:

$$\ell\gamma\exp(\gamma/2) = r\gamma\exp(\gamma/2)\sin\lambda = -(1/2\pi)\int_0^\infty d\ell'\int_0^\infty \ell' S_\gamma(\ell',z')\ \ln\big(1/|\vec r-\vec{r'}|\big)\ dz'$$
$$= -(1/2\pi)\int_0^\infty dr'\int_0^{2\pi} r'^2\sin\lambda'\ S_\gamma(r',\lambda')\ \ln\big(1/|\vec r-\vec{r'}|\big)\ d\lambda', \tag{4.2.191}$$

where Komatsu et al. (1989a) analytically continue $S_\gamma(r',\lambda')$ into the range $\pi<\lambda'\le 2\pi$, by defining $S_\gamma(r',\lambda'-\pi) = S_\gamma(r',\lambda')$. After multiplication by $r\sin\lambda\cos\varphi$, Eq. (4.2.181) can be written as [cf. Eq. (B.39)]

$$\nabla^2(r\omega\sin\lambda\cos\varphi) = r\sin\lambda\cos\varphi\ S_\omega, \tag{4.2.192}$$

or in integral form via Eq. (4.2.188):

$$r\omega \sin\lambda \cos\varphi = -(1/4\pi) \int_0^\infty dr' \int_0^\pi d\lambda' \int_0^{2\pi} r'^3 \sin^2\lambda' \cos\varphi' \ S_\omega(r',\lambda') \ d\varphi'/|\vec{r}-\vec{r'}|. \tag{4.2.193}$$

The distance $|\vec{r}-\vec{r'}|$ from Eq. (4.2.188) can be expanded as (cf. Eqs. (3.1.44), (3.1.48)-(3.1.52), Komatsu et al. 1989a)

$$\ln|\vec{r}-\vec{r'}| = \ln[r^2 + r'^2 - 2rr'\cos(\lambda'-\lambda)]^{1/2}$$
$$= g(r,r') - \sum_{j=1}^{\infty}(1/j) \ h_j(r,r') \ [\cos(j\lambda)\cos(j\lambda') + \sin(j\lambda)\sin(j\lambda')], \tag{4.2.194}$$

where

$$g(r,r') = \begin{cases} \ln r \\ \ln r' \end{cases} ; \qquad h_j(r,r') = \begin{cases} (r'/r)^j \\ (r/r')^j \end{cases} \quad \text{if} \quad \begin{matrix} r \geq r' \\ r \leq r' \end{matrix} \tag{4.2.195}$$

and [cf. (Eq. (3.1.53)]

$$1/|\vec{r}-\vec{r'}| = (1/r)\sum_{j=0}^{\infty}\sum_{k=0}^{j} f_j(r,r') \ \big[2(j-k)!/\delta_k(j+k)!\big] \ P_j^k(\cos\lambda) \ P_j^k(\cos\lambda')\cos[k(\varphi-\varphi')], \tag{4.2.196}$$

with

$$f_j(r,r') = \begin{cases} (1/r)(r'/r)^j \\ (1/r')(r/r')^j \end{cases} \quad \text{if} \quad \begin{matrix} r \geq r' \\ r \leq r' \end{matrix} ; \qquad \delta_k = \begin{cases} 2 \\ 1 \end{cases} \quad \text{if} \quad \begin{matrix} k = 0 \\ k > 0 \end{matrix} \tag{4.2.197}$$

We insert Eqs. (4.2.194)-(4.2.197) into Eqs. (4.2.189), (4.2.191), (4.2.193), taking into account the symmetry properties of trigonometric and Legendre functions. Eq. (4.2.189) reads

$$2\nu - \gamma = -\exp(-\gamma/2) \int_0^\infty dr' \int_0^{\pi/2} r'^2 S_\nu(r',\lambda') \left[\sum_{j=0}^{\infty} f_{2j}(r,r') \ P_{2j}(\cos\lambda) \ P_{2j}(\cos\lambda')\right] \sin\lambda' \ d\lambda', \tag{4.2.198}$$

since $P_{2j+1}(\cos\lambda')$ contains only odd powers of $\cos\lambda'$, that cancel out when integrated over the interval $[0,\pi]$. The sole nonvanishing terms in Eq. (4.2.191) are of the form

$$r\gamma \sin\lambda = -(2/\pi)\exp(-\gamma/2) \int_0^\infty dr' \int_0^{\pi/2} r'^2 S_\gamma(r',\lambda')$$
$$\times\left\{\sum_{j=1}^{\infty}[1/(2j-1)] \ h_{2j-1}(r,r') \ \sin[(2j-1)\lambda] \ \sin[(2j-1)\lambda']\right\} \sin\lambda' \ d\lambda', \tag{4.2.199}$$

because products like $\cos(2j\lambda')\sin\lambda'$ and $\sin(2j\lambda')\sin\lambda'$ cancel out when integrated between 0 and 2π.

Likewise, Eq. (4.2.193) becomes

$$r\omega\sin\lambda = -\int_0^\infty dr' \int_0^{\pi/2} r'^3 S_\omega(r',\lambda')$$
$$\times\sum_{j=1}^{\infty} f_{2j-1}(r,r') \ P_{2j-1}^1(\cos\lambda) \ P_{2j-1}^1(\cos\lambda')\sin^2\lambda' \ d\lambda'/2j(2j-1), \tag{4.2.200}$$

where the factor $(j-k)!/(j+k)!$ from Eq. (4.2.196) is equal to $1/2j(2j-1)$ if $k=1$ and $j \to 2j-1$. In Eq. (4.2.193) the sole surviving Legendre polynomials, connected with the integration of $\cos\varphi'\cos[k(\varphi-\varphi')]$, are the associated polynomials $P_j^k(\cos\lambda')$ of index $k=1$, when

$$\int_0^{2\pi} \cos\varphi' \cos(\varphi-\varphi') \ d\varphi' = \pi\cos\varphi. \tag{4.2.201}$$

The integrals of $P_{2j}^1(\cos\lambda') = P_{2j}^1(\mu')$ vanish, because in virtue of Eq. (3.1.39) $P_{2j}^1(\mu') = (1-\mu'^2)^{1/2}\ dP_{2j}(\mu')/d\mu'$, and $dP_{2j}(\mu')/d\mu'$ contains only odd powers of $\mu' = \cos\lambda'$, which cancel out over the interval $[0,\pi]$.

Subsequently we derive the equation of hydrostatic equilibrium for relativistic, axially symmetric, rotating hydrostatic configurations [cf. Bardeen 1970, Eq. (23); Bardeen and Wagoner 1971, Eq. (II.19); Butterworth and Ipser 1976, Eq. (7c); Nishida et al. 1992, Eq. (3.9)]. We start with the vanishing divergence (4.1.14) of the energy-momentum tensor

$$\nabla_k T_j^k = \nabla_k[(P+\varepsilon_r)u_j u^k - P\delta_j^k] = \nabla_k[(P+\varepsilon_r)u^k]u_j + (P+\varepsilon_r)u^k\ \nabla_k u_j - \nabla_j P = 0. \qquad (4.2.202)$$

The covariant divergence of u^k (e.g. Landau and Lifschitz 1987)

$$\nabla_k u^k = (-g)^{-1/2}\ \partial[(-g)^{1/2}u^k]/\partial x^k, \qquad (4.2.203)$$

vanishes for our particular metric (4.2.149), where all quantities depend only on $x^1 = r$ and $x^2 = \lambda$. Also $g = g_{11}g_{22}(g_{00}g_{33} - g_{03}^2) = -r^4\sin^2\lambda\exp(4\kappa+2\gamma)$, and $u^1, u^2 = 0$. The second term in Eq. (4.2.202) can be transformed as follows:

$$u^k\ \nabla_k u_j = (\partial u_j/\partial x^k - \Gamma_{jk}^\ell u_\ell)u^k = u^k\ \partial u_j/\partial x^k - (1/2)(\partial g_{jm}/\partial x^k + \partial g_{km}/\partial x^j - \partial g_{jk}/\partial x^m)u^k u^m$$
$$= u^k\ \partial u_j/\partial x^k - (1/2)u^k u^m\ \partial g_{km}/\partial x^j. \qquad (4.2.204)$$

Since for the considered particular coordinate dependences we have $\nabla_k[(P+\varepsilon_r)u^k] = 0$ and $u^k\ \partial u_j/\partial x^k = 0$, we can write instead of Eq. (4.2.202), (cf. Komatsu et al. 1989a, Nishida et al. 1992):

$$(1/2)(P+\varepsilon_r)u^k u^m\ \partial g_{km}/\partial x^j + \partial P/\partial x^j = (1/2)(P+\varepsilon_r)(u^0)^2(\partial g_{00}/\partial x^j + 2\Omega\ \partial g_{03}/\partial x^j$$
$$+\Omega^2\ \partial g_{33}/\partial x^j) + \partial P/\partial x^j = (1/2)(P+\varepsilon_r)(u^0)^2\big[\partial(g_{00} + 2\Omega g_{03} + \Omega^2 g_{33})/\partial x^j$$
$$-2(g_{03} + \Omega g_{33})\ \partial\Omega/\partial x^j\big] + \partial P/\partial x^j = (1/2)(P+\varepsilon_r)(u^0)^2\big[\partial(u^0)^{-2}/\partial x^j$$
$$+2(\Omega-\omega)r^2\sin^2\lambda\exp(2\gamma-2\nu)\ \partial\Omega/\partial x^j\big] + \partial P/\partial x^j$$
$$= (P+\varepsilon_r)\big\{\partial\nu/\partial x^j - [v/(1-v^2)]\ \partial v/\partial x^j + [v^2/(1-v^2)(\Omega-\omega)]\ \partial\Omega/\partial x^j\big\} + \partial P/\partial x^j$$
$$= (P+\varepsilon_r)[\nabla\nu - v\ \nabla v/(1-v^2) + v^2\ \nabla\Omega/(1-v^2)(\Omega-\omega)] + \nabla P = 0, \qquad (4.2.205)$$

where we have used Eqs. (4.2.149), (4.2.150), (4.2.165). The integrability condition of Eq. (4.2.205) is that the factor near $\nabla\Omega$ is a function of Ω only (Butterworth and Ipser 1976):

$$f(\Omega) = v^2/(1-v^2)(\Omega-\omega) = r^2(\Omega-\omega)\sin^2\lambda\exp(2\gamma-4\nu)/[1 - r^2(\Omega-\omega)^2\sin^2\lambda\exp(2\gamma-4\nu)]$$
$$= -u^0 u_3 = -u^t u_\varphi. \qquad (4.2.206)$$

Komatsu et al. (1989a) take

$$f(\Omega) = A^2(\Omega_0 - \Omega), \qquad (A, \Omega_0 = \mathrm{const}). \qquad (4.2.207)$$

Ω_0 is the angular velocity along the rotation axis $\lambda = 0, \pi$. We insert into Eq. (4.2.206) for $f(\Omega)$ from Eq. (4.2.207), and observe that as $A \to \infty$, the finiteness of $f(\Omega)$ requires that $\Omega_0 - \Omega \to 0$, i.e. $\Omega = \Omega_0 = \mathrm{const}$ (rigid rotation). In the other limiting case $A \to 0$, it would seem from Eq. (4.2.206) that $\omega \to \Omega$, which is clearly impossible, since this would lead to the absurd condition that the dragging potential ω is equal to the angular velocity Ω, even if gravity is weak (ω small), whereas Ω could be large. Thus, $f(\Omega) = A^2(\Omega_0 - \Omega)$ cannot vanish, so $\Omega_0 - \Omega \propto 1/A^2$ if $A \to 0$.

In the Newtonian limit we have $v \ll c = 1$ and $\nu, \omega, \gamma, \kappa \approx 0$. In this case Eqs. (4.2.206) and (4.2.207) yield

$$f(\Omega) = A^2(\Omega_0 - \Omega) \approx v^2/\Omega = \Omega r^2\sin^2\lambda \quad \text{or} \quad \Omega/\Omega_0 \approx A^2/(A^2 + r^2\sin^2\lambda), \quad (v \ll 1). \qquad (4.2.208)$$

In the Newtonian limit from Eq. (4.2.208) we get again rigid rotation ($\Omega = \Omega_0$) if $A \to \infty$, whereas in the case $A \to 0$ we have $\Omega r^2\sin^2\lambda = \Omega\ell^2 \approx A^2\Omega_0 = \mathrm{const}$. This implies rotation with constant specific angular momentum (cf. Eq. (3.8.84) and Fig. 4.2.7).

Taking the light velocity as unit, we write $\varepsilon_r = c^2\varrho_r = \varrho_r$, and the polytropic equation (4.1.1) can be written as

$$P = K\varepsilon_r^{1+1/n}, \qquad (c = 1). \tag{4.2.209}$$

If Eq. (4.2.206) subsists, the hydrostatic equation (4.2.205) can be integrated at once in virtue of Eq. (4.2.165), (Butterworth and Ipser 1976, Komatsu et al. 1989a):

$$-\ln[u^0/(u^0)_p] + \int_0^P dP'/(P' + \varepsilon_r) + \int_{\Omega_0}^{\Omega} f(\Omega')\, d\Omega'$$
$$= -\ln u^0 - \nu_p + (n+1)\ln(1 + P/\varepsilon_r) + \int_{\Omega_0}^{\Omega} f(\Omega')\, d\Omega' = 0. \tag{4.2.210}$$

$(u^0)_p = \exp(-\nu_p)$ and Ω_0 denote the values of $u^0 = 1/(1-v^2)^{1/2}\exp\nu$ and Ω at the surface pole of the rotating polytrope, where $P, \lambda, v = 0$. For constant angular velocity $\Omega = \Omega_0$ the hydrostatic equation (4.2.205) can be transformed further, by inserting for u^0 from Eq. (4.2.165), (Butterworth 1976)

$$P = \varepsilon_r\big[\exp(\nu_p - \nu)/(1-v^2)^{1/2}\big]^{1/(n+1)} - 1, \qquad (\Omega = \mathrm{const}), \tag{4.2.211}$$

and finally, by eliminating ε_r with the aid of Eq. (4.2.209):

$$P = K^{-n}\big\{\big[\exp(\nu_p - \nu)/(1-v^2)^{1/2}\big]^{1/(n+1)} - 1\big\}^{n+1}. \tag{4.2.212}$$

In general relativity some arbitrariness occurs concerning the definition of energy, angular momentum, and volume of a configuration, since in the general case no energy conservation equations exist for field and matter, both (Landau and Lifschitz 1987). The proper volume (volume of rest) of a rotating, axisymmetric relativistic polytrope may be defined as (Stephani 1977, Komatsu et al. 1989a)

$$V_1 = \int_{V_1} u^0(-g)^{1/2}\, dx^1dx^2dx^3 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\lambda \int_0^{r_1(\lambda)} u^0\exp(2\kappa + \gamma)\, r^2\sin\lambda\, dr$$
$$= 2\pi\int_0^{\pi} d\lambda \int_0^{r_1(\lambda)} (1-v^2)^{-1/2}\exp(2\kappa + \gamma - \nu)\, r^2\sin\lambda\, dr. \tag{4.2.213}$$

Integration proceeds over the spacelike hypersurface $x^0 = \mathrm{const}$, and we have inserted for u^0 from Eq. (4.2.165), $[g = -r^4\sin^2\lambda\exp(4\kappa + 2\gamma)]$. In the static nonrotating case we have $g = g_{00}g_{11}g_{22}g_{33}$, $u^\alpha = 0$, and Eq. (4.1.12) becomes $u^0u_0 = g_{00}(u^0)^2 = 1$, or $u^0 = (g_{00})^{-1/2}$. The proper volume (4.2.213) turns into the proper volume of the Schwarzschild sphere [cf. Eq. (4.1.53)]:

$$V_1 = \int_{V_1} (-g_{11}g_{22}g_{33})^{1/2}\, dx^1dx^2dx^3 = 4\pi\int_0^{r_1} \exp(2\kappa + \gamma - \nu)\, r^2\, dr, \qquad (\Omega = 0). \tag{4.2.214}$$

In the same way as in Eq. (4.2.213) we write the rest energy E and the proper energy E_{pr} of the rotating relativistic polytrope under the form $(c = 1)$

$$E = M_1 = \int_{V_1} \varepsilon u^0(-g)^{1/2}\, dx^1dx^2dx^3 = 2\pi\int_0^{\pi} d\lambda\int_0^{r_1(\lambda)} \varepsilon(1-v^2)^{-1/2}\exp(2\kappa+\gamma-\nu)\, r^2\sin\lambda\, dr$$
$$= 2\pi\int_0^{\pi} d\lambda\int_0^{r_1(\lambda)} [\varepsilon_r/(1 + P/\varepsilon_r)^n](1-v^2)^{-1/2}\exp(2\kappa+\gamma-\nu)\, r^2\sin\lambda\, dr, \tag{4.2.215}$$

$$E_{pr} = M_{pr} = \int_{V_1} \varepsilon_r u^0(-g)^{1/2}\, dx^1dx^2dx^3 = E + \int_{V_1} \varepsilon^{(int)}u^0(-g)^{1/2}\, dx^1dx^2dx^3$$
$$= 2\pi\int_0^{\pi} d\lambda\int_0^{r_1(\lambda)} \varepsilon_r(1-v^2)^{-1/2}\exp(2\kappa+\gamma-\nu)\, r^2\sin\lambda\, dr. \tag{4.2.216}$$

M_1 and M_{pr} denote the rest mass and proper mass of the rotating polytrope, respectively. In Eq. (4.2.215) we have also used the relationship $\varepsilon = \varepsilon_r/(1 + P/\varepsilon_r)^n$, which follows from the combination of Eq. (4.2.209) with the isentropic first law of thermodynamics (4.1.57):

$$d\varrho/\varrho = d\varepsilon/\varepsilon = d\varepsilon_r/(P + \varepsilon_r) = d\varepsilon_r/\varepsilon_r(1 + K\varepsilon_r^{1/n}). \tag{4.2.217}$$

This equation integrates at once:

$$\varepsilon = C\varepsilon_r/(1 + K\varepsilon_r^{1/n})^n = C\varepsilon_r/(1 + P/\varepsilon_r)^n, \qquad (C = \mathrm{const}). \tag{4.2.218}$$

The integration constant is determined by observing that near the surface, where relativistic effects are small, we must have $\varepsilon_r \approx \varepsilon$ and $P = 0:\ C = 1$, and [cf. Eqs. (4.1.59)-(4.1.64)]

$$\varepsilon = \varepsilon_r/(1 + P/\varepsilon_r)^n. \tag{4.2.219}$$

With the light velocity taken as unit ($c = 1$), the total angular momentum J round the rotation axis of the polytrope may be represented as (cf. Hartle and Sharp 1967, Bardeen and Wagoner 1971)

$$J = -\int_{V_1} T_3^0(-g)^{1/2}\ dx^1dx^2dx^3 = -\int_{V_1}(P+\varepsilon_r)u_3u^0(-g)^{1/2}\ dx^1dx^2dx^3 = \int_{M_1} j\ dM$$
$$= -\int_{M_1}[(P+\varepsilon_r)u_3/\varepsilon]\ dM = 2\pi\int_0^\pi d\lambda\int_0^{r_1(\lambda)}(P+\varepsilon_r)[v/(1-v^2)]\exp(2\kappa+2\gamma-2\nu)\ r^3\sin^2\lambda\ dr. \tag{4.2.220}$$

$j = -(P + \varepsilon_r)u_3/\varepsilon$ may be regarded as the angular momentum per unit rest mass; $dM = \varepsilon u^0(-g)^{1/2}\ dx^1dx^2dx^3 = \varepsilon(1 - v^2)^{-1/2}\exp(2\kappa + \gamma - \nu)\ r^2\sin\lambda\ dr\ d\lambda\ d\varphi$ is the rest mass element, and u^0, u_3 are inserted from Eqs. (4.2.165), (4.2.167), respectively. In the nonrelativistic limit the quantity $P + \varepsilon_r$ turns into the rest energy density ε, whereas u_3 from Eq. (4.2.167) becomes $u_3 = -v^2\exp\nu/(\Omega-\omega)(1-v^2)^{1/2} = -vr\sin\lambda\exp(\gamma-\nu)/(1-v^2)^{1/2} \approx -vr\sin\lambda,\ (\nu,\gamma,v\to 0)$. So, $j \approx -u_3$ is just equal to the axial Newtonian angular momentum $vr\sin\lambda$ per unit rest mass.

The total relativistic energy of the stationary rotating polytrope – including the energy of the stationary gravitational field – can be determined from Tolman's formula through the energy-momentum tensor T_j^k alone (cf. Eq. (4.1.69); Landau and Lifschitz 1987, §105; Nishida et al. 1992):

$$E_{r1} = M_{r1} = \int_{V_1}(T_0^0 - T_1^1 - T_2^2 - T_3^3)(-g)^{1/2}\ dx^1dx^2dx^3$$
$$= 2\pi\int_0^\pi d\lambda\int_0^{r_1(\lambda)}\big\{[(P+\varepsilon_r)/(1-v^2)][1+v^2(\Omega+\omega)/(\Omega-\omega)]+2P\big\}\exp(2\kappa+\gamma)\ r^2\sin\lambda\ dr$$
$$= 2\pi\int_0^\pi d\lambda\int_0^{r_1(\lambda)}\big\{[(P+\varepsilon_r)/(1-v^2)][1+v^2+2\omega vr\sin\lambda\exp(\gamma-2\nu)]+2P\big\}$$
$$\times\exp(2\kappa+\gamma)\ r^2\sin\lambda\ dr, \qquad [c=1;\ r\sin\lambda\exp(\gamma-2\nu) = v/(\Omega-\omega)]. \tag{4.2.221}$$

We have substituted Eqs. (4.1.11), (4.2.165), (4.2.167):

$$T_0^0 = (P+\varepsilon_r)[1+\omega v^2/(\Omega-\omega)]/(1-v^2) - P; \quad T_1^1 = T_2^2 = -P;$$
$$T_3^3 = -(P+\varepsilon_r)\Omega v^2/(\Omega-\omega)(1-v^2) - P. \tag{4.2.222}$$

The relativistic energy (4.2.221) becomes (Bardeen 1970, Nishida et al. 1992)

$$E_{r1} = M_{r1} = (1/2)\int_0^\pi d\lambda\int_0^{r_1(\lambda)}\big\{\nabla\cdot(\exp\gamma\ \nabla\nu) - (1/2)r^2\sin^2\lambda\exp(3\gamma-4\nu)\ \nabla\omega\cdot\nabla\omega$$
$$-(1/2)\omega\ \nabla\cdot\big[r^2\sin^2\lambda\exp(3\gamma-4\nu)\ \nabla\omega\big]\big\}r^2\sin\lambda\ dr$$
$$= (1/2)\int_0^\pi d\lambda\int_0^{r_1(\lambda)}\nabla\cdot\big[\exp\gamma\ \nabla\nu - (1/2)r^2\sin^2\lambda\exp(3\gamma-4\nu)\ \omega\ \nabla\omega\big]r^2\sin\lambda\ dr, \tag{4.2.223}$$

if we insert from the basic equations (4.2.176), (4.2.177).

An equivalent equation has been given by Bardeen and Wagoner [1971, Eq. (II.28)], as an integral

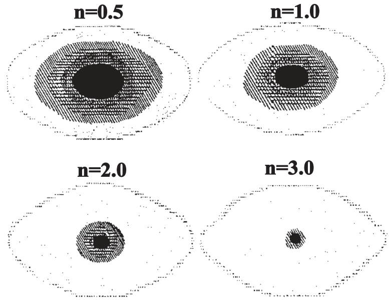

Fig. 4.2.1 Contour plots of isobaric surfaces ($P=$ const) for critical rotation with $K=P/\varepsilon_r^{1+1/n}=0.1$. Equally shaded areas represent a 20% range of pressure. If $n<2.5$, the relativistic polytropes are more spherical, and if $n>2.5$, they are more flattened in comparison to their Newtonian counterparts having the same angular velocity and central rest mass density (Butterworth 1976).

over the rest mass M_1 of the stationary rotating polytrope

$$\begin{aligned} E_{r1} = M_{r1} &= \int_{M_1} [(3P+\varepsilon_r)/\varepsilon u^0 + 2\Omega j]\ dM \\ &= \int_{V_1} [(P+\varepsilon_r) - 2\Omega(P+\varepsilon_r)u_3u^0 + 2P](-g)^{1/2}\ dx^1dx^2dx^3 \\ &= 2\pi\int_0^\pi d\lambda \int_0^{r_1(\lambda)} \big\{(P+\varepsilon_r)[1+2\Omega v^2/(\Omega-\omega)(1-v^2)] + 2P\big\} \exp(2\kappa+\gamma)\ r^2\sin\lambda\ dr \\ &= 2\pi\int_0^\pi d\lambda \int_0^{r_1(\lambda)} \big\{[(P+\varepsilon_r)/(1-v^2)][1+v^2(\Omega+\omega)/(\Omega-\omega)] + 2P\big\} \exp(2\kappa+\gamma)\ r^2\sin\lambda\ dr, \end{aligned} \tag{4.2.224}$$

which is the same as Eq. (4.2.221). Komatsu et al. (1989a) define, in analogy to the nonrelativistic case, the relativistic kinetic energy of rotation of the polytrope as [cf. Eq. (4.2.220)]

$$E_{r,kin} = (1/2)\int_{M_1} \Omega\ dJ = \pi\int_0^\pi d\lambda \int_0^{r_1(\lambda)} \Omega(P+\varepsilon_r)[v/(1-v^2)]\exp(2\kappa+2\gamma-2\nu)\ r^3\sin^2\lambda\ dr. \tag{4.2.225}$$

The total relativistic energy of a stationary polytrope (including the stationary gravitational field energy) is equal to the sum of proper energy, gravitational energy, and rotational energy [cf. Eqs. (2.6.98), (4.1.73), Butterworth 1976, Eq. (38)]:

$$E_{r1} = E_{pr} + W_r + E_{r,kin}. \tag{4.2.226}$$

The mean relativistic energy density is given by

$$\varepsilon_{r,m} = M_{pr}/V_1, \tag{4.2.227}$$

where proper mass and proper volume may be inserted via Eqs. (4.2.216) and (4.2.213), respectively.

Butterworth (1976) seems to have been the first who computed fully relativistic polytropes of arbitrary high uniform rotation with polytropic indices $n=0.5, 0.8, 1, 1.5, 2, 2.5, 3$ (Fig. 4.2.1). He integrates the basic equations (4.2.176)-(4.2.179) for the metric functions $\nu, \omega, \gamma, \kappa$ with a relativistic generalization of Stoeckly's (1965) method (cf. Sec. 3.8.2).

Butterworth and Ipser (1976) have performed a comprehensive study of the homogeneous polytrope $n=0$. The eccentricity e is calculated according to

$$e = (r_{re}^2 - r_{rp}^2)^{1/2}/r_{re}, \tag{4.2.228}$$

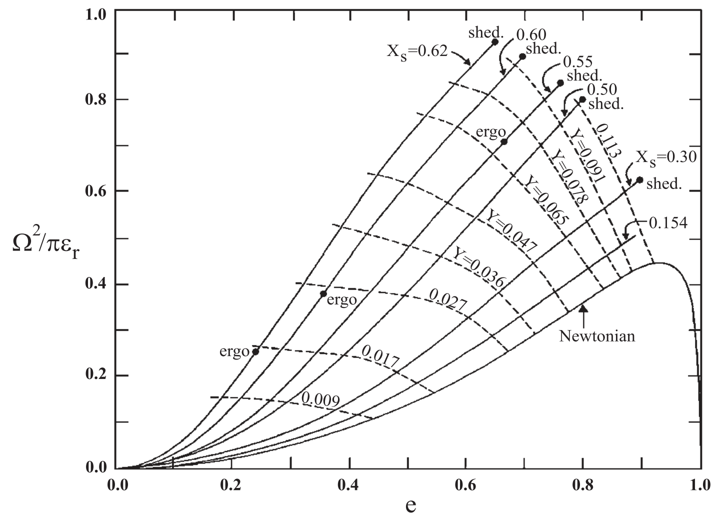

Fig. 4.2.2 Plot of angular velocity Ω versus eccentricity e for homogeneous polytropes $n = 0$. The bottom solid curve is the Newtonian Maclaurin sequence. The other solid curves are connected with particular values of X_S from Eq. (4.2.230). Dashed curves are obtained for a constant value of the rotation parameter $Y = J^2\varepsilon_r^{1/3}/M_1^{10/3}$ from Eq. (4.2.231). At the points marked with "shed", mass shedding from the equatorial bulge occurs. Above the points marked "ergo", ergoregions appear on each sequence, where any observer must rotate with positive angular velocity with respect to infinity (Butterworth and Ipser 1976).

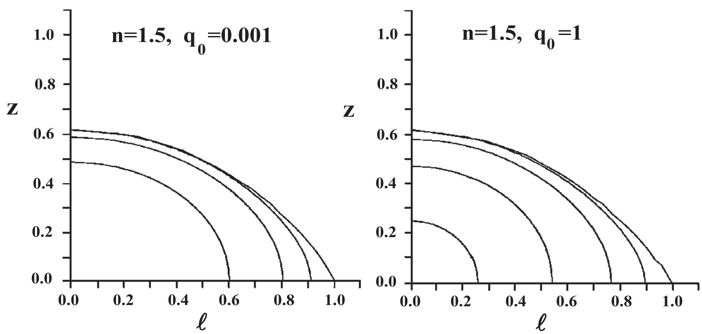

Fig. 4.2.3 Contours of constant relativistic energy density ε_r in the meridional plane of a uniformly and critically rotating polytropic $n = 1.5$ spheroid if $q_0 = 0.001$ (Newtonian limit), and $q_0 = 1$. The relativistic energy density ε_r changes by a factor 10 between successive contours (Komatsu et al. 1989a).

where $\gamma_{rr} = -g_{rr} = -g_{11} = \exp(2\kappa)$, and

$$r_{re} = \int_0^{r_1(\mu=0)} \exp\kappa\, dr; \quad r_{rp} = \int_0^{r_1(\mu=1)} \exp\kappa\, dr, \tag{4.2.229}$$

are the true relativistic equatorial and polar radii, respectively [cf. Eqs. (4.1.47), (4.2.149)].

From the metric (4.2.149) we observe that $g_{00} = \exp(2\nu) - \omega^2\exp(2\gamma - 2\nu)\, r^2\sin^2\lambda$ can have either sign. But in regions where $g_{00} < 0$, the time axis x^0 – the line $r, \lambda, \varphi = \text{const}$ – is not a timelike direction ($ds^2 = g_{00}\, dt^2 < 0$). Regions where $g_{00} < 0$ are called ergoregions. The principal property of an ergoregion

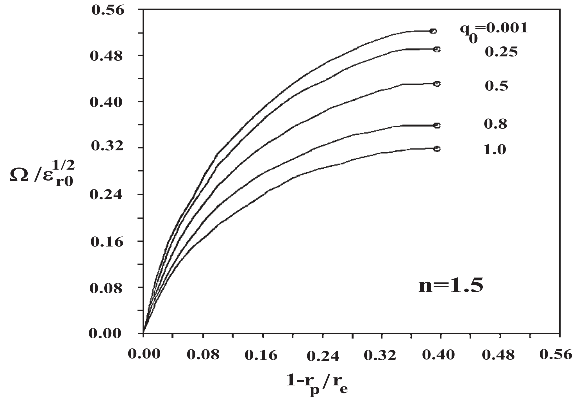

Fig. 4.2.4 Angular velocity versus ratio between polar and equatorial radial coordinate r_p/r_e for $n = 1.5$ spheroids. All sequences for various values of the relativity parameter q_0 terminate at critical rotation Ω_c (open circles), (Komatsu et al. 1989a).

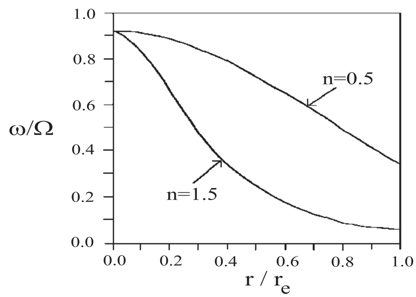

Fig. 4.2.5 Dragging of inertial frames ω/Ω in the equatorial plane plotted against the relative equatorial coordinate radius r/r_e for uniformly and critically rotating spheroids with polytropic index $n = 0.5$ and 1.5, ($q_0 = 1$), (Komatsu et al. 1989a).

is that motion with $\varphi =$ const, $d\varphi/dt = 0$ – as seen from infinity – is impossible: All particles must rotate. No static observer with $\Omega = d\varphi/dt = u^3/u^0 = 0$ and $u^1, u^2 = 0$ can exist. It is for this reason that the concept of the "locally nonrotating observer" has been introduced, who rotates with angular velocity $d\varphi/dt = \omega$ with respect to infinity (Eq. (4.2.150), Bardeen 1970, Bardeen and Wagoner 1971). Note, that inside the Schwarzschild gravitational radius (4.1.28) motion with $r =$ const is impossible in the spherically symmetric field, whereas for the gravitational field of rotating bodies $\varphi =$ const is not allowed inside an ergoregion, while $r =$ const is permitted (Landau and Lifschitz 1987, §104).

The behaviour of general relativistic Maclaurin spheroids shown in Fig. 4.2.2, is quite different from Newtonian Maclaurin spheroids, where $\Omega^2/\pi\varepsilon$ – equal to $\Omega^2/\pi G\varrho$ if $c, G = 1$ – reaches a maximum of 0.4494 at an eccentricity of about $e = 0.9299$ (Sec. 3.2). The delicate balance between gravitational and centrifugal forces, which permits Newtonian Maclaurin spheroids to exist for all values of e, is destroyed by relativistic effects. Butterworth and Ipser (1976) build up sequences of rotating relativistic Maclaurin spheroids by starting with the Schwarzschild sphere, specified by the value

$$X_S = 1 - (1 - 2M_{r1}/r_1)^{1/2}, \qquad (c, G = 1). \tag{4.2.230}$$

While X_S is a measure of the strength of relativity, the parameter $Y = J^2 \varepsilon_r^{1/3} / M_1^{10/3}$ characterizes

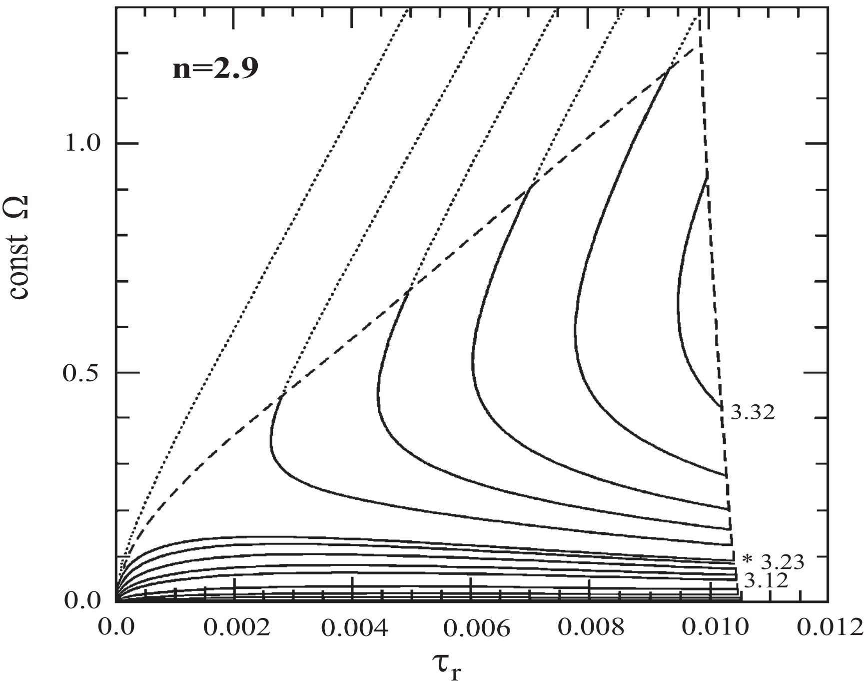

Fig. 4.2.6 Angular velocity Ω measured from infinity versus ratio $\tau_r = E_{r,kin}/|W_r|$ from Eqs. (4.1.73), (4.2.225) for the normal and supramassive equilibrium sequence of uniformly rotating, relativistic $n = 2.9$ polytropes with constant rest mass along each curve. The dashed line on the right is the mass loss limit, the dashed line on the left the stability limit against quasiradial oscillations of the supramassive sequence. Dotted curves are unstable. Three curves are labeled by the value of their constant dimensionless rest mass; the curve *3.23 – highlighted with an asterisk – is the delimitation between the two sequences (Cook et al. 1994).

the amount of rotation, since in the Newtonian limit

$$Y = J^2\varepsilon_r^{1/3}/M_1^{10/3} = J^2\varepsilon^{1/3}/M_1^{10/3} \propto (\Omega^2 M_1^{10/3}/\varrho^{4/3})(\varrho^{1/3}/M_1^{10/3}) = \Omega^2/\varrho,$$
$$(n = 0;\ c, G = 1;\ \varepsilon_r = \varepsilon = \varrho). \qquad (4.2.231)$$

Using the self-consistent field method [Eqs. (3.8.82)-(3.8.89)], the characteristics of uniformly rotating, relativistic polytropic spheroids have been calculated by Komatsu et al. (1989a) for polytropic indices $n = 0.5, 1.5, 3$, the relativity parameter from Eq. (4.1.31) being $q_0 = P_0/\varepsilon_{r0} = 0.001, 0.25, 0.5, 0.8, 1$ (Figs. 4.2.3-4.2.5). No figures have been published for the case $n = 3$, when numerical instabilities prevented calculations with $q_0 > 0.6$. The calculations have been done for an almost constant angular velocity $\Omega = \Omega_c$, determined by $A\varepsilon_{r0}^{1/2} = 100$ [Eq. (4.2.207)] for almost all models.

In the strong gravity case ($q_0 = 1$) the mass is more concentrated towards the centre, as compared to the Newtonian limit ($q_0 = 0.001$). Critical rotation occurs at nearly the same ratio between critical polar and equatorial coordinate ($r_{cp}/r_{ce} \approx 0.6$) for all spheroidal models depicted in Table 4.2.3, excepting the case $n = 3,\ q_0 = 0.6:\ r_{cp}/r_{ce} \approx 0.7$.

The critical angular velocity Ω_c can be evaluated in the spherical Schwarzschild approximation, since the mass in the equatorial bulge is too small to affect gravity. The effects of general relativity are incorporated by including the factor f_r in the simple Newtonian relationship $\Omega_c^2 = M_1/r_{ce}^3$ for a critically rotating sphere (Komatsu et al. 1989a):

$$\Omega_c = (f_r M_{r1}/r_{ce}^3)^{1/2}, \qquad (G = 1). \qquad (4.2.232)$$

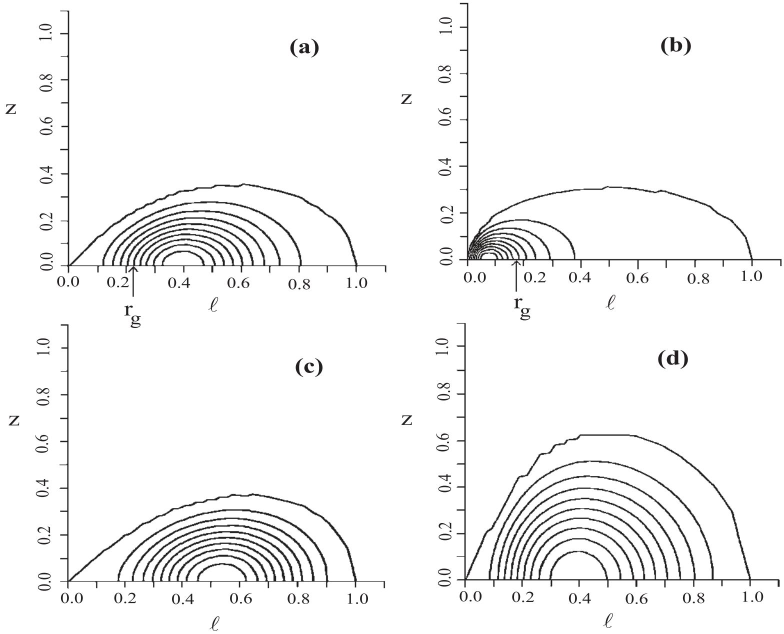

Fig. 4.2.7 Meridional cross-section of relativistic energy density ε_r for four ringlike structures if $n = 1.5$. Model (a): Strong gravity ($q_0 = 0.1$) with nearly rigid rotation ($A = 0.6$). Model (b): Strong gravity ($q_0 = 0.4$) with differential rotation ($A = 0.4$). Model (c): Newtonian limit ($q_0 = 0.001$) with nearly rigid rotation ($A = 0.065$). Model (d): Newtonian limit ($q_0 = 0.001$) with differential rotation ($A = 0.0125$). Contours are linearly spaced by $0.1\varepsilon_{r0}$, and the gravitational radius $r_g = 2M_{r1}$ is indicated for the strong gravity models (a) and (b). Strong gravity gathers matter towards the rotation axis in models (a) and (b), (Komatsu et al. 1989b).

Table 4.2.3 Parameters for critically and uniformly rotating, relativistic polytropic spheroids according to Komatsu et al. (1989a): r_{cp}/r_{ce} – ratio between critical polar and equatorial coordinate, $\beta_{rc} = \Omega_c^2/2\pi G\varrho_{r0}$, $(G, c = 1)$ – critical rotation parameter from Eq. (4.2.126), $\tau_{rc} = E_{r,kin}/|W_r|$ – critical ratio between relativistic rotational and gravitational energy from Eqs. (4.1.73), (4.2.225), v_{ce} – critical proper velocity at the equator with respect to the zero angular momentum observer [Eq. (4.2.150)]. $aE + b$ means $a \times 10^b$.

n	q_0	r_{cp}/r_{ce}	β_{rc}	τ_{rc}	v_{ce}
0.5	1	0.58	1.78 E−1	1.57 E−1	0.658
1.5	0.25	0.60	3.85 E−2	5.11 E−1	0.446
1.5	0.5	0.60	2.96 E−2	4.30 E−2	0.487
1.5	0.8	0.61	2.06 E−2	3.50 E−2	0.495
1.5	1	0.60	1.56 E−2	3.08 E−2	0.481
3	0.6	0.70	3.39 E−7	5.75 E−5	–

If $n = 3$, the mass of rotationally distorted, relativistic spheres is extremely concentrated towards the centre. Mass loss occurs at an early stage, when the critical relativistic rotation parameter is $\beta_{rc} = \Omega_c^2/2\pi\varepsilon_{r0} = \Omega_c^2/2\pi\varrho_{r0} = 3.39\times10^{-7}$, $(q_0 = 0.6;\ c, G = 1;\ \tau_r = E_{r,kin}/|W_r| = 5.75\times10^{-5})$, whereas in the Newtonian limit $q_0 \to 0$ Komatsu et al. (1989a) obtain $\beta_c = 4.02\times10^{-3}$ and $\tau = E_{kin}/|W| = 8.94\times10^{-3}$ (cf. Tables 3.8.1, 4.2.3, 5.8.2). No ergoregions appear in the calculations of Butterworth (1976) and Komatsu et al. (1989a) for polytropic relativistic spheroids with $0 < n \leq 3$, unlike to the case $n = 0$ from Fig. 4.2.2.

The relationship between the angular velocity Ω measured from infinity and the angular momentum J from Eq. (4.2.220) has been studied by Cook et al. (1992, 1994) for hydrostatic equilibrium sequences of uniformly and rapidly rotating, relativistic polytropic spheroids ($0.5 \leq n \leq 2.9$) with the isentropic

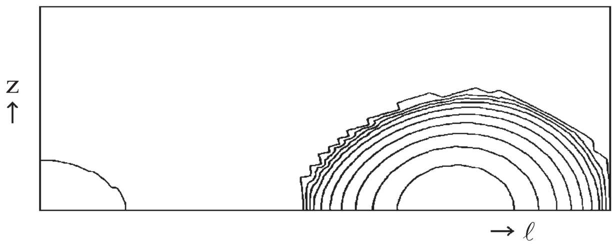

Fig. 4.2.8 Relativistic energy density contours for a uniformly rotating star (on the left) with polytropic index $n = 0.5$, and for the surrounding ring with polytropic index $n = 1$ (on the right). The relativity parameter of the ring is $q_0 = 0.01$, and the ratio between maximum relativistic energy density of star and ring is 500. The rotation parameter of the ring from Eq. (4.2.207) is $A = 0.2$ (Nishida et al. 1992).

equation of state (4.1.83), and constant baryon rest mass M_1. Even rapidly rotating Newtonian stars can *spin up* (Ω increases) as they lose energy and angular momentum via secular processes like stellar winds, electromagnetic or gravitational radiation: The sign of the rate of change of Ω depends on the variation of J and I_Ω both, since $\Omega = J/I_\Omega$, [I_Ω – moment of inertia with respect to rotation axis from Eq. (3.1.85)]. Indeed, Shapiro et al. (1990) found that a constant Newtonian mass with $n \approx 3$, ($n < 3$) can increase its angular velocity when angular momentum is lost.

Relativistic polytropic spheroids exhibit the same behaviour as Newtonian ones along the so-called "normal sequence". But there occurs also a uniquely relativistic effect along the "supramassive sequence", when the rest mass of the rotating polytrope exceeds the maximum stable rest mass of the nonrotating polytrope. Both sequences terminate on the maximum angular momentum end (the maximum of $\tau_r = E_{r,kin}/|W_r|$ in Fig. 4.2.6) by mass loss. The most massive polytropes of the supramassive sequence exhibit mass shedding also on their lower angular momentum end, rather than terminating as quasiradially unstable objects, like the less massive members.

Komatsu et al. (1989b) calculate differentially rotating, relativistic configurations for polytropic indices $n = 0.5$ and 1.5, when there appear besides spheroids also ringlike structures and ergoregions (Fig. 4.2.7). The models of Komatsu et al. (1989b) are specified by four parameters: The polytropic index n, the polar versus equatorial coordinate ratio r_p/r_e, the rotation parameter A from Eq. (4.2.207), and the relativity parameter $q_0 = P_0/\varepsilon_{r0}$, where P_0 and ε_{r0} now denote the maximum pressure and the maximum relativistic energy density appearing in the configuration.

Nishida et al. (1992) have numerically evaluated the structure of a star-toroid system, extending the pioneering work of Bardeen and Wagoner (1971) on relativistic thin disks. Relativistic toroidal structures round compact objects are relevant in the modelling of accretion disks, quasars, and active galactic nuclei. Such structures may also form during contraction of a close neutron binary, if one component is destroyed by tidal forces. Nishida et al. (1992) take into account for the first time the finite thickness of the relativistic disks, while the central object is different from a black hole.

As the gravity of the polytropic toroid deforms the central, rapidly rotating polytropic star, less angular velocity is needed for the central star to shed mass from its equator. Due to dragging of inertial frames by the rotating massive toroid, the angular velocity and the angular momentum can have opposite signs.

5 STABILITY AND OSCILLATIONS

5.1 Definitions and General Considerations

A somewhat comprehensive presentation of polytropic stability and oscillations would require a separate book, as this vast topic leads to sophisticated and cumbersome calculations, with manifold conclusions. The best example in this respect is perhaps Chandrasekhar's (1969) textbook, dealing merely with a subclass of constant density ($n = 0$) configurations – the incompressible homogeneous ellipsoids ($n = 0;\ \Gamma_1 = \infty$).

It seems difficult to formulate an entirely general and consistent definition of stability, but the problem becomes somewhat simpler, when defining the stability of equilibrium states, as will be relevant for our purposes. It is well known that for conservative (dissipationless) mechanical systems, having a finite number of degrees of freedom, the necessary and sufficient condition for stable equilibrium demands for the potential energy Φ to be an absolute minimum (theorem of Lejeune-Dirichlet). This means that if the initial displacements of the system are sufficiently small, all distances with respect to equilibrium will remain always small (Ledoux 1958). The generalization of this stability definition to continuous media with an infinite number of degrees of freedom, can be effected by defining the integral

$$\int_m V\ dm, \tag{5.1.1}$$

as a measure of virtual displacements of all mass elements dm, where V is the volume swept up by the mass element dm during the course of time. The distance $|S - S_0|$ between two configurations S and S_0 of the system (S_0 – equilibrium configuration) is taken as the minimum of the integral (5.1.1) for all virtual displacements leading from state S_0 to S. The equilibrium state S_0 is said to be stable in the sense of Wavre, if for an arbitrary small number ε_1 we can always define another small number ε_2 in such a way, that any displacement of the system with $|S - S_0| < \varepsilon_2$ and with kinetic energy smaller than ε_2, will lead to motions obeying the condition $|S - S_0| < \varepsilon_1$.

Depending on the time scale of stable (unstable) motions, and on the overall physical conditions in a configuration, we may distinguish various kinds of stability (instability). A system is dynamically (ordinarily) unstable if infinitesimal perturbations increase exponentially. Dynamical instability represents increasing departures from hydrostatic equilibrium (Cox 1980, p. 47), and the rate of departure from equilibrium is in general rapid. Under some circumstances unstable oscillations will continue to grow, until dissipation (friction) or other nonlinear factors become important enough to limit the oscillations to some finite amplitude: In this case the configuration is said to be pulsationally (vibrationally) unstable, or overstable in Eddington's terminology.

Another kind of aperiodic instability is the so-called secular instability, representing in stars increasing departures from thermal equilibrium. Unlike dynamical (ordinary) instability, which sets in independently of dissipative processes, the secular instability only manifests if some dissipative mechanism is operative; the e-folding time of secular instability is directly proportional to the efficiency of dissipative processes, and it may be regarded as a special case of pulsational instability, when dynamical acceleration terms can be neglected (cf. Ledoux 1958, Chandrasekhar 1969, p. 95; Cox 1980, p. 321). Secular instability is a milder form of instability, that could arise if dissipation is taken into account. The lunar orbit, for instance, is dynamically stable, but secularly unstable, due to the action of tidal friction (Lyttleton 1953, p. 113). The relationship between secular and dynamical instability may be casted into the following four propositions: (i) If an equilibrium configuration is secularly stable, it is also dynamically stable. (ii) If it is secularly unstable, it may be dynamically stable or unstable. (iii) If it is dynamically stable, it may be secularly stable or unstable. (iv) If it is dynamically unstable. it is necessarily secularly unstable.

Another type of instability – gravitational instability – is intimately connected with the following question: When can a fluid mass start contracting under its own gravitation? This is also the fundamental question of the origin of stars, if one thinks that they are formed by contraction of dilute interstellar matter subject to an external pressure (cf. Sec. 5.4).

Two types of description – called Eulerian and Lagrangian – can be used to analyze the oscillations of a fluid about an equilibrium state. The Eulerian approach regards the spatial coordinates $\vec{r} = \vec{r}(x_1, x_2, x_3)$ and the time t as independent variables, the position vector $\vec{r}$ referring to the arbitrary location of the observation point, not to a particular fluid element followed in its motion. The state of motion is described ab initio by the velocity field (Tassoul 1978)

$$\vec{v} = \vec{v}(\vec{r}, t), \tag{5.1.2}$$

at location $\vec{r}$ in the instant t. The state of the system is completely determined by the additional knowledge of density, pressure, potential, etc., as functions of $\vec{r}$ and t.

The Lagrangian description labels each fluid element by its initial position $\vec{r}_i = \vec{r}_i(x_{i1}, x_{i2}, x_{i3})$ at the initial moment, say $t = 0$. The position vector $\vec{r}$ at some subsequent instant t is no longer an independent variable, but a function of the independent variables $\vec{r}_i$ and t :

$$\vec{r} = \vec{r}(\vec{r}_i, t). \tag{5.1.3}$$

In view of Eq. (5.1.3), any quantity F which is a function of the Eulerian independent variables $\vec{r}$ and t, is also a function of the Lagrangian independent variables $\vec{r}_i$ and t, or vice versa.

By definition, the velocity of a particular fluid element in Lagrangian description is

$$\vec{v} = \vec{v}[\vec{r}(\vec{r}_i, t), t] = \vec{v}(\vec{r}_i, t) = [\partial\vec{r}(\vec{r}_i, t)/\partial t]_{\vec{r}_i=\mathrm{const}}, \tag{5.1.4}$$

where the partial derivative indicates that differentiation must be carried out by holding $\vec{r}_i$ constant. The time derivative of the Eulerian coordinates $\vec{r} = \vec{r}(x_1, x_2, x_3)$ has in general no meaning, as $\vec{r}$ labels merely the space, where the motion of the fluid, defined by $\vec{v}(\vec{r}, t)$, takes place. In the definition (5.1.4) the position vector $\vec{r}$ must be considered always in Lagrangian description $\vec{r} = \vec{r}(\vec{r}_i, t)$. Once the Eulerian velocity $\vec{v}(\vec{r}, t) = \vec{v}[\vec{r}(\vec{r}_i, t), t]$ is known, integration of Eq. (5.1.4) yields the Lagrangian solution (5.1.3). The connection between Eulerian and Lagrangian description is provided by Eq. (5.1.4).

In Eulerian coordinates the partial time derivative of a function F has the meaning $\partial F(\vec{r}, t)/\partial t = [\partial F(\vec{r}, t)/\partial t]_{\vec{r}=\mathrm{const}}$, i.e. it is the rate of change of F apparent to a fixed observer at position vector $\vec{r} = \mathrm{const}$. On the other hand, the partial temporal derivative for a certain mass element amounts in Lagrangian coordinates to $\partial F(\vec{r}_i, t)/\partial t = [\partial F(\vec{r}_i, t)/\partial t]_{\vec{r}_i=\mathrm{const}}$. Just the same rate of change is measured by the material derivative $DF(\vec{r}, t)/Dt$ from Eq. (5.1.6), as we follow a particular mass element along its path, by noting that $\vec{r}$ and t are formally regarded as independent variables (Tassoul 1978, Cox 1980):

$$DF(\vec{r}, t)/Dt = [\partial F(\vec{r}_i, t)/\partial t]_{\vec{r}_i=\mathrm{const}} \qquad \text{or} \qquad (DF/Dt)_{Eulerian} = (\partial F/\partial t)_{Lagrangian}. \tag{5.1.5}$$

Physically, this is quite different from $[\partial F(\vec{r}, t)/\partial t]_{\vec{r}=\mathrm{const}}$. Eq. (5.1.5) shows that the material derivative in Eulerian description $DF(\vec{r}, t)/Dt$ becomes equal to $\partial F(\vec{r}_i, t)/\partial t$ in Lagrangian description. In Eulerian variables the material (Stokes) derivative is equal to [cf. Eqs. (B.23)-(B.25)]

$$\begin{aligned} &DF[\vec{r}(x_1, x_2, x_3), t]/Dt = \partial F/\partial t + (\partial F/\partial x_1)\ \partial x_1/\partial t + (\partial F/\partial x_2)\ \partial x_2/\partial t + (\partial F/\partial x_3)\ \partial x_3/\partial t \\ &= \partial F/\partial t + \vec{v}\cdot\nabla F; \qquad D\vec{F}/Dt = \partial\vec{F}/\partial t + (\vec{v}\cdot\nabla)\vec{F}, \\ &(\vec{F} = F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3;\ v_k = \partial x_k(\vec{r}_i, t)/\partial t;\ k = 1, 2, 3). \end{aligned} \tag{5.1.6}$$

In the case of a vector function $\vec{F}$ the nabla operator in the term $(\vec{v}\cdot\nabla)\vec{F}$ acts on both, the vector components F_k and the unit vectors $\vec{e}_k$ along the coordinate axes [cf. Eqs. (B.23)-(B.25)].

The difference between Eulerian and Lagrangian description becomes even more apparent for the acceleration, which in Eulerian variables is

$$D\vec{v}(\vec{r}, t)/Dt = \partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v}, \tag{5.1.7}$$

and in Lagrangian ones

$$\partial\vec{v}(\vec{r}_i, t)/\partial t = \partial^2\vec{r}(\vec{r}_i, t)/\partial t^2. \tag{5.1.8}$$

Let us consider in the Lagrangian description a particular fluid element in an unperturbed flow (e.g. equilibrium state) defined by the radius vector $\vec{r}_u$, which will be a function of its initial position $\vec{r}_i$ from Eq. (5.1.3):

$$\vec{r}_u = \vec{r}_u(\vec{r}_i, t). \tag{5.1.9}$$

Let us also consider a perturbed flow with radius vector $\vec{r}$ given by Eq. (5.1.3). At time t, the Lagrangian displacement of the same fluid element with respect to its unperturbed position is defined by (e.g. Cox 1980)

$$\Delta\vec{r} = \Delta\vec{r}(\vec{r}_i, t) = \vec{r}(\vec{r}_i, t) - \vec{r}_u(\vec{r}_i, t) = \vec{r} - \vec{r}_u. \tag{5.1.10}$$

Since $\vec{r}_i$ is connected with $\vec{r}_u$ through Eq. (5.1.9), we could equally well regard $\Delta\vec{r}$ as a function of $\vec{r}_u$ and t :

$$\Delta\vec{r} = \Delta\vec{r}(\vec{r}_u, t) = \vec{r}(\vec{r}_u, t) - \vec{r}_u = \vec{r} - \vec{r}_u. \tag{5.1.11}$$

If the Lagrangian displacement is considered in the Eulerian description of the fluid, we have

$$\Delta\vec{r} = \Delta\vec{r}(\vec{r}, t) = \vec{r} - \vec{r}_u(\vec{r}, t) = \vec{r} - \vec{r}_u, \tag{5.1.12}$$

by replacing $\vec{r}_i$ in Eq. (5.1.10) with the (arbitrary) point of observation $\vec{r}$ from Eq. (5.1.3). Let $F(\vec{r}, t)$ and $F_u(\vec{r}, t)$ be the values of a physical quantity (e.g. pressure, density, potential) in the perturbed flow and in the unperturbed one. The Eulerian variation of F is defined by

$$\delta F = \delta F(\vec{r}, t) = F(\vec{r}, t) - F_u(\vec{r}, t), \tag{5.1.13}$$

representing the difference of the quantity F between the perturbed and unperturbed flow, observed simultaneously at the *same radius vector* $\vec{r}$. Alternatively, the perturbed flow can also be described by specifying at each moment the Lagrangian displacement $\Delta\vec{r}(\vec{r}_i, t)$ which a certain mass element experiences in the perturbed flow relative to the same mass element in the unperturbed flow. In this way we define the Lagrangian displacement of F by

$$\Delta F = \Delta F(\vec{r}_i, t) = F[\vec{r}_i + \Delta\vec{r}(\vec{r}_i, t), t] - F_u(\vec{r}_i, t), \tag{5.1.14}$$

representing the change of F observed simultaneously for the *same mass element* in the perturbed and unperturbed flow. Since $\vec{r}$ and $\vec{r}_u$ both are connected to the initial radius vector $\vec{r}_i$ of the same mass element through Eqs. (5.1.3), (5.1.9), we can replace $F_u(\vec{r}_i, t)$ by $F_u(\vec{r}_u, t)$, and $F[\vec{r}_i + \Delta\vec{r}(\vec{r}_i, t), t] = F[\vec{r}_u + \Delta\vec{r}(\vec{r}_u, t), t]$ by $F(\vec{r}, t)$ via Eqs. (5.1.10), (5.1.11). So, we are sure that we are indeed comparing the properties of the same mass element in the two flows, and the Lagrangian variation of F becomes

$$\Delta F = F(\vec{r}, t) - F_u(\vec{r}_u, t). \tag{5.1.15}$$

To first order the Eulerian and Lagrangian variations (5.1.13) and (5.1.15) can be related together, by adding and subtracting in Eq. (5.1.15) the same quantity $F(\vec{r}_u, t)$:

$$\begin{aligned} \Delta F &= F(\vec{r}, t) - F_u(\vec{r}_u, t) = F[\vec{r}_u + \Delta\vec{r}(\vec{r}_u, t),\ t] - F(\vec{r}_u, t) + F(\vec{r}_u, t) - F_u(\vec{r}_u, t) \\ &= \Delta\vec{r}(\vec{r}_u, t) \cdot \nabla F(\vec{r}_u, t) + \delta F(\vec{r}_u, t) \qquad \text{or} \qquad \Delta F = \delta F + \Delta\vec{r} \cdot \nabla F, \end{aligned} \tag{5.1.16}$$

where $F[\vec{r}_u + \Delta\vec{r}(\vec{r}_u, t), t] = F(\vec{r}_u, t) + \Delta\vec{r}(\vec{r}_u, t) \cdot \nabla F(\vec{r}_u, t)$. If F is a vector $\vec{F}$, Eq. (5.1.16) reads

$$\Delta\vec{F} = \delta\vec{F} + (\Delta\vec{r} \cdot \nabla)\vec{F}. \tag{5.1.17}$$

It is evident from the definition (5.1.13), that to first order in the disturbances, the Eulerian variation commutes with partial derivation, since it is a variation of a function F at the same location and moment:

$$\delta(\partial F/\partial t) = \partial \delta F/\partial t; \qquad \delta(\partial F/\partial x_k) = \partial \delta F/\partial x_k, \qquad (k = 1, 2, 3). \tag{5.1.18}$$

In contrast, the Lagrangian change Δ does not commute with partial derivation (e.g. Chandrasekhar 1969):

$$\begin{aligned} \Delta(\partial F/\partial x_k) &= \delta(\partial F/\partial x_k) + \Delta\vec{r} \cdot \nabla(\partial F/\partial x_k) = \partial \delta F/\partial x_k + \Delta\vec{r} \cdot \nabla(\partial F/\partial x_k) \\ &= \partial(\Delta F - \Delta\vec{r} \cdot \nabla F)/\partial x_k + \Delta\vec{r} \cdot \nabla(\partial F/\partial x_k) = \partial \Delta F/\partial x_k - (\partial \Delta\vec{r}/\partial x_k) \cdot \nabla F, \end{aligned} \tag{5.1.19}$$

$$\begin{aligned} \Delta(\partial F/\partial t) &= \delta(\partial F/\partial t) + \Delta\vec{r} \cdot \nabla(\partial F/\partial t) = \partial(\Delta F - \Delta\vec{r} \cdot \nabla F)/\partial t + \Delta\vec{r} \cdot \nabla(\partial F/\partial t) \\ &= \partial \Delta F/\partial t - (\partial \Delta\vec{r}/\partial t) \cdot \nabla F. \end{aligned} \tag{5.1.20}$$

An important consequence of Eqs. (5.1.19), (5.1.20) is that the Lagrangian variation ΔF commutes with the material derivative (5.1.6) up to first order:

$$\Delta(DF/Dt) = \Delta(\partial F/\partial t) + \Delta(\vec{v}\cdot\nabla F) = \partial\Delta F/\partial t - (\partial\Delta\vec{r}/\partial t)\cdot\nabla F + \Delta\vec{v}\cdot\nabla F + \vec{v}\cdot\Delta(\nabla F)$$
$$= \partial\Delta F/\partial t - (\partial\Delta\vec{r}/\partial t)\cdot\nabla F + (\partial\Delta\vec{r}/\partial t)\cdot\nabla F + [(\vec{v}\cdot\nabla)\Delta\vec{r}]\cdot\nabla F + \vec{v}\cdot\nabla\Delta F$$
$$-[(\vec{v}\cdot\nabla)\Delta\vec{r}]\cdot\nabla F = \partial\Delta F/\partial t + \vec{v}\cdot\nabla\Delta F = D(\Delta F)/Dt. \qquad (5.1.21)$$

Here, we have replaced $\Delta(\nabla F)$, $\Delta(\partial F/\partial t)$, $\Delta\vec{v}$ according to Eqs. (5.1.19), (5.1.20), (5.1.22), respectively. In virtue of Eqs. (5.1.4), (5.1.5), (5.1.10), (5.1.15) we deduce for the Lagrangian velocity variation

$$\Delta\vec{v} = \vec{v}(\vec{r},t) - \vec{v}_u(\vec{r}_u,t) = [\partial\vec{r}(\vec{r}_i,t)/\partial t]_{\vec{r}_i=\mathrm{const}} - [\partial\vec{r}_u(\vec{r}_i,t)/\partial t]_{\vec{r}_i=\mathrm{const}}$$
$$= [\partial\Delta\vec{r}(\vec{r}_i,t)/\partial t]_{\vec{r}_i=\mathrm{const}} = D[\Delta\vec{r}(\vec{r},t)]/Dt = \partial[\Delta\vec{r}(\vec{r},t)]/\partial t + [\vec{v}(\vec{r},t)\cdot\nabla]\ \Delta\vec{r}(\vec{r},t). \qquad (5.1.22)$$

On the other side, the Eulerian velocity change $\delta\vec{v}$ is via Eq. (5.1.17) equal to

$$\delta\vec{v} = \Delta\vec{v} - (\Delta\vec{r}\cdot\nabla)\vec{v} = D(\Delta\vec{r})/Dt - (\Delta\vec{r}\cdot\nabla)\vec{v} = \partial\Delta\vec{r}/\partial t + (\vec{v}\cdot\nabla)\Delta\vec{r} - (\Delta\vec{r}\cdot\nabla)\vec{v}. \qquad (5.1.23)$$

In the important particular case when the unperturbed flow is an equilibrium configuration ($\vec{v}_u = 0$), we have from Eqs. (5.1.22) and (5.1.23) up to the first order in $\Delta\vec{r}, \vec{v}$:

$$\delta\vec{v} \approx \Delta\vec{v} = \vec{v} = D(\Delta\vec{r})/Dt \approx \partial\Delta\vec{r}/\partial t, \qquad (|\Delta\vec{r}|, |\vec{v}| \approx 0;\ \vec{v}_u = 0). \qquad (5.1.24)$$

The operators δF and DF/Dt do not commute, and from Eq. (5.1.16) we get with Eq. (5.1.21), $(F \to DF/Dt)$:

$$\delta(DF/Dt) = \Delta(DF/Dt) - \Delta\vec{r}\cdot\nabla(DF/Dt) = D(\Delta F)/Dt - \Delta\vec{r}\cdot\nabla(DF/Dt). \qquad (5.1.25)$$

In problems with only one degree of freedom the Lagrangian description of oscillations is preferable, while in problems with more degrees of freedom the Eulerian form is more convenient.

The study of oscillations may be divided conceptually into two categories: (i) The linear theory, when the amplitude of oscillations is assumed infinitely small, so that a first order approximation of the basic equations is permissible. (ii) The nonlinear theory, when the oscillation amplitude may be of any size.

It is usually assumed that the solutions of the linear theory approximate closely enough the actual solution, to reveal the trend of motion in the vicinity of an equilibrium state. If a configuration is found to be unstable in the linear theory, then oscillations will grow in amplitude until nonlinear processes limit the amplitude, or until the configuration disrupts (Cox and Giuli 1968).

In linear theory the Lagrangian displacement $\Delta\vec{r}$, ($|\Delta\vec{r}| \ll 1$) is generally expressed in spherical coordinates as [Eq. (B.34)]

$$\Delta\vec{r}(\vec{r},t) = \Delta\vec{r}(r,\lambda,\varphi,t) = \Delta r(r,\lambda,\varphi,t)\ \vec{e}_r + r\ \Delta\lambda(r,\lambda,\varphi,t)\ \vec{e}_\lambda + r\sin\lambda\ \Delta\varphi(r,\lambda,\varphi,t)\ \vec{e}_\varphi,$$
$$[\vec{r} = \vec{r}(r,\lambda,\varphi)]. \qquad (5.1.26)$$

$\Delta r,\ r\ \Delta\lambda$, and $r\sin\lambda\ \Delta\varphi$ are the components of $\Delta\vec{r}$ along the coordinate axes.

In addition to the spheroidal (polar, even-parity) modes there exists also a second class – the toroidal (axial, odd-parity) modes – characterized by $\Delta r(r,\lambda,\varphi,t) = 0$ [see Eq. (5.8.166)]. It is usually assumed that spheroidal and toroidal modes form a complete set – each perturbation of a physical fluid characteristic (position, velocity, pressure, density, potential) can be represented as an infinite sum of such modes. Each mode is treated as if only that one mode existed. To facilitate the mathematical treatment, the radial component Δr of the Lagrangian displacement vector $\Delta\vec{r}$ is assumed under the form of a spheroidal mode with separated variables:

$$\Delta r(r,\lambda,\varphi,t) = \Delta r(r)\ Y_j^k(\lambda,\varphi)\exp(i\sigma t) = \Delta r(r)\ P_j^k(\cos\lambda)\exp(ik\varphi)\exp(i\sigma t), \quad \big(j = 0,1,2,3,...;$$
$$k = -j,-j+1,...j-1,j; \quad P_j^0 = P_j; \quad P_j^{-k}(\cos\lambda) = (j-k)!\ P_j^k(\cos\lambda)/(j+k)!\big). \qquad (5.1.27)$$

$P_j^k(\cos\lambda) \propto P_j^{-k}(\cos\lambda)$ are associated Legendre polynomials from Eqs. (3.1.38)-(3.1.41), (Hobson 1931, p. 99, Abramowitz and Stegun 1965). The angular oscillation frequency is denoted by σ, and $\exp(i\sigma t) = \cos(\sigma t) + i\sin(\sigma t)$, $\exp(ik\varphi) = \cos(k\varphi) + i\sin(k\varphi)$. The natural number j defines the latitudinal order of the disturbance (j – latitudinal number), and the integer k defines its azimuthal order (k – azimuthal number). The spherical harmonic $Y_j^k(\lambda, \varphi) = P_j^k(\cos\lambda)\exp(ik\varphi)$ is an eigenfunction of the equation [cf. Eqs. (3.1.41), (3.4.9)]

$$(1/\sin\lambda)\ \partial(\sin\lambda\ \partial Y_j^k/\partial\lambda)/\partial\lambda + (1/\sin^2\lambda)\ \partial^2 Y_j^k/\partial\varphi^2 = -j(j+1)Y_j^k. \qquad (5.1.28)$$

Equations for the tangential Lagrangian displacements $r\ \Delta\lambda$ and $r\sin\lambda\ \Delta\varphi$ are provided by Eqs. (5.2.75), (5.2.76). Assuming also for $\Delta\lambda$ and $\Delta\varphi$ the same time dependence as for Δr, we get by virtue of Eqs. (5.1.26), (5.1.27):

$$\Delta\vec{r}(\vec{r}, t) = \Delta\vec{r}(\vec{r})\exp(i\sigma t). \qquad (5.1.29)$$

If $\vec{v}_u = 0$, the Eulerian and Lagrangian velocity variation is via Eqs. (5.1.24), (5.1.29) equal to

$$\delta\vec{v} \approx \Delta\vec{v} = \vec{v} \approx \partial\Delta\vec{r}/\partial t = i\sigma\ \Delta\vec{r}(\vec{r})\exp(i\sigma t) = i\sigma\ \Delta\vec{r}(\vec{r}, t). \qquad (5.1.30)$$

The Eulerian and Lagrangian variations of pressure P, density ϱ, and gravitational potential Φ are assumed under the same form as in Eq. (5.1.27):

$$\delta F, \Delta F = F(r)\ Y_j^k(\lambda, \varphi)\exp(i\sigma t) = F(r)\ P_j^k(\cos\lambda)\exp(ik\varphi)\exp(i\sigma t), \qquad (F = P, \varrho, \Phi). \quad (5.1.31)$$

Generally, the angular oscillation frequency σ can be a complex number, and the real part of σ leads in the time-dependent factor $\exp(i\sigma t)$ to sinusoidal oscillations of $\Delta\vec{r}$ with finite amplitudes. Likewise, if the imaginary part of σ is positive, damping of oscillations takes place with respect to the time. If $\sigma = 0$, the modes from Eqs. (5.1.29) and (5.1.31) are time-independent: Neutral (marginal) stability occurs, leading in the course of perturbations to another equilibrium configuration (Tassoul 1978). Once the imaginary part of a particular eigenfrequency σ is negative, the configuration becomes unstable, due to the growing exponential factor $\exp(i\sigma t)$.

The boundary conditions at the centre and surface imply that solutions exist only for certain values of σ^2 – the eigenvalues of the problem. The displacements $\Delta\vec{r}$ corresponding to the eigenvalues σ are the eigenfunctions of the problem at hand.

So-called standing waves occur if the phase $\alpha(r)$ of $\Delta r(r)$, or if $\Delta r(r)$ itself is purely real, where the radial perturbation $\Delta r(r, t)$ is represented as

$$\Delta r(r, t) = \Delta r(r)\exp(i\sigma t) = |\Delta r(r)|\exp[i\alpha(r)]\exp(i\sigma t), \qquad (5.1.32)$$

and perfect reflection of the standing wave takes place on the surface.

5.2 Basic Equations

The relevant equations of an inviscid polytropic fluid are the mass conservation equation (equation of continuity), the equation of momentum conservation (equation of motion), and the equation of energy conservation. The equation of continuity obeys in Eulerian description the well known forms (e.g. Ledoux and Walraven 1958, Cox 1980):

$$\partial\varrho/\partial t + \nabla\cdot(\varrho\vec{v}) = 0, \tag{5.2.1}$$

$$D\varrho/Dt + \varrho\ \nabla\cdot\vec{v} = 0, \tag{5.2.2}$$

$$D(dm)/Dt = D(\varrho\ dV)/Dt = 0. \tag{5.2.3}$$

To obtain the Lagrangian description of Eqs. (5.2.1)-(5.2.3), we regard the expression (5.1.3) for the Lagrangian radius vector $\vec{r}$ as a continuous transformation of variables from the initial coordinates (x_{i1}, x_{i2}, x_{i3}) to (x_1, x_2, x_3). The volume integral of $F(\vec{r}, t)$ changes by the coordinate transformation (5.1.3) as (e.g. Smirnow 1967)

$$\int_V F(\vec{r}, t)\ dV = \int_{V_i} JF(\vec{r}_i, t)\ dV_i. \tag{5.2.4}$$

$dV_i = dx_{i1}dx_{i2}dx_{i3}$ is the volume element occupied by the considered fluid particle at the initial moment $t = 0$. The motion of a fluid particle takes place according to the coordinate transformation (5.1.3)

$$x_k = x_k(x_{i1}, x_{i2}, x_{i3}), \qquad (k = 1, 2, 3), \tag{5.2.5}$$

and Eq. (5.2.4) can be deduced at once by inserting the differential of Eq. (5.2.5) $dx_k = (\partial x_k/\partial x_{i\ell})\ dx_{i\ell}$ into the volume element $dV = dx_1dx_2dx_3$, where summation over the repeated index ℓ is to be understood. The Jacobian of the transformation (5.2.5) is given by the determinant

$$J = J(\vec{r}_i, t) = |\partial(x_1, x_2, x_3)/\partial(x_{i1}, x_{i2}, x_{i3})|. \tag{5.2.6}$$

The time derivative of J involves factors of the form $\partial^2 x_k/\partial x_{i\ell}\partial t = \partial v_k/\partial x_{i\ell} = (\partial v_k/\partial x_j)\ \partial x_j/\partial x_{i\ell}$. If these factors are multiplied with the remaining factors of the determinant (5.2.6), we observe that (cf. Ledoux and Walraven 1958, Tassoul 1978):

$$\partial J(\vec{r}_i, t)/\partial t = J(\vec{r}_i, t)\ \nabla\cdot\vec{v}, \qquad (\vec{v} = (\partial\vec{r}/\partial t)_{\vec{r}_i=\mathrm{const}};\ \nabla\cdot\vec{v} = \partial v_k/\partial x_k). \tag{5.2.7}$$

We get the Lagrangian description of the continuity equation, if we multiply its Eulerian form (5.2.2) by $J(\vec{r}_i, t)$, using also Eqs. (5.1.5) and (5.2.7), $(\varrho(\vec{r}, t) \to \varrho(\vec{r}_i, t);\ D\varrho(\vec{r}_i, t)/Dt = \partial\varrho(\vec{r}_i, t)/\partial t)$:

$$\begin{aligned} &J(\vec{r}_i, t)\ D\varrho(\vec{r}_i, t)/Dt + J(\vec{r}_i, t)\ \varrho(\vec{r}_i, t)\ \nabla\cdot\vec{v} = J(\vec{r}_i, t)\ \partial\varrho(\vec{r}_i, t)/\partial t + \varrho(\vec{r}_i, t)\ \partial J(\vec{r}_i, t)/\partial t \\ &= \partial[\varrho(\vec{r}_i, t)\ J(\vec{r}_i, t)]/\partial t = 0 \quad \text{or} \quad \varrho J = \varrho_i. \end{aligned} \tag{5.2.8}$$

$\varrho_i = \varrho(\vec{r}_i, 0)$ denotes the fluid density at the initial moment $t = 0$ when $J = |\partial(x_{i1}, x_{i2}, x_{i3}) /\partial(x_{i1}, x_{i2}, x_{i3})| = 1$. In view of Eq. (5.2.7), the first part of Eq. (5.2.8) can also be written as (Cox 1980)

$$\begin{aligned} &J(\vec{r}_i, t)\ \partial\varrho(\vec{r}_i, t)/\partial t + \varrho(\vec{r}_i, t)\ \partial J(\vec{r}_i, t)/\partial t = J(\vec{r}_i, t)\ \partial\varrho(\vec{r}_i, t)/\partial t + J(\vec{r}_i, t)\ \varrho(\vec{r}_i, t)\ \nabla\cdot\vec{v} = 0 \\ &\text{or} \quad \partial\varrho/\partial t + \varrho\ \nabla\cdot\vec{v} = 0, \end{aligned} \tag{5.2.9}$$

which is the Lagrangian form of the Eulerian continuity equation (5.2.2).

The Eulerian form of the equation of motion (equation of momentum conservation) for the ideal fluid without magnetic fields can be found by particularization of Eq. (2.1.1), $(\vec{v}_0 \to \vec{v};\ \vec{H}, \tau = 0;\ \vec{F} = \nabla\Phi)$:

$$\varrho\ D\vec{v}(\vec{r}, t)/Dt = \varrho\ D\vec{v}/Dt = \varrho\ \partial\vec{v}/\partial t + \varrho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \varrho\ \nabla\Phi. \tag{5.2.10}$$

To obtain the Lagrangian equation of motion, we have simply to replace $(D\vec{v}/Dt)_{Eulerian}$ by $(\partial\vec{v}/\partial t)_{Lagrangian}$, as already outlined in Eqs. (5.1.5), (5.1.7), (5.1.8):

$$\varrho\ \partial\vec{v}(\vec{r}_i,t)/\partial t = \varrho\ \partial^2\vec{r}(\vec{r}_i,t)/\partial t^2 = -\nabla P + \varrho\ \nabla\Phi. \tag{5.2.11}$$

We turn the nabla operators to Lagrangian coordinates by writing $\nabla P = (\partial P/\partial x_k)\vec{e}_k = (\partial P/\partial x_{i\ell})(\partial x_{i\ell}/\partial x_k)\vec{e}_k$, and an analogous expression for $\nabla\Phi$. Thus, Eq. (5.2.11) finally becomes

$$\varrho\ \partial^2\vec{r}/\partial t^2 = -(\partial P/\partial x_{i\ell})\ \nabla x_{i\ell} + \varrho(\partial\Phi/\partial x_{i\ell})\ \nabla x_{i\ell}, \tag{5.2.12}$$

where again, summation over ℓ is to be understood.

The foregoing equations of mass and momentum conservation should also be augmented by the energy conservation equation, which is a fairly simple matter for the reversible (quasistatic) polytropic processes considered in this book (see Sec. 1.1). For reversible processes Eq. (1.1.4) writes $dQ = T\ dS$, and the first law of thermodynamics (the conservation of thermal energy alone) from Eq. (1.1.3) becomes for the mass unit

$$T\ dS = dQ = dU + P\ d(1/\varrho), \qquad (m = 1;\ \varrho = 1/V). \tag{5.2.13}$$

T, S, Q, and U are the temperature, entropy, quantity of heat energy, and internal energy of the fluid. If we consider the conservation of thermal energy in the course of time, as we follow the fluid motion, we have to replace the differentials in Eq. (5.2.13) by the material derivative, dividing Eq. (5.2.13) by Dt :

$$\begin{aligned} &T\ DS/Dt = DQ/Dt = DU/Dt + P\ D(1/\varrho)/Dt = \partial U/\partial t + \vec{v}\cdot\nabla U + (P/\varrho)\ \nabla\cdot\vec{v} \quad \text{or} \\ &\varrho T(\partial S/\partial t + \vec{v}\cdot\nabla S) = \partial(\varrho U)/\partial t + \nabla\cdot(\varrho U\vec{v}) + P\ \nabla\cdot\vec{v}. \end{aligned} \tag{5.2.14}$$

Another useful form of the first law of thermodynamics (5.2.13) may be derived by regarding the internal energy as a function of pressure and density $U = U(P,\varrho)$, when the temperature T has been eliminated through the equation of state $T = T(P,\varrho)$. Eq. (5.2.13) becomes

$$T\ dS = dQ = (\partial U/\partial P)_\varrho\ dP + [(\partial U/\partial\varrho)_P - P/\varrho^2]\ d\varrho. \tag{5.2.15}$$

In the adiabatic case Eq. (5.2.15) is equal to zero ($dS, dQ = 0$), and we obtain for the generalized adiabatic exponent from Eq. (1.3.23):

$$\Gamma_1 = (d\ln P/d\ln\varrho)_S = \varrho[P/\varrho^2 - (\partial U/\partial\varrho)_P]/P(\partial U/\partial P)_\varrho. \tag{5.2.16}$$

Inserting this into Eq. (5.2.15), we get

$$dQ = P(\partial U/\partial P)_\varrho\ (d\ln P - \Gamma_1\ d\ln\varrho). \tag{5.2.17}$$

Using the definition of the specific heat at constant volume $c_V = (dQ/dT)_V = (dQ/dT)_\varrho = (\partial U/\partial T)_\varrho,\ (\varrho = 1/V)$, and the notation (1.3.3), we have

$$(\partial U/\partial P)_\varrho = (\partial U/\partial T)_\varrho\ (\partial T/\partial P)_\varrho = c_V T/P\chi_T = 1/\varrho(\Gamma_3 - 1), \tag{5.2.18}$$

where we have inserted from Eq. (1.3.14) for the generalized adiabatic exponent $\Gamma_3 - 1 = P\chi_T/c_V\varrho T$, $(c = dQ/dT = 0)$. Thus, another form of Eq. (5.2.17) is

$$[\varrho(\Gamma_3 - 1)/P]\ dQ = d\ln P - \Gamma_1\ d\ln\varrho, \qquad (\varrho = 1/V). \tag{5.2.19}$$

If we apply the first law of thermodynamics (5.2.19) to a certain mass element along its path, we have to replace in Eulerian description the differentials by material derivatives (Tassoul 1978, Cox 1980):

$$[\varrho(\Gamma_3 - 1)/P]\ DQ/Dt = D\ln P/Dt - \Gamma_1\ D\ln\varrho/Dt. \tag{5.2.20}$$

For adiabatic reversible oscillations ($S, Q = \text{const}$), Eq. (5.2.20) becomes

$$DP/Dt = (\Gamma_1 P/\varrho)\ D\varrho/Dt. \tag{5.2.21}$$

The material derivative in Eulerian description DF/Dt is by virtue of Eq. (5.1.5) equal to the partial time derivative $\partial F/\partial t$ in Lagrangian description, so the energy equation (5.2.20) writes in Lagrangian variables (cf. Ledoux and Walraven 1958, §84):

$$[\varrho(\Gamma_3 - 1)/P]\ \partial Q/\partial t = \partial \ln P/\partial t - \Gamma_1\ \partial \ln \varrho/\partial t. \tag{5.2.22}$$

The quoted general equations form a system of nonlinear partial differential equations that can be solved only in very particular cases. In linear perturbation theory we can neglect all powers higher than the first, and the relevant perturbed equations take a somewhat simpler form. The linear Eulerian variations of velocity, pressure, density, and potential are [cf. Eq. (5.1.13)]:

$$\vec{v}(\vec{r},t) = \vec{v}_u(\vec{r},t) + \delta\vec{v}(\vec{r},t);\quad \varrho(\vec{r},t) = \varrho_u(\vec{r},t) + \delta\varrho(\vec{r},t);$$
$$P(\vec{r},t) = P_u(\vec{r},t) + \delta P(\vec{r},t);\quad \Phi(\vec{r},t) = \Phi_u(\vec{r},t) + \delta\Phi(\vec{r},t). \tag{5.2.23}$$

u-indexed values characterize the unperturbed flow (equilibrium state). Eq. (5.2.23) is inserted into the Eulerian equation of continuity (5.2.1), taking into account that the unperturbed values satisfy Eq. (5.2.1), $(\partial\varrho_u/\partial t + \nabla\cdot(\varrho_u\vec{v}_u) = 0)$:

$$\partial\delta\varrho/\partial t + \nabla\cdot(\varrho_u\ \delta\vec{v} + \vec{v}_u\ \delta\varrho) = \partial\delta\varrho/\partial t + \nabla\cdot(\varrho\ \delta\vec{v} + \vec{v}\ \delta\varrho) = 0. \tag{5.2.24}$$

We have dropped the index u, since in the linear approximation perturbed and unperturbed quantities differ only by small first order quantities. If the unperturbed fluid is in hydrostatic equilibrium, we find in this important special case

$$\partial\delta\varrho/\partial t + \nabla\cdot(\varrho_u\vec{v}) = 0, \qquad (\vec{v}_u = 0;\ |\vec{v}| \approx 0;\ \vec{v} \approx \delta\vec{v}). \tag{5.2.25}$$

This equation can be transformed further, by writing via Eq. (5.1.24) $\varrho_u\vec{v} \approx \varrho_u\ \partial\Delta\vec{r}/\partial t = \partial(\varrho_u\ \Delta\vec{r})/\partial t,\ (\partial\varrho_u/\partial t = 0)$:

$$\partial\delta\varrho/\partial t + \nabla\cdot[\partial(\varrho_u\ \Delta\vec{r})/\partial t] = \partial\delta\varrho/\partial t + \partial[\nabla\cdot(\varrho_u\ \Delta\vec{r})]/\partial t = 0. \tag{5.2.26}$$

We integrate with respect to the time, the integration constant becoming zero due to the obvious initial condition $\delta\varrho = 0$ if $\Delta\vec{r} = 0$:

$$\delta\varrho + \nabla\cdot(\varrho\ \Delta\vec{r}) = 0, \qquad (|\vec{v}| \approx 0;\ \varrho \approx \varrho_u). \tag{5.2.27}$$

Turning with Eq. (5.1.16) to the Lagrangian description, we get

$$\delta\varrho + \nabla\cdot(\varrho\ \Delta\vec{r}) = \delta\varrho + \nabla\varrho\cdot\Delta\vec{r} + \varrho\ \nabla\cdot\Delta\vec{r} = \Delta\varrho + \varrho\ \nabla\cdot\Delta\vec{r} = 0, \qquad (|\vec{v}| \approx 0). \tag{5.2.28}$$

The Eulerian perturbation of the equation of motion (5.2.10) writes analogously to Eq. (5.2.24):

$$\partial\delta\vec{v}/\partial t + (\delta\vec{v}\cdot\nabla)\vec{v}_u + (\vec{v}_u\cdot\nabla)\delta\vec{v} = (\delta\varrho/\varrho_u^2)\ \nabla P_u - (1/\varrho_u)\ \nabla\delta P + \nabla\delta\Phi \quad \text{or}$$
$$\partial\delta\vec{v}/\partial t + (\delta\vec{v}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\delta\vec{v} = (\delta\varrho/\varrho^2)\ \nabla P - (1/\varrho)\ \nabla\delta P + \nabla\delta\Phi = \delta[-(1/\varrho)\ \nabla P + \nabla\Phi]. \tag{5.2.29}$$

If $\vec{v}_u = 0$, Eq. (5.2.29) becomes [$\vec{v} = \delta\vec{v} \approx \partial\Delta\vec{r}/\partial t$; Eqs. (5.1.23), (5.1.24)]:

$$\partial\delta\vec{v}/\partial t = \partial\vec{v}/\partial t = \partial^2\Delta\vec{r}/\partial t^2 = \delta[(-1/\varrho)\ \nabla P + \nabla\Phi], \qquad (|\vec{v}| \approx 0). \tag{5.2.30}$$

The Lagrangian variation of various fluid characteristics is by virtue of Eqs. (5.1.11) and (5.1.15) equal to

$$\vec{r} = \vec{r}_u + \Delta\vec{r};\quad \vec{v}(\vec{r},t) = \vec{v}_u(\vec{r}_u,t) + \Delta\vec{v};\quad P(\vec{r},t) = P_u(\vec{r}_u,t) + \Delta P;\quad \varrho(\vec{r},t) = \varrho_u(\vec{r}_u,t) + \Delta\varrho;$$
$$\Phi(\vec{r},t) = \Phi_u(\vec{r}_u,t) + \Delta\Phi;\ \ Q(\vec{r},t) = Q_u(\vec{r}_u,t) + \Delta Q;\ \ \Gamma_k(\vec{r},t) = \Gamma_{ku}(\vec{r}_u,t) + \Delta\Gamma_k,\quad (k = 1,2,3). \tag{5.2.31}$$

As the Lagrangian equation of motion (5.2.12) is generally fairly complicated [cf. Eq. (5.2.48)], we consider simply the Lagrangian variation of the Eulerian equation (5.2.10) by using Eqs. (5.1.16), (5.1.21), (5.1.22):

$$\Delta(D\vec{v}/Dt) = D(\Delta\vec{v})/Dt = D^2(\Delta\vec{r})/Dt^2 = \Delta[-(1/\varrho)\ \nabla P + \nabla\Phi]$$
$$= \delta[-(1/\varrho)\ \nabla P + \nabla\Phi] + (\Delta\vec{r}\cdot\nabla)[-(1/\varrho)\ \nabla P + \nabla\Phi]. \tag{5.2.32}$$

The equivalence of Eq. (5.2.29) with Eq. (5.2.32) can be shown at once, by using Eqs. (5.1.16) and (5.2.10) to transform the last term of Eq. (5.2.32), (Cox 1980):

$$(\Delta\vec{r}\cdot\nabla)[-(1/\varrho)\ \nabla P+\nabla\Phi]=(\Delta\vec{r}\cdot\nabla)\ D\vec{v}/Dt. \tag{5.2.33}$$

Introducing Eq. (5.2.33) into Eq. (5.2.32), we have

$$\Delta(D\vec{v}/Dt)-(\Delta\vec{r}\cdot\nabla)(D\vec{v}/Dt)=\delta(D\vec{v}/Dt)=\delta(\partial\vec{v}/\partial t)+\delta[(\vec{v}\cdot\nabla)\vec{v}]=\partial\delta\vec{v}/\partial t+(\delta\vec{v}\cdot\nabla)\vec{v}$$
$$+(\vec{v}\cdot\delta\nabla)\vec{v}=\partial\delta\vec{v}/\partial t+(\delta\vec{v}\cdot\nabla)\vec{v}+(\vec{v}\cdot\nabla)\delta\vec{v}=\delta[-(1/\varrho)\ \nabla P+\nabla\Phi], \tag{5.2.34}$$

which is just the same as Eq. (5.2.29).

Following a certain mass element along its motion, we apply the Lagrangian variation (5.1.21) to the equation of thermal energy conservation (5.2.20):

$$[\Delta\Gamma_3/(\Gamma_{3u}-1)+\Delta\varrho/\varrho_u-\Delta P/P_u][(\Gamma_{3u}-1)\varrho_u/P_u](DQ/Dt)_u+[(\Gamma_{3u}-1)\varrho_u/P_u]\ D(\Delta Q)/Dt$$
$$=D(\Delta P/P_u)/Dt-\Gamma_{1u}\ D(\Delta\varrho/\varrho_u)/Dt-\Delta\Gamma_1\ (D\ln\varrho/Dt)_u. \tag{5.2.35}$$

If the system in its unperturbed state is in hydrostatic and thermal equilibrium ($\vec{v}_u=0$), we have $(DQ/Dt)_u,\ (D\varrho/Dt)_u=0$. In the first order linear approximation Eq. (5.2.35) takes the form

$$[(\Gamma_{3u}-1)\varrho_u/P_u]\ D(\Delta Q)/Dt=[(\Gamma_{3u}-1)\varrho_u/P_u]\ D(T\ \Delta S)/Dt$$
$$=D(\Delta P/P_u)/Dt-\Gamma_{1u}\ D(\Delta\varrho/\varrho_u)/Dt,\qquad(\vec{v}_u=0;\ T\ \Delta S=\Delta Q). \tag{5.2.36}$$

If the motions are adiabatic (isentropic) we have $\Delta S,\Delta Q=0$. The equation of thermal energy conservation (5.2.36) becomes simply

$$D(\Delta P/P_u)/Dt=\Gamma_{1u}\ D(\Delta\varrho/\varrho_u)/Dt,\qquad(\vec{v}_u,\Delta S,\Delta Q=0). \tag{5.2.37}$$

We may trivially integrate this equation with respect to the time, replacing up to the first order $P_u,\varrho_u,\Gamma_{1u}$ with P,ϱ,Γ_1, respectively:

$$\Delta P=(\Gamma_1 P/\varrho)\ \Delta\varrho=-\Gamma_1 P\ \nabla\cdot\Delta\vec{r},\qquad(Q,S=\mathrm{const}). \tag{5.2.38}$$

We have also inserted from the continuity equation (5.2.28), and the integration constant is zero, due to the condition $\Delta P=0$ if $\Delta\varrho=0$.

The Eulerian adiabatic pressure change is via Eq. (5.1.16) equal to (Tassoul 1978)

$$\delta P=\Delta P-\Delta\vec{r}\cdot\nabla P=-\Gamma_1 P\ \nabla\cdot\Delta\vec{r}-\Delta\vec{r}\cdot\nabla P,\qquad(\vec{v}_u=0). \tag{5.2.39}$$

Because the Eulerian variation δ commutes by virtue of Eq. (5.1.18) with ∇, we apply this variation to Poisson's equation (2.1.4):

$$\nabla^2(\delta\Phi)=-4\pi G\ \delta\varrho. \tag{5.2.40}$$

A further important simplification – which appears adequate for the theoretical treatment of most pulsating stars – concerns radial, spherically symmetric oscillations with displacements as given by Eq. (5.1.32): $\Delta r(r,t)=\Delta r(r)\exp(i\sigma t)$. Since the mass m inside radial coordinate is conserved, we can write in Lagrangian description

$$m=\int_0^{r_i}4\pi\varrho_i(r',0)\ {r'}^2\ dr'=\int_0^r 4\pi\varrho(r',t)\ {r'}^2\ dr', \tag{5.2.41}$$

or in differential form

$$dm=4\pi\varrho_i(r_i,0){r_i}^2\ dr_i=4\pi\varrho(r,t)r^2\ dr;\quad \partial r/\partial r_i=r_i^2\varrho_i/r^2\varrho;\quad \partial r/\partial m=1/4\pi\varrho r^2. \tag{5.2.42}$$

To obtain the linearized mass conservation equation for radial oscillations in Lagrangian description, we insert the Lagrangian variations (5.2.31) and

$$\eta=\Delta r/r_u;\quad r=r_u(1+\eta),\qquad(\eta\ll 1), \tag{5.2.43}$$

into the mass conservation equation (5.2.42):

$$\partial r/\partial m = (\partial r/\partial r_u)\ \partial r_u/\partial m = (1/4\pi\varrho_u r_u^2)\ \partial[r_u(1+\eta)]/\partial r_u = 1/4\pi\varrho r^2$$
$$= 1/4\pi r_u^2(\varrho_u + \Delta\varrho)(1+\eta)^2. \tag{5.2.44}$$

Performing the elementary derivation and expanding up to the first order, we get

$$\Delta\varrho/\varrho_u = -3\eta - r_u\ \partial\eta/\partial r_u \quad \text{or} \quad \Delta\varrho/\varrho = -3\eta - r\ \partial\eta/\partial r. \tag{5.2.45}$$

The Lagrangian equation of motion (5.2.12) becomes for spherically symmetric radial motion, by using Eq. (5.2.42), $(\partial\Phi/\partial r = -Gm(r)/r^2)$:

$$\partial^2 r/\partial t^2 = [-(1/\varrho)\ \partial P/\partial r_i + \partial\Phi/\partial r_i]\ \partial r_i/\partial r = -4\pi r^2\ \partial P/\partial m - Gm(r)/r^2. \tag{5.2.46}$$

The linearized Lagrangian equation of perturbed radial motion is obtained in the same way as the linearized mass conservation equation (5.2.45), by inserting Eqs. (5.2.43), (5.2.44) into Eq. (5.2.46), where we assume

$$\partial r_u/\partial t,\ \partial^2 r_u/\partial t^2 = 0, \qquad (\vec{v}_u = 0). \tag{5.2.47}$$

Eq. (5.2.46) becomes

$$\partial^2(r_u + \Delta r)/\partial t^2 = \partial^2\Delta r/\partial t^2 = r_u\ \partial^2\eta/\partial t^2 = -[(r_u+\Delta r)^2/\varrho_u r_u^2]\ \partial(P_u + \Delta P)/\partial r_u$$
$$-Gm/(r_u+\Delta r)^2 = -(2\Delta r/\varrho_u r_u)\ dP_u/dr_u - (1/\varrho_u)\ \partial\Delta P/\partial r_u + 2Gm\ \Delta r/r_u^3$$
$$= 4Gm\eta/r_u^2 - (1/\varrho_u)\ \partial\Delta P/\partial r_u = -(4\eta/\varrho_u)\ dP_u/dr_u - (1/\varrho_u)\ \partial\Delta P/\partial r_u$$
$$= -(4\eta/\varrho_u)\ dP_u/dr_u - (1/\varrho_u)(\Delta P/P_u)\ dP_u/dr_u - (P_u/\varrho_u)\ \partial(\Delta P/P_u)/\partial r_u. \tag{5.2.48}$$

We have used Eq. (5.2.47) and the equation of hydrostatic equilibrium $(1/\varrho_u)\ dP_u/dr_u = -Gm/r_u^2$. The derivation of $\partial\Delta P/\partial r_u$ has been performed according to $\partial\Delta P/\partial r_u = \partial[P_u(\Delta P/P_u)]/\partial r_u$.

The perturbed linearized energy conservation equation in Lagrangian description can be obtained from Eq. (5.2.36) via Eq. (5.1.5), $(D\ /Dt \to \partial\ /\partial t)$:

$$[(\Gamma_{3u}-1)\varrho_u/P_u]\ \partial\Delta Q/\partial t = [(\Gamma_{3u}-1)\varrho_u/P_u]\ \partial(T\ \Delta S)/\partial t = \partial(\Delta P/P_u)/\partial t - \Gamma_{1u}\ \partial(\Delta\varrho/\varrho_u)/\partial t. \tag{5.2.49}$$

The perturbed linearized Lagrangian equations of mass, momentum, and energy conservation [Eqs. (5.2.45), (5.2.48), (5.2.49)] may be combined into a single equation (5.2.56) governing linear, radial nonadiabatic oscillations. We rewrite Eq. (5.2.48) under the form

$$r_u\ \partial^2\eta/\partial t^2 = -4\pi r_u^2[4\eta\ dP_u/dm + (\Delta P/P_u)\ dP_u/dm + P_u\ \partial(\Delta P/P_u)/\partial m]. \tag{5.2.50}$$

Taking the material derivative of Eq. (5.2.50) in the Lagrangian description from Eq. (5.1.5), we get

$$r_u\ \partial^3\eta/\partial t^3 = -4\pi r_u^2\{4(\partial\eta/\partial t)\ dP_u/dm + [\partial(\Delta P/P_u)/\partial t]\ dP_u/dm + P_u\ \partial^2(\Delta P/P_u)/\partial m\partial t\}. \tag{5.2.51}$$

$\partial(\Delta P/P_u)/\partial t$ can now be eliminated with the energy equation (5.2.49):

$$r_u\ \partial^3\eta/\partial t^3 = -4\pi r_u^2\{4(\partial\eta/\partial t)\ dP_u/dm + \Gamma_{1u}[\partial(\Delta\varrho/\varrho_u)/\partial t]\ dP_u/dm$$
$$+P_u\ \partial[\Gamma_{1u}\ \partial(\Delta\varrho/\varrho_u)/\partial t]/\partial m + \partial[(\Gamma_{3u}-1)\varrho_u\ \partial\Delta Q/\partial t]/\partial m\}, \tag{5.2.52}$$

where the last term is $[(\Gamma_{3u}-1)\varrho_u/P_u](\partial\Delta Q/\partial t)\ dP_u/dm + P_u\ \partial\{[(\Gamma_{3u}-1)\varrho_u/P_u]\ \partial\Delta Q/\partial t\}/\partial m$.

Up to the first order we get from Eq. (5.2.45)

$$\partial(\Delta\varrho/\varrho_u)/\partial t = -3\ \partial\eta/\partial t - 4\pi\varrho_u r_u^3\ \partial^2\eta/\partial m\partial t. \tag{5.2.53}$$

We insert into Eq. (5.2.52), and find after some obvious algebra:

$$r_u\ \partial^3\eta/\partial t^3 = 4\pi r_u^2\{(\partial\eta/\partial t)\ d[(3\Gamma_{1u}-4)P_u]/dm + 3\Gamma_{1u}P_u\ \partial^2\eta/\partial m\partial t$$
$$+\partial(4\pi\Gamma_{1u}r_u^3 P_u\varrho_u\ \partial^2\eta/\partial m\partial t)/\partial m - \partial[(\Gamma_{3u}-1)\varrho_u\ \partial\Delta Q/\partial t]/\partial m\}. \tag{5.2.54}$$

We note the identity

$$\partial(4\pi\Gamma_{1u}r_u^6 P_u\varrho_u\ \partial^2\eta/\partial m\partial t)/\partial m = 4\pi r_u^3\ \partial(\Gamma_{1u}r_u^3 P_u\varrho_u\ \partial^2\eta/\partial m\partial t)/\partial m + 3\Gamma_{1u}r_u^3 P_u\ \partial^2\eta/\partial m\partial t,$$
$$(\partial r_u^3/\partial m = 3/4\pi\varrho_u), \tag{5.2.55}$$

so the second and third term in Eq. (5.2.54) can be put together, and we obtain the final expression obeyed by $\eta = \Delta r/r_u \approx \Delta r/r$ for linear, radial and nonadiabatic oscillations:

$$\partial^3\eta/\partial t^3 = 4\pi r_u(\partial\eta/\partial t)\ d[(3\Gamma_{1u}-4)P_u]/dm + (1/r_u^2)\ \partial(16\pi^2\Gamma_{1u}r_u^6 P_u\varrho_u\ \partial^2\eta/\partial m\partial t)/\partial m$$
$$-4\pi r_u\ \partial[(\Gamma_{3u}-1)\varrho_u\ \partial\Delta Q/\partial t]/\partial m. \tag{5.2.56}$$

For small adiabatic oscillations ($Q=$ const) the last term disappears. We integrate in this case Eq. (5.2.56) with respect to the time, the integration constant being equal zero, i.e. $\eta = 0$ corresponds to the unperturbed hydrostatic state of the system:

$$\partial^2\eta/\partial t^2 = 4\pi r_u\eta\ d[(3\Gamma_{1u}-4)P_u]/dm + (1/r_u^2)\ \partial(16\pi^2\Gamma_{1u}r_u^6 P_u\varrho_u\ \partial\eta/\partial m)/\partial m. \tag{5.2.57}$$

We now assume a standing wave solution of Eq. (5.2.57) under the form (5.1.32):

$$\eta(r,t) = \Delta r(r,t)/r_u = [\Delta r(r)/r_u]\exp(i\sigma t) = \eta(r)\exp(i\sigma t); \quad \eta(r) = \Delta r(r)/r_u. \tag{5.2.58}$$

Eq. (5.2.57) becomes equal to the linear adiabatic wave equation, by suppressing the common factor $\exp(i\sigma t)$ and the index u, $(\partial\ /\partial m = (1/4\pi\varrho_u r_u^2)\ \partial\ /\partial r_u)$:

$$d(\Gamma_1 P r^4\ d\eta/dr)/dr + \eta\{r^3\ d[(3\Gamma_1-4)P]/dr + \sigma^2\varrho r^4\} = 0. \tag{5.2.59}$$

We may eliminate the pressure gradient by the equation of hydrostatic equilibrium $dP/dr = -G\varrho m(r)/r^2$:

$$d\eta^2/dr^2 + \left[4/r + (1/\Gamma_1)\ d\Gamma_1/dr - G\varrho\ m(r)/Pr^2\right]\ d\eta/dr$$
$$+\left[(4/\Gamma_1 - 3)G\varrho\ m(r)/Pr^3 + (3/\Gamma_1 r)\ d\Gamma_1/dr + \sigma^2\varrho/\Gamma_1 P\right]\eta = 0. \tag{5.2.60}$$

The boundary conditions usually imposed on Eqs. (5.2.59) or (5.2.60) at the centre $r=0$ and at the surface $r=r_1$ are (Ledoux and Walraven 1958, Cox 1980)

$$\Delta r = r\eta = 0, \qquad (r=0), \tag{5.2.61}$$

as demanded by spherical symmetry, and

$$\Delta P = 0, \qquad (r=r_1). \tag{5.2.62}$$

Eq. (5.2.62) means that the total surface pressure remains always zero as we follow the motion of the surface during oscillations.

Because of the factors $4/r$ and $(3/\Gamma_1 r)\ d\Gamma_1/dr$ in Eq. (5.2.60), the point $r=0$ is a first-order pole of the coefficients of $d\eta/dr$ and η, respectively. Close to the surface $\varrho/P = \varrho_0/P_0\theta$, $(P = K\varrho^{1+1/n} = P_0\theta^{n+1};\ \varrho = \varrho_0\theta^n)$ tends toward infinity if $0 \le n < 5$, and the coefficients of $d\eta/dr$ and η become singular again. We have to seek solutions η that are regular (continuous solutions with continuous derivatives) at the points $r=0$ and $r=r_1$.

In virtue of Eq. (5.1.16) the outer boundary condition (5.2.62) can also be written under the form

$$\Delta P = \delta P + (dP/dr)\ \Delta r = 0, \qquad (r=r_1). \tag{5.2.63}$$

If the density vanishes on the surface, then $(dP/dr)_{r=r_1} = -G\varrho_1 m(r_1)/r_1^2 = 0$ and $\Delta P = \delta P = 0$ at $r=r_1$. Eq. (5.2.62) can be transformed further, by replacing ΔP with the adiabatic form (5.2.38) of the energy equation, and by inserting for $\Delta\varrho/\varrho$ from Eq. (5.2.45):

$$\Delta P = \Gamma_1 P\ \Delta\varrho/\varrho = -\Gamma_1 P(3\eta + r\ d\eta/dr) = 0, \qquad (r=r_1). \tag{5.2.64}$$

For the polytropic indices of practical interest ($0 \le n < 5$) we can obtain another explicit expression of the surface boundary condition, suitable for practical applications. To this end we solve the perturbed Lagrangian equation of motion (5.2.48) for $\partial(\Delta P/P)/\partial r$, by dropping the index u :

$$\partial(\Delta P/P)/\partial r = -(d\ln P/dr)\big[\varrho r(\partial^2\eta/\partial t^2)/(dP/dr) + 4\eta + \Delta P/P\big]$$
$$= -(d\ln P/dr)\big[-r^3(\partial^2\eta/\partial t^2)/Gm + 4\eta + \Delta P/P\big]. \qquad (5.2.65)$$

Near the surface we have $d(\ln P)/dr = -Gm\varrho/r^2P \propto \varrho/P = \varrho_0/P_0\theta \to \infty,\ (0 \le n < 5)$, and since $\partial(\Delta P/P)/\partial r$ has to remain finite, we conclude that the last factor from Eq. (5.2.65) must be very small near the boundary, i.e.

$$-r^3(\partial^2\eta/\partial t^2)/Gm + 4\eta + \Delta P/P \approx 0, \qquad (0 \le n < 5;\ r \approx r_1). \qquad (5.2.66)$$

Assuming further the temporal dependences under the form $\eta(r,t) = \eta(r)\exp(i\sigma t)$, $\Delta P(r,t) = \Delta P(r)\exp(i\sigma t)$, and dropping the common factor $\exp(i\sigma t)$, Eq. (5.2.66) becomes (Sterne 1937, Cox 1980):

$$\Delta P/P = -\eta(r^3\sigma^2/Gm + 4), \qquad (0 \le n < 5;\ r = r_1). \qquad (5.2.67)$$

Inserting the adiabatic change (5.2.38) and the mass conservation equation (5.2.45), we get the boundary condition (5.2.62) under the final form

$$r\ d\eta/dr = \eta[(4/\Gamma_1 - 3) + r^3\sigma^2/\Gamma_1 Gm], \qquad (0 \le n < 5;\ r = r_1). \qquad (5.2.68)$$

In fact, we may obtain this equation at once from Eq. (5.2.60), by preserving near the surface $r = r_1$ only the dominant terms containing $\varrho/P \to \infty$ [cf. Ledoux and Walraven 1958, Eq. (58.17)].

Eq. (5.2.59) or (5.2.60) is a second order linear homogeneous equation, so its solution must involve two constants of integration, with one constant of integration remaining arbitrary [if η is a solution, then so is $C\eta,\ (C = \mathrm{const})$]. The remaining integration constant can clearly be used to satisfy only one boundary condition, and the sole disposable additional parameter is the angular oscillation frequency σ, which has to be varied until the second boundary condition is satisfied. Hence, only certain eigenfrequencies (eigenvalues, characteristic values) $\sigma_0, \sigma_1, \sigma_2, ...$, and the corresponding eigensolutions (proper solutions) $\eta_0, \eta_1, \eta_2, ...$ can satisfy Eq. (5.2.59). Since this equation is of the Sturm-Liouville type, its infinite discrete set of real eigenvalues can be ordered by increasing values of $\sigma_k^2,\ (\sigma_0^2 < \sigma_1^2 < \sigma_2^2 < ...)$, (e.g. Ledoux and Walraven 1958). For the k-th mode Eq. (5.2.59) may be written under the concise form

$$L(\eta_k) = \sigma_k^2\eta_k, \qquad (5.2.69)$$

where the linear operator is via Eq. (5.2.59) equal to

$$L(\eta) = -(1/\varrho r^4)\ d(\Gamma_1 P r^4\ d\eta/dr)/dr - (\eta/\varrho r)\ d[(3\Gamma_1 - 4)P]/dr, \qquad (\eta(r) = \Delta r(r)/r \ll 1). \qquad (5.2.70)$$

The eigenfunction η_0 corresponding to the smallest eigenvalue σ_0 will be called the fundamental mode. The eigenfunctions $\eta_1, \eta_2, ...$ are called, following Eddington, the first, second, etc. overtone (harmonic).

We now turn to the principal linear equations for the study of *nonradial* oscillations of an undistorted sphere. At first we write out the equation of continuity (5.2.28) in spherical coordinates:

$$\delta\varrho/\varrho + (\Delta r/\varrho)\ \partial\varrho/\partial r + (1/r^2)\ \partial(r^2\ \Delta r)/\partial r + (1/r\sin\lambda)\ \partial(r\sin\lambda\ \Delta\lambda)/\partial\lambda$$
$$+(1/r\sin\lambda)\ \partial(r\sin\lambda\ \Delta\varphi)/\partial\varphi = 0. \qquad (5.2.71)$$

The linearized momentum equation suitable for the study of small nonradial oscillations would be given for instance by Eq. (5.2.30):

$$\partial^2\Delta\vec{r}/\partial t^2 = (\delta\varrho/\varrho^2)\ \nabla P - (1/\varrho)\ \nabla\delta P + \nabla\delta\Phi, \qquad (|\vec{v}| \approx 0). \qquad (5.2.72)$$

We insert for the Lagrangian displacement via Eq. (5.1.29), and for the relevant Eulerian perturbations from Eq. (5.1.31), suppressing the common factor $\exp(i\sigma t)$:

$$\sigma^2\ \Delta\vec{r} = -(\delta\varrho/\varrho^2)\ \nabla P + (1/\varrho)\ \nabla\delta P - \nabla\delta\Phi = \delta[(1/\varrho)\ \nabla P - \nabla\Phi]. \qquad (5.2.73)$$

We write out the components up to the first order ($\delta\varrho\ \nabla P = \delta\varrho\ dP_u/dr$) :

$$\sigma^2\ \Delta r = -(\delta\varrho/\varrho_u^2)\ dP_u/dr + (1/\varrho_u)\ \partial\delta P/\partial r - \partial\delta\Phi/\partial r, \tag{5.2.74}$$

$$\sigma^2 r\ \Delta\lambda = (1/r)\ \partial(\delta P/\varrho_u + \delta\Phi)/\partial\lambda, \tag{5.2.75}$$

$$\sigma^2 r \sin\lambda\ \Delta\varphi = (1/r\sin\lambda)\ \partial(\delta P/\varrho_u + \delta\Phi)/\partial\varphi. \tag{5.2.76}$$

Integration with respect to t of the equation of thermal energy conservation (5.2.36) yields

$$[(\Gamma_3 - 1)\varrho/P]\ \Delta Q = \Delta(\ln P) - \Gamma_1\ \Delta(\ln\varrho), \tag{5.2.77}$$

after interchanging in our linear approximation perturbed and unperturbed quantities, when multiplied by small first order terms. Using Eq. (5.1.16), we find from Eq. (5.2.77)

$$(\Gamma_3 - 1)\varrho(\delta Q + \Delta\vec{r}\cdot\nabla Q) = \delta P + \Delta\vec{r}\cdot\nabla P - (\Gamma_1 P/\varrho)(\delta\varrho + \Delta\vec{r}\cdot\nabla\varrho), \tag{5.2.78}$$

and finally, up to the first order:

$$(\Gamma_3 - 1)\varrho(\delta Q + \Delta r\ \partial Q/\partial r) = \delta P + \Delta r\ \partial P/\partial r - (\Gamma_1 P/\varrho)(\delta\varrho + \Delta r\ \partial\varrho/\partial r). \tag{5.2.79}$$

The equation of motion (5.2.73) can also be brought into another form, by inserting for $\delta\varrho/\varrho$ from Eq. (5.2.78):

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) + (\delta P/\varrho^2)\ \nabla\varrho - (\delta\varrho/\varrho^2)\ \nabla P = \nabla(\delta P/\varrho - \delta\Phi)$$
$$+[(1/\varrho)\ \nabla\varrho - (1/\Gamma_1 P)\ \nabla P][\delta P/\varrho + (1/\varrho)\ \Delta\vec{r}\cdot\nabla P] + [(\Gamma_3 - 1)/\Gamma_1 P](\delta Q + \Delta\vec{r}\cdot\nabla Q)\ \nabla P. \tag{5.2.80}$$

From the continuity equation (5.2.28) we have

$$\Delta\varrho = \delta\varrho + \Delta\vec{r}\cdot\nabla\varrho = -\varrho\ \nabla\cdot\Delta\vec{r}, \tag{5.2.81}$$

which is substituted into Eq. (5.2.80) via Eq. (5.2.78)

$$\delta P/\varrho + (1/\varrho)\ \Delta\vec{r}\cdot\nabla P = -(\Gamma_1 P/\varrho)\ \nabla\cdot\Delta\vec{r} + (\Gamma_3 - 1)(\delta Q + \Delta\vec{r}\cdot\nabla Q), \tag{5.2.82}$$

to obtain eventually

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) - [(1/\varrho)\ \nabla\varrho - (1/\Gamma_1 P)\ \nabla P](\Gamma_1 P/\varrho)\ \nabla\cdot\Delta\vec{r}$$
$$+[(\Gamma_3 - 1)/\varrho](\delta Q + \Delta\vec{r}\cdot\nabla Q)\ \nabla\varrho. \tag{5.2.83}$$

The important quantity

$$\vec{A} = (1/\varrho)\ \nabla\varrho - (1/\Gamma_1 P)\ \nabla P = [1 - (1 + 1/n)/\Gamma_1]\ \nabla(\ln\varrho), \qquad (P = K\varrho^{1+1/n}), \tag{5.2.84}$$

occurring in the preceding equations turns for radial symmetry just into the well known K. Schwarzschild criterion of convective stability [e.g. Schwarzschild 1958, Eq. (7.1)]:

$$A = (1/\varrho)\ d\varrho/dr - (1/\Gamma_1 P)\ dP/dr = [1 - (1 + 1/n)/\Gamma_1]\ d\ln\varrho/dr, \qquad (P = K\varrho^{1+1/n}). \tag{5.2.85}$$

$A < 0$ demands stability against convection, whereas $A \geq 0$ requires instability against convective motions. Since generally $d\varrho/dr < 0$, convective stability demands $\Gamma_1 > 1 + 1/n$ or $n > 1/(\Gamma_1 - 1)$ if $n > 0$ (see Sec. 5.5).

In the case of adiabatic nonradial oscillations Eqs. (5.2.80) and (5.2.83) simplify to

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) + \vec{A}[\delta P/\varrho + (1/\varrho)\ \Delta\vec{r}\cdot\nabla P] = \nabla(\delta P/\varrho - \delta\Phi) - (\Gamma_1 P\vec{A}/\varrho)\ \nabla\cdot\Delta\vec{r}$$
$$= \nabla\chi - (\Gamma_1 P\vec{A}/\varrho)\ \nabla\cdot\Delta\vec{r}, \qquad (\chi = \delta P/\varrho - \delta\Phi;\ S, Q = \text{const}). \tag{5.2.86}$$

We assume for $\Delta r, \delta P, \delta\varrho, \delta\Phi, \chi$ series expansions into spherical surface harmonics, analogously to Eqs. (5.1.27), (5.1.31). For instance

$$\Delta r(r,\lambda,\varphi) = \Delta r(r)\ Y_j^k(\lambda,\varphi) = \Delta r(r)\ P_j^k(\cos\lambda)\exp(ik\varphi); \quad \chi(r,\lambda,\varphi) = \chi(r)\ Y_j^k(\lambda,\varphi)$$
$$= \chi(r)\ P_j^k(\cos\lambda)\exp(ik\varphi), \qquad (j = 0,1,2,...;\ k = -j,-j+1,...j-1,j). \tag{5.2.87}$$

The vector $\vec{A}$ can be equated in our linear approximation to its unperturbed value, which possesses only the radial component A from Eq. (5.2.85), due to the assumption of an unperturbed sphere. So, the scalar components of Eq. (5.2.86) are

$$\sigma^2\ \Delta r(r,\lambda,\varphi) = \partial\chi(r,\lambda,\varphi)/\partial r - (A\Gamma_1 P/\varrho)\ \nabla\cdot\Delta\vec{r}(r,\lambda,\varphi), \tag{5.2.88}$$

$$\sigma^2 r\ \Delta\lambda(r,\lambda,\varphi) = (1/r)\ \partial\chi(r,\lambda,\varphi)/\partial\lambda = [\chi(r)/r]\ \partial Y_j^k(\lambda,\varphi)/\partial\lambda, \tag{5.2.89}$$

$$\sigma^2 r\sin\lambda\ \Delta\varphi(r,\lambda,\varphi) = (1/r\sin\lambda)\ \partial\chi(r,\lambda,\varphi)/\partial\varphi$$
$$= [\chi(r)/r\sin\lambda]\ \partial Y_j^k(\lambda,\varphi)/\partial\varphi = ik\ \chi(r,\lambda,\varphi)/r\sin\lambda. \tag{5.2.90}$$

From Eqs. (5.1.16), (5.2.81), (5.2.88)-(5.2.90) we get up to the first order via Eqs. (5.1.26), (5.1.28):

$$-\Delta\varrho/\varrho = -\delta\varrho/\varrho - (1/\varrho)\ \Delta\vec{r}\cdot\nabla\varrho \approx -\delta\varrho/\varrho - (\Delta r/\varrho)\ d\varrho/dr = \nabla\cdot\Delta\vec{r}$$
$$= (1/r^2)\ \partial(r^2\ \Delta r)/\partial r + [\chi(r)/\sigma^2 r^2\sin\lambda]\ \partial(\sin\lambda\ \partial Y_j^k/\partial\lambda)/\partial\lambda + [\chi(r)/\sigma^2 r^2\sin^2\lambda]\ \partial^2 Y_j^k/\partial\varphi^2$$
$$= (1/r^2)\ \partial(r^2\ \Delta r)/\partial r - [j(j+1)/\sigma^2 r^2]\ \chi(r,\lambda,\varphi) = (1/r^2)\ \partial(r^2\ \Delta r)/\partial r$$
$$-[j(j+1)/\sigma^2 r^2](\delta P/\varrho - \delta\Phi). \tag{5.2.91}$$

The energy equation (5.2.79) becomes with Eq. (5.2.81) equal to

$$\delta P + \Delta r\ \partial P/\partial r = (\Gamma_1 P/\varrho)(\delta\varrho + \Delta r\ \partial\varrho/\partial r) = -\Gamma_1 P\ \nabla\cdot\Delta\vec{r}, \qquad (S, Q = \text{const}), \tag{5.2.92}$$

or via Eq. (5.2.85)

$$\delta P = \Gamma_1 P(\delta\varrho/\varrho + A\ \Delta r). \tag{5.2.93}$$

For perturbations like those in Eq. (5.2.87), the Poisson equation (5.2.40) writes as

$$\nabla^2\delta\Phi = (1/r^2)\ \partial(r^2\ \partial\delta\Phi/\partial r)/\partial r + (1/r^2\sin\lambda)\ \partial(\sin\lambda\ \partial\delta\Phi/\partial\lambda)/\partial\lambda + (1/r^2\sin^2\lambda)\ \partial^2\delta\Phi/\partial\varphi^2$$
$$= (1/r^2)\ \partial(r^2\ \partial\delta\Phi/\partial r)/\partial r - [j(j+1)/r^2]\ \delta\Phi = -4\pi G\ \delta\varrho = 4\pi G[\varrho\ \nabla\cdot\Delta\vec{r} + \Delta\vec{r}\cdot\nabla\varrho]$$
$$= 4\pi G[\varrho\ \nabla\cdot\Delta\vec{r} + \Delta r\ \partial\varrho/\partial r] = 4\pi G\varrho(A\ \Delta r - \delta P/\Gamma_1 P). \tag{5.2.94}$$

We have replaced at first $\delta\varrho$ by the equation of continuity (5.2.81), and then by the adiabatic energy conservation equation (5.2.79):

$$\delta\varrho = \varrho\{\delta P/\Gamma_1 P + \Delta r\ [(1/\Gamma_1 P)\ \partial P/\partial r - (1/\varrho)\ \partial\varrho/\partial r]\} = \varrho(\delta P/\Gamma_1 P - A\ \Delta r), \qquad (S, Q = \text{const}). \tag{5.2.95}$$

Pekeris [1938, Eq. (18)] was the first who carried out completely the elimination of variables between Eqs. (5.2.88)-(5.2.94), obtaining a fourth order homogeneous equation in $\nabla\cdot\vec{v}$ (cf. Sec. 5.5.1 and Hurley et al. 1966).

It should be noted that the azimuthal index k of the spherical harmonic Y_j^k does not appear in the equation for small nonradial oscillations of a static sphere, so these oscillations are degenerate with respect to k. For each value of j there exist according to Eq. (3.1.41) $2j+1$ spherical harmonics $Y_j^k(\lambda,\varphi) = P_j^k(\cos\lambda)\exp(ik\varphi)$, $(k = -j,-j+1,...j-1,j)$, but all of the $2j+1$ eigenvalues σ_k^2 are exactly the same in absence of other perturbations, such as rotation, magnetic fields, etc.

Eqs. (5.2.71), (5.2.88)-(5.2.90), (5.2.92), (5.2.94) are the basic equations to be solved for linear, nonradial adiabatic oscillations. They can be brought into a form more suitable for discussion, by inserting into Eq. (5.2.92) for $\nabla \cdot \Delta\vec{r}$ from Eq. (5.2.91), and by replacing χ with $\delta P/\varrho - \delta\Phi$ via Eq. (5.2.86):

$$(\delta P + \Delta r\ \partial P/\partial r)/\Gamma_1 P = -(1/r^2)\ \partial(r^2\ \Delta r)/\partial r + [j(j+1)/\sigma^2 r^2](\delta P/\varrho - \delta\Phi). \tag{5.2.96}$$

Introducing into Eq. (5.2.96) expansions of the form (5.2.87), the common factor $Y_j^k(\lambda,\varphi)$ cancels out, and all quantities become functions only of r :

$$d(r^2\ \Delta r)/dr - [G\varrho\ m(r)/r^2\Gamma_1 P]r^2\ \Delta r = \left[j(j+1)/\sigma^2 - (\varrho r^2/\Gamma_1 P)\right]\ \delta P/\varrho - j(j+1)\ \delta\Phi/\sigma^2. \tag{5.2.97}$$

Replacing in Eq. (5.2.88) $\nabla \cdot \Delta\vec{r}$ via Eq. (5.2.92), and dropping the common factor $Y_j^k(\lambda,\varphi)$, we get

$$d(\delta P/\varrho)/dr + A\ \delta P/\varrho = [\sigma^2 + AGm(r)/r^2]\ \Delta r + d\delta\Phi/dr, \qquad (\partial P/\partial r \approx -Gm(r)\ \varrho/r^2). \tag{5.2.98}$$

Defining the new variables

$$y = y(r) = \delta P(r)/\varrho(r) \quad \text{and} \quad u = u(r) = r^2\ \Delta r(r), \tag{5.2.99}$$

we cast Eqs. (5.2.97), (5.2.98), (5.2.94) into the form of a fourth order differential system (e.g. Ledoux and Walraven 1958):

$$du/dr - G\varrho\ m(r)\ u/r^2\Gamma_1 P = [j(j+1)\sigma^2 - \varrho r^2/\Gamma_1 P]\ y - j(j+1)\ \delta\Phi/\sigma^2, \tag{5.2.100}$$

$$dy/dr + Ay = [\sigma^2 + AG\ m(r)/r^2]u/r^2 + d\delta\Phi/dr, \tag{5.2.101}$$

$$(1/r^2)\ d(r^2\ d\delta\Phi/dr)/dr - [j(j+1)/r^2]\ \delta\Phi = 4\pi G\varrho(Au/r^2 - \varrho y/\Gamma_1 P). \tag{5.2.102}$$

Since Eqs. (5.2.100)-(5.2.102) apply to any value of j, they must describe as a special case linear, adiabatic radial oscillations, characterized by $j = 0$, which have already been discussed in Eqs. (5.2.41)-(5.2.70). Therefore, we shall restrict ourselves in the following mainly to the case $j > 0$.

Eqs. (5.2.100)-(5.2.102) have to satisfy certain boundary conditions at $r = 0$ and $r = r_1$. Pressure P, density ϱ, and adiabatic index Γ_1, all approach finite values at the centre $r = 0$. The quantities $m(r)/r^2 = 4\pi\varrho r/3$, $dP/dr = -G\varrho\ m(r)/r^2 = 4\pi G\varrho^2 r/3$, $d\varrho/dr = (d\varrho/dP)\ dP/dr$, and $A = (1/\varrho)\ d\varrho/dr - (1/\Gamma_1 P)\ dP/dr$ are proportional to r in a spherical polytrope with $0 < n < 5$. We now assume series expansions near the centre of the form

$$u = r^a \sum_{\ell=0}^{\infty} u_\ell r^\ell; \quad y = r^b \sum_{\ell=0}^{\infty} y_\ell r^\ell; \quad \delta\Phi = r^c \sum_{\ell=0}^{\infty} \varphi_\ell r^\ell, \qquad (r \approx 0). \tag{5.2.103}$$

Finiteness of $u = r^2\ \Delta r,\ y$, and $\delta\Phi$ at the centre demands that $a \geq 2$ and $b, c \geq 0$. To get a relationship among the exponents a, b, c, we insert Eq. (5.2.103) into the basic equations (5.2.100)-(5.2.102), and obtain for the lowest exponent

$$\begin{aligned}&au_0 r^{a-1} = [j(j+1)/\sigma^2](y_0 r^b - \varphi_0 r^c); \quad by_0 r^{b-1} = \sigma^2 u_0 r^{a-2} + c\varphi_0 r^{c-1};\\ &[c(c+1) - j(j+1)]\varphi_0 r^{c-2} = f u_0 r^{a-1} - (4\pi G\varrho^2 y_0/\Gamma_1 P)r^b.\end{aligned} \tag{5.2.104}$$

We have neglected $\varrho r^2/\Gamma_1 P$ with respect to $j(j+1)/\sigma^2$, and $AGm(r)/r^2$ with respect to σ^2. There is also $\lim_{r\to 0}(4\pi G\varrho A) = fr$, $(f = \text{const})$. From Eq. (5.2.104) we get two relationships among the exponents: $a - 1 = b = c$ and $a - 1 = b = c - 2$. These two conditions can be reconciled if (Cox 1980, p. 227)

$$a - 1 = b = c, \qquad (j > 0), \tag{5.2.105}$$

and if the factor near r^{c-2} from the last equation (5.2.104) vanishes:

$$c(c+1) = j(j+1). \tag{5.2.106}$$

The roots are $c_1 = j$ and $c_2 = -j-1$. The second root has to be discarded, in order to avoid singularities at the centre, and Eq. (5.2.103) takes the form

$$\Delta r = u/r^2 = r^{j-1}\sum_{\ell=0}^{\infty} u_\ell r^\ell; \quad y = \delta P/\varrho = r^j \sum_{\ell=0}^{\infty} y_\ell r^\ell; \quad \delta\Phi = r^j \sum_{\ell=0}^{\infty} \varphi_\ell r^\ell, \qquad (r \approx 0;\ j > 0). \tag{5.2.107}$$

From these equations we observe that for nonradial oscillations ($j > 0$) the variations δP and $\Delta P = \delta P + (dP/dr)\Delta r$ must vanish at the centre, opposite to the radial case $j = 0$ from Eq. (5.2.61). We have $\Delta r = 0$ at the origin $r = 0$ only if $j > 1$. The dipole mode $j = 1$ – when Eq. (5.2.107) yields $\Delta r(0) \neq 0$ – corresponds to a displacement of the geometrical centre of the configuration. In the case of an incompressible fluid ($\Gamma_1 = \infty$) this amounts to a bodily translation of the entire configuration, but for compressible fluids ($\Gamma_1 \neq \infty$) the case $j = 1$ may also correspond to a geometrical displacement, which leaves the centre of mass of the configuration unaltered (cf. Cox 1980, p. 229). We also note from the two first equations (5.2.104) that

$$\sigma^2 u_0 = j(y_0 - \varphi_0), \qquad (a - 1 = b = c = j). \tag{5.2.108}$$

For given σ and j the values of the two undetermined constants, say y_0 and φ_0, must be so, that the two subsequent surface boundary conditions (5.2.109) and (5.2.114) are satisfied. However, since the fourth order system (5.2.100)-(5.2.102) is homogeneous, one constant is left arbitrary, and we are lead again to an eigenvalue problem with the eigenvalue σ.

On the surface, the pressure is generally assumed to vanish during oscillations, and this requires – as in the radial case – that

$$\Delta P = \delta P + \Delta\vec{r}\cdot\nabla P = 0, \qquad (r = r_1). \tag{5.2.109}$$

Remembering that all unperturbed values depend on r only, we obtain in our linear approximation if $0 < n < 5$, $(\varrho(r_1) = 0;\ \Delta r(r_1) = \Delta r_1 = \text{finite})$:

$$\delta P = \Delta P - \Delta\vec{r}\cdot\nabla P = -(dP/dr)\ \Delta r = G\varrho(r_1)\ m(r_1)\ \Delta r_1/r_1^2 = 0, \qquad (r = r_1). \tag{5.2.110}$$

The second surface boundary condition concerns the continuity of the Eulerian surface perturbation $\delta\Phi$ of the gravitational potential. Let us denote by Φ, Φ_u and Φ_e, Φ_{ue} the perturbed and unperturbed values of the internal and external potential, and by $\delta\Phi, \delta\Phi_e$ the corresponding Eulerian variations. Continuity of the gravitational potential and of its radial derivative across the surface point $r_1 + \Delta r_1$ requires that

$$\Phi(r_1 + \Delta r_1, \lambda, \varphi) = \Phi_e(r_1 + \Delta r_1, \lambda, \varphi); \quad (\partial\Phi/\partial r)_{r=r_1+\Delta r_1} = (\partial\Phi_e/\partial r)_{r=r_1+\Delta r_1},$$
$$(\Phi = \Phi_u + \delta\Phi;\ \Phi_e = \Phi_{ue} + \delta\Phi_e). \tag{5.2.111}$$

We now approximate the surface values of the internal and external potential by their values on the unperturbed spherical surface $r = r_1$ (cf. Chandrasekhar 1933d). Eq. (5.2.111) becomes

$$\Phi_u(r_1, \lambda, \varphi) + \Delta r_1\ (d\Phi_u/dr)_{r=r_1} + \delta\Phi = \Phi_{ue}(r_1, \lambda, \varphi) + \Delta r_1\ (d\Phi_{ue}/dr)_{r=r_1} + \delta\Phi_e;$$
$$(d\Phi_u/dr)_{r=r_1} + \Delta r_1\ (d^2\Phi_u/dr^2)_{r=r_1} + (\partial\delta\Phi/\partial r)_{r=r_1}$$
$$= (d\Phi_{ue}/dr)_{r=r_1} + \Delta r_1\ (d^2\Phi_{ue}/dr^2)_{r=r_1} + (\partial\delta\Phi_e/\partial r)_{r=r_1}. \tag{5.2.112}$$

Equating terms of the same order, we get

$$\delta\Phi(r_1, \lambda, \varphi) = \delta\Phi_e(r_1, \lambda, \varphi); \quad \Delta r_1\ (d^2\Phi_u/dr^2)_{r=r_1} + (\partial\delta\Phi/\partial r)_{r=r_1}$$
$$= \Delta r_1\ (d^2\Phi_{ue}/dr^2)_{r=r_1} + (\partial\delta\Phi_e/\partial r)_{r=r_1}. \tag{5.2.113}$$

As shown by Eq. (3.1.58), the perturbed external potential $\delta\Phi_e$ corresponding to the surface harmonic $Y_j^k(\lambda, \varphi) = P_j^k(\cos\lambda)\exp(ik\varphi)$ has the form $\delta\Phi_e = F(\lambda, \varphi)/r^{j+1}$, and $\partial\delta\Phi_e/\partial r = -(j+1)\ F(\lambda, \varphi)/r^{j+2} = -[(j+1)/r]\ \delta\Phi_e = -[(j+1)/r]\ \delta\Phi$. Besides, the equations for the unperturbed potential $\nabla^2\Phi_u = (1/r^2)\ d(r^2 d\Phi_u/dr)/dr = -4\pi G\varrho_u$ and $\nabla^2\Phi_{ue} = 0$ yield $(d^2\Phi_u/dr^2)_{r=r_1} - (d^2\Phi_{ue}/dr^2)_{r=r_1} =$

$-4\pi G\varrho_u(r_1)$, because $(d\Phi_u/dr)_{r=r_1} = (d\Phi_{ue}/dr)_{r=r_1}$. Thus, the boundary conditions (5.2.113) take the final form ($\varrho(r_1) \approx \varrho_u(r_1) = 0$ if $0 < n < 5$; Ledoux and Walraven 1958, §75):

$$\delta\Phi(r_1,\lambda,\varphi) = \delta\Phi_e(r_1,\lambda,\varphi) = F(\lambda,\varphi)/r_1^{j+1};$$
$$(\partial\delta\Phi/\partial r)_{r=r_1} + (j+1)\ \delta\Phi(r_1,\lambda,\varphi)/r_1 = 4\pi G\varrho(r_1)\ \Delta r_1. \qquad (5.2.114)$$

In the case of a spherical polytrope the surface pressure condition (5.2.109) can be brought into a more explicit form by transforming Eq. (5.2.32) analogously to Eqs. (5.2.48) and (5.2.65):

$$D^2\Delta\vec{r}/Dt^2 = \partial^2\Delta\vec{r}/\partial t^2 = (\delta\varrho/\varrho^2)(dP/dr)\ \vec{e}_r - (1/\varrho)\ \nabla\delta P + (\Delta r/\varrho^2)(d\varrho/dr)(dP/dr)\ \vec{e}_r$$
$$-(\Delta r/\varrho)(d^2P/dr^2)\ \vec{e}_r + \nabla\delta\Phi + \Delta r\ (d^2\Phi/dr^2)\ \vec{e}_r = -(\nabla\cdot\Delta\vec{r})(1/\varrho)(dP/dr)\ \vec{e}_r - (1/\varrho)\ \nabla\Delta P$$
$$+(1/\varrho)\ \nabla(\Delta\vec{r}\cdot\nabla P) - (\Delta r/\varrho)(d^2P/dr^2)\ \vec{e}_r + \nabla\delta\Phi - 2(\Delta r/r)(d\Phi/dr)\ \vec{e}_r - 4\pi G\varrho\ \Delta r\ \vec{e}_r$$
$$= [-(1/r^2)\ \partial(r^2\ \Delta r)/\partial r + j(j+1)\chi/\sigma^2r^2](1/\varrho)(dP/dr)\ \vec{e}_r - (P/\varrho)\ \nabla(\Delta P/P) - (\Delta P/P)(\nabla P/\varrho)$$
$$+[\partial(\Delta r)/\partial r](1/\varrho)(dP/dr)\ \vec{e}_r + \nabla\delta\Phi - 2\eta(d\Phi/dr)\ \vec{e}_r - 4\pi G\varrho\ \Delta r\ \vec{e}_r. \qquad (5.2.115)$$

Assuming, as usual, the time dependence of Lagrangian variations under the form $\Delta F(\vec{r},t) = \Delta F(\vec{r})\exp(i\sigma t)$, and considering only the radial component of Eq. (5.2.115), we get

$$\nabla(\Delta P/P) = (\varrho/P)\big\{\big[-4\eta + j(j+1)\chi/\sigma^2r^2 - \Delta P/P\big]\ d\Phi/dr + \sigma^2r\eta + \partial\delta\Phi/\partial r - 4\pi G\varrho r\eta\big\}. \qquad (5.2.116)$$

Since $\nabla(\Delta P/P)$ must remain finite, and since $\varrho/P = \varrho_0/P_0\theta \to \infty$ if $r \to r_1$ and $0 \le n < 5$, the second factor in Eq. (5.2.116) must be very small or zero at the boundary (Cox 1980):

$$\Delta P/P = [-4 + (\sigma^2r - 4\pi G\varrho r)/(d\Phi/dr)]\eta + j(j+1)\chi/\sigma^2r^2 + (\partial\delta\Phi/\partial r)\big/(d\Phi/dr)$$
$$= (-4 - \sigma^2r^3/Gm + 4\pi\varrho r^3/m)\eta + j(j+1)(\delta P/\varrho - \delta\Phi)/\sigma^2r^2 - (r^2/Gm)\ \partial\delta\Phi/\partial r, \qquad (r = r_1). \qquad (5.2.117)$$

With the boundary condition on the potential (5.2.114) we can eliminate $\partial(\delta\Phi)/\partial r$, and substitute $\delta P/\varrho$ by $-(\Delta r/\varrho)\ dP/dr = Gm\eta/r$ via the boundary condition (5.2.109):

$$\Delta P/P = [-4 - \sigma^2r^3/Gm + j(j+1)Gm/\sigma^2r^3]\eta$$
$$-[j(j+1)Gm/\sigma^2r^3 - j - 1]r\ \delta\Phi/Gm, \qquad (r = r_1). \qquad (5.2.118)$$

The basic equations (5.2.100)-(5.2.102) simplify considerably in the Cowling (1941) approximation by neglecting $\delta\Phi$ if $j > 0$:

$$du/dr - G\varrho\ m(r)\ u/r^2\Gamma_1P = P^{-1/\Gamma_1}\ d(uP^{1/\Gamma_1})/dr = [j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1P]y, \quad (\Gamma_1 = \mathrm{const}), \qquad (5.2.119)$$

$$dy/dr + Ay = dy/dr + [(1/\varrho)\ d\varrho/dr - (1/\Gamma_1P)\ dP/dr]y$$
$$= (P^{1/\Gamma_1}/\varrho)\ d(y\varrho P^{-1/\Gamma_1})/dr = [\sigma^2 + AGm(r)/r^2]u/r^2, \qquad (\Gamma_1 = \mathrm{const}). \qquad (5.2.120)$$

According to Eqs. (5.2.107) and (5.2.109) the boundary conditions satisfied by Eqs. (5.2.119) and (5.2.120) are

$$r = 0:\ u = r^2\ \Delta r = 0; \quad y = \delta P/\varrho = 0. \qquad r = r_1:\ u_1 = r_1^2\ \Delta r_1 = \mathrm{finite};$$
$$y_1 = (\delta P)_{r=r_1}/\varrho_1 = -(\Delta r_1/\varrho_1)(dP/dr)_{r=r_1} = Gm(r_1)\ \Delta r_1/r_1^2 = \mathrm{finite}. \qquad (5.2.121)$$

The neglect of $\delta\Phi$ can be justified by regarding $\delta\Phi$ as some kind of average over the whole polytrope, smoothing out local perturbations. Calculations by Cowling (1941) and Kopal (1949) for the standard model $n = 3$ (cf. Ledoux and Walraven 1958, p. 523), and by Robe (1968a) indicate that the Cowling approximation for polytropes is generally in error by at most 20% for the eigenvalues σ^2, being often much more accurate (Cox 1980, p. 251).

The final form of Eqs. (5.2.119) and (5.2.120) is obtained with the new variables

$$v = uP^{1/\Gamma_1} = r^2P^{1/\Gamma_1}\Delta r; \quad w = y\varrho P^{-1/\Gamma_1} = P^{-1/\Gamma_1}\ \delta P, \qquad (5.2.122)$$

obeying the boundary conditions

$$r = 0: \ v, w = 0 \quad \text{and} \quad r = r_1: \ v_1 = 0; \quad w_1 = y_1 \varrho_1 P_1^{-1/\Gamma_1} = y_1 K^{-1/\Gamma_1} \varrho^{1-(1+1/n)/\Gamma_1}. \qquad (5.2.123)$$

Since $\varrho \to 0$ at the finite boundary if $0 < n < 5$, we observe that $w_1 \to 0$ if $\Gamma_1 > 1 + 1/n$ or if $A = (1/\varrho)\, d\varrho/dr - (1/\Gamma_1 P)\, dP/dr = (1/\varrho)(d\varrho/dr)[1 - (1 + 1/n)/\Gamma_1] < 0$, $(d\varrho/dr < 0)$.

With the notations (5.2.122), the equations (5.2.119) and (5.2.120) of nonradial adiabatic oscillations take in the Cowling approximation the final form

$$dv/dr = [j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1 P] P^{2/\Gamma_1} w/\varrho, \qquad (\Gamma_1 = \text{const}), \qquad (5.2.124)$$

$$dw/dr = [\sigma^2 + AGm(r)/r^2] \varrho v/P^{2/\Gamma_1} r^2, \qquad (\Gamma_1 = \text{const}). \qquad (5.2.125)$$

Nodes of w or v (points where w or v vanish) correspond to extrema ($dv/dr = 0$ or $dw/dr = 0$) of v or w, respectively. If w or v is eliminated between Eqs. (5.2.124) and (5.2.125), we obtain equivalent second order equations:

$$d\big\{\varrho(dv/dr)/P^{2/\Gamma_1}[j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1 P]\big\}/dr = [\sigma^2 + AGm(r)/r^2]\varrho v/P^{2/\Gamma_1} r^2, \quad (\Gamma_1 = \text{const}), \qquad (5.2.126)$$

$$d\big\{P^{2/\Gamma_1} r^2 (dw/dr)/\varrho[\sigma^2 + AGm(r)/r^2]\big\}/dr = [j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1 P] P^{2/\Gamma_1} w/\varrho, \quad (\Gamma_1 = \text{const}). \qquad (5.2.127)$$

Dividing Eqs. (5.2.124) and (5.2.125), we observe that the singular points $dv/dw = 0/0$ are given by

$$j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1 P = 0, \qquad (5.2.128)$$

$$\sigma^2 + AG\, m(r)/r^2 = 0. \qquad (5.2.129)$$

The singular points are regular – series solutions of the form (5.3.24) exist in these points. If $\sigma \gg 1$, Eq. (5.2.126) tends towards the Sturm-Liouville type (5.2.69), [cf. Ledoux 1974, Eqs. (10), (11)] with a spectrum of infinitely increasing eigenvalues σ^2. The corresponding eigenfunctions where termed by Cowling (1941) p-modes (pressure modes, pulsation modes, acoustic modes), and are characterized by relatively large Eulerian pressure and density variations during oscillations.

If $\sigma \ll 1$, Eq. (5.2.127) tends towards the Sturm-Liouville type (5.2.69) with the parameter $1/\sigma^2$, the corresponding spectrum tending to zero as the order of the modes increases. The corresponding eigenfunctions are called g-modes (gravity modes, convective modes), and are chiefly horizontal, with small pressure variations.

Cowling (1941) also distinguishes a unique fundamental f-mode (Kelvin mode), which exists only if $j \geq 2$. This mode is sometimes called pseudo-Kelvin mode, as it is similar to the well known Kelvin mode $\sigma_f^2 = (8\pi G\varrho/3) j(j-1)/(2j+1)$ from Eq. (5.5.26) for the compressible or incompressible ($\Gamma_1 = \infty$; $\Delta\varrho, \delta\varrho, \nabla \cdot \Delta\vec{r} = 0$) homogeneous sphere ($n = 0$). For sufficiently simple models (e.g. polytropes) the eigenvalue σ_f^2 of the fundamental mode is intermediate between the eigenvalues of the g_1^+ and p_1-mode (see Fig. 5.2.1 and Cox 1980, p. 240). Generally, no simple and unique method of ordering is available for the eigenvalues σ^2 of nonradial oscillations.

Since the homogeneous compressible model ($n = 0$) admits an analytic solution (Sec. 5.5.1, Pekeris 1938), we confine ourselves in Eqs. (5.2.130)-(5.2.133) below, to polytropic indices $0 < n < 5$. Integral expressions for the eigenvalues of p and g-modes can be obtained if we multiply Eq. (5.2.126) by v, and integrate the left-hand side by parts with the boundary conditions (5.2.123):

$$\int_0^{r_1} \varrho(dv/dr)^2\, dr/P^{2/\Gamma_1}\big[j(j+1)/\sigma^2 - \varrho r^2/\Gamma_1 P\big] + \sigma^2 \int_0^{r_1} \varrho v^2\, dr/r^2 P^{2/\Gamma_1}$$
$$+ \int_0^{r_1} AG\varrho\, m(r)\, v^2\, dr/r^4 P^{2/\Gamma_1} = 0. \qquad (5.2.130)$$

If $\sigma^2 \gg 1$, the term $j(j+1)/\sigma^2$ in the first integral is negligible, and the second integral, containing $\sigma^2 v^2$, becomes generally much larger than the last integral containing only the factor v^2. Thus (Ledoux and Walraven 1958)

$$\sigma^2 = \sigma_p^2 \approx \left[\int_0^{r_1} \Gamma_1 P^{1-2/\Gamma_1}(dv/dr)^2\ dr/r^2\right] \Big/ \int_0^{r_1} \varrho v^2\ dr/r^2 P^{2/\Gamma_1}, \qquad (\sigma^2 \gg 1), \qquad (5.2.131)$$

showing that σ_p^2 is certainly positive, increasing indefinitely with the order of the modes.

In the same manner we obtain from Eqs. (5.2.123) and (5.2.127), $(P^{2/\Gamma_1} w(dw/dr)/\varrho A \propto \varrho \to 0$ if $r \to r_1$ and $0 < n < 5)$:

$$\int_0^{r_1} P^{2/\Gamma_1} r^2 (dw/dr)^2\ dr/\varrho[\sigma^2 + AGm(r)/r^2] + [j(j+1)/\sigma^2] \int_0^{r_1} P^{2/\Gamma_1} w^2\ dr/\varrho$$
$$- \int_0^{r_1} P^{2/\Gamma_1 - 1} r^2 w^2\ dr/\Gamma_1 = 0. \qquad (5.2.132)$$

If $|\sigma^2| \ll 1$, the term σ^2 in the first integral is negligible, and the second integral, containing w^2/σ^2, becomes generally much larger than the last integral containing only the factor w^2. Thus:

$$\sigma^2 = \sigma_g^2 \approx -\left[j(j+1)\int_0^{r_1} P^{2/\Gamma_1} w^2\ dr/\varrho\right] \Big/ \int_0^{r_1} P^{2/\Gamma_1} r^4 (dw/dr)^2\ dr/AG\varrho\ m(r), \qquad (|\sigma^2| \ll 1). \qquad (5.2.133)$$

This approximate integral expression shows that two branches of gravity modes occur: g^+-modes if $\sigma_g^2 > 0,\ (A < 0)$, and g^--modes if $\sigma_g^2 < 0,\ (A > 0)$. It has been demonstrated by Lebovitz (1965) that if the polytrope is convectively stable throughout $(A < 0)$, then g^--modes do not exist at all, and only stable g^+-modes are present. Conversely, if the polytrope is convectively unstable throughout $(A > 0)$, then only unstable g^--modes exist. If A changes sign in the polytrope, then both g^+ and g^--modes exist (cf. Fig. 5.2.1). If $A = 0$ in any finite subinterval of the radius, there exist neutral modes $(\sigma_g = 0)$. If $A = 0$ everywhere, then $\sigma_g = 0$ for all g-modes, i.e. such modes do not exist at all (e.g. Cox 1980, p. 247).

Fig. 5.2.1 shows schematically the eigenvalues of linear adiabatic oscillations. It should be noted that no radial analogues of the f and g-modes exist. The p-modes $(j > 0)$ are the nonradial counterparts of the radial modes $(j = 0)$. The unique fundamental or Kelvin f-mode exists only if $j \geq 2$.

The order of radial modes $(j = 0)$ starts with 0, $(\sigma_0, \sigma_1, \sigma_2, ...)$, while the lowest mode of nonradial oscillations $(j \geq 1)$ corresponds to 1, $(\sigma_1, \sigma_2, \sigma_3, ...)$, (see Fig. 5.2.1 if $j = 0$ and $j \geq 1$).

For completeness we write down the basic equations (5.2.126) and (5.2.127) in polytropic variables (cf. Secs. 2.6.1-2.6.3):

$$r = [(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi = [(n+1)P_0/4\pi G\varrho_0^2]^{1/2}\xi = \alpha\xi; \quad P = K\varrho_0^{1+1/n}\theta^{n+1} = P_0\theta^{n+1};$$
$$\varrho = \varrho_0\theta^n; \quad m = 4\pi\varrho_0\alpha^3\xi^2(-\theta'); \quad A = (1/\varrho)\ d\varrho/dr - (1/\Gamma_1 P)\ dP/dr$$
$$= [(d\theta/d\xi)/\alpha\theta][n - (n+1)/\Gamma_1]; \quad v = \xi^2(\Delta r/\alpha)\ \theta^{(n+1)/\Gamma_1}; \quad w = (\delta P/P_0)\ \theta^{-(n+1)/\Gamma_1}, \qquad (5.2.134)$$

where we have properly scaled Δr and δP. If we further define

$$\kappa = (n+1)\sigma^2/4\pi G\varrho_0; \quad Q = 2(n+1)/\Gamma_1 - n; \quad f = \theta^{-Q}[\kappa + (Q-n)(n+1)(d\theta/d\xi)^2/2\theta];$$
$$g = \theta^Q[j(j+1)/\kappa - \xi^2/\Gamma_1\theta], \qquad (5.2.135)$$

Eqs. (5.2.126) and (5.2.127) become eventually equal to

$$d[(1/g)\ dv/d\xi]/d\xi = fv/\xi^2, \qquad (5.2.136)$$

$$d[(\xi^2/f)\ dw/d\xi]/d\xi = gw. \qquad (5.2.137)$$

Actual pulsations in real stars certainly involve nonlinear effects. The relevant equations for the study of nonlinear oscillations are the full equations of conservation of mass, momentum, and thermal energy: Eqs. (5.2.1)-(5.2.3), (5.2.10), (5.2.20) in Eulerian description, and Eqs. (5.2.8), (5.2.12), (5.2.22)

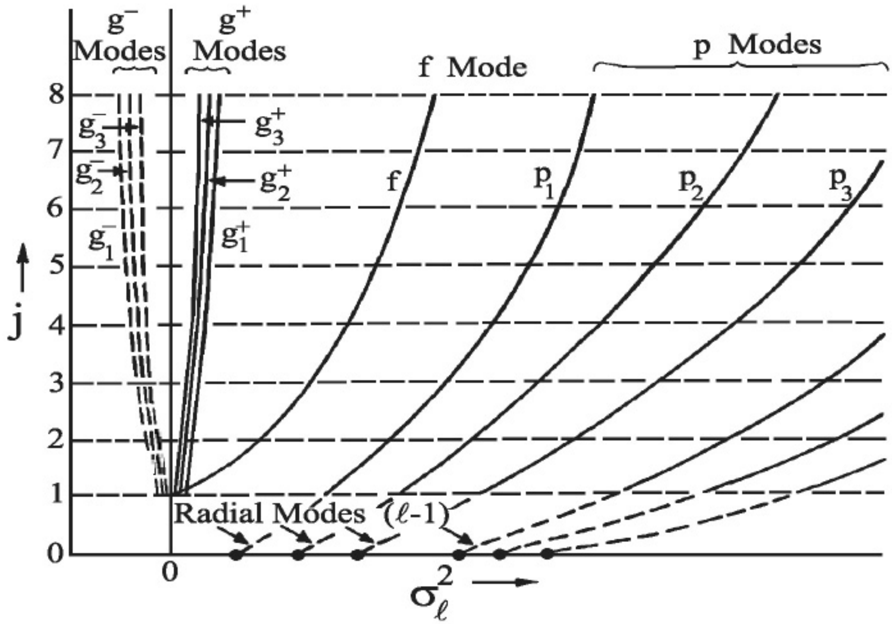

Fig. 5.2.1 Schematic view of the eigenvalue spectrum for radial ($j = 0$) and nonradial ($j \geq 1$) linear adiabatic oscillations. Shown are the four types of nonradial spheroidal modes (p, f, g^+, g^-) – as classified by Cowling (1941) for various orders of the modes ℓ versus the spherical harmonic index j (Cox 1980).

in Lagrangian form. Generally, the study of oscillations in polytropes is confined to the linear, first order theory. The only exactly integrable cases in the nonlinear theory occur for radial oscillations, when the radial and temporal variables are separable:

$$r(r_u, t) = X(r_u)\, W(t). \qquad (5.2.138)$$

The two analytically integrable cases are a general model with $\Gamma_1 = 4/3$ (homologous contraction with neutral stability) and the homogeneous polytrope $n = 0$ (cf. Sec. 5.3.2 for linear oscillations), (Ledoux and Walraven 1958, Rosseland 1964, Chap. VII, Cox 1980).

5.3 Radial Oscillations of Polytropic Spheres

5.3.1 Approximate Values of the Fundamental Oscillation Frequency

The theory of linear adiabatic oscillations provides information concerning the dynamical (ordinary) stability of a configuration, as can already be seen from a quite simple, somewhat unrealistic example, namely if the perturbed motion is homologous, i.e. if $\eta = \eta(t) = (\Delta r/r)\exp(i\sigma t) = \mathrm{const}\exp(i\sigma t)$, and the motion is independent of r or m. Then, the sole subsisting term in Eq. (5.2.59) is the last one, and if moreover the adiabatic index Γ_1 is constant, we get with the equation of hydrostatic equilibrium:

$$\sigma^2 = -[(3\Gamma_1 - 4)/\varrho r]\ dP/dr = (3\Gamma_1 - 4)Gm(r)/r^3, \qquad (\Delta r/r = \mathrm{const};\ \Gamma_1 = \mathrm{const}). \tag{5.3.1}$$

Oscillatory motions $\eta = \mathrm{const}\exp(i\sigma t)$ exist only if $\sigma^2 > 0$, i.e. if $\Gamma_1 > 4/3$. Dynamical instability occurs if $\sigma^2 < 0$, $(\Gamma_1 < 4/3)$. Since σ^2 has to be a constant, adiabatic and homologous motions with $\Gamma_1 \neq 4/3$ occur only if $m(r)/r^3 = \mathrm{const}$, i.e. for the constant density polytrope $n = 0$. In the special case $\Gamma_1 = 4/3$, we have $\sigma^2 = 0$ and $\eta = \mathrm{const}$: Any configuration can expand or contract homologously into a new equilibrium state. The system is neutrally stable (see Sec. 5.1).

A more refined first order determination of the fundamental angular oscillation frequency σ^2 for a rotating configuration can be obtained from the virial theorem for the case of quasiradial oscillations (mainly radial displacements), defined by the conditions $\partial\delta f/\partial\varphi = 0$ and $\partial\delta f/\partial\lambda \propto \Omega^2$, where $\vec{\Omega} = \vec{\Omega}(t) = \vec{\Omega}(0,0,\Omega)$, $(\Omega \ll 1)$ is the angular rotation velocity, and δf an Eulerian perturbation (Ledoux and Walraven 1958, Cox 1980). Combining Eqs. (2.6.80) and (3.1.85), the scalar virial theorem becomes with respect to (x_1, x_2, x_3)-axes, rotating together with the polytropic fluid round the x_3-axis:

$$(1/2)\ d^2I/dt^2 = 2E_{kin} + W + 3\int_V P\ dV + \int_M \Omega^2\ell^2\ dm, \qquad (\ell^2 = x_1^2 + x_2^2). \tag{5.3.2}$$

Let us consider the first variation (5.8.4) of Eq. (5.3.2) for the quasiradial displacement (Ledoux 1945, Ledoux and Walraven 1958)

$$\eta(r,t) = (\Delta r/r)\exp(i\sigma t) = \eta(r)\exp(i\sigma t). \tag{5.3.3}$$

By virtue of Eqs. (5.8.1)-(5.8.10) we get

$$\delta^*(d^2I/dt^2) = d^2(\delta^* I)/dt^2 = 2\ d^2\Big(\int_M r\ \Delta r\ dm\Big)\Big/dt^2 = 2\ d^2\Big(\int_M r^2\eta(r,t)\ dm\Big)\Big/dt^2$$
$$= -2\sigma^2\int_M r^2\eta(r,t)\ dm, \tag{5.3.4}$$

$$\delta^* E_{kin} = \int_M v\ \Delta v\ dm = O(\eta^2), \qquad (v_u = 0;\ v = \Delta v \approx \partial\Delta r/\partial t = i\sigma\ \Delta r = i\sigma r\eta), \tag{5.3.5}$$

$$\delta^* W = -\delta^*\int_M (Gm/r)\ dm = -\int_M Gm\ \Delta(1/r)\ dm = \int_M [Gm\ \eta(r,t)/r]\ dm. \tag{5.3.6}$$

In order to transform the pressure integral, we use the adiabatic energy equation (5.2.38), the equation of continuity (5.2.45), the radial component of the hydrostatic equation (3.1.16) $\partial P/\partial r = \partial\Phi/\partial r + \Omega^2\varrho r\sin^2\lambda \approx -Gm\varrho/r^2 + \Omega^2\varrho r\sin^2\lambda$, and the fact that $P = 0$ on the surface:

$$\delta^*\int_V P\ dV = \int_M \Delta(P/\varrho)\ dm = \int_M (\Delta P/\varrho - P\ \Delta\varrho/\varrho^2)\ dm = \int_M (\Gamma_1 - 1)(P/\varrho^2)\ \Delta\varrho\ dm$$
$$= -\int_V (\Gamma_1 - 1)P(3\eta + r\ \partial\eta/\partial r)\ dV = -3\int_V (\Gamma_1 - 1)P\eta\ dV - (\Gamma_1 - 1)Pr\eta\Big|_0^V$$
$$+ \int_V (d\Gamma_1/dr)Pr\eta\ dV + \int_V (\Gamma_1 - 1)(dP/dr)r\eta\ dV + 3\int_V (\Gamma_1 - 1)P\eta\ dV$$
$$= \int_M \eta(d\Gamma_1/dr)(Pr/\varrho)\ dm - G\int_M \eta[(\Gamma_1 - 1)m/r]\ dm + \int_M \eta(\Gamma_1 - 1)\Omega^2\ell^2\ dm. \tag{5.3.7}$$

To get the variation of the last integral in Eq. (5.3.2), we observe that for a quasiradial perturbation all terms in the equation of motion (3.1.12) lie in the meridian plane, excepting the vectors

$$(d\vec{\Omega}/dt)\times\vec{r} + 2(\vec{\Omega}\times\vec{v}) = 0, \tag{5.3.8}$$

which are perpendicular to this plane, and must accordingly vanish identically. In spherical coordinates we have $\vec{r} = \vec{r}(r,0,0)$, $\vec{v} = \vec{v}(dr/dt, r\ d\lambda/dt, 0)$, $\vec{\Omega} = \vec{\Omega}(\Omega\cos\lambda, -\Omega\sin\lambda, 0)$, and Eq. (5.3.8) turns into

$$r\sin\lambda\ d\Omega/dt + 2\Omega(r\cos\lambda\ d\lambda/dt + \sin\lambda\ dr/dt) = 0, \tag{5.3.9}$$

or

$$(1/\Omega)\ d\Omega = -(2/r)\ dr - 2\cot\lambda\ d\lambda, \tag{5.3.10}$$

which integrates to

$$\Omega r^2\sin^2\lambda = \mathrm{const}. \tag{5.3.11}$$

This means nothing else than conservation of angular momentum during quasiradial oscillations. In a first approximation these oscillations are mainly radial, with $\sin^2\lambda = \mathrm{const}$. Eq. (5.3.11) becomes $\Omega r^2 = \mathrm{const}$, and its Lagrangian variation is

$$\Delta\Omega/\Omega = -2\Delta r/r = -2\eta. \tag{5.3.12}$$

We are now in position to determine the first variation (5.8.7) of

$$\delta^*\int_M \Omega^2\ell^2\ dm = \int_M \Delta(\Omega^2\ell^2)\ dm = \int_M \Omega\ell^2\ \Delta\Omega\ dm = -2\int_M \eta\Omega^2\ell^2\ dm, \tag{5.3.13}$$

taking into account that we have $\Omega r^2\sin^2\lambda = \Omega\ell^2 = \mathrm{const}$ via Eq. (5.3.11). The equations (5.3.4), (5.3.5), (5.3.6), (5.3.7), (5.3.13) are now combined to get the first order variation of Eq. (5.3.2):

$$-\sigma^2\int_M \eta r^2\ dm = -\int_M \eta(3\Gamma_1-4)(Gm/r)\ dm + 3\int_M \eta(d\Gamma_1/dr)(P/\varrho)r\ dm$$
$$+\int_M (3\Gamma_1-5)\eta\Omega^2\ell^2\ dm, \tag{5.3.14}$$

or

$$\sigma^2 = \left[\int_M \eta(3\Gamma_1-4)(Gm/r)\ dm - 3\int_M \eta(d\Gamma_1/dr)(P/\varrho)r\ dm + \int_M \eta(5-3\Gamma_1)\Omega^2r^2\sin^2\lambda\ dm\right]$$
$$\Big/\int_M \eta r^2\ dm. \tag{5.3.15}$$

If we take as a first guess $\eta(r) = \Delta r/r = \mathrm{const}$, Eq. (5.3.15) assumes for the fundamental radial oscillation frequency the simple form (cf. Eqs. (5.7.40), (5.8.134), Tassoul 1978, §14.2)

$$\sigma^2 = \left\{\int_M [(3\Gamma_1-4)Gm(r)/r + (5-3\Gamma_1)\Omega^2r^2\sin^2\lambda]\ dm\right\}\Big/\int_M r^2\ dm$$
$$= [(3\Gamma_1-4)|W| + 2(5-3\Gamma_1)E_{kin}]/I, \qquad (\eta,\Gamma_1 = \mathrm{const}), \tag{5.3.16}$$

where E_{kin} denotes the kinetic energy of rotation. This approximate expression already offers a fairly good approximation, as long as the mass concentration in the polytrope is not too high, i.e. for polytropes with indices $n\leq 3$. Indeed, Table 1 of Ledoux and Pekeris (1941), Table 2 of Rosseland (1964), and Table 8.2 of Cox (1980) show that in the *nonrotating* case ($E_{kin},\Omega = 0$) the ratio between the approximate and exact fundamental oscillation periods is 0.996 if $n = 1.5$, ($\Gamma_1 = 5/3$), 0.964 if $n = 2$, ($\Gamma_1 = 1.428$), 0.957 if $n = 3$, ($\Gamma_1 = 5/3$), and 0.682 if $n = 4$, ($\Gamma_1 = 1.428$).

If we take $\Gamma_1,\Omega,\varrho = \mathrm{const}$, we obtain from Eq. (5.3.16) for the constant density polytrope:

$$\sigma^2 = (3\Gamma_1-4)(4\pi G\varrho/3) + (5-3\Gamma_1)(2\Omega^2/3), \qquad (n=0). \tag{5.3.17}$$

In absence of rotation we recover the elementary relationship (5.3.1).

Care should be exercised, when adopting Ledoux's (1945) formula (5.3.16): W and I refer to the *rotating* configuration, containing terms of order $\beta = \Omega^2/2\pi G\varrho_0$, as emphasized by Tassoul (1978). If we wish to put into evidence rotational effects with respect to the *nonrotating* sphere with the same central density ϱ_0, the density should be calculated with Eq. (3.2.44), for instance: $\varrho = \varrho_0\Theta^n(\xi,\mu) \approx \varrho_0[\theta^n(\xi) + \beta n\theta^{n-1}(\xi)\ \psi_0(\xi)]$, neglecting angular dependences. If this is properly effected, it turns out that $\omega^2 = \sigma^2/4\pi G\varrho_0 = 0.15 - 0.13\beta,\ (n = 1.5;\ \Gamma_1 = 5/3)$ and $0.082 - 0.73\beta,\ (n = 3;\ \Gamma_1 = 5/3)$, i.e. slow rotation reduces (destabilizes) the squared angular oscillation frequency with respect to that of the nonrotating polytrope (Cowling and Newing 1949, Table 1). An indiscriminate use of Eq. (5.3.16) would suggest that the eigenvalues σ^2 are unaffected by rotation if $\Gamma_1 = 5/3$. In fact, if $\varrho = \varrho_0\Theta^n \approx \varrho_0 = \mathrm{const}$, as for nearly homogeneous polytropes with $n \approx 0$, this holds true, since the above mentioned correction effects are minor (cf. Eqs. (5.7.40), (5.8.134), Chandrasekhar and Lebovitz 1962c).

5.3.2 Linear Radial Oscillations of the Constant Density Polytrope $n = 0$

Perhaps the most simple model suitable for analytic treatment is the homogeneous compressible sphere $(n = 0;\ \Gamma_1 \neq \infty)$, (e.g. Sterne 1937, Ledoux 1946, Vaughan 1972). If the adiabatic index Γ_1 is constant, the equation of small, radial adiabatic oscillations (5.2.60) particularizes to

$$d^2\eta/dr^2 + (4/r - G\varrho m/Pr^2)\ d\eta/dr + [(4/\Gamma_1 - 3)G\varrho m/Pr^3 + \sigma^2\varrho/\Gamma_1 P]\eta = 0, \qquad (\eta = \Delta r/r). \qquad (5.3.18)$$

We introduce the dimensionless distance $x = r/r_1,\ (0 \le x \le 1)$, and Eq. (5.3.18) turns into

$$d^2\eta/dx^2 + (1/x)(4 - G\varrho m/Pr)\ d\eta/dx + (\varrho r_1/P)[(4/\Gamma_1 - 3)Gm/r^2x + \sigma^2 r_1/\Gamma_1]\eta = 0. \qquad (5.3.19)$$

The outer boundary condition (5.2.68) writes

$$d\eta/dx = \eta[(4/\Gamma_1 - 3) + r^3\sigma^2/\Gamma_1 Gm], \qquad (r = r_1;\ x = 1). \qquad (5.3.20)$$

For the constant density polytrope $n = 0$ we have $\varrho = \varrho_m = \mathrm{const},\ m = 4\pi\varrho r^3/3,\ P = (2\pi G\varrho^2/3)(r_1^2 - r^2)$, and Eqs. (5.3.19), (5.3.20) read

$$d\eta^2/dx^2 + [(4 - 6x^2)/x(1 - x^2)]\ d\eta/dx + J\eta/(1 - x^2) = 0, \qquad (5.3.21)$$

$$d\eta/dx = J\eta/2, \qquad (x = 1), \qquad (5.3.22)$$

where

$$J = 3\sigma^2/2\pi\Gamma_1 G\varrho + 2(4/\Gamma_1 - 3). \qquad (5.3.23)$$

Because Eq. (5.3.21) is homogeneous, we can arbitrarily set $\eta(0) = 1$. Eq. (5.3.21) has regular singularities at $x = 0$ and $x = 1$. We apply the Frobenius method, seeking series solutions of the form (Ledoux and Walraven 1958, p. 460, Vaughan 1972)

$$\eta = x^q \sum_{\ell=0}^{\infty} a_\ell x^\ell, \qquad (a_0 \neq 0). \qquad (5.3.24)$$

The requirement that Eq. (5.3.24) is a solution of Eq. (5.3.21) demands that the coefficients of successive powers of x have to vanish. Near the singular point $x = 0$, Eq. (5.3.21) takes the form $d^2\eta/dx^2 + (4/x)\ d\eta/dx + J\eta = 0$, and the vanishing of the coefficient of x^{q-2} (the lowest power of x) demands that

$$a_0[q(q-1) + 4q] = 0 \qquad \text{or} \qquad q^2 + 3q = 0, \qquad (a_0 \neq 0). \qquad (5.3.25)$$

Among the two roots ($q_1 = 0$; $q_2 = -3$) of the indicial equation (5.3.25) only $q_1 = 0$ fulfils the condition that $\eta(0) = 1$ if $a_0 = 1$. It is found by substitution that odd terms $a_{2\ell+1}$ vanish, and the even coefficients satisfy the recurrence formula

$$a_{2\ell+2} = a_{2\ell}[(2\ell)^2 + 10\ell - J]/(2\ell+2)(2\ell+5), \qquad (\ell = 0, 1, 2, ...;\ a_0 = 1). \tag{5.3.26}$$

The boundary condition (5.3.22) will be satisfied if we choose J as to make the series solution (5.3.24) terminate with $a_{2\ell}$ (cf. Sterne 1937, Ledoux 1946, Vaughan 1972):

$$J = 2\ell(2\ell+5). \tag{5.3.27}$$

This value of J yields by virtue of Eqs. (5.3.24), (5.3.26), (5.3.27) all radial modes of the homogeneous sphere $n = 0$:

$$\ell = 0: \ J = 0 \quad \text{and} \quad \eta_0 = 1; \qquad \ell = 1: \ J = 14 \quad \text{and} \quad \eta_1 = 1 - 7x^2/5;$$
$$\ell = 2: \ J = 36 \quad \text{and} \quad \eta_2 = 1 - 18x^2/5 + 99x^4/35, \quad \text{etc.} \tag{5.3.28}$$

The relative displacement η_ℓ vanishes between 0 and 1 at ℓ different values of x. The period of the ℓ-th mode is $\Pi = 2\pi/\sigma$, where

$$\sigma^2 = \sigma_\ell^2 = (2\pi\Gamma_1 G\varrho/3)[J + 2(3 - 4/\Gamma_1)], \tag{5.3.29}$$

via Eq. (5.3.23). The ℓ-th mode is stable if $\sigma^2 > 0$, which is equivalent to

$$J + 2(3 - 4/\Gamma_1) > 0 \quad \text{or} \quad \Gamma_1 > 4/[\ell(2\ell+5) + 3], \qquad (\ell = 0, 1, 2, ...). \tag{5.3.30}$$

J is given by Eq. (5.3.27). If $\Gamma_1 = 4/3$ and $\ell = 0$, ($J = 0$), the oscillation frequency σ^2 from Eq. (5.3.29) is zero, and the period of oscillations is infinite, showing that the model is in neutral equilibrium with respect to the fundamental mode. If $\Gamma_1 < 4/3$, the model is unstable with respect to the fundamental mode. Eq. (5.3.29) also shows that instability must first arise from the fundamental mode as Γ_1 is decreasing below 4/3, since the first mode becomes unstable only if $\Gamma_1 < 2/5$, and the second mode if $\Gamma_1 < 4/21$. The effect of an increase of Γ_1 is always to diminish the oscillation period $\Pi = 2\pi/\sigma$.

5.3.3 Linear Radial Oscillations of the Roche Model

As we have already mentioned [see Eqs. (3.2.69), (3.6.25)], the Roche model resembles to some extent the limiting Schuster-Emden polytrope of index $n = 5$, although the oscillatory behaviour is generally quite different (see Sec. 5.3.4). To obtain the relevant equations (5.3.19), (5.3.20) for the Roche model, Sterne (1937) assumes that the density outside the central point mass is distributed as $\varrho = a/r^2$, ($a =$ const). The small constant a will be subsequently allowed to approach zero, so the whole mass $m = 4\pi\varrho_m r_1^3/3$ of the sphere will be concentrated at the centre, as in a genuine Roche model. If for the moment $a \neq 0$, the central mass is equal to $(4\pi\varrho_m r_1^3/3)(1 - 3a/\varrho_m r_1^2)$, because the mass inside radius r is just

$$m = (4\pi\varrho_m r_1^3/3)(1 - 3a/\varrho_m r_1^2) + 4\pi\int_0^r \varrho r^2\, dr = (4\pi\varrho_m r_1^3/3)(1 - 3a/\varrho_m r_1^2) + 4\pi a r. \tag{5.3.31}$$

From the equation of hydrostatic equilibrium $dP/dr = -Gm\varrho/r^2$ we get for the unperturbed pressure at radius r

$$P = (4\pi G\varrho_m a r_1^3/9)(1 - 3a/\varrho_m r_1^2)(r^{-3} - r_1^{-3}) + 2\pi G a^2(r^{-2} - r_1^{-2}), \qquad (P = 0 \quad \text{if} \quad r = r_1). \tag{5.3.32}$$

Now, if $a \to 0$, we get

$$\varrho/P = 9/4\pi G\varrho_m r_1^3 r^2(r^{-3} - r_1^{-3}) = 9x/4\pi G\varrho_m r_1^2(1 - x^3), \tag{5.3.33}$$

and Eq. (5.3.19) reads eventually

$$d^2\eta/dx^2 + [(1-4x^3)/x(1-x^3)]\ d\eta/dx + [Jx + 3(4/\Gamma_1 - 3)/x^2]\ \eta/(1-x^3), \tag{5.3.34}$$

$$J = 9\sigma^2/4\pi G\Gamma_1\varrho_m. \tag{5.3.35}$$

The boundary condition (5.3.20) turns into

$$d\eta/dx = [J + 3(4/\Gamma_1 - 3)]\ \eta/3, \qquad (x = 1). \tag{5.3.36}$$

Eq. (5.3.34) has regular singularities at $x = 0$ and $x = 1$. At $x = 0$, the indicial equation for the series solution (5.3.24) is obtained in the same way as Eq. (5.3.25):

$$q(q-1) + q + 3(4/\Gamma_1 - 3) = 0. \tag{5.3.37}$$

The solutions of Eq. (5.3.37) are $q_{1,2} = \pm[3(3-4/\Gamma_1)]^{1/2}$, and the eigenfunction (5.3.24) becomes with the relevant root q_1 equal to

$$\eta = x^{q_1}\sum_{\ell=0}^{\infty} a_\ell x^\ell, \qquad (q_1 = [3(3-4/\Gamma_1)]^{1/2};\ a_0 = 1). \tag{5.3.38}$$

By substitution into Eq. (5.3.34) we find that the only nonvanishing coefficients are

$$a_{3\ell+3} = a_{3\ell}[(3\ell+q_1)(3\ell+q_1+3) - J]/[(3\ell+q_1+3)^2 + 3(4/\Gamma_1 - 3)], \qquad (\ell = 0, 1, 2, ...). \tag{5.3.39}$$

A convergent solution at the surface is only obtained if the series is cutted at $a_{3\ell}$, which is equivalent to (Sterne 1937)

$$J = (3\ell+q_1)(3\ell+q_1+3), \qquad (\ell = 0, 1, 2, ...). \tag{5.3.40}$$

The eigenvalues of the angular oscillation frequency are accordingly

$$\sigma^2 = 4\pi G\Gamma_1\varrho_m(3\ell+q_1)(3\ell+q_1+3)/9. \tag{5.3.41}$$

Obviously, if $\Gamma_1 < 4/3$ the root q_1 becomes imaginary, and σ from Eq. (5.3.41) is a complex number, allowing for unstable modes. If $\Gamma_1 = 4/3$, $(q_1 = 0)$, all modes are stable, excepting for the fundamental mode $\ell = 0$, possessing neutral stability. If $\Gamma_1 > 4/3$, all modes are stable.

5.3.4 The Schuster-Emden Polytrope $n = 5$

Below, we expand on Owen's (1957) result that this polytrope – unlike the Roche model – is incapable of performing either radial ($j = 0$) or nonradial ($j > 0$) oscillations of finite period, provided that $w(r_1) = 0$, which implies $A < 0$ or $\Gamma_1 > 1 + 1/n = 6/5$ near the boundary [see Eq. (5.2.123)].

Instead of v from Eq. (5.2.134) we introduce the variable z by

$$z = v/\xi^2 = (\Delta r/\alpha)\ \theta^{(n+1)/\Gamma_1}. \tag{5.3.42}$$

In the Cowling approximation the basic equations (5.2.124), (5.2.125) become via Eqs. (5.2.134), (5.2.135), (5.3.42) equal to

$$d(\xi^2 z)/d\xi = gw, \tag{5.3.43}$$

$$dw/d\xi = fz. \tag{5.3.44}$$

We reduce the range of ξ from $0 \le \xi \le \infty$ to $0 \le \zeta \le \pi/2$ by the change of variable

$$\xi = 3^{1/2}\tan\zeta. \tag{5.3.45}$$

The Schuster-Emden integral (2.3.90) assumes the form

$$\theta = \cos\zeta, \qquad (n = 5). \tag{5.3.46}$$

Taking into account the notations (5.2.135), the equations (5.3.43) and (5.3.44) turn into, $(dz/d\xi = (\cos^2\zeta/3^{1/2})\ dz/d\zeta)$:

$$dz/d\zeta + 2z\big/\sin\zeta\cos\zeta = Bw, \tag{5.3.47}$$

$$dw/d\zeta = Cz, \tag{5.3.48}$$

where

$$B = 3^{-1/2}(\cos\zeta)^Q[j(j+1)/\kappa\sin^2\zeta - 3/\Gamma_1\cos^3\zeta];$$
$$C = 3^{1/2}(\cos\zeta)^{-Q-2}[\kappa + (Q-n)(n+1)\sin^2\zeta\cos^3\zeta/6]. \tag{5.3.49}$$

We insert into Eq. (5.3.47) for z and $dz/d\zeta$ from Eq. (5.3.48):

$$d^2w/d\zeta^2 + \big\{d\big[\ln(\tan^2\zeta/C)\big]/d\zeta\big\}\ dw/d\zeta - BCw = 0. \tag{5.3.50}$$

Near the origin $\zeta = 0$ we have $B = 3^{-1/2}j(j+1)/\kappa\zeta^2,\ C = 3^{1/2}\kappa$. The coefficients of $dw/d\zeta$ and w in Eq. (5.3.50) become $2/\zeta$ and $-j(j+1)/\zeta^2$, respectively. Near the boundary, where $\zeta_1 = \pi/2$, the leading parts of the same coefficients are $-Q/(\pi/2-\zeta)$ and $3\kappa/\Gamma_1(\pi/2-\zeta)^5$. Thus, Eq. (5.3.50) has a regular singularity at the origin $\zeta = 0$, and an irregular singularity at the boundary $\zeta_1 = \pi/2$, because $BC \propto (\pi/2-\zeta)^{-5}$.

This means that there exist no solutions of the polytrope $n = 5$ which satisfy the boundary condition $w(\zeta_1) = 0,\ [A < 0$ or $\Gamma_1 > 1 + 1/n = 6/5$; cf. Eq. (5.2.123)]. Only if $\Gamma_1 = \infty$ (incompressible fluid), we have $BC \propto j(j+1)\big/(\pi/2-\zeta)^2$, and regular solutions occur, but without much practical interest. Note, that the previous considerations apply only if $w(\zeta_1) = 0$.

5.3.5 Linear Radial Oscillations of Polytropes with $0 \le n < 5$

Analytical solutions are not available for general polytropic index $0 < n < 5$, and we must resort to numerical integration of Eqs. (5.2.60) or (5.3.18) with the boundary conditions (5.2.61), and (5.2.62) or (5.2.68). Two practical methods have so far been devised for the numerical integration of Eq. (5.2.60): Fitting techniques and matrix methods (Cox 1980). Fitting techniques have been used almost exclusively until the past years, and are based on integrations starting concomitantly outwards and inwards from the centre and from the surface, respectively, for a trial value of the eigenfrequency σ. Due to the singularities of the coefficients of $d\eta/dr$ and η at the centre $r = 0$ and at the surface $r = r_1$, the solutions have to be expressed round these points by series of the form (5.3.24). At an intermediary point, where the outward and inward integrations meet, η and $d\eta/dr$ should be continuous. If this condition is not satisfied, the whole process has to be renewed for another value of σ, until the continuity condition is satisfied within the desired precision (Ledoux and Walraven 1958).

Calculations concerning radial oscillations of polytropes have been effected mainly by Eddington (1918, 1959), Edgar (1933), Kluyver (1935, 1936, 1938), and Schwarzschild (1941) for the standard model with polytropic index $n = 3$, Miller (1929) if $n = 2, 4$, Cowling (1934) and Lucas (1956) if $n = 1.5$, Sterne (1937) and Vaughan (1972) if $n = 0$, Ledoux (1946) if $n = 0, 3$, Hurley et al. (1966) if $n = 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.25, 3.5, 4$, Simon (1971), Simon and Sastri (1972) if $n = 1.5, 2, 3, 4.5$. Dynamically unstable modes if $\Gamma_1 < 4/3$ have been calculated by Van der Borght (1968) for the polytropes $n = 1.5, 2, 2.5, 3, 4$.

Perhaps the simplest way to obtain the famous period-mean density relation is to start with Eq. (5.3.1), where the period of oscillation is $\Pi = 2\pi/\sigma$, and the mean density is simply $\varrho_m = 3m(r_1)/4\pi r_1^3$:

$$\Pi^2\varrho_m = 3\pi/G(3\Gamma_1 - 4) = \text{const.} \tag{5.3.51}$$

Table 5.3.1 Solutions of the radial adiabatic wave equation (5.3.18) for various polytropic indices if $\Gamma_1 = 5/3$. The squared dimensionless oscillation frequency ω^2 is from Eq. (5.3.52), (ω_0 – fundamental eigenfrequency; ω_1, ω_2 – eigenfrequencies of first and second overtone). $\Pi_0 = 2\pi/\sigma_0$ and $\Pi_1 = 2\pi/\sigma_1$ denotes the fundamental period and the period of the first overtone, respectively. The value of $Q_{d0} = \Pi_0(\varrho_m/\varrho_{m\odot})^{1/2} = 0.1159(-\theta_1'/\xi_1)^{1/2}/\omega_0$ is taken from Eq. (5.3.54), and $\Pi_1/\Pi_0 = \omega_0/\omega_1$. The ratio between surface and central value of the fundamental mode is denoted by η_{0s}/η_{0c}, ($\eta_0 = \Delta r/r$). And $a + b$ means $a \times 10^b$.

n	ω_0^2	ω_1^2	ω_2^2	Q_{d0}	Π_1/Π_0	η_{0s}/η_{0c}
0	3.333−1	4.222+0	1.033+1	0.1159	0.281	1
0.5	2.471−1	2.190+0	5.141+0	0.0993	0.336	–
1	1.917−1	1.225+0	2.744+0	0.0843	0.396	1.24
1.5	1.506−1	6.975−1	1.479+0	0.0705	0.465	1.42
2	1.170−1	3.901−1	7.771−1	0.0579	0.548	2.37
2.5	8.683−2	2.086−1	3.859−1	0.0469	0.645	–
3	5.693−2	1.045−1	1.752−1	0.0381	0.738	2.24+1
3.5	2.756−2	4.625−2	6.994−2	0.0326	0.772	2.55+2
4	8.112−3	1.336−2	1.985−2	0.0298	0.779	5.95+3
4.5	9.262−4	1.467−3	2.134−3	0.0279	0.795	–

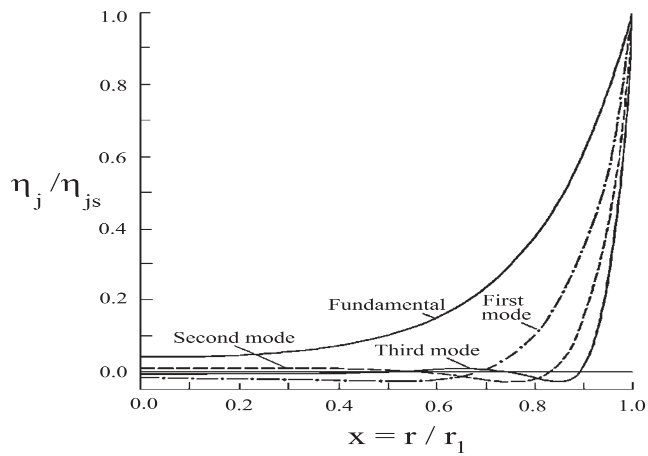

Fig. 5.3.1 Relative amplitudes η_j/η_{js}, $(j = 0, 1, 2, 3)$ of the first four modes (fundamental and first three overtones) as a function of the dimensionless radius $x = r/r_1$ for the standard model $n = 3$, $(\Gamma_1 = 5/3)$, (Ledoux and Walraven 1958, Fig. 38).

We now introduce the dimensionless, squared angular oscillation frequency ω by

$$\omega^2 = \sigma^2/4\pi G\varrho_0 = 3\sigma^2(-\theta_1')/4\pi G\varrho_m\xi_1 = \sigma^2 r_1^3(-\theta_1')/Gm_1\xi_1 = 3\pi(-\theta_1')/G\Pi^2\varrho_m\xi_1 = \pi/G\Pi^2\varrho_0. \tag{5.3.52}$$

m_1 is the total mass and r_1 the radius of the polytrope. The central density ϱ_0 is converted to mean density ϱ_m by Eq. (2.6.27): $\varrho_0 = \xi_1\varrho_m/3(-\theta_1')$. Many authors (e.g. Ledoux and Walraven 1958, Cox 1980) use instead of the eigenfrequency (5.3.52) the value $\sigma^2 r_1^3/Gm_1 = 3\sigma^2/4\pi G\varrho_m$. As seen from Eq. (5.3.52), the transformation to our eigenfrequency $\omega^2 = \sigma^2/4\pi G\varrho_0$ is effected simply by multiplication with the factor $-\theta_1'/\xi_1 = \varrho_m/3\varrho_0$.

Let us denote by

$$Q = \Pi(\varrho_m/\varrho_{m\odot})^{1/2}, \tag{5.3.53}$$

the so-called pulsation constant, where $\varrho_{m\odot} = 1.41$ g/cm^3 is the mean density of the Sun. Combining

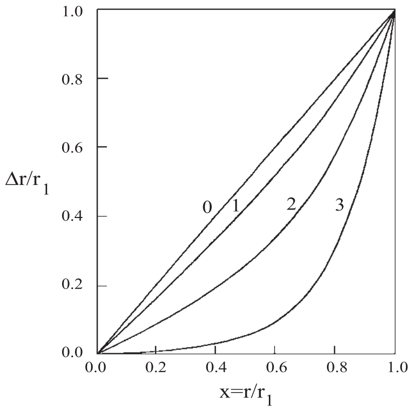

Fig. 5.3.2 Fundamental mode of radial oscillations for polytropic spheres of index $n = 0, 1, 2, 3$ if $\Gamma_1 = 5/3$. The displacements $\Delta r/r_1 = r\eta/r_1$ are normalized to unity at $x = r/r_1 = 1$, and the curves are labeled by the values of the polytropic index n (Tassoul 1978).

Eqs. (5.3.52) and (5.3.53), we obtain for Q, as expressed in days

$$Q_d = \Pi(\varrho_m/\varrho_{m\odot})^{1/2}/86400 = [3\pi(-\theta_1'/G\varrho_{m\odot}\xi_1)]^{1/2}/86400\omega$$
$$= 0.1159(-\theta_1'/\xi_1)^{1/2}/\omega = 0.0669(\varrho_m/\varrho_0)^{1/2}/\omega. \qquad (5.3.54)$$

It turns out from the numerical calculations (see Table 5.3.1) that Q does not change drastically with polytropic index (by a factor of 4). Q also changes due to nonadiabatic and nonlinear effects, but only by a few percent (e.g. Cox 1980).

In Table 5.3.1 all eigenvalues ω^2, together with the fundamental pulsation constant $Q_{d0} = \Pi_0(\varrho_m/\varrho_{m\odot})^{1/2} = 0.1159(-\theta_1'/\xi_1)^{1/2}/\omega_0$ and $\Pi_1/\Pi_0 = \omega_0/\omega_1$, are transformed from Table 1 of Hurley et al. (1966), excepting for $n = 4.5$ quoted from Table 2 of Simon and Sastri (1972). As the entries from Hurley et al. (1966) are in units of $4\pi G\varrho_0/(n+1)$, they have to be divided by $n+1$, in order to get ω^2 from Eq. (5.3.52) and Table 5.3.1. The surface to centre ratio of the fundamental mode η_{0s}/η_{0c} is taken from Cox (1980, Table 8.1). The surface to centre ratio of the first overtone η_{1s}/η_{1c} has been listed by Ledoux and Walraven (1958 Table 12): $\eta_{1s}/\eta_{1c} = -0.4, -3.39, -59.12$ if $n = 0, 1.5, 3$, respectively (cf. Fig. 5.3.1 if $n = 3$). Small differences in the third digit occur in some places among the values quoted by different authors.

The value of $Q_{d0} = 0.0381$ for the standard model $n = 3$ agrees well with the observed mean value for classical Cepheids $Q_{obs} = 0.0365$, as quoted by Ledoux and Walraven (1958, §98). For the bump Cepheids Whitney (1983) constructs polytropic models with index $2.5 < n < 3$, that have a period ratio between fundamental and second overtone of $\Pi_0/\Pi_2 = \omega_2/\omega_0 = 2$, in close agreement with observation and more sophisticated models.

5.3.6 Nonlinear, Second Order Radial Oscillations of Polytropes

Because the observed light and velocity curves of Cepheid variables are not of a simple trigonometric form, it is tempting to seek more elaborate solutions. One way would be to consider a nonlinear theory, although only the inclusion of nonadiabatic effects (H- and He-envelope ionization zones) offers a satisfactory explanation of most observed pulsational characteristics (e.g. Cox 1980). With the integrated adiabatic energy equation (5.2.22) $P = P_i(\varrho/\varrho_i)^{\Gamma_1}$, $(Q, \Gamma_1 = \mathrm{const})$, and with the mass conservation equation (5.2.42), the Lagrangian equation of radial motion (5.2.46) can be written as (e.g. Rosseland 1964, Chap. VII)

$$\partial^2 r/\partial t^2 = -(r^2/\varrho_i r_i^2)\ \partial\{P_i[r_i^2/r^2\ (\partial r/\partial r_i)]^{\Gamma_1}\}/\partial r_i - Gm(r_i)/r^2. \qquad (5.3.55)$$

$r = r(r_i, t)$ is the sole dependent variable, and the equilibrium pressure at initial state P_i satisfies the equation of hydrostatic equilibrium

$$dP_i/dr_i = -G\varrho_i\ m(r_i)/r_i^2. \tag{5.3.56}$$

The radius vector of an oscillating mass element is written under the form

$$r = r_i(1 + \Delta r/r_i) = r_i(1 + \eta), \qquad (\eta = \Delta r/r_i), \tag{5.3.57}$$

which yields

$$\partial^2 r/\partial t^2 = r_i\ \partial^2\eta/\partial t^2; \quad \partial r/\partial r_i = 1 + \eta + r_i\ \partial\eta/\partial r_i. \tag{5.3.58}$$

Then, the equation of motion (5.3.55) reads

$$[\varrho_i r_i/(1+\eta)^2]\ \partial^2\eta/\partial t^2 = -\partial\big[P_i(1+\eta)^{-2\Gamma_1}(1 + \eta + r_i\ \partial\eta/\partial r_i)^{-\Gamma_1}\big]/\partial r_i - G\varrho_i\ m(r_i)/r_i^2(1+\eta)^4. \tag{5.3.59}$$

We now particularize this quite general radial equation by considering η as a second order quantity

$$\varrho_i r_i(1 - 2\eta)\ \partial^2\eta/\partial t^2 = L(\eta) + Q(\eta^2), \qquad (O(\eta^3) = 0), \tag{5.3.60}$$

where L contains linear terms in $\eta,\ \partial\eta/\partial r_i,\ \partial^2\eta/\partial r_i^2$, and Q their quadratic combinations. Let us suppose that the solution of the first order part of Eq. (5.3.60)

$$\varrho_i r_i\ \partial^2\eta/\partial t^2 = L(\eta), \tag{5.3.61}$$

is given by the real part of the solution (5.2.58):

$$\eta(r_i, t) = X(r_i)\cos(\sigma t). \tag{5.3.62}$$

In seeking the solution of the second order equation (5.3.60), we choose its form equal to (Eddington 1919)

$$\eta(r_i, t) = X(r_i)\cos(\sigma t) + Y(r_i)\cos(2\sigma t) + Z(r_i). \tag{5.3.63}$$

Inserting this into Eq. (5.3.60), and taking into account that Y and Z are of second order, we obtain

$$\sigma^2\varrho_i r_i\big\{-X\cos(\sigma t) - 4Y\cos(2\sigma t) + X^2[1 + \cos(2\sigma t)]\big\} = L(X)\cos(\sigma t) + L(Y)\cos(2\sigma t) + L(Z)$$
$$+Q(X)\ [1 + \cos(2\sigma t)]/2, \qquad (\cos^2(\sigma t) = [1 + \cos(2\sigma t)]/2). \tag{5.3.64}$$

Equating to zero the cosines of different angles, we arrive at

$$L(Z) = \sigma^2\varrho_i r_i X^2 - Q(X)/2, \tag{5.3.65}$$

$$L(X) = -\sigma^2\varrho_i r_i X, \tag{5.3.66}$$

$$L(Y) + 4\sigma^2\varrho_i r_i Y = \sigma^2\varrho_i r_i X^2 - Q(X)/2, \tag{5.3.67}$$

where Eq. (5.3.66) is equivalent to (5.3.61).

Eddington (1919) and Kluyver (1935) carried out the integration of Eqs. (5.3.66), (5.3.67) for the standard model $n = 3$, while Simon (1971), and Simon and Sastri (1972) have integrated Eqs. (5.3.65)-(5.3.67) if $n = 1.5, 2, 3, 4.5$.

The function $Y(r_i)$ determines the amplitude of the second order pulsation, while $Z(r_i)$ shows how much the centre of pulsation is displaced by second order terms. Quantitatively, the results of numerical integrations are not satisfactory for Cepheids, since the second order amplitude of the radial velocity $|(\partial\eta/\partial t)_2| \propto |-2Y| = 0.033$ is no longer a small correction to the first order amplitude $|(\partial\eta/\partial t)_1| \propto |-X| = 0.06,\ (n = 3)$, (Rosseland 1964, Table 6).

Resonances occur for the displacement (5.3.63) whenever $4\sigma_0^2 = \sigma_k^2$, $(k = 1, 2, 3, ...)$, as can be seen by expanding Y and Z as a linear combination of the complete set of eigenfunctions η_k (Simon and Sastri 1972):

$$Y(r_i) = \sum_{k=0}^{\infty} b_k\eta_k(r_i); \quad Z(r_i) = \sum_{k=0}^{\infty} c_k\eta_k(r_i), \qquad (b_c, c_k = \mathrm{const}). \tag{5.3.68}$$

We write down Eqs. (5.3.67), (5.3.65) with $\sigma = \sigma_0$, and insert for $L(Y) = -\sigma^2\varrho_i r_i Y$ and $L(Z) = -\sigma^2\varrho_i r_i Z$ according to the first order approximation (5.3.66), $(X \to Y, Z)$. Eqs. (5.3.67) and (5.3.65) become, respectively:

$$\varrho_i r_i Y(4\sigma_0^2 - \sigma^2) = \varrho_i r_i(4\sigma_0^2 - \sigma^2)\sum_{k=0}^{\infty} b_k\eta_k = \sigma_0^2\varrho_i r_i X^2 - Q(X)/2, \tag{5.3.69}$$

$$\sigma^2\varrho_i r_i Z = \sigma^2\varrho_i r_i\sum_{k=0}^{\infty} c_k\eta_k = -\sigma_0^2\varrho_i r_i X^2 + Q(X)/2. \tag{5.3.70}$$

The eigenfunctions η_k are orthogonal, i.e. (Ledoux and Walraven 1958, Cox 1980):

$$\int_{m_1} \eta_k\eta_\ell^* r^2\ dm = \delta_{k\ell}\int_{m_1} |\eta_k|^2 r^2\ dm. \tag{5.3.71}$$

$\delta_{k\ell}$ means the Kronecker delta, η_ℓ^* is the complex conjugate of η_ℓ, and integration is carried out over the whole mass m_1. Our eigenfunctions η_k from Eq. (5.3.63) are all real $(\eta_k = \eta_k^*)$, and because Eqs. (5.3.65)-(5.3.67) are homogeneous, we may normalize the integral on the right-hand side of Eq. (5.3.71) to 1, fixing in this way the undetermined constant C of the solution $C\eta_k(r_i)$.

If we multiply Eqs. (5.3.69), (5.3.70) by $4\pi r_i^3\eta_k\ dr_i$, bearing in mind the orthogonality condition (5.3.71) and the normalization condition, the following equations for the coefficients b_k and c_k are obtained at once $(\sigma \to \sigma_k)$:

$$b_k = [4\pi/(4\sigma_0^2 - \sigma_k^2)]\int_{m_1} r_i^3\eta_k[\sigma_0^2\varrho_i r_i X - Q(X)/2]\ dr_i, \tag{5.3.72}$$

$$c_k = -(4\pi/\sigma_k^2)\int_{m_1} r_i^3\eta_k[\sigma_0^2\varrho_i r_i X - Q(X)/2]\ dr_i. \tag{5.3.73}$$

Indeed, if $4\sigma_0^2 \approx \sigma_k^2$ the coefficient b_k and $Y(r_i)$ become very large, and near-resonant oscillations occur in the model.

The statistical response to stochastic, nonlinear radial oscillations in polytropes $(n = 1.5, 2.5, 3)$ has been studied by Chaudhary et al. (1995), and Das et al. (1996).

5.3.7 Linear Radial Oscillations of Composite Polytropes

In this place we briefly present the radial pulsations of a composite polytrope consisting of a polytropic core with index $n_c = 1$, surrounded by an envelope of index $n_e = 5$, obeying Srivastava's (1962) solution (2.3.42), (see Eqs. (2.8.49)-(2.8.52), Murphy and Fiedler 1985a, b). The linear adiabatic wave equation (5.2.60) becomes with the polytropic variables (2.6.18), (5.2.134) equal to

$$d^2\eta/d\xi^2 + [4/\xi + (n+1)\theta'/\theta]\ d\eta/d\xi + [(3 - 4/\Gamma_1)(n+1)\theta'/\xi\theta + (n+1)\omega^2/\Gamma_1\theta]\ \eta = 0,$$
$$(\eta = \Delta r/r = \Delta\xi/\xi;\ \Gamma_1 = \mathrm{const};\ \omega^2 = \sigma^2/4\pi G\varrho_0). \tag{5.3.74}$$

Table 5.3.2 Fractional radius ξ_i/ξ_{e1} at the core-envelope interface, ratio ϱ_0/ϱ_m of central to mean density, squared dimensionless eigenfrequencies $\omega_0^2 = \sigma_0^2/4\pi G\varrho_0,\ \omega_1^2 = \sigma_1^2/4\pi G\varrho_0,\ (\Gamma_1 = 5/3)$ of the radial fundamental mode and the first overtone, ratio $\Pi_1/\Pi_0 = \omega_0/\omega_1$ between pulsation period of first overtone and fundamental period for the five composite $n_c = 1, n_e = 5$ polytropes from Table 2.8.2. The eigenfrequencies $\sigma^2 r_1^3/Gm_1 = 3\sigma^2/4\pi G\varrho_m$ of Murphy and Fiedler (1985a, b) are connected to $\omega^2 = \sigma^2/4\pi G\varrho_0$ by multiplication with $\varrho_m/3\varrho_0$. $a + b$ means $a \times 10^b$.

Model	ξ_i/ξ_{e1}	ϱ_0/ϱ_m	ω_0^2	ω_1^2	Π_1/Π_0
1	0.317	63.02	9.968−2	1.397−1	0.845
2	0.469	21.04	1.734−1	3.296−1	0.725
3	0.557	13.24	1.829−1	5.357−1	0.584
4	0.745	6.332	1.904−1	1.083+0	0.419
5	0.933	3.797	1.916−1	1.223+0	0.396

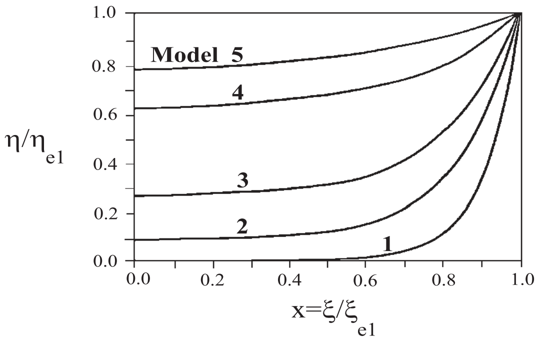

Fig. 5.3.3 Relative amplitudes $\eta/\eta_{e1} = \eta(\xi)/\eta(\xi_{e1})$ of fundamental modes as a function of the fractional radius $x = r/r_{e1} = \xi/\xi_{e1}$ for the five models from Table 5.3.2 if $\Gamma_1 = 5/3$ (Murphy and Fiedler 1985b).

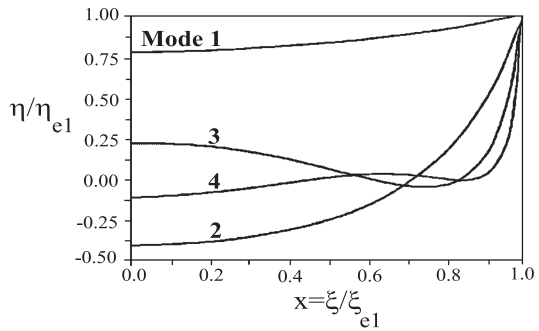

Fig. 5.3.4 Same as Fig. 5.3.3 for the first four modes of Model 5 from Table 5.3.2. Mode 1 is equal to the curve labeled Model 5 in Fig. 5.3.3 (Murphy and Fiedler 1985b).

Near the origin the coefficient of $d\eta/d\xi$ in Eq. (5.3.74) assumes the singular form $4/\xi$. From the requirement that η is regular everywhere (η must be finite, together with its derivatives), we observe that $(4/\xi)\ d\eta/d\xi$ must be finite at the origin, i.e. $d\eta/d\xi \propto \xi^s,\ (s \geq 1)$ or

$$d\eta/dr \propto d\eta/d\xi = 0, \qquad (r, \xi = 0). \tag{5.3.75}$$

This is taken to be the central boundary condition on η. The surface condition (5.2.68) assumes in polytropic variables the form

$$\xi\ d\eta/d\xi = \eta[(4/\Gamma_1 - 3) + \omega^2\xi_{e1}/\Gamma_1(-\theta'_{e1})], \qquad (\xi = \xi_{e1}). \tag{5.3.76}$$

Eq. (5.3.74) has been integrated numerically by inserting for θ the analytic solutions (2.8.49) and (2.8.50), subject to the boundary conditions (5.3.75) and (5.3.76), (Murphy and Fiedler 1985b). For the five models from Table 2.8.2 and Fig. 2.8.4, where most of the mass is contained in the core ($q \approx 1$), the relative displacement is shown in Fig. 5.3.3 as a function of fractional radius $x = r/r_{e1} = \xi/\xi_{e1}$.

5.3.8 Radial Oscillations of Isothermal Spheres $n = \pm\infty$

Due to their infinite mass and extension these spheres have to be cutted at some finite radius $r = r_1 = \alpha\xi_1$. Eigenvalues of the fundamental mode and of the first overtone, together with the eigenfunctions, have been calculated by Taff and Van Horn (1974) if $2 \le \xi_1 \le 90$. Introducing from Eqs. (2.6.2), (2.6.4), (2.6.6), (2.6.19) into the fundamental wave equation (5.2.60), we obtain the isothermal counterpart of Eq. (5.3.74):

$$d^2\eta/d\xi^2 + (4/\xi - \theta')\ d\eta/d\xi + [(4/\Gamma_1 - 3)\theta'/\xi + \omega^2/\Gamma_1]\ \eta = 0,$$
$$(\eta = \Delta r/r = \Delta\xi/\xi;\ \Gamma_1 = \text{const};\ n = \pm\infty). \tag{5.3.77}$$

The boundary conditions completing the definition of this eigenvalue problem are given by Eq. (5.3.75), and by $\Delta P = K\Delta\varrho = 0$ at $\xi = \xi_1$ [cf. Eq. (5.2.62)]. This amounts in virtue of Eq. (5.2.45) to $d\ln\eta/d\ln\xi = (\xi/\eta)\ d\eta/d\xi = -3,\ [(\Delta\varrho/\varrho)_{\xi=\xi_1} = 0;\ \varrho(\xi_1) \neq 0]$. We ignore Yabushita's (1968) results due to the comments of Taff and Van Horn (1974).

5.4 Instability of Truncated Polytropes

5.4.1 The Jeans Criterion for the Infinite Homogeneous Medium

Let us suppose that the medium is at rest, except for a small velocity $\vec{v}(\vec{r},t)$ due to wave motions. Neglecting squares and products of small quantities and of their derivatives, the hydrodynamic equation (5.2.29) becomes (e.g. Bonnor 1957, Chandrasekhar 1981)

$$\partial\delta\vec{v}/\partial t \approx \partial\vec{v}/\partial t = (\delta\varrho/\varrho^2)\ \nabla P - (1/\varrho)\ \nabla\delta P + \nabla\delta\Phi = -(1/\varrho)\ \nabla\delta P + \nabla\delta\Phi,$$
$$(\delta\vec{v} \approx \Delta\vec{v} = \vec{v} - \vec{v}_u = \vec{v};\ \nabla P \approx 0). \tag{5.4.1}$$

The other relevant equations are the continuity equation (5.2.25)

$$\partial\delta\varrho/\partial t = -\nabla\cdot(\varrho\vec{v}) = -\varrho\ \nabla\cdot\vec{v}, \qquad (\nabla\varrho \approx 0), \tag{5.4.2}$$

Poisson's equation (5.2.40)

$$\nabla^2\delta\Phi = -4\pi G\ \delta\varrho, \tag{5.4.3}$$

and the adiabatic energy equation (5.2.38)

$$\Delta P = (\Gamma_1 P/\varrho)\ \Delta\varrho. \tag{5.4.4}$$

To the first order the Lagrangian changes in Eq. (5.4.4) can be replaced in virtue of Eq. (5.1.16) with Eulerian changes, since $\nabla P, \nabla\varrho \approx 0$:

$$\delta P = (\Gamma_1 P/\varrho)\ \delta\varrho = a^2\ \delta\varrho. \tag{5.4.5}$$

By Eq. (2.1.49)

$$a^2 = (\partial P/\partial\varrho)_{S=\mathrm{const}} = \Gamma_1 P/\varrho \approx \mathrm{const}, \tag{5.4.6}$$

is the velocity of propagation (adiabatic sound velocity) of a small fluctuation in the density ϱ. This velocity is independent of the wave vector $\vec{j}$, where

$$j = |\vec{j}| = 2\pi/L, \tag{5.4.7}$$

denotes the wave number, and L the wavelength. We insert Eq. (5.4.5) into Eq. (5.4.1), taking its divergence:

$$\varrho\ \partial(\nabla\cdot\vec{v})/\partial t = -a^2\ \nabla^2\delta\varrho + \varrho\ \nabla^2\delta\Phi, \qquad (\nabla a \approx 0). \tag{5.4.8}$$

Making use of Eqs. (5.4.2) and (5.4.3), we find

$$\partial^2\delta\varrho/\partial t^2 = a^2\ \nabla^2\delta\varrho + 4\pi G\varrho\ \delta\varrho. \tag{5.4.9}$$

For a density perturbation of the form

$$\delta\varrho \propto \exp[i(\vec{j}\cdot\vec{r} + \sigma t)], \qquad [\vec{j} = \vec{j}(j_x, j_y, j_z);\ \vec{r} = \vec{r}(x,y,z)], \tag{5.4.10}$$

we obtain for the eigenvalue σ, after suppressing the common factor $\delta\varrho$:

$$\sigma^2 = a^2 j^2 - 4\pi G\varrho. \tag{5.4.11}$$

The propagation velocity a_J of these waves (Jeans velocity) is

$$a_J = r/t = -\sigma/j = a(1 - 4\pi G\varrho/a^2 j^2)^{1/2}. \tag{5.4.12}$$

Instability requires $\sigma^2 < 0$, or

$$j < (2/a)(\pi G\varrho)^{1/2} = j_J. \tag{5.4.13}$$

This is precisely the delimitation at which the Jeans velocity a_J starts up to become imaginary. If we turn with $j = 2\pi/L$ to the dimension of unstable fragments, Eq. (5.4.13) becomes [see Eq. (5.4.61)]

$$L > a(\pi/G\varrho)^{1/2} = L_J. \tag{5.4.14}$$

A mass of gas with dimensions larger than L_J will allow for time-growing disturbances of the type (5.4.10), and will therefore be unstable.

The Jeans criterion (5.4.14) for the gravitational instability of an infinite homogeneous medium is unaffected by the presence of uniform rotation Ω and/or of a uniform magnetic field (e.g. Chandrasekhar 1981, §120). Only in the special case, when the direction of wave propagation is exactly perpendicular to the rotation axis, the instability condition (5.4.14) becomes $\pi^2a^2/L^2 - \pi G\varrho + \Omega^2 < 0$, and instability along this *particular direction* cannot occur if $\Omega^2 > \pi G\varrho$. Some other extensions have been provided for an inhomogeneous, nonuniformly rotating medium (Anand and Kushwaha 1962b, Simon 1962, Safronov 1969), for the case when viscosity and thermal conduction are included (Kato and Kumar 1960), or when the energy balance equation replaces the assumption of adiabatic (isothermal) compression (Kegel and Traving 1976).

5.4.2 Radial Gravitational Instability of Polytropes Under External Pressure

The essence of this instability can be outlined from the virial theorem for a sphere (2.6.94), neglecting magnetic energy U_m and the kinetic energy of internal mass motions E_{kin} (McCrea 1957, Penston 1969):

$$(1/2)\ d^2I/dt^2 = W + 3V(P - P_S), \qquad \Big(I = \int_M r^2\ dM\Big). \tag{5.4.15}$$

The external pressure P_S has been replaced according to Eq. (2.6.96), and the internal energy U is approximated via Eq. (2.6.95) by the average pressure P_m inside the sphere:

$$(\Gamma - 1)U = \int_V P\ dV = P_mV. \tag{5.4.16}$$

Equilibrium configurations have $d^2I/dt^2 = 0$, or

$$P_S = W/3V + P_m. \tag{5.4.17}$$

The gravitational energy (2.6.133) of a truncated polytropic sphere writes with Eqs. (2.6.1), (2.6.18) as

$$W = 16\pi^2G\alpha^5\varrho_0^2(\pm\xi^3\theta^{n+1} + 3\xi^2\theta\theta' + 3\xi^3\theta'^2)/(n-5) = (\pm\theta^{n+1}/\theta'^2 + 3\theta/\xi\theta' + 3)GM^2/(n-5)r$$
$$= -AGM^2/r, \qquad (N = 3;\ n \neq -1, \pm\infty), \tag{5.4.18}$$

where $[\Gamma(1/2)]^3/\Gamma(3/2) = 2\pi$ via Eq. (C.11), and $\xi_0 \to 0,\ \xi_1 \to \xi$.

Likewise, the gravitational energy (2.6.148) of a truncated isothermal sphere is

$$W = 16\pi^2G\alpha^5\varrho_0^2[\xi^3\exp(-\theta) - 3\xi^2\theta'] = [(1/\theta'^2)\exp(-\theta) - 3/\xi\theta']GM^2/r = -AGM^2/r,$$
$$(N = 3;\ n = \pm\infty), \tag{5.4.19}$$

where we have inserted for r and M from Eqs. (2.6.1), (2.6.19). In a first approximation the virial coefficient A may be regarded as constant (Horedt 1970, Table I).

The average adiabatic pressure change in a sphere occurs according to Eq. (1.3.28):

$$P_m = BV^{-\Gamma_1} = B(3/4\pi r^3)^{\Gamma_1}, \qquad (\Gamma_1' = \Gamma_1;\ B = \mathrm{const}). \tag{5.4.20}$$

We insert Eqs. (5.4.18)-(5.4.20) into Eq. (5.4.17):

$$P_S = -AGM^2/4\pi r^4 + B(3/4\pi r^3)^{\Gamma_1} = -C/r^4 + D/r^{3\Gamma_1}, \qquad (A, C, D = \mathrm{const} > 0). \qquad (5.4.21)$$

A stable truncated sphere of constant mass M should expand by Δr if the external pressure diminishes by the amount ΔP_S : Stability demands $\Delta P_S/\Delta r < 0$. Conversely, the truncated sphere becomes unstable and starts collapse if its radius shrinks by $\Delta r < 0$ after an external pressure release $\Delta P_S < 0: \ \Delta P_S/\Delta r > 0$. Neutral stability – at the verge of instability – occurs if

$$\Delta P_S/\Delta r = 4C/r^5 - 3\Gamma_1 D/r^{3\Gamma_1+1} = 0 \quad \mathrm{or} \quad r = r_{cr} = (3\Gamma_1 D/4C)^{1/(3\Gamma_1-4)}$$
$$= (3/4\pi)^{\Gamma_1/(3\Gamma_1-4)}(3\pi\Gamma_1 B/AGM^2)^{1/(3\Gamma_1-4)}; \quad P_S = P_{S,cr} = (4C/3\Gamma_1 D)^{4/(3\Gamma_1-4)}(D-C). \qquad (5.4.22)$$

If $\Gamma_1 > 4/3$ and $r \to 0$, the leading term in Eq. (5.4.21) is $D/r^{3\Gamma_1}$, showing that strongly compressed spheres are stable: $\Delta P_S/\Delta r \approx -3\Gamma_1 D r^{-3\Gamma_1-1} < 0$. However, for extended and weakly compressed spheres with $\Gamma_1 > 4/3$ our simplified approach from Eq. (5.4.21) is not suitable, since it would lead to a vanishing, unphysical negative boundary pressure: $P_S \approx -C/r^4$ if $r \to \infty$ and $\Gamma_1 > 4/3$.

Instability only occurs if $\Gamma_1 < 4/3$, like for a vanishing external pressure [Eq. (2.6.101)]. The extended, weakly compressed sphere is stable: $\Delta P_S/\Delta r \approx -3\Gamma_1 D r^{-3\Gamma_1-1} < 0, \ (r \to \infty)$. As the equilibrium external pressure P_S increases, and the sphere becomes increasingly compressed, the critical radius r_{cr} and the maximum equilibrium boundary pressure $P_{S,cr}$ are attained. If the sphere is further compressed $(r < r_{cr})$, no stable equilibrium configurations subsist, because in this case the external equilibrium pressure (5.4.21) is $P_S < P_{S,cr}$, and $\Delta P_S/\Delta r > 0, \ (\Gamma_1 < 4/3)$. The strongly compressed sphere is unstable: $\Delta P_S/\Delta r \approx 4C/r^5 > 0, \ (r \to 0)$. Likewise, no stable sphere can exist if $P_S > P_{S,cr}, \ (\Gamma_1 < 4/3)$, because the equilibrium equation (5.4.21) cannot be satisfied for any value of r (McCrea 1957).

These qualitative findings will be further substantiated for truncated polytropic slabs, cylinders, and spheres. Polytropic stellar cores and polytropic interstellar clouds are subject to the external pressure from the stellar envelope and from the intercloud medium, respectively. In some of these incomplete polytropes there occurs radial gravitational instability if the external pressure exceeds a certain critical limit. In a stable, radially symmetric configuration the pressure P increases as the radial distance r decreases. Thus, if we apply a small quasistatic disturbance Δr to the constant mass M at radial distance r, the corresponding Lagrangian pressure change ΔP should have opposite sign in a stable hydrostatic configuration, i.e. $\Delta P/\Delta r < 0$. Conversely, if (Ebert 1955, Bonnor 1956, 1958, Spitzer 1968)

$$(\Delta P/\Delta r)_{M=\mathrm{const}} > 0, \qquad (5.4.23)$$

a radially symmetric, hydrostatic configuration with constant mass and constant polytropic index becomes unstable against radial disturbances. Radius, pressure, and mass of an incomplete N-dimensional polytrope are given by [cf. Eqs. (2.6.1), (2.6.3), (2.6.12)]

$$r = \alpha\xi = [\pm(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi; \quad P = K\varrho_0^{1+1/n}\theta^{n+1};$$
$$M = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}(\mp\theta') = \mathrm{const}, \qquad (N = 1, 2, 3, ...; \ n \neq -1, \pm\infty), \qquad (5.4.24)$$

and [cf. Eqs. (2.6.2), (2.6.4), (2.6.13)]

$$r = \alpha\xi = (K/4\pi G\varrho_0)^{1/2}\xi; \quad P = K\varrho = K\varrho_0\exp(-\theta);$$
$$M = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}\theta' = \mathrm{const}, \qquad (N = 1, 2, 3, ...; \ n = \pm\infty). \qquad (5.4.25)$$

Because the mass M of the incomplete polytrope remains constant under a disturbance Δr, we can eliminate the variable central density ϱ_0 between the relationships in Eqs. (5.4.24) and (5.4.25), respectively. Radius r and external pressure P of the incomplete polytrope will remain functions of a single variable ξ, aside of an insignificant constant factor:

$$\alpha \propto \varrho_0^{(1-n)/2n}; \quad M \propto \varrho_0^{[N(1-n)+2n]/2n}\xi^{N-1}(\mp\theta') = \mathrm{const}; \quad \varrho_0 \propto$$
$$\xi^{2n(1-N)/[N(1-n)+2n]}(\mp\theta')^{-2n/[N(1-n)+2n]}; \quad r \propto r^* = \xi^{(n+1)/[N(1-n)+2n]}(\mp\theta')^{(n-1)/[N(1-n)+2n]};$$
$$P \propto P^* = \xi^{2(1-N)(n+1)/[N(1-n)+2n]}(\mp\theta')^{-2(n+1)/[N(1-n)+2n]}\theta^{n+1}, \quad (N = 1, 2, 3, ...; \ n \neq -1, \pm\infty), \qquad (5.4.26)$$

$$\alpha \propto \varrho_0^{-1/2};\quad M \propto \varrho_0^{1-N/2}\xi^{N-1}\theta' = \mathrm{const};\quad \varrho_0 \propto \xi^{2(1-N)/(2-N)}\theta'^{-2/(2-N)};\quad r \propto r^*$$
$$= \xi^{1/(2-N)}\theta'^{1/(2-N)};\quad P \propto P^* = \xi^{2(1-N)/(2-N)}\theta'^{-2/(2-N)}\exp(-\theta),\quad (N = 1, 3, 4, ...;\ n = \pm\infty). \tag{5.4.27}$$

For the cases of practical interest the dimensionless external pressure P^* acting on the surface at dimensionless distance r^* is (Viala and Horedt 1974a):

$$N = 1\quad \text{(slabs)}:\quad r^* = \xi(\mp\theta')^{(n-1)/(n+1)};\quad P^* = (\mp\theta')^{-2}\theta^{n+1}\quad \text{if}\quad n \neq -1, \pm\infty.$$
$$r^* = \xi\theta';\quad P^* = \theta'^{-2}\exp(-\theta)\quad \text{if}\quad n = \pm\infty, \tag{5.4.28}$$

$$N = 2\quad \text{(cylinders)}:\quad r^* = \xi^{(n+1)/2}(\mp\theta')^{(n-1)/2};\quad P^* = \xi^{-n-1}(\mp\theta')^{-n-1}\theta^{n+1}\quad \text{if}\quad n \neq -1, \pm\infty, \tag{5.4.29}$$

$$N = 3\quad \text{(spheres)}:\quad r^* = \xi^{(n+1)/(3-n)}(\mp\theta')^{(n-1)/(3-n)};$$
$$P^* = \xi^{-4(n+1)/(3-n)}(\mp\theta')^{-2(n+1)/(3-n)}\theta^{n+1}\quad \text{if}\quad n \neq -1, \pm\infty.$$
$$r^* = \xi^{-1}\theta'^{-1};\quad P^* = \xi^4\theta'^2\exp(-\theta)\quad \text{if}\quad n = \pm\infty. \tag{5.4.30}$$

We have excluded from Eqs. (5.4.27) and (5.4.29) the case $N = 2,\ n = \pm\infty$ (the isothermal cylinder), because its mass per unit length [cf. Eqs. (2.3.48), (2.6.17)]

$$M = 2\pi\varrho_0\alpha^2\xi\theta' = (K/2G)\xi\theta' = 2K\xi^2/G(8 + \xi^2),\qquad (N = 2;\ n = \pm\infty), \tag{5.4.31}$$

is independent of the variable central density ϱ_0, and our elimination procedure of ϱ_0 breaks down. However, by assuming for the moment $N \neq 2$, and turning in Eq. (5.4.33) to the limit $N \to 2$, we observe at once that $(\Delta P/\Delta r)_{M=\mathrm{const}} \propto -2/\xi < 0$ throughout [cf. Eq. (5.4.37)], i.e. the infinitely long isothermal cylinder is *gravitationally* stable against radial perturbations (cf. McCrea 1957, Sec. 10).

We now investigate the sign of $(\Delta P/\Delta r)_{M=\mathrm{const}}$, which is equivalent to the sign of $\Delta P^*/\Delta r^*$:

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto \Delta P^*/\Delta r^* = (\Delta P^*/\Delta\xi)/(\Delta r^*/\Delta\xi) \propto (n+1)\{2\theta^{n+1} \pm [N(1-n) + 2n]\theta'^2\}$$
$$/\{(1-n)\xi\theta^n \pm [N(1-n) + 2n]\theta'\},\qquad (N = 1, 2, 3, ...;\ n \neq -1, \pm\infty), \tag{5.4.32}$$

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto \Delta P^*/\Delta r^* = (\Delta P^*/\Delta\xi)/(\Delta r^*/\Delta\xi) \propto [-2\exp(-\theta) + (N-2)\theta'^2]$$
$$/[\xi\exp(-\theta) - (N-2)\theta'],\qquad (N = 1, 2, 3, ...;\ n = \pm\infty), \tag{5.4.33}$$

where we have inserted for θ'' from Eqs. (2.1.14) and (2.1.21), respectively. For small values of the surface radial coordinate ξ, we get from the foregoing equations [$\theta \approx 1 \mp \xi^2/2N$, Eq. (2.4.21)]

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto -(n+1)N/n\xi,\qquad (\xi \approx 0;\ N = 1, 2, 3, ...;\ n \neq -1, \pm\infty), \tag{5.4.34}$$

and [$\theta \approx \xi^2/2N$, Eq. (2.4.36)]

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto -N/\xi,\qquad (\xi \approx 0;\ N = 1, 2, 3, ...;\ n = \pm\infty). \tag{5.4.35}$$

Eqs. (5.4.32) and (5.4.33) become for the cases of practical interest equal to

$$N = 1\quad \text{(slab)}:\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto (n+1)[2\theta^{n+1} \pm (n+1)\theta'^2]/[(1-n)\xi\theta^n \pm (n+1)\theta']\quad \text{if}$$
$$n \neq -1, \pm\infty;\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto [-2\exp(-\theta) - \theta'^2]/[\xi\exp(-\theta) + \theta']\quad \text{if}\quad n = \pm\infty, \tag{5.4.36}$$

$$N = 2\quad \text{(cylinder)}:\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto (n+1)(2\theta^{n+1} \pm 2\theta'^2)/[(1-n)\xi\theta^n \pm 2\theta']\quad \text{if}$$
$$n \neq -1, \pm\infty;\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto -2/\xi\quad \text{if}\quad n = \pm\infty, \tag{5.4.37}$$

$$N = 3\quad \text{(sphere)}:\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto (n+1)[2\theta^{n+1} \pm (3-n)\theta'^2]/[(1-n)\xi\theta^n \pm (3-n)\theta']\quad \text{if}$$
$$n \neq -1, \pm\infty;\quad (\Delta P/\Delta r)_{M=\mathrm{const}} \propto [-2\exp(-\theta) + \theta'^2]/[\xi\exp(-\theta) - \theta']\quad \text{if}\quad n = \pm\infty. \tag{5.4.38}$$

We now discuss separately the unstable states of incomplete polytropes under external pressure for the cases $N = 1,2,3$. We have to exclude from our considerations the constant density polytrope ($n = 0$), because in this case $\theta = 1 - \xi^2/2N$ [(Eq. (2.3.5)], and we observe from Eq. (5.4.26) that $\Delta r/\Delta\xi \propto \Delta r^*/\Delta\xi \equiv 0$. The sign of the derivative $(\Delta P/\Delta r)_{M=\mathrm{const}} = \pm\infty$ remains undefined. In fact, as will be shown below, the polytropic index $n = 0$ separates just the gravitationally stable polytropes with $n > 0$ from the unstable ones with $n < 0$.

(i) $-1 < n < 0;\ N = 1,2,3.$ The upper sign holds in Eq. (5.4.32), so the numerator is always positive in this case. To show the same also for the denominator, we apply Bonnor's (1958) reasoning, supposing the contrary:

$$(1-n)\xi\theta^n + [N(1-n)+2n]\theta' \le 0, \qquad (-1 < n < 0). \tag{5.4.39}$$

Inserting for θ^n and θ' from Eqs. (2.6.3) and (2.6.27), we get

$$(1-n)\xi\varrho/\varrho_0 - [N(1-n)+2n]\xi\varrho_m/N\varrho_0 \le 0, \tag{5.4.40}$$

or

$$\varrho \le [N(1-n)+2n]\varrho_m/N(1-n) = [1+2n/N(1-n)]\varrho_m < \varrho_m, \tag{5.4.41}$$

since the factor in the brackets is < 1 if $n < 0$. But for polytropic indices $-1 < n < 0$ the density increases outwards and $\varrho > \varrho_m$, contradicting the finding from Eq. (5.4.41). Thus, the denominator in Eq. (5.4.32) has to be positive throughout, and $(\Delta P/\Delta r)_{M=\mathrm{const}} > 0$. We conclude that polytropes with indices $-1 < n < 0$ are always gravitationally unstable under external pressure.

(ii) Polytropic slabs ($N = 1;\ -\infty \le n < -1$ and $0 < n \le \infty$). If $n = \pm\infty$, Eq. (5.4.36) shows at once that $(\Delta P/\Delta r)_{M=\mathrm{const}} < 0$, $(\theta' > 0)$, i.e. the isothermal slab is gravitationally stable against radial disturbances. The same result follows from an inspection of $(\Delta P/\Delta r)_{M=\mathrm{const}}$ if $-\infty < n < -1$ and $1 \le n < \infty$. If $0 < n < 1$, we proceed with the denominator of Eq. (5.4.36) in the same manner as in Eqs. (5.4.39)-(5.4.41), by supposing that

$$(1-n)\xi\theta^n + (1+n)\theta' \ge 0, \qquad (N = 1;\ 0 < n < 1). \tag{5.4.42}$$

After insertion of $\theta^n = \varrho/\varrho_0$ and $\theta' = -\xi\varrho_m/\varrho_0$ we obtain

$$\varrho \ge [(1+n)/(1-n)]\varrho_m > \varrho_m, \qquad (N = 1;\ 0 < n < 1), \tag{5.4.43}$$

which is clearly impossible. Thus, polytropic slabs are throughout gravitationally stable under external pressure, excepting if $-1 < n < 0$.

(iii) Polytropic cylinders ($N = 2;\ -\infty \le n < -1$ and $0 < n \le \infty$). If $n = \pm\infty$, we have already seen in Eq. (5.4.37) that $(\Delta P/\Delta r)_{M=\mathrm{const}} \propto -2/\xi < 0$, i.e. the isothermal cylinder is stable. Likewise, if $1 \le n < \infty$ the derivative (5.4.37) is negative. In the case $0 < n < 1$ we assume again that the denominator from Eq. (5.4.37) would be positive

$$(1-n)\xi\theta^n + 2\theta' \ge 0, \qquad (N = 2;\ 0 < n < 1), \tag{5.4.44}$$

from which we would obtain $(\theta^n = \varrho/\varrho_0;\ \theta' = -\xi\varrho_m/2\varrho_0)$

$$\varrho \ge \varrho_m/(1-n) > \varrho_m, \qquad (N = 2;\ 0 < n < 1), \tag{5.4.45}$$

and this is clearly impossible. Thus, polytropic cylinders of index $0 < n \le \infty$ are gravitationally stable under external pressure: $(\Delta P/\Delta r)_{M=\mathrm{const}} < 0$.

If $-\infty < n < -1$, the derivative $(\Delta P/\Delta r)_{M=\mathrm{const}}$ from Eq. (5.4.34) is negative for small values of ξ, hence these cylinders are stable, as long as $\xi \approx 0$. If $-\infty < n < -1$, polytropic cylinders extend to infinity, and the Lane-Emden function obeys the asymptotic expansion from Eq. (2.4.83). We insert θ from Eq. (2.4.83) and its derivative into Eq. (5.4.37), the principal terms associated with $\xi^{2/(1-n)}$ canceling exactly:

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto (n+1)\xi^{(1+n)/(1-n)}[c_1\sin F(\xi) + c_2\cos F(\xi)]/[c_3\sin F(\xi) + c_4\cos F(\xi)],$$
$$(\xi \to \infty;\ N = 2;\ -\infty < n < -1;\ F(\xi) = 2(-n)^{1/2}\ln\xi/(1-n) + c_5;\ c_1,c_2,c_3,c_4,c_5 = \mathrm{const}). \tag{5.4.46}$$

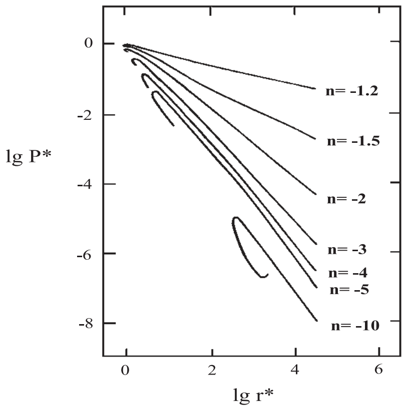

Fig. 5.4.1 Dimensionless external pressure P^* and corresponding radius r^* from Eq. (5.4.29) for unstable cylinders ($N = 2$) with polytropic index $-\infty < n < -1$. The parts of the curves on the left of the maximum of P^* represent unstable states, and cannot occur in practice. Only the first half of the spiral round the singular point (5.4.54) is represented at the scale of the figure (Viala and Horedt 1974a).

Because of the trigonometric terms, the derivative (5.4.46) takes positive and negative values if $\xi \to \infty$, so gravitational instability of cylinders with index $-\infty < n < -1$ must occur for some values of ξ.

(iv) Polytropic spheres ($N = 3$; $-\infty \leq n < -1$ and $0 < n \leq \infty$). If $1 \leq n \leq 3$, Eq. (5.4.38) yields $(\Delta P/\Delta r)_{M=\mathrm{const}} < 0$. To prove stability also in the case $0 < n < 1$, we apply the reasoning from Eqs. (5.4.39)-(5.4.45), by assuming

$$(1-n)\xi\theta^n + (3-n)\theta' \geq 0, \qquad (N = 3;\ 0 < n < 1). \tag{5.4.47}$$

With $\theta^n = \varrho/\varrho_0$ and $\theta' = -\xi\varrho_m/3\varrho_0$ we obtain

$$\varrho \geq (3-n)\varrho_m/3(1-n) > \varrho_m, \qquad (N = 3;\ 0 < n < 1), \tag{5.4.48}$$

which is clearly impossible. So, polytropic spheres are gravitationally stable if $0 < n \leq 3$. If $3 < n < 5$, we observe from Eq. (5.4.30) that $P \propto P^* \to 0$ if $\xi \to 0$ and $\xi \to \xi_1$, where ξ_1 is the first zero of the Lane-Emden function θ. Since for small values of ξ the derivative $(\Delta P/\Delta r)_{M=\mathrm{const}}$ is negative via Eq. (5.4.34), and since $P \geq 0$, at least one maximum of P^* must occur between 0 and ξ_1 where $(\Delta P/\Delta r)_{M=\mathrm{const}} = 0$. So, portions of the curve $P(r)$ exist having $(\Delta P/\Delta r)_{M=\mathrm{const}} > 0$. Spherical polytropes under external pressure are unstable if $3 < n < 5$ (Fig. 5.4.3).

In the particular case $n = 5$ we find, by inserting the Schuster-Emden integral (2.3.90) into Eq. (5.4.38)

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto 2(9-\xi^2)/\xi(1+\xi^2/3)^{1/2}(\xi^2/3-5), \qquad (N = 3;\ n = 5), \tag{5.4.49}$$

showing that the curve $P(r)$ has a maximum at $\xi = 3$, and if $\xi > 3$, unstable states occur having $(\Delta P/\Delta r)_{M=\mathrm{const}} > 0$.

If $-\infty < n < -1$ and $5 < n < \infty$, the polytropic spheres extend to infinity (Sec. 2.6.8), and θ is of the form (2.4.88). We insert θ and its derivative into Eq. (5.4.38), the principal terms associated with $\xi^{2/(1-n)}$ canceling exactly:

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto (n+1)\xi^{(1+n)/(1-n)}[c_1 \sin G(\xi) + c_2 \cos G(\xi)]/[c_3 \sin G(\xi) + c_4 \cos G(\xi)],$$
$$(\xi \to \infty;\ N = 3;\ -\infty < n < -1 \text{ and } 5 < n < \infty;\ G(\xi) = (7n^2 - 22n - 1)^{1/2} \ln \xi/2(1-n) + c_5;$$
$$c_1, c_2, c_3, c_4, c_5 = \mathrm{const}). \tag{5.4.50}$$

Because of the trigonometric terms, the derivative (5.4.50) takes positive and negative values if $\xi \to \infty$, so gravitational instability of spheres with polytropic index $-\infty < n < -1$ and $5 < n < \infty$ must occur for some values of ξ.

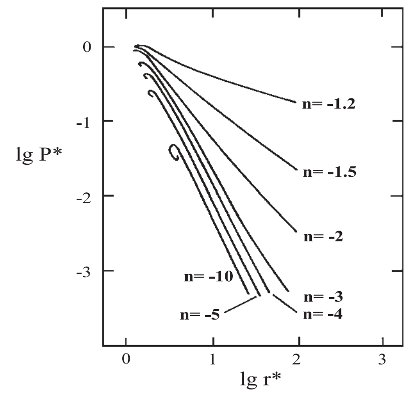

Fig. 5.4.2 Same as Fig. 5.4.1, but for unstable spheres ($N = 3$) with polytropic index $-\infty < n < -1$. The upper ends of the curves spiral round the singular points (5.4.55), (Viala and Horedt 1974a).

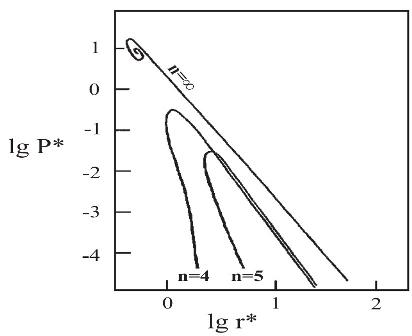

Fig. 5.4.3 Same as Figs. 5.4.1 and 5.4.2, but for unstable spheres ($N = 3$) with polytropic index $3 < n < \infty$ and $n = \pm\infty$. If $n = \pm\infty$, the curve spirals round the singular point (5.4.56), (Horedt 1970). See also the schematic Fig. 17 by Stahler (1983).

In the particular case $n = \pm\infty$, ($N = 3$) the asymptotic solution is of the form (2.4.104). We proceed exactly as for Eqs. (5.4.46) and (5.4.50):

$$(\Delta P/\Delta r)_{M=\mathrm{const}} \propto (1/\xi)[c_1 \sin H(\xi) + c_2 \cos H(\xi)]\big/[c_3 \sin H(\xi) + c_4 \cos H(\xi)],$$
$$(\xi \to \infty;\ N = 3;\ n = \pm\infty;\ H(\xi) = (7^{1/2}/2) \ln \xi + c_5;\ c_1, c_2, c_3, c_4, c_5 = \mathrm{const}). \qquad (5.4.51)$$

Like in the cases $-\infty < n < 1$ and $5 < n < \infty$, the derivative (5.4.51) takes positive and negative values as $\xi \to \infty$, so the isothermal sphere is gravitationally unstable for some values of ξ.

Summarizing the above findings, we see that radial gravitational instability under external pressure of polytropic slabs, cylinders, and spheres occurs for *any* value of P if $-1 < n < 0$. Above certain values of the external pressure, truncated polytropic cylinders become unstable if $-\infty < n < -1$, while truncated polytropic spheres become unstable if $-\infty \le n < -1$ and $3 < n \le \infty$. Otherwise, these configurations are stable under external pressure against radial perturbations (for slabs if $-\infty \le n < -1$ and $0 < n \le \infty$, for cylinders if $0 < n < \infty$ and $n = \pm\infty$, and for spheres if $0 < n \le 3$).

Table 5.4.1 Critical values (corresponding to the maximum of P^* from Figs. 5.4.1-5.4.3) at which gravitational instability under external pressure sets in, according to Bonnor (1956, 1958), Horedt (1970), Shu et al. (1972), and Viala and Horedt (1974a). Values of the Lane-Emden function are found by interpolation from the tables of Viala and Horedt (1974b) and Horedt (1986b). In the last column we have quoted the numerical factor $c_J = [-(n+1)N\xi_c\theta_c'/4\pi]^{1/2}$ from the Jeans criterion (5.4.59) if $n \neq \pm\infty$, and $c_J = (N\xi_c\theta_c'/4\pi)^{1/2}$ from Eq. (5.4.60) if $n = \pm\infty$. $a + b$ means $a \times 10^b$.

n	ξ_c	θ_c	θ_c'	r_c^*	P_c^*	c_J
			$N = 2$			
−10	3.10	1.55	1.41−1	2.94+2	1.13−5	7.91−1
−5	3.70	2.01	2.49−1	4.73	4.41−2	7.66−1
−4	3.90	2.21	3.02−1	2.59	1.51−1	7.50−1
−3	4.30	2.60	3.84−1	1.58	4.03−1	7.25−1
−2	4.95	3.38	5.44−1	1.12	7.97−1	6.55−1
−1.5	5.50	4.19	6.99−1	1.02	9.58−1	5.53−1
−1.2	6.10	5.41	8.50−1	9.98−1	1.00	4.06−1
			$N = 3$			
−10	2.05	1.27	1.31−1	3.40	5.09−2	7.60−1
−5	2.90	1.57	2.03−1	1.94	2.81−1	7.50−1
−4	3.30	1.74	2.34−1	1.69	4.23−1	7.44−1
−3	3.85	2.02	2.86−1	1.47	6.41−1	7.25−1
−2	4.80	2.63	3.90−1	1.29	9.15−1	6.69−1
−1.5	5.70	3.35	4.92−1	1.22	1.01	5.79−1
−1.2	6.60	4.19	5.98−1	1.20	1.02	4.34−1
4	3.50	3.73−1	−1.21−1	1.07	3.70−1	7.11−1
4.5	3.20	4.46−1	−1.25−1	1.80	7.20−2	7.25−1
5	3.00	5.00−1	−1.25−1	2.37	3.17−2	7.33−1
6	2.70	5.81−1	−1.22−1	3.28	1.29−2	7.42−1
±∞	6.50	2.66	3.75−1	4.10−1	1.76+1	7.63−1

Maloney (1988), and McLaughlin and Pudritz (1996) find that polytropic spheres of index $-\infty < n < -1$ are throughout *stable* under external pressure, if isothermal perturbations ($T_0 =$ const) are assumed instead of adiabatic ones. But at least for the considered interstellar molecular clouds such an assumption is physically untenable (Curry and McKee 2000).

Horedt (1973) finds for weakly distorted polytropic models (rotationally and tidally distorted spheres, slowly rotating cylinders, polytropic rings) and for weakly relativistic spheres, that gravitational instability under external pressure occurs in a first approximation for the same polytropic indices as for the undistorted Newtonian models. Horedt's (1973) results are not applicable if $N = 3$, $n \approx 3$, since he neglects the dependence of the small parameters $\beta = \Omega^2/2\pi G\varrho_0$ and $q_0 = K\varrho_{r0}^{1/n}/c^2$ on the central density [cf. Eq. (5.12.63)]. As should be expected on general grounds, the values of ξ for which instability of weakly distorted polytropes under external pressure occurs, differ only by first order quantities from those quoted in Table 5.4.1.

For small values of ξ the radius and external pressure from Eq. (5.4.26) become ($\theta \approx 1 \mp \xi^2/2N$)

$$r \propto r^* \approx (N^{1-n}\xi^{2n})^{1/[N(1-n)+2n]}; \quad P \propto P^* \approx (N\xi^{-N})^{2(n+1)/[N(1-n)+2n]}, \quad (\xi \approx 0;\ n \neq -1, \pm\infty), \tag{5.4.52}$$

and from Eq. (5.4.27), ($\theta \approx \xi^2/2N$)

$$r \propto r^* \approx (N^{-1}\xi^2)^{1/(2-N)}; \quad P \propto P^* \approx (N\xi^{-N})^{2/(2-N)}, \qquad (\xi \approx 0;\ N \neq 2;\ n = \pm\infty). \tag{5.4.53}$$

If $\xi \to \infty$, we find from Eqs. (2.4.83), (5.4.29) for the unstable cylinders with indices $-\infty < n < -1$ that r^* and P^* spiral round the singular points (see Fig. 5.4.1)

$$r_s^* = [2/(1-n)]^{(n+1)/2}; \quad P_s^* = [2/(1-n)]^{-n-1}, \qquad (N = 2;\ \xi \to \infty;\ -\infty < n < -1), \tag{5.4.54}$$

where $\Delta P^*/\Delta r^* = 0/0$. Similarly, in the spherical case one observes from Eqs. (2.4.88), (5.4.30) that r^* and P^* spiral round the singular points [see Fig. 5.4.2, and Fig. 17 by Stahler (1983)]

$$r_s^* = \big\{\pm 2^n(n-3)[\mp(1-n)]^{-n-1}\big\}^{1/(3-n)}; \quad P_s^* = \big\{\pm 2^{-3}(n-3)^{-1}[\mp(1-n)]^4\big\}^{(n+1)/(3-n)},$$
$$(N = 3;\ \xi \to \infty;\ -\infty < n < -1 \text{ and } 5 < n < \infty). \tag{5.4.55}$$

In the particular case $n = \pm\infty$ we get from Eqs. (2.4.104), (5.4.30), (see Fig. 5.4.3)

$$r_s^* = 1/2; \quad P_s^* = 8, \qquad (N = 3;\ \xi \to \infty;\ n = \pm\infty). \tag{5.4.56}$$

Eqs. (5.4.34), (5.4.35) show that the derivative $(\Delta P/\Delta r)_{M=\mathrm{const}}$ is negative outside the interval $n \in (-1, 0)$, so these polytropes are stable as long as $\xi \approx 0$. From Eqs. (5.4.52), (5.4.53) we also observe that for unstable polytropes $r \propto r^* \to \infty$ and $P \propto P^* \to 0$ if $\xi \approx 0$. As ξ increases (r^* decreases), the external pressure increases for unstable polytropes up to the critical maximum value $P_c \propto P_c^*$, corresponding to the dimensionless coordinate $\xi = \xi_c$. If ξ increases further, the external pressure P^* decreases, and the configuration becomes unstable for some values of ξ if $\xi > \xi_c$ (Figs. 5.4.1-5.4.3). The critical coordinate ξ_c is just given by the first zero of $\Delta P/\Delta r$ from Eqs. (5.4.32), (5.4.33), i.e. by the first root of $2\theta^{n+1}(\xi_c) \pm [N(1-n) + 2n]\theta'^2(\xi_c) = 0$ if $n \neq -1, \pm\infty$, and $-2\exp[\theta(\xi_c)] + (N-2)\theta'^2(\xi_c) = 0$ if $n = \pm\infty$.

For polytropic indices $-\infty < n < -1$, $(N = 2)$, and $-\infty \le n < -1,\ 3 < n \le \infty$, $(N = 3)$ there exists a relationship similar to the Jeans criterion (5.4.14). Since the equations (5.4.26), (5.4.27) obtained for the boundary radius r are much more complicated than the equivalent original equations (5.4.24), (5.4.25), we use the latter ones, the critical boundary radius r_c being simply obtained by inserting $\xi = \xi_c$:

$$r_c = [\pm(n+1)K/4\pi G\varrho_0^{1-1/n}]^{1/2}\xi_c,$$
$$(-\infty < n < -1 \text{ if } N = 2, 3, \quad \text{and} \quad 3 < n < \infty \text{ if } N = 3), \tag{5.4.57}$$

$$r_c = (K/4\pi G\varrho_0)^{1/2}\xi_c, \qquad (n = \pm\infty;\ N = 3). \tag{5.4.58}$$

The polytropes become unstable when $r > r_c$. If polytropic matter also obeys the perfect gas equation $P_0 = \mathcal{R}\varrho_0 T_0/\mu = K\varrho_0^{1+1/n}$, i.e. $K = \mathcal{R}T_0/\mu\varrho_0^{1/n}$, the foregoing equations can be brought into the form (T_0 = central temperature)

$$r_c = [\pm(n+1)\mathcal{R}T_0/4\pi G\mu\varrho_0]^{1/2}\xi_c = [-(n+1)N\xi_c\theta'(\xi_c)/4\pi]^{1/2}(\mathcal{R}T_0/G\mu\varrho_m)^{1/2},$$
$$(-\infty < n < -1 \text{ if } N = 2, 3, \quad \text{and} \quad 3 < n < \infty \text{ if } N = 3), \tag{5.4.59}$$

and

$$r_c = (\mathcal{R}T_0/4\pi G\mu\varrho_0)^{1/2}\xi_c = [N\xi_c\theta'(\xi_c)/4\pi]^{1/2}(\mathcal{R}T_0/G\mu\varrho_m)^{1/2}, \qquad (n = \pm\infty,\ N = 3), \tag{5.4.60}$$

via Eqs. (2.6.27), (2.6.28). The last two equations are very similar to the Jeans criterion (5.4.14) for the uniform isothermal medium composed of a perfect gas, when the adiabatic exponent (2.1.51) takes its isothermal value $\Gamma_1 = 1 + 1/n = 1,\ (n = \pm\infty)$:

$$r_c = L_J/2 = (\pi\Gamma_1 P_0/4G\varrho_0^2)^{1/2} = (\pi^{1/2}/2)(\mathcal{R}T_0/G\mu\varrho_0)^{1/2} = (\pi^{1/2}/2)(\mathcal{R}T_0/G\mu\varrho_m)^{1/2}$$
$$= 0.89(\mathcal{R}T_0/G\mu\varrho_m)^{1/2}, \qquad (\Gamma_1 = 1;\ \varrho_0 = \varrho_m = \mathrm{const}). \tag{5.4.61}$$

The factors c_J near $(\mathcal{R}T_0/G\mu\varrho_m)^{1/2}$ in Eqs. (5.4.59), (5.4.60) are shown in the last column of Table 5.4.1. Indeed, the quoted values of c_J are of the same order of magnitude as the factor 0.89 in Eq. (5.4.61).

In the spherical case $N = 3$ we may derive from Eqs. (5.4.59)-(5.4.61) a critical mass-radius relationship by substituting for $\mathcal{R}T_0/\mu = P_0/\varrho_0 = \mp 3P_c\theta_c^{-n-1}\theta_c'/\xi_c\varrho_m$ if $n \neq -1, \pm\infty$, and $\mathcal{R}T_0/\mu = P_0/\varrho_0 = 3P_c\theta_c'\exp\theta_c/\xi_c\varrho_m$ if $n = \pm\infty$ (Chièze 1987, Yabushita 1992):

$$M_c = [\pm 4\pi(n+1)\theta_c'^2/G\theta_c^{n+1}]^{1/2}P_c^{1/2}r_c^2, \qquad (N = 3;\ -\infty < n < -1;\ 3 < n < \infty), \tag{5.4.62}$$

$$M_c = (4\pi\theta_c'^2\exp\theta_c/G)^{1/2}P_c^{1/2}r_c^2, \qquad (N = 3;\ n = \pm\infty), \tag{5.4.63}$$

$$M_c = (2/3)(\pi^3/G)^{1/2}P_c^{1/2}r_c^2, \qquad (\Gamma_1 = 1;\ \varrho_0 = \varrho_m = \mathrm{const}). \tag{5.4.64}$$

Masses larger than M_c are gravitationally unstable under external pressure. Curry and McKee (2000) find for their composite polytropic models (Sec. 2.8.1) that the pressure drop P_0/P_c of critical composite isothermal models is limited by the square (≈ 200) of the Bonnor-Ebert value ≈ 14 for single isothermal spheres: With $\xi_c = 6.5$ from Table 5.4.1 we get $P_0/P_c = \exp\theta_c = 14.3$ for the Bonnor-Ebert value. Arbitrarily large pressure drops are possible if nonisentropic composite polytropes are considered ($\Gamma_1 \neq 1 + 1/n$). Similar results are also obtained for the so-called multipressure polytropes of McKee and Holliman (1999), when the total pressure is given by the sum of partial pressures

$$P = \sum_j P_j = \sum_j K_j \varrho^{1+1/n_j}, \qquad (K_j, n_j = \mathrm{const}), \tag{5.4.65}$$

where ϱ denotes the total mass density in the polytrope.

Kimura (1981b) – using uncommon notations – has considered besides the familiar Lane-Emden functions discussed above, also the stability under external pressure of other types of solutions found in Sec. 2.7.

5.5 Nonradial Oscillations of Polytropic Spheres

5.5.1 Nonradial Oscillations of the Homogeneous Polytrope $n = 0$

Pekeris (1938) seems to have been the first who treated the problem of nonradial oscillations of a compressible sphere for the simplest case $n = 0$, as earlier work by Emden (1907) was vitiated by inconsistent approximations concerning the equation of continuity (see Kopal 1949). Later on, Cowling (1941) correctly formulated Emden's idea of ignoring the influence of the perturbed gravitational potential $\delta\Phi$ (Cowling approximation). Further studies on the nonradial oscillations of polytropic spheres are due to Kopal (1949), Sauvenier-Goffin (1951), Owen (1957), Hurley et al. (1966), and Robe (1968a, 1974). The results of Robe (1968a) have shown that all p, g, and f-modes continue to exist also if $n > 3.25$, contrary to Owen's (1957) suggestion.

One of the simplest models that can be solved analytically is the constant density polytrope $n = 0$ (Pekeris 1938, Ledoux and Walraven 1958, Hurley et al. 1966). Inserting the Eulerian change (5.2.23) into the adiabatic energy conservation equation (5.2.21), we get

$$[1/(P_u + \delta P)]\ D(P_u + \delta P)/Dt = [\Gamma_1/(\varrho_u + \delta\varrho)]\ D(\varrho_u + \delta\varrho)/Dt, \tag{5.5.1}$$

or with Eqs. (5.1.6), (5.1.24), (5.2.25) up to the first order $[\vec{v} = \vec{v}(v_r, v_\lambda, v_\varphi) \approx \delta\vec{v};\ \vec{v}_u = 0;\ P_u = P_u(r);\ \varrho_u = \varrho_u(r)]$:

$$\partial\delta P/\partial t + v_r\ dP_u/dr = (\Gamma_1 P_u/\varrho_u)(\partial\delta\varrho/\partial t + v_r\ d\varrho_u/dr) = -\Gamma_1 P_u\ \nabla\cdot\vec{v}. \tag{5.5.2}$$

Since the unperturbed pressure satisfies the hydrostatic equation

$$dP_u/dr = -G\varrho_u\ m(r_u)/r_u^2 = -g\varrho_u, \qquad (\vec{g} = -\nabla\Phi_u;\ g = G\ m(r_u)/r_u^2), \tag{5.5.3}$$

we obtain from Eq. (5.5.2):

$$\partial\delta P/\partial t = \varrho_u g v_r - \Gamma_1 P_u\ \nabla\cdot\vec{v}, \qquad (\Gamma_1 = \mathrm{const}). \tag{5.5.4}$$

The spherical components of the Eulerian equation of motion (5.2.29) are

$$\varrho_u\ \partial v_r/\partial t = -\partial\delta P/\partial r - g\ \delta\varrho + \varrho_u\ \partial\delta\Phi/\partial r, \tag{5.5.5}$$

$$\varrho_u\ \partial v_\lambda/\partial t = (1/r)\ \partial(-\delta P + \varrho_u\ \delta\Phi)/\partial\lambda, \tag{5.5.6}$$

$$\varrho_u\ \partial v_\varphi/\partial t = (1/r\sin\lambda)\ \partial(-\delta P + \varrho_u\ \delta\Phi)/\partial\varphi. \tag{5.5.7}$$

Following Pekeris (1938), we derive Eq. (5.5.4) with respect to r

$$\partial^2\delta P/\partial r\partial t = g v_r\ d\varrho_u/dr + \varrho_u v_r\ dg/dr + \varrho_u g\ \partial v_r/\partial r + \Gamma_1\varrho_u g\ \nabla\cdot\vec{v} - \Gamma_1 P_u\ \partial(\nabla\cdot\vec{v})/\partial r, \tag{5.5.8}$$

and insert for $\partial^2\delta P/\partial r\partial t$ from the derivative with respect to t of Eq. (5.5.5), where we assume for v_r and $\delta\Phi$ a time dependence of the form (5.1.31), $(\partial\varrho_u/\partial t, \partial g/\partial t = 0;\ \partial^2 v_r/\partial t^2 = -\sigma^2 v_r;\ \partial\delta\Phi/\partial t = i\sigma\ \delta\Phi)$:

$$i\sigma\ \partial\delta\Phi/\partial r = -(\Gamma_1 P_u/\varrho_u)\ \partial(\nabla\cdot\vec{v})/\partial r + (\Gamma_1 - 1)g\ \nabla\cdot\vec{v} + g\ \partial v_r/\partial r + (dg/dr - \sigma^2)v_r. \tag{5.5.9}$$

To eliminate $\delta\Phi$ from Eq. (5.5.9), we derive at first Eq. (5.2.91) with respect to the time, inserting for $\partial\delta P/\partial t$ via Eq. (5.5.4), and for $j(j+1)\ \delta\Phi$ via Eqs. (5.1.28), (5.1.31), (B.39), $(\partial\Delta\vec{r}/\partial t = \vec{v};\ \partial\Delta r/\partial t = v_r)$:

$$\begin{aligned}\partial(\nabla\cdot\Delta\vec{r})/\partial t &= \nabla\cdot\vec{v} = (1/r^2)\ \partial(r^2 v_r)/\partial r - [j(j+1)/\sigma^2 r^2][\partial(\delta P/\varrho_u)/\partial t - \partial\delta\Phi/\partial t]\\ &= (1/r^2)\ \partial(r^2 v_r)/\partial r - [j(j+1)/\sigma^2 r^2][g v_r - (\Gamma_1 P_u/\varrho_u)\ \nabla\cdot\vec{v}] + ij(j+1)\ \delta\Phi/\sigma r^2\\ &= (1/r^2)\ \partial(r^2 v_r)/\partial r - [j(j+1)/\sigma^2 r^2][g v_r - (\Gamma_1 P_u/\varrho_u)\ \nabla\cdot\vec{v}]\\ &\quad -(i/\sigma)\ \nabla^2\delta\Phi + (i/\sigma r^2)\ \partial(r^2\ \partial\delta\Phi/\partial r)/\partial r.\end{aligned} \tag{5.5.10}$$

Then, we eliminate $\partial(r^2\ \partial\delta\Phi/\partial r)/\partial r$ between Eqs. (5.5.9) and (5.5.10) by multiplying Eq. (5.5.9) with r^2, and deriving with respect to r. The Laplacian $\nabla^2\delta\Phi$ in Eq. (5.5.10) is replaced with the time derivative of Poisson's equation (5.2.40), where $\partial\delta\varrho/\partial t$ is replaced via Eq. (5.2.25):

$$\partial(\nabla^2\delta\Phi)/\partial t = i\sigma\ \nabla^2\delta\Phi = -4\pi G\ \partial\delta\varrho/\partial t = 4\pi G(v_r\ d\varrho_u/dr + \varrho_u\ \nabla\cdot\vec{v}). \tag{5.5.11}$$

We will also replace $4\pi G\varrho_u$ via the unperturbed Poisson equation $-\nabla^2\Phi_u = \nabla\cdot\vec{g} = (1/r^2)\ d(r^2 g)/dr = dg/dr + 2g/r = 4\pi G\varrho_u$, and d^2g/dr^2 by $4\pi G\ d\varrho_u/dr - (2/r)\ dg/dr + 2g/r^2$. On account of the described eliminations we obtain an equation in $\nabla\cdot\vec{v}$ and v_r :

$$\begin{aligned}&(\Gamma_1 P_u/\varrho_u)\ \partial^2(\nabla\cdot\vec{v})/\partial r^2 + [\Gamma_1\ d(P_u/\varrho_u)/dr - (\Gamma_1-1)g + 2\Gamma_1 P_u/\varrho_u r]\ \partial(\nabla\cdot\vec{v})/\partial r\\ &+[\sigma^2 + (2-\Gamma_1)(dg/dr+2g/r) - j(j+1)\Gamma_1 P_u/\varrho_u r^2]\ \nabla\cdot\vec{v}\\ &= g\ \partial^2 v_r/\partial r^2 + (2\ dg/dr + 2g/r)\ \partial v_r/\partial r + [2-j(j+1)]\ v_r g/r^2.\end{aligned} \tag{5.5.12}$$

A second equation in the same variables can be found by differentiating Eq. (5.5.10) with respect to r, and eliminating $\partial\delta\Phi/\partial r$ between Eqs. (5.5.9) and (5.5.10):

$$\begin{aligned}&r^2\ \partial(\nabla\cdot\vec{v})/\partial r + \big\{2r - j(j+1)[(\Gamma_1-1)g + \Gamma_1 P_u/\varrho_u]/\sigma^2\big\}\ \nabla\cdot\vec{v}\\ &= r^2\ \partial^2 v_r/\partial r^2 + 4r\ \partial v_r/\partial r + [2-j(j+1)]\ v_r.\end{aligned} \tag{5.5.13}$$

The cumbersome elimination of v_r between Eqs. (5.5.12) and (5.5.13), as effected by Pekeris (1938), will not be reproduced, and we turn directly to the particular case $n=0$. In this circumstance we have: $\omega^2 = \sigma^2/4\pi G\varrho;\ g = 4\pi G\varrho r/3;\ P_u = (2\pi/3)G\varrho^2(r_1^2 - r^2)$. The right-hand sides of Eqs. (5.5.12) and (5.5.13) become, respectively

$$\begin{aligned}&(4\pi G\varrho r/3)\big\{\partial^2 v_r/\partial r^2 + (4/r)\ \partial v_r/\partial r + [2-j(j+1)]\ v_r/r^2\big\} \quad \text{and}\\ &r^2\big\{\partial^2 v_r/\partial r^2 + (4/r)\ \partial v_r/\partial r + [2-j(j+1)]\ v_r/r^2\big\}.\end{aligned} \tag{5.5.14}$$

Thus, on eliminating the brackets between Eqs. (5.5.12) and (5.5.13), we find a homogeneous equation in $\nabla\cdot\vec{v}$:

$$\begin{aligned}&(r_1^2 - r^2)\ \partial^2(\nabla\cdot\vec{v})/\partial r^2 + [(2r_1^2 - 6r^2)/r]\ \partial(\nabla\cdot\vec{v})/\partial r\\ &+\big[6\omega^2/\Gamma_1 + 8/\Gamma_1 - 6 - 2j(j+1)/3\Gamma_1\omega^2 - j(j+1)(r_1^2 - r^2)/r^2\big]\ \nabla\cdot\vec{v} = 0.\end{aligned} \tag{5.5.15}$$

We introduce the dimensionless radial coordinate $x = r/r_1$, by observing from the relationship (B.37) of the divergence in spherical coordinates that $\nabla\cdot\vec{v}(r,\lambda,\varphi) = (1/r_1)\ \nabla\cdot\vec{v}(x,\lambda,\varphi)$. We also separate the angular part of $\nabla\cdot\vec{v}$ according to Eq. (5.1.31): $\nabla\cdot\vec{v}(x,\lambda,\varphi) = [\nabla\cdot\vec{v}(x)]\ Y_j^k(\lambda,\varphi)$. So, all partial derivatives in Eq. (5.5.15) can be replaced by ordinary derivatives:

$$\begin{aligned}&(1-x^2)\ d^2(\nabla\cdot\vec{v})/dx^2 + [(2-6x^2)/x]\ d(\nabla\cdot\vec{v})/dx\\ &+\big[6\omega^2/\Gamma_1 + 8/\Gamma_1 - 6 - 2j(j+1)/3\Gamma_1\omega^2 - j(j+1)(1-x^2)/x^2\big]\ \nabla\cdot\vec{v} = 0.\end{aligned} \tag{5.5.16}$$

Assuming for $\nabla\cdot\vec{v}(x)$ a solution of the form (5.3.24), we get for the indicial equation of the lowest power x^{q-2} :

$$q^2 + q - j(j+1) = 0. \tag{5.5.17}$$

Since $\nabla\cdot\vec{v}(x)$ has to be finite at the origin, only $q_1 = j$ subsists among the two roots j and $-j-1$ of Eq. (5.5.17), and $\nabla\cdot\vec{v}(x)$ is of the form

$$\nabla\cdot\vec{v}(x) = x^j\sum_{\ell=0}^{\infty} a_\ell x^\ell = x^j F(x). \tag{5.5.18}$$

We substitute Eq. (5.5.18) into Eq. (5.5.16), obtaining

$$\begin{aligned}&(1-x^2)\ d^2F/dx^2 + (2/x)[j+1-(j+3)x^2]\ dF/dx + BF = 0;\\ &B = 6\omega^2/\Gamma_1 + 8/\Gamma_1 - 6 - 2j(j+1)/3\Gamma_1\omega^2 - 4j.\end{aligned} \tag{5.5.19}$$

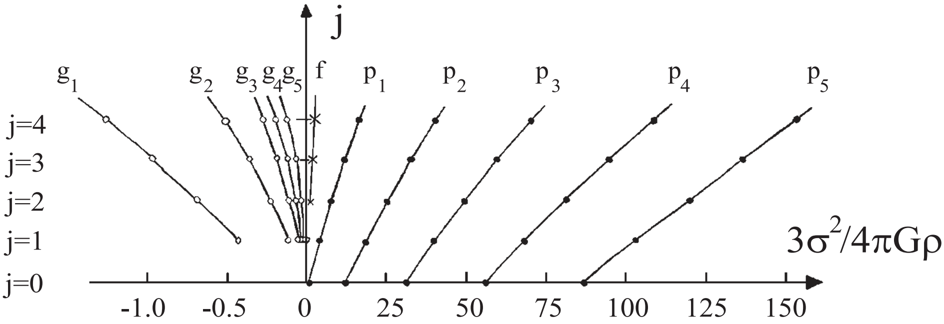

Fig. 5.5.1 Distribution of squared normalized eigenfrequencies $3\omega^2 = 3\omega^2(j,\ell) = 3\sigma^2/4\pi G\varrho = \sigma^2 r_1^3/Gm_1$ for the p-modes [dots, Eq. (5.5.23)], f-modes [crosses, Eq. (5.5.26)], and g-modes [circles, Eq. (5.5.24)] of the homogeneous compressible model $n = 0$, $(\Gamma_1 = 5/3)$. The purely radial modes are shown for $j = 0$ via Eq. (5.3.29). Note the different scales on the left and right of the ordinate axis (Ledoux 1974).

Inserting for F from Eq. (5.5.18), we find the recurrence relation

$$a_{\ell+2} = a_\ell[\ell(\ell+5+2j) - B]/(\ell+2)(\ell+3+2j); \quad a_0 \neq 0; \quad a_1 = 0. \tag{5.5.20}$$

Hence, odd indexed coefficients $a_{2\ell+1}$ are zero, and the series for F possesses only even powers of x. At the outer boundary $x = 1$ the quotient criterion yields $\lim_{\ell\to\infty}(a_{2\ell+2}/a_{2\ell}) = 1$, and F will not converge (cf. Vaughan 1972). In order to make $\nabla\cdot\vec{v}$ from Eq. (5.5.18) finite at $x = 1$, we have to cut the series at a_ℓ, i.e.

$$B = \ell(\ell+5+2j), \qquad (\ell = 0, 2, 4, ...). \tag{5.5.21}$$

Inserting for B from (5.5.19), we write Eq. (5.5.21) under the form

$$D_\ell = 3\omega^2/2 - j(j+1)/6\omega^2 = -2 + (\Gamma_1/4)[\ell(\ell+5+2j) + 6 + 4j], \quad (\ell = 0, 2, 4, ...;\ D_{2\ell+1} = 0). \tag{5.5.22}$$

Eq. (5.5.22) can be solved for the eigenvalues ω^2, and yields two sets of dimensionless angular oscillation frequencies ω^2, defining the p-modes

$$\omega_p^2 = \sigma_p^2/4\pi G\varrho = \left\{D_\ell + [D_\ell^2 + j(j+1)]^{1/2}\right\}/3, \qquad (\ell = 0, 2, 4, ...), \tag{5.5.23}$$

and the g-modes

$$\omega_g^2 = \sigma_g^2/4\pi G\varrho = \left\{D_\ell - [D_\ell^2 + j(j+1)]^{1/2}\right\}/3, \qquad (\ell = 0, 2, 4, ...). \tag{5.5.24}$$

For given ℓ and j the eigenvalues ω_p^2 of p-modes are always positive: The p-modes are always stable. The eigenvalues ω_g^2 of g-modes are always negative, which corresponds to instability, connected to the fact that $A = -(1/\Gamma_1 P)\, dP/dr > 0$ for the polytrope $n = 0$ [cf. Eq. (5.2.133)]. Obviously, if $j = 0$ (radial oscillations), then $\omega^2 = \omega_p^2 = 2D_\ell/3$, and with $\ell \to 2\ell$, we recover just the eigenvalues $\sigma^2 = 4\pi G\varrho\omega^2$ from Eq. (5.3.29). If $\ell \gg 1$, the two sets of eigenvalues from Eqs. (5.5.23) and (5.5.24) take for fixed j the simple forms

$$\omega_p^2 = \Gamma_1\ell^2/6 + 2j(j+1)/3\Gamma_1\ell^2 \quad \text{and} \quad \omega_g^2 = -2j(j+1)/3\Gamma_1\ell^2, \qquad (\ell \gg 1;\ D_\ell \approx \Gamma_1\ell^2/4). \tag{5.5.25}$$

The first spectrum of eigenvalues ω_p^2 tends to infinity if $\ell \to \infty$, while the second spectrum ω_g^2 decreases towards zero if $\ell \to \infty$ (cf. Eq. (5.2.126) if $\sigma, \omega \gg 1$, and Eq. (5.2.127) if $\sigma, \omega \ll 1$).

The existence of the stable fundamental (Kelvin) mode for the compressible homogeneous polytrope with eigenvalue

$$\omega_f^2 = \sigma_f^2/4\pi G\varrho = 2j(j-1)/3(2j+1), \qquad (n=0;\ j=2,3,4,...), \tag{5.5.26}$$

has been established by Chandrasekhar [1964a, Eq. (53)]. This mode has the same form as the Kelvin mode for the incompressible homogeneous sphere $\Gamma_1 = \infty,\ n = 0$ (Tassoul 1978). An approximate analytical expression for the $j = 2$ Kelvin mode has been devised by Lai et al. [1993, Eq. (5.21)] with their ellipsoidal energy variational method (cf. Eq. (5.5.26) if $n = 0,\ j = 2$) : $\omega_f^2 = [12\xi_1^3\theta_1'^2/5(5-n)]\Big/\int_0^{\xi_1}\theta^n\xi^4\ d\xi,\ (0 \le n \lesssim 2)$.

As expected on general grounds (Chandrasekhar and Lebovitz 1963b, 1964, Dixit et al. 1980), the g-modes from Tables 5.5.1, 5.5.2 become unstable ($\sigma^2 < 0$) for values of the generalized adiabatic exponent Γ_1 which are lower than the convective instability limit from Eqs. (5.2.85) and (5.2.134): $A > 0$ if $\Gamma_1 < 1 + 1/n$, i.e. $n < 1.5$ if $\Gamma_1 = 5/3$.

5.5.2 Nonradial Oscillations of Polytropes if $0 < n < 5$

Robe (1968a, see Table 5.5.2) seems to have effected the most comprehensive numerical integration of the system (5.2.100)-(5.2.102) for the oscillations of a polytropic sphere if $j = 2$ and $\Gamma_1 = 5/3$. The components of the displacement vector $\Delta\vec{r}(r,\lambda,\varphi)$ are via Eqs. (5.1.26), (5.1.27), (5.2.87)-(5.2.90), (5.2.99) equal to

$$\Delta r = [u(r)/r^2]\ Y_j^k(\lambda,\varphi); \quad r\ \Delta\lambda = [\chi(r)/\sigma^2 r]\ \partial Y_j^k(\lambda,\varphi)/\partial\lambda;$$
$$r\sin\lambda\ \Delta\varphi = [\chi(r)/\sigma^2 r\sin\lambda]\ \partial Y_j^k(\lambda,\varphi)/\partial\varphi. \tag{5.5.27}$$

The Eulerian density variation is in virtue of Eqs. (5.2.95), (5.2.99) equal to

$$\delta\varrho(r)/\varrho(r) = \varrho(r)\ y(r)/\Gamma_1 P(r) - A(r)\ \Delta r(r),$$
$$[\delta\varrho(r,\lambda,\varphi) = \delta\varrho(r)\ Y_j^k(\lambda,\varphi);\ \delta P(r,\lambda,\varphi) = \delta P(r)\ Y_j^k(\lambda,\varphi)], \tag{5.5.28}$$

where the convective discriminant A from Eq. (5.2.85) has already been quoted in Eq. (5.2.134). The mass of the oscillating polytrope is constant, so the central boundary condition (5.2.107) for the radial displacement reads

$$\Delta r = u(r)/r^2 = 0, \qquad (j \ge 2;\ r = 0). \tag{5.5.29}$$

The surface boundary condition (5.2.109) can be written under the equivalent form

$$\Delta P = \varrho y + (dP/dr)\ \Delta r = 0, \qquad (r = r_1), \tag{5.5.30}$$

while the surface boundary condition on the potential (5.2.114) becomes

$$(d\delta\Phi/dr)_{r=r_1} + (j+1)\ \delta\Phi(r_1)/r_1 = 4\pi G\varrho(r_1)\ \Delta r_1, \qquad [\delta\Phi(r,\lambda,\varphi) = \delta\Phi(r)\ Y_j^k(\lambda,\varphi)]. \tag{5.5.31}$$

Robe (1968a) found that the Cowling approximation (neglect of $\delta\Phi$) yields very satisfactory results for higher order modes and for the more condensed polytropes ($n \gtrsim 3$), as should be expected on general grounds.

Robe (1974) has also studied nonradial oscillations of incompressible polytropic spheres ($\Gamma_1 = \infty;\ j = 2;\ A = (1/\varrho)\ d\varrho/dr < 0;\ n = 1,2,3,4$), when according to Eqs. (5.2.21), (5.2.77) $\Delta\varrho = 0$ if $Q = \mathrm{const}$, and all p-modes are suppressed by incompressibility. All f and g-modes of the considered incompressible polytropes are stable ($\omega^2 > 0$). The eigenvalues of all p-modes tend to infinity and their oscillation periods become zero if $\Gamma_1 \to \infty$, as shown in Table 5.5.4 if $n = 3$.

Sobouti (1977a, 1980) develops a Rayleigh-Ritz variational scheme [cf. Eqs. (5.7.41)-(5.7.46)] for isolating normal modes of a polytropic structure. Although the numerical exploration of Sobouti (1977b) seems to be the most complete one, we omit his tables, as they generally lack convergence for the p_3, p_4-modes.

Table 5.5.1 Normalized squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of the first three p and g-modes for polytropic spheres if $j = 1$ and $\Gamma_1 = 5/3$ (Hurley et al. 1966, Robe and Brandt 1966, Cox 1980, Christensen-Dalsgaard and Mullan 1994). The entries quoted by the authors have been divided by $n + 1$. Values for $n = 4$ are ignored, as discrepancies appear among the entries of Hurley et al. (1966), and Robe and Brandt (1966). Unstable g^--modes occur if $A > 0$ or $n < 1/(\Gamma_1 - 1) = 1.5$. And $a + b$ means $a \times 10^b$.

n	p_3	p_2	p_1	g_1	g_2	g_3
0	1.37+1	6.48+0	1.58+0	−1.40−1	−3.43−2	−1.62−2
1	–	1.81+0	5.77−1	−1.70−2	−6.35−3	–
1.5	1.90+0	1.01+0	3.68−1	0	0	0
2	9.87−1	5.49−1	2.30−1	8.57−3	3.96−3	2.28−3
2.5	4.84−1	2.84−1	1.35−1	1.31−2	6.41−3	3.79−3
3	2.15−1	1.33−1	7.02−2	1.55−2	7.91−3	4.77−3
3.5	7.98−2	5.17−2	2.96−2	1.68−2	8.78−3	5.38−3

Table 5.5.2 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of p, f, g-modes for polytropic spheres if $j = 2$ and $\Gamma_1 = 5/3$ (Robe 1968a, Tassoul 1978, Table 6.1; Cox 1980, Tables 17.1, 17.2; Christensen-Dalsgaard and Mullan 1994). Unstable g^--modes occur if $A > 0$ or $n < 1/(\Gamma_1 - 1) = 1.5$. And $a + b$ means $a \times 10^b$.

Mode	$n = 0$	$n = 1$	$n = 1.5$	$n = 2$	$n = 3$	$n = 4$
p_6	5.53+1	1.32+1	6.71 + 0	3.32+0	6.45−1	5.64−2
p_5	4.03+1	9.74+0	5.00 + 0	2.50+0	4.96−1	4.68−2
p_4	2.76+1	6.80+0	3.53 + 0	1.79+0	3.66−1	4.11−2
p_3	1.70+1	4.34+0	2.30+0	1.19+0	2.55−1	3.37−2
p_2	8.74+0	2.38+0	1.31+0	7.04−1	1.64−1	2.72−2
p_1	2.79+0	9.43−1	5.72−1	3.38−1	9.39−2	2.26−2
f	2.67−1	1.52−1	1.18−1	9.10−2	5.03−2	1.84−2
g_1	−2.39−1	−3.07−2	0	1.65−2	3.02−2	1.48−2
g_2	−7.62−2	−1.40−2	0	8.68−3	1.74−2	1.23−2
g_3	−3.91−2	−8.13−3	0	5.38−3	1.12−2	9.64−3
g_4	−2.42−2	−5.34−3	0	3.67−3	7.82−3	8.23−3
g_5	−1.65−2	−3.79−3	0	2.66−3	5.76−3	6.84−3
g_6	−1.20−2	−2.84−3	0	2.03−3	4.42−3	5.40−3

Table 5.5.3 Normalized squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of the fundamental (Kelvin) f-mode for polytropic spheres if $j = 2, 3, 4$ and $\Gamma_1 = 5/3$ according to Hurley et al. (1966) and Managan (1986, Table 5 if $n = 1$). The values for $j = 2$ compare favourably with the corresponding f-modes from Table 5.5.2. Values for $n = 0$ are from Eq. (5.5.26). $a + b$ means $a \times 10^b$.

j	$n = 0$	$n = 1$	$n = 1.5$	$n = 2$	$n = 3$	$n = 3.5$
$j = 2$	2.67−1	1.52−1	1.18−1	9.13−2	5.03−2	3.53−2
$j = 3$	5.71−1	2.89−1	2.08−1	1.45−1	5.79−2	4.43−2
$j = 4$	8.89−1	4.14−1	2.84−1	1.86−1	6.18−2	4.92−2

Introducing into Eqs. (5.2.124), (5.2.125) the critical acoustic frequency S_ℓ (Scuflaire 1974, Cox 1980)

$$S_\ell^2 = j(j+1)\Gamma_1 P/\varrho r^2, \tag{5.5.32}$$

and the so-called Brunt-Väisälä frequency N

$$N^2 = (1/\varrho)(dP/dr)[(1/\varrho)\ d\varrho/dr - (1/\Gamma_1 P)\ dP/dr] = -AG\ m(r)/r^2, \tag{5.5.33}$$

we get

$$dv/dr = (S_\ell^2/\sigma^2 - 1)P^{2/\Gamma_1 - 1} r^2 w/\Gamma_1, \tag{5.5.34}$$

$$dw/dr = -(\sigma^2/N^2 - 1)AG\varrho\ m(r)\ v/P^{2/\Gamma_1} r^4. \tag{5.5.35}$$

The curves S_ℓ^2 and N^2 versus fractional radius are shown in Fig. 5.5.4 for the polytrope $n = 4$ if $\Gamma_1 = 5/3$ and $j = 2$. Scuflaire (1974) distinguishes two regions in a polytrope: The outer pressure

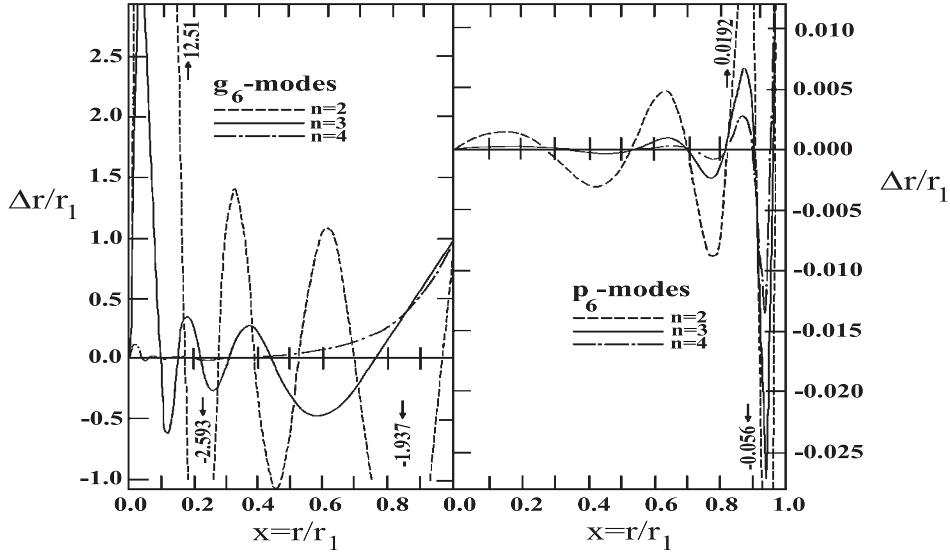

Fig. 5.5.2 Relative radial displacement $\Delta r/r_1,\ (\Delta r_1/r_1 = 1)$ of g_6, p_6-modes for the second harmonic $j = 2$ in polytropes with indices $n = 2, 3, 4$, and $\Gamma_1 = 5/3$. Numbers near arrows indicate the ordinate of an extremum (Robe 1968a, Ledoux 1974).

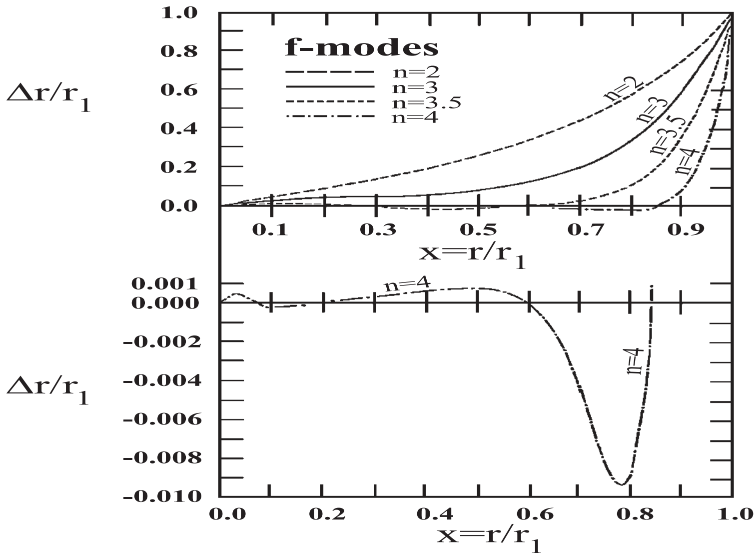

Fig. 5.5.3 Same as Fig. 5.5.2 for the f-modes ($n = 2, 3, 3.5, 4;\ j = 2;\ \Gamma_1 = 5/3$). If $n \gtrsim 3.25$, the f-mode acquires extra nodes. The enlarged displacement for the polytrope $n = 4$ is shown on the bottom (Robe 1968a).

(acoustic) P-region, where all modes look like p-pressure modes, and an inner gravity G-region, where oscillations behave like g-gravity modes. The curve $N^2(r)$ possesses two well defined extrema for the

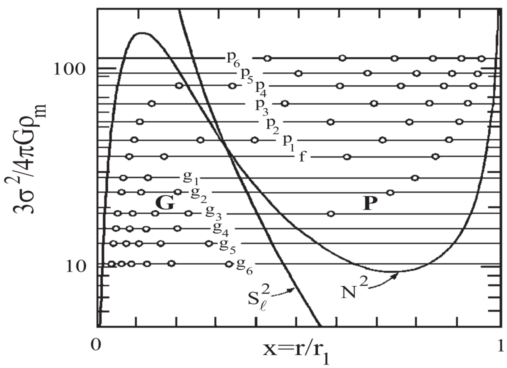

Fig. 5.5.4 Squared, dimensionless angular oscillation frequency $3\sigma^2/4\pi G\varrho_m = \sigma^2 r_1^3/Gm_1 = 3\varrho_0\omega^2/\varrho_m$ versus fractional radial distance $x = r/r_1$ for a polytrope of index $n = 4$. Shown are the squares of the two frequencies S_ℓ, N from Eqs. (5.5.32), (5.5.33) if $j = 2$ and $\Gamma_1 = 5/3$. Horizontal lines represent $3\sigma^2/4\pi G\varrho_m$, and open circles depict the nodes $\Delta r(x) = 0$ of a certain mode. Gravity and pressure (acoustic) regions are denoted by G and P, respectively (Scuflaire 1974).

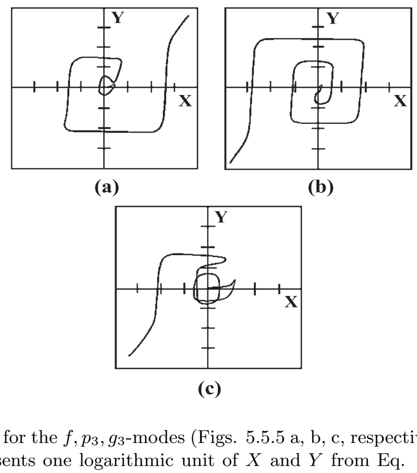

Fig. 5.5.5 Phase diagrams for the f, p_3, g_3-modes (Figs. 5.5.5 a, b, c, respectively) if $n = 4,\ j = 2,\ \Gamma_1 = 5/3$. Each division of the axes represents one logarithmic unit of X and Y from Eq. (5.5.36). $r = 0$ corresponds to $X, Y = 0$, and r increases as one follows the curve away from the origin (Scuflaire 1974).

centrally condensed polytrope $n = 4$ in Fig. 5.5.4.

In the same important paper Scuflaire (1974) introduces so-called phase diagrams, representing on a logarithmic scale the quantities

$$X = \pm \lg(1 + |\Delta r|/r_1); \quad Y = \pm \lg[1 + r_1|\delta P|/G\varrho\ m(r_1)]. \tag{5.5.36}$$

The signs of the logarithms are the same as those of Δr and δP, respectively. The phase diagrams are in fact plots of v versus w [see Eq. (5.2.134)], with the radial distance r as a parameter. While

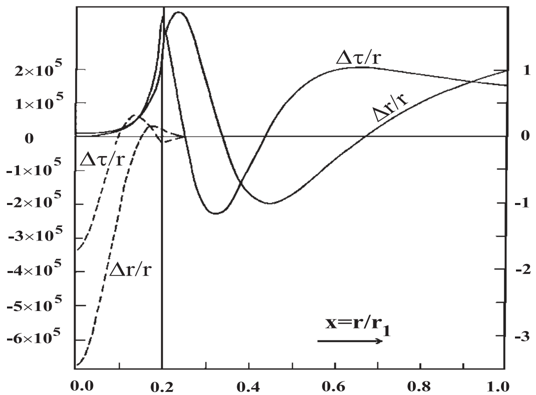

Fig. 5.5.6 Radial $\Delta r/r$ and tangential $\Delta\tau/r$ relative displacements of the g_2-mode belonging to a composite $n_c = 0,\ n_e = 3$ polytrope with $\Gamma_1 = 5/3,\ j = 2,\ (\Delta r_1/r_1 = 1)$. Ordinates on the left refer to the unstable g_2^--mode (dashed curves), and ordinates on the right to the stable g_2^+-mode (solid curves). The core-envelope interface is located at $x_i = 0.2$ (Ledoux and Smeyers 1966).

v is proportional to the radial component Δr of the Lagrangian displacement $\Delta\vec{r}$, the function w is in the Cowling approximation ($\delta\Phi = 0$) proportional to the tangential component $\Delta\tau$ of the Lagrangian displacement $\Delta\vec{r}$, as will be outlined below. If we consider a vector

$$\vec{\tau} = [1/Y_j^k(\lambda,\varphi)][(\partial Y_j^k/\partial\lambda)\ \vec{e}_\lambda + (1/\sin\lambda)(\partial Y_j^k/\partial\varphi)\ \vec{e}_\varphi], \tag{5.5.37}$$

entirely located in the tangential plane (perpendicular to the radius vector $\vec{r}$), we may write Eqs. (5.1.26) and (5.2.88)-(5.2.90) under the form

$$\Delta\vec{r} = \Delta r\ \vec{e}_r + r\ \Delta\lambda\ \vec{e}_\lambda + r\sin\lambda\ \Delta\varphi\ \vec{e}_\varphi = \Delta r\ \vec{e}_r + [\chi(r,\lambda,\varphi)/\sigma^2 r]\ \vec{\tau},$$
$$[\chi(r,\lambda,\varphi) = \chi(r)\ Y_j^k(\lambda,\varphi)]. \tag{5.5.38}$$

Thus, the tangential displacement $\Delta\tau$ of $\Delta\vec{r}$ is equal to

$$\Delta\tau = \chi(r,\lambda,\varphi)/\sigma^2 r \propto \delta P/\varrho - \delta\Phi, \qquad (\chi = \delta P/\varrho - \delta\Phi), \tag{5.5.39}$$

which in the Cowling approximation is just $\Delta\tau \propto \delta P/\varrho \propto w$: The phase diagrams from Fig. 5.5.5 may be regarded as scaled plots of the radial versus the tangential component of the Lagrangian displacement $\Delta\vec{r}$. The representative point in Fig. 5.5.5 starts in the first quadrant, and crosses the Y-axis (w or $\Delta\tau$-axis) as r increases from 0 to r_1. The order of the mode is equal to the number of clockwise (counterclockwise) crossings of the Y-axis, minus the number of counterclockwise (clockwise) crossings. In this scheme there is assigned to the f-mode (Fig. 5.5.5 a) the number 0, to the p_ℓ-mode (Fig. 5.5.5 b) the number ℓ, and to the g_ℓ-mode (Fig. 5.5.5 c) the number $-\ell$ of crossings, the plus sign belonging to counterclockwise crossings, and the minus sign to clockwise crossings.

The behaviour of g-modes is particularly intriguing if $A > 0$ in the core, and $A < 0$ in the envelope, as it is the case for a composite polytropic model consisting of a homogeneous compressible core ($n_c = 0$) and a $n_e = 3$ envelope (Fig. 5.5.6, Ledoux and Smeyers 1966, Tassoul 1978). As was first shown by Ledoux and Smeyers (1966), for each value of j two infinite discrete spectra of g-modes exist: The stable g^+-modes correspond to gravity oscillations in the radiative zone ($n_e = 3$), and the unstable g^--modes describe convective currents in the $n_c = 0$ core. To illustrate these findings, Goossens and Smeyers (1974) have studied numerically several spectra ($j = 1, 2, 3, 5$) of g-modes for a composite polytropic model that consists of a convectively stable $n = 3$ core and $n = 3$ envelope, separated by an intermediate, convectively unstable zone with polytropic index $n = 1$, $(A > 0,\ \Gamma_1 = 5/3)$. The p, f, g-modes in composite spherical polytropes ($n_c = 0.5, 1.5;\ n_e = 3, 4$) have been determined by Mohan and Singh (1981).

It should be stressed that the azimuthal spherical harmonic index k from the spherical harmonic $Y_j^k(\lambda,\varphi) = P_j^k(\cos\lambda)\exp(ik\varphi)$ never appears in any of the equations of linear nonradial oscillations of

Table 5.5.4 Variation of normalized squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ with changing adiabatic exponent Γ_1 for the polytrope $n = 3$, and the spherical harmonic index $j = 2$ (Robe 1974). If $\Gamma_1 = 5/3$, the entries coincide with the corresponding ones from Table 5.5.2. $a + b$ means $a \times 10^b$.

Mode	$\Gamma_1 = 5/3$	$\Gamma_1 = 5$	$\Gamma_1 = 100$	$\Gamma_1 = \infty$
p_4	3.66−1	1.25+0	2.65+1	∞
p_3	2.55−1	8.96−1	1.93+1	∞
p_2	1.64−1	6.00−1	1.32+1	∞
p_1	9.39−2	3.61−1	8.22+0	∞
f	5.03−2	1.03−1	1.18−1	1.19−1
g_1	3.02−2	6.65−2	7.80−2	7.86−2
g_2	1.74−2	4.52−2	5.42−2	5.47−2
g_3	1.12−2	3.21−2	3.93−2	3.97−2
g_4	7.82−3	2.38−2	2.96−2	2.99−2

spherical polytropes. The eigenfrequencies ω are therefore $(2j + 1)$-fold degenerate for each mode, when k assumes the $2j + 1$ distinct values $k = -j, -j + 1, ...j - 1, j$ depicted in Eq. (5.1.27) for the associated Legendre polynomial P_j^k.

5.6 Stability and Oscillations of Polytropic Cylinders

5.6.1 The Incompressible Homogeneous Cylinder ($n = 0;\ \Gamma_1 = \infty$)

The normal modes of oscillations and the stability of incompressible homogeneous cylinders ($\Gamma_1 = \infty;\ \Delta\varrho, \delta\varrho = 0$) in the presence of magnetic fields have been investigated by Chandrasekhar and Fermi (1953). Because of the reduced practical importance of incompressible objects we confine ourselves merely to a brief summary of their results for the nonmagnetic incompressible case. The Lagrangian displacement of the perturbed boundary $\ell(\ell_1, \varphi, z, t)$ is considered under the form

$$\Delta\ell_1(\ell_1, \varphi, z, t) = \ell(\ell_1, \varphi, z, t) - \ell_1 = \Delta\ell_1(t)\exp[i(k\varphi + jz)] = \Delta\ell_1 \exp[i(\sigma t + k\varphi + jz)],$$
$$(\Delta\ell_1(t) = \Delta\ell_1 \exp(i\sigma t);\ \Delta\ell_1 = \mathrm{const}), \tag{5.6.1}$$

where ℓ, φ, z are cylindrical coordinates, ℓ_1 is the radius of the nonoscillating cylinder, σ the angular oscillation frequency, j (real number) the wave number of the disturbance along the longitudinal z-axis of the cylinder, and k – the azimuthal wavenumber – is an integer, in order to have an oscillating solution in the φ-direction with a period of 2π. The circular frequencies obtained for k and $-k$ differ only by sign, so we will generally consider only nonnegative values $k = 0, 1, 2, 3, ...$

The perturbed Laplace and Poisson equations have the form of a Laplace equation, since by assumption $\delta\varrho = 0:\ \nabla^2\delta\Phi_e = 0,\ \nabla^2\delta\Phi = 0$. The solution of the cylindrical Laplace equation (B.48)

$$\nabla^2 f(\ell, \varphi, z) = (1/\ell)\ \partial(\ell\ \partial f/\partial\ell)/\partial\ell + (1/\ell^2)\ \partial^2 f/\partial\varphi^2 + \partial^2 f/\partial z^2 = 0, \tag{5.6.2}$$

is given in terms of separated variables $f(\ell, \varphi, z) = f_1(\ell)\ f_2(\varphi)\ f_3(z)$ by (e.g. Smirnow 1967, Vol. 3, §153)

$$f(\ell, \varphi, z) = [B_1 I_k(j\ell) + B_2 K_k(j\ell)]\exp[i(k\varphi + jz)], \qquad (j \neq 0;\ B_1, B_2 = \mathrm{const}). \tag{5.6.3}$$

This solution particularizes to $f(\ell, z) = [B_1 I_0(j\ell) + B_2 K_0(j\ell)]\exp(ijz)$ if $f(\ell, z) = f_1(\ell)\ f_3(z)$, $(j \neq 0;\ k = 0)$. In the particular case $f(\ell, \varphi) = f_1(\ell)\ f_2(\varphi)$ the solution is

$$f(\ell, \varphi) = (B_1\ell^k + B_2\ell^{-k})\exp(ik\varphi), \qquad (j = 0;\ k \neq 0), \tag{5.6.4}$$

and finally, if $f(\ell) = f_1(\ell)$ we recover the simple form (2.6.36):

$$f(\ell) = B_1 \ln\ell + B_2, \qquad (j, k = 0). \tag{5.6.5}$$

In Eq. (5.6.3) I_k and K_k denote the modified Bessel (cylindrical) functions of order k of the first and second kind, respectively. The asymptotic expansions of the modified Bessel functions are (e.g. Abramowitz and Stegun 1965)

$$I_k(j\ell) = (1/2\pi j\ell)^{1/2}\exp(j\ell); \quad K_k(j\ell) = (\pi/2j\ell)^{1/2}\exp(-j\ell), \qquad (\ell \to \infty). \tag{5.6.6}$$

To assure the finiteness of the Eulerian perturbation $\delta\Phi_e(\ell, \varphi, z, t) = \Delta\ell_1(t)\ f(\ell, \varphi, z)$ of the external potential at infinity, we have to put $B_1 = 0$ in Eq. (5.6.3). The Eulerian perturbation is therefore of the form $\delta\Phi_e(\ell, \varphi, z, t) = \delta\Phi_e(\ell, \varphi, z)\exp(i\sigma t) = B_2\ \Delta\ell_1\ K_k(j\ell)\exp[i(\sigma t + k\varphi + jz)]$, and the perturbed external potential writes (Chandrasekhar 1981):

$$\Phi_e(\ell, \varphi, z, t) = \Phi_{ue}(\ell) + \delta\Phi_e(\ell, \varphi, z, t) = C_1 \ln\ell + C_2 + B_2\ \Delta\ell_1(t)\ K_k(j\ell)\exp[i(k\varphi + jz)],$$
$$(C_1, C_2 = \mathrm{const}), \tag{5.6.7}$$

where we have inserted for the unperturbed external potential $\Phi_{ue} = f(\ell)$ the solution (5.6.5).

For the unperturbed homogeneous cylinder the internal gravitational potential Φ_u can be deduced at once from the elementary integration of Poisson's equation

$$\nabla^2\Phi_u = (1/\ell)\ d(\ell\ d\Phi_u/d\ell)/d\ell = -4\pi G\varrho. \tag{5.6.8}$$

With the conditions $\Phi_u(0) = 0$ and $(d\Phi_u/d\ell)_{\ell=0} = 0$, the integral of Eq. (5.6.8) is

$$\Phi_u = -\pi G\varrho\ell^2, \qquad (\varrho = \text{const}). \tag{5.6.9}$$

To assure the finiteness of the Eulerian perturbation $\delta\Phi(\ell,\varphi,z,t) = \Delta\ell_1(t)\ f(\ell,\varphi,z)$ of the internal potential at the origin, we have to put $B_2 = 0$ in Eq. (5.6.3), since $K_k(j\ell) \to \infty$ if $\ell \to 0$ (Spiegel 1968). Similarly to the external potential we write $\delta\Phi(\ell,\varphi,z,t) = \delta\Phi(\ell,\varphi,z)\exp(i\sigma t) = B_1\ \Delta\ell_1\ I_k(j\ell)\exp[i(\sigma t+ k\varphi + jz)]$, and obtain for the perturbed internal potential

$$\Phi(\ell,\varphi,z,t) = \Phi_u(\ell) + \delta\Phi(\ell,\varphi,z,t) = -\pi G\varrho\ell^2 + B_1\ \Delta\ell_1(t)\ I_k(j\ell)\exp[i(k\varphi + jz)]. \tag{5.6.10}$$

With Eqs. (5.6.7) and (5.6.10) we get from the continuity of internal and external potentials, and of their radial derivatives across the perturbed boundary $\ell(\ell_1,\varphi,z,t) = \ell_1 + \Delta\ell_1(\ell_1,\varphi,z,t)$ [cf. Eqs. (5.2.111)-(5.2.113)]:

$$\begin{aligned}
&C_1\ln\ell_1 + C_2 + \Delta\ell_1(t)\ [C_1/\ell_1 + B_2K_k(j\ell_1)]\exp[i(k\varphi+jz)] = -\pi G\varrho\ell_1^2\\
&+\Delta\ell_1(t)\ [-2\pi G\varrho\ell_1 + B_1I_k(j\ell_1)]\exp[i(k\varphi+jz)];\quad C_1/\ell_1 + \Delta\ell_1(t)\ [-C_1/\ell_1^2 + jB_2K_k'(j\ell_1)]\\
&\times\exp[i(k\varphi+jz)] = -2\pi G\ell_1 + \Delta\ell_1(t)\ [-2\pi G\varrho + jB_1I_k'(j\ell_1)]\exp[i(k\varphi+jz)],\\
&[I_k'(j\ell) = dI_k(j\ell)/d(j\ell);\ K_k'(j\ell) = dK_k(j\ell)/d(j\ell)].
\end{aligned} \tag{5.6.11}$$

As we do not need to be further concerned with the additive constant C_2, we observe merely that $C_1 = -2\pi G\varrho\ell_1^2$. The first order terms connected with $\Delta\ell_1(t)$ in Eq. (5.6.11) yield

$$B_1I_k(j\ell_1) = B_2K_k(j\ell_1);\quad B_1I_k'(j\ell_1) = B_2K_k'(j\ell_1) + 4\pi G\varrho/j. \tag{5.6.12}$$

The modified Bessel functions of the first and second kind are both solutions of the equation (Smirnow 1967)

$$(1/\ell)\ d[\ell\ df(j\ell)/d\ell]/d\ell = (j^2 + k^2/\ell^2)\ f(j\ell). \tag{5.6.13}$$

We multiply this equation successively by $K_k(j\ell)$ and $I_k(j\ell)$, inserting for $f(j\ell) = I_k(j\ell)$ and $f(j\ell) = K_k(j\ell)$, respectively. The difference is

$$K_k(j\ell)\ d[j\ell\ I_k'(j\ell)]/d\ell - I_k(j\ell)\ d[j\ell\ K_k'(j\ell)]/d\ell = 0. \tag{5.6.14}$$

Eq. (5.6.14) is equivalent to

$$K_k(j\ell)\ I_k'(j\ell) - I_k(j\ell)\ K_k'(j\ell) = C/j\ell, \qquad (C = 1), \tag{5.6.15}$$

where the integration constant $C = 1$ can be obtained by inserting the asymptotic expansions (5.6.6) into Eq. (5.6.15). With Eq. (5.6.15) we obtain at once from Eq. (5.6.12) the relevant integration constant B_1 :

$$B_1 = 4\pi G\varrho\ K_k(j\ell_1)/j\big[K_k(j\ell_1)\ I_k'(j\ell_1) - I_k(j\ell_1)\ K_k'(j\ell_1)\big] = 4\pi G\varrho\ell_1K_k(j\ell_1), \qquad (\varrho = \text{const}). \tag{5.6.16}$$

Thus, the Eulerian variation of the internal gravitational potential (5.6.10) is

$$\delta\Phi = \Phi - \Phi_u = \Phi + \pi G\varrho\ell^2 = 4\pi G\varrho\ell_1\ \Delta\ell_1(t)\ K_k(j\ell_1)\ I_k(j\ell)\exp[i(k\varphi + jz)]. \tag{5.6.17}$$

The basic equations governing the oscillatory motion of the incompressible cylinder are given by Eqs. (5.2.24), (5.2.30), (5.2.40):

$$\begin{aligned}
&\nabla\cdot\delta\vec v = 0;\quad \partial\delta\vec v/\partial t = \nabla(-\delta P/\varrho + \delta\Phi) = -\nabla\chi;\quad \nabla^2\delta\Phi = 0,\\
&(n = 0;\ \Gamma_1 = \infty;\ \Delta\varrho, \delta\varrho = 0;\ \chi = \delta P/\varrho - \delta\Phi).
\end{aligned} \tag{5.6.18}$$

Observing that $\nabla\cdot\delta\vec v = 0$, and taking the divergence of $\nabla\chi$, we find

$$\nabla^2\chi = 0, \tag{5.6.19}$$

having a solution analogous to Eq. (5.6.10):

$$\chi = \chi(\ell,\varphi,z,t) = \chi_0\ \Delta\ell_1(t)\ I_k(j\ell)\exp[i(k\varphi + jz)], \qquad (\chi_0 = \mathrm{const}). \tag{5.6.20}$$

The temporal dependence of all functions enters through the factor $\exp(i\sigma t)$. Integrating $\nabla\chi$ with respect to the time, we get from the equation of motion (5.6.18)

$$\delta\vec{v} = -(\chi_0\ \Delta\ell_1/i\sigma)\exp(i\sigma t)\ \nabla\{I_k(j\ell)\exp[i(k\varphi + jz)]\}, \qquad [\Delta\ell_1(t) = \Delta\ell_1\exp(i\sigma t)]. \tag{5.6.21}$$

In particular, the ℓ-component of Eq. (5.6.21) is

$$\delta v_\ell = -[\chi_0 j\ \Delta\ell_1\ I_k'(j\ell)/i\sigma]\exp[i(\sigma t + k\varphi + jz)]. \tag{5.6.22}$$

The radial velocity component must be compatible with the form (5.6.1) of the boundary surface:

$$\delta v_\ell \approx \partial(\ell-\ell_1)/\partial t = \partial[\Delta\ell_1(\ell_1,\varphi,z,t)]/\partial t = i\sigma\ \Delta\ell_1\exp[i(\sigma t + k\varphi + jz)]$$
$$= -[\chi_0 j\ \Delta\ell_1\ I_k'(j\ell_1)/i\sigma]\exp[i(\sigma t + k\varphi + jz)], \tag{5.6.23}$$

or

$$\sigma^2 = \chi_0 j I_k'(j\ell_1). \tag{5.6.24}$$

From the hydrostatic equation $\varrho\ \nabla\Phi_u = \nabla P_u$ and from Poisson's equation (5.6.8) we obtain

$$\varrho\ \nabla^2\Phi_u = \nabla^2 P_u = (1/\ell)\ d(\ell\ dP_u/d\ell)/d\ell = -4\pi G\varrho^2, \qquad (\varrho = \mathrm{const}), \tag{5.6.25}$$

and the unperturbed pressure becomes

$$P_u = \pi G\varrho^2(\ell_1^2 - \ell^2), \qquad (n = 0;\ (dP_u/d\ell)_{\ell=0} = 0;\ P_u(\ell_1) = 0). \tag{5.6.26}$$

The vanishing of the pressure $P(\ell)$ on the perturbed boundary (5.6.1) of the incompressible cylinder demands

$$P(\ell,\varphi,z,t) = P_u(\ell) + \delta P(\ell,\varphi,z,t) = \pi G\varrho^2(\ell_1^2-\ell^2) + \delta P(\ell,\varphi,z,t) \approx -2\pi G\varrho^2\ell_1\ \Delta\ell_1(\ell_1,\varphi,z,t)$$
$$+\delta P(\ell_1,\varphi,z,t) = -2\pi G\varrho^2\ell_1\ \Delta\ell_1(t)\exp[i(k\varphi + jz)] + \delta P(\ell_1,\varphi,z,t) = 0, \qquad (\ell\approx\ell_1). \tag{5.6.27}$$

On the other hand, we observe from Eqs. (5.6.17), (5.6.20) that

$$\chi(\ell_1,\varphi,z,t) = \chi_0\ \Delta\ell_1(t)\ I_k(j\ell_1)\exp[i(k\varphi + jz)] = \delta P(\ell_1,\varphi,z,t)/\varrho - \delta\Phi(\ell_1,\varphi,z,t)$$
$$= \delta P(\ell_1,\varphi,z,t)/\varrho - 4\pi G\varrho\ell_1\ \Delta\ell_1(t)\ K_k(j\ell_1)\ I_k(j\ell_1)\exp[i(k\varphi + jz)]. \tag{5.6.28}$$

Eliminating $\delta P(\ell_1,\varphi,z,t)$ between Eqs. (5.6.27) and (5.6.28), we get

$$4\pi G\varrho\ell_1[I_k(j\ell_1)\ K_k(j\ell_1) - 1/2] = -\chi_0\ I_k(j\ell_1), \tag{5.6.29}$$

and inserting for χ_0 from Eq. (5.6.24):

$$\omega^2 = \sigma^2/4\pi G\varrho = -[j\ell_1\ I_k'(j\ell_1)/I_k(j\ell_1)][I_k(j\ell_1)\ K_k(j\ell_1) - 1/2],$$
$$(n = 0;\ \varrho = \varrho_m = \varrho_0;\ \Gamma_1 = \infty;\ \Delta\varrho, \delta\varrho = 0). \tag{5.6.30}$$

In analogy to Eq. (5.3.52) for the spherical case, and in accordance with Eq. (5.6.30), we introduce the dimensionless, squared angular oscillation frequency for *polytropic* cylinders of index n by

$$\omega^2 = \sigma^2/4\pi G\varrho_0, \qquad [\varrho_0 = \varrho(0,\varphi,z)], \tag{5.6.31}$$

where ϱ_0 denotes the unperturbed density along the cylindrical axis (the central density of the cylinder).

From the theory of the zeros of modified Bessel functions follows that the sole positive zero $j\ell_1 = 1.0668$ of $I_k(j\ell_1)\ K_k(j\ell_1) - 1/2$ occurs if $k = 0$ in Eq. (5.6.30), (Abramowitz and Stegun 1965). Moreover, $I_k(j\ell_1)\ K_k(j\ell_1) < 1/2$ if $k > 0$, as it is obvious for instance from the expansions of I_k, K_k near 0 or ∞ : The homogeneous incompressible cylinder is stable ($\sigma^2 > 0$) for all nonaxisymmetric modes $k \neq 0$. If $k = 0$, the homogeneous incompressible cylinder exhibits an axisymmetric "varicose, (sausage)" gravitational instability ($\sigma^2 < 0$) for all values of $j\ell_1$ obeying the inequality (Chandrasekhar and Fermi 1953)

$$0 < j\ell_1 < j_c\ell_1 = 1.0668, \qquad (k = 0;\ n = 0). \tag{5.6.32}$$

The corresponding unstable wavelengths of the varicose deformations are

$$z = 2\pi/j > 2\pi/j_c = 2\pi\ell_1/1.0668, \qquad (k = 0;\ n = 0). \tag{5.6.33}$$

In the range $0 < j\ell_1 < j_c\ell_1 = 1.0668$ the dimensionless angular oscillation frequency ω^2 attains a minimum $\omega_m^2 = -0.0603$ at $j_m\ell_1 = 0.580$, while neutral stability ($\sigma^2 = 0$) takes place for the endpoints of the instability interval (5.6.32), (see Fig. 5.6.1). Since $\Delta\ell_1(t) = \Delta\ell_1 \exp(i\sigma t)$, the mode of most rapid amplitude growth occurs just at the minimum value $\omega_m^2 = \sigma_m^2/4\pi G\varrho = -0.0603$. For this mode of maximum instability the infinitely long, homogeneous incompressible cylinder will break up gravitationally into pieces of axial length $z_m = 2\pi/j_m = 2\pi\ell_1/0.580$.

5.6.2 The Compressible Homogeneous Cylinder ($n = 0$; $\Gamma_1 \neq \infty$)

The radial oscillations of compressible homogeneous cylinders have been studied by Chandrasekhar and Fermi (1953), while Simon (1963) has investigated the occurrence of neutral modes ($\sigma = 0$) for perturbations $\propto \exp[i(\sigma t + k\varphi + jz)]$, [cf. Eqs. (5.6.74)-(5.6.79)]. The relevant general equations are given by the equation of continuity (5.2.28), the equation of motion (5.2.73) or (5.2.86), the adiabatic ($Q =$ const) energy conservation equation (5.2.78), and Poisson's equation (5.2.40). The components of the Lagrangian displacement $\Delta\vec{r}(\vec{r}, t)$ are in cylindrical coordinates equal to $\Delta\ell(\vec{r}, t), \ell\ \Delta\varphi(\vec{r}, t), \Delta z(\vec{r}, t)$. As in the spherical case, the temporal dependence of all quantities occurs through the factor $\exp(i\sigma t)$, and assuming the spatial decomposition of $\Delta\ell, \delta P, \delta\varrho$, and $\delta\Phi$ through the factor $\cos(k\varphi)\cos(jz)$ – which is contained in the more general perturbation $\exp[i(k\varphi + jz)]$ – we get for $\Delta\ell$, and for the Eulerian perturbations of P, ϱ, Φ (Simon 1963, Ostriker 1964c, Robe 1967):

$$\begin{aligned}
&\Delta\ell(\vec{r}, t) = \Delta\ell(\vec{r}) \exp(i\sigma t) = \Delta\ell(\ell) \cos(k\varphi) \cos(jz) \exp(i\sigma t); \quad \delta P(\vec{r}, t) = \delta P(\ell) \cos(k\varphi) \cos(jz) \\
&\times \exp(i\sigma t); \quad \delta\varrho(\vec{r}, t) = \delta\varrho(\ell) \cos(k\varphi) \cos(jz) \exp(i\sigma t); \quad \delta\Phi(\vec{r}, t) = \delta\Phi(\ell) \cos(k\varphi) \cos(jz) \exp(i\sigma t); \\
&\delta P/\varrho - \delta\Phi = \chi(\ell, \varphi, z, t) = \chi(\ell) \cos(k\varphi) \cos(jz) \exp(i\sigma t).
\end{aligned} \tag{5.6.34}$$

If we project the equations of motion (5.2.80), (5.2.86) onto cylindrical axes, taking into account that $\vec{A} = \vec{A}(A, 0, 0),\ A = [1 - (1 + 1/n)/\Gamma_1]\ d\ln\varrho/d\ell$, we get up to the first order:

$$\begin{aligned}
&\sigma^2\ \Delta\ell(\ell, \varphi, z) = \partial\chi(\ell, \varphi, z)/\partial\ell + (\delta P/\varrho^2)\ \partial\varrho/\partial\ell - (\delta\varrho/\varrho^2)\ \partial P/\partial\ell = \cos(k\varphi) \cos(jz)\ d\chi(\ell)/d\ell \\
&+A[\delta P/\varrho + (\Delta\ell/\varrho)\ \partial P/\partial\ell]; \\
&\sigma^2\ell\ \Delta\varphi(\ell, \varphi, z) = (1/\ell)\ \partial\chi(\ell, \varphi, z)/\partial\varphi = -(k/\ell)\chi(\ell) \sin(k\varphi) \cos(jz); \\
&\sigma^2\ \Delta z(\ell, \varphi, z) = \partial\chi(\ell, \varphi, z)/\partial z = -j\chi(\ell) \cos(k\varphi) \sin(jz).
\end{aligned} \tag{5.6.35}$$

Ostriker (1964c) and Robe (1967) introduce new variables, analogously to Eq. (5.2.99) for the spherical case:

$$u = u(\ell) = \ell\ \Delta\ell(\ell) \quad \text{and} \quad y = y(\ell) = \delta P(\ell)/\varrho(\ell). \tag{5.6.36}$$

In the particular case $\varrho =$ const, ($n = 0$) Ostriker (1964c) takes the divergence and the curl of the equation of motion (5.2.73):

$$\sigma^2\ \nabla \cdot \Delta\vec{r} = -(1/\varrho^2)\ \nabla\delta\varrho \cdot \nabla P - (\delta\varrho/\varrho^2)\ \nabla^2 P + (1/\varrho)\ \nabla^2\delta P - \nabla^2\delta\Phi, \tag{5.6.37}$$

$$\sigma^2\ \nabla \times \Delta\vec{r} = -(1/\varrho^2)\ \nabla\delta\varrho \times \nabla P, \tag{5.6.38}$$

where $\nabla \times \nabla f = 0$ [cf. Eqs. (B.45), (B.47)].

Eq. (5.6.37) can be simplified with the aid of Eqs. (5.2.28), (5.2.40), (5.6.25), (5.6.26):

$$\sigma^2\ \delta\varrho(\vec r)/\varrho = -2\pi G\ell\ \partial[\delta\varrho(\vec r)]/\partial\ell - (1/\varrho)\ \nabla^2[\delta P(\vec r)] - 8\pi G\ \delta\varrho(\vec r). \qquad (5.6.39)$$

This equation can be transformed further via Eqs. (B.48), (5.6.34), by simplifying with the common factor $\cos(j\varphi)\cos(kz)$:

$$\sigma^2\ \delta\varrho(\ell)/\varrho = -2\pi G\ell\ d[\delta\varrho(\ell)]/d\ell - (1/\varrho\ell)\ d\{\ell\ d[\delta P(\ell)]/d\ell]\}/d\ell$$
$$+(k^2/\ell^2+j^2)\ \delta P(\ell)/\varrho - 8\pi G\ \delta\varrho(\ell). \qquad (5.6.40)$$

The ℓ-component of Eq. (5.6.38) vanishes, while the φ- and z-components both are equal to

$$\sigma^2\ \Delta\ell(\ell) - d\chi(\ell)/d\ell = \sigma^2 u(\ell)/\ell - d\chi(\ell)/d\ell = 2\pi G\ell\ \delta\varrho(\ell), \qquad (5.6.41)$$

as can be seen by inserting for $\Delta\ell, \Delta\varrho$ from Eq. (5.6.34), for $\Delta\varphi, \Delta z$ from Eq. (5.6.35), and for $\nabla P \approx dP_u/d\ell = -2\pi G\varrho^2\ell$ from Eq. (5.6.26).

Via Eqs. (5.6.34)-(5.6.36) we may rewrite the continuity equation (5.2.28), and the adiabatic thermal energy conservation equation (5.2.78) under the form

$$\delta\varrho(\vec r)/\varrho(\vec r) + \nabla\cdot\Delta\vec r = \big[\delta\varrho(\ell)/\varrho + (1/\ell)\ du/d\ell - (k^2/\ell^2+j^2)\ \chi(\ell)/\sigma^2\big]\cos(k\varphi)\cos(jz) = 0, \qquad (5.6.42)$$

$$\delta P + \Delta\vec r\cdot\nabla P - \Gamma_1 P\ \delta\varrho/\varrho = \big[\delta P(\ell) - 2\pi G\varrho^2 u - \pi\Gamma_1 G\varrho(\ell_1^2-\ell^2)\ \delta\varrho(\ell)\big]\cos(k\varphi)\cos(jz) = 0. \qquad (5.6.43)$$

Four relations between the unknown functions $\delta P, \delta\varrho, u, \chi$ are provided by Eqs. (5.6.40)-(5.6.43). If we now define the dimensionless quantities

$$\omega^2 = \sigma^2/4\pi G\varrho;\quad x=\ell/\ell_1;\quad \varepsilon(\ell)=\delta\varrho(\ell)/\varrho;\quad \delta p(\ell) = \delta P(\ell)/2\pi G\varrho^2\ell_1^2,\qquad (\varrho_0=\varrho_m=\varrho), \qquad (5.6.44)$$

Eqs. (5.6.40)-(5.6.43) become

$$2(\omega^2+2)\varepsilon + x\ d\varepsilon/dx + d^2\delta p/dx^2 + (1/x)\ d\delta p/dx - (k^2/x^2+j^2\ell_1^2)\ \delta p = 0, \qquad (5.6.45)$$

$$2\omega^2[u/x - (1/\sigma^2)\ d\chi/dx] - \ell_1^2 x\varepsilon = 0, \qquad (5.6.46)$$

$$\ell_1^2\varepsilon + (1/x)\ du/dx - (k^2/x^2+j^2\ell_1^2)\chi/\sigma^2 = 0, \qquad (5.6.47)$$

$$2\ \delta p - \Gamma_1(1-x^2)\varepsilon - 2u/\ell_1^2 = 0. \qquad (5.6.48)$$

Ostriker (1964c) has provided several exact analytical solutions for the compressible homogeneous cylinder ($n=0;\ \Gamma_1\neq\infty$).

(i) Nonaxisymmetric Modes in the (ℓ,φ)-Plane $(k=0,1,2,...;\ j=0;\ \varepsilon,\omega^2\neq 0)$. In this case we can reduce Eqs. (5.6.45)-(5.6.48) to a single second order equation, as in the case of the compressible homogeneous sphere from Sec. 5.5.1. We eliminate χ by inserting the derivative of Eq. (5.6.47) into Eq. (5.6.46):

$$d^2u/dx^2 + (1/x)\ du/dx - k^2u/x^2 + (2+k^2/2\omega^2)\ell_1^2\varepsilon + \ell_1^2 x\ d\varepsilon/dx = 0,\qquad (j=0). \qquad (5.6.49)$$

To get a second order equation in ε, we substitute δp from Eq. (5.6.48) into Eq. (5.6.45):

$$2(\omega^2+2)\varepsilon + x\ d\varepsilon/dx + d^2[\Gamma_1(1-x^2)\varepsilon/2]/dx^2 + (1/x)\ d[\Gamma_1(1-x^2)\varepsilon/2]/dx$$
$$-k^2\Gamma_1(1-x^2)\varepsilon/2x^2 = -(1/\ell_1^2)[d^2u/dx^2 + (1/x)\ du/dx - k^2u/x^2]. \qquad (5.6.50)$$

We insert for the right-hand side of Eq. (5.6.50) from Eq. (5.6.49), (Ostriker 1964c):

$$(1-x^2)\ d^2\varepsilon/dx^2 + (1/x-5x)\ d\varepsilon/dx + (B-k^2/x^2)\varepsilon = 0,\qquad (j=0), \qquad (5.6.51)$$

where

$$B = (4/\Gamma_1)[\omega^2 + 1 - \Gamma_1 + k^2(\Gamma_1 - 1/\omega^2)/4]. \tag{5.6.52}$$

Inserting for ε an expansion of the form (5.3.24), we get for the indicial equation of the lowest power x^{q-2} :

$$q(q-1) + q - k^2 = 0. \tag{5.6.53}$$

We get $q = \pm k$, where we have to take $q = k$ in order to have $\varepsilon = \delta\varrho/\varrho$ finite along the axis $x = 0$. The series solution for ε becomes

$$\varepsilon = \varepsilon(x) = x^k \sum_{m=0}^{\infty} a_m x^m, \qquad (0 \leq x \leq 1;\ k = 0, 1, 2, 3, ...;\ j = 0). \tag{5.6.54}$$

The coefficients satisfy the recurrence relation

$$a_{2m+2} = a_{2m}[(2m+k)(2m+k+4) - B]/4(m+1)(m+k+1); \quad a_0 \neq 0; \quad a_{2m+1} = 0,$$
$$(m = 0, 1, 2, ...). \tag{5.6.55}$$

Like in the spherical case from Eq. (5.5.21) we have to cut the series at a_{2m} :

$$B = (2m+k)(2m+k+4), \qquad (m = 0, 1, 2, ...). \tag{5.6.56}$$

Inserting for B from Eq. (5.6.52), we get

$$D_m = \omega^2 - k^2/4\omega^2 = \Gamma_1(m+1)(m+k+1) - 1. \tag{5.6.57}$$

Eq. (5.6.57) can be solved for the dimensionless angular oscillation frequency ω^2, yielding two sets of eigenvalues, defining the stable p-modes (Robe 1967)

$$\omega_p^2 = \sigma^2/4\pi G\varrho = [D_m + (D_m^2 + k^2)^{1/2}]/2 > 0, \qquad (m = 0, 1, 2, ...;\ k \neq 0;\ j = 0), \tag{5.6.58}$$

and the unstable g-modes

$$\omega_g^2 = \sigma^2/4\pi G\varrho = [D_m - (D_m^2 + k^2)^{1/2}]/2 < 0, \qquad (m = 0, 1, 2, ...;\ k \neq 0;\ j = 0). \tag{5.6.59}$$

In the particular case $k = 0$ the g-modes disappear, and the p-modes become stable, purely radial modes, since we have $\Gamma_1 \geq 1$ from Sec. 1.7 (Simon 1963):

$$\omega_p^2 = D_m = \Gamma_1(m+1)^2 - 1 \geq 0, \qquad (m = 0, 1, 2, ...;\ j, k = 0). \tag{5.6.60}$$

Like in the spherical case, the eigenvalues (5.6.58)-(5.6.60) increase with increasing m, the eigenvalues (5.6.59) being negative. The mode of maximum instability occurs for the g-mode with the lowest eigenvalue σ^2, $(\sigma^2 < 0)$, i.e. if $m = 0$:

$$\omega_g^2 = \{\Gamma_1(k+1) - 1 - [(\Gamma_1(k+1) - 1)^2 + k^2]^{1/2}\}/2, \qquad (k = 1, 2, 3, ...;\ j, m = 0). \tag{5.6.61}$$

(ii) Nonaxisymmetric Modes in the (ℓ, φ)-Plane $(k = 0, 1, 2, ...;\ j, \varepsilon = 0;\ \omega^2 \neq 0)$. Note, that our a priori assumption $\varepsilon(\ell) \propto \delta\varrho(\ell) = 0$ does not imply $\Gamma_1 = \infty$, as in the incompressible case (Simon 1963).

The equation of continuity (5.2.28), the equation of motion (5.2.73), the adiabatic energy equation (5.2.78), and Poisson's equation (5.2.40) become, respectively

$$\varepsilon(\ell) \propto \delta\varrho(\ell)\cos(k\varphi) = \delta\varrho(\vec{r}) = -\varrho\ \nabla\cdot\Delta\vec{r} = 0; \quad \sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) = \nabla\chi;$$
$$\delta P + (dP/d\ell)\ \Delta\ell = 0; \quad \nabla^2\delta\Phi = 0. \tag{5.6.62}$$

By virtue of Eq. (5.6.34) we have $\chi(\vec{r}) = \chi(\ell)\cos(k\varphi)\cos(jz)$, so we can write for $\Delta\vec{r}$ from Eq. (5.6.62)

$$\Delta\vec{r}(\ell, \varphi, z) = \nabla[\chi(\ell)\cos(k\varphi)]/\sigma^2 = \nabla[L(\ell)\cos(k\varphi)], \qquad (j = 0;\ L(\ell) = \chi(\ell)/\sigma^2). \tag{5.6.63}$$

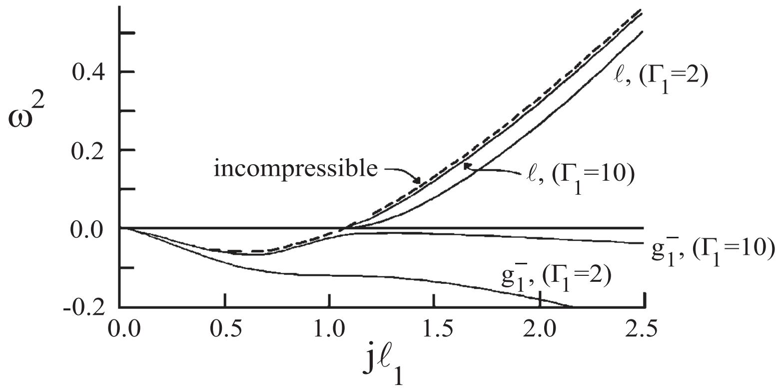

Fig. 5.6.1 Dimensionless squared angular oscillation frequency $\omega^2 = \sigma^2/4\pi G\varrho$ of the axisymmetric unstable g_1^--mode, and of the stable ℓ-mode for the homogeneous compressible cylinder $(n, k = 0)$ in the case of large values of the generalized adiabatic exponent $(\Gamma_1 = 2, 10)$. The broken curve shows the eigenfrequencies of the single, unstable axisymmetric mode for the incompressible homogeneous cylinder $(n, k = 0;\ \Gamma_1 = \infty)$ from Eq. (5.6.30). See end of Sec. 5.6.3 for further explanations (Ostriker 1964c).

We insert the expansions (5.6.34), (5.6.63) into Eq. (5.6.62), the individual harmonics satisfying the equations

$$\nabla \cdot \Delta\vec{r} = \nabla^2[L(\ell)\cos(k\varphi)] = [(1/\ell)\ d(\ell\ dL/d\ell)/d\ell - k^2 L/\ell^2]\cos(k\varphi) = 0;$$
$$\sigma^2 L(\ell) = \chi(\ell) = \delta P(\ell)/\varrho - \delta\Phi(\ell); \quad \delta P(\ell) + (dP/d\ell)\ \Delta\ell = \delta P(\ell) - 2\pi G\varrho^2\ell\ dL/d\ell = 0;$$
$$\nabla^2\delta\Phi = (1/\ell)\ d(\ell\ d\delta\Phi/d\ell)/d\ell - k^2\ \delta\Phi/\ell^2 = 0. \tag{5.6.64}$$

The solutions of the two Laplace equations for $L(\ell)\cos(k\varphi)$ and $\delta\Phi(\ell)\cos(k\varphi)$ are given by Eq. (5.6.4), where nonsingularity at the origin demands $B_2 = 0$:

$$L(\ell) = L(\ell_1)\ (\ell/\ell_1)^k; \quad \delta\Phi(\ell) = B_1\ell^k. \tag{5.6.65}$$

To determine the constant B_1, we require continuity of the internal potential

$$\Phi(\ell) = \Phi_u(\ell) + \delta\Phi(\ell) = -\pi G\varrho\ell^2 + B_1\ell^k, \tag{5.6.66}$$

and of its radial derivative with the external potential

$$\Phi_e(\ell) = \Phi_{ue}(\ell) + \delta\Phi_e(\ell) = C_1\ln\ell + C_2 + B_2\ell^{-k}, \tag{5.6.67}$$

on the boundary $\ell = \ell_1 + \Delta\ell_1$ of the perturbed cylinder [cf. Eq. (5.6.11)]. Since in our first order theory $\Phi \approx \Phi_u$ and $\Phi_e \approx \Phi_{ue}$, we infer that the constants B_1, B_2 are of first order, and

$$\Phi(\ell) = \Phi(\ell_1) + \Delta\ell_1\ (d\Phi/d\ell)_{\ell=\ell_1} = -\pi G\varrho\ell_1^2 + B_1\ell_1^k - 2\pi G\varrho\ell_1\ \Delta\ell_1$$
$$= \Phi_e(\ell) = \Phi_e(\ell_1) + \Delta\ell_1\ (d\Phi_e/d\ell)_{\ell=\ell_1} = C_1\ln\ell_1 + C_2 + B_2\ell_1^{-k} + C_1\ \Delta\ell_1/\ell_1, \tag{5.6.68}$$

$$(d\Phi/d\ell)_{\ell=\ell_1+\Delta\ell_1} = -2\pi G\varrho\ell_1 + kB_1\ell_1^{k-1} - 2\pi G\varrho\ \Delta\ell_1$$
$$= (d\Phi_e/d\ell)_{\ell=\ell_1+\Delta\ell_1} = C_1/\ell_1 - kB_2\ell_1^{-k-1} - C_1\ \Delta\ell_1/\ell_1^2. \tag{5.6.69}$$

Equating zeroth and first order terms in Eqs. (5.6.68), (5.6.69), we get

$$C_1 = -2\pi G\varrho\ell_1^2; \quad B_1\ell_1^k = B_2\ell_1^{-k}; \quad kB_1\ell_1^{k-1} = -kB_2\ell_1^{-k-1} + 4\pi G\varrho\ \Delta\ell_1. \tag{5.6.70}$$

Hence, $B_1 = 2\pi G\varrho\ \Delta\ell_1/k\ell_1^{k-1}$. The Eulerian correction to the internal potential is analogous to Eq. (5.6.17):

$$\delta\Phi(\ell) = \Phi(\ell) - \Phi_u(\ell) = B_1\ell^k = 2\pi G\varrho\ell^k\ \Delta\ell_1/k\ell_1^{k-1} = 2\pi G\varrho L(\ell_1)\ (\ell/\ell_1)^k. \tag{5.6.71}$$

We have replaced $\Delta\ell_1$ via Eqs. (5.6.63), (5.6.65) with

$$\Delta\ell(\ell) = (dL/d\ell) = kL(\ell_1)\ \ell^{k-1}/\ell_1^k;\quad \Delta\ell(\ell_1) = \Delta\ell_1 = kL(\ell_1)/\ell_1, \qquad (k \geq 1). \tag{5.6.72}$$

The characteristic frequencies σ are determined by substituting Eqs. (5.6.65), (5.6.71), and the third equation (5.6.64) into the second equation (5.6.64):

$$\sigma^2 = 2\pi G\varrho(k-1), \qquad (\varepsilon, \delta\varrho, j = 0;\ k = 1, 2, 3, ...). \tag{5.6.73}$$

If $k = 0$, we have in virtue of Eq. (5.6.65) $L(\ell) = \mathrm{const}$, and $\Delta\vec{r}(\ell, \varphi, z) = 0$ via Eq. (5.6.63). Neutral stability ($\sigma = 0$) occurs if $k = 1$, and if $k > 1$, all our divergence-free modes ($\nabla \cdot \Delta\vec{r} = 0$) are stable ($\sigma^2 > 0$).

(iii) Neutral Modes ($\sigma, \omega = 0;\ k = 0, 1, 2, ...;\ j \neq 0$). From Eq. (5.6.38) we observe that $\nabla\delta\varrho \times \nabla P = 0$, or in our first order approximation $\nabla\delta\varrho \times \nabla P_u = 0$. Thus, $\nabla\delta\varrho$ is parallel to ∇P_u, and since ∇P_u is directed along the ℓ-axis, the sole nonzero component of $\nabla\delta\varrho$ is $\partial\delta\varrho/\partial\ell \neq 0$. Therefore: $\delta\varrho = \delta\varrho(\ell, t)$. But the Eulerian perturbation (5.6.34) implies that $\delta\varrho(\vec{r}, t) = \delta\varrho(\ell, \varphi, z, t)$ is also a function of φ and z, $(j, k \neq 0)$, which contradicts our earlier finding that $\delta\varrho = \delta\varrho(\ell, t)$. The sole remaining possibility is that $\delta\varrho(\ell, t) = 0$, $(\varepsilon = 0)$, and therefore also $\delta\varrho(\vec{r}, t) \propto \delta\varrho(\ell, t) = 0$. From the continuity equation (5.2.28) we observe that in this case the oscillatory motion must be divergence-free: $\nabla \cdot \Delta\vec{r} = 0$. The energy equation is the same as Eq. (5.6.62), namely

$$\delta P + (dP/d\ell)\ \Delta\ell = \delta P - 2\pi G\varrho^2\ell\ \Delta\ell = 0, \tag{5.6.74}$$

where $\delta P, \Delta\ell$ are given by Eq. (5.6.34). Because $\delta\varrho(\vec{r}, t) = 0$, the Eulerian variation of the potential is given by Eq. (5.6.17), and since the equation of motion (5.2.73) is simply $\delta P = \varrho\ \delta\Phi$, we find by inserting from Eqs. (5.6.17), (5.6.74):

$$\Delta\ell(\ell) = \delta P/2\pi G\varrho^2\ell = 2\ell_1\ \Delta\ell_1\ K_k(j\ell_1)\ I_k(j\ell)/\ell, \qquad (\sigma = 0;\ \Gamma_1 \neq \infty). \tag{5.6.75}$$

Consistency at the unperturbed surface $\ell = \ell_1$ requires

$$I_k(j\ell_1)\ K_k(j\ell_1) = 1/2 \qquad (\Delta\ell(\ell_1) = \Delta\ell_1). \tag{5.6.76}$$

Like in the incompressible case [$\Gamma_1 = \infty$; Eq. (5.6.30)], this equation has a positive solution only if $k = 0$:

$$I_0(j\ell_1)\ K_0(j\ell_1) = 1/2 \quad \text{if} \quad j\ell_1 = 1.0668, \qquad (\sigma, k = 0). \tag{5.6.77}$$

The corresponding eigenfunction (5.6.75) is

$$\Delta\ell(\ell) = 1.0668\ \Delta\ell_1\ I_0(j\ell)/j\ell\ I_0(1.0668), \qquad (\sigma, k = 0). \tag{5.6.78}$$

There is still another way of satisfying Eq. (5.6.75), namely we may have

$$\Delta\ell(\ell) = \Delta\ell_1 = 0, \qquad (\sigma = 0). \tag{5.6.79}$$

This is not necessarily a trivial solution, since $\ell\ \Delta\varphi(\ell, \varphi, z)$ and $\Delta z(\ell, \varphi, z)$ do not need to be zero; they must satisfy only the divergence-free condition (5.6.62): $\nabla \cdot \Delta\vec{r} = (1/\ell)\ \partial(\ell\ \Delta\varphi)/\partial\varphi + \partial\Delta z/\partial z = 0$.

These are the analytical solutions so far found by Ostriker (1964c) for the compressible homogeneous cylinder. The discussion of his numerical results will be deferred to the subsequent general case.

Table 5.6.1 Dimensionless squared oscillation frequency $\omega^2 = \sigma^2/4\pi G\varrho_0$ of the fundamental mode ω_0^2, and of the first three overtones $\omega_1^2, \omega_2^2, \omega_3^2$ for radial oscillations ($j, k = 0$; $\Gamma_1 = 5/3$). The eigenvalues of the homogeneous cylinder ($n = 0$) are from Eq. (5.6.60), (Simon 1963, Robe 1967). $a + b$ means $a \times 10^b$.

n	ω_0^2	ω_1^2	ω_2^2	ω_3^2
0	6.67−1	5.67+0	1.40+1	2.57+1
1	4.63−1	2.27+0	4.99+0	8.62+0
3	2.94−1	7.02−1	1.26+0	1.95+0
6	1.38−1	1.97−1	2.71−1	3.63−1

Table 5.6.2 Normalized squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of nonaxisymmetric p, f, g-modes ($k = 2$; $j = 0$; $\Gamma_1 = 5/3$) for polytropic cylinders of index $n = 1, 3, 6$ (Robe 1967). $a + b$ means $a \times 10^b$.

n	g_3	g_2	g_1	f	p_1	p_2	p_3
1	−9.10 − 3	−1.58 − 2	−3.50 − 2	3.46−1	1.88+0	4.52+0	8.11+0
3	1.44−2	2.31−2	4.39−2	2.22−1	6.39−1	1.18+0	1.87+0
6	2.31−2	3.61−2	6.46−2	1.30−1	1.78−1	2.54−1	3.49−1

5.6.3 General Case $0 < n < \infty$

The most comprehensive numerical study of compressible cylinders with general polytropic index $0 < n < \infty$ seems to have been effected by Robe (1967). The equation of continuity (5.2.28) assumes in cylindrical coordinates the form

$$-\delta\varrho/\varrho - (\Delta\ell/\varrho)\ \partial\varrho/\partial\ell = \nabla\cdot\Delta\vec r = (1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell + (1/\ell)\ \partial(\ell\ \Delta\varphi)/\partial\varphi + \partial\Delta z/\partial z$$
$$= (1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell - (k^2/\ell^2 + j^2)\ \chi(\ell,\varphi,z)/\sigma^2, \qquad (5.6.80)$$

where we have used the equations of motion (5.6.35). The adiabatic ($Q =$ const) energy equation (5.2.78) can be written in cylindrical coordinates via Eq. (5.6.80) as

$$(\delta P + \Delta\ell\ \partial P/\partial\ell)/\Gamma_1 P = \delta\varrho/\varrho + (\Delta\ell/\varrho)\ \partial\varrho/\partial\ell$$
$$= -(1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell + (k^2/\ell^2 + j^2)(\delta P/\varrho - \delta\Phi)/\sigma^2, \qquad (\chi = \delta P/\varrho - \delta\Phi). \qquad (5.6.81)$$

If expansions of the form (5.6.34) are employed in Eq. (5.6.81), the common factor $\cos(k\varphi)\cos(jz)$ cancels out:

$$d(\ell\ \Delta\ell)/d\ell + \ell\ \Delta\ell\ (dP/d\ell)/\Gamma_1 P = [(k^2 + j^2\ell^2)/\sigma^2\ell - \varrho\ell/\Gamma_1 P]\ \delta P/\varrho - (k^2 + j^2\ell^2)\ \delta\Phi/\sigma^2\ell. \qquad (5.6.82)$$

The first equation (5.6.35) reads

$$d(\delta P/\varrho)/d\ell + A\ \delta P/\varrho = [\sigma^2 - (A/\varrho)\ dP/d\ell]\ \Delta\ell + d\delta\Phi/d\ell, \qquad (5.6.83)$$

by suppressing $\cos(k\varphi)\cos(jz)$, and rearranging the terms.

Poisson's equation (5.2.40) becomes in cylindrical coordinates via Eqs. (5.2.28), (5.2.95), (5.6.34) equal to

$$\nabla^2\delta\Phi = (1/\ell)\ \partial(\ell\ \partial\delta\Phi/\partial\ell)/\partial\ell + (1/\ell^2)\ \partial^2\delta\Phi/\partial\varphi^2 + \partial^2\delta\Phi/\partial z^2$$
$$= (1/\ell)\ \partial(\ell\ \partial\delta\Phi/\partial\ell)/\partial\ell - (k^2/\ell^2 + j^2)\ \delta\Phi = -4\pi G\ \delta\varrho = 4\pi G(\varrho\ \nabla\cdot\Delta\vec r + \Delta\vec r\cdot\nabla\varrho)$$
$$= 4\pi G(\varrho\ \nabla\cdot\Delta\vec r + \Delta\ell\ \partial\varrho/\partial\ell) = 4\pi G\varrho(A\ \Delta\ell - \delta P/\Gamma_1 P). \qquad (5.6.84)$$

With the notations (5.6.36) the basic equations (5.6.82)-(5.6.84) write as (Robe 1967)

$$du/d\ell + u(dP/d\ell)/\Gamma_1 P = [(k^2 + j^2\ell^2)/\sigma^2\ell - \varrho\ell/\Gamma_1 P]y - (k^2 + j^2\ell^2)\ \delta\Phi/\sigma^2\ell, \qquad (5.6.85)$$

$$dy/d\ell + Ay = [\sigma^2 - (A/\varrho)\ dP/d\ell]\ u/\ell + d\delta\Phi/d\ell, \qquad (5.6.86)$$

Table 5.6.3 Normalized squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ for axisymmetric p, ℓ, g-modes ($k = 0$; $\Gamma_1 = 5/3$) of polytropic cylinders (Ostriker 1964c, Robe 1967). If $n = 6$, $k^* = 4$, we may read in Robe's Table V for consistency reasons the values $gr_1 = 0.99212$ and $pr_2 = 1.400$ as 0.09212 and 1.100, respectively. If $j\ell_1 = 0$, the eigenvalues of p_1, p_2-modes coincide with the radial eigenvalues ω_0^2, ω_1^2 from Table 5.6.1. $a + b$ means $a \times 10^b$.

$j\ell_1$	g_2	g_1	ℓ	p_1	p_2
			$n = 0$		
0	–	–	–	6.67−1	5.67
0.2	−2.57−3	−3.28−2	–	6.81−1	5.68
0.6	−1.62−2	−1.18−1	–	7.72−1	5.74
1.2	−2.25−2	−1.66−1	9.40−3	1.05+0	5.97
2	−4.36−2	−2.33−1	2.52−1	1.63+0	6.52
3	−7.80−2	−3.54−1	7.42−1	2.78+0	7.61
			$n = 1$		
0	–	–	–	4.63−1	2.27
0.2	−2.54−4	−1.61−2	–	–	–
1	−5.52−3	−8.66−2	–	5.41−1	2.33
2	−8.77−3	−2.93−2	2.54−2	7.31−1	2.51
4	−2.70−2	−6.03−2	5.91−1	1.53+0	3.27
10	−9.31−2	−1.61−1	2.21+0	5.45+1	8.97
			$n = 3$		
0	–	–	–	2.94−1	7.02−1
0.5	3.90−4	8.62−4	−2.12−2	–	–
1	1.58−3	3.54−3	−4.75−2	3.07−1	7.11−1
3	1.49−2	3.76−2	−2.81−2	3.71−1	7.82−1
6	7.16−2	3.57−1	–	6.30−1	1.04+0
12	1.36−1	7.55−1	–	1.51+0	2.25+0
			$n = 6$		
0	–	–	–	1.38−1	1.97−1
1	7.22−4	1.31−3	−1.58−2	1.38−1	1.98−1
4	1.18−2	2.30−2	−5.82−2	1.42−1	2.08−1
6	2.63−2	5.43−2	−4.43−2	1.48−1	2.18−1
15	1.12−1	1.91−1	–	3.10−1	4.22−1

$$(1/\ell)\ d(\ell\ d\delta\Phi/d\ell)/d\ell - (k^2/\ell^2 + j^2)\ \delta\Phi = 4\pi G\varrho(Au/\ell - \varrho y/\Gamma_1 P). \tag{5.6.87}$$

We have discarded the common factor $\cos(k\varphi)\cos(jz)$ in Eq. (5.6.84). The solutions of Eqs. (5.6.85)-(5.6.87) must be continuous in the interval $0 \le \ell \le \ell_1$, and have to satisfy on the outer boundary the conditions (5.2.109) and (5.2.113), respectively, where we have to replace r, λ by ℓ, z :

$$\Delta P = \delta P + \Delta\vec{r}\cdot\nabla P = 0 \qquad \text{or} \qquad [\varrho y + (u/\ell)\ dP/d\ell]_{\ell=\ell_1} = 0, \tag{5.6.88}$$

$$\delta\Phi(\ell_1, \varphi, z) = \delta\Phi_e(\ell_1, \varphi, z); \quad \Delta\ell_1\ (d^2\Phi_u/d\ell^2)_{\ell=\ell_1} + (\partial\delta\Phi/\partial\ell)_{\ell=\ell_1}$$
$$= \Delta\ell_1\ (d^2\Phi_{ue}/d\ell^2)_{\ell=\ell_1} + (\partial\delta\Phi_e/\partial\ell)_{\ell=\ell_1}. \tag{5.6.89}$$

By the same arguments as subsequent to Eq. (5.2.113) we have $[d^2(\Phi_u - \Phi_{ue})/d\ell^2]_{\ell=\ell_1} = -4\pi G\varrho_u(\ell_1)$, and the second equation (5.6.89) writes

$$(\partial\delta\Phi/\partial\ell)_{\ell=\ell_1} = (\partial\delta\Phi_e/\partial\ell)_{\ell=\ell_1} + 4\pi G\varrho(\ell_1)\ \Delta\ell_1. \tag{5.6.90}$$

Since the Eulerian perturbation of the external potential has to satisfy the Laplace equation $\nabla^2\delta\Phi_e = 0$, its solution is given by Eqs. (5.6.3), (5.6.4), where $B_1 = 0$, in order to avoid an infinite value of $\delta\Phi_e$ as $\ell \to \infty$. Thus

$$\delta\Phi_e(\ell) = B_2 K_k(j\ell) \quad \text{if} \quad j, k \ne 0, \quad \text{and} \quad \delta\Phi_e(\ell) = B_2\ell^{-k} \quad \text{if} \quad j = 0,\ k \ne 0. \tag{5.6.91}$$

The outer boundary condition (5.6.90) turns into

$$(\partial\delta\Phi/\partial\ell)_{\ell=\ell_1} = \delta\Phi(\ell_1)\ (dK_k/d\ell)_{\ell=\ell_1}/K_k(j\ell_1) + 4\pi G\varrho(\ell_1)\ \Delta\ell_1 \quad \text{if} \quad j, k \ne 0, \quad \text{and}$$
$$(\partial\delta\Phi/\partial\ell)_{\ell=\ell_1} = -k\ \delta\Phi(\ell_1)/\ell_1 + 4\pi G\varrho(\ell_1)\ \Delta\ell_1 \quad \text{if} \quad j = 0,\ k \ne 0, \tag{5.6.92}$$

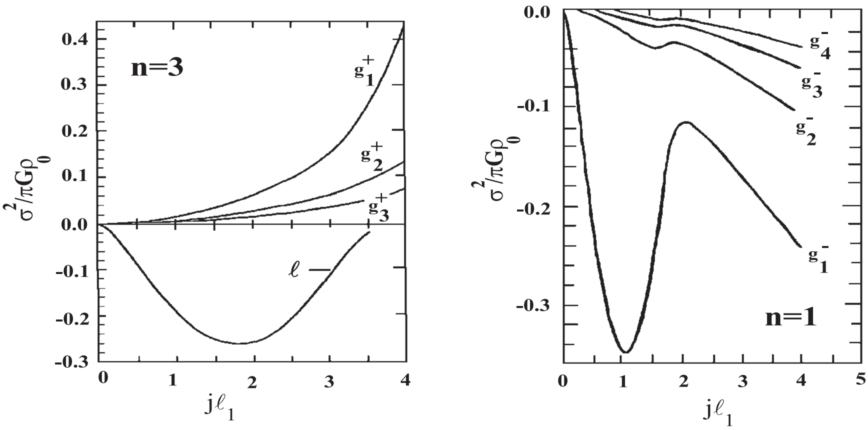

Fig. 5.6.2 Left-hand side: Squared normalized angular oscillation frequency $\sigma^2/\pi G\varrho_0$ of axisymmetric stable gravity modes ($A < 0$), and of the unstable ℓ-mode, leading to gravitational (varicose) instability of a polytropic cylinder with $n = 3$, $\Gamma_1 = 5/3$, $k = 0$. The figure for the polytropic index $n = 6$ is analogous. Right-hand side: Squared dimensionless angular oscillation frequency $\sigma^2/\pi G\varrho_0$ for axisymmetric unstable gravity modes of the polytropic cylinder $n = 1$, ($A > 0$; $\Gamma_1 = 5/3$; $k = 0$), (Robe 1967).

by inserting from Eq. (5.6.91) $\delta\Phi(\ell_1) = \delta\Phi_e(\ell_1)$, and $B_2 = \delta\Phi(\ell_1)/K_k(j\ell_1)$ or $B_2 = \delta\Phi(\ell_1)/\ell_1^k$.

The outer boundary conditions in the particular case $j, k = 0$, corresponding to purely radial oscillations along the ℓ-direction, are given by Eq. (5.6.97).

From Eqs. (5.6.34) and (5.6.35) we may distinguish three principal types of oscillations for cylinders: (i) Radial oscillations if $j, k = 0$. (ii) Nonaxisymmetric oscillations if $j = 0$ and $k \neq 0$. (iii) Axisymmetric oscillations if $j \neq 0,\ k = 0$. Evaluation of the general case $j, k \neq 0$ is even more involved.

(i) Radial Modes ($j, k = 0$). The eigenvalues of radial modes for the homogeneous compressible cylinder ($n = 0$; $\Gamma_1 \neq \infty$) are given by Eq. (5.6.60). For other polytropic indices we must resort to numerical integrations of the relevant equations (5.6.85)-(5.6.87), which can be reduced to a second order system if $j, k = 0$, by inserting into Poisson's equation (5.2.40) the equation of continuity (5.2.28):

$$\nabla^2\Phi = (1/\ell)\ d(\ell\ d\Phi/d\ell)/d\ell = -4\pi G\ \delta\varrho = (4\pi G/\ell)\ d(\varrho\ell\ \Delta\ell)/d\ell. \tag{5.6.93}$$

We integrate with the initial conditions $(d\delta\Phi/d\ell)_{\ell=0} = 0,\ \Delta\ell(0) = 0$, [cf. Eq. (5.2.61)]:

$$d\delta\Phi/d\ell = 4\pi G\varrho\ \Delta\ell(\ell). \tag{5.6.94}$$

With this equation we can eliminate $\delta\Phi$ from Eqs. (5.6.85)-(5.6.87), to obtain the second order system $(j, k = 0)$:

$$du/d\ell + u\ (dP/d\ell)/\Gamma_1 P = -\varrho\ell y/\Gamma_1 P, \tag{5.6.95}$$

$$dy/d\ell + Ay = [\sigma^2 - (A/\varrho)\ dP/d\ell + 4\pi G\varrho]\ u/\ell. \tag{5.6.96}$$

The boundary conditions are [cf. Eq. (5.2.121) and Robe (1967)]:

$$u(0) = 0,\ y(0) = 0 \quad \text{and} \quad u(\ell_1),\ y(\ell_1) = \text{finite}. \tag{5.6.97}$$

The eigenvalues of the fundamental mode and of the first three overtones are shown in Table 5.6.1.

(ii) Nonaxisymmetric Modes ($j = 0$; $k \neq 0$). These oscillations possess – like in the spherical case – the infinite discrete spectrum of pressure p-modes, with eigenvalues approaching infinity, the single fundamental f-mode, and the infinite discrete spectrum of gravity g-modes, with eigenvalues approaching zero. Like in the spherical case (Eq. (5.2.133), Sec. 5.5.1), the gravity modes are stable ($\sigma^2 > 0$) if $A < 0$, and unstable ($\sigma^2 < 0$) if $A > 0$. For the usual value $\Gamma_1 = 5/3$, unstable g-modes occur if we have $A > 0$ or $n < 1/(\Gamma_1 - 1) = 1.5$ in Eq. (5.10.2). Like for a sphere, the p-pressure modes turn into the radial modes if $k = 0$ (cf. Eqs. (5.6.58), (5.6.60) for the compressible homogeneous cylinder).

(iii) Axisymmetric Modes ($j \neq 0;\ k = 0$). These modes show the same two types of p and g-modes as exhibited by the nonradial modes of spheres and the nonaxisymmetric modes of cylinders: If $A < 0$ or $n > 1/(\Gamma_1 - 1)$, all axisymmetric p or g-modes are stable, but instead of the f-mode the single curious ℓ-mode (longitudinal mode) appears, leading to gravitational (varicose) instability if $A < 0$, ($n > 1.5$ if $\Gamma_1 = 5/3$), (Table 5.6.3, Fig. 5.6.2 on the left). If $A > 0$, ($n < 1.5$ if $\Gamma_1 = 5/3$), the axisymmetric g-modes become unstable (Table 5.6.3, Fig. 5.6.2 on the right), but the single ℓ-mode is now stable, at least for the values of $j\ell_1$ covered by the numerical exploration of Robe (1967, Table VIII): $j\ell_1 > 1.81$ if $n = 1$. As found by Ostriker (1964c), the stable ℓ-mode of the compressible homogeneous cylinder commences at $j\ell_1 = 1.0668$ (Eq. (5.6.77), Table 5.6.3, Fig. 5.6.1). In fact, as $\Gamma_1 \to \infty$, the stable ℓ-mode of the compressible homogeneous cylinder approaches the stable part of the single ℓ-mode of the incompressible homogeneous cylinder ($j\ell_1 > 1.0668$), while the unstable gravity g^--modes from Eq. (5.6.59) approach the unstable part of this single ℓ-mode, occurring if $0 < j\ell_1 < j_c\ell_1 = 1.0668$ (Eq. (5.6.32), Fig. 5.6.1). For the homogeneous compressible cylinder Ostriker (1964c) has depicted also neutral n-modes ($\sigma = 0$), touched in Eqs. (5.6.74)-(5.6.79). The pressure p-modes and the gravity g-modes have been called by Ostriker (1964c) r-(radial-like) modes and c-(convective) modes, respectively.

5.6.4 The Nonrotating Isothermal Cylinder $n = \pm\infty$

Isothermal cylinders extend to infinity (Secs. 2.6.8, 3.9.1), and in order to prevent these objects from expanding beyond a given radius $\ell_1 = \alpha\xi_1$, Hansen et al. (1976) invoke an artificial external pressure.

The cylindrical counterpart of the fundamental isothermal wave equation (5.3.77) was written down by Hansen et al. (1976) for radial oscillations:

$$d^2\eta/d\xi^2 + (3/\xi - \theta')\ d\eta/d\xi + [2(1/\Gamma_1 - 1)\theta'/\xi + \omega^2/\Gamma_1]\ \eta = 0,$$
$$(j, k = 0;\ \eta = \Delta\ell/\ell = \Delta\xi/\xi;\ \Gamma_1 = \mathrm{const};\ n = \pm\infty). \qquad (5.6.98)$$

The boundary conditions are analogous to those from Eqs. (5.3.75), (5.3.77): $d\eta/d\xi = 0$ if $\xi = 0$, and $d\ln\eta/d\ln\xi = (\xi/\eta)\ d\eta/d\xi = -2$ if $\xi = \xi_1$, as results from the continuity equation (5.2.45) for a cylinder:

$$\Delta\varrho/\varrho = -2\eta - \ell\ \partial\eta/\partial\ell, \qquad [(\Delta\varrho/\varrho)_{\xi=\xi_1} = 0;\ \varrho(\xi_1) \neq 0;\ \partial\ln\ell_u/\partial m = 1/2\pi\varrho_u\ell_u]. \qquad (5.6.99)$$

The results of numerical integrations of Eqs. (5.6.85)-(5.6.87) for the nonrotating isothermal cylinder in the case of radial ($j, k = 0$) and nonaxisymmetric ($j = 0,\ k = 2$) oscillations are depicted in Fig. 5.6.3. The Schwarzschild discriminant (5.10.2) becomes in the isothermal case

$$A = (1 - 1/\Gamma_1)\ d\ln\varrho/d\ell, \qquad (n = \pm\infty;\ 1 < \Gamma_1 \le \infty). \qquad (5.6.100)$$

If $A < 0$, and no density inversions $d\varrho/d\ell > 0$ occur – unlike to the rotating isothermal cylinder from Sec. 3.9.1 and Fig. 3.9.1 – no convectively unstable g^--modes are present in the truncated, nonrotating isothermal cylinder, and all p, f, g-modes are stable ($\sigma^2, \omega^2 > 0$, Fig. 5.6.3). The simple ordering $\sigma^2(g_1^+) < \sigma^2(f) < \sigma^2(p_1)$, depicted for the spherical case, is preserved for all values of the surface radius $\ell_1 = \alpha\xi_1$.

Stodółkiewicz (1963) has found for axisymmetric perturbations ($j \neq 0,\ k = 0$) of the form (5.11.126) that the isothermal cylinder becomes unstable along its axis if the vertical wavelength exceeds

$$L_c = 3.94(2K/\pi G\varrho_0)^{1/2} = 3.94(M_1/\pi\varrho_0)^{1/2}, \qquad (5.6.101)$$

as can be inferred from Eqs. (5.11.142), (5.11.157) for a vanishing magnetic field $v_B = 0$.

The finite mass per unit length of the complete isothermal cylinder extending to infinity is $M_1 = 2K/G = 2\mathcal{R}T/G\mu$ (cf. Eq. (2.6.188); $P_0 = \mathcal{R}T\varrho_0/\mu$). If the mass per unit length M exceeds the hydrostatic equilibrium mass M_1, the cylinder contracts radially, and if $M < M_1$ the cylinder expands (cf. Safronov 1969, Tomisaka 1995). This is most easily seen from the Lagrangian equation of motion (5.11.82):

$$\partial^2\ell/\partial t^2 = -2\pi\ell\ \partial P/\partial m - 2G\ m(\ell)/\ell, \qquad (H = 0). \qquad (5.6.102)$$

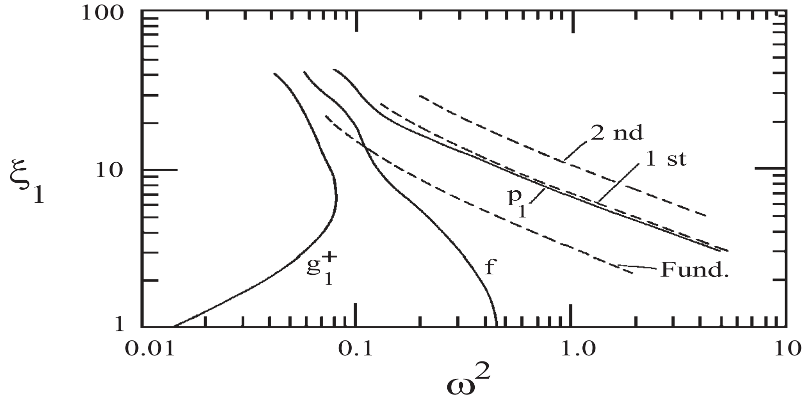

Fig. 5.6.3 Dimensionless angular oscillation frequency $\omega^2 = \sigma^2/4\pi G\varrho_0$ for radial modes (fundamental and first two overtones; $j, k = 0$; $\Gamma_1 = 5/3$; dashed curves), and for nonaxisymmetric modes ($j = 0,\ k = 2$; $\Gamma_1 = 5/3$; g_1^+, f, p_1-modes; solid curves) as a function of radial surface coordinate $\xi_1 = \ell_1/\alpha$ in the truncated isothermal cylinder $n = \pm\infty$ (Hansen et al. 1976).

If the mass $m(\ell)$ inside distance ℓ is larger than the hydrostatic equilibrium mass obtained for $\partial^2\ell/\partial t^2 = 0$, we have $\partial^2\ell/\partial t^2 < 0$ – contraction starts with inward directed negative acceleration. Conversely, if $m(\ell)$ is smaller than the equilibrium mass, there results $\partial^2\ell/\partial t^2 > 0$, and the isothermal cylinder expands.

5.7 Oscillations and Stability of Rotationally and Tidally Distorted Polytropic Spheres

5.7.1 The Rotating Homogeneous Sphere $n = 0$

Small oscillations of the uniformly rotating, incompressible and homogeneous sphere have already been investigated over hundred years ago by Bryan (1889). The theorem of von Zeipel (e.g. Tassoul 1978) shows the incompatibility of uniform rotation with radiative equilibrium (nuclear energy generation) in real stars, but the velocity of meridional currents is generally low, so that uniform rotation remains an adequate working hypothesis. A rigorous discussion of stability requires the proof of completeness for the normal modes, but since the problems encountered in this way – especially when rotation is present – are so formidable, one is led to more indirect methods of solution. Some of them are (e.g. Lebovitz 1967, Tassoul 1978): Investigation of exchange of stability (bifurcation points) due to Poincaré (Secs. 3.2, 3.8.4, 5.8.1, 6.4.3), the energy principle, the variational method (Secs. 5.7.2, 5.7.3, 5.11.2, 5.12.4), the virial theorem (Sec. 5.8), static and quasistatic methods (Sec. 5.12.1), etc.

The equations of motion with respect to axes rotating uniformly with the equilibrium configuration are given by Eq. (3.1.12), where the angular velocity $\vec{\Omega}$ is stationary, and magnetic as well as viscous forces are absent:

$$\varrho\ D\vec{v}/Dt = -\varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\varrho\ \vec{\Omega} \times \vec{v} - \nabla P + \varrho\ \nabla\Phi. \tag{5.7.1}$$

We project this equation onto spherical (r, λ, φ)-axes $[\vec{v} = \vec{v}(v_r, v_\lambda, v_\varphi);\ v_r = dr/dt;\ v_\lambda = r\ d\lambda/dt;\ v_\varphi = r\sin\lambda\ d\varphi/dt;\ \vec{\Omega} = \vec{\Omega}(\Omega\cos\lambda, -\Omega\sin\lambda, 0)]$:

$$\begin{aligned} &Dv_r/Dt - 2\Omega v_\varphi \sin\lambda = -(1/\varrho)\ \partial P/\partial r + \partial\Phi/\partial r + (1/2)\ \partial(\Omega^2 r^2 \sin^2\lambda)/\partial r;\\ &Dv_\lambda/Dt - 2\Omega v_\varphi \cos\lambda = -(1/\varrho r)\ \partial P/\partial\lambda + (1/r)\ \partial\Phi/\partial\lambda + (1/2r)\ \partial(\Omega^2 r^2 \sin^2\lambda)/\partial\lambda;\\ &Dv_\varphi/Dt + 2\Omega(v_r \sin\lambda + v_\lambda\cos\lambda) = -(1/\varrho r\sin\lambda)\ \partial P/\partial\varphi + (1/r\sin\lambda)\ \partial\Phi/\partial\varphi. \end{aligned} \tag{5.7.2}$$

If the initial state is an equilibrium configuration ($\vec{v}_u = 0$), we can write in the first order linear approximation by virtue of Eqs. (5.1.24), (5.1.29), (5.1.30): $D(\delta\vec{v})/Dt \approx D(\Delta\vec{v})/Dt = D^2(\Delta\vec{r})/Dt^2 \approx \partial^2\Delta\vec{r}/\partial t^2 = -\sigma^2\ \Delta\vec{r}$. We can also insert for the velocity components $\delta\vec{v} \approx \vec{v}$ via Eq. (5.1.30): $v_r = i\sigma\ \Delta r,\ v_\lambda = i\sigma r\ \Delta\lambda,\ v_\varphi = i\sigma r\sin\lambda\ \Delta\varphi$. We now apply the Eulerian changes (5.2.23) to v, ϱ, P, Φ, obtaining the first order Eulerian variation of Eq. (5.7.2) under the form (e.g. Ledoux and Walraven 1958)

$$\sigma^2\ \Delta r + 2i\sigma\Omega r\sin^2\lambda\ \Delta\varphi = (1/\varrho)\ \partial\delta P/\partial r - (\delta\varrho/\varrho^2)\ \partial P/\partial r - \partial\delta\Phi/\partial r, \tag{5.7.3}$$

$$\sigma^2 r\ \Delta\lambda + 2i\sigma\Omega r\sin\lambda\cos\lambda\ \Delta\varphi = (1/\varrho r)\ \partial\delta P/\partial\lambda - (\delta\varrho/r\varrho^2)\ \partial P/\partial\lambda - (1/r)\ \partial\delta\Phi/\partial\lambda, \tag{5.7.4}$$

$$\begin{aligned} &\sigma^2 r\sin\lambda\ \Delta\varphi - 2i\sigma\Omega(\sin\lambda\ \Delta r + r\cos\lambda\ \Delta\lambda) = (1/\varrho r\sin\lambda)\ \partial\delta P/\partial\varphi - (\delta\varrho/r\varrho^2\sin\lambda)\ \partial P/\partial\varphi\\ &-(1/r\sin\lambda)\ \partial\delta\Phi/\partial\varphi. \end{aligned} \tag{5.7.5}$$

The unperturbed equilibrium values P_u, ϱ_u, Φ_u satisfy the equation of hydrostatic equilibrium $[v_r, v_\lambda, v_\varphi = 0$ in Eq. (5.7.2)]:

$$(1/\varrho_u)\ \nabla P_u = \nabla[\Phi_u + (1/2)\Omega^2 r^2 \sin^2\lambda] = \nabla\Phi_{tot}. \tag{5.7.6}$$

Note, that the Eulerian change $\delta(\Omega^2 r^2 \sin^2\lambda/2)$ of the centrifugal potential is zero, as it is an extrinsic quantity depending solely on position (cf. Clement 1964, Chandrasekhar 1969, p. 29). For the homogeneous compressible model ($\varrho = \mathrm{const};\ n = 0$) we can solve Eqs. (5.7.3)-(5.7.5) for the three separate components $\Delta r,\ r\ \Delta\lambda,\ r\sin\lambda\ \Delta\varphi$ of $\Delta\vec{r}$. For instance, the radial component Δr is obtained by eliminating $\Delta\varphi$ between Eqs. (5.7.3), (5.7.5), and confining to third order terms of the form

$\Omega^2\ \Delta\vec{r} \approx (\Omega/\sigma)^2[\nabla\chi - (\delta\varrho/\varrho^2)\ \nabla P],\ (|\Delta\vec{r}|, \Omega \ll 1;\ \chi = \delta P/\varrho - \delta\Phi)$. Proceeding in a similar manner for the other two components of $\Delta\vec{r}$, the result can be written in vectorial form as (cf. Ledoux and Walraven 1958)

$$(\sigma^2 - 4\Omega^2)\ \Delta\vec{r} = \nabla\chi - (\delta\varrho/\varrho^2)\ \nabla P - (4\vec{\Omega}/\sigma^2)\{\vec{\Omega}\cdot[\nabla\chi - (\delta\varrho/\varrho^2)\ \nabla P]\}$$
$$+(2i/\sigma)\{\vec{\Omega}\times[\nabla\chi - (\delta\varrho/\varrho^2)\ \nabla P]\}, \qquad (\Omega \ll 1;\ n = 0). \tag{5.7.7}$$

We take the divergence of Eq. (5.7.7) by using $\nabla\cdot\Delta\vec{r} = -\delta\varrho/\varrho$ from the continuity equation (5.2.28), and $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b}),\ \nabla\cdot(\nabla\times\vec{a}) = 0,\ \nabla\cdot(f\vec{a}) = f\ \nabla\cdot\vec{a} + \nabla f\cdot\vec{a},\ \nabla\times(f\vec{a}) = f\ \nabla\times\vec{a} + \nabla f\times\vec{a},\ \nabla\times\nabla f = 0:$

$$(\sigma^2 - 4\Omega^2)\ \delta\varrho/\varrho = -\nabla^2\chi + (\delta\varrho/\varrho^2)\ \nabla^2 P + (1/\varrho^2)\ \nabla\delta\varrho\cdot\nabla P$$
$$+(4/\sigma^2)[L(\chi) - (\delta\varrho/\varrho^2)L(P) - (1/\varrho^2)(\vec{\Omega}\cdot\nabla P)(\vec{\Omega}\cdot\nabla\delta\varrho)] - (2i/\sigma\varrho^2)\ \vec{\Omega}\cdot(\nabla\delta\varrho\times\nabla P), \tag{5.7.8}$$

where

$$L(f) = \vec{\Omega}\cdot\nabla\big[\Omega\cos\lambda\ \partial f/\partial r - (\Omega/r)\sin\lambda\ \partial f/\partial\lambda\big] = \Omega^2\big[\cos^2\lambda\ \partial^2 f/\partial r^2 + (1/r)\sin^2\lambda\ \partial f/\partial r$$
$$+(1/r^2)\sin 2\lambda\ \partial f/\partial\lambda - (1/r)\sin 2\lambda\ \partial^2 f/\partial r\partial\lambda + (1/r^2)\sin^2\lambda\ \partial f^2/\partial\lambda^2\big]. \tag{5.7.9}$$

Eq. (5.7.8) leads to the distinction of two main cases:

(i) Axisymmetric Oscillations ($\partial\delta f/\partial\varphi = 0$). In this case $\partial\delta\varrho/\partial\varphi = 0$, and the vector $\nabla\delta\varrho$ is located in the meridional plane. The mixed vector product $\vec{\Omega}\cdot(\nabla\delta\varrho\times\nabla P)$ from Eq. (5.7.8) vanishes, as its vectors are coplanar in the meridian plane: For axisymmetric oscillations the effects of rotation are proportional to Ω^2.

Inertial forces will always induce some motions along the λ and φ-axis, but Eq. (5.7.10) shows that the assumptions of quasiradial motion, as defined subsequently to Eq. (5.3.1), will be verified. Indeed, we may neglect the right-hand sides of Eqs. (5.7.4) and (5.7.5), as they are by assumption at least of order Ω^2; then, these equations can be written as

$$r\ \Delta\lambda = 4\Omega^2\sin\lambda\cos\lambda\ \Delta r/(\sigma^2 - 4\Omega^2\cos^2\lambda); \quad r\sin\lambda\ \Delta\varphi = 2i\Omega\ \Delta r\sin\lambda/\sigma, \qquad (\sigma \gg \Omega), \tag{5.7.10}$$

by inserting $\Delta\varphi$ from Eq. (5.7.5) into Eq. (5.7.4), and neglecting $\Delta\lambda,\ [\Delta\lambda = O(\Omega^2\ \Delta r)]$ with respect to Δr in Eq. (5.7.5). If we further assume $\eta = \Delta r/r = \mathrm{const}$ – as for the fundamental radial mode $\ell = 0$ from Eq. (5.3.28) – we obtain from the continuity equation (5.2.28) if $\varrho = \mathrm{const}$:

$$\delta\varrho = -\varrho\ \nabla\cdot\Delta\vec{r} = -(\varrho/r^2)\ d(\eta r^3)/dr = -3\varrho\eta = \mathrm{const}. \tag{5.7.11}$$

With the foregoing simplifications $\eta, \varrho, \delta\varrho = \mathrm{const}$, all terms associated with $\nabla\delta\varrho$ vanish in Eq. (5.7.8), and we are left with $(\delta P/\varrho = \Gamma_1 P\ \delta\varrho/\varrho^2 - (\Delta r/\varrho)\ dP/dr,\ \delta\varrho/\varrho = -3\Delta r/r,\ \sigma^2\ \Delta r \approx d(\chi - P\ \delta\varrho/\varrho^2)/dr,\ P = 2\pi G\varrho^2(r_1^2 - r^2)/3,\ \nabla^2(r^2\sin^2\lambda) = 4)$:

$$(\sigma^2 - 4\Omega^2)\ \delta\varrho/\varrho = \nabla^2\delta\Phi - \nabla^2(\delta P/\varrho) + (\delta\varrho/\varrho^2)\ \nabla^2 P + (4/\sigma^2)[L(\chi) - (\delta\varrho/\varrho^2)L(P)] \approx -4\pi G\ \delta\varrho$$
$$+(1-\Gamma_1)(\delta\varrho/\varrho^2)\ \nabla^2 P + (1/\varrho)\ \nabla^2(\Delta r\ dP/dr) + (4\Omega^2/\sigma^2)[\cos^2\lambda\ d^2(\chi - P\ \delta\varrho/\varrho^2)/dr^2$$
$$+(1/r)\sin^2\lambda\ d(\chi - P\ \delta\varrho/\varrho^2)/dr] = -4\pi G\ \delta\varrho + (1-\Gamma_1)(\delta\varrho/\varrho)\ \nabla^2[\Phi + (\Omega^2/2)r^2\sin^2\lambda]$$
$$-(\delta\varrho/3\varrho^2)\ \nabla^2(r\ dP/dr) + 4\Omega^2[\cos^2\lambda\ d\Delta r/dr + \sin^2\lambda\ \Delta r/r] = 4\pi G(\Gamma_1 - 2)\ \delta\varrho$$
$$+2\Omega^2(1-\Gamma_1)\ \delta\varrho/\varrho + (\delta\varrho/3\varrho)\ \nabla^2(4\pi G\varrho r^2/3) - 4\Omega^2\ \delta\varrho/3\varrho - (4\Omega^2/3\sigma^2)(\sigma^2\ \delta\varrho/\varrho)$$
$$= 4\pi G(\Gamma_1 - 4/3)\ \delta\varrho + 2\Omega^2(1/3 - \Gamma_1)\ \delta\varrho/\varrho - 16\Omega^2\pi G(\Gamma_1 - 4/3)\ \delta\varrho/3\sigma^2. \tag{5.7.12}$$

In the last term we have inserted the zero order approximation of $\sigma^2 = 4\pi G\varrho(\Gamma_1 - 4/3)$ from Eq. (5.3.1), and Eq. (5.7.12) becomes eventually (Ledoux and Walraven 1958):

$$\sigma^2 = 4\pi G\varrho(\Gamma_1 - 4/3) + 2\Omega^2(7/3 - \Gamma_1) - 16\Omega^2\pi G\varrho(\Gamma_1 - 4/3)/3\sigma^2, \qquad (n = 0). \tag{5.7.13}$$

One solution of this biquadratic equation is $\sigma^2 \propto \Omega^2$, which must be discarded, as it does not satisfy the assumption (5.7.10) that $\sigma^2 \gg \Omega^2$. The other solution is, up to terms of order Ω^2, equal to Eq. (5.3.17).

(ii) First Order Nonaxisymmetric Oscillations. Keeping only terms in Ω, Eq. (5.7.8) turns into

$$\sigma^2\ \delta\varrho/\varrho = -4\pi G\ \delta\varrho - (\Gamma_1/\varrho^2)\ \nabla^2(\delta\varrho/P) + (1/\varrho)\ \nabla^2(\Delta r\ dP/dr) + (\delta\varrho/\varrho^2)\ \nabla^2 P$$
$$+(1/\varrho^2)(\partial\delta\varrho/\partial r)\ dP/dr + (2i\Omega\sin\lambda/\sigma\varrho^2)[(1/r\sin\lambda)\ \partial\delta\varrho/\partial\varphi]\ dP/dr. \qquad (5.7.14)$$

Since [cf. Eq. (5.2.87)]

$$\delta\varrho(r,\lambda,\varphi) = \delta\varrho(r)\ Y_j^k(\lambda,\varphi) = \delta\varrho(r)\ P_j^k(\cos\lambda)\exp(ik\varphi), \qquad (5.7.15)$$

we have $\partial\delta\varrho(r,\lambda,\varphi)/\partial\varphi = ik\ \delta\varrho(r,\lambda,\varphi)$. Eq. (5.7.14) reads after some rearrangements for the homogeneous sphere $n = 0$:

$$\nabla^2(r\ \Delta r) = -3\sigma^2\ \delta\varrho/4\pi G\varrho^2 - 6\ \delta\varrho/\varrho - (\Gamma_1/2\varrho)\ \nabla^2[(r_1^2 - r^2)\ \delta\varrho] - (r/\varrho)\ \partial\delta\varrho/\partial r + 2k\Omega\ \delta\varrho/\sigma\varrho,$$
$$(dP/dr = -4\pi G\varrho^2 r/3;\ \nabla^2 P = -4\pi G\varrho^2). \qquad (5.7.16)$$

On the other hand, we can write Eq. (5.7.8) – up to the first order in Ω – under the form ($\nabla\chi^2 = (1/r^2)\ \partial(r^2\ \partial\chi/\partial r)/\partial r - j(j+1)\chi/r^2$, $(\delta\varrho/\varrho^2)\ dP/dr \approx -\sigma^2\ \Delta r + \partial\chi/\partial r)$:

$$\sigma^2\ \delta\varrho/\varrho = -\nabla^2\chi + (\delta\varrho/\varrho^2)\ \nabla^2 P + (1/\varrho^2)(\partial\delta\varrho/\partial r)\ dP/dr - (2k\Omega\ \delta\varrho/\sigma\varrho^2 r)\ dP/dr$$
$$= -(1/r^2)\ \partial\big[(r^2\ \delta\varrho/\varrho^2)\ dP/dr + \sigma^2 r^2\ \Delta r + 2i\sigma\Omega r^3\sin^2\lambda\ \Delta\varphi\big]/\partial r + j(j+1)\chi/r^2 - 4\pi G\ \delta\varrho$$
$$+(1/\varrho^2)(\partial\delta\varrho/\partial r)\ dP/dr + 2k\Omega\sigma\ \Delta r/r - (2k\Omega/\sigma r)\ \partial\chi/\partial r = (\delta\varrho/\varrho^2 r^2)\ d(4\pi G\varrho^2 r^3/3)/dr$$
$$-(\sigma^2/r^2)\ \partial(r^2\ \Delta r)/\partial r + (2k\Omega/\sigma r^2)\ \partial(r\chi)/\partial r + j(j+1)\chi/r^2 - 4\pi G\ \delta\varrho + 2k\Omega\sigma\ \Delta r/r$$
$$-(2k\Omega/\sigma r)\ \partial\chi/\partial r = -(\sigma^2/r^2)\ \partial(r^2\ \Delta r)/\partial r + j(j+1)\chi/r^2 + (2k\Omega/\sigma r)(\sigma^2\ \Delta r + \chi/r). \qquad (5.7.17)$$

We have inserted the spherical approximation (5.2.90) for $\Delta\varphi$. To the same order of approximation the radial component of Eq. (5.7.7) is

$$\sigma^2\ \Delta r = \partial\chi/\partial r - (\delta\varrho/\varrho^2)\ dP/dr + 2k\Omega\chi/\sigma r. \qquad (5.7.18)$$

Solving Eq. (5.7.17) for χ, and introducing into Eq. (5.7.18), we obtain

$$[j(j+1) + 2k\Omega/\sigma]\sigma^2\ \Delta r - \sigma^2\ \partial^2(r^2\ \Delta r)/\partial r^2 - (\sigma^2/\varrho)\ \partial(r^2\ \delta\varrho)/\partial r + 2k\Omega\sigma\ \partial(r\ \Delta r)/\partial r$$
$$-4\pi G[j(j+1) + 2k\Omega/\sigma]r\ \delta\varrho/3 - (2k\Omega\sigma/r)\ \partial(r^2\ \Delta r)/\partial r - 2k\Omega\sigma r\ \delta\varrho/\varrho = \sigma^2[j(j+1)\ \Delta r$$
$$-\partial^2(r^2\ \Delta r)/\partial r^2] - 2\sigma^2 r\ \delta\varrho/\varrho - (\sigma^2 r^2/\varrho)\ \partial\delta\varrho/\partial r - 4\pi G j(j+1)r\ \delta\varrho/3$$
$$-2k\Omega\sigma(4\pi G r\ \delta\varrho/3\sigma^2 + r\ \delta\varrho/\varrho) = 0. \qquad (5.7.19)$$

With Eqs. (5.1.27), (5.1.28) we get

$$\nabla^2(r\ \Delta r) = (1/r^2)\ \partial[r^2\ \partial(r\ \Delta r)/\partial r]/\partial r - j(j+1)\ \Delta r/r = 2\Delta r/r + 4\ \partial\Delta r/\partial r + r\ \partial^2\Delta r/\partial r^2$$
$$-j(j+1)\ \Delta r/r = (1/r)\ \partial^2(r^2\ \Delta r)/\partial r^2 - j(j+1)\ \Delta r/r, \qquad (5.7.20)$$

casting Eq. (5.7.19) into the final form

$$\nabla^2(r\ \Delta r) + 2\ \delta\varrho/\varrho + (r/\varrho)\ \partial\delta\varrho/\partial r + 4\pi G j(j+1)\ \delta\varrho/3\sigma^2 + (2k\Omega/\sigma)(4\pi G\ \delta\varrho/3\sigma^2 + \delta\varrho/\varrho) = 0. \qquad (5.7.21)$$

We may now insert $\nabla^2(r\ \Delta r)$ from Eq. (5.7.16) into Eq. (5.7.21):

$$\nabla^2[(r_1^2 - r^2)\ \delta\varrho] + (\delta\varrho/\Gamma_1)\big[3\sigma^2/2\pi G\varrho + 8 - 8\pi G\varrho j(j+1)/3\sigma^2 - (4k\Omega/\sigma)(2 + 4\pi G\varrho/3\sigma^2)\big] = 0. \qquad (5.7.22)$$

The Laplacian $\nabla^2[(r_1^2 - r^2)\ \delta\varrho]$ can be written via Eqs. (B.39), (5.1.28) as $(1/r^2)\ \partial\{r^2\ \partial[(r_1^2 - r^2)\ \delta\varrho]/\partial r\}/\partial r - j(j+1)(r_1^2 - r^2)\ \delta\varrho/r^2$, by decomposing $\delta\varrho$ like in Eq. (5.7.15). We suppress in Eq. (5.7.22) the common factor $Y_j^k(\lambda,\varphi)$, replacing in this way – as in Eq. (5.5.16) – all partial derivatives of $\delta\varrho$ by ordinary derivatives:

$$(1 - x^2)\ d^2\delta\varrho/dx^2 + [(2 - 6x^2)/x]\ d\delta\varrho/dx + \big[3\sigma^2/2\pi G\varrho\Gamma_1 + 8/\Gamma_1 - 6 - 8\pi G\varrho j(j+1)/3\sigma^2\Gamma_1$$
$$-j(j+1)(1 - x^2)/x^2 - (8k\Omega/\Gamma_1\sigma)(1 + 2\pi G\varrho/3\sigma^2)\big]\ \delta\varrho = 0, \qquad (n = 0;\ x = r/r_1). \qquad (5.7.23)$$

The eigensolutions are again polynomials of the form (5.3.24), their coefficients obeying the recurrence formula (5.5.20); however, B is different:

$$B = 3\sigma^2/2\pi G\varrho\Gamma_1 + 8/\Gamma_1 - 6 - 8\pi G\varrho j(j+1)/3\sigma^2\Gamma_1 - 4j - (8k\Omega/\Gamma_1\sigma)(1 + 2\pi G\varrho/3\sigma^2)$$
$$= \ell(\ell + 5 + 2j), \qquad (\ell = 0, 2, 4, ...). \tag{5.7.24}$$

A first order approximation to the solution of this fifth order equation can be obtained at once by writing $\sigma = \sigma_0 + \varepsilon$, where $\varepsilon = O(\Omega) \ll 1$ and $\sigma_0^2 = 4\pi G\varrho\omega_0^2$ is the solution of the eigenvalue equation (5.5.22) for the nonrotating sphere. We insert

$$\big[8/\Gamma_1 - 6 - \ell(\ell + 5 + 2j) - 4j\big]\sigma_0^2 = 8\pi G\varrho j(j+1)/3\Gamma_1 - 3\sigma_0^4/2\pi G\varrho\Gamma_1, \tag{5.7.25}$$

from Eq. (5.5.22) into Eq. (5.7.24), and finally obtain for the eigenvalues of p and g-modes in a compressible homogeneous rotating sphere (Ledoux and Walraven 1958):

$$\sigma = \sigma_0 + \varepsilon = \sigma_0 + k\Omega(3\sigma_0^2/2\pi G\varrho + 1)\big/\big[9\sigma_0^4/16\pi^2G^2\varrho^2 + j(j+1)\big]$$
$$= \sigma_0 + k\Omega(6\omega_0^2 + 1)/[9\omega_0^4 + j(j+1)], \qquad (n = 0;\ j = 0, 1, 2, ...;\ k = -j, -j+1, ...j-1, j). \tag{5.7.26}$$

Up to the first order in Ω the equation for the fundamental (Kelvin) mode is simply (Smeyers and Denis 1971)

$$\sigma = \sigma_0 + k\Omega/j, \qquad (n = 0;\ j = 2, 3, 4, ...;\ k = -j, -j+1, ...j-1, j). \tag{5.7.27}$$

The eigenvalue $\sigma_0^2 = 4\pi G\varrho\omega_f^2$ is given by Eq. (5.5.26).

Thus, in the presence of rotation, the $(2j+1)$-fold degeneracy of the nonrotating case from Eqs. (5.5.23), (5.5.24), (5.5.26) disappears completely. The splitting of eigenfrequencies exhibited by Eqs. (5.7.26), (5.7.27) is for slow rotation symmetrical about the eigenfrequency σ_0, that corresponds either to $k = 0$, or to no rotation ($\Omega = 0$). The frequency difference between successive sublevels increases linearly with increasing rotation speed.

For quasiradial oscillations Sidorov [1982, Eq. (30), $m \to 2(\ell+1)$] obtains the following simple second order formula for the eigenfrequencies in a compressible homogeneous sphere

$$\sigma^2 = (2\pi\Gamma_1 G\varrho/3)[2\ell(\ell+5) + 2(3 - 4/\Gamma_1)] + (2\Omega^2/3)[5 - (\ell+1)(2\ell+3)\Gamma_1],$$
$$(n = 0;\ \ell = 0, 1, 2, 3, ...), \tag{5.7.28}$$

which coincides with Sterne's (1937) formula (5.3.29) in the case of no rotation ($\Omega = 0$), and with Ledoux' (1945) formula (5.3.17) for the fundamental quasiradial mode $\ell = 0$. Another paper by Sidorov (1981) seems to be less conclusive, as he applies Eq. (5.3.17) to polytropic indices $n \neq 0$, introducing a posteriori into the rough estimates (5.3.15)-(5.3.17) correction terms of order Ω^2.

5.7.2 The Cowling-Newing Variational Approach

Eq. (5.7.2) writes with Eq. (5.1.30) for general three-dimensional oscillations as

$$\varrho[\sigma^2\ \Delta r + r\sin^2\lambda(\Omega^2 + 2i\sigma\Omega\ \Delta\varphi)] = \partial P/\partial r - \varrho\ \partial\Phi/\partial r;$$
$$\varrho r^2[\sigma^2\ \Delta\lambda + \sin\lambda\cos\lambda(\Omega^2 + 2i\sigma\Omega\ \Delta\varphi)] = \partial P/\partial\lambda - \varrho\ \partial\Phi/\partial\lambda;$$
$$\varrho r\sin\lambda[\sigma^2 r\sin\lambda\ \Delta\varphi - 2i\sigma\Omega(\sin\lambda\ \Delta r + r\cos\lambda\ \Delta\lambda)] = \partial P/\partial\varphi - \varrho\ \partial\Phi/\partial\varphi. \tag{5.7.29}$$

On multiplying the equations (5.7.29) with the complex conjugate displacements $\Delta r^*, \Delta\lambda^*, \Delta\varphi^*$, adding together, and integrating over the volume V of the rotating polytrope, Cowling and Newing

(1949) obtain the Rayleigh principle for the determination of eigenvalues under the form

$$\sigma^2\int_V \varrho(\Delta r\ \Delta r^* + r^2\ \Delta\lambda\ \Delta\lambda^* + r^2\sin^2\lambda\ \Delta\varphi\ \Delta\varphi^*)\ dV + 2i\sigma\Omega\int_V \varrho\big[r\sin^2\lambda(\Delta r^*\ \Delta\varphi - \Delta r\ \Delta\varphi^*)$$
$$+r^2\sin\lambda\cos\lambda(\Delta\lambda^*\ \Delta\varphi - \Delta\lambda\ \Delta\varphi^*)\big]\ dV = \int_V \big[\Delta r^*(\partial P/\partial r - \varrho\ \partial\Phi/\partial r - \Omega^2\varrho r\sin^2\lambda)$$
$$+\Delta\lambda^*(\partial P/\partial\lambda - \varrho\ \partial\Phi/\partial\lambda - \Omega^2\varrho r^2\sin\lambda\cos\lambda) + \Delta\varphi^*(\partial P/\partial\varphi - \varrho\ \partial\Phi/\partial\varphi)\big]\ dV$$
$$\approx \int_V [\Delta r^*\ \Delta(\partial P/\partial r - \varrho\ \partial\Phi/\partial r) + \Delta\lambda^*\ \Delta(\partial P/\partial\lambda - \varrho\ \partial\Phi/\partial\lambda) + \Delta\varphi^*\ \Delta(\partial P/\partial\varphi - \varrho\ \partial\Phi/\partial\varphi)]\ dV$$
$$= \sigma_0^2\int_V \varrho(\Delta r\ \Delta r^* + r^2\ \Delta\lambda\ \Delta\lambda^* + r^2\sin^2\lambda\ \Delta\varphi\ \Delta\varphi^*)\ dV. \qquad (5.7.30)$$

By virtue of the hydrostatic equation (5.7.6), the expression $\nabla P - \varrho\ \nabla[\Phi + (1/2)\Omega^2 r^2\sin^2\lambda]$ has been replaced in Eq. (5.7.30) by its Lagrangian variation, since its *unperturbed* value vanishes [cf. Eqs. (5.1.15), (5.2.32)]. We have further neglected in a first approximation the centrifugal term involving Ω^2, by observing that the last integral in Eq. (5.7.30) represents just Rayleigh's principle for the nonrotating polytropic sphere. Eq. (5.7.30) can be written equivalently as

$$\sigma^2 + 2\Omega A\sigma = \sigma_0^2;\quad A = \left(i\int_V \varrho\big[r\sin^2\lambda(\Delta r^*\ \Delta\varphi - \Delta r\ \Delta\varphi^*) + r^2\sin\lambda\cos\lambda(\Delta\lambda^*\ \Delta\varphi\right.$$
$$\left.-\Delta\lambda\ \Delta\varphi^*)\big]\ dV\right)\Big/\int_V \varrho(\Delta r\ \Delta r^* + r^2\ \Delta\lambda\ \Delta\lambda^* + r^2\sin^2\lambda\ \Delta\varphi\ \Delta\varphi^*)\ dV. \qquad (5.7.31)$$

The solution of the second order equation (5.7.31) is $\sigma_{1,2} = -\Omega A \pm (\sigma_0^2 + \Omega^2 A^2)^{1/2}$, and since Ω is a small first order quantity, there has to be $\sigma \approx \sigma_0$. Thus, the relevant solution of Eq. (5.7.30) is

$$\sigma \approx \sigma_0 - \Omega A, \qquad (\Omega \ll 1). \qquad (5.7.32)$$

Since Ω is small, we can use in Eq. (5.7.31) the displacement values for the nonrotating polytrope from Eqs. (5.2.87)-(5.2.90):

$$\Delta r(r,\lambda,\varphi) = \Delta r(r)\ P_j^k(\cos\lambda)\exp(ik\varphi);\quad \Delta\lambda(r,\lambda,\varphi) = [\chi(r)/\sigma^2 r^2][dP_j^k(\cos\lambda)/d\lambda]\exp(ik\varphi);$$
$$\Delta\varphi(r,\lambda,\varphi) = ik[\chi(r)/\sigma^2 r^2][P_j^k(\cos\lambda)/\sin^2\lambda]\exp(ik\varphi). \qquad (5.7.33)$$

If we insert these displacements into Eq. (5.7.31), we observe that in the numerator the common factor k appears in the terms associated with $\Delta\varphi, \Delta\varphi^*$. Consequently, the eigenvalues (5.7.32) for the rotating polytrope can be written under the form (Cowling and Newing 1949)

$$\sigma = \sigma_0 + k\Omega B, \qquad (j = 0, 1, 2, ...;\ k = -j, -j+1, ...j-1, j). \qquad (5.7.34)$$

B denotes the remaining part of A from Eq. (5.7.31). Thus, the splitting of eigenvalues due to rotation subsists also for heterogeneous polytropes, and the $(2j+1)$-fold degeneracy from the nonrotating case is lifted completely. This splitting of the eigenfrequencies may offer an explanation for the so-called beat phenomenon between two oscillations of very close periods, observed in many β Cephei stars (e.g. Ledoux and Walraven 1958, p. 580). However, this explanation does not work for Cepheids of the β Canis Majoris type, for which lifting of degeneracy due to rotation may be invoked (see Sec. 5.7.3).

For quasiradial oscillations Cowling and Newing (1949) postulate $\Delta\lambda, \Delta\varphi \equiv 0$, and Eq. (5.7.30) becomes

$$\sigma^2\int_V \varrho(\Delta r)^2\ dV = \int_V \varrho\ \Delta r\ \Delta\big[(1/\varrho)\ \partial P/\partial r - \partial\Phi/\partial r - \Omega^2 r\sin^2\lambda\big]\ dV. \qquad (5.7.35)$$

We have $\Delta(\varrho\ \Delta V) = 0$ due to mass conservation [cf. Eq. (5.2.42)], and $\Delta r^* = \Delta r$, since Δr from Eq. (5.1.27) is assumed to be independent of φ. The actual polytrope, distorted by rotation, is approximated with a polytrope expanded spherically by the radial centrifugal force $\Omega^2\varrho r\sin^2\lambda$. The Lagrangian variation of $\Omega^2 r\sin^2\lambda$ is transformed by taking into account conservation of angular momentum from Eq. (5.3.12):

$$\Delta(\Omega^2 r\sin^2\lambda) = 2\Omega\ \Delta\Omega\ r\sin^2\lambda + \Omega^2\ \Delta r\sin^2\lambda = -3\Omega^2\ \Delta r\sin^2\lambda, \qquad (\lambda = \text{const}). \qquad (5.7.36)$$

In our quasiradial approximation we can take $\partial\Phi/\partial r = -Gm/r^2$, and $\Delta(\partial\Phi/\partial r) = 2Gm\ \Delta r/r^3$, ($m =$ const). Eq. (5.7.35) reads via Eqs. (5.1.16), (5.2.28), (5.7.36) as

$$\sigma^2 \int_V \varrho\ (\Delta r)^2\ dV = \int_V \varrho\ \Delta r\ \{\delta[(1/\varrho)\ \partial P/\partial r] + \Delta r\ \partial[(1/\varrho)\ \partial P/\partial r]/\partial r - \Delta(\partial\Phi/\partial r)$$
$$-\Delta(\Omega^2 r \sin^2\lambda)\}\ dV = \int_V \varrho\ \Delta r\ \big[-(1/\varrho^2)(\partial P/\partial r)(\delta\varrho + \Delta r\ \partial\varrho/\partial r) + (1/\varrho)\ \partial\delta P/\partial r$$
$$+(\Delta r/\varrho)\ \partial^2 P/\partial r^2 - 2Gm\ \Delta r/r^3 + 3\Omega^2\ \Delta r \sin^2\lambda\big]\ dV = \int_V \varrho\ \Delta r\ \big[-(\Delta\varrho/\varrho^2)\ \partial P/\partial r$$
$$+(1/\varrho)\ \partial\Delta P/\partial r - (1/\varrho)(\partial P/\partial r)\ \partial\Delta r/\partial r - 2Gm\ \Delta r/r^3 + 3\Omega^2\ \Delta r \sin^2\lambda\big]\ dV$$
$$= \int_V \Delta r\ [(2\Delta r/r)\ \partial P/\partial r + \partial\Delta P/\partial r - 2Gm\varrho\ \Delta r/r^3 + 3\Omega^2\varrho\ \Delta r \sin^2\lambda]\ dV. \qquad (5.7.37)$$

This equation can be transformed further via Eq. (3.1.16), by inserting for $-Gm/r^2 = \partial\Phi/\partial r$:

$$\sigma^2 \int_V \varrho\ (\Delta r)^2\ dV = \int_V \Delta r[(4\ \Delta r/r)\ \partial P/\partial r + \partial\Delta P/\partial r + \Omega^2\varrho\ \Delta r \sin^2\lambda]\ dV. \qquad (5.7.38)$$

The terms involving P are now integrated by parts, taking into account Eqs. (5.2.28), (5.2.38), (5.2.63):

$$\int_V \varrho\ (\Delta r)^2(\sigma^2 - \Omega^2\sin^2\lambda)\ dV = 4\pi(\sigma^2 - 2\Omega^2/3) \int_V \varrho\ (\Delta r)^2 r^2\ dr$$
$$= -4\pi \int_V \{4P\ \partial[r(\Delta r)^2]/\partial r + \Delta P\ \partial(r^2\ \Delta r)/\partial r\}\ dr$$
$$= 4\pi \int_V [-4P(\Delta r)^2 - 8Pr\ \Delta r\ \partial\Delta r/\partial r + r^2\ \Delta P\ \Delta\varrho/\varrho]\ dr$$
$$= \int_V P[12(\Delta r/r)^2 + 8(\Delta r/r)(\Delta\varrho/\varrho) + \Gamma_1(\Delta\varrho/\varrho)^2]\ dV. \qquad (5.7.39)$$

If $\eta = \Delta r/r =$ const, Eqs. (5.2.28) and (5.2.38) yield: $\Delta P/P = \Gamma_1\ \Delta\varrho/\varrho = -3\Gamma_1\ \Delta r/r$. Eq. (5.7.37) becomes

$$\sigma^2 \int_V \varrho r^2\ dV = \int_V [(3\Gamma_1 - 4)Gm\varrho/r + (5 - 3\Gamma_1)\Omega^2\varrho r^2 \sin^2\lambda]\ dV. \qquad (5.7.40)$$

We have again inserted from Eq. (3.1.16) for $\partial P/\partial r$, the equation (5.7.40) being identical to Eq. (5.3.16). The angular velocity Ω appears explicitly and implicitly in Eqs. (5.7.39) and (5.7.40) through the values of P and ϱ for the rotating polytrope. Cowling and Newing (1949) have taken the values of Δr and $\Delta\varrho$ for the nonrotating polytrope, while P and ϱ were always calculated for the rotating polytrope according to Chandrasekhar's (1933a) first order theory from Eq. (3.2.44) with $A_2 = 0$. Eq. (5.7.39) yields for the eigenvalue of the fundamental quasiradial mode: $\sigma^2/4\pi G\varrho_0 = 0.15 - 0.15\beta,\ (n = 1.5)$, and $0.057 - 1.92\beta,\ (n = 3;\ \Gamma_1 = 5/3;\ \beta = \Omega^2/2\pi G\varrho_0)$, (cf. Secs. 5.3.1, 5.7.3). The rough equations (5.3.16), (5.7.40) yield considerably different results if $n = 3:\ \sigma^2/4\pi G\varrho_0 = 0.15 - 0.13\beta,\ (n = 1.5)$, and $0.082 - 0.73\beta,\ (n = 3)$. These few results indicate that for quasiradial oscillations and $\Gamma_1 = 5/3,\ n = 1.5, 3$, the eigenvalues σ^2 of the fundamental quasiradial mode are reduced by slow rotation, keeping in mind the comments subsequent to Eq. (5.3.17).

5.7.3 Variational Approach of Clement

Following the theory of Chandrasekhar (1964a), the variational principle has been extended by Clement (1964, 1965, 1967, 1984, 1986) to rotating polytropes. Via Eq. (5.1.24) we have $v_i \approx \partial\Delta x_i/\partial t$, $[\Delta\vec{r} = \Delta\vec{r}(\Delta x_1, \Delta x_2, \Delta x_3);\ \vec{v} = \vec{v}(v_1, v_2, v_3)]$, and the linearized Eulerian change of the equation of motion (3.1.79) becomes

$$\partial^2\Delta x_i/\partial t^2 = -(1/\varrho)\ \partial\delta P/\partial x_i + (\delta\varrho/\varrho^2)\ \partial P/\partial x_i + \partial\delta\Phi/\partial x_i + 2\varepsilon_{ijk}\Omega_k\ \partial\Delta x_j/\partial t,$$
$$(i, j, k = 1, 2, 3), \qquad (5.7.41)$$

where the Eulerian change of the centrifugal potential $|\vec{\Omega}\times\vec{r}|^2/2$ is zero [cf. Eqs. (5.7.2)-(5.7.5)]. In absence of rotation it would be sufficient to consider the temporal dependence of the Lagrangian displacement $\Delta\vec{r}$ under the form $\Delta\vec{r}(\vec{r},t) = \Delta\vec{r}(\vec{r})\exp(i\sigma t)$. But as shown by Eq. (5.7.43), rotation couples the two components $\cos(\sigma t)$ and $i\sin(\sigma t)$ of $\exp(i\sigma t)$, and therefore Clement (1964) decomposes the Eulerian variations of r, P, ϱ, Φ as

$$\Delta x_i(\vec{r},t) = \Delta x_{i+}(\vec{r})\cos(\sigma t) + \Delta x_{i-}(\vec{r})\sin(\sigma t); \quad \delta P(\vec{r},t) = \delta P_+(\vec{r})\cos(\sigma t) + \delta P_-(\vec{r})\sin(\sigma t);$$
$$\delta\varrho(\vec{r},t) = \delta\varrho_+(\vec{r})\cos(\sigma t) + \delta\varrho_-(\vec{r})\sin(\sigma t); \quad \delta\Phi(\vec{r},t) = \delta\Phi_+(\vec{r})\cos(\sigma t) + \delta\Phi_-(\vec{r})\sin(\sigma t). \tag{5.7.42}$$

Inserting this first order variations into Eq. (5.7.41), collecting together terms in $\cos(\sigma t)$ and $\sin(\sigma t)$, and suppressing the common factors, we obtain two linearized equations of motion:

$$\sigma^2\ \Delta x_{i\pm} = (1/\varrho)\ \partial\delta P_\pm/\partial x_i - (\delta\varrho_\pm/\varrho^2)\ \partial P/\partial x_i - \partial\delta\Phi_\pm/\partial x_i \mp 2\sigma\varepsilon_{ijk}\Omega_k\ \Delta x_{j\mp}. \tag{5.7.43}$$

Thus, rotation induces through the Coriolis term $-2\vec{\Omega}\times\vec{v}$ a coupling between the two amplitudes Δx_{i+} and Δx_{i-}, that vanishes in absence of rotation, when $\Omega = 0$. The outer boundary conditions assumed by Clement (1964) are $\varrho, P = 0$ at $r = r_1$. From Eq. (5.2.110) follows $\partial P/\partial x_i = 0$ and $\delta P_\pm = 0$ at $r = r_1$. Let us now consider Eq. (5.7.43) as belonging to an eigenvalue $\sigma^{(\alpha)}$ with the eigensolution $\Delta\vec{r}_\pm^{(\alpha)}$, and let us multiply Eq. (5.7.43) successively with the components of another eigenfunction $\Delta\vec{r}_\pm^{(\beta)}$, belonging to another mode with the eigenvalue $\sigma^{(\beta)}$. Summing the products together, and integrating over the volume of the configuration, we get

$$[\sigma^{(\alpha)}]^2\int_V \varrho\ \Delta\vec{r}_\pm^{(\alpha)}\cdot\Delta\vec{r}_\pm^{(\beta)}\ dV = \int_V \big[\Delta\vec{r}_\pm^{(\beta)}\cdot\nabla\delta P_\pm^{(\alpha)} - (\delta\varrho_\pm^{(\alpha)}/\varrho)\ \Delta\vec{r}_\pm^{(\beta)}\cdot\nabla P - \varrho\ \Delta\vec{r}_\pm^{(\beta)}\cdot\nabla\delta\Phi_\pm^{(\alpha)}$$
$$\mp 2\sigma^{(\alpha)}\varrho\ \varepsilon_{ijk}\Omega_k\ \Delta x_{i\pm}^{(\beta)}\ \Delta x_{j\mp}^{(\alpha)}\big]\ dV. \tag{5.7.44}$$

We integrate the first and third term by parts, taking into account the boundary conditions $\delta P_\pm, \varrho = 0$; we also substitute for $\delta\varrho_\pm$ from the continuity equation (5.2.28), and for $\delta P_\pm$ from the adiabatic energy equation (5.2.39):

$$[\sigma^{(\alpha)}]^2\int_V \varrho\ \Delta\vec{r}_\pm^{(\alpha)}\cdot\Delta\vec{r}_\pm^{(\beta)}\ dV = \int_V \big\{\big[\Delta\vec{r}_\pm^{(\alpha)}\cdot\nabla P + \Gamma_1 P(\nabla\cdot\Delta\vec{r}_\pm^{(\alpha)})\big](\nabla\cdot\Delta\vec{r}_\pm^{(\beta)}) + \big[(1/\varrho)$$
$$\times(\Delta\vec{r}_\pm^{(\alpha)}\cdot\nabla\varrho) + \nabla\cdot\Delta\vec{r}_\pm^{(\alpha)}\big](\Delta\vec{r}_\pm^{(\beta)}\cdot\nabla P) + \delta\Phi_\pm^{(\alpha)}\ \nabla\cdot(\varrho\ \Delta\vec{r}_\pm^{(\beta)}) \pm 2\sigma^{(\alpha)}\varrho\ \varepsilon_{ijk}\Omega_j\ \Delta x_{i\pm}^{(\beta)}\ \Delta x_{k\mp}^{(\alpha)}\big\}\ dV$$
$$= \int_V \big[(\Delta\vec{r}_\pm^{(\alpha)}\cdot\nabla P)(\nabla\cdot\Delta\vec{r}_\pm^{(\beta)}) + (\Delta\vec{r}_\pm^{(\beta)}\cdot\nabla P)(\nabla\cdot\Delta\vec{r}_\pm^{(\alpha)}) + \Gamma_1 P(\nabla\cdot\Delta\vec{r}_\pm^{(\alpha)})(\nabla\cdot\Delta\vec{r}_\pm^{(\beta)})$$
$$+(1/\varrho)(\Delta\vec{r}_\pm^{(\alpha)}\cdot\nabla\varrho)(\Delta\vec{r}_\pm^{(\beta)}\cdot\nabla P) - \delta\varrho_\pm^{(\beta)}\delta\Phi_\pm^{(\alpha)} \mp 2\sigma^{(\alpha)}\varrho\ \varepsilon_{ijk}\Omega_k\ \Delta x_{i\pm}^{(\beta)}\ \Delta x_{j\mp}^{(\alpha)}\big]\ dV. \tag{5.7.45}$$

Recall that according to Eqs. (3.1.23), (3.1.24), (5.7.6) equipotential surfaces ($\Phi_{tot} = \mathrm{const}$) coincide with isopycnic ($\varrho = \mathrm{const}$) and isobaric ($P = \mathrm{const}$) surfaces, and therefore pressure and density can be considered as functions only of the total potential $P = P(\Phi_{tot})$, $\varrho = \varrho(\Phi_{tot})$. If $\Omega_k = 0$, the right-hand side of Eq. (5.7.45) is manifestly symmetric in α, β, since instead of ∇P and $\nabla\varrho$ we can write $(dP/d\Phi_{tot})\ \nabla\Phi_{tot}$ and $(d\varrho/d\Phi_{tot})\ \nabla\Phi_{tot}$, respectively. If $\Omega_k = 0$, the difference between Eq. (5.7.45) and the analogous equation written down for $[\sigma^{(\beta)}]^2$ is just

$$\{[\sigma^{(\alpha)}]^2 - [\sigma^{(\beta)}]^2\}\int_V \varrho\ \Delta\vec{r}_\pm^{(\alpha)}\cdot\Delta\vec{r}_\pm^{(\beta)}\ dV = 0 \quad \text{or} \quad \int_V \varrho\ \Delta\vec{r}_\pm^{(\alpha)}\cdot\Delta\vec{r}_\pm^{(\beta)}\ dV = 0, \quad (\alpha\neq\beta;\ \Omega_k = 0). \tag{5.7.46}$$

Setting $\alpha = \beta$ in Eq. (5.7.45), suppressing the distinguishing superscripts, and adding together the distinct equations for $\Delta\vec{r}_+$ and $\Delta\vec{r}_-$, we obtain a variational base for determining the characteristic eigenvalues. σ^2 is stationary with respect to arbitrary, independent variations of $\Delta\vec{r}_+$ and $\Delta\vec{r}_-$ (Clement 1964):

$$\sigma^2\int_V \varrho[(\Delta\vec{r}_+)^2 + (\Delta\vec{r}_-)^2]\ dV = \int_V \big\{2[(\Delta\vec{r}_+\cdot\nabla P)(\nabla\cdot\Delta\vec{r}_+) + (\Delta\vec{r}_-\cdot\nabla P)(\nabla\cdot\Delta\vec{r}_-)]$$
$$+\Gamma_1 P[(\nabla\cdot\Delta\vec{r}_+)^2 + (\nabla\cdot\Delta\vec{r}_-)^2] + (1/\varrho)[(\Delta\vec{r}_+\cdot\nabla\varrho)(\Delta\vec{r}_+\cdot\nabla P) + (\Delta\vec{r}_-\cdot\nabla\varrho)(\Delta\vec{r}_-\cdot\nabla P)]$$
$$-\delta\varrho_+\ \delta\Phi_+ - \delta\varrho_-\ \delta\Phi_- - 4\sigma\varrho\ \varepsilon_{ijk}\Omega_k\ \Delta x_{i+}\ \Delta x_{j-}\big\}\ dV. \tag{5.7.47}$$

Introducing the differential operator

$$L(\Delta\vec{r}) = -\nabla\delta P(\Delta\vec{r}) + [\delta\varrho(\Delta\vec{r})/\varrho]\ \nabla P + \varrho\ \nabla\delta\Phi(\Delta\vec{r}), \tag{5.7.48}$$

the two equations (5.7.44) become, when added together:

$$\sigma^2 \int_V \varrho[(\Delta\vec{r}_+)^2 + (\Delta\vec{r}_-)^2]\ dV = -\int_V [\Delta\vec{r}_+ \cdot L(\Delta\vec{r}_+) + \Delta\vec{r}_- \cdot L(\Delta\vec{r}_-) + 4\sigma\Omega\varrho\ (\Delta\vec{r}_+ \times \Delta\vec{r}_-)_3]\ dV,$$
$$[\alpha = \beta;\ \vec{\Omega} = \vec{\Omega}(0, 0, \Omega)]. \tag{5.7.49}$$

The subscript 3 denotes the component along $\vec{\Omega}$, i.e. along the x_3-axis. The expression of the eigenvalue σ is assumed up to the second order in Ω under the form

$$\sigma^2 = \sigma_0^2 + 2\Omega\sigma_1^2 + 2\Omega^2\sigma_2^2. \tag{5.7.50}$$

σ_0 denotes, as previously, the eigenvalue of the nonrotating polytrope.

Clement (1965) considers *nonaxisymmetric* perturbations up to the first order in Ω, $(\sigma_2 = 0)$, and in this case pressure and density distributions are unaffected by rotation, as seen for instance from the expansion (3.2.44) of the fundamental polytropic function $\Theta : P = P(r),\ \varrho = \varrho(r)$. The Lagrangian displacements $\Delta\vec{r}_+$ and $\Delta\vec{r}_-$ are expanded as

$$\Delta\vec{r}_+ = A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)} + \Omega\ \Delta\vec{r}_{1+};\quad \Delta\vec{r}_- = C\ \Delta\vec{r}_0^{(1)} + D\ \Delta\vec{r}_0^{(2)} + \Omega\ \Delta\vec{r}_{1-}. \tag{5.7.51}$$

A, B, C, D are variational parameters, and the real Lagrangian displacement $\Delta\vec{r}_\pm$ has been split into its zero order $\Delta\vec{r}_0^{(1)}, \Delta\vec{r}_0^{(2)}$ and first order $\Delta\vec{r}_{1\pm}$ parts, the azimuthal dependence of $\Delta\vec{r}_0^{(1)}$ and $\Delta\vec{r}_0^{(2)}$ entering through the factors $\cos(k\varphi)$ and $\sin(k\varphi)$, respectively: $\Delta\vec{r}_0^{(1)} = \Delta\vec{r}_0\cos(k\varphi),\ \Delta\vec{r}_0^{(2)} = \Delta\vec{r}_0\sin(k\varphi)$. Due to this representation the following relationships subsist, where integration can be taken over the spherical volume:

$$\int_V \varrho(r)\ [\Delta\vec{r}_0^{(1)}(r, \lambda, \varphi)]^2\ dV = \int_V \varrho(r)\ [\Delta\vec{r}_0(r, \lambda)]^2\cos^2(k\varphi)\ dV$$
$$= \int_V \varrho(r)\ [\Delta\vec{r}_0(r, \lambda)]^2\sin^2(k\varphi)\ dV = \int_V \varrho(r)\ [\Delta\vec{r}_0^{(2)}(r, \lambda, \varphi)]\ dV \qquad \text{and}$$
$$\int_V \varrho(r)\ \Delta\vec{r}_0^{(1)}(r, \lambda, \varphi)\ \Delta\vec{r}_0^{(2)}(r, \lambda, \varphi)\ dV = \int_V \varrho(r)\ [\Delta\vec{r}_0(r, \lambda)]^2\sin(k\varphi)\cos(k\varphi)\ dV = 0. \tag{5.7.52}$$

We insert Eqs. (5.7.50), (5.7.51) into the variational criterion (5.7.47), taking into account Eq. (5.7.52). The first order terms yield

$$2\int_V \varrho\big\{\sigma_1^2(A^2 + B^2 + C^2 + D^2)(\Delta\vec{r}_0^{(1)})^2 + \sigma_0^2\big[\Delta\vec{r}_{1+} \cdot (A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)})$$
$$+\Delta\vec{r}_{1-} \cdot (C\ \Delta\vec{r}_0^{(1)} + D\ \Delta\vec{r}_0^{(2)})\big]\big\}\ dV = -\int_V \big\{[A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)}] \cdot L(\Delta\vec{r}_{1+})$$
$$+[C\ \Delta\vec{r}_0^{(1)} + D\ \Delta\vec{r}_0^{(2)}] \cdot L(\Delta\vec{r}_{1-}) + \Delta\vec{r}_{1+} \cdot L[A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)}] + \Delta\vec{r}_{1-} \cdot L[C\ \Delta\vec{r}_0^{(1)} + D\ \Delta\vec{r}_0^{(2)}]$$
$$+4\sigma_0\varrho(AD - BC)[\Delta\vec{r}_0^{(1)} \times \Delta\vec{r}_0^{(2)}]_3\big\}\ dV. \tag{5.7.53}$$

From Eqs. (5.7.43) and (5.7.48) we observe that

$$\sigma_0^2\varrho\ \Delta\vec{r} = -L(\Delta\vec{r}), \tag{5.7.54}$$

and therefore

$$\int_V \big\{[A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)}] \cdot L(\Delta\vec{r}_{1+}) + L[A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)}] \cdot \Delta\vec{r}_{1+}\big\}\ dV$$
$$= -2\sigma_0^2 \int_V \varrho[A\ \Delta\vec{r}_0^{(1)} + B\ \Delta\vec{r}_0^{(2)}] \cdot \Delta\vec{r}_{1+}\ dV, \tag{5.7.55}$$

with a similar equation for C, D, and $\Delta\vec{r}_{1-}$. Thus, all terms containing $\Delta\vec{r}_{1+}, \Delta\vec{r}_{1-}$ vanish, and the sole surviving terms in Eq. (5.7.53) are

$$\sigma_1^2(A^2+B^2+C^2+D^2)\int_V \varrho(\Delta\vec{r}_0^{(1)})^2\, dV = 2\sigma_0(BC-AD)\int_V \varrho[\Delta\vec{r}_0^{(1)}\times\Delta\vec{r}_0^{(2)}]_3\, dV. \tag{5.7.56}$$

We require σ_1 to be stationary with respect to variations of the parameters A, B, C, D. Differentiating with respect to these coefficients, we obtain two equations in A and D

$$A\sigma_1^2\int_V \varrho[\Delta\vec{r}_0^{(1)}]^2\, dV + D\sigma_0\int_V \varrho[\Delta\vec{r}_0^{(1)}\times\Delta\vec{r}_0^{(2)}]_3\, dV = 0;$$
$$D\sigma_1^2\int_V \varrho[\Delta\vec{r}_0^{(1)}]^2\, dV + A\sigma_0\int_V \varrho[\Delta\vec{r}_0^{(1)}\times\Delta\vec{r}_0^{(2)}]_3\, dV = 0, \tag{5.7.57}$$

and two similar equations in B and C. The homogeneous system (5.7.57) has nontrivial solutions σ_0, σ_1 if its determinant A^2-D^2 is zero, or if $A=\pm D$ and $B=\pm C$; hence, the first order change of the eigenvalue due to rotation is

$$\sigma_1^2 = \pm\sigma_0\int_V \varrho[\Delta\vec{r}_0^{(1)}\times\Delta\vec{r}_0^{(2)}]_3\, dV\Big/\int_V \varrho[\Delta\vec{r}_0^{(1)}]^2\, dV. \tag{5.7.58}$$

The Cartesian x_3-component of the vector product from Eq. (5.7.58) is converted into spherical displacements with the transformation matrix [cf. Eq. (3.8.135); $\vec{Z}=\vec{Z}(Z_1,Z_2,Z_3)=\vec{Z}(Z_r,Z_\lambda,Z_\varphi)$]

$$\begin{bmatrix} Z_1\\ Z_2\\ Z_3\end{bmatrix} = \begin{bmatrix} \sin\lambda\cos\varphi & \cos\lambda\cos\varphi & -\sin\varphi\\ \sin\lambda\sin\varphi & \cos\lambda\sin\varphi & \cos\varphi\\ \cos\lambda & -\sin\lambda & 0\end{bmatrix}\begin{bmatrix} Z_r\\ Z_\lambda\\ Z_\varphi\end{bmatrix}. \tag{5.7.59}$$

The Cartesian and spherical components of the Lagrangian displacement are denoted by $\Delta\vec{r}_0^{(i)}(x_1,x_2,x_3) = \Delta\vec{r}_0^{(i)}\big[\Delta x_{01}^{(i)}(x_1,x_2,x_3), \Delta x_{02}^{(i)}(x_1,x_2,x_3), \Delta x_{03}^{(i)}(x_1,x_2,x_3)\big]$ and $\Delta\vec{r}_0^{(i)}(r,\lambda,\varphi) = \Delta\vec{r}_0^{(i)}\big[\Delta r_{0r}^{(i)}(r,\lambda,\varphi), \Delta r_{0\lambda}^{(i)}(r,\lambda,\varphi), \Delta r_{0\varphi}^{(i)}(r,\lambda,\varphi)\big]$, $(i=1,2)$, respectively. We have

$$Z_3 = [\Delta\vec{r}_0^{(1)}\times\Delta\vec{r}_0^{(2)}]_3 = \Delta x_{01}^{(1)}\ \Delta x_{02}^{(2)} - \Delta x_{02}^{(1)}\ \Delta x_{01}^{(2)} = Z_r\cos\lambda - Z_\lambda\sin\lambda$$
$$= [\Delta r_{0\lambda}^{(1)}\ \Delta r_{0\varphi}^{(2)} - \Delta r_{0\varphi}^{(1)}\ \Delta r_{0\lambda}^{(2)}]\cos\lambda + [\Delta r_{0r}^{(1)}\ \Delta r_{0\varphi}^{(2)} - \Delta r_{0\varphi}^{(1)}\ \Delta r_{0r}^{(2)}]\sin\lambda. \tag{5.7.60}$$

The azimuthal displacement $\Delta r_{0\varphi}^{(i)} = r\sin\lambda\ \Delta\varphi^{(i)}$ is given by Eq. (5.7.33), and contains k as a factor. Consequently, the vectorial product from Eqs. (5.7.58), (5.7.60) contains k as a factor, and for axisymmetric oscillations, if $k=0$, the first order correction σ_1^2 in Eq. (5.7.50) vanishes. We arrive at the important result that in this special case the perturbation of the eigenvalues is proportional to Ω^2, $(\sigma_1=0)$. For axisymmetric perturbations $(k=0)$ Clement (1984) takes the Lagrangian displacement under the form

$$\Delta\vec{r}(r,\lambda,t) = \Delta\vec{r}(r,\lambda)\exp(i\sigma t). \tag{5.7.61}$$

For nonrotating polytropes the azimuthal component $r\sin\lambda\ \Delta\varphi$ of $\Delta\vec{r}$ is zero if $k=0$ [see Eqs. (5.2.90), (5.7.33)]. But for rotating polytropes the azimuthal displacement is generally nonzero, even if $k=0$; this can be seen at once from Eq. (5.7.5), when $P, \delta P, \delta\Phi$ are independent of the azimuth angle φ :

$$\sigma r\sin\lambda\ \Delta\varphi = 2i\Omega(\sin\lambda\ \Delta r + r\cos\lambda\ \Delta\lambda) = 2i\Omega\ \Delta r_\ell. \tag{5.7.62}$$

Δr_ℓ is just the Lagrangian displacement along the distance $\ell = r\sin\lambda$ from the rotation axis, consistent with conservation of angular momentum.

For *axisymmetric* perturbations $k=0$ all quantities are expanded up to order Ω^2 :

$$\varrho = \varrho_{(0)} + \Omega^2\varrho_{(2)};\quad \delta\varrho = \delta\varrho_{(0)} + \Omega^2\ \delta\varrho_{(2)};\quad P = P_{(0)} + \Omega^2 P_{(2)};\quad \delta P = \delta P_{(0)} + \Omega^2\ \delta P_{(2)};$$
$$\Phi = \Phi_{(0)} + \Omega^2\Phi_{(2)};\quad \delta\Phi = \delta\Phi_{(0)} + \Omega^2\delta\Phi_{(2)};\quad \sigma^2 = \sigma_0^2 + 2\Omega^2\sigma_2^2;$$
$$\Delta\vec{r}_+ = A\ \Delta\vec{r}_0 + \Omega\ \Delta\vec{r}_{1+} + \Omega^2\ \Delta\vec{r}_{2+};\quad \Delta\vec{r}_- = B\ \Delta\vec{r}_0 + \Omega\ \Delta\vec{r}_{1-} + \Omega^2\ \Delta\vec{r}_{2-}. \tag{5.7.63}$$

The zero order approximation $\varrho_{(0)}, P_{(0)}$ should not be confused with the somewhat similar notation ϱ_0, P_0, used for central values of density and pressure.

It will be obvious from Eqs. (5.7.65), (5.7.66) that the variational parameters A and B are not coupled together, as they are in the first order theory by the Coriolis term from Eq. (5.7.56). Therefore, we may omit without loss of generality the $\pm$ index in the zero order displacement $\Delta\vec{r}_0$ from Eq. (5.7.63). Also $\Delta\vec{r}_{2\pm}$ will be ignored at the outset, since all terms containing $\Delta\vec{r}_{2\pm}$ drop out, just as $\Delta\vec{r}_{1\pm}$ has dropped out from the first order equation (5.7.53). Continuing with the separation procedure, we split the operator (5.7.48) into its zeroth and second order parts:

$$L(\Delta\vec{r}) = L_0(\Delta\vec{r}) + \Omega^2 L_2(\Delta\vec{r}) = -\nabla\delta P_{(0)} + [\delta\varrho_{(0)}/\varrho_{(0)}]\ \nabla P_{(0)} + \varrho_{(0)}\ \nabla\delta\Phi_{(0)} + \Omega^2\big\{ -\nabla\delta P_{(2)}$$
$$+[1/\varrho_{(0)}^2][\varrho_{(0)}\ \delta\varrho_{(2)} - \varrho_{(2)}\ \delta\varrho_{(0)}]\ \nabla P_{(0)} + [\delta\varrho_{(0)}/\varrho_{(0)}]\ \nabla P_{(2)} + \varrho_{(2)}\ \nabla\delta\Phi_{(0)} + \varrho_{(0)}\ \nabla\delta\Phi_{(2)}\big\}. \quad (5.7.64)$$

Inserting now Eqs. (5.7.63), (5.7.64) into the variational equation (5.7.49), dropping terms containing $\Delta\vec{r}_{2\pm}$, and collecting together terms with Ω^2, we obtain similarly to Eq. (5.7.56):

$$2\sigma_2^2(A^2+B^2)\int_V \varrho_{(0)}(\Delta\vec{r}_0)^2\ dV = -\int_V \big\{4\sigma_0\varrho_{(0)}[A(\Delta\vec{r}_0\times\Delta\vec{r}_{1-})_3 - B(\Delta\vec{r}_0\times\Delta\vec{r}_{1+})_3]$$
$$+(A^2+B^2)[\sigma_0^2\varrho_{(2)}(\Delta\vec{r}_0)^2 + \Delta\vec{r}_0\cdot L_2(\Delta\vec{r}_0)] + \sigma_0^2\varrho_{(0)}[(\Delta\vec{r}_{1+})^2 + (\Delta\vec{r}_{1-})^2]$$
$$+\Delta\vec{r}_{1+}\cdot L_0(\Delta\vec{r}_{1+}) + \Delta\vec{r}_{1-}\cdot L_0(\Delta\vec{r}_{1-})\big\}\ dV. \quad (5.7.65)$$

We again require σ_2 to be stationary with respect to the variational parameters A, B, and differentiate with respect to A and B :

$$2A\sigma_2^2\int_V \varrho_{(0)}(\Delta\vec{r}_0)^2\ dV = -\int_V \big\{2\sigma_0\varrho_{(0)}(\Delta\vec{r}_0\times\Delta\vec{r}_{1-})_3 + A[\sigma_0^2\varrho_{(2)}(\Delta\vec{r}_0)^2 + \Delta\vec{r}_0\cdot L_2(\Delta\vec{r}_0)]\big\}\ dV;$$
$$2B\sigma_2^2\int_V \varrho_{(0)}(\Delta\vec{r}_0)^2\ dV = -\int_V \big\{2\sigma_0\varrho_{(0)}(\Delta\vec{r}_0\times\Delta\vec{r}_{1+})_3 + B[\sigma_0^2\varrho_{(2)}(\Delta\vec{r}_0)^2 + \Delta\vec{r}_0\cdot L_2(\Delta\vec{r}_0)]\big\}\ dV. \quad (5.7.66)$$

Since the variational parameters A and B are not coupled, we can simplify considerably the equations, by taking without loss of generality $A=1,\ B=0,\ \Delta\vec{r}_{1+}=0$; the displacements from Eq. (5.7.63) become

$$\Delta\vec{r}_+ = \Delta\vec{r}_0;\quad \Delta\vec{r}_- = \Omega\ \Delta\vec{r}_{1-} = \Omega\ \Delta\vec{r}_1, \quad (5.7.67)$$

where we have suppressed the minus subscript from $\Delta\vec{r}_{1-}$. The variational equation (5.7.65) for σ_2 simplifies to (Clement 1965)

$$2\sigma_2^2\int_V \varrho_{(0)}(\Delta\vec{r}_0)^2\ dV = -\int_V \big[4\sigma_0\varrho_{(0)}(\Delta\vec{r}_0\times\Delta\vec{r}_1)_3 + \sigma_0^2\varrho_{(2)}(\Delta\vec{r}_0)^2 + \Delta\vec{r}_0\cdot L_2(\Delta\vec{r}_0)$$
$$+\sigma_0^2\varrho_{(0)}(\Delta\vec{r}_1)^2 + \Delta\vec{r}_1\cdot L_0(\Delta\vec{r}_1)\big]\ dV. \quad (5.7.68)$$

In our special case ($k=0$) the first order Lagrangian displacement $\Delta\vec{r}_1$ can be expressed in terms of the zero order displacement for the nonrotating polytrope. Indeed, if $k=0$, the azimuthal component $\Delta r_{0\varphi}$ of $\Delta\vec{r}_0$ is zero by virtue of Eq. (5.7.33), and the vector product from Eq. (5.7.68) becomes via Eq. (5.7.60) equal to ($\Delta\vec{r}_0^{(1)}\rightarrow\Delta\vec{r}_0;\ \Delta r_{0\varphi}^{(1)}\rightarrow\Delta r_{0\varphi}=0;\ \Delta\vec{r}_0^{(2)}\rightarrow\Delta\vec{r}_1$)

$$Z_3 = (\Delta\vec{r}_0\times\Delta\vec{r}_1)_3 = \Delta r_{1\varphi}(\Delta r_{0\lambda}\cos\lambda + \Delta r_{0r}\sin\lambda) = \Delta r_{1\varphi}\ \Delta r_{0\ell}. \quad (5.7.69)$$

Let us assume for the moment that Eq. (5.7.68) does not contain the linear term in $\Delta\vec{r}_1$ from the vector product, but only the square terms connected with $(\Delta\vec{r}_1)^2$ and $\Delta\vec{r}_1\cdot L_0(\Delta\vec{r}_1)$. The variation of this equation with respect to the components $\Delta r_{1r}, \Delta r_{1\lambda}, \Delta r_{1\varphi}$ of the Lagrangian displacement $\Delta\vec{r}_1$ would lead to three homogeneous equations in terms of these components. The requirement that they yield nontrivial solutions leads to the vanishing of the determinant of this system, i.e. to a characteristic equation for σ_0^2. But since σ_0^2 has already been determined from the zero order equations, we would have to require that $\Delta\vec{r}_1=0$. However, the presence of the term $(\Delta\vec{r}_0\times\Delta\vec{r}_1)_3$ changes this conclusion, because a system of nonhomogeneous equations would result. This term consists by virtue of Eq. (5.7.69) only of $\Delta r_{1\varphi}\ \Delta r_{0\ell}$, so we must conclude that $\Delta\vec{r}_1$ consists only of $\Delta r_{1\varphi}$, i.e.

$$|\Delta\vec{r}_1| = \Delta r_{1\varphi}(r,\lambda). \quad (5.7.70)$$

There also subsists $\Delta\vec{r}_1 \cdot L_0(\Delta\vec{r}_1) = 0$, because all relevant quantities are functions only of r and λ. The vector $\Delta\vec{r}_1$ possesses only the azimuthal component $\Delta r_{1\varphi}$. In this way Eq. (5.7.68) simplifies to

$$\int_V [2\sigma_2^2 \varrho_{(0)}(\Delta\vec{r}_0)^2 + \sigma_0^2 \varrho_{(2)}(\Delta\vec{r}_0)^2 + \Delta\vec{r}_0 \cdot L_2(\Delta\vec{r}_0)]\ dV = -\int_V \sigma_0 \varrho_{(0)}(4\Delta r_{0\ell}\ \Delta r_{1\varphi} + \sigma_0\ \Delta r_{1\varphi}^2)\ dV. \tag{5.7.71}$$

The right-hand side should be stationary with respect to $\Delta r_{1\varphi}$. If c_i denote the coefficients of some series expansion of $\Delta r_{1\varphi}$, the derivative of this integral with respect to c_i should be zero:

$$-\int_V 2\sigma_0 \varrho_{(0)}(\partial\Delta r_{1\varphi}/\partial c_i)(2\ \Delta r_{0\ell} + \sigma_0\ \Delta r_{1\varphi}) = 0. \tag{5.7.72}$$

Therefore

$$\Delta r_{1\varphi} = -2\ \Delta r_{0\ell}/\sigma_0, \tag{5.7.73}$$

and the equation for the second order correction σ_2 finally becomes

$$2\sigma_2^2 \int_V \varrho_{(0)}(\Delta\vec{r}_0)^2\ dV = \int_V \big[-\sigma_0^2 \varrho_{(2)}(\Delta\vec{r}_0)^2 - \Delta\vec{r}_0 \cdot L_2(\Delta\vec{r}_0) + 4\varrho_{(0)}(\Delta r_{0\ell})^2\big]\ dV. \tag{5.7.74}$$

The determination of σ_2 by this equation requires a unique specification of $\Delta\vec{r}_0$, which is not possible if in the nonrotating case two eigenvalues $\sigma_{0\alpha}$ and $\sigma_{0\beta}$, $(\alpha \neq \beta)$ are the same. In this case an accidental degeneracy occurs in the eigenvalues of two different eigenfunctions $\Delta\vec{r}_{0\alpha}$ and $\Delta\vec{r}_{0\beta}$. The Lagrangian displacement $\Delta\vec{r}_0$ will be some unknown linear combination of these two eigenfunctions

$$\Delta\vec{r}_0 = E\ \Delta\vec{r}_{0\alpha} + F\ \Delta\vec{r}_{0\beta};\quad \sigma_0 = \sigma_{0\alpha} = \sigma_{0\beta}, \qquad (E, F = \mathrm{const}). \tag{5.7.75}$$

The coefficients E, F, and the actual lifting of degeneracy by rotation will now be determined with the aid of the variational equation (5.7.74). It will be shown that rotation leads to two distinct modes $\Delta\vec{r}_\alpha$ and $\Delta\vec{r}_\beta$ with slightly different oscillation frequencies σ_α and σ_β, $[\sigma_\alpha = \sigma_\beta + O(\Omega^2)]$. We substitute Eq. (5.7.75) into Eq. (5.7.74):

$$\begin{aligned} &E^2 \int_V \big[2\sigma_2^2 \varrho_{(0)}(\Delta\vec{r}_{0\alpha})^2 + \sigma_0^2 \varrho_{(2)}(\Delta r_{0\alpha})^2 + \Delta\vec{r}_{0\alpha} \cdot L_2(\Delta\vec{r}_{0\alpha}) - 4\varrho_{(0)}(\Delta r_{0\ell\alpha})^2\big]\ dV + 2EF \\ &\times \int_V \big\{\sigma_0^2 \varrho_{(2)}\ \Delta\vec{r}_{0\alpha} \cdot \Delta\vec{r}_{0\beta} + (1/2)[\Delta\vec{r}_{0\alpha} \cdot L_2(\Delta\vec{r}_{0\beta}) + \Delta\vec{r}_{0\beta} \cdot L_2(\Delta\vec{r}_{0\alpha})] - 4\varrho_{(0)}\ \Delta r_{0\ell\alpha}\ \Delta r_{0\ell\beta}\big\}\ dV \\ &+F^2 \int_V \big[2\sigma_2^2 \varrho_{(0)}(\Delta\vec{r}_{0\beta})^2 + \sigma_0^2 \varrho_{(2)}(\Delta\vec{r}_{0\beta})^2 + \Delta\vec{r}_{0\beta} \cdot L_2(\Delta\vec{r}_{0\beta}) - 4\varrho_{(0)}(\Delta r_{0\ell\beta})^2\big]\ dV = 0. \end{aligned} \tag{5.7.76}$$

The integral of $\varrho_{(0)}\ \Delta\vec{r}_{0\alpha} \cdot \Delta\vec{r}_{0\beta}$ is zero via Eq. (5.7.46), and has accordingly been omitted in Eq. (5.7.76). Let us denote the coefficient of $2EF$ by $C_{\alpha\beta}$ and

$$C_\alpha = 2\int_V \varrho_{(0)}(\Delta\vec{r}_{0\alpha})^2\ dV;\quad C_\beta = 2\int_V \varrho_{(0)}(\Delta\vec{r}_{0\beta})^2\ dV. \tag{5.7.77}$$

The last three terms from the coefficients of E^2 and F^2 are just equal to the left-hand side of Eq. (5.7.74), if degeneracy of σ_0^2 would not occur. Let us denote this value of σ_2 by $\sigma_{2\alpha}$ and $\sigma_{2\beta}$, respectively. With these notations Eq. (5.7.76) becomes

$$(\sigma_2^2 - \sigma_{2\alpha}^2)C_\alpha E^2 + 2C_{\alpha\beta}EF + (\sigma_2^2 - \sigma_{2\beta}^2)C_\beta F^2 = 0. \tag{5.7.78}$$

We differentiate this equation with respect to the variational parameters E and F, requiring again stationarity of σ_2 with respect to variations of E and F :

$$(\sigma_2^2 - \sigma_{2\alpha}^2)C_\alpha E + C_{\alpha\beta}F = 0;\quad C_{\alpha\beta}E + (\sigma_2^2 - \sigma_{2\beta}^2)C_\beta F = 0. \tag{5.7.79}$$

The vanishing of the determinant of this homogeneous system leads to the characteristic equation for σ_2^2 :

$$\sigma_2^4 - (\sigma_{2\alpha}^2 + \sigma_{2\beta}^2)\sigma_2^2 + (\sigma_{2\alpha}^2\sigma_{2\beta}^2 - C_{\alpha\beta}^2/C_\alpha C_\beta) = 0. \tag{5.7.80}$$

The two roots $\sigma_{2,1}^2, \sigma_{2,2}^2$ will lift the degeneracy of the two characteristic values $\sigma_{0\alpha} = \sigma_{0\beta}$ that occurs in absence of rotation. If $C_{\alpha\beta} = 0$, we have simply $\sigma_{2,1}^2 = \sigma_{2\alpha}^2$ and $\sigma_{2,2}^2 = \sigma_{2\beta}^2$.

Chandrasekhar's (1933a) first order theory for rotating polytropes can now be applied to Eq. (5.7.74) for the evaluation of the eigenvalues of the fundamental radial r-mode $(j, k = 0)$, and of the Kelvin f-mode $(j = 2,\ k = 0)$. Recall that our rotating and nonrotating polytropes possess the same central density ϱ_0, and the same polytropic constant K, but not the same mass and volume (cf. Sec. 3.2); it is by no means obvious that the characteristic frequencies of oscillations in a rotating and nonrotating polytrope with the same ϱ_0, K, and n are strictly comparable (Chandrasekhar and Lebovitz 1962d, §IX, Chandrasekhar and Lebovitz 1968). In the present approximation, when all terms of order Ω^4 are neglected, it will suffice to extend the range of integration only over the volume V_0 of the undistorted polytrope, provided the integrand vanishes over the outer boundary. To see this, we write a certain physical quantity $F(r, \lambda, \varphi)$ under the form (5.7.63): $F(r, \lambda, \varphi) = F_0(r, \lambda, \varphi) + \Omega^2 F_2(r, \lambda, \varphi)$. By virtue of Eq. (3.2.45) the outer boundary is equal to $R_1(\lambda) = r_1 + \Omega^2 f(\lambda)$, where r_1 is the radius of the undistorted Lane-Emden sphere. We have

$$\int_V F(r, \lambda, \varphi)\ dV = \int_{V_0} F(r, \lambda, \varphi)\ dV + \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\lambda\ d\lambda \int_{r_1}^{R_1(\lambda)} F(r, \lambda, \varphi)\ r^2\ dr$$
$$= \int_{V_0} F(r, \lambda, \varphi)\ dV + \Omega^2 r_1^2 \int_0^{2\pi} d\varphi \int_0^{\pi} f(\lambda)\ F_0(r_1, \lambda, \varphi)\ \sin\lambda\ d\lambda + O(\Omega^4)$$
$$= \int_{V_0} F(r, \lambda, \varphi)\ dV + \Omega^2 \int_S f(\lambda)\ F_0(r_1, \lambda, \varphi)\ dS + O(\Omega^4). \qquad (5.7.81)$$

Therefore, if $F_0(r_1, \lambda, \varphi)$ vanishes over the surface S (as $\varrho_{(0)}$ and $P_{(0)}$ do if $0 < n < 5$), the difference between the integrals over V and V_0 is of order Ω^4 or smaller, and all integrations can be effected over the spherical volume V_0 of the undistorted polytrope.

From an inspection of Eqs. (5.5.27) or (5.7.33) results that the spherical components of the Lagrangian displacement $\Delta\vec{r}(r, \lambda, \varphi)$ can be written as [Chandrasekhar 1964a, Eq. (31)]

$$\Delta r = [u(r)/r^2]\ Y_j^k(\lambda, \varphi); \quad r\ \Delta\lambda = [1/j(j+1)r](dw/dr)\ \partial Y_j^k(\lambda, \varphi)/\partial\lambda;$$
$$r \sin\lambda\ \Delta\varphi = [1/j(j+1)r\sin\lambda](dw/dr)\ \partial Y_j^k(\lambda, \varphi)/\partial\varphi. \qquad (5.7.82)$$

The two unknown radial functions u and w are often assumed under the trial form (e.g. Clement 1965, 1967, Robe and Brandt 1966, Sood and Trehan 1972a, b, Miketinac 1974)

$$u(r) = ar^3 + br^5; \quad w(r) = ar^3 + cr^5, \qquad (a, b, c = \mathrm{const}). \qquad (5.7.83)$$

The constants a, b, c are variational parameters, to be determined from the variational principle. With the aid of Eqs. (3.1.41), (5.1.27), (5.7.82) the divergence of the Lagrangian displacement becomes $\nabla\cdot\Delta\vec{r} = (Y_j^k/r^2)\ d(u-w)/dr$. Since this divergence has to vanish at the origin, the variational parameter a must be the same in both functions u and w (Chandrasekhar and Lebovitz 1964). As pointed out by Simon (1969), the rotational corrections σ_2^2 to the eigenvalue of the fundamental radial r-mode ($j = 0,\ \Gamma_1 = 5/3$) for the $n = 3$ polytrope differ grossly among various authors, while the eigenvalues σ_0^2 for the nonrotating polytropic sphere are in satisfactory agreement, excepting for the virial results of Chandrasekhar and Lebovitz (1962d). The calculated eigenvalues are $\omega^2 = \sigma^2/4\pi G\varrho_0 = (\sigma_0^2 + 2\Omega^2\sigma_2^2)/4\pi G\varrho_0 = \omega_0^2 + \beta\sigma_2^2 = 0.057 - 1.92\beta$ (Cowling and Newing 1949), $0.082 - 0.737\beta$ (Chandrasekhar and Lebovitz 1962d), $0.060 - 1.43\beta$ (Clement 1965, and Table 5.7.1), $0.057 - 3.55\beta$ (Occhionero 1967b, 1968), $0.057 - 1.93\beta$ (Chandrasekhar and Lebovitz 1968), $0.057 - 1.93\beta$ (Simon 1969), $0.057 - 1.89\beta$ (Saio 1981).

For the polytrope $n = 3$ Saio [1981, Eq. (56)] obtains for the f-mode with $j = 2$ the value $\omega^2 = 0.0503 - 0.185\beta$, whereas the value of Clement (1965) from Table 5.7.1 is $0.0493 - 0.127\beta$.

The value Γ_{1d} of Γ_1 for which accidental degeneracy appears between the radial r-mode and the Kelvin f-mode is in a first approximation equal to $\Gamma_{1d} = 8/5 = 1.6$ (cf. Sec. 5.8.1, Chandrasekhar and Lebovitz 1962b, c, d), and has been found by Chandrasekhar and Lebovitz (1964) in a second approximation to be close to 1.6 if $n = 3.25:\ \Gamma_{1d} = 1.57$. In fact, from the more exact calculations by Hurley et al. (1966) it emerges that $\Gamma_{1d} = 1.96$ if $n = 3.25$. Table 5.7.2 shows the splitting of eigenfrequencies occurring at accidental degeneracy of σ_0^2 for uniformly and differentially rotating polytropes. The rotation parameter $\beta_1 = \Omega^2(\Xi_1)/2\pi G\varrho_0$ refers to the equatorial angular velocity at the surface $\xi = \Xi_1(\lambda)$, the angular velocity of differential rotation being given by Eq. (3.5.8).

Table 5.7.1 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0 = (\sigma_0^2 + 2\Omega^2\sigma_2^2)/4\pi G\varrho_0 = \omega_0^2 + \beta\sigma_2^2$, $(\omega_0^2 = \sigma_0^2/4\pi G\varrho_0;\ \beta = \Omega^2/2\pi G\varrho_0)$ for the fundamental quasiradial $r(j=0)$-mode and the fundamental (Kelvin) $f(j=2)$-mode in the presence of slow uniform rotation if $\Gamma_1 = 5/3$ (Clement 1965). Note however, that for faster rotation rates β the quasiradial eigenvalue $r(j=0)$ increases with increasing angular speed if $n=1$ (Clement 1984, Table 7A), and remains nearly constant if $n=2$ (Clement 1986, Fig. 1).

n	$r(j=0)$	$f(j=2)$
1	$0.192 - 0.0681\beta$	$0.152 + 0.564\beta$
1.5	$0.151 - 0.162\beta$	$0.118 + 0.495\beta$
2	$0.117 - 0.336\beta$	$0.0911 + 0.388\beta$
3	$0.0602 - 1.43\beta$	$0.0493 - 0.127\beta$
3.5	$0.0346 - 2.89\beta$	$0.0312 - 0.931\beta$

Table 5.7.2 Adiabatic index Γ_{1d} at which accidental degeneracy occurs between the fundamental quasiradial $r(j=0)$-mode and the fundamental (Kelvin) $f(j=2)$-mode (cf. Table 5.11.4, Hurley et al. 1966). The last two columns show the two degenerate split eigenfrequencies $\omega^2 = \sigma_0^2/4\pi G\varrho_0 + \beta_1\sigma_{2,1}^2$ and $\sigma_0^2/4\pi G\varrho_0 + \beta_1\sigma_{2,2}^2$, $(\beta_1 = \Omega^2(\Xi_1)/2\pi G\varrho_0)$ from Eq. (5.7.80) for uniformly and differentially rotating polytropes (Clement 1965, 1967).

n	Γ_{1d}	ω_{unif}^2	ω_{diff}^2
1	1.596	$0.152 + 0.653\beta_1$	–
		$0.152 - 0.0725\beta_1$	–
1.5	1.592	$0.117 + 0.593\beta_1$	–
		$0.117 - 0.148\beta_1$	–
2	1.586	$0.0901 + 0.506\beta_1$	$0.0901 + 4.41\beta_1$
		$0.0901 - 0.305\beta_1$	$0.0901 - 0.470\beta_1$
2.5	1.579	$0.0664 + 0.383\beta_1$	$0.0664 + 7.89\beta_1$
		$0.0664 - 0.641\beta_1$	$0.0664 - 1.42\beta_1$
3	1.581	$0.0460 + 0.185\beta_1$	$0.0460 + 17.2\beta_1$
		$0.0460 - 1.66\beta_1$	$0.0460 - 5.51\beta_1$
3.25	1.961	$0.0596 + 0.201\beta_1$	$0.0596 + 60.3\beta_1$
		$0.0596 - 6.30\beta_1$	$0.0596 - 20.1\beta_1$

Note, that the eigenvalues from Table 5.7.1 are obtained in absence of degeneracy. At degeneracy $(\Gamma_1 = \Gamma_{1d})$ rotation mixes the two basic eigenfunctions $\Delta\vec{r}_{0\alpha}$, $\Delta\vec{r}_{0\beta}$ together with the corresponding second order corrections of the eigenvalues $\sigma_{2\alpha}^2, \sigma_{2\beta}^2$ into the two new combinations from Eqs. (5.7.75) and (5.7.80), respectively.

It is also observed from Table 5.7.2 that the relative splitting of the eigenvalues is about 10 times larger in the case of differential rotation in comparison to uniform rotation. The two slightly different eigenfrequencies ω_{diff}^2 could provide an explanation for the so-called beat phenomenon in β Canis Majoris stars (Chandrasekhar and Lebovitz 1962e, Clement 1965, 1967). This hypothesis has the advantage that the nonradial, second harmonic $f(j=2)$-mode does not require a separate mechanism for its excitation, but is naturally coupled by rotation to the purely radial $r(j=0)$-mode; this coupling produces two distinct modes of oscillation, both of which are nonradial. β CMa has a beat period of 49 days which can be interpreted as resulting from the interference of two slightly different sinusoidal oscillations with periods of 6^h and $6^h\ 2^m$ (Ledoux and Walraven 1958, p. 398).

Clement (1984, 1986) calculates axisymmetric normal modes of rigidly and rapidly rotating polytropes up to the break-up rotational velocity if $n = 1, 2, 3$. The Lagrangian *axisymmetric* displacement is given by $\Delta\vec{r}(r,\lambda,t) = \Delta\vec{r}(r,\lambda)\exp(i\sigma t)$, and the equation of motion (5.7.41) can be written under the form

$$\sigma^2\ \Delta\vec{r} - (1/\varrho)\ \nabla\delta P + (\delta\varrho/\varrho^2)\ \nabla P + \nabla\delta\Phi + 2i\sigma\Omega(\Delta x_2\ \vec{e}_1 - \Delta x_1\ \vec{e}_2) = 0,$$
$$[\Delta\vec{r} = \Delta\vec{r}(\Delta x_1, \Delta x_2, \Delta x_3);\ \vec{\Omega} = \vec{\Omega}(0,0,\Omega)], \tag{5.7.84}$$

where $\vec{e}_i$ are the unit vectors along the coordinate axes x_i, $(i = 1,2,3)$. We take the scalar product of Eq. (5.7.84) with $\Delta\vec{r}$, and integrate over the volume of the polytrope, by observing that the Coriolis term vanishes:

$$\int_V \left[\sigma^2\varrho\ (\Delta\vec{r})^2 - \Delta\vec{r}\cdot\nabla\delta P + (\delta\varrho/\varrho)\ \Delta\vec{r}\cdot\nabla P + \varrho\ \Delta\vec{r}\cdot\nabla\delta\Phi\right]\ dV = 0. \tag{5.7.85}$$

We transform the last three terms exactly as in Eqs. (5.7.45)-(5.7.47), turn to spherical coordinates, and eliminate $\sigma\Delta r_\varphi = 2i\Omega\ \Delta r_\ell$ via Eq. (5.7.62):

$$\int_V \{\sigma^2\varrho[(\Delta r)^2 + (r\ \Delta\lambda)^2] - 4\Omega^2\varrho(\Delta r_\ell)^2 - 2(\Delta\vec r\cdot\nabla P)\ \nabla\cdot\Delta\vec r - \Gamma_1 P(\nabla\cdot\Delta\vec r)^2$$
$$-(1/\varrho)(\Delta\vec r\cdot\nabla\varrho)\ \Delta\vec r\cdot\nabla P + \delta\varrho\ \delta\Phi\}\ dV = \int_V \{\sigma^2\varrho[(\Delta r)^2 + (r\ \Delta\lambda)^2] - 4\Omega^2\varrho(\Delta r_\ell)^2$$
$$-2\varrho(\Delta\vec r\cdot\nabla\Phi_{tot})\ \nabla\cdot\Delta\vec r - \Gamma_1 P(\nabla\cdot\Delta\vec r)^2 - (\Delta\vec r\cdot\nabla\Phi_{tot})^2\ d\varrho/d\Phi_{tot} - \delta\Phi[\varrho\ \nabla\cdot\Delta\vec r$$
$$+(\Delta\vec r\cdot\nabla\Phi_{tot})\ d\varrho/d\Phi_{tot}]\}\ dV = 0. \qquad (5.7.86)$$

To obtain the second integral, we have substituted for $\delta\varrho$ from Eq. (5.2.28), for $\nabla P = \varrho\ \nabla\Phi_{tot}$ from Eq. (5.7.6), and for $\nabla\varrho = (d\varrho/d\Phi_{tot})\ \nabla\Phi_{tot}$ from $\varrho = \varrho(\Phi_{tot})$. Clement (1986) incorporates into the variational principle (5.7.86) a trial Lagrangian displacement, written as a sum of p and g-type basis vectors in the following way (cf. Sobouti 1977a, 1980):

$$\Delta\vec r(r,\lambda) = \sum_{i=0}^{I}\left(\sum_{j=0}^{i+1} a_{ij}\ \Delta\vec r_{p,ij} + \sum_{j=1}^{i+1} b_{ij}\ \Delta\vec r_{g,ij}\right), \qquad (a_{ij}, b_{ij} = \mathrm{const}). \qquad (5.7.87)$$

The basis vectors are forced to lie in the meridian plane (φ = const), the Δr_φ-component being eliminated from the beginning by virtue of Eq. (5.7.62):

$$\Delta\vec r_{p,ij} = \Delta\vec r_{p,ij}\{[(2i+2)P_{2j},\ dP_{2j}/d\lambda,\ 0]r^{2i+1}\};$$
$$\Delta\vec r_{g,ij} = \Delta\vec r_{g,ij}\{[c_1P_{2j} + c_2\ dP_{2j}/d\lambda,\ c_3P_{2j} + c_4\ dP_{2j}/d\lambda,\ 0]r^{2i}\}. \qquad (5.7.88)$$

The coefficients $c_k,\ (k = 1,2,3,4)$ depend on the unperturbed pressure, density, and local gravity. Since rotation mixes the angular dependence described by various Legendre polynomials $P_j(\cos\lambda)$, the latitudinal dependence of a particular eigenfunction $\Delta\vec r$ is no longer described by a single $P_j(\cos\lambda)$, although the eigenfunctions – being represented in Eqs. (5.7.87) and (5.7.88) as a linear combination of r^i and $P_j(\cos\lambda)$ – still belong to a definite order j.

As shown by Eqs. (5.7.3)-(5.7.5), (5.7.62), no purely radial modes exist in the presence of rotation, so our terminology "quasiradial modes" refers in fact to oscillations that would be radial in absence of rotation. Quite generally – including also nonaxisymmetric modes – Clement (1984) assigns to each mode three indices: The first is the radial order ("quantum number") of the mode, the second and the third, added in parentheses, define the latitudinal order j and the azimuthal order k from $P_j^k(\cos\lambda)\exp(ik\varphi)$. The order of the radial (quasiradial) modes starts with 0, i.e. $r_0(0,0),\ r_1(0,0),\ r_2(0,0), ...$ (cf. Fig. 5.2.1, Table 5.3.1); the latitudinal and azimuthal indices – which are always zero – are omitted: $r_0, r_1, r_2, ...$ correspond to the eigenvalues $\sigma_0, \sigma_1, \sigma_2, ...$, and to the dimensionless eigenfrequencies $\omega_0, \omega_1, \omega_2, ...$ The f-modes have only one radial order for any combination of the angular indices j and k, and will therefore be denoted by $f(j,k)$. The radial order of the p and g-modes starts with 1, i.e. $p_1(j,k),\ p_2(j,k),\ p_3(j,k), ...,$ and $g_1(j,k),\ g_2(j,k),\ g_3(j,k), ...,$ respectively. For axisymmetric modes we have always $k = 0$, and this index will generally be omitted, in order to shorten the notations.

As a general trend, the axisymmetric r, p, and f-modes are "destabilized" by rapid rotation in the sense that their eigenvalues σ^2 are decreased if $1 \le n \le 3$ and $\Gamma_1 = 5/3$, excepting for some low order r and f-modes, which are rather insensitive to rotation, especially if $n = 2$ (Clement 1986; cf. also comments to Fig. 5.8.2). On the other hand, the axisymmetric g-modes are generally stabilized due to rotation (cf. Tassoul 1978, §14.5) in the sense that their eigenvalues σ^2 are increased if $1 \le n \le 3$ and $\Gamma_1 = 5/3$, excepting for some low order $g_1(j)$-modes, which decrease for fast rotation if $n = 3$, behaving like p-modes (Fig. 5.7.1). In particular, the $g_1(2)$-mode (that is unstable ($\sigma^2 < 0$) in absence of rotation if $n = 1,\ \Gamma_1 = 5/3$), acquires stability ($\sigma^2 > 0$) by a small rotation (Clement 1984). The influence of fast rotation on the eigenvalues is quite pronounced, attaining over 30% for some modes.

Another salient feature of the computations of Clement (1984, 1986) is the absence of degeneracies. Even though many curves in Fig. 5.7.1 cross, there are only the *mode characteristics* which intersect, and not the eigenfrequencies ω. This avoided degeneracies for two modes having nearly the same eigenvalues is clearly illustrated on the larger scale of Fig. 5.7.2 for the near degeneracy of the $p_1(2)$ and $f(8)$-mode. The eigenvalues never cross, and only a mode mixing takes place, with exchange of mode characteristics belonging to another sequence of eigenvalues. This behaviour is just the same as the lifting of degeneracy due to slow rotation, discussed previously in this section (Table 5.7.2).

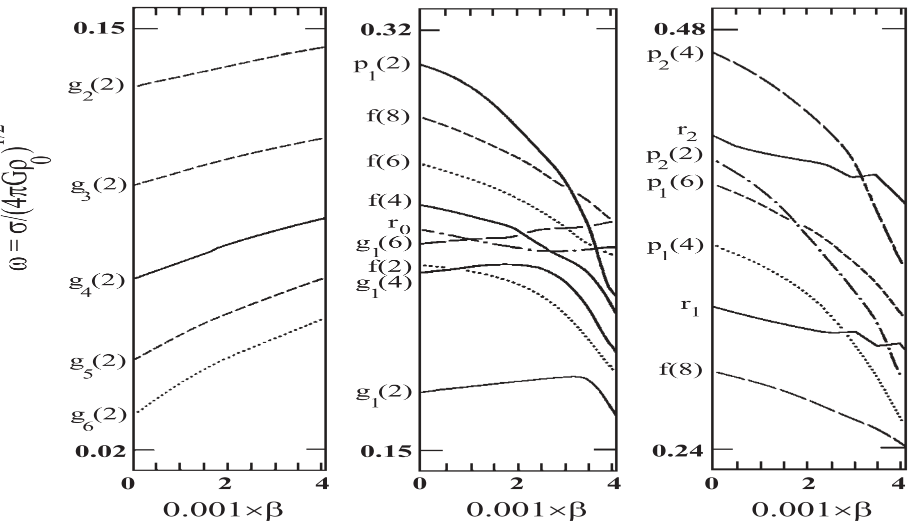

Fig. 5.7.1 Normalized eigenfrequencies $\omega = \sigma/(4\pi G\varrho_0)^{1/2}$ of a rotating $n = 3$ polytrope as a function of the rotation parameter $\beta = \Omega^2/2\pi G\varrho_0$, $(\Gamma_1 = 5/3)$, (cf. Tables 5.3.1, 5.5.2, 5.5.3 if $\beta, \Omega = 0$), (Clement 1986).

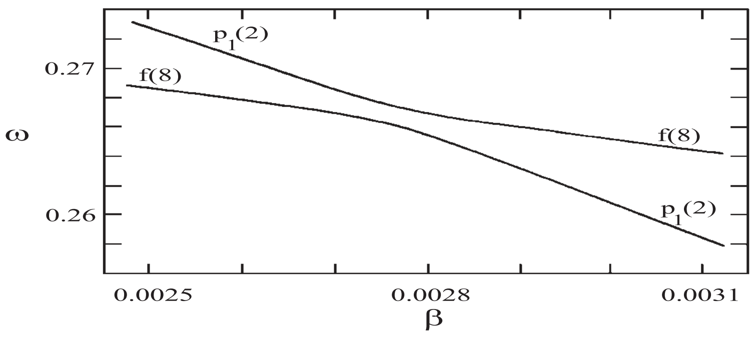

Fig. 5.7.2 Same as Fig. 5.7.1 in a large-scale view, showing the avoided degeneracy between the $f(8)$ and $p_1(2)$ eigenfrequencies. The two sequences of eigenfrequencies do not intersect, only the mode characteristics propagate along the other sequence (Clement 1986).

Note, that for axisymmetric oscillations one of the Solberg-Høiland conditions for dynamical stability of a differentially rotating configuration becomes equal to (e.g. Tassoul 1978, Sec. 7.3; Robe 1979)

$$(1/\ell^3)\ \partial[\ell^4\Omega^2(\ell,z)]/\partial\ell + (\vec{A}\cdot\nabla P)/\varrho > 0, \qquad (\delta P, \delta\Phi \equiv 0). \tag{5.7.89}$$

$\vec{A}$ is the Schwarzschild discriminant from Eq. (5.2.84). Since for barotropes (including polytropes) we have $\Omega = \Omega(\ell)$ by virtue of Eq. (3.1.11), this equation turns into

$$2[\Omega(\ell)/\ell]\ d[\ell^2\Omega(\ell)]/d\ell + (\vec{A}\cdot\nabla P)/\varrho = 4\Omega^2(\ell) + 2\ell\ \Omega(\ell)\ d\Omega(\ell)/d\ell + (\vec{A}\cdot\nabla P)/\varrho > 0. \tag{5.7.90}$$

$\vec{A} = 0$ subsists on isentropic surfaces $S =$ const, $(\Gamma_1 = 1 + 1/n)$. And in convectively stable regions we have $\vec{A}\cdot\nabla P > 0$. In both cases the stability condition (5.7.90) turns into Eq. (3.5.1).

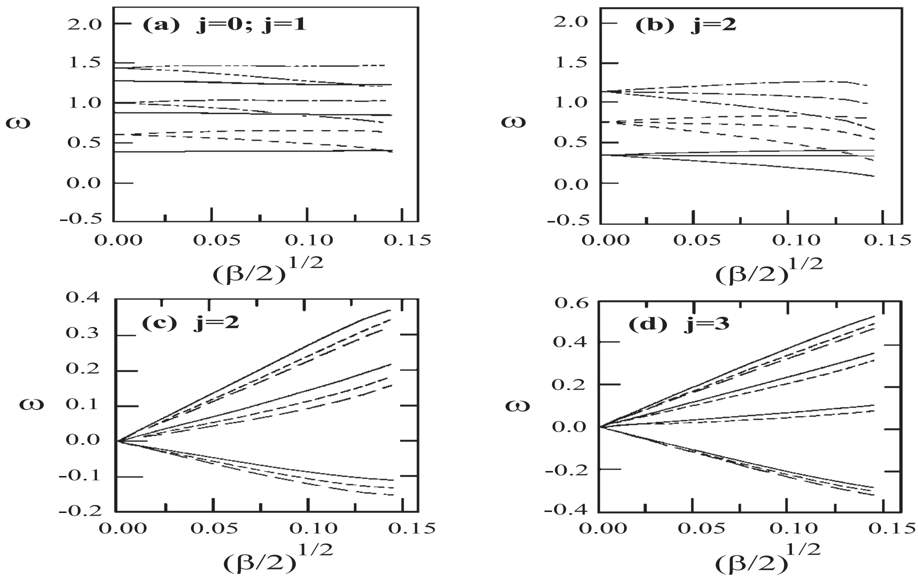

Fig. 5.7.3 Normalized eigenfrequencies $\omega = \sigma/(4\pi G\varrho_0)^{1/2}$ of nonaxisymmetric modes for the rotating isentropic polytrope ($\beta = \Omega^2/2\pi G\varrho_0,\ n = 1.5,\ \Gamma_1 = 5/3$). (a) Solid lines show with increasing ordinate the eigenvalues of the quasiradial r_0, r_1, r_2-modes ($j = 0$), respectively. Dashed lines show with increasing ordinate the $p_1(1,1),\ p_1(1,-1)$-modes, lines with one short dash are for the $p_2(1,1),\ p_2(1,-1)$-modes, and lines with three short dashes are for the $p_3(1,1),\ p_3(1,-1)$-modes. (b) Solid curves are for the fundamental $f(2,k)$-modes, dashed curves are for $p_1(2,k)$, and curves with a short dash are for $p_2(2,k)$, the inexact lines for $p_3(2,k)$ being omitted. Lowest curve in each group is for $k = 2$, middle curve for $k = 0$, and upper curve for $k = -2$. (c) Gravity g-modes: $g_1(2,k)$ – solid curves, $g_2(2,k)$ – short-dashed curves, $g_3(2,k)$ – long-dashed curves if $k = 2, 0, -2$, respectively. (d) Same as in (c) but for $j = 3$, the values of k being now $3, 1, -1, -3$ for the lowest group to the highest group, respectively (Managan 1986).

In Eqs. (5.7.14)-(5.7.27) we have briefly discussed the nonaxisymmetric oscillations ($k \neq 0$) of the homogeneous polytrope $n = 0$. So far, nonaxisymmetric modes in rotating polytropes with indices $n \neq 0$ seem to have been examined to some extent only by Managan (1986) for isentropic polytropes ($\Gamma_1 = 1+1/n$) of index $n = 1$ and 1.5 within the relativistic context of gravitational radiation reaction. Fig. 5.7.3 shows the splitting due to rotation of the nonaxisymmetric modes, calculated in a nonrotating inertial frame with the Lagrangian displacement $\Delta\vec{r}(\vec{r},t) = f(\ell,z)\exp[i(\sigma t + k\varphi)]$, where ℓ, φ, z are cylindrical coordinates. Since the basis functions adopted by Managan (1986) are equatorially symmetric, it is obvious from the representation (3.1.39) of associated Legendre polynomials that only modes with even values of $j + k$, ($|k| \leq j$) will result, i.e. if $j = 0$ the sole choice is $k = 0$; if $j = 1$ we have $k = \pm 1$; if $j = 2$ there is $k = 0, \pm 2$; if $j = 3$ we have $k = \pm 1, \pm 3$, etc. As noted in Sec. 5.5.2 (Table 5.5.2), all eigenfrequencies of g-modes are zero in the isentropic case $\Gamma_1 = 5/3,\ n = 1.5$ for the nonrotating polytropic sphere. For moderate rotation rates the changes of nonaxisymmetric modes are proportional to $k\Omega$ [Eq. (5.7.34)], and the influence of second order terms becomes noticeable only near the critical angular velocity of break-up Ω_c; treating rotation with linear perturbation theory would be a good approximation, except near the end of the rotation sequence. Modes which are radial in absence of rotation (r-modes) are nearly insensitive to rotation. For the same values of k Managan (1986) found also several avoided degeneracies, as in the axisymmetric case studied by Clement (1984, 1986).

Further results on the eigenfrequencies of rotating polytropes will be presented in Sec. 5.8.2, within the context of the virial method.

5.7.4 Rotational and Tidal Effects

An important problem was formulated by Roche during the years 1847-1850 (Chandrasekhar 1969), concerning the critical minimum distance D_c between two masses before disruption of the primary mass M due to tidal action from the secondary mass M'. Initially, this "double star problem" was formulated for incompressible, constant density configurations $n = 0$. Lai et al. (1993) have extended this approach to isentropic, compressible ellipsoidal polytropes $n = 1/(\Gamma_1 - 1) \neq 0$ (cf. Sec. 3.8.5). Their energy variational method is suited best for polytropic indices $0 \leq n \lesssim 2$, such as neutron stars, white or brown dwarfs. The evolution of close binaries under the combined influence of viscosity and gravitational radiation reaction can also be followed up (e.g. Secs. 5.8.3, 5.8.4, Lai et al. 1994b, Lai and Shapiro 1995).

The equidensity, or equivalently the isobaric surfaces are assumed in the method of Lai et al. (1993) to be self-similar ellipsoids – as in the incompressible approach. The total mass of the ellipsoidal polytrope is equal to the mass inside a polytropic sphere of radius $r_s = (a_1a_2a_3)^{1/3}$, where a_1, a_2, a_3 are the three principal axes of the ellipsoidal polytrope. Note, that if $n \neq 0$, the surface radius r_s is *not equal* to the radius r_1 of a polytropic sphere with the same polytropic index n and polytropic constant K (Lai et al. 1994a, b). We may distinguish several types of binary models, supplemented by various mixed forms (Lai et al. 1993):

(i) The classical Roche problem, when an ellipsoidal polytrope M is considered instead of the incompressible primary, the secondary M' being simply a point mass. If not stated explicitly otherwise, the masses $M,\ M'$ are assumed to rotate synchronously in a circular orbit of semimajor axis D. The spin and orbital angular velocities are equal to Ω, obeying Kepler's modified law (3.1.93): The additional factor ε amounts up to 0.2 if $n = 0$, correcting Chandrasekhar's (1969) result (Lai et al. 1993, Figs. 10, 17; Lai et al. 1994a, b). If $n = 1.5$, the maximum correction shrinks to $\varepsilon = 0.06$ (Rasio and Shapiro 1995, Fig. 3).

(ii) As a special case of the Roche problem Chandrasekhar (1969) considers the somewhat artificial Jeans problem – a nonrotating tidal configuration – when the incompressible primary M is subject to the tidal action of a point mass secondary M' [see Sec. 3.3, Eq. (5.7.96)].

(iii) The classical Darwin problem of two synchronously rotating, homogeneous congruent ellipsoids ($q = M'/M = 1$) has been extended by Lai et al. (1993) to ellipsoidal polytropes.

(iv) The same has been done by Lai et al. (1993) also with the Riemann problem (Darwin-Riemann problem if $M = M'$, Roche-Riemann problem if M' is a point mass), allowing for nonsynchronous rotation of the ellipsoidal polytropes: In addition to rigid background rotation with orbital angular velocity $\vec{\Omega}$ there exist also internal fluid motions of velocity $\vec{v}$ with constant vorticity $\nabla \times \vec{v}$ parallel to $\vec{\Omega}$, $(\vec{\Omega} = \Omega\vec{e}_z \propto |\nabla \times \vec{v}|\vec{e}_z;\ v_z = 0)$. The shape of the binary components is stationary as seen from a frame rotating with angular velocity Ω. The fluid velocity as seen from an inertial frame is

$$\vec{v}_i = \vec{v} + \vec{\Omega} \times \vec{r}, \qquad (\Omega = \mathrm{const}). \tag{5.7.91}$$

The velocity circulation along the equator C_{equ} of the ellipsoid is by the Stokes theorem [Eq. (B.47)]:

$$C = \oint_{C_{equ}} \vec{v}_i \cdot d\vec{C}_{equ} = \int_{S_{equ}} (\nabla \times \vec{v}_i) \cdot d\vec{S}_{equ} = \pi a_1 a_2 (|\nabla \times \vec{v}| + 2\Omega),$$
$$(\vec{\Omega} = \Omega\vec{e}_z \propto |\nabla \times \vec{v}|\vec{e}_z;\ \vec{S}_{equ} = \pi a_1 a_2 \vec{e}_z). \tag{5.7.92}$$

In the case of a single homogeneous body the condition $\nabla \times \vec{v} = 0$ leads to the Maclaurin-Jacobi sequence, and $\vec{\Omega} = 0$ to the Dedekind sequence (see Sec. 3.2). In the special case $\nabla \times \vec{v}_i = 0$ or $|\nabla \times \vec{v}| = -2\Omega$, $(C = 0)$ we get irrotational sequences, which could be of interest for the late stages of neutron binaries, when the viscosity of neutron matter is too low to synchronize spin and orbital angular velocity: The binary orbit shrinks mainly due to the emission of gravitational waves (Sec. 5.8.4). Since for inviscid fluids irrotational motion is conserved under the action of potential forces like gravitational radiation reaction (Landau and Lifshitz 1959), the assumption $\nabla \times \vec{v}_i = 0$, $(\vec{v}_i = \nabla f)$ is a realistic one for a close neutron star binary (Uryū and Eriguchi 1996, 1998; Taniguchi and Nakamura 2000a, b; Uryū et al. 2000; Taniguchi et al. 2001). Note, that we are speaking about the last 15 minutes (about 16000 orbital periods) of inspiral of a neutron star binary (Duez et al. 2001).

In the ellipsoidal energy variational method of Lai et al. (1993) the *sufficient* condition for the onset of secular and dynamical instability along a binary sequence is determined by an analogue of the static (turning point) method, to be sketched in Sec. 5.12.1. For secular instability to arise there must be present a dissipative mechanism such as ordinary fluid viscosity [conserving the angular momentum J of the system in absence of external torques, but not the velocity circulation (5.7.92)], or gravitational radiation [conserving C but not J (cf. Miller 1974, and Sec. 5.8.4)]. The instability points are presumed to be dynamical, whenever a turning point occurs on a hydrostatic equilibrium sequence with J and C both held fixed. The binary then evolves on a dynamical time scale due to orbital instability. A third instability point is the Roche limit below which no hydrostatic equilibrium sequence can exist, because tidal disruption or Roche lobe overflow with mass transfer at the inner Lagrangian point sets in. All these instability points occur at certain separation distances between M and M'.

Lai et al. (1994b, §3.3) distinguish five cases concerning the existence of instability points prior to contact, when the close binary system diminishes its separation distance due to viscous friction and/or gravitational radiation: (i) No stability limits or Roche limit occur prior to contact. (ii) Only a secular stability limit is encountered. (iii) Secular stability limit and Roche limit are present. (iv) Secular and dynamical stability limits occur. (v) All three stability limits are present prior to contact, the Roche limit being reached before or after the dynamical stability limit. As a general rule, secular instability always occurs at a distance larger than the dynamical instability distance and the Roche limit.

In the special case of a single rotating mass M, $(q = M'/M = 0)$ no essentially new outcomes result from the ellipsoidal energy variational method. The *sufficient* condition for the onset of secular and dynamical instability of polytropic Maclaurin ellipsoids $(n \neq 0)$ is independent of the polytropic index n, occurring at the same eccentricity e as for the incompressible homogeneous ellipsoid $n = 0$ (cf. Secs. 3.2, 5.8.2, 6.4.3): $e = 0.81267,\ \tau = E_{kin}/|W| = 0.1375$, and $e = 0.95289,\ \tau = E_{kin}/|W| = 0.2738$, respectively.

A salient new finding of Lai et al. (1993) corrects an earlier result of Chandrasekhar (1969) concerning the secular stability limit, which occurs in Roche binaries before the Roche limit is reached, and in Darwin ellipsoids $(q = M'/M = 1)$ prior to contact if $n < 2$. If $n > 2$, the secular stability distance is located inside the Darwin ellipsoids.

The Roche binaries remain dynamically stable all the way up to the Roche limit if $n \gtrsim 1.7$. If $0 < n \lesssim 1.7$, the dynamical stability limit is reached before the Roche limit, unless the mass ratio q is below a certain limit: $q_{max} = 250$ if $n = 0$, and $q_{max} = 2.5$ if $n = 1.5$. Congruent Darwin ellipsoids become dynamically unstable prior to contact if $n < 0.7$. No Roche limit is reached by the Darwin ellipsoids, because if $q = 1$ no mass transfer can take place through the inner Lagrangian point, and there occurs merely mass shedding through the outer Lagrangian points. If $q \neq 1$, a Roche limit may exist prior to contact, the hydrostatic equilibrium sequences terminating at the onset of Roche lobe overflow with mass transfer at the inner Lagrangian point (Lai et al. 1994b, Rasio and Shapiro 1995).

Synchronization of spin and orbital angular velocities is assumed to be caused by viscous dissipation, preserving the *total* angular momentum of the binary (no external forces). As synchronization is approached, viscous dissipation falls to zero, and the orbit (separation distance) will cease to evolve unless the system is not losing angular momentum by some additional mechanism like emission of gravitational radiation (e.g. Landau and Lifschitz 1987, §110; Kuznetsov et al. 1998 if $n = 1.5,\ q = 1$). Due to gravitational radiation the orbit shrinks, viscous dissipation being negligible during this phase, until the secular instability distance is approached.

During the quasiequilibrium evolution two extreme regimes of interest occur (cf. Taniguchi et al. 2001). If viscosity is too low to maintain synchronization – like in close neutron star binaries – the orbit shrinks due to emission of gravitational waves, and evolution takes place towards the irrotational sequence, as already outlined subsequently to Eq. (5.7.92). The orbit decays until a dynamical instability $(n \lesssim 0.6)$ or Roche lobe overflow $(n \gtrsim 0.6)$ occurs. For compact binary stars the general relativistic orbital instability, appearing first at orbital separation of $D \approx (6-10)G(M_{r1} + M'_{r1})/c^2$, must be added to this scenario (Sec. 5.12.6, Lai et al. 1994a, Sec. 5, Wilson and Mathews 1995, Lombardi et al. 1997).

If, on the other side, viscosity dominates – as in binaries containing at least one nondegenerate component – the secularly unstable, polytropic binary will at first be driven away from synchronization by viscous forces until the two masses coalesce or reach a new stable synchronized state. If in the latter case the system is losing angular momentum by some additional process (e.g. gravitational radiation), the orbit decays, approaching again the secular stability limit. The orbital decay is so fast that viscosity can no longer maintain synchronization, driving the system away from synchronization. The final coalescence is driven almost entirely by internal viscous dissipation, with almost no loss of total angular momentum (Hachisu and Eriguchi 1984c if $n = 0$, Lai et al. 1994b, c).

When a dynamical stability limit is approached during secular orbital decay, the evolution becomes much faster and numerical calculations are needed to follow up the subsequent evolution (Rasio and

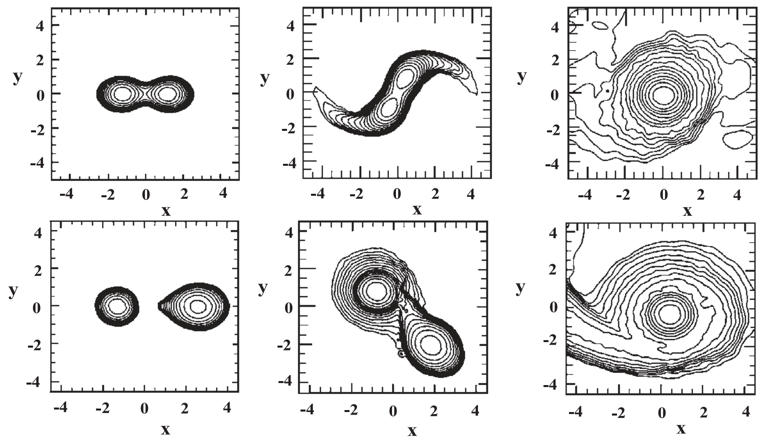

Fig. 5.7.4 Evolution during several orbital periods of a $n = 1.5$ binary with mass ratio $q = M'/M = 1$ (top) and $q = 0.5$ (bottom), (Rasio and Shapiro 1995).

Shapiro 1994, 1995). A dynamically unstable binary coalesces in a few orbital periods, forming a rapidly rotating spheroid surrounded by a thick disk of shock-heated material. This is illustrated in the upper part of Fig. 5.7.4 for two congruent, dynamically unstable Darwin polytropes, when all stability limits are reached in deep contact, well below the distance of first contact (see also Ruffert et al. (1997), and Faber and Rasio (2000, Fig. 2) for the relativistic case). The lower part of Fig. 5.7.4 exhibits the evolution during several orbital periods of a $q = 0.5$ binary, which is secularly and dynamically stable up to the Roche limit, when catastrophic mass transfer sets in.

The Roche limit D_c in a binary system can be calculated with the dimensionless critical rotation parameter

$$\beta_c = \Omega_c^2/2\pi G\varrho_0 = (1+\varepsilon)(M+M')/2\pi\varrho_0 D_c^3 \quad \text{or} \quad D_c = [(1+\varepsilon)(M+M')/2\pi\beta_c\varrho_0]^{1/3}. \qquad (5.7.93)$$

Ω_c means the synchronous critical angular velocity (3.1.93), and ϱ_0 is the central density of the mass M. Eq. (5.7.93) turns for two homogeneous masses with the same density into

$$D_c = [2(1+\varepsilon)(r^3 + r'^3)/3\beta_c]^{1/3}, \qquad (n = 0;\ \varrho = \varrho' = \varrho_0 = \text{const}). \qquad (5.7.94)$$

$(r^3 + r'^3)^{1/3} = [3(M+M')/4\pi\varrho]^{1/3}$ is the radius of the coalesced homogeneous mass $M + M'$.

If $M = 4\pi\varrho r^3/3 \ll M' = 4\pi\varrho' r'^3/3$, $(q \gg 1)$, e.g. small satellite M rotating round a homogeneous planet M', the Roche limit (5.7.93) becomes (Jeans 1919, Chandrasekhar 1969)

$$D_c = (2\varrho'/3\beta_c\varrho)^{1/3} r' = (2\varrho'/3 \times 0.045046\varrho)^{1/3} r' = 2.4552(\varrho'/\varrho)^{1/3} r' = 2.4552(M'/M)^{1/3} r,$$
$$(\varepsilon, n = 0;\ \beta_c = 0.045046;\ q = M'/M \gg 1;\ r \ll r'). \qquad (5.7.95)$$

In the opposite limit, if $M \gg M'$, $(q \approx 0)$ the ellipsoidal binary turns into a single mass M – the combined Maclaurin-Jacobi sequence. The Maclaurin sequence is pertinent prior to the onset of bifurcation, and the Jacobi sequence subsequently to that point. At the bifurcation point, where the Jacobi sequence branches off, the Maclaurin sequence becomes secularly unstable, while the new Jacobi sequence remains secularly stable (Secs. 3.2, 5.8.2, 6.4.3, Christodoulou et al. 1995a).

In the purely tidal Jeans problem there exist always two biaxial ellipsoidal equilibrium forms of the homogeneous mass M, as long as $GM'/D^3 \leq 0.125536\pi G\varrho$. The one equilibrium form of the Jeans ellipsoid with eccentricity $0 \leq e \leq 0.8830265$ is stable, the other one with eccentricity $0.8830265 < e \leq 1$ is unstable. If $GM'/D^3 > 0.125536\pi G\varrho$, no equilibrium ellipsoid of M is possible, and the limiting Roche distance for the incompressible Jeans problem is (Chandrasekhar 1969)

$$D_c = (M'/0.125536\pi\varrho)^{1/3} = (4\varrho'/3 \times 0.125536\varrho)^{1/3} r' = 2.1981(\varrho'/\varrho)^{1/3} r', \qquad (n = 0). \qquad (5.7.96)$$

This value is comparable to the Roche limit (5.7.95). In the Roche problem (M' – point mass) the maximum value Ω_{max} of the synchronous angular velocity of the binary is attained at a distance D slightly *larger* than the Roche limit. This corrects Chandrasekhar's (1969, Table XVIII) result that his maximum value of the rotation parameter $\beta_{max} = \Omega_{max}^2/2\pi G\varrho_0 = 0.070661,\ (q = 1)$ is just equal to the critical value β_c at the Roche limit (5.7.94). Instead, Lai et al. (1993, p. 228) take $\varepsilon = \varepsilon(D)$, and find $\beta_{max} = 0.07635,\ (q = 1)$, whereas the critical value (5.7.93) amounts only to $\beta_c = 0.07550$, with the corrected Roche limit (5.7.94) equal to

$$D_c = 2.1533(r^3 + r'^3)^{1/3}, \qquad (\varepsilon \neq 0;\ n = 0;\ \varrho = \varrho' = \varrho_0;\ r = r';\ q = 1). \tag{5.7.97}$$

If we replace all ellipsoidal masses with point masses (Roche model), we have found [Eqs. (3.6.36)-(3.6.41)] that $\beta_{cm} = 0.065844$ if $q = \infty$, $\beta_{cm} = 0.072267$ if $q = 1$, and $\beta_{cm} = 0.36074$ if $q = 0$, which are of the same order of magnitude as the previously noted values of β_c, $(n = 0)$. Note, that $\beta_{cm} = \Omega_c^2/2\pi G\varrho_m = \Omega_c^2/2\pi G\varrho_0 = \beta_c$ for homogeneous masses ($n = 0;\ \varrho = \varrho_0 = \varrho_m$). The maximum value β_{max} for Maclaurin ellipsoids $(q, n = 0)$ is 0.2247 (Sec. 3.2).

Gingold and Monaghan (1979b) have numerically confirmed for $n = 0.5, 1.5$ polytropes the approximate validity of the classical formulas (5.7.93)-(5.7.97) for the minimum distance of a binary system (Roche limit) in the Roche and Darwin problems.

The rotational and tidal splitting of oscillation frequencies can give important clues to the identification of modes in Cepheids and binary stars. The hydrostatic models of Chandrasekhar's (1933a-d) first order perturbation theory have been used by Saio (1981) to study the oscillations of a synchronously rotating polytropic binary, by expressing the distorted equipotential surface according to the level surface theory from Sec. 3.7 under the form

$$r = s[1 + \varepsilon(s, \lambda, \varphi)], \qquad (\varepsilon \ll 1). \tag{5.7.98}$$

s denotes the average radius of a level surface. The equidistant separation of oscillation frequencies from Eq. (5.7.34) is broken by second order effects in the angular velocity Ω. Unfortunately, it seems not possible to get a clear insight upon tidal effects from Saio's (1981) tables.

For their hydrostatic polytropic models Mohan and Saxena (1983) use the simple averaging technique of Kippenhahn and Thomas (1970), in conjunction with the approximation of level surfaces by Roche equipotentials (e.g. Kopal 1978). An equipotential surface is defined by $\Phi_{tot} =$ const, and its expression in the Roche approximation is given by Eq. (3.6.25) in a system rotating with angular velocity Ω round the primary M. The mean value f_m of a function $f(\vec{r})$ over an equipotential surface S_Φ is obviously

$$f_m = f(\Phi_{tot}) = (1/S_\Phi)\int_{\Phi_{tot}=\mathrm{const}} f(\vec{r})\ dS; \quad S_\Phi = \int_{\Phi_{tot}=\mathrm{const}} dS. \tag{5.7.99}$$

If we denote by dn the distance along the normal $\vec{n}$ between two neighboring equipotential surfaces Φ_{tot} and $\Phi_{tot} + d\Phi_{tot}$, we may define – corresponding to the usual definition of the acceleration of gravity – the function [cf. Eqs. (3.1.22), (3.1.27)]

$$g = d\Phi_{tot}/dn. \tag{5.7.100}$$

The volume element between two equipotential surfaces is

$$dV_\Phi = \int_{\Phi_{tot}=\mathrm{const}} dn\ dS = d\Phi_{tot}\int_{\Phi_{tot}=\mathrm{const}} dS/g = (1/g)_m S_\Phi\ d\Phi_{tot}. \tag{5.7.101}$$

The total volume of the configuration may be defined as a function of the radius r_Φ by $V_\Phi = 4\pi r_\Phi^3/3$ – in analogy to the volume of a sphere. The surface S_Φ is generally not equal to $4\pi r_\Phi^2$, so we define the function

$$u = u(\Phi_{tot}) = S_\Phi/4\pi r_\Phi^2, \tag{5.7.102}$$

together with the function

$$w = -(1/g)_m GM_\Phi/r_\Phi^2, \tag{5.7.103}$$

where the mass between two neighboring equipotentials is

$$dM_\Phi = \varrho_\Phi\ dV_\Phi = 4\pi\varrho_\Phi r_\Phi^2\ dr_\Phi. \tag{5.7.104}$$

Obviously, for a sphere we have $u = 1$ and $w = 1$, since $\Phi_{tot} = GM_\Phi/r_\Phi$ and $g = -GM_\Phi/r_\Phi^2$. From Eqs. (5.7.101)-(5.7.104) we get

$$d\Phi_{tot} = (d\Phi_{tot}/dV_\Phi)\ dV_\Phi = dM_\Phi/[(1/g)_m\varrho_\Phi S_\Phi] = -GM_\Phi\ dM_\Phi/4\pi\varrho_\Phi r_\Phi^4 uw = -GM_\Phi\ dr_\Phi/r_\Phi^2 uw. \tag{5.7.105}$$

Since the pressure P_Φ on an equipotential surface is a function only of Φ_{tot}, the equation of hydrostatic equilibrium $\nabla P_\Phi = \varrho_\Phi\ \nabla\Phi_{tot}$ can be written under the form

$$dP_\Phi/d\Phi_{tot} = \varrho_\Phi \quad \text{or} \quad dP_\Phi/dr_\Phi = -GM_\Phi\varrho_\Phi/r_\Phi^2 uw. \tag{5.7.106}$$

Taking the derivative of $(r_\Phi^2 uw/\varrho_\Phi)\ dP_\Phi/dr_\Phi$, we obtain at once the equivalent on a level surface of Poisson's equation (2.1.4):

$$d[(r_\Phi^2 uw/\varrho_\Phi)\ dP_\Phi/dr_\Phi]/dr_\Phi = d(r_\Phi^2 uw\ d\Phi_{tot}/dr_\Phi)/dr_\Phi = -G\ dM_\Phi/dr_\Phi = -4\pi G\varrho_\Phi r_\Phi^2. \tag{5.7.107}$$

If we define in the usual manner [cf. Eqs. (2.1.10), (2.1.13)]

$$r_\Phi = \alpha\xi_\Phi; \quad P = P_0\theta_\Phi^{n+1}; \quad \varrho = \varrho_0\theta_\Phi^n, \qquad \big(\alpha^2 = (n+1)P_0/4\pi G\varrho_0^2\big), \tag{5.7.108}$$

the dimensionless polytropic variables ξ_Φ and θ_Φ on a level surface, Eq. (5.7.107) reads

$$d[\xi_\Phi^2 uw\ d\theta_\Phi/d\xi_\Phi]/d\xi_\Phi = -\xi_\Phi^2\theta_\Phi^n, \tag{5.7.109}$$

which in the spherically symmetric case $u, w = 1$ turns into the familiar Lane-Emden equation (2.1.14).

Mohan and Saxena (1983, 1985) approximate the level surfaces of a polytrope of index n by Roche equipotentials – a satisfactory approximation for centrally condensed polytropes ($3 \lesssim n \leq 5$), since the point mass Roche model closely approximates the polytrope $n = 5$.

Kopal (1978, p. 323) shows that on the surface of a Roche equipotential the spherical coordinates r, λ, φ are connected through the relationship

$$r = Dr_0\{1 + r_0^3[qP_2(\cos\lambda) + (q+1)\sin^2\lambda/2] + ...\}, \qquad (q = M'/M). \tag{5.7.110}$$

The dimensionless parameter r_0 is given by

$$r_0 = 1/(\Phi_{tot}D/GM - q), \tag{5.7.111}$$

which for the undistorted sphere ($\Omega, q = 0;\ \Phi_{tot} = GM/r$) is just equal to $r_0 = r/D$. Mohan and Saxena (1985) express the radius r_Φ of a sphere that is topologically equivalent to the distorted equipotential surface under the form

$$r_\Phi = Dr_0[1 + (q+1)r_0^3/3 + ...], \tag{5.7.112}$$

and solve numerically the eigenvalue problem, determining eigenfrequencies of quasiradial and nonradial axisymmetric modes ($j = 0, 2;\ k = 0$) of rotationally and tidally distorted polytropes. Because of the smallness of the mass ratio ($q \leq 0.2$), and due to the relatively large separation distance D, tidal effects are minor as compared to rotational ones. The gravity g-eigenfrequencies of Mohan and Saxena (1985, 1990) are generally decreased due to rotation, contrary to the findings presented in Figs. 5.7.1, 5.7.3.

Denis (1972) has investigated nonaxisymmetric tidal modes of the compressible homogeneous polytrope $n = 0,\ \Gamma_1 = 5/3$. The stable eigenfrequencies ($\sigma^2 > 0$) of p and f-modes are generally destabilized due to tidal action, excepting for the case $j = |k|$. Tidal perturbations of eigenvalues for the fundamental Kelvin f-mode are of the same order of magnitude as rotational perturbations, and tidal action is just as effective as rotation in lifting the degeneracy between the fundamental radial $r(j = 0,\ k = 0)$-mode, and the fundamental Kelvin $f(j = 2,\ k = 0)$-mode occurring at $\Gamma_1 = 1.6$. The tidal influence on the eigenvalues of the unstable g-modes ($\sigma^2 < 0$, Table 5.5.2) is noticeably less than the rotational effects on a spheroid with the same oblateness. The splitting of degenerate modes is amplified by tidal action.

5.8 The Virial Method for Rotating Polytropes

5.8.1 Stability and Oscillations with the Second Order Virial Equations

As shown by Clement (1964) the variational equations discussed in Secs. 5.7.2 and 5.7.3 contain implicitly the virial equations of all orders. For displacements expressed in Cartesian form the virial and variational methods are equivalent, but the variational formulation has the advantage that it admits a spherical harmonic analysis of *all* normal modes, whereas from the virial equations of a given order one can extract only a limited number of overtones. While the scalar virial theorem (5.3.2) allows a simple description of the fundamental radial mode [see Eqs. (5.3.2)-(5.3.17)], the second order virial equations yield in the limit of zero rotation the eigenvalues of the radial $r(j=0)$ and of the fundamental Kelvin $f(j=2)$-mode [Eqs. (5.8.74)-(5.8.84)]. Similarly, the third order virial theorem describes oscillations belonging to the first $Y_1^k(\lambda,\varphi),\ (k=\pm1)$ and third order $Y_3^k(\lambda,\varphi),\ (k=\pm1,\pm2,\pm3)$ spherical harmonics in the nonrotating case. Higher order virial equations provide higher order modes, but this road becomes rapidly unpracticable [there is only one zeroth order (scalar) virial equation, but there are three first order, nine second order, eighteen third order, and thirty fourth order virial equations (e.g. Chandrasekhar 1969, Tassoul 1978)]. Nevertheless, the virial method provides a powerful, sophisticated tool to investigate small oscillations of rotating bodies and their secular stability due to some dissipative mechanism, such as viscosity, gravitational radiation reaction, etc.

The first variation δ^*H of the integral

$$H=\int_{V_u}F_u(\vec r_u,t)\ dV_u, \tag{5.8.1}$$

is defined by

$$\delta^*H=\delta^*\int_{V_u}F_u(\vec r_u,t)\ dV_u=\int_V F(\vec r,t)\ dV-\int_{V_u}F_u(\vec r_u,t)\ dV_u, \tag{5.8.2}$$

and should not be confused with the Eulerian perturbation denoted by δ. The functions $F(\vec r,t)$ and $F_u(\vec r_u,t)$ denote any attribute (such as velocity, density, pressure, gravitational potential) of a fluid element in the perturbed and unperturbed flow, respectively. The perturbed volume $V=V_u+\Delta V$ is derived from the unperturbed volume V_u by subjecting its boundary to the Lagrangian displacement (5.1.11) $\Delta\vec r(\vec r_u,t)=\vec r(\vec r_u,t)-\vec r_u$, which projects onto Cartesian axes as $\Delta x_k(x_{u1},x_{u2},x_{u3},t)=x_k(x_{u1},x_{u2},x_{u3},t)-x_{uk},\ (k=1,2,3)$. By virtue of Eqs. (5.2.4)-(5.2.6) the integral over V in Eq. (5.8.2) can be transformed into an integral over V_u

$$\delta^*H=\int_{V_u}[JF(\vec r,t)-F_u(\vec r_u,t)]\ dV_u=\int_{V_u}[(1+\nabla\cdot\Delta\vec r)\ F(\vec r,t)-F_u(\vec r_u,t)]\ dV_u$$
$$=\int_{V_u}[\Delta F(\vec r,t)+F(\vec r,t)\ \nabla\cdot\Delta\vec r]\ dV_u,\qquad (dV=J\ dV_u), \tag{5.8.3}$$

or up to the first order of smallness

$$\delta^*H=\delta^*\int_V F(\vec r,t)\ dV=\int_V[\Delta F(\vec r,t)+F(\vec r,t)\ \nabla\cdot\Delta\vec r]\ dV=\int_V(\Delta F+F\ \nabla\cdot\Delta\vec r)\ dV. \tag{5.8.4}$$

The Jacobian of the considered transformation $x_k=x_{uk}+\Delta x_k$ is up to the first order equal to

$$J=|\partial(x_1,x_2,x_3)/\partial(x_{u1},x_{u2},x_{u3})|\approx1+\partial\Delta x_1/\partial x_{u1}+\partial\Delta x_2/\partial x_{u2}+\partial\Delta x_3/\partial x_{u3}=1+\nabla\cdot\Delta\vec r. \tag{5.8.5}$$

If $F=\varrho$, we obtain with the continuity equation (5.2.28):

$$\delta^*\int_V\varrho(\vec r,t)\ dV=\delta^*\int_V\varrho\ dV=\int_V(\Delta\varrho+\varrho\ \nabla\cdot\Delta\vec r)\ dV=0. \tag{5.8.6}$$

And if $F \to \varrho F$, we have for any attribute F

$$\delta^* \int_V \varrho F\ dV = \int_V [\Delta(\varrho F) + \varrho F\ \nabla \cdot \Delta\vec{r}]\ dV = \int_V [\varrho\ \Delta F + F(\Delta\varrho + \varrho\ \nabla \cdot \Delta\vec{r})]\ dV = \int_V \varrho\ \Delta F\ dV. \tag{5.8.7}$$

In particular, if F is an extrinsic attribute, i.e. a quantity specified simply by virtue of its location (like the centrifugal potential), then its Eulerian variation δF is zero (cf. Clement 1964, Chandrasekhar 1969, p. 29), and

$$\delta^* \int_V \varrho F\ dV = \int_V \varrho\ \Delta F\ dV = \int_V \varrho(\delta F + \Delta\vec{r} \cdot \nabla F)\ dV = \int_V \varrho(\Delta\vec{r} \cdot \nabla F)\ dV. \tag{5.8.8}$$

We have already used in Eq. (2.6.59) the obvious formula

$$d\left[\int_V \varrho(\vec{r},t)\ F(\vec{r},t)\ dV\right]\Big/dt = d\left[\int_M F(\vec{r},t)\ dM\right]\Big/dt = \int_M [DF(\vec{r},t)/Dt]\ dM$$
$$= \int_V \varrho(\vec{r},t)\ [DF(\vec{r},t)/Dt]\ dV = \int_V \varrho(\vec{r},t)\ [\partial F(\vec{r},t)/\partial t + v_i\ \partial F(\vec{r},t)/\partial x_i]\ dV, \qquad (v_i = dx_i/dt), \tag{5.8.9}$$

since, due to mass conservation, there is

$$d\left[\int_V \varrho(\vec{r},t)\ dV\right]\Big/dt = d\left(\int_M dM\right)\Big/dt = dM/dt = 0. \tag{5.8.10}$$

The virial equations are in fact no more than the moments of the relevant hydrodynamical equations. The scalar (zero order) virial equation (2.6.80), (3.1.85), or (5.3.2) can be obtained at once by contracting the second order virial equations (2.6.79), (3.1.83), or (3.1.84). The first order virial equations are simply obtained by integrating the equations of motion (3.1.79) over the volume V of the configuration (Chandrasekhar and Lebovitz 1963a, Chandrasekhar 1969):

$$\int_V \varrho\ (Dv_i/Dt)\ dV = d\left(\int_V \varrho v_i\ dV\right)\Big/dt = -\int_V (\partial P/\partial x_i)\ dV + \int_V \varrho(\partial\Phi/\partial x_i)\ dV +$$
$$(1/2)\int_V \left\{\partial\left[(\Omega_2 x_3 - \Omega_3 x_2)^2 + (\Omega_3 x_1 - \Omega_1 x_3)^2 + (\Omega_1 x_2 - \Omega_2 x_1)^2\right]/\partial x_i\right\}\ dV + 2\varepsilon_{ijk}\Omega_k \int_V \varrho v_j\ dV$$
$$= \Omega^2 I_i - \Omega_i\Omega_k I_k + 2\varepsilon_{ijk}\Omega_k \int_V \varrho v_j\ dV, \qquad (i,j,k = 1,2,3;\ \vec{\Omega} = \vec{\Omega}(\Omega_1,\Omega_2,\Omega_3) = \mathrm{const}). \tag{5.8.11}$$

The integrals of ∇P and $\nabla\Phi$ vanish because we have in virtue of Eq. (2.6.61)

$$\int_V (\partial P/\partial x_i)\ dV = \int_S P\ dS_i = 0, \qquad (P(S) = 0), \tag{5.8.12}$$

and via Eq. (2.6.62)

$$\int_V \varrho(\vec{r})\ [\partial\Phi(\vec{r})/\partial x_i]\ dV = \int_V \varrho(\vec{r})\ dV\ \partial\left[\int_V \varrho(\vec{r'})\ dV'/|\vec{r} - \vec{r'}|\right]\Big/\partial x_i$$
$$= \int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ (x_i - x_i')\ dV\ dV'/|\vec{r} - \vec{r'}|^3 = 0, \qquad [\vec{r} = \vec{r}(x_1,x_2,x_3);\ \vec{r'} = \vec{r'}(x_1',x_2',x_3')]. \tag{5.8.13}$$

In Eq. (5.8.11) we have introduced the first order moments of the density distribution

$$I_i = \int_V \varrho x_i\ dV. \tag{5.8.14}$$

Quite generally, we may consider moments of density distribution of various orders by writing

$$I_{ijk...} = \int_V \varrho x_i x_j x_k...\ dV. \tag{5.8.15}$$

The second order moment I_{ij} (the moment of inertia tensor) has already been considered in Eq. (2.6.74). The first variation of the moments of density distribution is via Eq. (5.8.7) equal to

$$\delta^* I_{ijk...} = \int_V \varrho\ \Delta(x_i x_j x_k...)\ dV = \int_V \varrho(\Delta x_i\ x_j x_k... + \Delta x_j\ x_i x_k... + \Delta x_k\ x_i x_j... + ...)\ dV$$
$$= U_{i;jk...} + U_{j;ik...} + U_{k;ij...} + ... = \int_V \varrho\ \Delta x_\ell\ [\partial(x_i x_j x_k...)/\partial x_\ell]\ dV = U_{ijk...} \qquad (5.8.16)$$

To shorten the notations we have introduced the symmetrized quantity

$$U_{ijk...} = \int_V \varrho\ \Delta x_\ell\ [\partial(x_i x_j x_k...)/\partial x_\ell]\ dV$$
$$= \int_V \varrho\ (\Delta x_i\ x_j x_k... + \Delta x_j\ x_i x_k... + \Delta x_k\ x_i x_j... + ...)\ dV, \qquad (5.8.17)$$

and the unsymmetrized notation

$$U_{i;jk...} = \int_V \varrho\ \Delta x_i\ x_j x_k...\ dV. \qquad (5.8.18)$$

The Lagrangian variation Δx_k of the Cartesian coordinate x_k is given by [cf. Eqs. (5.1.10)-(5.1.12)]

$$\Delta x_k(x_{u1}, x_{u2}, x_{u3}, t) = x_k(x_{u1}, x_{u2}, x_{u3}, t) - x_{uk};$$
$$\Delta\vec{r} = \Delta\vec{r}[\Delta x_1(x_{u1}, x_{u2}, x_{u3}, t),\ \Delta x_2(x_{u1}, x_{u2}, x_{u3}, t),\ \Delta x_3(x_{u1}, x_{u2}, x_{u3}, t)]. \qquad (5.8.19)$$

x_{uk} denotes the Cartesian coordinate of a mass element in the unperturbed flow. If the direction of the angular velocity $\vec{\Omega}$ is chosen along the x_3-axis, Eq. (5.8.11) takes the form $[\vec{\Omega} = \vec{\Omega}(0, 0, \Omega)]$

$$d\left(\int_V \varrho v_k\ dV\right)\Big/dt = \Omega^2(I_k - \delta_{k3} I_3) + 2\Omega\varepsilon_{kj3}\int_V \varrho v_j\ dV. \qquad (5.8.20)$$

δ_{k3} denotes the Kronecker delta. If no relative motions are present in the uniformly rotating frame of reference ($v_k = v_{uk} = 0$), we infer from Eq. (5.8.20) that

$$I_1 = I_2 = 0, \qquad (v_k = 0). \qquad (5.8.21)$$

Now let an initial state in which hydrostatic equilibrium prevails ($v_{uk} = 0$) be slightly perturbed by the Lagrangian displacement $\Delta x_k = x_k - x_{uk}$. The velocity v_k of the perturbed mass element is via Eq. (5.1.22) equal to $v_k = v_{uk} + \Delta v_k = \Delta v_k = D(\Delta x_k)/Dt$. Inserting for x_k and v_k into Eq. (5.8.20), and taking into account the hydrostatic form

$$I_k - \delta_{k3} I_3 = \int_V \varrho(x_{uk} - \delta_{k3} x_{u3})\ dV = 0, \qquad (v_{uk} = 0), \qquad (5.8.22)$$

of this equation, we get

$$d\left\{\int_V \varrho[D(\Delta x_i)/Dt]\ dV\right\}\Big/dt = d^2\left(\int_V \varrho\ \Delta x_i\ dV\right)\Big/dt^2 = \Omega^2\int_V \varrho(\Delta x_i - \delta_{i3}\ \Delta x_3)\ dV$$
$$+2\Omega\varepsilon_{ij3}\int_V \varrho[D(\Delta x_j)/Dt]\ dV = \Omega^2\int_V \varrho(\Delta x_i - \delta_{i3}\ \Delta x_3)\ dV + 2\Omega\varepsilon_{ij3}\ d\left(\int_V \varrho\ \Delta x_j\ dV\right)\Big/dt. \qquad (5.8.23)$$

With the symmetrized quantity from Eq. (5.8.17)

$$U_i = \int_V \varrho\ \Delta x_i\ dV, \qquad (5.8.24)$$

this amounts to

$$d^2 U_i/dt^2 = \Omega^2(U_i - \delta_{i3} U_3) + 2\Omega\varepsilon_{ij3}\ dU_j/dt. \qquad (5.8.25)$$

The general solution of the first order virial equations (5.8.25) can readily be written down, but they are not relevant for a discussion of the oscillations and the stability of the system, as they are in no way dependent on the particular constitution of the system. We can therefore assume without loss of generality that

$$U_i = \int_V \varrho\ \Delta x_i\ dV \equiv 0. \tag{5.8.26}$$

It may however be noted that Eq. (5.8.25) allows solutions for U_1 and U_2, which are periodic with a frequency of Ω. The meaning of assumption (5.8.26) is that we are considering the polytrope in a frame of reference whose origin is permanently located in the centre of mass of the system, which implies that $I_i = 0$ [Eq. (5.8.14)]. Since the only motion which the centre of mass of a self-gravitating system effects, is a uniform motion, no generality is lost by assumption (5.8.26), (Chandrasekhar and Lebovitz 1963a).

Let us now consider the first variation of the second order virial equations (3.1.83). We get a set of nine equations

$$\delta^*\left[d\left(\int_V \varrho x_i v_j\ dV\right)\Big/dt\right] = 2\delta^* E_{ij} + \delta^* W_{ij} + \delta_{ij}\ \delta^*\int_V P\ dV + \Omega^2 U_{ij} - \Omega_j\Omega_k U_{ik}$$
$$+2\varepsilon_{jk\ell}\Omega_\ell\ \delta^*\int_V \varrho x_i v_k\ dV, \qquad (i,j,k,\ell = 1,2,3), \tag{5.8.27}$$

where we have used Eqs. (5.8.16), (5.8.17) to denote the variation $\delta^* I_{ij} = U_{ij}$. The variation of the gravitational potential energy tensor can be found by applying an obvious generalization of Eq. (5.8.8) to the definition (2.6.71) of W_{ij} (Chandrasekhar 1969, p. 34):

$$\delta^* W_{ij} = -(1/2)\ \delta^*\int_V \varrho(\vec r)\ \Phi_{ij}(\vec r)\ dV = -(G/2)\ \delta^*\int_V\int_V \varrho(\vec r)\ \varrho(\vec{r'})\ (x_i - x_i')(x_j - x_j')\ dV\ dV'$$
$$/|\vec r - \vec{r'}|^3 = -(G/2)\int_V \varrho(\vec r)\ \Delta x_k(\vec r)\ dV\ \partial\left[\int_V \varrho(\vec{r'})\ (x_i - x_i')(x_j - x_j')\ dV'/|\vec r - \vec{r'}|^3\right]\Big/\partial x_k$$
$$-(G/2)\int_V \varrho(\vec{r'})\ \Delta x_k(\vec{r'})\ dV'\ \partial\left[\int_V \varrho(\vec r)\ (x_i - x_i')(x_j - x_j')\ dV/|\vec r - \vec{r'}|^3\right]\Big/\partial x_k'$$
$$= -\int_V \varrho(\vec r)\ \Delta x_k(\vec r)\ [\partial\Phi_{ij}(\vec r)/\partial x_k]\ dV, \qquad (\partial\varrho(\vec{r'})/\partial x_k,\ \partial\varrho(\vec r)/\partial x_k' = 0). \tag{5.8.28}$$

The first variation of the kinetic energy tensor follows readily from its definition (2.6.57):

$$\delta^* E_{ij} = (1/2)\ \delta^*\int_V \varrho v_i v_j\ dV = (1/2)\int_V \varrho(v_i\ \Delta v_j + v_j\ \Delta v_i)\ dV$$
$$= (1/2)\int_V \varrho\big[v_i\ D(\Delta x_j)/Dt + v_j\ D(\Delta x_i)/Dt\big]\ dV. \tag{5.8.29}$$

The left-hand side of Eq. (5.8.27) is evaluated by observing that the operations of δ^* and $d\ /dt$ outside an integral are permutable:

$$\delta^*\left[d\left(\int_V \varrho x_i v_j\ dV\right)\Big/dt\right] = d\left(\delta^*\int_V \varrho x_i v_j\ dV\right)\Big/dt = d\left(\int_V \varrho x_i\ \Delta v_j\ dV\right)\Big/dt$$
$$+d\left(\int_V \varrho v_j\ \Delta x_i\ dV\right)\Big/dt = d\left\{\int_V \varrho\big[D(x_i\ \Delta x_j)/Dt - v_i\ \Delta x_j\big]\ dV\right\}\Big/dt$$
$$+d\left(\int_V \varrho v_j\ \Delta x_i\ dV\right)\Big/dt = d^2\left(\int_V \varrho x_i\ \Delta x_j\ dV\right)\Big/dt^2 + d\left[\int_V \varrho(v_j\ \Delta x_i - v_i\ \Delta x_j)\ dV\right]\Big/dt$$
$$= d^2 U_{j;i}/dt^2 + d\left[\int_V \varrho(v_j\ \Delta x_i - v_i\ \Delta x_j)\ dV\right]\Big/dt. \tag{5.8.30}$$

It remains to consider the variation of the pressure integral via Eqs. (5.2.38) and (5.8.4):

$$\delta^*\int_V P\ dV = \int_V (\Delta P + P\ \nabla\cdot\Delta\vec r)\ dV = -(\Gamma_1 - 1)\int_V P\ \nabla\cdot\Delta\vec r\ dV. \tag{5.8.31}$$

If the polytrope is initially in hydrostatic equilibrium ($v_{uk}=0$), the small oscillation velocity v_k is by virtue of Eq. (5.1.22) equal to its Lagrangian change Δv_k, and the first variation (5.8.29) of the kinetic energy is of second order, and can accordingly be neglected, as well as the last integral on the right-hand side of Eq. (5.8.30). Inserting Eqs. (5.8.28)-(5.8.31) into Eq. (5.8.27), we obtain up to the first order (Chandrasekhar and Lebovitz 1962b)

$$d^2\left(\int_V \varrho x_i\ \Delta x_j\ dV\right)\Big/dt^2 = -\int_V \varrho\ \Delta x_k\ (\partial\Phi_{ij}/\partial x_k)\ dV - (\Gamma_1-1)\ \delta_{ij}\int_V P\ \nabla\cdot\Delta\vec{r}\ dV$$
$$+\Omega^2\int_V (x_i\ \Delta x_j + x_j\ \Delta x_i)\ dV - \Omega_j\Omega_k\int_V \varrho(x_k\ \Delta x_i + x_i\ \Delta x_k)\ dV$$
$$+2\varepsilon_{jk\ell}\Omega_\ell\ d\left(\int_V \varrho x_i\ \Delta x_k\ dV\right)\Big/dt, \tag{5.8.32}$$

the last integral being transformed analogously to Eq. (5.8.30).

The simplest trial eigenfunction that suggests itself in the present context is

$$\Delta x_j = \Delta x_j(x_1,x_2,x_3,t) = L_{jk}x_k\exp(i\sigma t), \qquad (j=1,2,3). \tag{5.8.33}$$

The nine coefficients L_{jk} of this linear transformation play the role of variational parameters, which have to be determined by the nine equations (5.8.32). A partial justification of the substitution (5.8.33) comes from the fact that the results so obtained become exact in the limit of the constant density polytrope $n=0$ – the compressible Maclaurin spheroid (Chandrasekhar and Lebovitz 1962c, Sec. III). They should therefore be a good approximation if the central condensation is not too high ($n \lesssim 2.5$) as shown by comparing the eigenvalues ω_{zr}^2 and ω_{zf}^2 from Table 5.8.1 if $\beta=0,\ \Gamma_1=5/3$, with the eigenvalues ω_0^2 from Table 5.3.1, and with the eigenvalues of the fundamental Kelvin $f(j=2)$-mode from Table 5.5.3, respectively.

A trial Lagrangian displacement for the third order virial equations would be

$$\Delta x_i = L_{ijk}x_jx_k + L_i, \qquad (L_{ijk}=L_{ikj};\ i,j,k=1,2,3), \tag{5.8.34}$$

amounting to a total of 21 parameters, the three L_i's being eliminated by the first order virial theorem (5.8.25), (Chandrasekhar and Lebovitz 1963b, p. 192; Chandrasekhar 1969).

Inserting into Eq. (5.8.32) the trial variation (5.8.33), we obtain after some algebra

$$\sigma^2L_{jk}I_{ik} + 2i\sigma\Omega\varepsilon_{jk3}L_{k\ell}I_{i\ell} + \Omega^2(L_{jk}I_{ik}+L_{ik}I_{jk}) - \Omega^2\delta_{j3}(L_{3k}I_{ik}+L_{ik}I_{3k})$$
$$-L_{k\ell}W_{\ell k;ij} + \delta_{ij}L_{kk}\Pi = 0, \qquad (L_{kk} = \mathrm{Tr}\ L_{ij} = L_{11}+L_{22}+L_{33}), \tag{5.8.35}$$

where

$$\Pi = -(\Gamma_1-1)\int_V P\ dV, \tag{5.8.36}$$

and

$$W_{k\ell;ij} = \int_V \varrho x_k(\partial\Phi_{ij}/\partial x_\ell)\ dV. \tag{5.8.37}$$

The direction of the rotation axis has been chosen along the x_3-axis $[\vec{\Omega}=\vec{\Omega}(0,0,\Omega)]$. If use is made of the equilibrium condition (3.1.87), we can write instead of Eq. (5.8.36):

$$\Pi = (\Gamma_1-1)W_{33}. \tag{5.8.38}$$

Chandrasekhar and Lebovitz (1962a) also introduce the so-called superpotential

$$\chi(\vec{r}) = -G\int_V \varrho(\vec{r'})\ |\vec{r}-\vec{r'}|\ dV'. \tag{5.8.39}$$

Its derivatives are

$$\partial\chi/\partial x_i = -G\int_V \varrho(\vec{r'})\ (x_i-x_i')\ dV'/|\vec{r}-\vec{r'}|; \quad \partial^2\chi/\partial x_i\partial x_j = -G\delta_{ij}\int_V \varrho(\vec{r'})\ dV'/|\vec{r}-\vec{r'}|$$
$$+G\int_V \varrho(\vec{r'})\ (x_i-x_i')(x_j-x_j')\ dV'/|\vec{r}-\vec{r'}|^3. \tag{5.8.40}$$

We replace the integrals by the definitions (2.6.62) and (2.6.63) of Φ and Φ_{ij}, respectively:

$$\Phi_{ij} = \delta_{ij}\Phi + \partial^2\chi/\partial x_i \partial x_j. \tag{5.8.41}$$

Contracting this equation, we find

$$\nabla^2\chi = -2\Phi \quad \text{and} \quad \nabla^4\chi = -2\ \nabla^2\Phi = 8\pi G\varrho. \tag{5.8.42}$$

As defined by Eq. (5.8.37), the tensor $W_{k\ell;ij}$ is clearly symmetric in its pair of indices i, j, and its contraction with respect to this pair

$$W_{k\ell;ii} = \int_V \varrho x_k(\partial\Phi/\partial x_\ell)\ dV = -(1/2)\int_V \varrho\Phi_{k\ell}\ dV = W_{k\ell} = W_{\ell k} = W_{\ell k;ii}, \tag{5.8.43}$$

is symmetric in k and ℓ, by virtue of Eq. (2.6.71). However, the uncontracted tensor, in general, is not symmetric in its first pair of indices. Indeed, substituting for Φ_{ij} its explicit expression (2.6.63), we obtain

$$\begin{aligned}
W_{k\ell;ij} &= G\int_V \varrho(\vec{r})\ x_k\ dV\ \partial\Big[\int_V \varrho(\vec{r'})\ (x_i - x_i')(x_j - x_j')\ dV'/|\vec{r} - \vec{r'}|^3\Big]\Big/\partial x_\ell \\
&= G\ \delta_{\ell i}\int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ x_k(x_j - x_j')\ dV\ dV'/|\vec{r} - \vec{r'}|^3 \\
&+ G\ \delta_{\ell j}\int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ x_k(x_i - x_i')\ dV\ dV'/|\vec{r} - \vec{r'}|^3 \\
&- 3G\int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ x_k(x_\ell - x_\ell')(x_i - x_i')(x_j - x_j')\ dV\ dV'/|\vec{r} - \vec{r'}|^5.
\end{aligned} \tag{5.8.44}$$

Interchanging primed and unprimed variables in this equation, and adding together, we find with the notation (2.6.71):

$$\begin{aligned}
2W_{k\ell;ij} &= -2\delta_{\ell i}W_{kj} - 2\delta_{\ell j}W_{ki} \\
&- 3G\int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ (x_k - x_k')(x_\ell - x_\ell')(x_i - x_i')(x_j - x_j')\ dV\ dV'/|\vec{r} - \vec{r'}|^5,
\end{aligned} \tag{5.8.45}$$

the last integral being completely symmetric in all four indices. Several identities can be deduced from this equation, by suspending for the moment the summation convention over repeated indices:

$$W_{kk;ii} = W_{ii;kk}; \quad W_{ij;ij} - W_{ji;ij} = W_{ij;ij} - W_{ji;ji} = W_{jj} - W_{ii}, \qquad (\text{no summation over } i, j, k). \tag{5.8.46}$$

If $i \neq j$ and $k = i,\ \ell = j$, another identity follows by inserting the representation (5.8.41) into Eq. (5.8.37), and suspending again the summation convention:

$$\begin{aligned}
W_{ij;ij} &= \int_V \varrho x_i(\partial\Phi_{ij}/\partial x_j)\ dV = \int_V \varrho x_i[\partial(\partial^2\chi/\partial x_i\partial x_j)/\partial x_j]\ dV = \int_V \varrho x_i[\partial(\partial^2\chi/\partial x_j^2)/\partial x_i]\ dV \\
&= \int_V \varrho x_i[\partial(\Phi_{jj} - \Phi)/\partial x_i]\ dV = W_{ii;jj} - W_{ii}, \qquad (i \neq j;\ \text{no summation over } i, j).
\end{aligned} \tag{5.8.47}$$

Next, by taking into account the expression (2.6.71)

$$\begin{aligned}
W_{ij} &= \int_V \varrho(\vec{r})\ x_i[\partial\Phi(\vec{r})/\partial x_j]\ dV = -(1/2)\int_V \varrho(\vec{r})\ \Phi_{ij}(\vec{r})\ dV \\
&= -(G/2)\int_V\int_V \varrho(\vec{r})\ \varrho(\vec{r'})\ (x_i - x_i')(x_j - x_j')\ dV\ dV'/|\vec{r} - \vec{r'}|^3,
\end{aligned} \tag{5.8.48}$$

of the potential energy tensor, we obtain from various contractions of Eq. (5.8.45):

$$W_{k\ell;ii} = W_{k\ell}; \quad W_{kk;ij} = W_{ij}; \quad W_{kj;ij} = -W_{ki}; \quad W_{i\ell;ij} = 2W_{\ell j} - \delta_{\ell j}W. \tag{5.8.49}$$

Many components of the tensor $W_{k\ell;ij}$ vanish if the system has triplanar symmetry, i.e. if

$$\varrho(x_1,x_2,x_3)=\varrho(-x_1,x_2,x_3)=\varrho(x_1,-x_2,x_3)=\varrho(x_1,x_2,-x_3). \tag{5.8.50}$$

If the density distribution has this property, the moment of inertia tensor (2.6.74) is clearly diagonal: $I_{ij}=0$ if $i\neq j$. From Eqs. (2.6.62) and (5.8.39), defining Φ and χ, it follows that if ϱ is an even function of the coordinates – as assumed by Eq. (5.8.50) – then so are Φ and χ. From Eq. (5.8.41) we now infer that Φ_{ij} is an odd function of the coordinates x_i, x_j if $i\neq j: \Phi_{ij}(x_i)=-\Phi_{ij}(-x_i),\ \Phi_{ij}(x_j)=-\Phi_{ij}(-x_j)$. And Φ_{ii} is an even function in all three coordinates if $i=j$. Considering now the gravitational energy tensor from Eq. (5.8.48), we conclude from the symmetry properties of Φ_{ij} that $W_{ij}=0$ if $i\neq j$. Thus, the potential energy tensor W_{ij} and the moment of inertia tensor I_{ij} can be brought simultaneously to the diagonal form if the polytrope has triplanar symmetry.

Turning now our attention to the supermatrix (5.8.37), we observe that when $i=j$, the integral is odd in the two coordinates x_k and x_ℓ if $k\neq\ell$, and $W_{k\ell;ij}$ will consequently vanish. Also, if $k\neq\ell$, and one of these indices is not equal to either i or j, the integral (5.8.37) will again be odd in two of the three coordinates, and $W_{k\ell;ij}$ vanishes. The only circumstance when $W_{k\ell;ij}$ will not vanish identically in the case of triplanar symmetry, is when the integrand is even in all three Cartesian coordinates. When $i=j$, this happens if $k=\ell$; and when $i\neq j$, this can happen only if the pair of indices (k,ℓ) coincides with the pair (i,j) or (j,i). Thus

$$W_{k\ell;ij}\neq 0 \quad \text{if } i=j \text{ and } k=\ell; \quad \text{if } i\neq j \text{ and } k=i,\ \ell=j; \quad \text{if } i\neq j \text{ and } k=j,\ \ell=i. \tag{5.8.51}$$

If the polytrope possesses besides triplanar symmetry also axial symmetry about the x_3-axis, the density distribution is of the form

$$\varrho(\ell,z)=\varrho(\ell,-z), \tag{5.8.52}$$

where $\ell=(x_1^2+x_2^2)^{1/2},\ \varphi,\ z=x_3$ denote cylindrical coordinates. From Eqs. (2.6.62), (5.8.39) results that Φ and χ are independent of φ. Considering the diagonal components of the gravitational potential energy tensor, we get

$$\begin{aligned} W_{11}&=\int_V \varrho x_1(\partial\Phi/\partial x_1)\ dV=\int_V \varrho x_1^2(\partial\Phi/\partial\ell)\ d\ell\ d\varphi\ dz=\int_V \varrho\ell^2\cos^2\varphi\ (\partial\Phi/\partial\ell)\ d\ell\ d\varphi\ dz\\ &=\pi\int_0^\infty \ell^2\ d\ell\int_{-\infty}^\infty \varrho(\partial\Phi/\partial\ell)\ dz, \qquad (\partial\ell/\partial x_1=x_1/\ell=\cos\varphi). \end{aligned} \tag{5.8.53}$$

Obviously, for W_{22} we obtain just the same equation: $W_{11}=W_{22}$. However

$$W_{33}=\int_V \varrho x_3(\partial\Phi/\partial x_3)\ dV=2\pi\int_0^\infty \ell\ d\ell\int_{-\infty}^\infty \varrho z(\partial\Phi/\partial z)\ dz, \qquad (x_3\equiv z), \tag{5.8.54}$$

will be, in general, different from $W_{11}=W_{22}$. From the equality of W_{11} and W_{22} follows that $W_{12;12}=W_{21;12}=W_{21;21}$ in virtue of Eq. (5.8.46). Considering now the nonvanishing elements of $W_{k\ell;ij}$ systematically, we observe that after integration over φ, the integrals

$$\begin{aligned} W_{11;11}&=\int_V \varrho x_1(\partial\Phi_{11}/\partial x_1)\ dV=\int_V \varrho x_1[\partial(\Phi+\partial^2\chi/\partial x_1^2)/\partial x_1]\ dV,\\ W_{22;22}&=\int_V \varrho x_2(\partial\Phi_{22}/\partial x_2)\ dV=\int_V \varrho x_2[\partial(\Phi+\partial^2\chi/\partial x_2^2)/\partial x_2]\ dV, \end{aligned} \tag{5.8.55}$$

lead to identical expressions: $W_{11;11}=W_{22;22}$. Similar relationships will be summarized below in Eq. (5.8.60). In the case of axial symmetry another less obvious identity subsists:

$$W_{12;12}=(W_{11;11}-W_{11;22})/2. \tag{5.8.56}$$

This can be established by observing that

$$\begin{aligned} W_{k\ell;ij}&=\int_V \varrho x_k(\partial\Phi_{ij}/\partial x_\ell)\ dV=\int_V \big[\partial(\varrho x_k\Phi_{ij})/\partial x_\ell-(\partial\varrho/\partial x_\ell)x_k\Phi_{ij}-\varrho\ \delta_{k\ell}\Phi_{ij}\big]\ dV\\ &=\int_S \varrho x_k\Phi_{ij}\ dS_\ell-\int_V[(\partial\varrho/\partial x_\ell)x_k\Phi_{ij}+\varrho\ \delta_{k\ell}\Phi_{ij}]\ dV=-\int_V[(\partial\varrho/\partial x_\ell)x_k\Phi_{ij}+\varrho\ \delta_{k\ell}\Phi_{ij}]\ dV, \end{aligned} \tag{5.8.57}$$

provided the density ϱ vanishes on the boundary of V. Thus, using Eqs. (5.8.41), (5.8.57), we get

$$W_{12;12} = \int_V \varrho x_1(\partial\Phi_{12}/\partial x_2)\, dV = -\int_V (\partial\varrho/\partial x_2)x_1\Phi_{12}\, dV$$
$$= -\int_V x_1(\partial\varrho/\partial x_2)(\partial^2\chi/\partial x_1\partial x_2)\, dV = -\int_V (x_1^2x_2^2/\ell)(\partial\varrho/\partial\ell)\{\partial[(1/\ell)\ \partial\chi/\partial\ell]/\partial\ell\}\, d\ell\, d\varphi\, dz$$
$$= -(\pi/4)\int_0^\infty \ell^3\, d\ell \int_{-\infty}^{\infty} (\partial\varrho/\partial\ell)\{\partial[(1/\ell)\ \partial\chi/\partial\ell]/\partial\ell\}\, dz. \tag{5.8.58}$$

Similarly

$$W_{11;11} - W_{11;22} = \int_V \varrho x_1[\partial(\partial^2\chi/\partial x_1^2 - \partial^2\chi/\partial x_2^2)/\partial x_1]\, dV$$
$$= -\int_V x_1(\partial\varrho/\partial x_1)(\partial^2\chi/\partial x_1^2 - \partial^2\chi/\partial x_2^2)\, dV - \int_V \varrho(\partial^2\chi/\partial x_1^2 - \partial^2\chi/\partial x_2^2)\, dV$$
$$= -\int_V (x_1^2/\ell)(\partial\varrho/\partial\ell)\big\{\partial[(x_1/\ell)\ \partial\chi/\partial\ell]/\partial x_1 - \partial[(x_2/\ell)\ \partial\chi/\partial\ell]/\partial x_2\big\}\, dV$$
$$= -\int_V (1/\ell^2)(x_1^4 - x_1^2x_2^2)(\partial\varrho/\partial\ell)\{\partial[(1/\ell)\ \partial\chi/\partial\ell]/\partial\ell\}\, dV$$
$$= -\int_0^\infty \ell^3\, d\ell \int_0^{2\pi} (\cos^4\varphi - \sin^2\varphi\cos^2\varphi)\, d\varphi \int_{-\infty}^{\infty} (\partial\varrho/\partial\ell)\{\partial[(1/\ell)\ \partial\chi/\partial\ell]/\partial\ell\}\, dz$$
$$= -(\pi/2)\int_0^\infty \ell^3\, d\ell \int_{-\infty}^{\infty} (\partial\varrho/\partial\ell)\{\partial[(1/\ell)\ \partial\chi/\partial\ell]/\partial\ell\}\, dz, \tag{5.8.59}$$

the integral of $\varrho(\partial^2\chi/\partial x_1^2 - \partial^2\chi/\partial x_2^2)$ being exactly zero. Comparing the end results of Eqs. (5.8.58) and (5.8.59), we obtain just Eq. (5.8.56).

Summarizing the findings from Eqs. (5.8.46)-(5.8.49), (5.8.51), (5.8.55)-(5.8.59) for the case of axial symmetry, we express the nonvanishing elements of the supermatrix $W_{k\ell;ij}$ in terms of four of them, denoted subsequently by A, B, C, D (Chandrasekhar and Lebovitz 1962a):

$$W_{11;11} = W_{22;22} = A;\quad W_{11;22} = W_{22,11} = B;\quad W_{11;33} = W_{22;33} = W_{33;11} = W_{33;22} = C;$$
$$W_{33;33} = D;\quad W_{12;12} = W_{21;12} = W_{21;21} = (W_{11;11} - W_{22;11})/2 = (A - B)/2;\quad W_{13;13} = W_{23;23}$$
$$= W_{11;33} - W_{11} = C - W_{11};\quad W_{31;13} = W_{32;23} = W_{11;33} - W_{33} = C - W_{33}. \tag{5.8.60}$$

Moreover, three independent relationships exist among the four basic components A, B, C, D. This can be shown by expressing the elements of three relationships resulting from Eq. (5.8.49)

$$W_{11;ii} = W_{11;11} + W_{11;22} + W_{11;33} = W_{11};\quad W_{kk;33} = W_{11;33} + W_{22;33} + W_{33;33} = W_{33};$$
$$W_{1j;1j} = W_{11;11} + W_{12;12} + W_{13;13} = -W_{11}, \tag{5.8.61}$$

by the components A, B, C, D from Eq. (5.8.60):

$$A + B + C = W_{11};\quad 2C + D = W_{33};\quad 3A - B + 2C = 0. \tag{5.8.62}$$

In the case of spherical symmetry, the number of independent elements is further reduced, because in this case

$$A = D;\quad B = C,\qquad (W_{11} = W_{22} = W_{33} = W/3), \tag{5.8.63}$$

as may be seen, for instance, by inserting Eq. (5.8.41) into the definition (5.8.37), and integrating over the angular spherical coordinates λ and φ, $[\Phi = \Phi(r),\ \chi = \chi(r)]$.

From Eqs. (5.8.62) and (5.8.63) we obtain for a sphere

$$A = D = -W/15;\quad B = C = W/5. \tag{5.8.64}$$

With the previously established symmetries of $I_{ij}, W_{ij}, W_{k\ell;ij}$ for axisymmetric configurations, the virial equations (5.8.35) take the explicit form

$$\sigma^2 L_{11}I_{11} + 2i\sigma\Omega L_{21}I_{11} + 2\Omega^2 L_{11}I_{11} - (L_{11}W_{11;11} + L_{22}W_{22;11} + L_{33}W_{33;11})$$
$$+\Pi(L_{11} + L_{22} + L_{33}) = 0, \tag{5.8.65}$$

$$\sigma^2L_{22}I_{11}-2i\sigma\Omega L_{12}I_{11}+2\Omega^2L_{22}I_{11}-(L_{11}W_{22;11}+L_{22}W_{11;11}+L_{33}W_{33;11})$$
$$+\Pi(L_{11}+L_{22}+L_{33})=0, \qquad (5.8.66)$$

$$\sigma^2L_{33}I_{33}-(L_{11}+L_{22})W_{33;11}-L_{33}W_{33;33}+\Pi(L_{11}+L_{22}+L_{33})=0, \qquad (5.8.67)$$

$$\sigma^2L_{21}I_{11}-2i\sigma\Omega L_{11}I_{11}+(\Omega^2I_{11}-W_{12;12})(L_{12}+L_{21})=0, \qquad (5.8.68)$$

$$\sigma^2L_{12}I_{11}+2i\sigma\Omega L_{22}I_{11}+(\Omega^2I_{11}-W_{12;12})(L_{12}+L_{21})=0, \qquad (5.8.69)$$

$$\sigma^2L_{31}I_{11}-L_{13}W_{31;13}-L_{31}W_{13;13}=0, \qquad (5.8.70)$$

$$\sigma^2L_{32}I_{11}-L_{23}W_{31;13}-L_{32}W_{13;13}=0, \qquad (5.8.71)$$

$$\sigma^2L_{13}I_{33}+2i\sigma\Omega L_{23}I_{33}+(\Omega^2I_{33}-W_{31;13})L_{13}+(\Omega^2I_{11}-W_{13;13})L_{31}=0, \qquad (5.8.72)$$

$$\sigma^2L_{23}I_{33}-2i\sigma\Omega L_{13}I_{33}+(\Omega^2I_{33}-W_{31;13})L_{23}+(\Omega^2I_{11}-W_{13;13})L_{32}=0. \qquad (5.8.73)$$

Before proceeding to the general evaluation of Eqs. (5.8.65)-(5.8.73), we consider the special case $\Omega=0$, when the unperturbed polytrope is spherical, and

$$I_{11}=I_{22}=I_{33}=I/3; \quad W_{11}=W_{22}=W_{33}=W/3. \qquad (5.8.74)$$

Only three distinct components of the supermatrix $W_{k\ell;ij}$ exist in this case by virtue of Eqs. (5.8.60), (5.8.64):

$$W_{11;11}=A=-W/15; \quad W_{11;22}=B=W/5; \quad W_{12;12}=(W_{11;11}-W_{11;22})/2=-2W/15. \qquad (5.8.75)$$

The equations (5.8.68)-(5.8.73) governing the nondiagonal elements of L_{ij} become

$$\sigma^2IL_{21}/3+2W(L_{12}+L_{21})/15=0; \quad \sigma^2IL_{12}/3+2W(L_{12}+L_{21})/15=0;$$
$$\sigma^2IL_{32}/3+2W(L_{23}+L_{32})/15=0; \quad \sigma^2IL_{23}/3+2W(L_{23}+L_{32})/15=0;$$
$$\sigma^2IL_{13}/3+2W(L_{31}+L_{13})/15=0; \quad \sigma^2IL_{31}/3+2W(L_{31}+L_{13})/15=0. \qquad (5.8.76)$$

These three pairs of homogeneous equations have nontrivial solutions if their equal determinants $(\sigma^2I/3)(\sigma^2I/3+4W/15)$ are zero, i.e. if

$$\sigma^2=0 \quad \text{and} \quad \sigma^2=-4W/5I, \qquad (5.8.77)$$

each eigenvalue being repeated three times. The first eigenvalue implies $L_{12}=-L_{21},\ L_{23}=-L_{32},\ L_{13}=-L_{31}$, the second one $L_{12}=L_{21},\ L_{23}=L_{32},\ L_{13}=L_{31}$. Turning next to Eqs. (5.8.65)-(5.8.67), we have

$$\sigma^2IL_{11}/3+WL_{11}/15-W(L_{22}+L_{33})/5+\Pi(L_{11}+L_{22}+L_{33})=0, \qquad (5.8.78)$$

$$\sigma^2IL_{22}/3+WL_{22}/15-W(L_{33}+L_{11})/5+\Pi(L_{11}+L_{22}+L_{33})=0, \qquad (5.8.79)$$

$$\sigma^2IL_{33}/3+WL_{33}/15-W(L_{11}+L_{22})/5+\Pi(L_{11}+L_{22}+L_{33})=0. \qquad (5.8.80)$$

Recall that now $\Pi=(\Gamma_1-1)W/3$ via Eq. (5.8.38). Subtracting Eqs. (5.8.78)-(5.8.80) among themselves, we get the three equations

$$\sigma^2I(L_{11}-L_{22})/3+4W(L_{11}-L_{22})/15=0; \quad \sigma^2I(L_{22}-L_{33})/3+4W(L_{22}-L_{33})/15=0;$$
$$\sigma^2I(L_{33}-L_{11})/3+4W(L_{33}-L_{11})/15=0. \qquad (5.8.81)$$

If $L_{11} \neq L_{22} \neq L_{33}$, the root of these equations is

$$\sigma^2 = -4W/5I, \tag{5.8.82}$$

which is only of multiplicity two, because only two of the three equations are linearly independent. Finally, adding together the three equations (5.8.78)-(5.8.80), we obtain

$$\sigma^2 I(L_{11}+L_{22}+L_{33})/3 + (L_{11}+L_{22}+L_{33})(-W/3+3\Pi) = (\Gamma_1 - 4/3)W(L_{11}+L_{22}+L_{33}), \tag{5.8.83}$$

with the eigenvalue

$$\sigma^2 = (4-3\Gamma_1)W/I, \qquad (L_{11}+L_{22}+L_{33} \neq 0). \tag{5.8.84}$$

The foregoing discussion exhibits the eigenvalues of two fundamental modes of oscillation of a sphere: Eq. (5.8.84) is just the eigenvalue of the fundamental mode of radial (or quasiradial) oscillation from Eq. (5.3.16). And the root $\sigma^2 = -4W/5I$ from Eqs. (5.8.77) and (5.8.82) (of multiplicity 5) is just equal to the eigenvalue of the fundamental (Kelvin) f-mode, belonging to the five second order spherical harmonics $Y_2^0, Y_2^{\pm 1}, Y_2^{\pm 2}$. If $n=0$, we have $I = 4\pi\varrho r^5/5,\ W = -16\pi^2 G\varrho^2 r^5/15,\ \sigma^2 = -4W/5I = 16\pi G\varrho/15$, and we recover with $\omega^2 = \sigma^2/4\pi G\varrho = 4/15$ just the $f(j=2)$-mode from Eq. (5.5.26). In addition, we have three neutral modes $\sigma^2 = 0$, belonging to the triple root (5.8.77), and corresponding to rotations about three principal axes. And finally, if $\Gamma_1 = 1.6$ the two nonvanishing eigenvalues $-4W/5I$ and $(4-3\Gamma_1)W/I$ coincide, and we have a case of accidental degeneracy (cf. Sec. 5.7.3).

Actually, the virial equations lead to oscillations with angular dependences corresponding in the nonrotating limit ($\Omega = 0$) to tesseral [$k = \pm 1, \pm 2, ... \pm (j-1)$], sectorial ($k = \pm j$), and zonal ($k = 0$) spherical harmonics of the form $Y_j^k(\lambda, \varphi) = P_j^k(\cos\lambda)\exp(ik\varphi)$. Subsequently, these three kinds of modes will be discussed separately for the rotational case within the context of the second order virial theorem (Chandrasekhar and Lebovitz 1962b).

(i) The Tesseral (Transverse-Shear) Modes. We observe that Eqs. (5.8.70)-(5.8.73) – involving $L_{13}, L_{23}, L_{31}, L_{32}$ – are independent of Eqs. (5.8.65)-(5.8.69), so the remaining variational parameters can be set equal to zero in the corresponding Lagrangian displacements (5.8.33), which resume to

$$\Delta x_1(x_1,x_2,x_3) = L_{13}x_3; \quad \Delta x_2(x_1,x_2,x_3) = L_{23}x_3; \quad \Delta x_3(x_1,x_2,x_3) = L_{31}x_1 + L_{32}x_2. \tag{5.8.85}$$

We have suppressed the time-dependent factor $\exp(i\sigma t)$. The oscillations (5.8.85) are characterized by a relative shearing of the northern and southern hemisphere, the motions being antisymmetric (in x_3) with respect to the equatorial plane. The four equations (5.8.70)-(5.8.73) are written in matricial form as

$$\begin{bmatrix} -\sigma^2 I_{11} + W_{13;13} & 0 & W_{31;13} & 0 \\ 0 & -\sigma^2 I_{11} + W_{13;13} & 0 & W_{31;13} \\ -\Omega^2 I_{11} + W_{13;13} & 0 & (-\sigma^2 - \Omega^2)I_{33} + W_{31;13} & -2i\sigma\Omega I_{33} \\ 0 & -\Omega^2 I_{11} + W_{13;13} & 2i\sigma\Omega I_{33} & (-\sigma^2 - \Omega^2)I_{33} + W_{31;13} \end{bmatrix}$$
$$\cdot \begin{bmatrix} L_{31} \\ L_{32} \\ L_{13} \\ L_{23} \end{bmatrix} = 0. \tag{5.8.86}$$

This homogeneous system has nontrivial solutions if its determinant is zero; solving the determinant, we get the characteristic equation for the determination of the four eigenvalues σ^2 :

$$\big\{(\sigma^2 I_{11} - W_{13;13})[(\sigma^2+\Omega^2)I_{33} - W_{31;13}] + (\Omega^2 I_{11} - W_{13;13})W_{31;13}\big\}^2$$
$$= 4\sigma^2\Omega^2 I_{33}^2(\sigma^2 I_{11} - W_{13;13})^2, \tag{5.8.87}$$

or

$$(\sigma^2 - W_{13;13}/I_{11})(\sigma^2 + \Omega^2 - W_{31;13}/I_{33}) + (W_{31;13}/I_{33})(\Omega^2 - W_{13;13}/I_{11})$$
$$= \pm 2\sigma\Omega(\sigma^2 - W_{13;13}/I_{11}). \tag{5.8.88}$$

The occurrence of the double sign on the right-hand side means that the tesseral modes have a doublet character, with Ω and $-\Omega$ playing equivalent roles, as in the normal Zeeman effect. Eq. (5.8.88) can be factorized further:

$$(\sigma \mp \Omega)^2(\sigma^2 - W_{13;13}/I_{11}) - (W_{31;13}/I_{33})(\sigma^2 - \Omega^2)$$
$$= (\sigma \mp \Omega)[(\sigma \mp \Omega)(\sigma^2 - W_{13;13}/I_{11}) - (W_{31;13}/I_{33})(\sigma \pm \Omega)] = 0. \quad (5.8.89)$$

Therefore

$$\sigma = \pm\Omega, \quad (5.8.90)$$

is an eigenvalue. The remaining six eigenvalues result from the expression in the bracket of Eq. (5.8.89):

$$\sigma^3 \mp \Omega\sigma^2 - (W_{13;13}/I_{11} + W_{31;13}/I_{33})\sigma \pm \Omega(W_{13;13}/I_{11} - W_{31;13}/I_{33}) = 0. \quad (5.8.91)$$

With the substitution

$$\zeta = \sigma \mp \Omega/3, \quad (5.8.92)$$

Eq. (5.8.91) is brought to the reduced form

$$\zeta^3 - (\Omega^2/3 + W_{13;13}/I_{11} + W_{31;13}/I_{33})\zeta \mp 2\Omega[\Omega^2 + 9(2W_{31;13}/I_{33} - W_{13;13}/I_{11})]/27. \quad (5.8.93)$$

The necessary and sufficient condition for real roots is the nonpositiveness of the cubic discriminant, which is equivalent to

$$27(\Omega^2/3 + W_{13;13}/I_{11} + W_{31;13}/I_{33})^3 \geq \Omega^2[\Omega^2 + 9(2W_{31;13}/I_{33} - W_{13;13}/I_{11})]^2,$$
$$(\Omega^2/3 + W_{13;13}/I_{11} + W_{31;13}/I_{33} \geq 0). \quad (5.8.94)$$

For small rotation rates the roots of Eq. (5.8.91) can be evaluated algebraically:

$$\sigma_1 = (W_{13;13}/I_{11} + W_{31;13}/I_{33})^{1/2} \pm (\Omega/2)[1 - (W_{13;13}/I_{11} - W_{31;13}/I_{33})$$
$$/(W_{13;13}/I_{11} + W_{31;13}/I_{33})]; \quad \sigma_2 = -(W_{13;13}/I_{11} + W_{31;13}/I_{33})^{1/2}$$
$$\pm(\Omega/2)[1 - (W_{13;13}/I_{11} - W_{31;13}/I_{33})/(W_{13;13}/I_{11} + W_{31;13}/I_{33})];$$
$$\sigma_3 = \pm\Omega(W_{13;13}/I_{11} - W_{31;13}/I_{33})/(W_{13;13}/I_{11} + W_{31;13}/I_{33}), \qquad (\Omega \approx 0). \quad (5.8.95)$$

These formulas exhibit the doublet character of the tesseral modes. For small rotation rates ($\Omega \approx 0$) we observe from Eqs. (5.8.60), (5.8.63), (5.8.64) that $W_{13;13} = W_{31;31} = -2W/15$, $(W < 0)$, so the tesseral modes certainly start being stable (σ real). In the nonrotating spherical limit we have $\sigma_{1,2} = \pm(-4W/5I)^{1/2}$; the corresponding tesseral modes reduce to the f-mode belonging to the tesseral harmonics $Y_2^{\pm 1}(\lambda, \varphi)$, (Tassoul 1978).

(ii) The Sectorial (Toroidal or Barlike) Modes. Returning to the remaining equations (5.8.65)-(5.8.69), we obtain, on subtracting Eq. (5.8.66) from Eq. (5.8.65):

$$\sigma^2 I_{11}(L_{11} - L_{22}) + 2i\sigma\Omega I_{11}(L_{12} + L_{21}) + 2\Omega^2 I_{11}(L_{11} - L_{22}) - (W_{11;11} - W_{22;11})(L_{11} - L_{22}) = 0. \quad (5.8.96)$$

This can be rewritten via Eq. (5.8.56) as

$$[\sigma^2 I_{11} + 2(\Omega^2 I_{11} - W_{12;12})](L_{11} - L_{22}) + 2i\sigma\Omega I_{11}(L_{12} + L_{21}) = 0. \quad (5.8.97)$$

Next, by addition of Eqs. (5.8.68) and (5.8.69), we get

$$\sigma^2 I_{11}(L_{12} + L_{21}) - 2i\sigma\Omega I_{11}(L_{11} - L_{22}) + 2(\Omega^2 I_{11} - W_{12;12})(L_{12} + L_{21}) = 0, \quad (5.8.98)$$

or

$$-2i\sigma\Omega I_{11}(L_{11} - L_{22}) + [\sigma^2 I_{11} + 2(\Omega^2 I_{11} - W_{12;12})](L_{12} + L_{21}) = 0. \quad (5.8.99)$$

Eqs. (5.8.97), (5.8.99) are homogeneous in the two parameters $(L_{11}-L_{22})$ and $(L_{21}+L_{12})$, so their characteristic determinant has to vanish in the case of nontrivial solutions:

$$[\sigma^2 I_{11}+2(\Omega^2 I_{11}-W_{12;12})]^2-4\sigma^2\Omega^2 I_{11}^2=0, \tag{5.8.100}$$

or

$$\big[\sigma^2 I_{11}-2\sigma\Omega I_{11}+2(\Omega^2 I_{11}-W_{12;12})\big]\big[\sigma^2 I_{11}+2\sigma\Omega I_{11}+2(\Omega^2 I_{11}-W_{12;12})\big]=0. \tag{5.8.101}$$

The four solutions of this equation are

$$\sigma_1=\Omega+(2W_{12;12}/I_{11}-\Omega^2)^{1/2};\quad \sigma_2=\Omega-(2W_{12;12}/I_{11}-\Omega^2)^{1/2};$$
$$\sigma_3=-\Omega+(2W_{12;12}/I_{11}-\Omega^2)^{1/2};\quad \sigma_4=-\Omega-(2W_{12;12}/I_{11}-\Omega^2)^{1/2}, \tag{5.8.102}$$

and neutral stability $(\sigma=0)$ of $\sigma_{2,3}$ occurs if $\Omega^2=W_{12;12}/I_{11}$. But this equality [see Eq. (5.8.108)] holds just at the point of bifurcation between axially symmetric configurations (e.g. Maclaurin spheroids) and configurations with triplanar symmetry (e.g. Jacobi ellipsoids). This can be shown by inserting into the virial equilibrium equation (3.1.87)

$$W_{11}+\Omega^2 I_{11}=W_{22}+\Omega^2 I_{22}=W_{33}, \tag{5.8.103}$$

the relationships (5.8.47):

$$W_{11}=W_{11;22}-W_{12;12};\quad W_{22}=W_{11;22}-W_{21;12}. \tag{5.8.104}$$

Eq. (5.8.103) reduces to

$$-W_{12;12}+\Omega^2 I_{11}=-W_{21;12}+\Omega^2 I_{22}=W_{33}-W_{11;22}. \tag{5.8.105}$$

These equations can be satisfied in two ways (Chandrasekhar and Lebovitz 1962a). We may require that the polytrope has axial symmetry, the angular velocity Ω being determined from the last equality in Eq. (5.9.105):

$$W_{12;12}=W_{21;12}\quad\text{and}\quad I_{11}=I_{22}. \tag{5.8.106}$$

The other possibility results by requiring

$$\Omega^2 I_{11}=W_{12;12};\quad \Omega^2 I_{22}=W_{21;12}\quad\text{and}\quad W_{33}=W_{11;22}. \tag{5.8.107}$$

This second requirement can be satisfied only when Ω^2 exceeds a certain critical value, because if $\Omega^2\to 0$ we have $W_{12;12}=-2W/15$ via Eq. (5.8.75), and it is not possible to satisfy the first two equalities in Eq. (5.8.107). Therefore, if $\Omega\approx 0$, the configuration must be axisymmetric, but as we proceed along the sequence with increasing Ω, a point may be reached where

$$\Omega^2 I_{11}=\Omega^2 I_{22}=W_{12;12}=W_{21;12}. \tag{5.8.108}$$

At this point it will become possible to satisfy Eq. (5.8.108) for the first time. For larger values of Ω both conditions (5.8.106) and (5.8.107) could possibly be fulfilled; and this is just what happens at a bifurcation point. The occurrence of the eigenvalues $\sigma_{2,3}=0$ at $\Omega^2=W_{12;12}/I_{11}$ simply means that at the bifurcation point (should one occur) the associated neutral mode carries the axially symmetric configuration into one with genuine triplanar symmetry. The foregoing discussion rests on the assumption that $W_{12;12}$ is positive, as Ω^2 and I_{ij} are certainly positive. $W_{12;12}$ is positive for a sphere, and it is numerically verified that $W_{12;12}$ is positive for rotating polytropes (Chandrasekhar and Lebovitz 1962d, Tassoul and Ostriker 1970).

Note, that a bifurcation point is always a point of neutral stability $(\sigma=0)$, but not any neutral point needs to be a bifurcation point (Chandrasekhar 1969, p. 90).

In the nonrotating spherical limit we have $W_{12;12}=-2W/15$, and from Eq. (5.8.102) we recover Eqs. (5.8.77) and (5.8.82): $\sigma^2=-4W/5I>0$. The corresponding sectorial modes reduce to the f-mode belonging to the sectorial harmonics $Y_2^{\pm 2}(\lambda,\varphi)$. The sectorial modes (5.8.102) are definitely stable if $\Omega\to 0$. If $W_{12;12}$ remains positive, then at $\Omega^2>2W_{12;12}/I_{11}$ the eigenvalues (5.8.102) become complex numbers, and the polytrope is dynamically unstable.

To determine the eigenvalues of the sectorial modes, we have used only the four equations (5.8.65), (5.8.66), (5.8.68), (5.8.69). In order to satisfy the whole set of nine equations (5.8.65)-(5.8.73), we have to put $L_{13}, L_{31}, L_{23}, L_{32}, L_{33} = 0$. Moreover, if $L_{33} = 0$, Eq. (5.8.67) can be satisfied only if $L_{11} + L_{22} = 0$ or $L_{11} = -L_{22}$. And with this finding the difference of Eqs. (5.8.68) and (5.8.69) becomes $\sigma^2 I_{11}(L_{12} - L_{21}) = 0$, requiring $L_{12} = L_{21}$ if $\sigma \neq 0$. Thus, the Lagrangian displacements (5.8.33) appropriate to sectorial modes are

$$\Delta x_1(x_1, x_2, x_3) = L_{11}x_1 + L_{12}x_2; \quad \Delta x_2(x_1, x_2, x_3) = L_{12}x_1 - L_{11}x_2; \quad \Delta x_3(x_1, x_2, x_3) = 0. \tag{5.8.109}$$

The predominant feature of these oscillations is to transform an axially symmetric polytrope into a genuine triplanar body, while preserving its plane of symmetry. All motions are parallel and symmetrical with respect to the equatorial symmetry plane.

(iii) The Zonal (Pulsation) Modes. Eqs. (5.8.87) and (5.8.100) for the tesseral and sectorial modes determine six of the nine eigenvalues σ^2 belonging to the virial equations (5.8.65)-(5.8.73). Of the five equations (5.8.65)-(5.8.69) we have considered only two linear combinations of them. It remains to consider three other linear combinations for the determination of the remaining three eigenvalues σ^2. Subtracting Eq. (5.8.69) from Eq. (5.8.68), we obtain

$$\sigma[\sigma I_{11}(L_{21} - L_{12}) - 2i\Omega I_{11}(L_{11} + L_{22})] = 0. \tag{5.8.110}$$

Next, subtracting Eq. (5.8.67) twice from the sum of Eqs. (5.8.65) and (5.8.66), we find

$$\sigma^2[I_{11}(L_{11} + L_{22}) - 2I_{33}L_{33}] + 2i\sigma\Omega I_{11}(L_{21} - L_{12}) + 2\Omega^2 I_{11}(L_{11} + L_{22})$$
$$-2(W_{33;11} - W_{33;33})L_{33} - (W_{11;11} + W_{22;11} - 2W_{33;11})(L_{11} + L_{22}) = 0. \tag{5.8.111}$$

Eq. (5.8.67) will be retained as the third equation. The zonal modes are symmetric with respect to the axis of rotation: $L_{11} = L_{22}$ (Chandrasekhar and Lebovitz 1962b, c, Chandrasekhar 1969, Tassoul 1978). With $L_{11} = L_{22}$ we get from Eq. (5.8.96) $L_{12} = -L_{21}$ if $\sigma \neq 0$. To satisfy the remaining virial equations (5.8.70)-(5.8.73), we have to assume $L_{13}, L_{31}, L_{23}, L_{32} = 0$. The Lagrangian displacements (5.8.33) appropriate to zonal pulsations are therefore

$$\Delta x_1(x_1, x_2, x_3) = L_{11}x_1 + L_{12}x_2; \quad \Delta x_2(x_1, x_2, x_3) = -L_{12}x_1 + L_{11}x_2; \quad \Delta x_3(x_1, x_2, x_3) = L_{33}x_3. \tag{5.8.112}$$

These displacements represent pulsations that preserve both, the planar and axial symmetry of the polytrope. If we ignore the neutral mode $\sigma = 0$ in Eq. (5.8.110), we can eliminate $L_{21} - L_{12} \neq 0$ in Eq. (5.8.111), by making use of Eq. (5.8.110):

$$(-\sigma^2 I_{11} + 2\Omega^2 I_{11} + W_{11;11} + W_{22;11} - 2W_{33;11})(L_{11} + L_{22}) + 2(\sigma^2 I_{33} + W_{33;11} - W_{33;33})L_{33} = 0. \tag{5.8.113}$$

With the relationships from Eqs. (5.8.60), (5.8.62), and with the equilibrium condition $W_{11} + \Omega^2 I_{11} = W_{33}$ from Eq. (5.8.103), we can transform a part of the coefficient of $L_{11} + L_{22}$ as

$$2\Omega^2 I_{11} + W_{11;11} + W_{22;11} - 2W_{33;11} = 2\Omega^2 I_{11} + A + B - 2C$$
$$= 2\Omega^2 I_{11} + W_{11} - 3C = \Omega^2 I_{11} + W_{33} - 3C = \Omega^2 I_{11} - C + D. \tag{5.8.114}$$

Inserting also $W_{33;11} = C$ and $W_{33;33} = D$ into the remaining coefficients of Eqs. (5.8.113) and (5.8.67), we find, respectively:

$$(-\sigma^2 I_{11} + \Omega^2 I_{11} - C + D)(L_{11} + L_{22}) + 2(\sigma^2 I_{33} + C - D)L_{33} = 0, \tag{5.8.115}$$

$$(\Pi - C)(L_{11} + L_{22}) + (\sigma^2 I_{33} + \Pi - D)L_{33} = 0. \tag{5.8.116}$$

For nontrivial solutions the determinant of this system has to vanish, leading to the characteristic equation

$$I_{11}I_{33}\sigma^4 + [(\Pi - D)I_{11} + (2\Pi - C - D)I_{33} - \Omega^2 I_{11}I_{33}]\sigma^2$$
$$-\Omega^2 I_{11}(\Pi - D) + (3\Pi - 2C - D)(C - D) = 0. \tag{5.8.117}$$

Eq. (5.8.117) predicts a coupling between radial and nonradial modes of oscillation, similar to Eq. (5.7.43) for the variational method. In the nonrotating spherical case Eq. (5.8.117) becomes

$$I^2\sigma^4/9+(3\Pi-C-2D)I\sigma^2/3+(3\Pi-2C-D)(C-D)=0, \qquad (I_{11}=I_{33}=I/3;\ \Omega=0). \tag{5.8.118}$$

The roots of this equation are via Eqs. (5.8.38), (5.8.64) equal to

$$\sigma_r^2=3(-3\Pi+2C+D)/I=(4-3\Gamma_1)W/I; \quad \sigma_f^2=3(D-C)/I=-4W/5I, \qquad (\Omega=0). \tag{5.8.119}$$

The two squared eigenvalues (5.8.119) just agree with the eigenvalue (5.8.84) of the fundamental radial mode, and with the eigenvalue (5.8.82) of the nonradial fundamental (Kelvin) f-mode, belonging to the zonal harmonic $Y_2^0(\lambda,\varphi)=P_2(\cos\lambda)$. We now see from Eq. (5.8.117) that rotation couples these modes. And if $\Gamma_1=1.6$ the degeneracy which exists in absence of rotation is lifted by the presence of rotation, as already discussed in Sec. 5.7.3 within the context of the variational approach.

5.8.2 Application to Polytropes

In the case of rotationally distorted polytropes ($0\le n<5$) the previous equations can be used to derive concrete results. For elucidation, the first order perturbation method of Chandrasekhar (1933a) from Sec. 3.2 is employed by Chandrasekhar and Lebovitz (1962d). As we have outlined subsequently to Eq. (5.8.50), the moment of inertia tensor I_{ij} and the potential energy tensor W_{ij} can be brought simultaneously to the diagonal form in the case of triplanar symmetry; if the polytrope also possesses axial symmetry, we have additionally $I_{11}=I_{22},\ W_{11}=W_{22}$. With the dimensionless variables from Eq. (3.2.1) the two distinct components of I_{ij} are

$$I_{11}=I_{22}=\int_V \varrho x_1^2\,dV=\int_V \varrho r^4\cos^2\varphi\ \sin^3\lambda\ dr\ d\lambda\ d\varphi=2\pi\varrho_0\alpha^5\int_0^1(1-\mu^2)\ d\mu\int_0^{\Xi_1(\mu)}\Theta^n\xi^4\ d\xi;$$

$$I_{33}=\int_V \varrho x_3^2\,dV=\int_V \varrho r^4\cos^2\lambda\ \sin\lambda\ dr\ d\lambda\ d\varphi=4\pi\varrho_0\alpha^5\int_0^1\mu^2\ d\mu\int_0^{\Xi_1(\mu)}\Theta^n\xi^4\ d\xi, \quad (\mu=\cos\lambda). \tag{5.8.120}$$

Using the first order solution of the fundamental function (3.2.44), we can replace the integration limit $\Xi_1(\mu)$ by ξ_1, as in Eqs. (3.2.58), (3.2.66):

$$I_{11}=I_{22}=2\pi\varrho_0\alpha^5\int_0^{\xi_1}\xi^4\ d\xi\int_0^1\big\{\theta^n+\beta n\theta^{n-1}[\psi_0+A_2\psi_2P_2(\mu)]\big\}(1-\mu^2)\ d\mu$$

$$=(4\pi\varrho_0\alpha^5/3)\int_0^{\xi_1}(\theta^n+\beta n\theta^{n-1}\psi_0-\beta nA_2\theta^{n-1}\psi_2/5)\xi^4\ d\xi;$$

$$I_{33}=4\pi\varrho_0\alpha^5\int_0^{\xi_1}\xi^4\ d\xi\int_0^1\big\{\theta^n+\beta n\theta^{n-1}[\psi_0+A_2\psi_2P_2(\mu)]\big\}\mu^2\ d\mu$$

$$=(4\pi\varrho_0\alpha^5/3)\int_0^{\xi_1}(\theta^n+\beta n\theta^{n-1}\psi_0+2\beta nA_2\theta^{n-1}\psi_2/5)\xi^4\ d\xi, \qquad [P_2(\mu)=(3\mu^2-1)/2]. \tag{5.8.121}$$

The trace of the tensor I_{ij} (the moment of inertia) is then given by

$$I=2I_{11}+I_{33}=4\pi\varrho_0\alpha^5\int_0^{\xi_1}(\theta^n+\beta n\theta^{n-1}\psi_0)\xi^4\ d\xi. \tag{5.8.122}$$

The nonvanishing components of the gravitational energy tensor can readily be inferred from Eq. (3.1.88):

$$W_{11}=W_{22}=(-\Phi_pM+n\Omega^2I_{11})/(5-n)=(-\Phi_pM+2\beta\pi nG\varrho_0I_{11})/(5-n);$$

$$W_{33}=(-\Phi_pM+5\Omega^2I_{11})/(5-n)=(-\Phi_pM+10\beta\pi G\varrho_0I_{11})/(5-n), \qquad (I_{11}=I_{22}). \tag{5.8.123}$$

The gravitational potential at the poles of the polytrope Φ_p is obtained via Eqs. (3.2.39), (3.2.43):

$$\Phi_p(\Xi_1,\pm 1)=(n+1)K\varrho_0^{1/n}(c_0+\beta c_{10})=(n+1)K\varrho_0^{1/n}\{-\xi_1\theta'(\xi_1)+\beta[\xi_1^2/2-\psi_0(\xi_1)-\xi_1\psi_0'(\xi_1)]\}. \tag{5.8.124}$$

Inserting Eqs. (3.2.58), (5.8.124), (5.8.126) into Eq. (5.8.123), we find eventually

$$W_{11}=[4\pi(n+1)K\varrho_0^{1+1/n}\alpha^3/(5-n)]\Big\{-\xi_1^3\theta'^2(\xi_1)+\beta\xi_1^2\theta'(\xi_1)\ [5\xi_1^2/6-\psi_0(\xi_1)-2\xi_1\psi_0'(\xi_1)]$$
$$+(\beta n/6)\int_0^{\xi_1}\theta^n\xi^4\ d\xi\Big\};\quad W_{33}=[4\pi(n+1)K\varrho_0^{1+1/n}\alpha^3/(5-n)]\Big\{-\xi_1^3\theta'^2(\xi_1)$$
$$+\beta\xi_1^2\theta'(\xi_1)\ [5\xi_1^2/6-\psi_0(\xi_1)-2\xi_1\psi_0'(\xi_1)]+(5\beta/6)\int_0^{\xi_1}\theta^n\xi^4\ d\xi\Big\}. \tag{5.8.125}$$

Consistently with our order of approximation we have used for the moments of inertia the spherical value

$$I/3=I_{11}=I_{22}=I_{33}=(4\pi\varrho_0\alpha^5/3)\int_0^{\xi_1}\theta^n\xi^4\ d\xi. \tag{5.8.126}$$

For axisymmetric configurations all distinct elements of the supermatrix $W_{k\ell;ij}$ can be expressed in terms of one of them. We chose $W_{12;12}$ as this element, expressing the others in terms of it via Eqs. (5.8.60), (5.8.62). We integrate the last integral from Eq. (5.8.58) by parts, and transform to spherical coordinates r,μ :

$$W_{12;12}=(\pi/4)\int_{-z_1}^{z_1}dz\int_0^{\ell_1(z)}\varrho\big\{\partial\big[\ell^3\ \partial\big((1/\ell)\ \partial\chi/\partial\ell\big)/\partial\ell\big]/\partial\ell\big\}\ d\ell$$
$$=(\pi\varrho_0\alpha^3/4)\int_{-1}^{1}d\mu\int_0^{\Xi_1(\mu)}(\Theta^n/\ell)\big\{\partial\big[\ell^3\ \partial\big((1/\ell)\ \partial\chi/\partial\ell\big)/\partial\ell\big]/\partial\ell\big\}\xi^2\ d\xi,$$
$$(\ell\ d\ell\ dz\to\alpha^3\xi^2\ d\xi\ d\mu;\ 0<n<5). \tag{5.8.127}$$

With $r=\alpha\xi,\ \ell=\alpha s=\alpha\xi\sin\lambda,\ z=\alpha\zeta=\alpha\xi\cos\lambda$ from Eqs. (3.5.5), (3.5.11), and defining the new operators

$$D_\xi\chi=(1/s)\ \partial\chi/\partial s=(1/s)[(\partial\chi/\partial\xi)\ \partial\xi/\partial s+(\partial\chi/\partial\mu)\ \partial\mu/\partial s]=(1/\xi)\ \partial\chi/\partial\xi-(\mu/\xi^2)\ \partial\chi/\partial\mu;$$
$$D_\mu\chi=(1/\xi)\ \partial\chi/\partial\mu,\qquad \big(\xi^2=s^2+\zeta^2;\ \lambda=\arctan(s/\zeta);\ \partial\xi/\partial s=s/\xi;$$
$$\partial\mu/\partial s=-\sin\lambda\ \partial\lambda/\partial s=-\zeta\sin\lambda/\xi^2=-\mu s/\xi^2\big), \tag{5.8.128}$$

the partial derivatives with respect to ℓ are transformed into derivatives with respect to ξ and μ. To this end we observe that the Laplacian (5.8.42) can be written under the form

$$\nabla^2\chi=(\xi^2\ D_{\xi\xi}\chi+2\xi\mu\ D_{\xi\mu}\chi+D_{\mu\mu}\chi+3D_\xi\chi)/\alpha^2=-2\Phi, \tag{5.8.129}$$

where

$$D_{\xi\xi}\chi=(1/\xi)\ \partial D_\xi\chi/\partial\xi-(\mu/\xi^2)\ \partial D_\xi\chi/\partial\mu=-(1/\xi^3)\ \partial\chi/\partial\xi+(1/\xi^2)\ \partial^2\chi/\partial\xi^2$$
$$+(3\mu/\xi^4)\ \partial\chi/\partial\mu-(2\mu/\xi^3)\ \partial^2\chi/\partial\xi\partial\mu+(\mu^2/\xi^4)\ \partial^2\chi/\partial\mu^2;\quad D_{\xi\mu}\chi=(1/\xi)\ \partial D_\xi\chi/\partial\mu$$
$$=(1/\xi^2)\ \partial^2\chi/\partial\xi\partial\mu-(\mu/\xi^3)\ \partial^2\chi/\partial\mu^2-(1/\xi^3)\ \partial\chi/\partial\mu;\quad D_{\mu\mu}=(1/\xi^2)\ \partial^2\chi/\partial\mu^2. \tag{5.8.130}$$

Since by virtue of Eq. (5.8.128) there subsists $(1/\ell)\ \partial\chi/\partial\ell=(1/\alpha^2)\ D_\xi\chi$, we can write instead of Eq. (5.8.127):

$$W_{12;12}=(\pi\varrho_0\alpha/4)\int_{-1}^{1}d\mu\int_0^{\Xi_1(\mu)}\Theta^n\ D_\xi\big[\xi^4(1-\mu^2)^2\ D_{\xi\xi}\chi\big]\xi^2\ d\xi$$
$$=-(\pi\varrho_0\alpha/4)\int_{-1}^{1}d\mu\int_0^{\Xi_1(\mu)}\Theta^n\ D_\xi\big[\xi^2(1-\mu^2)^2(2\alpha^2\Phi+2\xi\mu\ D_{\xi\mu}\chi+D_{\mu\mu}\chi+3D_\xi\chi)\big]\xi^2\ d\xi. \tag{5.8.131}$$

Table 5.8.1 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of tesseral $(\omega_{t1}, \omega_{t2}, \omega_{t3})$, sectorial (ω_s), and zonal modes $(\omega_{zr}, \omega_{zf})$, obtained with the second order virial equations for rotating polytropes from Eqs. (5.8.89), (5.8.102), (5.8.117). ω_{zr} denotes the eigenfrequency belonging to the mode which is radial in absence of rotation (cf. ω_0 in Table 5.3.1 if $\beta = 0$), and ω_{zf} is the eigenfrequency of the fundamental (Kelvin) f-mode belonging to the zonal harmonic $P_2(\mu)$, (cf. Table 5.5.3 if $j = 2,\ \beta = 0$), (Chandrasekhar and Lebovitz 1962d). Note, that tesseral eigenfrequencies are unsquared

n	$\omega_{t1,t2}$	ω_{t3}	ω_s^2	ω_{zr}^2	ω_{zf}^2
1	$\pm 0.39 \pm 0.35\beta^{1/2}$	$\pm 0.033\beta^{1/2}$	$0.16 \pm 0.56\beta^{1/2}$	$(3\Gamma_1 - 4)(0.19 - 0.39\beta) + \beta/3$	$0.16 + 0.58\beta$
1.5	$\pm 0.35 \pm 0.35\beta^{1/2}$	$\pm 0.12\beta^{1/2}$	$0.12 \pm 0.50\beta^{1/2}$	$(3\Gamma_1 - 4)(0.16 - 0.47\beta) + \beta/3$	$0.12 + 0.53\beta$
2	$\pm 0.32 \pm 0.35\beta^{1/2}$	$\pm 0.34\beta^{1/2}$	$0.10 \pm 0.45\beta^{1/2}$	$(3\Gamma_1 - 4)(0.13 - 0.58\beta) + \beta/3$	$0.10 + 0.45\beta$
3	$\pm 0.26 \pm 0.35\beta^{1/2}$	$\pm 1.9\beta^{1/2}$	$0.065 \pm 0.36\beta^{1/2}$	$(3\Gamma_1 - 4)(0.082 - 1.1\beta) + \beta/3$	$0.065 + 0.15\beta$
3.5	$\pm 0.23 \pm 0.35\beta^{1/2}$	$\pm 4.7\beta^{1/2}$	$0.051 \pm 0.32\beta^{1/2}$	$(3\Gamma_1 - 4)(0.064 - 1.7\beta) + \beta/3$	$0.051 - 0.22\beta$

After some lengthy transformations Chandrasekhar and Lebovitz (1962d) obtain $W_{12;12}$ in terms of ordinary integrals

$$W_{12;12} = 4\pi(n+1)K\varrho_0^{1+1/n}\alpha^3\Big\{2\xi_1^3\theta'^2(\xi_1)/5(5-n) + \beta\Big[(8/105)\int_0^{\xi_1}\theta^n\xi^4\ d\xi$$
$$+(2/5)\int_0^{\xi_1}\theta^n\psi_0\xi^2\ d\xi - (A_2/35)\int_0^{\xi_1}n\theta^{n-1}\psi_2\xi^2(2\theta + 2c_0 + 3h_0'/\xi)\ d\xi\Big]\Big\}. \tag{5.8.132}$$

ψ_0, ψ_2, c_0, A_2 are from Eqs. (3.2.43), (3.2.44), and $h_0' = dh_0(\xi)/d\xi$ is the derivative of the dimensionless zero order (spherically symmetric) term from the expansion of the superpotential (5.8.39):

$$\chi(\xi,\mu) = (n+1)K\varrho_0^{1/n}\alpha^2\{h_0(\xi) + \beta[h_{10}(\xi) + h_{12}(\xi)\ P_2(\mu)]\}. \tag{5.8.133}$$

The eigenvalues of tesseral, sectorial, and zonal modes are given by Eqs. (5.8.89), (5.8.102), and (5.8.117), respectively. In accordance with the previous general discussion it is obvious from Table 5.8.1 that for slowly rotating polytropes all tesseral and sectorial modes are stable ($\sigma^2 > 0$), while the zonal mode ω_{zr}, which is radial in absence of rotation, becomes unstable if $\Gamma_1 < 4/3$, ($\beta \ll 1$). As expected from the comment subsequent to Eq. (5.8.33), the eigenfrequencies ω_{zr}^2 and ω_{zf}^2 compare favourably to the corresponding eigenfrequencies $\omega^2(j=0)$ and $\omega^2(j=2)$ from Table 5.7.1 for the centrally less condensed polytropes $n = 1, 1.5, 2$.

We have noted subsequently to Eq. (5.8.33) that the virial results become exact for the constant density polytrope $n = 0$. As an illustration, Fig. 5.8.1 shows the eigenfrequencies $\sigma/(\pi G\varrho_0)^{1/2}$ of the incompressible ($\Gamma_1 = \infty$) Maclaurin ellipsoid. Only the sectorial modes ω_s become dynamically unstable (σ^2 complex) for fast rotation ($\beta = \Omega^2/2\pi G\varrho_0 = 0.22011$) and large eccentricity ($e = 0.95289$) at the point O_2, where $\Omega^2 = 2W_{12;12}/I_{11}$ [(cf. Eq. (5.8.102)].

The findings of Darwin, Poincaré, Jeans, Cartan, Chandrasekhar (1969), Christodoulou et al. (1995a), and others, concerning the dynamical and secular stability of incompressible Maclaurin and Jacobi ellipsoids, may be briefly summarized as follows: As the eccentricity $e = (1 - a_3^2/a_1^2)^{1/2}$, ($a_1 = a_2$) of the incompressible Maclaurin ellipsoid increases, a first neutral point occurs for the sectorial modes at $e = 0.81267,\ \beta = 0.18711,\ \tau_b = E_{kin}/|W| = 0.1375$, where the Jacobi ellipsoids branch off (cf. Secs. 3.2, 3.8.4, 5.7.4, 6.1.8, 6.4.3). At this point the Maclaurin ellipsoids become secularly unstable, and at $e = 0.95289,\ \beta = 0.22011,\ \tau_d = 0.2738$ they become dynamically unstable (Fig. 5.8.1, Lyttleton 1953, Chandrasekhar 1969, Tassoul 1978). At $a_2/a_1 = 0.4322,\ a_3/a_1 = 0.3451,\ \beta = 0.14201$ a first neutral point occurs along the Jacobi sequence, where the secularly unstable pear-shaped configurations branch off. In an important paper Christodoulou et al. (1995a) have shown that the Jacobi sequence remains secularly and dynamically stable – even at the bifurcation of the dumbbell-binary sequence (Fig. 3.8.3) – contrary to earlier beliefs.

Further neutral points along the incompressible Maclaurin sequence, belonging to the third-harmonic modes occur at $e = 0.89926,\ \beta = 0.22007$, and $e = 0.96937,\ \beta = 0.20707$; other neutral points along the Maclaurin and Jacobi sequences, belonging to the fourth harmonics, can be located with the aid of the fourth order virial equations (Sec. 3.8.4; Chandrasekhar 1969, p. 128).

Concerning the influence of rotation on the dynamical instability of the fundamental quasiradial r-mode, the values of Clement (1965) from Table 5.7.1 indicate that the squared eigenfrequency of the

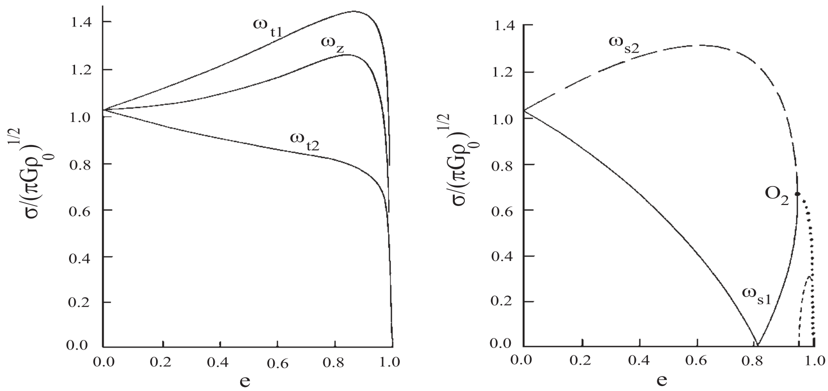

Fig. 5.8.1 Dimensionless eigenfrequencies $\sigma/(\pi G\varrho_0)^{1/2}$ of tesseral $(\omega_{t1}, \omega_{t2})$ and zonal (ω_z) modes (left-hand side), and of sectorial modes $(\omega_{s1}, \omega_{s2}$; right-hand side) belonging to second order harmonic oscillations $(Y_2^k(\lambda, \varphi)$ if $\Omega = 0)$ of the incompressible Maclaurin ellipsoid $(\Gamma_1 = \infty)$. The bifurcation point $(\omega_{s1} = 0)$ occurs at $e = 0.81267$, and ω_{s1} increases again up to 0.66349 at the point of onset of dynamical instability O_2. The other eigenfrequency ω_{s2} (long-dashed curve) increases from 1.0328 up to 1.30 at $e \approx 0.6$, and afterwards falls to 0.66349 at O_2. The dotted line shows the real part of ω_{s1}, ω_{s2}, which decreases from 0.66349 at $e = 0.95289$ to 0 at $e = 1$. And the short-dashed line is the imaginary part of ω_{s1}, ω_{s2}, which changes from 0 at O_2 to 0.23529 at $e = 0.999$ (Chandrasekhar 1969).

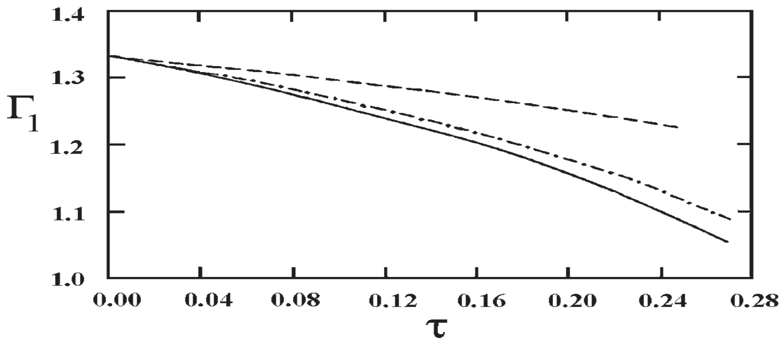

Fig. 5.8.2 For adiabatic indices Γ_1 located below the curves, there occurs dynamical instability of the mode, which in absence of rotation becomes the fundamental radial mode with eigenvalue $\sigma^2 = (3\Gamma_1 - 4)|W|/I$. The differentially rotating polytropic sequences with $n = 0,\ n' = 0$ (solid line), $n = 1.5,\ n' = 0$ (dashed-dotted curve), and $n = 3,\ n' = 0$ (broken curve) are plotted as a function of the parameter $\tau = E_{kin}/|W|$ from Eqs. (3.1.35), (3.1.36). The meaning of the polytropic indices n, n' has already been explained in Sec. 3.8.4, Fig. 3.8.2 (Ostriker and Bodenheimer 1973).

r-mode is decreased (destabilized) due to slow rotation even if $n = 1, 1.5, 2,\ (\Gamma_1 = 5/3)$. The more recent calculations of Clement (1984, Table 7A; 1986, Figs. 1, 2, p_{10}-mode) show that for *fast* rotation the eigenfrequency ω^2 of the r-mode is increased if $n = 1$; it remains approximately constant if $n = 2$, and decreases if $n = 3$, increasing however moderately as limiting rotation is approached (Fig. 5.7.1). For the compressible Maclaurin spheroid $(n = 0)$ Chandrasekhar and Lebovitz (1962c, Table 2A and Fig. 2) have shown that for slow rotation (moderate eccentricity) ω^2 increases if $4/3 \le \Gamma_1 \lesssim 1.5$, and is nearly constant if $\Gamma_1 = 1.6, 5/3$, in agreement with the theoretical estimate from Eq. (5.3.16) if $n \approx 0$, as emphasized subsequently to Eq. (5.3.17). Instability against quasiradial disturbances demands $\sigma^2 < 0$

in Eq. (5.3.16), or

$$\Gamma_1 < 4/3 - 2E_{kin}/3|W| = 4/3 - 2\tau/3, \qquad (\tau = E_{kin}/|W| \approx 0). \tag{5.8.134}$$

This equation is in concordance with Fig. 5.8.2, apparently because the differentially rotating polytropes of Ostriker and Bodenheimer (1973) are initially homogeneous $n' = 0$, and closely mimic the instability properties of Maclaurin ellipsoids ($n = 0$).

In the case of disturbances with *finite* amplitudes no clear-cut distinction exists between stable and unstable quasiradial modes, as in the case of infinitesimal oscillations: Finite disturbances with adiabatic indices ranging between Γ_1 from Eq. (5.8.134) and $\Gamma_1 = 4/3$ can grow (the polytrope becomes metastable) whenever the energy input sustaining the oscillations exceeds a certain limit (Tassoul 1978, Chap. 14).

5.8.3 Secular Instability by Viscous Dissipation

Problems associated with secular stability are often considered within the context of virial equations or variational methods. We elucidate the meaning of secular instability, whose growth rate depends on the magnitude of dissipative forces (e.g. viscosity, dissipation of gravitational radiation energy), by writing down the Narvier-Stokes equations (3.1.12) of the viscous fluid in a frame rotating with constant angular velocity $\vec{\Omega}$:

$$\varrho\ D\vec{v}/Dt = -\varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\varrho\ \vec{\Omega} \times \vec{v} - \nabla P + \varrho\vec{F} + \nabla \cdot \tau. \tag{5.8.135}$$

By preserving in Eq. (3.5.56) only the coefficient of shear (dynamic) viscosity μ, the stress tensor τ assumes the components

$$\tau_{ij} = \mu[\partial v_i/\partial x_j + \partial v_j/\partial x_i - (2\delta_{ij}/3)\ \partial v_k/\partial x_k], \tag{5.8.136}$$

so we can write for the components of $-\nabla P + \nabla \cdot \tau$:

$$\begin{aligned} &-\delta_{ij}\ \partial P/\partial x_j + \partial\tau_{ij}/\partial x_j = -\partial P_{ij}/\partial x_j \\ &= \partial\big\{-\delta_{ij}P + \mu[\partial v_i/\partial x_j + \partial v_j/\partial x_i - (2\delta_{ij}/3)\ \partial v_k/\partial x_k]\big\}\big/\partial x_j, \qquad (P_{ij} = \delta_{ij}P - \tau_{ij}). \end{aligned} \tag{5.8.137}$$

The body force $\vec{F}$ is assumed to contain the internal gravitational potential $\nabla\Phi$, and the tide-generating external potential $\nabla\Phi_t$ of a secondary. We express $-\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ in terms of the centrifugal potential $\nabla\Phi_f$, $[\Phi_f = \Omega^2(x_1^2 + x_2^2)/2$ if $\vec{\Omega} = \vec{\Omega}(0,0,\Omega)]$. Eq. (5.8.135) becomes (Robe 1969)

$$\varrho\ Dv_i/Dt = \varrho\ \partial\Phi_{tot}/\partial x_i - \partial P_{ij}/\partial x_j - 2\varrho(\vec{\Omega} \times \vec{v})_i, \qquad (\Phi_{tot} = \Phi + \Phi_t + \Phi_f), \tag{5.8.138}$$

where summation over the repeated index j is to be understood. We multiply Eq. (5.8.138) with v_i and add together, by using Eq. (5.8.9):

$$d\left[\int_V (\varrho v^2/2)\ dV\right]\bigg/dt = \int_V (\varrho v_i\ \partial\Phi_{tot}/\partial x_i - v_i\ \partial P_{ij}/\partial x_j)\ dV. \tag{5.8.139}$$

The stress tensor acting on the free surface S of the polytrope must vanish, hence (Landau and Lifshitz 1959, Tassoul 1978)

$$(n_i P_{ij})_S = 0, \tag{5.8.140}$$

$\vec{n}(n_1, n_2, n_3)$ denoting the outer normal to the surface. With

$$v_i\ \partial P_{ij}/\partial x_j = \partial(v_i P_{ij})/\partial x_j - P_{ij}\ \partial v_i/\partial x_j, \tag{5.8.141}$$

and via Eq. (2.6.61), the stress integral in Eq. (5.8.139) can be transformed as

$$\begin{aligned} &-\int_V v_i(\partial P_{ij}/\partial x_j)\ dV = \int_V P_{ij}(\partial v_i/\partial x_j)\ dV - \int_S v_i P_{ij} n_j\ dS = \int_V P_{ij}(\partial v_i/\partial x_j)\ dV \\ &= \int_V \bigg\{P\ \nabla \cdot \vec{v} - \mu\bigg[2(\partial v_1/\partial x_1)^2 + 2(\partial v_2/\partial x_2)^2 + 2(\partial v_3/\partial x_3)^2 + (1/2)\sum_{i,j=1}^{3}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)^2 \\ &-(2/3)(\nabla \cdot \vec{v})^2\bigg]\bigg\}\ dV = \int_V P\ \nabla \cdot \vec{v}\ dV - D_R, \end{aligned} \tag{5.8.142}$$

where D_R denotes the Rayleigh dissipation function. The total potential energy W_{tot} of the polytrope is the sum of the potential energies associated with the gravitational, tidal, and centrifugal potentials:

$$W_{tot} = W + W_t + W_f = -(1/2)\int_V \varrho\Phi\ dV - \int_V \varrho(\Phi_t + \Phi_f)\ dV. \tag{5.8.143}$$

Taking into account that the total potential Φ_{tot} does not depend explicitly on time ($\partial\Phi_{tot}/\partial t = 0$), we can write for the change (5.8.9) of the gravitational energy (2.6.65):

$$dW/dt = -(1/2)\ d\Big(\int_V \varrho\Phi\ dV\Big)\Big/dt = -(G/2)\ d\Big[\int_V\int_V \varrho(\vec r)\ \varrho(\vec{r'})\ dV\ dV'/|\vec r - \vec{r'}|\Big]\Big/dt$$
$$= -(G/2)\int_V\int_V \varrho(\vec r)\ \varrho(\vec{r'})\ \big\{[\partial(1/|\vec r - \vec{r'}|)/\partial x_i]\ dx_i/dt + [\partial(1/|\vec r - \vec{r'}|)/\partial x_i']\ dx_i'/dt\big\}\ dV\ dV'$$
$$= -G\int_V\int_V \varrho(\vec r)\ \varrho(\vec{r'})\ [\partial(1/|\vec r - \vec{r'}|)/\partial x_i]v_i\ dV\ dV' = -\int_V \varrho v_i(\partial\Phi/\partial x_i)\ dV. \tag{5.8.144}$$

The derivative of the potential energy of the tidal and centrifugal potential is similar:

$$dW_t/dt = -d\Big(\int_V \varrho\Phi_t\ dV\Big)\Big/dt = -\int_V \varrho v_i(\partial\Phi_t/\partial x_i)\ dV;$$
$$dW_f/dt = -d\Big(\int_V \varrho\Phi_f\ dV\Big)\Big/dt = -\int_V \varrho v_i(\partial\Phi_f/\partial x_i)\ dV, \tag{5.8.145}$$

and therefore

$$dW_{tot}/dt = d(W + W_t + W_f)/dt = -\int_V \varrho v_i(\partial\Phi_{tot}/\partial x_i)\ dV. \tag{5.8.146}$$

Since the left-hand side of Eq. (5.8.139) is just the time derivative of the kinetic energy E_{kin}, we can write via Eqs. (5.8.142), (5.8.146):

$$d(E_{kin} + W_{tot})/dt = \int_V P\ \nabla\cdot\vec v\ dV - D_R. \tag{5.8.147}$$

For an incompressible fluid (ϱ = const; $\Gamma_1 = \infty$), Eq. (5.2.1) yields $\nabla\cdot\vec v = 0$, and Eq. (5.8.147) becomes

$$d(E_{kin} + W_{tot})/dt = -D_R \le 0, \qquad (\varrho = \text{const};\ \Gamma_1 = \infty), \tag{5.8.148}$$

since the dissipation function (5.8.142) is now a sum of squares.

Consider now a configuration which is initially at rest (in stable or unstable equilibrium) with respect to the rotating coordinate system, i.e. $E_{kin} = 0$, W_{tot} = const, and let us apply a small change $\Delta E_{kin} + \Delta W_{tot}$ to the system. ΔE_{kin} is always positive, since initially $E_{kin} = 0$, the kinetic energy being always nonnegative. We have

$$d(\Delta E_{kin} + \Delta W_{tot})/dt = -D_R \le 0, \qquad (\varrho = \text{const};\ \Delta E_{kin} > 0). \tag{5.8.149}$$

If W_{tot} possesses an absolute minimum, then any variation ΔW_{tot} can be only positive, and $\Delta E_{kin} + \Delta W_{tot}$ must be positive. But because the time derivative of this quantity is by virtue of Eq. (5.8.149) negative, the whole variation $\Delta E_{kin} + \Delta W_{tot} > 0$ will continuously decrease with time, and the system will eventually return to its initial position. The system is said to be secularly stable, as well as dynamically stable. On the other hand, if W_{tot} is no longer an absolute minimum, then we may always chose a change in such a way that $\Delta W_{tot} < 0$, and at the same time also $\Delta E_{kin} + \Delta W_{tot} < 0$. But because the derivative (5.8.149) of this negative quantity is negative, $\Delta E_{kin} + \Delta W_{tot}$ decreases still further (increases in absolute value), and the configuration will depart more and more from its initial position: The system is secularly unstable, and it may be dynamically stable or unstable (cf. Secs. 3.2, 5.1; Lyttleton 1953, Ledoux 1958, Robe 1969, Tassoul 1978).

The small viscous stress term (5.8.136) has been considered by Tassoul and Ostriker (1970) in their study on the secular stability of viscous polytropes. The second order virial equations (5.8.35) now read via Eqs. (5.8.151), (5.8.152) as

$$\sigma^2 L_{jk}I_{ik} + 2i\sigma\Omega\varepsilon_{jk3}L_{k\ell}I_{i\ell} + \Omega^2(L_{jk}I_{ik} + L_{ik}I_{jk}) - \Omega^2\delta_{j3}(L_{3k}I_{ik} + L_{ik}I_{3k})$$
$$-L_{\ell k}W_{k\ell;ij} + \delta_{ij}L_{kk}\Pi - i\sigma\eta M_1[L_{ij} + L_{ji} - (2\delta_{ij}/3)L_{kk}] = 0. \tag{5.8.150}$$

The kinematic viscosity η is connected to shear (dynamic) viscosity μ by $\mu = \varrho\eta$, and M_1 denotes the total mass of the polytrope. The viscous term in Eq. (5.8.150) arises if we repeat the derivation of Eq. (3.1.83), by starting with the Narvier-Stokes equations (5.8.135) instead of the Eulerian equations (3.1.79). The virial of the viscous pressure force is

$$\int_V x_i(\partial\tau_{jk}/\partial x_k)\ dV = \int_V [\partial(x_i\tau_{jk})/\partial x_k - \tau_{ij}]\ dV = \int_S x_i\tau_{jk}n_k\ dS - \int_V \tau_{ij}\ dV = -\int_V \tau_{ij}\ dV, \tag{5.8.151}$$

where we have used the Gauss theorem (2.6.61) and the boundary condition (5.8.140). The required first variation (5.8.7) of Eq. (5.8.151) results by inserting for the stress tensor from Eq. (5.8.136):

$$\begin{aligned}
&\delta^*\int_V \tau_{ij}\ dV = \delta^*\int_V \varrho\eta[\partial v_i/\partial x_j + \partial v_j/\partial x_i - (2\delta_{ij}/3)\ \partial v_k/\partial x_k]\ dV \\
&= \int_V \varrho\eta\ \Delta[\partial v_i/\partial x_j + \partial v_j/\partial x_i - (2\delta_{ij}/3)\ \partial v_k/\partial x_k]\ dV + \int_V \varrho\ \Delta\eta\ [\partial v_i/\partial x_j + \partial v_j/\partial x_i \\
&-(2\delta_{ij}/3)\ \partial v_k/\partial x_k]\ dV \approx \eta\int_V \varrho\big[\partial\Delta v_i/\partial x_j + \partial\Delta v_j/\partial x_i - (2\delta_{ij}/3)\ \partial\Delta v_k/\partial x_k]\ dV \\
&\approx \eta\int_V \varrho\big\{\partial[D(\Delta x_i)/Dt]/\partial x_j + \partial[D(\Delta x_j)/Dt]/\partial x_i - (2\delta_{ij}/3)\ \partial[D(\Delta x_k)/Dt]/\partial x_k\big\}\ dV \\
&\approx \eta\int_V \varrho\big\{\partial[\partial\Delta x_i/\partial x_j + \partial\Delta x_j/\partial x_i - (2\delta_{ij}/3)\ \partial\Delta x_k/\partial x_k]/\partial t\big\}\ dV = i\sigma\eta\int_V \varrho[\partial\Delta x_i/\partial x_j \\
&+\partial\Delta x_j/\partial x_i - (2\delta_{ij}/3)\ \partial\Delta x_k/\partial x_k]\ dV = i\sigma\eta M_1[L_{ij} + L_{ji} - (2\delta_{ij}/3)L_{kk}], \qquad (\Delta\eta \ll \eta).
\end{aligned} \tag{5.8.152}$$

To derive this final form we have considered Eqs. (5.1.19), (5.1.24), (5.8.33), neglecting second order products of $\Delta\eta,\ \partial\Delta x_i/\partial x_j,\ \partial v_i/\partial x_j$.

In the presence of viscosity the trial displacements (5.8.33) do not satisfy all the conditions (5.8.140) imposed on the stress tensor P_{ij} at the distorted boundary. But these displacements may still be used in the present connection, provided we restrict our attention to small deviations from the inviscid solution, i.e. to a small kinematic viscosity η (Rosenkilde 1967). Since second order harmonic deformations of the tesseral and zonal type are damped by viscosity (Tassoul and Ostriker 1970), we concentrate on the sectorial modes, proceeding with Eq. (5.8.150) in the same way as for the inviscid sectorial modes (5.8.97) and (5.8.99). We obtain the homogeneous system

$$[\sigma^2 I_{11} + 2(\Omega^2 I_{11} - W_{12;12}) - 2i\sigma\eta M_1](L_{11} - L_{22}) + 2i\sigma\Omega I_{11}(L_{12} + L_{21}) = 0, \tag{5.8.153}$$

$$-2i\sigma\Omega I_{11}(L_{11} - L_{22}) + [\sigma^2 I_{11} + 2(\Omega^2 I_{11} - W_{12;12}) - 2i\sigma\eta M_1](L_{12} + L_{21}) = 0. \tag{5.8.154}$$

The determinant of this homogeneous system has to vanish for nontrivial solutions:

$$\begin{aligned}
&[\sigma^2 I_{11} + 2\sigma(-\Omega I_{11} - i\eta M_1) + 2(\Omega^2 I_{11} - W_{12;12})] \\
&\times[\sigma^2 I_{11} + 2\sigma(\Omega I_{11} - i\eta M_1) + 2(\Omega^2 I_{11} - W_{12;12})] = 0.
\end{aligned} \tag{5.8.155}$$

Restricting ourselves to the first bracket (the last one is simply obtained by reversing the sense of rotation from Ω to $-\Omega$), we seek the solution under the perturbed form $\sigma_\eta = \sigma_2 + \Delta\sigma_\eta$, where σ_2 is the eigenvalue of the sectorial mode (5.8.102) that vanishes at the bifurcation point $\Omega^2 = W_{12;12}/I_{11}$, and $\Delta\sigma_\eta$ is the small correction due to viscosity. The first bracket yields up to the first order (Rosenkilde 1967, Chandrasekhar 1969, §37, Tassoul and Ostriker 1970):

$$\begin{aligned}
&\Delta\sigma_\eta = i\sigma_2\eta M_1/I_{11}(\sigma_2 - \Omega) = i\eta M_1[(2W_{12;12}/I_{11} - \Omega^2)^{1/2} - \Omega]/I_{11}(2W_{12;12}/I_{11} - \Omega^2)^{1/2}, \\
&(\eta \ll 1).
\end{aligned} \tag{5.8.156}$$

This equation, representing the contribution of the secular viscous stress term to the relevant sectorial eigenvalue σ_2, shows that the oscillations $\exp(i\ \Delta\sigma_\eta\ t)$ are damped prior to the bifurcation point $\Omega^2 = W_{12;12}/I_{11}$, but they grow between the bifurcation point and the point of onset of dynamical instability,

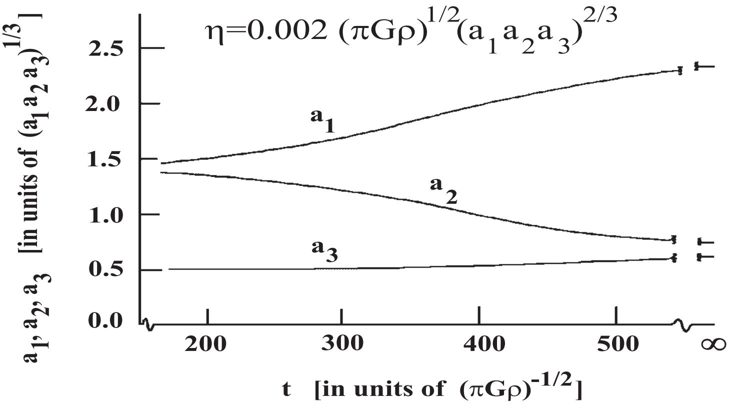

Fig. 5.8.3 Principal axes of a perturbed viscid Maclaurin ellipsoid ($n = 0$) as it evolves monotonically during the course of time into the corresponding Jacobi ellipsoid (Press and Teukolsky 1973).

when $W_{12;12}/I_{11} < \Omega^2 < 2W_{12;12}/I_{11}$. In this range of Ω the viscous part $i\ \Delta\sigma_\eta$ is positive and $\exp(i\ \Delta\sigma_\eta\ t)$ increases; the polytrope is secularly unstable, with an e-folding time scale of

$$t_\eta = 1/i\ \Delta\sigma_\eta = I_{11}(2W_{12;12}/I_{11} - \Omega^2)^{1/2}/\eta M_1[\Omega - (2W_{12;12}/I_{11} - \Omega^2)^{1/2}]. \tag{5.8.157}$$

Obviously, the characteristic time scale of viscous secular instability decreases from ∞ at the bifurcation point $\Omega^2 = W_{12;12}/I_{11}$ to 0 at the point of dynamical instability $\Omega^2 = 2W_{12;12}/I_{11}$, should this point ever been reached by the uniformly rotating polytrope.

In the case of a homogeneous Maclaurin ellipsoid ($n = 0$) we have $I_{ij} = \delta_{ij}M_1a_i^2/5$, ($M_1 = 4\pi\varrho a_1a_2a_3/3$), and Eq. (5.8.156) turns into Eq. (135), §37 from Chandrasekhar (1969; $W_{12;12}/I_{11} \to 2B_{11}$). As already mentioned at the end of Sec. 3.8.8, the ratio $\tau = E_{kin}/|W|$ ranges for the incompressible Maclaurin spheroid from $\tau = 0$ (spherical body) to $\tau = 0.5$ [an infinitely thin disk having $a_1 = a_2 = \infty$; $a_3 = 0$; $\beta = 0$; Eqs. (5.10.217)-(5.10.223)], and for the incompressible Jacobi ellipsoids from $\tau_b = 0.1375$ at the bifurcation point to $\tau = 0.5$ (infinitely thin needle having $a_1 = \infty$; $a_2 = a_3 = 0$; $\beta = 0$). If $0 \leq \tau \leq \tau_b$, the Maclaurin spheroids are the only possible equilibrium figures, whereas in the range $\tau_b < \tau \leq 0.5$ to each value of τ there correspond two ellipsoidal figures of equilibrium: Maclaurin and Jacobi ellipsoids. Since the total mechanical energy $E_{kin} + W$ is smaller in a triplanar Jacobi ellipsoid as compared to an axisymmetric Maclaurin ellipsoid having the same mass, density, and angular momentum, we may expect that under the influence of viscous dissipation a Maclaurin spheroid with $\tau > \tau_b$ will gradually evolve by relative internal motions into a rigidly rotating Jacobi ellipsoid, where viscosity ceases to play any role (Tassoul 1978). This has indeed been shown by direct numerical integration of the Narvier-Stokes equations (5.8.138), (Press and Teukolsky 1973, and Fig. 5.8.3). If the combined influence of viscosity and gravitational radiation reaction is taken into account, the triaxial Jacobi ellipsoid loses angular momentum by gravitational radiation, and evolves towards a stable member of the Maclaurin sequence having $\tau < \tau_b$ (Detweiler and Lindblom 1977).

Eq. (5.8.156) shows that if a bifurcation point occurs in an axisymmetric, uniformly rotating polytropic sequence, the polytrope will always become secularly unstable when $\tau > \tau_b$ or $\Omega^2 > W_{12;12}/I_{11}$. As noted at the end of Sec. 3.8.1, bifurcation points always occur in polytropes with index $0 \leq n \leq 0.808$ if $\tau = \tau_b \approx 0.14$: We have $\tau_b = 0.1375$ and 0.1374 if $n = 0$ and 0.6, respectively (Tassoul and Ostriker 1970). Axisymmetric polytropes are therefore always secularly unstable beyond the bifurcation point. More centrally condensed polytropes with index $n > 0.808$ never reach a value of $\tau = 0.14$, because equatorial mass loss already occurs at $\tau_c = 1.2 \times 10^{-1},\ 5.95 \times 10^{-2},\ 9.00 \times 10^{-3},\ 1.19 \times 10^{-3}$ if $n = 1, 1.5, 3, 4$, respectively (Secs. 3.8.8, 6.1.8, Fig. 3.8.10, Table 5.8.2, Tassoul and Ostriker 1970, Hachisu 1986a). In other words, the amount of rotational energy E_{kin} that a uniformly rotating, centrally condensed polytrope can store, is not very large as compared to its gravitational energy $|W|$; only in the small range $0 \leq \tau \ll 0.5$ we can construct centrally condensed polytropic equilibrium figures, and no polytrope with $n > 0.808$ is able to sustain enough rotational kinetic energy to reach the bifurcation value $\tau_b \approx 0.14$.

As remarked by Tassoul (1978, p. 267), the close resemblance between the Maclaurin sequence and the differentially rotating polytropic models of Bodenheimer and Ostriker (1973; cf. Fig. 3.8.2), and Ostriker and Bodenheimer (1973; cf. Fig. 5.8.2) may be due to the fact that their inner level surfaces never deviate greatly from Maclaurin spheroids. For these somewhat artificial models, bifurcation due to the barlike (sectorial, toroidal) σ_2-mode from Eq. (5.8.102) occurs always at $\tau_b \approx 0.14$ (if bifurcation occurs at all). At the bifurcation point ($\sigma = 0$) these polytropes become secularly unstable to nonaxisymmetric barlike disturbances. The onset of dynamical instability to barlike oscillations – when the eigenvalues (5.8.102) become complex – takes place at $\tau_d = 0.26 \pm 0.02$. Higher order modes have been included in the more elaborate investigations of the nonaxisymmetric secular (Imamura et al. 1995) and dynamical (Toman et al. 1998, Imamura et al. 2000) instability of differentially rotating polytropic models.

5.8.4 Secular Instability Due to Gravitational Radiation Reaction

Chandrasekhar (1970) discovered that gravitational radiation reaction, i.e. reaction of the fluid elements to the emission of gravitational waves, can cause a nonaxisymmetric normal mode with an $\exp(ik\varphi)$, $(k \geq 2)$ azimuthal dependence to become secularly unstable (Chandrasekhar-Friedman-Schutz instability). This weak secular effect sets in at certain nonaxisymmetric points of bifurcation ($\tau = \tau_b$) along a sequence of axisymmetric configurations. After exhausting its nuclear fuel a star may collapse to a white dwarf ($1.5 \leq n \leq 3$, Sec. 1.6) or to a neutron star ($0.5 \lesssim n \lesssim 1$, Imamura et al. 1985), and the resulting compact object may be rapidly rotating, having $\tau = E_{kin}/|W| > \tau_b$. Secular instabilities caused by viscosity or gravitational radiation reaction could drive the object away from the axisymmetric state to a nonaxisymmetric configuration, whereby kinetic rotational energy would be dissipated, until a secularly stable state is reached (Lindblom and Detweiler 1977).

A necessary condition for the onset of secular instability to gravitational radiation reaction is the occurrence of a neutral mode ($\sigma = 0$) in the inertial frame, where the object is considered (Managan 1986). Instability due to viscosity sets in when a mode has zero frequency $\sigma = 0$ in a frame corotating with the polytrope. Because the gravitational radiation emitted by a secularly unstable, nonaxisymmetric mode will carry off angular momentum (e.g. Landau and Lifschitz 1987), the final rotation rate of a compact object [white dwarf, neutron star (pulsar)] is in principle limited by this instability, provided the magnetic field is as ineffective in spinning down the object, as seems to be the case for millisecond pulsars (Friedman 1983). The neutral modes of nonaxisymmetric perturbations determine the critical angular velocities at which secular instabilities caused by dissipative processes first occur, and constitute upper limits to the rotation rates of compact objects, attracting much interest since the discovery of rapidly rotating pulsars.

The Burke-Thorne formalism includes the general relativistic effect of gravitational radiation reaction on an otherwise Newtonian system, by adding to the Newtonian gravitational potential Φ, the gravitational radiation-reaction potential (e.g. Miller 1973)

$$\Phi_g = -(G/15c^5)\ d^5[x_i x_j(3I_{ij} - \delta_{ij}I_{\ell\ell})]/dt^5, \qquad (I_{\ell\ell} = I_{11} + I_{22} + I_{33}). \tag{5.8.158}$$

The equations of motion of the polytrope in an inertial frame are then

$$\varrho\ Dv_i/Dt = -\partial P/\partial x_i + \varrho\ \partial(\Phi + \Phi_g)/\partial x_i = -\partial P/\partial x_i + \varrho\ \partial\Phi/\partial x_i$$
$$-(2\varrho G/15c^5)(3x_j\ d^5I_{ij}/dt^5 - x_i\ d^5I_{\ell\ell}/dt^5). \tag{5.8.159}$$

The contribution made by gravitational radiation reaction to the second order virial equations (3.1.83) is given by

$$\int_V x_k(\partial\Phi_g/\partial x_i)\ dV = -(2G/15c^5)\int_V (3x_jx_k\ d^5I_{ij}/dt^5 - x_ix_k\ d^5I_{\ell\ell}/dt^5)\varrho\ dV$$
$$= -(2G/15c^5)(3I_{jk}\ d^5I_{ij}/dt^5 - I_{ik}\ d^5I_{\ell\ell}/dt^5). \tag{5.8.160}$$

where the fifth time derivative of the moments of inertia can be taken outside the volume integral. The small radiation-reaction term (5.8.160) is simply added to the virial equations (3.1.83). Again, tesseral and zonal modes are unaffected or damped by gravitational radiation reaction, and we concentrate ourselves on the characteristic equation for the sectorial (bar) modes, i.e. the first bracket of Eq. (5.8.101) in the nonrelativistic limit $D = 0$ (Miller 1973):

$$\sigma^2 - 2\sigma\Omega + 2(\Omega^2 - W_{12;12}/I_{11}) + 4iD(2\Omega - \sigma)^5/5 = 0; \quad D = (G/c^5)\int_V x_1^2 \varrho \, dV = GI_{11}/c^5. \tag{5.8.161}$$

Letting $\sigma = \sigma_1 + \Delta\sigma_g,\ (|\Delta\sigma_g| \ll \sigma_1)$, where σ_1 is the sectorial eigenvalue from Eq. (5.8.102), we get

$$\Delta\sigma_g = 2iD(2\Omega - \sigma_1)^5/5(\Omega - \sigma_1) = 2iD[(2W_{12;12}/I_{11} - \Omega^2)^{1/2} - \Omega]^5/5(2W_{12;12}/I_{11} - \Omega^2)^{1/2}, \tag{5.8.162}$$

with a characteristic e-folding time of the gravitational radiation-reaction instability equal to

$$t_g = 1/i\ \Delta\sigma_g = 5(2W_{12;12}/I_{11} - \Omega^2)^{1/2}/2D[\Omega - (2W_{12;12}/I_{11} - \Omega^2)^{1/2}]^5. \tag{5.8.163}$$

It is seen at once that $i\ \Delta\sigma_g \le 0$ if $0 \le \Omega^2 \le W_{12;12}/I_{11}$, and $i\ \Delta\sigma_g > 0$ if $W_{12;12}/I_{11} < \Omega^2 < 2W_{12;12}/I_{11}$. In the latter case $\exp(i\ \Delta\sigma_g\ t)$ grows continuously, and the polytrope becomes secularly unstable to gravitational radiation reaction via the sectorial σ_1-mode, whereas secular instability due to viscosity occurs via the other sectorial σ_2-mode from Eq. (5.8.102). The variation of t_g with Ω is perfectly analogous to that of t_η from Eq. (5.8.157).

Thus, all uniformly rotating polytropes with index $n < 0.808$, exhibiting a second harmonic neutral sectorial mode ($P_2^{\pm 2}(\cos\lambda)\exp(\pm 2i\varphi)$ if $\Omega = 0$), that deforms them into a triaxial configuration (Sec. 3.8.1, James 1964), are secularly unstable to gravitational radiation reaction beyond the bifurcation point.

For the rotating, incompressible, constant density ellipsoids ($n = 0$) there emerges from the second order virial equations the following evolutionary picture: Only the *unperturbed* Maclaurin and Dedekind ellipsoids (Sec. 3.2, Chandrasekhar 1969) do not emit gravitational radiation, because I_{ij} is constant for these ellipsoids as seen in an inertial frame, and therefore $\Phi_g = 0$. All other classical ellipsoids must emit gravitational radiation, lose energy and angular momentum, and thereby evolve. Since evolution must ultimately proceed towards a nonradiating state, and since the triaxial Dedekind ellipsoid has a lower mechanical energy than the Maclaurin spheroid, the gravitational radiation driven evolution of the *secularly unstable* Maclaurin ellipsoid is towards the Dedekind ellipsoid. The combined secular effects of viscosity and gravitational radiation tend to cancel each other, and the sequence of secularly stable Maclaurin spheroids reaches in this case past the classical bifurcation point, up to a new bifurcation point, which is determined by the relative strength of both viscosity and gravitational radiation reaction (Detweiler and Lindblom 1977, Lindblom and Detweiler 1977, Shapiro and Teukolsky 1983, §7.3).

These findings have been confirmed for rotating $0 < n < 1.25$ polytropes (resembling neutron stars) with an independent method by Ipser and Lindblom (1990, 1991). They investigate $2 \le j = k \le 6$ modes, which reduce in the nonrotating limit to the Kelvin f-modes [Eq. (5.7.27)]; these modes – together with the rotational modes to be mentioned below – are responsible for the gravitational radiation induced instability.

The general relativistic calculations of Yoshida and Eriguchi (1997) with the Cowling approximation [Eqs. (5.2.119)-(5.2.121)] agree in the nonrelativistic limit with the entries from Table 5.8.2 only for the $n = 1.5$ polytrope [see also Yoshida and Eriguchi (1995, Table 2)]. More recently, Stergioulas and Friedman (1998) have shown that the fundamental $f(j = k = 2)$ bar mode becomes unstable in relativistic polytropes with a softer equation of state ($n \le 1.3$) than the stiffer Newtonian limit $n < 0.808$ of James (1964, and Sec. 3.8.1). For instance, in the isentropic $n = 1/(\Gamma_1 - 1) = 1$ polytrope the f-modes become unstable at $\tau = 0.065,\ 0.045,\ 0.035,\ 0.025$ if $j = k = 2, 3, 4, 5$, respectively. The Newtonian quotes from Table 5.8.2 are considerably larger if $j = k = 3, 4, 5$. Cutler and Lindblom (1992) did not find the $j = k = 2$ bar mode to be unstable in the rotating post Newtonian $n = 1$ polytrope – opposite to the previously mentioned fully relativistic calculations of Stergioulas and Friedman (1998).

In general relativity the onset of gravitational radiation driven instability by the $j = k = 2$ bar mode at $n \le 1.3$, no longer coincides with the onset of viscosity induced instability. The latter seems to have a critical relativistic polytropic index only slightly larger than the Newtonian value $n = 0.808$ of James (1964), (Bonazzola et al. 1996).

Mainly for two reasons the secular instability under the influence of gravitational radiation reaction is physically interesting only for low values of k in nonaxisymmetric modes with an azimuthal dependence $\exp(ik\varphi)$: (i) If k increases, the growth time scale of the instability rapidly becomes too large to be of any

Table 5.8.2 Bifurcation values τ_b of $\tau = E_{kin}/|W|$ when $\sigma = 0$, and secular instability due to gravitational radiation reaction becomes effective in uniformly rotating, axisymmetric Newtonian polytropes under the influence of nonaxisymmetric perturbations with an azimuthal dependence $\exp(ik\varphi)$, $(j = k)$. The entries are a compilation of the values listed by Imamura et al. (1985), Managan (1985), Stergioulas and Friedman (1998). The value 0.14 for $n = 0.5$, $k = 2$ is extrapolated from Tassoul and Ostriker (1970). Shown is also the critical value $\tau_c = E_{kin}/|W|$ against equatorial mass loss, as mentioned in Secs. 3.8.8, 5.8.3, 6.1.8, and depicted in Fig. 3.8.10. $a + b$ means $a \times 10^b$.

n	τ_c	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0	–	1.38−1	9.91−2	7.71−2	6.29−2
0.5	1.9−1	1.4−1	9.6−2	6.8−2	–
1	1.2−1	–	7.92−2	5.79−2	4.62−2
1.5	5.95−2	–	5.61−2	4.33−2	3.36−2
2	–	–	3.35−2	2.81−2	2.28−2
3	9.00−3	–	9.0−3	8.3−3	7.5−3

physical interest in neutron stars or other compact objects. (ii) If dissipation by viscosity is comparable to energy loss by gravitational radiation, the viscosity damps modes with $k \gtrsim 5$. Therefore, Imamura et al. (1985) and Managan (1985, 1986) have calculated for nonaxisymmetric modes with $k = 2, 3, 4, 5$ the critical values $\tau = \tau_b$ at the bifurcation points $\sigma = 0$, when axisymmetric polytropes become secularly unstable to gravitational radiation reaction. The bifurcation values τ_b decrease as k increases, so secular instabilities occur for $k > 2$, even if $n > 0.808$ (Table 5.8.2). As outlined previously, secular instabilities due to gravitational radiation reaction occur in *Newtonian* polytropes for the $j = k = 2$ barlike mode only if $n < 0.808$.

We have already noted that the trial eigenfunctions (5.8.33) provide only approximate results for uniformly rotating polytropes if $n \neq 0$. Besides, for differentially rotating configurations these eigenfunctions do not satisfy the requirement that the Lagrangian perturbation of the velocity circulation

$$\Delta \oint_C v_i \; dx_i = 0, \tag{5.8.164}$$

vanishes for any curve C on an isentropic fluid surface (Bardeen et al. 1977). However, Durisen and Imamura (1981) have shown that the induced errors are only about 1-7% for the differentially rotating polytropes considered by Bodenheimer and Ostriker (1973), and Ostriker and Bodenheimer (1973).

The so-called rotational modes are also susceptible to the Chandrasekhar-Friedman-Schutz instability (e.g. Lockitch and Friedman 1999, Yoshida and Lee 2000a, b). Practically all discussions of nonaxisymmetric oscillations are based on the polar (even-parity, spheroidal) modes

$$\Delta r = R_{jk}(r)\; Y_j^k(\lambda, \varphi) \exp(i\sigma t); \quad \Delta\lambda = [S_{jk}(r)/r^2][\partial Y_j^k(\lambda, \varphi)/\partial\lambda] \exp(i\sigma t);$$
$$\Delta\varphi = [S_{jk}(r)/r^2 \sin^2\lambda][\partial Y_j^k(\lambda, \varphi)/\partial\varphi] \exp(i\sigma t), \tag{5.8.165}$$

from Eqs. (5.2.87)-(5.2.90), (5.5.27), (5.7.82).

A second class of axial (odd-parity, toroidal) modes can be found with the separation (e.g. Cox 1980, p. 222)

$$\Delta r = 0; \quad \Delta\lambda = [T_{jk}(r)/r^2 \sin\lambda][\partial Y_j^k(\lambda, \varphi)/\partial\varphi] \exp(i\sigma t);$$
$$\Delta\varphi = -[T_{jk}(r)/r^2 \sin\lambda][\partial Y_j^k(\lambda, \varphi)/\partial\lambda] \exp(i\sigma t). \tag{5.8.166}$$

For a spherical mass the polar modes can be divided into the well-known subclasses of p and f-modes (pressure as the dominant restoring force), and g-modes (gravity dominated modes), (see Sec. 5.2).

In a nonrotating Newtonian equilibrium sphere the axial modes (5.8.166) are time independent ($\sigma = 0$), having vanishing Lagrangian and Eulerian perturbations of pressure (density) and gravitational potential (e.g. Lockitch and Friedman 1999). Only the axial velocity perturbations are nonzero in a nonrotating sphere, giving rise to a slow twisting of the system. Rotation mixes the polar and axial contributions to the velocity perturbation. In spherical coordinates the Eulerian velocity perturbation is given by Eq. (5.1.23), and assumes for uniform rotation and $\Delta\vec{r}(\ell, \varphi, z, t) \propto \exp[i(\sigma t + k\varphi)]$ the form (cf. Eq. (5.9.43) if $\Omega \neq$ const) :

$$\delta\vec{v} = i(\sigma + k\Omega)\; \Delta\vec{r}. \tag{5.8.167}$$

The rotational modes are induced by the Coriolis force [the last term in Eq. (3.1.79)], and generally possess comparable axial and polar velocity components. These hybrid rotational modes are termed inertial modes or *generalized* r-modes. A subclass of the rotational modes are the so-called *pure* r-modes, when the velocity perturbations (5.8.167) are dominated by axial Lagrangian displacements of the form (5.8.166).

The secular instability due to gravitational radiation reaction of *rotational* modes – when $\sigma(\sigma+k\Omega) < 0$ – has been examined by Lockitch and Friedman [1999, Eq. (55)], and Yoshida and Lee (2000a, Eqs. (43), (44); 2000b) for homogeneous $n = 0$, and neutron star-like $n = 1$ Newtonian polytropes. Among the most unstable rotational modes in the presence of viscous damping are the r-modes with $2 \leq j = |k| \lesssim 10$, the strongest appearing to be $j = |k| = 2$ mode. Eigenvalues of r-modes in differentially rotating polytropes ($n = 0.5, 1, 1.5$) have been calculated by Karino et al. (2001).

The secular and dynamical evolution of polytropic, ellipsoidal close binaries (neutron star binary, black hole - neutron star binary, brown and white dwarf binary), including the influence of viscosity and gravitational radiation reaction, has been followed up by Lai et al. (1994b), and Lai and Shapiro (1995), (see also Sec. 5.7.4).

5.9 Stability and Oscillations of Rotating Polytropic Cylinders

5.9.1 The Homogeneous, Uniformly Rotating Cylinder $n = 0$

Small linearized oscillations are obtained if we project the equations of motion (5.7.1) onto uniformly rotating cylindrical (ℓ, φ, z)-axes $[\vec{v} = \vec{v}(v_\ell, v_\varphi, v_z);\ \vec{\Omega} = \vec{\Omega}(0, 0, \Omega)]$:

$$Dv_\ell/Dt - 2\Omega v_\varphi = -(1/\varrho)\ \partial P/\partial\ell + \partial\Phi/\partial\ell + \Omega^2\ell;$$
$$Dv_\varphi/Dt + 2\Omega v_\ell = -(1/\varrho\ell)\ \partial P/\partial\varphi + (1/\ell)\ \partial\Phi/\partial\varphi;$$
$$Dv_z/Dt = -(1/\varrho)\ \partial P/\partial z + \partial\Phi/\partial z. \qquad (5.9.1)$$

We apply the Eulerian variations (5.2.23) to this system in the same way as for Eqs. (5.7.3)-(5.7.5), $(\delta(\Omega^2\ell) = 0)$, (Robe 1968b, Hansen et al. 1976):

$$\sigma^2\ \Delta\ell + 2i\sigma\Omega\ell\ \Delta\varphi = (1/\varrho)\ \partial\delta P/\partial\ell - (\delta\varrho/\varrho^2)\ \partial P/\partial\ell - \partial\delta\Phi/\partial\ell, \qquad (5.9.2)$$

$$\sigma^2\ell\ \Delta\varphi - 2i\sigma\Omega\ \Delta\ell = (1/\varrho\ell)\ \partial\delta P/\partial\varphi - (\delta\varrho/\varrho^2\ell)\ \partial P/\partial\varphi - (1/\ell)\ \partial\delta\Phi/\partial\varphi, \qquad (5.9.3)$$

$$\sigma^2\ \Delta z = (1/\varrho)\ \partial\delta P/\partial z - (\delta\varrho/\varrho^2)\ \partial P/\partial z - \partial\delta\Phi/\partial z. \qquad (5.9.4)$$

We have taken into account that for small oscillations – when the unperturbed configuration is in hydrostatic equilibrium $(\vec{v}_u = 0;\ \delta\vec{v} \approx \Delta\vec{v} = \vec{v})$ – we have via Eqs. (5.1.24), (5.1.29), (5.1.30):

$$D(\delta\vec{v})/Dt \approx D(\Delta\vec{v})/Dt = D^2(\Delta\vec{r})/Dt^2 \approx \partial^2\Delta\vec{r}/\partial t^2 = -\sigma^2\ \Delta\vec{r}. \qquad (5.9.5)$$

The basic equations of linear oscillations (5.9.2)-(5.9.4) have been derived by using the small cylindrical displacements

$$\Delta\vec{r}(\ell, \varphi, z, t) = \Delta\vec{r}[\Delta\ell(\ell, \varphi, z, t),\ \ell\ \Delta\varphi(\ell, \varphi, z, t),\ \Delta z(\ell, \varphi, z, t)] = \Delta\vec{r}(\ell, \varphi, z)\exp(i\sigma t);$$
$$\Delta\ell(\ell, \varphi, z, t) = \Delta\ell(\ell)\exp[i(\sigma t + k\varphi + jz)], \qquad (5.9.6)$$

and the small velocity components (5.1.30)

$$\delta v_\ell \approx v_\ell = d\ell/dt = \Delta v_\ell \approx \partial\Delta\ell/\partial t = i\sigma\Delta\ell; \quad \delta v_\varphi \approx v_\varphi = \ell\ d\varphi/dt = \Delta v_\varphi \approx \partial(\ell\ \Delta\varphi)/\partial t = i\sigma\ell\ \Delta\varphi;$$
$$\delta v_z \approx v_z = dz/dt = \Delta v_z \approx \partial\Delta z/\partial t = i\sigma\Delta z. \qquad (5.9.7)$$

The separation of variables for the Eulerian variation of pressure, density, and internal gravitational potential is assumed under the form (cf. Eq. (5.6.34), Cretin and Tassoul 1965, Robe 1968b)

$$\delta P(\ell, \varphi, z, t) = \delta P(\ell)\exp[i(\sigma t + k\varphi + jz)]; \quad \delta\varrho(\ell, \varphi, z, t) = \delta\varrho(\ell)\exp[i(\sigma t + k\varphi + jz)];$$
$$\delta\Phi(\ell, \varphi, z, t) = \delta\Phi(\ell)\exp[i(\sigma t + k\varphi + jz)]. \qquad (5.9.8)$$

σ, j, k have the same meaning as in Sec. 5.6, and it suffices to consider only nonnegative values of the integer azimuthal number k.

In their study on the nonaxisymmetric oscillations of a homogeneous, uniformly rotating cylinder, Cretin and Tassoul (1965) consider only oscillations having $j = 0$. We are left with the equations of motion (5.9.2) and (5.9.3), where the term $(\delta\varrho/\varrho^2\ell)\ \partial P/\partial\varphi$ is negligible, since the unperturbed hydrostatic pressure is independent of φ : $P_u = P_u(\ell)$. We eliminate $\Delta\varphi$ between Eqs. (5.9.2) and (5.9.3), taking into account the representation (5.9.8), and simplifying with the common factor $\exp[i(\sigma t + k\varphi)]$:

$$(\sigma^2 - 4\Omega^2)\ \Delta\ell = d(\delta P/\varrho - \delta\Phi)/d\ell + (2k\Omega/\sigma\ell)(\delta P/\varrho - \delta\Phi) - (\delta\varrho/\varrho^2)\ dP/d\ell. \qquad (5.9.9)$$

With the stated assumptions, the equation of continuity (5.6.80), the adiabatic energy equation (5.6.81), and Poisson's equation (5.6.84) take the simplified form

$$\delta\varrho/\varrho+(1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell+(1/\ell)\ \partial(\ell\ \Delta\varphi)/\partial\varphi=0, \tag{5.9.10}$$

$$\delta P+\Delta\ell\ dP/d\ell=\Gamma_1 P\ \delta\varrho/\varrho, \tag{5.9.11}$$

$$(1/\ell)\ d(\ell\ d\delta\Phi/d\ell)/d\ell-k^2\ \delta\Phi/\ell^2=-4\pi G\ \delta\varrho, \tag{5.9.12}$$

suppressing in Eqs. (5.9.11), (5.9.12) the common factor $\exp[i(\sigma t+k\varphi)]$. Let us now eliminate $\ell\ \Delta\varphi$ between Eqs. (5.9.3) and (5.9.10):

$$\delta\varrho/\varrho+(1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell+(1/\sigma^2\ell)\ \partial[2i\sigma\Omega\ \Delta\ell+(1/\ell)\ \partial(\delta P/\varrho-\delta\Phi)/\partial\varphi]/\partial\varphi=0, \tag{5.9.13}$$

or

$$\sigma^2[d\Delta\ell/d\ell+(\Delta\ell/\ell)(1-2k\Omega/\sigma)+\delta\varrho/\varrho]-(k^2/\ell^2)(\delta P/\varrho-\delta\Phi)=0, \tag{5.9.14}$$

omitting again in Eq. (5.9.14) the common factor $\exp[i(\sigma t+k\varphi)]$.

Now, $\delta P/\varrho-\delta\Phi$ and its derivative can be eliminated between Eqs. (5.9.9) and (5.9.14):

$$(1/\ell)\ d[\ell\ d(\ell\ \Delta\ell)/d\ell]/d\ell-k^2\ \Delta\ell/\ell+\ell\ d(\delta\varrho/\varrho)/d\ell+(\delta\varrho/\varrho)[2k^2\pi G\varrho(1-\beta)/\sigma^2+2+2k\Omega/\sigma]=0, \tag{5.9.15}$$

where $dP/d\ell=-2\pi G\varrho^2\ell(1-\beta)$ in virtue of Eq. (3.9.13). We insert from Eq. (5.9.11) for δP into Eqs. (5.9.9), (5.9.14), obtaining in this way $\delta\Phi$ and $d\delta\Phi/d\ell$ as a function of $\delta\varrho, \Delta\ell$, and their derivatives. Inserting for $\delta\Phi$ into Poisson's equation (5.9.12), and eliminating $d[\ell\ d(\ell\ \Delta\ell)/d\ell]/d\ell-k^2\ \Delta\ell$ via Eq. (5.9.15), Cretin and Tassoul (1965) get after some lengthy algebra an equation which is formally identical to Eq. (5.6.51), as obtained by Ostriker (1964c):

$$(1-x^2)\ d^2\varepsilon/dx^2+(1/x-5x)\ d\varepsilon/dx+(B-k^2/x^2)\ \varepsilon=0,$$
$$(n=0;\ j=0;\ k=0,1,2,...;\ x=\ell/\ell_1;\ \varepsilon=\delta\varrho/\varrho;\ \varrho=\varrho_0=\mathrm{const}). \tag{5.9.16}$$

The constant

$$B=(4/\Gamma_1)\{[\omega^2+(1-\Gamma_1)+\beta(\Gamma_1-2)]/(1-\beta)+(k^2/4)[\Gamma_1-(1-\beta)/\omega^2-2^{5/2}\beta^{1/2}/k\omega]\},$$
$$(\omega^2=\sigma^2/4\pi G\varrho;\ \beta=\Omega^2/2\pi G\varrho), \tag{5.9.17}$$

is equal to the constant from Eq. (5.6.52) if the cylinder is nonrotating ($\beta=0$). The solution of Eq. (5.9.16) is given as in the nonrotating case by Eqs. (5.6.53)-(5.6.56), where B should be taken from Eq. (5.9.17). However, the fourth order equation (5.6.57) for the determination of the dimensionless oscillation frequency $\omega=\sigma/(4\pi G\varrho)^{1/2}$ is now obtained by equating Eq. (5.6.56) with Eq. (5.9.17):

$$\omega^4-[\Gamma_1(1-\beta)(m+1)(m+k+1)+2\beta-1]\ \omega^2-2^{1/2}k\beta^{1/2}(1-\beta)\ \omega-k^2(1-\beta)^2/4=0,$$
$$(n=0;\ j=0;\ k=0,1,2,...;\ m=0,1,2,...). \tag{5.9.18}$$

The oscillation frequencies of the quasiradial modes ($k=0$) are obtained at once from the previous equation

$$\omega^2=\sigma^2/4\pi G\varrho=\Gamma_1(1-\beta)(m+1)^2+2\beta-1, \qquad (n=0;\ j,k=0;\ m=0,1,2,...), \tag{5.9.19}$$

becoming equal to Eq. (5.6.60) in the nonrotating case $\beta=0$. Obviously, these radial modes are always stable, since $0\leq\beta\leq 1$ [cf. Eq. (3.9.12)], and $\Gamma_1\geq 1$ (cf. Sec. 1.7). The condition $\sigma^2\geq 0$ implies

$$\Gamma_1\geq 1\geq(1-2\beta)/(1-\beta)(m+1)^2, \qquad (n,j,k=0;\ 0\leq\beta\leq 1). \tag{5.9.20}$$

Cretin and Tassoul (1965) solve numerically Eq. (5.9.18) for nonaxisymmetric oscillations $j=0,\ k\neq 0$. The case $k=\pm 1$ presents little interest, as it amounts to a simple displacement of the cylindrical axis. Two of the four oscillation frequencies $\sigma_1/(2\pi G\varrho)^{1/2},\ \sigma_2/(2\pi G\varrho)^{1/2}$ are real numbers of different

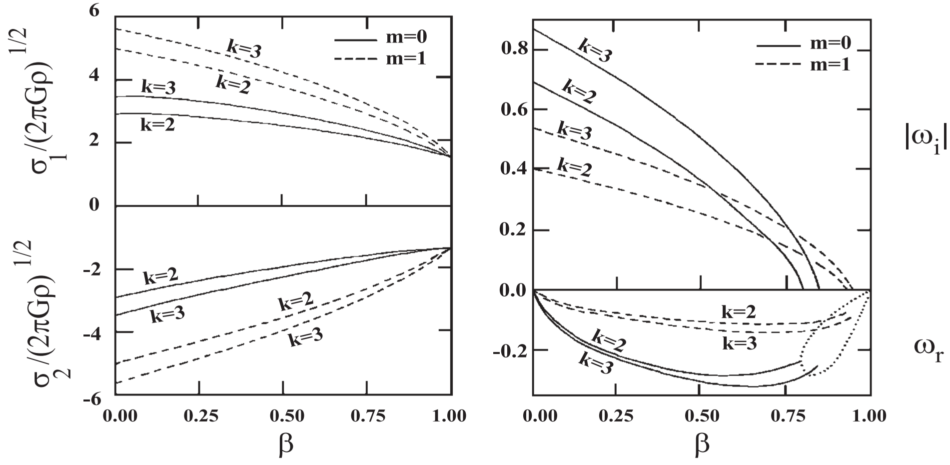

Fig. 5.9.1 Normalized eigenfrequencies for nonaxisymmetric oscillations of the homogeneous, uniformly rotating cylinder if $\Gamma_1 = 5/3$, $n = 0$, $\varrho = \varrho_0 =$ const, $\beta = \Omega^2/2\pi G\varrho$. Left-hand side: The stable p-modes $\sigma_1/(2\pi G\varrho)^{1/2}$ and $\sigma_2/(2\pi G\varrho)^{1/2}$ from Eq. (5.9.18) if $k = 2, 3$ and $m = 0$ (solid curves), $m = 1$ (broken curves). Right-hand side: The real (bottom) and imaginary (top) parts of the dimensionless eigenfrequencies $\sigma_{3,4}/(2\pi G\varrho)^{1/2} = \omega_r + i\omega_i$ for the g-modes if $k = 2, 3$ and $m = 0, 1$ with $\beta \leq \beta_s$. Solid curves are for $m = 0$, broken ones for $m = 1$. The two dotted curves, originating at $\beta_s = 0.795$, represent the stable real eigenfrequencies ω_r if $k = 2,\ m = 0$, and $0.795 \leq \beta \leq 1$ (Cretin and Tassoul 1965).

sign, corresponding to the stable p-modes from Eq. (5.6.58) if $\beta = 0$. Their limiting value $\pm 2^{1/2}$ if $\beta \to 1$ results at once from Eq. (5.9.18) with $\sigma_{1,2}/(2\pi G\varrho)^{1/2} = 2^{1/2}\omega$. The two remaining frequencies $\sigma_{3,4}/(2\pi G\varrho)^{1/2}$ – reducing in the nonrotating case to the purely imaginary unstable g-modes (5.6.59) – are now complex numbers, as long as the rotation parameter $\beta = \Omega^2/2\pi G\varrho$ is below some limiting value β_s, depending on Γ_1, k, and m. If $\Gamma_1 = 5/3$, Cretin and Tassoul (1965) obtain: $\beta_s = 0.795$ if $k = 2,\ m = 0$; $\beta_s = 0.845$ if $k = 3,\ m = 0$; $\beta_s = 0.925$ if $k = 2,\ m = 1$; $\beta_s = 0.940$ if $k = 3,\ m = 1$. The g-modes are stable if $\beta_s \leq \beta \leq 1$: The oscillation frequencies are real numbers, their common limiting value 0 being approached as $\beta \to 1$ (Fig. 5.9.1).

5.9.2 Uniformly Rotating Cylinders with Polytropic Index $0 < n \leq \infty$

Introducing the Eulerian perturbations (5.2.23) into the equation of motion (5.7.1), we obtain via Eqs. (5.1.24), (5.9.5) the basic equation of small adiabatic oscillations of a rotating cylinder in a frame rotating at uniform angular speed Ω (e.g. Robe 1968b):

$$\partial^2\Delta\vec{r}/\partial t^2 = -(1/\varrho)\ \nabla\delta P + (\delta\varrho/\varrho^2)\ \nabla P + \nabla\delta\Phi - 2\vec{\Omega} \times (\partial\Delta\vec{r}/\partial t), \quad (\delta[\vec{\Omega} \times (\vec{\Omega} \times \vec{r})] = 0). \quad (5.9.21)$$

Inserting for $\partial\Delta\vec{r}/\partial t = i\sigma\ \Delta\vec{r}$, and for the pressure and potential terms from Eq. (5.2.86), we get

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) + \vec{A}[\delta P/\varrho + (1/\varrho)(\Delta\vec{r} \cdot \nabla P)] + 2i\sigma\ \vec{\Omega} \times \Delta\vec{r}. \quad (5.9.22)$$

The projection of this equation onto the coordinate axes yields, with A given by Eq. (5.10.2), $[\vec{A} = \vec{A}(A, 0, 0);\ P = P(\ell)]$:

$$\sigma^2\ \Delta\ell = \partial(\delta P/\varrho)/\partial\ell - \partial\delta\Phi/\partial\ell + A\ \delta P/\varrho + (A\ \Delta\ell/\varrho)\ dP/d\ell - 2i\sigma\Omega\ell\ \Delta\varphi, \quad (5.9.23)$$

$$\sigma^2\ell\ \Delta\varphi = (1/\ell)\ \partial(\delta P/\varrho)/\partial\varphi - (1/\ell)\ \partial\delta\Phi/\partial\varphi + 2i\sigma\Omega\ \Delta\ell = (ik/\ell)(\delta P/\varrho - \delta\Phi) + 2i\sigma\Omega\ \Delta\ell, \quad (5.9.24)$$

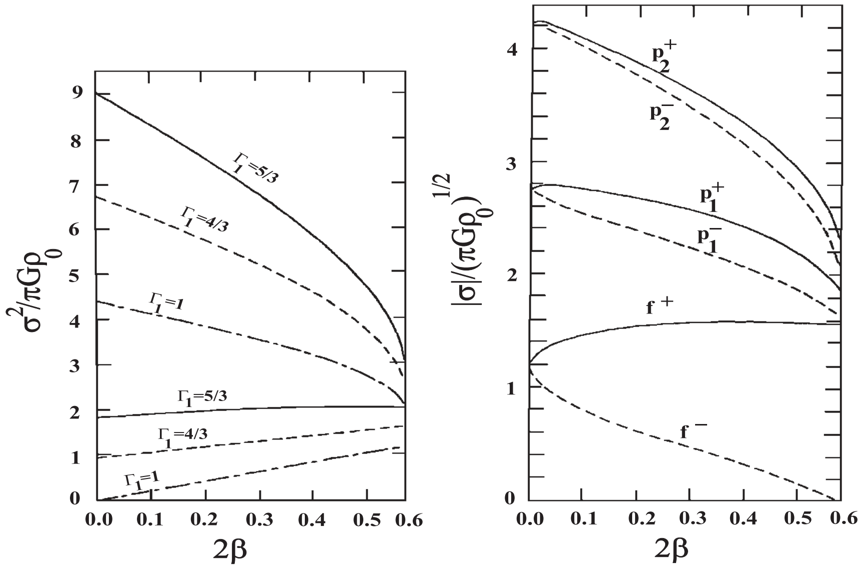

Fig. 5.9.2 Left-hand side: Stable quasiradial modes ($j, k = 0;\ \beta = \Omega^2/2\pi G\varrho_0$) of the uniformly rotating cylinder with polytropic index $n = 1$. The three lower curves represent the squared normalized eigenfrequencies $\sigma_0^2/\pi G\varrho_0$ of the fundamental quasiradial mode for different adiabatic exponents, the three upper curves are for the first overtone $\sigma_1^2/\pi G\varrho_0$. Right-hand side: Normalized eigenfrequencies $|\sigma|/(\pi G\varrho_0)^{1/2}$ of nonaxisymmetric stable $p^\pm$ and $f^\pm$-modes of the uniformly rotating cylinder if $n = 1,\ j = 0,\ k = 2,\ \Gamma_1 = 5/3,\ \beta = \Omega^2/2\pi G\varrho_0$ (Robe 1968b).

$$\sigma^2\ \Delta z = \partial(\delta P/\varrho)/\partial z - \partial\delta\Phi/\partial z = ij(\delta P/\varrho - \delta\Phi). \tag{5.9.25}$$

Like in Eq. (5.9.9) we eliminate $\Delta\varphi$ between Eqs. (5.9.23) and (5.9.24):

$$\partial(\delta P/\varrho)/\partial\ell = [\sigma^2 - 4\Omega^2 - (A/\varrho)\ dP/d\ell]\ \Delta\ell - A\ \delta P/\varrho + \partial\delta\Phi/\partial\ell - (2k\Omega/\sigma\ell)(\delta P/\varrho - \delta\Phi). \tag{5.9.26}$$

To get a further basic equation, we may eliminate $\delta\varrho$ between the continuity equation (5.2.28) and the adiabatic energy equation (5.2.78) if $Q = \text{const}$:

$$\delta P/\Gamma_1 P + (\Delta\vec{r}\cdot\nabla P)/\Gamma_1 P + \nabla\cdot\Delta\vec{r} = \delta P/\Gamma_1 P + (\Delta\ell/\Gamma_1 P)\ dP/d\ell$$
$$+(1/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell + (1/\ell)\ \partial(\ell\ \Delta\varphi)/\partial\varphi + \partial\Delta z/\partial z = 0. \tag{5.9.27}$$

We substitute for $\ell\ \Delta\varphi$ and Δz from Eqs. (5.9.24) and (5.9.25), ($\partial\Delta\ell/\partial\varphi = ik\ \Delta\ell;\ \partial\Delta z/\partial z = ij\ \Delta z$) :

$$\partial(\ell\ \Delta\ell)/\partial\ell = [2k\Omega/\sigma\ell - (1/\Gamma_1 P)\ dP/d\ell]\ \ell\ \Delta\ell - \ell\ \delta P/\Gamma_1 P + (k^2 + j^2\ell)(\delta P/\varrho - \delta\Phi)/\sigma^2\ell. \tag{5.9.28}$$

If we insert the decompositions from Eqs. (5.9.6), (5.9.8) into Eqs. (5.9.28), (5.9.26), (5.6.84), suppressing the common factor $\exp[i(k\varphi + jz)]$, we obtain with the variables $u(\ell) = \ell\ \Delta\ell(\ell),\ y(\ell) = \delta P(\ell)/\varrho(\ell)$ from Eq. (5.6.36) the three basic equations for the determination of the unknowns $u, y, \delta\Phi$:

$$du/d\ell = [2k\Omega/\sigma\ell - (1/\Gamma_1 P)\ dP/d\ell]\ u - \ell\varrho y/\Gamma_1 P + (k^2 + j^2\ell)(y - \delta\Phi)/\sigma^2\ell, \tag{5.9.29}$$

$$dy/d\ell = [\sigma^2 - 4\Omega^2 - (A/\varrho)\ dP/d\ell]\ u/\ell - Ay + d\delta\Phi/d\ell - (2k\Omega/\sigma\ell)(y - \delta\Phi), \tag{5.9.30}$$

$$(1/\ell)\ d(\ell\ d\delta\Phi/d\ell)/d\ell - (k^2/\ell^2 + j^2)\ \delta\Phi = 4\pi G\varrho(Au/\ell - \varrho y/\Gamma_1 P). \tag{5.9.31}$$

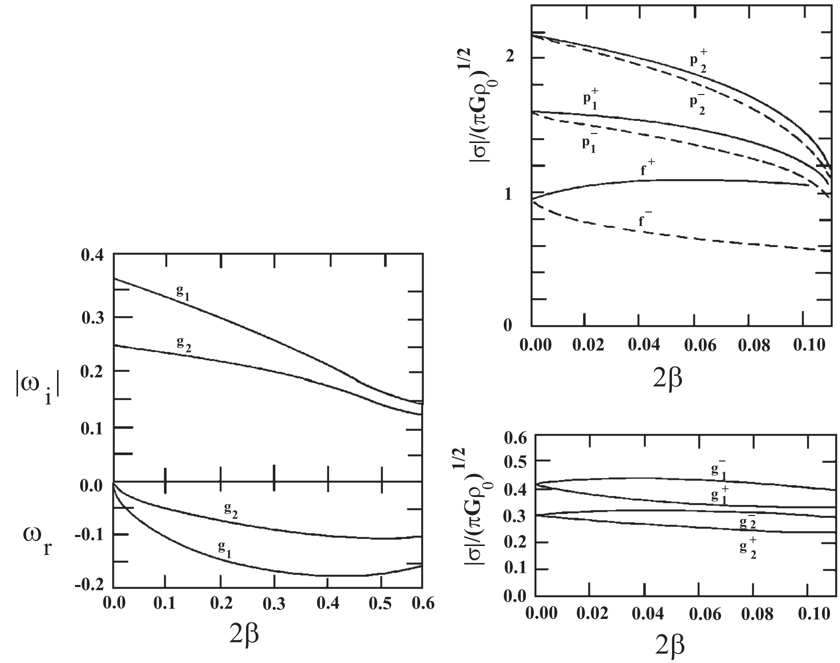

Fig. 5.9.3 Left-hand side: Complex normalized eigenfrequencies $\omega = (\sigma_r + i\sigma_i)/(\pi G\varrho_0)^{1/2} = \omega_r + i\omega_i$ of unstable g_1 and g_2-modes for the uniformly rotating cylinder if $n = 1,\ j = 0,\ k = 2,\ \Gamma_1 = 5/3,\ \beta = \Omega^2/2\pi G\varrho_0$. Right-hand side: Normalized eigenfrequencies $|\sigma|/(\pi G\varrho_0)^{1/2}$ of stable $p^\pm, f^\pm, g^\pm$-modes for the uniformly rotating cylinder if $n = 3,\ j = 0,\ k = 2,\ \Gamma_1 = 5/3,\ \beta = \Omega^2/2\pi G\varrho_0$ (Robe 1968b).

Like in the nonrotating case, the boundary conditions are given by Eqs. (5.6.88), (5.6.92).

Robe (1968b) has solved numerically the system (5.9.29)-(5.9.31) for the polytropic indices $n = 1, 3, 6$, calculating eigenvalues of quasiradial ($j, k = 0$) and nonaxisymmetric ($j = 0;\ k = 2$) oscillations (Figs. 5.9.2, 5.9.3). The quasiradial oscillations become strictly radial in absence of rotation, and they always conserve the axial symmetry of the rotating cylinder. The elimination of the potential $\delta\Phi$ from Eqs. (5.9.29)-(5.9.31) proceeds for quasiradial oscillations ($j, k = 0$) exactly as in the nonrotating case [Eqs. (5.6.93)-(5.6.96)], and we are left with the system

$$du/d\ell + u\ (dP/d\ell)/\Gamma_1 P = -\varrho\ell y/\Gamma_1 P, \tag{5.9.32}$$

$$dy/d\ell + Ay = [\sigma^2 - 4\Omega^2 - (A/\varrho)\ dP/d\ell + 4\pi G\varrho]\ u/\ell. \tag{5.9.33}$$

If $n = 1$, ($\Gamma_1 = 1, 4/3, 5/3$; Fig. 5.9.2), the stability of the fundamental quasiradial mode is always enhanced with increasing rotation speed, just as for the homogeneous cylinder. Eq. (5.9.19) becomes $\omega^2 = \Gamma_1 - 1 + (2 - \Gamma_1)\beta$ if $n = 0$ and $m = 0$. On the other hand, if $n = 3$ and 6, ($\Gamma_1 = 5/3$), rotation decreases (destabilizes) the eigenvalues of quasiradial modes (Robe 1968b).

Like in the spherical case, rotation lifts the degeneracy of the nonrotating eigenvalues ($\sigma_+^2 = \sigma_-^2$) with respect to the azimuthal coordinate φ, yielding two distinct eigenvalues $\sigma_+^2 \neq \sigma_-^2$ for the p, f, and g-modes of the rotating cylinder if $k = 2,\ j = 0$. Like in the nonrotating case from Sec. 5.6, the g-modes become unstable if $A > 0$, which amounts to $n < 1.5$ if $\Gamma_1 = 5/3$. The left-hand side of Fig. 5.9.3 shows the complex eigenvalues if $n = 1$ as a function of $2\beta = \Omega^2/\pi G\varrho_0$. Although the imaginary part diminishes continuously with increasing rotation, there is no value of β, ($\beta \leq \beta_c = 0.287$ if $n = 1$, Table 3.9.1) compatible with hydrostatic equilibrium, for which the g-mode becomes stable (σ – real), quite opposite to the homogeneous cylinder (Fig. 5.9.1), which constitutes probably a singular case (Robe 1968b). The p and f-modes are always stable ($\sigma^2 > 0$), as well as the g-modes if $A < 0$.

The nonaxisymmetric oscillations of uniformly rotating, truncated isothermal cylinders ($n = \pm\infty$) have been considered by Hansen et al. (1976) if $\beta = 0.05,\ \Gamma_1 = 5/3,\ j = 0,\ k = 2$. Besides stable p, f, and g^+-modes – as for the nonrotating isothermal cylinder from Fig. 5.6.3 – they find also an unstable g^--mode with complex eigenvalues, changing in a curious manner as the radius of the cutted cylinder increases form $\xi_1 = 14$ (when the first density inversion occurs in Fig. 3.9.1 if $\beta = 0.05$) to $\xi_1 = 58.8$.

5.9.3 Differentially Rotating Cylinders

As far as I know, oscillations of differentially rotating polytropic cylinders have been investigated by Robe (1979), Veugelen (1985b, c), Ishibashi and Ando (1985, 1986). The rotation law adopted by Robe (1979) is [cf. Eq. (3.9.15)]: $\Omega(\ell) = a_0/(1 + a_1\ell^2),\ (n = 1;\ a_0 > 0;\ a_1 \geq 0;\ a_0, a_1 = \mathrm{const})$. Veugelen's (1985b) rotation law is

$$\Omega(\ell) = \Omega_0[1 - (1 - \Omega_s/\Omega_0)\ell^2/\ell_1^2], \qquad (n = 3). \tag{5.9.34}$$

Ω_0 and Ω_s are the angular velocities on the axis and at the surface, respectively.

Since we are considering a medium in nonuniform rotation, the relevant equations are written in an inertial frame, with the velocity of the initial unperturbed equilibrium state equal to $\vec{v}_u = \vec{v}_u[0, \ell\Omega(\ell), 0]$. Taking into account the material derivative (B.51), and inserting into the equation of motion (5.2.10) the Eulerian perturbations (5.2.23), we get up to the first order ($\vec{v} = \vec{v}(\delta v_\ell, \ell\Omega + \delta v_\varphi, \delta v_z)$, Robe 1979):

$$\begin{aligned}&\partial\delta v_\ell/\partial t + \Omega\ \partial\delta v_\ell/\partial\varphi - 2\Omega\ \delta v_\varphi = (\delta\varrho/\varrho^2)\ \partial P/\partial\ell - (1/\varrho)\ \partial\delta P/\partial\ell + \partial\delta\Phi/\partial\ell\\ &= -\partial(\delta P/\varrho - \delta\Phi)/\partial\ell - A\ \delta P/\varrho - (A\ \Delta\ell/\varrho)\ dP/d\ell,\end{aligned} \tag{5.9.35}$$

$$\partial\delta v_\varphi/\partial t + \Omega\ \partial\delta v_\varphi/\partial\varphi + [\Omega + d(\ell\Omega)/d\ell]\ \delta v_\ell = (\delta\varrho/\varrho^2\ell)\ \partial P/\partial\varphi - (1/\varrho\ell)\ \partial\delta P/\partial\varphi + (1/\ell)\ \partial\delta\Phi/\partial\varphi, \tag{5.9.36}$$

$$\partial\delta v_z/\partial t + \Omega\ \partial\delta v_z/\partial\varphi = (\delta\varrho/\varrho^2)\ \partial P/\partial z - (1/\varrho)\ \partial\delta P/\partial z + \partial\delta\Phi/\partial z. \tag{5.9.37}$$

The continuity equation (5.2.1) becomes in the same way via Eq. (B.46)

$$\partial\delta\varrho/\partial t + \Omega\ \partial\delta\varrho/\partial\varphi + (1/\ell)\ \partial(\varrho\ell\ \delta v_\ell)/\partial\ell + (1/\ell)\ \partial(\varrho\ \delta v_\varphi)/\partial\varphi + \partial(\varrho\ \delta v_z)/\partial z = 0, \tag{5.9.38}$$

while the adiabatic energy equation (5.2.78)

$$\delta P + \Delta\vec{r}\cdot\nabla P = (\Gamma_1 P/\varrho)(\delta\varrho + \Delta\vec{r}\cdot\nabla\varrho), \qquad (Q = \mathrm{const}), \tag{5.9.39}$$

and Poisson's equation (5.2.40) preserve their usual form. Up to now, we have not assumed, as did Robe (1967, 1968b, 1979), Hansen et al. (1976), that $\Delta\varphi$ and Δz change in the same way as $\Delta\ell$ with the common factor $\exp[i(\sigma t + k\varphi + jz)]$. To make further progress, we will now assume besides the decompositions (5.9.6), (5.9.8) that

$$\begin{aligned}&\Delta\vec{r}(\ell,\varphi,z,t) = \Delta\vec{r}(\ell)\exp[i(\sigma t + k\varphi + jz)]; \quad \Delta\ell(\ell,\varphi,z,t) = \Delta\ell(\ell)\exp[i(\sigma t + k\varphi + jz)];\\ &\Delta\varphi(\ell,\varphi,z,t) = \Delta\varphi(\ell)\exp[i(\sigma t + k\varphi + jz)]; \quad \Delta z(\ell,\varphi,z,t) = \Delta z(\ell)\exp[i(\sigma t + k\varphi + jz)],\end{aligned} \tag{5.9.40}$$

and

$$\delta\vec{v}(\ell,\varphi,z,t) = \delta\vec{v}(\ell)\exp[i(\sigma t + k\varphi + jz)]. \tag{5.9.41}$$

Since $|\vec{v}_u| = \ell\Omega(\ell)$ is no longer small, the Eulerian velocity perturbation $\delta\vec{v}$ is related to the Lagrangian displacement vector $\Delta\vec{r}$ by Eq. (5.1.23) – where all derivatives act in the inertial, cylindrical curvilinear coordinate system from Eq. (B.44) – and we get in virtue of Eq. (5.1.23):

$$\begin{aligned}&\delta\vec{v} = \partial\Delta\vec{r}/\partial t + \Omega\ \partial(\Delta\ell\ \vec{e}_\ell + \ell\ \Delta\varphi\ \vec{e}_\varphi + \Delta z\ \vec{e}_z)/\partial\varphi - [d(\ell\Omega)/d\ell]\ \Delta\ell\ \vec{e}_\varphi - \Omega\ell\ \Delta\varphi\ \partial\vec{e}_\varphi/\partial\varphi\\ &= (\partial\Delta\ell/\partial t + \Omega\ \partial\Delta\ell/\partial\varphi)\ \vec{e}_\ell + \big[\partial(\ell\ \Delta\varphi)/\partial t + \Omega\ \partial(\ell\ \Delta\varphi)/\partial\varphi - \ell(d\Omega/d\ell)\ \Delta\ell\big]\ \vec{e}_\varphi\\ &+(\partial\Delta z/\partial t + \Omega\ \partial\Delta z/\partial\varphi)\ \vec{e}_z.\end{aligned} \tag{5.9.42}$$

Using the decomposition (5.9.40), the components of the Eulerian velocity perturbation are

$$\delta v_\ell = i(\sigma + k\Omega)\ \Delta\ell; \quad \delta v_\varphi = i(\sigma + k\Omega)\ell\ \Delta\varphi - \ell(d\Omega/d\ell)\ \Delta\ell; \quad \delta v_z = i(\sigma + k\Omega)\ \Delta z. \tag{5.9.43}$$

With the obvious assumption that the pressure in the unperturbed state depends only on the radial cylindrical coordinate ℓ, the equation of hydrostatic equilibrium (3.1.2) reduces to

$$g = (1/\varrho)\ dP/d\ell = d\Phi/d\ell + \Omega^2\ell. \tag{5.9.44}$$

g denotes, as in Eq. (3.1.20), the effective gravity along the cylindrical radius. Eqs. (5.9.35)-(5.9.39) write in terms of the decompositions from Eqs. (5.9.8), (5.9.40), (5.9.43), by dropping the common factor $\exp[i(\sigma t + k\varphi + jz)]$:

$$[(\sigma + k\Omega)^2 - 2\ell\Omega\ d\Omega/d\ell - (A/\varrho)\ dP/d\ell]\ \Delta\ell + 2i\Omega(\sigma + k\Omega)\ell\ \Delta\varphi$$
$$= d(\delta P/\varrho)/d\ell + A\ \delta P/\varrho - d\delta\Phi/d\ell, \tag{5.9.45}$$

$$(\sigma + k\Omega)^2\ell\ \Delta\varphi - 2i\Omega(\sigma + k\Omega)\ \Delta\ell = (ik/\ell)(\delta P/\varrho - \delta\Phi), \tag{5.9.46}$$

$$(\sigma + k\Omega)^2\ \Delta z = ik(\delta P/\varrho - \delta\Phi), \tag{5.9.47}$$

$$\delta\varrho + (1/\ell)\ d(\varrho\ell\ \Delta\ell)/d\ell - k\varrho\ \Delta\ell\ (d\Omega/d\ell)/(\sigma + k\Omega) + i\varrho(k\ \Delta\varphi + j\ \Delta z) = 0, \tag{5.9.48}$$

$$\delta P + \Delta\ell\ dP/d\ell = (\Gamma_1 P/\varrho)(\delta\varrho + \Delta\ell\ d\varrho/d\ell). \tag{5.9.49}$$

Proceeding now exactly in the same manner as with Eqs. (5.9.26), (5.9.28), we obtain [cf. Sung 1974, Eqs. (22), (23)]

$$(\sigma + k\Omega)^2\ d(\ell\ \Delta\ell)/d\ell = (\sigma + k\Omega)[2k\Omega + k\ell\ d\Omega/d\ell - (\sigma + k\Omega)\ell(dP/d\ell)/\Gamma_1 P]\ \Delta\ell$$
$$+[k^2/\ell + j^2\ell - (\sigma + k\Omega)^2\ell\varrho/\Gamma_1 P]\ \delta P/\varrho - (k^2/\ell + j^2\ell)\ \delta\Phi, \tag{5.9.50}$$

$$(\sigma + k\Omega)\ d(\delta P/\varrho)/d\ell = (\sigma + k\Omega)[(\sigma + k\Omega)^2 - (1/\ell^3)\ d(\ell^4\Omega^2)/d\ell - (A/\varrho)\ dP/d\ell]\ \Delta\ell$$
$$-[2k\Omega/\ell + (\sigma + k\Omega)A]\ \delta P/\varrho + 2k\Omega\ \delta\Phi/\ell + (\sigma + k\Omega)\ d\delta\Phi/d\ell, \tag{5.9.51}$$

which should be supplemented by the perturbed Poisson equation (5.6.84)

$$(1/\ell)\ d(\ell\ d\delta\Phi/d\ell)/d\ell - (k^2/\ell^2 + j^2)\ \delta\Phi = 4\pi G\varrho(A\ \Delta\ell - \delta P/\Gamma_1 P), \tag{5.9.52}$$

in order to obtain a fourth order system for the determination of all relevant quantities. The boundary conditions that have to be satisfied by the system (5.9.50)-(5.9.52) are the finiteness of all physical quantities at the origin, while at the free surface the Lagrangian pressure variation must vanish, and the perturbation of the gravitational potential and of its derivatives must be continuous.

Veugelen (1985b) has studied nonaxisymmetric oscillations ($j = 0,\ k = 2$) of differentially rotating cylinders with polytropic index $n = 3$, taking in Eq. (5.9.34) the values: (i) $0 \leq \beta = \Omega_0^2/2\pi G\varrho_0 \leq 0.05$, $(\Omega_s/\Omega_0 = 1)$, and $\beta = 0.05$, $(0.1 \leq \Omega_s/\Omega_0 \leq 1)$; (ii) $\beta = 0.25$, $(\Omega_s/\Omega_0 = 0.2)$. Only differentially rotating equilibrium models exist for case (ii), (for uniform rotation $\beta \leq 0.0547$, Table 3.9.1). For case (i) the sufficient stability criterion of Sung (1974)

$$(A/\varrho)\ dP/d\ell + [(j^2/\ell^3)\ d(\ell^4\Omega^2)/d\ell - (k^2/4)(d\Omega/d\ell)^2]/(j^2 + k^2/\ell^2) \geq 0, \tag{5.9.53}$$

is everywhere satisfied, and consequently there appear only stable p, f, and g-modes (Fig. 5.9.4), while for case (ii) Veugelen (1985b) finds an unstable mode with complex conjugate eigenfrequency $\sigma/(\pi G\varrho_0)^{1/2} = -0.8521 \pm i0.0060$. Veugelen (1985a) has remarked that Sung (1974) derived Eq. (5.9.53) in the Cowling approximation $\delta\Phi \equiv 0$, so gravitational instabilities may exist, even if Eq. (5.9.53) is satisfied (cf. the ℓ-mode in Fig. 5.6.2).

In a nonrotating cylinder the propagation sense of a perturbation $\Delta\ell(\ell)\exp[i(\sigma t+k\varphi)]$ is symmetrically with respect to the neutral oscillation frequency $\sigma = 0$. The angular velocity of wave propagation in the

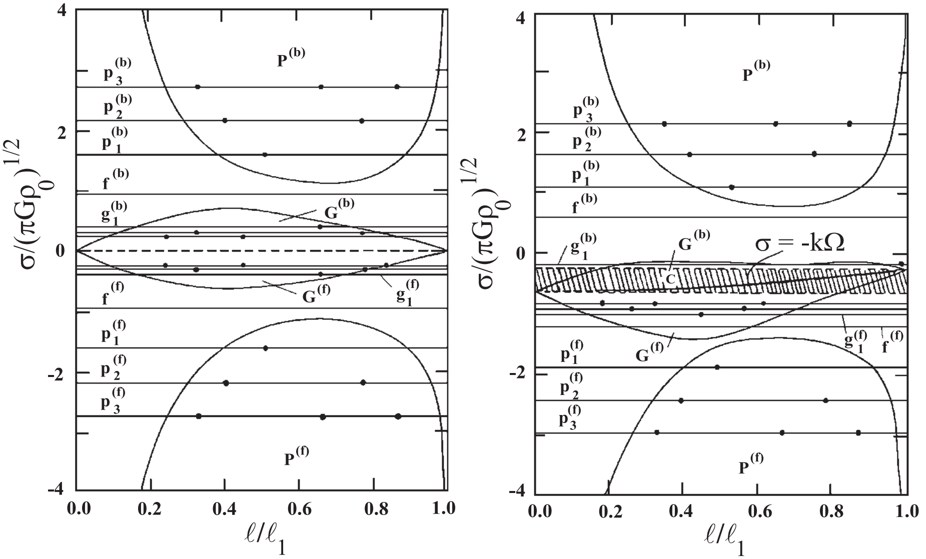

Fig. 5.9.4 Propagation diagram for polytropic cylinders with $n = 3$, $\Gamma_1 = 5/3$, $j = 0$, $k = 2$. Left-hand side: Nonrotating cylinder. Right-hand side: Differentially rotating cylinder with $\Omega_0^2/2\pi G\varrho_0 = 0.05$, $\Omega_s/\Omega_0 = 0.4$. Shown are the P and G-regions together with the hatched C-region of the continuous eigenvalue spectrum between $\sigma = -k\Omega_0$ and $\sigma = -k\Omega_s$. The line $\sigma = -k\Omega$ depicts the corotation surface. The dimensionless eigenfrequencies $\sigma/(\pi G\varrho_0)^{1/2}$ of stable p, f, g-modes are represented by straight lines (g_2, g_3-modes not indicated). Relevant quantities are indexed by (f) or (b), as to whether they belong to the region of forward or backward propagating waves (prograde or retrograde modes). Dots indicate zeros of the eigenfunction $\Delta\ell(\ell/\ell_1)$, (Veugelen 1985b).

azimuthal φ-direction is just equal to $\varphi/t = -\sigma/k$ (e.g. Gerthsen et al. 1977), and propagation of the oscillation occurs in the retrograde ($\varphi < 0$) direction if $\sigma > 0$, and in the prograde trigonometric sense ($\varphi > 0$) if $\sigma < 0$. In the case of a differentially rotating cylinder, the *relative* angular velocity φ'/t of azimuthal wave propagation with respect to fluid particles rotating with the equilibrium angular velocity Ω is now $\varphi'/t = -\sigma/k - \Omega = \varphi/t - \Omega$. The wave is running ahead ($\varphi' > 0$) of the equilibrium rotation Ω if $\sigma/k + \Omega < 0$, and backward ($\varphi' < 0$) if $\sigma/k + \Omega > 0$.

Veugelen (1985b, c) distinguishes a P-region with high frequency pressure (acoustic) waves (stable p modes), and a G-region with low frequency gravity waves (stable g-modes), in the same manner as for the nonradial oscillations of a sphere (cf. Sec. 5.5.2, Fig. 5.5.4). The f-modes are surface modes located between the frequencies of p and g-modes (see Fig. 5.9.4). For case (ii), when the differential rotation speed increases, some modes acquire a mixed character, the mode classification becoming sometimes only indicative.

For the rotating, isentropic polytropic cylinder $n = 3$, ($\Gamma_1 = 1 + 1/n = 4/3$) Veugelen (1985c) has identified a region of propagation of so-called rotational modes [r-modes, Rossby modes, inertial modes; see Eq. (5.8.166)]. But these modes may be regarded simply as forward propagating (prograde) g-modes ($g^{(f)}$-modes), since the eigenfrequencies of rotational modes merge into $g^{(f)}$-modes as the cylinders depart from isentropy (adiabaticity). When a rotating polytropic cylinder becomes isentropic (adiabatic), we have $A \to 0$, and the eigenfrequencies of backward propagating (retrograde) $g^{(b)}$-modes tend to zero, while the eigenfrequencies of prograde $g^{(f)}$-modes tend to the eigenfrequencies of rotational r-modes. This degeneracy is a special feature of the two-dimensional oscillations considered in this section and in Sec. 5.10.6. It should not subsist for three-dimensional oscillations (Iye 1984).

In addition to the discrete spectrum of eigenvalues there exists for differentially rotating cylinders also a continuous spectrum of eigenvalues, because the system (5.9.50)-(5.9.52) possesses a regular singularity at $\ell = \ell_c$, where $\sigma + k\Omega(\ell_c) = 0$. This relationship defines the surface of corotation, i.e. the surface where

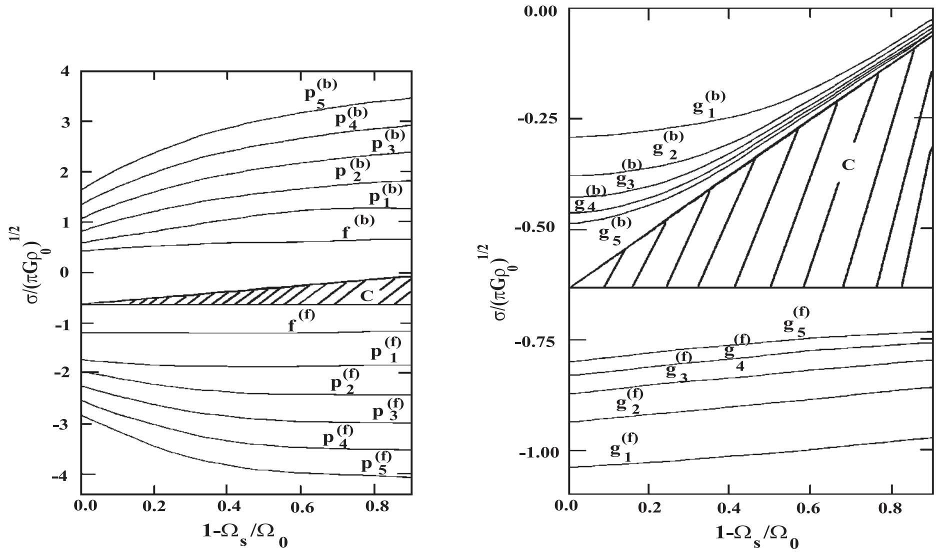

Fig. 5.9.5 Normalized eigenfrequencies $\sigma/(\pi G\varrho_0)^{1/2}$ of p and f-modes (on the left), and of g-modes (on the right), as a function of the degree of differential rotation $1-\Omega_s/\Omega_0$ [Eq. (5.9.34)] if $n=3$, $\Gamma_1=5/3$, $\Omega_0^2/2\pi G\varrho_0=0.05$, $j=0$, $k=2$. The hatched area depicts the C-region of the continuous eigenvalue spectrum between $\sigma=-k\Omega_0$ and $\sigma=-k\Omega_s$ (Veugelen 1985b).

the angular azimuthal velocity φ/t of wave propagation in the *inertial* frame just equals the local angular speed $\Omega(\ell_c)$ of rotating fluid particles $\varphi/t=-\sigma/k=\Omega(\ell_c)$. With the matrix

$$T=\big[(\sigma+k\Omega)\ell\ \Delta\ell,\ \delta P/\varrho,\ \delta\Phi,\ \ell\ d\delta\Phi/d\ell\big], \tag{5.9.54}$$

the fourth order system (5.9.50)-(5.9.52) can be written in matricial form as

$$(\sigma+k\Omega)\ dT/d\ell=BT, \tag{5.9.55}$$

where the elements of matrix B are obviously

$$\begin{aligned}
&B_{11}=2k\Omega/\ell+k\ d\Omega/d\ell-(\sigma+k\Omega)(dP/d\ell)/\Gamma_1P;\quad B_{12}=k^2/\ell+j^2\ell-(\sigma+k\Omega)^2\ell\varrho/\Gamma_1P;\\
&B_{13}=-k^2/\ell-j^2\ell;\quad B_{21}=(\sigma+k\Omega)^2/\ell-(1/\ell^4)\ d(\ell^4\Omega^2)/d\ell-(A/\varrho\ell)\ dP/d\ell;\\
&B_{22}=-2k\Omega/\ell-(\sigma+k\Omega)A;\quad B_{23}=2k\Omega/\ell;\quad B_{24}=(\sigma+k\Omega)/\ell;\quad B_{34}=(\sigma+k\Omega)/\ell;\\
&B_{41}=4\pi G\varrho A;\quad B_{42}=-4\pi G\varrho^2\ell(\sigma+k\Omega)/\Gamma_1P;\quad B_{43}=(\sigma+k\Omega)(k^2/\ell+j^2\ell);\\
&B_{14},B_{31},B_{32},B_{33},B_{44}=0.
\end{aligned} \tag{5.9.56}$$

Note, that $(\sigma+k\Omega)\ell\ \Delta\ell=-i\ell\ \delta v_\ell$ via Eq. (5.9.43), so all components of T are Eulerian perturbations. Expanding $\sigma+k\Omega(\ell)$ round the singular point ℓ_c, we obtain $\sigma+k\Omega(\ell)\approx k(\ell-\ell_c)(d\Omega/d\ell)_{\ell=\ell_c}$, and the matricial system (5.9.55) becomes up to the first order

$$dT/d\ell=B_0T/k(\ell-\ell_c)\ (d\Omega/d\ell)_{\ell=\ell_c}. \tag{5.9.57}$$

B_0 is the zero order approximation of B, obtained simply by putting $\ell=\ell_c$ and $\sigma+k\Omega(\ell_c)=0$. Assuming for the elements of T solutions of the form (5.3.24), the vanishing of the lowest power $(\ell-\ell_c)^{q-1}$ in the homogeneous system (5.9.57) demands the vanishing of the determinant $|B_0/k(d\Omega/d\ell)_{\ell=\ell_c}-qI|$, in order to obtain nontrivial solutions. I denotes the unit matrix. The roots of the determinant can easily be found with the elements of matrix B_0 :

$$q_{1,2} = 1/2 \pm \left\{1/4 - [(j^2 + k^2/\ell^2)(A/\varrho)\ dP/d\ell + (j^2/\ell^3)\ d(\ell^4\Omega^2)/d\ell]/k^2(d\Omega/d\ell)^2\right\}^{1/2}_{\ell=\ell_c}; \quad q_{3,4} = 0. \tag{5.9.58}$$

After sophisticated evaluations Veugelen (1985b) has found that differentially rotating cylinders are stable against perturbations of the type (5.9.8), (5.9.40) if $\mathrm{Re}(q_1), \mathrm{Re}(q_2) > 0$. And these conditions, when inserted into Eq. (5.9.58), demand that

$$(j^2 + k^2/\ell^2)(A/\varrho)\ dP/d\ell + (j^2/\ell^3)\ d(\ell^4\Omega^2)/d\ell > 0. \tag{5.9.59}$$

Thus, the cylinders are stable for eigenvalues σ associated with the continuous spectrum, if both, the Schwarzschild criterion $A < 0$ of convective stability (5.2.84), and the Rayleigh criterion (3.5.1), $(d(\ell^2\Omega)/d\ell > 0)$ are verified concomitantly.

For the $n = 1$ differentially rotating cylinder considered by Robe (1979), the convectively unstable g_1-mode of the uniformly rotating cylinder (left-hand side of Fig. 5.9.3) may become totally dominant.

For a differential rotation law of the form $\Omega = a\ell^b$, $(a, b > 0;\ a, b = \mathrm{const})$ Balbinski (1984) has obtained an analytic solution for the oscillations of an incompressible homogeneous cylinder ($n = 0;\ \Gamma_1 = \infty$), investigating the continuous spectrum of its eigenfrequencies. Oscillations in polytropic cylinders without self-gravity, and with special differential rotation laws of the form (6.4.153) have been studied among others by Glatzel (1987), Sozou (1988), Sozou and Wilkinson (1989).

5.10 Stability and Oscillations of Rotating Slabs and Disks

5.10.1 Introduction

Much of current research on the stability and oscillations of polytropic disks has been motivated by the astronomical phenomena associated with barred spiral and normal spiral galaxies, accretion disks, and planetary (protoplanetary) rings. The type of galaxy corresponding to normal spirals (e.g. our own galaxy) has a stellar component and a gaseous component. Fortunately, the gas has such a small mass that its contribution to an approximate treatment of global galactic dynamics can be neglected. The mass of a spiral galaxy is distributed into a spheroidal component (the nuclear bulge and the galactic halo, which may be approximated by a rigid sphere) and the disk component. In some cases the spheroidal component may contain as little as 25% of the total mass of a galaxy, and this helps justify the disk approximation for spiral galaxies. The circularly rotating, zero thickness disk may be regarded as the lowest order approximation of a whole spiral galaxy containing stars and gas.

The basic equations governing the behaviour of the stellar component are the Boltzmann equation in a collisionless stellar system, and the Poisson equation for the gravitational potential. This kind of approach is investigated by direct many-body numerical simulations, while a much simpler way is the description of stars and gas by a fluid dynamical model. Of course, a fluid with pressure is at the best only a crude approximation of a collisionless stellar system. The pressure is merely introduced to model and simulate all the effects of random velocities of the stars. Unfortunately, there has not been much linkage between studies of galactic disks with the aid of stellar dynamics (adopting a discrete particle description of the disk) and those adopting a hydrodynamic model, describing the disk as a fluid continuum (Hunter 1972, Lin and Lau 1979, Aoki et al. 1979). We will exclusively concentrate on the fluid dynamical model of disks. In fact, the oscillations of stellar disks composed of discrete particles show striking differences as compared to those of fluid disks. For instance, in the short wavelength regime none of the p or g-modes exist in collisionless stellar disks. As substitutes for these fluid disk modes there emerges in collisionless stellar disks a number of modes inherent to the large degree of freedom in the velocity distribution of stars in phase space (Iye 1984).

The situation becomes even more frustrating due to a missing concordance between studies on rotating gaseous disks and those on rotating gaseous stars: A common classification of modes is generally not attempted. For instance, Aoki et al. (1979) introduce classes of B and S-modes in their investigation on the global stability of polytropic disks. For all the above reasons this section appears to be one of the most fragmentary in this book, especially as a multitude of incoherent results exist, rather than a straightforward theory concerning the local and global stability of polytropic (barotropic) disks. As reviewed for instance by Hunter (1972) and Toomre (1977), a further drawback of most approaches is the fact that they just select and discuss modes that have certain desirable properties (e.g. spirals), without showing why just these particular solutions should be significant and dominant.

The study of rotating polytropic disks is further complicated by the fact that three distinct forces are of comparable magnitude, and need to be considered simultaneously: Gravitation, pressure (velocity dispersion), and rotation. In sharp contrast to a sphere, the gravitational field of a disk cannot be computed as if the whole mass outside a given distance from its centre did not exist; the mass distribution over the whole disk must be taken into account. Magnetic forces are almost certainly small as compared with the overall gravitational force, and will be neglected, although they may be very well comparable to the self-gravitation of a local mass concentration, such as a spiral arm (Mestel 1963, Hunter 1972, Spitzer 1978).

The inner parts of disk-shaped galaxies are observed to rotate with higher angular velocity $\Omega(\ell)$ than the outer parts, i.e. in many cases the linear circular velocity $v(\ell) = \ell\ \Omega(\ell)$ is observed to be nearly constant over large distances ℓ, the cylindrical coordinate ℓ being measured from the rotation axis. Therefore, differential rotation with outward decreasing angular velocity must be considered for a more realistic picture of galactic disks (Aoki et al. 1979, Lin and Lau 1979). Provided that $\vec{A}\cdot\nabla P = A\ dP/d\ell \geq 0$, Rayleigh's stability condition against *axisymmetric* disturbances in the circularly rotating disk is given by the Solberg-Høiland criterion on surfaces of constant entropy S (cf. Eqs. (3.5.1), (5.7.90), (6.4.160); Tassoul 1978, Sec. 7.3):

$$\kappa^2 = (1/\ell^3)\ d[\ell^4\Omega^2(\ell)]/d\ell = 2[\Omega(\ell)/\ell]\ d[\ell^2\Omega(\ell)]/d\ell = 4\Omega^2(\ell) + 2\ell\ \Omega(\ell)\ d\Omega(\ell)/d\ell > 0. \qquad (5.10.1)$$

The ℓ-component of the Schwarzschild discriminant (5.2.84), belonging to the constraint $A\ dP/d\ell \geq 0$, is

$$A = (1/\varrho)\ d\varrho/d\ell - (1/\Gamma_1 P)\ dP/d\ell = [1 - (n+1)/n\Gamma_1]\ d\ln\varrho/d\ell, \qquad (P = K\varrho^{1+1/n}). \qquad (5.10.2)$$

The epicyclic frequency κ decreases from $\kappa = 2\Omega(\ell)$ for uniform rotation ($\Omega(\ell) = \mathrm{const}$) to $\kappa = \Omega(\ell)$ for Keplerian rotation $\Omega(\ell) = (GM_1/\ell^3)^{1/2}$, and becomes zero if $\Omega(\ell) = 0$, or if the angular momentum per unit mass $\ell^2\Omega(\ell) = \mathrm{const}$ (Spitzer 1978). Condition (5.10.1) simply states that the specific angular momentum $\ell^2\Omega(\ell)$ must necessarily increase outwards, or equivalently, that the angular velocity should not decrease too fast, as the distance ℓ from the rotation axis increases. If $\kappa^2 \leq 0$, a mass element displaced from its circular rotation will not experience a restoring force (Hunter 1972). If $\kappa^2 > 0$, the real quantity κ is equal (when pressure forces are neglected) to the circular oscillation frequency in the inertial frame of a particle in a nearly circular orbit, moving exclusively under the influence of an axially symmetric gravitational field, and being displaced from its circular equilibrium position ℓ_0 to $\ell = \ell_0 + \Delta\ell_0 \sin(\kappa t)$, $(\ell_0, \Delta\ell_0 = \mathrm{const},\ |\Delta\ell_0| \ll \ell_0)$, (Chandrasekhar 1960, p 156). The orbit described by the particle in the plane of the disk under the exclusive influence of an axisymmetric potential is an elliptic epicycle. It should be noted that, although the epicyclic parameter κ appears in the theory, the epicyclic motion itself is not properly simulated in the fluid dynamical approach of polytropic disks (Lin and Lau 1979).

The stability against nonaxisymmetric perturbations and the fragmentation of polytropic tori (rings, annuli) will be touched in Sec. 6.4.3.

Two lines of approach can be used to study the stability of gaseous disks: The local analysis and the global one. The local analysis (asymptotic theory) is confined to perturbations with wavelengths that are short compared to the distance ℓ from the rotation axis of the considered local region. Let us assume the spatial and temporal dependence of a perturbation under the usual form $\exp[i(\sigma t + \vec{j}\cdot\vec{r})]$, where $|\vec{j}| = 2\pi/L$ denotes the wave number, and L the wavelength. The local short wavelength approximation demands that [Lin and Lau 1979, Eq. (32)]

$$L = 2\pi/|\vec{j}| \ll \ell \qquad \mathrm{or} \qquad |\vec{j}|\ell \gg 1. \qquad (5.10.3)$$

Local theories, like the density wave theory, cannot be applied to very long wavelengths strictly; for instance, even if a disk is locally stable, it is not assured that the disk is also stable on a global scale, and this urges the need for analyzing barotropic (polytropic) disks also on a global scale, either by direct numerical particle simulation (which will be ignored), or by global linear mode analysis, as summarized in Secs. 5.10.5 and 5.10.6.

5.10.2 Stability of the Nonrotating Isothermal Slab

The surface density Σ contained within height $-z$ and z is

$$\Sigma = \Sigma(\ell,\varphi,z) = \int_{-z}^{z} \varrho(\ell,\varphi,z')\ dz' = 2\int_0^z \varrho(\ell,\varphi,z')\ dz', \qquad (5.10.4)$$

and the total surface density of a slab is

$$\Sigma_1 = \Sigma_1(\ell,\varphi) = \int_{-\infty}^{\infty} \varrho(\ell,\varphi,z')\ dz' = 2\int_0^{\infty} \varrho(\ell,\varphi,z')\ dz', \qquad (5.10.5)$$

where in virtue of Eqs. (2.3.65), (2.6.177)-(2.6.180) we have $\varrho = 0$ outside the finite boundary $\pm z_1$ of a polytropic slab with polytropic index $-1 < n < \infty$. If $-\infty < n < -1$ and $n = \pm\infty$, the slab has infinite extension in the z-direction (Sec. 2.6.8).

For a polytropic slab with *uniform* density in the ℓ, φ-directions, the equation of hydrostatic equilibrium (2.1.3) writes as

$$(1/\varrho)\ dP/dz = K(1+1/n)\varrho^{1/n-1}\ d\varrho/dz = d\Phi/dz, \qquad (n \neq -1). \tag{5.10.6}$$

If we insert into Poisson's equation (2.1.4), we get

$$\nabla^2\Phi = d^2\Phi/dz^2 = d[K(1+1/n)\varrho^{(1-n)/n}\ d\varrho/dz]/dz = -4\pi G\varrho. \tag{5.10.7}$$

In the case of homogeneous slabs these two equations integrate at once:

$$\nabla\Phi = d\Phi/dz = -4\pi G\varrho z; \quad P = P_0 - 2\pi G\varrho^2 z^2 = P_0 - \pi G\Sigma^2/2,$$
$$(n=0;\ \varrho = \text{const};\ |z| \leq z_1;\ (d\Phi/dz)_{z=0} = 0;\ P(0) = P_0). \tag{5.10.8}$$

For other values of the polytropic index Goldreich and Lynden-Bell (1965a) introduce the auxiliary variable

$$\mu = (2\pi G/K)^{1/2}\varrho_0^{(1-n)/2n}\int_0^z \varrho^{(n-1)/n}\ dz', \tag{5.10.9}$$

and Eq. (5.10.7) becomes

$$d^2\varrho/d\mu^2 = -2\varrho_0^{(n-1)/n}\varrho^{1/n}/(1+1/n). \tag{5.10.10}$$

ϱ_0 denotes the density in the central plane (plane of symmetry, midplane) $z = 0$ of the slab. With the initial condition $(d\varrho/d\mu)_{\mu=0} = 0$ this equation can be easily integrated after multiplication with $d\varrho/d\mu$ [cf. Eq. (2.3.51)]:

$$(d\varrho/d\mu)^2 + [4\varrho_0^{(n-1)/n}/(1+1/n)^2](\varrho^{1+1/n} - \varrho_0^{1+1/n}) = 0. \tag{5.10.11}$$

With the substitution

$$\cos^2 X = (\varrho/\varrho_0)^{1+1/n}, \tag{5.10.12}$$

Eq. (5.10.11) may be written under the form

$$\mu = \int_0^X \cos^{(n-1)/(n+1)} X'\ dX'. \tag{5.10.13}$$

In the particular case $n = 1$, Eqs. (5.10.9) and (5.10.13) become

$$\mu = (2\pi G/K)^{1/2} z = X = \arccos(\varrho/\varrho_0), \tag{5.10.14}$$

or [cf. Eqs. (2.3.26), (2.3.59)]:

$$\varrho = \varrho_0 \cos\mu = \varrho_0 \cos[(2\pi G/K)^{1/2} z], \qquad (n=1). \tag{5.10.15}$$

In the isothermal case considered by Ledoux (1951), we find similarly

$$\mu = \sin X = (1 - \varrho/\varrho_0)^{1/2}; \quad \varrho = \varrho_0(1-\mu^2), \qquad (n = \pm\infty), \tag{5.10.16}$$

and by differentiation of Eq. (5.10.9)

$$d\mu/dz = (2\pi G\varrho_0/K)^{1/2}(1-\mu^2), \qquad (n = \pm\infty). \tag{5.10.17}$$

This equation integrates with the obvious condition $\mu = 0$ if $z = 0$:

$$\mu = \tanh[(2\pi G\varrho_0/K)^{1/2} z]; \quad \varrho = \varrho_0(1-\mu^2) = \varrho_0\{1 - \tanh^2[(2\pi G\varrho_0/K)^{1/2} z]\}$$
$$= \varrho_0/\cosh^2[(2\pi G\varrho_0/K)^{1/2} z], \qquad (n = \pm\infty). \tag{5.10.18}$$

If we insert from Eqs. (2.1.18) and (2.1.20) $\varrho = \varrho_0 \exp(-\theta)$ and $z = (K/4\pi G\varrho_0)^{1/2}\xi$, respectively, we recover Eq. (2.3.65).

For the nonrotating isothermal slab $n = \pm\infty$ studied by Ledoux (1951), the adiabatic equation of thermal energy conservation (5.2.78) writes in our one-dimensional case ($Q = \mathrm{const};\ P = K\varrho$) :

$$\delta P = \Gamma_1 P\ \delta\varrho/\varrho + \Delta\vec{r}\cdot(\Gamma_1 P\ \nabla\varrho/\varrho - \nabla P) = K\Gamma_1\ \delta\varrho + K(\Gamma_1 - 1)\ \Delta z\ d\varrho/dz. \tag{5.10.19}$$

For adiabatic oscillations the linearly perturbed Eulerian equation of motion (5.2.80) reads

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi) + (\delta P/\varrho)[(1/\varrho)\ \nabla\varrho - (1/\Gamma_1 P)\ \nabla P] + (1/\varrho)(\Delta\vec{r}\cdot\nabla P)[(1/\varrho)\ \nabla\varrho$$
$$-(1/\Gamma_1 P)\ \nabla P] = \nabla(\delta P/\varrho - \delta\Phi) + (1 - 1/\Gamma_1)(\nabla\varrho/\varrho^2)(\delta P + \Delta\vec{r}\cdot\nabla P), \qquad (\nabla P = K\ \nabla\varrho). \tag{5.10.20}$$

To simplify the problem, Ledoux (1951) assumes an isentropic isothermal slab $\Gamma_1 = 1 + 1/n = 1,\ (n = \pm\infty)$, [cf. Eq. (2.1.51)]. The two relevant equations (5.10.19) and (5.10.20) become in this case

$$\delta P = K\ \delta\varrho, \tag{5.10.21}$$

$$\sigma^2\ \Delta\vec{r} = \nabla(\delta P/\varrho - \delta\Phi). \tag{5.10.22}$$

Goldreich and Lynden-Bell (1965a) have shown the eigenvalues σ^2 of rotating polytropic slabs to be always real; we may therefore determine the stability limits by examining the behaviour of modes having σ close to the neutral stability limit $\sigma = 0$. In this particular case, Eq. (5.10.22) integrates to $\delta P/\varrho = \delta\Phi$ with the boundary condition $\delta P = 0$ if $\delta\Phi = 0$. Poisson's equation (5.2.40) is equal to

$$\nabla^2\delta\Phi = \nabla^2(\delta P/\varrho) = -4\pi G\ \delta P/K, \qquad (\sigma = 0). \tag{5.10.23}$$

Let us consider a perturbation along the x-direction of a Cartesian frame:

$$\delta P/\varrho = h(z)\exp(ijx). \tag{5.10.24}$$

Inserting this into Eq. (5.10.23), we get by suppressing the common factor $\exp(ijx)$:

$$d^2h/dz^2 + (4\pi G\varrho/K - j^2)h = 0. \tag{5.10.25}$$

We turn with Eq. (5.10.17) to the variable μ and introduce for ϱ from Eq. (5.10.16):

$$d^2h/d\mu^2 - [2\mu/(1-\mu^2)]\ dh/d\mu + [2/(1-\mu^2) - \eta^2/(1-\mu^2)^2]\ h = 0, \qquad (\eta^2 = j^2K/2\pi G\varrho_0). \tag{5.10.26}$$

If we put in Eq. (3.1.41) $j = 1$ and $k = \eta$, we recover just Eq. (5.10.26), where k is now a real number, instead of being an integer or zero. If in Eq. (3.1.41) μ, j, k assume any real or complex values, the associated Legendre polynomials $P_j^k(\mu)$ are called spherical harmonics of the general type. If μ is real and $-1 \le \mu \le 1$, the general solution of Eq. (3.1.41) is given by the generalized spherical harmonic (e.g. Hobson 1931, §§144, 145)

$$P_j^k(\mu) = C_1[(1+\mu)/(1-\mu)]^{k/2}\ F[-j; j+1; 1-k; (1-\mu)/2]$$
$$+C_2[(1-\mu)/(1+\mu)]^{k/2}\ F[-j; j+1; 1-k; (1+\mu)/2], \qquad (C_1, C_2 = \mathrm{const}), \tag{5.10.27}$$

where (e.g. Smirnow 1967)

$$F(\alpha,\beta,\gamma,\zeta) = 1 + \alpha\beta\zeta/1!\gamma + \alpha(\alpha+1)\beta(\beta+1)\zeta^2/2!\gamma(\gamma+1)$$
$$+ ... + [\alpha(\alpha+1)...(\alpha+m-1)\beta(\beta+1)...(\beta+m-1)\zeta^m/m!\gamma(\gamma+1)...(\gamma+m-1)] + ..., \tag{5.10.28}$$

is the hypergeometric function. In our special case from Eq. (5.10.27) we have $\alpha = -j = -1,\ \beta = j+1 = 2,\ \gamma = 1-k = 1-\eta,\ \zeta = (1\pm\mu)/2$, and

$$h = C_1'[(1+\mu)/(1-\mu)]^{\eta/2}(\eta-\mu) + C_2'[(1-\mu)/(1+\mu)]^{\eta/2}(\eta+\mu),$$
$$[C_1' = C_1/(\eta-1);\ C_2' = C_2/(\eta-1)]. \tag{5.10.29}$$

Since h must remain finite at the boundary of the slab, we must have $\eta - \mu = 0$ if $\mu = \Sigma/\Sigma_1 \to 1$ [see Eq. (5.10.32)]. The critical values are

$$\mu = \eta_c = j_c(K/2\pi G\varrho_0)^{1/2} = 1 \quad \text{or} \quad L_c = 2\pi/j_c = (2\pi K/G\varrho_0)^{1/2}, \qquad (\sigma = 0;\ n = \pm\infty). \tag{5.10.30}$$

This is equal to the critical Jeans wavelength (5.4.14), if the density of the uniform medium ϱ is replaced by the halved density $\varrho_0/2$ in the symmetry plane of the slab, and $a^2 = \Gamma_1 P/\varrho = P_0/\varrho_0 = K$, $(\Gamma_1 = 1)$. If $n = \pm\infty$, we observe from Eqs. (5.10.4), (5.10.9) that

$$\mu = (2\pi G/K\varrho_0)^{1/2} \int_0^z \varrho(z')\ dz' = (\pi G/2K\varrho_0)^{1/2}\Sigma, \qquad (z \geq 0;\ n = \pm\infty). \tag{5.10.31}$$

Eq. (5.10.18) shows that $\varrho = 0$ and $z = \infty$ if $\mu = 1$. In this case Σ in Eq. (5.10.31) becomes just equal to the total surface density Σ_1, i.e. equal to the total mass per unit surface:

$$\Sigma_1 = (2K\varrho_0/\pi G)^{1/2}; \quad \mu = \Sigma/\Sigma_1. \tag{5.10.32}$$

The critical wavelength (5.10.30) assumes the form

$$L_c = \pi\Sigma_1/\varrho_0, \qquad (n = \pm\infty). \tag{5.10.33}$$

If $\sigma \neq 0$, we take the divergence of Eq. (5.10.22)

$$\nabla \cdot \{\varrho[\nabla(\delta P/\varrho - \delta\Phi)]\} = \sigma^2\ \nabla \cdot (\varrho\ \Delta\vec{r}) = -\sigma^2\ \delta\varrho = -\sigma^2\ \delta P/K, \tag{5.10.34}$$

by using the continuity equation (5.2.28). Since $\varrho = \varrho(z)$, we get explicitly

$$(d\varrho/dz)\ \partial\delta\Phi/\partial z - (d\varrho/dz)\ \partial(\delta P/\varrho)/\partial z + \varrho[\nabla^2\delta\Phi - \nabla^2(\delta P/\varrho)] = \sigma^2\ \delta P/K, \tag{5.10.35}$$

and by applying the Laplace operator to eliminate $\delta\Phi$ via Eq. (5.10.23):

$$\nabla^2\big\{\varrho[\nabla^2(\delta P/\varrho) + (4\pi G/K + \sigma^2/K\varrho)\ \delta P]/(d\varrho/dz)\big\} + \partial\big[\nabla^2(\delta P/\varrho) + 4\pi G\ \delta P/K\big]/\partial z = 0. \tag{5.10.36}$$

If $\sigma \to 0$, we observe that the general solution of this equation is arbitrarily close to that of the equation

$$\nabla^2(\delta P/\varrho) + (4\pi G\varrho/K + \sigma^2/K)(\delta P/\varrho) = 0, \qquad (\sigma \approx 0). \tag{5.10.37}$$

With a perturbation of the form (5.10.24) this equation is similar to Eq. (5.10.26):

$$d^2h/d\mu^2 - [2\mu/(1-\mu^2)]\ dh/d\mu + [2/(1-\mu^2) - {\eta'}^2/(1-\mu^2)^2]\ h = 0,$$
$$({\eta'}^2 = (j^2K - \sigma^2)/2\pi G\varrho_0). \tag{5.10.38}$$

The solution of this equation is given by Eq. (5.10.29) if η is replaced by η'. For finite solutions we must have [cf. Eq. (5.10.30)]

$$\mu^2 = {\eta'}^2 = (j^2K - \sigma^2)/2\pi G\varrho_0 = 1, \tag{5.10.39}$$

or

$$\sigma^2 = 2\pi G\varrho_0(j^2K/2\pi G\varrho_0 - 1) = 2\pi G\varrho_0(2\pi K/G\varrho_0 L^2 - 1), \qquad (\sigma \approx 0), \tag{5.10.40}$$

which shows that the isothermal slab is stable ($\sigma^2 > 0$) if $L < (2\pi K/G\varrho_0)^{1/2}$, and unstable ($\sigma^2 < 0$) if $L > (2\pi K/G\varrho_0)^{1/2}$ with respect to isentropic perturbations ($\Gamma_1 = 1$) parallel to the equatorial plane of the slab.

It should be remarked that the limiting case $j = 0$, $(L = \infty)$ cannot be obtained by extrapolation of the previous results, because it amounts to purely vertical oscillations [$\delta P/\varrho = h(z)$], so we have to proceed ex novo. Eq. (5.10.22) writes for purely vertical oscillations as

$$\sigma^2\ \Delta z = (1/\varrho)\ d\delta P/dz - (\delta\varrho/\varrho^2)\ dP/dz - d\delta\Phi/dz. \tag{5.10.41}$$

The three terms on the right-hand side can be written out explicitly. At first we observe from the continuity equation (5.2.28) that

$$\delta\varrho = -\nabla \cdot (\varrho\ \Delta\vec{r}) = -d(\varrho\ \Delta z)/dz = -\varrho\ d\Delta z/dz - \Delta z\ d\varrho/dz. \tag{5.10.42}$$

And from Eq. (5.10.21) we obtain, by introducing for $\delta\varrho$:

$$\delta P = -K\varrho\ d\Delta z/dz - K\ \Delta z\ d\varrho/dz. \tag{5.10.43}$$

Poisson's equation (5.2.40) reads

$$(1/4\pi G)\ d^2\delta\Phi/dz^2 = -\delta\varrho = d(\varrho\ \Delta z)/dz, \tag{5.10.44}$$

and can be integrated with $d\delta\Phi/dz = 0$ if $\varrho = 0$:

$$d\delta\Phi/dz = 4\pi G\varrho\ \Delta z. \tag{5.10.45}$$

We introduce Eqs. (5.10.6), (5.10.7), (5.10.42), (5.10.43), (5.10.45) into Eq. (5.10.41), and get after some algebra the equation governing oscillations in the z-direction of stratification:

$$d^2\Delta z/dz^2 + (1/\varrho)(d\varrho/dz)\ d\Delta z/dz + \sigma^2\ \Delta z/K = 0. \tag{5.10.46}$$

We multiply this equation with $\varrho\ \Delta z$, by observing that the first two terms can then be written as $\Delta z\ d(\varrho\ d\Delta z/dz)/dz$. Integrating this expression by parts between $z = \pm\infty$, we obtain eventually

$$-\varrho\ \Delta z\ d\Delta z/dz\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varrho\ (d\Delta z/dz)^2\ dz = (\sigma^2/K)\int_{-\infty}^{\infty} \varrho\ (\Delta z)^2\ dz. \tag{5.10.47}$$

If $z \to \pm\infty$, we must have $\delta\varrho/\varrho \ll 1,\ \Delta z =$ finite, and $(1/\varrho)\ d\varrho/dz \propto (1/\varrho)\ d\varrho/d\xi = -d\theta/d\xi = -2^{1/2}$ [cf. Eq. (2.3.62)]. From Eq. (5.10.42) we observe that $d\Delta z/dz$ must remain finite too, and because $\varrho \to 0$ if $z \to \pm\infty$, the first term in Eq. (5.10.47) cancels. The eigenvalue σ^2 is strictly positive, and no instability occurs for strictly vertical oscillations (Ledoux 1951). This result helps justify the zero thickness approximation, when the whole mass of a slab is assumed to be concentrated with surface density Σ in the central plane of the disk $z = 0$: No instability is ignored if the details of the structure in the vertical z-direction are neglected (Hunter 1972).

5.10.3 Stability and Oscillations of Uniformly Rotating Polytropic Slabs

Fricke (1954) attempted to generalize Ledoux' (1951) treatment to a rotating isothermal slab, but due to false approximations his results are now recognized to be untenable (Goldreich and Lynden-Bell 1965a).

The equation of motion (5.7.1) in a system rotating with angular velocity $\vec{\Omega} = \vec{\Omega}(0,0,\Omega),\ (\Omega = \mathrm{const})$ can be written under the form

$$D\vec{v}/Dt = \Omega^2\vec{\ell} - 2\vec{\Omega} \times \vec{v} - (1/\varrho)\ \nabla P + \nabla\Phi, \tag{5.10.48}$$

where $\vec{\ell} = \vec{\ell}(x,y,0),\ \ell^2 = x^2 + y^2,\ P = P(z),\ \varrho = \varrho(z),\ \Phi = \Phi(z)$. Goldreich and Lynden-Bell (1965a) consider, analogously to Ledoux (1951), isentropic polytropic slabs, when $\vec{A}$ from Eqs. (5.2.84), (5.10.2) is zero, and $\Gamma_1 = 1 + 1/n$. In this case the adiabatic energy conservation equation (5.2.95) becomes

$$\delta P = [(1 + 1/n)P/\varrho]\ \delta\varrho, \tag{5.10.49}$$

and the linearized Eulerian variation of Eq. (5.10.48) reads (cf. Eq. (5.2.86) if $\vec{A} = 0;\ \delta(\Omega^2\vec{\ell}) = 0$) :

$$\delta(D\vec{v}/Dt) \approx \delta(\partial\vec{v}/\partial t) = \partial\delta\vec{v}/\partial t \approx \partial\vec{v}/\partial t = -2\vec{\Omega} \times \delta\vec{v} - \nabla(\delta P/\varrho - \delta\Phi) \approx -2\vec{\Omega} \times \vec{v} - \nabla\chi,$$
$$(\delta\vec{v} \approx \vec{v} \approx 0;\ \chi = \delta P/\varrho - \delta\Phi). \tag{5.10.50}$$

Since axial symmetry is assumed, Goldreich and Lynden-Bell (1965a) take the perturbations of physical quantities along the Cartesian x-axis under the particular form [cf. Eq. (5.10.24)]

$$\Delta\vec{r}(x,z,t)=\Delta\vec{r}(z)\exp[i(\sigma t+jx)];\quad \delta\vec{v}(x,z,t)\approx\vec{v}(x,z,t)=\vec{v}(z)\exp[i(\sigma t+jx)];\quad \chi(x,z,t)$$
$$=\delta P(x,z,t)/\varrho(z)-\delta\Phi(x,z,t)=\chi(z)\exp[i(\sigma t+jx)];\quad \delta\varrho(x,z,t)=\delta\varrho(z)\exp[i(\sigma t+jx)]. \tag{5.10.51}$$

We insert this form of the perturbations into the equation of motion (5.10.50), suppressing the common factor $\exp[i(\sigma t+jx)]$:

$$i\sigma v_x(z)-2\Omega v_y(z)=-ij\chi(z), \tag{5.10.52}$$

$$i\sigma v_y(z)+2\Omega v_x(z)=0, \tag{5.10.53}$$

$$i\sigma v_z(z)=-d\chi(z)/dz. \tag{5.10.54}$$

The perturbed continuity equation (5.2.25) becomes

$$i\sigma\ \delta\varrho(z)+ij\varrho(z)\ v_x(z)+d[\varrho(z)\ v_z(z)]/dz=0, \tag{5.10.55}$$

and Poisson's equation (5.2.40) writes

$$d^2\delta\Phi/dz^2-j^2\ \delta\Phi(z)=-4\pi G\ \delta\varrho(z). \tag{5.10.56}$$

Eliminating v_y between Eqs. (5.10.52) and (5.10.53), we obtain

$$(4\Omega^2-\sigma^2)\ v_x(z)=\sigma j\chi(z), \tag{5.10.57}$$

and the continuity equation (5.10.55) writes via Eq. (5.10.57) as

$$i\sigma\ \delta\varrho(z)+i\sigma j^2\varrho(z)\ \chi(z)/(4\Omega^2-\sigma^2)+d(\varrho v_z)/dz=0. \tag{5.10.58}$$

The fourth order equation for $\delta\Phi$ found by elimination of $\delta\varrho$ and v_z between Eqs. (5.10.54), (5.10.56), (5.10.58) is not very enlightening, and will consequently be omitted.

The boundary condition for the pressure on the finite, oscillating boundary

$$Z_1(x,z,t)=z_1+\Delta z_1(x,z,t)=z_1+\Delta z_1(z)\exp[i(\sigma t+jx)], \tag{5.10.59}$$

is $P(Z_1)=0$. To obtain a condition – similar to Eq. (5.2.63) – for the Eulerian pressure perturbation δP we observe from the equation of hydrostatic equilibrium that $dP/dz=\varrho\ d\Phi/dz$. From $\Delta P=\delta P+(dP/dz)\ \Delta z=0$ we get at $z=z_1$:

$$\delta P=-(dP/dz)\ \Delta z=-\varrho\ \Delta z\ d\Phi/dz,\qquad (z=z_1). \tag{5.10.60}$$

The inner and outer gravitational potential, and the corresponding derivatives with respect to z must be equal on the oscillating boundary $Z_1=z_1+\Delta z_1$ [cf. Eqs. (5.2.111)-(5.2.114)]:

$$\Phi(x,z_1+\Delta z_1)=\Phi_u(x,z_1+\Delta z_1)+\delta\Phi=\Phi_u(x,z_1)+\Delta z_1\ (d\Phi_u/dz)_{z=z_1}+\delta\Phi$$
$$=\Phi_e(x,z_1+\Delta z_1)=\Phi_{ue}(x,z_1+\Delta z_1)+\delta\Phi_e=\Phi_{ue}(x,z_1)+\Delta z_1\ (d\Phi_{ue}/dz)_{z=z_1}+\delta\Phi_e, \tag{5.10.61}$$

$$(\partial\Phi/\partial z)_{z=z_1+\Delta z_1}=(d\Phi_u/dz)_{z=z_1}+\Delta z_1\ (d^2\Phi_u/dz^2)_{z=z_1}+(\partial\delta\Phi/\partial z)_{z=z_1}$$
$$=(\partial\Phi_e/\partial z)_{z=z_1+\Delta z_1}=(d\Phi_{ue}/dz)_{z=z_1}+\Delta z_1\ (d^2\Phi_{ue}/dz^2)_{z=z_1}+(\partial\delta\Phi_e/\partial z)_{z=z_1}. \tag{5.10.62}$$

The indices u and e denote unperturbed and external values, respectively. Subtracting Laplace's equation $\nabla^2\Phi_{ue}=0$ from Poisson's equation $\nabla^2\Phi_u=-4\pi G\varrho_u$, we get $(d^2\Phi_u/dz^2)_{z=z_1}-(d^2\Phi_{ue}/dz^2)_{z=z_1}=-4\pi G\varrho_u(z_1)$, and since $(d\Phi_u/dz)_{z=z_1}=(d\Phi_{ue}/dz)_{z=z_1}$, we obtain from Eqs. (5.10.61), (5.10.62):

$$\delta\Phi(x,z_1)=\delta\Phi_e(x,z_1);\quad (\partial\delta\Phi/\partial z)_{z=z_1}-(\partial\delta\Phi_e/\partial z)_{z=z_1}=4\pi G\varrho(z_1)\ \Delta z_1. \tag{5.10.63}$$

With $\delta\Phi_e(x,z,t) = \delta\Phi_e(z)\exp[i(\sigma t + jx)]$ obeying the same dependence as in Eq. (5.10.51), a solution of Laplace's equation $\nabla^2\delta\Phi_e(x,z,t) = \partial^2\delta\Phi_e/\partial x^2 + \partial^2\delta\Phi_e/\partial z^2 = 0$ is of the form

$$\delta\Phi_e = C\exp[-|jz| + i(\sigma t + jx)], \qquad (C = \mathrm{const};\ |z| \geq z_1), \tag{5.10.64}$$

dying away if $z \to \pm\infty$. The boundary condition (5.10.63) takes with this equation its final form

$$(\partial\delta\Phi/\partial z)_{z=z_1} + |j|\ \delta\Phi_e(z_1) = (\partial\delta\Phi/\partial z)_{z=z_1} + |j|\ \delta\Phi(z_1) = 4\pi G\varrho(z_1)\ \Delta z_1. \tag{5.10.65}$$

Generally, the density vanishes on the surface: $\varrho(z_1) = 0$. The nonexistence of an edge invalidates our previous arguments in the case of an isothermal slab $n = \pm\infty$, so we have to derive in this case a new boundary condition by integrating the continuity equation (5.10.58):

$$i\sigma\int_{-\infty}^{\infty}\delta\varrho\ dz + [i\sigma j^2/(4\Omega^2 - \sigma^2)]\int_{-\infty}^{\infty}\varrho\chi\ dz + \varrho v_z\Big|_{-\infty}^{\infty} = 0. \tag{5.10.66}$$

The last term represents the mass flow in the z-direction through the infinity points, and has to vanish accordingly. The boundary condition for the infinite isothermal slab writes therefore

$$i\sigma\ \delta\Sigma_1 = i\sigma\int_{-\infty}^{\infty}\delta\varrho\ dz = -[i\sigma j^2/(4\Omega^2 - \sigma^2)]\int_{-\infty}^{\infty}\varrho\chi\ dz, \qquad (n = \pm\infty). \tag{5.10.67}$$

Goldreich and Lynden-Bell (1965a) discuss at first the particular case $j = 0$ in Eq. (5.10.51). In this case Eqs. (5.10.52), (5.10.53) become

$$i\sigma v_x(z) - 2\Omega v_y(z) = 0; \quad 2\Omega v_x(z) + i\sigma v_y(z) = 0. \tag{5.10.68}$$

For nontrivial solutions the determinant $\sigma^2 - 4\Omega^2$ of this system has to be zero. But as σ^2 is always real, the value $\sigma^2 = 4\Omega^2 \neq 0$ is not associated with neutral (marginal) stability, and the modes $j = 0$ are stable. Only if $\Omega = 0$ these modes are unstable for the nonrotating isothermal slab, since in this case $L = 2\pi/|j| = \infty$ is larger than the critical Jeans wavelength (5.10.30). If $\sigma^2 \neq 4\Omega^2$, we must have $v_x, v_y = 0$, and the oscillations reduce to purely vertical motions in the z-direction. After some lengthy evaluations Goldreich and Lynden-Bell (1965a) are able to show that rotating isentropic polytropic slabs are stable if $v_x, v_y = 0,\ j = 0$, and $\sigma^2 \neq 4\Omega^2$, excepting for the trivial case of a disturbance that imparts a small velocity v_z to the slab as a whole.

The overall run of the results if $j \neq 0$ can be easiest exemplified by the isentropic, incompressible homogeneous slab ($n = 0,\ \Gamma_1 = \infty$). The outcomes are remarkably insensitive to the value of the polytropic index (e.g. $n = 1, \pm\infty$), because the pressure perturbation δP is related linearly to the density disturbance $\delta\varrho$ [see Eq. (5.10.49)], unlike the unperturbed pressure and density, which satisfy the power relationship $P_u = K\varrho_u^{1+1/n}$. Larson (1985, Table 1) has extended this analysis to negative polytropic indices $n = -4/3, -2, -3$.

If $n = 0,\ \varrho = \mathrm{const},\ \delta\varrho = 0$, the perturbed continuity equation (5.10.55) is equal to

$$\nabla\cdot\vec{v} = ijv_x + dv_z/dz = 0. \tag{5.10.69}$$

We eliminate the velocities among Eqs. (5.10.54), (5.10.57), (5.10.69):

$$d^2\chi/dz^2 - j^2\chi/(1 - 4\Omega^2/\sigma^2) = 0. \tag{5.10.70}$$

The solution of this second order equation with constant coefficients can be written under the form

$$\chi = C_1\exp[jz/(1 - 4\Omega^2/\sigma^2)^{1/2}] + C_2\exp[-jz/(1 - 4\Omega^2/\sigma^2)^{1/2}] = A\cosh(\alpha jz)/\cosh(\alpha jz_1)$$
$$+B\sinh(\alpha jz)/\sinh(\alpha jz_1), \qquad \big(\alpha = (1 - 4\Omega^2/\sigma^2)^{-1/2};\ C_1, C_2, A, B = \mathrm{const}\big). \tag{5.10.71}$$

Since $\delta\varrho = 0$ by assumption, the right-hand side of Eq. (5.10.56) becomes zero, and Poisson's equation integrates analogously to Eq. (5.10.70):

$$\delta\Phi = C\cosh(jz)/\cosh(jz_1) + D\sinh(jz)/\sinh(jz_1), \qquad (C, D = \mathrm{const}). \tag{5.10.72}$$

The solution of the problem consists of a superposition of antisymmetrical modes if $A, C = 0$, and of symmetrical modes if $B, D = 0$.

(i) $A, C = 0$. The boundary condition (5.10.65) on the potential becomes ($\varrho = \varrho_0 = \varrho(z_1) = \mathrm{const}$) :

$$Dj\coth(jz_1) + D|j| = 4\pi G\varrho\ \Delta z_1. \tag{5.10.73}$$

And from Eqs. (5.10.71), (5.10.72) we get

$$\delta P/\varrho = \chi(z) + \delta\Phi = B\sinh(\alpha jz)\big/\sinh(\alpha jz_1) + D\sinh(jz)\big/\sinh(jz_1). \tag{5.10.74}$$

The boundary condition on the pressure (5.10.60) therefore reads as

$$(\delta P/\varrho)_{z=z_1} = B + D = -\Delta z_1\ (d\Phi/dz)_{z=z_1} = 4\pi G\varrho z_1\ \Delta z_1. \tag{5.10.75}$$

Finally, Eqs. (5.10.54), (5.10.71) yield

$$i\sigma v_z = i\sigma\ D(\Delta z)/Dt \approx i\sigma\ \partial\Delta z/\partial t = -\sigma^2\ \Delta z = -d\chi/dz = -B\alpha j\cosh(\alpha jz)\big/\sinh(\alpha jz_1), \tag{5.10.76}$$

which amounts on the boundary to

$$\sigma^2\ \Delta z_1 = B\alpha j\coth(\alpha jz_1). \tag{5.10.77}$$

To proceed further, let us eliminate the constant D between Eqs. (5.10.73) and (5.10.75):

$$B = 4\pi G\varrho_0 z_1\ \Delta z_1\big\{1 - 1/[jz_1\coth(jz_1) + |j|z_1]\big\}. \tag{5.10.78}$$

Substituting this value into Eq. (5.10.77), and performing separately the calculations for the two cases $|j| = j$ if $j \geq 0$, and $|j| = -j$ if $j < 0$, we obtain the final dispersion relation:

$$\sigma^2 = 4\pi G\varrho\alpha jz_1\coth(\alpha jz_1)\big\{1 - [1 - \exp(-2|j|z_1)]/2|j|z_1\big\}. \tag{5.10.79}$$

We observe that $\coth(iU) = -i\cot U$, and $1 - [1 - \exp(-Y)]/Y > 0$ if $Y = 2|j|z_1 > 0$. Hence, σ^2 could only be negative if α is imaginary, but in this case σ^2 should be positive by virtue of Eq. (5.10.71): $\alpha^2 = 1/(1 - 4\Omega^2/\sigma^2)$. Thus, we arrive at a contradiction, showing that σ^2 is nonnegative, and the homogeneous isentropic slab is stable against antisymmetrical modes with $A, C = 0$. Expanding Eq. (5.10.79) if $jz_1 \approx 0$ we get $\sigma^2 \approx 4\pi G\varrho|j|z_1$, showing that neutral stability $\sigma = 0$ only occurs if $j = 0$, and we have previously mentioned that this merely amounts to a displacement of the slab as a whole.

(ii) $B, D = 0$. The calculations for these symmetrical modes are analogous to Eqs. (5.10.73)-(5.10.79). The dispersion relation becomes

$$\sigma^2 = 4\pi G\varrho\alpha jz_1\tanh(\alpha jz_1)\ \big\{1 - [1 + \exp(-2|j|z_1)]/2|j|z_1\big\}. \tag{5.10.80}$$

Near the point of neutral stability we have $\alpha \propto \sigma \approx 0$, and

$$\alpha jz_1\tanh(\alpha jz_1) \approx \alpha^2 j^2 z_1^2 = j^2 z_1^2/(1 - 4\Omega^2/\sigma^2), \qquad (\alpha \approx 0). \tag{5.10.81}$$

The dispersion relation (5.10.80) takes the simplified form

$$\sigma^4 - 4\Omega^2\sigma^2 = 4\pi G\varrho\sigma^2 j^2 z_1^2\big\{1 - [1 + \exp(-2|j|z_1)]/2|j|z_1\big\}, \tag{5.10.82}$$

showing that either $\sigma^2 = 0$ for all wave numbers j, or

$$\sigma^2/\pi G\varrho = 4\Omega^2/\pi G\varrho + 2|j|z_1\big[2|j|z_1 - 1 - \exp(-2|j|z_1)\big], \qquad (\sigma \approx 0). \tag{5.10.83}$$

The eigenvalue σ^2 takes its lowest value (the slab is most unstable), if the last term becomes minimum, and this occurs when its derivative is zero

$$4|j|z_1 - 1 + (2|j|z_1 - 1)\exp(-2|j|z_1) = 0, \tag{5.10.84}$$

i.e. if

$$2|j|z_1 = 2|j_c|z_1 = 0.607, \qquad (n = 0). \tag{5.10.85}$$

For this most unstable slab the critical value Ω_c of Ω is obtained by putting $\sigma \approx 0$ in Eq. (5.10.83):

$$4\Omega_c^2/\pi G\varrho = 0.569, \qquad (n = 0). \tag{5.10.86}$$

The slab is stable ($\sigma^2 > 0$) if $4\Omega^2/\pi G\varrho > 0.569$, and unstable ($\sigma^2 < 0$) if $4\Omega^2/\pi G\varrho < 0.569$. Wavelengths larger than the critical wavelength

$$L_c = 2\pi/|j_c| = 4\pi z_1/0.607, \qquad (n = 0), \tag{5.10.87}$$

are unstable, and it is seen that unstable wavelengths are larger than a few times the thickness of the slab.

If the dispersion relation (5.10.83) is solved for general $|j|z_1$ – provided that $\sigma = 0$ – we get two critical wavenumbers j_c, $(0 < 2|j_c|z_1 \lesssim 1.2)$, including the most unstable value (5.10.85). Oscillations with wave vectors of magnitude between the two critical values $|j_c|$ are unstable, and are bounded in both directions of $|j_c|$ by stable wave numbers. This behaviour is a direct consequence of the gravitational stratification of the rotating isentropic slabs, and is contrasted to some extent to that of the rotating, homogeneous infinite medium, when only a single critical wavelength occurs, and all wavenumbers having

$$j_c^2 = 4\pi^2/L^2 < 4(\pi G\varrho - \Omega^2)/a^2, \tag{5.10.88}$$

are unstable in the direction perpendicular to the rotation axis (e.g. Sec. 5.4.1, Chandrasekhar 1981, §120).

A shortcoming of the analysis of Goldreich and Lynden-Bell (1965a) – which is shared with many others – is that the possibility of multiple roots of σ in the dispersion relations, leading to solutions of the form (*polynomial in* t) $\exp(i\sigma t)$, is considered only for the neutral case $\sigma = 0$.

Nonlinear waves in polytropic slabs have been studied by Qian and Spiegel (1994), including the influence of an external halo potential $\Phi_e \propto z^2$.

5.10.4 Zero Thickness Disks

As we have already stressed, there seem to appear no significant mechanical effects depending on the detailed structure of slabs and disks along the vertical z-direction. We therefore neglect in most of the subsequent discussion the vertical extension of the slabs. All motions are supposed to occur only in the central plane $z = 0$ of the system. Gravitational forces are overestimated by the assumption of an infinitesimal thickness of the slab, and the zero thickness approximation is not valid for wavelengths comparable to the real thickness of the configuration (cf. Hunter 1972).

In order to connect the two-dimensional pressure P_Σ acting in the (x, y)-plane of the zero thickness disk to its surface density Σ from Eq. (5.10.4), we consider a highly flattened slab. The high degree of flattening means that the structure changes most rapidly in the z-direction normal to the central plane of the slab: $\partial\Phi/\partial z \gg \partial\Phi/\partial x, \partial\Phi/\partial y$. Thus, we can approximate Poisson's equation (2.1.4) by

$$\partial^2\Phi(x, y, z)/\partial z^2 = -4\pi G\varrho(x, y, z), \tag{5.10.89}$$

and if $v_z = 0$ the vertical equilibrium component of the equation of motion (5.10.48) becomes with the polytropic equation of state

$$(1/\varrho)\ \partial P/\partial z = K(1 + 1/n)\varrho^{-1+1/n}\ \partial\varrho/\partial z = (n + 1)K\ \partial\varrho^{1/n}/\partial z = \partial\Phi/\partial z. \tag{5.10.90}$$

Inserting the derivative of Eq. (5.10.90) into Eq. (5.10.89), we obtain after multiplication with $\partial\varrho^{1/n}/\partial z$

$$(n + 1)K(\partial\varrho^{1/n}/\partial z)\ \partial^2\varrho^{1/n}/\partial z^2 = [(n + 1)K/2]\ \partial[(\partial\varrho^{1/n}/\partial z)^2]/\partial z = -4\pi G\varrho\ \partial\varrho^{1/n}/\partial z$$
$$= -(4\pi G/n)\varrho^{1/n}\ \partial\varrho/\partial z = -[4\pi G/(n + 1)]\ \partial\varrho^{1+1/n}/\partial z. \tag{5.10.91}$$

Integration of Eq. (5.10.91) with the central boundary condition $(\partial\varrho/\partial z)_{z=0} = 0$ yields

$$(1 + 1/n)^2 K\varrho^{2/n-2}(\partial\varrho/\partial z)^2 = 8\pi G\big(\varrho_0^{1+1/n} - \varrho^{1+1/n}\big), \tag{5.10.92}$$

and

$$dz = -(1+1/n)(K/8\pi G)^{1/2}\varrho^{1/n-1}\ d\varrho/\big(\varrho_0^{1+1/n} - \varrho^{1+1/n}\big)^{1/2}, \qquad (\partial\varrho/\partial z \le 0). \tag{5.10.93}$$

The total surface density (5.10.5) is equal to (we will omit throughout the index 1 in the zero thickness approximation)

$$\Sigma = \Sigma(\lambda,\varphi) = (1+1/n)(K/2\pi G)^{1/2}\int_0^{\varrho_0} \varrho^{1/n}\ d\varrho/\big(\varrho_0^{1+1/n} - \varrho^{1+1/n}\big)^{1/2}$$
$$= (2K/\pi G)^{1/2}\varrho_0^{(n+1)/2n} = (2P_0/\pi G)^{1/2}. \tag{5.10.94}$$

Analogously to the two-dimensional surface density (5.10.5) we define via Eqs. (2.1.6), (5.10.93) the cumulative two-dimensional pressure P_Σ acting per unit length in the plane of a polytropic zero thickness disk [cf. Hunter 1972, Eq. (2.11)]:

$$P_\Sigma = P_\Sigma(\lambda,\varphi) = 2\int_0^{z_1} P\ dz = (1+1/n)(K^3/2\pi G)^{1/2}\int_0^{\varrho_0} \varrho^{2/n}\ d\varrho/\big(\varrho_0^{1+1/n} - \varrho^{1+1/n}\big)^{1/2}$$
$$= (K^3/2\pi G)^{1/2}\varrho_0^{(n+3)/2n}\int_0^1 t^{1/(n+1)}(1-t)^{-1/2}\ dt = (K^3/2\pi G)^{1/2}\varrho_0^{(n+3)/2n}$$
$$\times B[(n+2)/(n+1), 1/2] = (K^3/2G)^{1/2}\varrho_0^{(n+3)/2n}\Gamma[(n+2)/(n+1)]/\Gamma[(3n+5)/2(n+1)],$$
$$(-\infty \le n < -2;\ -1 < n \le \infty;\ K^{3/2}\varrho_0^{(n+3)/2n} = P_0^{3/2}/\varrho_0;\ t = (\varrho/\varrho_0)^{1+1/n};\ \Gamma(1/2) = \pi^{1/2}). \tag{5.10.95}$$

Euler's beta function has been denoted by (cf. Eqs. (2.3.56), (2.3.57); Smirnow 1967)

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\ dt = \Gamma(p)\ \Gamma(q)/\Gamma(p+q), \qquad (p,q>0), \tag{5.10.96}$$

and $\Gamma(x)$ means the gamma function from Eq. (C.9). Substituting into Eq. (5.10.95) for ϱ_0 from Eq. (5.10.94), we finally obtain the desired relationship between surface density and two-dimensional pressure in the (λ,φ) or (x,y)-plane of a zero thickness disk [cf. Eqs. (5.10.224)-(5.10.225)]:

$$P_\Sigma = 2^{-(n+2)/(n+1)}\pi^{(n+3)/2(n+1)}G^{1/(n+1)}K^{n/(n+1)}\ \Gamma[(n+2)/(n+1)]\ \Sigma^{(n+3)/(n+1)}$$
$$/\Gamma[(3n+5)/2(n+1)] = K_\Sigma\Sigma^{1+1/n_\Sigma}, \qquad (K_\Sigma, n_\Sigma = \mathrm{const};\ n_\Sigma = (n+1)/2;\ n = 2n_\Sigma - 1;$$
$$-\infty \le n_\Sigma < -0.5;\ 0 < n_\Sigma \le \infty;\ -\infty \le n < -2;\ -1 < n \le \infty). \tag{5.10.97}$$

If $n = 1, \pm\infty$, the exponent $(n+3)/(n+1) = 1 + 1/n_\Sigma$ of Σ is the same as in the three-dimensional polytropic law $P = K\varrho^{1+1/n}$; we have $P_\Sigma = K_\Sigma\Sigma$ if $n_\Sigma, n = \pm\infty$, and $P_\Sigma = K_\Sigma\Sigma^2$ if $n_\Sigma, n = 1$. There is $n > n_\Sigma$ if $n_\Sigma > 1$, and $n < n_\Sigma$ if $n_\Sigma < 1$. Note, that the two-dimensional pressure P_Σ acts isotropically per unit length in the (x,y), (ℓ,φ)-plane of the zero thickness disk, whereas ordinary three-dimensional pressure P acts in space per unit surface.

The basic equations of zero thickness disks are simply obtained by replacing the three-dimensional volume density $\varrho(x,y,z) = \varrho(\ell,\varphi,z)$ with the two-dimensional surface density $\Sigma(x,y) = \Sigma(\ell,\varphi)$, where some care has to be exercised with Poisson's equation (2.1.4), taking in the plane $z = 0$ of the zero thickness disk the form

$$\nabla^2\Phi(\ell,\varphi) = (1/\ell)\ \partial(\ell\ \partial\Phi/\partial\ell)/\partial\ell + (1/\ell^2)\ \partial^2\Phi/\partial\varphi^2 = -4\pi G\Sigma(\ell,\varphi)\ \delta_D(z). \tag{5.10.98}$$

The one-dimensional Dirac function $\delta_D(z)$ is defined as

$$\delta_D(z) = 0 \quad \text{if} \quad z \ne 0; \qquad \int_{z=0_-}^{z=0_+} \delta_D(z)\ dz = 1, \tag{5.10.99}$$

where 0_+ and 0_- denote the z-coordinates at the upper and lower border of the central symmetry plane $z = 0$. Outside the central plane the gravitational potential satisfies Laplace's equation

$$\nabla^2\Phi_e(x,y,z) = \nabla^2\Phi_e(\lambda,\varphi,z) = 0 \quad \text{if} \quad z \ne 0. \tag{5.10.100}$$

The boundary condition on the potential of a zero thickness disk can be derived by integrating Poisson's equation (5.10.89) between the boundaries z_1 and $-z_1$ of a disk with small thickness $2z_1 \approx 0$, where $\partial\Phi/\partial x, \partial\Phi/\partial y \ll \partial\Phi/\partial z$ and $(\partial\Phi/\partial z)_{z=0} = 0$:

$$\int_0^{z_1} [\partial(\partial\Phi/\partial z)/\partial z]\; dz = (\partial\Phi/\partial z)_{z=z_1} = -4\pi G \int_0^{z_1} \varrho\; dz = -2\pi G\Sigma;$$
$$\int_{-z_1}^{0} [\partial(\partial\Phi/\partial z)/\partial z]\; dz = -(\partial\Phi/\partial z)_{z=-z_1} = -4\pi G \int_{-z_1}^{0} \varrho\; dz = -2\pi G\Sigma. \qquad (5.10.101)$$

Letting now $z_1 \rightarrow 0_+,\ -z_1 \rightarrow 0_-$, we get

$$(\partial\Phi/\partial z)_{z=0_+} = -2\pi G\Sigma; \quad (\partial\Phi/\partial z)_{z=0_-} = 2\pi G\Sigma, \qquad (5.10.102)$$

and it is seen that, while the potential is required to be continuous across the infinitely thin disk, its derivative is discontinuous, changing sign. The continuity equation (5.2.1) becomes simply

$$\partial\Sigma/\partial t + (1/\ell)\; \partial(\ell\Sigma v_\ell)/\partial\ell + (1/\ell)\; \partial(\Sigma v_\varphi)/\partial\varphi = 0, \qquad [\varrho(\ell,\varphi,z) \rightarrow \Sigma(\ell,\varphi);\ \vec{v} = \vec{v}(v_\ell, v_\varphi, 0)]. \qquad (5.10.103)$$

The equation of motion (5.2.10) writes in an inertial frame as [cf. Eq. (B.51)]

$$Dv_\ell/Dt = \partial v_\ell/\partial t + v_\ell\; \partial v_\ell/\partial\ell + (v_\varphi/\ell)\; \partial v_\ell/\partial\varphi - v_\varphi^2/\ell = -(1/\Sigma)\; \partial P_\Sigma/\partial\ell + \partial\Phi/\partial\ell, \qquad (5.10.104)$$

$$Dv_\varphi/Dt = \partial v_\varphi/\partial t + v_\ell\; \partial v_\varphi/\partial\ell + (v_\varphi/\ell)\; \partial v_\varphi/\partial\varphi + v_\ell v_\varphi/\ell = -(1/\ell\Sigma)\; \partial P_\Sigma/\partial\varphi + (1/\ell)\; \partial\Phi/\partial\varphi. \qquad (5.10.105)$$

Sometimes, the enthalpy from Eq. (3.8.82) is introduced for a two-dimensional fluid $P_\Sigma = P_\Sigma(\Sigma)$:

$$dH = dP_\Sigma/\Sigma \quad \text{or} \quad H = H(P_\Sigma) = \int_0^{P_\Sigma} dP_\Sigma/\Sigma(P_\Sigma), \qquad (\nabla H = (dH/dP_\Sigma)\; \nabla P_\Sigma = (1/\Sigma)\; \nabla P_\Sigma). \qquad (5.10.106)$$

Usually, the hydrostatic equilibrium state of zero thickness disks is assumed under the form of axisymmetric, steady circular motion: $v_\ell, v_z = 0,\ v_\varphi = \ell\Omega(\ell)$. In this case Eq. (5.10.104) is equal to

$$-[1/\Sigma(\ell)]\; dP_\Sigma/d\ell + d\Phi/d\ell + \ell\Omega^2(\ell) = 0. \qquad (5.10.107)$$

The first order Eulerian perturbations of Eqs. (5.10.98), (5.10.100), (5.10.102) become (cf. Hunter 1972, Aoki et al. 1979, Lin and Lau 1979, Iye 1984):

$$\nabla^2\delta\Phi = -4\pi G\; \delta_D(z)\; \delta\Sigma, \qquad (5.10.108)$$

$$\nabla^2\delta\Phi_e = 0 \quad \text{if} \quad z \neq 0; \qquad (\partial\delta\Phi/\partial z)_{z=0_+} - (\partial\delta\Phi/\partial z)_{z=0_-} = -4\pi G\; \delta\Sigma. \qquad (5.10.109)$$

Because the perturbed velocity components in an inertial cylindrical frame are

$$\vec{v} = \vec{v}(\delta v_\ell, \ell\Omega(\ell) + \delta v_\varphi, 0), \qquad (5.10.110)$$

the linearized Eulerian perturbation of the continuity equation (5.10.103) reads

$$\partial\delta\Sigma/\partial t + \Omega(\ell)\; \partial\delta\Sigma/\partial\varphi + (1/\ell)\; \partial(\ell\Sigma\; \delta v_\ell)/\partial\ell + (\Sigma/\ell)\; \partial\delta v_\varphi/\partial\varphi = 0, \qquad [\Sigma = \Sigma(\ell)]. \qquad (5.10.111)$$

Likewise, we derive the linear perturbed equations of motion from Eqs. (5.10.104) and (5.10.105) in a cylindrical inertial frame $[\Sigma = \Sigma(\ell);\ P_\Sigma = P_\Sigma(\ell);\ \Phi = \Phi(\ell)]$:

$$\partial\delta v_\ell/\partial t + \Omega(\ell)\; \partial\delta v_\ell/\partial\varphi - 2\Omega(\ell)\; \delta v_\varphi = (\delta\Sigma/\Sigma^2)\; dP_\Sigma/d\ell - (1/\Sigma)\; \partial\delta P_\Sigma/\partial\ell + \partial\delta\Phi/\partial\ell$$
$$= (a_\Sigma^2\; \delta\Sigma/\Sigma^2)\; d\Sigma/d\ell - (a_\Sigma^2/\Sigma)\; \partial\delta\Sigma/\partial\ell - (\delta\Sigma/\Sigma)\; da_\Sigma^2/d\ell + \partial\delta\Phi/\partial\ell, \qquad (5.10.112)$$

$$\partial\delta v_\varphi/\partial t + \Omega(\ell)\ \partial\delta v_\varphi/\partial\varphi + [2\Omega(\ell) + \ell\ d\Omega(\ell)/d\ell]\ \delta v_\ell = -(1/\ell\Sigma)\ \partial\delta P_\Sigma/\partial\varphi + (1/\ell)\ \partial\delta\Phi/\partial\varphi$$
$$= -(a_\Sigma^2/\ell\Sigma)\ \partial\delta\Sigma/\partial\varphi + (1/\ell)\ \partial\delta\Phi/\partial\varphi, \quad (5.10.113)$$

where $\delta P_\Sigma(\ell,\varphi) = a_\Sigma^2(\ell)\ \delta\Sigma(\ell,\varphi)$, (see below). We have introduced the two-dimensional adiabatic sound velocity in the infinitely thin plane of the disk for adiabatic oscillations with two-dimensional adiabatic exponent $\Gamma_{1\Sigma} = (\Sigma/P_\Sigma)(\partial P_\Sigma/\partial\Sigma)_S$, [cf. Eqs. (1.3.1), (2.1.49)]:

$$a_\Sigma^2 = (\partial P_\Sigma/\partial\Sigma)_{S=\mathrm{const}} = \Gamma_{1\Sigma}P_\Sigma/\Sigma. \quad (5.10.114)$$

It should be stressed that the parts of Eqs. (5.10.112), (5.10.113) containing a_Σ are valid only for an isentropic polytrope ($P_\Sigma = K_\Sigma\Sigma^{1+1/n_\Sigma}$; $\Gamma_{1\Sigma} = 1 + 1/n_\Sigma$), as tacitly assumed by Hunter (1972). In this case the adiabatic energy equation (5.10.226) simplifies to $\delta P_\Sigma = a_\Sigma^2\ \delta\Sigma$, because by virtue of Eqs. (5.1.16), (5.10.114), (5.10.224)-(5.10.227) we have $\Delta P_\Sigma = \Delta\ell\ dP_\Sigma/d\ell + \delta P_\Sigma = (1 + 1/n_\Sigma)(P_\Sigma/\Sigma)\ \Delta\ell\ d\Sigma/d\ell + \delta P_\Sigma = (\Gamma_{1\Sigma}P_\Sigma/\Sigma)\ \Delta\ell\ d\Sigma/d\ell + \delta P_\Sigma = a_\Sigma^2\ \Delta\ell\ d\Sigma/d\ell + \delta P_\Sigma$ on the one side, and $\Delta P_\Sigma = a_\Sigma^2\ \Delta\Sigma = a_\Sigma^2\ \Delta\ell\ d\Sigma/d\ell + a_\Sigma^2\ \delta\Sigma$ on the other side. Comparing these two relationships for ΔP_Σ, we just get $\delta P_\Sigma = a_\Sigma^2\ \delta\Sigma$. The sound velocity a_Σ replaces in this context the adiabatic index $\Gamma_{1\Sigma}$, familiar from previous sections.

Investigations on the local stability of zero thickness disks (including spiral density wave theory) are concerned with the short wavelength regime, when the Wentzel-Kramers-Brillouin-Jeffreys (WKBJ) approximation can be applied, i.e. when the proportional change of wavelength within a local wavelength is small.

In order to determine the general nature of solutions of a zero thickness disk in the short wavelength regime, we derive subsequently with the stroke of a pen an important local dispersion relationship. Whereas Goldreich and Lynden-Bell (1965b) consider a local comoving frame with sheared axes, and Hunter (1972) introduces a local inertial Cartesian frame, we consider simply the perturbed motion in the neighborhood of a certain fixed point with polar coordinates ℓ_0, φ_0 in the inertial frame. Let $\ell = \ell_0 + s$ and φ be the polar coordinates of a neighboring point, where $s = \ell - \ell_0 \ll \ell_0,\ \varphi - \varphi_0 \ll \varphi_0$. The angular velocity of the unperturbed fluid at point (ℓ,φ) is $\Omega(\ell) = \Omega(\ell_0 + s) = \Omega(s)$, ($\ell_0 = \mathrm{const}$). All other *unperturbed* physical quantities are approximated with constants in the vicinity of ℓ_0, φ_0 : $P_{u\Sigma}, \Sigma_u, \Phi_u, a_{u\Sigma} = \mathrm{const}$. The perturbations in the (ℓ,φ)-plane can therefore be assumed under the form

$$\delta P_\Sigma(s,\varphi,t) = C_P \exp[i(\sigma t + js + k\varphi)]; \quad \delta\Sigma(s,\varphi,t) = C_\Sigma \exp[i(\sigma t + js + k\varphi)];$$
$$\delta\Phi(s,\varphi,t) = C_\Phi \exp[i(\sigma t + js + k\varphi)]; \quad \delta v_\ell(s,\varphi,t) = C_\ell \exp[i(\sigma t + js + k\varphi)];$$
$$\delta v_\varphi(s,\varphi,t) = C_\varphi \exp[i(\sigma t + js + k\varphi)], \qquad (C_P, C_\Sigma, C_\Phi, C_\ell, C_\varphi = \mathrm{const}). \quad (5.10.115)$$

j is the radial wave number, and k denotes the azimuthal wave number [cf. Eq. (5.6.1)]. In our local approximation we consider $j = \mathrm{const}$. In fact, the perturbations (5.10.115) are similar to those used with the WKBJ method in density wave theory for Φ [e.g. Lin and Lau 1979, Eq. (29)]:

$$\delta\Phi(s,\varphi,t) = \delta\Phi(s)\exp\left[i\left(\sigma t + \int_0^s j(s')\ ds' + k\varphi\right)\right]. \quad (5.10.116)$$

By assumption $j(s'), \delta\Phi(s) \approx \mathrm{const}$ if $s \ll \ell_0$, and Eq. (5.10.116) assumes the form (5.10.115). Since $\ell_0 = \mathrm{const}$, we turn in Eqs. (5.10.111)-(5.10.113) to the new variables s, φ, and get up to the first order $(ds = d\ell;\ \ell \approx \ell_0)$:

$$\partial\delta\Sigma/\partial t + \Omega(s)\ \partial\delta\Sigma/\partial\varphi + \Sigma[\partial\delta v_\ell/\partial s + (1/\ell_0)\ \partial\delta v_\varphi/\partial\varphi] = 0, \quad (5.10.117)$$

$$\partial\delta v_\ell/\partial t + \Omega(s)\ \partial\delta v_\ell/\partial\varphi - 2\Omega(s)\ \delta v_\varphi = -(a_\Sigma^2/\Sigma)\ \partial\delta\Sigma/\partial s + \partial\delta\Phi/\partial s, \quad (5.10.118)$$

$$\partial\delta v_\varphi/\partial t + \Omega(s)\ \partial\delta v_\varphi/\partial\varphi + [2\Omega(s) + \ell_0\ d\Omega(s)/ds]\ \delta v_\ell = -(a_\Sigma^2/\ell_0\Sigma)\ \partial\delta\Sigma/\partial\varphi + (1/\ell_0)\ \partial\delta\Phi/\partial\varphi. \quad (5.10.119)$$

The two equations (5.10.109) take in the vicinity of ℓ_0, φ_0 the form

$$\nabla^2\delta\Phi_e = (1/\ell)\ \partial(\ell\ \partial\delta\Phi_e/\partial\ell)/\partial\ell + (1/\ell^2)\ \partial^2\delta\Phi_e/\partial\varphi^2 + \partial^2\delta\Phi_e/\partial z^2$$
$$\approx \partial^2\delta\Phi_e/\partial s^2 + (1/\ell_0^2)\ \partial^2\delta\Phi_e/\partial\varphi^2 + \partial^2\delta\Phi_e/\partial z^2 = 0 \quad \text{if} \quad z \neq 0;$$
$$(\partial\delta\Phi/\partial z)_{z=0_+} - (\partial\delta\Phi/\partial z)_{z=0_-} = -4\pi G\ \delta\Sigma, \quad (5.10.120)$$

and we observe that a solution of both equations (5.10.120) is given with the local perturbations (5.10.115) by [cf. Eq. (5.10.64)]

$$\delta\Phi = \delta\Phi_e = [2\pi G C_\Sigma/(j^2 + k^2/\ell_0^2)^{1/2}]\exp[i(\sigma t + js + k\varphi) - (j^2 + k^2/\ell_0^2)^{1/2}|z|]. \tag{5.10.121}$$

This yields in the plane of the disk the local relationship [Bardeen 1975, Lin and Lau 1979, Eq. (D14)]

$$\delta\Phi = [2\pi G C_\Sigma/(j^2 + k^2/\ell_0^2)^{1/2}]\exp[i(\sigma t + js + k\varphi)] = [2\pi G/(j^2 + k^2/\ell_0^2)^{1/2}]\ \delta\Sigma,$$
$$[z = 0;\ \Phi_u, \Sigma_u = \text{const};\ C_\Phi = 2\pi G C_\Sigma/(j^2 + k^2/\ell_0^2)^{1/2}]. \tag{5.10.122}$$

Now, we insert the perturbations (5.10.115), (5.10.122) into Eqs. (5.10.117)-(5.10.119), to obtain the homogeneous system

$$j\Sigma\ \delta v_\ell + k\Sigma\ \delta v_\varphi/\ell_0 + [\sigma + k\Omega(s)]\ \delta\Sigma = 0;$$
$$i[\sigma + k\Omega(s)]\ \delta v_\ell - 2\Omega(s)\ \delta v_\varphi + i\big[a_\Sigma^2 j/\Sigma - 2\pi G j/(j^2 + k^2/\ell_0^2)^{1/2}\big]\ \delta\Sigma = 0;$$
$$[2\Omega(s) + \ell_0\ d\Omega(s)/ds]\ \delta v_\ell + i[\sigma + k\Omega(s)]\ \delta v_\varphi + i\big[a_\Sigma^2 k/\ell_0\Sigma - 2\pi G k/\ell_0(j^2 + k^2/\ell_0^2)^{1/2}\big]\ \delta\Sigma = 0. \tag{5.10.123}$$

Nonzero solutions only occur if the determinant of this system vanishes, yielding a cubic dispersion relation for σ [Hunter 1972, Eq. (5.8); Lin and Lau 1979, Eq. (D15)]:

$$(\sigma + k\Omega)[(\sigma + k\Omega)^2 - 4\Omega^2 - 2\ell_0\Omega\ d\Omega/ds] + [2\pi G(j^2 + k^2/\ell_0^2)^{1/2}\Sigma - a_\Sigma^2(j^2 + k^2/\ell_0^2)]$$
$$\times[\sigma + k\Omega + ijk(d\Omega/ds)/(j^2 + k^2/\ell_0^2)] = 0. \tag{5.10.124}$$

An instructive and important particular case occurs for axisymmetric oscillations with $k = 0$ (cf. Eq. (5.10.88) for the infinite, uniformly rotating, isothermal medium):

$$\sigma^2 = \kappa^2 - 2\pi G j\Sigma + a_\Sigma^2 j^2, \qquad (k = 0), \tag{5.10.125}$$

where $\kappa^2 = 4\Omega^2 + 2\ell_0\Omega\ d\Omega/ds$ is approximately equal to Eq. (5.10.1). The minimum of σ^2 occurs if $d\sigma^2/dj = 0$ or $j = \pi G\Sigma/a_\Sigma^2$. Introducing this into Eq. (5.10.125), we observe that the zero thickness disk is stable for all axisymmetric wavelengths if $\sigma^2 \geq 0$, or

$$a_\Sigma = [(dP/d\Sigma)_S]^{1/2} \geq \pi G\Sigma/\kappa, \qquad (k = 0). \tag{5.10.126}$$

If the velocity of sound is equated to the root-mean-square radial velocity $\overline{v_r}$ of stars with a Schwarzschild distribution of velocities, the local stability condition in a thin, pure stellar disk is nearly equal to Eq. (5.10.126), (Toomre 1977: $\overline{v_r} \geq 3.36 G\Sigma/\kappa$). This kind of agreement should not blind us to the fact that fundamental differences exist between stellar and gaseous rotating disk systems (Hunter 1972).

Care must be taken in interpreting Eq. (5.10.124), especially in the full nonaxisymmetric case, because a root σ with negative imaginary part found for some real values of j and k, can equally well be interpreted as σ being real with j and k complex. By virtue of Eq. (5.10.115) this latter case would merely correspond to a steady stable wave with spatially varying amplitude. Such ambiguities can be avoided only by a full local analysis of the WKBJ type (Hunter 1972).

According to Eqs. (5.10.116) and (5.10.122) the Eulerian surface density perturbation can be written as

$$\delta\Sigma(s,\varphi,t) = \delta\Sigma(s)\exp\left[i\left(\sigma t + \int_0^s j(s')\ ds' + k\varphi\right)\right], \tag{5.10.127}$$

which reduces to the surface density perturbation (5.10.115) if $\delta\Sigma(s) = C_\Sigma = \text{const}$, $j(s') = j = \text{const}$. We separate the real and imaginary parts of the eigenvalue by writing $\sigma = \sigma_r + i\sigma_i$. Eq. (5.10.127) assumes the form

$$\delta\Sigma(s,\varphi,t) = \delta\Sigma(s)\exp(-\sigma_i t)\exp\left[i\left(\sigma_r t + \int_0^s j(s')\ ds' + k\varphi\right)\right]. \tag{5.10.128}$$

The angular velocity of propagation of the wave pattern is just equal to $\Omega_p = \varphi/t = -\sigma_r/k$ (Gerthsen et al. 1977). The maxima of the surface density perturbation (5.10.128) occur if

$$\sigma_r t + \int_0^s j(s')\ ds' + k\varphi = k(\varphi + \sigma_r t/k) + \int_0^s j(s')\ ds' = 2m\pi, \qquad (m = 0, 1, 2, 3, ...). \quad (5.10.129)$$

At a fixed moment t_0 this equation represents a k-armed spiral density wave

$$k(\varphi + \sigma_r t_0/k - 2m\pi/k) = k(\varphi - \varphi_0) = -\int_0^s j(s')\ ds',$$
$$(\varphi_0 = 2m\pi/k - \sigma_r t_0/k;\ m = 0, 1, 2, ...k-1;\ k = 1, 2, 3, ...), \quad (5.10.130)$$

and it is seen that the m-th arm of the spiral originates at polar angle $\varphi_0 = 2m\pi/k - \sigma_r t_0/k$, where $\varphi = \varphi_0$ if $s = 0$. The density waves are trailing $d\varphi = -j(s)\ ds/k < 0$ if $j > 0$, and leading $d\varphi > 0$ if $j < 0$. The validity of the local approximation depends on the constraint (5.10.3): $|j|\ell_0 \gg 1$. For moderate values of k this condition is equivalent to the requirement of a small pitch angle p of the spiral arms, i.e. a tightly wound spiral [Lin and Lau 1979, Eq. (31a)]:

$$|\tan p| = (1/\ell_0)\ |ds/d\varphi| = |k/j\ell_0| \ll 1, \qquad (d\varphi/ds = -j/k). \quad (5.10.131)$$

Thus, for small pitch angles of the spiral density wave (5.10.128) we can neglect k^2/ℓ_0^2 with respect to j^2 in the local dispersion relationship (5.10.124). With this constraint in mind, we write the last term of Eq. (5.10.124) under the form

$$ijk(d\Omega/ds)/(j^2 + k^2/\ell_0^2) = (ijk/\ell_0)(\kappa^2/2\Omega - 2\Omega)/(j^2 + k^2/\ell_0^2) \approx (ik/j\ell_0)\ O(\Omega), \quad (5.10.132)$$

since $\kappa^2 \approx O(\Omega^2)$ for a wide range of rotation laws [cf. Eq. (5.10.1)]. Thus, in the last bracket of Eq. (5.10.124) the term (5.10.132) is negligible with respect to $k\Omega$, and the local dispersion relation (5.10.124) assumes a form suitable to spiral density wave theory (Hunter 1972, Bardeen 1975, Lin and Lau 1979):

$$(\sigma + k\Omega)^2 = \kappa^2 - 2\pi G j\Sigma + a_\Sigma^2 j^2, \qquad (j^2 \gg k^2/\ell_0^2), \quad (5.10.133)$$

with two solutions for the radial wave number

$$j_{1,2} = \{\pi G\Sigma \pm [\pi^2 G^2 \Sigma^2 + a_\Sigma^2(\sigma + k\Omega)^2 - a_\Sigma^2 \kappa^2]^{1/2}\}/a_\Sigma^2. \quad (5.10.134)$$

If the constraint (5.10.131) is fulfilled, the system (5.10.123) writes as (Bardeen 1975)

$$\delta\Sigma = -(j\Sigma\ \delta v_\ell + k\Sigma\ \delta v_\varphi/\ell_0)/(\sigma + k\Omega);$$
$$\delta v_\ell = \delta\Sigma\ [(2\pi G - a_\Sigma^2 j/\Sigma)(\sigma + k\Omega) + 2i\Omega a_\Sigma^2 k/\ell_0\Sigma]/[(\sigma + k\Omega)^2 - \kappa^2];$$
$$\delta v_\varphi = \delta\Sigma\ [-a_\Sigma^2 k(\sigma + k\Omega)/\ell_0\Sigma + i\kappa^2(2\pi G - a_\Sigma^2 j/\Sigma)/2\Omega]/[(\sigma + k\Omega)^2 - \kappa^2]. \quad (5.10.135)$$

Thus, in our rudimentary theory, the surface density and velocity perturbations, both become singular in three cases, namely when $\sigma + k\Omega = 0$, and $\sigma + k\Omega = \pm\kappa$. Restricting to the real part $\sigma_r = -k\Omega_p$ of the eigenvalue σ, we observe that $(\sigma_r + k\Omega)/\kappa = k(\Omega - \Omega_p)/\kappa$ can be regarded as the dimensionless frequency of encounter of disk material – traveling at angular speed Ω – with the spiral pattern of a k-armed spiral, rotating with angular velocity (pattern frequency) $\Omega_p = -\sigma_r/k$. The singularity of the eigenfunctions occurring at radius $\ell = \ell_0 + s$, where $\sigma_r + k\Omega = k(\Omega - \Omega_p) = 0$, is called the corotation resonance (Sec. 5.9.3), and the other two singularities occurring at $\sigma_r + k\Omega = k(\Omega - \Omega_p) = \pm\kappa$ are called the outer and inner Lindblad resonance, respectively. These resonances act as extra boundary conditions on the eigenfunctions. It is seen from Eq. (5.10.134) that at the Lindblad resonances one radial wave number falls to zero. If

$$-[\kappa^2 - (\pi G\Sigma/a_\Sigma)^2]^{1/2} < \sigma + k\Omega < [\kappa^2 - (\pi G\Sigma/a_\Sigma)^2]^{1/2}, \quad (5.10.136)$$

the two radial wave numbers $j_{1,2}$ become complex.

With respect to the existence of stable spiral modes in rotating fluids Lynden-Bell and Ostriker (1967) have enounced their "antispiral theorem": Stable normal modes (σ – real) with spiral structure are possible only in the case of degenerate eigenvalues (linearly independent modes having the same eigenvalue σ), and in this case the spiral eigenfunctions always occur in pairs, one leading ($j < 0$) and the other trailing ($j > 0$). It is always possible to chose the stable normal modes of oscillations in such a way that none of them has spiral structure.

According to Hunter (1972) the proof of the antispiral theorem relies simply on the fact that when the surface density perturbation $\delta\Sigma(\ell, \varphi, t) = \delta\Sigma(\ell)\exp[i(\sigma t + k\varphi)]$ is a solution of the hydrodynamic equations (5.10.103)-(5.10.105), there also exists a solution $\delta\Sigma'(\ell, \varphi, t) = \overline{\delta\Sigma}(\ell)\exp[i(\bar{\sigma}t + k\varphi)]$ of the complex conjugate type, where $\sigma = \bar{\sigma}$, since we consider only stable modes with σ being real. Because the complex conjugate modes $\delta\Sigma(\ell)$ and $\overline{\delta\Sigma}(\ell)$ possess the same eigenvalue $\sigma = \bar{\sigma}$, they are degenerate and linearly independent. The complex function $\delta\Sigma(\ell)$ can be written under the form $\delta\Sigma(\ell) = |\delta\Sigma(\ell)|\exp[i\arg\delta\Sigma(\ell)]$, where the modulus $|\delta\Sigma(\ell)|$ and the argument $\arg\delta\Sigma(\ell)$ are real functions. Thus, we may write $\delta\Sigma(\ell, \varphi, t) = |\delta\Sigma(\ell)|\exp\{i[\sigma t + k\varphi + \arg\delta\Sigma(\ell)]\}$ and $\delta\Sigma'(\ell, \varphi, t) = |\delta\Sigma(\ell)|\exp\{i[\sigma t + k\varphi - \arg\delta\Sigma(\ell)]\}$, where $\sigma = \bar{\sigma}$, $|\delta\Sigma(\ell)| = |\overline{\delta\Sigma}(\ell)|$, $\arg\delta\Sigma(\ell) = -\arg\overline{\delta\Sigma}(\ell)$. If $\arg\delta\Sigma(\ell) \neq \text{const}$, these are spirals [cf. Eq. (5.10.127)]. If $\delta\Sigma$ is of the leading type, its pair $\delta\Sigma'$ is of the trailing type, and vice versa, their arguments having opposite signs.

We can always chose the linearly independent set of eigenfunctions $\delta\Sigma(\ell)+\overline{\delta\Sigma}(\ell)$ and $[\delta\Sigma(\ell)-\overline{\delta\Sigma}(\ell)]/i$, which are real and stable by construction; they are of the nonspiral type, as their arguments are 0 or π, i.e. independent of ℓ. This completes the brief proof of the antispiral theorem. The existence of spiral modes is not disproved by the antispiral theorem, and it is not a severe restriction in practice (Hunter 1972, Bardeen 1975).

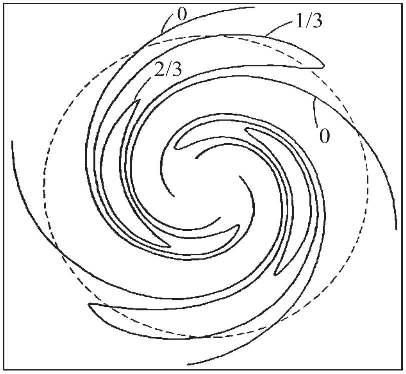

Fig. 5.10.1 Fundamental unstable spiral mode of the perturbed surface density $\delta\Sigma$ in a zero thickness disk. Numbers denote fractions of the maximum surface density perturbation $\delta\Sigma_{max}$. The dashed circle is the corotation circle $\sigma_r + k\Omega(\ell) = 0$ (Lin and Lau 1979).

More sophisticated approximations to the density wave theory in infinitely thin disks have been adopted for instance by Lau and Bertin (1978) in order to elucidate the "winding dilemma" in spiral galaxies, i.e. the seemingly long-lived nature of spiral arms in the presence of differential galactic rotation. Introducing a "radiation condition" (as widely used in plasma physics) at $[\sigma_r + k\Omega(\ell)]/\kappa(\ell) = 0.5$, which implicitly favours trailing spiral waves, Lau and Bertin (1978) obtain a number of discrete unstable spiral modes. Fig. 5.10.1 shows the perturbed surface density $\delta\Sigma$ of the fundamental mode. While Goldreich and Lynden-Bell (1965b) consider the amplification of local, unstable, transient shearing wavelets as an explanation of spiral structure in galaxies, the discrete, unstable, trailing spiral modes are maintained in the theory of Lau and Bertin (1978) by an outward transport of angular momentum. Unfortunately, these exciting problems are beyond the scope of this book (see Binney and Tremaine 1987, Chap. 6).

While this subsection has been mainly concerned with the local stability of zero thickness disks, the next subsections briefly present also some results of global stability analysis.

5.10.5 Hunter's Pressure-free, Zero Thickness Disk

Disks with pressure and pressure-free disks are often designated as "hot" and "cold" disks, respectively. This misleading nomenclature stems from an undue extrapolation of the perfect gas law (1.2.5), ignoring pressure-density relationships in cold matter at zero temperature. The simplest pressure-less model is the uniformly rotating, zero thickness disk with surface density given by Eq. (5.10.156). This is the only one for which a global, fully analytical solution for all modes is available also for disks with pressure (Iye 1978, 1984).

The most straightforward (though generally not the most convenient) expression for the potential of an infinitely thin disk is in polar coordinates equal to (cf. Eqs. (3.1.47), (3.1.48); Mestel 1963)

$$\Phi(\vec{r}) = G\int_V \Sigma(\vec{r'})\ dV'/|\vec{r}-\vec{r'}| = G\int_0^{2\pi} d\varphi' \int_0^\infty \Sigma(\ell',\varphi')\ \ell'\ d\ell'/(\ell^2+\ell'^2-2\ell\ell'\cos\gamma)^{1/2}$$
$$= \Phi(\ell,\varphi), \qquad (\vec{r}=\vec{r}(\ell,\varphi);\ dV'=\ell'\ d\ell'\ d\varphi'), \tag{5.10.137}$$

where γ is the angle between $\vec{r}$ and $\vec{r'}$. While Yabushita (1969) and others seek a solution of Poisson's equation (5.10.98) in terms of Bessel and associated Bessel functions, Hunter (1963) expresses the gravitational potential in terms of Legendre polynomials, introducing oblate spheroidal coordinates (u, v, φ), which are related to Cartesian coordinates by (e.g. Spiegel 1968)

$$x = \ell_1 \cosh u \cos v \cos\varphi; \quad y = \ell_1 \cosh u \cos v \sin\varphi; \quad z = \ell_1 \sinh u \sin v. \tag{5.10.138}$$

ℓ_1 is identified with the finite radius $\ell_1 = (x_1^2+y_1^2)^{1/2}$ of the unperturbed disk. The new coordinates are restricted to vary in the range $0 \le u \le \infty$, $-\pi/2 \le v \le \pi/2$, the azimuth angle φ changing as usually between 0 and 2π. Eq. (5.10.138) becomes with the new notations $\zeta = \sinh u$, $\mu = \sin v$ equal to

$$x = \ell_1(1+\zeta^2)^{1/2}(1-\mu^2)^{1/2}\cos\varphi; \quad y = \ell_1(1+\zeta^2)^{1/2}(1-\mu^2)^{1/2}\sin\varphi; \quad z = \ell_1\zeta\mu, \tag{5.10.139}$$

where $0 \le \zeta \le \infty$ and $-1 \le \mu \le 1$. The surfaces $\zeta = \text{const}$ and $\mu = \text{const}$ are respectively confocal ellipsoids and hyperboloids of revolution about the z-axis. The plane $z = 0$ of the zero thickness disk is represented by $\zeta = 0$ if $\ell = (x^2+y^2)^{1/2} = \ell_1(1+\zeta^2)^{1/2}(1-\mu^2)^{1/2} \in [0, \ell_1]$. The value of the coordinate μ differs in sign on the upper and lower side of the disk, but not in magnitude. The (x, y)-plane outside the disk where $\ell > \ell_1$ has $\mu = 0$, $(z = 0)$. On the z-axis we have $\mu = 1$ if $z > 0$, and $\mu = -1$ if $z < 0$.

The orthogonal line element in oblate spheroidal coordinates is given by

$$ds^2 = dx^2+dy^2+dz^2 = \ell_1^2\big[(\zeta^2+\mu^2)\ d\zeta^2/(1+\zeta^2) + (\zeta^2+\mu^2)\ d\mu^2/(1-\mu^2)$$
$$+(1+\zeta^2)(1-\mu^2)\ d\varphi^2\big]. \tag{5.10.140}$$

The Laplace equation [cf. Eq. (B.21)]

$$\nabla^2\Phi_e = [1/\ell_1^2(\zeta^2+\mu^2)]\big\{\partial[(1+\zeta^2)\ \partial\Phi_e/\partial\zeta]/\partial\zeta + \partial[(1-\mu^2)\ \partial\Phi_e/\partial\mu]/\partial\mu$$
$$+[(\zeta^2+\mu^2)/(1+\zeta^2)(1-\mu^2)]\ \partial^2\Phi_e/\partial\varphi^2\big\} = 0, \tag{5.10.141}$$

has single-valued separable solutions of the form (e.g. Lamb 1945, §§107, 109)

$$p_j^m(\zeta)\ P_j^m(\mu)\exp(im\varphi); \quad q_j^m(\zeta)\ P_j^m(\mu)\exp(im\varphi). \tag{5.10.142}$$

$P_j^m(\mu)$ are associated Legendre polynomials, and p_j^m, q_j^m are two independent solutions of the equation

$$d[(1+\zeta^2)\ d\chi/d\zeta]/d\zeta + [m^2/(1+\zeta^2) - j(j+1)]\ \chi = 0, \qquad (\chi = p_j^m, q_j^m). \tag{5.10.143}$$

This equation becomes identical to the associated Legendre equation (3.1.41) if ζ is replaced by $i\zeta$, and one real solution of Eq. (5.10.143) is therefore

$$p_j^m(\zeta) = i^{m-j}P_j^m(i\zeta) = [(2j-1)!!\ (1+\zeta^2)^{m/2}/(j-m)!][\zeta^{j-m} + (j-m)(j-m-1)\zeta^{j-m-2}$$
$$/2(2j-1)+...], \qquad [(2j-1)!! = 1\times 3\times ...\times(2j-3)(2j-1)]. \tag{5.10.144}$$

At large distances from the origin $r = (x^2+y^2+z^2)^{1/2} = \ell_1(1-\mu^2+\zeta^2)^{1/2} \approx \ell_1\zeta$ we have $p_j^m \propto \zeta^j \propto r^j$, which diverges if $r \to \infty$, so the other solution is pertinent for the external potential:

$$q_j^m(\zeta) = [(-1)^m (j+m)!\; p_j^m(\zeta)/(j-m)!] \int_\zeta^\infty d\zeta'/(1+\zeta'^2)[p_j^m(\zeta')]^2. \quad (5.10.145)$$

This solution satisfies Laplace's equation (5.10.141) at all points outside the disk, and decays as r^{-j-1} at large distances from the disk. Therefore, Hunter (1963) considers the external potential under the form

$$\Phi_e(\zeta,\mu,\varphi) = \sum_{j,m=0}^{\infty} A_{jm}[q_j^m(\zeta)/q_j^m(0)]\; P_j^m(\mu)\exp(im\varphi), \qquad (A_{jm} = \mathrm{const}). \quad (5.10.146)$$

The external potential has to be continuous across the disk, when $\mu \to -\mu$, so P_j^m has to be an even function of μ, i.e. $j-m$ has to be an even number (e.g. Spiegel 1968). Near the disk we have $\zeta \approx 0$, and Eq. (5.10.139) yields $x^2+y^2 = \ell^2 \approx \ell_1^2(1-\mu^2)$. The boundary condition (5.10.102) obtained if $\ell,\mu,\varphi = \mathrm{const}$, reads as

$$[(1/\ell_1\mu)\; \partial\Phi_e/\partial\zeta]_{\zeta=0_+} - [(1/\ell_1\mu)\; \partial\Phi_e/\partial\zeta]_{\zeta=0_-} = -4\pi G\Sigma(\mu,\varphi),$$
$$[(\partial\Phi_e/\partial z)_{z=0_\pm} = (\partial\Phi/\partial z)_{z=0_\pm}], \qquad (\partial\zeta/\partial z = 1/\ell_1\mu). \quad (5.10.147)$$

Eq. (5.10.139) yields for the zero thickness disk $\zeta = 0$:

$$\mu = (1-\ell^2/\ell_1^2)^{1/2}; \quad \ell^2 = x^2+y^2 = \ell_1^2(1-\mu^2). \quad (5.10.148)$$

We consider only the positive sign of μ, adopting μ as a dimensionless measure of the distance $\ell = \ell_1(1-\mu^2)^{1/2}$ of a disk point from the origin. Since $P_j^m(\mu)$ is an even function of μ if $j-m$ is an even number, the restriction $0 \le \mu \le 1$ instead of $-1 \le \mu \le 1$ has no practical significance. Taking into account that $\mu \to -\mu$ on the lower side $\zeta = 0_-$ of the disk, insertion of Eq. (5.10.146) into Eq. (5.10.147) yields

$$\Sigma(\mu,\varphi) = \sum_{j,m=0}^{\infty} A_{jm}[-\pi(dq_j^m/d\zeta)_{\zeta=0}/q_j^m(0)]\; P_j^m(\mu)\exp(im\varphi)/2\pi^2 G\ell_1\mu,$$
$$(j-m \text{ even};\ 0 \le \mu \le 1). \quad (5.10.149)$$

The result

$$\gamma_j^m = -q_j^m(0)/\pi(dq_j^m/d\zeta)_{\zeta=0} = (j+m)!(j-m)!/2^{2j+1}\{[(j+m)/2]!\}^2\{[(j-m)/2]!\}^2,$$
$$(j-m \text{ even}), \quad (5.10.150)$$

has been obtained by Hunter (1963) after lengthy evaluations. Assuming a representation of Σ under the form

$$\Sigma(\mu,\varphi) = (1/\mu)\sum_{j,m=0}^{\infty} c_{jm}P_j^m(\mu)\exp(im\varphi), \qquad (j-m \text{ even};\ c_{jm} = \mathrm{const}), \quad (5.10.151)$$

and combining with Eqs. (5.10.149), (5.10.150), we get $A_{jm} = 2\pi^2 G\ell_1 c_{jm}\gamma_j^m$. The external potential (5.10.146) becomes on the disk surface at $\zeta = 0_\pm$ equal to

$$\Phi_e = 2\pi^2 G\ell_1 \sum_{j,m=0}^{\infty} c_{jm}\gamma_j^m P_j^m(\mu)\exp(im\varphi). \quad (5.10.152)$$

For an axisymmetric density distribution Eq. (5.10.151) simplifies to

$$\Sigma(\mu) = (1/\mu)\sum_{j=0}^{\infty} c_{2j}P_{2j}(\mu), \qquad (m=0;\ c_{2j} = \mathrm{const}), \quad (5.10.153)$$

and the surface density of the disk becomes infinite at its outer edge $\mu = 0$, unless the coefficients c_{2j} satisfy the identity

$$\sum_{j=0}^{\infty} c_{2j} P_{2j}(0) = 0. \tag{5.10.154}$$

The total mass of the finite disk is

$$M_1 = 2\pi \int_0^{\ell_1} \Sigma(\ell)\ \ell\ d\ell = 2\pi\ell_1^2 \int_0^1 \Sigma(\mu)\ \mu\ d\mu = 2\pi\ell_1^2 \int_0^1 \sum_{j=0}^{\infty} c_{2j} P_{2j}(\mu)\ d\mu = 2\pi\ell_1^2 c_0. \tag{5.10.155}$$

The simplest surface density model of the form (5.10.153) is that with only c_0 and c_2 nonzero. Eq. (5.10.154) yields $c_2 = 2c_0,\ (P_0(\mu) = 1,\ P_2(0) = -1/2)$, which is inserted together with Eq. (5.10.155) into Eq. (5.10.153):

$$\Sigma(\ell) = 3M_1\mu/2\pi\ell_1^2 = (3M_1/2\pi\ell_1^2)(1 - \ell^2/\ell_1^2)^{1/2}. \tag{5.10.156}$$

The perturbed surface density is considered under the form

$$\begin{aligned}\Sigma(\ell,\varphi,t) &= (3M_1/2\pi\ell_1^2)[1 - \ell^2/L_1^2(\varphi,t)]^{1/2} + \Sigma^*(\ell,\varphi,t) = 3M_1\eta/2\pi\ell_1^2 + \Sigma^*(\eta,\varphi,t)\\ &= 3M_1\eta/2\pi\ell_1^2 + \Sigma^*(\eta)\exp[i(\sigma t + k\varphi)], \qquad (\Sigma^* \ll \Sigma),\end{aligned} \tag{5.10.157}$$

where we have introduced the perturbed outer boundary of the disk

$$L_1(\varphi,t) = \ell_1 + \varepsilon(\varphi,t) = \ell_1 + \varepsilon\exp[i(\sigma t + k\varphi)], \qquad (|\varepsilon| \ll \ell_1), \tag{5.10.158}$$

avoiding the occurrence of imaginary surface densities in Eq. (5.10.157) by using L_1 instead of ℓ_1. It should be stressed that Σ^* is not equal to the Eulerian perturbation $\delta\Sigma$, since the leading term in Eq. (5.10.157) depends on the perturbed quantity $L_1(\varphi,t)$, the unperturbed surface density being given by Eq. (5.10.156). We have also introduced the new variable η, connected with μ from Eq. (5.10.148) by

$$\eta = (1 - \ell^2/L_1^2)^{1/2} \approx \mu + \varepsilon(1-\mu^2)/\mu\ell_1 = \mu + \ell^2\varepsilon/\mu\ell_1^3. \tag{5.10.159}$$

The leading term of the gravitational potential due to the surface density distribution (5.10.156) can be found by flattening the homogeneous Maclaurin ellipsoid

$$x^2/(c^2+\ell_1^2) + y^2/(c^2+\ell_1^2) + z^2/c^2 = \ell^2/(c^2+\ell_1^2) + z^2/c^2 = 1, \tag{5.10.160}$$

to a zero thickness disk, by letting $c \to 0,\ \varrho \to \infty$, and taking into account that mass conservation implies $\varrho a_3 = 3M_1/4\pi a_1 a_2$ or $\varrho c = 3M_1/4\pi(c^2+\ell_1^2)$, where the semimajor axes are $a_1 = a_2 = (c^2+\ell_1^2)^{1/2}$ and $a_3 = c = a_1(1-e^2)^{1/2} = (c^2+\ell_1^2)^{1/2}(1-e^2)^{1/2}$. Indeed, if $c, z \to 0$, the surface density of the ellipsoid (5.10.160) writes

$$\Sigma = 2\lim_{z\to 0}(\varrho z) = 2\lim_{c\to 0}\{\varrho c[1 - \ell^2/(c^2+\ell_1^2)]^{1/2}\} = (3M_1/2\pi\ell_1^2)(1-\ell^2/\ell_1^2)^{1/2} = 3M_1\mu/2\pi\ell_1^2, \tag{5.10.161}$$

which is just identical to the unperturbed surface density (5.10.156). The unperturbed internal potential of the ellipsoid (5.10.160) is (Chandrasekhar 1969, §18)

$$\Phi_u = \pi G\varrho\{A_1[2(c^2+\ell_1^2) - \ell^2] + A_3(c^2 - z^2)\}, \tag{5.10.162}$$

where A_1, A_3 are given by Eq. (5.10.221). If we flatten the ellipsoid (5.10.160), its unperturbed potential (5.10.162) becomes

$$\begin{aligned}&\Phi_u = 3\pi GM_1(2\ell_1^2 - \ell^2)/8\ell_1^3 = 3\pi GM_1(1+\mu^2)/8\ell_1, \qquad (c, z \to 0;\ e \to 1;\ \varrho \to \infty;\\ &M_1 \to 4\pi\varrho\ell_1^3(1-e^2)^{1/2}/3;\ \varrho A_1 \to \pi\varrho(1-e^2)^{1/2}/2 \to 3M_1/8\ell_1^3;\ A_3 \to 2).\end{aligned} \tag{5.10.163}$$

If we now flatten the slightly different ellipsoid

$$\ell^2/(c^2+L_1^2) + z^2/c^2 = 1, \tag{5.10.164}$$

having the same mass M_1 as the ellipsoid (5.10.160), we observe that its surface density

$$\Sigma = 2\lim_{z\to 0}(\varrho z) = 2\lim_{c\to 0}\{\varrho c[1-\ell^2/(c^2+L_1^2)]^{1/2}\} = (3M_1/2\pi\ell_1^2)(1-\ell^2/L_1^2)^{1/2} = 3M_1\eta/2\pi\ell_1^2, \qquad (5.10.165)$$

is just the leading term of the perturbed surface density (5.10.157). Hunter (1963) decomposes the gravitational potential of the perturbed zero thickness disk into three parts: (i) The unperturbed potential (5.10.163) of the unperturbed disk. (ii) A small perturbation Φ_ε stemming from the difference between the surface densities (5.10.165) and (5.10.156). (iii) A small perturbation Φ^* corresponding to the surface density perturbation Σ^* in Eq. (5.10.157).

The potential Φ_ε is just equal to the potential of the flattened thin ellipsoidal shell ($c\to 0$), contained between the two ellipsoids (5.10.164) and (5.10.160). The normal thickness h of this shell is obtained from $h=\Delta\ell\cos\alpha_n$. The angle of the exterior normal with the ℓ-axis is denoted by α_n, where $\tan\alpha_n = -d\ell/dz = z(c^2+\ell_1^2)/c^2\ell$. The difference $\Delta\ell = \varepsilon\ell_1\ell/(c^2+\ell_1^2)$ is obtained by differentiating Eq. (5.10.164): $2\ell\ \Delta\ell/(c^2+L_1^2) - 2\ell^2 L_1\ \Delta L_1/(c^2+L_1^2)^2 = 0$ with $c, z=$ const, and $\varepsilon = \Delta L_1 = L_1-\ell_1,\ (L_1\approx\ell_1)$. Thus:

$$h = \Delta\ell\cos\alpha_n = \Delta\ell/(1+\tan^2\alpha_n)^{1/2} = c\varepsilon\ell_1\ell^2/(c^2+\ell_1^2)[(c^2+\ell_1^2)^2-\ell_1^2\ell^2]^{1/2}. \qquad (5.10.166)$$

The external gravitational potential of the ellipsoidal shell with variable thickness $h\propto\varepsilon(\varphi,t)$ can be represented by a series of the form (5.10.146)

$$\Phi_{\varepsilon e}(\zeta,\mu,\varphi,t) = \sum_{j,m=0}^{\infty} A_{jm}(t)\ q_j^m(\zeta)\ P_j^m(\mu)\exp(im\varphi), \qquad (5.10.167)$$

while for the internal potential Φ_ε the functions $q_j^m(\zeta)$ have to be replaced by $p_j^m(\zeta)$, which are continuous together with their derivatives throughout the interior of the ellipsoid:

$$\Phi_\varepsilon = \sum_{j,m=0}^{\infty} B_{jm}(t)\ p_j^m(\zeta)\ P_j^m(\mu)\exp(im\varphi). \qquad (5.10.168)$$

We observe that the ellipsoid (5.10.160) becomes in oblate spheroidal coordinates (ζ,μ,φ) equal to

$$(c^2+\ell_1^2\mu^2)(\ell_1^2\zeta^2-c^2)=0, \qquad (5.10.169)$$

and is determined by the level surface of the oblate spheroidal coordinate $\ell_1^2\zeta^2-c^2=0$ or $\zeta=c/\ell_1$, since $c^2+\ell_1^2\mu^2>0$. Continuity of the internal and external gravitational potential in the thin shell at $\zeta=\zeta_c=c/\ell_1=$ const requires that

$$\Phi_{\varepsilon e}(\zeta_c,\mu,\varphi,t) = \Phi_\varepsilon(\zeta_c,\mu,\varphi,t) \quad\text{or}\quad A_{jm}(t)\ q_j^m(\zeta_c) = B_{jm}(t)\ p_j^m(\zeta_c). \qquad (5.10.170)$$

Another condition concerning the derivatives of Φ_ε and $\Phi_{\varepsilon e}$ comes from the integration of Poisson's equation (2.1.4) in oblate spheroidal coordinates:

$$\partial[(1+\zeta^2)\ \partial\Phi_\varepsilon/\partial\zeta]/\partial\zeta + \partial[(1-\mu^2)\ \partial\Phi_\varepsilon/\partial\mu]/\partial\mu + [(\zeta^2+\mu^2)/(1+\zeta^2)(1-\mu^2)]\ \partial^2\Phi_\varepsilon/\partial\varphi^2$$
$$= -4\pi G\varrho\ell_1^2(\zeta^2+\mu^2). \qquad (5.10.171)$$

We integrate across the small thickness h of the ellipsoidal shell between $\zeta_c=c/\ell_1$ and $\zeta_c+\Delta\zeta$, if $d\mu, d\varphi=0$:

$$[(1+\zeta^2)\ \partial\Phi_\varepsilon/\partial\zeta]_{\zeta=\zeta_c+\Delta\zeta} - [(1+\zeta^2)\ \partial\Phi_\varepsilon/\partial\zeta]_{\zeta=\zeta_c} = [(1+\zeta^2)\ \partial\Phi_{\varepsilon e}/\partial\zeta]_{\zeta=\zeta_c+\Delta\zeta}$$
$$-[(1+\zeta^2)\ \partial\Phi_\varepsilon/\partial\zeta]_{\zeta=\zeta_c} = -4\pi G\varrho\ell_1^2(\zeta_c^2+\mu^2)\ \Delta\zeta. \qquad (5.10.172)$$

The derivatives of the internal and external potential must be equal at the outer boundary of the shell: $(\partial\Phi_\varepsilon/\partial\zeta)_{\zeta=\zeta_c+\Delta\zeta} = (\partial\Phi_{\varepsilon e}/\partial\zeta)_{\zeta=\zeta_c+\Delta\zeta}$. From the line element (5.10.140) it is seen that in this case

$$\Delta s = h = \ell_1[(\zeta_c^2+\mu^2)/(1+\zeta_c^2)]^{1/2}\ \Delta\zeta, \qquad (\Delta\mu,\Delta\varphi=0), \qquad (5.10.173)$$

while Eq. (5.10.166) becomes $[\ell^2=\ell_1^2(1+\zeta_c^2)(1-\mu^2);\ (c^2+\ell_1^2)^2-\ell_1^2\ell^2=\ell_1^4(1+\zeta_c^2)(\zeta_c^2+\mu^2)]$:

$$h = c\varepsilon\ell_1(1+\zeta_c^2)^{1/2}(1-\mu^2)/(c^2+\ell_1^2)(\zeta_c^2+\mu^2)^{1/2}. \qquad (5.10.174)$$

Equating Eqs. (5.10.173) and (5.10.174), we find

$$\Delta\zeta = c\varepsilon(1+\zeta_c^2)(1-\mu^2)/(c^2+\ell_1^2)(\zeta_c^2+\mu^2). \tag{5.10.175}$$

Inserting into Eq. (5.10.172) for $\partial\Phi_{\varepsilon e}/\partial\zeta,\ \partial\Phi_\varepsilon/\partial\zeta,\ \Delta\zeta$ from Eqs. (5.10.167), (5.10.168), (5.10.175), respectively, we get at $\zeta_c = c/\ell_1$:

$$(1-\mu^2)\ \varepsilon(\varphi,t) = [(c^2+\ell_1^2)/4\pi G\varrho c\ell_1^2] \sum_{j,m=0}^{\infty} \left[B_{jm}(t)\ (dp_j^m/d\zeta)_{\zeta=\zeta_c} - A_{jm}(t)\ (dq_j^m/d\zeta)_{\zeta=\zeta_c+\Delta\zeta}\right]$$
$$\times P_j^m(\mu)\exp(im\varphi). \tag{5.10.176}$$

Let us denote by $D_{jm}(t)$ the coefficient of $P_j^m(\mu)\exp(im\varphi)$:

$$(1-\mu^2)\ \varepsilon(\varphi,t) = \sum_{j,m=0}^{\infty} D_{jm}(t)\ P_j^m(\mu)\exp(im\varphi). \tag{5.10.177}$$

$j-m$ must be even, since $1-\mu^2$ is an even function of μ. Equating the coefficients of the two identical representations (5.10.176) and (5.10.177), we obtain

$$B_{jm}(t) = [4\pi G\varrho c\ell_1^2\ q_j^m(\zeta_c)\ D_{jm}(t)/(c^2+\ell_1^2)]/[(dp_j^m/d\zeta)_{\zeta=\zeta_c}\ q_j^m(\zeta_c) - p_j^m(\zeta_c)\ (dq_j^m/d\zeta)_{\zeta=\zeta_c+\Delta\zeta}], \tag{5.10.178}$$

where we have inserted for A_{jm} from Eq. (5.10.170). Now we flatten the ellipsoidal shell into a zero thickness disk, by making $\Delta\zeta,\zeta_c,c\to 0,\ \varrho c\to 3M_1/4\pi\ell_1^2$, and taking into account that via Eq. (5.10.144) we have $(dp_j^m/d\zeta)_{\zeta_c=0} = 0$ if $j-m$ is even. Eq. (5.10.168) becomes

$$\Phi_\varepsilon(\mu,\varphi,t) = \sum_{j,m=0}^{\infty} (3\pi GM_1\gamma_j^m/\ell_1^2)\ D_{jm}(t)\ P_j^m(\mu)\exp(im\varphi), \qquad (\zeta=0;\ j-m\ \text{even}). \tag{5.10.179}$$

To our degree of accuracy we can replace μ from Eq. (5.10.148) by η from Eq. (5.10.159), and if we introduce for $\varepsilon(\varphi,t)$ from Eq. (5.10.158) into Eq. (5.10.177), we get

$$(1-\eta^2)\ \varepsilon\exp[i(\sigma t+k\varphi)] = \sum_{j,m=0}^{\infty} D_{jm}(t)\ P_j^m(\eta)\exp(im\varphi), \qquad (\mu=\eta), \tag{5.10.180}$$

observing that the required form of the expansion is one which contains only terms having $m\equiv k$, i.e.

$$(1-\eta^2)\ \varepsilon\exp[i(\sigma t+k\varphi)] = \sum_{j=0}^{\infty} D_{jk}(t)\ P_j^k(\eta)\exp(ik\varphi), \qquad (j-k\ \text{even}). \tag{5.10.181}$$

This requires $D_{jk}(t) = \varepsilon d_j\exp(i\sigma t),\ (d_j = \text{const})$. Putting $m\equiv k$ in Eq. (5.10.179), we get eventually

$$\Phi_\varepsilon(\eta,\varphi,t) = \sum_{j=0}^{\infty} (3\pi GM_1\varepsilon\ d_j\gamma_j^k/\ell_1^2)\ P_j^k(\eta)\exp[i(\sigma t+k\varphi)]. \tag{5.10.182}$$

The constant ε is arbitrary, since perturbations of all amplitudes are possible, provided that $|\varepsilon|\ll\ell_1$.

The first order part Φ^* of the perturbed potential corresponding to $\Sigma^*(\eta,\varphi,t) = \Sigma^*(\eta)\exp[i(\sigma t+k\varphi)]$ from Eq. (5.10.157) is obtained with a surface density expansion of the form (5.10.151):

$$\Sigma^*(\eta) = (1/\eta)\sum_{j=0}^{\infty}\alpha_j P_j^k(\eta), \qquad (\alpha_j = \text{const};\ j-k\ \text{even}). \tag{5.10.183}$$

To avoid a singularity at the edge $\eta = 0$, we impose the restriction

$$\sum_{j=0}^{\infty}\alpha_j P_j^k(0) = 0. \tag{5.10.184}$$

The potential $\Phi^*(\eta,\varphi,t)=\Phi^*(\eta)\exp[i(\sigma t+k\varphi)]$ corresponding to the density field (5.10.183) results analogously to the potential (5.10.152):

$$\Phi^*(\eta,\varphi,t)=2\pi^2G\ell_1\sum_{j=0}^{\infty}\alpha_j\gamma_j^kP_j^k(\eta)\exp[i(\sigma t+k\varphi)]. \tag{5.10.185}$$

We are now looking for separable solutions of Eqs. (5.10.103)-(5.10.105) having the form

$$\begin{aligned}\Phi(\eta,\varphi,t)&=\Phi_u(\ell)+\Phi_\varepsilon(\eta,\varphi,t)+\Phi^*(\eta,\varphi,t)=\Phi_u(\ell)+[\Phi_\varepsilon(\eta)+\Phi^*(\eta)]\exp[i(\sigma t+k\varphi)]\\&=3\pi GM_1(2\ell_1^2-\ell^2)/8\ell_1^3+\sum_{j=0}^{\infty}(3\pi GM_1\varepsilon d_j/\ell_1^2+2\pi^2G\ell_1\alpha_j)\gamma_j^kP_j^k(\eta)\exp[i(\sigma t+k\varphi)],\end{aligned} \tag{5.10.186}$$

$$\begin{aligned}&v_\ell(\eta,\varphi,t)=v_\ell^*(\eta,\varphi,t)=v_\ell^*(\eta)\exp[i(\sigma t+k\varphi)];\quad v_\varphi(\eta,\varphi,t)=\Omega\ell+v_\varphi^*(\eta,\varphi,t)\\&=\Omega\ell+v_\varphi^*(\eta)\exp[i(\sigma t+k\varphi)],\end{aligned} \tag{5.10.187}$$

where the constant angular velocity of unperturbed rotation follows from Eqs. (5.10.107) and (5.10.163) if $P_\Sigma=0,\ v_\varphi=\Omega\ell,\ \Phi=\Phi_u$:

$$v_\varphi^2/\ell=\Omega^2\ell=-\partial\Phi_u/\partial\ell\quad\text{or}\quad\Omega=(3\pi GM_1/4\ell_1^3)^{1/2}. \tag{5.10.188}$$

The first order equations governing small oscillations are obtained by inserting Eqs. (5.10.157), (5.10.159), (5.10.186), (5.10.187) into Eqs. (5.10.103)-(5.10.105), $(d\eta\approx-(1-\eta^2)^{1/2}\ d\ell/\ell_1\eta)$:

$$\begin{aligned}&[3M_1(1-\eta^2)/2\pi\ell_1^3\eta](\partial\varepsilon/\partial t+\Omega\ \partial\varepsilon/\partial\varphi)+\partial\Sigma^*/\partial t+\Omega\ \partial\Sigma^*/\partial\varphi\\&+[3M_1(1-\eta^2)^{1/2}/2\pi\ell_1^3][(2\eta^2-1)v_\ell^*/\eta(1-\eta^2)-\partial v_\ell^*/\partial\eta+\eta(\partial v_\varphi^*/\partial\varphi)/(1-\eta^2)]=0,\end{aligned} \tag{5.10.189}$$

$$\partial v_\ell^*/\partial t+\Omega\ \partial v_\ell^*/\partial\varphi-2\Omega v_\varphi^*=-[(1-\eta^2)^{1/2}/\eta\ell_1]\ \partial(\Phi_\varepsilon+\Phi^*)/\partial\eta, \tag{5.10.190}$$

$$\partial v_\varphi^*/\partial t+\Omega\ \partial v_\varphi^*/\partial\varphi+2\Omega v_\ell^*=[1/\ell_1(1-\eta^2)^{1/2}]\ \partial(\Phi_\varepsilon+\Phi^*)/\partial\varphi. \tag{5.10.191}$$

We introduce the $\exp[i(\sigma t+k\varphi)]$-dependence of the perturbations into Eqs. (5.10.190) and (5.10.191):

$$\begin{aligned}&v_\ell^*(\eta)=\{i/\ell_1[(\sigma+k\Omega)^2-4\Omega^2]\}\{[(\sigma+k\Omega)(1-\eta^2)^{1/2}/\eta]\ d(\Phi_\varepsilon+\Phi^*)/d\eta\\&-2k\Omega(\Phi_\varepsilon+\Phi^*)/(1-\eta^2)^{1/2}\},\end{aligned} \tag{5.10.192}$$

$$\begin{aligned}&v_\varphi^*(\eta)=\{[1/\ell_1[(\sigma+k\Omega)^2-4\Omega^2]\}\{k(\sigma+k\Omega)(\Phi_\varepsilon+\Phi^*)/(1-\eta^2)^{1/2}\\&-[2\Omega(1-\eta^2)^{1/2}/\eta]\ d(\Phi_\varepsilon+\Phi^*)/d\eta\}.\end{aligned} \tag{5.10.193}$$

These two equations are substituted together with the $\exp[i(\sigma t+k\varphi)]$-dependence of $\Sigma^*(\eta,\varphi,t)$ and $\varepsilon(\varphi,t)$ into the continuity equation (5.10.189), to give

$$\begin{aligned}&\eta\Sigma^*(\eta)+3\varepsilon M_1(1-\eta^2)/2\pi\ell_1^3+\big\{3M_1/2\pi\ell_1^4[(\sigma+k\Omega)^2-4\Omega^2]\big\}\\&\times\big\{d[(\eta^2-1)\ d(\Phi_\varepsilon+\Phi^*)/d\eta]/d\eta+[k^2/(1-\eta^2)-k^2+2k\Omega/(\sigma+k\Omega)](\Phi_\varepsilon+\Phi^*)\big\}=0.\end{aligned} \tag{5.10.194}$$

To obtain the final form of the dispersion relation, we insert for $\eta\Sigma^*(\eta),\ 1-\eta^2,\ 3\pi GM_1/4\ell_1^3,\ d[(\eta^2-1)\ d(\Phi_\varepsilon+\Phi^*)/d\eta]/d\eta,\ \Phi_\varepsilon+\Phi^*$ from Eqs. (5.10.183), (5.10.181), (5.10.188), (3.1.41), (5.10.186), respectively:

$$\begin{aligned}&\sum_{j=0}^{\infty}\Big[(\alpha_j+3\varepsilon M_1d_j/2\pi\ell_1^3)\big\{1+4\Omega^2\gamma_j^k[j(j+1)-k^2+2k\Omega/(\sigma+k\Omega)]\\&/[(\sigma+k\Omega)^2-4\Omega^2]\big\}\ P_j^k(\eta)\Big]=0.\end{aligned} \tag{5.10.195}$$

Either the first factor $\alpha_j + 3\varepsilon M_1 d_j/2\pi\ell_1^3$ vanishes for all j, but this possibility can be ruled out since it would imply via Eq. (5.10.184)

$$\sum_{j=0}^{\infty}\alpha_j P_j^k(0) = -(3\varepsilon M_1/2\pi\ell_1^3)\sum_{j=0}^{\infty} d_j P_j^k(0) = 0, \tag{5.10.196}$$

while Eq. (5.10.181) requires

$$\sum_{j=0}^{\infty} d_j P_j^k(0) = 1. \tag{5.10.197}$$

Eqs. (5.10.196) and (5.10.197) can be reconciled only in the trivial case $\varepsilon = 0$. We are left with the possibility that the first factor in Eq. (5.10.195) vanishes for all j, except for some single value $j = p$:

$$\alpha_j + 3\varepsilon M_1 d_j/2\pi\ell_1^3 = 0 \quad \text{if} \quad j \neq p. \tag{5.10.198}$$

If Eq. (5.10.198) holds, the eigenvalue σ is chosen in such a way that for $j = p$ the second factor in Eq. (5.10.195) vanishes:

$$1 + 4\Omega^2\gamma_p^k[p(p+1) - k^2 + 2k\Omega/(\sigma + k\Omega)]/[(\sigma + k\Omega)^2 - 4\Omega^2] = 0, \qquad (j = p). \tag{5.10.199}$$

The restrictions (5.10.184), (5.10.197) are satisfied by taking

$$\alpha_p = (3\varepsilon M_1/2\pi\ell_1^3)[1/P_p^k(0) - d_p], \qquad (j = p). \tag{5.10.200}$$

The other α_j's are given by Eq. (5.10.198). With the values of α_j from Eqs. (5.10.198), (5.10.200), all perturbed quantities contain only the Legendre polynomial $P_p^k(\eta)$, and can now be determined by using Eqs. (5.10.157), (5.10.181), (5.10.183), (5.10.185)-(5.10.187):

$$\Sigma = (3M_1/2\pi\ell_1^2)\big\{\eta + \varepsilon\ [(\eta^2 - 1)/\eta\ell_1 + P_p^k(\eta)/\eta\ell_1 P_p^k(0)]\exp[i(\sigma t + k\varphi)]\big\}, \tag{5.10.201}$$

$$\Phi = (3\pi G M_1/\ell_1^2)\big\{(2\ell_1^2 - \ell^2)/8\ell_1 + \varepsilon\gamma_p^k P_p^k(\eta)\exp[i(\sigma t + k\varphi)]/P_p^k(0)\big\}, \tag{5.10.202}$$

$$\begin{aligned} v_\ell = {} & 3\pi i\varepsilon G M_1\gamma_p^k\exp[i(\sigma t + k\varphi)]\big\{[(\sigma + k\Omega)(1 - \eta^2)^{1/2}/\eta]\ dP_p^k/d\eta - 2k\Omega\ P_p^k(\eta)/(1 - \eta^2)^{1/2}\big\} \\ & /\ell_1^3[(\sigma + k\Omega)^2 - 4\Omega^2]\ P_p^k(0), \end{aligned} \tag{5.10.203}$$

$$\begin{aligned} v_\varphi = {} & \Omega\ell + 3\pi\varepsilon G M_1\gamma_p^k\exp[i(\sigma t + k\varphi)]\big\{k(\sigma + k\Omega)\ P_p^k(\eta)/(1 - \eta^2)^{1/2} \\ & -[2\Omega(1 - \eta^2)^{1/2}/\eta]\ dP_p^k/d\eta\big\}/\ell_1^3[(\sigma + k\Omega)^2 - 4\Omega^2]\ P_p^k(0). \end{aligned} \tag{5.10.204}$$

$p - k$ is an even number in virtue of Eq. (5.10.181). In the case of axisymmetric oscillations ($k = 0$) the cubic equation (5.10.199) for the determination of the eigenvalues degenerates into the quadratic

$$\sigma^2 = 4\Omega^2[1 - p(p+1)\gamma_p^0] = 4\Omega^2\{1 - p(p+1)(p!)^2/2^{2p+1}[(p/2)!]^4\}, \qquad (k = 0;\ p = 2, 4, 6, ...). \tag{5.10.205}$$

No solution is possible if $p = k = 0$, since this would give $\sigma^2 = 4\Omega^2$, and in this case the velocity disturbances v_ℓ^*, v_φ^* take the undetermined form 0/0. If $p = 2$, we have $\sigma = \pm\Omega$, corresponding to stable sinusoidal oscillations. If $p = 4, 6, 8, ...$, all squared eigenvalues (5.10.205) are negative, decreasing monotonically with increasing p, and unstable modes occur. The instabilities become progressively more violent as p increases. Using Wallis-Stirling's formula (e.g. Smirnow 1967)

$$\begin{aligned} & \pi = \lim_{n\to\infty}(1/n)[(2n)!!/(2n-1)!!]^2 = \lim_{n\to\infty} 2^{4n}(n!)^4/n[(2n)!]^2 = \lim_{p\to\infty} 2^{2p+1}[(p/2)!]^4/p(p!)^2, \\ & (p = 2n;\ (2n)!! = 2n \times (2n-2)... \times 2;\ (2n-1)!! = (2n-1)\times(2n-3)... \times 3 \times 1), \end{aligned} \tag{5.10.206}$$

Eq. (5.10.205) writes for large values of p as

$$\sigma^2 = -4p\Omega^2/\pi, \qquad (k = 0;\ p \to \infty;\ p \text{ even}). \tag{5.10.207}$$

A significant feature of these instabilities is that they represent motions which tend to break up the disk into a number of concentric rings. This can be seen from Eq. (5.10.203) – the η-dependence of the radial velocity v_ℓ being of the form

$$v_\ell \propto [(1-\eta^2)^{1/2}/\eta]\ dP_p(\eta)/d\eta, \qquad (k = 0;\ p \text{ even}). \tag{5.10.208}$$

This function vanishes at $\eta = \pm 1$, and has $p/2 - 1$ roots in the range $0 \le \eta < 1$, since $dP_p/d\eta$ is involved instead of P_p (cf. Hobson 1931, Chap. IX). The radial velocity has thus $p/2 - 1$ changes of sign in space at any instant. Only if $p = 2$, there are no such changes and the motion is stable.

In the particular case $p = k$ we can find analytic expressions for the nonaxisymmetric disturbances, because the dispersion relation (5.10.199) reads

$$(\sigma + p\Omega)[(\sigma + p\Omega)^2 - 4\Omega^2] + 4p\Omega^2\gamma_p^p(\sigma + p\Omega + 2\Omega) = 0 \quad \text{or}$$
$$(\sigma + p\Omega + 2\Omega)[(\sigma + p\Omega)^2 - 2\Omega(\sigma + p\Omega) + 4p\Omega^2\gamma_p^p] = 0. \tag{5.10.209}$$

One eigenvalue $\sigma = -(p+2)\Omega$ is always real, and the other two are

$$\sigma = \Omega[(1-p) \pm (1 - 4p\gamma_p^p)^{1/2}]; \quad \gamma_p^p = (2p)!/2^{2p+1}(p!)^2 = (2p-1)!!/2(2p)!!, \qquad (p = k). \tag{5.10.210}$$

If $p = 1$, we have $p\gamma_p^p = 1/4$, and neutral stability ($\sigma = 0$) occurs, amounting to a lateral displacement of the disk in its plane $z = 0$. Since $p\gamma_p^p$ increases with p, the roots (5.10.210) are complex conjugate if $p > 1$, and one of them gives an oscillation with exponentially growing amplitude, implying unstable oscillations if $p = k > 1$. The unstable first order quantities from Eqs. (5.10.201)-(5.10.204) are of the form

$$f(\eta)\exp\{\Omega t(4p\gamma_p^p - 1)^{1/2} + i[\Omega(1-p)t + p\varphi]\}, \qquad (p = k > 1), \tag{5.10.211}$$

and these disturbances propagate angularly in the direction of rotation of the disk with angular velocity $\varphi/t = (p-1)\Omega/p$, slower than the angular speed of the disk.

In the general case, if $p > k,\ (p - k$ even), the dispersion relation (5.10.199) can be written under the form

$$[(\sigma + k\Omega)/2\Omega]^3 + [(\sigma + k\Omega)/2\Omega][\gamma_p^k(p^2 + p - k^2) - 1] + k\gamma_p^k = 0. \tag{5.10.212}$$

The coefficients of this cubic equation are real, so one root is always real (e.g. Smirnow 1967), and we show subsequently that the other two roots are complex conjugate, implying instability also in the general case $p > k > 0$. The cubic equation $x^3 + ax + b = 0$ has two complex conjugate roots if its discriminant $a^3/27 + b^2/4$ is positive. In the present context this amounts to

$$-a^3/27 < b^2/4 \quad \text{or} \quad [1 - \gamma_p^k(p^2 + p - k^2)]^3/27 < k^2(\gamma_p^k)^2/4. \tag{5.10.213}$$

Now $\gamma_p^k(p^2 + p - k^2) = \gamma_p^k[(p+k)(p-k) + p]$ is always positive, and a simple ratio test shows that it increases monotonically with $p + k$, if $p - k = S$ is kept fixed. The minimum value of $\gamma_p^k(p^2 + p - k^2)$ is therefore obtained if $k = 1$ and $p = S + k = S + 1$. We have

$$\gamma_p^k(p^2 + p - k^2) = [S(S+2) + S + 1](S+2)!S!/2^{2S+3}[(S/2)!]^2\{[(S+2)/2]!\}^2. \tag{5.10.214}$$

This expression however increases with increasing S, and since $S = p - k,\ (p > k \ge 1)$ must be an even number, the minimum value of S is 2. Eq. (5.10.214) becomes equal to 33/32, and $a = 1/32 > 0$, all other values of a being positive too. The inequality (5.10.213) is always satisfied, and one eigenvalue of the dispersion relation (5.10.212) gives rise to unstable oscillations.

Thus, the sole stable modes of the uniformly rotating, pressure-free, zero thickness disk are the axisymmetric oscillations associated with $P_2^0(\eta)$, (Eq. (5.10.205) if $k = 0,\ p = 2$), and the nonaxisymmetric oscillations associated with $P_1^1(\eta)$, (Eq. (5.10.210) if $k = p = 1$). All other modes become unstable for certain eigenvalues (Hunter 1963, Iye 1984). The surface density perturbations (5.10.201) show no spiral structure at all [cf. Eq. (5.10.128)]. Thus, *differentially* rotating, zero thickness disks have to be considered if one wants perturbations with leading (trailing) spiral patterns. While Hunter (1969) finds only leading spirals in his local short wavelength analysis of the WKBJ type, Yabushita (1969) obtains also trailing spirals in his global analysis of differentially rotating, pressure-free, zero thickness disks.

A sequence of differentially rotating equilibrium models can be constructed with the formula (cf. Hunter 1972)

$$v^2(\ell) = \ell^2\Omega^2(\ell) = (2N+1)\pi G M_1(1-\mu^{2N})/4N\ell_1, \qquad (\mu^2 = 1-\ell^2/\ell_1^2;\ N=1,2,3,...). \tag{5.10.215}$$

If $N = 1$, we recover $\Omega^2 = 3\pi G M_1/4\ell_1^3$ for the uniformly rotating disk from Eq. (5.10.188). If $N \to \infty$, we get the so-called Mestel (1963) zero thickness disk with uniform rotation velocity

$$v^2 = \pi G M_1/2\ell_1 = \text{const}; \quad \Sigma(\ell) = M_1 \arcsin\mu/2\pi\ell_1^2(1-\mu^2)^{1/2}. \tag{5.10.216}$$

The eigenfrequencies of higher order modes in zero thickness, pressure-free disks are often very close together, i.e. they belong to a continuum (Hunter 1972). Another, quite different class of continuous modes is formed by the corotating continuum that spans the range $-k\Omega(0) \le \sigma \le -k\Omega(\ell_1)$, $[\Omega(0) \ge \Omega(\ell_1)]$, as already revealed for differentially rotating cylinders (Sec. 5.9.3), and in the local analysis of disks from Sec. 5.10.4.

In pressure-free disks the stabilizing effect of pressure (of random stellar motions) is neglected – an effect becoming increasingly important at short wavelengths – so we are lead to the stability study of disks with pressure.

5.10.6 Global Stability of Thin Disks with Pressure

Attempts have been made to connect the stability of high-eccentricity Maclaurin ellipsoids with the stability of thin stellar disks (e.g. Ostriker and Peebles 1973). The ratio $\tau = E_{kin}/|W|$ from Eq. (3.1.35) can easily be evaluated for Maclaurin ellipsoids (Chandrasekhar 1969, p. 57):

$$E_{kin} = (\Omega^2/2)\int_V \varrho\ell^2\, dV = \Omega^2 I_{11} = (8\pi^2 G\varrho^2 a_1^5/15)[(1/e^3)(1-e^2)(3-2e^2)\arcsin e$$
$$-3(1-e^2)^{3/2}/e^2], \qquad (a_1 = a_2;\ a_3 = a_1(1-e^2)^{1/2};\ I_{11} = I_{22}). \tag{5.10.217}$$

The moment of inertia tensor (2.6.74) and the mass M_1 of the Maclaurin spheroid are given by

$$I_{jk} = M_1 a_j^2\delta_{jk}/5 = 4\pi\varrho a_1a_2a_3a_j^2\delta_{jk}/15; \quad M_1 = 4\pi\varrho a_1a_2a_3/3, \tag{5.10.218}$$

the angular velocity being equal to

$$\Omega^2 = 2\pi G\varrho[(1/e^3)(1-e^2)^{1/2}(3-2e^2)\arcsin e - 3(1-e^2)/e^2]. \tag{5.10.219}$$

The gravitational energy tensor (2.6.71) of the Maclaurin spheroid is

$$W_{jk} = -2\pi G\varrho A_j I_{jk}, \tag{5.10.220}$$

where

$$A_1 = A_2 = (1/e^3)(1-e^2)^{1/2}\arcsin e - (1-e^2)/e^2; \quad A_3 = 2/e^2 - (2/e^3)(1-e^2)^{1/2}\arcsin e. \tag{5.10.221}$$

The gravitational energy reads

$$W = W_{11} + W_{22} + W_{33} = -16\pi^2 G\varrho^2 a_1^5(1-e^2)\arcsin e/15e, \tag{5.10.222}$$

and

$$\tau = E_{kin}/|W| = (1/2)[(3-2e^2)/e^2 - 3(1-e^2)^{1/2}/e\arcsin e]. \tag{5.10.223}$$

As mentioned in Sec. 5.8.2 the incompressible Maclaurins become secularly unstable for the sectorial (toroidal, barlike) modes at a value of $\tau = 0.1375$. And at nearly the same value of τ the pressure-free stellar disks of Ostriker and Peebles (1973) become neutrally stable, exhibiting barlike instabilities if $\tau \geq 0.14$, although these disks satisfy throughout the local stability criterion (5.10.126), when the sound velocity a_Σ is replaced by the radial velocity dispersion $\overline{v_r}$. A nonrotating spherical halo with mass over two times the disk mass can stabilize the disk as a whole, because it merely increases the total potential energy $|W|$, diminishing the value of τ below the secular instability limit (e.g. Bardeen 1975).

The pressure in the plane of the zero thickness disk is assumed to obey the polytropic law (5.10.97):

$$P_\Sigma = K_\Sigma \Sigma^{1+1/n_\Sigma}, \qquad (K_\Sigma, n_\Sigma = \mathrm{const}). \tag{5.10.224}$$

Comparing Eq. (5.10.97) with Eq. (5.10.224), we obtain the relationship between the polytropic indices and the polytropic constants of the two- and three-dimensional fluid medium:

$$n_\Sigma = (n+1)/2; \quad n = 2n_\Sigma - 1; \quad K_\Sigma = 2^{-(n+2)/(n+1)} \pi^{(n+3)/2(n+1)} G^{1/(n+1)} K^{n/(n+1)}$$
$$\times \Gamma[(n+2)/(n+1)]/\Gamma[(3n+5)/2(n+1)]. \tag{5.10.225}$$

The adiabatic Lagrangian pressure-density perturbations are obtained from Eq. (5.2.38) by replacing ϱ and Γ_1 with Σ and $\Gamma_{1\Sigma}$, respectively:

$$\Delta P_\Sigma = (\Gamma_{1\Sigma} P_\Sigma/\Sigma)\ \Delta\Sigma = a_\Sigma^2\ \Delta\Sigma, \qquad (Q = \mathrm{const}). \tag{5.10.226}$$

Analogously to the three-dimensional case (2.1.51) the two-dimensional adiabatic index becomes in the case of an isentropic, zero thickness disk equal to

$$\Gamma_{1\Sigma} = 1 + 1/n_\Sigma; \quad n_\Sigma = 1/(\Gamma_{1\Sigma} - 1), \qquad (S = \mathrm{const}). \tag{5.10.227}$$

Combining Eqs. (2.1.51), (5.10.225), (5.10.227), we obtain the relationship between the two- and three-dimensional adiabatic index (Iye 1984, Laughlin et al. 1998):

$$\Gamma_{1\Sigma} = 1 + 2/(n+1) = 3 - 2/\Gamma_1, \qquad [S = \mathrm{const};\ n = 1/(\Gamma_1 - 1)]. \tag{5.10.228}$$

A special isentropic disk model due to Toomre (1963) with pressure $P_\Sigma = K_\Sigma \Sigma^{4/3}$, and infinite radius has been considered by Aoki et al. (1979) in their global stability study of zero thickness disks:

$$\Phi = (GM_1/c)[(1-x)/2]^{1/2} = (GM_1/c)/(1 + \ell^2/c^2)^{1/2}, \tag{5.10.229}$$

$$\Sigma = (M_1/2\pi c^2)[(1-x)/2]^{3/2} = (M_1/2\pi c^2)/(1 + \ell^2/c^2)^{3/2}, \tag{5.10.230}$$

where

$$x = (\ell^2/c^2 - 1)/(\ell^2/c^2 + 1) = (\ell^2 - c^2)/(\ell^2 + c^2), \qquad (-1 \leq x \leq 1), \tag{5.10.231}$$

and c is a positive scaling factor.

The mass of these Toomre disks is finite and equal to M_1, while their radius ℓ_1 extends to infinity, similarly to the spherical polytropes having polytropic index $n = 5$. The equation of hydrostatic equilibrium (5.10.107) becomes for these disks equal to $(\ell^2 = c^2(1+x)/(1-x);\ dx/d\ell = (1+x)^{1/2}(1-x)^{3/2}/c)$

$$v_\varphi^2 = (\ell/\Sigma)\ dP_\Sigma/d\ell - \ell\ d\Phi/d\ell = (GM_1/2^{3/2}c)(1-x)^{1/2}(1+x)$$
$$\times\{1 - (M_1/2\pi c^2)^{1/n_\Sigma}[3cK_\Sigma(1 + 1/n_\Sigma)/GM_1][(1-x)/2]^{(3-n_\Sigma)/2n_\Sigma}\}. \tag{5.10.232}$$

If $n_\Sigma = 3$, as considered by Aoki et al. (1979), this yields

$$\Omega^2 = v_\varphi^2/\ell^2 = (GM_1/2^{3/2}c^3)(1-x)^{3/2}[1 - 4cK_\Sigma(M_1/2\pi c^2)^{1/3}/GM_1]. \tag{5.10.233}$$

The global stability of polytropic, uniformly rotating, zero thickness disks with finite pressure has been studied by Iye (1978) and Takahara (1978) if $n_\Sigma = 0.5$, $n = 0$, ("Maclaurin disks"), where Takahara considers throughout isentropic disks ($n_\Sigma = 0.5;\ \Gamma_{1\Sigma} = 1 + 1/n_\Sigma = 3;\ \Gamma_1 = \infty$), [see Eqs. (5.10.262)-(5.10.275)]. Pressure effects stabilize the disks especially in the short wavelength regime. There is no mode showing spiral patterns in the case of uniform rotation, suggesting that differential rotation is essential to the existence of growing spiral modes. The results of Iye (1978; $n_\Sigma = 0.75$; $n = 0.5$; $\Gamma_{1\Sigma} = 1 + 1/n_\Sigma = 7/3$; $\Gamma_1 = 3$), Takahara (1978; $n_\Sigma = 0.5$; $\Gamma_{1\Sigma} = 3$), Aoki et al. (1979; $n_\Sigma = 3$; $n = 5$; $\Gamma_{1\Sigma} = 1 + 1/n_\Sigma = 4/3$; $\Gamma_1 = 6/5$), and Ambastha and Varma (1983; $n_\Sigma = 0.75$; $\Gamma_{1\Sigma} = 7/3$) concerning differentially rotating isentropic disks with $\Gamma_{1\Sigma} = 1 + 1/n_\Sigma$ may be briefly summarized as follows: Ambastha and Varma (1983) found some open, tightly wrapped, leading spiral modes in pressure-free disks, which appear less pronounced in the studies of Iye (1978) and Aoki et al. (1979). As pressure effects become more important, the leading spiral modes become gradually trailing, and not only two-armed, but also one- and multi-armed spiral modes, as well as ring modes are expected to grow, even for low-pressure models.

In the *short* wavelength regime Iye (1984) has derived a quartic dispersion relation, showing the properties of $p, g^{(f)}$, and $g^{(b)}$-modes (cf. Sec. 5.9.3). Assuming for the Eulerian perturbations expansions of the form

$$\delta\Sigma(\ell,\varphi,t) = \delta\Sigma(\ell)\exp[i(\sigma t + k\varphi)]; \quad \delta P_\Sigma(\ell,\varphi,t) = \delta P_\Sigma(\ell)\exp[i(\sigma t + k\varphi)]; \quad \delta\Phi \equiv 0;$$
$$\delta v_\ell(\ell,\varphi,t) = \delta v_\ell(\ell)\exp[i(\sigma t + k\varphi)]; \quad \delta v_\varphi(\ell,\varphi,t) = \delta v_\varphi(\ell)\exp[i(\sigma t + k\varphi)], \tag{5.10.234}$$

the perturbed equations (5.10.111)-(5.10.113) become

$$i(\sigma + k\Omega)\ \delta\Sigma + (1/\ell)\ d(\ell\Sigma\ \delta v_\ell)/d\ell + ik\Sigma\ \delta v_\varphi/\ell = 0, \tag{5.10.235}$$

$$i(\sigma + k\Omega)\ \delta v_\ell - 2\Omega\ \delta v_\varphi = (\delta\Sigma/\Sigma^2)\ dP_\Sigma/d\ell - (1/\Sigma)\ d\delta P_\Sigma/d\ell, \tag{5.10.236}$$

$$i(\sigma + k\Omega)\ \delta v_\varphi + (\kappa^2/2\Omega)\ \delta v_\ell = -ik\ \delta P_\Sigma/\ell\Sigma, \tag{5.10.237}$$

in the Cowling approximation $\delta\Phi \equiv 0$, which is justified at short wavelengths.

Via Eq. (5.1.16) we get up to the first order

$$D(\Delta P_\Sigma)/Dt = D(\delta P_\Sigma + \Delta\ell\ dP_\Sigma/d\ell)/Dt \approx \partial\delta P_\Sigma/\partial t + (v_\varphi/\ell)\ \partial\delta P_\Sigma/\partial\varphi + (D\Delta\ell/Dt)(dP_\Sigma/d\ell)$$
$$= i(\sigma + k\Omega)\ \delta P_\Sigma + \delta v_\ell\ dP_\Sigma/d\ell, \tag{5.10.238}$$

and a similar equation for $D(\Delta\Sigma)/Dt$. Then, the adiabatic energy equation (5.10.226) becomes in the linear approximation equal to [cf. Eqs. (5.2.35)-(5.2.39)]

$$D(\Delta P_\Sigma)/Dt = D(a_\Sigma^2\ \Delta\Sigma)/Dt \approx a_\Sigma^2\ D(\Delta\Sigma)/Dt = (\Gamma_{1\Sigma}P_\Sigma/\Sigma)\ D(\Delta\Sigma)/Dt \quad \text{or}$$
$$i(\sigma + k\Omega)\ \delta P_\Sigma + \delta v_\ell\ dP_\Sigma/d\ell = a_\Sigma^2[i(\sigma + k\Omega)\ \delta\Sigma + \delta v_\ell\ d\Sigma/d\ell]. \tag{5.10.239}$$

For nonaxisymmetric oscillations ($k \neq 0$) and away from corotation ($\sigma + k\Omega \neq 0$, Sec. 5.9.3), the elimination of δP_Σ and $\delta\Sigma$ among Eqs. (5.10.235)-(5.10.237), (5.10.239) yields a set of two first order ordinary differential equations:

$$d\delta v_\ell/d\ell = [\kappa^2\ell(\sigma + k\Omega)/2k\Omega a_\Sigma^2 - (1/a_\Sigma^2\Sigma)\ dP_\Sigma/d\ell - 1/\ell]\ \delta v_\ell$$
$$+(i\ell/k)[(\sigma + k\Omega)^2/a_\Sigma^2 - k^2/\ell^2]\ \delta v_\varphi = H\ \delta v_\ell + J\ \delta v_\varphi, \tag{5.10.240}$$

$$d\delta v_\varphi/d\ell = (ik/\ell)\big\{1 + \kappa^4\ell^2/4k^2\Omega^2a_\Sigma^2 + [\kappa^2\ell/2k\Omega(\sigma + k\Omega)][2A + d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]$$
$$-[A/\Sigma(\sigma + k\Omega)^2]\ dP_\Sigma/d\ell\big\}\ \delta v_\ell - [\kappa^2\ell(\sigma + k\Omega)/2k\Omega a_\Sigma^2 + A + 1/\ell]\ \delta v_\varphi = L\ \delta v_\ell + M\ \delta v_\varphi. \tag{5.10.241}$$

In this context A denotes the Schwarzschild discriminant for the two-dimensional medium [cf. Eqs. (5.2.85), (5.10.2)]:

$$A = (1/\Sigma)\ d\Sigma/d\ell - (1/\Gamma_{1\Sigma}P_\Sigma)\ dP_\Sigma/d\ell = [1 - (n_\Sigma + 1)/n_\Sigma\Gamma_{1\Sigma}]\ d\ln\Sigma/d\ell, \quad \big(P_\Sigma = K_\Sigma\Sigma^{1+1/n_\Sigma}\big). \tag{5.10.242}$$

Iye (1984) introduces new functions p and q

$$\delta v_\ell = p(\ell)\exp\left[\int_0^\ell H(\ell')\,d\ell'\right];\quad \delta v_\varphi = q(\ell)\exp\left[\int_0^\ell M(\ell')\,d\ell'\right], \tag{5.10.243}$$

from which the new canonical set is obtained:

$$dp/d\ell = J\,q(\ell)\exp\left[\int_0^\ell (-H+M)\,d\ell'\right];\quad dq/d\ell = L\,p(\ell)\exp\left[\int_0^\ell (H-M)\,d\ell'\right]. \tag{5.10.244}$$

If we assume that the radial dependence of p and q has in the short wavelength regime the form [cf. Eq. (5.10.116)]

$$p(\ell) = P\exp\left[i\int_0^\ell j(\ell')\,d\ell'\right];\quad q(\ell) = Q\exp\left[i\int_0^\ell j(\ell')\,d\ell'\right],\qquad (P,Q=\mathrm{const}), \tag{5.10.245}$$

Eq. (5.10.244) becomes

$$ij(\ell)\,p(\ell) - J\,q(\ell)\exp\left[\int_0^\ell (-H+M)\,d\ell'\right] = 0;\quad L\,p(\ell)\exp\left[\int_0^\ell (H-M)\,d\ell'\right] - ij(\ell)\,q(\ell) = 0, \tag{5.10.246}$$

where $j(\ell)$, $(|j(\ell)|\gg 1)$ is the radial wave number. Nontrivial solutions of p and q occur if the determinant $j^2(\ell)+JL$ of this system is zero, i.e. if the quartic dispersion relation

$$j^2(\ell) = -JL = [(\sigma+k\Omega)^2/a_\Sigma^2 - k^2/\ell^2]\{1+\kappa^4\ell^2/4k^2\Omega^2a_\Sigma^2 + [\kappa^2\ell/2k\Omega(\sigma+k\Omega)]$$
$$\times[2A + d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell] - [A/\Sigma(\sigma+k\Omega)^2]\,dP_\Sigma/d\ell\},\qquad (k\neq 0), \tag{5.10.247}$$

is satisfied in a local region. To obtain the particular form of the dispersion relation for axisymmetric oscillations ($k=0$), we have to proceed ex novo, and get from Eqs. (5.10.237), (5.10.239):

$$\delta v_\varphi = (i\kappa^2/2\sigma\Omega)\,\delta v_\ell;\quad \delta\Sigma = (1/a_\Sigma^2)\,\delta P_\Sigma + i[(1/\sigma)\,d\Sigma/d\ell - (1/a_\Sigma^2\sigma)\,dP_\Sigma/d\ell]\,\delta v_\ell. \tag{5.10.248}$$

Inserting this into Eqs. (5.10.235), (5.10.236), we obtain the axisymmetric counterparts of Eqs. (5.10.240) and (5.10.241):

$$d\delta v_\ell/d\ell = -[(1/\Sigma a_\Sigma^2)\,dP_\Sigma/d\ell + 1/\ell]\,\delta v_\ell - (i\sigma/\Sigma a_\Sigma^2)\,\delta P_\Sigma = H\,\delta v_\ell + J\,\delta P_\Sigma, \tag{5.10.249}$$

$$d\delta P/d\ell = i[-\sigma\Sigma + \kappa^2\Sigma/\sigma - (1/\sigma\Sigma a_\Sigma^2)\,(dP_\Sigma/d\ell)^2 + (1/\sigma\Sigma)(d\Sigma/d\ell)(dP_\Sigma/d\ell)]\,\delta v_\ell$$
$$+(1/\Sigma a_\Sigma^2)(dP_\Sigma/d\ell)\,\delta P_\Sigma = L\,\delta v_\ell + M\,\delta P_\Sigma. \tag{5.10.250}$$

With the new functions

$$\delta v_\ell = p(\ell)\exp\left[\int_0^\ell H(\ell')\,d\ell'\right];\quad \delta P = q(\ell)\exp\left[\int_0^\ell M(\ell')\,d\ell'\right], \tag{5.10.251}$$

we obtain exactly in the same way as in Eqs. (5.10.244)-(5.10.247) the local dispersion relation for axisymmetric oscillations [cf. Iye 1984, Eq. (4.8)]:

$$j^2(\ell) = -JL = (1/a_\Sigma^2)\{\sigma^2 - \kappa^2 - (dP_\Sigma/d\ell)[(1/\Sigma^2)\,d\Sigma/d\ell - (1/\Sigma^2a_\Sigma^2)\,dP_\Sigma/d\ell]\}$$
$$= (1/a_\Sigma^2)[\sigma^2 - \kappa^2 - (A/\Sigma)\,dP_\Sigma/d\ell],\qquad (k=0). \tag{5.10.252}$$

If the unperturbed pressure and surface density are constant, we have $A=0$ in Eq. (5.10.252), and if we neglect self-gravity in the density wave equation (5.10.125), the two equations coincide.

The dispersion relation (5.10.247) allows for an approximate solution in the short wavelength regime at high angular oscillation frequencies $\sigma+k\Omega$, as observed in the local corotating frame:

$$j^2 \approx [(\sigma+k\Omega)^2/a_\Sigma^2 - k^2/\ell^2](1+\kappa^4\ell^2/4k^2\Omega^2a_\Sigma^2) - (A/\Sigma a_\Sigma^2)\,dP_\Sigma/d\ell,\qquad (|j|,|\sigma+k\Omega|\gg 1). \tag{5.10.253}$$

The eigenvalues are approximately [Iye 1984, Eqs. (4.14a), (4.14b)]:

$$\sigma \approx -k\Omega \pm \left\{a_\Sigma^2 k^2/\ell^2 + [a_\Sigma^2 j^2 + (A/\Sigma)\ dP_\Sigma/d\ell]/(1+\kappa^4\ell^2/4k^2\Omega^2 a_\Sigma^2)\right\}^{1/2}$$
$$\approx -k\Omega \pm 2jk\Omega a_\Sigma^2/(4k^2\Omega^2 a_\Sigma^2 + \kappa^4\ell^2)^{1/2}, \qquad (j^2 \gg k^2/\ell^2). \qquad (5.10.254)$$

If the epicyclic frequency κ is zero, Eq. (5.10.254) shows that the only characteristic quantity of the oscillation frequency is the sound speed a_Σ. We can then identify these high-frequency modes with the well known p-modes (pressure modes, acoustic modes, sound waves) restored by pressure. From Eq. (5.10.254) it is obvious that one of the pressure modes propagates forward ($p^{(f)}$-mode if $\sigma/k+\Omega = -\varphi/t+\Omega<0$), and the other backward ($p^{(b)}$-mode if $\sigma/k+\Omega>0$) with respect to the local corotating frame (see Sec. 5.9.3). The oscillation frequencies of axisymmetric p-modes are given by Eq. (5.10.252).

Another set of short wavelength solutions is found from Eq. (5.10.247) for a nonisentropic fluid ($A \neq 0$) in the low frequency regime $|\sigma + k\Omega| \ll 1$:

$$j^2 \approx -(A/\Sigma a_\Sigma^2)\ dP_\Sigma/d\ell - (k^2/\ell^2)\left\{1+\kappa^4\ell^2/4k^2\Omega^2 a_\Sigma^2 - [A/\Sigma(\sigma+k\Omega)^2]\ dP_\Sigma/d\ell\right\},$$
$$(|j| \gg 1;\ |\sigma+k\Omega| \ll 1), \qquad (5.10.255)$$

or

$$\sigma \approx -k\Omega \pm \left\{[(k^2A/\ell^2\Sigma)\ dP_\Sigma/d\ell]/[(A/\Sigma a_\Sigma^2)\ dP_\Sigma/d\ell + j^2 + k^2/\ell^2 + \kappa^4/4\Omega^2 a_\Sigma^2]\right\}^{1/2}$$
$$\approx -k\Omega \pm (k/\ell)\{[(A/\Sigma)\ dP_\Sigma/d\ell]/(j^2+k^2/\ell^2)\}^{1/2} = -k\Omega \pm (kN/\ell)/(j^2+k^2/\ell^2)^{1/2}. \qquad (5.10.256)$$

Here we have inserted the two-dimensional Brunt-Väisälä frequency [cf. Eqs. (5.5.33), (5.10.107), (5.10.242)]

$$N^2 = (1/\Sigma)(dP_\Sigma/d\ell)[d\ln\Sigma/d\ell - (1/\Gamma_{1\Sigma})\ d\ln P_\Sigma/d\ell] = (A/\Sigma)\ dP_\Sigma/d\ell = A[d\Phi/d\ell + \ell\Omega^2(\ell)]. \qquad (5.10.257)$$

Eq. (5.10.256) shows that effective gravity (buoyancy) from Eq. (3.1.20) is the restoring force of these g-modes, and their pattern frequency $\Omega_p = -\sigma/k$ (see Sec. 5.10.4) is almost independent of k, ($j^2 \gg k^2/\ell^2$). If the polytrope is isentropic ($A=0,\ \Gamma_{1\Sigma} = 1+1/n_\Sigma$), the frequencies $\sigma/k+\Omega$ of the g-modes in the local comoving frame merge to zero in the first approximation from Eq. (5.10.256). The forward and backward propagating $g^{(f)}, g^{(b)}$-modes (prograde or retrograde g-modes, as to whether $\sigma/k+\Omega<0$ or >0) are essentially nonaxisymmetric oscillations ($k \neq 0$), and for axisymmetric oscillations ($k=0$) no g-modes exist, or else the g-modes become neutral $\sigma=0$.

In a second approximation we preserve in the fundamental dispersion relation (5.10.247) the term in $(\sigma+k\Omega)^{-1}$, neglecting however for conciseness $(A/\Sigma a_\Sigma^2)\ dP_\Sigma/d\ell + \kappa^4/4\Omega^2 a_\Sigma^2$ with respect to j^2+k^2/ℓ^2, as compared to Eq. (5.10.255):

$$j^2 \approx -(k^2/\ell^2) - [k\kappa^2/2\Omega\ell(\sigma+k\Omega)][2A + d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]$$
$$+[k^2A/\ell^2\Sigma(\sigma+k\Omega)^2]\ dP_\Sigma/d\ell, \qquad (|j| \gg 1;\ |\sigma+k\Omega| \ll 1). \qquad (5.10.258)$$

Solving this second order equation with respect to $\sigma + k\Omega$, we get eventually

$$\sigma \approx -k\Omega - (k\kappa^2/2\Omega\ell)[2A + d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]/2(j^2+k^2/\ell^2) \pm \left\{(k\kappa^2/2\Omega\ell)^2[2A\right.$$
$$\left.+d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]^2 + 4(j^2+k^2/\ell^2)(k^2A/\ell^2\Sigma)\ dP_\Sigma/d\ell\right\}^{1/2}/2(j^2+k^2/\ell^2), \qquad (5.10.259)$$

which becomes in the isentropic limit

$$\sigma \approx -k\Omega - \left\{(k\kappa^2/2\Omega\ell)[d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]\right.$$
$$\left.\pm(k\kappa^2/2\Omega\ell)[d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]\right\}/2(j^2+k^2/\ell^2), \qquad (A=0;\ |j|\gg 1;\ |\sigma+k\Omega| \ll 1). \qquad (5.10.260)$$

This equation shows that the prograde $g^{(f)}$-mode (the "r-mode") has the oscillation frequency

$$\sigma \approx -k\Omega - (k\kappa^2/2\Omega\ell)[d\ln(\kappa^2/2\Omega)/d\ell - d\ln\Sigma/d\ell]/(j^2+k^2/\ell^2), \qquad (\sigma/k+\Omega<0), \qquad (5.10.261)$$

while the retrograde $g^{(b)}$-mode becomes neutral ($\sigma + k\Omega = 0$) in the comoving frame. The essential restoring force of the $g^{(f)}$-mode in the case of an isentropic disk is the Coriolis force. The changeover of $g^{(f)}$ and $g^{(b)}$-modes into a rotational $g^{(f)} = r$-mode and a trivial neutral $g^{(b)}$-mode in the limit $A \to 0$ is a special feature of the two-dimensional oscillations considered here and in Sec. 5.9.3. This degeneracy should not occur for three-dimensional oscillations of thick disks (Iye 1984).

Long wavelength oscillations are generally more complicated to study as compared to short wavelength oscillations, because of global couplings between p and g-modes. Only uniformly rotating Maclaurin disks ($\Omega = \text{const},\ n = 0,\ n_\Sigma = 0.5$) have been solved analytically for long wavelength modes. The analysis of such modes in terms of propagation and phase diagrams has been effected by Schutz and Verdaguer (1983), and Verdaguer (1983). For uniformly rotating Maclaurin disks the spatial eigenfunctions are similar in stellar disks and gas disks in the short wavelength limit. But for differentially rotating disks they may differ strikingly.

Schutz and Bowen (1983), Schutz and Verdaguer (1983), and Verdaguer (1983) have studied thick isentropic disks having $n = 0$ and 2, ($n_\Sigma = 0.5, 1.5;\ \Gamma_{1\Sigma} = 1 + 1/n_\Sigma = 3, 5/3;\ \Gamma_1 = 1 + 1/n = \infty, 1.5$) in an approximation due to Bardeen (1975): The vertical structure of the thick disk is included up to the first order in its thickness. The disk is always assumed to be in hydrostatic equilibrium in the vertical z-direction, and this reduces the normal mode analysis to two-dimensional oscillations in the (ℓ, φ)-plane, as already considered in most parts of this section. Schutz and Bowen (1983) start with Hunter's (1963) infinitely thin, uniformly rotating, pressure-free disk, and compute first order corrections when hydrodynamic pressure is taken into account. The surface density (5.10.156), the mass (5.10.188), and the gravitational potential (5.10.163) of the unperturbed zero thickness disk become in terms of the variable μ from Eq. (5.10.148) and of the angular speed $\Omega_u^2 = 3\pi G M_1/4\ell_1^3$ from Eq. (5.10.188) equal to

$$\Sigma = 2\Omega_u^2 \ell_1 \mu/\pi^2 G = 2\Omega_u^2 \ell_1 (1 - \ell^2/\ell_1^2)^{1/2}/\pi^2 G; \quad M_1 = 4\Omega_u^2 \ell_1^3/3\pi G;$$
$$\Phi_u = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 = \Omega_u^2 (2\ell_1^2 - \ell^2)/2, \qquad (z = 0;\ \mu = (1 - \ell^2/\ell_1^2)^{1/2}). \tag{5.10.262}$$

The derivatives $\partial\Phi_u/\partial z$ at the upper and lower border of the equatorial plane are given by Eq. (5.10.102), and the Taylor expansion of this equation yields for the external potential near the unperturbed, pressure-free, zero thickness disk:

$$\Phi_{ue}(\mu, z) = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 - 2\pi G \Sigma |z| + O(z^2), \qquad (z \approx 0). \tag{5.10.263}$$

A disk with finite half-thickness $h = h(\mu)$, $(h \ll \ell_1)$ and finite pressure has the surface density from Eq. (5.10.161)

$$\Sigma(\mu) = \int_{-h}^{h} \varrho\, dz = 2\varrho\, h(\mu) \quad \text{or} \quad h(\mu) = \Sigma(\mu)/2\varrho = 3M_1\mu/4\pi\ell_1^2\varrho = \Omega_u^2 \ell_1 \mu/\pi^2 G\varrho,$$
$$(n = 0;\ \varrho = \text{const}), \tag{5.10.264}$$

hence the same mass M_1 and radius ℓ_1 as the pressure-free, zero thickness disk. The internal gravitational potential is obtained by integrating Eq. (5.10.89) twice:

$$\Phi(\mu, z) = \Phi(\mu, 0) - 2\pi G \varrho z^2, \qquad [(\partial\Phi/\partial z)_{z=0} = 0;\ \varrho = \text{const}]. \tag{5.10.265}$$

Equating at the boundary $z = h$ the external potential (5.10.263) of the pressure-free, zero thickness disk to the internal potential (5.10.265) of the thick disk with pressure, we find

$$\Phi_{ue}(\mu, h) = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 - 2\pi G \Sigma h = \Phi(\mu, h) = \Phi(\mu, 0) - 2\pi G \varrho h^2 = \Phi(\mu, 0) - \pi G \Sigma h \quad \text{or}$$
$$\Phi(\mu, 0) = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 - \pi G \Sigma h, \tag{5.10.266}$$

and

$$(\partial\Phi_{ue}/\partial z)_{z=h} = (\partial\Phi/\partial z)_{z=h} = -2\pi G \Sigma. \tag{5.10.267}$$

Eq. (5.10.265) turns into

$$\Phi(\mu, z) = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 - \pi G \Sigma h - 2\pi G \varrho z^2 = \Omega_u^2 \ell_1^2 (1 + \mu^2)/2 - \pi G \Sigma^2/2\varrho - 2\pi G \varrho z^2,$$
$$(n = 0;\ \varrho = \text{const}). \tag{5.10.268}$$

This derivation seems not quite convincing, as it matches the external potential $\Phi_{ue}(\mu, h)$ of a zero thickness disk with the internal potential $\Phi(\mu, h)$ of a thick disk at its boundary $z = h$. The equation of hydrostatic equilibrium (5.10.90) in the vertical z-direction yields

$$\partial P/\partial z = \varrho\ \partial\Phi/\partial z = -4\pi G\varrho^2 z \quad \text{or} \quad P(\mu, z) = 2\pi G\varrho^2(h^2 - z^2), \qquad (P(\mu, h) = 0;\ \varrho = \text{const}), \tag{5.10.269}$$

and

$$P(\mu, 0) = 2\pi G\varrho^2 h^2 = \pi G\Sigma^2/2. \tag{5.10.270}$$

If $\vec{v} = 0$, Eq. (5.10.48) turns into the equation of hydrostatic equilibrium, whose ℓ-component is

$$\Omega^2\ell - (1/\varrho)\ \partial P/\partial\ell + \partial\Phi/\partial\ell = 0. \tag{5.10.271}$$

We insert Eqs. (5.10.268)-(5.10.270) into this equation, and get a relationship between the angular velocity Ω_u of the pressure-free, zero thickness disk and the constant angular velocity Ω of a disk with constant volume density, and finite thickness and pressure:

$$\Omega^2 = \Omega_u^2(1 - 8\Omega_u^2/\pi^3 G\varrho) = \Omega_u^2(1 - 8R/\pi). \tag{5.10.272}$$

We have introduced the aspect ratio R between the two semimajor axes of the disk, taking into account that $\mu = 1$ along the positive z-axis via Eq. (5.10.139):

$$R = a_3/a_1 = (1 - e^2)^{1/2} = h(1)/\ell_1 = \Omega_u^2/\pi^2 G\varrho \ll 1. \tag{5.10.273}$$

The ratio (3.1.35) for this special disk is calculated as

$$\tau = E_{kin}/|W| = (\pi - 8R)/2(\pi - R) = [\pi - 8(1 - e^2)^{1/2}]/2[\pi - (1 - e^2)^{1/2}],$$
$$(n = 0;\ \varrho, \Omega = \text{const}), \tag{5.10.274}$$

where

$$E_{kin} = (1/2)\int_{M_1} \Omega^2\ell^2\ dM = 2\pi\varrho\Omega^2 \int_0^{\ell_1} h\ell^3\ d\ell = 4\Omega_u^4\ell_1^5(\pi - 8R)/15\pi^2 G;$$
$$W = -(1/2)\int_{M_1} \Phi\ dM = 8\Omega_u^4\ell_1^5(-\pi + R)/15\pi^2 G. \tag{5.10.275}$$

Since $0 \leq \tau \leq 0.5$, the aspect ratio R changes between $R = 0$, $(\tau = 0.5;\ e = 1;$ pressure-free, zero thickness disk) and $R = \pi/8 = 0.39$, $(\tau = 0;\ e = 0.92)$. Comparison of Eq. (5.10.274) with the corresponding ratio (5.10.223) for Maclaurin ellipsoids shows that these particular disks are a good approximation to the Maclaurins merely if $1 \leq e \lesssim 0.989$, $(0 \leq R \leq 0.15)$. However, some similarities occur between the stability behaviour of the constant density disks envisaged by Schutz and Bowen (1983) and the Maclaurin ellipsoids, already sketched in Secs. 5.8.3, 5.8.4: Secular instability against ordinary viscosity or gravitational radiation reaction occurs for the sectorial harmonic if $\tau = 0.15,\ R = 0.29,\ e = 0.96$, and dynamical instability if $\tau = 0.29,\ R = 0.20,\ e = 0.98$.

In the same approximation Schutz and Verdaguer (1983), and Verdaguer (1983) have studied an isentropic disk with finite height and polytropic index $n = 2$. The equation of hydrostatic equilibrium (5.10.107) becomes $(n = 2;\ n_\Sigma = 1.5;\ \Gamma_1 = 1 + 1/n = 1.5;\ \Gamma_{1\Sigma} = 1 + 1/n_\Sigma = 5/3)$

$$\ell\Omega^2 = (\pi G K^2/16)^{1/3} B(1/3, 1/2)\ d\Sigma^{2/3}/d\ell - d\Phi/d\ell. \tag{5.10.276}$$

We have inserted for P_Σ via Eq. (5.10.97), taking into account Eqs. (5.10.96) and (C.11):

$$P_\Sigma = 2^{-4/3}\pi^{1/3}G^{1/3}K^{2/3}B(4/3, 1/2)\ \Sigma^{5/3} = 2^{-1/3}\pi^{1/3}G^{1/3}K^{2/3}B(1/3, 1/2)\ \Sigma^{5/3}/5. \tag{5.10.277}$$

The numerical evaluation of the perturbed hydrodynamical equations by Schutz and Verdaguer (1983) shows that – as in other isentropic disks and cylinders – two types of modes arise: p-modes with forward and backward propagating $p^{(f)}$ and $p^{(b)}$-modes, and the $g^{(f)} = r$-modes. Apparently, a continuous spectrum exists, dominated by $g^{(f)}$-modes, covering the range of Ω in the disk, and having corotation

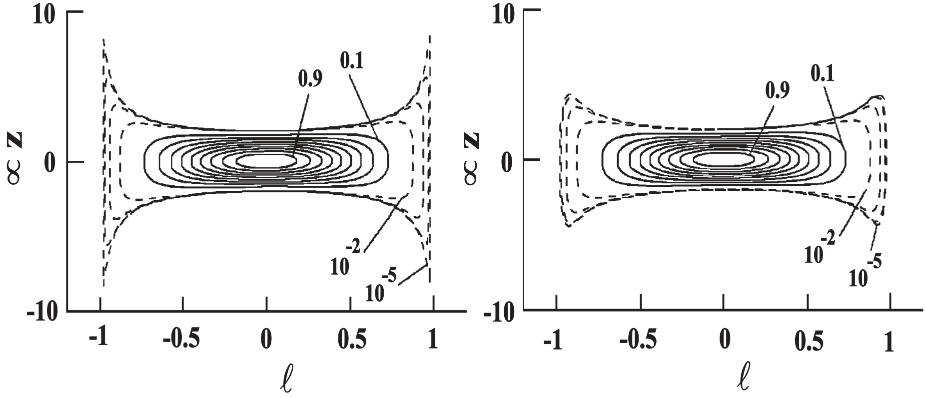

Fig. 5.10.2 Equidensity contours for a $n = 1.5$ polytropic disk of surface density $\Sigma(\ell) = \Sigma(0)\ [1-(\ell/\ell_1)^2]^{5/2}$ with uncorrected diverging edge (on the left), and corrected boundary layer (on the right), (Balmforth et al. 1993).

points, where the eigenfunctions may exhibit singular behaviour. Dynamical instability sets in when a paired $p^{(f)}$ and $p^{(b)}$-mode acquires the same real eigenvalue, becoming afterwards complex as τ increases. By contrast, the $g^{(f)}$-modes occur singly rather than in pairs, and are always stable in uniformly rotating disks. The results concerning instability to gravitational radiation reaction of the $n = 2$ disks are more conclusive, showing that – similarly to the Maclaurin ellipsoids from Sec. 5.8.4 – secular instability of the fundamental $p^{(b)}$-mode sets in at $\tau = 0.12$, and dynamical instability at $\tau \approx 0.27$. The $p^{(f)}$ and $g^{(f)}$-modes are always secularly stable against gravitational radiation reaction.

Balmforth et al. (1993) have constructed equilibrium structures of polytropic disks with finite thickness, and a surface density law $\Sigma(\ell) = \Sigma(0)\ [1 - (\ell/\ell_1)^2]^q,\ (q = \mathrm{const})$. If $n \geq 1$, the disks flare out near the rim $\ell = \ell_1$, the vertical extension becoming infinite. This diverging shape is corrected with the double approximation technique from Sec. 3.6, by assuming the outer, low-density layers to rotate mainly under the influence of the centrally condensed regions (Fig. 5.10.2). A three-dimensional linear analysis of the gravitational instability of such disks has been undertaken by Balmforth et al. (1995).

Hayashi et al. (1982) have studied the stability of a particular, analytic isothermal disk with constant rotation velocity $\Omega\ell = \mathrm{const}$, having the density distribution

$$\varrho(\ell, z) = C^2\mathcal{R}T/2\pi\mu G \cosh^2\{C \ln[(\ell^2 + z^2)^{1/2}/\ell + z/\ell]\}, \qquad (C, T = \mathrm{const}). \tag{5.10.278}$$

The structure of relativistic, uniformly rotating, zero thickness disks has been investigated by Bardeen and Wagoner (1971), using the formalism outlined at the beginning of Sec. 4.2.6. As this matter has no direct bearing on the polytropic equation of state, it will not be pursued further.

In conclusion, our present knowledge concerning the stability and oscillations of polytropic disks seems to be in an incomplete stage.

5.11 Stability and Oscillations of Magnetopolytropes

5.11.1 The Virial Theorem for Spheroidal Magnetopolytropes

The study of stability and oscillations in infinitely conducting magnetopolytropes is a first stage in the attempt to solve the problem of magnetic fields in more realistic stellar models. The investigations on this subject may be divided into two classes (Goossens and Veugelen 1978):

(i) The first class deals with the influence of magnetic fields on the oscillations of a nonmagnetic star. As these oscillations are governed by pressure and gravitational forces, their stability is generally not much altered, even by the occurrence of strong magnetic fields (ratio between magnetic and potential energy $U_m/|W| \approx 0.25$; Monaghan 1968, Miketinac 1974, Tassoul 1978). Actually, strong observational evidence exists that magnetic fields in stars are generally weak, and most investigations are based on this fact. It is not likely that new instabilities can be induced through magnetic fields on this kind of oscillations.

(ii) The second class deals with instabilities introduced by magnetic fields in zones that are stably stratified in absence of any magnetic field. These motions are of typically hydromagnetic nature, and are associated with motions that are mainly governed by magnetic forces. These local hydromagnetic instabilities depend more on topology than on the strength of the magnetic field (Tayler 1973). As far as I know, only the paper by Goossens and Veugelen (1978) studies the second class of instabilities in polytropes, so this section is concerned almost exclusively with instabilities belonging to class (i).

The overall conditions for dynamical instability of a magnetohydrostatic configuration with zero surface pressure have already been written down in Eq. (2.6.99), and become for a sphere with $d^2I/dt^2, E_{kin}, P_S = 0$ equal to

$$(3\Gamma - 4)[-3GM_1^2/(5-n)r_1 + pr_1^3H_m^2/6] < 0, \qquad (-1 < n < 5;\ \Gamma > 1). \tag{5.11.1}$$

We have inserted for the gravitational and magnetic energy from Eqs. (2.6.137) and (2.6.78), respectively. H_m^2 denotes the mean square magnetic field intensity in the polytrope.

Provided that $\Gamma > 4/3$, as required for dynamical stability of a nonmagnetic configuration, the condition imposed by the virial theorem for the dynamical stability of a spherical magnetopolytrope is according to Eq. (5.11.1) equal to (Spitzer 1978, Chandrasekhar 1981)

$$H_m < (3M_1/r_1^2)[2G/p(5-n)]^{1/2} = 4\pi\varrho_m r_1[2G/p(5-n)]^{1/2}, \qquad (\Gamma > 4/3). \tag{5.11.2}$$

If a magnetic, infinitely long cylinder is in hydrostatic equilibrium, the virial theorem (2.6.106) becomes per unit length:

$$2(\Gamma-1)U + 2U_m - GM_1^2 = 0, \qquad (d^2I/dt^2, E_{kin}, P_S = 0). \tag{5.11.3}$$

By virtue of Eq. (2.6.93) we have $\Gamma > 1$, and therefore $(\Gamma-1)U > 0$. A necessary requirement to satisfy the equilibrium condition (5.11.3) is therefore $2U_m - GM_1^2 < 0$ or (Chandrasekhar and Fermi 1953)

$$H_m < (2M_1/\ell_1)(G/p)^{1/2} = 2\pi\varrho_m\ell_1(G/p)^{1/2}, \tag{5.11.4}$$

where Eq. (2.6.78) reads for a cylinder of radius ℓ_1 per unit length as

$$U_m = pH_m^2\ell_1^2/8. \tag{5.11.5}$$

The stability condition (5.11.1) can also be obtained from the contraction of the virial equations for small oscillations, which have already been derived for a nonmagnetic configuration in Sec. 5.8.1. We start with Eq. (2.6.79), where it is often assumed that the two last surface integrals vanish, and this assumption may require to place the surface S at infinity. In this case the surface integrals will vanish, because the magnetic field of any isolated object must decrease at least as rapidly as a dipole field, i.e. as r^{-3} (cf. Eq. (3.10.25), Chandrasekhar 1981). However, it is often convenient to let the surface S coincide with the natural boundary of the configuration, and then, in general, the surface integrals must be retained in Eq. (2.6.79). The tensor H_{ij} will refer only to the magnetic field interior to S. As emphasized by Ledoux and Walraven (1958) the physical meaning of boundary conditions in magnetohydrodynamics is sometimes elusive, as dictated by the mathematical tractability of the problem.

By virtue of Eq. (2.6.76) the left-hand sides of Eqs. (2.6.73) and (2.6.79) are equal among each other, as well as their first order variations calculated in Eqs. (5.8.16) and (5.8.30), respectively. Thus, the first order variation of Eqs. (2.6.73) or (2.6.79) becomes

$$(1/2)\ d^2\delta^* I_{ij}/dt^2 = (1/2)\ d^2\left[\int_V \varrho(x_i\ \Delta x_j + x_j\ \Delta x_i)\ dV\right]\Big/ dt^2 = \delta^*\left[d\left(\int_V \varrho x_i v_j\ dV\right)\Big/ dt\right]$$
$$\approx d^2\left(\int_V \varrho x_i\ \Delta x_j\ dV\right)\Big/ dt^2 = 2\ \delta^* E_{ij} + \delta^* W_{ij} + \delta_{ij}\ \delta^*\int_V P\ dV + \delta_{ij}\ \delta^*\int_V (pH_kH_k/8\pi)\ dV$$
$$-2\ \delta^* H_{ij} - \delta^* S_{ij}, \tag{5.11.6}$$

where S_{ij} denotes the surface integral

$$S_{ij} = \int_S x_i(P + pH_kH_k/8\pi)\ dS_j - (p/4\pi)\int_S x_iH_jH_k\ dS_k. \tag{5.11.7}$$

$dS_i = n_i\ dS$ is the projection perpendicular to the x_i-axis of the surface element dS, and n_i denotes the projection of the exterior normal $\vec{n} = \vec{n}(n_1, n_2, n_3)$ on the x_i-axis.

The first order variations $\delta^* W_{ij},\ \delta^* E_{ij},\ \delta^* I_{ij}$ have already been written down in Eqs. (5.8.28)-(5.8.30) if the configuration is initially in hydrostatic equilibrium. The pressure integral in Eq. (5.11.6) can be transformed with the help of the hydromagnetic equilibrium equation (2.1.1), $(\vec{v}, \tau = 0)$, by evaluating $(\nabla\times\vec{H})\times\vec{B}$ via Eqs. (2.6.52), (2.6.54):

$$\partial P/\partial x_k = \varrho\ \partial\Phi/\partial x_k + (p/4\pi)[\partial(H_kH_\ell)/\partial x_\ell - (1/2)\ \partial(H_\ell H_\ell)/\partial x_k]. \tag{5.11.8}$$

Integrating Eq. (5.8.31) by parts, we get

$$\delta^*\int_V P\ dV = -(\Gamma_1 - 1)\int_V P\ \nabla\cdot\Delta\vec{r}\ dV = (\Gamma_1 - 1)\int_V \Delta x_k(\partial P/\partial x_k)\ dV$$
$$-(\Gamma_1 - 1)\int_S P\ \Delta x_k\ dS_k = (\Gamma_1 - 1)\int_V \Delta x_k\{\varrho\ \partial\Phi/\partial x_k + (p/4\pi)[\partial(H_kH_\ell)/\partial x_\ell$$
$$-(1/2)\ \partial(H_\ell H_\ell)/\partial x_k]\}\ dV - (\Gamma_1 - 1)\int_S P\ \Delta x_k\ dS_k = (\Gamma_1 - 1)\int_V\{\varrho\ \Delta x_k\ \partial\Phi/\partial x_k$$
$$+(p/4\pi)[-H_kH_\ell\ \partial\Delta x_k/\partial x_\ell + (1/2)H_\ell H_\ell\ \partial\Delta x_k/\partial x_k]\}\ dV$$
$$-(\Gamma_1 - 1)\int_S (P + pH_\ell H_\ell/8\pi)\ \Delta x_k\ dS_k + p(\Gamma_1 - 1)\int_S H_kH_\ell\ \Delta x_k\ dS_\ell/4\pi. \tag{5.11.9}$$

To obtain the variation of the magnetic integrals, we have to calculate the Lagrangian change $\Delta\vec{H}$ of the magnetic field intensity, by observing that Eq. (3.10.12) writes in the case of high electrical conductivity as (Alfvén and Fälthammar 1963)

$$\vec{E} = -(p/c)\ \vec{v}\times\vec{H}. \tag{5.11.10}$$

$\vec{v}$ is the fluid velocity with respect to a laboratory frame, and if the unperturbed fluid is in equilibrium, then $\vec{v}_u = 0$, and $\vec{v} = \vec{v}_u + \Delta\vec{v} \approx \vec{v}_u + \delta\vec{v} = \delta\vec{v}$ is always a small first order quantity, so $\vec{v}\times\delta\vec{H} \approx \delta\vec{v}\times\delta\vec{H}$ is of second order and negligible. Therefore, the Eulerian variation of Eq. (5.11.10) is

$$\delta\vec{E} = -(p/c)\ \delta(\vec{v}\times\vec{H}) = -(p/c)(\delta\vec{v}\times\vec{H} + \vec{v}\times\delta\vec{H}) \approx -(p/c)\ \delta\vec{v}\times\vec{H}. \tag{5.11.11}$$

Taking the Eulerian variation of Maxwell's equation (3.10.1), we get

$$\delta(\nabla\times\vec{E}) = \nabla\times\delta\vec{E} = -(p/c)\ \partial\delta\vec{H}/\partial t, \tag{5.11.12}$$

and combining with Eq. (5.11.11):

$$\partial\delta\vec{H}/\partial t = -(c/p)\ \nabla\times\delta\vec{E} \approx \nabla\times(\delta\vec{v}\times\vec{H}) \approx \nabla\times[(\partial\Delta\vec{r}/\partial t)\times\vec{H}] \approx \partial[\nabla\times(\Delta\vec{r}\times\vec{H})]/\partial t. \tag{5.11.13}$$

By virtue of Eq. (3.10.14) the quantity $\Delta\vec r\times(\partial\vec H/\partial t)=\Delta\vec r\times[\nabla\times(\vec v\times\vec H)]\approx\Delta\vec r\times[\nabla\times(\delta\vec v\times\vec H)]$ is a negligible second order quantity. Integrating Eq. (5.11.13) with respect to the time, we obtain the Eulerian variation of the magnetic field intensity (Chandrasekhar and Fermi 1953, Eqs. (130)-(131); Chandrasekhar 1981)

$$\delta\vec H=\nabla\times(\Delta\vec r\times\vec H)=(\vec H\cdot\nabla)\Delta\vec r-(\Delta\vec r\cdot\nabla)\vec H-(\nabla\cdot\Delta\vec r)\vec H;$$
$$\delta H_j=\partial(H_k\ \Delta x_j-H_j\ \Delta x_k)/\partial x_k,\qquad (\nabla\cdot\vec H=\partial H_k/\partial x_k=0),\tag{5.11.14}$$

by using the well known vectorial identity

$$\nabla\times(\vec a\times\vec b)=(\vec b\cdot\nabla)\vec a+(\nabla\cdot\vec b)\vec a-(\vec a\cdot\nabla)\vec b-(\nabla\cdot\vec a)\vec b.\tag{5.11.15}$$

The Lagrangian variation writes via Eqs. (5.1.17), (5.11.14) as

$$\Delta\vec H=\nabla\times(\Delta\vec r\times\vec H)+(\Delta\vec r\cdot\nabla)\vec H,\tag{5.11.16}$$

and its components are

$$\Delta H_j=\Delta x_j\ \partial H_k/\partial x_k+H_k\ \partial\Delta x_j/\partial x_k-\Delta x_k\ \partial H_j/\partial x_k-H_j\ \partial\Delta x_k/\partial x_k+\Delta x_k\ \partial H_j/\partial x_k$$
$$=H_k\ \partial\Delta x_j/\partial x_k-H_j\ \partial\Delta x_k/\partial x_k,$$
$$(\Delta\vec r=\Delta\vec r(\Delta x_1,\Delta x_2,\Delta x_3);\ \ \vec H=\vec H(H_1,H_2,H_3);\ \ j,k=1,2,3).\tag{5.11.17}$$

Using this relation, the variation (5.8.4) of the magnetic tensor (2.6.77) becomes

$$\delta^*H_{ij}=(p/8\pi)\int_V[\Delta(H_iH_j)+H_iH_j\ \nabla\cdot\Delta\vec r]\ dV=(p/8\pi)\int_V(H_i\ \Delta H_j+H_j\ \Delta H_i$$
$$+H_iH_j\ \partial\Delta x_k/\partial x_k)\ dV=(p/8\pi)\int_V[H_k(H_i\ \partial\Delta x_j/\partial x_k+H_j\ \partial\Delta x_i/\partial x_k)-H_iH_j\ \partial\Delta x_k/\partial x_k]\ dV,\tag{5.11.18}$$

$$\delta^*H_{ii}=(p/8\pi)\ \delta^*\int_V H_iH_i\ dV=(p/8\pi)\int_V(2H_iH_k\ \partial\Delta x_i/\partial x_k-H_iH_i\ \partial\Delta x_k/\partial x_k)\ dV.\tag{5.11.19}$$

The variation (5.8.4) of the surface integral (5.11.7) is somewhat more involved, and we find by completing the derivation of Trehan and Billings (1971):

$$\delta^*S_{ij}=\delta^*\int_V[\partial(x_iP+px_iH_kH_k/8\pi)/\partial x_j-(p/4\pi)\ \partial(x_iH_jH_k)/\partial x_k]\ dV$$
$$=\int_V\big\{\Delta[\partial(x_iP+px_iH_kH_k/8\pi)/\partial x_j]\ dV+[\partial(x_iP+px_iH_kH_k/8\pi)/\partial x_j]\ \partial\Delta x_\ell/\partial x_\ell$$
$$-(p/4\pi)\ \Delta[\partial(x_iH_jH_k)/\partial x_k]-(p/4\pi)[\partial(x_iH_jH_k)/\partial x_k]\ \partial\Delta x_\ell/\partial x_\ell\big\}\ dV$$
$$=\int_V\{\partial[\Delta(x_iP+px_iH_kH_k/8\pi)]/\partial x_j-(p/4\pi)\ \partial[\Delta(x_iH_jH_k)]/\partial x_k$$
$$-[\partial(x_iP+px_iH_kH_k/8\pi)/\partial x_\ell]\ \partial\Delta x_\ell/\partial x_j+(p/4\pi)[\partial(x_iH_jH_k)/\partial x_\ell]\ \partial\Delta x_\ell/\partial x_k$$
$$+[\partial(x_iP+px_iH_kH_k/8\pi)/\partial x_j]\ \partial\Delta x_\ell/\partial x_\ell-(p/4\pi)[\partial(x_iH_jH_k)/\partial x_k]\ \partial\Delta x_\ell/\partial x_\ell\big\}\ dV$$
$$=\int_S\Delta(x_iP+px_iH_kH_k/8\pi)\ dS_j-(p/4\pi)\int_S\Delta(x_iH_jH_k)\ dS_k$$
$$+\int_V\big\{-\partial[(x_iP+px_iH_kH_k/8\pi)\ \partial\Delta x_\ell/\partial x_j]/\partial x_\ell+\partial[(x_iP+px_iH_kH_k/8\pi)\ \partial\Delta x_\ell/\partial x_\ell]/\partial x_j$$
$$+(p/4\pi)\ \partial(x_iH_jH_k\ \partial\Delta x_\ell/\partial x_k)/\partial x_\ell-(p/4\pi)\ \partial(x_iH_jH_k\ \partial\Delta x_\ell/\partial x_\ell)/\partial x_k\big\}\ dV$$
$$=\int_S[\Delta(x_iP+px_iH_kH_k/8\pi)+(x_iP+px_iH_kH_k/8\pi)\ \partial\Delta x_\ell/\partial x_\ell]\ dS_j$$
$$+\int_S[-x_i(P+pH_kH_k/8\pi)\ \partial\Delta x_\ell/\partial x_j+(p/4\pi)x_iH_jH_k\ \partial\Delta x_\ell/\partial x_k]\ dS_\ell-\int_S(p/4\pi)[\Delta(x_iH_jH_k)$$
$$+x_iH_jH_k\ \partial\Delta x_\ell/\partial x_\ell]\ dS_k=\int_S\big[P\ \Delta x_i+x_iP+(p/8\pi)(H_kH_k\ \Delta x_i+2x_iH_kH_\ell\ \partial\Delta x_k/\partial x_\ell$$
$$-x_iH_kH_k\ \partial\Delta x_\ell/\partial x_\ell)\big]\ dS_j+\int_S\big[-x_i(P+pH_\ell H_\ell/8\pi)\ \partial\Delta x_k/\partial x_j$$

$$+(p/4\pi)(-H_jH_k\ \Delta x_i - x_iH_kH_\ell\ \partial\Delta x_j/\partial x_\ell + x_iH_jH_k\ \partial\Delta x_\ell/\partial x_\ell)\big]\ dS_k. \qquad (5.11.20)$$

We have used Eq. (5.1.19) and the surface condition (5.2.63): $(\Delta P)_S = 0$. Some further simplifications of Eq. (5.11.20) occur if the fluid pressure on the surface is zero, or if the fluid is incompressible: $\Gamma_1 = \infty,\ \Delta\varrho = 0,\ \nabla\cdot\Delta\vec{r} = 0$.

Eq. (5.11.6) reads, by taking into account Eqs. (5.8.28)-(5.8.30), (5.11.9), (5.11.18), (5.11.19), (Chandrasekhar 1981):

$$d^2\Big(\int_V \varrho x_i\ \Delta x_j\ dV\Big)\Big/dt^2 = -\sigma^2\int_V \varrho x_i\ \Delta x_j\ dV = -\int_V \varrho\ \Delta x_k\ (\partial\Phi_{ij}/\partial x_k)\ dV$$
$$+\delta_{ij}\int_V \big\{(\Gamma_1-1)\varrho\ \Delta x_k\ \partial\Phi/\partial x_k + [p(\Gamma_1-2)/4\pi][(1/2)H_\ell H_\ell\ \partial\Delta x_k/\partial x_k - H_kH_\ell\ \partial\Delta x_k/\partial x_\ell]\big\}\ dV$$
$$+(p/4\pi)\int_V [H_iH_j\ \partial\Delta x_k/\partial x_k - H_k(H_i\ \partial\Delta x_j/\partial x_k + H_j\ \partial\Delta x_i/\partial x_k)]\ dV$$
$$+\delta_{ij}(\Gamma_1-1)\int_S [-(P+pH_\ell H_\ell/8\pi)\ \Delta x_k + pH_kH_\ell\ \Delta x_\ell/4\pi]\ dS_k - \delta^*S_{ij}. \qquad (5.11.21)$$

The temporal dependence of Δx_i has been assumed under the familiar form $\Delta x_i(x_1,x_2,x_3,t) = \Delta x_i(x_1,x_2,x_3)\exp(i\sigma t)$. Eq. (5.11.21) becomes, by ignoring the surface integrals and contracting with respect to the indices i,j :

$$-\sigma^2\int_V \varrho x_k\ \Delta x_k\ dV = (3\Gamma_1-4)\int_V \big\{\varrho\ \Delta x_k\ \partial\Phi/\partial x_k + (p/4\pi)[(1/2)H_\ell H_\ell\ \partial\Delta x_k/\partial x_k$$
$$-H_kH_\ell\ \partial\Delta x_\ell/\partial x_k]\big\}\ dV. \qquad (5.11.22)$$

To get an estimate of the eigenvalue σ, the simplest trial displacement is $\Delta x_i = \mathrm{const}\ x_i$. Eq. (5.11.22) yields

$$\sigma^2\int_V \varrho r^2\ dV = -(3\Gamma_1-4)\int_V [\varrho x_k\ \partial\Phi/\partial x_k + (p/8\pi)H_kH_k]\ dV, \qquad (5.11.23)$$

or by inserting Eqs. (2.6.65), (2.6.75), (2.6.78):

$$\sigma^2 = -(3\Gamma_1-4)(W+U_m)/I. \qquad (5.11.24)$$

This is equivalent to the stability conditions (2.6.99) or (5.11.2), since stability requires $\sigma^2 > 0$ or $U_m < |W|,\ (W<0)$ if $\Gamma_1 > 4/3$.

A more refined trial displacement would be given by Eq. (5.8.33), yielding sufficient accuracy for spheroidal polytropes with polytropic index $0 \le n \lesssim 2.5$. Eqs. (5.11.6) or (5.11.21) write, by preserving the expression (5.8.31) for the pressure integral (Trehan and Billings 1971):

$$d^2\Big(\int_V \varrho x_i\ \Delta x_j\ dV\Big)\Big/dt^2 = (1/2)\ d^2\delta^*I_{ij}/dt^2 = \int_V \big\{-\varrho\ \Delta x_k\ \partial\Phi_{ij}/\partial x_k$$
$$+\delta_{ij}[-(\Gamma_1-1)P\ \partial\Delta x_k/\partial x_k + (p/8\pi)(2H_kH_\ell\ \partial\Delta x_k/\partial x_\ell - H_kH_k\ \partial\Delta x_\ell/\partial x_\ell)]$$
$$+(p/4\pi)[H_iH_j\ \partial\Delta x_k/\partial x_k - H_k(H_i\ \partial\Delta x_j/\partial x_k + H_j\ \partial\Delta x_i/\partial x_k)]\big\}\ dV - \delta^*S_{ij}. \qquad (5.11.25)$$

With the displacement (5.8.33) we are lead to

$$\sigma^2L_{jk}I_{ik} - L_{k\ell}W_{\ell k;ij} + \delta_{ij}(L_{kk}\Pi + 2L_{k\ell}H_{k\ell} - L_{\ell\ell}H_{kk}) + 2(L_{kk}H_{ij} - L_{jk}H_{ik} - L_{ik}H_{jk})$$
$$-\int_S [PL_{ik}x_k + Px_i + (p/8\pi)(L_{i\ell}x_\ell H_kH_k + 2L_{k\ell}x_iH_kH_\ell - L_{\ell\ell}x_iH_kH_k)]\ dS_j$$
$$+\int_S [L_{kj}x_i(P+pH_\ell H_\ell/8\pi) + (p/4\pi)(L_{i\ell}x_\ell H_jH_k + L_{j\ell}x_iH_kH_\ell - L_{\ell\ell}x_iH_jH_k)]\ dS_k = 0. \qquad (5.11.26)$$

Instead of Eq. (5.8.38) we now get from Eq. (2.6.79) for the equilibrium value of the pressure integral (5.8.36) the relationship $(d^2I_{ij}/dt^2, E_{ij} = 0)$

$$\Pi = -(\Gamma_1-1)\int_V P\ dV = (\Gamma_1-1)(W_{11} - H_{11} + H_{22} + H_{33} - S_{11})$$
$$= (\Gamma_1-1)(W_{22} + H_{11} - H_{22} + H_{33} - S_{22}) = (\Gamma_1-1)(W_{33} + H_{11} + H_{22} - H_{33} - S_{33}), \qquad (5.11.27)$$

or

$$-3\int_V P\ dV = W + H_{kk} - S_{kk} = W + U_m - S_{kk}. \tag{5.11.28}$$

Similarly to Eqs. (5.8.65)-(5.8.73), the explicit form of the nine virial equations (5.11.26) becomes for axial symmetry about the x_3-axis equal to ($H_{ij} = 0$ if $i \neq j;\ I_{11} = I_{22};\ H_{11} = H_{22};\ S_{ij} = 0$) :

$$\sigma^2 L_{11} I_{11} - (L_{11} W_{11;11} + L_{22} W_{22;11} + L_{33} W_{33;11}) + \Pi(L_{11} + L_{22} + L_{33})$$
$$+2H_{11}(L_{22} - L_{11}) + H_{33}(L_{33} - L_{11} - L_{22}) = 0, \tag{5.11.29}$$

$$\sigma^2 L_{22} I_{11} - (L_{11} W_{22;11} + L_{22} W_{11;11} + L_{33} W_{33;11}) + \Pi(L_{11} + L_{22} + L_{33})$$
$$+2H_{11}(L_{11} - L_{22}) + H_{33}(L_{33} - L_{11} - L_{22}) = 0, \tag{5.11.30}$$

$$\sigma^2 L_{33} I_{33} - (L_{11} + L_{22}) W_{33;11} - L_{33} W_{33;33} + \Pi(L_{11} + L_{22} + L_{33})$$
$$+H_{33}(L_{11} + L_{22} - L_{33}) - 2H_{11} L_{33} = 0, \tag{5.11.31}$$

$$\sigma^2 L_{21} I_{11} - W_{12;12}(L_{12} + L_{21}) - 2H_{11}(L_{12} + L_{21}) = 0, \tag{5.11.32}$$

$$\sigma^2 L_{12} I_{11} - W_{12;12}(L_{12} + L_{21}) - 2H_{11}(L_{12} + L_{21}) = 0, \tag{5.11.33}$$

$$\sigma^2 L_{31} I_{11} - L_{13} W_{31;13} - L_{31} W_{13;13} - 2(L_{31} H_{11} + L_{13} H_{33}) = 0, \tag{5.11.34}$$

$$\sigma^2 L_{32} I_{11} - L_{23} W_{31;13} - L_{32} W_{13;13} - 2(L_{23} H_{33} + L_{32} H_{11}) = 0, \tag{5.11.35}$$

$$\sigma^2 L_{13} I_{33} - L_{13} W_{31;13} - L_{31} W_{13;13} - 2(L_{13} H_{33} + L_{31} H_{11}) = 0, \tag{5.11.36}$$

$$\sigma^2 L_{23} I_{33} - L_{23} W_{31;13} - L_{32} W_{13;13} - 2(L_{23} H_{33} + L_{32} H_{11}) = 0. \tag{5.11.37}$$

The nine eigenvalues σ^2 of the various oscillation modes are obtained analogously to Eqs. (5.8.85)-(5.8.119), (Chandrasekhar and Lebovitz 1962b, Anand and Kushwaha 1962a, Anand 1969, Trehan and Billings 1971).

(i) The Tesseral (Transverse-Shear) Modes. Adding together Eqs. (5.11.34), (5.11.35) and (5.11.36), (5.11.37), respectively, we get a homogeneous system in the variables $L_{13} + L_{23}$ and $L_{31} + L_{32}$:

$$(-\sigma^2 I_{33} + W_{31;13} + 2H_{33})(L_{13} + L_{23}) + (W_{13;13} + 2H_{11})(L_{31} + L_{32}) = 0, \tag{5.11.38}$$

$$(W_{31;13} + 2H_{33})(L_{13} + L_{23}) + (-\sigma^2 I_{11} + W_{13;13} + 2H_{11})(L_{31} + L_{32}) = 0. \tag{5.11.39}$$

Nontrivial solutions are obtained if the determinant of this system vanishes, leading to the dispersion relation

$$\sigma^2[\sigma^2 - (W_{31;13} + 2H_{33})/I_{33} - (W_{13;13} + 2H_{11})/I_{11}] = 0. \tag{5.11.40}$$

The eigenvalue $\sigma^2 = 0$ implies neutral stability, while the other eigenvalue is obviously

$$\sigma^2 = (W_{31;13} + 2H_{33})/I_{33} + (W_{13;13} + 2H_{11})/I_{11}. \tag{5.11.41}$$

These are two of the four tesseral eigenvalues, and the other two are exactly the same; they can be obtained for instance by subtraction of Eqs. (5.11.35) and (5.11.37) from Eqs. (5.11.34) and (5.11.36), respectively.

(ii) The Sectorial (Toroidal or Barlike) Modes. Returning to the remaining five equations (5.11.29)-(5.11.33), we obtain, on subtracting Eq. (5.11.30) from Eq. (5.11.29), and taking into account Eq. (5.8.60):

$$(\sigma^2 I_{11} - W_{11;11} - W_{22;11} - 4H_{11})(L_{11} - L_{22}) = (\sigma^2 I_{11} - 2W_{12;12} - 4H_{11})(L_{11} - L_{22}) = 0. \tag{5.11.42}$$

If $L_{11} \neq L_{22}$, the corresponding sectorial eigenvalue is

$$\sigma^2 = (2W_{12;12} + 4H_{11})/I_{11}. \tag{5.11.43}$$

Another sectorial mode is obtained by adding together Eqs. (5.11.32) and (5.11.33):

$$(\sigma^2 I_{11} - 2W_{12;12} - 4H_{11})(L_{12} + L_{21}) = 0. \tag{5.11.44}$$

And if $L_{12} \neq -L_{21}$ the sectorial eigenvalue resulting from the first factor is just the same as in Eq. (5.11.43), so this eigenvalue is repeated twice.

(iii) The Zonal (Pulsation) Modes. The remaining three modes of oscillation are obtained by subtracting Eq. (5.11.32) from Eq. (5.11.33):

$$\sigma^2 I_{11}(L_{12} - L_{21}) = 0. \tag{5.11.45}$$

If $L_{12} \neq L_{21}$, neutral stability $\sigma^2 = 0$ occurs. The two other eigenvalues may be obtained by adding together Eqs. (5.11.29) and (5.11.30):

$$(\sigma^2 I_{11} - W_{11;11} - W_{11;22} + 2\Pi - 2H_{33})(L_{11} + L_{22}) + 2(-W_{33;11} + \Pi + H_{33})L_{33} = 0,$$
$$(W_{22;11} = W_{11;22}). \tag{5.11.46}$$

Eq. (5.11.46) minus twice Eq. (5.11.31) leads to

$$(\sigma^2 I_{11} - W_{11;11} - W_{11;22} + 2W_{33;11} - 4H_{33})(L_{11} + L_{22})$$
$$+2(-\sigma^2 I_{33} - W_{33;11} + W_{33;33} + 2H_{11} + 2H_{33})L_{33} = 0. \tag{5.11.47}$$

By virtue of Eqs. (5.8.62) and (5.11.27) we replace in Eq. (5.11.47) the sum

$$W_{11;11} + W_{11;22} = W_{33;11} + W_{33;33} + W_{11} - W_{33} = W_{33;11} + W_{33;33} + 2H_{11} - 2H_{33}, \tag{5.11.48}$$

to obtain

$$(\sigma^2 I_{11} + W_{33;11} - W_{33;33} - 2H_{11} - 2H_{33})(L_{11} + L_{22})$$
$$+2(-\sigma^2 I_{33} - W_{33;11} + W_{33;33} + 2H_{11} + 2H_{33})L_{33} = 0. \tag{5.11.49}$$

Eq. (5.11.31) reads

$$(-W_{33;11} + \Pi + H_{33})(L_{11} + L_{22}) + (\sigma^2 I_{33} - W_{33;33} + \Pi - 2H_{11} - H_{33})L_{33} = 0. \tag{5.11.50}$$

Eqs. (5.11.49) and (5.11.50) yield the dispersion relationship (Trehan and Billings 1971)

$$\sigma^4 I_{11} I_{33} + \sigma^2 [I_{11}(-W_{33;33} + \Pi - 2H_{11} - H_{33}) + I_{33}(-W_{33;11} - W_{33;33} + 2\Pi - 2H_{11})]$$
$$+(W_{33;11} - W_{33;33} - 2H_{11} - 2H_{33})(-2W_{33;11} - W_{33;33} + 3\Pi - 2H_{11} + H_{33}) = 0. \tag{5.11.51}$$

This equation furnishes two coupled zonal eigenvalues. For a spherical configuration with a magnetic field satisfying $H_{11} = H_{22} = H_{33} = H_{kk}/3 = U_m/3$, the coupled modes from Eq. (5.11.51) become uncoupled and equal to (cf. Eqs. (5.8.64), (5.8.119); Anand 1969)

$$\sigma_r^2 = (4 - 3\Gamma_1)(W + U_m)/I; \quad \sigma_f^2 = -4(W/5 - U_m)/I. \tag{5.11.52}$$

In the spherical case Eqs. (5.11.41) and (5.11.43) also reduce to σ_f^2. Hence, we conclude that if $H_{11} = H_{22} = H_{33}$, there are three neutral modes $\sigma^2 = 0$, five nonradial modes σ_f^2, and one radial mode σ_r^2. Like in the rotating case, the magnetic field lifts the accidental degeneracy occurring in a first

Table 5.11.1 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of tesseral ω_t, sectorial ω_s, and zonal ω_{zr}, ω_{zf}-modes of magnetopolytropes pervaded by the toroidal field (3.10.98): $H_\varphi = C\varrho r \sin\lambda,\ h = C^2/16\pi^2 G$ (Anand 1969).

n	ω_t^2	ω_s^2	ω_{zr}^2	ω_{zf}^2
1	$0.16 + 93h$	$0.16 + 0.096h$	$(3\Gamma_1 - 4)(0.19 - 2.1h)$	$0.16 + 5.4h$
1.5	$0.12 + 1.2h$	$0.12 - 0.39h$	$(3\Gamma_1 - 4)(0.16 - 0.34h)$	$0.12 - 1.9h$
2	$0.10 - 0.50h$	$0.10 - 0.30h$	$(3\Gamma_1 - 4)(0.13 - 0.61h)$	$0.10 - 0.43h$
3	$0.065 - 0.21h$	$0.065 - 0.26h$	$(3\Gamma_1 - 4)(0.082 - 0.45h)$	$0.065 - 0.48h$
3.5	$0.051 - 0.10h$	$0.051 - 0.23h$	$(3\Gamma_1 - 4)(0.064 - 0.37h)$	$0.051 - 0.44h$

Table 5.11.2 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ of tesseral ω_t, sectorial ω_s, and zonal ω_{zr}, ω_{zf}-modes in magnetopolytropes pervaded by the poloidal field (3.10.145)-(3.10.150), $(\nabla \times \vec{H} = b_1\varrho(\ell, z)\ \ell\ \vec{e}_\varphi;\ \varepsilon = b_1^2\alpha^2/16\pi^2 G)$, (Trehan and Billings 1971). Note, that ω_t, ω_s are unsquared.

n	ω_t	ω_s	ω_{zr}^2	ω_{zf}^2
1	$0.39 + 0.78\varepsilon$	$0.39 - 0.17\varepsilon$	$(3\Gamma_1 - 4)0.19(1 + 0.21\varepsilon)$	$0.16(1 + 5.6\varepsilon)$
1.5	$0.35 + 0.79\varepsilon$	$0.35 - 0.056\varepsilon$	$(3\Gamma_1 - 4)0.16(1 + 0.51\varepsilon)$	$0.12(1 + 6.1\varepsilon)$
2	$0.32 + 0.79\varepsilon$	$0.32 + 0.030\varepsilon$	$(3\Gamma_1 - 4)0.13(1 + 0.82\varepsilon)$	$0.10(1 + 6.6\varepsilon)$
3	$0.26 + 0.80\varepsilon$	$0.26 + 0.16\varepsilon$	$(3\Gamma_1 - 4)0.082(1 + 1.6\varepsilon)$	$0.065(1 + 7.9\varepsilon)$
3.5	$0.23 + 0.81\varepsilon$	$0.23 + 0.23\varepsilon$	$(3\Gamma_1 - 4)0.064(1 + 2.3\varepsilon)$	$0.051(1 + 8.9\varepsilon)$
4	$0.19 + 0.84\varepsilon$	$0.19 + 0.31\varepsilon$	$(3\Gamma_1 - 4)0.047(1 + 3.5\varepsilon)$	$0.038(1 + 10\varepsilon)$

approximation at $\Gamma_1 = 1.6$ between the eigenvalues σ_r and σ_f of the nonmagnetic sphere (Anand 1969, Fahlman 1971).

Anand (1969) has calculated the eigenvalues for a toroidal magnetic field given by Eq. (3.10.98), (Table 5.11.1). His equilibrium model is practically identical to that of Sinha (1968a), (see Sec. 3.10.4). Anand (1969) takes $h = C^2/4\pi^2 G$ instead of $h = C^2/16\pi^2 G$ from Eq. (3.10.101). The toroidal field has a destabilizing influence on the oscillation frequencies (σ^2 is decreased), excepting for some modes in the polytropes $n = 1$ and 1.5.

As in the rotational case, the difference between the less precise virial method and the variational approach is rather large for appreciable central condensation of the polytrope ($n \gtrsim 2.5$). The radial eigenfrequency $\omega_{zr}^2 = 0.065 - 0.31h$ obtained by Roxburgh and Durney (1967) with a variational approach if $n = 3,\ \Gamma_1 = 5/3$ differs from the corresponding entry $\omega_{zr}^2 = 0.082 - 0.45h$ in Table 5.11.1.

For a poloidal field of the form (3.10.145), (3.10.147) Trehan and Billings (1971) calculate the characteristic eigenfrequencies resulting from the virial theorem (Table 5.11.2). The poloidal field (3.10.145) has a stabilizing influence on the eigenfrequencies, excepting for the polytropes $n = 1$ and 1.5, when the eigenvalues of sectorial modes are decreased. The fundamental radial mode is always stable, if the usual stability criterion $\Gamma_1 > 4/3$ is satisfied.

Using the magnetopolytropic equilibrium models from Eqs. (3.10.223)-(3.10.239), Sood and Trehan (1975) find from the second order virial equations that the combined poloidal and toroidal field (3.10.227) of Trehan and Uberoi (1972) decreases the eigenvalues ω_t and ω_{zr}, while ω_s, ω_{zf} are increased. Opposite to the polytropes from Table 5.11.2 with a purely poloidal field, the magnetic shift always decreases with increasing n. The magnetic field vanishes on the surface.

Das and Tandon (1977a) include uniform rotation and determine with the second order virial equations the tesseral, sectorial, and zonal eigenvalues of $n = 1, 1.5, 2, 3$ magnetopolytropes with a general field vanishing on the surface. Especially the frequencies of tesseral and zonal modes decrease or increase in a complicated manner, depending on the interdependence of polytropic index, field strength, and rotation rate.

5.11.2 Variational Approach to the Oscillations of Spheroidal Magnetopolytropes

Repeating the transformations (2.6.51)-(2.6.55), the equation of motion (2.1.1) becomes for an inviscid fluid

$$\varrho\ Dv_i/Dt = -\partial P/\partial x_i + \varrho\ \partial\Phi/\partial x_i + (p/4\pi)[\partial(H_iH_j)/\partial x_j - (1/2)\ \partial H^2/\partial x_i], \qquad (i,j=1,2,3). \tag{5.11.53}$$

The first order Eulerian perturbation of this equation is (Kovetz 1966, Chandrasekhar 1981)

$$\varrho\ \partial^2\Delta x_i/\partial t^2 = -\sigma^2\varrho\ \Delta x_i = -\partial\delta P/\partial x_i + \delta\varrho\ \partial\Phi/\partial x_i + \varrho\ \partial\delta\Phi/\partial x_i$$
$$+(p/4\pi)[\partial(H_i\ \delta H_j + H_j\ \delta H_i)/\partial x_j - \partial(H_j\ \delta H_j)/\partial x_i], \tag{5.11.54}$$

where $Dv_i/Dt = D\Delta v_i/Dt = D^2\Delta x_i/Dt^2 \approx \partial^2\Delta x_i/\partial t^2$, by virtue of Eq. (5.1.24).

Up to now it was not necessary to consider boundary conditions on magnetopolytropes to some extent (cf. Sec. 3.10), but subsequently we remove this omission. First, the gravitational field and its normal derivative have to be continuous across the boundary surface S.

Second, the normal component of the magnetic induction vector is continuous across the boundary. This can be shown at once by integrating the Maxwell equation (3.10.1) $\nabla\cdot\vec{B}=0$ over the small volume V_ε of a thin shell element, having the small area S_ε. We apply the Gauss divergence theorem (e.g. Eq. (2.6.61), Bronstein and Semendjajew 1985):

$$\int_{V_\varepsilon}(\nabla\cdot\vec{B})\ dV = \int_{S_\varepsilon}(\vec{B}\cdot\vec{n})\ dS = 0. \tag{5.11.55}$$

When the thickness of the shell element approaches zero, its volume vanishes, and the surface integral reduces to an integral over the interior and exterior side of the small surface S_ε, becoming $(\vec{B}\cdot\vec{n}+\vec{B}_e\cdot\vec{n}_e)S_\varepsilon = 0$, where for the moment, e-indexed quantities denote exterior values. Since the two surface normals are obviously related by $\vec{n}=-\vec{n}_e$, we finally have (Stratton 1941, Parks 1991)

$$(\vec{B}-\vec{B}_e)\cdot\vec{n}=0 \quad \text{or} \quad (\vec{H}-\vec{H}_e)\cdot\vec{n}=0, \qquad (\vec{B}=p\vec{H}). \tag{5.11.56}$$

Third, the tangential component of the electric field is continuous across the boundary. This can be seen by integrating the Maxwell equation (3.10.1) $\nabla\times\vec{E}=-(1/c)\ \partial\vec{B}/\partial t$ over a vertical cross-section S_ε through a thin small shell element, and applying the Stokes theorem (e.g. Bronstein and Semendjajew 1985):

$$\int_{S_\varepsilon}[\nabla\times\vec{E}+(1/c)\ \partial\vec{B}/\partial t]\cdot\vec{n}\ dS = \int_{C_\varepsilon}\vec{E}\cdot\vec{\tau}\ d\ell + (1/c)\int_{S_\varepsilon}(\partial\vec{B}/\partial t)\cdot\vec{n}\ dS = 0. \tag{5.11.57}$$

When the thickness of the vertical cross-section tends to zero, its surface S_ε degenerates into its contour C_ε, the surface integrals becoming zero, while the curvilinear integral equals $(\vec{E}\cdot\vec{\tau}+\vec{E}_e\cdot\vec{\tau}_e)C_\varepsilon/2=0$. Since the two unit tangent vectors along the contour C_ε are related by $\vec{\tau}=-\vec{\tau}_e$, the final result is

$$(\vec{E}-\vec{E}_e)\cdot\vec{\tau}=0 \quad \text{or} \quad (\vec{E}-\vec{E}_e)\times\vec{n}=0. \tag{5.11.58}$$

The components P_{ij} of the material and magnetic stress tensor (2.6.81) can be expressed in terms of the components $P_i = P_{ij}n_j$ of three stress vectors acting on the three coordinate planes which are perpendicular to the components n_1, n_2, n_3 of the outer normal $\vec{n}$ (e.g. Roberts 1967, Tassoul 1978). P_{ij} is the component of the stress tensor acting along the coordinate direction x_i upon the coordinate plane belonging to the component n_j of the exterior normal $\vec{n}$.

And fourth, the stress vector P_i, acting on the boundary, has to be continuous across S [cf. Eq. (6.3.163)]:

$$P_i = P_{ij}n_j = [\delta_{ij}(P+pH^2/8\pi)+(p/4\pi)H_iH_j]n_j = (P+pH^2/8\pi)n_i + (p/4\pi)H_iH_jn_j$$
$$= (P_e+pH_e^2/8\pi)n_i + (p/4\pi)H_{ei}H_{ej}n_j, \quad \text{or}$$
$$(P+pH^2/8\pi)\ \vec{n} + (p/4\pi)(\vec{H}\cdot\vec{n})\ \vec{H} = (P_e+pH_e^2/8\pi)\ \vec{n} + (p/4\pi)(\vec{H}_e\cdot\vec{n})\ \vec{H}_e. \tag{5.11.59}$$

We decompose this continuity condition into components perpendicular and tangent to the boundary surface, by scalar multiplication with $\vec{n}$ and $\vec{\tau}$, taking into account that $\vec{n}\cdot\vec{n}=1,\ \vec{n}\cdot\vec{\tau}=0,\ \vec{H}\cdot\vec{n}=\vec{H}_e\cdot\vec{n}$ (Parks 1991):

$$P+pH^2/8\pi=P_e+pH_e^2/8\pi \quad \text{and} \quad (\vec{H}\cdot\vec{n})[(\vec{H}-\vec{H}_e)\cdot\vec{\tau}]=0. \tag{5.11.60}$$

Instead of the above tangential boundary condition we may derive an equivalent condition, by observing that the continuity of $P+pH^2/8\pi$ in Eqs. (5.11.59), (5.11.60) requires $(\vec{H}\cdot\vec{n})\ \vec{H}=(\vec{H}_e\cdot\vec{n})\ \vec{H}_e$, or via Eq. (5.11.56)

$$(\vec{H}\cdot\vec{n})(\vec{H}-\vec{H}_e)=0. \tag{5.11.61}$$

Either one of the factors $\vec{H}\cdot\vec{n}$ or $\vec{H}-\vec{H}_e$ must be zero, the boundary conditions (5.11.60) being always satisfied. If $\vec{H}-\vec{H}_e=0$, the boundary conditions (5.11.60) simplify to $H=H_e,\ P=P_e$ (Roberts 1967, Chap. 4). These boundary conditions will be substantiated further in Sec. 6.3.4, in connection with the magnetohydrodynamic case, when $\vec{v}\neq 0$.

While the gas pressure P_e outside the boundary S is generally identical zero, the magnetic field has to satisfy the vacuum equations [cf. Eq. (3.10.1)]

$$\nabla\cdot\vec{H}_e=0;\quad \nabla\times\vec{H}_e=0. \tag{5.11.62}$$

Since the boundary conditions (5.11.60) and (5.11.61) have to hold also for perturbed quantities at the perturbed surface, their Lagrangian variation has to vanish, and Eq. (5.11.54) has to be solved subject to the boundary conditions (e.g. Kovetz 1966, Chandrasekhar 1981)

$$\Delta[P-P_e+(p/8\pi)(H^2-H_e^2)]=0;\quad \Delta[(\vec{H}\cdot\vec{n})(\vec{H}-\vec{H}_e)]=0. \tag{5.11.63}$$

Let us consider two eigenvalues $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$ with the corresponding eigenfunctions $\Delta\vec{r}^{(\alpha)}=\Delta\vec{r}^{(\alpha)}(\Delta x_1^{(\alpha)},\Delta x_2^{(\alpha)},\Delta x_3^{(\alpha)})$ and $\Delta\vec{r}^{(\beta)}=\Delta\vec{r}^{(\beta)}(\Delta x_1^{(\beta)},\Delta x_2^{(\beta)},\Delta x_3^{(\beta)})$. We multiply Eq. (5.11.54), belonging to the eigenvalue $\sigma^{(\alpha)}$ by $\Delta x_i^{(\beta)}$, and integrate over the volume V occupied by the fluid (cf. Sec. 5.7.3):

$$[\sigma^{(\alpha)}]^2\int_V \varrho\ \Delta x_i^{(\alpha)}\ \Delta x_i^{(\beta)}\ dV=\int_V \Delta x_i^{(\beta)}\big\{\partial\delta P/\partial x_i-\delta\varrho\ \partial\Phi/\partial x_i-\varrho\ \partial\delta\Phi/\partial x_i+(p/4\pi)$$
$$\times[-\delta H_j\ \partial H_i/\partial x_j-H_j\ \partial\delta H_i/\partial x_j+\partial(H_j\ \delta H_j)/\partial x_i]\big\}\ dV,\qquad (\partial\delta H_j/\partial x_j=\delta(\partial H_j/\partial x_j)=0). \tag{5.11.64}$$

To get the more refined forms (5.11.70) and (5.11.74) of this equation, as used in many variational calculations (e.g. Singh and Tandon 1969, Trehan and Uberoi 1972, Sood and Trehan 1972a, Grover et al. 1973, Miketinac 1974), we replace at first $\delta\varrho$ from the equation of continuity (5.2.28)

$$\delta\varrho=-\Delta x_i\ \partial\varrho/\partial x_i-\varrho\ \partial\Delta x_i/\partial x_i, \tag{5.11.65}$$

and δP from Eq. (5.2.39)

$$\delta P=-\Delta x_i\ \partial P/\partial x_i-\Gamma_1 P\ \partial\Delta x_i/\partial x_i, \tag{5.11.66}$$

and integrate by parts the first and the two last terms on the right-hand side of Eq. (5.11.64), similarly to Eqs. (5.7.44)-(5.7.47):

$$[\sigma^{(\alpha)}]^2\int_V \varrho\ \Delta x_i^{(\alpha)}\ \Delta x_i^{(\beta)}\ dV=\int_S \Delta x_i^{(\beta)}[(\delta P+pH_j\ \delta H_j/4\pi)n_i-pH_j\ \delta H_i\ n_j/4\pi]\ dS$$
$$+\int_V\big\{\Delta x_i^{(\alpha)}(\partial\Delta x_j^{(\beta)}/\partial x_j)\ \partial P/\partial x_i+(\partial\Delta x_i^{(\alpha)}/\partial x_i)(\partial\Delta x_j^{(\beta)}/\partial x_j)\Gamma_1 P$$
$$+\Delta x_i^{(\alpha)}\Delta x_j^{(\beta)}(\partial\varrho/\partial x_i)\ \partial\Phi/\partial x_j+(\partial\Delta x_i^{(\alpha)}/\partial x_i)\ \Delta x_j^{(\beta)}\varrho\ \partial\Phi/\partial x_j-\Delta x_i^{(\beta)}\varrho\ \partial\delta\Phi/\partial x_i$$
$$+(p/4\pi)[-\Delta x_i^{(\beta)}\ \delta H_j\ \partial H_i/\partial x_j+(\partial\Delta x_i^{(\beta)}/\partial x_j)H_j\ \delta H_i-(\partial\Delta x_i^{(\beta)}/\partial x_i)H_j\ \delta H_j]\big\}\ dV. \tag{5.11.67}$$

We replace δH_i via Eq. (5.11.14), and the term $(\partial\Delta x_i^{(\alpha)}/\partial x_i)\ \Delta x_j^{(\beta)}\varrho\ \partial\Phi/\partial x_j$ via the equilibrium equation (5.11.53) if $Dv_i/Dt = 0$:

$$[\sigma^{(\alpha)}]^2 \int_V \varrho\ \Delta x_i^{(\alpha)}\ \Delta x_i^{(\beta)}\ dV = \int_S \Delta x_i^{(\beta)}[(\delta P + pH_j\ \delta H_j/4\pi)n_i - pH_j\ \delta H_i\ n_j/4\pi]\ dS$$
$$+\int_V \big\{[\Delta x_i^{(\alpha)}\ \partial\Delta x_j^{(\beta)}/\partial x_j + (\partial\Delta x_j^{(\alpha)}/\partial x_j)\ \Delta x_i^{(\beta)}]\ \partial P/\partial x_i + (\partial\Delta x_i^{(\alpha)}/\partial x_i)(\partial\Delta x_j^{(\beta)}/\partial x_j)\Gamma_1 P$$
$$+\Delta x_i^{(\alpha)}\ \Delta x_j^{(\beta)}(\partial\varrho/\partial x_i)\ \partial\Phi/\partial x_j - \Delta x_i^{(\beta)}\varrho\ \partial\delta\Phi/\partial x_i + (p/4\pi)\big[(\partial\Delta x_i^{(\alpha)}/\partial x_i)(\partial\Delta x_j^{(\beta)}/\partial x_j)H^2$$
$$+(1/2)(\Delta x_i^{(\beta)}\ \partial\Delta x_j^{(\alpha)}/\partial x_j + \Delta x_i^{(\alpha)}\ \partial\Delta x_j^{(\beta)}/\partial x_j)\ \partial H^2/\partial x_i$$
$$+(\partial\Delta x_i^{(\alpha)}/\partial x_j)H_j(\partial\Delta x_i^{(\beta)}/\partial x_k)H_k - \big((\partial\Delta x_j^{(\alpha)}/\partial x_i)\ \partial\Delta x_k^{(\beta)}/\partial x_k$$
$$+(\partial\Delta x_k^{(\alpha)}/\partial x_k)\ \partial\Delta x_j^{(\beta)}/\partial x_i\big)H_iH_j - (\partial\Delta x_i^{(\alpha)}/\partial x_j)\ \Delta x_k^{(\beta)}H_j\ \partial H_k/\partial x_i$$
$$-\Delta x_i^{(\alpha)}(\partial\Delta x_k^{(\beta)}/\partial x_j)H_j\ \partial H_k/\partial x_i + \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}(\partial H_j/\partial x_i)\ \partial H_k/\partial x_j\big]\big\}\ dV. \quad (5.11.68)$$

The three last terms can be brought into their final form by observing that

$$\int_V \big\{-H_j(\partial H_k/\partial x_i)[(\partial\Delta x_i^{(\alpha)}/\partial x_j)\ \Delta x_k^{(\beta)} + \Delta x_i^{(\alpha)}\ \partial\Delta x_k^{(\beta)}/\partial x_j]$$
$$+\Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}(\partial H_j/\partial x_i)\ \partial H_k/\partial x_j\big\}\ dV = -\int_S \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}H_j(\partial H_k/\partial x_i)n_j\ dS$$
$$+\int_V \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}[H_j\ \partial^2 H_k/\partial x_i\partial x_j + (\partial H_j/\partial x_i)\ \partial H_k/\partial x_j]\ dV$$
$$= -\int_S \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}H_j(\partial H_k/\partial x_i)n_j\ dS + \int_V \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}[\partial(H_j\ \partial H_k/\partial x_j)/\partial x_i]\ dV. \quad (5.11.69)$$

Replacing δH_i in the surface integral (5.11.68) via Eq. (5.11.14), we obtain the final form of the variational principle (e.g. Singh and Tandon 1969, Sood and Trehan 1972a, Chandrasekhar 1981):

$$[\sigma^{(\alpha)}]^2 \int_V \varrho\ \Delta x_i^{(\alpha)}\ \Delta x_i^{(\beta)}\ dV = \int_S \Delta x_i^{(\beta)}[\delta(P + pH^2/8\pi)n_i + (p\vec{H}\cdot\vec{n}/4\pi)(H_i\ \partial\Delta x_k^{(\alpha)}/\partial x_k$$
$$-H_k\ \partial\Delta x_i^{(\alpha)}/\partial x_k)]\ dS + \int_V \big\{[\Delta x_i^{(\alpha)}\ \partial\Delta x_j^{(\beta)}/\partial x_j + (\partial\Delta x_j^{(\alpha)}/\partial x_j)\ \Delta x_i^{(\beta)}]\ \partial P/\partial x_i$$
$$+(\partial\Delta x_i^{(\alpha)}/\partial x_i)(\partial\Delta x_j^{(\beta)}/\partial x_j)\Gamma_1 P + \Delta x_i^{(\alpha)}\ \Delta x_j^{(\beta)}(\partial\varrho/\partial x_i)\ \partial\Phi/\partial x_j$$
$$-\Delta x_i^{(\beta)}\varrho\ \partial\delta\Phi/\partial x_i + (p/4\pi)\big[(\partial\Delta x_i^{(\alpha)}/\partial x_i)(\partial\Delta x_j^{(\beta)}/\partial x_j)H^2$$
$$+(1/2)(\Delta x_i^{(\beta)}\ \partial\Delta x_j^{(\alpha)}/\partial x_j + \Delta x_i^{(\alpha)}\ \partial\Delta x_j^{(\beta)}/\partial x_j)\ \partial H^2/\partial x_i$$
$$+(\partial\Delta x_i^{(\alpha)}/\partial x_j)H_j(\partial\Delta x_i^{(\beta)}/\partial x_k)H_k - \big((\partial\Delta x_j^{(\alpha)}/\partial x_i)\ \partial\Delta x_k^{(\beta)}/\partial x_k$$
$$+(\partial\Delta x_k^{(\alpha)}/\partial x_k)\ \partial\Delta x_j^{(\beta)}/\partial x_i\big)H_iH_j + \Delta x_i^{(\alpha)}\ \Delta x_k^{(\beta)}\ \partial(H_j\ \partial H_k/\partial x_j)/\partial x_i\big]\big\}\ dV. \quad (5.11.70)$$

According to the definition (5.1.13) of the Eulerian perturbation, and via Eqs. (5.2.28), (5.8.1)-(5.8.4), (5.8.8), we can write (e.g. Cox 1980)

$$\delta\Phi(\vec{r}) = \Phi(\vec{r}) - \Phi_u(\vec{r}) = G\int_V \varrho(\vec{r'})\ dV'/|\vec{r}-\vec{r'}| - G\int_{V_u} \varrho_u(\vec{r'_u})\ dV'_u/|\vec{r}-\vec{r'_u}|$$
$$= \delta^* \int_V G\varrho(\vec{r'})\ dV'/|\vec{r}-\vec{r'}| = G\int_V \varrho(\vec{r'})\ \Delta x_j'^{(\alpha)}[\partial(1/|\vec{r}-\vec{r'}|)/\partial x_j']\ dV'$$
$$= G\int_S \varrho(\vec{r'})\ \Delta x_j'^{(\alpha)} n_j'\ dS'/|\vec{r}-\vec{r'}| - G\int_V \{\partial[\varrho(\vec{r'})\ \Delta x_j'^{(\alpha)}]/\partial x_j'\}\ dV'/|\vec{r}-\vec{r'}|$$
$$= G\int_S \varrho(\vec{r'})\ \Delta x_j'^{(\alpha)} n_j'\ dS'/|\vec{r}-\vec{r'}| + G\int_V \delta\varrho(\vec{r'})\ dV'/|\vec{r}-\vec{r'}|. \quad (5.11.71)$$

The term in Eq. (5.11.70) containing $\partial\delta\Phi/\partial x_i$ takes the symmetrical form

$$\int_V \Delta x_i^{(\beta)}\varrho(\vec{r})\ [\partial\delta\Phi(\vec{r})/\partial x_i]\ dV = G\int_V\int_V \Delta x_i^{(\beta)}\ \Delta x_j'^{(\alpha)}\varrho(\vec{r})\ \varrho(\vec{r'})\ [\partial^2(1/|\vec{r}-\vec{r'}|)/\partial x_i\partial x_j']\ dV\ dV'$$
$$= G\int_V\int_V \{\partial[\varrho(\vec{r})\ \Delta x_i^{(\beta)}]/\partial x_i\}\{\partial[\varrho(\vec{r'})\ \Delta x_j'^{(\alpha)}]/\partial x_j'\}\ dV\ dV'/|\vec{r}-\vec{r'}|$$
$$= G\int_V\int_V \delta\varrho(\vec{r})\ \delta\varrho(\vec{r'})\ dV\ dV'/|\vec{r}-\vec{r'}|, \tag{5.11.72}$$

via Eqs. (5.2.28), (5.11.71), and after integrating by parts. Note, that the two last equalities are only valid if the density vanishes on the surface.

We write down Eq. (5.11.70) with vanishing surface integral for the eigenvalue $\sigma^{(\beta)}$ and take the difference of these two equations. The right-hand side of this difference vanishes, as it is symmetrical in the indices α and β, and therefore [cf. Eq. (5.7.46)]

$$\{[\sigma^{(\alpha)}]^2-[\sigma^{(\beta)}]^2\}\int_V \varrho\ \Delta x_i^{(\alpha)}\ \Delta x_i^{(\beta)}\ dV = 0 \quad \text{if} \quad \alpha\neq\beta, \tag{5.11.73}$$

establishing the self-adjoint character of the problem (Chandrasekhar 1981). If $\alpha=\beta$, we may replace $\Delta x_i^{(\beta)}$ by the complex conjugate $\overline{\Delta x_i^{(\alpha)}}$ of $\Delta x_i^{(\alpha)}$, and observe that the imaginary parts in Eq. (5.11.70) vanish, so σ^2 is always real. A necessary and sufficient condition for stability is that $\sigma^2>0$.

The surface integral in Eq. (5.11.70) vanishes if $\delta(P+pH^2/4\pi)=0$ and $\vec{n}\cdot\vec{H}=0$ on S, for instance. In this case Eq. (5.11.70) becomes, by taking $\Delta x_i^{(\beta)}=\Delta x_i^{(\alpha)}$ and dropping the superscript (α), (Singh and Tandon 1969, Trehan and Uberoi 1972, Miketinac 1974):

$$\sigma^2\int_V \varrho\ |\Delta\vec{r}|^2\ dV = \int_V \big\{2(\nabla\cdot\Delta\vec{r})\ \Delta\vec{r}\cdot\nabla P+\Gamma_1 P(\nabla\cdot\Delta\vec{r})^2+(\Delta\vec{r}\cdot\nabla\varrho)\ \Delta\vec{r}\cdot\nabla\Phi$$
$$-\varrho\ \Delta\vec{r}\cdot\nabla\delta\Phi+(p/4\pi)[(\nabla\cdot\Delta\vec{r})^2H^2+(\nabla\cdot\Delta\vec{r})\ \Delta\vec{r}\cdot\nabla H^2+(H_j\ \partial\Delta x_i/\partial x_j)^2$$
$$-2(\nabla\cdot\Delta\vec{r})(\partial\Delta x_j/\partial x_i)H_iH_j+\Delta x_i\ \Delta x_k\ \partial(H_j\ \partial H_k/\partial x_j)/\partial x_i]\big\}\ dV. \tag{5.11.74}$$

Magnetopolytropic calculations obtained with the variational method are more exact in comparison to the virial method, but otherwise yield similar results. Eq. (5.11.74) has been used to study the axisymmetric oscillations of polytropes having $n=0,\ j=2$ (Singh and Tandon 1969), and $n=1,\ j=1,2,3$ (Grover et al. 1973) with a *poloidal and toroidal* magnetic field of the form (3.10.225), (3.10.227). Trial Lagrangian displacements are taken under a form similar to Eq. (5.7.83). The stable p and f-modes are "stabilized" by the considered field, in the sense that their eigenvalues σ^2, $(\sigma^2>0)$ are increased. In absence of magnetic fields, and for the considered values of Γ_1 and $n=0,1$, all g-modes are unstable (see Tables 5.5.1, 5.5.2). If $n=1$, the negative eigenvalues σ^2 of the principal unstable g_1-mode are decreased, destabilizing even more the magnetopolytrope. On the other hand, positive magnetic corrections are acquired by the g_1-mode if $n=0,\ j=2$, and the g_2-mode if $n=1$. Grover et al. (1973) have also found a stable, purely hydromagnetic mode if $n=1,\ j=1$, which becomes neutral $(\sigma=0)$ for a vanishing magnetic field.

Comprehensive studies based on Eq. (5.11.74) have been effected by Sood and Trehan (1972a, b) concerning quasiradial and nonradial axisymmetric oscillations with a *toroidal* field of the form (3.10.98). Calculations have been done if $1\leq n\leq 3.5$ for deformations associated with the Legendre polynomials $P_1(\mu),P_2(\mu),P_3(\mu)$, i.e. for latitudinal harmonic indices $j=1,2,3$, and for trial displacements of the form (5.7.83). The squared eigenfrequencies of $p_1(j=1,2,3)$, $f(j=1,2,3)$, and $g_1(j=1)$-modes are always decreased by the presence of the toroidal magnetic field – the configuration is "destabilized". If $n<1.5$, $(\Gamma_1=5/3)$, the unstable $g_1(j=2,3)$-modes become less unstable with positive magnetic corrections to σ^2 (Table 5.11.3).

The same approach has been implemented by Fahlman (1971) to investigate the influence of the poloidal field (3.10.202) on the radial $r(j=0)$ and the fundamental (Kelvin) $f(j=2)$-mode. In fact, the r and f-modes investigated by Fahlman (1971) correspond in his first approximation just to the zonal eigenfrequencies ω_{zr} and ω_{zf} already quoted in Table 5.11.2. Goossens' (1977) magnetic corrections of these modes for a $n=3$ polytrope are not in agreement with those from Table 5.11.2 and Fahlman (1971).

Trehan and Uberoi (1972) have examined the fundamental quasiradial $r(j=0)$-mode of magnetopolytropes, pervaded by a combined toroidal and poloidal field of the form (3.10.225), (3.10.227), while

Table 5.11.3 Dimensionless squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ for magnetopolytropes with the toroidal field (3.10.98) $H_\varphi = C\varrho r \sin\lambda,\ h = C^2/16\pi^2 G$ for axisymmetric r, p_1, f, g_1-modes ($j = 0, 2;\ \Gamma_1 = 5/3$). Numbers in parentheses denote powers of 10, the magnetic corrections of Sood and Trehan (1972b) being multiplied with $4/(n+1)$, (cf. Tables 5.3.1, 5.5.2 if $h = 0$).

n	$r(j=0)$	$f(j=2)$	$p_1(j=2)$	$g_1(j=2)$
1	$1.92(-1) - 5.07(-1)h$	$1.52(-1) - 2.78(-1)h$	$9.79(-1) - 2.44h$	$-2.85(-2) + 3.76(-2)h$
1.5	$1.51(-1) - 4.92(-1)h$	$1.18(-1) - 3.01(-1)h$	$6.29(-1) - 1.81h$	$5.28(-3)h$
2	$1.17(-1) - 4.53(-1)h$	$9.11(-2) - 2.89(-1)h$	$4.09(-1) - 1.27h$	$1.32(-2) - 3.22(-2)h$
3	$6.02(-2) - 3.03(-1)h$	$4.93(-2) - 2.15(-1)h$	$1.67(-1) - 5.52(-1)h$	$1.46(-2) - 5.65(-2)h$
3.5	$3.45(-2) - 1.88(-1)h$	$3.12(-2) - 1.56(-1)h$	$1.03(-1) - 3.46(-1)h$	$1.01(-2) - 4.43(-2)h$

Table 5.11.4 Adiabatic index Γ_{1d} at which accidental degeneracy occurs between the radial $r(j = 0)$ and the fundamental Kelvin $f(j = 2)$-mode (cf. Table 5.7.2, Chandrasekhar and Lebovitz 1964), as well as the corresponding split squared eigenfrequencies $\omega^2 = \sigma^2/4\pi G\varrho_0$ are shown for the toroidal magnetic field (3.10.98), [Sood and Trehan 1972b; magnetic corrections are multiplied with $4/(n + 1)$].

n	Γ_{1d}	$r(j=0)$	$f(j=2)$
1	1.596	$0.152 + 2.43h$	$0.152 - 1.75h$
1.5	1.592	$0.117 + 1.73h$	$0.117 - 1.05h$
2	1.586	$0.0900 + 1.27h$	$0.0900 - 0.640h$
3	1.572	$0.0462 + 0.581h$	$0.0462 - 0.147h$

Billings et al. (1973) calculate the eigenfrequencies of the fundamental (Kelvin) mode $f(j = 2)$. Their stable squared eigenfrequencies are decreased for the $r(j = 0)$-mode, and increased for the $f(j = 2)$-mode.

The splitting of eigenfrequencies occurring at accidental degeneracy of the nonmagnetic modes has been calculated by Sood and Trehan (1972b) for the toroidal field (3.10.98), (Table 5.11.4), and by Fahlman (1971) for the poloidal field (3.10.202), in the same way as in Eqs. (5.7.75)-(5.7.80) for the rotational case. The values of Γ_{1d} at which degeneracy occurs have been taken from Chandrasekhar and Lebovitz (1964). These values agree for the considered polytropic indices with the more exact evaluations of Hurley et al. (1966) from Table 5.7.2. These authors were not able to locate values of astrophysical interest for Γ_{1d} if $n > 3.25$; therefore the entries in Table 5.11.4 are limited to $1 \le n \le 3$.

Using trial displacements of the form (5.7.83)

$$u(r) = r^2\ \Delta r(r) = ar^{j+1} + br^{j+3};\quad w(r) = ar^{j+1} + cr^{j+3},\qquad (j = 1, 2, 3, ...;\ a, b, c = \mathrm{const}), \tag{5.11.75}$$

for each latitudinal harmonic index j, the variational principle (5.11.74) assumes the form (Miketinac 1974)

$$\sigma^2(A_{11}a^2 + 2A_{12}ab + 2A_{13}ac + A_{22}b^2 + 2A_{13}bc + A_{33}c^2)$$
$$= B_{11}a^2 + 2B_{12}ab + 2B_{13}ac + B_{22}b^2 + 2B_{23}bc + B_{33}c^2, \tag{5.11.76}$$

with the known constants $A_{ik}, B_{ik},\ (A_{ik} = A_{ki};\ B_{ik} = B_{ki};\ i, k = 1, 2, 3)$.

The eigenvalues must be constant with respect to variations $\Delta a, \Delta b, \Delta c$ of the variational parameters a, b, c. Changing successively only one of these parameters, we get three homogeneous equations with the unknowns a, b, c, and for nontrivial solutions the determinant of this homogeneous system has to vanish:

$$\begin{vmatrix} \sigma^2 A_{11} - B_{11} & \sigma^2 A_{12} - B_{12} & \sigma^2 A_{13} - B_{13} \\ \sigma^2 A_{12} - B_{12} & \sigma^2 A_{22} - B_{22} & \sigma^2 A_{23} - B_{23} \\ \sigma^2 A_{13} - B_{13} & \sigma^2 A_{23} - B_{23} & \sigma^2 A_{33} - B_{33} \end{vmatrix} = 0. \tag{5.11.77}$$

This cubic equation in σ^2 yields for each harmonic index j the eigenvalues of pressure (acoustic) p-modes, fundamental (Kelvin) f-modes ($j \ge 2$), and gravity (convective) g-modes. Miketinac (1974) has calculated these modes for $n = 1.5, 3,\ (j = 1, 2, 3;\ \Gamma_1 = 5/3)$, and a strong toroidal field (3.10.98). The run of the eigenfrequencies for a weak field is similar to the results of Sood and Trehan (1972b), excepting for some g-modes if $j = 1, 2$. In Miketinac's (1974) calculations all squared eigenfrequencies decrease with increasing field strength, excepting for most g-modes, which increase at first and afterwards decrease, becoming unstable at maximum considered field strength.

Nonlinear radial oscillations in *spherical* magnetopolytropes with the toroidal field (3.10.98) have been considered by Das et al. (1994).

The oscillations of force-free $n = 1, 2$ magnetopolytropes have been studied by Nasiri and Sobouti (1989). In this case $\vec{B} \propto \nabla \times \vec{B}$, and the magnetic force term in Eq. (3.10.9) vanishes ($B = pH$). The magnetopolytrope remains spherical, and besides the familiar p and g-modes there appear toroidal t-modes, representing mainly standing hydromagnetic waves.

5.11.3 Hydromagnetic Instabilities in Toroidal Fields

As already stressed at the beginning, these local instabilities cannot be detected by the low order virial and variational techniques ($j = 1, 2, 3$) presented in previous subsections. The stability of these local hydromagnetic oscillations is generally investigated by using the energy principle already outlined subsequently to Eq. (5.8.149), including in the total energy E also the magnetic energy U_m (e.g. Roberts 1967, Chap. 8):

$$E = W + U + U_m + U_{me}. \tag{5.11.78}$$

W denotes the gravitational energy, and U the internal energy (2.6.95). U_m and U_{me} is the magnetic energy (2.6.78) of the internal and external magnetic field, respectively. If a Lagrangian displacement $\Delta\vec{r}$ exists, for which the corresponding variation $\delta^* E(\Delta\vec{r})$ is negative, we say the system is unstable. To settle stability, E has to be expanded to second order in $\Delta\vec{r}$, because first order terms vanish for equilibrium configurations. We quote only the final result, composed of two volume integrals over the internal and external domain V and V_e, which are separated by the surface S, over which the third integral is extended (Goossens and Veugelen 1978):

$$\delta^* E = (1/2) \int_V \big[(p/4\pi)(\delta\vec{H})^2 - (p/4\pi)(\nabla \times \vec{H}) \cdot (\delta\vec{H} \times \Delta\vec{r}) + \Gamma_1 P (\nabla \cdot \Delta\vec{r})^2$$
$$+ (\nabla \cdot \Delta\vec{r})\ \Delta\vec{r} \cdot \nabla P + \nabla \cdot (\varrho\ \Delta\vec{r})(\Delta\vec{r} \cdot \nabla\Phi) - \varrho\ \Delta\vec{r} \cdot \nabla\delta\Phi\big]\ dV + (p/8\pi) \int_{V_e} (\delta H_e)^2\ dV + (1/2) \times$$
$$\int_S \big\{ -\vec{n} \cdot \nabla[P - P_e + (p/8\pi)(H^2 - H_e^2)](\Delta\vec{r} \cdot \vec{n})^2 + (p/4\pi)(\vec{H} \cdot \vec{n})\ \Delta r_i\ \Delta r_j\ \partial(H_i - H_{ei})/\partial x_j \big\}\ dS. \tag{5.11.79}$$

The nonaxisymmetric Lagrangian displacements of the infinitely conducting medium are taken by Goossens and Veugelen (1978) in spherical coordinates equal to

$$\Delta r = R(r, \lambda) \exp(ik\varphi); \quad r\ \Delta\lambda = S(r, \lambda) \exp(ik\varphi); \quad r \sin\lambda\ \Delta\varphi = iT(r, \lambda) \exp(ik\varphi), \tag{5.11.80}$$

assuming some special trial functions for R, S, T. The equilibrium model with the toroidal field (3.10.98), ($h = 2.5 \times 10^{-4}, 1.25 \times 10^{-4}$) is quite similar to Sinha's (1968a) model already presented in Eqs. (3.10.98)-(3.10.137). The evaluations are lengthy and complicated, and will not be reproduced. For the polytrope $n = 3$ the only unstable oscillations are associated with the azimuthal number $k = \pm 1$, and are localized in the immediate vicinity of the centre of the polytrope, revealing the local nature of these hydromagnetic instabilities. The most violent instabilities are characterized by small vertical wavelengths for perturbations near the z-axis, and by small horizontal wavelengths for other perturbations. For typical *Ap* stars the e-folding time of the instabilities is of order 50 to 500 days.

5.11.4 Stability and Oscillations of Cylindrical Magnetopolytropes

The radial oscillations of an axially symmetric, infinitely conducting cylinder, pervaded by an axial magnetic field $\vec{H} = \vec{H}[0, 0, H_z(\ell)]$ have been investigated by Chandrasekhar and Fermi (1953) in a similar manner as those of the nonmagnetic sphere by Ledoux (1945; see also Sec. 5.3.1). The radial oscillations are considered in Lagrangian description, i.e. for the same mass element. The equivalent forms of the equation of continuity (5.2.42) and of motion (5.2.46) are devised in cylindrical (ℓ, φ, z)-coordinates by

$$\partial\ell/\partial\ell_i = \ell_i\varrho_i/\ell\varrho; \quad \partial\ell^2/\partial m = 1/\pi\varrho, \tag{5.11.81}$$

$$\begin{aligned}\partial^2\ell/\partial t^2 &= [-(1/\varrho)\ \partial P/\partial\ell_i - (p/8\pi\varrho)\ \partial H^2/\partial\ell_i + \partial\Phi/\partial\ell_i]\ \partial\ell_i/\partial\ell \\ &= -2\pi\ell[\partial P/\partial m + (p/8\pi)\ \partial H^2/\partial m] - 2Gm(\ell)/\ell, \qquad (H_z = H),\end{aligned} \tag{5.11.82}$$

where via Eq. (2.6.30) $F = \partial\Phi/\partial\ell = -2Gm(\ell)/\ell$. Also, in virtue of Eqs. (2.1.1), (B.47): $(\nabla \times \vec{H}) \times \vec{H} = -H(dH/d\ell)\ \vec{e}_\ell = -(1/2)(dH^2/d\ell)\ \vec{e}_\ell$. The equilibrium equation is obtained from Eq. (5.11.82) if $\partial^2\ell/\partial t^2 = 0$. Eqs. (5.11.81), (5.11.82) become with the Lagrangian variations $\ell = \ell_u + \Delta\ell,\ P = P_u + \Delta P,\ \varrho = \varrho_u + \Delta\varrho,\ H_z = H = H_u + \Delta H$ equal to

$$\partial(\ell\ \Delta\ell)/\partial m = -\Delta\varrho/2\pi\varrho^2, \tag{5.11.83}$$

$$\begin{aligned}\partial^2\Delta\ell/\partial t^2 &= -2\pi\ \Delta\ell[\partial P/\partial m + (p/8\pi)\ \partial H^2/\partial m] - 2\pi\ell[\partial\Delta P/\partial m + (p/4\pi)\ \partial(H\ \Delta H)/\partial m] \\ &+2Gm(\ell)\ \Delta\ell/\ell^2 = -2\pi\ell\ \partial[\Delta P + (p/4\pi)H\ \Delta H]/\partial m + 4Gm(\ell)\ \Delta\ell/\ell^2.\end{aligned} \tag{5.11.84}$$

We insert for $\Delta P = -(\Gamma_1 P/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell$ via the adiabatic relationship (5.2.38), and for $\Delta H = -(1/\ell)\ \partial(\ell H\ \Delta\ell)/\partial\ell + \Delta\ell\ \partial H/\partial\ell = -(H/\ell)\ \partial(\ell\ \Delta\ell)/\partial\ell$ via Eqs. (5.11.16), (B.47):

$$\partial^2\Delta\ell/\partial t^2 = 4\pi^2\ell\ \partial[\varrho(\Gamma_1 P + pH^2/4\pi)\ \partial(\ell\ \Delta\ell)/\partial m]/\partial m + 4Gm\ \Delta\ell/\ell^2. \tag{5.11.85}$$

If $\Delta\ell(\ell, t) = \Delta\ell(\ell)\exp(i\sigma t)$, Eq. (5.11.85) becomes, by suppressing the factor $\exp(i\sigma t)$:

$$(\sigma^2 + 4Gm/\ell^2)\ \Delta\ell = -4\pi^2\ell\ d[\varrho(\Gamma_1 P + pH^2/4\pi)\ d(\ell\ \Delta\ell)/dm]/dm. \tag{5.11.86}$$

The boundary conditions (5.2.61) $\Delta\ell(0) = 0$, and (5.11.63) $\Delta[P - P_e + (p/8\pi)(H^2 - H_e^2)] = 0$ at $\ell = \ell_1$, in conjunction with Eq. (5.11.86), will determine a sequence of distinct eigenvalues, the eigenfunctions $\Delta\ell^{(\alpha)}, \Delta\ell^{(\beta)}$ belonging to different eigenvalues $\sigma^{(\alpha)}, \sigma^{(\beta)}$ being orthogonal. This can be shown by writing Eq. (5.11.86) twice for the eigenvalues $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$, multiplying by $\Delta\ell^{(\beta)}$ and $\Delta\ell^{(\alpha)}$, respectively, and integrating the difference of the products over the mass per unit length of the cylinder:

$$\begin{aligned}&\{[\sigma^{(\alpha)}]^2 - [\sigma^{(\beta)}]^2\}\int_0^{M_1}\Delta\ell^{(\alpha)}\ \Delta\ell^{(\beta)}\ dm = -4\pi^2\int_0^{M_1}\{\ell\ \Delta\ell^{(\beta)}\ d[\varrho(\Gamma_1 P + pH^2/4\pi) \\ &\times d(\ell\ \Delta\ell^{(\alpha)})/dm] - \ell\ \Delta\ell^{(\alpha)}\ d[\varrho(\Gamma_1 P + pH^2/4\pi)\ d(\ell\ \Delta\ell^{(\beta)})/dm]\} = 0, \qquad (\alpha \neq \beta).\end{aligned} \tag{5.11.87}$$

To obtain this result we have integrated by parts, taking into account that $\Delta\ell^{(\alpha)}(0), \Delta\ell^{(\beta)}(0) = 0$, and $P(M_1), H(M_1) = H_e = 0$, since the total hydrostatic pressure $P + pH^2/8\pi$ – including the magnetic pressure $pH^2/8\pi$ – is assumed to vanish on the cylindrical surface [cf. Eqs. (5.7.43)-(5.7.46)].

In view of the orthogonality property (5.11.87), the eigenvalues can be determined by a variational approach [cf. Eq. (5.11.74)]. We multiply Eq. (5.11.86) by $\Delta\ell$, and integrate over the mass per unit length:

$$\begin{aligned}&\int_0^{M_1}(\sigma^2 + 4Gm/\ell^2)\ (\Delta\ell)^2\ dm = -4\pi^2\int_0^{M_1}\ell\ \Delta\ell\ d\big[\varrho(\Gamma_1 P + pH^2/4\pi)\ d(\ell\ \Delta\ell)/dm\big] \\ &= -2\pi^2\varrho(\Gamma_1 P + pH^2/4\pi)\ d(\ell\ \Delta\ell)^2/dm\Big|_0^{M_1} + 4\pi^2\int_0^{M_1}\varrho(\Gamma_1 P + pH^2/4\pi)\ [d(\ell\ \Delta\ell)/dm]^2\ dm,\end{aligned} \tag{5.11.88}$$

or

$$\sigma^2\int_0^{M_1}(\Delta\ell)^2\ dm = \int_0^{M_1}\{[(\Gamma_1 P + pH^2/4\pi)/\varrho\ell^2][d(\ell\ \Delta\ell)/d\ell]^2 - (4Gm/\ell^2)\ (\Delta\ell)^2\}\ dm. \tag{5.11.89}$$

As outlined subsequently to Eq. (5.3.16), the trial Lagrangian displacement $\Delta\ell/\ell=$ const yields a fairly good approximation for the fundamental radial eigenfrequency:

$$\sigma^2\int_0^{M_1}\ell^2\ dm=8\pi\int_0^{\ell_1}(\Gamma_1P+pH^2/4\pi)\ell\ d\ell-2GM_1^2=-4\pi\int_0^{\ell_1}\ell^2\ d(\Gamma_1P+pH^2/4\pi)-2GM_1^2$$
$$=p(\Gamma_1/2-1)\int_0^{\ell_1}\ell^2\ dH^2+2(\Gamma_1-1)GM_1^2=p(2-\Gamma_1)\int_0^{\ell_1}H^2\ell\ d\ell+2(\Gamma_1-1)GM_1^2$$
$$=4(2-\Gamma_1)U_m+2(\Gamma_1-1)GM_1^2=4[(\Gamma_1-1)^2U+U_m]. \tag{5.11.90}$$

To derive this equation, we have considered the hydromagnetic equilibrium equation (5.11.82) $dP=-2Gm\varrho\ d\ell/\ell-p\ dH^2/8\pi,\ (\partial^2\ell/\partial t^2=0)$, integrating by parts, and inserting from Eqs. (5.11.3), (5.11.5). Since $U,U_m>0$, the infinitely long cylinder pervaded by a longitudinal magnetic field of the form $H=H_z(\ell)$ is stable against radial perturbations.

Other studies concerning stability and oscillations of cylindrical magnetopolytropes seem to have been confined to the quite particular cases of homogeneous ($n=0$) and isothermal ($n=\pm\infty$) cylinders.

Concerning homogeneous cylinders Chandrasekhar and Fermi (1953) have investigated the axisymmetric oscillations of an incompressible cylinder pervaded by a constant axial field ($n=0;\ k=0;\ \Gamma_1=\infty;\ H_z=H=$ const), while Simon (1958) and Tassoul (1963) have extended the problem to general oscillations ($j,k\neq0$), and to uniform rotation, respectively.

Taking into account that $\delta[\vec\Omega\times(\vec\Omega\times\vec r)]=0$, the Eulerian variation of the hydromagnetic equation of motion (3.1.12) becomes in a frame rotating uniformly with the cylinder ($v_{tr},\tau=0;\ p=1$) :

$$\partial^2\Delta\vec r/\partial t^2+2\vec\Omega\times(\partial\Delta\vec r/\partial t)=\nabla(\delta\Phi-\delta P/\varrho)+(1/4\pi\varrho)(\nabla\times\delta\vec H)\times\vec H,$$
$$(n=0;\ \Gamma_1=\infty;\ \vec\Omega=\vec\Omega(0,0,\Omega);\ \vec H=\vec H(0,0,H);\ \delta\varrho=0;\ \Omega,\varrho,H,=\mathrm{const}). \tag{5.11.91}$$

For Eulerian perturbations of the form (5.9.8), (5.9.40), (5.11.14) – after suppressing the common factor $\exp[i(\sigma t+k\varphi+jz)]$ – this equation becomes

$$\sigma^2\ \Delta\vec r-2i\sigma\vec\Omega\times\Delta\vec r=\nabla\chi-(1/4\pi\varrho)\ \nabla\times[\nabla\times(\Delta\vec r\times\vec H)]\times\vec H,\qquad(\chi=\delta P/\varrho-\delta\Phi). \tag{5.11.92}$$

The continuity equation (5.2.28) reads $\nabla\cdot\Delta\vec r=0,\ (\Gamma_1=\infty;\ \delta\varrho=0)$, or explicitly [cf. Eqs. (5.9.40), (B.46)]:

$$d\Delta\ell/d\ell+\Delta\ell/\ell+ik\ \Delta\varphi+ij\ \Delta z=0. \tag{5.11.93}$$

The Eulerian perturbation (5.11.14) of the constant axial field writes with the help of Eqs. (5.11.93), (B.47) as

$$\delta\vec H(\ell)=\nabla\times(\Delta\vec r\times\vec H)=ijH\ \Delta\vec r(\ell), \tag{5.11.94}$$

and the perturbed equation of motion (5.11.92) becomes eventually (Tassoul 1963)

$$\sigma^2\ \Delta\ell+2i\sigma\Omega\ell\ \Delta\varphi=d\chi/d\ell+(H^2/4\pi\varrho)(j^2\ \Delta\ell+ij\ d\Delta z/d\ell), \tag{5.11.95}$$

$$\sigma^2\ell\ \Delta\varphi-2i\sigma\Omega\ \Delta\ell=ik\chi/\ell+(H^2/4\pi\varrho)(j^2\ell\ \Delta\varphi-jk\ \Delta z/\ell), \tag{5.11.96}$$

$$\sigma^2\ \Delta z=ij\chi. \tag{5.11.97}$$

Δz and its derivative can be eliminated at once among Eqs. (5.11.95)-(5.11.97):

$$\sigma^2[(\sigma^2-j^2v_B^2)\ \Delta\ell+2i\sigma\Omega\ell\ \Delta\varphi]=(\sigma^2-j^2v_B^2)\ d\chi/d\ell, \tag{5.11.98}$$

$$\sigma^2[(\sigma^2-j^2v_B^2)\ell\ \Delta\varphi-2i\sigma\Omega\ \Delta\ell]=ik(\sigma^2-j^2v_B^2)\chi/\ell, \tag{5.11.99}$$

where we have introduced the Alfvén velocity (3.10.254) if $p=1:\ \ v_B=H/(4\pi\varrho)^{1/2}$. We ignore the trivial eigenvalue $\sigma^2=j^2v_B^2$, which implies $\Delta\ell,\Delta\varphi=0$, and $\Delta z(\ell)=i\chi(\ell)/jv_B^2$.

If $\sigma^2 \neq j^2 v_B^2$, we get for $\Delta\ell$ and $\Delta\varphi$, by solving the system (5.11.98)-(5.11.99):

$$\sigma^2(1-\alpha^2)\ \Delta\ell = d\chi/d\ell + k\alpha\chi/\ell, \tag{5.11.100}$$

$$\sigma^2(1-\alpha^2)\ell\ \Delta\varphi = i(k\chi/\ell + \alpha\ d\chi/d\ell), \tag{5.11.101}$$

with the notation

$$\alpha = 2\sigma\Omega/(\sigma^2 - j^2 v_B^2). \tag{5.11.102}$$

We insert Eqs. (5.11.97), (5.11.100), (5.11.101) into the continuity equation (5.11.93), observing that χ satisfies just the modified Bessel equation

$$d^2\chi/d\ell^2 + (1/\ell)\ d\chi/d\ell - (\kappa^2 + k^2/\ell^2)\chi = 0, \qquad (\kappa \neq 0), \tag{5.11.103}$$

where

$$\kappa^2 = j^2[1 - 4\sigma^2\Omega^2/(\sigma^2 - j^2 v_B^2)^2] = j^2(1-\alpha^2). \tag{5.11.104}$$

The solution of Eq. (5.11.103) is given by the modified Bessel functions $I_k(\kappa\ell)$ and $K_k(\kappa\ell)$, $(\kappa \neq 0)$, of the first and second kind, respectively [cf. Eq. (5.6.3)]. Since the azimuthal number k is an integer, we have $I_k = I_{-k},\ K_k = K_{-k}$ (e.g. Spiegel 1968), so the notations $I_{|k|}, K_{|k|}$ are superfluous (Simon 1958, Tassoul 1963).

The Eulerian perturbation of Poisson's equation (5.2.40) is

$$\nabla^2\delta\Phi = d^2\delta\Phi/d\ell^2 + (1/\ell)\ d\delta\Phi/d\ell - (j^2 + k^2/\ell^2)\ \delta\Phi = 0, \qquad (\delta\varrho = 0), \tag{5.11.105}$$

and the solutions of this modified Bessel equation are given by $I_k(j\ell)$ and $K_k(j\ell)$.

The functions $\chi, \delta\Phi$ must be independent of $K_k(\kappa\ell), K_k(j\ell)$, respectively, in order to assure the finiteness of the Eulerian perturbations along the axis $\ell = 0$:

$$\chi(\ell) = C_1 I_k(\kappa\ell);\quad \delta\Phi = C_2 I_k(j\ell);\quad \delta P/\varrho = C_1 I_k(\kappa\ell) + C_2 I_k(j\ell), \qquad (C_1, C_2 = \mathrm{const};\ \kappa, j \neq 0). \tag{5.11.106}$$

If $j = 0,\ k \neq 0$ (nonaxisymmetric oscillations), the function χ becomes [cf. Eq. (5.6.4)]

$$\chi(\ell) = C_1\ell^{|k|};\quad \delta\Phi = C_2\ell^{|k|};\quad \delta P/\varrho = (C_1 + C_2)\ell^{|k|}. \tag{5.11.107}$$

The perturbation of the external potential $\delta\Phi_e$ is determined by a Laplace equation analogous to Eq. (5.11.105), but now $I_k(j\ell)$ has to be discarded, in order to assure a vanishing perturbation at infinity:

$$\delta\Phi_e = C_3 K_k(j\ell),\quad (C_3 = \mathrm{const};\ j \neq 0)\quad \text{and}\quad \delta\Phi_e = C_3\ell^{-|k|},\quad (j = 0). \tag{5.11.108}$$

The perturbed external field is determined by the Eulerian perturbations of Eq. (5.11.62):

$$\nabla\cdot\delta\vec{H}_e = 0;\quad \nabla\times\delta\vec{H}_e = 0. \tag{5.11.109}$$

But $\nabla\times\delta\vec{H}_e = 0$ implies that $\delta\vec{H}_e$ can be derived from a scalar magnetic potential $h_e = h_e(\ell,\varphi,z,t) = h(\ell)\exp[i(\sigma t + k\varphi + jz)]$:

$$\delta\vec{H}_e = H\ \nabla h_e;\quad \nabla\cdot\delta\vec{H}_e = H\ \nabla\cdot(\nabla h_e) = H\ \nabla^2 h_e = 0. \tag{5.11.110}$$

h_e satisfies the Laplace equation $\nabla^2 h_e = d^2h_e/d\ell^2 + (1/\ell)\ dh_e/d\ell - (j^2 + k^2/\ell)h_e = 0$ with the solution

$$h_e = C_4 K_k(j\ell),\quad (C_4 = \mathrm{const};\ j \neq 0)\quad \text{and}\quad h_e = C_4\ell^{-|k|},\quad (j = 0). \tag{5.11.111}$$

The four constants C_i obey four boundary conditions, two of which are related to the continuity of the gravitational potential and of its radial derivative [cf. Eqs. (5.6.89), (5.6.90), (5.11.100)]:

$$\delta\Phi(\ell_1) = C_2 I_k(j\ell_1) = \delta\Phi_e(\ell_1) = C_3 K_k(j\ell_1), \tag{5.11.112}$$

$$(d\delta\Phi/d\ell)_{\ell=\ell_1} = C_2[dI_k(j\ell)/d\ell]_{\ell=\ell_1} = (d\delta\Phi_e/d\ell)_{\ell=\ell_1} + 4\pi G\varrho\ \Delta\ell_1$$
$$= C_3[dK_k(j\ell)/d\ell]_{\ell=\ell_1} + [4C_1\pi G\varrho/\sigma^2(1-\alpha^2)]\{[dI_k(\kappa\ell)/d\ell]_{\ell=\ell_1} + k\alpha\ I_k(\kappa\ell_1)/\ell_1\}. \quad (5.11.113)$$

The continuity across the boundary of the radial component of the Eulerian perturbations $\delta\vec{H}$ and $\delta\vec{H}_e$ from Eqs. (5.11.94), (5.11.111) yields

$$\delta H_\ell(\ell_1) = ijH\ \Delta\ell_1 = [C_1 ijH/\sigma^2(1-\alpha^2)]\{[dI_k(\kappa\ell)/d\ell]_{\ell=\ell_1} + k\alpha\ I_k(\kappa\ell_1)/\ell_1\}$$
$$= \delta H_{e\ell}(\ell_1) = H(dh_e/d\ell)_{\ell=\ell_1} = C_4 H[dK_k(j\ell)/d\ell]_{\ell=\ell_1}. \quad (5.11.114)$$

By virtue of Eq. (5.1.17) the first equation (5.11.63) becomes

$$\delta P + \Delta\vec{r}\cdot\nabla P + \vec{H}\cdot\delta\vec{H}/4\pi = \vec{H}_e\cdot\delta\vec{H}_e/4\pi, \qquad (p=1;\ P_e=0;\ H, H_e = \mathrm{const}). \quad (5.11.115)$$

We insert for the unperturbed pressure gradient from Eq. (3.9.13): $dP/d\ell = -2\pi G\varrho^2(1-\beta)\ell$, $(\beta = \Omega^2/2\pi G\varrho)$. Since $\vec{H} = \vec{H}(0,0,H)$, $\vec{H}_e = \vec{H}_e(0,0,H)$, we get

$$\delta P/\varrho - 2\pi G\varrho(1-\beta)\ell\ \Delta\ell + ijH^2\ \Delta z/4\pi\varrho = H\ \delta H_{ez}/4\pi\varrho = H^2(\partial h_e/\partial z)/4\pi\varrho = ijH^2 h_e/4\pi\varrho. \quad (5.11.116)$$

This equation transforms at $\ell = \ell_1$ into

$$C_1 I_k(\kappa\ell_1) + C_2 I_k(j\ell_1) - [2C_1\pi G\varrho(1-\beta)\ell_1/\sigma^2(1-\alpha^2)]\{[dI_k(\kappa\ell)/d\ell]_{\ell=\ell_1} + k\alpha\ I_k(\kappa\ell_1)/\ell_1\}$$
$$-C_1 j^2H^2 I_k(\kappa\ell_1)/4\pi\varrho\sigma^2 = C_4 ijH^2 K_k(j\ell_1)/4\pi\varrho, \quad (5.11.117)$$

after inserting Eqs. (5.11.97), (5.11.100), (5.11.106), (5.11.111).

Nontrivial values of the constants C_i in the four boundary conditions (5.11.112)-(5.11.114) and (5.11.117) require the vanishing of the determinant of the homogeneous system formed by these four equations. This yields after some algebra the dispersion relationship (Tassoul 1963)

$$(\sigma^2 - j^2v_B^2)(1-\alpha^2) = [\kappa\ell_1 I_k'(\kappa\ell_1)/I_k(\kappa\ell_1) + k\alpha][2\pi G\varrho(1-\beta) - 4\pi G\varrho\ I_k(j\ell_1)\ K_k(j\ell_1)$$
$$-jv_B^2 K_k(j\ell_1)/\ell_1 K_k'(j\ell_1)], \qquad (j\neq 0;\ n=0;\ \Gamma_1=\infty;\ H=\mathrm{const};\ \beta=\Omega^2/2\pi G\varrho), \quad (5.11.118)$$

where we have used Eq. (5.6.15), and a prime denotes derivation with respect to the argument $\kappa\ell_1$ or $j\ell_1$, respectively [cf. Eq. (5.6.11)].

Due to the argument $\kappa\ell_1$ from Eq. (5.11.104), this dispersion relation is equivalent to an algebraic equation of infinite degree in σ.

By virtue of Eq. (5.6.15) the dispersion relation (5.11.118) particularizes at once for the nonrotating cylinder (Simon 1958):

$$\sigma^2 = [j\ell_1 I_k'(j\ell_1)/I_k(j\ell_1)][2\pi G\varrho - 4\pi G\varrho\ I_k(j\ell_1)\ K_k(j\ell_1) - H^2/4\pi\varrho\ell_1^2 I_k'(j\ell_1)\ K_k'(j\ell_1)],$$
$$(\Omega, \alpha = 0;\ j\neq 0). \quad (5.11.119)$$

It has been outlined subsequently to Eq. (5.6.30) that $1/2 - I_k(j\ell_1)\ K_k(j\ell_1) > 0$ if $k > 0$, so σ^2 is positive, and the nonrotating, homogeneous, incompressible cylinder with constant axial field is stable if $k \neq 0$. The magnetic field strengthens the stability, because σ^2 increases due to the presence of the constant axial field. If $k = 0$, this cylinder exhibits an axisymmetric "varicose" gravitational instability, as in the nonmagnetic case (5.6.32). Taking into account that $I_0' = I_1$ and $K_0' = -K_1$ (e.g. Spiegel 1968), Eq. (5.11.119) becomes in this particular case equal to (Chandrasekhar 1981)

$$\omega^2 = \sigma^2/4\pi G\varrho = j\ell_1 I_1(j\ell_1)\ [1/2 - I_0(j\ell_1)\ K_0(j\ell_1)]/I_0(j\ell_1) + jH^2/16\pi^2 G\varrho^2\ell_1 I_0(j\ell_1)\ K_1(j\ell_1),$$
$$(\Omega = 0;\ j\neq 0;\ k = 0). \quad (5.11.120)$$

The critical wavenumber $j = j_c$ at which instability sets in will be determined by the root of the equation $\sigma^2 = 0$, or by

$$I_0(j_c\ell_1)\ K_0(j_c\ell_1) - 1/2 = H^2/16\pi^2 G\varrho^2\ell_1^2 I_1(j_c\ell_1)\ K_1(j_c\ell_1). \quad (5.11.121)$$

For an assigned value of $H/4\pi G^{1/2}\varrho\ell_1$ this equation allows a single positive root $j_c\ell_1$, and the nonrotating cylinder is unstable ($\sigma^2 < 0$) for all varicose deformations if $0 < j < j_c$, and stable if $j > j_c$. In the

Table 5.11.5 Nonrotating, homogeneous incompressible cylinder with constant axial field ($n, \Omega, k = 0;\ \Gamma_1 = \infty;\ H = \mathrm{const}$). The wave numbers j_c and j_m at onset of instability ($\sigma = 0$) and at maximum instability ($\sigma = \sigma_m$) are tabulated for an assigned value of $H/4\pi G^{1/2}\varrho\ell_1$. The dimensionless squared eigenfrequency corresponding to maximum instability is denoted by $\omega_m^2 = \sigma_m^2/4\pi G\varrho$ (Chandrasekhar 1981). $a + b$ means $a \times 10^b$.

$H/4\pi G^{1/2}\varrho\ell_1$	$j_c\ell_1$	$j_m\ell_1$	ω_m^2
0	1.067	0.580	−6.03−2
0.25	0.7899	0.452	−4.16−2
0.5	0.4460	0.266	−1.66−2
0.75	0.2205	0.134	−4.48−3
1	0.0910	0.055	−7.73−4

unstable range $0 < j < j_c$ the negative eigenvalue σ^2 attains a minimum σ_m^2 for a wavenumber j_m. The minimum of σ_m^2 (maximum of $|\sigma_m^2|$) corresponds just to the mode of most rapid amplitude growth, i.e. to the mode of maximum instability, when the cylinder breaks up into pieces of axial length $z_m = 2\pi/j_m$. Table 5.11.5 exhibits the strong tendency towards stabilization exerted by the constant axial field, in the sense that the eigenvalue of maximum instability $|\sigma_m^2|$ is rapidly decreased as the field intensity increases.

In the particular case $j = 0,\ (\Omega \neq 0)$ we have to proceed ex novo (Simon 1958), the relevant Eulerian perturbations being given by Eqs. (5.11.107), (5.11.108), (5.11.110). The four boundary conditions (5.11.112)-(5.11.114), and (5.11.117) become

$$C_2\ell_1^{|k|} = C_3\ell_1^{-|k|};\ C_2|k|\ell_1^{|k|-1} = -C_3|k|\ell_1^{-|k|-1} + [4C_1\pi G\varrho/\sigma^2(1-\alpha^2)](|k|\ell_1^{|k|-1} + k\alpha\ell_1^{|k|-1});$$
$$(C_1 + C_2)\ell_1^{|k|} - [2C_1\pi G\varrho(1-\beta)\ell_1/\sigma^2(1-\alpha^2)](|k|\ell_1^{|k|-1} + k\alpha\ell_1^{|k|-1}) = 0;\quad C_4 = 0,\qquad (j = 0). \tag{5.11.122}$$

If $j = 0$, the outer magnetic field H_e remains constant during these oscillations, because we infer from Eq. (5.11.114) that $\delta H_{e\ell}(\ell_1) = 0$. The vanishing determinant of the homogeneous system (5.11.122) yields the cubic dispersion relation [cf. Tassoul 1963, Eq. (52)]

$$\sigma^2 - 4\Omega^2 - [2\pi G\varrho(|k|-1) - |k|\Omega^2][1 + 2\ \mathrm{sign}(k)\ \Omega/\sigma] = 0,\qquad (j = 0). \tag{5.11.123}$$

In the nonrotating case this reduces to Simon's (1958) dispersion relation for nonaxisymmetric oscillations

$$\sigma^2 = 2\pi G\varrho(|k|-1),\qquad (\Omega, j = 0). \tag{5.11.124}$$

These oscillations are independent of the constant axial field, and are stable, because $\sigma^2 \geq 0$ if $|k| = 1, 2, 3, ...$

The case $k, j, \Omega = 0$ amounts to radial oscillations, which have been shown in Eq. (5.11.90) to be stable.

Summarizing, the nonrotating, incompressible homogeneous cylinder – pervaded by a constant axial field ($n, \Omega = 0;\ \Gamma_1 = \infty;\ H = \mathrm{const}$) – remains always stable if $k \neq 0$ or $j, k = 0$ (Simon 1958). If $k = 0$ and $0 < j < j_c$, this cylinder exhibits the axisymmetric "varicose" instability from Eqs. (5.6.30)-(5.6.33), and (5.11.120).

The uniformly rotating magnetic cylinder becomes unstable ("kink instability") for all wavenumbers $0 < j < j_c$ if the rotation $\beta = \Omega^2/2\pi G\varrho_0$ exceeds a certain critical value. The critical wavenumber j_c at which instability sets in is determined with the aid of Eq. (5.6.15) by introducing the condition of neutral stability $\sigma = 0,\ (\alpha = 0;\ \kappa = j)$ in Eq. (5.11.118):

$$(1-\beta)/2 - I_k(j_c\ell_1)\ K_k(j_c\ell_1) = H^2/16\pi^2 G\varrho^2\ell_1^2 I_k'(j_c\ell_1)\ K_k'(j_c\ell_1). \tag{5.11.125}$$

This instability is enhanced by uniform rotation, whereas the uniform axial field acts against this instability (cf. Eq. (5.11.120) if $\Omega, k = 0$).

Singh and Tandon (1968) have investigated axisymmetric oscillations ($k = 0$) of the nonrotating, compressible homogeneous cylinder ($n, \Omega = 0;\ \Gamma_1 \neq \infty$), pervaded by an axial field of the form $\vec{H} = \vec{H}[0, 0, H_0(1 - \ell^2/\ell_1^2)],\ (H_0 = \mathrm{const})$. The variational approach considered by Singh and Tandon (1968) shows that the particular field stabilizes this cylinder for weak fields, and destabilizes the configuration (σ^2 is decreased) for stronger fields. In the terminology of Sec. 5.6.3 the most unstable mode is the g_1-mode if $\Gamma_1 = 5/3$.

The nonaxisymmetric stability of uniformly rotating, compressible homogeneous cylinders ($j, n = 0$; $\Omega = \mathrm{const}$; $\Gamma_1 \neq \infty$) with an axial field of the form $H^2 = H_0^2 + (H_s^2 - H_0^2)\ell^2/\ell_1^2$ has been studied by Vandakurov and Kolesnikova (1966) if $H_0, H_s = \mathrm{const}$, and the surface field H_s remains constant in the exterior vacuum region. Radial oscillations ($j, k = 0$) are always stable, whereas some nonaxisymmetric modes ($j = 0$; $k = 1, 2, 3, ...$) increase exponentially with time.

For a nonrotating isothermal cylinder Stodólkiewicz (1963) has investigated the occurrence of neutral stability ($\sigma = 0$) against isentropic axisymmetric oscillations of the form

$$\Delta\vec{r} = \Delta\vec{r}(\ell, z, t) = \Delta\vec{r}(\ell)\exp[i(\sigma t + jz)], \qquad (n = \pm\infty;\ k = 0;\ \Gamma_1 = 1 + 1/n = 1), \qquad (5.11.126)$$

the Eulerian perturbations of relevant physical quantities obeying the same dependence. The Alfvén velocity (3.10.254) is assumed constant throughout the isothermal cylinder

$$|\vec{H}| = H = (4\pi\varrho)^{1/2}v_B, \qquad (v_B = \mathrm{const};\ \vec{B} = p\vec{H};\ p = 1;\ H \propto \varrho^{1/2}), \qquad (5.11.127)$$

and the squared sound velocity (2.1.49) is $a^2 = P/\varrho = K$, ($\Gamma_1 = 1$). Stodólkiewicz (1963) considers (i) a toroidal field

$$\vec{H} = \vec{H}[0, H_\varphi(\ell), 0]; \quad \nabla\times\vec{H} = [0, 0, (1/\ell)\ d(\ell H_\varphi)/d\ell]; \quad (\nabla\times\vec{H})\times\vec{H}$$
$$= -[(H_\varphi/\ell)\ d(\ell H_\varphi)/d\ell]\ \vec{e}_\ell = -[(1/2\ell^2)\ d(\ell^2 H_\varphi^2)/d\ell]\ \vec{e}_\ell = -[(2\pi v_B^2/\ell^2)\ d(\ell^2\varrho)/d\ell]\ \vec{e}_\ell, \qquad (5.11.128)$$

and (ii) an axial field

$$\vec{H} = \vec{H}[0, 0, H_z(\ell)]; \quad \nabla\times\vec{H} = (0, -dH_z/d\ell, 0); \quad (\nabla\times\vec{H})\times\vec{H} = -(H_z\ dH_z/d\ell)\ \vec{e}_\ell$$
$$= -(1/2)(dH_z^2/d\ell)\ \vec{e}_\ell = -2\pi v_B^2(d\varrho/d\ell)\ \vec{e}_\ell. \qquad (5.11.129)$$

Substituting into the equilibrium equation (3.10.15) consecutively Eqs. (5.11.128) and (5.11.129), we get, respectively

$$d\Phi/d\ell = (1/\varrho)\ dP/d\ell + (v_B^2/2\varrho\ell^2)\ d(\varrho\ell^2)/d\ell = K\ d\ln\varrho/d\ell + (v_B^2/2)\ d\ln(\varrho\ell^2)/d\ell$$
$$= (K + v_B^2/2)\ d\ln\varrho/d\ell + v_B^2\ d\ln\ell/d\ell, \qquad (5.11.130)$$

$$d\Phi/d\ell = (1/\varrho)\ dP/d\ell + (v_B^2/2\varrho)\ d\varrho/d\ell = (K + v_B^2/2)\ d\ln\varrho/d\ell. \qquad (5.11.131)$$

Taking the divergence (B.46) of these equations, and substituting into Poisson's equation (2.1.4), we obtain the same equation, since $\nabla^2\ln\ell = 0$:

$$\nabla^2\Phi = (K + v_B^2/2)\ \nabla^2\ln\varrho = (K + v_B^2/2)(1/\ell)\ d(\ell\ d\ln\varrho/d\ell)/d\ell = -4\pi G\varrho. \qquad (5.11.132)$$

With the usual transformations (2.1.18), (2.1.20)

$$\varrho = \varrho_0\exp(-\Theta); \quad P = P_0\exp(-\Theta); \quad \ell = [(K + v_B^2/2)/4\pi G\varrho_0]^{1/2}\xi, \qquad (5.11.133)$$

Eq. (5.11.132) takes the form (2.3.85), with the solution (2.3.48): $\exp(-\Theta) = (1 + \xi^2/8)^{-2}$. Turning back to physical variables, we get the equilibrium density distribution of the magnetic cylinder with $|\vec{H}| = H_\varphi(\ell)$ or $|\vec{H}| = H_z(\ell)$:

$$\varrho = \varrho_0/[1 + \pi G\varrho_0\ell^2/(2K + v_B^2)]^2 \propto H^2, \qquad (n = \pm\infty;\ v_B = \mathrm{const}). \qquad (5.11.134)$$

The first order Eulerian perturbation of the equation of motion (2.1.1) becomes

$$\partial^2\Delta\vec{r}/\partial t^2 = -\sigma^2\ \Delta\vec{r} = -K\ \nabla(\delta\varrho/\varrho) + \nabla\delta\Phi - (\delta\varrho/4\pi\varrho^2)(\nabla\times\vec{H})\times\vec{H}$$
$$+(1/4\pi\varrho)[(\nabla\times\vec{H})\times\delta\vec{H} + (\nabla\times\delta\vec{H})\times\vec{H}], \qquad (p = 1;\ \tau = 0). \qquad (5.11.135)$$

(i) $\vec{H} = \vec{H}[0, H_\varphi(\ell), 0]$. The Eulerian perturbation (5.11.14) has the components $\delta\vec{H} = \delta\vec{H}[0, \delta H_\varphi(\ell, z), 0]$, and Eq. (5.11.135) writes with Eq. (5.11.128) in the case of neutral stability $\sigma = 0$ as

$$\nabla(-\delta P/\varrho + \delta\Phi) + (H_\varphi\ \delta\varrho/4\pi\varrho^2\ell)[d(\ell H_\varphi)/d\ell]\ \vec{e}_\ell - (1/4\pi\varrho\ell)[\delta H_\varphi\ d(\ell H_\varphi)/d\ell$$
$$+H_\varphi\ \partial(\ell\ \delta H_\varphi)/\partial\ell]\ \vec{e}_\ell - (1/4\pi\varrho)(H_\varphi\ \partial\delta H_\varphi/\partial z)\ \vec{e}_z = -\nabla\chi + [(\delta\varrho/8\pi\varrho^2\ell^2)\ d(\ell^2 H_\varphi^2)/d\ell]\ \vec{e}_\ell$$
$$-(1/4\pi\varrho\ell^2)\big\{[\partial(\ell^2 H_\varphi\ \delta H_\varphi)/\partial\ell]\ \vec{e}_\ell + [\partial(\ell^2 H_\varphi\ \delta H_\varphi)/\partial z]\ \vec{e}_z\big\} = -\nabla\chi + (v_B^2\ \delta\varrho/2\varrho^2\ell^2)\ \nabla(\ell^2\varrho)$$
$$-(1/4\pi\varrho\ell^2)\ \nabla(\ell^2 H_\varphi\ \delta H_\varphi) = 0, \qquad (\chi = \delta P/\varrho - \delta\Phi). \qquad (5.11.136)$$

Integration of the z-component of this equation with the boundary condition $\vec{H}=0$ if $\varrho=0$, ($H\propto\varrho^{1/2}$) leads to

$$H_\varphi\ \delta H_\varphi=-4\pi\varrho\chi. \qquad (5.11.137)$$

Substitution of Eq. (5.11.137) into the ℓ-component of Eq. (5.11.136) yields

$$[d(\ell^2\varrho)/d\ell][\chi+(v_B^2/2)\ \delta\varrho/\varrho]=0, \qquad (5.11.138)$$

or

$$\delta\Phi=(K+v_B^2/2)\ \delta\varrho/\varrho. \qquad (5.11.139)$$

Taking into account the perturbed Poisson equation (5.2.40), we find

$$\nabla^2(\delta\varrho/\varrho)=-4\pi G\ \delta\varrho/(K+v_B^2/2), \qquad (5.11.140)$$

with the equilibrium density resulting from Eq. (5.11.134). The previous equation becomes for a perturbation of the form $\delta\varrho(\ell,z)=\delta\varrho(\ell)\exp(ijz)$ equal to

$$(1/\ell)\ d[\ell\ d(\delta\varrho/\varrho)/d\ell]/d\ell+[4\pi G\varrho/(K+v_B^2/2)-j^2]\ \delta\varrho/\varrho=0. \qquad (5.11.141)$$

The critical maximum wave number $j=j_c$ has been determined numerically by Stodólkiewicz (1963) from Eq. (5.11.141) with the Rayleigh-Ritz method. The cylinder becomes unstable along its axis for wavelengths exceeding the critical length

$$L_c=2\pi/j_c=3.94[(2K+v_B^2)/\pi G\varrho_0]^{1/2}, \qquad (5.11.142)$$

and this value differs in the nonmagnetic case $v_B=0$ from the critical length (5.10.30) of the isothermal slab only in the numerical coefficient preceding $(2K/G\varrho_0)^{1/2}$. The particular assumptions of case (i) stabilize the cylinder: The critical wavelength increases with increasing field strength.

(ii) $\vec{H}=\vec{H}[0,0,H_z(\ell)]$. This case is slightly more involved, since the Eulerian perturbations (5.11.14) of the field are now $\delta\vec{H}=\delta\vec{H}[\delta H_\ell(\ell,z),0,\delta H_z(\ell,z)]$. We insert Eq. (5.11.129) into Eq. (5.11.135) with $\sigma=0:$

$$\nabla(-\delta P/\varrho+\delta\Phi)+(v_B^2\ \delta\varrho/2\varrho)\ \nabla\ln\varrho+(1/4\pi\varrho)\{[-\partial(H_z\ \delta H_z)/\partial\ell+\partial(H_z\ \delta H_\ell)/\partial z]\ \vec{e}_\ell$$
$$+\delta H_\ell\ (dH_z/d\ell)\ \vec{e}_z\}=0. \qquad (5.11.143)$$

To proceed further, we introduce the auxiliary vectorial function $\vec{S}=\vec{S}[S_\ell(\ell,z),0,S_z(\ell,z)]$, defined by

$$H_z\ \delta H_\ell=\partial S_\ell/\partial z;\quad H_z\ \delta H_z=S_z. \qquad (5.11.144)$$

Eq. (5.11.143) reads in terms of this new function as

$$-\nabla\chi+(v_B^2\ \delta\varrho/2\varrho)\ \nabla\ln\varrho+(1/4\pi\varrho)\big[(-\partial S_z/\partial\ell+\partial^2S_\ell/\partial z^2)\ \vec{e}_\ell+(d\ln H_z/d\ell)(\partial S_\ell/\partial z)\ \vec{e}_z\big]=0. \qquad (5.11.145)$$

The z-component of this equation integrates with the boundary conditions $dH_z/d\ell\propto\varrho^{-1/2}\ d\varrho/d\ell=0$ if $\varrho=0:$

$$S_\ell\ d\ln H_z/d\ell=4\pi\varrho\chi. \qquad (5.11.146)$$

Substituting Eq. (5.11.144) into $\nabla\cdot\delta\vec{H}=0$, we get

$$(1/\ell)\ \partial[(\ell/H_z)\ \partial S_\ell/\partial z]/\partial\ell+(1/H_z)\ \partial S_z/\partial z=\partial[(1/\ell)\ \partial(\ell S_\ell/H_z)/\partial\ell+S_z/H_z]/\partial z=0, \qquad (5.11.147)$$

and after integration with respect to $z:$

$$-S_\ell\ d\ln H_z/d\ell+(1/\ell)\ \partial(\ell S_\ell)/\partial\ell+S_z=0. \qquad (5.11.148)$$

Introducing S_ℓ from Eq. (5.11.146) into Eq. (5.11.148), we obtain

$$S_z = 4\pi\varrho\chi - (1/\ell)\ \partial[4\pi\ell\varrho\chi/(d\ln H_z/d\ell)]/\partial\ell. \tag{5.11.149}$$

We substitute S_ℓ and S_z from Eqs. (5.11.146) and (5.11.149) into the ℓ-component of Eq. (5.11.145):

$$\partial\chi/\partial\ell - (v_B^2\ \delta\varrho/2\varrho)\ d\ln\varrho/d\ell + (1/\varrho)\ \partial\big\{\varrho\chi - (1/\ell)\ \partial[\ell\varrho\chi/(d\ln H_z/d\ell)]/\partial\ell\big\}/\partial\ell$$
$$+j^2\chi/(d\ln H_z/d\ell) = 0. \tag{5.11.150}$$

We effect the derivation, using Eq. (B.48):

$$\chi\ \partial\{\varrho - (1/\ell)\ \partial[\ell\varrho/(d\ln H_z/d\ell)]/\partial\ell\}/\partial\ell + 2(\partial\chi/\partial\ell)\{\varrho - \partial[\varrho/(d\ln H_z/d\ell)]/\partial\ell\}$$
$$-\varrho\ \nabla^2\chi/(d\ln H_z/d\ell) - (v_B^2\ \delta\varrho/2)\ d\ln\varrho/d\ell = 0. \tag{5.11.151}$$

From Eqs. (5.11.127), (5.11.134) we find the relationships

$$\varrho/(d\ln H_z/d\ell) = 2\varrho^2/(d\varrho/d\ell); \quad \varrho - (1/\ell)\ d[\ell\varrho/(d\ln H_z/d\ell)]/d\ell = 0;$$
$$\varrho - d[\varrho/(d\ln H_z/d\ell)]/d\ell = 2\varrho^2/(\ell\ d\varrho/d\ell). \tag{5.11.152}$$

Eq. (5.11.151) transforms with these equations into

$$\nabla^2\chi - (2/\ell)\ \partial\chi/\partial\ell + (v_B^2/4\varrho^3)(d\varrho/d\ell)^2\ \delta\varrho = 0. \tag{5.11.153}$$

Poisson's equation (5.2.40) now becomes

$$\nabla^2\delta\Phi = \nabla^2(\delta P/\varrho) - \nabla^2\chi = K\ \nabla^2(\delta\varrho/\varrho) + (v_B^2/4\varrho^3)(d\varrho/d\ell)^2\ \delta\varrho - (2/\ell)\ \partial\chi/\partial\ell = -4\pi G\ \delta\varrho. \tag{5.11.154}$$

On the other hand, we can express $(1/\ell)\ \partial\chi/\partial\ell$ with the aid of Eqs. (5.11.153), (B.48) as

$$\nabla^2[(1/\ell)\ \partial\chi/\partial\ell] = (1/\ell)\ \partial[\nabla^2\chi - (2/\ell)\ \partial\chi/\partial\ell]/\partial\ell = -(1/\ell)\ \partial[(v_B^2/4\varrho^3)(d\varrho/d\ell)^2\ \delta\varrho]/\partial\ell. \tag{5.11.155}$$

Taking the Laplacian of Eq. (5.11.154), and combining with Eq. (5.11.155), we get eventually the fourth order equation for $\delta\varrho$, depending on the parameter v_B^2/K :

$$\nabla^2\{K\ \nabla^2(\delta\varrho/\varrho) + [4\pi G + (v_B^2/4\varrho^3)(d\varrho/d\ell)^2]\ \delta\varrho\} + (1/\ell)\ \partial[(v_B^2/2\varrho^3)(d\varrho/d\ell)^2\ \delta\varrho]/\partial\ell = 0. \tag{5.11.156}$$

The critical maximum value j_c of the wave number j appearing in the Laplacian $\nabla^2(\delta\varrho/\varrho) = (1/\ell)\ \partial[\ell\ \partial(\delta\varrho/\varrho)/\partial\ell]/\partial\ell - j^2\ \delta\varrho/\varrho$ has been determined by Stodólkiewicz (1963), and the critical wavelength above which instability occurs, becomes

$$L_c = 2\pi/j_c = 3.94[(2K + v_B^2)/\pi G\varrho_0]^{1/2}\ F(v_B^2/K), \tag{5.11.157}$$

with $F(v_B^2/K)$ being a decreasing function of v_B^2/K : $F(0) = 1,\ F(\infty) = 0$.

For the particular assumptions of case (ii) Stodólkiewicz (1963) finds that the critical wavelength L_c remains nearly constant, changing with respect to its nonmagnetic value ($v_B = 0$) by a factor between 1.01 if $v_B^2/K \approx 0$ and 0.83 if $v_B^2/K = \infty$. This behaviour of the compressible isothermal cylinder with axial field differs from that of the homogeneous incompressible cylinder considered in Table 5.11.5. As explained by Stodólkiewicz (1963), this is due to the fact that the axial field affects only the equilibrium structure (5.11.134) of the compressible cylinder and not that of the incompressible one. More recently, Tomisaka (1995) has performed gravitational collapse calculations of isothermal cylindrical clouds permeated by an axial magnetic field.

5.11.5 Stability and Oscillations of the Magnetic Isothermal Slab

The density distribution along the z-axis of the uniformly rotating, magnetic isothermal slab can be found from the equation of hydrostatic equilibrium (3.1.12) with $\vec{v}, \vec{v}_{tr}, \tau = 0;\ \Omega = \mathrm{const};\ \vec{F} = \nabla\Phi;\ \vec{B} = p\vec{H} = \vec{H},\ (p = 1)$:

$$\nabla P = \varrho\ \nabla\Phi + (1/4\pi)(\nabla \times \vec{H}) \times \vec{H} - \varrho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}). \tag{5.11.158}$$

For a constant axial field $\vec{H} = \vec{H}(0, 0, H_z),\ H_z = H = \mathrm{const}$, and constant angular velocity $\vec{\Omega} = \vec{\Omega}(0, 0, \Omega)$, the projection of Eq. (5.11.158) along the z-axis becomes simply $dP/dz = K\ d\varrho/dz = \varrho\ d\Phi/dz$, with the solution (2.3.80) or (5.10.18): $\varrho = \varrho_0/\cosh^2[(2\pi G\varrho_0/K)^{1/2}z],\ (n = \pm\infty)$.

If the magnetic field is parallel to the symmetry plane and directed along the y-axis $\vec{H} = \vec{H}[0, H_y(z), 0]$, the z-component of the equation of hydrostatic equilibrium writes

$$dP/dz = K\ d\varrho/dz = \varrho\ d\Phi/dz - H_y\ (dH_y/dz)/4\pi, \qquad (n = \pm\infty). \tag{5.11.159}$$

If we now assume, following Pacholczyk (1963) and Stodólkiewicz (1963), that the Alfvén speed (3.10.254) is constant throughout the isothermal slab, we get with Eq. (5.11.127):

$$(K + v_B^2/2)\ d\ln\varrho/dz = d\Phi/dz. \tag{5.11.160}$$

With the transformations $\varrho = \varrho_0 \exp(-\theta),\ z = [(K+v_B^2/2)/4\pi G\varrho_0]^{1/2}\xi$, the derivative of this equation becomes with Poisson's equation $d^2\Phi/dz^2 = -4\pi G\varrho$ just equal to Eq. (2.3.50) with the solution (2.3.65):

$$\varrho = \varrho_0/\cosh^2\{[2\pi G\varrho_0/(K + v_B^2/2)]^{1/2}z\}, \qquad [n = \pm\infty;\ H_y^2(z) \propto \varrho(z)]. \tag{5.11.161}$$

As mentioned subsequently to Eq. (5.11.158), one of the simplest cases occurs if the uniformly rotating isothermal slab is permeated by the constant axial field $H_z = H = \mathrm{const}$, and if all isentropic $(\Gamma_1 = 1 + 1/n = 1)$ Eulerian perturbations are in Cartesian coordinates of the form

$$\delta f(x, z, t) = \delta f(z) \exp[i(\sigma t + jx)]. \tag{5.11.162}$$

Since the slab is symmetrical in the (x, y)-plane, no direction in this plane is preferred, and we may choose without loss of generality the perturbations under the previous form. The first order Eulerian perturbation of the equation of motion (3.1.12) in a Cartesian frame rotating with uniform angular speed $\vec{\Omega}(0, 0, \Omega)$ is

$$\delta(D\vec{v}/Dt) \approx D(\delta\vec{v})/Dt \approx D\vec{v}/Dt = \partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} \approx \partial\vec{v}/\partial t = (\delta\varrho/\varrho^2)\ \nabla P - (1/\varrho)\ \nabla\delta P$$
$$+(1/4\pi\varrho)(\nabla \times \delta\vec{H}) \times \vec{H} + \nabla\delta\Phi - 2\vec{\Omega} \times \vec{v}, \qquad (P = K\varrho;\ \vec{v}_{tr}, \tau = 0;\ p = 1;\ \vec{\Omega} = \mathrm{const};$$
$$H_z = H = \mathrm{const};\ \vec{v} \approx \delta\vec{v};\ \delta[\vec{\Omega} \times (\vec{\Omega} \times \vec{r})] = 0). \tag{5.11.163}$$

In virtue of Eqs. (5.11.13), (5.11.162) we get for the Eulerian perturbation of the constant magnetic field $\partial\delta\vec{H}/\partial t = i\sigma\ \delta\vec{H} = \nabla \times (\delta\vec{v} \times \vec{H}) \approx \nabla \times (\vec{v} \times \vec{H})$. Likewise

$$i\sigma(\nabla \times \delta\vec{H}) = \nabla \times [\nabla \times (\vec{v} \times \vec{H})] = -H(\partial^2 v_y/\partial z^2)\ \vec{e}_x + H(\partial^2 v_x/\partial z^2 - j^2 v_x)\ \vec{e}_y$$
$$+ijH(\partial v_y/\partial z)\ \vec{e}_z. \tag{5.11.164}$$

Suppressing the common factor $\exp[i(\sigma t + kx)]$, the equation of motion (5.11.163) becomes via Eq. (5.11.164), (Nakamura 1983):

$$\sigma^2 v_x = (H^2/4\pi\varrho)(-d^2 v_x/dz^2 + j^2 v_x) - j\sigma\chi - 2i\sigma\Omega v_y, \tag{5.11.165}$$

$$\sigma^2 v_y = -(H^2/4\pi\varrho)\ d^2 v_y/dz^2 + 2i\sigma\Omega v_x, \tag{5.11.166}$$

$$i\sigma v_z = -d\chi/dz, \tag{5.11.167}$$

where

$$\chi = \delta P/\varrho - \delta\Phi = K\ \delta\varrho/\varrho - \delta\Phi = \chi(z)\exp[i(\sigma t + jx)], \tag{5.11.168}$$

and $(\delta\varrho/\varrho^2)\ \nabla P - (1/\varrho)\ \nabla\delta P = -K\ \nabla(\delta\varrho/\varrho) = -\nabla(\delta P/\varrho)$. The velocity of sound (2.1.49) is in the isentropic limit (2.1.51) just equal to $a^2 = K$. The continuity equation (5.2.25) becomes in our particular case

$$i\sigma\ \delta\varrho + ij\varrho v_x + d(\varrho v_z)/dz = 0, \tag{5.11.169}$$

and Poisson's equation (5.2.40) reads

$$d^2\delta\Phi/dz^2 - j^2\ \delta\Phi = -4\pi G\ \delta\varrho. \tag{5.11.170}$$

The natural boundary conditions on a physical variable g are $dg/dz \to 0$ if $z \to \pm\infty$, and Nakamura (1983) has shown that σ^2 is real in this case. The stability criterion of the rotating slab can therefore be found by examining the point of neutral (marginal) stability $\sigma = 0$. We now assume that all Eulerian perturbations $\delta f(z)$ from Eq. (5.11.162) are analytic near $\sigma = 0$, allowing for a power series:

$$\delta f(z,\sigma) = \sum_{\ell=0}^{\infty} f^{(\ell)}(z)\ \sigma^\ell, \qquad (\sigma \approx 0). \tag{5.11.171}$$

The zero order terms in σ from Eq. (5.11.165) give

$$d^2v_x^{(0)}/dz^2 - j^2v_x^{(0)} = 0, \tag{5.11.172}$$

with the elementary solution

$$v_x^{(0)}(z) = C_1\exp(jz) + C_2\exp(-jz), \qquad (C_1, C_2 = \mathrm{const}). \tag{5.11.173}$$

Only $v_x^{(0)} \equiv 0$ does satisfy the boundary conditions $dv_x^{(0)}(z)/dz = 0$ if $z = \pm\infty$. The zero order terms in Eq. (5.11.167) yield $d\chi^{(0)}/dz = 0$ or $\chi^{(0)}(z) = \mathrm{const}$. Since v_x is at least of order σ, the zeroth and first order terms in σ from Eq. (5.11.166) give $d^2v_y^{(0)}/dz^2,\ d^2v_y^{(1)}/dz^2 = 0$. Due to the boundary conditions at infinity we have $dv_y^{(0)}/dz,\ dv_y^{(1)}/dz = \mathrm{const} = 0$, and consequently $v_y^{(0)}, v_y^{(1)} = \mathrm{const}$. The σ^2-terms in Eq. (5.11.166) yield

$$v_y^{(0)} = -(H^2/4\pi\varrho)\ d^2v_y^{(2)}/dz^2 + 2i\Omega v_x^{(1)}. \tag{5.11.174}$$

Multiplying with ϱ and integrating, we get

$$v_y^{(0)}\int_{-\infty}^{\infty}\varrho\ dz = v_y^{(0)}\Sigma_1 = v_y^{(0)}(2K\varrho_0/\pi G)^{1/2} = 2i\Omega\int_{-\infty}^{\infty}\varrho v_x^{(1)}\ dz. \tag{5.11.175}$$

Σ_1 is given by Eq. (5.10.32), and the integral over the magnetic term vanishes because $(dv_y^{(2)}/dz)_{z=\pm\infty} = 0$. The first order terms in σ from Eq. (5.11.165) yield

$$\begin{aligned} &d^2v_x^{(1)}/dz^2 - j^2v_x^{(1)} = -(4\pi\varrho/H^2)(j\chi^{(0)} + 2i\Omega v_y^{(0)}) \\ &= -\{4\pi\varrho_0/H^2\cosh^2[(2\pi G\varrho_0/K)^{1/2}z]\}(j\chi^{(0)} + 2i\Omega v_y^{(0)}), \end{aligned} \tag{5.11.176}$$

where the unperturbed density ϱ is determined by Eq. (5.11.158), and is for our constant axial field just equal to the unperturbed solution (2.3.80) or (5.10.18) of the isothermal slab. The integral of the second order nonhomogeneous equation (5.11.176) is

$$\begin{aligned} v_x^{(1)} = {} & [2\pi\varrho_0/jH^2](j\chi^{(0)} + 2i\Omega v_y^{(0)})\Big\{\exp(-jz)\int_{-\infty}^{z}\exp(jz')\ dz'/\cosh^2[(2\pi G\varrho_0/K)^{1/2}z'] \\ &+ \exp(jz)\int_{z}^{\infty}\exp(-jz')\ dz'/\cosh^2[(2\pi G\varrho_0/K)^{1/2}z']\Big\}. \end{aligned} \tag{5.11.177}$$

Nakano and Nakamura (1978) insert this solution into Eq. (5.11.175) to obtain

$$v_y^{(0)} = (2i\Omega/\Sigma_1)\int_{-\infty}^{\infty} \varrho v_x^{(1)}\ dz = 8\pi i\Omega\varrho_0(j\chi^{(0)} + 2i\Omega v_y^{(0)})\eta\ D(\eta)/j^2H^2, \qquad (n=\pm\infty;\ \sigma\approx 0), \tag{5.11.178}$$

where

$$\eta^2 = j^2K/2\pi G\varrho_0; \quad D(\eta) = 1-\eta+(\eta^2/2)\int_0^\infty s\exp[-(\eta/2+1)s]\ ds/[1-\exp(-s)]. \tag{5.11.179}$$

A second order equation for $\delta\varrho^{(0)}$ results from the zeroth approximation in σ of Poisson's equation (5.11.170) by inserting $\delta\Phi^{(0)} = \delta P^{(0)}/\varrho - \chi^{(0)} = K\ \delta\varrho^{(0)}/\varrho - \chi^{(0)}$:

$$K\ d^2(\delta\varrho^{(0)}/\varrho)/dz^2 + 4\pi G\ \delta\varrho^{(0)} - j^2K\ \delta\varrho^{(0)}/\varrho = -j^2\chi^{(0)}. \tag{5.11.180}$$

With the new variable $\mu = \tanh[(2\pi G\varrho_0/K)^{1/2}z]$ from Eq. (5.10.18), this equation transforms into (cf. Eq. (5.10.26) if $\chi^{(0)}=0$) :

$$d^2(\delta\varrho^{(0)}/\varrho)/d\mu^2 - [2\mu/(1-\mu^2)]\ d(\delta\varrho^{(0)}/\varrho)/d\mu + [2/(1-\mu^2) - \eta^2/(1-\mu^2)^2]\ \delta\varrho^{(0)}/\varrho$$
$$= -\eta^2\chi^{(0)}/K(1-\mu^2)^2. \tag{5.11.181}$$

Its solution is

$$\delta\varrho^{(0)}/\varrho = -[\eta\chi^{(0)}/2K(1-\eta^2)]\Big\{(\eta+\mu)[(1-\mu)/(1+\mu)]^{\eta/2}\int_{-1}^{\mu}[(\eta-\mu')/(1-\mu'^2)][(1+\mu')$$
$$/(1-\mu')]^{\eta/2}\ d\mu' + (\eta-\mu)[(1+\mu)/(1-\mu)]^{\eta/2}\int_{\mu}^{1}[(\eta+\mu')/(1-\mu'^2)][(1-\mu')/(1+\mu')]^{\eta/2}\ d\mu'\Big\},$$
$$(n=\pm\infty;\ \sigma\approx 0). \tag{5.11.182}$$

Another relationship between $\chi^{(0)}$ and $v_y^{(0)}$ can be derived from the first order terms of the continuity equation (5.11.169)

$$\int_{-\infty}^{\infty}(\delta\varrho^{(0)} + j\varrho v_x^{(1)})\ dz = 0, \qquad [(\varrho v_z)_{z=\pm\infty} = 0], \tag{5.11.183}$$

which has been integrated by Nakano and Nakamura (1978) via Eqs. (5.11.177) and (5.11.182):

$$\chi^{(0)}\eta\ E(\eta) = (32\pi\varrho_0K^3/GH^4)^{1/2}(j\chi^{(0)} + 2i\Omega v_y^{(0)})\ D(\eta), \qquad [E(\eta) = [2\eta + D(\eta)]/(1-\eta^2)]. \tag{5.11.184}$$

Eqs. (5.11.178) and (5.11.184) form a linear homogeneous system with respect to $\chi^{(0)}$ and $v_y^{(0)}$, having nontrivial solutions if its determinant vanishes, and this requirement leads to

$$H^2/8\pi K\varrho_0 = H^2/4\pi^2G\Sigma_1^2 = D(\eta)/E(\eta) - (\Omega^2/\pi G\varrho_0)\ D(\eta)/\eta, \qquad (n=\pm\infty;\ \sigma\approx 0). \tag{5.11.185}$$

This dispersion relation for the critical values of $\eta \propto j$ includes the two limiting cases $\Omega = 0$ (Nakano and Nakamura 1978) and $H=0$ (Sec. 5.10.3, Goldreich and Lynden-Bell 1965a). For certain values of the rotation parameter $2\beta = \Omega^2/\pi G\varrho_0$ and of the magnetic field $H^2/4\pi^2G\Sigma_1^2$ there exists – as in the nonmagnetic case from Sec. 5.10.3 – an intermediate range of wavelengths $L = 2\pi/j \propto 2\pi/\eta$, where the isothermal, uniformly rotating, magnetic slab becomes unstable. If $D(1/E - 2\beta/\eta) < 0$, the disk remains stable, since $H^2/4\pi^2G\Sigma_1^2$ cannot become negative. Rotation has a stabilizing effect on this magnetic slab.

Only a single critical wavelength exists in the nonrotating limit $\Omega, \beta = 0$, and the magnetic slab becomes unstable only if $H^2/4\pi^2G\Sigma_1^2 = D/E < 1$, because the calculations of Nakano and Nakamura (1978) show that $D/E < 1$ for any $\eta > 0$. No unstable modes exist if $\eta > 1$. The critical wavelength $L_c = 2\pi/j_c \propto 2\pi/\eta_c$ decreases as the magnetic field intensity H decreases, but it never becomes less than its value $L_c = (2\pi K/G\varrho_0)^{1/2}$ from Eq. (5.10.30) for the nonmagnetic isothermal slab, when $\eta_c = 1$ and $j_c = (2\pi G\varrho_0/K)^{1/2}$.

For a magnetic field of the form $\vec{H} = \vec{H}[0, H_y(z), 0]$ the equilibrium density distribution is given by Eq. (5.11.161). Pacholczyk (1963) assumes perturbations of the form (5.11.162), propagating perpendicularly to the magnetic field, so the Eulerian perturbation (5.11.14) has by assumption the components $\delta\vec{H} = \delta\vec{H}[0, \delta H_y(x, z), 0]$. After effecting the vector operations, the (x, z)-components of the equation of motion (5.11.135) turn into $[\Delta\vec{r} = \Delta\vec{r}(\Delta x, \Delta y, \Delta z)]$

$$-\sigma^2\ \Delta x = -K\ \partial(\delta\varrho/\varrho)/\partial x + \partial\delta\Phi/\partial x - (H_y/4\pi\varrho)\ \partial\delta H_y/\partial x, \tag{5.11.186}$$

$$-\sigma^2\ \Delta z = -K\ \partial(\delta\varrho/\varrho)/\partial z + \partial\delta\Phi/\partial z + (H_y\ \delta\varrho/4\pi\varrho^2)\ dH_y/dz$$
$$-(1/4\pi\varrho)(\delta H_y\ dH_y/dz + H_y\ \partial\delta H_y/\partial z). \tag{5.11.187}$$

The further treatment of this system proceeds quite analogously to the cylindrical case from Eqs. (5.11.136)-(5.11.142). We rewrite Eq. (5.11.186) for the case of neutral stability $\sigma = 0$:

$$ij[\chi + (1/4\pi\varrho)H_y\ \delta H_y] = 0, \tag{5.11.188}$$

or

$$H_y\ \delta H_y = -4\pi\varrho\chi. \tag{5.11.189}$$

The magnetic field is taken under the form (5.11.127)

$$|\vec{H}(z)| = H_y(z) = [4\pi\varrho(z)]^{1/2}v_B, \qquad (v_B = \mathrm{const}), \tag{5.11.190}$$

and Eq. (5.11.187) becomes, by inserting Eqs. (5.11.189), (5.11.190):

$$-\partial\chi/\partial z + (\delta\varrho/8\pi\varrho^2)\ dH_y^2/dz - (1/4\pi\varrho)\ \partial(H_y\ \delta H_y)/\partial z = -\partial\chi/\partial z + (v_B^2/2\varrho^2)\ \delta\varrho\ d\varrho/dz$$
$$+(1/\varrho)\ \partial(\varrho\chi)/\partial z = [d(\ln\varrho)/dz][(v_B^2/2)\ \delta\varrho/\varrho + \chi] = 0, \qquad (\sigma = 0). \tag{5.11.191}$$

This amounts to $(v_B^2/2)\ \delta\varrho/\varrho + \chi = 0$ or

$$\delta\Phi = K\ \delta\varrho/\varrho - \chi = (K + v_B^2/2)\ \delta\varrho/\varrho, \tag{5.11.192}$$

and Poisson's equation (5.2.40) yields

$$\nabla^2(\delta\varrho/\varrho) = -4\pi G\ \delta\varrho/(K + v_B^2/2). \tag{5.11.193}$$

Putting $h(x, z, t) = \delta\varrho(x, z, t)/\varrho(z) = [\delta\varrho(z)/\varrho(z)]\exp[i(\sigma t + jx)]$, this equation assumes just the form (5.10.25) if $v_B \propto |\vec{H}| = 0$:

$$dh^2/dz^2 + [4\pi G\varrho/(K + v_B^2/2) - j^2]h = 0. \tag{5.11.194}$$

The critical wavenumber j_c and the critical wavelength L_c are just equal to the solution (5.10.30) of the nonmagnetic isothermal slab if $v_B = 0$:

$$j_c = [2\pi G\varrho_0/(K + v_B^2/2)]^{1/2}; \quad L_c = 2\pi/j_c = [2\pi(K + v_B^2/2)/G\varrho_0]^{1/2}, \qquad (\sigma = 0;\ n = \pm\infty). \tag{5.11.195}$$

For all wavelengths larger than L_c the magnetic isothermal slab considered by Pacholczyk (1963) is unstable, and the particular field $H_y(z)$ stabilizes the slab, increasing the critical wavelength if the perturbation (5.11.162) propagates perpendicularly to the magnetic field $H_y(z)$.

For perturbations $\delta f(y, z, t)$ running parallel to the magnetic field $H_y(z)$, Stodólkiewicz [1963, Eq. (10.101)] obtains the critical wavelength

$$L_c = (2\pi K/G\varrho_0)^{1/2}[(1 + v_B^2/2K)/(1 + v_B^2/K)]^{1/2}, \qquad (\sigma = 0;\ n = \pm\infty), \tag{5.11.196}$$

showing that L_c remains nearly constant, decreasing in comparison to its nonmagnetic value $v_B = 0$ at most by the factor $1/2^{1/2} = 0.71$ if $v_B^2/K \to \infty$.

5.12 Stability and Oscillations of Relativistic Polytropes

5.12.1 The Static Method

The static (energy, turning point) method – originally devised by Zeldovich – has been extensively used for discussing the stability of relativistic (very dense and/or supermassive) stars. Its usefulness stems from the fact that it avoids the detailed calculation of radial modes of oscillation (e.g. Zeldovich and Novikov 1971, Chap. 10; Tassoul 1978, Sec. 6.8). Mass and/or mean density of relativistic bodies are considerably larger than for Newtonian objects, so we make a brief survey of the limits which determine the three final stages of stellar evolution predicted by theory: White (black) dwarfs, neutron stars (pulsars), and black holes. We also touch the stability of supermassive objects (stars) with masses between the Oppenheimer-Volkoff (1939) limit for cold objects of about 2 $M_\odot$ ($M_\odot$ – solar mass) and $10^9 M_\odot$ (e.g. Fowler 1964, Zeldovich and Novikov 1971, Shapiro and Teukolsky 1983). Such supermassive objects are thought to occur in the centre of radiogalaxies and quasars.

At first we recapitulate elementary estimates of three time scales occurring in stellar evolution.

(i) Free-fall Time. The radial form of the equation of motion (5.2.10) is

$$\varrho\, d^2r_1/dt^2 = -dP/dr_1 - G\varrho M_1/r_1^2, \qquad (M_1 = \mathrm{const}), \tag{5.12.1}$$

where M_1 and r_1 is the mass and radius of a sphere. With pressure forces being neglected this equation is integrated after multiplication by dr_1/dt :

$$dr_1/dt = -(2GM_1)^{1/2}(1/r_1 - 1/r_{10})^{1/2}, \tag{5.12.2}$$

where the obvious initial condition $dr_1/dt = 0$ if $r_1 = r_{10}$ is inserted. To obtain the free-fall time t_f this equation is integrated again from $r_1 = r_{10}$ if $t = 0$, to $r_1 = 0$ if $t = t_f$:

$$(2GM_1/r_{10})^{1/2}t_f = \int_0^{r_{10}} r_1^{1/2}\ dr_1/(r_{10} - r_1)^{1/2} = 2r_{10}\int_0^{\pi/2} \sin^2 x\ dx = \pi r_{10}/2,$$
$$(\sin^2 x = r_1/r_{10}). \tag{5.12.3}$$

Introducing the initial mean density of the sphere $\varrho_{m0} = 3M_1/4\pi r_{10}^3$, we observe that the free fall time

$$t_f = (3\pi/32G\varrho_{m0})^{1/2}, \tag{5.12.4}$$

depends only on the initial mean density of the sphere. t_f amounts to about 2×10^6 yr for an interstellar cloud of mean density $\varrho_{m0} = 10^{-22}$ g cm^{-3}, and is ridiculously small $t_f = 1800$ s for the Sun ($\varrho_{m0} =$ 1.4 g cm^{-3}).

(ii) The so-called Kelvin-Helmholtz time scale t_K (gravitational contraction time scale or thermal time scale) is the e-folding time scale for radius changes in a gravitationally contracting star with pressure forces being present:

$$1/t_K = |d\ln r_1/dt|. \tag{5.12.5}$$

Integrating from er_1 to r_1 with $t_K = $ const, there results just $t = t_K$. For a nonmagnetic polytropic sphere we have from Eqs. (2.6.98), (2.6.137):

$$E = (3\Gamma - 4)W/3(\Gamma - 1) = -(3\Gamma - 4)qGM_1^2/3(\Gamma - 1)r_1, \qquad (U_m = 0;\ q = 3/(5 - n);\ \Gamma > 4/3). \tag{5.12.6}$$

Without nuclear energy sources the total luminosity of the contracting star is just

$$L = -dE/dt = -(3\Gamma - 4)qGM_1^2(dr_1/dt)/3(\Gamma - 1)r_1^2. \tag{5.12.7}$$

The time required for the star to contract with constant luminosity from infinity to radius r_1 is obtained by integrating Eq. (5.12.7), or by inserting for $-dr_1/dt$ from Eq. (5.12.5):

$$t = t_K = (3\Gamma - 4)qGM_1^2/3(\Gamma - 1)Lr_1, \qquad (L, \Gamma = \text{const}). \tag{5.12.8}$$

We have $\varepsilon^{(kin)} = 3(\gamma - 1)\varepsilon^{(int)}/2$ by Eq. (1.7.58), and the total thermal energy (2.6.90) becomes via Eq. (2.6.95) equal to $E_{th} = 3(\gamma - 1)U/2$. If the star is composed of noninteracting, nonrelativistic particles ($\gamma = 5/3$), this amounts to $E_{th} = U$. If $\Gamma = \gamma = 5/3$, the virial theorem (2.6.97) yields: $2U + W = 2E_{th} + W = 2E_{th} - qGM_1^2/r_1 = 0$, $(U_m = 0)$. Thus, with $E_{th} = qGM_1^2/2r_1$, the Kelvin time scale (5.12.8) for normal stars is a thermal time scale, equivalent to their "cooling time" (Cox and Giuli 1968):

$$t_K = 2(3\Gamma - 4)E_{th}/3(\Gamma - 1)L \approx E_{th}/L. \tag{5.12.9}$$

Taking $\gamma = \Gamma = 5/3$ (all internal energy $U = E_{th}$ is assumed under the form of nonrelativistic, translational kinetic energy of particle motion) we get for the Sun $t_K = E_{th}/L \approx 3 \times 10^7$ yr, representing its interval of existence without nuclear energy sources.

(iii) The third relevant time scale is the nuclear time scale – the interval required for the properties of a star to change significantly as a result of nuclear burning:

$$t_N = E_N/L. \tag{5.12.10}$$

E_N denotes the reserve of nuclear energy of the star. For complete burning of a hydrogen core of mass $0.1M_\odot$ into ^{4}He the energy release is $0.007c^2$ per gram, and t_N becomes for the Sun equal to about 10^{10} yr. Thus, at least for solar type stars we have $t_f \ll t_K \ll t_N$ (Cox and Giuli 1968, Zeldovich and Novikov 1971).

Evolution – involving thermal and nuclear processes, accretion or mass loss – produces changes in the equilibrium configuration. And often these changes involve instabilities leading to a disruption of hydrostatic equilibrium and to catastrophic phenomena. The essence of the static method relies upon the fact that the total energy is a minimum for stable equilibria, and fails to be so for unstable equilibria (e.g. Lyttleton 1953, Tassoul 1978).

Let us write down within Newtonian gravitation the equation (2.6.98) for the total energy of a nonmagnetic star ($U_m = 0$) :

$$E = W + U = -\int_{V_1} GM(r)\ \varrho\ dV/r + \int_{V_1} \varepsilon^{(int)}\ dV = -3GM_1^2/(5-n)r_1 + \int_{M_1} \varepsilon^{(int)}\ dM/\varrho. \tag{5.12.11}$$

We have inserted for the gravitational energy W from Eqs. (2.6.70), (2.6.137), and for the internal energy U from Eq. (2.6.95). At constant entropy the first law of thermodynamics (1.1.3) becomes for the unit of mass

$$P = \varrho^2(\partial U/\partial\varrho)_{S=\text{const}} = \varrho^2[\partial(\varepsilon^{(int)}/\varrho)/\partial\varrho]_{S=\text{const}}. \tag{5.12.12}$$

Integration of Eq. (1.3.23) with a constant adiabatic index Γ_1 yields

$$P = K\varrho^{\Gamma_1}, \qquad (K, \Gamma_1, S = \text{const}). \tag{5.12.13}$$

Inserting this into Eq. (5.12.12), and integrating again, we obtain

$$\varepsilon^{(int)}/\varrho = [K/(\Gamma_1 - 1)]\varrho^{\Gamma_1 - 1} + \text{const} = P/\varrho(\Gamma_1 - 1) + \text{const}. \tag{5.12.14}$$

We introduce an average value $\varepsilon^{(int)}/\varrho = [K/(\Gamma_1 - 1)]\varrho_m^{\Gamma_1 - 1} + \text{const}$ of the specific internal energy over the sphere. The total energy (5.12.11) becomes in terms of mean density $\varrho_m = 3M_1/4\pi r_1^3$ equal to

$$E = -6^{2/3}\pi^{1/3}G\varrho_m^{1/3}M_1^{5/3}/(5-n) + [K/(\Gamma_1 - 1)]\varrho_m^{\Gamma_1 - 1}M_1 + \text{const}\ M_1. \tag{5.12.15}$$

Following Zeldovich and Novikov (1971) we will demonstrate that hydrostatic equilibrium necessarily prevails at an extremum of total energy. To this end we calculate the first variation (5.8.7) of the total energy (5.12.11):

$$\delta^* E = -G\int_{V_1} M(r)\ \Delta(1/r)\ \varrho\ dV + \int_{M_1} \Delta(\varepsilon^{(int)}/\varrho)\ dM, \qquad (\Delta M(r) \equiv 0). \tag{5.12.16}$$

For a certain unit mass element we can replace the partial derivative in Eq. (5.12.12) by the Lagrangian variation

$$\Delta(\varepsilon^{(int)}/\varrho) = (P/\varrho^2)\ \Delta\varrho, \qquad (m = 1;\ S = \mathrm{const}), \tag{5.12.17}$$

and the second integral in Eq. (5.12.16) is via Eq. (5.2.28) equal to

$$\int_{M_1} \Delta(\varepsilon^{(int)}/\varrho)\ dM = \int_{M_1} (P/\varrho^2)\ \Delta\varrho\ dM = -\int_{M_1} (P/\varrho)(\nabla\cdot\Delta\vec{r})\ dM$$
$$= -4\pi\int_0^{r_1} P[d(r^2\ \Delta r)/dr]\ dr = -4\pi r^2 P\ \Delta r\Big|_0^{r_1} + 4\pi\int_0^{r_1} r^2(dP/dr)\ \Delta r\ dr = \int_{V_1}(dP/dr)\ \Delta r\ dV. \tag{5.12.18}$$

Eq. (5.12.16) now reads

$$\delta^* E = \int_{V_1} [G\varrho M(r)/r^2 + dP/dr]\ \Delta r\ dV, \tag{5.12.19}$$

and the extremum of E is obtained by putting $\delta^* E = 0$, which is equivalent to the equation of hydrostatic equilibrium $dP/dr = -G\varrho M(r)/r^2$. The equilibrium state $\delta^* E = 0$ is stable if the extremum of E is a minimum, i.e. if $\delta^{*2}E > 0$. If $\delta^{*2}E < 0$, the extremum is a maximum, and the sphere is unstable. At an extremum of E we have $(\partial E/\partial\varrho_m)_{M_1,S=\mathrm{const}} = 0$ in virtue of Eq. (5.12.15), and this amounts to

$$(\partial E/\partial\varrho_m)_{M_1,S} = C_1\varrho_m^{-2/3}M_1^{5/3} + C_2\varrho_m^{\Gamma_1-2}M_1 = 0, \qquad (C_1, C_2 = \mathrm{const}). \tag{5.12.20}$$

From this we obtain

$$M_1 = \mathrm{const}\ \varrho_m^{(3/2)(\Gamma_1-4/3)}, \tag{5.12.21}$$

and the sign of the derivative

$$(\partial M_1/\partial\varrho_m)_{S=\mathrm{const}} = \mathrm{const}\ (\Gamma_1 - 4/3)\varrho_m^{(3/2)(\Gamma_1-2)}, \qquad (\mathrm{const} > 0), \tag{5.12.22}$$

is the same as the sign of $\Gamma_1 - 4/3$. As shown by Eq. (2.6.99), the stable equilibrium of a nonmagnetic configuration ($U_m = 0$) requires $\Gamma = \Gamma_1 > 4/3$, and therefore Eq. (5.12.22) implies $(\partial M_1/\partial\varrho_m)_S > 0$ for stability, and $(\partial M_1/\partial\varrho_m)_S < 0$ for instability. The calculation of $(\partial M_1/\partial\varrho_m)_S$ involves the comparison of two stellar models, both of which have the same entropy and equation of state, but with slightly differing masses. The criterion is a natural one: In a stable state, any addition of mass from outside would cause a contraction and an increased mean density ϱ_m. Hence, for stability ∂M_1 and $\partial\varrho_m$ should have the same sign.

Another condition of stability results from

$$(\partial M_1/\partial r)_S < 0, \tag{5.12.23}$$

which implies that with addition of mass from outside ($\partial M_1 > 0$) each internal mass element at radius r must move toward the centre, i.e. $\partial r < 0$. The stability condition (5.12.23) will not be satisfied unless $(\partial M_1/\partial\varrho_m)_S > 0$, as can be seen by inserting for ∂r the expression

$$dr_1 = (1/36\pi)^{1/3}\big(M_1^{-2/3}\ dM_1/\varrho_m^{1/3} - M_1^{1/3}\ d\varrho_m/\varrho_m^{4/3}\big), \tag{5.12.24}$$

resulting from differentiation of $r_1 = (3M_1/4\pi\varrho_m)^{1/3}$. The stability condition (5.12.23) writes

$$(\partial r_1/\partial M_1)_S \propto M_1^{-2/3}/\varrho_m^{1/3} - (M_1^{1/3}/\varrho_m^{4/3})(\partial\varrho_m/\partial M_1)_S < 0, \tag{5.12.25}$$

which can only be satisfied if $(\partial M_1/\partial\varrho_m)_S > 0$. At least for polytropes we have via Eq. (2.6.27) $\varrho_m = 3\varrho_0(\mp d\theta/d\xi)_{\xi=\xi_1}/\xi_1 = \mathrm{const}\ \varrho_0$, and the condition of stability $(\partial M_1/\partial\varrho_m)_S > 0$ can also be expressed in terms of the central density ϱ_0 :

$$(\partial M_1/\partial\varrho_0)_{S=\mathrm{const}} > 0. \tag{5.12.26}$$

Hydrostatic equilibrium models located on the descending branch of the curve $M_1(\varrho_0)$, where $(\partial M_1/\partial\varrho_0)_S < 0$, are unstable.

In general relativity the condition of equilibrium is obtained in a similar manner, by calculating the extremum of relativistic mass M_{r1} from Eq. (4.1.23) with respect to relativistic central density ϱ_{r0}. Like in Eq. (5.12.19), the extremum of relativistic mass (relativistic energy) is calculated from

$$\delta^* M_{r1} = 4\pi\ \delta^* \int_0^{r_1} \varrho_r r^2\ dr = 0, \tag{5.12.27}$$

under the condition that the total number of baryons (number of nucleons (protons, neutrons) and hyperons)

$$N_B = \int_{V_1} n_d\ dV = 4\pi \int_0^{r_1} n_d r^2 \exp(\kappa/2)\ dr, \tag{5.12.28}$$

remains constant, or $\delta^* N_B = 0$, where the proper volume element is given by Eq. (4.1.24), and n_d denotes the baryon number density. Instead of the number of baryons we can also use the rest mass of a sphere M_1 from Eq. (4.1.76), by substituting via Eq. (1.2.5) $n_d = \varrho/\mu H = \varrho N_A/\mu \approx \varrho N_A$, where the mean molecular weight is $\mu \approx 1$ for baryons consisting mostly of neutrons and protons: $M_1 \approx N_B/N_A$.

Zeldovich and Novikov (1971) argue that the stability condition (5.12.26) is also valid in general relativity:

$$(\partial M_{r1}/\partial\varrho_{r0})_{S=\mathrm{const}} > 0. \tag{5.12.29}$$

The extrema of M_{r1} and N_B occur at the same relativistic central density ϱ_{r0}, if $(\partial M_{r1}/\partial\varrho_{r0})_{S=\mathrm{const}} = 0$. In the immediate vicinity, to the left and right of the extremal central density ϱ_{r0}, we may select two models with slightly different central densities $\varrho_{r0}^{(1)}$ and $\varrho_{r0}^{(2)}$, but with the same number of baryons N_B. The density distribution of one of these models can be represented as a perturbation of the other model:

$$\varrho_{r0}^{(2)} = \varrho_{r0}^{(1)} + \delta\varrho_{r0} \exp(i\sigma t). \tag{5.12.30}$$

Since the perturbation $\propto \exp(i\sigma t)$ which transforms one stationary equilibrium model into the other must be independent of time, we must have $\sigma = 0$ at the extremum $\partial M_{r1}/\partial\varrho_{r0} = 0$ of the $M_{r1}(\varrho_{r0})$-curve. On the one side of the extremum – where $\partial M_{r1}/\partial\varrho_{r0} > 0$ – we have $\sigma^2 > 0$, and the equilibrium is stable, whereas on the other side the equilibrium is unstable, and $\partial M_{r1}/\partial\varrho_{r0} < 0$, $(\sigma^2 < 0)$.

In spite of its simplicity, the static method is not free of shortcomings (Tassoul 1978): First, the method is restricted to real values of σ^2, as it pinpoints exchanges of stability from positive to negative values of σ^2, and vice versa. Second, rotation for instance, couples various p, f, and g-modes, so the correct identification of a neutral (marginal) mode $\sigma^2 = 0$ may be somewhat ambiguous.

The first law of thermodynamics and the equation of hydrostatic equilibrium complicates considerably if special relativistic effects are considered [see Eqs. (1.1.3), (4.1.27), (4.1.56), (4.1.57)]. Fortunately, the rest mass approximation of mass density is valid for most observed objects (even for neutron stars, see Fig. 5.12.1). For cold, degenerate matter this can be seen by equating the pressure (1.7.34) of a completely degenerate, nonrelativistic neutron gas to the pressure (1.6.7) of the extremely relativistic neutron gas

$$5.461 \times 10^9 (\varrho/\mu_n)^{5/3} = 1.244 \times 10^{15} (\varrho/\mu_n)^{4/3} \quad \text{or} \quad \varrho = 1.2 \times 10^{16}\ \mathrm{g\ cm^{-3}}. \tag{5.12.31}$$

Thus, only well above nuclear densities of 2×10^{14} g cm^{-3} the effects of special relativity on mass density are no longer negligible.

Quite generally, we have $\varrho_r \not\approx \varrho$, whenever the density $\varepsilon^{(int)}/c^2$, corresponding to the internal energy density $\varepsilon^{(int)}$ from Eq. (1.2.16) or (4.1.2) is not much smaller than the rest mass density

$$\varrho = \varepsilon/c^2 = n_d m_p, \tag{5.12.32}$$

where n_d is the number density, and m_p the particle rest mass. In the nondegenerate perfect gas region special relativistic effects are important whenever the internal energy per particle $e^{(int)} = fkT/w$ from Eq. (1.7.59) is comparable to or larger than the particle rest energy $m_p c^2$.

Incidentally, when general relativistic effects become important, the relevant densities (5.12.33) for solar mass objects are of the same order of magnitude as those from Eq. (5.12.31). As results from the

Schwarzschild metric (4.1.20), general relativistic effects are dominant if $2GM_{r1}/c^2r_1 \approx 1$, and hydrostatic equilibrium configurations cannot exist if the surface radial coordinate of a sphere r_1 is smaller than its Schwarzschild radius r_g :

$$r_1 < r_g = 2GM_{r1}/c^2 \quad \text{or} \quad \varrho_m \approx 3M_{r1}/4\pi r_1^3 > 3M_{r1}/4\pi r_g^3 = 3c^6/32\pi G^3 M_{r1}^2$$
$$= 1.84 \times 10^{16}(M_\odot/M_{r1})^2. \tag{5.12.33}$$

If $M_\odot \approx M_{r1}$, this is indeed of the same order of magnitude as Eq. (5.12.31).

5.12.2 Stability and Maximum Masses of Cold Spheres

As mentioned in the previous subsection two final stages of stellar evolution can be approximated by spheres composed of degenerate electrons (white, black dwarfs) and degenerate neutrons (neutron stars, pulsars). The equations of state (1.6.6) and (1.6.7) of completely degenerate electron (neutron) matter, as well as general relativistic effects, determine certain limiting (maximum) masses of hydrostatic equilibrium configurations of white dwarfs (neutron stars).

Let us consider at first the well known limiting mass of white dwarf stars. Typical central rest mass densities ϱ_0/μ_e of white dwarfs are between about 1.23×10^5 and 2.48×10^{11} g cm^{-3}, the lower limit corresponding to a nonrelativistically degenerate white dwarf of mass $M_1 = 0.88M_\odot/\mu_e^2$, and the upper one to a relativistically degenerate white dwarf of mass $M_1 = 5.75M_\odot/\mu_e^2$ (Zeldovich and Novikov 1971, Table 10). The mean molecular weight per free ionization electron is given by Eq. (1.7.23): $1 \leq \mu_e \leq 2$.

The Newtonian equation of hydrostatic equilibrium $dP/dr = -G\varrho M(r)/r^2$ can be combined with the continuity of mass $dM(r) = 4\pi\varrho r^2\ dr$ to yield

$$d[(r^2/\varrho)\ dP/dr]/dr = -4\pi G\varrho r^2. \tag{5.12.34}$$

The basic structure equation of white dwarfs is obtained by substituting the equations of state (1.6.1) and (1.6.4):

$$d[(r^2/x^3)\ df(x)/dr]/dr = -4\pi GB^2x^3r^2/A. \tag{5.12.35}$$

We insert for $df(x)$ from Eq. (1.6.8):

$$d[r^2\ d(x^2+1)^{1/2}/dr]/dr = -\pi GB^2x^3r^2/2A. \tag{5.12.36}$$

With the new auxiliary variable $y^2 = x^2 + 1$ this equation reads

$$d(r^2\ dy/dr)/dr = -\pi GB^2r^2(y^2-1)^{3/2}/2A. \tag{5.12.37}$$

We now define a normalized parameter Y and a dimensionless radial distance ζ by

$$y = y_0Y; \quad r = \kappa\zeta = (2A/\pi G)^{1/2}\zeta/By_0 = (6^{1/2}N_A/8\pi m_e\mu_e y_0)(h^3/cG)^{1/2}\zeta$$
$$= 7.776 \times 10^8\zeta/\mu_e y_0\ [\text{cm}], \tag{5.12.38}$$

where N_A is Avogadro's number, m_e the electron rest mass, h Planck's constant, and $x_0,\ y_0 = (x_0^2+1)^{1/2}$ denote central values of x, y, respectively. Eq. (5.12.37) becomes

$$d(\zeta^2\ dY/d\zeta)/d\zeta = -\zeta^2(Y^2 - 1/y_0^2)^{3/2}. \tag{5.12.39}$$

The conditions at the centre $\zeta = 0$ are clearly $Y = 1$ and $dY/d\zeta = 0$. At the surface boundary $\zeta = \zeta_1$ the density has to vanish: $x \propto \varrho^{1/3} = 0,\ y = 1$, and $Y(\zeta_1) = 1/y_0$. The solutions of Eq. (5.12.39) form a one parameter family, depending on the value of y_0, i.e. on the central density $\varrho_0 = Bx_0^3 = B(y_0^2-1)^{3/2}$.

In the nonrelativistic limit we have $x \to 0$ or $y \to 1$, and

$$Y = [(1+x^2)/(1+x_0^2)]^{1/2} \approx 1 + x^2/2 - x_0^2/2; \quad dY/d\zeta \approx (1/2)\ dx^2/d\zeta;$$
$$Y^2 - 1/y_0^2 = x^2/y_0^2 \approx x^2, \qquad (x \approx 0;\ y_0 \approx 1). \tag{5.12.40}$$

The structure equation of a completely degenerate, nonrelativistic electron gas becomes

$$d(\zeta^2\ dx^2/d\zeta)/d\zeta = -2\zeta^2(x^2)^{3/2}, \qquad (x \approx 0). \tag{5.12.41}$$

With the change of the independent variable $\zeta = \eta/2^{1/2}$ this turns just into the equation (2.3.87) of a polytropic sphere with index $n = 1.5$:

$$d(\eta^2\ dx^2/d\eta)/d\eta = -\eta^2(x^2)^{3/2}, \qquad (x \approx 0). \tag{5.12.42}$$

In the extreme relativistic limit we have $x_0, y_0 \to \infty$, and the white dwarf equation (5.12.39) turns into that of a polytrope with index $n = 3$:

$$d(\zeta^2\ dY/d\zeta)/d\zeta = -\zeta^2 Y^3, \qquad (x \to \infty;\ 1/y_0 \approx 0). \tag{5.12.43}$$

The mass inside radius r is simply

$$M(r) = 4\pi \int_0^r \varrho(r')\ r'^2\ dr' = 4\pi\kappa^3 \int_0^\zeta \varrho(\zeta')\ \zeta'^2\ d\zeta'. \tag{5.12.44}$$

However, in virtue of Eqs. (1.6.4), (5.12.38) we get

$$\varrho/\varrho_0 = x^3/x_0^3 = [(y^2-1)/(y_0^2-1)]^{3/2} = y_0^3(Y^2 - 1/y_0^2)^{3/2}/(y_0^2-1)^{3/2}, \tag{5.12.45}$$

and Eq. (5.12.44) reads as

$$\begin{aligned} M(\zeta) &= [4\pi\kappa^3\varrho_0 y_0^3/(y_0^2-1)^{3/2}] \int_0^\zeta \zeta'^2[Y^2(\zeta') - 1/y_0^2]^{3/2}\ d\zeta' \\ &= -(2^{7/2}/\pi^{1/2}B^2)(A/G)^{3/2} \int_0^\zeta d(\zeta'^2\ dY/d\zeta') = -(2^{7/2}/\pi^{1/2}B^2)(A/G)^{3/2}\zeta^2\ dY/d\zeta. \end{aligned} \tag{5.12.46}$$

We have inserted for κ from Eq. (5.12.38), for $\varrho_0 = Bx_0^3 = B(y_0^2-1)^{3/2}$ from Eq. (1.6.4), and for $Y^2 - 1/y_0^2$ from the hydrostatic equation (5.12.39).

The monotonic increase of total mass with increasing central density ϱ_0 (or with increasing y_0) can be understood from a simple order-of-magnitude evaluation (cf. Cox and Giuli 1968): From the equation of hydrostatic equilibrium there results $P_0 \propto GM_1\varrho_0/r_1$, and inserting for r_1 from $\varrho_0 \propto \varrho_m \propto M_1/r_1^3$, we get $P_0 \propto GM_1^{2/3}\varrho_0^{4/3}$. Also, the pressure of completely degenerate electrons can be approximated by $P_0 \propto \varrho_0^{1+1/n}$, where $1.5 \le n \le 3$. Equating this with the hydrostatic pressure, we get $M_1^{2/3} \propto \varrho_0^{1/n-1/3}$, showing that mass is an increasing function of central density, provided that $0 < n < 3$.

Hence, the Chandrasekhar limiting mass of a white dwarf occurs if its central density ϱ_0 tends to infinity ($x_0, y_0 \to \infty$), and this takes place just for the extremely relativistic case with the structure equation (5.12.43) equal to a $n = 3$ polytrope. The Chandrasekhar limiting mass is obtained from Eq. (5.12.46) if $\zeta^2\ dY/d\zeta$ just assumes the surface value $\xi_1^2\theta_1' = -2.018$ of the $n = 3$ polytrope:

$$\begin{aligned} M_1 = M_{Ch} &= -(2^{7/2}/\pi^{1/2}B^2)(A/G)^{3/2}(-\zeta^2\ dY/d\zeta)_{\zeta=\zeta_1} = (6^{1/2}/8\pi)(ch/G)^{3/2}(N_A/\mu_e)^2 \\ &\times(-\zeta^2\ dY/d\zeta)_{\zeta=\zeta_1} = 5.836/\mu_e^2, \qquad [(-\zeta^2\ dY/d\zeta)_{\zeta=\zeta_1} = 2.018;\ x_0, y_0 = \infty], \end{aligned} \tag{5.12.47}$$

where M_{Ch} is in solar units. For massive white dwarfs the hydrogen content is likely to be small, so $\mu_e \approx 2$ [Eq. (1.7.23)], and the limiting mass of completely degenerate configurations is about $M_{Ch} = 1.46$ solar masses. This limiting mass would be attained for an infinite central density $\varrho_0 = B(y_0^2-1)^{3/2}$ and zero radius $r_1 = (2A/\pi G)^{1/2}\zeta_1/By_0 = 0,\ (y_0 = \infty)$.

As real matter, including black holes, is not likely to reach infinite density, the Chandrasekhar mass (5.12.47) is a limiting mass, which cannot be attained in practice by white dwarfs. Moreover, for densities above about 10^9 g cm^{-3} the energies of degenerate electrons become large enough to initiate inverse β-decays in iron stars, i.e. electron capture by a nucleus according to the equation (Sec. 1.7, Zeldovich and Novikov 1971, Chap. 6)

$$(A, Z) + e^- = (A, Z-1) + \nu, \tag{5.12.48}$$

where A and Z mean atomic mass and charge number, e^- denotes an electron, and ν a freely escaping neutrino. Disintegration of neutron-rich nuclei produces free, degenerate nonrelativistic neutrons. The equation of state in the density interval $10^{11} - 10^{13}$ g cm^{-3} can possibly be approximated with the equation of state (1.7.34) of a completely degenerate, nonrelativistic neutron gas. At densities of about 1.15×10^9 g cm^{-3} there starts the transformation (5.12.48) of $^{56}_{26}$Fe into $^{56}_{25}$Mn, $^{56}_{24}$Cr, ending possibly with such exotic nuclei as $^{56}_{12}$Mg, $(A/Z = 4.67)$, when formation of free neutrons begins by the reaction $^{56}_{12}\mathrm{Mg} + e^- =^{53}_{11}\mathrm{Na} + 3n + \nu$. Another exotic nucleus that could form during neutronization of matter is $^{8}_{2}$He, $(A/Z = 4)$. During neutronization with constant atomic mass number, the mean molecular weight per free ionization electron from Eq. (1.7.21) $\mu_e = A/Z,\ (i = 1)$ increases from about 2 to about 4. And the rest mass density (1.6.4) of the degenerate electron gas increases during this phase transition at the same rate, since $\varrho \propto \mu_e = A/Z$. For a certain chemical composition neutronization is primarily controlled by pressure, and will, of course, start at the centre, forming at first a small, growing, neutron-rich core.

Ramsey (1950) and Lighthill (1950) have found that the small core of a sphere becomes unstable if the density jump between core and envelope is larger than 1.5, i.e. if

$$k = \varrho_c/\varrho_e = A_cZ_e/A_eZ_c > 1.5. \tag{5.12.49}$$

A simple picture of the instability of small cores, that can be handled algebraically, results with the assumption of incompressible matter, possessing constant core and envelope density:

$$\varrho_c = k\varrho_e, \qquad (\varrho_c, \varrho_e = \mathrm{const}). \tag{5.12.50}$$

The pressure on the core boundary is just equal to the critical pressure P_{cr} at which a phase transition occurs from envelope density ϱ_e to core density $\varrho_c = k\varrho_e$. This pressure is obtained by integrating the equation of hydrostatic equilibrium between core radius r_c and envelope radius r_e, the latter being just equal to the total radius r_1 of the star:

$$dP/dr = -G\varrho M(r)/r^2 = -(4\pi G\varrho_e/3r^2)[\varrho_c r_c^3 + \varrho_e(r^3 - r_c^3)], \qquad (r_c \le r \le r_e). \tag{5.12.51}$$

We get

$$P_{cr} = P(r_c) = (2\pi G\varrho_e^2/3)[r_e^2 - r_c^2 + 2(k-1)r_c^3(1/r_c - 1/r_e)]. \tag{5.12.52}$$

It is convenient to introduce a star whose central pressure is just P_{cr}, i.e. a star without core. Eq. (5.12.52) reads with $r_c = 0$:

$$P_{cr} = P(0) = 2\pi G\varrho_e^2 r_{cr}^2/3, \qquad (r_e = r_{cr};\ r_c = 0), \tag{5.12.53}$$

where r_{cr} denotes the radius of a star with central pressure P_{cr}. If there is a core, the central core pressure $P(0) = P_{c0}$ can be obtained in terms of $P_{cr} = P(r_c)$, by integrating the hydrostatic equilibrium equation $dP/dr = -G\varrho M(r)/r^2 = -4\pi G\varrho_c^2 r/3$ over the core:

$$P_{c0} - P_{cr} = 2\pi G\varrho_c^2 r_c^2/3 \quad \text{or} \quad P_{c0} = P_{cr}[1 + (kr_c/r_{cr})^2]. \tag{5.12.54}$$

The total mass of this composite model is obviously

$$M_e = (4\pi\varrho_e/3)[r_e^3 + (k-1)r_c^3]. \tag{5.12.55}$$

The condition (5.12.52) that the core boundary is at critical pressure P_{cr} can be written via Eq. (5.12.53) as

$$r_e^2 - 2(k-1)r_c^3/r_e + (2k-3)r_c^2 - r_{cr}^2 = 0. \tag{5.12.56}$$

Eqs. (5.12.54)-(5.12.56) contain all physical assumptions, and we have to find out the conditions on the variables M_e, r_e, r_c, P_{c0} which are compatible with these equations. We investigate the sign of the derivative dM_e/dP_{c0}, i.e. the shape of the curve total mass versus central pressure. We have

$$dM_e/dP_{c0} = (dM_e/dr_c)(dr_c/dP_{c0}) = r_{cr}^2(dM_e/dr_c)/2k^2P_{cr}r_c. \tag{5.12.57}$$

Since dr_c/dP_{c0} from Eq. (5.12.54) is invariably positive, it will suffice to investigate the sign of dM_e/dr_c. The latter derivative is by virtue of Eqs. (5.12.55), (5.12.56) equal to

$$dM_e/dr_c = 4\pi\varrho_e[r_e^2\ dr_e/dr_c + (k-1)r_c^2]$$
$$= 4\pi\varrho_e(M_{cr}/M_e)(r_e/r_{cr})^3 r_e r_c[3-2k+4(k-1)r_c/r_e+(k-1)^2(r_c/r_e)^4], \qquad (5.12.58)$$

where r_{cr} is defined by Eq. (5.12.53), and $M_{cr} = 4\pi\varrho_e r_{cr}^3/3$ is the mass of a coreless sphere with central pressure just equal to the critical pressure (5.12.53). The derivative (5.12.58) takes negative values if $r_c/r_e \approx 0$ and $k = \varrho_c/\varrho_e > 1.5$, and the configuration becomes accordingly unstable.

If there is no core, the central pressure is given by Eq. (5.12.53) with $P_{cr} \rightarrow P_{c0},\ r_{cr} \rightarrow r_e$:

$$P_{c0} = 2\pi G\varrho_e^2 r_e^2/3 = G(\pi\varrho_e^4 M_e^2/6)^{1/3}. \qquad (5.12.59)$$

In this case the central pressure P_{c0} increases continuously with increasing mass: $dM_e/dP_{c0} > 0$. The portions of the (M_e, P_{c0})-curve with negative derivative belong to unstable configurations, because in this circumstance the central pressure decreases as mass and radius of the configuration increase: $dM_e/dP_{c0} < 0$. The total mass of spheres possessing a core – after attaining a certain minimum at $dM_e/dP_{c0} = 0$, $[dM_e/dr_c = 0$ in Eq. (5.12.58)] – will again increase with increasing pressure ($dM_e/dP_{c0} > 0$), since for moderate ratios r_c/r_e the central pressure is given approximately by Eq. (5.12.59); this can be shown by integration of the hydrostatic equilibrium equation between $r = 0$ and $r = r_e$:

$$P_{c0} = (2\pi G\varrho_e^2/3)[r_e^2 + (k^2-1)r_c^2 + 2(k-1)r_c^3(1/r_c - 1/r_e)]. \qquad (5.12.60)$$

Thus, if portions occur on the (M_e, P_{c0})-curve with negative derivative dM_e/dP_{c0}, $(dM_e/dr_c < 0)$, there will exist for the same mass M_e at least three distinct equilibrium configurations: Two stable spheres (one without core having $P_{c0} < P_{cr}$, and the other one with a core having $P_{c0} > P_{cr}$), and a third unstable sphere with a small core and intermediate central pressure having $k > 1.5$, $[dM_e/dr_c < 0$ in Eq. (5.12.58)].

Lighthill [1950, see also Seidov (1967)] has shown that Ramsey's (1950) results are also valid for compressible bodies, in particular for polytropes. For isentropic polytropes with a phase jump of density there exists a maximum value k_m of $k = \varrho_c/\varrho_e$ below which the sphere is always stable: $k_m = 3/2, 1.46, 1.33, 1.20, 1.09, 1.00$ if the polytropic index is $n = 1/(\Gamma_1 - 1) = 0, 1, 1.5, 2, 2.5, 3$, respectively (Blinnikov 1975, Bisnovatyi-Kogan 2002, Sec. 12.4.3). The isentropic $n = 3$, $(\Gamma_1 = 4/3)$ polytrope is neutrally stable [Eq. (5.3.1)], so any phase transition leads to instability.

As we have already outlined subsequently to Eq. (5.12.48), the ratio (5.12.49) becomes during neutronization $\gtrsim 2$, so instability of the neutron core will eventually set in as its size grows.

Another kind of instability arises from the effects of general relativity on the adiabatic exponents (4.1.65). Chandrasekhar (1964b, 1965b; see also Sec. 5.12.4) has shown that the Newtonian value $\Gamma_1 < 4/3$ from Eq. (5.3.1) – as required for dynamical instability – is increased by general relativistic effects, i.e. radial instability is enhanced in the post Newtonian approximation. If $\Gamma_1 \approx 4/3$, $(\Gamma_1 > 4/3)$, dynamical instability against radial oscillations occurs at a radius smaller than

$$r_1 = 2C(n)\ GM_{r1}/c^2(\Gamma_1 - 4/3), \qquad (\Gamma_1 \approx 4/3;\ \Gamma_1 > 4/3), \qquad (5.12.61)$$

where $C(n)$ is of order unity and depends on the polytropic index n [see Eq. (5.12.144)]. As obvious from the external Schwarzschild metric (4.1.20), the effects of relativity become important if the mass is compressed close to its Schwarzschild radius (4.1.28). Otherwise, the metric remains nearly Galilean. However, if the adiabatic exponent Γ_1 is close to its critical value 4/3, as required for transition from stability to instability, the effects of general relativity – as described by Eq. (5.12.61) – become marked, even if the radius of the sphere is much larger than its Schwarzschild radius.

For an isentropic relativistic polytrope Bludman (1973a) has numerically calculated the values of the relativity parameter $q_0 = P_0/\varepsilon_{r0} = K\varrho_{r0}^{1/n}/c^2$ from Eq. (4.1.31) for which the fundamental radial mode becomes unstable. Eqs. (4.1.37), (4.1.38) are integrated to calculate the total relativistic mass (4.1.49):

$$M_{r1} = 4\pi\varrho_{r0}\alpha^3\eta(\xi_1) = [(n+1)^{3/2}K^{n/2}c^{3-n}/(4\pi)^{1/2}G^{3/2}]q_0^{(3-n)/2}\eta(\xi_1) = \mathrm{const}\ q_0^{(3-n)/2}\eta(\xi_1). \qquad (5.12.62)$$

We have used Eqs. (4.1.29), (4.1.31), where $\eta(\xi_1)$ depends implicitly on q_0. By virtue of Eq. (5.12.29) the polytrope starts instability against radial oscillations if $(\partial M_{r1}/\partial\varrho_{r0})_{S=\mathrm{const}} = 0$, and the corresponding critical values of q_0 are found to be $q_{0,cr} = 0, 0.097, 0.42, 1.24, \infty$ if $n = 3, 2, 1, 0.5, 0$, respectively. If $0.926 < n < 3$, there subsists in Bludman's (1973a) isentropic polytropes the delimitation

(4.1.68): $q_{0,cr} \leq 1/\Gamma_{r1} = n/(n+1)$, $(S = \mathrm{const})$. If $q_{0,cr} > n/(n+1)$, the isentropic polytrope would have a superluminal core (sound velocity larger than light velocity), and this would happen if $0 < n < 0.926$, $(\Gamma_{r1} > 2.08;\ q_{0,cr} > 0.48)$. The critical values $\Gamma_1 = 4/3$ or $n = 1/(\Gamma_1 - 1) = 3$ required in Newtonian gravitation for unstable oscillations are changed by general relativistic effects to [Bludman 1973a, Eq. (3.4)]:

$$\Gamma_{r1} = 4/3 + 1.73q_{0,cr};\quad n = 1/(\Gamma_{r1} - 1) = 3 - 15.57q_{0,cr},\qquad (q_{0,cr} \approx 0). \tag{5.12.63}$$

This relationship is of the same form as Eqs. (5.12.61), (5.12.144), since these can be transformed with the aid of Eq. (4.1.49) into Eq. (5.12.63):

$$\Gamma_1 = 4/3 + 2C(n)\ GM_{r1}/c^2 r_1 = 4/3 + 2C(n)\ (n+1)\eta(\xi_1)\ q_{0,cr}/\xi_1 \approx 4/3 + \mathrm{const}\ q_{0,cr} > 4/3. \tag{5.12.64}$$

Eq. (5.12.61) is pertinent for relativistically degenerate white dwarfs with central densities larger than $1.5\times10^{10} - 3\times10^{10}$ g cm^{-3} (depending on chemical composition), when the adiabatic exponent approaches $\Gamma_1 = 4/3$ (see Table 1.7.2). However, for atomic weight numbers of order $A \approx 20$, as occurring in white dwarfs, the critical density for neutronization – when instability of the neutron core takes place according to Eq. (5.12.49) – is about an order of magnitude lower (Zeldovich and Novikov 1971, Table 11). For nuclei lighter than $^{16}_{8}O$ the story would be different, because in this case the critical density of neutronization is generally larger than the density required for instability due to general relativity – but light nuclei will not exist at the prevailing high densities. Thus, the stability limit of white dwarfs apparently occurs due to neutronization [$k > 1.5$ in Eq. (5.12.49)], rather than because of general relativistic effects described by Eq. (5.12.61). The maximum stable mass of a white dwarf is about $1.2M_\odot$ (see first peak in Fig. 5.12.1), somewhat smaller than the Chandrasekhar limit (5.12.47) of about $1.46M_\odot$, $(\mu_e = 2)$. On the descending branches of the $M_{r1}(\varrho_{r0})$-curve from Fig. 5.12.1 – when $(\partial M_{r1}/\partial \varrho_{r0})_S < 0$ – all equilibrium configurations are unstable by virtue of Eq. (5.12.29).

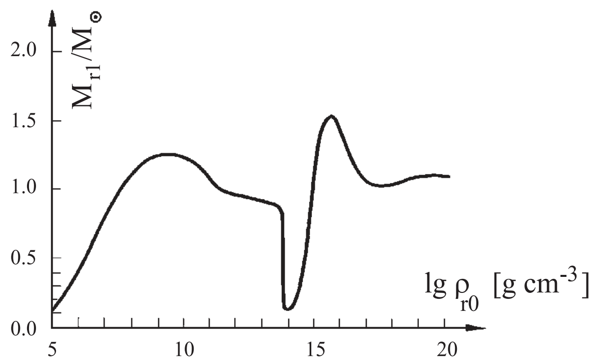

Fig. 5.12.1 Mass M_{r1} of a cold star $(T, S = 0)$ in hydrostatic equilibrium as a function of relativistic central density ϱ_{r0}. Below nuclear densities of 2×10^{14} g cm^{-3} we have $M_{r1} \approx M_1$, $\varrho_{r0} \approx \varrho_0$ according to Eq. (5.12.31). First peak is the Chandrasekhar limiting mass of a white dwarf M_{Ch}, second peak is the Oppenheimer-Volkoff maximum mass of a neutron star M_{OV}. The present curve corresponds to the curve labeled $T = 0,\ S = 0$ on the schematic figure 5.12.3 (Zeldovich and Novikov 1971, Fig. 39; see also Bisnovatyi-Kogan 2002, Fig. 11.1).

At densities larger than about 1.5×10^{12} g cm^{-3} [cf. Eq. (1.7.33)], the equation of state (1.7.34) of a nonrelativistic degenerate neutron gas becomes appropriate, up to densities $\approx 2.8 \times 10^{13}$ g cm^{-3} (Eq. (1.7.35), domain 5 in Fig. 1.7.1). Oppenheimer and Volkoff (1939) have integrated the general relativistic equation of spherical hydrostatic equilibrium (4.1.27) with the equation of state (1.7.34), obtaining a maximum mass of $M_{OV} = 0.71\ M_\odot$ for a star that can be in hydrostatic equilibrium (second peak in Fig. 5.12.1). With purely Newtonian gravitation the Chandrasekhar limit (5.12.47) of a neutron star (a $n = 3$ polytrope) would be considerably larger $M_1 = 5.836/\mu_n^2 = 5.836\ M_\odot$, $(\mu_n \approx 1)$. Calculations with more refined equations of state yield masses in the range $M_{OV} = 0.6 - 2.7\ M_\odot$ for the Oppenheimer-Volkoff limit (Cox and Giuli 1968, Shapiro and Teukolsky 1983, Table 9.1). The polytropic index n decreases from 3.48 to 0.32 for polytropic models of neutron stars with masses ranging from 0.131 to $2.1 M_\odot$ (Lai et al. 1994a).

Cold stars with larger masses cannot be in hydrostatic equilibrium. They undergo hydrodynamic relativistic collapse, described by the concept of a black hole, although no such structures have unequivocally been discovered. The assumption of black holes would solve the dilemma of cold stars with masses larger than M_{OV}. For the external distant observer the collapse into a black hole stops at the Schwarzschild radius (4.1.28) after infinite time, whereas the comoving mass reaches after *finite* proper time the singularity $r = 0$, attaining perhaps infinite density. As nature tends to avoid singularities, it is questionable whether such hypothetical objects like black holes can ever come to existence.

Uniform rotation does not alter significantly the above conclusions concerning cold spherical matter (Zeldovich and Novikov 1971, Landau and Lifschitz 1987).

5.12.3 Stability of Massive Hot Polytropes

General relativistic effects are not only important for cold masses near and beyond the Oppenheimer-Volkoff limit, but also for hot supermassive objects. From Fig. 5.12.3 it is apparent that stable or unstable hydrostatic solutions exist only inside the domain that is bounded by the ordinate M_{r1}, and the two lines aa' and $T, S = 0$. Below, we determine an elementary lower limit for the mass of a star with pressure and internal energy mainly due to radiation. As already mentioned in Sec. 1.4, the mass density is given by the nonrelativistic gas density ϱ – the equivalent mass density $\varrho_r = aT^4/c^2$ of radiation being generally negligible. Via Eq. (C.11) we get from Eq. (2.6.25) the central pressure of a spherical $n = 3$ polytrope:

$$P_0 = (GM_1^2/r_1^4)/4\pi(n+1)(d\theta/d\xi)^2_{\xi=\xi_1} = (4\pi)^{1/3}GM_1^{2/3}\varrho_m^{4/3}/3^{4/3}(n+1)(d\theta/d\xi)^2_{\xi=\xi_1}$$
$$= (4\pi)^{1/3}GM_1^{2/3}\varrho_0^{4/3}/(n+1)\xi_1^{4/3}(-d\theta/d\xi)^{2/3}_{\xi=\xi_1} = 0.36GM_1^{2/3}\varrho_0^{4/3},$$
$$(1-\beta = P_r/P \approx 1;\ N = 3;\ n = 1/(\Gamma_1 - 1) = 3). \qquad (5.12.65)$$

We have taken $n = 3$, $\Gamma_1 = 1 + 1/n = 4/3$, since this is the appropriate value for an isentropic configuration where radiation pressure prevails [Eq. (1.4.6)]. We equate the central pressure (5.12.65) to the total pressure (1.4.11) at the centre

$$\mathcal{R}\varrho_0 T_0/\mu + aT_0^4/3 = 0.36GM_1^{2/3}\varrho_0^{4/3}, \qquad (n = 3), \qquad (5.12.66)$$

and get the remarkable result that the total mass of a spherical $n = 3$ polytrope is uniquely related to the parameter $T_0/\varrho_0^{1/3}$:

$$(\mathcal{R}/\mu)(T_0/\varrho_0^{1/3}) + (a/3)(T_0/\varrho_0^{1/3})^4 = 0.36GM_1^{2/3}, \qquad (n = 3). \qquad (5.12.67)$$

Then, a star for which the central values of gas and radiation pressure are just equal, has the mass

$$M_1 = (2\mathcal{R}T_0/0.36G\mu\varrho_0^{1/3})^{3/2} = [2 \times 6^{1/2}/(0.36G)^{3/2}a^{1/2}](\mathcal{R}/\mu)^2 = 52.6M_\odot/\mu^2,$$
$$(\varrho_0 = a\mu T_0^3/3\mathcal{R}). \qquad (5.12.68)$$

For completely ionized material we have $1/2 \leq \mu \leq 2$ by virtue of Eq. (1.7.17). Pressure and internal energy of radiation will be dominant over plasma contributions for hydrostatic configurations larger than about 100 solar masses (above the broken line in Fig. 5.12.3).

If radiation pressure is dominant, the adiabatic index is close to its critical value for instability $\Gamma_1 = 4/3$ (Secs. 1.4, 1.7). The gas (plasma) makes a positive contribution to stability, since $\Gamma_{1g} \geq 4/3$ [Eqs. (1.7.59), (1.7.60)], whereas positron-electron pairs and general relativistic effects enhance instability: In regions where $e^{\pm}$-pairs are dominant, the adiabatic exponent Γ_1 is generally below its critical value 4/3 [Eq. (1.7.63)]. General relativistic effects increase the critical Newtonian value – below which instability occurs – from $\Gamma_1 = 4/3$ to its value (5.12.64). In the zeroth approximation a massive hot star can be considered as a radiation dominated body with energy density and specific internal energy given by [cf. Eqs. (1.4.1), (1.4.13)]

$$\varepsilon_r = \varrho U = \varepsilon^{(int)} = aT^4; \quad U = \varepsilon_r/\varrho = aT^4/\varrho. \qquad (5.12.69)$$

The specific entropy of radiation is calculated from its definition (1.1.4), with the first law of thermodynamics (1.1.2) applied to constant proper volume:

$$dS = dQ/T = dU/T = 4aT^2/\varrho, \qquad (dV = 0). \tag{5.12.70}$$

This integrates to

$$S = 4aT^3/3\varrho, \qquad (m = 1). \tag{5.12.71}$$

The energy density of radiation is expressed in terms of entropy as

$$\varepsilon_r = \varepsilon^{(int)} = (3S\varrho/4)^{4/3}/a^{1/3}, \qquad (m = 1), \tag{5.12.72}$$

and the total energy (5.12.11) of the supermassive object becomes at constant entropy just equal to that of an isentropic $n = 3$ polytrope:

$$\begin{aligned} E &= \int_{V_1} \varepsilon^{(int)}\, dV - \int_{M_1} GM(r)\, dM/r = [\pi(3S\varrho_0)^{4/3}/(4a)^{1/3}]\alpha^3 \int_0^{\xi_1} \xi^2\theta^4\, d\xi - 3GM_1^2/2r_1 \\ &= \pi\alpha^3(3S\varrho_0)^{4/3}(2/a)^{1/3}\xi_1^3{\theta_1'}^2 - 3^{2/3}\pi^{1/3}M_1^{5/3}\varrho_m^{1/3}/2^{1/3} \\ &= (1/32a)^{1/3}M_1\varrho_0^{1/3}(3S)^{4/3}\xi_1(-\theta_1') - 3\pi^{1/3}M_1^{5/3}\varrho_0^{1/3}[(-\theta_1')/\xi_1]^{1/3}/2^{1/3}, \qquad (n = 3;\ S = \mathrm{const}), \end{aligned} \tag{5.12.73}$$

where we have taken into account Eqs. (2.6.18), (2.6.27), (2.6.137), (2.6.159).

For hydrostatic equilibrium we must have $(\partial E/\partial \varrho_0)_{S,M_1=\mathrm{const}} = 0$ via Eq. (5.12.20), which amounts to

$$(3/16a)^{1/3}S^{4/3}\xi_1^{4/3}(-\theta_1')^{2/3} - \pi^{1/3}M_1^{2/3} = 0 \qquad \text{or} \qquad E = 0. \tag{5.12.74}$$

In a more general Newtonian approximation Fowler (1964) starts with Eq. (5.12.11), by inserting for $\varepsilon^{(int)}$ from Eq. (2.6.92):

$$\begin{aligned} E &= \int_{V_1} P\, dV/(\Gamma - 1) - \int_{M_1} GM(r)\, dM/r = PV/(\Gamma - 1)\Big|_0^{V_1} - [1/(\Gamma - 1)] \int_{V_1} V\, dP + W \\ &= -[1/3(\Gamma - 1)] \int_{V_1} 4\pi G\varrho M(r)\, r\, dr + W = (3\Gamma - 4)W/3(\Gamma - 1), \qquad (P(V_1) = 0;\ \Gamma = \mathrm{const}). \end{aligned} \tag{5.12.75}$$

For a gas-radiation mixture the quantity Γ can be roughly approximated with the adiabatic index Γ_1 [cf. Eqs. (1.4.20), (2.6.93)], and it is apparent that at temperatures above 10^9 K, electron-positron pair formation can led to unstable stars with total positive energy ($E > 0$), since in virtue of Eq. (1.7.63) the minimum value of Γ_1 is $1.22 < 4/3$.

For a mixture of perfect gas and radiation we find via Eqs. (1.2.5), (1.4.11), (1.4.13), (1.4.14), (1.7.59), (2.6.93), (5.12.69):

$$\begin{aligned} &\Gamma - 1 = P/\varepsilon^{(int)} = P_g/\beta[P_g/(\gamma_g - 1) + 3P_r] = 1/\beta[f/w + 3(1 - \beta)/\beta], \\ &(\beta = P_g/P;\ f \geq 3;\ 1 \leq w \leq 2). \end{aligned} \tag{5.12.76}$$

For a completely ionized plasma we have $f = 3$, and the quantity $w = 1 + (1 - v^2/c^2)^{1/2}$ from Eq. (1.7.53) is equal to 2 in the nonrelativistic limit $v \approx 0$, and approaches 1 in the extreme relativistic case $v \approx c$. The total energy of the sphere becomes

$$E = \beta(1 - f/3w)W = \beta(1 - 1/w)W, \qquad (f = 3). \tag{5.12.77}$$

Thus, the total energy of a classical star is bounded by $E = W/2$, ($w = 2,\ \beta = 1$; small stars with nonrelativistic ionized gas without radiation pressure) and $E = 0$ (massive stars with dominant radiation pressure $\beta = 0$).

In the post Newtonian approximation Fowler (1964) calculates the total energy (4.1.79), which is in this case just equal but opposite in sign to the binding energy (4.1.77), i.e. equal to the difference between relativistic energy and rest energy:

$$E = -E_b = c^2(M_{r1} - M_1). \tag{5.12.78}$$

In virtue of Eqs. (2.6.92), (4.1.2), (4.1.71), (4.1.72), (5.12.73) this equation can be written in the post Newtonian approximation as

$$E = 4\pi c^2 \int_0^{r_1} \{\varrho_r - \varrho/[1-2GM_r(r)/c^2r]^{1/2}\}r^2\ dr$$
$$= 4\pi c^2 \int_0^{r_1} \varrho_r\{1-[1-2GM_r(r)/c^2r]^{-1/2}\}r^2\ dr + 4\pi \int_0^{r_1} \varepsilon^{(int)}[1-2GM_r(r)/c^2r]^{-1/2}r^2\ dr$$
$$\approx -4\pi \int_0^{r_1} GM_r(r)\ \varrho_r r\ dr - (6\pi G^2/c^2)\int_0^{r_1} M^2(r)\ \varrho\ dr$$
$$+4\pi \int_0^{r_1} P[1+GM(r)/c^2r]r^2\ dr/(\Gamma-1) = -\int_{M_{r1}} GM_r(r)\ dM_r/r - (6\pi G^2/c^2)\int_0^{r_1} M^2(r)\ \varrho\ dr$$
$$+4\pi r^3 P[1+GM(r)/c^2r]/3(\Gamma-1)\Big|_0^{r_1} - [4\pi/3(\Gamma-1)]\Big\{\int_0^{r_1} r^3[1+GM(r)/c^2r]\ dP$$
$$-(G/c^2)\int_0^{r_1} PM(r)\ r\ dr + (4\pi G/c^2)\int_0^{r_1} P\varrho r^4\ dr\Big\}, \qquad (\varrho_r \approx \varrho;\ M_r \approx M), \tag{5.12.79}$$

where in the relativistic terms we can replace M_r and ϱ_r by their Newtonian values M and ϱ, respectively.

To evaluate dP we consider a post Newtonian approximation of the relativistic equation of hydrostatic equilibrium (4.1.27) with the factor $[1-2GM_r(r)/c^2r]^{-3/2}$ on its right-hand side:

$$[1-2GM_r(r)/c^2r]^{-1/2}\ dP/dr \approx [1+GM(r)/c^2r]\ dP/dr \approx -G[\varrho_r M_r(r)/r^2 + 4\pi P\varrho r/c^2$$
$$+PM(r)/c^2r^2 + 3G\varrho M^2(r)/c^2r^3], \qquad (\varrho_r \approx \varrho;\ M_r \approx M). \tag{5.12.80}$$

Insertion into Eq. (5.12.79) yields after some algebra (Fowler 1964)

$$E = (3\Gamma-4)W_r/3(\Gamma-1) + [2(5-3\Gamma)\pi G^2/c^2(\Gamma-1)]\int_0^{r_1} \varrho M^2(r)\ dr$$
$$+[8\pi G/3c^2(\Gamma-1)]\int_0^{r_1} PM(r)\ r\ dr, \qquad (\Gamma = \mathrm{const}), \tag{5.12.81}$$

where

$$W_r = -\int_{M_{r1}} GM_r(r)\ dM_r/r, \tag{5.12.82}$$

is the relativistic counterpart (4.1.73) of the Newtonian gravitational energy. It is clear that the small post Newtonian corrections given by the two integrals in Eq. (5.12.81), can markedly influence the stability of a configuration only if the term $(3\Gamma-4)W_r/3(\Gamma-1)$ is close to zero, i.e. if $\Gamma \approx 4/3$. Inserting $\Gamma = 4/3$ into the relativistic corrections, and using for $(3\Gamma-4)/3(\Gamma-1)$ its value from Eq. (5.12.76), we obtain

$$E = \beta(1-1/w)W_r + (6\pi G^2/c^2)\int_0^{r_1} \varrho M^2(r)\ dr + (8\pi G/c^2)\int_0^{r_1} PM(r)\ r\ dr, \quad (\Gamma \approx 4/3;\ f=3). \tag{5.12.83}$$

For nondegenerate matter and negligible contribution of electron-positron pairs we get with Eqs. (1.4.11), (1.4.14)

$$P = P_r/(1-\beta) = aT^4/3(1-\beta) = P_g/\beta = \mathcal{R}\varrho T/\mu\beta \quad \text{or} \quad T = [3\mathcal{R}(1-\beta)/a\mu\beta]^{1/3}\varrho^{1/3}, \tag{5.12.84}$$

and

$$P = \mathcal{R}\varrho T/\mu\beta = [3(1-\beta)(\mathcal{R}/\mu)^4/a\beta^4]^{1/3}\varrho^{4/3} = K\varrho^{4/3}. \tag{5.12.85}$$

If $\beta = \mathrm{const}$ throughout the configuration, its radial structure is that of a $n=3$ polytrope. The mass (2.6.18) of such a polytrope is given by the Bialobjesky-Eddington equation

$$M_1 = 4\pi[(n+1)K/4\pi G]^{3/2}\varrho_0^{(3-n)/2n}\xi_1^2(-\theta_1') = 4[3(1-\beta)(\mathcal{R}/\mu)^4/\pi a G^3\beta^4]^{1/2}\xi_1^2(-\theta_1'),$$
$$(n=3;\ \beta = \mathrm{const}). \tag{5.12.86}$$

For vanishing gas pressure this amounts to (cf. Eq. (5.12.68), Chandrasekhar 1939, Table 6)

$$\beta \approx (2\mathcal{R}/\mu)(3/\pi a G^3)^{1/4}\xi_1(-\theta_1')^{1/2}/M_1^{1/2} = (4.28/\mu)(M_\odot/M_1)^{1/2},$$
$$(\beta = \mathrm{const} \approx 0;\ n = 3;\ M_1 \gtrsim 10^3\ M_\odot). \qquad (5.12.87)$$

As noted subsequently to Eq. (1.7.32), protons become relativistic only at temperatures of order 10^{13} K, so we can take in Eq. (5.12.83) the nonrelativistic value $w = 2$. For massive stars we have in virtue of Eq. (5.12.87) $\beta \approx 0$, and if we take $f = 3$, as for complete ionization, we get from Eq. (5.12.76) $\Gamma \approx 4/3$ and $n = 1/(\Gamma - 1) \approx 3$, $(S = \mathrm{const})$. Further, βW_r is already of first order of smallness, and in the considered post Newtonian approximation we can replace βW_r by the Newtonian value $\beta W = -3\beta G M_1^2/2r_1$ (see Eq. (2.6.137) if $n = 3$). Eq. (5.12.83) writes in polytropic variables as

$$E = -3\beta G M_1^2/4r_1 + (96\pi^3 G^2 \varrho_0^3 \alpha^7/c^2)\left[\int_0^{\xi_1} \xi^4\theta^3\theta'^2\ d\xi - (1/3)\int_0^{\xi_1} \xi^3\theta^4\theta'\ d\xi\right]$$
$$= -3\beta G M_1^2/4r_1 + (G^2M_1^3/c^2r_1^2)[1/2\xi_1^4(-\theta_1')^3]\left(3\int_0^{\xi_1} \xi^4\theta^3\theta'^2\ d\xi - \int_0^{\xi_1} \xi^3\theta^4\theta'\ d\xi\right)$$
$$= -3\beta G M_1^2/4r_1 + 5.1\ G^2M_1^3/c^2r_1^2 = -0.32\ \beta G M_1^{5/3}\varrho_0^{1/3} + 0.93\ G^2 M_1^{7/3}\varrho_0^{2/3}/c^2,$$
$$(S = \mathrm{const};\ \beta \approx 0;\ n = 3;\ \Gamma = 1 + 1/n = 4/3;\ r_1^3 \approx 40.6\ M_1/\pi\varrho_0). \qquad (5.12.88)$$

The two integrals in this equation have been evaluated by Fowler (1964) if $n = 3$. The relativistic correction to the hydrostatic equilibrium energy is positive.

By virtue of Eq. (5.12.33) general relativistic effects become important if $r_1 \approx r_g = 2GM_{r1}/c^2$ or if $\varrho_m = 1.84 \times 10^{16}(M_\odot/M_1)^2$, $(M_{r1} \approx M_1)$. If $M_1 \approx 10^8 M_\odot$, we get quite normal nonrelativistic stellar densities of order $\varrho_m \approx 2$ g cm^{-3}, $\varrho_0 \approx 100$ g cm^{-3}, $(n = 3)$.

In any case, the configuration becomes unstable if its total energy E is positive, and this occurs in virtue of Eq. (5.12.88) when the relativistic correction term $5.1\ G^2M_1^3/c^2r_1^2$ is comparable to the Newtonian energy $3\beta G M_1^2/4r_1$:

$$r_g/r_1 = 2GM_1/c^2r_1 = 3\beta/10.2 \approx 0.29\beta = (1.26/\mu)(M_\odot/M_1)^{1/2}, \qquad (M_{r1} \approx M_1). \qquad (5.12.89)$$

The corresponding average density $\varrho_m = 3M_1/4\pi r_1^3$ required for instability would be

$$\varrho_m \approx (3/16\pi)(c^2/G\mu)^3(M_\odot^3/M_1^7)^{1/2} = (3.69 \times 10^{16}/\mu^3)(M_\odot/M_1)^{7/2}\ [\mathrm{g\ cm^{-3}}]. \qquad (5.12.90)$$

But according to Zeldovich and Novikov (1971) instability already starts at the last equilibrium state, beyond the point where the equilibrium energy (5.12.88) possesses its minimum, i.e. when $dE/d\varrho_0 = 0$, or if via Eq. (5.12.87):

$$\varrho_0 = 5.09 \times 10^{-3}(\beta c^2/G)^3/M_1^2 = (2.46 \times 10^{17}/\mu^3)(M_\odot/M_1)^{7/2}\ [\mathrm{g\ cm^{-3}}];$$
$$\varrho_m = (4.54 \times 10^{15}/\mu^3)(M_\odot/M_1)^{7/2}\ [\mathrm{g\ cm^{-3}}], \qquad (n = 3). \qquad (5.12.91)$$

For higher densities the polytrope becomes unstable. The critical central temperature of the last equilibrium state is via Eqs. (5.12.84), (5.12.87), (5.12.91) equal to

$$T_0 \approx (3\mathcal{R}/a\mu\beta)^{1/3}\varrho_0^{1/3} = (1.23 \times 10^{13}/\mu)M_\odot/M_1, \qquad (n = 3), \qquad (5.12.92)$$

and it is seen that for massive hot objects the minimum equilibrium energy (5.12.88) of the last stable state $dE/d\varrho_0 = 0$ does not depend on mass:

$$E = -9 \times 10^{53}/\mu^2\ [\mathrm{erg}], \qquad (n = 3;\ \beta \propto M_1^{-1/2}). \qquad (5.12.93)$$

Eq. (5.12.92) shows that the central temperature is insufficient to assure hydrogen burning ($T_0 \approx 8 \times 10^7$ K) and other nuclear reactions if $M_1 > 3 \times 10^5\ M_\odot$, where $\mu = 1/2$, as for completely ionized hydrogen. We conclude that massive objects with $M_1 \gtrsim 10^5\ M_\odot$ and central densities exceeding the equilibrium value (5.12.91) are gravitationally contracting, and the contraction time scale should be not considerably larger than the free-fall time scale, which is about 7.5×10^{-3} yr if $M_1 = 10^6\ M_\odot$, and 25 yr if $M_1 = 10^8\ M_\odot$. These time scales are completely insufficient to assure a lifetime of $10^5 - 10^6$ yr,

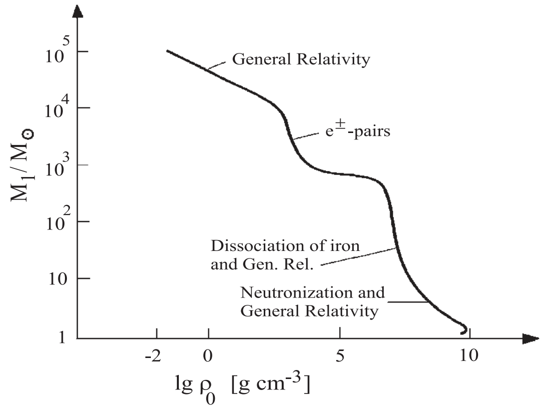

Fig. 5.12.2 Delimitation line between stable (domain on the left) and unstable (domain on the right) hydrostatic equilibrium configurations for an iron star, corresponding to curve bb' on the schematic figure 5.12.3, ($\varrho_{r0} \to \varrho_0$; $M_{r1} \to M_1$). Principal causes of instability in the domain on the right are indicated on the curve (Zeldovich and Novikov 1971, Fig. 55).

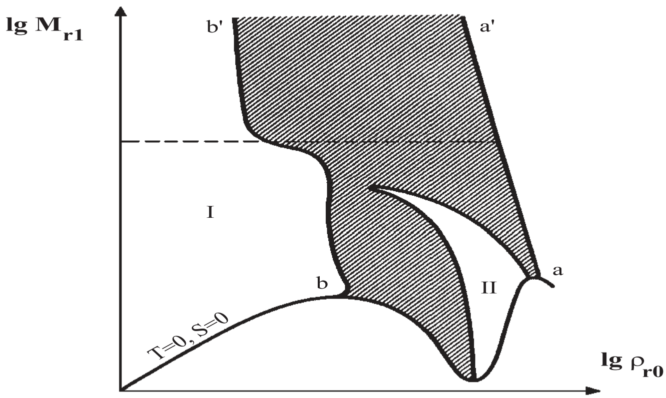

Fig. 5.12.3 Schematic view of possible hydrostatic equilibrium configurations of a sphere having mass M_{r1} and central density ϱ_{r0}. The line aa' is the black hole limit $M_{r1} = 1.36 \times 10^8 M_\odot/\varrho_{r0}^{1/2}$, ($\varrho_{r0} \approx \varrho_m$) from Eq. (5.12.33); to the right of this line no hydrostatic equilibrium structures can exist. The line $T, S = 0$ corresponds to Fig. 5.12.1; below this line no hydrostatic equilibrium structures can exist. Line bb' corresponds to Fig. 5.12.2, and the hatched area consequently exhibits unstable equilibrium configurations. Domain I represents stable hydrostatic equilibrium configurations, imparted by the broken line into a lower – plasma pressure dominated domain, and an upper – radiation pressure dominated domain. Domain II represents stable neutron star-like equilibrium configurations (Zeldovich and Novikov 1971, Fig. 34).

associated with quasars and similar radio sources. Rotation, possibly assisted by internal turbulence and convection, has been suggested to impede rapid gravitational collapse of massive objects (Sec. 5.12.6, Fowler 1964).

The central temperature (5.12.92) increases as $1/M_1$, and becomes with decreasing mass sufficiently large ($T_0 \approx 0.1 m_e c^2/k \approx 6 \times 10^8$ K; $M_1 \approx 4 \times 10^4\ M_\odot$; $\mu = 1/2$) for creation of a significant number of electron-positron pairs, causing instability since $\Gamma_1 < 4/3$. The instability effects due to general relativity and $e^\pm$-pairs are of the same order of magnitude if $M_1 \approx 8000\ M_\odot$ for iron stars, and $M_1 \approx 30000\ M_\odot$ for hydrogen stars – on the average about $10^4\ M_\odot$ (Zeldovich and Novikov 1971, Chap. 10.14).

Below, we sketch the basic causes of instability in an iron star, which has exhausted its nuclear energy supplies (Fig. 5.12.2). Neutronization of iron [transformation of ${}^{56}_{26}$Fe into ${}^{56}_{12}$Mg by a chain of reactions described by Eq. (5.12.48)] induces instability through the instability of small cores in cold stars of solar mass order [Eq. (5.12.49)], as well as in hotter objects up to masses of about $5M_\odot$ (Fig. 5.12.2). General relativistic effects play also a role through Eq. (5.12.61), if the iron star is composed of relativistically degenerate matter having $\Gamma_1 \approx 4/3$ (see Table 1.7.2).

For larger masses, between about 5 and 500 $M_\odot$, the instability of iron stars is likely to be induced by the dissociation of iron according to the reaction ${}^{56}_{26}\mathrm{Fe} = 13\ {}^4_2\mathrm{He} + 4\ {}^1_0\mathrm{n}$, (${}^1_0$n denotes a neutron). It is well known that in dissociation zones the average adiabatic exponent Γ_1 is generally less than 4/3, which implies instability. A similar phenomenon also occurs in protostars at temperatures of about 1800 K, when H_2 dissociates into hydrogen atoms (e.g. Cox and Giuli 1968). Effects of general relativity can also cause instability, since the mass-radius values in this mass range are appropriate to Eq. (5.12.61).

If the mass of the iron star increases from 5 to about 500 $M_\odot$, the role of general relativistic effects, described by Eq. (5.12.61), decreases. Between about $10^3 - 10^4\ M_\odot$ the stability loss of the iron star occurs mainly due to creation of electron-positron pairs in this radiation pressure dominated region [$\Gamma_1 < 4/3$, Eqs. (1.5.8), (1.7.63)].

If $M \gtrsim 10^5\ M_\odot$, the critical state of neutral stability of the supermassive iron star is again controlled by the effects of general relativity described by Eq. (5.12.61), ($\Gamma_1 \approx 4/3$, see Fig. 5.12.2).

A certain mean temperature of the star $T_m = T_m(M_{r1}, \varrho_{r0})$ corresponds to each value in the (M_{r1}, ϱ_{r0})-plane of Figs. 5.12.2, 5.12.3. The strong bend of the stable radiation pressure domain in these figures is due to the fact that in the region $\Gamma_1 \approx 4/3$ small general relativistic effects [Eq. (5.12.61)] or electron-positron pairs [Eq. (1.7.63)] led to unstable equilibrium configurations.

5.12.4 Radial Oscillations of Relativistic Polytropes

With the metric (4.1.3) the contravariant spatial velocity v^α is defined as the change of coordinate x^α, $(\alpha = 1,2,3)$ during the interval of proper time $d\tau$ (e.g. Landau and Lifschitz 1987, §§84, 88):

$$v^\alpha = dx^\alpha/d\tau = c\ dx^\alpha/g_{00}^{1/2}(dx^0 + g_{0\beta}\ dx^\beta/g_{00}), \qquad [c\ d\tau = g_{00}^{1/2}(dx^0 + g_{0\beta}\ dx^\beta/g_{00})]. \tag{5.12.94}$$

The lowering of the index of v^α proceeds with the metric tensor $\gamma_{\alpha\beta}$ of the spatial line element

$$d\ell^2 = \gamma_{\alpha\beta}\ dx^\alpha dx^\beta = (-g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00})\ dx^\alpha dx^\beta, \tag{5.12.95}$$

and the squared spatial velocity is [cf. Eq. (4.2.39)]

$$v^2 = v_\alpha v^\alpha = \gamma_{\alpha\beta}v^\alpha v^\beta = \gamma_{\alpha\beta}\ dx^\alpha\ dx^\beta/d\tau^2 = (d\ell/d\tau)^2. \tag{5.12.96}$$

Hence, the line element of spacetime can be written as

$$ds^2 = g_{ik}\ dx^i dx^k = g_{00}(dx^0 + g_{0\alpha}\ dx^\alpha/g_{00})^2 - d\ell^2 = g_{00}(dx^0 + g_{0\alpha}\ dx^\alpha/g_{00})^2(1 - v^2/c^2). \tag{5.12.97}$$

The four-velocity is found by combining Eqs. (5.12.94) and (5.1.2.97):

$$u^0 = dx^0/ds = dx^0/g_{00}^{1/2}(dx^0 + g_{0\alpha}\ dx^\alpha/g_{00})(1 - v^2/c^2)^{1/2} = (1/g_{00}^{1/2} - g_{0\alpha}v^\alpha/cg_{00})$$
$$/(1 - v^2/c^2)^{1/2};\ u^\alpha = dx^\alpha/ds = dx^\alpha/g_{00}^{1/2}(dx^0 + g_{0\beta}\ dx^\beta/g_{00})(1 - v^2/c^2)^{1/2} = v^\alpha/c(1 - v^2/c^2)^{1/2}. \tag{5.12.98}$$

Since spherical symmetry is preserved during purely radial oscillations, the unperturbed and perturbed metric is given by equations of the form (4.1.5):

$$ds_u^2 = \exp[\nu_u(r)]\ dt^2 - \exp[\kappa_u(r)]\ dr^2 - r^2(d\lambda^2 + \sin\lambda\ d\varphi^2), \tag{5.12.99}$$

$$ds^2 = \exp[\nu(r,t)]\ dt^2 - \exp[\kappa(r,t)]\ dr^2 - r^2(d\lambda^2 + \sin\lambda\ d\varphi^2). \tag{5.12.100}$$

The components $u^\lambda = u^2,\ u^\varphi = u^3$ of the four-velocity vanish identically, since the oscillations proceed radially. The other components are via Eqs. (5.12.98)-(5.12.100) up to the first order in v/c equal to

$$u^t = u^0 = g_{00}^{-1/2} = g_{tt}^{-1/2} = \exp(-\nu/2); \quad u_t = u_0 = g_{00}u^0 = g_{tt}u^t = \exp(\nu/2);$$
$$u^r = u^1 = g_{00}^{-1/2}\ dx^1/dx^0 = g_{tt}^{-1/2}\ dr/dt = \exp(-\nu/2)\ dr/dt \approx \exp(-\nu_u/2)\ dr/dt;$$
$$u_r = u_1 = g_{11}u^1 = g_{rr}u^r \approx -\exp(-\nu_u/2 + \kappa_u)\ dr/dt, \qquad [dr/dt = dx^1/dx^0 = O(v/c)]. \quad (5.12.101)$$

Note, that the sole subsisting components of the spatial velocity (5.12.94) are the radial ones: $v^r = v^1 = (c/g_{00}^{1/2})\ dx^1/dx^0 = (c/g_{tt}^{1/2})\ dr/dt = c\exp(-\nu/2)\ dr/dt$, and $v_r = v_1 = \gamma_{11}v^1 = -g_{11}v^1 = -(cg_{rr}/g_{tt}^{1/2})\ dr/dt = c\exp(-\nu/2+\kappa)\ dr/dt$, $(v = (v_rv^r)^{1/2} = c(-g_{rr}/g_{tt})^{1/2}\ dr/dt)$. This should be contrasted to Chandrasekhar's (1964b) definition $v = c\ dr/dt$ of the radial velocity.

The mixed components of the energy-momentum tensor (4.1.11) are up to the first order in v/c equal to

$$T_t^t = \varepsilon_r; \quad T_r^r = T_\lambda^\lambda = T_\varphi^\varphi = -P; \quad T_t^r = (P+\varepsilon_r)u_tu^r = (P_u + \varepsilon_{ru})\ dr/dt;$$
$$T_r^t = (P+\varepsilon_r)u_ru^t = -(P_u+\varepsilon_{ru})\exp(-\nu_u + \kappa_u)\ dr/dt. \quad (5.12.102)$$

Let us denote the various Eulerian perturbations by (Stergioulas and Friedman 1998)

$$\delta P(r,t) = P(r,t) - P_u(r); \quad \delta\varepsilon_r(r,t) = \varepsilon_r(r,t) - \varepsilon_{ru}(r); \quad \delta\nu(r,t) = \nu(r,t) - \nu_u(r);$$
$$\delta\kappa(r,t) = \kappa(r,t) - \kappa_u(r). \quad (5.12.103)$$

The (r,r) and (t,t)-components of the field equations (4.1.4), appropriate to the metric (5.12.100), are similar to Eqs. (4.1.6) and (4.1.7):

$$\exp(-\kappa)\ [(1/r)\ \partial\nu/\partial r + 1/r^2] - 1/r^2 = 8\pi GP/c^4, \quad (5.12.104)$$

$$\exp(-\kappa)\ [(1/r)\ \partial\kappa/\partial r - 1/r^2] + 1/r^2 = -(1/r^2)\ \partial[r\exp(-\kappa)]/\partial r + 1/r^2 = 8\pi G\varepsilon_r/c^4. \quad (5.12.105)$$

Adding together, we get the useful relationship

$$[\exp(-\kappa)/r]\ \partial(\nu+\kappa)/\partial r = 8\pi G(P+\varepsilon_r)/c^4. \quad (5.12.106)$$

The (t,r)-component $R_t^r = 8\pi GT_t^r/c^4$ of the field equations (4.1.4) assumes the form

$$[\exp(-\kappa)/r]\ \partial\kappa/\partial t = -[8\pi G(P_u+\varepsilon_{ru})/c^4]\ dr/dt. \quad (5.12.107)$$

As the fourth field equation Chandrasekhar (1964b) takes the r-component of the identity (4.1.14)

$$\nabla_iT_r^i = \partial T_r^t/\partial t + \partial T_r^r/\partial r + (1/2)(T_r^r - T_t^t)\ \partial\nu/\partial r + (T_r^t/2)(\partial\nu/\partial t + \partial\kappa/\partial t)$$
$$+(1/r)(2T_r^r - T_\lambda^\lambda - T_\varphi^\varphi) = 0, \qquad [T_r^t = -T_t^r\exp(-\nu_u+\kappa_u)], \quad (5.12.108)$$

instead of the two equal (λ,λ) and (φ,φ)-components of the field equations (4.1.4). Some of the nonvanishing Christoffel symbols (4.1.15) are required for the calculation of the covariant derivative (5.12.108), (Landau and Lifschitz 1987):

$$\Gamma_{tt}^t = (1/2)\ \partial\nu/\partial t;\ \Gamma_{tt}^r = (1/2)\exp(\nu-\kappa)\ \partial\nu/\partial r;\ \Gamma_{rt}^t = (1/2)\ \partial\nu/\partial r;\ \Gamma_{rt}^r = (1/2)\ \partial\kappa/\partial t;$$
$$\Gamma_{rr}^t = (1/2)\exp(-\nu+\kappa)\ \partial\kappa/\partial t;\ \Gamma_{rr}^r = (1/2)\ \partial\kappa/\partial r;\ \Gamma_{r\lambda}^\lambda = \Gamma_{r\varphi}^\varphi = 1/r;\ \Gamma_{\lambda\lambda}^r = -r\exp(-\kappa);$$
$$\Gamma_{\lambda\varphi}^\varphi = \cot\lambda;\ \Gamma_{\varphi\varphi}^r = -r\sin^2\lambda\exp(-\kappa);\ \Gamma_{\varphi\varphi}^\lambda = -\sin\lambda\cos\lambda. \quad (5.12.109)$$

The linearization of Eqs. (5.12.104), (5.12.105), (5.12.107), (5.12.108) with the perturbations (5.12.103) is straightforward:

$$[\exp(-\kappa_u)/r](\partial\delta\nu/\partial r - \delta\kappa\ d\nu_u/dr) - [\exp(-\kappa_u)/r^2]\ \delta\kappa = 8\pi G\ \delta P/c^4, \quad (5.12.110)$$

$$\partial[r\exp(-\kappa_u)\ \delta\kappa]/\partial r = 8\pi Gr^2\ \delta\varepsilon_r/c^4, \quad (5.12.111)$$

$$[\exp(-\kappa_u)/r]\ \partial\delta\kappa/\partial t = -[8\pi G(P_u+\varepsilon_{ru})/c^4]\ \delta(dr/dt) = -[8\pi G(P_u+\varepsilon_{ru})/c^4][dr/dt-(dr/dt)_u]$$
$$= -[8\pi G(P_r+\varepsilon_r)/c^4]\ dr/dt, \qquad [(dr/dt)_u = 0], \tag{5.12.112}$$

$$\exp(-\nu_u+\kappa_u)(P_u+\varepsilon_{ru})\ d^2r/dt^2 + \partial\delta P/\partial r + (1/2)(P_u+\varepsilon_{ru})\ \partial\delta\nu/\partial r$$
$$+(1/2)(\delta P+\delta\varepsilon_r)\ d\nu_u/dr = 0. \tag{5.12.113}$$

To obtain Eq. (5.12.113) from Eq. (5.12.108) we have taken into account Eq. (4.1.18)

$$dP_u/dr = -(P_u/2+\varepsilon_{ru}/2)\ d\nu_u/dr, \tag{5.12.114}$$

and the fact that $\partial\nu/\partial t + \partial\kappa/\partial t = \partial\delta\nu/\partial t + \partial\delta\kappa/\partial t$ is of the first order of smallness.

Eq. (5.12.112) can be integrated directly, to give the perturbation of κ :

$$\delta\kappa = -8\pi Gr(P_u+\varepsilon_{ru})(r-r_u)\exp\kappa_u/c^4 = 8\pi Gr(P_u+\varepsilon_{ru})\ \Delta r(r,t)\exp\kappa_u/c^4. \tag{5.12.115}$$

We have replaced the variation $r-r_u$ of the radial coordinate by the Lagrangian displacement $\Delta r(r,t) = \Delta r(r)\exp(i\sigma t)$. The Eulerian variation of the energy density results by insertion of Eq. (5.12.115) into Eq. (5.12.111):

$$\delta\varepsilon_r = -(1/r^2)\ \partial[r^2(P_u+\varepsilon_{ru})\ \Delta r]/\partial r = -\Delta r\ d\varepsilon_{ru}/dr - \Delta r\ dP_u/dr - [(P_u+\varepsilon_{ru})/r^2]\ \partial(r^2\ \Delta r)/\partial r$$
$$= -\Delta r\ d\varepsilon_{ru}/dr - [(P_u+\varepsilon_{ru})\exp(\nu_u/2)/r^2]\ \partial[r^2\exp(-\nu_u/2)\ \Delta r]/\partial r. \tag{5.12.116}$$

The last term has been obtained by inserting for dP_u/dr from Eq. (5.12.114). The variation of ν is found from Eq. (5.12.110), after substitution of $\delta\kappa$ from Eq. (5.12.115), and of $r\exp\kappa_u$ from Eq. (5.12.106):

$$\partial\delta\nu/\partial r = (8\pi Gr\exp\kappa_u/c^4)[\delta P-(P_u+\varepsilon_{ru})(d\nu_u/dr+1/r)\ \Delta r]$$
$$= [\delta P/(P_u+\varepsilon_{ru}) - (d\nu_u/dr+1/r)\ \Delta r]\ d(\nu_u+\kappa_u)/dr. \tag{5.12.117}$$

All perturbations in Eq. (5.12.103) are now taken under the form $\delta F(r,t) = \delta F(r)\exp(i\sigma t)$. After simplification with $\exp(i\sigma t)$, Eq. (5.12.113) is rewritten with $d^2r/dt^2 = d^2(r-r_u)/dt^2 = d^2\Delta r/dt^2 = D^2\Delta r/Dt^2 \approx \partial^2\Delta r/\partial t^2 = -\sigma^2\ \Delta r(r)\exp(i\sigma t)$ as

$$\sigma^2\exp(-\nu_u+\kappa_u)(P_u+\varepsilon_{ru})\ \Delta r(r) = d\delta P(r)/dr + \delta P(r)\ d(\nu_u+\kappa_u/2)/dr$$
$$-(1/2)(P_u+\varepsilon_{ru})(d\nu_u/dr+1/r)\ \Delta r(r)\ d(\nu_u+\kappa_u)/dr + [\delta\varepsilon_r(r)/2]\ d\nu_u/dr, \tag{5.12.118}$$

where we have substituted for $\partial\delta\nu/\partial r$ via Eq. (5.12.117).

In order to express δP in terms of Δr and of the unperturbed variables, we derive at first a relationship between the two adiabatic exponents from Eq. (4.1.65), (Glass and Harpaz 1983)

$$\Gamma_{r1}/\Gamma_1 = (d\ln\varrho/d\ln\varepsilon_r)_S = (\varepsilon_r/\varrho)(\partial\varrho/\partial\varepsilon_r)_S = \varepsilon_r/(P+\varepsilon_r), \tag{5.12.119}$$

which is obtained with the isentropic form of the first law of thermodynamics (4.1.57).

Like in the nonrelativistic case [Eqs. (5.2.21), (5.2.37), (5.2.39)] we get from Eq. (4.1.65)

$$D\ln P/Dt - \Gamma_1\ D\ln\varrho/Dt = 0 \quad \text{and} \quad D\ln P/Dt - \Gamma_{r1}\ D\ln\varepsilon_r/Dt = 0, \qquad (S=\text{const}), \tag{5.12.120}$$

where we have replaced in Eulerian description the differentials by material derivatives. We apply the Lagrangian variation to Eq. (5.12.120), and get after integration with respect to t

$$\Delta P/P = \Gamma_1\ \Delta\varrho/\varrho \quad \text{and} \quad \Delta P/P = \Gamma_{r1}\ \Delta\varepsilon_r/\varepsilon_r = \Gamma_{r1}\ \Delta\varrho_r/\varrho_r. \tag{5.12.121}$$

The Eulerian pressure variations are via Eq. (5.1.16) equal to

$$\delta P = \Gamma_1 P\ \delta\varrho/\varrho + (\Gamma_1 P\ \nabla\varrho/\varrho - \nabla P)\cdot\Delta\vec{r} = \Gamma_1 P\ \delta\varrho/\varrho + (P/\varrho)(\Gamma_1-1-1/n)\ \nabla\varrho\cdot\Delta\vec{r}, \tag{5.12.122}$$

for an equation of state $P = K\varrho^{1+1/n}$, and

$$\delta P = \Gamma_{r1}P\ \delta\varepsilon_r/\varepsilon_r + (\Gamma_{r1}P\ \nabla\varepsilon_r/\varepsilon_r - \nabla P)\cdot\Delta\vec{r} = \Gamma_{r1}P\ \delta\varepsilon_r/\varepsilon_r + (P/\varepsilon_r)(\Gamma_{r1}-1-1/n)\ \nabla\varepsilon_r\cdot\Delta\vec{r}, \tag{5.12.123}$$

for an equation of state $P = K\varepsilon_r^{1+1/n}$ or $P = K\varrho_r^{1+1/n}$, $(\varepsilon_r = c^2\varrho_r)$.

For isentropic polytropes we have $\Gamma_1 = 1+1/n$ or $\Gamma_{r1} = 1+1/n$, respectively. The Eulerian pressure variations (5.12.122), (5.12.123) simplify in this particular case to (Yoshida and Eriguchi 1997)

$$\delta P = \Gamma_1 P\ \delta\varrho/\varrho, \qquad (P = K\varrho^{1+1/n} = K\varrho^{\Gamma_1}), \tag{5.12.124}$$

$$\delta P = \Gamma_{r1} P\ \delta\varrho_r/\varrho_r = \Gamma_{r1} P\ \delta\varepsilon_r/\varepsilon_r = \Gamma_1 P\ \delta\varepsilon_r/(P+\varepsilon_r), \qquad (P = K\varepsilon_r^{1+1/n} = K\varepsilon_r^{\Gamma_{r1}}). \tag{5.12.125}$$

We eventually get via Eqs. (5.12.116), (5.12.119), (5.12.121), [Chandrasekhar 1964b, Eq. (53)]:

$$\delta P = \Delta P - \Delta r\ dP/dr \approx (\Gamma_{r1}P/\varepsilon_r)\ \Delta\varepsilon_r - \Delta r\ dP_u/dr \approx (\Gamma_{r1}P_u/\varepsilon_{ru})(\delta\varepsilon_r + \Delta r\ d\varepsilon_{ru}/dr)$$
$$-\Delta r\ dP_u/dr = -[\Gamma_1 P_u \exp(\nu_u/2)/r^2]\ \partial[r^2\exp(-\nu_u/2)\ \Delta r]/\partial r - \Delta r\ dP_u/dr. \tag{5.12.126}$$

The pulsation equation (5.12.118) takes in virtue of Eqs. (5.12.116), (5.12.126) the lengthy form

$$\sigma^2\exp(-\nu_u+\kappa_u)(P_u+\varepsilon_{ru})\ \Delta r = -d(\Delta r\ dP_u/dr)/dr - \Delta r\ (dP_u/dr)\ d(\nu_u+\kappa_u/2)/dr$$
$$-\exp(-\nu_u-\kappa_u/2)\ d\{\exp(\nu_u+\kappa_u/2)[\Gamma_1 P_u\exp(\nu_u/2)/r^2]\ d[r^2\exp(-\nu_u/2)\ \Delta r]/dr\}/dr$$
$$-(1/2)(P_u+\varepsilon_{ru})(d\nu_u/dr+1/r)\ \Delta r\ d(\nu_u+\kappa_u)/dr - (1/2)(d\nu_u/dr)\{(2/r)(P_u+\varepsilon_{ru})\ \Delta r$$
$$+d[(P_u+\varepsilon_{ru})\ \Delta r]/dr\}. \tag{5.12.127}$$

The right-hand side of this equation becomes after substitution of dP_u/dr from Eq. (5.12.114) equal to

$$[(P_u+\varepsilon_{ru})\ \Delta r/2][d^2\nu_u/dr^2 - (1/2)(d\nu_u/dr)\ d\kappa_u/dr - (1/r)\ d\kappa_u/dr - (3/r)\ d\nu_u/dr]$$
$$-\exp(-\nu_u-\kappa_u/2)\ d\{(\Gamma_1 P_u/r^2)\exp(3\nu_u/2+\kappa_u/2)\ d[r^2\exp(-\nu_u/2)\ \Delta r]/dr\}/dr. \tag{5.12.128}$$

This expression can be transformed further with the aid of Eqs. (4.1.8), (5.12.114):

$$(P_u+\varepsilon_{ru})\ \Delta r[8\pi G P_u\exp\kappa_u/c^4 - (1/4)(d\nu_u/dr)^2 - (2/r)\ d\nu_u/dr]$$
$$-\exp(-\nu_u-\kappa_u/2)\ d\{(\Gamma_1 P_u/r^2)\exp(3\nu_u/2+\kappa_u/2)\ d[r^2\exp(-\nu_u/2)\ \Delta r]/dr\}/dr$$
$$= \Delta r[8\pi G P_u(P_u+\varepsilon_{ru})\exp\kappa_u/c^4 - (dP_u/dr)^2/(P_u+\varepsilon_{ru}) + (4/r)\ dP_u/dr]$$
$$-\exp(-\nu_u-\kappa_u/2)\ d\{(\Gamma_1 P_u/r^2)\exp(3\nu_u/2+\kappa_u/2)\ d[r^2\exp(-\nu_u/2)\ \Delta r]/dr\}/dr. \tag{5.12.129}$$

Eq. (5.12.127) takes via Eq. (5.12.129) the final form

$$\sigma^2\exp(-\nu_u+\kappa_u)(P_u+\varepsilon_{ru})\ \Delta r = \Delta r[8\pi G P_u(P_u+\varepsilon_{ru})\exp\kappa_u/c^4 - (dP_u/dr)^2/(P_u+\varepsilon_{ru})$$
$$+(4/r)\ dP_u/dr] - \exp(-\nu_u-\kappa_u/2)\ d\{(\Gamma_1 P_u/r^2)\exp(3\nu_u/2+\kappa_u/2)\ d[r^2\exp(-\nu_u/2)\ \Delta r]/dr\}/dr. \tag{5.12.130}$$

A variational base for determining the eigenvalue σ^2 with the boundary conditions (5.2.61), (5.2.63) is provided by multiplication of Eq. (5.12.130) with Δr, and with the invariant four-dimensional volume element $(-g)^{1/2}\ dx^0\ dx^1\ dx^2\ dx^3 = r^2\exp(\nu_u/2+\kappa_u/2)\sin\lambda\ dt\ dr\ d\lambda\ d\varphi$. Integration yields after simplification with $\sin\lambda\ dt\ d\lambda\ d\varphi$, and by suppressing the index u :

$$\sigma^2\int_0^{r_1}\exp(-\nu/2+3\kappa/2)(P+\varepsilon_r)(\Delta r)^2 r^2\ dr = \int_0^{r_1}[8\pi G P(P+\varepsilon_r)\exp(\nu/2+3\kappa/2)/c^4$$
$$-\exp(\nu/2+\kappa/2)(dP/dr)^2/(P+\varepsilon_r) + (4/r)\exp(\nu/2+\kappa/2)\ dP/dr](\Delta r)^2 r^2\ dr$$
$$+\int_0^{r_1}\exp(3\nu/2+\kappa/2)(\Gamma_1 P/r^2)\{d[r^2\exp(-\nu/2)\ \Delta r]/dr\}^2\ dr. \tag{5.12.131}$$

The last term has been obtained by partial integration. Eq. (5.12.131) expresses a minimal (not merely an extremal) principle. The sufficient condition for dynamical instability is the vanishing of the right-hand side under neutral stability $\sigma = 0$.

Associated with the variational condition (5.12.131) is the orthogonality condition

$$\{[\sigma^{(\alpha)}]^2 - [\sigma^{(\beta)}]^2\}\int_0^{r_1}\exp(-\nu/2+3\kappa/2)(P+\varepsilon_r)r^2\ \Delta r^{(\alpha)}\ \Delta r^{(\beta)}\ dr = 0, \qquad (\alpha\neq\beta), \tag{5.12.132}$$

which is obtained similarly to Eqs. (5.7.43)-(5.7.46).

With the equation of state (4.1.1) we get from the polytropic relationships (4.1.29):

$$P = P_0\theta_r^{n+1} = P_0(\varrho_r/\varrho_{r0})^{1+1/n} = P_0(\varepsilon_r/\varepsilon_{r0})^{1+1/n} = q_0\varepsilon_{r0}^{-1/n}\varepsilon_r^{1+1/n};$$
$$\alpha^2 = (n+1)P_0/4\pi G\varrho_{r0}^2 = (n+1)q_0c^4/4\pi G\varepsilon_{r0}, \qquad (-1 < n < \infty). \tag{5.12.133}$$

Insertion into Eq. (5.12.131) yields the variational relationship for radial oscillations of a relativistic polytropic sphere:

$$(\sigma^2\alpha^2/q_0)\int_0^{\xi_1}\exp(-\nu/2+3\kappa/2)\theta_r^n(1+q_0\theta_r)(\Delta r)^2\xi^2\ d\xi = 2(n+1)\int_0^{\xi_1}\{q_0\exp(\nu/2+3\kappa/2)\theta_r^{2n+1}$$
$$\times(1+q_0\theta_r) + 2\exp(\nu/2+\kappa/2)(\theta_r^n/\xi)(d\theta_r/d\xi)[1-(n+1)q_0\xi(d\theta_r/d\xi)/4(1+q_0\theta_r)]\}(\Delta r)^2\xi^2\ d\xi$$
$$+\Gamma_1\int_0^{\xi_1}\exp(3\nu/2+\kappa/2)\theta_r^{n+1}\{d[\xi^2\exp(-\nu/2)\ \Delta r]/d\xi\}^2\ d\xi/\xi^2. \tag{5.12.134}$$

The two metric coefficients are via Eqs. (4.1.25), (4.1.34), (4.1.39), (5.12.133) equal to

$$\exp(-\kappa) = 1 - 2GM_{r1}u/c^2r = 1 - 8\pi G\varrho_{r0}\alpha^2\eta(\xi)/c^2\xi = 1 - 2(n+1)q_0\eta(\xi)/\xi;$$
$$\exp\nu = (1-2GM_{r1}/c^2r_1)/(1+q_0\theta_r)^{2(n+1)} = [1-2(n+1)q_0\eta(\xi_1)/\xi_1]/(1+q_0\theta_r)^{2(n+1)}. \tag{5.12.135}$$

These expressions take in the post Newtonian limit the form

$$\exp(-\kappa) \approx 1 + 2(n+1)q_0\xi\theta_r'; \quad \exp\nu \approx 1 - 2(n+1)q_0(\theta_r - \xi_1\theta_{r1}'),$$
$$(q_0 \approx 0;\ \theta_r = \theta + O(q_0);\ \eta = -\xi^2\theta' + O(q_0) = -\xi^2\theta_r' + O(q_0);\ \theta_{r1}' = (d\theta_r/d\xi)_{\xi=\xi_1}). \tag{5.12.136}$$

As we know from Newtonian theory (e.g. Eq. (5.3.16) if $\Gamma_1 = \mathrm{const}$, $\Omega = 0$), dynamical stability ($\sigma^2 > 0$) requires $\Gamma_1 > 4/3$. In the post Newtonian approximation the relevant adiabatic exponent will be $\Gamma_1 > 4/3 + O(q_0)$. We take as in Sec. 5.3.1 for the Lagrangian displacement a trial function of the form $\Delta r \propto r \propto \xi$. If we insert this together with Eq. (5.12.136) and $\Gamma_1 \approx 4/3$ into the basic equation (5.12.134), we get after some algebra up to the first order

$$(\sigma^2\alpha^2/q_0)\int_0^{\xi_1}\theta_r^n\{1+q_0[\theta_r - 3(n+1)\xi\theta_r' + (n+1)(\theta_r - \xi_1\theta_{r1}')]\}\xi^4\ d\xi$$
$$= 9(\Gamma_1 - 4/3)\int_0^{\xi_1}\xi^2\theta_r^{n+1}\ d\xi + 4(n+1)q_0\int_0^{\xi_1}[\xi_1\theta_{r1}'\ d(\xi^3\theta_r^{n+1})/d\xi - d(\xi^3\theta_r^{n+2})/d\xi$$
$$+(1/2)\xi^4\theta_r^{2n+1} - (5/4)(n+1)\xi^4\theta_r^n{\theta_r'}^2]\ d\xi, \qquad (\Gamma_1 \approx 4/3;\ q_0 \approx 0). \tag{5.12.137}$$

Up to the first order in q_0 we can replace throughout on the right-hand side the relativistic Lane-Emden function with its Newtonian counterpart, and the first zero of θ_r with the first zero of θ. The criterion of onset of dynamical instability is obtained with the condition of neutral stability $\sigma = 0$, and Eq. (5.12.137) becomes

$$9(\Gamma_1 - 4/3)\int_0^{\xi_1}\xi^2\theta^{n+1}\ d\xi = 4(n+1)q_0\int_0^{\xi_1}[5(n+1)\theta^n\theta'^2/4 - \theta^{2n+1}/2]\xi^4\ d\xi. \tag{5.12.138}$$

This is equivalent to Eq. (94) of Chandrasekhar (1964b), and can be reduced to Eq. (77) of Chandrasekhar (1965b) after integration by parts, making frequent use of the Lane-Emden equation $(\xi^2\theta')' = -\xi^2\theta^n$. We have

$$\int_0^{\xi_1}\xi^2\theta^{n+1}\ d\xi = \xi^3\theta^{n+1}/3\Big|_0^{\xi_1} - [(n+1)/3]\int_0^{\xi_1}\xi^3\theta^n\theta'\ d\xi = [(n+1)/6]\int_0^{\xi_1}d(\xi^2\theta')^2/\xi$$
$$= [(n+1)/6]\xi_1^3{\theta_1'}^2 + [(n+1)/6]\int_0^{\xi_1}\xi^2\theta'\ d\theta = [(n+1)/6]\xi_1^3{\theta_1'}^2 + [(n+1)/6]\xi\theta\theta'\Big|_0^{\xi_1}$$
$$+[(n+1)/6]\int_0^{\xi_1}\xi^2\theta^{n+1}\ d\xi. \tag{5.12.139}$$

Equating the first and last expression we get [cf. Eq. (2.6.159)]

$$\int_0^{\xi_1} \xi^2\theta^{n+1}\ d\xi = (n+1)\xi_1^3{\theta_1'}^2/(5-n), \qquad (-1 < n \le 5). \tag{5.12.140}$$

The integrals on the right-hand side of Eq. (5.12.138) are transformed as

$$\begin{aligned}
&\int_0^{\xi_1} \xi^4\theta^n\theta'^2\ d\xi = -\int_0^{\xi_1} (\xi^2\theta')^2\ d(\xi^2\theta')/\xi^2 = -\xi_1^4{\theta_1'}^3/3 - (2/3)\int_0^{\xi_1} (\xi^2\theta')^3\ d\xi/\xi^3 = -\xi_1^4{\theta_1'}^3/3 \\
&-2\xi^3\theta\theta'^2/3\Big|_0^{\xi_1} + (2/3)\int_0^{\xi_1} \theta\ d[(\xi^2\theta')^2/\xi] = -\xi_1^4{\theta_1'}^3/3 - (2/3)\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi - (4/3)\int_0^{\xi_1} \xi^3\theta^{n+1}\theta'\ d\xi \\
&= -\xi_1^4{\theta_1'}^3/3 - (2/3)\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi + [4/(n+2)]\int_0^{\xi_1} \xi^2\theta^{n+2}\ d\xi = -\xi_1^4{\theta_1'}^3/3 - (2/3)\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi \\
&-[4/(n+2)]\int_0^{\xi_1} \theta^2\ d(\xi^2\theta') = -\xi_1^4{\theta_1'}^3/3 + [2(10-n)/3(n+2)]\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi,
\end{aligned} \tag{5.12.141}$$

and

$$\begin{aligned}
&\int_0^{\xi_1} \xi^4\theta^{2n+1}\ d\xi = -\int_0^{\xi_1} \xi^2\theta^{n+1}\ d(\xi^2\theta') = -\xi^4\theta^{n+1}\theta'\Big|_0^{\xi_1} + 2\int_0^{\xi_1} \xi^3\theta^{n+1}\theta'\ d\xi \\
&+(n+1)\int_0^{\xi_1} \xi^4\theta^n\theta'^2\ d\xi = -[6/(n+2)]\int_0^{\xi_1} \xi^2\theta^{n+2}\ d\xi + (n+1)\int_0^{\xi_1} \xi^4\theta^n\theta'^2\ d\xi \\
&= [6/(n+2)]\int_0^{\xi_1} \theta^2\ d(\xi^2\theta') + (n+1)\int_0^{\xi_1} \xi^4\theta^n\theta'^2\ d\xi = -[12/(n+2)]\int_0^{\xi} \xi^2\theta\theta'^2\ d\xi \\
&-(n+1)\xi_1^4{\theta_1'}^3/3 + [2(10-n)(n+1)/3(n+2)]\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi = -(n+1)\xi_1^4{\theta_1'}^3/3 \\
&-[2(n-8)(n-1)/3(n+2)]\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi.
\end{aligned} \tag{5.12.142}$$

We insert Eqs. (5.12.140)-(5.12.142) together with $q_0 = GM_{r1}\xi_1/(n+1)c^2r_1\eta(\xi_1) \approx GM_{r1}/(n+1)c^2r_1\xi_1(-\theta_1') = r_g/2(n+1)r_1\xi_1(-\theta_1')$ from Eq. (4.1.49) into Eq. (5.12.138), and obtain in the post Newtonian approximation the lower limit of Γ_1 required for dynamical stability:

$$\begin{aligned}
&\Gamma_1 = 4/3 + [(5-n)r_g/18r_1]\Big\{1 + [2(11-n)/(n+1)\xi_1^4(-{\theta_1'}^3)]\int_0^{\xi_1} \xi^2\theta\theta'^2\ d\xi\Big\} \\
&= 4/3 + C(n)\ r_g/r_1, \qquad (-1 < n \le 5;\ q_0, r_g/r_1 \ll 1).
\end{aligned} \tag{5.12.143}$$

Thus, general relativity induces dynamical instability against purely radial oscillations in a spherical polytrope if

$$\Gamma_1 < 4/3 + C(n)\ r_g/r_1, \qquad (-1 < n \le 5;\ \Gamma_1 \approx 4/3;\ q_0, r_g/r_1 \ll 1), \tag{5.12.144}$$

where the constant $C(n)$ equals 0.452, 0.565, 0.645, 0.751, 0.900, 1.124, 1.285, 1.500 if $n =$ 0, 1, 1.5, 2, 2.5, 3, 3.25, 3.5, respectively (Chandrasekhar 1964b, 1965b). General relativity tends to destabilize relativistic polytropic spheres: The stronger is gravity, the easier is radial collapse (Fig. 5.12.5).

Pandey et al. (1991) have calculated the onset of radial instability in relativistic, isentropic neutron star models. Radial instability always occurs for the considered models if $n = 1/(\Gamma_{r1} - 1) \gtrsim 2.5$, $(\Gamma_{r1} \lesssim 7/5)$, in accordance with Eq. (5.12.144). If $1 \lesssim n \lesssim 2.5$, radial instability sets in above certain values of the relativity parameter q_0. If $0.25 \le n \lesssim 1$, the models are radially stable.

The infinitely long, cylindrical relativistic polytropes considered by Scheel et al. (1993; see Sec. 4.1.5) are stable against purely radial perturbations – in contrast to spherical polytropes. These cylinders may not be suitable to get insight into the behaviour of *spherical* finite systems, due to the unrealistic assumption of an infinitely long configuration, with the metric (4.1.157) diverging at infinity.

5.12.5 Nonradial Oscillations of Relativistic Spheres

Most investigations on nonradial oscillations of relativistic polytropes assume isentropic configurations – the barotropic one-parameter equation of state is assumed for both the interior structure and the small-amplitude oscillations [Eqs. (4.1.65), (4.1.83)]:

$$(\varrho/P)(\partial P/\partial \varrho)_S = \Gamma_1 = 1 + 1/n; \quad P = K\varrho^{1+1/n} = K\varrho^{\Gamma_1}, \tag{5.12.145}$$

or

$$(\varrho_r/P)(\partial P/\partial \varrho_r)_S = (\varepsilon_r/P)(\partial P/\partial \varepsilon_r)_S = \Gamma_{r1} = 1 + 1/n;$$
$$P = K\varrho_r^{1+1/n} = K\varrho_r^{\Gamma_{r1}} \quad \text{or} \quad P = K\varepsilon_r^{1+1/n} = K\varepsilon_r^{\Gamma_{r1}}. \tag{5.12.146}$$

In a nonrotating sphere the nonradial oscillation modes are classified into two decoupled sets of polar (spheroidal, even parity) and axial (toroidal, odd parity) modes [Eqs. (5.8.165), (5.8.166)]. Rotation mixes these two types, and an oscillation generally contains contributions from both the polar and axial modes (Lockitch and Friedman 1999, Yoshida and Lee 2000a). Pure polar fluid modes exist for all conceivable models. The existence of nondegenerate ($\sigma \neq 0$) purely axial fluid modes relies upon rotation, magnetic field or nonisotropic pressure (Andersson et al. 1996).

The three families of purely polar fluid modes, discussed so far, are the familiar p, f, and g-modes. In isentropic Newtonian spheres the polar g-modes are degenerate to zero frequency $\sigma = 0$; they become time-independent, neutrally stable convective currents, being of no great interest (Tables 5.5.1, 5.5.2 if $n = 1/(\Gamma_1 - 1) = 1.5$). The same is true for the purely axial r-modes in a nonrotating sphere (Sec. 5.8.4). This degeneracy to zero frequency in isentropic spheres is lifted by rotation, the corresponding modes in rotating isentropic stars becoming hybrids of polar and axial perturbations, although isentropic rotating Newtonian stars retain a vestigial set of purely axial fluid r-modes (those having $j = k$ in the spherical harmonic Y_j^k). The existence of purely axial fluid modes in *relativistic* stars with *isotropic* pressure is not conclusive (Lockitch and Friedman 1999).

In general relativity a configuration possesses also modes associated with the gravitational field itself: Spacetime modes which are gravitational wave modes – the so-called w-modes. These purely relativistic modes can be of the polar and axial type, which however are qualitatively similar (Andersson et al. 1995). In nonrotating relativistic spheres the gravitational wave modes always damp the fluid oscillations, being often the dominant dissipation mechanism. However, with rotation included, the relativistic polytropes are susceptible to the Chandrasekhar-Friedman-Schutz instability, driven by gravitational radiation reaction, as mentioned in Sec. 5.8.4.

The purely radial modes ($j = 0$) discussed in the previous section, as well as the dipole modes ($j = 1$, Sec. 5.2.) are special cases in general relativity; henceforth they will not be considered as they cannot emit gravitational radiation (Thorne and Campolattaro 1967, p. 596).

The unperturbed metric is given again by Eq. (5.12.99), while the perturbed metric amounts to

$$ds^2 = (g_{\ell m} + \delta g_{\ell m})\, dx^\ell\, dx^m, \tag{5.12.147}$$

where $\delta g_{\ell m}$ are the Eulerian perturbations of the metric coefficients. Further, the ten metric perturbations can be split into polar and axial contributions:

$$\delta g_{\ell m} = \delta g_{\ell m}^{polar} + \delta g_{\ell m}^{axial}. \tag{5.12.148}$$

The ten field equations (4.1.4) can be written with the contraction $R = -8\pi GT/c^4$ as

$$R_{\ell m} = (8\pi G/c^4)(T_{\ell m} - g_{\ell m}T/2), \tag{5.12.149}$$

having the Eulerian perturbations

$$\delta R_{\ell m} = (8\pi G/c^4)(\delta T_{\ell m} - T\ \delta g_{\ell m}/2 - g_{\ell m}\ \delta T/2). \tag{5.12.150}$$

The ten unknowns $\delta g_{\ell m}$ have to be supplemented with the spatial Lagrangian displacement vector $\Delta\vec{r}$ and the energy density perturbation $\delta\varepsilon$. However, because of the identity (4.1.14), only ten of these fourteen variables will be independent, involving different possible sets of perturbation equations and gauge choices.

(i) Polar Modes. The *contravariant* Lagrangian displacement can be considered under the familiar form of spherical harmonics Y_j^k (Eqs. (5.2.87), (5.2.89), (5.2.90), (5.5.27), (5.8.165); Andersson et al. 1995):

$$\Delta r = r^{j-1}\exp[-\kappa(r)/2]\ W(r)\ Y_j^k(\lambda,\varphi)\ \exp(i\sigma t);\quad \Delta\lambda = -r^{j-2}V(r)\ (\partial Y_j^k/\partial\lambda)\ \exp(i\sigma t);$$
$$\Delta\varphi = -(r^{j-2}/\sin^2\lambda)\ V(r)\ (\partial Y_j^k/\partial\varphi)\ \exp(i\sigma t). \tag{5.12.151}$$

The perturbations $\delta g_{\ell m}$ have been expanded by Thorne and Campolattaro (1967) under the form of scalar, vectorial, or tensorial spherical harmonics, transforming as scalars, vectors, or tensors under (λ,φ)-rotations, respectively. With the Regge-Wheeler gauge some of the polar metric perturbations can be annulled, removing the arbitrariness in the coordinate system. The perturbed metric (5.12.147) takes the simplified form

$$ds^2 = \exp\nu(r)\ [1 + r^jH_0(r)\ Y_j^k(\lambda,\varphi)\ \exp(i\sigma t)]\ dt^2 + 2i\sigma r^{j+1}H_1(r)\ Y_j^k(\lambda,\varphi)\ \exp(i\sigma t)\ dt\ dr -$$
$$\exp\kappa(r)[1 - r^jH_0(r)\ Y_j^k(\lambda,\varphi)\ \exp(i\sigma t)]\ dr^2 - r^2[1 - r^jK(r)\ Y_j^k(\lambda,\varphi)\ \exp(i\sigma t)](d\lambda^2 + \sin^2\lambda\ d\varphi^2), \tag{5.12.152}$$

where the three small perturbation functions H_0, H_1, K have to be determined from the perturbed field equations (5.12.150). The perturbed curvature tensor $\delta R_{\ell m}$ can be calculated with $\delta g_{\ell m}$ from the perturbations of the Christoffel symbols (4.1.15). The perturbed energy-momentum tensor (5.12.155), (5.12.156), (5.12.161) is computed with the four-velocity (5.12.98), which amounts for the displacements (5.12.151) to

$$u^t = dt/ds \approx g_{tt}^{-1/2} \approx \exp(-\nu/2)[1 - (r^jH_0/2)\ Y_k^j \exp(i\sigma t)];$$
$$u^r = \delta u^r = dr/ds = (dr/dt)\ dt/ds \approx g_{tt}^{-1/2}\ dr/dt \approx g_{tt}^{-1/2}\ \partial\Delta r/\partial t \approx i\sigma\exp(-\nu/2)\ \Delta r;$$
$$u^\lambda = \delta u^\lambda = d\lambda/ds = (d\lambda/dt)\ dt/ds \approx g_{tt}^{-1/2}\ d\lambda/dt \approx g_{tt}^{-1/2}\ \partial\Delta\lambda/\partial t \approx i\sigma\exp(-\nu/2)\ \Delta\lambda;$$
$$u^\varphi = \delta u^\varphi = d\varphi/ds = (d\varphi/dt)\ dt/ds \approx g_{tt}^{-1/2}\ d\varphi/dt \approx g_{tt}^{-1/2}\ \partial\Delta\varphi/\partial t \approx i\sigma\exp(-\nu/2)\ \Delta\varphi,$$
$$(dx^\alpha/dt = d(x^\alpha - x_u^\alpha)/dt = d\Delta x^\alpha/dt = D\Delta x^\alpha/Dt \approx \partial\Delta x^\alpha/\partial t). \tag{5.12.153}$$

And the covariant four-velocities (4.1.12) are up to the first order:

$$u_t = g_{tt}^{1/2} + g_{tr}\ \delta u^r \approx g_{tt}^{1/2};\quad u_r = g_{rr}\ \delta u^r + g_{tr}g_{tt}^{-1/2};\quad u_\lambda = g_{\lambda\lambda}\ \delta u^\lambda;\quad u_\varphi = g_{\varphi\varphi}\ \delta u^\varphi. \tag{5.12.154}$$

The Eulerian perturbations

$$\delta T_\ell^m = (\delta\varepsilon_r + \delta P)u_\ell u^m + (\varepsilon_r + P)(u_\ell\ \delta u^m + u^m\ \delta u_\ell) - \delta_\ell^m\ \delta P, \tag{5.12.155}$$

of the stress-energy tensor (4.1.11) become up to the first order

$$\delta T_t^t = \delta\varepsilon_r;\quad \delta T_r^r = \delta T_\lambda^\lambda = \delta T_\varphi^\varphi = -\delta P;\quad \delta T_a^b = (\varepsilon_r + P)u_a u^b \quad\text{if}\quad a \neq b \quad\text{and}\quad a = t \text{ or } b = t;$$
$$\delta T_\alpha^\beta = 0 \quad\text{if}\quad \alpha \neq \beta, \qquad (\alpha,\beta = r,\lambda,\varphi). \tag{5.12.156}$$

For nonrotating spheres it suffices to specialize to the particular choice $Y_j^0(\lambda,\varphi) = P_j(\cos\lambda),\ (j \geq 2)$ of the spherical harmonics. The modes having $k \neq 0$ can be obtained with a suitable rotation about the centre. The perturbed field equations (5.12.150) can be written in condensed form as two coupled wave equations for a barotropic fluid $P(\varepsilon_r)$, (Andersson et al. 1996):

$$(\delta P/\delta\varepsilon_r)\ \partial^2 Y/\partial t^2 - \partial^2 Y/\partial r^{*2} = F(Y, Z, \partial Y/\partial r^*, \partial Z/\partial r^*);$$
$$\partial^2 Z/\partial t^2 - \partial^2 Z/\partial r^{*2} = G(Y, Z, \partial Y/\partial r^*, \partial Z/\partial r^*), \qquad [Y = Y(\delta g_{tt}, \delta g_{\lambda\lambda});\ Z = Z(\delta g_{tt}, \delta g_{\lambda\lambda})], \tag{5.12.157}$$

the first one for the polar fluid oscillations, and the second one for the polar gravitational waves. The so-called tortoise coordinate is defined by

$$\partial\ /\partial r^* = \exp[(\nu - \kappa)/2]\ \partial\ /\partial r, \tag{5.12.158}$$

which outside the relativistic sphere can easily be integrated via Eq. (4.1.20): $r^* = r + (2GM_{r1}/c^2)\ln(c^2r/2GM_{r1} - 1),\ (r > r_1)$.

(ii) Axial Modes. In the nonrotating relativistic sphere axial modes do not couple to fluid oscillations. Thorne and Campolattaro (1967) take the *covariant* Lagrangian displacement under the form

$$\Delta r = 0;\quad \Delta\lambda = -[U(r)/\sin\lambda](\partial Y_j^k/\partial\varphi)\exp(i\sigma t);\quad \Delta\varphi = U(r)\sin\lambda(\partial Y_j^k/\partial\lambda)\exp(i\sigma t), \tag{5.12.159}$$

where the transition from the contravariant axial Lagrangian displacements (5.8.166) to the present covariant ones proceeds with the spatial metric tensor (5.12.95): $\Delta x_\alpha = \gamma_{\alpha\beta}\ \Delta x^\beta \approx -g_{\alpha\alpha}\ \Delta x^\alpha,\ (g_{\lambda\lambda} = -r^2;\ g_{\varphi\varphi} = -r^2\sin^2\lambda)$.

With the Regge-Wheeler gauge only $\delta g_{t\lambda}, \delta g_{t\varphi}, \delta g_{r\lambda}, \delta g_{r\varphi}$ subsist, and the perturbed metric (5.12.147) assumes the form

$$\begin{aligned} &ds^2 = \exp[\nu(r)]\ dt^2 - \exp[\kappa(r)]\ dr^2 - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2)\\ &+2\exp(i\sigma t)\{-[h_0(r)/\sin\lambda](\partial Y_j^k/\partial\varphi)\ dt\ d\lambda + h_0(r)\sin\lambda\ (\partial Y_j^k/\partial\lambda)\ dt\ d\varphi\\ &-[h_1(r)/\sin\lambda](\partial Y_j^k/\partial\varphi)\ dr\ d\lambda + h_1(r)\sin\lambda\ (\partial Y_j^k/\partial\lambda)\ dr\ d\varphi\}, \end{aligned} \tag{5.12.160}$$

with the two small perturbation functions h_0, h_1.

Pressure and density are unchanged under purely axial perturbations ($\delta P, \delta\varepsilon_r = 0$), and the surviving components of the perturbed energy-momentum tensor $\delta T_{\ell m} = (P + \varepsilon_r)(u_\ell\ \delta u_m + u_m\ \delta u_\ell) - P\ \delta g_{\ell m}$ are in the particular case $k = 0$ equal to

$$\begin{aligned} &\delta T_{t\varphi} = \delta T_{\varphi t} = [i\sigma(P + \varepsilon_r)\ U(r)\exp(i\sigma t) - Ph_0]\sin\lambda\ dP_j(\cos\lambda)/d\lambda;\\ &\delta T_{r\varphi} = \delta T_{\varphi r} = -Ph_1\sin\lambda\ dP_j(\cos\lambda)/d\lambda, \qquad (\delta u_t = 0;\ \delta u_\varphi = u_\varphi), \end{aligned} \tag{5.12.161}$$

with the covariant four-velocities (Thorne and Campolattaro 1967)

$$\begin{aligned} &u_t = g_{t\ell}u^\ell = g_{t\ell}\ dx^\ell/ds \approx g_{tt}\ dt/ds = g_{tt}u^t \approx g_{tt}^{1/2};\quad u_r, u_\lambda = 0;\quad u_\varphi = g_{\varphi\ell}u^\ell = g_{\varphi\ell}\ dx^\ell/ds\\ &= dx_\varphi/ds = (dx_\varphi/dt)\ dt/ds \approx u^t\ \partial\Delta x_\varphi/\partial t = i\sigma u^t\ \Delta x_\varphi \approx i\sigma g_{tt}^{-1/2}\ \Delta\varphi,\\ &(k = 0;\ dx_\alpha/dt = d(x_\alpha - x_{\alpha u})/dt = d\Delta x_\alpha/dt = D\Delta x_\alpha/Dt \approx \partial\Delta x_\alpha/\partial t). \end{aligned} \tag{5.12.162}$$

The nonvanishing perturbations of the curvature tensor are $\delta R_{t\varphi}, \delta R_{r\varphi}, \delta R_{\lambda\varphi}$. An equation of the form (Andersson et al. 1996)

$$\partial^2 X/\partial t^2 - \partial^2 X/\partial r^{*2} + S(r)\ X = 0, \qquad [X = X(\delta g_{r\varphi})], \tag{5.12.163}$$

emerges from the perturbed field equations for the propagation of axial gravitational waves. In the present context no *axial fluid* modes exist.

The second equation (5.12.157) for the polar gravitational waves turns outside the polytrope into an equation similar to Eq. (5.12.163). The second order equation (5.12.163) admits two linearly independent solutions, which can be identified far away from the surface with an outgoing and ingoing gravitational wave, respectively. The quasinormal modes of purely polar oscillations are those for which there are no *incoming* gravitational waves: They represent the natural, free nonradial oscillations of a relativistic sphere. The complex eigenvalues σ involve damping of the polar fluid modes due to outgoing gravitational radiation, if their imaginary part – via $\exp(i\sigma t)$ – is positive. Conversely, if $\mathrm{Im}(\sigma) < 0$, the quasinormal oscillations of the relativistic polytropic sphere are unstable against nonradial oscillations.

The w-modes due to gravitational waves arising from polar and axial perturbations, as well as the fluid polar f, p_1, p_2, p_3-modes if $j = 2,\ k = 0$ have been computed by Andersson and Kokkotas (1998) for an isentropic $n = 1/(\Gamma_{r1} - 1) = 1$ polytrope with an equation of state $P = K\varepsilon_r^2,\ (K = 100\ \mathrm{km}^2; c, G = 1)$. Axial and polar w-modes are very similar, as shown on the upper left part of Fig. 5.12.4. As outlined subsequently to Eq. (4.1.80), the $n = 1$ polytrope becomes *radially* unstable if $r_1/M_{r1} < 3.62G/c^2$ or $M_{r1}/r_{r1} > 0.214c^2/G$, $[dM_{r1}/d\varepsilon_{r0} < 0$, Eq. (5.12.29)]. Andersson and Kokkotas (1998) quote $r_1/M_{r1} < 3.77G/c^2$, and Baumgarte et al. (1997) $M_{r1}/r_{r1} > 0.217c^2/G$ (cf. the broken curve in Fig. 5.12.5, and Fig. 4.1.3 if $n = 1$). The fluid f and p-modes are always stable, and less rapidly damped as compared to the gravitational wave w-modes (Fig. 5.12.4 on the left). They can become extremely long-lived as the compactness of the star increases (as r_1/M_{r1} decreases). And the $f, p_1, p_2, p_3,\ (j = 2)$

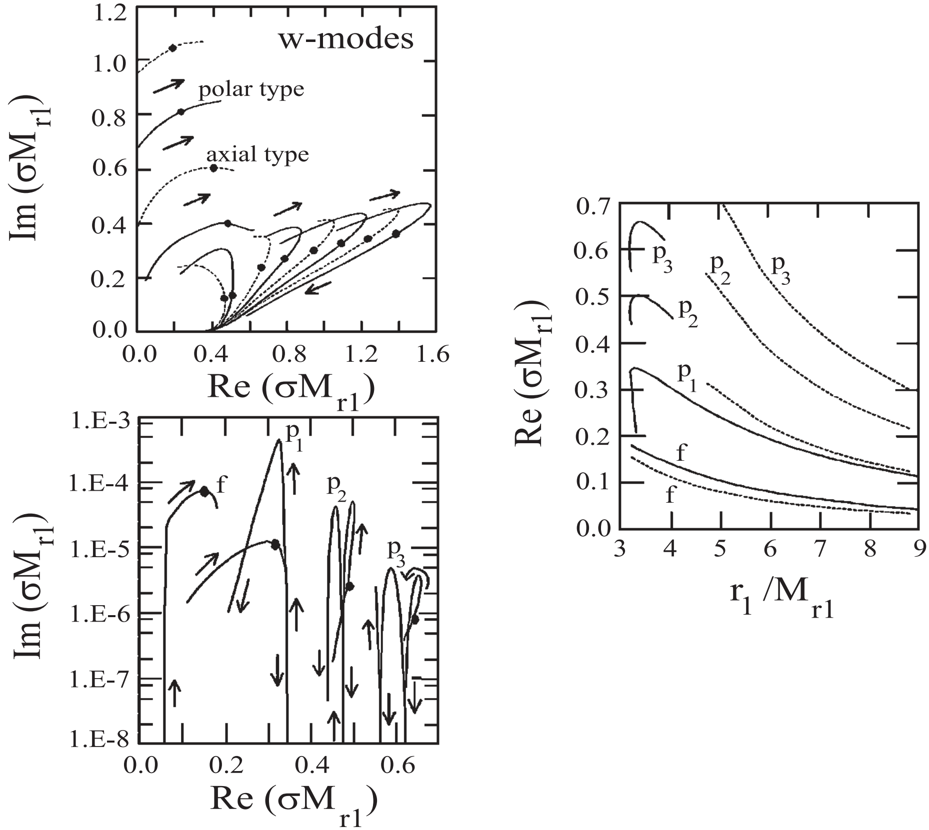

Fig. 5.12.4 w, f, and p-modes in the $n = 1/(\Gamma_{r1} - 1) = 1$ isentropic polytropic sphere. Left-hand side: Real versus imaginary part of the eigenvalue σ for the damped stable w-modes (upper left), and nonradial f, p_1, p_2, p_3, $(j = 2,\ k = 0)$ fluid modes (lower left). Solid curves are for polar type w-modes, dashed ones for the axial type. Arrows indicate increasing compactness r_1/M_{r1}, $(c, G = 1)$, ranging from 9 to 3.2. Points indicate onset of *radial instability* from Sec. 5.12.4. Right-hand side: Continuous curves show again the four nonradial fluid modes from the figure on the bottom left. Dotted lines depict the corresponding Newtonian modes of the homogeneous compressible Newtonian $n = 0$ polytrope via Eqs. (5.5.23), (5.5.26), (Andersson and Kokkotas 1998).

nonradial modes from Eqs. (5.5.23), (5.5.26) for the compressible $n = 0$ polytrope provide useful estimates of σ as long as $r_1/M_{r1} \gtrsim 5$ (up to neutron star compactness; see dotted lines on the right of Fig. 5.12.4). All g-modes have $\sigma = 0$ for the considered isentropic $n = 1$ polytrope. An alternative approach to the evolution of neutron star oscillations has been developed by Ruoff (2001).

5.12.6 Stability and Oscillations of Rotating Relativistic Polytropes

Most macroscopic kinetic motions (e.g. turbulence) attenuate rapidly under stellar conditions, while rotation – in absence of considerable dissipation – is prevented from being transformed into other forms of energy, since it is connected to a conserved quantity – the angular momentum. As already mentioned in repeated places (Secs. 3.8.1, 3.8.8, 5.8.3, 5.8.4), nonrelativistic nonaxisymmetric polytropes occur only for polytropic indices $0 \leq n \leq 0.808$. Axisymmetric and nonaxisymmetric polytropic sequences both terminate by equatorial mass loss if $n \gtrsim 0.1$ (e.g. Hachisu and Eriguchi 1982), and no equilibrium configurations exist if the angular momentum is larger than a certain critical value. The Maclaurin ellipsoids, resembling in a first approximation the $n = 0$ polytrope, and the Jacobi ellipsoids, obey no mass loss. A quantity of salient interest is the ratio (3.1.35) between rotational kinetic energy and gravitational energy: $\tau = E_{kin}/|W|$. This quantity changes between $\tau = 0$ and 0.5 for the Maclaurin ellipsoids, and between 0.1375 and 0.5 for the Jacobi ellipsoids [Eqs. (5.10.217)-(5.10.223)]. As the central mass concentration and the polytropic index of the polytrope increase, the maximum value of τ – just before onset of equatorial mass loss – decreases sharply (Table 5.8.2, Secs. 3.8.8, 5.8.3, Tassoul and Ostriker 1970, Hachisu 1986a). Rotating axisymmetric polytropes with index $0 \leq n \leq 0.808$ become secularly unstable at the bifurcation point, where the nonaxisymmetric sequence branches off at $\tau \approx 0.14$. Polytropes with $n > 0.808$ never reach a value of $\tau \approx 0.14$, since equatorial mass shedding already occurs at lower values of τ.

It is generally assumed that the angular velocity becomes rapidly uniform, due to a high viscosity of turbulent and magnetic momentum transport. Rapidly and uniformly rotating equilibrium stars are possible only in a narrow range of τ, and generally the influence of uniform rotation upon the overall characteristics of a star (mean density, pressure, energy, etc.) is not great.

Most stars can be approximated by polytropes of index $1.5 \leq n \leq 3$, and since τ_{max} is so small for such polytropes, we can preserve the Newtonian expression of the rotational energy, adding it to the post Newtonian expression (5.12.79) of the total energy

$$E = 4\pi c^2 \int_0^{r_1} \{\varrho_r - \varrho/[1 - 2GM_r(r)/c^2 r]^{1/2}\} r^2\ dr + (1/2) \int_{M_1} \Omega^2 r^2 \sin^2\lambda\ dM$$
$$\approx -\int_{M_{r1}} GM_r(r)\ dM_r/r - (6\pi G^2/c^2) \int_0^{r_1} M^2(r)\ \varrho\ dr + 4\pi \int_0^{r_1} P[1 + GM(r)/c^2 r] r^2\ dr/(\Gamma - 1)$$
$$+(4\pi\Omega^2/3) \int_0^{r_1} \varrho r^4\ dr. \qquad (5.12.164)$$

The virial theorem for a slowly rotating sphere in the post Newtonian approximation can be derived from the exact general relativistic equation of motion in the spherically symmetric case (cf. Fowler 1966):

$$[(\varrho_r + P/c^2)/\varrho]\ D[v(\varrho_r + P/c^2)/\varrho]/Dt = -(1/\varrho_r)(dP/dr)\big\{1 + [(\varrho_r + P/c^2)/\varrho]^2 v^2/c^2$$
$$-2GM_r(r)/rc^2\big\}/(1 + P/\varrho_r c^2) - GM_r(r)/r^2 - 4\pi GPr/c^2, \qquad [v = v(r)]. \qquad (5.12.165)$$

Since the initial configuration is in hydrostatic equilibrium, we can omit for a linear stability analysis the second order terms in v^2; moreover, since Dv/Dt is already a small first order quantity, the factor $(\varrho_r + P/c^2)/\varrho$ in Eq. (5.12.165) can be replaced by its zeroth approximation, i.e. by 1. Likewise, M_r and ϱ_r can be replaced in the small relativistic terms by their Newtonian values M and ϱ, respectively:

$$Dv/Dt = -(1/\varrho_r)(dP/dr)[1 - 2GM(r)/rc^2 - P/\varrho c^2] - GM_r(r)/r^2 - 4\pi GPr/c^2 + \Omega^2 r \sin^2\lambda,$$
$$(v \approx 0). \qquad (5.12.166)$$

In Eq. (5.12.166) we have added the radial Newtonian centrifugal acceleration from Eq. (5.7.2) – neglecting small Coriolis terms – to obtain the post Newtonian equation of *quasiradial* motion in a frame rotating at constant angular speed Ω with the considered configuration. In the small relativistic terms we replace dP/dr by its nonrotating Newtonian hydrostatic equivalent $-GM\varrho/r^2$:

$$Dv/Dt = -(1/\varrho_r)\ dP/dr - GM_r(r)/r^2 - [GM(r)/r^2][2GM(r)/rc^2 + P/\varrho c^2]$$
$$-4\pi GPr/c^2 + \Omega^2 r \sin^2\lambda. \qquad (5.12.167)$$

The virial theorem is obtained after multiplication by $r\ dM_r$ and integration over the whole star, where $dM_r = 4\pi\varrho_r r^2\ dr$ can be replaced by $dM = 4\pi\varrho r^2\ dr$ in the first order terms:

$$4\pi\int_0^{r_1}(Dv/Dt)\ \varrho r^3\ dr = 4\pi\int_0^{P_0} r^3\ dP + W_r - 4\pi\int_0^{r_1}[GM(r)/r][2GM(r)/rc^2 + P/\varrho c^2]\varrho r^2\ dr$$
$$-16\pi^2 G\int_0^{r_1} P\varrho r^4\ dr/c^2 + \int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM. \qquad (5.12.168)$$

The left-hand side can be brought to the form $(1/2)\ d^2I/dt^2$ from Eq. (2.6.80) or (5.3.2), by using Eqs. (2.6.56), (2.6.76):

$$4\pi\int_0^{r_1}(Dv/Dt)\ \varrho r^3\ dr = 4\pi\int_0^{r_1}[D(rv)/Dt - v^2]\varrho r^2\ dr \approx \int_{M_1}[D(rv)/Dt]\ dM$$
$$= d\left(\int_{M_1} rv\ dM\right)\Big/dt = \int_{M_1}[d(r\ dr/dt)/dt]\ dM = (1/2)\int_{M_1}(d^2r^2/dt^2)\ dM$$
$$= (1/2)\ d^2\left(\int_{M_1} r^2\ dM\right)\Big/dt^2 = (1/2)\ d^2I/dt^2, \qquad (dM/dt = 0). \qquad (5.12.169)$$

We integrate by parts the first and third integral on the right-hand side of Eq. (5.12.168):

$$\int_0^{P_0} r^3\ dP = -r^3P\Big|_0^{r_1} + 3\int_0^{r_1} Pr^2\ dr = 3\int_0^{r_1} Pr^2\ dr;$$
$$4\pi\int_0^{r_1} P\varrho r^4\ dr = \int_{M_1} Pr^2\ dM = Pr^2M(r)\Big|_0^{r_1} - \int_{M_1} M(r)\ d(r^2P)$$
$$= -\int_0^{P_0} r^2M(r)\ dP - 2\int_0^{r_1} PM(r)\ r\ dr = \int_0^{r_1} GM^2(r)\ \varrho\ dr - 2\int_0^{r_1} PM(r)\ r\ dr. \qquad (5.12.170)$$

The result is inserted into Eq. (5.12.168):

$$(1/2)\ d^2I/dt^2 = d\left(\int_{M_1} rv\ dM\right)\Big/dt = W_r + 12\pi\int_0^{r_1} Pr^2\ dr - (3G^2/c^2)\int_{M_1} M^2(r)\ dM/r^2$$
$$+(4\pi G/c^2)\int_0^{r_1} PM(r)\ r\ dr + \int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM. \qquad (5.12.171)$$

Under conditions of hydrostatic equilibrium we have $d^2I/dt^2 = 0$, and the right-hand side of Eq. (5.12.171) is zero, yielding a simple virial relationship. Another form of the post Newtonian equilibrium energy is obtained by eliminating the integral of the internal energy $4\pi\varepsilon^{(int)}r^2\ dr = 4\pi Pr^2\ dr/(\Gamma - 1)$ between Eqs. (5.12.164), (5.12.171), and replacing Γ by a suitable average:

$$E = (3\Gamma - 4)W_r/3(\Gamma - 1) + [(5 - 3\Gamma)/2(\Gamma - 1)](G/c)^2\int_{M_1} M^2(r)\ dM/r^2$$
$$+[8\pi G/3c^2(\Gamma - 1)]\int_0^{r_1} PM(r)\ r\ dr - [(5 - 3\Gamma)/6(\Gamma - 1)]\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM. \qquad (5.12.172)$$

Eq. (5.12.76) yields for a mixture of completely ionized, nonrelativistic gas and radiation

$$\Gamma = (8 - 3\beta)/3(2 - \beta), \qquad (f = 3;\ w = 2;\ \beta = 1 - P_r/P), \qquad (5.12.173)$$

and the relativistic equilibrium energy (5.12.172) becomes in this important particular case equal to

$$E = \beta W_r/2 + [3(1 - \beta)/2](G/c)^2\int_{M_1} M^2(r)\ dM/r^2 + [4\pi(2 - \beta)G/c^2]\int_0^{r_1} PM(r)\ r\ dr$$
$$-[(1 - \beta)/2]\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM. \qquad (5.12.174)$$

The approximate eigenvalue σ of the fundamental quasiradial mode of oscillation of the rotating relativistic polytrope can be found with the simple form of the Lagrangian displacement already employed in Eq. (5.3.1):

$$\eta = \Delta r(r,t)/r = [\Delta r(r)/r]\exp(i\sigma t) = (\Delta r_1/r_1)\exp(i\sigma t) = \mathrm{const}\ \exp(i\sigma t), \quad (\Delta r(r)/r = \mathrm{const}). \tag{5.12.175}$$

The first variation (5.8.4) of the left-hand side of Eq. (5.12.171) is in virtue of Eq. (5.8.30) equal to

$$(1/2)\ \delta^*(d^2I/dt^2) = \delta^*\Big[d\Big(\int_{M_1} rv\ dM\Big)\Big/dt\Big] \approx d^2\Big(\int_{M_1} r\ \Delta r(r,t)\ dM\Big)\Big/dt^2$$
$$= -(\sigma^2\ \Delta r_1/r_1)\exp(i\sigma t)\int_{M_1} r^2\ dM, \qquad (v = dr/dt \approx 0). \tag{5.12.176}$$

It will be shown in Eqs. (5.12.180)-(5.12.184) that the first variation of Eq. (5.12.171)

$$\sigma^2\ (\Delta r_1/r_1)\int_{M_1} r^2\ dM = \sigma^2\ I\ \Delta r_1/r_1 = -\delta^* W_r - 12\pi\ \delta^*\int_0^{r_1} Pr^2\ dr$$
$$+3(G/c)^2\ \delta^*\int_{M_1} M^2(r)\ dM/r^2 - (4\pi G/c^2)\ \delta^*\int_0^{r_1} PM(r)\ r\ dr - \delta^*\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM, \tag{5.12.177}$$

always contains $\eta \propto \exp(i\sigma t)$ as a factor, so we suppress it.

For adiabatic oscillations we have $\delta^* E = 0$ by definition, and the first variation of Eq. (5.12.164) is

$$\delta^* E = \delta^* W_r - (3G^2/2c^2)\ \delta^*\int_{M_1} M^2(r)\ dM/r^2 + 4\pi\ \delta^*\int_0^{r_1} P[1 + GM(r)/c^2 r]r^2\ dr/(\Gamma - 1)$$
$$+(1/2)\ \delta^*\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM = 0. \tag{5.12.178}$$

If we eliminate, as in Eqs. (5.12.164), (5.12.171), the integral of internal energy $4\pi P r^2\ dr/(\Gamma - 1)$ between Eqs. (5.12.177), (5.12.178), we get

$$\sigma^2 I\ \Delta r_1/r_1 = (3\Gamma - 4)\ \delta^* W_r + [3G^2(5 - 3\Gamma)/2c^2]\ \delta^*\int_{M_1} M^2(r)\ dM/r^2$$
$$+(8\pi G/c^2)\ \delta^*\int_0^{r_1} PM(r)\ r\ dr - [(5 - 3\Gamma)/2]\ \delta^*\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM. \tag{5.12.179}$$

The first order variations

$$\delta^* W_r = (\Delta r_1/r_1)\int_{M_{r1}} GM_r(r)\ dM_r/r;\ \ \delta^*\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM = -(2\Delta r_1/r_1)\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM, \tag{5.12.180}$$

have already been determined in Eqs. (5.3.6) and (5.3.13). Hence

$$\delta^* W_r/W_r = -\Delta r_1/r_1;\quad \delta^*\Big(\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM\Big)\Big/\int_{M_1}\Omega^2 r^2\sin^2\lambda\ dM = -2\ \Delta r_1/r_1. \tag{5.12.181}$$

It has already been shown in Eq. (5.12.88) that with the polytropic variables (2.6.1), (2.6.3), (2.6.18) we can express the integrals

$$A_1 = \int_{M_1} M^2(r)\ dM/r^2 = 64\pi^3\alpha^7\varrho_0^3\int_0^{\xi_1}\xi^4\theta^n\theta'^2\ d\xi = [M_1^3/r_1^2(-\xi_1^4\theta_1'^3)]\int_0^{\xi_1}\xi^4\theta^n\theta'^2\ d\xi, \tag{5.12.182}$$

$$A_2 = 4\pi\int_0^{r_1} PM(r)\ r\ dr = -16\pi^2 K\alpha^5\varrho_0^{2+1/n}\int_0^{\xi_1}\xi^3\theta^{n+1}\theta'\ d\xi$$
$$= -[64\pi^3 G\alpha^7\varrho_0^3/(n+1)]\int_0^{\xi_1}\xi^3\theta^{n+1}\theta'\ d\xi = -[GM_1^3/r_1^2(n+1)(-\xi_1^4\theta_1'^3)]\int_0^{\xi_1}\xi^3\theta^{n+1}\theta'\ d\xi, \tag{5.12.183}$$

by exhibiting M_1^3/r_1^2 as a factor. During quasiradial oscillations the sole variable in these integrals is the radius r_1, and the first variation of these integrals is via Eq. (5.8.2) equal to

$$\delta^* A_{1,2} = \mathrm{const}\ (1/r_1^2 - 1/r_{1u}^2) = -2\ \mathrm{const}\ \Delta r_1/r_1^3 = -2A_{1,2}\ \Delta r_1/r_1, \qquad (\Delta r = r_1 - r_{1u}), \tag{5.12.184}$$

where r_{1u} denotes the unperturbed radius.

The conserved angular momentum can be written as

$$J = \Omega \int_{M_1} r^2 \sin^2\lambda\ dM = (8\pi\Omega/3) \int_0^{r_1} \varrho r^4\ dr = (2\Omega/3) \int_{M_1} r^2\ dM = 2\Omega I/3$$
$$= k\Omega M_1 r_1^2 = \mathrm{const}, \qquad (k = 2I/3M_1 r_1^2). \tag{5.12.185}$$

k denotes the dimensionless gyration factor from Eq. (6.1.179) and Table 6.1.2. The Newtonian rotational energy reads

$$(\Omega^2/2) \int_{M_1} r^2 \sin^2\lambda\ dM = \Omega^2 I/3 = \Omega J/2 = J^2/2kM_1 r_1^2 \propto r_1^{-2}, \qquad (J, M_1 = \mathrm{const}). \tag{5.12.186}$$

We insert Eqs. (5.12.181), (5.12.184), (5.12.186) into Eq. (5.12.179), suppressing the common factor $\Delta r_1/r_1$:

$$\sigma^2 I = -(3\Gamma - 4)W_r - [G^2 M_1^3/c^2 r_1^2(-\xi_1^4 {\theta_1'}^3)]\Big\{3(5-3\Gamma) \int_0^{\xi_1} \xi^4\theta^n {\theta'}^2\ d\xi$$
$$-[4/(n+1)] \int_0^{\xi_1} \xi^3\theta^{n+1}\theta'\ d\xi\Big\} + (5-3\Gamma)J^2/kM_1 r_1^2. \tag{5.12.187}$$

As noted subsequently to Eq. (2.6.92), we have $\Gamma \approx \Gamma_1$ for a nondegenerate gas-radiation mixture without $e^{\pm}$-pairs. Neglecting the relativistic terms, we observe that Eq. (5.12.187) is equal to Eq. (5.3.15) if $\eta = \Delta r/r = \mathrm{const}$, and $\Gamma_1 = \Gamma = \mathrm{const}$.

For a massive star consisting of radiation-dominated nonrelativistic plasma, Eq. (5.12.173) yields $\Gamma = 4/3 + \beta/6$, $(\beta \approx 0)$, and Eq. (5.12.187) reads

$$\sigma^2 I = 3\beta G M_1^2/2(5-n)r_1 - [G^2 M_1^3/c^2 r_1^2(-\xi_1^4 {\theta_1'}^3)]\Big\{3 \int_0^{\xi_1} \xi^4\theta^n {\theta'}^2\ d\xi$$
$$-[4/(n+1)] \int_0^{\xi_1} \xi^3\theta^{n+1}\theta'\ d\xi\Big\} + J^2/kM_1 r_1^2 = r_1\ dE/dr_1,$$
$$\big(\beta = 1 - P_r/P \approx 0;\ \Gamma = 4/3 + \beta/6;\ W_r \approx W = -3GM_1^2/(5-n)r_1\big). \tag{5.12.188}$$

The last equality has been obtained by inserting into the equilibrium energy (5.12.174) the relationships (2.6.137), (5.12.182), (5.12.183), (5.12.186).

For massive stars with $\beta = \mathrm{const}$, the $n = 3$ polytrope yields a fairly accurate representation of their internal structure (Sec. 6.1.1), and Eq. (5.12.188) becomes with the values of the integrals from Eq. (5.12.88) equal to

$$\sigma^2 I = 3\beta G M_1^2/4r_1 - 5.1\ G^2 M_1^3/c^2 r_1^2 + J^2/kM_1 r_1^2,$$
$$[\beta \approx 0;\ \Gamma = 4/3 + \beta/6 \approx 4/3;\ n = 3 \approx 1/(\Gamma - 1)]. \tag{5.12.189}$$

From this approximate treatment it appears that if $n = 3$, $\Gamma \approx 4/3 = 1 + 1/n$, rotation has a stabilizing influence on the fundamental mode of quasiradial oscillations [Eq. (5.3.16)], while general relativity enhances instability (decreases σ^2). Thus, large enough rotation could prevent the general relativistic instability; rotation extends the mass and energy limits of stable oscillations triggered by hydrogen burning from $10^6\ M_\odot$ and 10^{58} erg to $10^8 M_\odot$ and 10^{60} erg, respectively (Fowler 1966).

A similar study has been undertaken by Durney and Roxburgh (1967), confined to massive polytropic stars with $n = 3$ and masses between about 10^6 and $10^{10}\ M_\odot$. In a rough approximation the angular term $1 - \mu^2$ in Eq. (3.1.17) can be averaged over the surface elements $-2\pi\ d\mu = 2\pi \sin\lambda\ d\lambda$ of the unit sphere. Taking also $\partial\Phi/\partial\mu = 0$ and $d\Phi/dr = -GM(r)/r^2$, the equation of hydrostatic equilibrium (3.1.17) reads

$$dP/dr = -GM(r)\ \varrho/r^2 + 2\Omega^2\varrho r/3, \tag{5.12.190}$$

amounting to an overall expansion of the slowly rotating sphere.

The slow rotation approximation demands that the centrifugal term $2\Omega^2\varrho r/3$ is much smaller than the gravitational attraction $-GM(r)\ \varrho/r^2$, or $v^2/c^2 = \Omega^2 r^2/c^2 \ll GM(r)/rc^2$, so that relativistic effects of rotation of order $\Omega^2 r^2/c^2$ are much smaller than those due to the spherically symmetric distribution of matter, and can be consistently neglected.

With a trial Lagrangian displacement $\Delta r(r,t) = (a+br+cr^2)\exp(i\sigma t),\ (a,b,c = \mathrm{const})$ the eigenvalue of the fundamental quasiradial displacement has been obtained by Durney and Roxburgh (1967) under the form of a linear combination of small quantities

$$\sigma^2/4\pi G\varrho_0 = 0.041\beta + 0.333\Omega^2/2\pi G\varrho_0 - 0.644K\varrho_0^{1/3}/c^2,$$
$$(\beta \approx 0;\ n = 3;\ \Gamma = 4/3 + \beta/6 \approx 4/3 = 1 + 1/n), \tag{5.12.191}$$

where $\beta = P_g/P$ denotes the ratio of gas to total pressure from Eq. (1.4.14), rather than the dimensionless rotation parameter $\Omega^2/2\pi G\varrho_0$ from Eq. (3.2.3). Instability ($\sigma^2 < 0$) occurs in the nonrotating case if

$$0.041\beta < 0.644K\varrho_0^{1/3}/c^2, \qquad (\Omega = 0). \tag{5.12.192}$$

The relativity parameter $q_0 = K\varrho_0^{1/3}/c^2 = P_0/\varrho_0 c^2,\ (\varrho_{r0} \approx \varrho_0)$ from Eq. (4.1.31) can be expressed in terms of mass and radius of the $n = 3$ polytrope according to Eqs. (5.12.65), (6.1.11), (6.1.12):

$$\varrho_0 = -\xi_1\varrho_m/3\theta_1' = 54.18 \times 3M_1/4\pi r_1^3; \quad P_0 = 11.05GM_1^2/r_1^4. \tag{5.12.193}$$

Hence

$$q_0 = P_0/\varrho_0 c^2 = 0.427 \times 2GM_1/r_1c^2 = 0.427r_g/r_1. \tag{5.12.194}$$

Inserting into Eq. (5.12.192), we find that stability of the nonrotating star is lost if [cf. Fowler 1966, Eq. (44)]

$$r_1 \le 6.71r_g/\beta, \qquad (\Omega = 0;\ \beta \approx 0;\ n = 3;\ \Gamma = 4/3 + \beta/6 \approx 4/3 = 1 + 1/n). \tag{5.12.195}$$

This value agrees quite well with $r_1 \le 6.73r_g/\beta$, obtained from Eq. (5.12.189) under the same assumptions. Since by virtue of Eq. (5.12.87) we have $\beta \approx 8.56(M_\odot/M_1)^{1/2}$ for a pure hydrogen star [$\mu = 1/2$, Eq. (1.7.16)], the ratio between gas pressure and total pressure becomes $\beta \approx 10^{-2} - 10^{-4}$ if $M_1/M_\odot = 10^6 - 10^{10}$. For maximum rotation we infer from Table 3.8.1 that $\Omega_c^2/2\pi G\varrho_0 \approx 0.004$ if $n = 3$, and therefore we may neglect β in Eq. (5.12.191) under this assumption, the quasiradial instability condition amounting to

$$q_0 = K\varrho_0^{1/3}/c^2 > 0.52\Omega_c^2/2\pi G\varrho_0 = 2.07 \times 10^{-3}, \qquad (\beta = 0;\ \sigma^2 < 0). \tag{5.12.196}$$

Inserting for q_0 from Eq. (5.12.194), quasiradial instability of the critically rotating star with vanishing gas pressure occurs if

$$r_1 < 206r_g, \qquad (\Omega_c^2/2\pi G\varrho_0 = 0.004;\ \beta \approx 0;\ n = 3;\ \Gamma \approx 4/3). \tag{5.12.197}$$

Papoyan et al. (1972) have determined the quasiradial pulsation frequencies of relativistic isentropic polytropes having $\Gamma_1 = 1 + 1/n$, incorporating a post Newtonian correction to the slow rotation of an otherwise spherical polytrope. From the coarse presentation of Papoyan et al. (1972) it seems that their definition [cf. Eqs. (4.1.85), (5.12.119)]

$$\Gamma_1 = 1 + 1/n = \Gamma_{r1}(P+\varepsilon_r)/\varepsilon_r = (d\ln P/d\ln\varepsilon_r)_S\ (P+\varepsilon_r)/\varepsilon_r = [(P/c^2 + \varrho_r)/P]\ dP/d\varrho_r, \tag{5.12.198}$$

of the adiabatic exponent would imply the equation of state (5.12.145), rather than Eq. (5.12.146). The rotation parameter of Papoyan et al. (1972) corresponds in the spherical Newtonian limit to $\eta(\xi_1)/2\xi_1^3 \approx -\theta_1'/2\xi_1 = M_1/8\pi r_1^3\varrho_0 = \Omega^2/8\pi G\varrho_0$, [$\Omega^2 = GM_1/r_1^3$, see Eq. (4.2.232)], and seems nearly equal to the critical values of Chandrasekhar (1933a, d), quoted in Table 3.8.1. In Fig. 5.12.5 rotation has a stabilizing influence if $2.5 \lesssim n < 3$, as noted previously (σ^2 increases); but it appears that it enhances the general relativistic instability if $1 < n \lesssim 2.5$. These rotational effects are approximated by Eq. (5.3.16) in the Newtonian limit if $\Gamma_1 = 1 + 1/n$.

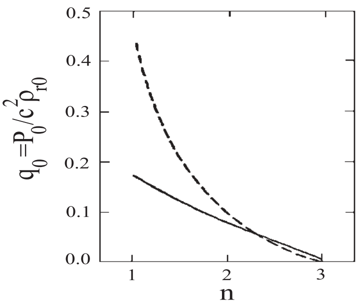

Fig. 5.12.5 Critical relativity parameter $q_0 = P_0/c^2\varrho_{r0}$ at the onset of instability against quasiradial oscillations as a function of isentropic polytropic index $n = 1/(\Gamma_1-1)$. Dashed curve corresponds to the nonrotating case [see also Fig. 3 of Glass and Harpaz (1983)], continuous line to nearly critical rotation. Stability subsists on the left of the curves, instability on the right. The region of Newtonian stability in the nonrotating case is the rectangle $0 \leq n \leq 3$ (Papoyan et al. 1972).

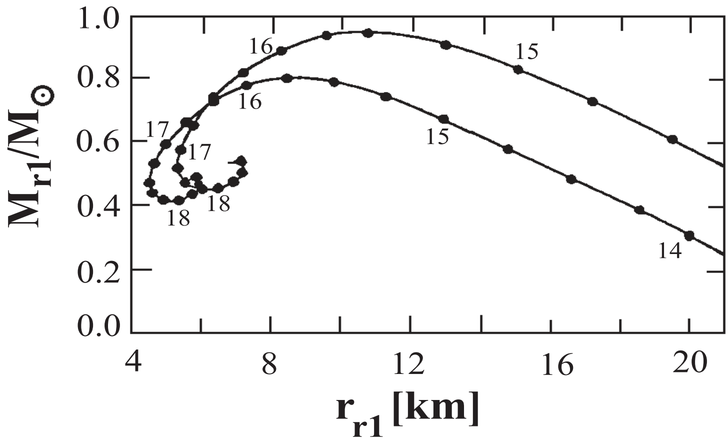

Fig. 5.12.6 Nonrotating (lower curve) and critically rotating (upper curve) polytropic hydrostatic equilibrium models calculated with the equation of state (5.12.201). M_{r1} and r_{r1} denotes the relativistic mass and radius of the neutron star. Stable equilibrium configurations are located to the right of the maximum mass. Numbers on the curves indicate powers of 10 of the central density ϱ_{r0} measured in g cm^{-3} (Hartle and Friedman 1975; see also Fig. 6.1.2).

Hartle and Munn (1975) have investigated the stability of quasiradial modes in a slowly rotating, relativistic $n = 1.5$ polytrope with equation of state (5.12.145). This task is undertaken with the static method outlined in Sec. 5.12.1 for the nonrotating case. The equation of state $P = K\varrho^{5/3}$, $(n = 1.5)$ is satisfied, among others, by a neutron star composed of a completely degenerate, nonrelativistic neutron gas [Eq. (1.7.34)]. The metric adopted by Hartle and Munn (1975) can be obtained from Eqs. (4.2.9), (4.2.149), giving up the condition $g_{11} = g_{22}$:

$$ds^2 = (g_{00} - g_{03}^2/g_{33})\ dt^2 + g_{11}\ dr^2 + g_{22}\ d\lambda^2 + g_{33}(d\varphi + g_{03}\ dt/g_{33})^2. \tag{5.12.199}$$

Adopting the condition that this metric should converge for $\Omega \to 0$ into the Schwarzschild metric

(4.1.5), we infer that $g_{00} = \exp\nu,\ g_{11} = -\exp\kappa,\ g_{22} = -r^2,\ g_{33} = -r^2\sin^2\lambda$ and

$$ds^2 = \exp\nu\ dt^2 - \exp\kappa\ dr^2 - r^2(d\lambda^2 + \sin^2\lambda\ d\varphi^2) + 2g_{03}\ d\varphi dt + O(\Omega^2). \qquad (5.12.200)$$

Due to rotation the relativistic central density ϱ_{r0} of equilibrium models changes, allowing for a 20% more massive stable mass, as compared to the nonrotating case (Fig. 5.12.6). Critically rotating equilibrium models have been calculated by Hartle and Friedman (1975), and Hartle and Munn (1975) for the isentropic polytrope $n = 1/(\Gamma_1 - 1) = 1.5$ with an equation of state of the form (4.1.86):

$$P = K\varrho^{5/3} = K\varepsilon^{5/3};\quad \varepsilon_r = CP^{3/5} + 3P/2\ [\mathrm{km}^{-2}],\qquad (c, G = 1;\ C = 0.3032). \qquad (5.12.201)$$

The onset of gravitational radiation driven instability due to the nonaxisymmetric f-mode ($2 \leq j = k \leq 5$) has been located in rapidly rotating relativistic polytropes by Stergioulas and Friedman (1998) if $n = 1, 1.5, 2$ (Table 5.8.2). Even with a suitably chosen gauge the perturbed metric still possesses six metric perturbation functions a, b, c, d, e, f, depending on r, λ, and $\exp(ik\varphi)$:

$$\begin{aligned} ds^2 &= \exp(2\nu)(1+2a)\ dt^2 - \exp(2\kappa)[(1+2b)\ dr^2 + r^2(1+2d)\ d\lambda^2] \\ &- \exp(2\mu)(1+2d)(d\varphi - \omega\ dt - c\ dr)^2 - 2e\ dt\ dr - 2f\ dt\ d\lambda. \end{aligned} \qquad (5.12.202)$$

The unperturbed metric ds_u^2 assumes the form (4.2.149) for vanishing metric perturbation functions. In the case of polar type perturbations (which are mixed by rotation with axial ones) the perturbed Einstein equations (5.12.150) have to be supplemented by the perturbed relativistic continuity equation $\delta(u_\ell\ \nabla_m T^{\ell m}) = 0$ (conservation of relativistic energy), and by the perturbed relativistic Euler equation of motion $\delta[(\delta_\ell^m - u_\ell u^m)\ \nabla_n T^{\ell n}] = 0$ (conservation of momentum). These two equations can be derived with the identity (4.1.14), which may be interpreted as a conservation equation for the energy-momentum of fluid matter alone, when contributions from the gravitational field itself can be neglected (Landau and Lifschitz 1987, §96). Multiplication of Eq. (4.1.14) by u_ℓ yields with the energy-momentum tensor (4.1.11) the conservation equation of relativistic energy:

$$\begin{aligned} u_\ell\ \nabla_m T^{\ell m} &= u^m\ \nabla_m(P+\varepsilon_r) + (P+\varepsilon_r)\ \nabla_m u^m + (P+\varepsilon_r)u^m u_\ell\ \nabla_m u^\ell - g^{\ell m}u_\ell\ \nabla_m P \\ &= u^m\ \nabla_m\varepsilon_r + (P+\varepsilon_r)\ \nabla_m u^m = 0. \end{aligned} \qquad (5.12.203)$$

To derive this equation, we have used $\nabla_m T^{\ell m} = g^{\ell n}\ \nabla_m T_n^m = 0,\ u_\ell u^\ell = 1,\ \nabla_m P = \partial P/\partial x^m$, and $u_\ell\ \nabla_m u^\ell = u^\ell\ \nabla_m u_\ell = 0$ from Eq. (6.4.88).

The relativistic Euler equation (6.4.90) can also be derived by multiplication of Eq. (4.1.14) with $\delta_\ell^m - u_\ell u^m$:

$$(\delta_\ell^m - u_\ell u^m)\ \nabla_n T^{\ell n} = (P+\varepsilon_r)u^n\ \nabla_n u^m - (g^{mn} - u^m u^n)\ \nabla_n P = 0. \qquad (5.12.204)$$

Yoshida (2001) has investigated the properties of rotational r-modes in slowly rotating relativistic polytropes (neutron stars with polytropic index $0 \leq n \lesssim 1.2$) for the purely axial perturbations from Eqs. (5.12.159), (5.12.160) with $h_1(r) = 0$. Bar-mode instabilities (sectorial or toroidal mode instabilities $\propto \exp(\pm ik\varphi);\ k = 2$) of differentially rotating $n = 1$ polytropes in the post Newtonian approximation show that general relativity enhances the dynamical bar-mode instability, similarly to the quasiradial mode instability from Fig. 5.12.5 (see also Sec. 6.1.6, (iii); Saijo et al. 2001).

The relativistic evolution of close neutron binaries near the innermost stable circular orbit (prior to dynamical orbital instability) has been followed up among others by Lombardi et al. (1997) if $n = 0.5$, 1, Baumgarte et al. (1997) if $n = 1$, Uryū et al. (2000) if $0.5 \leq n \leq 1.25$. The quasiequilibrium inspiral by emission of gravitational radiation is destroyed with the onset of relativistic gravitational instabilities at separation distances $(6-10)G(M_{r1} + M'_{r1})/c^2$, or when the Newtonian Roche limit of tidal disruption is reached at typical separation distances $(2-3)(r^3 + r'^3)^{1/3}$, (Sec. 5.7.4, Lai et al. 1993, 1994a, Wilson and Mathews 1995). The final merging of the two neutron stars takes place during a few orbital periods (time intervals of order 10 milliseconds, e.g. Fig. 5.7.4 top, Ruffert et al. 1997, Faber and Rasio 2000).

Using a nonpolytropic, more realistic equation of state Wilson and Mathews (1995) find that otherwise stable neutron stars may individually collapse to black holes when placed in close binary orbits.

6 FURTHER APPLICATIONS TO POLYTROPES

6.1 Applications to Stars and Stellar Systems

6.1.1 Eddington's Standard Model

The overwhelming parts of this book have direct relevance to stellar and galactic structure. In fact, prior to about 1930, most stellar models had more or less direct bearing on polytropes, and many analytic solutions to the internal constitution of stars are connected with polytropic relationships.

Eddington's standard model of radiative equilibrium results by division of the equation of radiative transfer

$$dP_r/dr = (4aT^3/3)\ dT/dr = -\kappa\varrho L(r)/4\pi cr^2, \qquad (P_r = aT^4/3), \tag{6.1.1}$$

with the equation of hydrostatic equilibrium for a sphere

$$dP/dr = d(P_g + P_r)/dr = -G\varrho M(r)/r^2. \tag{6.1.2}$$

This yields

$$dP_r/dP = (4aT^3/3)\ dT/dP = \kappa L(r)/4\pi cGM(r). \tag{6.1.3}$$

$P_g = \beta P$ and $P_r = (1-\beta)P$ denotes the gas and radiation pressure in the medium (Sec. 1.4), a is Stefan's radiation constant, c the velocity of light, $\kappa = \kappa(r)$ the opacity, and $L(r)$ the luminosity of the star at distance r from the centre (energy flux through a sphere of radius r), where

$$dL(r) = 4\pi\varepsilon(r)\ \varrho r^2\ dr. \tag{6.1.4}$$

It is implicitly assumed in Eqs. (6.1.1) and (6.1.4) that the whole luminosity $L(r)$ is due to radiative energy transport. Generally, the rate of energy flow $L(r)$ is due to energy transport by radiation, convection, conduction, and sometimes neutrinos. Eq. (6.1.4) represents simply the energy balance equation, when $dL(r)$ is the variation of energy flux through a shell of thickness dr, and $4\pi\varepsilon(r)\ \varrho r^2\ dr$ is just the energy produced per unit time within this shell, $\varepsilon(r)$ denoting the energy production per unit time and mass. Defining the quantity

$$\eta = \eta(r) = \varepsilon_m(r)/\varepsilon_{m1} = [L(r)/M(r)]/(L_1/M_1), \qquad [M_1 = M_1(r_1);\ L_1 = L(r_1);\ \varepsilon_{m1} = \varepsilon_m(r_1)], \tag{6.1.5}$$

we may write Eq. (6.1.3) under the form

$$dP_r/dP = \kappa(r)\ \eta(r)\ L_1/4\pi cGM_1. \tag{6.1.6}$$

$\varepsilon_m(r) = L(r)/M(r)$ and $\varepsilon_{m1} = \varepsilon_m(r_1) = L(r_1)/M(r_1)$ represent the average energy-generation rates per unit mass inside radius r and r_1, respectively, r_1 being the stellar radius. We integrate Eq. (6.1.6) between the surface where $P(r_1), P_r(r_1) \approx 0$, and some interior point $P = P(r), P_r = P_r(r)$:

$$P_r = (L_1/4\pi cGM_1)\int_0^P \kappa(r)\ \eta(r)\ dP = (L_1P/4\pi cGM_1)\left[(1/P)\int_0^P \kappa(r)\ \eta(r)\ dP\right]$$
$$= L_1P\ [\kappa(r)\ \eta(r)]_m/4\pi cGM_1. \tag{6.1.7}$$

The integral

$$[\kappa(r)\ \eta(r)]_m = (1/P)\int_0^P \kappa(r)\ \eta(r)\ dP, \tag{6.1.8}$$

is an average of $\kappa(r)\ \eta(r)$ with respect to total pressure in the region *exterior* to the radius r. Inserting for $P_r/P = 1-\beta,\ (\beta = P_g/P)$, Eq. (6.1.7) finally becomes

$$1-\beta = [\kappa(r)\ \eta(r)]_m L_1/4\pi c G M_1. \tag{6.1.9}$$

Eddington's fundamental assumption for the standard model is that $[\kappa(r)\ \eta(r)]_m = \text{const}$, i.e. $1-\beta = P_r/P$ is constant throughout the star: $\beta = \beta_0 = \text{const}$. The central value of the ratio gas pressure to total pressure is denoted by $\beta_0 = P_{g0}/P_0$. For a perfect gas-radiation mixture there subsist Eqs. (5.12.84), (5.12.85), showing that the chemically homogeneous standard model is just a polytrope of index $n = 3:$

$$P = \mathcal{R}\varrho T/\mu\beta_0 = [3(1-\beta_0)(\mathcal{R}/\mu)^4/a\beta_0^4]^{1/3}\varrho^{4/3} = K\varrho^{4/3}, \qquad (\beta_0, K, \mu = \text{const}). \tag{6.1.10}$$

The Bialobjesky-Eddington equation (5.12.86) permits the calculation of the ratio β_0 for a standard $n = 3$ star of given mean molecular weight and mass.

For the central pressure we get from Eq. (2.6.25)

$$P_0 = GM_1^2/4\pi(n+1)\theta_1'^2 r_1^4 = 11.05\ GM_1^2/r_1^4 = 1.24\times 10^{17}(M_1/M_\odot)^2/(r_1/r_\odot)^4\ [\text{dyne/cm}^2],$$
$$(n, N = 3;\ \Gamma(3/2) = \Gamma(1/2)/2 = \pi^{1/2}/2). \tag{6.1.11}$$

The ratio between central and mean density in the standard model is obtained at once from Eq. (2.6.27) and Table 2.5.2 if $N = 3:$

$$\varrho_0/\varrho_m = -\xi_1/3\theta_1' = 54.18, \qquad (n = 3). \tag{6.1.12}$$

The central temperature for a star composed of a perfect gas-radiation mixture is given by

$$T_0 = \mu\beta_0 P_0/\mathcal{R}\varrho_0 = -3\theta_1'\mu\beta_0 P_0/\xi_1\mathcal{R}\varrho_m = \mu\beta_0 GM_1/(n+1)\xi_1(-\theta_1')\mathcal{R}r_1 = 0.8543\ \mu\beta_0 GM_1/\mathcal{R}r_1$$
$$= 1.96\times 10^7\mu\beta_0(M_1/M_\odot)/(r_1/r_\odot)\ [\text{K}], \qquad (n = 3). \tag{6.1.13}$$

Much more sophisticated numerical models of the Sun, for instance, yield values of ϱ_0/ϱ_m and T_0/μ of the same order of magnitude (cf. Cox and Giuli 1968).

The gravitational energy of the standard model is obtained from Eq. (2.6.137) if $n = 3:$

$$W = -3GM_1^2/2r_1. \tag{6.1.14}$$

The mean temperature over the mass of the standard model is given by

$$T_m = (1/M_1)\int_{M_1} T\ dM = (\mu\beta_0/\mathcal{R}M_1)\int_{M_1} P\ dV = -\mu\beta_0 W/3\mathcal{R}M_1 = \mu\beta_0 GM_1/2\mathcal{R}r_1$$
$$= 1.15\times 10^7\mu\beta_0(M_1/M_\odot)/(r_1/r_\odot) = 0.5853\ T_0, \qquad (n = 3), \tag{6.1.15}$$

where we have used Eq. (3.1.85) in the nonrotating case $\Omega = 0:$

$$W = -3\int_{V_1} P\ dV. \tag{6.1.16}$$

With this equation the internal energy of the perfect gas-radiation mixture is given by [cf. Eq. (1.4.13)]

$$U = U_g + U_r = \int_{M_1} c_V T\ dM + \int_{V_1} aT^4\ dV = \int_{V_1}\{[\mathcal{R}/\mu(\gamma-1)]\varrho T + 3(1-\beta_0)P\}\ dV$$
$$= [\beta_0/(\gamma-1) + 3(1-\beta_0)]\int_{V_1} P\ dV = -[1+\beta_0(4-3\gamma)/3(\gamma-1)]W. \tag{6.1.17}$$

This is equivalent to the virial theorem for a hydrostatic equilibrium mass composed of a perfect gas-radiation mixture (cf. Eq. (2.6.100) if $\beta_0 = 1,\ \gamma = \Gamma$). The total energy becomes (cf. Eq. (2.6.98) if $\beta_0 = 1,\ \gamma = \Gamma,\ U_m = 0):$

$$E = U + W = -\beta_0(4-3\gamma)W/3(\gamma-1), \qquad (\gamma, \beta_0 = \text{const}). \tag{6.1.18}$$

A mass-radius-luminosity relation for the standard model can be obtained by assuming for the opacity an interpolation law of the form

$$\kappa = \kappa(r) = \kappa_{ef}\varrho^e T^f, \qquad (\kappa_{ef}, e, f = \mathrm{const}), \tag{6.1.19}$$

which turns into Kramers' opacity law if $e = 1,\ f = -3.5$. Using the central value $a\mu\beta_0 T_0^3/3\mathcal{R}(1-\beta_0)\varrho_0 = 1$ of Eq. (5.12.84), we get with Eq. (6.1.13):

$$\kappa_0 = \kappa(0) = \kappa_{ef}\varrho_0^e T_0^f = \kappa_{ef} a\mu\beta_0\varrho_0^{e-1}T_0^{f+3}/3\mathcal{R}(1-\beta_0)$$
$$= [\kappa_{ef} a\varrho_0^{e-1}/3(1-\beta_0)][GM_1/(n+1)\xi_1(-\theta_1')r_1]^{f+3}(\mu\beta_0/\mathcal{R})^{f+4}, \qquad (n=3). \tag{6.1.20}$$

The central opacity value $\kappa_0 = \kappa(0)$ can be brought into evidence in Eq. (6.1.9), by writing Eddington's assumption under the form

$$[\kappa(r)\ \eta(r)]_m = \mathrm{const} = [\kappa(0)\ \eta(0)]_m = \kappa_0 E_0, \tag{6.1.21}$$

where via Eq. (6.1.8), (Chandrasekhar 1939, pp. 243-245):

$$E_0 = [\kappa(0)\ \eta(0)]_m/\kappa(0) = (1/\kappa_0 P_0)\int_0^{P_0}\kappa(r)\ \eta(r)\ dP. \tag{6.1.22}$$

If P, ϱ, T belong to a polytrope composed of a perfect gas with negligible radiation pressure ($\beta_0 \approx 1$), we have in virtue of Eqs. (2.6.3), (2.6.7): $P = P_0\theta^{n+1},\ \varrho = \varrho_0\theta^n,\ T = T_0\theta$. Eq. (6.1.19) yields $\kappa/\kappa_0 = \theta^{ne+f}$, so Eq. (6.1.22) becomes

$$E_0 = (n+1)\int_0^1 \eta[r(\theta)]\ \theta^{n(1+e)+f}\ d\theta, \qquad [\beta = \beta_0 \approx 1;\ r = r(\theta)]. \tag{6.1.23}$$

If ε = const, as in the uniform energy source model from Sec. 6.1.2, we infer from Eq. (6.1.5): $\eta(r) = \eta[r(\theta)] = 1$. And Eq. (6.1.23) becomes with Kramers' opacity for the standard model:

$$E_0 = (n+1)\int_0^1 \theta^{n(1+e)+f}\ d\theta = (n+1)/[n(1+e)+f+1] = 8/7 = 1.14,$$
$$(n=3;\ \varepsilon = \mathrm{const};\ \beta_0 \approx 1;\ e=1;\ f=-3.5). \tag{6.1.24}$$

In the case of constant opacity $\kappa = \kappa_0,\ (e, f = 0)$ the previous equation yields $E_0 = 1$. Generally, the values of E_0 are between about 1 and 3.3 (Chandrasekhar 1939, Chap. IX).

We combine Eqs. (6.1.9), (6.1.20), and (6.1.21):

$$L_1 = 4\pi cGM_1(1-\beta_0)/\kappa_0 E_0 = [12\pi c(1-\beta_0)^2\varrho_0^{1-e}/a\kappa_{ef}E_0](\mathcal{R}/\mu\beta_0)^{f+4}$$
$$\times (GM_1)^{-f-2}[(n+1)r_1\xi_1(-\theta_1')]^{f+3}, \qquad (n=3). \tag{6.1.25}$$

Eliminating $1-\beta_0$ between Eqs. (5.12.86) and (6.1.25), we get the mass-radius-luminosity relationship of the standard model after insertion of $\varrho_0 = -\xi_1 M_1/4\pi\theta_1' r_1^3$ from Eq. (6.1.12):

$$L_1 = 4^{e+f-1}\pi^{e+2}ac(\mathcal{R}/\mu G\beta_0)^{f-4}M_1^{-e-f+3}r_1^{3e+f}\xi_1^{-e+f-4}(-\theta_1')^{e+f-2}/3\kappa_{ef}E_0$$
$$= \mathrm{const}\ M_1^{-e-f+3}r_1^{3e+f}(\mu\beta_0)^{-f+4}/\kappa_{ef}, \qquad (n=3). \tag{6.1.26}$$

All $n = 3$ polytropes of the standard model form a homologous family (see Sec. 2.2.1), and therefore E_0 is a pure number. The luminosity of all stars obeying the assumptions of the standard model changes according to a homology relationship, the homology constant depending only on the constants e and f.

The standard model in general relativity has been considered by Tooper (1966a), [cf. Eq. (4.1.87)].

6.1.2 Uniform Energy Source Model

If $\beta = \text{const}$, this model can be approximated with a polytrope. The uniform source model is not a good approximation for stars with strongly concentrated energy sources – Cowling's point source model from the next section constituting the opposite limiting case. The uniform source model is subject to analytical treatment and serves as a comparative base for other models with nonuniform distribution of energy sources. The main characteristic of the model constitutes the relationship

$$\varepsilon = L(r)/M(r) = L_1/M_1 = \text{const.} \tag{6.1.27}$$

It is always assumed that the uniform source model is convectively stable (in radiative equilibrium). In view of Eq. (6.1.29) this amounts to the condition that the absolute value $|dT/dr| = |dT/dr|_{rad}$ of the radiative temperature gradient (6.1.33) is always smaller than the absolute value $|dT/dr|_{ad}$ of the adiabatic temperature gradient (6.1.34).

From the condition of convective stability (5.2.85)

$$A < 0 \quad \text{or} \quad (d\ln\varrho/dr)/(d\ln P/dr) = d\ln\varrho/d\ln P > 1/\Gamma_1 = (\partial\ln\varrho/\partial\ln P)_S = (\partial\ln\varrho/\partial\ln P)_{ad},$$
$$(dP/dr < 0), \tag{6.1.28}$$

we can deduce the condition of convective stability of the temperature gradient in a *chemically uniform* region of a star (Cox and Giuli 1968, Chap. 13):

$$dT/dr > (dT/dr)_{ad} \quad \text{or} \quad |dT/dr| < |dT/dr|_{ad}, \qquad (dT/dr < 0). \tag{6.1.29}$$

$P = P(r)$, $\varrho = \varrho(r)$, $T = T(r)$ denote pressure, density, and temperature at radius r in the actual star, $d\ln\varrho/d\ln P = 1/\Gamma_1'$ is the actual density gradient (1.3.1), while $(\partial\ln\varrho/\partial\ln P)_{ad} = 1/\Gamma_1$ means the density change (1.3.23) in a hypothetical, adiabatically changing mass element, starting from the same initial conditions on P, ϱ, T at the same radius r. The pressure within the adiabatic mass element is assumed always equal to the actual pressure in the star, so the pressure change dP within radial distance dr is identical on both sides of Eq. (6.1.28): Only $d\varrho$ and dT assume different values on both sides of the inequalities (6.1.28), (6.1.29).

From Eq. (1.3.5) we get for the actual change of density and temperature over distance dr

$$d\ln\varrho/d\ln P = 1/\chi_\varrho - (\chi_T/\chi_\varrho)\ d\ln T/d\ln P, \tag{6.1.30}$$

where we use $\ln P(r)$ instead of r as the independent variable.

The density and temperature variation in the adiabatic mass element is – with the same equation of state – equal to

$$(d\ln\varrho/d\ln P)_{ad} = 1/\chi_\varrho - (\chi_T/\chi_\varrho)(d\ln T/d\ln P)_{ad}, \tag{6.1.31}$$

where in view of the small density and temperature variations $d\varrho, dT$ – starting from the same initial state – the derivatives χ_ϱ, χ_T assume the same values in the actual star and in the adiabatic mass element. Inserting Eqs. (6.1.30), (6.1.31) into Eq. (6.1.28), we get the stability condition against convective motions:

$$(P/T)\ dT/dP < [(P/T)\ dT/dP]_{ad}. \tag{6.1.32}$$

And the convective stability condition (6.1.29) is obtained at once after multiplication with $(T/P)\ dP/dr \approx (T_{ad}/P)\ dP/dr < 0$, taking into account that the adiabatic temperature T_{ad} differs from the actual temperature T at most by infinitesimal quantities of order dT.

If the actual temperature gradient dT/dr from Eq. (6.1.29) is given by the radiative temperature gradient (6.1.1), the latter can be expressed in terms of the effective polytropic index $n'_{rad} = 1/(\Gamma_2' - 1)$ from Eq. (1.3.26):

$$(dT/dr)_{rad} = -3\kappa\varrho L(r)/16\pi acT^3r^2 = [3\kappa L(r)/16\pi acGM(r)\ T^3]\ dP/dr$$
$$= [\kappa L(r)\ T/16\pi cGM(r)\ (1-\beta)P]\ dP/dr = [T/P(n'_{rad}+1)]\ dP/dr \quad \text{or}$$
$$(d\ln T/d\ln P)_{rad} = 1/(n'_{rad}+1) = (\Gamma_2'-1)/\Gamma_2', \qquad [n'_{rad} = -1 + 16\pi cGM(r)\ (1-\beta)/\kappa L(r)]. \tag{6.1.33}$$

And the effective polytropic index $n'_{ad} = 1/(\Gamma_2 - 1)$ corresponding to adiabatic (convective) equilibrium is obtained via Eqs. (1.3.23), (1.3.30):

$$(dT/dr)_{ad} = [T(\Gamma_2 - 1)/P\Gamma_2]\ dP/dr = [T/P(n'_{ad} + 1)]\ dP/dr \quad \text{or}$$
$$(d\ln T/d\ln P)_{ad} = 1/(n'_{ad} + 1) = (\Gamma_2 - 1)/\Gamma_2. \tag{6.1.34}$$

The convective stability condition on the effective polytropic index n'_{rad} results from Eq. (6.1.32), by substituting $[(P/T)\ dT/dP]_{rad} = 1/(n'_{rad} + 1)$ from Eq. (6.1.33) for $(P/T)\ dT/dP$, and $[(P/T)\ dT/dP]_{ad} = 1/(n'_{ad}+1)$ from Eq. (6.1.34): $n'_{rad} > n'_{ad}$, $(n'_{rad}, n'_{ad} > -1)$. This stability condition reads in terms of polytropic (adiabatic) exponents as: $\Gamma'_2 < \Gamma_2$, $(\Gamma'_2, \Gamma_2 > 1;\ \Gamma'_2 = 1 + 1/n'_{rad};\ \Gamma_2 = 1 + 1/n'_{ad})$.

If the inequalities (6.1.28) or (6.1.29) are not fulfilled, convection sets in, and the resulting temperature gradient is for almost all practical purposes only very slightly larger (in absolute value) than the absolute value of the adiabatic temperature gradient $|dT/dr|_{ad}$ (Cox and Giuli 1968).

The uniform source model approximates fairly well gravitationally contracting stars for which the energy production rate ε_g does not change too drastically over the stellar radius, because ε_g is directly proportional to the temperature $[\varepsilon_g \propto T \propto \theta,\ (0 \le \theta \le 1)]$, as will be shown in Eq. (6.1.46) below. Note, that we do not consider for the moment, as in Secs. 2.6.6 or 5.12.1, the gravitational energy release from a contracting star as a whole, but merely the local energy loss per unit time of a certain small, gravitationally contracting mass element. The first law of thermodynamics (1.1.3) is written for a unit mass in the Lagrangian representation (5.1.5) as

$$\partial Q/\partial t = \partial U/\partial t - (P/\varrho^2)\ \partial\varrho/\partial t, \qquad (m = \varrho V = 1). \tag{6.1.35}$$

We now regard the specific internal energy $U = \varepsilon^{(int)}/\varrho$ as a function of pressure and density, the temperature being eliminated through the equation of state $T = T(P, \varrho)$:

$$\partial Q/\partial t = (\partial U/\partial P)_\varrho\ \partial P/\partial t + (\partial U/\partial\varrho)_P\ \partial\varrho/\partial t - (P/\varrho^2)\ \partial\varrho/\partial t$$
$$= P\ (\partial U/\partial P)_\varrho \big\{\partial\ln P/\partial t + [\varrho(\partial U/\partial\varrho)_P - P/\varrho]\ (\partial\ln\varrho/\partial t)/P\ (\partial U/\partial P)_\varrho\big\}. \tag{6.1.36}$$

$\partial Q/\partial t = 0$ has to subsist for an adiabatic change, i.e. $\partial\ln P/\partial t + [\varrho(\partial U/\partial\varrho)_P - P/\varrho]\ (\partial\ln\varrho/\partial t)$ $/P(\partial U/\partial P)_\varrho = 0$. But from the adiabatic relationship (1.3.23) we have $\partial\ln P/\partial t = \Gamma_1\ \partial\ln\varrho/\partial t$, so the first adiabatic index Γ_1 can be identified with

$$\Gamma_1 = [P/\varrho - \varrho(\partial U/\partial\varrho)_P]/P(\partial U/\partial P)_\varrho. \tag{6.1.37}$$

The factor

$$P(\partial U/\partial P)_\varrho = P(\partial U/\partial T)_\varrho/(\partial P/\partial T)_\varrho = T(\partial U/\partial T)_\varrho/(\partial\ln P/\partial\ln T)_\varrho = c_V T/\chi_T, \tag{6.1.38}$$

is transformed via Eqs. (1.3.3), (1.3.12). The gravitational energy release per unit mass and time is therefore

$$\varepsilon_g = -\partial Q/\partial t = \partial L(r)/\partial M(r) = -(c_V T/\chi_T)[\partial\ln P/\partial t - \Gamma_1\ \partial\ln\varrho/\partial t]$$
$$= -(c_V T/\chi_T)[\partial\ln(P/\varrho^{\Gamma_1})/\partial t + (\partial\Gamma_1/\partial t)\ln\varrho], \tag{6.1.39}$$

where $dL(r)$ is the gravitational luminosity escaping from a shell of mass $dM(r) = 4\pi\varrho(r)\ r^2\ dr$.

An alternative form of this energy release is obtained if U is regarded as a function of ϱ and T :

$$\partial Q/\partial t = (\partial U/\partial T)_\varrho\ \partial T/\partial t + (\partial U/\partial\varrho)_T\ \partial\varrho/\partial t - (P/\varrho^2)\ \partial\varrho/\partial t$$
$$= T(\partial U/\partial T)_\varrho\big\{\partial\ln T/\partial t + [\varrho(\partial U/\partial\varrho)_T - P/\varrho]\ (\partial\ln\varrho/\partial t)/T(\partial U/\partial T)_\varrho\big\}. \tag{6.1.40}$$

From the adiabatic relationship (1.3.23) we get $\partial\ln T/\partial t = (\Gamma_3 - 1)\ \partial\ln\varrho/\partial t$, and analogously to Eq. (6.1.37):

$$\Gamma_3 - 1 = [P/\varrho - \varrho(\partial U/\partial\varrho)_T]/T(\partial U/\partial T)_\varrho. \tag{6.1.41}$$

Eq. (6.1.40) becomes

$$\varepsilon_g = -\partial Q/\partial t = -c_V T[\partial\ln T/\partial t + (\Gamma_3 - 1)\ \partial\ln\varrho/\partial t] = -c_V T[\partial\ln(T/\varrho^{\Gamma_3 - 1})/\partial t + (\partial\Gamma_3/\partial t)\ln\varrho]. \tag{6.1.42}$$

Likewise, if the thermodynamic changes occurring in the unit mass are reversible, we have in virtue of Eqs. (1.1.4), (6.1.39):

$$\varepsilon_g = -\partial Q/\partial t = -T\ \partial S/\partial t. \tag{6.1.43}$$

Thus, if $\Gamma_1, \Gamma_3 =$ const the local energy release per unit mass and time (6.1.39) or (6.1.42) is the result of departures from adiabaticity, since for an adiabatic change there is via Eq. (1.3.23) $P \propto \varrho^{\Gamma_1}$ and $T \propto \varrho^{\Gamma_3 - 1}$. The two equations (6.1.39) and (6.1.42) are merely alternative forms of the first law of thermodynamics, i.e. of the energy conservation equation (6.1.35). The energy release rate caused by gravitational contraction of the star as a whole is introduced through the simple, approximate assumption of uniform (homologous) contraction, already mentioned in Eqs. (2.6.196), (2.6.203), (2.6.205):

$$P(r) = (r/r_i)^{-4}P(r_i) = (r_1/r_{i1})^{-4}P(r_i); \quad \varrho(r) = (r/r_i)^{-3}\varrho(r_i) = (r_1/r_{i1})^{-3}\varrho(r_i),$$
$$[r = r(t);\ r_1 = r_1(t)]. \tag{6.1.44}$$

i-indexed quantities denote initial values of the radial distance r, and of the stellar radius r_1. We have

$$P(r)/[\varrho(r)]^{\Gamma_1} = \{P(r_i)/[\varrho(r_i)]^{\Gamma_1}\}(r_1/r_{i1})^{3\Gamma_1 - 4} = \mathrm{const}\ r_1^{3\Gamma_1 - 4}. \tag{6.1.45}$$

In the case of a perfect gas we have $\Gamma_1 = \gamma$, $\chi_T = 1$, and via Eq. (1.4.12): $c_V = \mathcal{R}/\mu(\gamma - 1)$ if $\Gamma_1 = \gamma =$ const. Inserting this together with Eq. (6.1.45) into Eq. (6.1.39), we get the local energy production rate per unit mass of a gravitationally contracting star composed of perfect gas with constant specific heats (e.g. Menzel et al. 1963, Cox and Giuli 1968):

$$\varepsilon_g = -(3\gamma - 4)c_V T(r)\ d\ln r_1/dt = -[(3\gamma - 4)\mathcal{R}T(r)/\mu(\gamma - 1)]\ d\ln r_1/dt, \qquad (\Gamma_1 = \gamma = \mathrm{const}). \tag{6.1.46}$$

Within our particular assumptions ε_g depends on the local temperature T at distance r, and on the instantaneous total radius of the contracting star. The connection with the gravitational energy production rate $-dE/dt$ of the whole contracting star is made at once by integration of ε_g over the whole star at a fixed moment, taking into account that dr_1/dt is independent of r :

$$L = -dE/dt = \int_{M_1} \varepsilon_g\ dM = -(3\gamma - 4)(d\ln r_1/dt)\int_{M_1} c_V T\ dM = -(3\gamma - 4)U\ d\ln r_1/dt$$
$$= [(3\gamma - 4)W/3(\gamma - 1)]\ d\ln r_1/dt = -(3\gamma - 4)GM_1^2(dr_1/dt)\big/(5 - n)(\gamma - 1)r_1^2$$
$$= -[(3\gamma - 4)/3(\gamma - 1)]\ dW/dt. \tag{6.1.47}$$

This equation is just identical to the luminosity (5.12.7) of a contracting star without nuclear energy sources, where $c_V T$, ($c_V =$ const) is the specific internal energy of a perfect gas from Eq. (1.2.19), U the total internal energy of a star from Eq. (2.6.100), and W the total gravitational energy (2.6.137) of a spherical polytrope.

We will solve the uniform source model for two somewhat particular cases. At first we consider an arbitrary value of $\beta = P_g/P$ and Kramers' opacity law

$$\kappa = \kappa_{ef}\varrho T^{-3.5}, \qquad (\kappa_{ef} = \mathrm{const}). \tag{6.1.48}$$

With

$$P = P_g + P_r = \beta P + (1 - \beta)P = \mathcal{R}\varrho T/\mu + aT^4/3, \tag{6.1.49}$$

Eq. (6.1.3) can be written under the form (Chandrasekhar 1939)

$$(3\mathcal{R}/a\mu)\ d(\varrho T)/dT^4 = 4\pi cGM(r)/\kappa L(r) - 1 = (4\pi cGM_1/\kappa_{ef}L_1)T^{3.5}/\varrho - 1. \tag{6.1.50}$$

With the auxiliary variable

$$y = \beta/(1 - \beta) = P_g/P_r = 3\mathcal{R}\varrho/a\mu T^3, \tag{6.1.51}$$

Eq. (6.1.50) reads

$$d(yT^4)/dT^4 = (12\pi cG\mathcal{R}M_1/a\mu\kappa_{ef}L_1)T^{1/2}/y - 1, \tag{6.1.52}$$

or

$$(yT/4)\ dy/dT = (12\pi cG\mathcal{R}M_1/a\mu\kappa_{ef}L_1)T^{1/2} - y(y+1). \tag{6.1.53}$$

With the new variable

$$x = (12\pi cG\mathcal{R}M_1/a\mu\kappa_{ef}L_1)T^{1/2}, \tag{6.1.54}$$

Eq. (6.1.53) becomes

$$(xy/8)\ dy/dx = x - y(y+1), \tag{6.1.55}$$

or with $s = 1/8 \ll 1$:

$$sxy\ dy/dx = x - y(y+1). \tag{6.1.56}$$

Assume a perturbation solution of the form

$$y(x,s) = y_0(x) + sy_1(x) + ..., \qquad (s \ll 1), \tag{6.1.57}$$

and equate the coefficients of equal powers of s :

$$x = y_0(y_0+1); \quad xy_0\ dy_0/dx = -y_1(2y_0+1), \tag{6.1.58}$$

or

$$y_0 = [-1 + (1+4x)^{1/2}]/2; \quad y_1 = -xy_0/(1+4x), \qquad (y_0 > 0). \tag{6.1.59}$$

Hence

$$y = y_0[1 - sx/(1+4x) + ...] = y_0[1 - x/32(x+1/4) + ...]. \tag{6.1.60}$$

Since x is very large, excepting perhaps near the surface when $T \to 0$, the bracket can be approximated by the limiting value 31/32 if $x \gg 1$, and we find eventually

$$y_0 = 32y/31; \quad x = y_0(y_0+1) = (32y/31)(32y/31+1). \tag{6.1.61}$$

Eq. (6.1.54) can now be written as

$$x = (12\pi cG\mathcal{R}M_1/a\mu\kappa_{ef}L_1)T^{1/2} = 32\beta(1+\beta/31)/31(1-\beta)^2. \tag{6.1.62}$$

The temperature can be eliminated between Eqs. (6.1.51) and (6.1.62), to obtain a relationship between ϱ and β

$$(4\pi cGM_1/\kappa_{ef}L_1)(3\mathcal{R}/a\mu)^{7/6}\varrho^{1/6} = 32\beta^{7/6}(1+\beta/31)/31(1-\beta)^{13/6}, \tag{6.1.63}$$

which differentiates to

$$(1/6)\ d\varrho/\varrho = [7/6 + \beta + \beta(1-\beta)/(\beta+31)]\ d\beta/\beta(1-\beta). \tag{6.1.64}$$

On the other hand, we get by logarithmic differentiation of Eq. (5.12.85)

$$dP/P = 4\ d\varrho/3\varrho + (3\beta-4)\ d\beta/3\beta(1-\beta). \tag{6.1.65}$$

Eliminating $d\beta/\beta$ between the two last equations, we obtain

$$dP/P = \{4/3 + (3\beta-4)/3[7 + 6\beta + 6\beta(1-\beta)/(31+\beta)]\}\ d\varrho/\varrho, \tag{6.1.66}$$

which can also be written under the form of the logarithmic differential of the local polytropic law $P = K\varrho^{1+1/n}$:

$$dP/P = (1+1/n)\ d\varrho/\varrho. \tag{6.1.67}$$

Dividing by Eq. (6.1.66), the effective polytropic index is given by [cf. Eq. (1.3.25)]

$$n = [(7+6\beta)(31+\beta)+6\beta(1-\beta)]/[(1+3\beta)(31+\beta)+2\beta(1-\beta)], \qquad [\beta=\beta(r)]. \qquad (6.1.68)$$

For negligible radiation pressure ($\beta \approx 1$) we have $n \approx 3.25$, and for negligible gas pressure $n \approx 7$, ($\beta \approx 0$). Note, that β, and therefore n is not rigorously constant for this uniform source model; if β would be exactly constant, Eq. (6.1.65) would turn into the standard model from Sec. 6.1.1: $dP/P = 4\ d\varrho/3\varrho,\ (n=3)$.

The uniform source model with Kramers' opacity and negligible radiation pressure can therefore be roughly approximated by a polytrope of nearly constant index $n = 3.25$. Its polytropic constant K can be obtained by equating $P = K\varrho^{1+1/3.25} = K\varrho^{1+4/13}$ with the exact equation (5.12.85), valid for any perfect gas-radiation mixture:

$$K = [3(1-\beta)(\mathcal{R}/\mu)^4/a\beta^4]^{1/3}\varrho^{1/39} \approx [3(1-\beta_0)(\mathcal{R}/\mu)^4/a\beta_0^4]^{1/3}\varrho_0^{1/39},$$
$$(n=3.25;\ \beta_0=\beta(0)\approx\beta(r)\approx 1). \qquad (6.1.69)$$

The mass of this $n = 3.25$ polytrope is via Eq. (2.6.18) equal to

$$M_1 = 4\pi\alpha^3\varrho_0\xi_1^2(-\theta_1') = 4\pi[(n+1)K/4\pi G]^{3/2}\varrho_0^{(3-n)/2n}\xi_1^2(-\theta_1')$$
$$= 4(4.25/4)^{3/2}[3(1-\beta_0)(\mathcal{R}/\mu)^4/\pi a G^3\beta_0^4]^{1/2}\xi_1^2(-\theta_1'), \qquad (\beta\approx\beta_0\approx 1). \qquad (6.1.70)$$

This equation differs from the mass (5.12.86) of the standard model $n = 3$ only by the factor (cf. Chandrasekhar 1939)

$$(4.25/4)^{3/2}(-\xi_1^2\theta_1')_{n=3.25}/(-\xi_1^2\theta_1')_{n=3} = 1.06. \qquad (6.1.71)$$

The central pressure and temperature of the $n = 3.25$ model is given by Eqs. (6.1.11) and (6.1.13), respectively, where $n = 3$ should be replaced by $n = 3.25$.

The value of E_0 from Eq. (6.1.24) becomes $E_0 = 17/16 = 1.0625$ if $n = 3.25$. Obviously, the considered special model $n = 3.25$ is very similar to the standard model $n = 3$, but is valid only for Kramers' opacity law, $\varepsilon = \text{const}$, and the restriction $\beta \approx 1$.

Therefore, we also consider another uniform source model with the opacity law (e.g. Cox and Giuli 1968)

$$\kappa = \kappa_{ef}' P^e T^{-e+f}, \qquad (\kappa_{ef}' = \text{const}), \qquad (6.1.72)$$

which turns for the perfect gas equation of state into the opacity law (6.1.19) with

$$\kappa_{ef}' = \kappa_{ef}(\mu/\mathcal{R})^e, \qquad (\beta \approx 1). \qquad (6.1.73)$$

Note, that the exponents e and f are generally different in the two opacity formulas (6.1.19) and (6.1.72); they only coincide if the perfect gas equation of state $P = \mathcal{R}\varrho T/\mu$ subsists.

Eq. (6.1.3) is for the uniform source model with the opacity law (6.1.72) equal to

$$dT/dP = 3\kappa L_1/16\pi acGM_1T^3 = 3\kappa_{ef}' L_1 P^e T^{-e+f-3}/16\pi acGM_1, \qquad (L(r)/M(r) = L_1/M_1), \qquad (6.1.74)$$

which can be integrated at once:

$$P^{e+1} = 16\pi acGM_1[(e+1)/(e-f+4)]T^{e-f+4}/3\kappa_{ef}'L_1 + C,$$
$$(e \neq -1;\ e-f+4 \neq 0;\ C = \text{const}), \qquad (6.1.75)$$

$$P^{e+1} = 16\pi acGM_1(e+1)\ln T/3\kappa_{ef}'L_1 + C, \qquad (e\neq -1;\ e-f+4 = 0). \qquad (6.1.76)$$

The last two equations become with the boundary condition $P = P_1$ if $T = T_1$ equal to

$$(P/P_1)^{e+1} = 1 + (16\pi acGM_1T_1^{e-f+4}/3\kappa_{ef}'L_1P_1^{e+1})[(e+1)/(e-f+4)][(T/T_1)^{e-f+4} - 1],$$
$$(e\neq -1;\ e-f+4 \neq 0), \qquad (6.1.77)$$

$$(P/P_1)^{e+1} = 1 + [16\pi acGM_1(e+1)/3\kappa'_{ef}L_1P_1^{e+1}]\ln(T/T_1), \qquad (e \neq -1;\ e-f+4=0). \quad (6.1.78)$$

If P_1, T_1 denote surface (photospheric) values of pressure and temperature, we observe that – as we move inwards – the values of $(P/P_1)^{e+1}$ and $(T/T_1)^{e-f+4}$ rapidly increase, since generally $e+1>0$ and $e-f+4>0$. Under these circumstances, Eqs. (6.1.77) and (6.1.78) rapidly converge to the so-called radiative zero solutions, obtained from Eqs. (6.1.75), (6.1.76) if $C=0$. The surface values P_1, T_1 of the radiative zero solution $C=0$ obey the relationships

$$3\kappa'_{ef}L_1P_1^{e+1}/16\pi acGM_1T_1^{e-f+4} = (e+1)/(e-f+4), \qquad (e\neq -1;\ e-f+4\neq 0), \qquad (6.1.79)$$

$$3\kappa'_{ef}L_1P_1^{e+1}/16\pi acGM_1\ln T_1 = e+1, \qquad (e\neq -1;\ e-f+4=0). \qquad (6.1.80)$$

And the radiative zero solution from Eq. (6.1.75) takes the polytropic form (1.3.26):

$$P = [16\pi(e+1)acGM_1/3(e-f+4)\kappa'_{ef}L_1]^{1/(e+1)}T^{(e-f+4)/(e+1)} = K'T^{n'+1},$$
$$(C=0;\ e\neq -1;\ e-f+4\neq 0), \qquad (6.1.81)$$

where

$$n' = n'_{rad} = (-f+3)/(e+1); \quad K' = [16\pi(e+1)acGM_1/3(e-f+4)\kappa'_{ef}L_1]^{1/(e+1)},$$
$$(e\neq -1;\ e-f+4\neq 0). \qquad (6.1.82)$$

The radiative gradient (6.1.33) is for the radiative zero solution (6.1.81) simply equal to

$$(d\ln T/d\ln P)_{rad} = (P/T)\ dT/dP = (e+1)/(e-f+4) = 1/(n'+1) = 3\kappa L_1P/16\pi acGM_1T^4. \qquad (6.1.83)$$

In virtue of Eq. (6.1.32) it is here assumed that the uniform source model is throughout in radiative equilibrium, i.e. the radiative polytropic index (6.1.82) is always larger than the adiabatic polytropic index: $n'_{rad} > n'_{ad} = 1/(\Gamma_2 - 1)$. If $\Gamma_2 = 5/3$, as for a completely ionized, nonrelativistic perfect gas, the condition is $n'_{rad} > 1.5$.

For a perfect gas with negligible radiation pressure ($\beta \approx 1$) the two polytropic indices n and n' from Eqs. (1.3.25) and (1.3.26) are equal among each other. In this particular case the two polytropic constants from $P = K\varrho^{1+1/n}$ and $P = K'T^{n'+1}$ are connected by

$$K' = (\mathcal{R}/\mu)^{n+1}/K^n, \qquad (n=n'), \qquad (6.1.84)$$

as may be seen at once by inserting $T = \mu P/\mathcal{R}\varrho = K\mu\varrho^{1/n}/\mathcal{R}$ from Eq. (2.6.7) into $P = K'T^{n+1} = K'(K\mu/\mathcal{R})^{n+1}\varrho^{1+1/n} = K\varrho^{1+1/n}$.

The luminosity of this particular uniform source model can be calculated by inserting

$$K = (4\pi)^{1/n}[G/(n+1)]\xi_1^{-1-1/n}(-\theta'_1)^{-1+1/n}M_1^{(n-1)/n}r_1^{(3-n)/n},$$
$$(N=3;\ [\Gamma(1/2)]^3/\Gamma(3/2) = 2\pi), \qquad (6.1.85)$$

from Eq. (2.6.21) into Eq. (6.1.84), with K' given by Eq. (6.1.82), and κ'_{ef} by Eq. (6.1.73):

$$L_1 = (4^{e+3}\pi^{e+2}ac/3\kappa_{ef})[(n+1)\mathcal{R}/\mu G]^{f-4}\xi_1^{-e+f-4}(-\theta'_1)^{e+f-2}M_1^{-e-f+3}r_1^{3e+f},$$
$$[\beta\approx 1;\ n=n'=(-f+3)/(e+1)]. \qquad (6.1.86)$$

For Kramers' opacity law we have $e=1,\ f=-3.5$, and $n=n'=3.25$, as it has already been found in Eq. (6.1.68) under the same assumptions. The luminosity (6.1.86) becomes in this special case equal to (Chandrasekhar 1939, Table 4, Cox and Giuli 1968)

$$L_1 = 1.37\times 10^{25}\mu^{7.5}(M_1/M_\odot)^{5.5}/\kappa_{ef}(r_1/r_\odot)^{1/2},$$
$$(\beta\approx 1;\ n=n'=3.25;\ \xi_1 = 8.01894;\ \theta'_1 = -3.03219\times 10^{-2}). \qquad (6.1.87)$$

In the constant opacity case we have $e, f = 0,\ \kappa_0 = \kappa_{ef},\ n=n'=3$, and the luminosity (6.1.86) turns into the luminosity (6.1.26) of the standard model:

$$L_1 = 38.9\ \mu^4(M_1/M_\odot)^3/\kappa_0, \qquad (\beta\approx 1;\ n=n'=3). \qquad (6.1.88)$$

It should be stressed that all uniform source models approximated with polytropes of nearly constant index $n' = n > 3$ appear to be not quite realistic, as they are likely to become unstable under the external pressure of their outer layers (Sec. 5.4.2). Moreover, uniform source models with nearly constant polytropic index $n > 5$, $[n \approx 7$ if $\beta \approx 0$ in Eq. (6.1.68)] extend to infinity with infinite mass – an additional decrease of their reliability.

6.1.3 Point Source (Cowling) Model

This model is from the standpoint of energy generation opposite to the previous uniform source model. The whole energy generation is limited to a core of radius r_c :

$$L_1 = L(r_c) = 4\pi \int_0^{r_c} \varepsilon(r)\ r^2\ dr, \qquad (r_c \ll r_1). \tag{6.1.89}$$

Point source models are composed of a radiative envelope and a small convective core, where all the energy generation takes place. The existence of a convective core results from the instability of the radiative gradient (6.1.33), as we move towards the centre of the star. The radiative zero solution (6.1.81) holds roughly in the outermost layers of a point source model, where $L(r) = L_1$ and $M(r) \approx M_1$. As already mentioned subsequently to Eq. (6.1.78), all solutions with $L(r)/M(r) = L_1/M_1 = \mathrm{const}$ tend simply to the radiative zero solution as we move inwards into the star, provided that $e+1 > 0,\ e-f+4 > 0$.

From the radiative gradient (6.1.83) follows that the relationship

$$\kappa P/T^4 = 16\pi(e+1)acGM_1/3(e-f+4)L_1 \approx \mathrm{const}, \qquad (r \approx r_1), \tag{6.1.90}$$

holds in the outermost radiative layers. The radiative gradient (6.1.33) subsists in the inner layers of a radiative envelope, where for the point source model $L(r) = L_1$:

$$(d\ln T/d\ln P)_{rad} = 1/(n'_{rad} + 1) = 3\kappa P L_1/16\pi acGT^4 M(r). \tag{6.1.91}$$

In virtue of Eq. (6.1.90) we have $\kappa P/T^4 \approx \mathrm{const}$, and the inward decrease of n'_{rad} from Eq. (6.1.91) is mainly due to the decrease of $M(r)$. The radiative polytropic index n'_{rad} eventually decreases below the adiabatic value n'_{ad}, when convection sets in [see Eq. (6.1.32)]. In the case of a completely ionized, perfect gas we have $\Gamma_1 = \Gamma_2 = 5/3$, and the convective core of a point source model is identical to an incomplete polytrope of index $n = n' = 1/(\Gamma_1 - 1) = 1.5$.

A simplified model with $\kappa = \mathrm{const}$, $(e, f = 0)$ has been amply discussed by Chandrasekhar (1939). This so-called complete point source model fully includes radiation pressure, but ignores the previously mentioned instability of the radiative gradient. In fact, the radiative gradient becomes unstable below $r_c/r_1 = 0.28$ if $\kappa = \mathrm{const}$, and $\beta \approx 1$ (Cox and Giuli 1968, Table 23.2). As the complete point source model has no direct bearing on polytropes, we turn directly to composite point source models with negligible radiation pressure ($P \approx \mathcal{R}\varrho T/\mu$), consisting of a convective core surrounded by a radiative envelope. The equations of stellar structure take a somewhat simpler form by introducing the dimensionless variables

$$r = \alpha\xi; \quad \varrho = \varrho_0\sigma; \quad T = T_0\tau; \quad M = m_0\psi, \qquad (\alpha, \varrho_0, T_0, m_0 = \mathrm{const}). \tag{6.1.92}$$

ϱ_0 and T_0 denote central values of density and temperature, respectively, and α is just equal to the distance scaling factor (2.1.13) of the Lane-Emden equation (2.1.14), as will be shown by Eqs. (6.1.95), (6.1.97). The equation of hydrostatic equilibrium

$$dP/dr = (\mathcal{R}/\mu)\ d(\varrho T)/dr = -G\varrho M(r)/r^2, \qquad (\beta \approx 1), \tag{6.1.93}$$

together with the mass conservation equation $dM(r)/dr = 4\pi\varrho r^2$ are brought to the dimensionless forms

$$d(\sigma\tau)/d\xi = -(n+1)\sigma\psi/\xi^2; \quad d\psi/d\xi = \sigma\xi^2, \tag{6.1.94}$$

where ϱ_0, T_0, m_0 are related by (Chandrasekhar 1939)

$$\mathcal{R}T_0/\mu = Gm_0/(n+1)\alpha = 4\pi G\varrho_0\alpha^2/(n+1); \quad m_0 = 4\pi\varrho_0\alpha^3,$$
$$\big(\alpha^2 = (n+1)\mathcal{R}T_0/4\pi G\mu\varrho_0 = (n+1)P_0/4\pi G\varrho_0^2\big). \tag{6.1.95}$$

The temperature gradient is only slightly superadiabatic in the convective core, and therefore the core can well be approximated with an incomplete (truncated) polytrope of index

$$n_{ad} = n'_{ad} = 1/(\Gamma_1 - 1) = 1/(\Gamma_2 - 1) = d\ln P/d\ln T - 1. \qquad (6.1.96)$$

In the convective core the appropriate solution is therefore

$$\sigma = \theta^n; \quad \tau = \theta; \quad \psi = -\xi^2\theta', \qquad (n = n_{ad} = \text{const}), \qquad (6.1.97)$$

where θ is the Lane-Emden function from Eqs. (2.6.3), (2.6.7), (2.6.18). In the radiative envelope the temperature gradient is governed by Eq. (6.1.1), which takes the form

$$d\tau/d\xi = -Q\sigma^{e+1}\tau^{f-3}/\xi^2, \qquad (Q = 3\kappa_{ef}L_1\varrho_0^{e+1}T_0^{f-4}/16\pi ac\alpha = \text{const};\ \kappa = \kappa_{ef}\varrho_0^eT_0^f\sigma^e\tau^f). \qquad (6.1.98)$$

Suppose that the convective core extends up to the dimensionless Lane-Emden coordinate $\xi = \xi_c$. At this point we have via Eqs. (6.1.97), (6.1.98):

$$\sigma_c = \theta_c^n; \quad \tau_c = \theta_c; \quad \psi_c = -\xi_c^2\theta'_c; \quad Q = \xi_c^2\theta_c^{3-f-n(1+e)}(-\theta'_c), \qquad (\theta'_c = (d\tau/d\xi)_{\xi=\xi_c}). \qquad (6.1.99)$$

With this value of Q we can numerically integrate Eqs. (6.1.94), (6.1.98) up to a surface coordinate $\xi = \xi_1$, where σ or τ becomes zero first. Now, for a physically significant solution σ and τ have to become zero simultaneously, which will generally not be the case for an arbitrarily assigned value of ξ_c. Therefore, ξ_c has to be adjusted until σ and τ tend to zero simultaneously.

The central temperature, density, and pressure of the point source model are obtained from Eq. (6.1.95):

$$T_0 = G\mu m_0/(n+1)\mathcal{R}\alpha = G\mu\xi_1M_1/(n+1)\mathcal{R}\psi_1r_1; \quad \varrho_0 = m_0/4\pi\alpha^3 = \xi_1^3M_1/4\pi\psi_1r_1^3 = \xi_1^3\varrho_m/3\psi_1;$$
$$P_0 = \mathcal{R}\varrho_0T_0/\mu = G\mu\xi_1^4M_1^2/4\pi(n+1)\mathcal{R}\psi_1^2r_1^4. \qquad (6.1.100)$$

Eqs. (6.1.98), (6.1.100) lead to the following mass-radius-luminosity relationship:

$$L_1 = 16\pi acQr_1T_0^{4-f}/3\kappa_{ef}\varrho_0^{e+1}\xi_1$$
$$= (4^{e+3}\pi^{e+2}acQ/3\kappa_{ef})[(n+1)\mathcal{R}/\mu G]^{f-4}\xi_1^{-3e-f}\psi_1^{e+f-3}M_1^{-e-f+3}r_1^{3e+f}. \qquad (6.1.101)$$

Thus, the luminosity of the point source model shows the same $\mu^{-f+4}M_1^{-e-f+3}r_1^{3e+f}$ dependence as the luminosities (6.1.26) and (6.1.86) of the standard and uniform source model, respectively.

For $n_{ad} = 1.5$, and constant opacity $\kappa = \kappa_{ef}$, $(e, f = 0)$ there results: $\xi_c/\xi_1 = r_c/r_1 = 0.283$, $\psi_c/\psi_1 = M_c/M_1 = 0.312$, $\varrho_0/\varrho_m = 19.8$, $T_0 = 1.76 \times 10^7$ K (Sun). For $n_{ad} = 1.5$, and Kramers' opacity law $(e = 1,\ f = -3.5)$ Cowling has found: $\xi_c = 1.19$, $\theta_c = 0.788$, $\theta'_c = -0.321$, $\psi_c = 0.453$, $r_c/r_1 = 0.169$, $M_c/M_1 = 0.145$, $\varrho_0/\varrho_m = 37.0$, $T_0 = 2.08 \times 10^7$ K (Sun), $Q = 0.197$ (Chandrasekhar 1939, Chap. IX; Cox and Giuli 1968, Table 23.2).

Although the mass concentration $\varrho_0/\varrho_m = 37.0$ of the point source model with Kramers' opacity is less than in the standard model [$\varrho_0/\varrho_m = 54.18$, Eq. (6.1.12)], its central temperature 2.08×10^7 K is somewhat larger than in the standard model [1.96×10^7 K, Eq. (6.1.13)] of a solar-type star.

6.1.4 Convective Model with Radiation Pressure

As a considerable restriction of this fully convective model it is assumed that the perfect gas component is completely ionized: $\gamma_g = c_{Pg}/c_{Vg} = 5/3$ (Henrich 1941). With this particular value of the adiabatic gas exponent, the first adiabatic index (1.4.20) becomes

$$\Gamma_1 = \beta + 2(4-3\beta)^2/3(8-7\beta) = (3\beta^2 + 24\beta - 32)/3(7\beta - 8), \qquad (\gamma_g = 5/3), \qquad (6.1.102)$$

and the adiabatic pressure-density relationship (1.3.23) turns into

$$dP/P = \Gamma_1\ d\varrho/\varrho = (3\beta^2 + 24\beta - 32)\ d\varrho/3\varrho(7\beta - 8), \qquad (P = \mathcal{R}\varrho T/\mu + aT^4/3;\ \gamma_g = 5/3). \qquad (6.1.103)$$

From the logarithmic differentiation of the general relationship (5.12.85) we get

$$dP/P = (3\beta - 4)\ d\beta/3\beta(1-\beta) + 4\ d\varrho/3\varrho. \tag{6.1.104}$$

Elimination of $d\varrho/\varrho$ between Eqs. (6.1.103) and (6.1.104) leads to

$$dP/P = (3\beta^2 + 24\beta - 32)\ d\beta/3\beta^2(\beta - 1) = [32/3 + 5/3y + 1/(y+1)]\ dy, \tag{6.1.105}$$

where

$$\beta = 1/(y+1); \qquad y = (1-\beta)/\beta = 1/\beta - 1. \tag{6.1.106}$$

On integration, Eq. (6.1.105) gives

$$P = P_0(y/y_0)^{5/3}[(1+y)/(1+y_0)]\exp[32(y-y_0)/3], \tag{6.1.107}$$

where zero indexed quantities denote central values. Insertion of Eq. (6.1.107) into Eq. (5.12.85) yields

$$\varrho = \varrho_0(y/y_0)\exp[8(y-y_0)], \tag{6.1.108}$$

and then, using Eq. (5.12.84):

$$T = T_0(y/y_0)^{2/3}\exp[8(y-y_0)/3]. \tag{6.1.109}$$

Eqs. (6.1.106)-(6.1.109) are the parametric equations of state of a mixture consisting of monoatomic gas and radiation in adiabatic equilibrium (Menzel et al. 1963). Obviously, only two of the four constants y_0, P_0, ϱ_0, T_0 are arbitrary.

Eqs. (6.1.107) and (6.1.108) are now inserted into the spherically symmetric form of Poisson's equation [Henrich 1941, Eq. (22)]:

$$(1/r^2)\ d(r^2\ d\Phi/dr)/dr = (1/r^2)\ d[(r^2/\varrho)\ dP/dr]/dr = [5P_0/2\varrho_0(y_0+1)r^2]\ d(r^2\ dF/dr)/dr$$
$$= -4\pi G\varrho = -4\pi G\varrho_0(y/y_0)\exp[8(y-y_0)], \tag{6.1.110}$$

where

$$F = (y/y_0)^{2/3}(1 + 8y/5)\exp[8(y-y_0)/3]. \tag{6.1.111}$$

The ratio P_0/ϱ_0 can be expressed with the equation of state

$$P_0/\varrho_0 = P_{g0}/\beta_0\varrho_0 = \mathcal{R}T_0/\beta_0\mu = \mathcal{R}T_0(1+y_0)/\mu, \tag{6.1.112}$$

and Eq. (6.1.110) becomes

$$(1/r^2)\ d(r^2\ dF/dr)/dr = -(8\pi G\mu\varrho_0/5\mathcal{R}T_0)(y/y_0)\exp[8(y-y_0)]. \tag{6.1.113}$$

This equation writes in dimensionless form as

$$(1/\eta^2)\ d(\eta^2\ dF/d\eta)/d\eta = -\sigma, \tag{6.1.114}$$

by substituting

$$r = A\eta; \quad A^2 = 5\mathcal{R}T_0/8\pi G\mu\varrho_0; \quad \varrho = \varrho_0\sigma = \varrho_0(y/y_0)\exp[8(y-y_0)]. \tag{6.1.115}$$

From Eqs. (6.1.111) and (6.1.115) we can easily check that for small y, y_0 we have

$$F \approx (y/y_0)^{2/3}; \quad \sigma \approx y/y_0, \qquad (y, y_0 \approx 0;\ \beta, \beta_0 \approx 1). \tag{6.1.116}$$

For large y, y_0 the variation of the functions F and σ is determined mainly by the exponentials

$$F \approx C_1\exp[8(y-y_0)/3]; \quad \sigma \approx C_2\exp[8(y-y_0)], \qquad (y, y_0 \to \infty;\ \beta, \beta_0 \approx 0;\ C_1, C_2 = \mathrm{const}). \tag{6.1.117}$$

Thus, inserting $F \approx (y/y_0)^{2/3} = \theta,\ \sigma \approx y/y_0 = \theta^{3/2}$ if $y, y_0 \to 0$, and $F \approx C_1 \exp[8(y-y_0)/3] = C_1\theta,\ \sigma \approx C_2 \exp[8(y-y_0)] = C_2\theta^3$ if $y, y_0 \to \infty$, we observe that in these two limiting cases the structure equation (6.1.114) reduces approximately to a Lane-Emden equation of polytropic index $n = 1.5$ and 3, respectively, the constants C_1, C_2 being included in a new distance scaling factor A.

The boundary conditions at the origin $\eta = 0$ are obviously

$$F = 1 + 8y_0/5; \quad dF/d\eta = 0; \quad \sigma = 1. \tag{6.1.118}$$

The condition $(dF/d\eta)_{\eta=0} = 0$ obtains if we expand F near the origin:

$$F = 1 + 8y_0/5 + a_1\eta + a_2\eta^2 + ... \approx 1 + 8y_0/5 - \eta^2/6, \qquad (\eta \approx 0). \tag{6.1.119}$$

The density $\varrho = \varrho_0\sigma$ becomes zero at the radius $r_1 = A\eta_1$ of the configuration. The mass of the convective model is simply

$$M_1 = \int_0^{r_1} 4\pi\varrho r^2\ dr = 4\pi A^3\varrho_0 \int_0^{\eta_1} \sigma\eta^2\ d\eta = 4\pi A^3\varrho_0\eta_1^2(-dF/d\eta)_{\eta=\eta_1}. \tag{6.1.120}$$

In virtue of Eqs. (5.12.84) and (6.1.115) we have

$$A^3\varrho_0 = (5\mathcal{R}T_0/8\pi G\mu)^{3/2}\varrho_0^{-1/2} = (5/8\pi G)^{3/2}(\mathcal{R}/\mu)^2[3(1-\beta_0)/a\beta_0]^{1/2}$$
$$= (5/8\pi G)^{3/2}(\mathcal{R}/\mu)^2(3y_0/a)^{1/2}. \tag{6.1.121}$$

So, we can write the mass (6.1.120) as (Henrich 1941, Menzel et al. 1963)

$$M_1 = (5/2)^{3/2}(Cy_0^{1/2}/\mu^2)\eta_1^2(-dF/d\eta)_{\eta=\eta_1} = 4.42(y_0^{1/2}M_\odot/\mu^2)\eta_1^2(-dF/d\eta)_{\eta=\eta_1},$$
$$(C = (3/4\pi aG^3)^{1/2}\mathcal{R}^2). \tag{6.1.122}$$

To get M_1 in the two limiting cases $y_0 \to 0$ and $y_0 \to \infty$, we write down at first an expression for the polytropic constant K, by equating the central polytropic pressure $P_0 = K\varrho_0^{1+1/n}$ to the pressure (5.12.85) of a perfect gas-radiation mixture:

$$K = [3(1-\beta_0)(\mathcal{R}/\mu)^4/a\beta_0^4]^{1/3}\varrho_0^{(n-3)/3n}. \tag{6.1.123}$$

Inserting for α from Eq. (2.1.13), and for K from Eq. (6.1.123), the polytropic mass (2.6.18) becomes in the two limiting cases $n = 1.5, 3$:

$$M_1 = 4\pi\alpha^3\varrho_0\xi_1^2(-\theta_1') = 4\pi[(n+1)K/4\pi G]^{3/2}\varrho_0^{(3-n)/2n}\xi_1^2(-\theta_1')$$
$$= (3/4\pi aG^3)^{1/2}(\mathcal{R}/\mu)^2(n+1)^{3/2}[(1-\beta_0)/\beta_0^4]^{1/2}\xi_1^2(-\theta_1')$$
$$= C(n+1)^{3/2}[y_0^{1/2}(1+y_0)^{3/2}/\mu^2]\xi_1^2(-\theta_1'), \qquad [\beta_0 = 1/(1+y_0)]. \tag{6.1.124}$$

Numerically this equation reads (Henrich 1941)

$$M_1 = 12y_0^{1/2}M_\odot/\mu^2, \quad (n = 1.5;\ y_0 \to 0) \qquad \text{and} \qquad M_1 = 18y_0^2M_\odot/\mu^2, \quad (n = 3;\ y_0 \to \infty). \tag{6.1.125}$$

The ratio of central to mean density for the two particular limiting cases $n = 1.5$, $(y_0 \to 0)$ and $n = 3$, $(y_0 \to \infty)$ is given by Eq. (2.6.27) if $N = 3$. For general y_0 one easily gets with Eq. (6.1.120):

$$\varrho_0/\varrho_m = -\eta_1/3(dF/d\eta)_{\eta=\eta_1}, \qquad (y_0 \neq 0, \infty). \tag{6.1.126}$$

And finally, the central temperature of the fully convective model with radiation pressure is obtained from Eq. (6.1.115) by inserting for $A^3\varrho_0$ from Eq. (6.1.121), for $A = r_1/\eta_1$ from Eq. (6.1.115), and for $y_0^{1/2}$ from Eq. (6.1.122):

$$T_0 = 8\pi\mu GA^2\varrho_0/5\mathcal{R} = (15y_0/8\pi Ga)^{1/2}\mathcal{R}\eta_1/\mu r_1 = 2\mu GM_1/5\mathcal{R}\eta_1(-dF/d\eta)_{\eta=\eta_1}r_1. \tag{6.1.127}$$

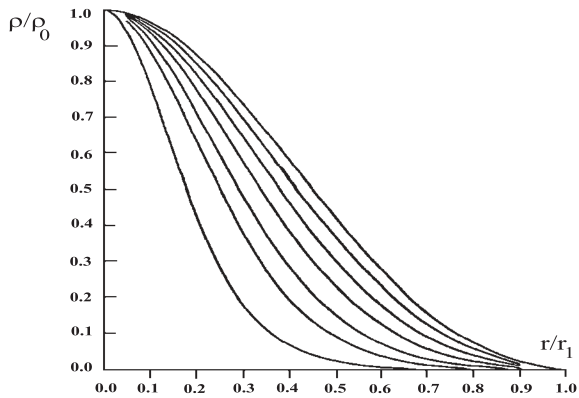

Fig. 6.1.1 Dimensionless density run in a fully convective model with radiation pressure and $\gamma_g = 5/3$. The seven curves from left to right are obtained for values of $y_0 = 1/\beta_0 - 1$ equal to $y_0 = \infty, 2, 1, 0.5, 0.25, 0.1, 0$, respectively, the values $y_0 = 0$ and $y_0 = \infty$ corresponding to the $n = 1.5$ and $n = 3$ polytrope (Henrich 1941).

For the two limiting polytropes $n = 1.5$ and $n = 3$ the central temperature results, for instance, from Eq. (6.1.13):

$$T_0 = \mu\beta_0 GM_1/(n+1)\xi_1(-\theta_1')\mathcal{R}r_1 = \mu GM_1/(y_0+1)(n+1)\xi_1(-\theta_1')\mathcal{R}r_1,$$
$$(n = 1.5,\ \beta_0 \to 1,\ y_0 \to 0 \quad \text{and} \quad n = 3,\ \beta_0 \to 0,\ y_0 \to \infty). \tag{6.1.128}$$

The numerical results of Henrich (1941) show that the two limiting polytropic models $n = 1.5$ and $n = 3$ include the other models obtained if $0 < y_0 < \infty$, $(0 < \beta_0 < 1)$, (see Fig. 6.1.1). For instance, from Eq. (6.1.125) results $M_1 = 0$ if $y_0 = 0$, and $M_1 = \infty$ if $y_0 = \infty$, while the limiting ratios ϱ_0/ϱ_m from Eq. (6.1.126) are included between the values $-\xi_1/3\theta_1' = 5.991$ and 54.18 of the $n = 1.5$ and $n = 3$ polytrope, respectively.

6.1.5 Convective Model with Negligible Radiation Pressure

Completely convective stars are represented by late M-dwarfs, gravitationally contracting pre-main sequence stars, supermassive objects (Sec. 5.12.3), and perhaps by some relativistically degenerate white dwarfs. In all these stars the adiabatic gradient is always smaller than the radiative gradient [see Eqs. (6.1.28)-(6.1.34)]:

$$(d\ln T/d\ln P)_{ad} = (\Gamma_2 - 1)/\Gamma_2 < (d\ln T/d\ln P)_{rad} = (\Gamma_2' - 1)/\Gamma_2'. \tag{6.1.129}$$

We also assume the perfect gas equation of state (1.2.5) to be valid throughout the star. In this case $\Gamma_1 = \Gamma_2 = \Gamma_3$, and if $\Gamma_2 = \text{const}$ we can integrate Eq. (6.1.129) at once:

$$P = K'T^{\Gamma_2/(\Gamma_2-1)} = K'T^{n'+1} = K'T^{n+1} = (\mathcal{R}/\mu)^{n+1}T^{n+1}/K^n,$$
$$(\beta \approx 1;\ K', K = \text{const};\ n' = n'_{ad} = 1/(\Gamma_2 - 1) = 1/(\Gamma_1 - 1) = n_{ad} = n). \tag{6.1.130}$$

Since the perfect gas is completely ionized, we have $\Gamma_1 = \Gamma_2 = 5/3$, and the structure of the completely convective star is approximately equal to that of a $n = 1/(\Gamma_1 - 1) = 1.5$ polytrope, like the convective core in the point source model from Sec. 6.1.3. Generally, we have $n, n' \approx 3$ in the radiative regions of the standard model, and of the uniform and point source model. The value of the constant K' from Eq. (6.1.130) is given by Eqs. (6.1.84), (6.1.85):

$$K' = (\mathcal{R}/\mu)^{n+1}/K^n = [(n+1)/G]^n(\mathcal{R}/\mu)^{n+1}\xi_1^{n+1}(-\theta_1)^{n-1}M_1^{1-n}r_1^{n-3}/4\pi, \qquad (n = 1.5). \tag{6.1.131}$$

The central pressure of the completely convective star is [cf. Eqs. (2.6.25), (6.1.11)]

$$P_0 = GM_1^2/4\pi(n+1)\theta_1'^2 r_1^4 = 0.77014\ GM_1^2/r_1^4$$
$$= 8.66 \times 10^{15}(M_1/M_\odot)^2/(r_1/r_\odot)^4\ [\mathrm{dyne/cm^2}], \qquad (n = 1.5). \tag{6.1.132}$$

The ratio central to mean density amounts to [cf. Eqs. (2.6.27), (6.1.12)]

$$\varrho_0/\varrho_m = -\xi_1/3\theta_1' = 5.991, \qquad (n = 1.5). \tag{6.1.133}$$

Likewise, the central temperature of the convective model is [cf. Eq. (6.1.13)]

$$T_0 = \mu GM_1/(n+1)\xi_1(-\theta_1')\mathcal{R}r_1 = 0.538\ \mu GM_1/\mathcal{R}r_1$$
$$= 1.23 \times 10^7 \mu(M_1/M_\odot)/(r_1/r_\odot)\ [\mathrm{K}], \qquad (n = 1.5;\ \beta_0 \approx 1). \tag{6.1.134}$$

A somewhat more explicit form of the adiabatic relationship (6.1.130) can be given in terms of the dimensionless Schwarzschild variables. From the equation of hydrostatic equilibrium $dP = -G\varrho M(r)\ dr/r^2$ it is seen that the dimension of P is $[P] = [GM_1^2/r_1^4]$, and the dimensionless pressure p is accordingly defined by

$$p = P/(GM_1^2/4\pi r_1^4). \tag{6.1.135}$$

Likewise, from the perfect gas law $P = \mathcal{R}\varrho T/\mu$ we deduce that the dimension of T is $[T] = [\mu P/\mathcal{R}\varrho] = [\mu P r_1^3/\mathcal{R}M_1] = [\mu GM_1/\mathcal{R}r_1]$, and the dimensionless temperature t is defined by the relationship

$$t = T/(\mu GM_1/\mathcal{R}r_1). \tag{6.1.136}$$

Eq. (6.1.130) reads via Eq. (6.1.131) in terms of the Schwarzschild variables p, t as

$$p = 4\pi G^n(\mu/\mathcal{R})^{n+1}K'M_1^{n-1}r_1^{3-n}t^{n+1} = (n+1)^n\xi_1^{n+1}(-\theta_1')^{n-1}t^{n+1} = Et^{n+1},$$
$$(E = (n+1)^n\xi_1^{n+1}(-\theta_1')^{n-1}). \tag{6.1.137}$$

With our value $n = 1.5$ of the polytropic index this amounts to

$$E = 4\pi G^{3/2}(\mu/\mathcal{R})^{5/2}K'M_1^{1/2}r_1^{3/2} = (2.5)^{3/2}\xi_1^{5/2}(-\theta_1')^{1/2} = 45.48, \qquad (n = 1.5). \tag{6.1.138}$$

If radiation pressure is included, as in the previous section, we obtain for a completely ionized gas by elimination of y/y_0 between Eqs. (6.1.107) and (6.1.109):

$$P = [P_0/T_0^{5/2}(1+y_0)\exp(4y_0)](1+y)\exp(4y)\ T^{5/2} = (K'/\beta)\exp[4(1/\beta - 1)]\ T^{5/2},$$
$$[\gamma_g = 5/3;\ y = 1/\beta - 1;\ K' = P_0/T_0^{5/2}(1+y_0)\exp(4y_0)]. \tag{6.1.139}$$

In the case of negligible radiation pressure we have $\beta = 1$, and Eq. (6.1.139) turns into Eq. (6.1.130). If we differentiate Eq. (6.1.130) or (6.1.137), we get

$$dT/dP = 1/(n+1)K'T^n; \quad dt/dp = 1/(n+1)Et^n, \tag{6.1.140}$$

showing that – for the same pressure change dP or dp – the temperature will increase faster and faster if the value of K' or E becomes smaller and smaller, as we move from the surface ($P, T \approx 0$) to the centre ($P = P_0,\ T = T_0$). Hence, in a convective envelope a certain temperature will be attained closer to the surface if K' or E take small values, and the temperature gradient $|dT/dr|$ between the bottom of the convective envelope and the centre will become more modest, the smaller the value of K' or E. But if the temperature gradient $|dT/dr|$ is small, the temperature is determined in virtue of Eq. (6.1.29) by radiative energy transport, rather than by convection, i.e. the convective envelope possesses a radiative core. Thus, the fractional depth of the convective envelope decreases as K or E becomes smaller and smaller. In the limit $K' \to 0$ or $E \to 0$, a steep temperature increase with pressure would occur almost instantaneously with depth, and the thickness of the convective envelope tends to zero, i.e. $K', E = 0$ amounts to a wholly radiative star, without convective envelope. The maximum value of E is given by Eq. (6.1.138), when the star is fully convective, without radiative core. Since a star cannot be more than completely convective, a star having $E > 45.48$ is located in Hayashi's "forbidden region" on a Hertzsprung-Russell diagram, and cannot be in hydrostatic equilibrium; it would adjust itself to an equilibrium structure roughly on the free-fall time scale (5.12.4), (Hayashi et al. 1962, Cox and Giuli 1968).

In contrast to convective (adiabatic) envelopes, where the polytropic constant K' is determined by the surface conditions and convection theory, the value of K' can simply be evaluated from the internal stellar structure in the case of *completely* convective stars with $n = \mathrm{const}$ [e.g. from the central condition $K' = P_0/T_0^{n+1}$, or from Eq. (6.1.131)]. Note, that K' is determined in a straightforward manner in the case of radiative stars [cf. Eq. (6.1.82)].

On the other side, the stellar surface boundary conditions on pressure and temperature have to be utilized for the determination of the total luminosity L_1 of completely convective stars, obeying the adiabatic relationship (6.1.130); this is not necessarily the case in stars which are in radiative equilibrium, when total mass, radius, and opacity are sufficient for evaluating L_1 [see Eqs. (6.1.25), (6.1.86), (6.1.101)]. The luminosity of fully convective stars can easily be computed if we go to the one place in the star where radiative transfer holds – the region above the photosphere. The material above the photosphere must be predominantly in radiative equilibrium, because the photosphere is by definition the "visible surface" at $r = r_1$ of a star, wherefrom energy is radiated into external space. The normal optical depth

$$\tau(r_1) = \int_{r_1}^{r_2} \kappa(r)\ \varrho(r)\ dr, \qquad (d\tau = -\kappa\varrho\ dr;\ \varrho(r_2), \tau(r_2) = 0;\ r_2 - r_1 \ll r_1), \tag{6.1.141}$$

of the narrow surface layer $r_2 - r_1$ must be minor, and Eddington assumes that radiation pressure is given for all optical depths by $P_r = aT^4/3$ from Eq. (6.1.1), a relationship which is actually valid only for large optical depths. For a grey atmosphere, the theory of radiative transfer yields the well known formula for the temperature change with atmospheric optical depth (Cox and Giuli 1968):

$$T^4 = T_e^4(1/2 + 3\tau/4). \tag{6.1.142}$$

T_e denotes the effective temperature occurring at the bottom of the photosphere at optical depth $\tau = \tau(r_1) = \tau_1 = 2/3$, $(T(r_1) = T_e)$. The outer boundary temperature at $r = r_2$ is equal to $T(r_2) = T_e/2^{1/4}$, $(\tau(r_2) = 0)$. To determine the photospheric pressure P_1, we integrate the equation of hydrostatic equilibrium:

$$P_1 = P(r_1) = P(r_2) + \int_{r_1}^{r_2} GM(r)\ \varrho(r)\ dr/r^2 \approx a[T(r_2)]^4/3 + (GM_1/r_1^2) \int_0^{\tau_1} d\tau/\kappa \approx 2\sigma T_e^4/3c$$
$$+2GM_1/3\kappa_1 r_1^2 = (2GM_1/3\kappa_1 r_1^2)(1 + \kappa_1 L_1/4\pi c GM_1) \approx 2GM_1/3\kappa_1 r_1^2, \qquad [\tau_1 = 2/3;\ \kappa_1 = \kappa(\tau_1)]. \tag{6.1.143}$$

We have replaced in Eq. (6.1.143) the net radiation flux from the photospheric surface unit by the flux $\sigma T_e^4 = L_1/4\pi r_1^2$, $[T_e = 2^{1/4}T(r_2)]$ of a black body, $\sigma = ac/4$ being the Stefan-Boltzmann constant. In terms of the effective temperature T_e the luminosity of a completely convective star is simply

$$L_1 = 4\pi\sigma r_1^2 T_e^4, \tag{6.1.144}$$

where the effective temperature can be determined for instance from $T_e^{n+1} = [T(r_1)]^{n+1} = P_1/K'$ with K' and P_1 given by Eqs. (6.1.131) and (6.1.143), respectively.

Convective envelopes occur if Eq. (6.1.130) holds only in the outer regions of the star. On the other hand, the same relationship also holds in the radiative envelopes of the uniform source model [Eq. (6.1.81)]. Thus, convective and radiative envelopes, both obey a polytropic relationship of the form (6.1.81) and (6.1.130), where $n' = n'_{ad} = 1/(\Gamma_2 - 1)$ and $n' = n'_{rad} = (-f+3)/(e+1)$ in the convective and radiative envelope, respectively. The polytropic constant K' has to be determined from the photospheric surface condition (6.1.143) in the case of convective envelopes: $P(r_1) = 2GM_1/3\kappa_1 r_1^2 = K'[T(r_1)]^{n'_{ad}+1} = K' T_e^{n'_{ad}+1}$. For radiative envelopes K' is given by Eq. (6.1.82) in the uniform source model.

6.1.6 Applications to Differentially Rotating Polytropes

(i) Differentially Rotating Sun. Geroyannis and Antonakopoulos (1980) have considered a solar model of polytropic index $n = 3.25$, constructed on the basis of the theory of differentially rotating polytropes with the rotation law (3.5.8), the central rotation period of the core being 1.8 days [cf. Dicke 1970, and next point (ii)]. Differential rotation traps a large amount of angular momentum into the core, increasing the total storage capacity of angular momentum as compared to uniform rotation by about two orders of magnitude [see also Mohan et al. (1992, Table V)].

(ii) Solar Polytropes and Perihelion Advance of Mercury. As will be obvious from the following, this item is closely connected to a nonuniformly (differentially) rotating Sun. The perihelion movement (perihelion precession) of Mercury's orbit is caused by three factors (e.g. Gerthsen et al. 1977): (i) Perturbations of the other planets. (ii) Oblateness of the rotating Sun. (iii) Relativistic effects due to general relativity, scalar-tensor theories, and other gravitation theories.

The perihelion advance in the orbital plane of the Keplerian ellipse due to general relativity is given by [e.g. Landau and Lifschitz 1987, Eq. (101.7)]

$$\Delta\omega_r = 6\pi GM_\odot/c^2a(1-e^2)\ \text{[radians per revolution period]}. \tag{6.1.145}$$

This amounts for Mercury to 43.0 seconds of arc per century. In Eq. (6.1.145) a and e denote the semimajor axis and eccentricity. The perturbations of the other planets on the perihelion movement of Mercury can be calculated quite exactly with Newtonian gravitation, leaving a residue which is just equal to the general relativistic perihelion advance (6.1.145). This implies that the perihelion perturbations $\Delta\omega$ from Eqs. (6.1.166)-(6.1.168) – caused by solar oblateness – are almost negligible within the limits of observational error. As it appears from Table 6.1.1, the oblateness resulting from uniformly rotating polytropic models of the Sun ($1 \leq n \leq 3.5$) is $f = (1.08 - 1.59) \times 10^{-5}$, which is 3-5 times smaller than the oblateness $f = (5 \pm 0.7) \times 10^{-5}$ measured by Dicke and Goldenberg (Fahlman et al. 1970). This large oblateness has been used as an argument in favour of a large influence of the solar quadrupole moment J_2 on the perihelion advance of Mercury, causing an 8% discrepancy ($\Delta\omega \approx 3.4''$ per century) with the general relativistic prediction (6.1.145), (Anand and Fahlman 1968, Fahlman et al. 1970). And this discrepancy may be used as an argument in favour of scalar-tensor theories of gravitation, like the Brans-Dicke theory. As it is not our scope to dispute the reliability of various gravitation theories – although Einstein's general relativity seems to be the most reliable one – we concentrate on the determination of Mercury's perihelion advance $\Delta\omega$ caused by oblate polytropic models of the Sun. Winer's (1966) first order theory appears to be sufficiently accurate.

As already shown by Eqs. (3.1.61)-(3.1.63), the quadrupole moment J_2 of the rotationally symmetric Sun is given by

$$\begin{aligned} M_\odot r_\odot^2 J_2 &= C - (A+B)/2 = \int_{V_1} [(x_1^2 + x_2^2) - (2x_3^2 + x_1^2 + x_2^2)/2]\varrho\ dV \\ &= \int_{V_1} [(x_1^2+x_2^2)/2 - x_3^2]\varrho\ dV = 2\pi \int_0^\pi [(1/2)\sin^2\lambda - \cos^2\lambda]\sin\lambda\ d\lambda \int_0^{r_1(\lambda)} r^4 \varrho(r,\lambda)\ dr \\ &= -2\pi \int_{-1}^1 P_2(\mu)\ d\mu \int_0^{r_1(\mu)} r^4\varrho(r,\mu)\ dr, \qquad (\mu = \cos\lambda;\ P_2(\mu) = (3\mu^2-1)/2). \end{aligned} \tag{6.1.146}$$

In our first order approximation we have equated the equatorial radius a_1 of the Sun from Eq. (3.1.63) to its mean radius $r_\odot = 6.96 \times 10^{10}$ cm.

Turning with Eqs. (3.2.1), (3.2.44) to the polytropic variables of Chandrasekhar's (1933a) first order perturbation theory, we get via Eqs. (3.2.58), (3.5.16):

$$\begin{aligned} M_\odot r_\odot^2 J_2 &= -2\pi\alpha^5\varrho_0 \int_{-1}^1 P_2(\mu)\ d\mu \int_0^{\Xi_1(\mu)} \xi^4\Theta^n(\xi,\mu)\ d\xi \\ &\approx \{2\pi\alpha^5\varrho_0\beta\xi_1^2/3[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\} \int_0^{\xi_1} n\theta^{n-1}(\xi)\ \psi_2(\xi)\ \xi^4\ d\xi. \end{aligned} \tag{6.1.147}$$

From Eq. (3.2.11) follows

$$n\xi^2\theta^{n-1}\psi_2 = 6\psi_2 - d(\xi^2\psi_2')/d\xi, \qquad (6.1.148)$$

which is inserted into the integral (6.1.147), and integrated by parts:

$$M_\odot r_\odot^2 J_2 = \{2\pi\alpha^5\varrho_0\beta\xi_1^2/3[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]\}\int_0^{\xi_1}[6\psi_2 - d(\xi^2\psi_2')/d\xi]\xi^2\ d\xi$$
$$= 2\pi r_\odot^5\varrho_0\beta[2\psi_2(\xi_1) - \xi_1\psi_2'(\xi_1)]/3[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)], \qquad (r_\odot = \alpha\xi_1). \qquad (6.1.149)$$

To make the connection with the perihelion advance of Mercury, we introduce the first approximation of the perturbation function due to solar oblateness (Stumpff 1965, p. 244, 603; Winer 1966, Anand and Fahlman 1968)

$$S = -G(C-A)P_2(\lambda)/\ell^3 = GM_\odot r_\odot^2 J_2(1 - 3\cos^2\lambda)/2\ell^3 = GM_\odot r_\odot^2 J_2[1 - 3\sin^2(\varphi+\omega)\sin^2 i]/2\ell^3, \qquad (6.1.150)$$

acting on a planet of negligible mass, having semimajor axis a, instantaneous distance ℓ from the Sun, eccentricity e, inclination i with respect to solar equatorial plane, polar angle λ with respect to solar rotation axis, perihelion longitude ω with respect to the line of nodes located in the solar equatorial plane, and true anomaly φ. The corresponding perihelion motion of Mercury due to solar oblateness is given by the perturbation equation

$$d\omega/dt = (GM_\odot a)^{-1/2}[\tan(i/2)\ (\partial S/\partial i)/(1-e^2)^{1/2} + (1-e^2)^{1/2}(\partial S/\partial e)/e]. \qquad (6.1.151)$$

A relationship between true anomaly and time is provided by Kepler's laws (e.g. Stumpff 1959):

$$\ell^2\ d\varphi/dt = \mathrm{const} = [GM_\odot a(1-e^2)]^{1/2}; \quad \ell = a(1-e^2)/(1+e\cos\varphi). \qquad (6.1.152)$$

If we insert for dt into Eq. (6.1.151), we get

$$d\omega/d\varphi = [\ell^2/GM_\odot a(1-e^2)^{1/2}][\tan(i/2)\ (\partial S/\partial i)/(1-e^2)^{1/2} + (1-e^2)^{1/2}(\partial S/\partial e)/e]. \qquad (6.1.153)$$

The evaluation of $\partial S/\partial i$ proceeds at once, whereas the calculation of $\partial S/\partial e$ is somewhat more involved, since ℓ and φ depend on e, but not on i. We write therefore

$$\partial S/\partial e = (\partial S/\partial\ell)\ \partial\ell/\partial e + (\partial S/\partial\varphi)\ \partial\varphi/\partial e, \qquad (6.1.154)$$

neglecting the slow variation of ω with respect to e.

As the calculation of $\partial\ell/\partial e$ and $\partial\varphi/\partial e$ is not quite trivial, it will be sketched below. The planetary radius vector can also be expressed with the aid of the so-called eccentric anomaly E :

$$\ell = a(1 - e\cos E). \qquad (6.1.155)$$

When calculating derivatives with respect to e, we neglect the slow variation of a with respect to e :

$$\partial\ell/\partial e = -a\cos E + ae\sin E\ \partial E/\partial e. \qquad (6.1.156)$$

The derivative $\partial E/\partial e$ follows from Kepler's equation (Stumpff 1959)

$$E - e\sin E = (GM_\odot/a^3)^{1/2}(t - t_0), \qquad (6.1.157)$$

as

$$\partial E/\partial e = \sin E/(1 - e\cos E) = a\sin E/\ell, \qquad (6.1.158)$$

by neglecting the slow variation of a and of the osculating epoch of perihelion passage t_0 with respect to e. Inserting Eq. (6.1.158) into Eq. (6.1.156), we get the desired relationship:

$$\partial\ell/\partial e = a(e - \cos E)/(1 - e\cos E) = -a\cos\varphi. \qquad (6.1.159)$$

Table 6.1.1 Solar rotation parameter $\beta = \Omega^2/2\pi G\varrho_0$ if $\Omega = 2.87 \times 10^{-6}$ rad/s (rotation period $T = 2\pi/\Omega = 25.34$ days), solar oblateness f, perihelion advance $\Delta\omega$ of Mercury due to solar oblateness, rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0$ of solar core, and rotation period $T_c = 2\pi/\Omega_c$ of solar core if $f = 5\times10^{-5}$. The entries are calculated for various polytropic indices n with Anand's (1968) second order theory of rotating polytropes (Anand and Fahlman 1968). $a + b$ means $a \times 10^b$.

n	1	1.5	2	3	3.5
$\beta = \Omega^2/2\pi G\varrho_0$	4.24−6	2.33−6	1.22−6	2.57−7	9.12−8
f	1.59−5	1.35−5	1.20−5	1.08−5	1.09−5
$\Delta\omega$ [arcsec/century]	0.46	0.25	0.13	0.025	0.017
$\beta_c = \Omega_c^2/2\pi G\varrho_0$	4.34−5	4.32−5	4.39−5	4.71−5	2.46−5
$T_c = 2\pi/\Omega_c$ [days]	5.9	4.4	3.2	1.4	1.2

the mass of the envelope amounts even for a $n = 1$ polytrope only to about 10% of solar mass. In absence of shear stresses, the envelope arranges itself so that its outer surface is an equipotential in the field formed jointly by its own rotation and the gravitational field of the solar core alone. We approximate the quadrupole moment J_2 of the whole Sun with that of its core J_{2c}, neglecting contributions from the small-mass envelope. Eq. (3.1.71) becomes

$$3J_{2c}/2 \approx 3J_2/2 = f - \Omega^2 r_\odot^3/2GM_\odot. \qquad (6.1.169)$$

We get $J_{2c} \approx J_2 \approx 2.64\times10^{-5}$, if we insert the observed values at the solar surface $f = 5\times10^{-5}$, $\Omega = 2.87 \times 10^{-6}$ rad/s. This approximate value of the solar quadrupole moment would cause in virtue of Eq. (6.1.166) a Mercurian perihelion advance of 3.4″ per century. And with this value we calculate from Eq. (6.1.167) the rotation parameter $\beta_c = \Omega_c^2/2\pi G\varrho_0$ and the rotation period $T_c = 2\pi/\Omega_c$ of the solar polytropic core, as resulting from first order theory (Table 6.1.1). The surface value $r_\odot = \alpha\xi_1$ should be replaced throughout by $r_c = 0.85r_\odot = 0.85\alpha\xi_1 = \alpha\xi_c$. It should however be emphasized that the stability of such a solar core-envelope configuration is questionable.

A solar model based on Stoeckly's (1965) differential rotation law (3.8.37) yields $f = 3.54 \times 10^{-5}$ if $n = 3$, and a negligible perihelion advance of at most $\Delta\omega = 0.12$ arcsec/century (Fahlman et al. 1970). But Stoeckly's differential rotation law would yield a solar rotation rate which is faster on the poles than at the equator – contrary to observation.

(iii) Differentially Rotating Stellar Cores, White Dwarfs, and Neutron Stars. From Fig. 5.12.1 it is obvious that between central densities of about 10^9 to 10^{14} g cm^{-3} no stable, nonrotating, zero temperature equilibrium configurations are possible, because the adiabatic index Γ_1 drops below its critical value 4/3. As seen by the rough equation (5.3.16), rotation would tend to stabilize the configuration, even for values $\Gamma_1 < 4/3$. This holds even more for differentially rotating polytropes, which are rotationally stable also for values of $\beta = \Omega^2/2\pi G\varrho_0$ larger than the critical value β_c of uniformly rotating polytropes (Figs. 3.8.2, 5.8.2). The differential rotation laws adopted by Eriguchi and Müller (1985b), and Müller and Eriguchi (1985) are similar to those from Eqs. (3.8.82)-(3.8.84):

$$\Omega(\ell) = \Omega(0)/(1 + \ell^2/A^2); \quad \Omega(\ell) = \Omega(0)/(1 + \ell/A), \qquad (A = \mathrm{const}). \qquad (6.1.170)$$

In the limit $\ell/A \gg 1$ these equations turn into the constant angular momentum law $\ell^2\Omega(\ell) = A^2\Omega(0)$, and into the constant velocity law $\ell\Omega(\ell) = A\Omega(0)$, respectively. If $\ell/A \ll 1$, the differential rotation becomes uniform, with angular velocity equal to $\Omega(0)$ – the angular velocity along the rotation axis.

A stellar core of density $\varrho \approx 10^9$ g cm^{-3}, obeying the differential rotation laws (6.1.170), will not collapse to neutron star densities ($\varrho \approx 10^{14}$ g cm^{-3}) if the initial value of $\tau = E_{kin}/|W|$ is *larger* than 0.01, 0.03, 0.08, if $\Gamma_1 = 1.30, 1.25, 1.20$, respectively (Eriguchi and Müller 1985b, Fig. 6, dash-dotted curve). The evolution of these dynamically stable cores to neutron star densities occurs on a secular time scale, through emission of gravitational radiation, losing in this way angular momentum (Sec. 5.8.4). On the other hand, nonaxisymmetric *dynamical* instabilities $\propto \exp(\pm ik\varphi)$, $(k = 1, 2, 3, 4)$ of a differentially rotating polytrope with $n = 1/(\Gamma_1-1) = 3.33$, $\Gamma_1 = 1.30$ at $\tau = 0.14,\ 0.18$ have been detected numerically by New and Centrella (2001, Figs. 8-12), and Centrella et al. (2001).

For the study of differentially rotating white dwarfs Müller and Eriguchi (1985) use a so-called piece-wise polytropic approximation, assuming that in each of $H - 1$ adjacent density intervals $[\varrho_i, \varrho_{i+1}]$, $(i = 1, 2, ...H - 1)$ the pressure is approximated by a polytropic isentropic law

$$P = K_i\varrho^{1+1/n_i} = K_i\varrho^{\Gamma_{1i}}; \qquad \varrho \in [\varrho_i, \varrho_{i+1}]. \qquad (6.1.171)$$

The last expression results simply from Eqs. (6.1.152) and (6.1.155):

$$\ell e\cos\varphi = a(1-e^2)-\ell = a(1-e^2)-a(1-e\cos E) \quad \text{or} \quad \ell\cos\varphi = a(\cos E - e). \qquad (6.1.160)$$

To get the derivative $\partial\varphi/\partial e$, we differentiate the last equation (6.1.160)

$$\cos\varphi\ \partial\ell/\partial e - \ell\sin\varphi\ \partial\varphi/\partial e = -a(\sin E\ \partial E/\partial e + 1), \qquad (6.1.161)$$

and insert for $\partial E/\partial e$ and $\partial\ell/\partial e$ from Eqs. (6.1.158) and (6.1.159), respectively:

$$a\cos^2\varphi + \ell\sin\varphi\ \partial\varphi/\partial e = a^2\sin^2 E/\ell + a. \qquad (6.1.162)$$

The connection between $\sin E$ and $\sin\varphi$ can be obtained with Eqs. (6.1.155), (6.1.160):

$$\ell^2\sin^2\varphi = \ell^2 - \ell^2\cos^2\varphi = a^2(1-e\cos E)^2 - a^2(\cos E - e)^2 = a^2(1-e^2)\sin^2 E. \qquad (6.1.163)$$

After simplification with $\sin\varphi$, Eq. (6.1.162) finally becomes (Stumpff 1965, p. 387, Kopal 1978, p. 204)

$$\partial\varphi/\partial e = \sin\varphi\ [a/\ell + 1/(1-e^2)] = \sin\varphi\ (2+e\cos\varphi)/(1-e^2). \qquad (6.1.164)$$

Returning with Eqs. (6.1.150), (6.1.154), (6.1.159), (6.1.164) to Eq. (6.1.153), we complete Eq. (10) of Winer (1966):

$$\begin{aligned} d\omega/d\varphi = &-[3r_\odot^2 J_2/2a(1-e^2)^{1/2}\ell]\big\{\tan(i/2)\sin 2i\sin^2(\varphi+\omega)/(1-e^2)^{1/2} \\ &+(1-e^2)^{1/2}[1-3\sin^2(\varphi+\omega)\sin^2 i](\partial\ell/\partial e)/e\ell + (1-e^2)^{1/2}\sin[2(\varphi+\omega)]\sin^2 i\ (\partial\varphi/\partial e)/e\big\} \\ &= [3r_\odot^2 J_2(1+e\cos\varphi)/2a^2(1-e^2)^2]\big\{-\tan(i/2)\sin 2i\sin^2(\varphi+\omega) \\ &+[1-3\sin^2(\varphi+\omega)\sin^2 i](1+e\cos\varphi)\cos\varphi/e - \sin[2(\varphi+\omega)]\sin^2 i\sin\varphi\ (2+e\cos\varphi)/e\big\}. \qquad (6.1.165)\end{aligned}$$

The integration over φ from 0 to 2π is tedious for the second order terms involving $\sin^2 i \approx 0.003$ [Anand and Fahlman 1968, Eq. (34)]:

$$\begin{aligned}\Delta\omega &= [3\pi r_\odot^2 J_2/a^2(1-e^2)^2]\{1-\sin^2 i\ [\cos i/(1+\cos i) + 3(1+2\sin^2\omega)/4 + 3\cos 2\omega/4]\} \\ &= [3\pi r_\odot^2 J_2/a^2(1-e^2)^2]\{1-\sin^2 i\ [\cos i/(1+\cos i)+3/2]\} \approx 3\pi r_\odot^2 J_2/a^2(1-e^2)^2. \qquad (6.1.166)\end{aligned}$$

If we insert for J_2 from Eq. (6.1.149), Mercury's perihelion advance per revolution period due to solar oblateness is obtained in a sufficiently accurate first approximation:

$$\Delta\omega = [2\pi^2 r_\odot^5\varrho_0\beta/M_\odot a^2(1-e^2)^2][2\psi_2(\xi_1) - \xi_1\psi_2'(\xi_1)]/[3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)]. \qquad (6.1.167)$$

We replace $3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)$ from the relationship (3.2.55) of the oblateness f, correcting for the missing factor 2 in Eq. (15) of Winer (1966), (cf. Anand and Fahlman 1968, Table V):

$$\Delta\omega = [8\pi^2 r_\odot^5\varrho_0 f\ (-\theta_1')/5M_\odot a^2(1-e^2)^2][2/\xi_1 - \psi_2'(\xi_1)/\psi_2(\xi_1)]. \qquad (6.1.168)$$

Anand and Fahlman (1968) have calculated for the observed solar rotation rate of $\Omega = 2.87\times 10^{-6}$ radians per second at 16° solar latitude, the dimensionless rotation parameter $\beta = \Omega^2/2\pi G\varrho_0$ of rotating polytropes by using Anand's (1968) second order theory. The results, together with the solar oblateness f and Mercury's perihelion advance $\Delta\omega$, are shown in Table 6.1.1. The differences between the second order theory of Anand (1968) and Winer's (1966) first order theory from Eqs. (6.1.167), (6.1.168) are modest, amounting merely to a decrease of about 5% in Winer's *corrected* results. As already stated, Mercury's perihelion advance due to solar oblateness from Table 6.1.1 is completely negligible as compared to the relativistic perihelion advance $\Delta\omega_r$ from Eq. (6.1.145). Li (1997a) quotes $f = 2.18\times 10^{-5}$ and $\Delta\omega = 0.1''$/century for a $n=3$ polytrope, instead of $f = 1.08\times 10^{-5}$ and $\Delta\omega = 0.025''$/century from Table 6.1.1.

The observed solar oblateness $f = (5\pm 0.7)\times 10^{-5}$ is about 3-5 times larger than the polytropic values quoted in Table 6.1.1, and this led Dicke (1970) to invoke a rapidly rotating solar core ($T_c \approx 1.8$ days), surrounded by a slowly rotating ($\Omega = 2.87\times 10^{-6}$ rad/s; $T = 2\pi/\Omega = 25.34$ days), virtually massless envelope, having negligible dynamical interaction with the core. If the core contains 85% of solar radius,

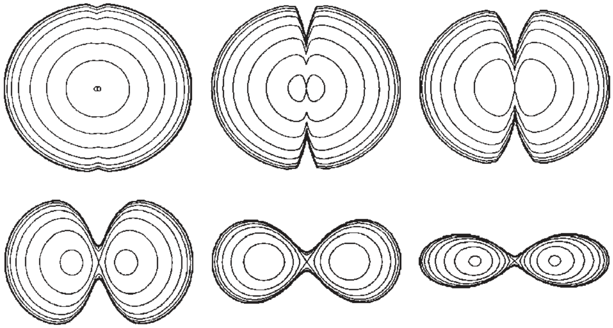

Fig. 6.1.2 Equidensity contours of differentially rotating, isentropic stellar cores with the first rotation law from Eq. (6.1.170), and τ increasing successively up to $\tau = 0.29$, ($\Gamma_1 = 1+1/n = 1.32$; $n = 3.125$; $A = 0.02r_c$; r_c – radius of nonrotating core), [Eriguchi and Müller 1985b; see also Fig. 7 of New and Centrella (2001)].

From the continuity of pressure on the margins of the density intervals one obtains $2(H-1)$ conditions for the $2H$ parameters K_i and n_i :

$$P(\varrho_i) = P_i = K_i\varrho_i^{1+1/n_i}; \quad P(\varrho_{i+1}) = P_{i+1} = K_i\varrho_{i+1}^{1+1/n_i}. \tag{6.1.172}$$

$2(H-1)$ parameters result immediately:

$$K_i = P_i/\varrho_i^{1+1/n_i}; \quad \Gamma_{1i} = 1 + 1/n_i = (\ln P_{i+1} - \ln P_i)\big/(\ln \varrho_{i+1} - \ln \varrho_i), \qquad (i = 1, 2, ...H-1). \tag{6.1.173}$$

If the value of n_H is prescribed in accordance with the equation of state, the corresponding polytropic constant results from $K_H = P_H/\varrho_H^{1+1/n_H}$.

Dynamically stable equilibrium models of white dwarfs exist up to central densities $\approx 10^{11}$ g cm^{-3} (without differential rotation $\approx 10^9$ g cm^{-3}), and with masses up to $\approx 1.7\ M_\odot$ (without differential rotation $\approx 1.0\ M_\odot$; see also Fig. 5.12.6).

6.1.7 Polytropic Planets

Öpik (1962) and Bobrov et al. (1978) have estimated the polytropic indices n of planetary models obeying the polytropic equation of state $P = K\varrho^{1+1/n}$. Öpik (1962) uses for his estimate the observed planetary ratio between oblateness $f = (a_1 - a_3)/a_1$ and the dimensionless rotation parameter $m = 3\Omega^2/4\pi G\varrho_m$ from Eq. (3.7.24), which is approximately equal to the ratio between centrifugal and gravitational force at the planetary equator. The parameter m is more suitable than the rotation parameter $\beta = \Omega^2/2\pi G\varrho_0$ for the comparison of rotating configurations having the same mass, but different rotation rates. If we express the mean density of the rotating planet to first order with the central density of a spherical planet, we get in virtue of Eq. (2.6.27)

$$m = 3\Omega^2/4\pi G\varrho_m = -\Omega^2\xi_1/4\pi G\varrho_0\theta_1' = -\beta\xi_1/2\theta_1'. \tag{6.1.174}$$

The oblateness f of a rotating planet has already been expressed by Eq. (3.2.55), and the ratio f/m reads accordingly

$$f/m = 5\psi_2(\xi_1)/2[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)], \qquad (0 \le n \le 5). \tag{6.1.175}$$

This value has already been tabulated by Chandrasekhar (1933d) for various polytropic indices, and Öpik (1962) has estimated by interpolation the mean polytropic index of planetary models build up with

Table 6.1.2 Gyration factor k from Eq. (6.1.179), (Motz 1952), gyration factor k_0 normalized to the gyration factor $k = 0.4$ of a constant density sphere $n = 0$, and oblateness parameter f/m according to Table 3.2.1 and Eq. (6.1.175). The last two lines are reproduced from Öpik's (1962) Table XII, showing the observed ratio $(f/m)_p$ for Earth, Mars, Jupiter, and Saturn, as well as the interpolated polytropic index n_p of the corresponding planetary polytrope.

n	0	1	1.5	2	3	4	5
k	0.400	0.261	0.205	0.157	0.0758	0.0236	0.000
k_0	1.000	0.653	0.513	0.393	0.190	0.0590	0.000
f/m	1.250	0.760	0.643	0.574	0.514	0.501	0.500

Planet	Earth	Mars	Jupiter	Saturn
$(f/m)_p$	0.97	1.14	0.77	0.69
n_p	0.51	0.26	0.98	1.25

the polytropic equation of state, by using the observed planetary ratio f/m (Table 6.1.2). This procedure is suitable for objects with polytropic index $0 \leq n \lesssim 1.5$ (e.g. planets), but not for most stars having $n \approx 3$, since in this case the ratio f/m is already close to its limiting value 0.5, obtained for the point mass model (Roche model), which is similar to the polytrope $n = 5$, as outlined subsequently to Eq. (3.2.69), (see case (ii) $n = 5$, below).

In the two particular cases $n = 0$ and $n = 5$ the ratio f/m can be calculated analytically.

(i) $\boldsymbol{n = 0}$**.** By virtue of Eq. (3.2.64) we have $f = 15\beta/8 = 15\Omega^2/16\pi G\varrho_m$, and $f/m = 5/4$. In fact, this value can be derived also directly from the more general first order equations (3.1.68), (3.1.69), expressing the integration constant by equatorial values at $r_1 = a_1$:

$$GM_1(1/a_1 - 1/r_1) + (GM_1J_2a_1^2/2)(1/a_1^3 - 1/r_1^3 + 3\cos^2\lambda/r_1^3) + (\Omega^2/2)(a_1^2 - r_1^2\sin^2\lambda) = 0. \tag{6.1.176}$$

Putting $r_1 = a_1$ in the factors near the small first order terms, this equation becomes

$$(r_1 - a_1)/a_1 + \cos^2\lambda\ (3J_2/2 + \Omega^2 a_1^3/2GM_1) = 0. \tag{6.1.177}$$

The gravitational moment J_2 is via Eqs. (3.1.72), (6.1.146) equal to

$$J_2 = -(2\pi/M_1a_1^2)\int_{-1}^{1}(3\mu^2/2 - 1/2)\,d\mu\int_0^{r_1(\mu)}\varrho_m r^4\,dr = 8\pi\varrho_m a_1^3 f/15M_1 \approx 2f/5,$$
$$[n = 0;\ r_1 = a_1(1 - f\mu^2)]. \tag{6.1.178}$$

With this value of the quadrupole moment we obtain from Eqs. (3.1.72), (6.1.177) just $f/m = 5/4$.

(ii) $\boldsymbol{n = 5}$**.** From Eq. (6.1.175) we get at once $f/m = 1/2$ in virtue of the asymptotic solutions (3.2.89).

With increasing mass concentration towards the centre, the oblateness parameter f/m changes between the limits 1.25, $(n = 0)$ and 0.5, $(n = 5)$.

The dimensionless gyration factor of a sphere k is quoted in Table 6.1.2 (Motz 1952):

$$k = 2I/3r_1^2M_1 = \int_{M_1}\ell^2\,dM/(r_1^2M_1) = (2\pi/r_1^2M_1)\int_0^{\pi}\sin^3\lambda\,d\lambda\int_0^{r_1}\varrho(r)\,r^4\,dr$$
$$= (8\pi/3r_1^2M_1)\int_0^{r_1}\varrho(r)\,r^4\,dr = [2/3\xi_1^4(-\theta_1')]\int_0^{\xi_1}\xi^4\theta^n\,d\xi$$
$$= [2/3\xi_1^4(-\theta_1')]\Big[\xi_1^4(-\theta_1') + 2\xi_1^3\theta_1 - 6\int_0^{\xi_1}\xi^2\theta\,d\xi\Big], \qquad [-1 < n \leq 5;\ N = 3;\ \xi^2\theta^n = -(\xi^2\theta')']. \tag{6.1.179}$$

I denotes the moment of inertia (2.6.75), and we have preserved the term $2\xi_1^3\theta_1$ in the partial integrations, since it does not vanish in the limiting case $k = 0$, $(n = 5;\ \xi_1 \to \infty)$ from Eq. (2.3.90).

The polytropic equation of state has been used by Bobrov et al. (1978) to calculate wholly polytropic models of the giant planets with the level surface theory (see Sec. 3.7). The models are required to fit the observed mass, radius, rotation period, and observed gravitational moments from Eq. (3.1.58). For Jupiter the values of the observed gravitational moments J_2, J_4, J_6 correspond best to a polytrope of index $n = 0.95$ (cf. Table 6.1.2) with polytropic constant $K = 1.99 \times 10^{12}$ [CGS units]. The values of the observed moments J_2, J_4 for Saturn, and J_2 for Uranus and Neptune are fitted by polytropes of index $n = 1.2 - 1.3,\ 0.9 - 1.6$, and 1.2 for Saturn, Uranus, and Neptune, respectively.

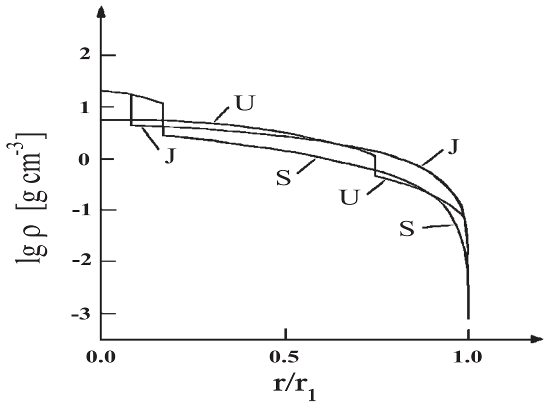

Fig. 6.1.3 Density run in models consisting of a rocky core surrounded by a polytropic envelope for Jupiter (J), Saturn (S), and Uranus (U), (Horedt and Hubbard 1983).

Two-layer models of Jupiter, Saturn, and Uranus have been calculated by Horedt and Hubbard (1983). The temperature independent equation of state for the rocky cores of the three giant planets is given by the "perturbed" polytropic equation of state (1.7.44). This equation results from a generalization

$$d\ln P/d\ln\varrho = X_0 + X_1\varrho + 2X_2\varrho^2 + 3X_3\varrho^3 + ... \qquad (6.1.180)$$

of the polytropic relationship $d\ln P/d\ln\varrho = 1 + 1/n$. Integration of Eq. (6.1.180) yields

$$\ln P = \ln K + X_0\ln\varrho + X_1\varrho + X_2\varrho^2 + X_3\varrho^3 + ... \qquad (6.1.181)$$

or (Slattery 1977)

$$P = K\varrho^{X_0}\exp(X_1\varrho + X_2\varrho^2 + X_3\varrho^3 + ...), \qquad (K, X_0, X_1, X_2, X_3, ... = \mathrm{const}). \qquad (6.1.182)$$

Hubbard (1974) has constructed an analytical model of Jupiter with polytropic index $n = 1$, and equation of state $P = 1.96 \times 10^{12}\varrho^2$ [dyne cm^{-2}] $= 1.96\varrho^2$ [Megabar], which fits approximately the observed gravitational moments [cf. Eqs. (3.2.48)-(3.2.54)].

A hot model of Jupiter, prompted by the large measured Jovian heat flux of about 10^4 erg/cm^2 s, has been proposed by Kozyrev (1977). The central temperature of $T_c = 165000$ K has been estimated from the ideal gas law (1.2.5), with central pressure and density determined from a polytropic sphere of index $n \approx 1$.

Geroyannis (1993b), Geroyannis and Valvi (1993, 1994), Geroyannis and Dallas (1994) estimate with the "global polytropic model" distances and masses in the planetary and Jovian system, extending the Lane-Emden equation (2.3.87) into the complex plane beyond its first zero ξ_1 [cf. Eq. (3.5.54)].

6.1.8 Fission Hypothesis of Rotating Polytropes

Magnetic braking of the rotation of pre-main sequence stars, approximated as fully convective $n = 1.5$ polytropes, has been calculated by Okamoto (1969, 1970). Opposite to this behaviour of magnetopolytropes is the assumed fission of a rapidly rotating polytrope into a binary star – an idea that goes back to the times of Darwin and Jeans (1919, 1961). The overall picture envisaged by most authors is a fully convective, gravitationally contracting, pre-main sequence star with polytropic index $n = 1/(\gamma - 1) = 1.5$, $(\gamma = 5/3)$, (cf. Sec. 6.1.5). The equation of state of a homogeneous $n = 0$ polytrope is also sometimes assumed, as it leads to the Maclaurin, Jacobi, Dedekind, and Riemann sequences of rotating ellipsoids. If we consider the picture of star formation from a collapsing interstellar cloud, there is plenty of angular momentum to be spent for the fission process. For a cloud corotating with the galaxy at velocity $v = 250$ km/s, and at distance $\ell = 10$ kpc from the galactic centre, we get the angular velocity $\Omega_1 = v/\ell = 8 \times 10^{-16} \approx 10^{-15}\ \mathrm{s}^{-1}$. Conservation of angular momentum of a typical cloud with radius $r_1 = 1$ pc would lead to a final equatorial rotation velocity of the star of

$$v_2 = \Omega_2 r_2 = \Omega_1 r_1^2 / r_2, \tag{6.1.183}$$

exceeding the velocity of light for a solar radius equal to r_2 (Horedt 1978a).

Arguments against and in favour of fission, as outlined by Lyttleton (1953), Roxburgh (1966b), Ostriker (1970), and Gingold and Monaghan (1978), may be summarized as follows:

(i) The classical view, envisaged by Darwin, of a disrupting pear-shaped configuration is untenable, because it is secularly unstable, as already outlined subsequently to Eqs. (3.2.61) and (5.8.133). Moreover, as calculated by Eriguchi et al. (1982), the pear-shaped sequence terminates by mass loss long before fission could occur. Instead, fission may occur along the so-called dumbbell sequence (Fig. 3.8.3), which bifurcates smoothly from the secularly and dynamically stable Jacobi sequence at the neutral point against fourth order harmonic disturbances, occurring if $a_2/a_1 = 0.2972$, $a_3/a_1 = 0.2575$, $\beta = \Omega^2/2\pi G\varrho_m = 0.106$ (Chandrasekhar 1969, p. 128, Christodoulou et al. 1995a).

(ii) In constructing protostars, ordinary viscosity seems not large enough to secure the apparition of secular instabilities on a time scale shorter than the contraction time. In this case the axisymmetric Maclaurin sequence determines the relevant configuration of a homogeneous contracting protostar, because beyond the first point of bifurcation, occurring at $\tau = E_{kin}/|W| = 0.1375$, the Maclaurin spheroids are merely secularly unstable, so they will not enter the Jacobi sequence at all, if viscosity is negligible. Instead, the homogeneous, gravitationally contracting protostar will continue to evolve with increasing τ as a Maclaurin ellipsoid, until at $\tau = 0.2738$ it becomes dynamically unstable. The only means of energy dissipation is by radiation, and further evolution of the contracting star could proceed beyond the point of dynamical instability of Maclaurin ellipsoids along the lower self-adjoint series of Riemann ellipsoids, when angular velocity $\vec{\Omega}$ and vorticity $\nabla \times \vec{v}$, both remain aligned along the rotation axis a_3 (S-type Riemann ellipsoids). The conjecture of the fission hypothesis is that ultimately there forms a pair of detached masses orbiting each other, since all contracting compressible Riemann ellipsoids of large enough angular momentum and small enough departure from axial symmetry become unstable to third order harmonic disturbances (Lebovitz 1972, 1974). Dynamically induced fission is possible here – but remains unproven.

(iii) Another objection against fission comes from the fact (see Sec. 3.8.1) that uniformly rotating polytropes with index $n > 0.808$ never reach a point of bifurcation $(\tau \approx 0.14)$, or even of dynamical instability $(\tau \approx 0.27)$, because they become rotationally unstable in the equatorial plane, leaving behind during contraction a disk of material in their equatorial plane. This argument has been refuted by Ostriker and Bodenheimer (1973), claiming that their inviscid differentially rotating polytropes closely resemble the instability behaviour of the homogeneous Maclaurin ellipsoids, with secular instability occurring at $\tau \approx 0.14$, and dynamical instability at $\tau \approx 0.26$, although this expectation has not been properly demonstrated (Figs. 3.8.2, 5.8.2 and Sec. 5.8.3; Tassoul 1978, p. 268).

(iv) Lyttleton (1953, p. 134) argues that the dynamics of the fission process can be examined without considering the effect of friction, and therefore the fission process must be time reversible. But if the direction of time is reversed in a binary system, it does not revert to a single star, but simply remains a binary with the orbital direction of motion reversed. Therefore, a single star cannot evolve into a binary system without dissipation. An argument against this reasoning is that the fission process is not dissipationless (energy conservative), and at least some energy is lost by radiation (Roxburgh 1966b, Ostriker 1970, Lebovitz 1972). But when the system is dissipative, Lyttleton's (1953) argument breaks down. A more convincing point has been raised by Gingold and Monaghan (1978), claiming that on a macroscopic scale the frictionless fission process involves vibrations, transferring energy amongst various modes, and the chance of returning exactly to the initial state is effectively excluded, the fission dynamics becoming irreversible.

(v) It should be noted that the concrete dynamics of the fission process has never been modeled, and even its strongest advocates admit that fission is completely inadequate to account for the formation of wide binaries, with semimajor axes exceeding about 1 AU (Jeans 1961, p. 311; Ostriker 1970, p. 149). As the semimajor axes of binaries range from 10^{-2} AU to more than 10^5 AU, with an average value of about 20 AU (Horedt 1978a), it would be from a more philosophical standpoint very disappointing to assume that in nature quite different processes are responsible for the formation of close binaries on the one side, and wide binaries on the other side.

The reader should weigh for his own the strength of the above arguments (i)-(v). A schematic view concerning the limits placed by equilibrium and stability requirements on the ratio $\tau = E_{kin}/|W|$ has been provided by Ostriker (1970). The total energy of a pressure-free system of N discrete particles, orbiting under the influence of their mutual gravitational interaction is composed of their gravitational energy from Eq. (2.6.67) and the kinetic energy E_{kin}. As outlined subsequently to Eq. (2.6.98) the condition of dynamical stability is

$$E = E_{kin} + W = \sum_{j=1}^{N}\left(m_j v_j^2/2 + \sum_{k=j+1}^{N} W_{jk}\right) = \sum_{j=1}^{N} m_j v_j^2/2 - \sum_{j=1}^{N}\sum_{k=j+1}^{N} G m_j m_k/|\vec{r}_j - \vec{r}_k| < 0. \tag{6.1.184}$$

Bound particle systems are therefore only possible if $E < 0$ or $E_{kin} < -W$, i.e. if $0 \leq \tau = E_{kin}/|W| < 1$. For rotating configurations in hydrostatic equilibrium the value of τ is $0 \leq \tau \leq 1/2$, by virtue of Eqs. (3.1.33)-(3.1.36). As outlined previously, dynamically and secularly stable rotating configurations without equatorial mass shedding exist in the range $0 \leq \tau \lesssim 1/4$ and $0 \leq \tau \lesssim 1/8$, respectively. As mentioned in Secs. 3.8.8 and 5.8.3, uniformly rotating, axisymmetric polytropes without equatorial mass loss occur if $0 \leq \tau \leq 0.32,\ 0.19,\ 0.12,\ 5.95\times10^{-2},\ 9.00\times10^{-3},\ 1.19\times10^{-3}$, where $n = 0.1$, 0.5, 1, 1.5, 3, 4, respectively.

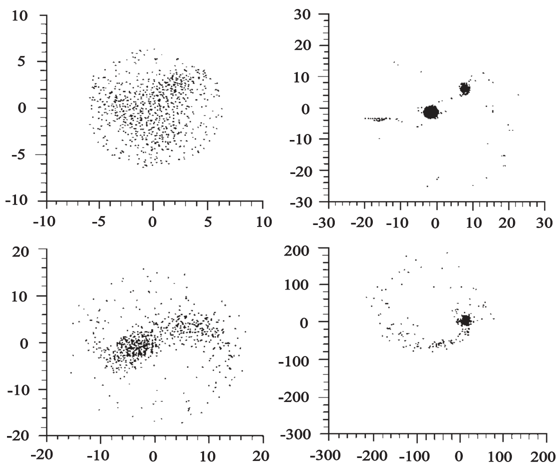

Fig. 6.1.4 Particle positions of a $n = 0.5$ polytrope if $T = (4\pi G\varrho_0/n)^{1/2}t = 60$ (upper left), and $T = 285$ (upper right). The $n = 1.5$ polytrope is shown at lower left ($T = 291$), and lower right ($T = 574$), (Gingold and Monaghan 1979a).

We conclude this section with a brief presentation of the numerical results of Gingold and Monaghan (1978, 1979a). With smoothed particle hydrodynamics, and using a number of up to 800 particles, these authors have calculated the evolution of an initially uniformly rotating sphere of polytropic index $n = 0.5$ and 1.5. The initial values of τ are $\tau = 1.3$ $(n = 0.5)$, and 2.1, $(n = 1.5)$. A radial damping term was introduced to remove radial motions, while conserving angular momentum. The $n = 0.5$ polytrope indeed exhibits fission – a result also confirmed by Lucy (1977) – whereas the $n = 1.5$ polytrope merely throws out a filamentary stream of matter ending probably as a diffuse disk surrounding a central star. This finding is somewhat similar to the outcomes of Durisen and Tohline (1980), showing that a rapidly rotating $n = 0.5$ and $n = 1.5$ polytrope with $\tau = 0.33$ ends up as a dynamically stable star with $\tau \approx 0.19$, ejecting some of its high angular momentum material via gravitational torques into an outer disk/ring.

6.1.9 Applications to Stellar Systems

Vandervoort (1980a) has considered an application to galactic bars of the nonaxisymmetric, triaxial polytropes with index $0.5 \leq n \leq 0.808$ (cf. Sec. 3.8.1). The isolating Jacobi integral of a star moving with respect to uniformly rotating x_1, x_2, x_3 coordinate axes, can be obtained from the general equation (3.1.12), when kinetic pressure forces arise exclusively from the macroscopic velocity $\vec{v}$ of the stars – the fluid pressure P of the "stellar gas" being neglected. Taking $\partial\ /\partial t, P, H, B, \tau, v_{tr} = 0$, and $\vec{\Omega} = \vec{\Omega}(0,0,\Omega),\ \Omega = \mathrm{const}$, Eq. (3.1.12) becomes

$$d\vec{v}/dt = \nabla\Phi - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - 2\vec{\Omega}\times\vec{v}. \qquad (6.1.185)$$

We multiply this equation scalarly by $\vec{v} = d\vec{r}/dt$ to obtain

$$\vec{v}\cdot(d\vec{v}/dt) = (1/2)\ dv^2/dt = d\Phi/dt + (\Omega^2/2)\ d\ell^2/dt, \qquad (\ell^2 = x_1^2 + x_2^2;\ \vec{v}\cdot(\vec{\Omega}\times\vec{v}) = 0). \quad (6.1.186)$$

This can be integrated for the motion of a star:

$$v^2/2 - \Phi(\vec{r}) - \Omega^2\ell^2/2 = \mathrm{const} = H. \qquad (6.1.187)$$

In order to get the polytropic density distributions (6.1.192) and (6.1.194), Vandervoort (1980a) assumes the distribution function (phase density function) of the stars over the spatial and velocity coordinates under the form (cf. Eq. (2.8.101), Hénon 1973):

$$f(\vec{r},\vec{v}) = C(H_0 - H)^{n-3/2} \quad \mathrm{if} \quad H \leq H_0 < 0 \qquad \mathrm{and} \qquad f(\vec{r},\vec{v}) = 0 \quad \mathrm{if} \quad H > H_0,$$
$$(C, H_0 = \mathrm{const};\ C > 0;\ H_0 < 0). \qquad (6.1.188)$$

n denotes the polytropic index of the spatial density distribution of stars. As will be obvious from Eq. (6.1.192) below, the requirement of a finite mass density imposes the restriction $n > 0.5$ on the polytropic index. In the limiting case $n = 0.5$, Vandervoort (1980a) takes the distribution function equal to

$$f(\vec{r},\vec{v}) = C\ \delta_D(H_0 - H), \qquad (6.1.189)$$

where $\delta_D(H_0 - H) = \delta_D(H - H_0)$ is Dirac's function from Eq. (5.10.99):

$$\delta_D(H_0 - H) = 0 \quad \mathrm{if} \quad H_0 - H \neq 0 \qquad \mathrm{and} \qquad \int_{H_0-H=0_-}^{H_0-H=0_+} \delta_D(H_0 - H)\ dH = 1. \qquad (6.1.190)$$

0_+ and 0_- denote infinitesimal positive and negative increments with respect to $H_0 - H = 0$. The local number density of stars n_d is obtained by integrating the distribution function f over the velocity space V_v [e.g. Ogorodnikov 1965, Eq. (2.1)]:

$$n_d = n_d(\vec{r}) = \int_{V_v} f(\vec{r},\vec{v})\ dV_v. \qquad (6.1.191)$$

And if m_m denotes the mean mass of a star, the local spread-out mass density of stars becomes with the distribution function (6.1.188):

$$\varrho = \varrho(\vec{r}) = n_d m_m = m_m\int_{V_v} f(\vec{r},\vec{v})\ dV_v = 4\pi m_m C\int_0^\infty (H_0 - H)^{n-3/2}v^2\ dv$$
$$= 4\pi m_m C\int_0^\infty (H_0 - v^2/2 + \Phi + \Omega^2\ell^2/2)^{n-3/2}v^2\ dv = 4\pi m_m C\psi^{n-3/2}\int_0^\infty (1 - v^2/2\psi)^{n-3/2}v^2\ dv$$
$$= 2^{5/2}\pi m_m C\psi^n\int_0^1 x^{1/2}(1-x)^{n-3/2}\ dx = 2^{5/2}\pi m_m C\ B(3/2, n-1/2)\ \psi^n = \varrho_0\psi^n,$$
$$(\psi = H_0 + \Phi + \Omega^2\ell^2/2 \geq 0;\ dV_v = 4\pi v^2\ dv;\ 0 \leq v^2/2\psi \equiv x \leq 1;$$
$$\varrho_0 = 2^{5/2}\pi m_m C\ B(3/2, n-1/2);\ n > 0.5). \qquad (6.1.192)$$

$B(p,q)$ denotes Euler's complete beta function from Eq. (5.10.96). Finiteness of the beta function demands $p,q>0$, i.e. $n>0.5$. If $n=0.5$, we get via the distribution function (6.1.189):

$$\varrho = n_d m_m = 4\pi m_m C\int_0^\infty \delta_D(H_0-H)\ v^2\ dv, \qquad (n=0.5). \tag{6.1.193}$$

Since $\delta_D(H_0-H)=0$ if $H_0-H\neq 0$, the above integral is limited to the immediate vicinity of $H_0-H=0$, i.e. in virtue of Eq. (6.1.187) to $v^2/2 = H+\Phi+\Omega^2\ell^2/2 \approx H_0+\Phi+\Omega^2\ell^2/2=\psi$. Thus, $v\approx 2^{1/2}\psi^{1/2}$ and $v\ dv = dH$, since Φ and $\Omega^2\ell^2/2$ are independent of v. Inserting into Eq. (6.1.193), we find

$$\varrho = 2^{5/2}\pi m_m C\psi^{1/2}\int_{H_0-H=0_-}^{H_0-H=0_+}\delta_D(H_0-H)\ dH = 2^{5/2}\pi m_m C\psi^{1/2} = \varrho_0\psi^{1/2},$$
$$(n=0.5;\ \psi=H_0+\Phi+\Omega^2\ell^2/2;\ \varrho_0=2^{5/2}\pi m_m C). \tag{6.1.194}$$

Hence, the density distribution over coordinate space forms the simple family

$$\varrho=\varrho(\vec r)=\varrho_0\psi^n(\vec r), \qquad (n\geq 0.5;\ \psi(\vec r)=H_0+\Phi(\vec r)+\Omega^2\ell^2/2;\ H_0,\varrho_0=\mathrm{const}). \tag{6.1.195}$$

And Poisson's equation (2.1.4) writes

$$\nabla^2\Phi(\vec r)=\nabla^2(\psi-H_0-\Omega^2\ell^2/2)=\nabla^2\psi-2\Omega^2=-4\pi G\varrho=-4\pi G\varrho_0\psi^n. \tag{6.1.196}$$

The kinetic pressure in the macroscopic "stellar gas" arises via Eqs. (5.10.96), (6.1.192), (C.11) from kinetic motions of the stars with mean square velocity v_m^2 :

$$v_m^2=\int_{V_v}v^2 f(\vec r,\vec v)\ dV_v\Big/\int_{V_v}f(\vec r,\vec v)\ dV_v=(4\pi m_m C/\varrho)\int_0^\infty v^4(H_0-H)^{n-3/2}\ dv$$
$$=(4\pi m_m C\psi^{n-3/2}/\varrho)\int_0^\infty(1-v^2/2\psi)^{n-3/2}v^4\ dv=(2^{7/2}\pi m_m C\psi^{n+1}/\varrho)\int_0^1 x^{3/2}(1-x)^{n-3/2}\ dx$$
$$=2\psi\ B(5/2,n-1/2)\big/B(3/2,n-1/2)=2\psi\ \Gamma(5/2)\ \Gamma(n+1)\big/\Gamma(3/2)\ \Gamma(n+2)=3\psi/(n+1),$$
$$(n>0.5). \tag{6.1.197}$$

If $n=0.5$, we find in the same manner via Eqs. (6.1.190), (6.1.193), (6.1.194):

$$v_m^2=(4\pi m_m C/\varrho)\int_0^\infty v^4\ \delta_D(H_0-H)\ dv=(2^{7/2}\pi m_m C\psi^{3/2}/\varrho)\int_{H_0-H=0_-}^{H_0-H=0_+}\delta_D(H_0-H)\ dH$$
$$=2\psi, \qquad (n=0.5). \tag{6.1.198}$$

The kinetic pressure is given by Eq. (1.7.37), and obeys the polytropic law

$$P=\varrho v_m^2/3=\varrho\psi/(n+1)=[1/(n+1)\varrho_0^{1/n}]\varrho^{1+1/n}=K\varrho^{1+1/n},$$
$$(v_m\ll c;\ n\geq 0.5;\ K=1/(n+1)\varrho_0^{1/n}). \tag{6.1.199}$$

The boundary of the configuration is the surface on which ϱ and ψ vanish, being determined by the condition $\psi=0$ or $H_0=-\Phi-\Omega^2\ell^2/2$, since $\psi=H_0+\Phi+\Omega^2\ell^2/2$.

As noted at the end of Sec. 5.8.3, axisymmetric polytropes with polytropic index $0\leq n\leq 0.808$ are always secularly unstable beyond the bifurcation point of the nonaxisymmetric sequence, occurring at $\tau_b\approx 0.14$. Vandervoort (1980a) speculates that the nonaxisymmetric configurations branching off at $\tau_b\approx 0.14$, resemble galactic bars. Indeed, if a_1,a_2,a_3, $(a_1\geq a_2\geq a_3)$ denote the semimajor axes of triaxial ellipsoids, Table 1 of Vandervoort (1980a) shows that for the limiting nonaxisymmetric polytrope of index $n=0.5$ and 0.7, the axis a_1 is about two times larger than a_2, and three times larger than a_3, exhibiting to some extent a barlike shape.

On the other hand, although triaxial polytropes with index $0.5\leq n\leq 0.808$ also resemble the shapes of some elliptical galaxies, they cannot account for the observed general absence of more rapidly rotating elliptical galaxies, because triaxial polytropes are rotating rapidly with $\beta_c=\Omega_c^2/2\pi G\varrho_0\approx 0.13$ [cf. Table 3.8.1, and Table 1 of Vandervoort (1980a)]. In this respect Caimmi (1980a, Fig. 2) has shown that the oblateness (ellipticity) and rotation rate of most ellipticals is consistent with axisymmetric (biaxial), rigidly rotating polytropes of index $3 < n < 5$. Massive halos may serve as an explanation for the few observation points located outside the limiting curves of these axisymmetric, uniformly rotating polytropes. Polytropes may also be suitable models for the central bulges of spiral galaxies. And a spherical polytrope of index $n = 4$ can closely approximate a centrally condensed globular galaxy (Alladin 1965, Subrahmanyam 1980, Chatterjee 1987, Subrahmanyam and Narasimhan 1989). Its polytropic gravitational potential, as tabulated by Limber (1961), follows from the integration of Eq. (2.1.7):

$$\Phi = (n+1)K\varrho^{1/n} + \mathrm{const} = (n+1)K\varrho_0^{1/n}\theta + \mathrm{const} = (n+1)K\varrho_0^{1/n}(-\xi_1\theta_1' + \theta)$$
$$= [GM_1/r_1(-\xi_1\theta_1')](-\xi_1\theta_1' + \theta) = (GM_1/r_1)(1 + 3\varrho_0\theta/\xi_1^2\varrho_m), \qquad (-1 < n \le 5;\ N = 3). \quad (6.1.200)$$

The integration constant has been determined via Eqs. (2.6.1), (2.6.18), (2.6.43) from the continuity of the potential across the boundary $\theta(\xi_1) = 0$ of the sphere: const $= \Phi(r_1) = \Phi_e(r_1) = GM_1/r_1 = -(n+1)K\varrho_0^{1/n}\xi_1\theta_1'$.

Oscillations of spherical galaxies with a polytropic density distribution ($0 \le n \le 4$) have been simulated by Namboodiri (2000), and collisions between spherical $n = 0, 4$ galaxies by Namboodiri et al. (2001).

The stability and normal modes of oscillation of collisionless polytropic stellar systems have been investigated by Samimi and Sobouti (1995) with the aid of the linearized Liouville equation: The oscillations are stable, with periods of the order of free fall time scales. The post Newtonian approximation of Liouville's equation in spherical polytropes yields a new sequence of relativistic modes, which are similar to toroidal modes [Eq. (5.8.166)], and degenerate to zero frequency in the Newtonian limit (Rezania and Sobouti 2000, Sobouti and Rezania 2000).

Caimmi and Dallaporta (1978), and Caimmi (1986) have considered galactic models based on two-component polytropes (cf. Sec. 2.8.2 for two-component isothermal spheres with $n = \pm\infty$). The first order perturbation theory of Chandrasekhar (1933a), as rectified by Chandrasekhar and Lebovitz (1962d), has been extended by Caimmi and Dallaporta (1978) to two coaxial, homocentric, uniformly rotating polytropes with *nonintersecting* boundary surfaces. The polytropic indices of the two components are denoted by n_I and n_{II}, and the two radii of the nonrotating spherical polytropes are a_I and a_{II}, ($a_I < a_{II}$).

In the region of the inner spheroid a_I both polytropic components are assumed to be present, while the outer region is filled only with component II. In spherical coordinates the equations of hydrostatic equilibrium to be satisfied by both polytropic spheroids are by virtue of Dalton's law of partial pressures equal to [cf. Eqs. (2.8.54), (3.8.2)]

$$\partial P_j/\partial r = \varrho_j\ \partial\Phi/\partial r + \varrho_j\Omega_j^2 r(1-\mu^2); \quad \partial P_j/\partial\mu = \varrho_j\ \partial\Phi/\partial\mu - \varrho_j\Omega_j^2 r^2\mu, \qquad (j = I, II). \quad (6.1.201)$$

Φ is the total internal gravitational potential, and P_j, ϱ_j, Ω_j denote pressure, density, and constant angular rotation velocity of the j-th component, respectively. Let us first consider the region of the inner spheroid, Poisson's equation (3.8.1) becoming

$$\partial(r^2\ \partial\Phi/\partial r)/\partial r + \partial[(1-\mu^2)\ \partial\Phi/\partial\mu]/\partial\mu = -4\pi G r^2(\varrho_I + \varrho_{II}), \qquad (r \le a_I). \quad (6.1.202)$$

We insert for the derivatives of Φ from Eq. (6.1.201), obtaining the two fundamental equations of the problem in the region I of the inner spheroid:

$$\partial[(r^2/\varrho_j)\ \partial P_j/\partial r]/\partial r + \partial\{[(1-\mu^2)/\varrho_j]\ \partial P_j/\partial\mu\}/\partial\mu = -4\pi G r^2(\varrho_I + \varrho_{II}) + 2\Omega_j^2 r^2, \quad (j = I, II). \quad (6.1.203)$$

Dimensionless variables are introduced in a similar manner as in Eqs. (3.2.1) and (3.2.3):

$$r = \alpha_I\xi_I = \alpha_{II}\xi_{II}; \quad \alpha_j = [(n_j+1)K_j/4\pi G\varrho_{0j}^{-1/n_j}(\varrho_{0I} + \varrho_{0II})]^{1/2}; \quad \varrho_j = \varrho_{0j}\Theta_j^{n_j};$$
$$P_j = K_j\varrho_{0j}^{1+1/n_j}\Theta_j^{n_j+1}; \quad \beta_j = \Omega_j^2/2\pi G(\varrho_{0I} + \varrho_{0II}) \ll 1, \qquad (j = I, II). \quad (6.1.204)$$

Eq. (6.1.203) becomes in the region I of the inner spheroid equal to

$$\partial(\xi_j^2\ \partial\Theta_j/\partial\xi_j)/\partial\xi_j + \partial[(1-\mu^2)\ \partial\Theta_j/\partial\mu]/\partial\mu = -\xi_j^2(\varrho_{0I}\Theta_I^{n_I} + \varrho_{0II}\Theta_{II}^{n_{II}})/(\varrho_{0I} + \varrho_{0II}) + \beta_j\xi_j^2,$$
$$(j = I, II;\ 0 \le r \le a_I). \quad (6.1.205)$$

In the nonrotating spherical case we denote $\Theta_j(\xi_j,\mu)$ by $\theta_j(\xi_j)$, and Eq. (6.1.205) turns into

$$d(\xi_j^2\ d\theta_j/d\xi_j)/d\xi_j = -\xi_j^2(\varrho_{0I}\theta_I^{n_I} + \varrho_{0II}\theta_{II}^{n_{II}})/(\varrho_{0I} + \varrho_{0II}), \qquad (r \le a_I;\ \beta_j = 0). \qquad (6.1.206)$$

Only the second component is present in region II outside the inner sphere:

$$d(\xi_{II}^2\ d\theta_{II}/d\xi_{II})/d\xi_{II} = -\xi_{II}^2\varrho_{0II}\theta_{II}^{n_{II}}/(\varrho_{0I} + \varrho_{0II}), \qquad (a_I \le r \le a_{II};\ \theta_I, \beta_{II} = 0). \qquad (6.1.207)$$

The foregoing two equations are the equivalents of the Lane-Emden equation (2.3.87) in the case of the considered spherical two-component polytrope.

If $r = a_I$, the first zero $\theta_I(\xi_{1I}) = 0$ of Eq. (6.1.206) occurs at ξ_{1I}, or at

$$\xi_{iII} = \alpha_I\xi_{1I}/\alpha_{II}, \qquad (r = \alpha_I\xi_I = \alpha_{II}\xi_{II}), \qquad (6.1.208)$$

in the dimensionless units of the outer spheroid II. If $r = a_{II}$, the first zero $\theta_{II}(\xi_{1II}) = 0$ of Eq. (6.1.207) occurs at ξ_{1II}, constituting the boundary surface of the spherical two-component polytrope.

In the dimensionless units of the rotating outer spheroid II the boundary $\Theta_I[\Xi_{iII}(\mu),\mu] = 0$ of the inner spheroid I is considered by Caimmi and Dallaporta (1978) under the form

$$\Xi_{iII}(\mu) = \xi_{iII} + (\beta_I - \beta_{II})\sum_{k=0}^{\infty} p_k P_k(\mu) + \beta_{II}\sum_{k=0}^{\infty} q_k P_k(\mu), \qquad (p_k, q_k = \text{const}), \qquad (6.1.209)$$

while the surface $\Theta_{II}(\Xi_{1II}(\mu),\mu) = 0$ of the two-component polytrope (of the outer spheroid II) is taken as

$$\Xi_{1II}(\mu) = \xi_{1II} + (\beta_I - \beta_{II})\sum_{k=0}^{\infty} s_k P_k(\mu) + \beta_{II}\sum_{k=0}^{\infty} t_k P_k(\mu), \qquad (s_k, t_k = \text{const}), \qquad (6.1.210)$$

where $P_k(\mu)$ denotes the Legendre polynomial of order k. Analogously to Eq. (3.1.74) we can integrate the set (6.1.201) to obtain the prime integral

$$\Phi = (n_j + 1)P_j/\varrho_j - \Omega_j^2 r^2(1 - \mu^2)/2 + \Phi_{pj}$$
$$= 4\pi G\alpha_j^2(\varrho_{0I} + \varrho_{0II})\{\Theta_j(\xi_j,\mu) - \beta_j\xi_j^2[1 - P_2(\mu)]/6\} + \Phi_{pj}, \qquad (j = I, II). \qquad (6.1.211)$$

Φ_{pj} denotes the value of the internal potential Φ at the poles of the respective spheroid. Writing Eq. (6.1.211) separately for $j = I, II$ at the pole $r = \alpha_{II}\Xi_{iII}(\pm 1)$ of the inner spheroid, and equating the results, we get

$$\Phi_{pI} = 4\pi G\alpha_{II}^2(\varrho_{0I} + \varrho_{0II})\ \Theta_{II}(\Xi_{iII}, \pm 1) + \Phi_{pII}, \qquad (\Theta_I(\Xi_{iII}, \pm 1) = 0). \qquad (6.1.212)$$

With the help of Eq. (6.1.212) we get from Eq. (6.1.211) the basic relationship inside spheroid I between the fundamental functions of the two spheroids in the system of units of the outer spheroid II:

$$\Theta_I(\xi_I,\mu) = \Theta_I(\xi_{II},\mu) = (\alpha_{II}/\alpha_I)^2\{\Theta_{II}(\xi_{II},\mu) + (\beta_I - \beta_{II})\xi_{II}^2[1 - P_2(\mu)]/6 - \Theta_{II}(\Xi_{iII}, \pm 1)\},$$
$$(\xi_I = \alpha_{II}\xi_{II}/\alpha_I). \qquad (6.1.213)$$

The initial conditions are, as for the one-component rotating polytrope, equal to

$$\Theta_j(0,\mu) = 1; \quad (\partial\Theta_j/\partial\xi_j)_{\xi_j=0} = 0, \qquad (j = I, II). \qquad (6.1.214)$$

If we take in Eq. (6.1.213) $\xi_{II} = 0$ and $\mu = \pm 1$, we get with the initial conditions $\Theta_j(0, \pm 1) = 1$:

$$\Theta_{II}(\Xi_{iII}, \pm 1) = 1 - (\alpha_I/\alpha_{II})^2. \qquad (6.1.215)$$

If $j = II$, Eq. (6.1.205) reads with the substitutions (6.1.213) and (6.1.215) as

$$\partial(\xi_{II}^2\ \partial\Theta_{II}/\partial\xi_{II})/\partial\xi_{II} + \partial[(1 - \mu^2)\ \partial\Theta_{II}/\partial\mu]/\partial\mu = -\xi_{II}^2\{\varrho_{0I}[(\alpha_{II}/\alpha_I)^2(\Theta_{II} - 1) + 1]^{n_I}$$
$$+\varrho_{0II}\Theta_{II}^{n_{II}}\}/(\varrho_{0I} + \varrho_{0II}) - \xi_{II}^4 n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2(\beta_I - \beta_{II})[1 - P_2(\mu)][(\alpha_{II}/\alpha_I)^2(\Theta_{II} - 1) + 1]^{n_I - 1}$$
$$/6(\varrho_{0I} + \varrho_{0II}) + \beta_{II}\xi_{II}^2. \qquad (6.1.216)$$

Following Chandrasekhar (1933a) we assume inside the inner spheroid I a solution of the form

$$\Theta_{II}(\xi_{II},\mu) = \theta_{II}(\xi_{II}) + (\beta_I - \beta_{II})\left[\varphi_0(\xi_{II}) + \sum_{k=1}^{\infty} A_k\varphi_k(\xi_{II})\ P_k(\mu)\right]$$

$$+\beta_{II}\left[\psi_0(\xi_{II}) + \sum_{k=1}^{\infty} B_k\psi_k(\xi_{II})\ P_k(\mu)\right], \qquad (A_k, B_k = \mathrm{const}). \qquad (6.1.217)$$

In the nonrotating case we have $\Theta_j(\xi_{II},\mu) = \theta_j(\xi_{II})$, and Eq. (6.1.213) becomes via Eq. (6.1.215) equal to

$$\theta_I(\xi_I) = \theta_I(\xi_{II}) = (\alpha_{II}/\alpha_I)^2[\theta_{II}(\xi_{II}) - 1] + 1, \qquad (\beta_j = 0). \qquad (6.1.218)$$

In the region between the inner spheroid I and the surface of spheroid II we have by assumption $\Theta_I(\xi_{II},\mu) = 0$, and Eq. (6.1.205) becomes

$$\partial(\xi_{II}^2\ \partial\Theta_{II}/\partial\xi_{II})/\partial\xi_{II} + \partial[(1-\mu^2)\ \partial\Theta_{II}/\partial\mu]/\partial\mu = -\xi_{II}^2\varrho_{0II}\Theta_{II}^{n_{II}}/(\varrho_{0I} + \varrho_{0II}) + \beta_{II}\xi_{II}^2, \qquad (6.1.219)$$

with an assumed solution

$$\Theta_{II}(\xi_{II},\mu) = \theta_{II}(\xi_{II}) + (\beta_I - \beta_{II})\left[\sigma_0(\xi_{II}) + \sum_{k=1}^{\infty} C_k\sigma_k(\xi_{II})\ P_k(\mu)\right]$$

$$+\beta_{II}\left[\tau_0(\xi_{II}) + \sum_{k=1}^{\infty} D_k\tau_k(\xi_{II})\ P_k(\mu)\right], \qquad (C_k, D_k = \mathrm{const}). \qquad (6.1.220)$$

Clearly, the gravitational potential in this region is given by Eq. (6.1.211) if $j = II$.

In the external region outside the boundary (6.1.210) of the two-component polytrope we have $\Theta_j(\xi,\mu) = 0$. The external potential is assumed under a form similar to Eq. (3.2.33):

$$\Phi_e = 4\pi G\alpha_{II}^2(\varrho_{0I} + \varrho_{0II})\left[g_0/\xi_{II} + (\beta_I - \beta_{II})\sum_{k=0}^{\infty} g_{1k}P_k(\mu)/\xi_{II}^{k+1} + \beta_{II}\sum_{k=0}^{\infty} h_{1k}P_k(\mu)/\xi_{II}^{k+1}\right],$$

$$(g_0, g_{1k}, h_{1k} = \mathrm{const}). \qquad (6.1.221)$$

We now substitute Eqs. (6.1.217) and (6.1.220) into Eqs. (6.1.216) and (6.1.219), respectively. Equating separately to zero the coefficients of $(\beta_I - \beta_{II})P_k(\mu)$ and $\beta_{II}P_k(\mu)$, we obtain a set of differential equations whose solutions allow us to determine all the unknown functions $\varphi_k, \psi_k, \sigma_k, \tau_k$. Inside spheroid I we get via Eq. (6.1.218)

$$d(\xi_{II}^2\ d\theta_{II}/d\xi_{II})/d\xi_{II} = -\xi_{II}^2\{\varrho_{0I}[(\alpha_{II}/\alpha_I)^2(\theta_{II} - 1) + 1]^{n_I} + \varrho_{0II}\theta_{II}^{n_{II}}\}/(\varrho_{0I} + \varrho_{0II})$$
$$= -\xi_{II}^2(\varrho_{0I}\theta_I^{n_I} + \varrho_{0II}\theta_{II}^{n_{II}})/(\varrho_{0I} + \varrho_{0II}), \qquad (6.1.222)$$

$$d(\xi_{II}^2\ d\varphi_0/d\xi_{II})/d\xi_{II} = -\xi_{II}^2\varphi_0[n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1} + n_{II}\varrho_{0II}\theta_{II}^{n_{II}-1}]/(\varrho_{0I} + \varrho_{0II})$$
$$-\xi_{II}^4 n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1}/6(\varrho_{0I} + \varrho_{0II}), \qquad (6.1.223)$$

$$d(\xi_{II}^2\ d\varphi_2/d\xi_{II})/d\xi_{II} - 6\varphi_2 = -\xi_{II}^2\varphi_2[n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1} + n_{II}\varrho_{0II}\theta_{II}^{n_{II}-1}]/(\varrho_{0I} + \varrho_{0II})$$
$$+\xi_{II}^4 n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1}/6A_2(\varrho_{0I} + \varrho_{0II}), \qquad (6.1.224)$$

$$d(\xi_{II}^2\ d\varphi_k/d\xi_{II})/d\xi_{II} - k(k+1)\varphi_k = -\xi_{II}^2\varphi_k[n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1} + n_{II}\varrho_{0II}\theta_{II}^{n_{II}-1}]$$
$$/(\varrho_{0I} + \varrho_{0II}), \qquad (k = 4, 6, 8, ...), \qquad (6.1.225)$$

$$d(\xi_{II}^2\ d\psi_0/d\xi_{II})/d\xi_{II} = -\xi_{II}^2\psi_0[n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1} + n_{II}\varrho_{0II}\theta_{II}^{n_{II}-1}]$$
$$/(\varrho_{0I} + \varrho_{0II}) + \xi_{II}^2, \qquad (6.1.226)$$

$$d(\xi_{II}^2\ d\psi_k/d\xi_{II})/d\xi_{II} - k(k+1)\psi_k = -\xi_{II}^2\psi_k[n_I\varrho_{0I}(\alpha_{II}/\alpha_I)^2\theta_I^{n_I-1} + n_{II}\varrho_{0II}\theta_{II}^{n_{II}-1}]$$
$$/(\varrho_{0I}+\varrho_{0II}), \qquad (k=2,4,6,...). \tag{6.1.227}$$

The initial conditions imposed on Eqs. (6.1.222)-(6.1.227) are

$$\theta_I(0),\ \theta_{II}(0)=1;\quad \theta_I'(0),\ \theta_{II}'(0),\ \varphi_k(0),\ \varphi_k'(0),\ \psi_k(0),\ \psi_k'(0)=0, \qquad (k=0,2,4,...). \tag{6.1.228}$$

At the boundary of spheroid I there is $\theta_I(\xi_{iII})=0$, and $\theta_{II}(\xi_{iII})=1-(\alpha_I/\alpha_{II})^2$ via Eq. (6.1.218). At the outer surface boundary we have $\theta_{II}(\xi_{1II})=0$. Eqs. (6.1.222)-(6.1.227) can be integrated numerically, and the solutions are functions only of α_{II}/α_I and $\varrho_{0I},\varrho_{0II}$.

In the region between spheroid I and the surface of the two-component polytrope we get analogously, by inserting Eq. (6.1.220) into Eq. (6.1.219):

$$d(\xi_{II}^2\ d\theta_{II}/d\xi_{II})/d\xi_{II} = -\varrho_{0II}\xi_{II}^2\theta_{II}^{n_{II}}/(\varrho_{0I}+\varrho_{0II}), \tag{6.1.229}$$

$$d(\xi_{II}^2\ d\sigma_k/d\xi_{II})/d\xi_{II} - k(k+1)\sigma_k = -n_{II}\varrho_{0II}\xi_{II}^2\theta_{II}^{n_{II}-1}\sigma_k/(\varrho_{0I}+\varrho_{0II}), \qquad (k=0,2,4,6,...), \tag{6.1.230}$$

$$d(\xi_{II}^2\ d\tau_0/d\xi_{II})/d\xi_{II} = -n_{II}\varrho_{0II}\xi_{II}^2\theta_{II}^{n_{II}-1}\tau_0/(\varrho_{0I}+\varrho_{0II}) + \xi_{II}^2, \tag{6.1.231}$$

$$d(\xi_{II}^2\ d\tau_k/d\xi_{II})/d\xi_{II} - k(k+1)\tau_k = -n_{II}\varrho_{0II}\xi_{II}^2\theta_{II}^{n_{II}-1}\tau_k/(\varrho_{0I}+\varrho_{0II}), \qquad (k=2,4,6,...). \tag{6.1.232}$$

The initial conditions of these equations are

$$\theta_{II}(0)=1;\quad \theta_{II}'(0),\ \sigma_k(0),\ \sigma_k'(0),\ \tau_k(0),\ \tau_k'(0)=0, \qquad (k=0,2,4,...). \tag{6.1.233}$$

We now turn to the calculation of the surface constants p_k, q_k, s_k, t_k from Eqs. (6.1.209) and (6.1.210), in analogy to Eqs. (3.2.34)-(3.2.38). At the boundary $\Xi_{iII}(\mu)$ of spheroid I, Eq. (6.1.213) becomes via Eqs. (6.1.209), (6.1.215), (6.1.217), (6.1.218) equal to

$$\Theta_I[\Xi_{iII}(\mu),\mu] = (\alpha_{II}/\alpha_I)^2\{\Theta_{II}[\Xi_{iII}(\mu),\mu] + (\beta_I-\beta_{II})\ \Xi_{iII}^2(\mu)\ [1-P_2(\mu)]/6 - 1 + (\alpha_I/\alpha_{II})^2\}$$
$$= (\alpha_{II}/\alpha_I)^2\bigg\{\theta_{II}[\Xi_{iII}(\mu)] + (\beta_I-\beta_{II})\bigg[\varphi_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty}A_k\varphi_k[\Xi_{iII}(\mu)]\ P_k(\mu)\bigg]$$
$$+\beta_{II}\bigg[\psi_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty}B_k\psi_k[\Xi_{iII}(\mu)]\ P_k(\mu)\bigg] + (\beta_I-\beta_{II})\ \Xi_{iII}^2(\mu)\ [1-P_2(\mu)]/6 - 1$$
$$+(\alpha_I/\alpha_{II})^2\bigg\} \approx (\alpha_{II}/\alpha_I)^2\bigg\{\theta_{II}(\xi_{iII}) + [\Xi_{iII}(\mu)-\xi_{iII}]\ \theta_{II}'(\xi_{iII})$$
$$+(\beta_I-\beta_{II})\bigg[\varphi_0(\xi_{iII}) + \sum_{k=1}^{\infty}A_k\varphi_k(\xi_{iII})\ P_k(\mu)\bigg] + \beta_{II}\bigg[\psi_0(\xi_{iII}) + \sum_{k=1}^{\infty}B_k\psi_k(\xi_{iII})\ P_k(\mu)\bigg]$$
$$+(\beta_I-\beta_{II})\xi_{iII}^2[1-P_2(\mu)]/6 - 1 + (\alpha_I/\alpha_{II})^2\bigg\} = \theta_I(\xi_{iII}) + \theta_I'(\xi_{iII})\bigg[(\beta_I-\beta_{II})\sum_{k=0}^{\infty}p_kP_k(\mu)$$
$$+\beta_{II}\sum_{k=0}^{\infty}q_kP_k(\mu)\bigg] + (\alpha_{II}/\alpha_I)^2\bigg\{(\beta_I-\beta_{II})\bigg[\varphi_0(\xi_{iII}) + \sum_{k=1}^{\infty}A_k\varphi_k(\xi_{iII})\ P_k(\mu)\bigg] + \beta_{II}\bigg[\psi_0(\xi_{iII})$$
$$+\sum_{k=1}^{\infty}B_k\psi_k(\xi_{iII})\ P_k(\mu)\bigg] + (\beta_I-\beta_{II})\xi_{iII}^2[1-P_2(\mu)]/6\bigg\} = 0, \qquad [\theta_I'(\xi_{iII}) = (\alpha_{II}/\alpha_I)^2\theta_{II}'(\xi_{iII})]. \tag{6.1.234}$$

To satisfy Eq. (6.1.234), the coefficients connected with the same $(\beta_I - \beta_{II})P_k(\mu)$ and $\beta_{II}P_k(\mu)$ must vanish:

$$\theta_I(\xi_{iII}) = \theta_I(\xi_{1I}) = 0; \quad p_0 = -(\alpha_{II}/\alpha_I)^2[\varphi_0(\xi_{iII}) + \xi_{iII}^2/6]/\theta_I'(\xi_{iII}) = -[\varphi_0(\xi_{iII}) + \xi_{iII}^2/6] \\ /\theta_{II}'(\xi_{iII}); \quad p_2 = -(\alpha_{II}/\alpha_I)^2[A_2\varphi_2(\xi_{iII}) - \xi_{iII}^2/6]/\theta_I'(\xi_{iII}) = -[A_2\varphi_2(\xi_{iII}) - \xi_{iII}^2/6]/\theta_{II}'(\xi_{iII}); \\ p_k = -(\alpha_{II}/\alpha_I)^2 A_k\varphi_k(\xi_{iII})/\theta_I'(\xi_{iII}) = -A_k\varphi_k(\xi_{iII})/\theta_{II}'(\xi_{iII}) \quad \text{if} \quad k = 4,6,8,...; \\ q_0 = -(\alpha_{II}/\alpha_I)^2\psi_0(\xi_{iII})/\theta_I'(\xi_{iII}) = -\psi_0(\xi_{iII})/\theta_{II}'(\xi_{iII}); \\ q_k = -(\alpha_{II}/\alpha_I)^2 B_k\psi_k(\xi_{iII})/\theta_I'(\xi_{iII}) = -B_k\psi_k(\xi_{iII})/\theta_{II}'(\xi_{iII}) \quad \text{if} \quad k = 2,4,6,... \qquad (6.1.235)$$

On the outer boundary of the two-component polytrope (spheroid II) we have by virtue of Eqs. (6.1.210) and (6.1.220)

$$\Theta_{II}[\Xi_{1II}(\mu),\mu] = \theta_{II}[\Xi_{1II}(\mu)] + (\beta_I - \beta_{II})\left\{\sigma_0[\Xi_{1II}(\mu)] + \sum_{k=1}^{\infty} C_k\sigma_k[\Xi_{1II}(\mu)]\ P_k(\mu)\right\} \\ +\beta_{II}\left\{\tau_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty} D_k\tau_k[\Xi_{1II}(\mu)]\ P_k(\mu)\right\} \approx \theta_{II}(\xi_{1II}) + [\Xi_{1II}(\mu) - \xi_{1II}]\theta_{II}'(\xi_{1II}) \\ +(\beta_I - \beta_{II})\left[\sigma_0(\xi_{1II}) + \sum_{k=1}^{\infty} C_k\sigma_k(\xi_{1II})\ P_k(\mu)\right] + \beta_{II}\left[\tau_0(\xi_{1II}) + \sum_{k=1}^{\infty} D_k\tau_k(\xi_{1II})\ P_k(\mu)\right] \\ = \theta_{II}(\xi_{1II}) + \theta_{II}'(\xi_{1II})\left[(\beta_I - \beta_{II})\sum_{k=0}^{\infty} s_kP_k(\mu) + \beta_{II}\sum_{k=0}^{\infty} t_kP_k(\mu)\right] \\ +(\beta_I - \beta_{II})\left[\sigma_0(\xi_{1II}) + \sum_{k=1}^{\infty} C_k\sigma_k(\xi_{1II})\ P_k(\mu)\right] + \beta_{II}\left[\tau_0(\xi_{1II}) + \sum_{k=1}^{\infty} D_k\tau_k(\xi_{1II})\ P_k(\mu)\right] = 0. \qquad (6.1.236)$$

Proceeding exactly as with Eq. (6.1.234), we get

$$\theta_{II}(\xi_{1II}) = 0; \quad s_0 = -\sigma_0(\xi_{1II})/\theta_{II}'(\xi_{1II}); \quad s_k = -C_k\sigma_k(\xi_{1II})/\theta_{II}'(\xi_{1II}); \\ t_0 = -\tau_0(\xi_{1II})/\theta_{II}'(\xi_{1II}); \quad t_k = -D_k\tau_k(\xi_{1II})/\theta_{II}'(\xi_{1II}), \qquad (k = 2,4,6,...). \qquad (6.1.237)$$

The continuity of density and of its first radial derivative on the interface $\Xi_{iII}(\mu)$ requires equality between Eqs. (6.1.217) and (6.1.220), as well as between the respective derivatives on the boundary $\Xi_{iII}(\mu)$ of spheroid I:

$$\Theta_{II}[\Xi_{iII}(\mu),\mu] = \theta_{II}[\Xi_{iII}(\mu)] + (\beta_I - \beta_{II})\left\{\varphi_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty} A_k\varphi_k[\Xi_{iII}(\mu)]\ P_k(\mu)\right\} \\ +\beta_{II}\left\{\psi_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty} B_k\psi_k[\Xi_{iII}(\mu)]\ P_k(\mu)\right\} = \theta_{II}[\Xi_{iII}(\mu)] + (\beta_I - \beta_{II})\left\{\sigma_0[\Xi_{iII}(\mu)]\right. \\ \left.+\sum_{k=1}^{\infty} C_k\sigma_k[\Xi_{iII}(\mu)]\ P_k(\mu)\right\} + \beta_{II}\left\{\tau_0[\Xi_{iII}(\mu)] + \sum_{k=1}^{\infty} D_k\tau_k[\Xi_{iII}(\mu)]\ P_k(\mu)\right\}. \qquad (6.1.238)$$

Since up to the first order we have $\varphi_k[\Xi_{iII}(\mu)] = \varphi_k(\xi_{iII}),\ \psi_k[\Xi_{iII}(\mu)] = \psi_k(\xi_{iII}),\ \sigma_k[\Xi_{iII}(\mu)] = \sigma_k(\xi_{iII}),\ \tau_k[\Xi_{iII}(\mu)] = \tau_k(\xi_{iII}),\ (k = 0,2,4,...)$, we get, by equating the corresponding terms of $(\beta_I - \beta_{II})P_k(\mu)$ and $\beta_{II}P_k(\mu)$:

$$\varphi_0(\xi_{iII}) = \sigma_0(\xi_{iII}); \quad \psi_0(\xi_{iII}) = \tau_0(\xi_{iII}); \quad A_k\varphi_k(\xi_{iII}) = C_k\sigma_k(\xi_{iII}); \\ B_k\psi_k(\xi_{iII}) = D_k\tau_k(\xi_{iII}), \qquad (k = 2,4,6,...). \qquad (6.1.239)$$

In the same way we obtain from the equality of the derivatives $[\partial\Theta_{II}(\xi_{II},\mu)/\partial\xi_{II}]_{\xi_{II}=\Xi_{iII}}$ of Eqs. (6.1.217) and (6.1.220):

$$\varphi_0'(\xi_{iII}) = \sigma_0'(\xi_{iII}); \quad \psi_0'(\xi_{iII}) = \tau_0'(\xi_{iII}); \quad A_k\varphi_k'(\xi_{iII}) = C_k\sigma_k'(\xi_{iII}); \quad B_k\psi_k'(\xi_{iII}) = D_k\tau_k'(\xi_{iII}). \qquad (6.1.240)$$

We equate analogously to Eqs. (3.2.39)-(3.2.47) the potentials (6.1.211), (6.1.221), and their derivatives on the boundaries $\Xi_{iII}(\mu)$ and $\Xi_{1II}(\mu)$ of spheroid I and II, respectively. After some algebra Eqs. (6.1.217), (6.1.213), (6.1.209), (6.1.211) take inside spheroid I the form

$$\Theta_{II}(\xi_{II},\mu) = \theta_{II}(\xi_{II}) + (\beta_I - \beta_{II})[\varphi_0(\xi_{II}) + A_2\varphi_2(\xi_{II})\ P_2(\mu)] + \beta_{II}[\psi_0(\xi_{II}) + B_2\psi_2(\xi_{II})\ P_2(\mu)], \tag{6.1.241}$$

$$\Theta_I(\xi_{II},\mu) = (\alpha_{II}/\alpha_I)^2\{\Theta_{II}(\xi_{II},\mu) + (\beta_I - \beta_{II})\xi_{II}^2[1 - P_2(\mu)]/6 - 1\} + 1, \tag{6.1.242}$$

$$\Xi_{iII}(\mu) = \xi_{iII} - (\beta_I - \beta_{II})\{\varphi_0(\xi_{iII}) + \xi_{iII}^2/6 + [A_2\varphi_2(\xi_{iII}) - \xi_{iII}^2/6]\ P_2(\mu)\}/\theta'_{II}(\xi_{iII})$$
$$-\beta_{II}[\psi_0(\xi_{iII}) + B_2\psi_2(\xi_{iII})\ P_2(\mu)]/\theta'_{II}(\xi_{iII}), \tag{6.1.243}$$

$$\Phi = 4\pi G\alpha_{II}^2(\varrho_{0I} + \varrho_{0II})\{\Theta_{II}(\xi_{II},\mu) - \beta_{II}\xi_{II}^2[1 - P_2(\mu)]/6\} + \Phi_{pII}. \tag{6.1.244}$$

In the region between spheroid I and II we have

$$\Theta_I(\xi_{II},\mu) \equiv 0; \quad \Theta_{II}(\xi_{II},\mu) = \theta_{II}(\xi_{II}) + (\beta_I - \beta_{II})[\sigma_0(\xi_{II}) + C_2\sigma_2(\xi_{II})\ P_2(\mu)]$$
$$+\beta_{II}[\tau_0(\xi_{II}) + D_2\tau_2(\xi_{II})\ P_2(\mu)], \tag{6.1.245}$$

$$\Xi_{1II}(\mu) = \xi_{1II} - (\beta_I - \beta_{II})[\sigma_0(\xi_{1II}) + C_2\sigma_2(\xi_{1II})]/\theta'_{II}(\xi_{1II})$$
$$-\beta_{II}[\tau_0(\xi_{1II}) + D_2\tau_2(\xi_{1II})]/\theta'_{II}(\xi_{1II}). \tag{6.1.246}$$

The internal potential in this region is given by Eq. (6.1.244), and the gravitational potential outside the polytrope writes via Eq. (6.1.221) as

$$\Phi_e = 4\pi G\alpha_{II}^2(\varrho_{0I} + \varrho_{0II})\{g_0/\xi_{II} + (\beta_I - \beta_{II})[g_{10}/\xi_{II} + g_{12}P_2(\mu)/\xi_{II}^3]$$
$$+\beta_{II}[h_{10}/\xi_{II} + h_{12}P_2(\mu)/\xi_{II}^3]\}. \tag{6.1.247}$$

The total mass of the two-component polytrope is given by $M = M_I + M_{II}$, where the mass $M_j,\ (j = I, II)$ of each component is in the same first order approximation as in Eq. (3.2.58), equal to

$$M_I = 2\pi\alpha_I^3\varrho_{0I}\int_0^{\xi_{1I}} \xi_I^2\ d\xi_I \int_{-1}^1 \Theta_I^{n_I}(\xi_I,\mu)\ d\mu = 2\pi\alpha_{II}^3\varrho_{0I}\int_0^{\xi_{iII}} \xi_{II}^2\ d\xi_{II}\int_{-1}^1 \Theta_I^{n_I}(\xi_{II},\mu)\ d\mu;$$
$$M_{II} = 2\pi\alpha_{II}^3\varrho_{0II}\int_0^{\xi_{1II}} \xi_{II}^2\ d\xi_{II}\int_{-1}^1 \Theta_{II}^{n_{II}}(\xi_{II},\mu)\ d\mu. \tag{6.1.248}$$

And finally, the angular momenta of the components are in the same approximation:

$$J_I = \Omega_I I_I = 2\pi\Omega_I\alpha_I^5\varrho_{0I}\int_0^{\xi_{1I}} \xi_I^4\ d\xi_I\int_{-1}^1 \Theta_I^{n_I}(\xi_I,\mu)\ (1 - \mu^2)\ d\mu$$
$$= 2\pi\Omega_I\alpha_{II}^5\varrho_{0I}\int_0^{\xi_{iII}} \xi_{II}^4\ d\xi_{II}\int_{-1}^1 \Theta_I^{n_I}(\xi_{II},\mu)\ (1 - \mu^2)\ d\mu;$$
$$J_{II} = \Omega_{II}I_{II} = 2\pi\Omega_{II}\alpha_{II}^5\varrho_{0II}\int_0^{\xi_{1II}} \xi_{II}^4\ d\xi_{II}\int_{-1}^1 \Theta_{II}^{n_{II}}(\xi_{II},\mu)\ (1 - \mu^2)\ d\mu, \tag{6.1.249}$$

where $I_j,\ (j = I, II)$ is the moment of inertia with respect to the rotation axis of the j-th component.

If the masses M_j and angular momenta J_j are supposed to be assigned, the four parameters $\beta_I, \beta_{II}, \varrho_{0I}, \varrho_{0II}$ can be determined for a given value of α_{II}/α_I, obtaining in this way the full solution for a certain physical configuration. Caimmi and Dallaporta (1978) identify spheroid II with the star-spheroidal component (halo) of a galaxy, and spheroid I with the disk component and the central bulge. The main properties of a $n_I = n_{II} = 1$ model sequence with varying mass ratio M_I/M_{II}, or changing total angular momentum $J = J_I + J_{II}$ are: The equatorial radius of the disk component (spheroid I) is smallest if M_I/M_{II} is largest, or the angular momentum J is smallest; the flattening of the disk component is independent of the ratio M_I/M_{II} and of J; the central density ϱ_{0I} and the angular velocity Ω_I is largest for the largest mass ratio M_I/M_{II} and for the smallest J. Caimmi (1986) finds that a vanishing spheroid I disk component may be efficient in flattening the outer halo component of spheroid II in the case of $n_I = n_{II} = 0$ two-component polytropes with energy dissipation and mass transfer from one component to the other.

An attempt to apply the theory of polytropic spheres to globular clusters (collisionless spherical stellar systems, as envisaged in Sec. 2.8.5) has been undertaken in 1913 by von Zeipel (see Ogorodnikov 1965, Table XI). The statistics of the distribution of stars in M2, M3, M13, and M15 would yield an average polytropic index of $n = 5.62$, and if the central parts are excluded, there would result $n \approx 5.03$. These values cannot be regarded as reliable, because polytropes having $n > 5$ have infinite mass and radius. Camm (1952) has investigated polytropic models of globular clusters, especially a modification of Plummer's $n = 5$ polytropic model [cf. Eq. (2.3.90)], involving angular momentum terms.

In globular clusters or galactic nuclei the tidal interaction during a sufficiently close encounter between two polytropic stars can divert enough orbital energy into nonradial stellar oscillations to bind the two stars into a close binary system (Press and Teukolsky 1977, Giersz 1986, Lee and Ostriker 1986, McMillan et al. 1987, Ray et al. 1987). About 40 captured $n = 3$ binaries could arise in a globular cluster with 10^4 stars (Ardakani and Sobouti 1990).

Cleary and Monaghan (1990) examine close three-body encounters between binaries and field stars, showing substantial differences between the point mass and the $n = 1.5$ polytropic approximation.

Head-on axisymmetric collisions between two identical $n = 1$ polytropes have been calculated by Rasio and Shapiro (1992, §4 and references therein). Sills and Lombardi (1997) argue that the use of polytropes as parent star models during collisions is likely to result in qualitatively mistaken outcomes for the structure of the merger.

6.2 Polytropic Atmospheres, Polytropic Clouds and Cores, Embedded Polytropes

6.2.1 Instabilities in Polytropic Atmospheres

As already evidenced by Skumanich (1955), convection in the outer solar region is turbulent in character (mixing length theory of convection), rather than laminar. Thus, the approximation of the outer convective layers of the Sun (and of other stars with outer convection zones) with a plane-parallel atmosphere of constant polytropic index is at the best a rough one. All physical quantities of the unperturbed steady state are assumed to depend only on atmospheric depth z, measured *downward* from the solar surface. In Cartesian coordinates the Eulerian perturbations of physical quantities are assumed under the form [cf. Eq. (5.10.51)]

$$\delta\vec{f}[\vec{r}(x,y,z),t] = \delta\vec{f}(z)\exp[i(\sigma t + j_x x + j_y y)]. \qquad (6.2.1)$$

The wave numbers of the horizontal wave vector along the x and y directions are j_x and j_y : $\vec{j} = \vec{j}(j_x, j_y, 0)$; $|\vec{j}| = j = (j_x^2 + j_y^2)^{1/2}$.

As an additional detriment of the polytropic approximation it turns out that the growth rate (the eigenvalue σ) of convective (gravity) g-modes tends to infinity with increasing wave number j (with decreasing eddy size $L = 2\pi/j$) like $j^{1/2}$. This holds for the most unstable mode – the fundamental mode – as well as for the overtones [Eqs. (6.2.63), (6.2.81), (6.2.82)]. The reason for this behaviour is the vanishing mass density at the top of the atmosphere. Since σ has no finite bound as the wave number increases, the smallest convective eddies are the most unstable ones. This is unlike to the behaviour in a homogeneous atmosphere ($\varrho =$ const), where the growth rate approaches a finite limit when the wave number increases (when the dimension of convective cells decreases), (e.g. Chandrasekhar 1981, Chap. II). Approximating the outermost convective regions of the Sun with a polytropic atmosphere, the above mentioned findings would imply that small-sized eddies are the most abundant and unstable ones – in apparent disagreement with the observed solar granulation pattern.

The basic equations of the problem at hand are the conservation of mass (5.2.1) or (5.2.2), and the equation of motion (5.2.10) in a constant gravity field of gravitational acceleration $g = GM_1/r_1^2$, with the z-axis directed *downward* from the free surface:

$$\varrho\ D\vec{v}/Dt = \varrho\ \partial\vec{v}/\partial t + \varrho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \varrho\ \nabla\Phi = -\nabla P + \varrho\vec{g}, \quad [\nabla\Phi = \vec{g} = \vec{g}(0,0,g)]. \qquad (6.2.2)$$

We have to add the energy conservation equation

$$\varrho\ DQ/Dt = \varrho\ DU/Dt + P\ \nabla\cdot\vec{v} - D_R. \qquad (6.2.3)$$

This equation can easily be obtained from Eq. (5.2.14), by subtracting also the dissipation function D_R considered in Eq. (5.8.142), to account for the negative work done per unit volume by viscosity forces. DQ/Dt denotes the change of heat energy per unit mass and time. In order to put separately into evidence heat changes DQ_c/Dt due to radiative (thermal) conductivity, we consider the Fourier heat conduction law for an optically thick fluid element

$$\varrho\ DQ_c/Dt = \nabla\cdot(\kappa\ \nabla T), \qquad (6.2.4)$$

where κ is the coefficient of radiative (thermal) conductivity. Subtracting Eqs. (6.2.3) and (6.2.4), we get

$$\varrho\ D(Q - Q_c)/Dt = \varrho\ DU/Dt - \nabla\cdot(\kappa\ \nabla T) + P\ \nabla\cdot\vec{v} - D_R, \qquad (6.2.5)$$

the term $D(Q - Q_c)/Dt$ representing the contribution per unit mass and time of additional heat sources or sinks.

The equation of state assumed for the polytropic atmosphere is both, that of a perfect gas and of a polytrope of index n :

$$P = \mathcal{R}\varrho T/\mu = K\varrho^{1+1/n}. \tag{6.2.6}$$

For a perfect gas with constant specific heats the internal energy per unit mass is via Eq. (1.2.19) equal to

$$U = c_V T, \qquad (c_V = \mathrm{const}), \tag{6.2.7}$$

and Eq. (6.2.5) becomes in the inviscid case ($D_R = 0$, Spiegel 1964):

$$\varrho\ D(Q - Q_c)/Dt = \varrho c_V\ DT/Dt - \nabla\cdot(\kappa\ \nabla T) + P\ \nabla\cdot\vec{v}. \tag{6.2.8}$$

If heat energy changes occur exclusively by conduction ($Q_c = Q$), the energy equation (6.2.8) is in the particular hydrostatic case ($\vec{v} = 0$) equal to the heat conduction equation (6.2.4):

$$\varrho c_V\ \partial T/\partial t = \varrho\ \partial U/\partial t = \nabla\cdot(\kappa\ \nabla T). \tag{6.2.9}$$

Eq. (6.2.2) reads in the hydrostatic case as

$$dP/dz = (1 + 1/n)K\varrho^{1/n}\ d\varrho/dz = \varrho g, \qquad (g = GM_1/r_1^2 = \mathrm{const}). \tag{6.2.10}$$

Integration with the surface conditions $\varrho = 0$ if $z = 0$ yields

$$\varrho = [g/(n + 1)K]^n z^n = Az^n, \qquad (A^{1/n} = g/(n + 1)K = \mathrm{const}). \tag{6.2.11}$$

The pressure becomes with the polytropic law (6.2.6) equal to

$$P = K\varrho^{1+1/n} = KA^{1+1/n}z^{n+1} = [Ag/(n + 1)]z^{n+1}, \tag{6.2.12}$$

while the temperature obeys the simple linear law

$$T = \mu P/\mathcal{R}\varrho = [\mu g/\mathcal{R}(n + 1)]z = (dT/dz)\ z = \beta z, \qquad (dT/dz = \beta = \mu g/\mathcal{R}(n + 1) = \mathrm{const}). \tag{6.2.13}$$

Thus, in a plane-parallel polytropic atmosphere, the atmospheric depth z formally assumes the role of the Lane-Emden function θ from Eqs. (2.6.3), (2.6.7).

Let us consider first, as an introductory exercise, the purely *vertical* propagation of waves in a polytropic plane-parallel atmosphere, the Lagrangian displacement $\Delta\vec{r}(\vec{r}, t)$ being equal to (Lamb 1945)

$$\Delta z(z, t) = \Delta z(z) \exp(i\sigma t). \tag{6.2.14}$$

It is advisable to work in this one-dimensional case with the Lagrangian equations, similarly to the purely radial motion considered in Sec. 5.2. The equation of motion (5.2.12) writes for vertical motion

$$\partial^2 z/\partial t^2 = -(1/\varrho)(\partial P/\partial z_i)\ \partial z_i/\partial z + g = -\partial P/\partial m + g, \qquad (dm = \varrho\ dz). \tag{6.2.15}$$

Let us consider the Lagrangian variations $\Delta z = z - z_u$, $\Delta P = P - P_u$, and insert them into Eq. (6.2.15), [cf. Eq. (5.2.48)]:

$$\partial^2(z_u + \Delta z)/\partial t^2 = \partial^2\Delta z/\partial t^2 = -\partial(P_u + \Delta P)/\partial m + g = -(1/\varrho_u)\ \partial(P_u + \Delta P)/\partial z_u + g$$
$$= -(1/\varrho_u)\ \partial\Delta P/\partial z_u \approx -(1/\varrho)\ \partial\Delta P/\partial z, \qquad (dP_u/dz_u = g\varrho_u;\ dm = \varrho\ dz = \varrho_u\ dz_u). \tag{6.2.16}$$

We have to add the continuity equation (5.2.28) along the vertical z-direction

$$\Delta\varrho = -\varrho\ \partial\Delta z/\partial z, \tag{6.2.17}$$

and the adiabatic energy equation (5.2.38)

$$\Delta P = -\Gamma_1 P\ \partial\Delta z/\partial z. \tag{6.2.18}$$

Substituting into Eq. (6.2.16), we find

$$\partial^2\Delta z/\partial t^2 = (\Gamma_1 P/\varrho)\ \partial^2\Delta z/\partial z^2 + (\Gamma_1/\varrho)(\partial P/\partial z)\ \partial\Delta z/\partial z = a^2\ \partial^2\Delta z/\partial z^2 + \Gamma_1 g\ \partial\Delta z/\partial z, \tag{6.2.19}$$

where $a = (\Gamma_1 P/\varrho)^{1/2} = (\Gamma_1 \mathcal{R}T/\mu)^{1/2}$ denotes the velocity of sound (2.1.49).

In the particular isothermal case ($T = \mathrm{const}$; $n = \pm\infty$) the equation of hydrostatic equilibrium

$$dP/\varrho = (\mathcal{R}T/\mu)\ d\varrho/\varrho = g\ dz, \qquad (T, g = \mathrm{const}), \tag{6.2.20}$$

integrates to

$$\varrho = C\exp(\mu g z/\mathcal{R}T), \qquad (C, T = \mathrm{const}), \tag{6.2.21}$$

where z is directed downwards. The sound velocity is constant, and Eq. (6.2.19) becomes for a perturbation of the form (6.2.14) equal to

$$a^2\ d^2\Delta z/dz^2 + \Gamma_1 g\ d\Delta z/dz + \sigma^2 = 0, \qquad (a^2 = \mathrm{const}). \tag{6.2.22}$$

The characteristic equation of this homogeneous second order differential equation with constant coefficients is

$$a^2 s^2 + \Gamma_1 g s + \sigma^2 = 0. \tag{6.2.23}$$

The two roots s_1 and s_2 of this equation yield the solution of Eq. (6.2.22) under the standard form

$$\Delta z(z,t) = [C_1\exp(s_1 z) + C_2\exp(s_2 z)]\exp(i\sigma t) = C_1\exp[i\sigma t - \Gamma_1 g z/2a^2 + (\Gamma_1^2 g^2 - 4\sigma^2 a^2)^{1/2} z$$
$$/2a^2] + C_2\exp[i\sigma t - \Gamma_1 g z/2a^2 - (\Gamma_1^2 g^2 - 4\sigma^2 a^2)^{1/2} z/2a^2], \qquad (C_1, C_2 = \mathrm{const}). \tag{6.2.24}$$

If $\sigma^2 < \Gamma_1^2 g^2/4a^2$, the whole spatial part $\Delta z(z)$ of the wave system (6.2.24) is real, leading to two standing waves in consequence of a harmonic plane source with a time factor $\exp(i\sigma t)$, [Eq. (5.1.32)]. On the other side, if $\sigma^2 > \Gamma_1^2 g^2/4a^2$ the imaginary parts $\exp\{i[\sigma t \pm (4\sigma^2 a^2 - \Gamma_1^2 g^2)^{1/2} z/2a^2]\}$ of Eq. (6.2.24) represent two stable wave systems ($\sigma^2 > 0$) with oscillations of the form $\exp[i(\sigma t \pm jz)]$, having a wave length $L = 2\pi/|j|$. The connection between the wave number j and the eigenvalue σ, ($\sigma^2 > 0$) is obtained by equating the wave number $\pm j$ with the wave number $\pm(4\sigma^2 a^2 - \Gamma_1^2 g^2)^{1/2}/2a^2$ (Lamb 1945):

$$\sigma^2 = j^2 a^2 + \Gamma_1^2 g^2/4a^2 = j^2 a^2 + (ag\mu/2\mathcal{R}T)^2, \qquad (\sigma^2 > \Gamma_1^2 g^2/4a^2;\ T = \mathrm{const}). \tag{6.2.25}$$

If the equilibrium temperature, instead of being uniform, increases downward with the uniform gradient from Eq. (6.2.13), the atmosphere has a polytropic structure. The velocity of sound is no longer constant and assumes the value

$$a^2 = \Gamma_1 P/\varrho = \Gamma_1\mathcal{R}T/\mu = \Gamma_1\mathcal{R}\beta z/\mu, \qquad (T = \beta z;\ dT/dz = \beta). \tag{6.2.26}$$

Lamb (1945) introduces the new variable

$$\tau = \int_0^z dz/a = (\mu/\Gamma_1\mathcal{R}\beta)^{1/2}\int_0^z dz/z^{1/2} = 2(\mu z/\Gamma_1\mathcal{R}\beta)^{1/2}; \quad z = \Gamma_1\mathcal{R}\beta\tau^2/4\mu. \tag{6.2.27}$$

τ is just the time that a point, moving with the local velocity of sound, would take to travel from the top $z = 0$ of the atmosphere to position z. With this new variable the equation of motion (6.2.19) writes

$$\partial^2\Delta z/\partial t^2 = \partial^2\Delta z/\partial\tau^2 + [(2n+1)/\tau]\ \partial\Delta z/\partial\tau, \qquad (g = \mathcal{R}\beta(n+1)/\mu). \tag{6.2.28}$$

If $\Delta z(\tau, t)$ is substituted via Eq. (6.2.14), we get after simplification with $\exp(i\sigma t)$ an equation similar to Eq. (2.3.7):

$$d^2\Delta z/d\tau^2 + [(2n+1)/\tau]\ d\Delta z/d\tau + \sigma^2\ \Delta z = 0. \tag{6.2.29}$$

With the change of variables outlined in Eq. (2.3.9)

$$x = \sigma\tau; \quad f = \tau^n\ \Delta z, \tag{6.2.30}$$

the second order equation (6.2.29) becomes equal to the Bessel equation

$$x^2\ d^2f/dx^2 + x\ df/dx + (x^2 - n^2)f = 0, \tag{6.2.31}$$

with the general solutions (2.3.10) and (2.3.11):

$$f(x) = C_1 J_n(x) + C_2 J_{-n}(x), \qquad (n \neq 0, 1, 2, 3, ...), \tag{6.2.32}$$

$$f(x) = C_1 J_n(x) + C_2 Y_n(x), \qquad (\text{all } n). \tag{6.2.33}$$

Then, the solutions of Eq. (6.2.29) are

$$\Delta z(\tau) = \tau^{-n}[C_1 J_n(\sigma\tau) + C_2 J_{-n}(\sigma\tau)], \qquad (n \neq 0, 1, 2, 3, ...), \tag{6.2.34}$$

$$\Delta z(\tau) = \tau^{-n}[C_1 J_n(\sigma\tau) + C_2 Y_n(\sigma\tau)], \qquad (\text{all } n). \tag{6.2.35}$$

The boundary condition (5.2.63) on the finite surface $\tau = 0$ or $z = 0$ writes via Eqs. (6.2.13), (6.2.18), (6.2.27) as

$$\Delta P = -\Gamma_1 P\ \partial\Delta z/\partial z = -[Ag\Gamma_1 z^{n+1}/(n+1)]\ \partial\Delta z/\partial z \propto \tau^{2n+1}\ \partial\Delta z/\partial\tau = 0, \qquad (\tau = 0). \tag{6.2.36}$$

Inserting the solutions (6.2.34), (6.2.35) with the series expansions (2.3.12), (2.3.13) into the boundary condition (6.2.36), the functions associated with C_2 yield a nonzero constant as $\tau \to 0$, contradicting the boundary condition (6.2.36): This implies $C_2 = 0$. Thus, the vertical oscillations in a polytropic atmosphere obey the law

$$\Delta z(z,t) = C_1(4\mu z/\Gamma_1\mathcal{R}\beta)^{-n/2} J_n[\sigma(4\mu z/\Gamma_1\mathcal{R}\beta)^{1/2}]\ \exp(i\sigma t). \tag{6.2.37}$$

We now turn to the consideration of *horizontal* disturbances propagating in the polytropic atmosphere according to the law (6.2.1). The linear Eulerian perturbations of the equation of motion (6.2.2) are by virtue of Eq. (5.1.24) equal to, ($\vec{v}_u = 0;\ \delta(D\vec{v}/Dt) \approx \Delta(D\vec{v}/Dt) = D(\Delta\vec{v}/Dt) = D\vec{v}/Dt \approx \partial\vec{v}/\partial t$) :

$$\varrho\ \partial v_x/\partial t = -\partial\delta P/\partial x; \quad \varrho\ \partial v_y/\partial t = -\partial\delta P/\partial y; \quad \varrho\ \partial v_z/\partial t = -\partial\delta P/\partial z + g\ \delta\varrho. \tag{6.2.38}$$

The continuity equation (5.2.25) writes

$$\partial\delta\varrho/\partial t + v_z\ d\varrho/dz = -\varrho\ \nabla\cdot\vec{v}. \tag{6.2.39}$$

The time derivative of the energy equation (5.2.39) becomes up to the first order in the case of adiabatic oscillations

$$\begin{aligned}\partial\delta P/\partial t &= -\Gamma_1 P\ \partial(\nabla\cdot\Delta\vec{r})/\partial t - (\partial\Delta\vec{r}/\partial t)\cdot\nabla P = -\Gamma_1 P\ \nabla\cdot\vec{v} - \varrho g v_z\\ &= -a^2\varrho\ \nabla\cdot\vec{v} - \varrho g v_z, \qquad (dP/dz = \varrho g).\end{aligned} \tag{6.2.40}$$

We eliminate the Eulerian perturbations $\delta P, \delta\varrho$ among Eqs. (6.2.38)-(6.2.40), by deriving Eq. (6.2.38) with respect to t, and inserting into Eqs. (6.2.39), (6.2.40):

$$\begin{aligned}&\partial^2 v_x/\partial t^2 = a^2\ \partial(\nabla\cdot\vec{v})/\partial x + g\ \partial v_z/\partial x; \quad \partial^2 v_y/\partial t^2 = a^2\ \partial(\nabla\cdot\vec{v})/\partial y + g\ \partial v_z/\partial y;\\ &\partial^2 v_z/\partial t^2 = \partial(a^2\ \nabla\cdot\vec{v} + g v_z)/\partial z + [(a^2/\varrho)\ d\varrho/dz - g]\ \nabla\cdot\vec{v} = a^2\ \partial(\nabla\cdot\vec{v})/\partial z + g\ \partial v_z/\partial z\\ &+(\Gamma_1 - 1)g\ \nabla\cdot\vec{v}, \qquad [(a^2/\varrho)\ d\varrho/dz = (\Gamma_1/\varrho)\ dP/dz - da^2/dz].\end{aligned} \tag{6.2.41}$$

Now, we derive these three equations with respect to x, y, z, respectively, and add together:

$$\begin{aligned}&\partial^2(\nabla\cdot\vec{v})/\partial t^2 = a^2\ \nabla^2(\nabla\cdot\vec{v}) + [da^2/dz + (\Gamma_1 - 1)g]\ \partial(\nabla\cdot\vec{v})/\partial z + g\ \nabla^2 v_z = a^2\ \nabla^2(\nabla\cdot\vec{v})\\ &+(da^2/dz + \Gamma_1 g)\ \partial(\nabla\cdot\vec{v})/\partial z + g\ \partial(\partial v_z/\partial y - \partial v_y/\partial z)/\partial y - g\ \partial(\partial v_x/\partial z - \partial v_z/\partial x)/\partial x.\end{aligned} \tag{6.2.42}$$

To write down the last expression, we have used the identity

$$\nabla^2 v_z - \partial(\nabla\cdot\vec{v})/\partial z = \partial(\partial v_z/\partial y - \partial v_y/\partial z)/\partial y - \partial(\partial v_x/\partial z - \partial v_z/\partial x)/\partial x, \tag{6.2.43}$$

where the two terms on the right-hand side are just the derivatives of the components of $\nabla \times \vec{v}$ along the x- and y-axis. By appropriately deriving Eq. (6.2.41), we get for the second temporal derivatives of these two components:

$$\partial^2(\partial v_z/\partial y - \partial v_y/\partial z)/\partial t^2 = -[da^2/dz - (\Gamma_1 - 1)g]\ \partial(\nabla \cdot \vec{v})/\partial y;$$
$$\partial^2(\partial v_x/\partial z - \partial v_z/\partial x)/\partial t^2 = [da^2/dz - (\Gamma_1 - 1)g]\ \partial(\nabla \cdot \vec{v})/\partial x. \qquad (6.2.44)$$

Deriving Eq. (6.2.42) twice with respect to the time, we finally obtain

$$\partial^4(\nabla \cdot \vec{v})/\partial t^4 = a^2\ \nabla^2[\partial^2(\nabla \cdot \vec{v})/\partial t^2] + (da^2/dz + \Gamma_1 g)\ \partial^3(\nabla \cdot \vec{v})/\partial t^2 \partial z$$
$$-g[da^2/dz - (\Gamma_1 - 1)g][\partial^2(\nabla \cdot \vec{v})/\partial x^2 + \partial^2(\nabla \cdot \vec{v})/\partial y^2]. \qquad (6.2.45)$$

Assuming for $\nabla \cdot \vec{v}$ a representation of the form (6.2.1), we find

$$\partial^2(\nabla \cdot \vec{v})/\partial x^2 + \partial^2(\nabla \cdot \vec{v})/\partial y^2 = -(j_x^2 + j_y^2)\ \nabla \cdot \vec{v} = -j^2\ \nabla \cdot \vec{v}, \qquad (j^2 = j_x^2 + j_y^2), \qquad (6.2.46)$$

and Eq. (6.2.45) turns after simplification with $\exp[i(\sigma t + j_x x + j_y y)]$ into (Lamb 1945)

$$a^2\ d^2(\nabla \cdot \vec{v})/dz^2 + (da^2/dz + \Gamma_1 g)\ d(\nabla \cdot \vec{v})/dz$$
$$+\{\sigma^2 - j^2 a^2 - (j^2 g/\sigma^2)[da^2/dz - (\Gamma_1 - 1)g]\}\ \nabla \cdot \vec{v} = 0. \qquad (6.2.47)$$

The sound velocity can be expressed with the aid of the temperature gradient (6.2.13) as

$$a^2 = \Gamma_1 P/\varrho = \Gamma_1 \mathcal{R}T/\mu = \Gamma_1 \mathcal{R}\beta z/\mu = \Gamma_1 g z/(n+1), \qquad (6.2.48)$$

and the coefficients of $d(\nabla \cdot \vec{v})/dz$ and $\nabla \cdot \vec{v}$ in Eq. (6.2.47) become, respectively

$$da^2/dz + \Gamma_1 g = (n+2)\Gamma_1 g/(n+1);$$
$$da^2/dz - (\Gamma_1 - 1)g = [\Gamma_1 g/(n+1)][1 - (n+1)(\Gamma_1 - 1)/\Gamma_1] = [\Gamma_1 g/(n+1)](1 - \beta_{ad}/\beta). \qquad (6.2.49)$$

The notation β_{ad} has been introduced for the adiabatic (isentropic) temperature gradient resulting from Eq. (6.2.13) if n is substituted with $n_{ad} = 1/(\Gamma_1 - 1)$:

$$\beta_{ad} = (dT/dz)_{ad} = \mu g/\mathcal{R}(n_{ad} + 1) = \mu g(\Gamma_1 - 1)/\mathcal{R}\Gamma_1. \qquad (6.2.50)$$

Eq. (6.2.47) reads with the help of Eqs. (6.2.48)-(6.2.50) as

$$z\ d^2(\nabla \cdot \vec{v})/dz^2 + (n+2)\ d(\nabla \cdot \vec{v})/dz$$
$$+j\{[(n+1)/\Gamma_1]\sigma^2/jg - jz + (jg/\sigma^2)(n\Gamma_1 - n - 1)/\Gamma_1\}\ \nabla \cdot \vec{v} = 0. \qquad (6.2.51)$$

With the substitution

$$\nabla \cdot \vec{v} = \psi(z)\exp(-jz), \qquad (6.2.52)$$

we can eliminate the variable z from the coefficient of $\nabla \cdot \vec{v}$ in Eq. (6.2.51), which then turns into the confluent hypergeometric equation (e.g. Abramowitz and Stegun 1965)

$$z\ d^2\psi/dz^2 + (n + 2 - 2jz)\ d\psi/dz + 2\alpha j\psi = 0, \qquad (6.2.53)$$

where

$$2\alpha = [(n+1)/\Gamma_1]\sigma^2/jg + [(n\Gamma_1 - n - 1)/\Gamma_1]jg/\sigma^2 - n - 2. \qquad (6.2.54)$$

Eq. (6.2.53) becomes, by imparting with $2j$ (Poyet 1983)

$$2jz\ d^2[\psi(2jz)]/d(2jz)^2 + (n + 2 - 2jz)\ d[\psi(2jz)]/d(2jz) + \alpha\ \psi(2jz) = 0, \qquad (6.2.55)$$

taking the form of Kummer's confluent hypergeometric equation

$$x\ d^2F/dx^2 + (c - x)\ dF/dx - bF = 0, \qquad (b, c = \text{const}), \qquad (6.2.56)$$

with one of the solutions equal to the confluent hypergeometric function [cf. Eq. (5.10.28)]:

$$F_1(b,c,x) = 1 + bx/1!c + b(b+1)x^2/2!c(c+1) + ... + b(b+1)...(b+m-1)x^m$$
$$/m!c(c+1)...(c+m-1) + ... \tag{6.2.57}$$

This series is convergent for all values of b, c, and x, provided that c is not a negative integer. The solution of Kummer's equation (6.2.55) is therefore

$$\psi = \psi(2jz) = F_1(-\alpha, n+2, 2jz) = 1 - \alpha(2jz)/1!(n+2) + \alpha(\alpha-1)(2jz)^2/2!(n+2)(n+3)$$
$$+ ... + (-1)^m\alpha(\alpha-1)...(\alpha-m+1)(2jz)^m/m!(n+2)(n+3)...(n+m+1) + ..., \tag{6.2.58}$$

which is finite, and equal to 1 as the surface of the polytropic atmosphere $z = 0$ is approached. The other independent solution of Kummer's equation (6.2.56) has a singularity at $z = 0$ if $c > 1$, i.e. if $n > -1$ (e.g. Abramowitz and Stegun 1965), and must accordingly be discarded.

$\nabla \cdot \vec{v}$ results from Eq. (6.2.52), once ψ is known from Eq. (6.2.55). The vertical velocity component v_z can be deduced by deriving the first two equations (6.2.41) with respect to x and y, respectively, and inserting the separable solution (6.2.1):

$$\partial^3 v_x/\partial t^2\partial x + \partial^3 v_y/\partial t^2\partial y = -\sigma^2(\nabla \cdot \vec{v} - \partial v_z/\partial z) = -j^2a^2\ \nabla \cdot \vec{v} - j^2 g v_z. \tag{6.2.59}$$

From the third equation (6.2.41) we get with the aid of Eq. (6.2.1):

$$\sigma^2 v_z = -a^2\ \partial(\nabla \cdot \vec{v})/\partial z - g\ \partial v_z/\partial z - (\Gamma_1 - 1)g\ \nabla \cdot \vec{v}. \tag{6.2.60}$$

$\partial v_z/\partial z$ is eliminated at once between Eqs. (6.2.59), (6.2.60), yielding with Eqs. (6.2.48), (6.2.52), (Lamb 1945, Spiegel and Unno 1962):

$$(\sigma^4 - j^2g^2)v_z = -\sigma^2a^2\ \partial(\nabla \cdot \vec{v})/\partial z - g(\sigma^2\Gamma_1 - j^2a^2)\ \nabla \cdot \vec{v}$$
$$= -[\Gamma_1 jg^2/(n+1)]\{(\sigma^2/jg)[z\ d\psi/dz + (n+1)\psi] - (1 + \sigma^2/jg)jz\psi\}\exp(-jz). \tag{6.2.61}$$

The boundary condition to be satisfied by this equation at the bottom $z = z_1$ of the polytropic atmosphere is obviously $v_z(z_1) = 0$, i.e. there is no mass flow across the lower boundary. This boundary condition reads via Eq. (6.2.61) as

$$z_1(d\psi/dz)_{z=z_1} + (n+1)\ \psi(z_1) = (1 + jg/\sigma^2)jz_1\psi(z_1). \tag{6.2.62}$$

As will be shown subsequently, the value of α from Eq. (6.2.54) can be determined analytically in the limiting case of wavelengths that are either very short or very long in comparison to the depth z, $(0 \le z \le z_1)$ in the polytropic atmosphere. This means that either $L = 2\pi/j \ll z$, $(jz \to \infty)$ or $L = 2\pi/j \gg z$, $(jz \to 0)$. Provided that α is known, the eigenvalues σ^2/jg can be determined from the quadratic equation (6.2.54), [cf. Spiegel 1964, Eq. (101) if $\alpha = 0$; Christensen-Dalsgaard 1980, Eqs. (4.8), (4.9)]:

$$(\sigma^2/jg)_{1,2} = [\Gamma_1(2\alpha+n+2)/2(n+1)]\{1 \pm [1 - 4(n+1)(n\Gamma_1 - n - 1)/\Gamma_1^2(2\alpha+n+2)^2]^{1/2}\}. \tag{6.2.63}$$

(i) Long wavelength case: $jz \approx 0$. Eq. (6.2.53) becomes after multiplication with z equal to

$$z^2\ d^2\psi/dz^2 + (n+2)z\ d\psi/dz + 2\alpha jz\psi = 0. \tag{6.2.64}$$

Eq. (6.2.64) is of the form (2.3.7), the relevant transformation (2.3.9) to the Bessel equation (2.3.8) being

$$z = \eta^2/8\alpha j; \quad \psi(z) = z^{-(n+1)/2}u(\eta). \tag{6.2.65}$$

Eq. (6.2.64) turns into

$$\eta^2\ d^2u/d\eta^2 + \eta\ du/d\eta + [\eta^2 - (n+1)^2]u = 0, \qquad (n > -1). \tag{6.2.66}$$

The solution (2.3.11) which is finite at the origin $\eta = 0$ is given by $u = C_1 J_{n+1}(\eta)$, $(C_1 = \text{const})$, and this yields via Eq. (6.2.65):

$$\psi(z) = C\eta^{-n-1} J_{n+1}(\eta), \qquad (C = (8\alpha j)^{(n+1)/2} C_1). \tag{6.2.67}$$

Inserting this solution of Eq. (6.2.64) into the boundary condition (6.2.62), we get

$$z_1 (d\psi/dz)_{z=z_1} + (n+1)\ \psi(z_1) = C\eta_1^{-n} J_n(\eta_1)/2 = (1 + jg/\sigma^2) j z_1 \psi(z_1)$$
$$= (1 + jg/\sigma^2) C\eta_1^{-n+1} J_{n+1}(\eta_1)/8\alpha, \tag{6.2.68}$$

or

$$J_n(\eta_1) = (1 + jg/\sigma^2)\eta_1 J_{n+1}(\eta_1)/4\alpha, \tag{6.2.69}$$

where we have used for the left-hand side of Eq. (6.2.68) the relationship (e.g. Spiegel 1968)

$$dJ_{n+1}/d\eta = J_n - (n+1) J_{n+1}/\eta. \tag{6.2.70}$$

In principle, the numerical calculation of α, η_1, and of the four eigenvalues σ is possible from the three equations (6.2.54), (6.2.65), (6.2.69) if n, Γ_1, z_1, and j are known. To make further analytical progress we assume, following Lamb (1945, p. 553), that $\sigma^2/jg \approx 0$. Eq. (6.2.54) becomes with this constraint

$$2\alpha = (n\Gamma_1 - n - 1) jg/\sigma^2\Gamma_1 = (\beta_{ad}/\beta - 1) jg/\sigma^2, \qquad (jz, \sigma^2/jg \approx 0). \tag{6.2.71}$$

The boundary condition (6.2.69) now reads

$$J_n(\eta_1) = jg\eta_1 J_{n+1}(\eta_1)/4\alpha\sigma^2 = \Gamma_1 \eta_1 J_{n+1}(\eta_1)/2(n\Gamma_1 - n - 1)$$
$$= \eta_1 J_{n+1}(\eta_1)/2(\beta_{ad}/\beta - 1), \qquad (jz, \sigma^2/jg \approx 0). \tag{6.2.72}$$

For instance, if $n = 6$ and $\Gamma_1 = 1.40$ Lamb (1945) finds for the lowest root of this transcendent equation $\eta_1 = 4.96$. This permits the calculation of $\alpha = \eta_1^2/8jz_1$ and of the eigenvalue σ^2/jg by virtue of Eqs. (6.2.65) and (6.2.71), respectively.

(ii) Short wavelength case: $jz \to \infty$. We introduce the notation

$$\zeta = 2jz \to \infty. \tag{6.2.73}$$

Eq. (6.2.61) reads with the solution (6.2.58):

$$(\sigma^4 - j^2 g^2) v_z = [\Gamma_1 jg^2/(n+1)]\big\{-(\sigma^2/jg)\big[\zeta\ dF_1(-\alpha, n+2, \zeta)/d\zeta + (n+1)\ F_1(-\alpha, n+2, \zeta)\big]$$
$$+ (1 + \sigma^2/jg)\zeta\ F_1(-\alpha, n+2, \zeta)/2\big\} \exp(-\zeta/2) = [\Gamma_1 jg^2/(n+1)]\big\{[\sigma^2\alpha\zeta/jg(n+2)]$$
$$\times F_1(-\alpha+1, n+3, \zeta) + [-(n+1)\sigma^2/jg + (1 + \sigma^2/jg)\zeta/2]\ F_1(-\alpha, n+2, \zeta)\big\} \exp(-\zeta/2), \tag{6.2.74}$$

where the differentiation rule

$$dF_1(b, c, x)/dx = (b/c)\ F_1(b+1, c+1, x), \tag{6.2.75}$$

can be derived at once from the series (6.2.57). In the limit $\zeta \to \infty$, the confluent hypergeometric function takes the asymptotic form (e.g. Abramowitz and Stegun 1965, Sec. 13.5.1)

$$F_1(b, c, x) = [\exp(i\pi b)\ x^{-b}\Gamma(c)/\Gamma(c-b)]\Big\{1 + \sum_{m=1}^{\infty} \big[b(b+1)...(b+m-1)(b-c+1)(b-c+2)$$
$$\times ...(b-c+m)(-x)^{-m}/m!\big]\Big\} + [x^{b-c} \exp x\ \Gamma(c)/\Gamma(b)]\Big\{1 + \sum_{m=1}^{\infty} \big[(c-b)(c-b+1)$$
$$\times ...(c-b+m-1)(-b+1)(-b+2)...(-b+m) x^{-m}/m!\big]\Big\}, \qquad (x \to \infty). \tag{6.2.76}$$

Retaining in Eq. (6.2.74) only the leading terms, we find via Eq. (C.11)

$$(\sigma^4 - j^2 g^2) v_z = [\Gamma_1 jg^2\ \Gamma(n+1)/2]\big[(1 + \sigma^2/jg) \exp(-i\pi\alpha)\ \zeta^{\alpha+1}/\Gamma(n + \alpha + 2)$$
$$+ (1 - \sigma^2/jg)\zeta^{-n-\alpha-1} \exp\zeta/\Gamma(-\alpha)\big] \exp(-\zeta/2), \qquad (\zeta \to \infty). \tag{6.2.77}$$

The boundary condition $v_z(\zeta_1) = 0,\ (z_1 = \zeta_1/2j)$ implies (Spiegel and Unno 1962)

$$\Gamma(-\alpha)/\Gamma(n+\alpha+2) = [(\sigma^2/jg - 1)/(\sigma^2/jg + 1)]\zeta_1^{-n-2\alpha-2}\exp\zeta_1\exp(i\pi\alpha). \tag{6.2.78}$$

The two sides of this boundary condition must be large, because $\exp\zeta_1 \rightarrow \infty$. By virtue of Euler's formula (e.g. Abramowitz and Stegun 1965)

$$\Gamma(x) = \lim_{N\rightarrow\infty} N!\ N^x/x(x+1)(x+2)...(x+N), \qquad (N = 0,1,2,3,...;\ x \neq 0,-1,-2,-3,...), \tag{6.2.79}$$

the gamma function is singular for negative integers, so α in Eq. (6.2.78) must be close to a positive integer, in order to make $\Gamma(-\alpha)$ large:

$$\alpha \approx 0,1,2,3,..., \qquad (\zeta \rightarrow \infty). \tag{6.2.80}$$

For the fundamental mode $\alpha \approx 0$ we find from Eq. (6.2.54), (Spiegel 1964):

$$(\sigma^2/jg)_{1,2} = [\Gamma_1(n+2)/2(n+1)]\left\{1 \pm \left[1 - 4(n+1)(n\Gamma_1 - n - 1)/\Gamma_1^2(n+2)^2\right]^{1/2}\right\},$$
$$(\alpha \approx 0;\ \zeta \rightarrow \infty). \tag{6.2.81}$$

If σ^2/jg is assumed small, we may neglect in Eq. (6.2.54) the term associated with σ^2/jg (cf. Skumanich 1955, Spiegel and Unno 1962, Spiegel 1964):

$$\sigma^2/jg = (n\Gamma_1 - n - 1)/\Gamma_1(2\alpha + n + 2), \qquad (\alpha \approx 0,1,2,3,...;\ \sigma^2/jg \approx 0;\ \zeta \rightarrow \infty). \tag{6.2.82}$$

From the Schwarzschild discriminant (5.2.85) follows that the convective stability condition $A < 0$ demands

$$n\Gamma_1 - n - 1 > 0 \quad \text{or} \quad \Gamma_1 > 1 + 1/n \quad \text{and} \quad n > 1/(\Gamma_1 - 1), \qquad (d\ln\varrho/dr < 0;\ n > 0;\ \Gamma_1 > 1). \tag{6.2.83}$$

This stability condition can be expressed – as in Eq. (6.1.29) – in terms of the polytropic and adiabatic temperature gradient β and β_{ad} from Eqs. (6.2.13) and (6.2.50), respectively:

$$(n + 1 - n\Gamma_1)/\Gamma_1 = 1 - (n+1)(\Gamma_1 - 1)/\Gamma_1 = 1 - \beta_{ad}/\beta < 0 \quad \text{or}$$
$$\beta = dT/dz < \beta_{ad} = (dT/dz)_{ad}, \qquad (dT/dz > 0;\ n > 0;\ \Gamma_1 > 1). \tag{6.2.84}$$

If $n > 0,\ \alpha \approx 0,1,2,3,...$, and if the adiabatic index Γ_1 changes in the common interval $[1, 5/3]$, (see Table 1.7.1), it turns out that the quantity under the square root in Eq. (6.2.63) is positive definite, so σ^2/jg cannot be complex conjugate.

Convective instability takes place if the actual temperature gradient is superadiabatic, i.e. larger than the adiabatic temperature gradient [Schwarzschild 1958, Eq. (7.2)]. If the atmosphere is assumed unstable against convection $[n < 1/(\Gamma_1 - 1)]$, the quantity under the square root in Eq. (6.2.63) is > 1, and one of the roots σ^2/jg becomes negative, giving rise to an unstable g-mode (gravity, convective mode). On the other hand, if the atmosphere is convectively stable $(n > 1/(\Gamma_1 - 1);\ n > 0;\ 1 < \Gamma_1 < 5/3)$, the positive quantity under the square root in Eq. (6.2.63) is < 1, and the eigenvalues σ^2/jg are always positive, giving rise to stable g (gravity) or p (pressure, acoustic) waves, depending on whether σ^2/jg is small or large [cf. Eqs. (5.2.126), (5.2.127) for the spherical case (Spiegel 1964)]. If the atmosphere is convectively unstable, the gravity waves are replaced by unstable convective g-modes, the p-modes (acoustic waves) being then the only form of stable wave motion.

The use of phase-integral methods for the description of short-wavelength p-modes yields not always satisfactory agreement with the exact dispersion relationship (6.2.54) if $\alpha = 0,1,2,3,...$ (Price 1992).

Our discussion has left out two types of modes. The one type – the f-mode – results if

$$\sigma^2 = jg. \tag{6.2.85}$$

In this case the velocity v_z from Eq. (6.2.61) is finite at the surface $z = 0$ only if ψ vanishes identically. The velocity $v_z = A\exp(-jz),\ (A = \text{const})$, resulting from Eq. (6.2.60) with $\psi \propto \nabla \cdot \vec{v} \equiv 0$, belongs essentially to a surface wave at $z \approx 0$ (Christensen-Dalsgaard 1980).

The other type of modes are axial (toroidal) modes, which are degenerate to zero frequency $\sigma = 0$, as in a nonrotating sphere (see Eq. (5.8.166), Poyet 1983).

Antia and Chitre (1978, 1979) have found that the linear Eulerian perturbation theory breaks down just at the boundary of a complete polytropic atmosphere with vanishing temperature at the surface, while the relative Lagrangian perturbations remain finite at the boundary: $\delta f/f \rightarrow \infty$ and $\Delta f/f =$ finite. This draw-back of the Eulerian treatment does not seem to be crucial, as all involved quantities (pressure, density, temperature) remain small, so their influence on the deeper and denser layers would be insignificant, similar to the objection of Smith (1975, 1976) concerning the break-down of Chandrasekhar's (1933a-d) perturbation theory near the boundary of a distorted polytrope (Sec. 3.2).

The thermal instability of a fluid layer with constant density ($n = 0$) has already been treated theoretically by Rayleigh, assuming constant viscosity and thermal conductivity (e.g. Chandrasekhar 1981, Chap. II). Unno et al. (1960) have abandoned the restriction of constant density, showing that the critical Rayleigh number for the occurrence of convective instability together with the associated horizontal wave number j depend very little on density variations in the atmosphere, provided that appropriate mean values of density and temperature are considered.

Allowing for thermal diffusion Spiegel (1964) and Jones (1976) have investigated the decay or amplification up to a certain limit (overstability, Sec. 5.1) of sound waves. Jones (1976) indicates that the results of Chitre and Gokhale (1973, 1975) concerning the overstability of horizontally propagating acoustic waves are incorrect, i.e. that such waves are stable.

The most complete study of overstabilities in a plane polytropic atmosphere under the influence of viscosity, thermal conduction, and a uniform vertical magnetic field has been undertaken by Lou (1990, 1991). Nonadiabatic acoustic overstabilities occur under the influence of viscosity and thermal conductivity in a polytropic atmosphere with subadiabatic (convectively stable) temperature gradient (Lou 1990). The problem complicates considerably in the presence of a constant vertical magnetic field: Since the vertical field enhances convective stability, magnetohydrodynamic overstabilities can occur over a wide range of subadiabatic as well as superadiabatic temperature gradients, while the atmosphere retains convective stability (Lou 1991, Bogdan and Cally 1997).

Solar p-modes (acoustic waves) and magnetic surface waves in a plane polytropic atmosphere truncated by a horizontal magnetic region have been investigated by Foullon (1999).

6.2.2 Polytropic Interstellar Clouds

About 10^7 yr after a dynamic shock, an interstellar cloud which does not collapse gravitationally, will reach a state of rough hydrostatic equilibrium (e.g. Shu et al. 1972). Such a cloud, heated by an external flux of energetic particles or photons, will not be isothermal ($n = \pm\infty$), since the denser regions near the centre must be cooler than the more tenuous regions near the surface if the flux is nearly uniform throughout the cloud. If the external flux is attenuated by absorbing agents, the central parts may be even cooler. Unno and Simoda (1963) have calculated the hydrostatic structure of an interstellar cloud heated by suprathermal particles, the cooling being mainly caused by ionized carbon. They found that a cloud of mass $M_1 = 1.1 \times 10^3\ M_\odot$, mean molecular weight $\mu = 1.5$, and mean temperature $T = 100$ K at an average number density of $n_d = 10\ \mathrm{cm}^{-3}$ is just at the verge of instability under the external pressure exerted by the intercloud medium (cf. Sec. 5.4.2, Table 5.4.1). The polytropic index of the cloud is $n \approx -3.32$, corresponding to a two-fold increase of temperature ($\propto \theta$) towards the surface, and to a decrease $\propto \theta^{-3.32}, \theta^{-2.32}$ of density and pressure, respectively.

Note, that Eddington (1931) was the first who has considered the theoretical possibility of negative polytropic indices.

Similar results have been reached by Shu et al. (1972) for polytropic indices $n = -1.3$, ($M_1 = 3000\ M_\odot$) and $n = -4$, ($M_1 = 120\ M_\odot$), the cloud being heated by a uniform flux of low-energy cosmic rays (30 K $\lesssim T \lesssim$ 200 K).

As shown by Kenyon and Starrfield (1979), Bok globules with central number densities $n_d = 10^5\ \mathrm{cm}^{-3}$, masses 25 $M_\odot \lesssim M_1 \lesssim$ 200 $M_\odot$, mean temperature 7.7μ [K] $\lesssim T_m \lesssim 22.3\mu$ [K], (mean molecular weight $\mu = 2$, as for molecular hydrogen), can also be approximated by incomplete polytropes of index $n \approx -2$.

The outer layers ($r \propto \xi \gg 1$) of interstellar clouds with polytropic indices $5 < n < \infty$ and $-\infty < n < -1$ can be described by the singular solutions (2.3.92) and (2.3.93): $\varrho \propto \theta^n \propto \xi^{-2n/(n-1)} \propto r^{-2n/(n-1)}$, (McKee and Holliman 1999, §3.1). However, the more refined structure of interstellar molecular clouds cannot be described adequately by simple polytropic spheres of negative index. Therefore, Curry and McKee (2000) adopt composite, nonisentropic polytropes ($\Gamma_1 \neq 1 + 1/n$; Secs. 2.8.1, 5.4.2), which reproduce much better the bulk properties (mass, radius, density contrast) of dense molecular cores and Bok globules.

A formal "logatropic" equation of state for the structure of giant molecular clouds can be obtained from the differential $dP = (1 + 1/n)K\varrho^{1/n}\,d\varrho = K_1\varrho^{1/n}\,d\varrho$, $(K_1 = (1 + 1/n)K)$ of the polytropic equation of state. If $n = -1$, this gives the isobaric $P = \mathrm{const}$, as outlined in Sec. 2.1. Only if K_1 is formally regarded as a nonzero constant, the integration yields the logatrope $P = P_0 + K_1 \ln(\varrho/\varrho_0)$, $(n = -1;\ K_1 \neq 0)$, (McLaughlin and Pudritz 1996).

Discussing the *local* polytropic index connected to the heating and cooling function of the interstellar gas in dense, cool interstellar clouds ($n_d = 10^2 - 10^5\ \mathrm{cm}^{-3}$, $T = 10 - 100$ K), Scalo et al. (1998) suggest values of $n = 1/(\Gamma_1' - 1)$ between -4.5 and -100 if $n_d \lesssim 10^3\ \mathrm{cm}^{-3}$, and between 2.5 and ∞ if $n_d \gtrsim 10^3\ \mathrm{cm}^{-3}$. Γ_1' is the polytropic exponent from Eq. (1.3.25).

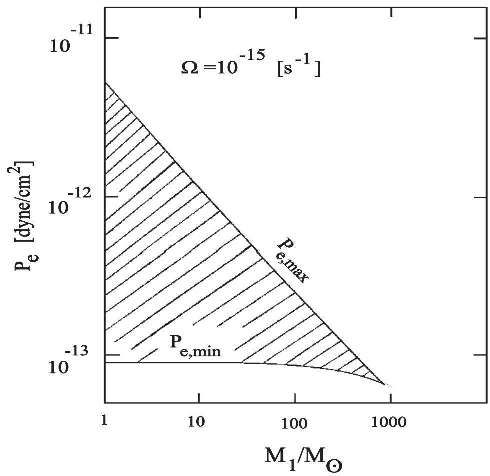

Fig. 6.2.1 Hydrostatic equilibrium models of rotating interstellar clouds with polytropic index $n = -3$ and angular velocity $\Omega = 10^{-15}\ \mathrm{s}^{-1}$ are possible only within the hatched area confined between the curves $P_e = P_{e,min}$ and $P_e = P_{e,max}$. Beyond a cloud mass of $M_1 \approx 830\ M_\odot$ no equilibrium states exist (Viala et al. 1978).

The quasistatic evolution of axisymmetric, uniformly rotating, polytropic interstellar clouds under external pressure has been calculated by Viala et al. (1978). The relevant integral of hydrostatic equilibrium (3.8.4) reads in cylindrical (ℓ, z)-coordinates

$$(n + 1)K\varrho^{1/n} = \Phi + \Omega^2\ell^2/2 + \mathrm{const}. \tag{6.2.86}$$

The integration constant const $= (n+1)K\varrho_0^{1/n}$ may be determined from the central condition $\Phi(0, 0) = 0$. Poisson's equation (2.1.4) writes as

$$(1/\ell)\ \partial(\ell\ \partial\Phi/\partial\ell)/\partial\ell + \partial^2\Phi/\partial z^2 = -4\pi G\varrho = -4\pi G\big\{[\Phi + \Omega^2\ell^2/2 + (n + 1)K\varrho_0^{1/n}]/(n + 1)K\big\}^n. \tag{6.2.87}$$

This equation becomes with the dimensionless polytropic variables [cf. Eqs. (3.2.1), (3.2.3)]

$$\ell = \alpha\eta;\quad z = \alpha\zeta;\quad \varrho = \varrho_0\Theta^n;\quad P = K\varrho_0^{1+1/n}\Theta^{n+1};\quad \chi = \Phi/(n + 1)K\varrho_0^{1/n};\quad \beta = \Omega^2/2\pi G\varrho_0,$$
$$(\alpha^2 = -(n + 1)K/4\pi G\varrho_0^{1-1/n};\ -\infty < n < -1), \tag{6.2.88}$$

equal to

$$(1/\eta)\ \partial(\eta\ \partial\chi/\partial\eta)/\partial\eta + \partial^2\chi/\partial\zeta^2 = (\chi - \beta\eta^2/4 + 1)^n. \tag{6.2.89}$$

The condition that a uniform external pressure P_e acts on the distorted boundary is expressed by virtue of Eqs. (6.2.86)-(6.2.89) as

$$\Theta_1^{n+1} = \Theta^{n+1}(\eta_1, \zeta_1) = P_e(\ell_1, z_1)/K\varrho_0^{1+1/n}$$
$$= \{[\Phi_1 + \Omega^2\ell_1^2/2 + (n+1)K\varrho_0^{1/n}]/(n+1)K\varrho_0^{1/n}\}^{n+1} = (\chi_1 - \beta\eta_1^2/4 + 1)^{n+1}, \qquad (6.2.90)$$

where the hydrostatic integral (6.2.86) writes via Eq. (6.2.88) as

$$\Theta(\eta, \zeta) = \chi(\eta, \zeta) - \beta\eta^2/4 + 1. \qquad (6.2.91)$$

The fundamental Lane-Emden function is known at once from Eq. (6.2.91), if Poisson's equation (6.2.89) is solved subject to the boundary condition (6.2.90). Because the results are not very sensitive to the polytropic index, equilibrium states have been calculated only for $n = -3$, in accordance to the previously quoted values of n in interstellar cloud models. Take the cloud mass M_1 and the angular velocity $\Omega = 10^{-15}\ \mathrm{s}^{-1}$ fixed. If the external pressure P_e decreases, the equilibrium radius r_1 increases (the central density ϱ_0 decreases), and below a certain minimum external pressure $P_{e,min}$ no equilibrium models will be possible, as the cloud will be disrupted due to the increasing centrifugal forces overcoming gravity in the equatorial plane. Nonrotating clouds do not exhibit this behaviour, as centrifugal forces are absent, and hydrostatic models exist however small there is P_e (cf. Sec. 5.4.2, Fig. 5.4.2).

On the other hand, keeping M_1 and Ω still constant, there exists also a maximum external pressure $P_{e,max}$ to which a nearly spherical, polytropic cloud can withstand if $-\infty < n < -1$. If $P_e > P_{e,max}$, the pressure gradient dP_e/dr_1 is positive, and the cloud will be unstable to gravitational collapse. Thus, equilibrium models of rotating, polytropic interstellar clouds are possible only between external pressures confined to the interval $P_{e,min} \le P \le P_{e,max}$. If $P_{e,min} > P_{e,max}$, no equilibrium states exist at all, unlike to the nonrotating case when $P_{e,min} \equiv 0$ (Fig. 6.2.1).

Saigo et al. (2000) have performed collapse calculations of axisymmetric cylindrical clouds with polytropic indices $-\infty < n < -5$ and $5 < n < \infty$. Such values are suggested by the delimitation $0.2 \le \Gamma_1' \le 1.4$ of the polytropic exponent (1.3.25) in interstellar clouds, corresponding to polytropic indices $n = 1/(\Gamma_1' - 1)$ in the intervals $-\infty < n < -1.25$ and $2.5 < n < \infty$ (Spaans and Silk 2000).

Closely connected to this topic is the radial gravitational collapse (the contraction) of an interstellar cloud to a polytropic star, as studied by McVittie (1956b) under the assumption that the radial velocity v of each mass element at instant t obeys the so-called linear-wave hypothesis, i.e. a class of flows where the velocity has the form

$$v = v(r,t) = dr/dt = [r/f(t)]\ df/dt \quad \text{or} \quad r = Cf(t), \qquad (C = \text{const}). \qquad (6.2.92)$$

f is an arbitrary function of time. Eq. (6.2.92) implies a homology transformation for the contraction of the sphere, the ratio $v/r = [1/f(t)]\ df/dt$ being independent of radial position r. To solve the problem, a multitude of functional transformations are introduced, the starting point being the radially symmetric equation of continuity in a N-dimensional space [cf. Eqs. (5.2.1), (C.13)]

$$\partial\varrho/\partial t + \nabla\cdot(\varrho v) = \partial\varrho/\partial t + r^{1-N}\ \partial(r^{N-1}\varrho v)/\partial r = 0, \qquad (N = 1, 2, 3, ...), \qquad (6.2.93)$$

and the equation of motion (5.2.10)

$$\varrho\ Dv/Dt = \varrho\ \partial v/\partial t + \varrho v\ \partial v/\partial r = -\partial P/\partial r + 4\pi G\varrho\ \partial\varphi/\partial r. \qquad (6.2.94)$$

All quantities are assumed to depend merely on the radial coordinate r and on time t. The internal gravitational potential Φ is written under the form

$$\Phi = \Phi(r,t) = 4\pi G\varphi(r,t), \qquad (6.2.95)$$

in order to simplify subsequent equations, like the Poisson equation (2.1.4), (C.15):

$$\varrho = -\nabla^2\varphi = -r^{1-N}\ \partial(r^{N-1}\ \partial\varphi/\partial r)/\partial r. \qquad (6.2.96)$$

A solution of Eqs. (6.2.93)-(6.2.96) in terms of φ can be found by inserting Eq. (6.2.96) into Eq. (6.2.93), (e.g. McVittie 1956a, Chaps. 6, 7):

$$\nabla^2\varphi_t = r^{1-N}\ \partial(r^{N-1}\varphi_{tr})/\partial r = -r^{1-N}\ \partial(r^{N-1}v\ \nabla^2\varphi)/\partial r, \qquad (6.2.97)$$

the subscripts denoting the respective derivatives. Integration with respect to r is immediate, yielding

$$v(r,t) = dr/dt = rf_t/f = -\varphi_{tr}/\nabla^2\varphi, \tag{6.2.98}$$

where a time-dependent integration function has been set identical zero, in order to simplify the result. The pressure is obtained by inserting Eqs. (6.2.96), (6.2.98) into Eq. (6.2.94):

$$\begin{aligned}\partial P/\partial r &= -4\pi G\varphi_r\ \nabla^2\varphi - \varphi_{ttr} + \varphi_{tr}[\partial(\nabla^2\varphi)/\partial t]/\nabla^2\varphi + \varphi_{tr}\ \partial(\varphi_{tr}/\nabla^2\varphi)/\partial r\\ &= -4\pi G[(N-1)\varphi_r^2/r + \varphi_r\varphi_{rr}] - \varphi_{ttr} + \partial(\varphi_{tr}^2/\nabla^2\varphi)/\partial r + (\varphi_{tr}/\nabla^2\varphi)[-\varphi_{trr} + \partial(\nabla^2\varphi)/\partial t].\end{aligned} \tag{6.2.99}$$

Observing that the last bracket is via Eq. (6.2.96) just equal to $(N-1)\varphi_{tr}/r$, the integration with respect to r yields

$$P(r,t) = D(t) - 2\pi G\varphi_r^2 - \varphi_{tt} + \varphi_{tr}^2/\nabla^2\varphi + (N-1)\int_0^r (\varphi_{tr'}^2/\nabla^2\varphi - 4\pi G\varphi_{r'}^2)\ dr'/r', \tag{6.2.100}$$

by including as an integration constant the arbitrary function of time $D(t)$. Since Eq. (6.2.98) is invariant to a scale change of r and f, we express φ with the aid of an arbitrary function $h(\zeta)$ of the conformal variable

$$\zeta = \zeta(r,t) = r/f(t). \tag{6.2.101}$$

It may be verified by direct insertion into Eq. (6.2.98) that φ can be taken under the form

$$\varphi(r,t) = \varphi(\zeta,t) = -Bf^{1-N}(t)\int_0^r h[r'/f(t)]\ dr' = -Bf^{2-N}(t)\int_0^\zeta h(\zeta')\ d\zeta', \qquad (B = \mathrm{const}), \tag{6.2.102}$$

where integration over r is effected as if $f(t)$ were a constant.

The various derivatives occurring in Eqs. (6.2.96), (6.2.100) are via Eq. (6.2.101) equal to

$$\begin{aligned}&\varphi_r = -Bf^{1-N}h;\quad \varphi_{rr} = -Bf^{-N}h_\zeta;\quad \varphi_{tr} = \varphi_{rt} = Bf^{-N}f_t[(N-1)h + \zeta h_\zeta];\quad \varphi_t = Bf^{1-N}f_t\\ &\times\left[(N-2)\int_0^\zeta h\ d\zeta' + \zeta h\right];\quad \varphi_{tt} = Bf^{-N}f_t^2\left[(-N^2+3N-2)\int_0^\zeta h\ d\zeta' + 2(1-N)\zeta h - \zeta^2 h_\zeta\right]\\ &+Bf^{1-N}f_{tt}\left[(N-2)\int_0^\zeta h\ d\zeta' + \zeta h\right],\qquad (h_\zeta = dh/d\zeta;\ \partial h[\zeta(r,t)]/\partial t = -\zeta h_\zeta f_t/f).\end{aligned} \tag{6.2.103}$$

Eqs. (6.2.96) and (6.2.100) become, respectively

$$\varrho = -\nabla^2\varphi = Bf^{1-N}(t)\ r^{1-N}\ \partial\{r^{N-1}h[r/f(t)]\}/\partial r = Bf^{-N}(t)\ [(N-1)h(\zeta)/\zeta + dh/d\zeta], \tag{6.2.104}$$

$$\begin{aligned}P = D(t) &- Bf^{1-N}(t)\ (d^2f/dt^2)\left[(N-2)\int_0^\zeta h(\zeta')\ d\zeta' + \zeta h(\zeta)\right]\\ &-2\pi GB^2f^{2-2N}(t)\left[2(N-1)\int_0^\zeta h^2(\zeta')\ d\zeta'/\zeta' + h^2(\zeta)\right].\end{aligned} \tag{6.2.105}$$

At moment t the outer boundary of the configuration is by virtue of Eq. (6.2.92) equal to $r_1 = Cf(t)$, and taking $C = 1$, the function $r_1 = f(t)$ represents just the surface of the configuration at instant t. The variable ζ therefore ranges from 0 to 1. Density and pressure are assumed to vanish on the surface $\zeta = \zeta(r_1,t) = r_1/f(t) = 1:$

$$(N-1)\ h(1) + (dh/d\zeta)_{\zeta=1} = 0, \tag{6.2.106}$$

$$\begin{aligned}D(t) = &Bf^{1-N}(t)\ (d^2f/dt^2)\left[(N-2)\int_0^1 h(\zeta')\ d\zeta' + h(1)\right]\\ &+2\pi GB^2f^{2-2N}(t)\left[2(N-1)\int_0^1 h^2(\zeta')\ d\zeta'/\zeta' + h^2(1)\right].\end{aligned} \tag{6.2.107}$$

The time can be normalized by introducing the dimensionless temporal variable

$$\tau = t/t_f, \tag{6.2.108}$$

equal to the time taken for the gas to move from the initial boundary radius $r_{1i} = r_1(0) = f(0)$ to the final equilibrium boundary radius $r_{1f} = r_1(t_f) = f(t_f)$. The boundary radius $r_1(t) = f(t)$ at instant t is replaced by the dimensionless function

$$R(\tau) = R(t/t_f) = r_1(t)/r_{1i} = f(t)/r_{1i}; \quad f(t) = r_{1i}R(\tau),$$
$$(r_1(t) = f(t);\ R(0) = 1;\ R(1) = r_{1f}/r_{1i}). \tag{6.2.109}$$

Thus, the physical characteristics of the collapsing ideal gas configuration from Eqs. (6.2.92), (6.2.104), (6.2.105) can be thrown into the form

$$v = v(\zeta,\tau) = \zeta\ df/dt = r_{1i}\zeta\ dR(t/t_f)/dt = (r_{1i}\zeta/t_f)\ dR(\tau)/d\tau, \tag{6.2.110}$$

$$\varrho = \varrho(\zeta,\tau) = Br_{1i}^{-N}R^{-N}(\tau)\ [(N-1)\ h(\zeta)/\zeta + dh/d\zeta], \tag{6.2.111}$$

$$P = P(\zeta,\tau) = (Br_{1i}^{2-N}/t_f^2)\ R^{1-N}(\tau)\ (d^2R/d\tau^2)\left[(N-2)\int_\zeta^1 h(\zeta')\ d\zeta' + h(1) - \zeta h(\zeta)\right]$$
$$+2\pi GB^2r_{1i}^{2-2N}R^{2-2N}(\tau)\left[2(N-1)\int_\zeta^1 h^2(\zeta')\ d\zeta'/\zeta' + h^2(1) - h^2(\zeta)\right]. \tag{6.2.112}$$

The identification of the unknown function $h(\zeta)$ depends on the nature of the final equilibrium configuration, rather than on the form assumed for $R(\tau)$. At the final instant $t = t_f$ the variable ζ from Eq. (6.2.101) becomes via Eq. (6.2.109) equal to

$$\zeta_f = \zeta_f(r_f, t_f) = r_f/f(t_f) = r_f/r_{1f} = r_f/r_{1i}R(1). \tag{6.2.113}$$

And since the final state is assumed to be an equilibrium state, velocity $v \propto dR/d\tau$ and acceleration $\propto d^2R/d\tau^2$ must be zero if $\tau = 1$ [cf. Eqs. (6.2.110), (6.2.130)]. The final equilibrium density and pressure are via Eqs. (6.2.111), (6.2.112) equal to $[r_{1f} = r_{1i}R(1)]$

$$\varrho_f = Br_{1f}^{-N}\left[(N-1)h(\zeta_f)/\zeta_f + (dh/d\zeta)_{\zeta=\zeta_f}\right], \tag{6.2.114}$$

$$P_f = 2\pi GB^2r_{1f}^{2-2N}\left[2(N-1)\int_{\zeta_f}^1 h^2(\zeta)\ d\zeta/\zeta + h^2(1) - h^2(\zeta_f)\right]. \tag{6.2.115}$$

As can be verified by direct derivation, these two equations are connected by the differential equation

$$d[(\zeta_f^{N-1}/\varrho_f)\ dP_f/d\zeta_f]/d\zeta_f = -4\pi Gr_{1f}^2\varrho_f\zeta_f^{N-1}. \tag{6.2.116}$$

This equation is just equal to the radially symmetric Poisson equation (2.1.4) $\nabla^2\Phi_f = -4\pi G\varrho_f$, if $\nabla\Phi_f$ is replaced by $(1/\varrho_f)\ \nabla P_f$ from the equation of hydrostatic equilibrium (2.1.3):

$$\nabla\cdot[(1/\varrho_f)\ \nabla P_f] = r_f^{1-N}\ d[(r_f^{N-1}/\varrho_f)\ dP_f/dr_f]/dr_f = -4\pi G\varrho_f, \qquad (r_f = r_{1f}\zeta_f). \tag{6.2.117}$$

We now assume that the final equilibrium state is a complete polytrope obeying the Lane-Emden equation (2.1.14), where ζ_f is connected to the Lane-Emden variable ξ_f by

$$\xi_f = \xi_1\zeta_f, \qquad (0 \le \zeta_f \le 1), \tag{6.2.118}$$

ξ_1 being the first zero of the Lane-Emden function $\theta(\xi_f) = \theta(\xi_1\zeta_f)$. At the origin we have $r_f = 0,\ \zeta_f = 0$, and $\xi_f = 0$. Denoting by $\zeta_{1f} = 1$ the value of $\zeta_f = r_f/r_{1f}$ on the surface $r_f = r_{1f}$, we get via Eq. (6.2.118): $\xi_{1f} = \xi_1\zeta_{1f} = \xi_1$. Thus, pressure and density from Eqs. (6.2.114), (6.2.115) take the polytropic form

$$\varrho_f = Br_{1f}^{-N}\theta^n(\xi_f); \quad P_f = K\varrho_f^{1+1/n} = K(Br_{1f}^{-N})^{1+1/n}\theta^{n+1}(\xi_f). \tag{6.2.119}$$

With the polytropic constant equal to

$$K = 4\pi GB^{1-1/n} r_{1f}^{2+N(1/n-1)}/(n+1)\xi_1^2, \tag{6.2.120}$$

Eq. (6.2.116) turns just into the Lane-Emden equation (2.1.14): $d(\xi_f^{N-1}\ d\theta/d\xi_f)/d\xi_f = -\xi_f^{N-1}\theta^n$, $(-1 < n < \infty)$. The densities (6.2.114) and (6.2.119) must be the same, so we get in virtue of Eqs. (2.1.14), (6.2.118):

$$\xi_1[(N-1)\ h(\xi_f)/\xi_f + dh/d\xi_f] = \xi_1\xi_f^{1-N}\ d(\xi_f^{N-1}h)/d\xi_f = \theta^n(\xi_f) = -\xi_f^{1-N}\ d(\xi_f^{N-1}\ d\theta/d\xi_f)/d\xi_f. \tag{6.2.121}$$

This yields

$$h(\xi_f) = -(1/\xi_1)\ d\theta/d\xi_f = -\theta'(\xi_f)/\xi_1. \tag{6.2.122}$$

In analogy to ξ_f from Eq. (6.2.118) we define the variable

$$\xi = \xi_1\zeta, \qquad (0 \le \zeta \le 1), \tag{6.2.123}$$

and observe that the form of h is preserved, since the range of its argument ξ, running from 0 to ξ_1, is not changed:

$$h(\xi) = -(1/\xi_1)\ d\theta/d\xi = -\theta'(\xi)/\xi_1. \tag{6.2.124}$$

Since h is now known, it is possible to evaluate density and pressure from Eqs. (6.2.111), (6.2.112) in terms of the Lane-Emden function $\theta(\xi)$:

$$\begin{aligned} \varrho = \varrho(\xi,\tau) &= Br_{1i}^{-N}R^{-N}(\tau)\ \xi_1[(N-1)\ h(\xi)/\xi + dh/d\xi] \\ &= -Br_{1i}^{-N}R^{-N}(\tau)\ \xi^{1-N}\ d(\xi^{N-1}\theta')/d\xi = Br_{1i}^{-N}R^{-N}(\tau)\ \theta^n(\xi), \end{aligned} \tag{6.2.125}$$

$$\begin{aligned} P = P(\xi,\tau) &= (Br_{1i}^{2-N}/\xi_1^2 t_f^2)\ R^{1-N}(\tau)\ (d^2R/d\tau^2)[(N-2)\ \theta(\xi) + \xi\theta'(\xi) - \xi_1\theta'(\xi_1)] \\ &+ 4\pi GB^2 r_{1i}^{2-2N}R^{2-2N}(\tau)\ \theta^{n+1}(\xi)/(n+1)\xi_1^2. \end{aligned} \tag{6.2.126}$$

The two brackets in Eq. (6.2.112) have been transformed according to

$$\begin{aligned} &(N-2)\int_\zeta^1 h(\zeta')\ d\zeta' + h(1) - \zeta h(\zeta) = -\xi_1^{-2}\left[(N-2)\int_\xi^{\xi_1}\theta'(\xi')\ d\xi' + \xi_1\theta'(\xi_1) - \xi\theta'(\xi)\right] \\ &= \xi_1^{-2}[(N-2)\ \theta(\xi) + \xi\theta'(\xi) - \xi_1\theta'(\xi_1)], \end{aligned} \tag{6.2.127}$$

$$\begin{aligned} &2(N-1)\int_\zeta^1 h^2(\zeta')\ d\zeta'/\zeta' + h^2(1) - h^2(\zeta) = 2\xi_1^{-2}\int_\xi^{\xi_1}\theta'(\xi')\big[(N-1)\ \theta'(\xi')/\xi' + \theta''(\xi')\big]\ d\xi' \\ &= 2\xi_1^{-2}\int_\xi^{\xi_1}[\xi'^{1-N}\theta'(\xi')]\ d[\xi'^{N-1}\theta'(\xi')] = -2\xi_1^{-2}\int_\xi^{\xi_1}\theta^n(\xi')\ \theta'(\xi')\ d\xi' = 2\theta^{n+1}(\xi)/(n+1)\xi_1^2. \end{aligned} \tag{6.2.128}$$

Since at the initial moment we have $R(0) = 1$, we infer from Eq. (6.2.125) that Br_{1i}^{-N} represents just the initial central density ϱ_{0i}, and $Br_{1i}^{-N}R^{-N}(1) = Br_{1f}^{-N}$ the final central density ϱ_{0f}, $(\varrho_{0f}/\varrho_{0i} = (r_{1i}/r_{1f})^N = R^{-N}(1);\ \theta(0) = 1)$. The final central pressure reads via Eq. (6.2.126) as: $P_{0f} = 4\pi G\varrho_{0f}^2 r_{1f}^2/(n+1)\xi_1^2 = K\varrho_{0f}^{1+1/n}$.

The temperature T is simply determined from the perfect gas law (1.2.5). The whole theory suffers from the fact that the function $R(\tau)$ is unknown, and may be determined from the first law of thermodynamics, i.e. from the rate (6.1.39) of gravitational energy generation ε_g per unit mass [McVittie 1956b, Eq. (3.04)]:

$$\begin{aligned} \varepsilon_g' &= \varrho\varepsilon_g = -[P/(\gamma-1)][\partial\ln P/\partial t - \gamma\ \partial\ln\varrho/\partial t] = -[1/(\gamma-1)t_f]\big\{(Br_{1i}^{2-N}/\xi_1^2 t_f^2) \\ &\times\big[R^{1-N}(\tau)\ d^3R/d\tau^3 + (1-N+N\gamma)R^{-N}(\tau)\ (dR/d\tau)(d^2R/d\tau^2)\big][(N-2)\ \theta(\xi) + \xi\theta'(\xi) \\ &-\xi_1\theta'(\xi_1)] + [4\pi GB^2 r_{1i}^{2-2N}\theta^{n+1}(\xi)/(n+1)\xi_1^2][(2-2N+N\gamma)R^{1-2N}(\tau)\ dR/d\tau]\big\}. \end{aligned} \tag{6.2.129}$$

We have inserted for ϱ and P from Eqs. (6.2.125) and (6.2.126), respectively. In the present case of a perfect gas the factor c_VT/χ_T from Eq. (6.1.39) becomes by virtue of Eqs. (1.2.22), (1.3.3) equal to $P/\varrho(\gamma-1),\ (\chi_T=1;\ \Gamma_1=\gamma=c_P/c_V=\mathrm{const})$.

McVittie (1956b) enumerates eight conditions which have to be fulfilled by the initial and final values of the dimensionless boundary radius $R(\tau)=r_1(t)/r_{1i}$ and by its first three derivatives. The initial and final values of $R(\tau)$ are clearly equal to $R(0)=1$ and $R(1)=r_{1f}/r_{1i}$, respectively, where generally $R(1)\ll 1,\ (r_{1f}\ll r_{1i})$. Since the initial and final state is one of rest $(v=0)$, the initial and final values of $dR/d\tau$ are via Eq. (6.2.110) equal to $R_\tau(0), R_\tau(1)=0$. The acceleration of a mass element at an arbitrary moment is

$$Dv/Dt=\partial v/\partial t+v\ \partial v/\partial r=rf_{tt}/f=\zeta f_{\tau\tau}/t_f^2=r_{1i}\zeta R_{\tau\tau}/t_f^2\propto R_{\tau\tau}. \qquad (6.2.130)$$

The initial state is not one of equilibrium, and the configuration just starts moving inward, with a negative acceleration proportional to $R_{\tau\tau}(0)=b<0$. The final state has been assumed to be a hydrostatic polytrope and therefore $R_{\tau\tau}(1)=0$. The collapse should be free of shock waves, i.e. without infinite pressure gradients and accelerations. This implies by virtue of Eq. (6.2.130) that $R_{\tau\tau}$ has no singularity in the interval $0\leq\tau\leq 1$.

The gravitational energy generation ε_g' is obviously zero at the beginning of contraction, but could be different from zero if the final state is one of quasistatic contraction. From Eq. (6.2.129) results $R_{\tau\tau\tau}(0)=0,\ R_{\tau\tau\tau}(1)=c\leq 0$. Obviously, these eight limiting conditions are very far from determining the function $R(\tau)$; a polynomial of the lowest order that verifies the eight limiting conditions has been selected by McVittie (1956b) as follows:

$$R(\tau)=1-[1-R(1)]\tau^2(8\tau^3-15\tau^2+10)/3;\quad R_{\tau\tau}(0)=b=-20[1-R(1)]/3;$$
$$R_{\tau\tau\tau}(1)=c=-40[1-R(1)],\qquad (R(1)=r_{1f}/r_{1i}<1). \qquad (6.2.131)$$

The acceleration, which is proportional to $R_{\tau\tau}=-20[1-R(1)](8\tau^3-9\tau^2+1)/3$, changes its sign (the collapse starts to be decelerated) at $\tau=(33^{1/2}+1)/16=0.42$, corresponding to the moment when the velocity $\propto R_\tau$ is greatest.

Ibáñez and Sigalotti (1983) have evaluated Eqs. (6.2.125), (6.2.126) at the initial and final moment if $N=1,2,3$, and $\gamma=5/3$. The flow remains always subsonic towards the centre, and becomes strongly supersonic towards the boundary. Pressure, density, and temperature increase during collapse in the spherical $N=3$ case by factors of about $10^{22}, 10^{16}$, and 10^6, respectively, in accordance to the estimates of McVittie (1956b). However, if $N=1,2$, the numerical results of Ibáñez and Sigalotti (1983) seem to be discordant.

6.2.3 Collapsing Polytropic Stellar Cores and Expanding Polytropes

The supernova progenitors, which are massive enough to ignite carbon nonexplosively, are thought to consist of a hot degenerate iron core of 1-3 $M_\odot$ with overlaying nuclear burning shells of lighter elements. And these cores resemble to some extent the structure of an isentropic $n=3,\ (\Gamma_1=1+1/n=4/3)$ polytrope (Goldreich and Weber 1980). When the collapsing core bounces to form a neutron star, the equation of state will become stiffer at densities $\gtrsim 10^{12}$ g cm^{-3}, resembling a nonrelativistically degenerate neutron gas, i.e. a $n=1.5$ polytrope [cf. Eqs. (1.7.33)-(1.7.34)]. However, the $n=3$ equation of state should be reasonable at lower densities, which means that the proposed model describes only the early stages of collapse, before a degenerate neutron gas or nuclear densities are reached.

The equation of motion (5.2.10) can be transformed with the vector identity $\nabla(v^2/2)=(\vec v\cdot\nabla)\vec v+\vec v\times(\nabla\times\vec v)$ into

$$D\vec v/Dt=\partial\vec v/\partial t+(\vec v\cdot\nabla)\vec v=\partial\vec v/\partial t+\nabla(v^2/2)-\vec v\times(\nabla\times\vec v)=-\nabla P/\varrho+\nabla\Phi=-\nabla H+\nabla\Phi,$$
$$(dH=dP/\varrho=4K\varrho^{-2/3}\ d\varrho/3;\ n=3). \qquad (6.2.132)$$

$H=4K\varrho^{1/3}$ denotes the enthalpy from Eq. (3.8.82). The collapse is assumed to be vorticity-free $\nabla\times\vec v=0$, and in this case the velocity may be obtained via Eq. (B.29) from a stream function $u=u(\vec r,t)$, where $\vec v=\nabla u,\ (\nabla\times\nabla u=0)$. Then, Eq. (6.2.132) can be integrated:

$$\partial u(\vec r,t)/\partial t+[\nabla u(\vec r,t)]^2/2+H(\vec r,t)-\Phi(\vec r,t)=0. \qquad (6.2.133)$$

The time-dependent scale factor $\alpha(t)$ adopted by Goldreich and Weber (1980) is given by Eq. (2.1.13), where the central density ϱ_0 is now a temporal function:

$$\alpha = \alpha(t) = (K/\pi G)^{1/2} \varrho_0^{-1/3}(t), \qquad (n = 3). \tag{6.2.134}$$

The radius vector $\vec{r}$ is scaled to the dimensionless Lane-Emden variable $\vec{\xi}$:

$$\vec{\xi} = \vec{\xi}(\vec{r}, t) = \vec{r}/\alpha(t); \quad \partial\vec{\xi}(\vec{r}, t)/\partial\vec{r} = 1/\alpha(t); \quad \partial\vec{\xi}(\vec{r}, t)/\partial t = -[\vec{r}/\alpha^2(t)]\ d\alpha/dt. \tag{6.2.135}$$

The equation of continuity (5.2.1), the equation of motion (6.2.133), and Poisson's equation (2.1.4) are transformed with this time-dependent scaling into

$$\begin{aligned} &\partial\varrho(\vec{r}, t)/\partial t + \nabla\cdot[\varrho(\vec{r}, t)\ \nabla u(\vec{r}, t)] = \partial\varrho(\vec{\xi}, t)/\partial t + \nabla\varrho(\vec{\xi}, t)\cdot(\partial\vec{\xi}/\partial t) + \nabla\cdot[\varrho(\vec{\xi}, t)\ \nabla u(\vec{\xi}, t)]/\alpha^2(t) \\ &= \partial\varrho(\vec{\xi}, t)/\partial t - \vec{r}\cdot\nabla\varrho(\vec{\xi}, t)\ (d\alpha/dt)/\alpha^2(t) + \nabla\cdot[\varrho(\vec{\xi}, t)\ \nabla u(\vec{\xi}, t)]/\alpha^2(t) = \partial\varrho(\vec{\xi}, t)/\partial t \\ &-\vec{\xi}\cdot\nabla\varrho(\vec{\xi}, t)\ (d\alpha/dt)/\alpha(t) + \nabla\varrho(\vec{\xi}, t)\cdot\nabla u(\vec{\xi}, t)/\alpha^2(t) + \varrho(\vec{\xi}, t)\ \nabla^2 u(\vec{\xi}, t)/\alpha^2(t) = 0, \end{aligned} \tag{6.2.136}$$

$$\begin{aligned} &\partial u(\vec{\xi}, t)/\partial t + \nabla u(\vec{\xi}, t)\cdot(\partial\vec{\xi}/\partial t) + [\nabla u(\vec{\xi}, t)]^2/2\alpha^2(t) + H(\vec{\xi}, t) - \Phi(\vec{\xi}, t) \\ &= \partial u(\vec{\xi}, t)/\partial t - \vec{\xi}\cdot\nabla u(\vec{\xi}, t)\ (d\alpha/dt)/\alpha(t) + [\nabla u(\vec{\xi}, t)]^2/2\alpha^2(t) + H(\vec{\xi}, t) - \Phi(\vec{\xi}, t) = 0, \end{aligned} \tag{6.2.137}$$

$$\nabla^2\Phi(\vec{r}, t) = \nabla^2\Phi(\vec{\xi}, t)/\alpha^2(t) = -4\pi G\varrho(\vec{\xi}, t), \tag{6.2.138}$$

where the functional dependence is considered under the form $f(\vec{r}, t) = f[\vec{\xi}(\vec{r}, t), t] = f(\vec{\xi}, t)$.

Next, we scale the density in terms of central density

$$\varrho = \varrho(\vec{\xi}, t) = \varrho_0(t)\ \theta^3(\vec{\xi}, t) = (K/\pi G)^{3/2}\alpha^{-3}(t)\ \theta^3(\vec{\xi}, t), \tag{6.2.139}$$

and the potential in terms of the square of central sound speed $a_0^2 = \Gamma_1 P_0/\varrho_0 = 4K\varrho_0^{1/3}/3,\ (\Gamma_1 = 4/3)$:

$$\Phi = \Phi(\vec{\xi}) = a_0^2(t)\ \psi(\vec{\xi}) = 4K\varrho_0^{1/3}(t)\ \psi(\vec{\xi})/3 = 4(K^3/\pi G)^{1/2}\psi(\vec{\xi})/3\alpha(t). \tag{6.2.140}$$

Goldreich and Weber (1980) assume the collapse to occur homologously, i.e. the relative mass distribution in the stellar core remains invariant during collapse, and any point of the collapsing core is located always at the same fraction of the total radius (Cox and Giuli 1968). Hence, the density profile inside the collapsing core should not evolve: $\partial(\varrho/\varrho_0)/\partial t \propto \partial\theta(\vec{\xi}, t)/\partial t = 0$. Such a homologous collapse is achieved by taking the velocity under the radial form (6.2.92):

$$\begin{aligned} &\vec{v} = \vec{v}(\vec{\xi}, t) = \vec{v}[\vec{\xi}(\vec{r}, t), t] = \vec{v}(v_\xi, v_\lambda, v_\varphi) = [\vec{r}/\alpha(t)]\ d\alpha(t)/dt = \vec{\xi}\ d\alpha/dt, \\ &[v_\xi = \xi\ d\alpha/dt;\ v_\lambda, v_\varphi = 0;\ \vec{\xi} = \vec{\xi}(\xi, 0, 0)]. \end{aligned} \tag{6.2.141}$$

The resulting velocity potential is

$$u = u(\vec{\xi}, t) = (\xi^2\alpha/2)\ d\alpha/dt, \qquad [\vec{v}(\vec{\xi}, t) = \nabla u(\vec{\xi}, t)/\alpha(t)]. \tag{6.2.142}$$

If this is inserted together with Eq. (6.2.139) into Eq. (6.2.136), we indeed obtain the homology condition $\partial\theta(\vec{\xi}, t)/\partial t = 0$. And the Eulerian equation of motion (6.2.137) turns with $H = 4K\varrho^{1/3}$ into

$$[\theta(\xi) - \psi(\xi)/3]/\xi^2 = -(\pi G/64K^3)^{1/2}\alpha^2(t)\ d^2\alpha/dt^2. \tag{6.2.143}$$

Both sides of this equation, depending separately on ξ and t, have to be equated with a constant, say $C/6$. We get

$$\psi(\xi) = 3\theta(\xi) - C\xi^2/2, \tag{6.2.144}$$

and

$$d^2\alpha/dt^2 = -4C(K^3/\pi G)^{1/2}/3\alpha^2(t). \tag{6.2.145}$$

Eq. (6.2.145) can be integrated after multiplication with $d\alpha/dt$:

$$(d\alpha/dt)^2 = 8C(K^3/\pi G)^{1/2}/3\alpha + D, \qquad (D = \mathrm{const}). \tag{6.2.146}$$

As seen from Eq. (6.2.134), we have $\alpha \to \infty$ if $\varrho_0 \to 0$ and $r_1 \propto \varrho_m^{-1/3} \propto \varrho_0^{-1/3} \to \infty$. Therefore, if $\alpha \to \infty$, the constant D is proportional to the square of the velocity at infinity $v_\xi = \xi\ d\alpha/dt$. If this velocity at infinity is zero, i.e. if $D = 0$, we can integrate Eq. (6.2.146) again:

$$\alpha^{3/2} = -(6C)^{1/2}(K^3/\pi G)^{1/4}t + E, \qquad (d\alpha/dt < 0;\ E = \mathrm{const};\ C > 0;\ D = 0). \tag{6.2.147}$$

Let us denote by $t = 0$ the moment at which the radius is zero: $r_1 \propto \varrho_0^{-1/3} \propto \alpha = 0$. Eq. (6.2.147) now reads

$$\alpha = (6C)^{1/3}(K^3/\pi G)^{1/6}(-t)^{2/3}, \tag{6.2.148}$$

the time decreasing from negative values ($t < 0$) to $t = 0$. In a realistic core collapse we should have $D \neq 0$, because the initial velocity $v_\xi \propto d\alpha/dt$ is zero at some finite α. However, the value of D should have little effect on the solution as $\alpha(t) \to 0$. Poisson's equation (6.2.138) becomes with Eqs. (6.2.139), (6.2.140), (6.2.144) equal to

$$(1/\xi^2)\ d(\xi^2\ d\theta/d\xi)/d\xi = -\theta^3 + C, \tag{6.2.149}$$

with the central initial conditions $\theta(0) = 1,\ \theta'(0) = 0$. In the limit $C = 0$, this reduces to the Lane-Emden equation (2.1.14) if $N, n = 3$.

In order to assure real values of $d\alpha/dt$, the constant C in Eq. (6.2.146) must be nonnegative if $D = 0$. There exists a maximum value of $C = C_m = 0.006544$ for which a physical solution subsists with vanishing density $\varrho \propto \theta^3 = 0$ on the surface (Goldreich and Weber 1980). Mathematically, this limit occurs when $\varrho \propto \theta^3$ becomes tangent to the axis $\theta = 0$, i.e. when $d\theta/d\xi$ vanishes just on the surface $\xi = \xi_1$ of the polytrope: $\theta'(\xi_1) = 0$. Analogously to Eq. (2.6.27), the ratio between mean and central density becomes via Eq. (6.2.149) equal to

$$\varrho_m = M_1/V_1 = \int_0^{\xi_1} 4\pi\varrho\xi^2\ d\xi/(4\pi\xi_1^3/3) = (3\varrho_0/\xi_1^3)\int_0^{\xi_1} \xi^2\theta^3\ d\xi = \varrho_0(C - 3\theta'(\xi_1)/\xi_1),$$
$$(\theta'(\xi_1) \leq 0). \tag{6.2.150}$$

Since $\theta'(\xi_1) = 0$ if $C = C_m$, we find in this limiting case: $C = C_m = \varrho_m/\varrho_0 = 0.006544$. Physically, the limiting value $C = C_m$ is just reached when the core surface $r = r_1$ is in free fall, i.e. when $(Dv_r/Dt)_{r=r_1} = -GM_1/r_1^2$. Inserting from Eq. (6.2.141) $v_\xi = \xi\ d\alpha/dt = (r/\alpha)\ d\alpha/dt = v_r$, we get

$$(Dv_r/Dt)_{r=r_1} = (\partial v_r/\partial t)_{r=r_1} + (v_r\ \partial v_r/\partial r)_{r=r_1} = (r_1/\alpha)\ d^2\alpha/dt^2 - (r_1/\alpha^2)(d\alpha/dt)^2$$
$$+(r_1/\alpha^2)(d\alpha/dt)^2 = (r_1/\alpha)\ d^2\alpha/dt^2 = -GM_1/r_1^2, \qquad (v_\lambda, v_\varphi = 0). \tag{6.2.151}$$

Inserting for α and $d^2\alpha/dt^2$ from Eqs. (6.2.134) and (6.2.145), we find

$$C = (3\alpha^3 M_1/4r_1^3)(\pi G^3/K^3)^{1/2} = 3M_1/4\pi\varrho_0 r_1^3 = \varrho_m/\varrho_0 = C_m, \tag{6.2.152}$$

which proves our previous affirmation.

With Eqs. (6.2.134), (6.2.135) the core mass reads

$$M_1 = 4\pi\varrho_m r_1^3/3 = 4\pi\varrho_m\alpha^3(t)\ \xi_1^3/3 = 4\pi\varrho_m\xi_1^3(K/\pi G)^{3/2}/3\varrho_0. \tag{6.2.153}$$

If C increases from its minimum value $C = 0$ to its maximum value C_m, the surface value ξ_1 of the radial Lane-Emden coordinate increases from 6.897 to 9.889, while the ratio ϱ_m/ϱ_0 decreases from 0.01846 (Table 2.5.2) to $C_m = 0.006544$. If M_1 and C are specified, Eq. (6.2.152) yields the value of the polytropic constant K, and it is found that K, or equivalently, the pressure P decreases by no more than 2.9% for the admissible range of C, $(0 \leq C \leq 0.006544)$. Thus, if the pressure at a given density is reduced by more than 2.9% with respect to its value for the neutrally (marginally) stable Lane-Emden polytrope ($n = 3,\ \Gamma_1 = 1 + 1/n = 4/3,\ C = 0$), no homologous collapse of the entire core is possible. But a less massive inner core can do so, while the remainder of the core is left behind, and this neglect of the pressure of the outer core [which is in free fall if $C = C_m$ by virtue of Eq. (6.2.152)] constitutes the main limitation of the Goldreich-Weber model (Yahil 1983). Low order radial and nonradial modes of the homologously collapsing $n = 3$ solutions have been calculated by Goldreich and Weber (1980), and are found to be essentially stable. But if $n = 1/(\Gamma_1 - 1) > 3$ the similarity solutions are unstable against nonradial (vortex) modes (Hanawa and Matsumoto 2000a). They are unstable too against bar modes $\propto Y_2^k(\lambda, \varphi)$ if $n > 10.3$, $(\Gamma_1 = 1 + 1/n < 1.097)$, (Hanawa and Matsumoto 2000b, Lai 2000).

Yahil (1983) has numerically integrated the Eulerian equation of motion for a self-similar core collapse with a polytropic equation of state if $3 \leq n \leq 5$. Collapse (contraction) of an isentropic polytrope requires via Eq. (2.6.100) $\Gamma = \Gamma_1 = 1 + 1/n < 4/3$ or $n > 3$. The essence of the considered self-similar model is the existence of only two dimensional parameters, i.e. the polytropic constant K and the gravitational constant G. Yahil (1983) works with the dimensionless parameter

$$X = X(r,t) = K^{-1/2}G^{1/2n}r\ (-t)^{1/n-1},$$
$$\big(n = 1/(\Gamma_1 - 1);\ [G] = [\mathrm{g^{-1}cm^3s^{-2}}];\ [K] = [\mathrm{g^{-1/n}cm^{2+3/n}s^{-2}}]\big), \tag{6.2.154}$$

where, as before, the origin of time is chosen to be the catastrophic moment at which the radius r_1 becomes zero $(t \leq 0)$. All hydrodynamic variables must be functions of X only, except for a dimensional scale factor (e.g. Sedov 1959):

$$\varrho(r,t) = G^{-1}(-t)^{-2}D_*(X);\quad v(r,t) = K^{1/2}G^{-1/2n}(-t)^{-1/n}U(X);$$
$$M(r,t) = K^{3/2}G^{-1-3/2n}(-t)^{1-3/n}m(X). \tag{6.2.155}$$

The relationship between the dimensionless functions $D_*(X)$ and $m(X)$ is obtained from the mass relationship

$$M(r,t) = 4\pi\int_0^r \varrho r'^2\ dr' = 4\pi K^{3/2}G^{-1-3/2n}(-t)^{1-3/n}\int_0^X D_*(X')\ X'^2\ dX'. \tag{6.2.156}$$

In virtue of Eqs. (6.2.155), (6.2.156) we have

$$m(X) = 4\pi\int_0^X D_*(X')\ X'^2\ dX'. \tag{6.2.157}$$

Eqs. (6.2.154), (6.2.155), (B.37) are substituted into the equation of continuity (5.2.1), and into the equation of motion (5.2.10), to obtain analogously to Eqs. (6.2.136), (6.2.137):

$$[U + (1 - 1/n)X]\ dD_*/dX + D_*\ dU/dX + 2D_* + 2\ D_*U/X = 0, \tag{6.2.158}$$

$$(1 + 1/n)D_*^{-1+1/n}\ dD_*/dX + [U + (1 - 1/n)X]\ dU/dX + U/n + m/X^2 = 0, \tag{6.2.159}$$

where again $f(r,t) = f[X(r,t),t] = f(X,t)$, and $\nabla P = (1 + 1/n)K\varrho^{1/n}\ \nabla\varrho,\ \nabla\Phi = -GM(r)/r^2$.

Yahil (1983) turns to a comoving frame which collapses exactly homologously, i.e. it is the noninertial frame where the fluid element remains stationary if $n = 3$, $(\Gamma_1 = 1 + 1/n = 4/3)$. To this end, we divide the dimensionless fluid velocity $U(X)$ into a homologous part $(1/n - 1)X$ plus the velocity $U_1(X)$ with respect to the homologous frame of the "zooming coordinates". $U_1(X)$ cancels in the case of homologous collapse when $n = 1/(\Gamma_1 - 1) = 3$ (cf. Eq. (6.2.164), Hanawa and Matsumoto 2000b):

$$U(X) = (1/n - 1)X + U_1(X) = (\Gamma_1 - 2)X + U_1(X). \tag{6.2.160}$$

Eqs. (6.2.158) and (6.2.159) become, respectively

$$U_1\ dD_*/dX + D_*\ dU_1/dX = (1 - 3/n)D_* - 2D_*U_1/X, \tag{6.2.161}$$

$$(1 + 1/n)D_*^{-1+1/n}\ dD_*/dX + U_1\ dU_1/dX = (1 - 1/n)X/n + (1 - 2/n)U_1 - m/X^2. \tag{6.2.162}$$

Eq. (6.2.161) can be written under the form

$$X^2\ d(D_*U_1) + 2D_*U_1X\ dX = (1 - 3/n)D_*X^2\ dX, \tag{6.2.163}$$

which is integrated via Eq. (6.2.157) to give

$$4\pi X^2 D_* U_1 = (1-3/n)m, \qquad (m(0)=0). \tag{6.2.164}$$

The Larson-Penston-Yahil solution of a contracting (collapsing) mass is given by the transonic solution of Eqs. (6.2.157), (6.2.162), (6.2.164), which is supersonic at large distances X (Lai 2000). The core mass splits into a homologously collapsing inner core, and an outer core that is supersonically infalling at about half the free-fall velocity (Yahil 1983). Nonradial perturbations are amplified during the subsequent accretion of the outer core (Lai and Goldreich 2000, Lai 2000). Post-collapse solutions subsist if $t > 0$ (Yahil 1983).

Similarity solutions for the gravitational collapse of polytropic gaseous spheres have also been considered by Suto and Silk (1988) under the restrictive condition that the similarity variable $\alpha(t)$ from Eq. (6.2.134) is proportional to t^c, ($c = 1$ and $c = 1 - 1/n$). And for the collapse of cylindrical polytropic clouds similarity solutions have been found by Kawachi and Hanawa (1998) if $-\infty < n < -1$.

Bonazzola and Marck (1993) estimate the gravitational radiation from $n = 1/(\Gamma_1 - 1) = 3$ collapsing cores.

A hydrostatic isentropic $n = 3$, $(\Gamma = \Gamma_1 = 1 + 1/n = 4/3)$ polytrope is by virtue of Eq. (2.6.100) in neutral equilibrium, and may be subject to contraction or expansion under the action of some perturbations (e.g. decrease of Γ_1 below the critical value 4/3, or sudden increase of total radiant energy due to nuclear reactions). The rapid expansion of a $n = 3$ polytrope is to some extent the opposite of the core collapse already discussed, and has been considered by Barnes and Boss (1984), by combining the previously outlined treatment of Goldreich and Weber (1980) and Yahil (1983). The lower mass limit of a massive, hot, radiation dominated object has been given by Eq. (5.12.68). And such an object can be approximated by a $n = 3$ polytrope at the verge of instability. Of course, a massive star with a primarily hydrogen core will not undergo catastrophic collapse like supernova progenitors, because of the different microphysics involved. Rather, it will undergo expansion if a small pressure excess subsists, giving the star a positive total energy, ensuring in this way that evolution progresses along the expansion branch of instability.

The similarity variable (6.2.154) is written by Barnes and Boss (1984) under the form

$$X \equiv \xi = \xi(r,t) = Ar/t^{2/3}, \qquad (n = 3;\ A = \mathrm{const}), \tag{6.2.165}$$

which will be shown later in Eq. (6.2.178) to be just equal to the dimensionless Lane-Emden distance (2.1.13). The hydrodynamic variables are defined analogously to (6.2.155):

$$\varrho(r,t) = D_*(\xi)/4\pi G t^2; \quad P(r,t) = r^2 p(\xi)/4\pi G t^4; \quad v(r,t) = rU(\xi)/t; \quad M(r,t) = r^3 m(\xi)/Gt^2. \tag{6.2.166}$$

The continuity equation (5.2.3), the equation of motion (5.2.10), and the energy equation (5.2.21) become for spherically symmetric expansion, respectively

$$DM/Dt = \partial M/\partial t + v\ \partial M/\partial r = 0, \qquad (\partial M/\partial r = 4\pi\varrho r^2), \tag{6.2.167}$$

$$Dv/Dt = \partial v/\partial t + v\ \partial v/\partial r = -(1/\varrho)\ \partial P/\partial r - GM/r^2, \tag{6.2.168}$$

$$D(P\varrho^{-4/3})/Dt = 0. \tag{6.2.169}$$

The mass (6.2.166) is solely a function of ξ :

$$M(r,t) = \xi^3 m(\xi)/GA^3 = M(\xi). \tag{6.2.170}$$

Therefore, the sphere $\xi = \mathrm{const}$ comoves with the expanding fluid, and

$$D\xi/Dt = 0 \quad \mathrm{or} \quad \partial(r/t^{2/3})/\partial t + v(r,t)\ \partial(r/t^{2/3})/\partial r = 0, \tag{6.2.171}$$

which amounts to

$$v(r,t) = 2r/3t; \quad U(\xi) = 2/3. \tag{6.2.172}$$

The similarity functions $p(\xi)$ and $m(\xi)$ can be expressed with the aid of the density profile $D_*(\xi)$ by using the polytropic law $P = K\varrho^{4/3}$, and the mass equation (6.2.156):

$$p(\xi) = KA^2 D_*^{4/3}(\xi)/(4\pi G)^{1/3}\xi^2; \quad m(\xi) = (1/\xi^3)\int_0^\xi D_*(\xi')\ \xi'^2\ d\xi'. \tag{6.2.173}$$

Thus, we are left to seek a differential equation for the similarity function $D_*(\xi)$. This can be achieved if we multiply Eq. (6.2.168) by r^2. The left-hand side becomes

$$r^2\ Dv/Dt = (2r^2/3)\ D(r/t)/Dt = (2r^2/3)(-r/t^2 + 2r/3t^2) = -2\xi^3/9A^3. \tag{6.2.174}$$

The pressure term can be transformed as follows:

$$(r^2/\varrho)\ \partial P/\partial r = 4Kr^2\ \partial\varrho^{1/3}/\partial r = [4K\xi^2 t^{2/3}/A^2(4\pi G)^{1/3}](dD_*^{1/3}/d\xi)\ \partial\xi/\partial r$$
$$= [4KD_{*0}^{1/3}\xi^2/A(4\pi G)^{1/3}]\ d\theta/d\xi = (D_{*0}\xi^2/A^3)\ d\theta/d\xi, \qquad [n=3;\ D_{*0} = D_*(0)]. \tag{6.2.175}$$

The scale factor from Eq. (6.2.165) has been specified according to

$$A = D_{*0}^{1/3}(4\pi G)^{1/6}/2K^{1/2}, \tag{6.2.176}$$

and the Lane-Emden function (2.1.10) is equal to

$$\theta = [\varrho(r,t)/\varrho(0,t)]^{1/3} = (\varrho/\varrho_0)^{1/3} = [D_*(\xi)/D_*(0)]^{1/3} = (D_*/D_{*0})^{1/3}. \tag{6.2.177}$$

With the specification (6.2.176) it is easy to show that the similarity variable (6.2.165) is just equal to the dimensionless Lane-Emden variable ξ from Eq. (2.1.13):

$$r = t^{2/3}\xi/A = 2K^{1/2}t^{2/3}\xi/D_{*0}^{1/3}(4\pi G)^{1/6} = (K/\pi G)^{1/2}\xi/\varrho_0^{1/3}, \qquad (n=3;\ D_{*0} = 4\pi G t^2\varrho_0). \tag{6.2.178}$$

Combining Eqs. (6.2.174), (6.2.175), the equation of motion (6.2.168) reads

$$2\xi^3/9A^3 = (D_{*0}\xi^2/A^3)\ d\theta/d\xi + GM. \tag{6.2.179}$$

If we derive this equation with respect to ξ, we get an equation analogous to Eq. (6.2.149):

$$(1/\xi^2)\ d(\xi^2\ d\theta/d\xi)/d\xi = -\theta^3 + 2/3D_{*0}, \qquad (C = 2/3D_{*0}). \tag{6.2.180}$$

The derivative $dM/d\xi$ has been inserted from

$$\partial M/\partial r = (dM/d\xi)\ \partial\xi/\partial r = (A/t^{2/3})\ dM/d\xi = 4\pi\varrho r^2 = D_*(\xi)\ \xi^2/GA^2t^{2/3} = D_{*0}\xi^2\theta^3/GA^2t^{2/3};$$
$$dM/d\xi = D_{*0}\xi^2\theta^3/GA^3. \tag{6.2.181}$$

As confirmed also by the numerical evaluations of Barnes and Boss (1984), the quantity $2/3D_{*0}$ is just equal to the constant C from Eq. (6.2.149). Physically admissible solutions occur if $D_{*0m} = 101.9 \leq D_{*0} \leq \infty$, corresponding to $0 \leq C \leq C_m = 0.006544$, $(D_{*0m} = 2/3C_m)$.

The profile of the Mach number

$$v/a = (2r/3t)/(4K\varrho^{1/3}/3)^{1/2} = (4\pi G)^{1/6}\xi/K^{1/2}AD_*^{1/6}(\xi) = 2\xi/(3D_{*0}\theta)^{1/2}, \qquad (\Gamma_1 = 4/3), \tag{6.2.182}$$

is obtained with the aid of Eqs. (2.1.49), (6.2.165), (6.2.176). All solutions with $D_{*0m} \leq D_{*0} \lesssim 10^4$ show a few percent of the total mass expanding at a significant fraction of the local sound speed, or even supersonically in the outermost layers of the star. The similarity solutions describe the explosive expansion of an isentropic $n = 1/(\Gamma_1 - 1) = 3$ polytrope.

It is also instructive to consider the velocity v_1 of the expanding surface as compared to the escape velocity

$$v_1/(2GM_1/r_1)^{1/2} = (2\xi_1^3/A^3GM_1)^{1/2}/3 = 1/[1 - 9D_{*0}\theta'(\xi_1)/2\xi_1]^{1/2}, \tag{6.2.183}$$

where r_1, v_1, and GM_1 have been replaced according to Eqs. (6.2.165), (6.2.172), and (6.2.179), respectively.

If $D_{*0} = \infty$, the surface velocity is zero, and Eq. (6.2.180) turns into the hydrostatic Lane-Emden equation (2.1.14). The maximum expansion velocity occurs if $D_{*0} = D_{*0m}$; in this case we have $\theta'(\xi_1) = 0$, and the polytropic surface always expands with the local velocity of escape, just analogously, but opposite in sign to the free-fall velocity of collapsing cores from Eq. (6.2.151) if $C = C_m$.

6.2.4 Embedded Polytropes

This topic has applications to interstellar (intergalactic) gas clouds within uniform background matter, as for example a uniform neutrino background or background matter composed of uniformly distributed stars or dusts (dark matter) in interstellar (intergalactic) space. Komatsu and Seljak (2001), for instance, use a polytropic relationship with $5 < n < 10$ for the density-temperature profile of hot gases in dark matter halos of clusters of galaxies. Let us denote by P and ϱ pressure and density of polytropic matter embedded within uniform background matter of density ϱ_b. The pressure equilibrium of the embedded polytropic matter is given by the hydrostatic equation (2.1.3)

$$\nabla P = dP/dr = K\ d\varrho^{1+1/n}/dr = K(1+1/n)\varrho^{1/n}\ d\varrho/dr = \varrho\ d\Phi/dr, \tag{6.2.184}$$

while Poisson's equation (2.1.4) for the total potential Φ now reads [cf. Eqs. (6.1.202), (C.15)]

$$\nabla^2\Phi = r^{1-N}\ d(r^{N-1}\ d\Phi/dr)/dr = r^{1-N}\ d[(r^{N-1}/\varrho)\ dP/dr]/dr = -4\pi G(\varrho+\varrho_b),$$
$$(N = 1, 2, 3, ...). \tag{6.2.185}$$

N denotes the dimension of the considered spatial geometry, and the total density at a given point is equal to the density $\varrho = \varrho(r)$ of the polytropic configuration plus the constant density ϱ_b of uniform background matter. Umemura and Ikeuchi (1986) tacitly ignore the problems resulting from Seeliger's paradox concerning uniform background matter of infinite extension and mass (e.g. Horedt 1971, 1989).

With the Lane-Emden variables (2.1.10) and (2.1.13) the equation (6.2.185) assumes a modified form of the Lane-Emden equation (2.1.14):

$$\xi^{1-N}\ d(\xi^{N-1}\ d\theta/d\xi)/d\xi = \mp\theta^n \mp \delta, \qquad (\delta = \varrho_b/\varrho_0 = \mathrm{const};\ n \neq -1, \pm\infty;\ \theta(0) = 1;\ \theta'(0) = 0), \tag{6.2.186}$$

where the upper sign holds if $-1 < n < \infty$, and the lower one if $-\infty < n < -1$. Incidentally, Eq. (6.2.186) is of the same form as the equilibrium equation (3.2.2) of an axisymmetric rotating polytrope if $N = 3$, $\Theta(\xi,\mu) \to \theta(\xi)$, and $\beta \to -\delta$.

By inserting Eqs. (2.1.18) and (2.1.20) into Eq. (6.2.185), we get in the special case $n = \pm\infty$:

$$\xi^{1-N}\ d(\xi^{N-1}\ d\theta/d\xi)/d\xi = \exp(-\theta) + \delta, \qquad (\delta = \varrho_b/\varrho_0 = \mathrm{const};\ n = \pm\infty;\ \theta(0), \theta'(0) = 0). \tag{6.2.187}$$

The series expansion of the Lane-Emden function near the origin can easily be deduced in the same way as effected for Eqs. (2.4.21) and (2.4.36), respectively:

$$\theta \approx 1 \mp (1+\delta)\xi^2/2N + n(1+\delta)\xi^4/2^3N(N+2) \mp [n^2N(1+\delta) + n(n-1)(N+2)(1+\delta)^2]\xi^6$$
$$/2^4 \times 3N^2(N+2)(N+4) + ..., \qquad (\xi \approx 0;\ n \neq -1, \pm\infty), \tag{6.2.188}$$

$$\theta \approx (1+\delta)\xi^2/2N - (1+\delta)\xi^4/2^3N(N+2) + [N(1+\delta) + (N+2)(1+\delta)^2]\xi^6$$
$$/2^4 \times 3N^2(N+2)(N+4) - ..., \qquad (\xi \approx 0;\ n = \pm\infty). \tag{6.2.189}$$

Particular solutions of Eq. (6.2.186) if $n = 0$ and 1 are obtained in the same manner as in Eqs. (2.3.5), (2.3.21), (2.3.22), (2.3.26):

$$\theta = 1 - (1+\delta)\xi^2/2N, \qquad (n = 0;\ N = 1, 2, 3, ...), \tag{6.2.190}$$

$$\theta = (1+\delta)(N/2-1)!\ (\xi/2)^{(2-N)/2}J_{(N-2)/2}(\xi) - \delta, \qquad (n = 1;\ N = 2, 4, 6, ...), \tag{6.2.191}$$

$$\theta = (1+\delta)(N-2)(N-4)...5 \times 3 \times 1 \times (-1)^{(N-3)/2}\ d^{(N-3)/2}(\sin\xi/\xi)/(\xi\ d\xi)^{(N-3)/2} - \delta,$$
$$(n = 1;\ N = 3, 5, 7, ...), \tag{6.2.192}$$

$$\theta = (1+\delta)\cos\xi - \delta, \qquad (n=1;\ N=1), \tag{6.2.193}$$

$$\theta = (1+\delta)\ J_0(\xi) - \delta, \qquad (n=1;\ N=2), \tag{6.2.194}$$

$$\theta = (1+\delta)\sin\xi/\xi - \delta, \qquad (n=1;\ N=3). \tag{6.2.195}$$

The mass of the polytrope, embedded in the uniform medium, is given by [cf. Eqs. (2.6.12), (2.6.13)]

$$M = \{2[\Gamma(1/2)]^N/\Gamma(N/2)\}\int_0^r \varrho r'^{N-1}\ dr' = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\int_0^\xi \theta^n\xi'^{N-1}\ d\xi'$$
$$= \{2[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}[\varrho_0(\mp d\theta/d\xi) - \varrho_b\xi/N], \qquad (n\neq -1,\pm\infty), \tag{6.2.196}$$

$$M = \{2\varrho_0[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\int_0^\xi \exp(-\theta)\ \xi'^{N-1}\ d\xi'$$
$$= \{2[\alpha\Gamma(1/2)]^N/\Gamma(N/2)\}\xi^{N-1}(\varrho_0\ d\theta/d\xi - \varrho_b\xi/N), \qquad (n=\pm\infty). \tag{6.2.197}$$

The mean density of the embedded polytrope turns out in a quite similar way as effected in Eqs. (2.6.27), (2.6.28):

$$\varrho_m = N\varrho_0(\mp d\theta/d\xi)/\xi - \varrho_b,\ (n\neq -1,\pm\infty) \qquad \text{and} \qquad \varrho_m = N\varrho_0(d\theta/d\xi)/\xi - \varrho_b,\ (n=\pm\infty). \tag{6.2.198}$$

Due to the nonzero density ratio $\delta = \varrho_b/\varrho_0$, the discussion of the finiteness of mass and radius of embedded polytropes is considerably simpler than without background matter (cf. Sec. 2.6.8). We formally integrate Eq. (6.2.186) between the limits ξ_c and ξ :

$$\theta(\xi) = \mp\int_{\xi_c}^{\xi} \xi'^{1-N}\ d\xi' \int_{\xi_c}^{\xi'} \theta^n(\xi'')\ \xi''^{N-1}\ d\xi'' \mp \delta\xi^2/2N + C\xi^{2-N}/(2-N) + D,$$
$$(n\neq -1,\pm\infty;\ \xi_c, C, D = \text{const};\ N\neq 2). \tag{6.2.199}$$

The logarithmic term $C\ln\xi$ appears instead of $C\xi^{2-N}/(2-N)$ in the particular case $N=2$, the final result (6.2.203) being otherwise the same.

We have to distinguish three different cases:

(i) $\boldsymbol{-1 < n < \infty}$. If we assume the radius ξ_1 of the embedded polytrope to be infinite, from Eq. (6.2.199) would result that $\theta_1 = \theta(\xi_1) = -\infty$, $(N\geq 1)$, contradicting the basic requirement $\theta(\xi)\geq 0$. Therefore, radius and mass (6.2.196) of the embedded polytrope, having $-1<n<\infty$, must be finite.

(ii) $\boldsymbol{-\infty < n < -1}$. In order to satisfy the requirement that $P\propto\theta^{n+1}\to 0$ at the boundary ξ_1 of the embedded polytrope, we must have $\theta(\xi_1)=\infty$, and from Eq. (6.2.199) follows at once that $\xi_1=\infty$. Although the radius of embedded polytropes with polytropic indices $-\infty<n<-1$ is infinite, we will show that their mass is often finite, analogously to the Schuster-Emden polytropes from Eq. (2.6.193), having $n=(N+2)/(N-2)$. Because $\theta^n/\delta\to 0$ if $\xi\to\infty$, we can always find a certain value ξ_c of ξ such that there subsists $\theta^n(\xi)/\delta < 1$ if $\xi>\xi_c$. And in this case we can write down the inequality

$$\int_{\xi_c}^{\xi} \xi'^{N-1}\ d\xi' = (\xi^N - \xi_c^N)/N > \int_{\xi_c}^{\xi} \theta^n\xi'^{N-1}\ d\xi'/\delta. \tag{6.2.200}$$

We multiply by $\xi^{1-N}\ d\xi$ and integrate again, assuming $\xi\to\infty$, $(\xi\gg\xi_c)$:

$$\delta\xi^2/2N - \delta\xi_c^N\xi^{2-N}/N(2-N) + E \approx \delta\xi^2/2N > \int_{\xi_c}^{\xi}\xi'^{1-N}\ d\xi'\int_{\xi_c}^{\xi'}\theta^n(\xi'')\ \xi''^{N-1}\ d\xi'',$$
$$(E=\text{const};\ N\geq 1;\ N\neq 2). \tag{6.2.201}$$

If $\xi\to\infty$, Eq. (6.2.199) writes as

$$\theta \approx \int_{\xi_c}^{\xi}\xi'^{1-N}\ d\xi'\int_{\xi_c}^{\xi'}\theta^n(\xi'')\ \xi''^{N-1}\ d\xi'' + \delta\xi^2/2N, \qquad (N\geq 1). \tag{6.2.202}$$

And if we now introduce the inequality (6.2.201) into Eq. (6.2.202), we find the delimitation

$$\theta < \delta\xi^2/N, \qquad (\xi \to \infty;\ \xi \gg \xi_c;\ -\infty < n < -1). \qquad (6.2.203)$$

In virtue of Eq. (6.2.196) the difference between the total mass $M_1 = M(\xi_1) = M(\infty)$ and the finite mass $M(\xi_c)$ contained within coordinate radius ξ_c is bounded by

$$M(\infty) - M(\xi_c) \propto \int_{\xi_c}^{\infty} \theta^n \xi^{N-1}\, d\xi < (\delta/N)^n \int_{\xi_c}^{\infty} \xi^{2n+N-1}\, d\xi = [(\delta/N)^n/(2n+N)]\xi^{2n+N}\Big|_{\xi_c}^{\infty},$$
$$(n \neq -N/2). \qquad (6.2.204)$$

The last integral is finite if $2n + N < 0$, and the total mass M_1 of embedded polytropes with indices $-\infty < n < -1$ is finite if $n < -N/2$, i.e. the mass of embedded polytropic slabs ($N = 1$) and cylinders ($N = 2$) is always finite, whereas embedded spheres ($N = 3$) certainly have finite mass if $-\infty < n < -1.5$.

With the initial conditions $\theta(0) = 1,\ \theta'(0) = 0$ we can also obtain an inferior delimitation of the Lane-Emden function

$$\theta > \delta\xi^2/2N, \qquad (\xi_c = 0;\ C = 0;\ D = 1;\ N \geq 1;\ -\infty < n < -1;\ n = \pm\infty), \qquad (6.2.205)$$

valid for any $\xi > 0,\ (\xi_c = 0)$, by observing that the integral in Eq. (6.2.199) is always positive. Eq. (6.2.205) is also valid if $n = \pm\infty$ by virtue of Eq. (6.2.206). And with the delimitation (6.2.205) we infer similarly to Eq. (6.2.204) that embedded spheres have infinite total mass if $-1.5 \leq n < -1$.

(iii) $\boldsymbol{n = \pm\infty}$. We formally integrate Eq. (6.2.187) between the limits 0 and ξ with the initial conditions $\theta(0), \theta'(0) = 0$:

$$\theta(\xi) = \int_0^{\xi} \xi'^{1-N}\, d\xi' \int_0^{\xi'} \exp[-\theta(\xi'')]\ \xi''^{N-1}\, d\xi'' + \delta\xi^2/2N. \qquad (6.2.206)$$

Since $\theta(\xi_1) \to \infty$, we conclude from Eq. (6.2.206) that $\xi_1 = \infty$, i.e. the radius of embedded isothermal configurations is infinite. The mass however, remains finite. Because the integral in Eq. (6.2.206) is always positive, we obtain the inferior delimitation (6.2.205). In virtue of Eqs. (6.2.197), (6.2.205) the total mass of the isothermal embedded polytrope is bounded by

$$M_1 = M(\infty) \propto \int_0^{\infty} \exp(-\theta)\ \xi^{N-1}\, d\xi < \int_0^{\infty} \exp(-\delta\xi^2/2N)\ \xi^{N-1}\, d\xi, \qquad (N \geq 1). \qquad (6.2.207)$$

The last integral can readily be calculated if N is an even number, and can be reduced to the error integral if N is odd. This integral is always finite, and the same is true for the total mass $M_1 = M(\infty)$ of embedded isothermal polytropes.

The stability of complete embedded polytropes can be investigated with the Zeldovich instability criterion (5.12.26). The explicit dependence of the masses (6.2.196) or (6.2.197) on central density ϱ_0 follows, if we observe from Eq. (2.1.13) that $\alpha \propto \varrho_0^{(1/n-1)/2}$, which becomes $\alpha \propto \varrho_0^{-1/2}$ if $n = \pm\infty$:

$$M_1 \propto \varrho_0^{1+N(1/n-1)/2} \xi_1^{N-1}(\mp\theta_1' - \varrho_b\xi_1/N\varrho_0), \qquad (n \neq -1, 0). \qquad (6.2.208)$$

Stability occurs if

$$(\partial M_1/\partial\varrho_0)_{S=\mathrm{const}} \propto [1 + N(1/n - 1)/2]\xi_1^{N-1}(\mp\theta_1' - \varrho_b\xi_1/N\varrho_0)$$
$$+\varrho_0\ d[\xi_1^{N-1}(\mp\theta_1' - \varrho_b\xi_1/N\varrho_0)]/d\varrho_0 > 0, \qquad (n \neq -1, 0), \qquad (6.2.209)$$

where ξ_1 and θ_1' depend on $\delta = \varrho_b/\varrho_0 \propto 1/\varrho_0$. Umemura and Ikeuchi (1986) have found by numerical integration that the last term in Eq. (6.2.209) is positive if $N = 1, 2, 3$ and $n = 1,\ 1.5,\ 2,\ 3,\ 4,\ 5,\ 10$, $(0 \leq \delta \leq 10)$. Thus, complete embedded polytropes are stable for the considered polytropic indices if $1 + N(1/n - 1)/2 > 0,\ (\mp\theta_1' - \varrho_b\xi_1/N\varrho_0 \propto M_1 > 0)$. If $N = 1, 2, 3$, this condition yields $(n + 1)/2n > 0,\ 1/n > 0,\ (3 - n)/2n > 0$, respectively. Embedded polytropic slabs ($N = 1$) and cylinders ($N = 2$) are always stable for the considered values of n, $(1 \leq n \leq 10)$, and δ, $(0 \leq \delta \leq 10)$, while the stability of embedded polytropic spheres ($N = 3$) is depicted in Fig. 6.2.2 for the considered values of n, δ.

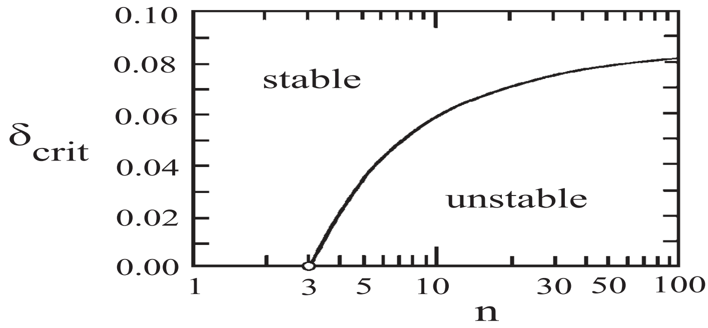

Fig. 6.2.2 Stabilization of otherwise unstable polytropic spheres having $3 < n \leq \infty$, due to the presence of uniform background matter, having a density ratio $\delta = \varrho_b/\varrho_0$ larger than the critical value δ_{crit}. If $n \to \infty$, the parameter δ_{crit} approaches 0.1 (Umemura and Ikeuchi 1986).

If $3 < n \leq \infty$, $(N = 3)$, Umemura and Ikeuchi (1986) have found numerically that a certain critical value $\delta_{crit} = \varrho_{b,crit}/\varrho_0$ exists above which polytropic spheres – that are unstable in absence of a background medium – become stable. A certain amount of background matter stabilizes polytropic spheres having $n > 3$ and $\delta > \delta_{crit}$ (Fig. 6.2.2). They are unstable if $0 \leq \delta < \delta_{crit}$ and $3 < n \leq \infty$.

This stabilization effect of background matter may be intuitively understood from the fact that the background medium smoothes out the huge density differences between centre and surface layers, making these polytropes "more homogeneous": The ratio ϱ_m/ϱ_0 between mean and central density is increased by the presence of background matter, and the density distribution of embedded polytropic spheres having $n > 3$ resembles the structure of stable polytropes $(n < 3)$ without background matter, provided that the density ϱ_b of background matter is sufficiently high.

The effect of a background medium on the radial stability of truncated (incomplete) polytropes under external pressure is similar. Incomplete embedded polytropes under external pressure must always be stable if the complete embedded polytrope with zero external pressure is stable. If the complete embedded polytrope is unstable, there results – in analogy to the stability of polytropes without background medium (Sec. 5.4.2) – that the incomplete embedded polytrope is radially stable under a fixed external pressure only up to a certain mass limit [the "Jeans mass", Eqs. (5.4.57)-(5.4.63)]. For larger masses the incomplete (truncated) polytrope will be radially unstable, like the corresponding complete embedded polytrope with the same n, ϱ_0, ϱ_b (Horedt 2000a).

In fact, from an inspection of Eqs. (5.12.19)-(5.12.26) we observe the equivalence – at least for polytropes – of the two criteria, viz. $(\partial M_1/\partial \varrho_0)_{S=\mathrm{const}} > 0$ [Eqs. (5.12.26), (6.2.209)] and $(\Delta P/\Delta r)_{M=\mathrm{const}} > 0$ [(Eq. (5.4.23)], which have been used to establish the stability of complete polytropes and the instability of incomplete polytropes, respectively.

Gerhard and Silk (1996) suggest that some of the unseen dark matter in the outer halos of galaxies may be in the form of cold dense gas clouds with mass $\approx 1\ M_\odot$, temperature $\lesssim 10$ K, and polytropic index $5 \leq n \leq 10$. In accordance with the previous findings these polytropic gas clouds can be stabilized against gravitational collapse if they are embedded in a background medium of small objects or particles.

6.3 Polytropic Winds

6.3.1 Summers' General Solution

In the next two sections 6.3 and 6.4 we discuss the radial expansion of a polytropic gas from a point source [solar, stellar, cometary, planetary, galactic wind (e.g. Holzer and Axford 1970)], as well as the related problem of accretion of polytropic gas onto a point mass with applications to stellar and black hole mass accretion from interstellar clouds (e.g. Foglizzo and Ruffert 1997, Font and Ibáñez 1998), the formation of comets (e.g. Fahr 1980, p. 273), the accretion wake model of galactic and quasar jets [motion of a galaxy (quasar) through intracluster gas (Yabushita 1979)], X-ray binaries [interaction of a compact component (neutron star, black hole) with the stellar wind originating from a giant component in a binary system when the giant star overflows its Roche lobe (e.g. Shapiro and Teukolsky 1983, Fig. 13.11)].

Generally, the polytropic gas is assumed to obey the perfect gas law (1.2.5) in the presence of a gravitating point mass. The polytropic law (2.1.6), to be used for the description of expansion/accretion flows, serves as an energy equation, which can be remarkably successful in modeling the physical phenomena occurring in the flow (e.g. heat conduction, viscosity, radiation losses, dissipation of shock waves, magnetic energy), whenever the precise energy transfer equations are not known or are intractable (Parker 1963, Summers 1980). If the flow is isentropic, we have by Eq. (1.3.30) $\Gamma_1 = 1 + 1/n$. If additionally, the isentropic flow material consists of perfect gas, we have [see Eqs. (1.3.24), (6.3.4)]

$$\gamma = c_P/c_V = \Gamma_1 = \Gamma_2 = \Gamma_3 = 1 + 1/n; \quad n = 1/(\gamma - 1), \qquad (P = \mathcal{R}\varrho T/\mu). \tag{6.3.1}$$

γ denotes the adiabatic exponent (1.2.32) of a perfect gas, and Γ_1 the adiabatic exponent for a general equation of state from Eqs. (1.3.23), (1.3.30). As we have already noted in Sec. 1.2, the perfect adiabatic gas is always isentropic [Eq. (1.2.41)]. The Mach number $M_A = v/a$ can be expressed for a perfect adiabatic (isentropic) gas under the form

$$M_A^2 = v^2/a^2 = \varrho v^2/\gamma P = \mu v^2/\gamma\mathcal{R}T = v^2/K(1+1/n)\varrho^{1/n} = \varrho v^2/(1+1/n)P$$
$$= \mu v^2/(1+1/n)\mathcal{R}T, \qquad (a^2 = \gamma P/\varrho;\ \gamma = 1 + 1/n), \tag{6.3.2}$$

where v denotes the flow velocity, and a the adiabatic sound velocity from Eq. (2.1.49).

Following Summers (1980) it seems useful to present at first the general topological properties of the relevant equations in a more mathematical form. Supplementation with specific boundary (initial) conditions leads then to the various astrophysical applications mentioned at the beginning. The influence of magnetic fields and shock waves is disregarded for the moment (see Secs. 6.3.3, 6.3.4).

For a polytropic perfect gas the equation of state is given by Eqs. (1.2.5) and (2.1.6):

$$P = K\varrho^{1+1/n} = \mathcal{R}\varrho T/\mu, \qquad (K, n = \mathrm{const}). \tag{6.3.3}$$

If the perfect gas is adiabatic, i.e. isentropic, we have by Eq. (1.2.32)

$$P = K\varrho^{1+1/n} = K\varrho^{\gamma} = \mathcal{R}\varrho T/\mu, \qquad (S = \mathrm{const};\ \gamma = 1 + 1/n). \tag{6.3.4}$$

If the physical processes occurring in the wind are adiabatic and reversible (no shock waves, for instance), the entropy remains constant. Indeed, Eq. (1.2.38) can be written in this case as $dS = c_V[dT/T - (\gamma - 1)\ d\varrho/\varrho]$, or after integration [cf. Holzer and Axford 1970, Eq. (2.5)]:

$$S = c_V \ln(T\varrho^{1-\gamma}) + \mathrm{const} = c_V \ln(\mu P\varrho^{-\gamma}/\mathcal{R}) + \mathrm{const} = c_V \ln(\mu K/\mathcal{R}) + \mathrm{const} = \mathrm{const},$$
$$(c_V, \gamma = \mathrm{const};\ \gamma = 1 + 1/n = c_P/c_V = 1 + \mathcal{R}/\mu c_V). \tag{6.3.5}$$

Note, that it is not required that the polytropic flow is isentropic, i.e. generally $\gamma \neq 1 + 1/n$. Isentropic flows [Eq. (6.3.4)] occur only as a particular case of the much more general polytropic flows.

Let us first consider the equation of motion (5.2.10) of a polytropic perfect gas along a stationary (steady) streamline, where s measures the distance along the flux tube:

$$D\vec{v}/Dt = v\ dv/ds = -(1/\varrho)\ dP/ds + d\Phi/ds = -(1+1/n)K\varrho^{1/n-1}\ d\varrho/ds + d\Phi/ds. \qquad (6.3.6)$$

Integration yields

$$v^2(s)/2 + (n+1)P(s)/\varrho(s) - \Phi(s) = v^2(s)/2 + (n+1)(P_0/\varrho_0)[\varrho(s)/\varrho_0]^{1/n} - \Phi(s)$$
$$= v_0^2/2 + (n+1)P_0/\varrho_0 - \Phi_0 = \mathrm{const}, \qquad (n \neq \pm\infty), \qquad (6.3.7)$$

where the integration constant takes in general different values on different streamlines (Landau and Lifshitz 1959, §9; Parker 1963), and the zero subscript denotes values at some reference point. If $n = \pm\infty$, the pressure term $(n+1)P/\varrho$ should be replaced by $K \ln\varrho = (P/\varrho)\ln\varrho = (P_0/\varrho_0)\ln\varrho = (\mathcal{R}T/\mu)\ln\varrho$, $(T = T_0 = \mu K/\mathcal{R} = \mathrm{const})$. If we denote by $A(s)$ the cross-section of a stream-tube, the mass flux conservation equation becomes

$$\varrho(s)\ v(s)\ A(s) = \varrho_0 v_0 A_0 = \mathrm{const}. \qquad (6.3.8)$$

For the sake of simplicity it is generally assumed that wind and accretion flows are basically radial. In fact, the effect of solar rotation, for instance, on the hydrodynamic expansion of the corona is slight, so the angular momentum imparted to the solar wind gas by the angular velocity of the Sun can be neglected in a first approximation (see however Sec. 6.3.3). The distance s along a streamline is approximated by the radial distance $s \approx r$. If the flow tube is strictly radial, the ratio of two cross-sections is equal to $A(s)/A_0 = (r/r_0)^2$, which may be generalized to $A(s)/A_0 = (r/r_0)^b$, where $b > 2$ and $b < 2$ corresponds to flux tubes diverging, respectively, more rapidly and less rapidly than radial ones [cf. Eqs. (6.3.95)-(6.3.110)].

For a spherically symmetric flow in the presence of a gravitating point mass M, the equation of motion (momentum conservation) reads (cf. Eq. (6.3.6) if $r \equiv s,\ \Phi = GM/r$) :

$$v\ dv/dr = (1/2)\ dv^2/dr = -(1/\varrho)\ dP/dr - GM/r^2 = -(1+1/n)K\varrho^{1/n-1}\ d\varrho/dr - GM/r^2$$
$$= -(\mathcal{R}/\mu\varrho)(T\ d\varrho/dr + \varrho\ dT/dr) - GM/r^2. \qquad (6.3.9)$$

The mass conservation equation (6.3.8) writes

$$F = \varrho v r^2 = \varrho_0 v_0 r_0^2. \qquad (6.3.10)$$

Eqs. (6.3.3), (6.3.9), (6.3.10) may be combined into a nondimensional form by introducing the new dimensionless quantities (Parker 1963, Brandt 1970, Summers 1980)

$$\lambda = \mu GM/\mathcal{R}T_0 r; \quad \psi = \psi(\lambda) = \mu v^2(\lambda)/\mathcal{R}T_0; \quad \tau = \tau(\lambda) = T(r)/T_0. \qquad (6.3.11)$$

The reciprocal distance $\lambda \propto 1/r$ should not be confused with the polar angle of a spherical coordinate system, while T_0 denotes the flow temperature at reference level $r = r_0$. From Eq. (6.3.10) we get by logarithmic differentiation $(1/\varrho)\ d\varrho/dr = -(1/2v^2)\ dv^2/dr - 2/r$, and from Eq. (6.3.11) $dr = -\mu GM\ d\lambda/\mathcal{R}T_0\lambda^2$. The desired dimensionless form of Eq. (6.3.9) is after insertion:

$$(1/2 - \tau/2\psi)\ d\psi/d\lambda = 1 - 2\tau/\lambda - d\tau/d\lambda, \qquad (n \neq 0). \qquad (6.3.12)$$

In the special case $n = 0$, $(\varrho = \mathrm{const})$ the equation of motion (6.3.9) reads

$$v\ dv/dr = (1/2)\ dv^2/dr = -(\mathcal{R}/\mu)\ dT/dr - GM/r^2, \qquad (n = 0). \qquad (6.3.13)$$

Inserting from Eq. (6.3.11), this equation takes the particular form

$$(1/2)\ d\psi/d\lambda = 1 - d\tau/d\lambda, \qquad (n = 0). \qquad (6.3.14)$$

From Eqs. (6.3.3) and (6.3.10) we obtain another relationship

$$\varrho^{1/n} = (\varrho_0 v_0 r_0^2/vr^2)^{1/n} = \mathcal{R}T/\mu K, \qquad (n \neq 0), \qquad (6.3.15)$$

taking the dimensionless form

$$\lambda^{-2/n}\psi^{1/2n}\tau = 1 \quad \text{or} \quad \psi = \lambda^4\tau^{-2n}, \qquad (n \neq 0, 3/2), \tag{6.3.16}$$

with the reference temperature suitably chosen as

$$T_0 = (\varrho_0 v_0 r_0^2)^{2/(2n-3)}(\mu/\mathcal{R})K^{2n/(2n-3)}(GM)^{-4/(2n-3)}, \qquad (n \neq 0, 3/2). \tag{6.3.17}$$

In the particular case $n = 0$ we obtain from the mass conservation equation (6.3.10): $vr^2 = v_0 r_0^2$, ($\varrho =$ const). The dimensionless equivalent of Eq. (6.3.16) is

$$\psi = \lambda^4, \qquad (n = 0). \tag{6.3.18}$$

The reference temperature T_0 from Eq. (6.3.17) has been chosen in such a way that $\psi/\lambda^4 = 1$:

$$T_0 = (v_0 r_0^2)^{-2/3}(\mu/\mathcal{R})(GM)^{4/3}, \qquad (n = 0;\ \varrho = \text{const}). \tag{6.3.19}$$

In the other special case $n = 3/2$ we observe, by raising Eq. (6.3.17) to the $(2n-3)$-th power, that T_0 is arbitrary if $n = 3/2$, and we get analogously to Eq. (6.3.16):

$$\lambda^{-4/3}\psi^{1/3}\tau = C \quad \text{or} \quad \psi = C^3\lambda^4\tau^{-3}, \qquad (n = 3/2), \tag{6.3.20}$$

where

$$C = (\varrho_0 v_0 r_0^2)^{2/3}K(GM)^{-4/3} = \text{const}, \qquad (n = 3/2). \tag{6.3.21}$$

The basic differential equations to be used for the topological study of the wind/accretion problem can be derived by inserting into Eq. (6.3.12) for τ and $d\tau/d\lambda$ from Eqs. (6.3.16) and (6.3.20), respectively:

$$(1/2)\left[1 - (1 + 1/n)\lambda^{2/n}\psi^{-1/2n-1}\right] d\psi/d\lambda = 1 - 2(1 + 1/n)\lambda^{2/n-1}\psi^{-1/2n}, \qquad (n \neq 0, 3/2), \tag{6.3.22}$$

$$(1/2)\left[1 - 5C/3)\lambda^{4/3}\psi^{-4/3}\right] d\psi/d\lambda = 1 - (10C/3)\lambda^{1/3}\psi^{-1/3}, \qquad (n = 3/2). \tag{6.3.23}$$

Clearly, Eq. (6.3.22) is also valid in the limiting isothermal case $n = \pm\infty$, yielding simply

$$(1/2)(1 - 1/\psi)\ d\psi/d\lambda = 1 - 2/\lambda, \qquad (n = \pm\infty). \tag{6.3.24}$$

In the hydrostatic case $v = 0$, ($\psi = 0$) we get from Eqs. (6.3.3), (6.3.9) the temperature run ($T\ d\varrho = n\varrho\ dT$; $(1/\varrho)\ dP/dr = (n+1)(\mathcal{R}/\mu)\ dT/dr = -GM/r^2$) :

$$d\tau/d\lambda = 1/(n+1) \quad \text{or} \quad \tau - \lambda/(n+1) = \varepsilon = \text{const}, \qquad (v, \psi = 0;\ n \neq -1). \tag{6.3.25}$$

The reference temperature T_0 is arbitrary in this particular case.

In the special case $n = 0$ we get for the temperature run by integration of Eq. (6.3.14) via Eq. (6.3.18):

$$\tau + \psi/2 - \lambda = \tau + \lambda^4/2 - \lambda = \varepsilon = \text{const}, \qquad (n = 0;\ \psi = \lambda^4). \tag{6.3.26}$$

Integration of Eqs. (6.3.22), (6.3.24) yields the so-called Bernoulli integrals

$$\psi/2 - \lambda + (n+1)(\lambda^2/\psi^{1/2})^{1/n} = \varepsilon = \text{const}, \qquad (n \neq 0, 3/2, \pm\infty), \tag{6.3.27}$$

$$\psi/2 - \lambda + \ln(\lambda^2/\psi^{1/2}) = \varepsilon = \text{const}, \qquad (n = \pm\infty). \tag{6.3.28}$$

The Bernoulli integral for the special case $n = 3/2$ is obtained by integration of Eq. (6.3.23):

$$\psi/2 - \lambda + (5C/2)(\lambda^2/\psi^{1/2})^{2/3} = \varepsilon = \text{const}, \qquad (n = 3/2;\ C = \text{const}). \tag{6.3.29}$$

The Bernoulli integrals (6.3.27)-(6.3.29) represent conservation of the total gas energy. Multiplying the Bernoulli integrals by $\mathcal{R}T_0/\mu$, it becomes obvious that the energy constant ε is composed of the kinetic energy per unit mass $v^2/2 \propto \psi/2$, the gravitational potential energy (2.6.69) of the mass unit $-GM/r \propto -\lambda$, and the heat function per unit mass H (the enthalpy), where $dH = dP/\varrho$, (cf. Eq. (3.8.82), Landau and Lifshitz 1959). If $n \neq \pm\infty$, we have $H = (n+1)P/\varrho = (n+1)\mathcal{R}T_0\tau/\mu = (n+1)\mathcal{R}T_0\lambda^{2/n}\psi^{-1/2n}/\mu$. If $n = \pm\infty$, there is $T = T_0$, and $H = (\mathcal{R}T_0/\mu)\ln\varrho + \mathrm{const} = -(\mathcal{R}T_0/\mu)\ln(r^2v) + \mathrm{const} = (\mathcal{R}T_0/\mu)\ln(\lambda^2/\psi^{1/2})$.

The analytic study of the topology of Eqs. (6.3.27)-(6.3.29) if $\lambda, \psi \geq 0$ involves the singular points $(d\psi/d\lambda = 0)$, the behaviour near the ψ-axis, where $\lambda \approx 0$, $(r \to \infty)$, and the asymptotic forms $\lambda \to \infty$, $(r \to 0)$, similarly to the topological survey of the Lane-Emden equation effected in Sec. 2.7. If we rewrite Eqs. (6.3.22) and (6.3.23) under the form

$$d\psi/d\lambda = 2\psi\ \left[\psi^{1/2n} - 2(1+1/n)\lambda^{2/n-1}\right]\Big/\left[\psi^{1/2n+1} - (1+1/n)\lambda^{2/n}\right], \qquad (n \neq 0, 3/2), \qquad (6.3.30)$$

$$d\psi/d\lambda = 2\psi\ \left[\psi^{1/3} - (10C/3)\lambda^{1/3}\right]\Big/\left[\psi^{4/3} - (5C/3)\lambda^{4/3}\right], \qquad (n = 3/2), \qquad (6.3.31)$$

it becomes obvious that the origin $O_s(0,0)$ is a singular point of Eqs. (6.3.30) and (6.3.31) if $n \neq -1, 0$. Figs. 6.3.1-6.3.6 exhibit that O_s is always a node. In the isothermal case $n = \pm\infty$, Eqs. (6.3.24), (6.3.30) become $d\psi/d\lambda = (2\psi/\lambda)(\lambda-2)/(\psi-1) = 0/0$ if $\lambda, \psi = 0$. Incidentally, if $n = -1$, Eq. (6.3.27) reads $\psi = 2\lambda + 2\varepsilon$, and possesses no singular points [cf. Eq. (6.3.61)]. Note, that this particular polytropic index leads to unphysical results for the *hydrostatic* structure of a $n = -1$ polytrope (cf. Sec. 2.1). And if $n = 0$, Eq. (6.3.18) shows that $d\psi/d\lambda = 4\lambda^3$, i.e. a singular point is missing too in this special case.

Another singular point $S_s(\lambda_s, \psi_s)$ results if we equate the two brackets of Eq. (6.3.30) to zero:

$$\lambda_s = 2^{(2n+1)/(2n-3)}(1+1/n)^{2n/(2n-3)}; \quad \psi_s = 2^{4/(2n-3)}(1+1/n)^{2n/(2n-3)} = \lambda_s/2,$$
$$(-\infty \leq n < -1;\ 0 < n \leq \infty;\ n \neq 3/2). \qquad (6.3.32)$$

Obviously, if $n = \pm\infty$, we have $\lambda_s = 2,\ \psi_s = 1$.

In conclusion, the singular point $O_s(0,0)$ exists whenever $n \neq -1, 0$, while the singular point $S_s(\lambda_s, \psi_s)$ exists only if $1 + 1/n > 0$, i.e. if $-\infty \leq n < -1,\ 0 < n \leq \infty,\ n \neq 3/2$. In the particular case of an isentropic perfect gas the singular point S_s is a sonic point, because the Mach number (6.3.2) is then just unity, as follows via Eqs. (6.3.11), (6.3.16), (6.3.32):

$$M_A^2 = \mu v_s^2/(1+1/n)\mathcal{R}T_s = \psi_s/(1+1/n)\tau_s = \lambda_s^{-2/n}\psi_s^{1+1/2n}/(1+1/n) = 1,$$
$$(\gamma = 1+1/n;\ -\infty \leq n < -1;\ 0 < n \leq \infty;\ n \neq 3/2). \qquad (6.3.33)$$

The nature of the solutions of Eq. (6.3.22) in the neighborhood of the singular point (6.3.32) – where the subsonic-supersonic transition may occur – is found by substituting $\lambda = \lambda_s + \lambda_1,\ \psi = \psi_s + \psi_1$ into Eq. (6.3.22), neglecting higher order terms:

$$d\psi_1/d\lambda_1 = [2\psi_1 + 2(n-2)\lambda_1]/[(2n+1)\psi_1 - 2\lambda_1]. \qquad (6.3.34)$$

Integration of this equation yields the conic

$$(2n+1)(\psi-\psi_s)^2 - 4(\psi-\psi_s)(\lambda-\lambda_s) - 2(n-2)(\lambda-\lambda_s)^2 = \mathrm{const}. \qquad (6.3.35)$$

If this second order curve is brought to the normal form $(\psi-\psi_s)^2/a_1^2 + (\lambda-\lambda_s)^2/a_2^2 = 1$ (e.g. Smirnow 1967), the minor axis a_2 is proportional to $2^{1/2}/\{5-[25+8n(2n-3)]^{1/2}\}^{1/2}$, which is real if $0 \leq n \leq 3/2$, and imaginary if $-\infty \leq n < 0$ and $3/2 < n \leq \infty$. Taking into account the domain of existence of the singular point S_s from Eq. (6.3.32), we infer that the integral curves are hyperbolas in the vicinity of the singular saddle point S_s if $-\infty \leq n < -1$ and $3/2 < n \leq \infty$, with two singular solutions passing through the saddle point (Figs. 6.3.3, 6.3.4, and Fig. 6.3.5 on the left). If $0 < n < 3/2$, the integral curves are ellipses, and S_s is a vortex point (a centre), without integral curves passing through the vortex (Fig. 6.3.1 on the left). The integration constant $\varepsilon = \varepsilon_s$ of the singular solution passing through S_s is found from Eqs. (6.3.27), (6.3.28), (6.3.32):

$$\varepsilon_s = [(2n-3)/2]\ 2^{4/(2n-3)}(1+1/n)^{2n/(2n-3)} = [(2n-3)/2]\ \psi_s = [(2n-3)/4]\ \lambda_s$$
$$\text{if} \quad -\infty < n < -1,\ 0 < n < \infty,\ n \neq 3/2; \qquad \varepsilon_s = \ln 4 - 3/2 \approx -0.114 \quad \text{if} \quad n = \pm\infty. \qquad (6.3.36)$$

Finally, let us discuss the behaviour of the integral curves if $\lambda \approx 0,\ (r \to \infty)$ and $\lambda \to \infty,\ (r \approx 0)$. Following Parker (1963), we will neglect in Eqs. (6.3.27)-(6.3.29) either the first or the third term

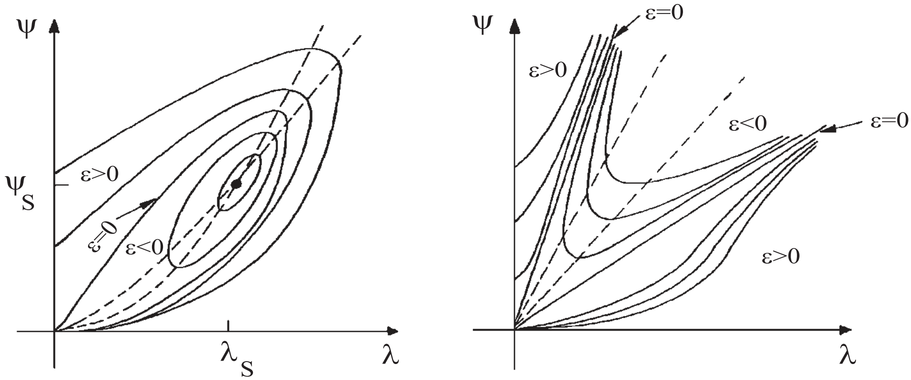

Fig. 6.3.1 Family of dimensionless velocity profiles ($\psi \propto v^2$) as a function of reciprocal distance ($\lambda \propto 1/r$) if $0 < n < 3/2$ (on the left), and $n = 3/2,\ 0 < C < 3/5 \times 2^{4/3} = 0.2381$ from Eq. (6.3.48), (on the right). The dot represents the singular vortex point S_s of coordinates (λ_s, ψ_s) with the energy constant $\varepsilon = \varepsilon_s < 0$. The domains where the energy constant ε takes positive, negative, and zero values are shown on all figures 6.3.1-6.3.6. Dashed lines represent the geometrical loci of the points on the solution curves where the velocity gradient $d\psi/d\lambda$ is zero or infinite (Summers 1980).

containing ψ. In this way we get at once a rough first approximation for $\psi = \psi(\lambda, \varepsilon)$, and we merely have to check the consistency of this approximation by inserting it into the neglected ψ-term.

Case 1, ($0 < n < 3/2$). (i) $\lambda \approx 0,\ \varepsilon > 0$. The singular point S_s is a vortex, and the solution curves take near the ψ-axis two limiting forms, which are obtained by neglecting the third and first term in Eq. (6.3.27), and inserting the resulting rough approximations $\psi \approx 2\varepsilon + 2\lambda$ and $\psi \approx [(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4$ into the third and first term of Eq. (6.3.27), respectively:

$$\psi = 2\varepsilon + 2\lambda - 2(n+1)(\lambda^2/\psi^{1/2})^{1/n} \approx 2\varepsilon + 2\lambda - 2^{1-1/2n}(n+1)[\lambda^2/(\varepsilon+\lambda)^{1/2}]^{1/n}$$
$$\approx 2\varepsilon + 2\lambda \approx 2\varepsilon > 0, \tag{6.3.37}$$

$$\psi = [(n+1)/(\varepsilon+\lambda-\psi/2)]^{2n}\lambda^4 \approx (n+1)^{2n}\lambda^4/\{\varepsilon+\lambda-[(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4/2\}^{2n}$$
$$\approx [(n+1)/\varepsilon]^{2n}\lambda^4 \approx 0. \tag{6.3.38}$$

(ii) $\lambda \approx 0,\ \varepsilon = 0$. In this special case the solution curve originates and terminates in the node O_s. The two branches are obtained by inserting the rough approximations $\psi \approx 2\lambda$ and $\psi \approx (n+1)^{2n}\lambda^{4-2n}$ into the third and first term of Eq. (6.3.27), respectively:

$$\psi = 2\lambda - 2(n+1)(\lambda^2/\psi^{1/2})^{1/n} \approx 2\lambda - 2^{1-1/2n}(n+1)\lambda^{3/2n} \approx 2\lambda \approx 0, \tag{6.3.39}$$

$$\psi = [(n+1)/(\lambda-\psi/2)]^{2n}\lambda^4 \approx \lambda^4\{(n+1)/\lambda[1-(n+1)^{2n}\lambda^{3-2n}/2]\}^{2n} \approx (n+1)^{2n}\lambda^{4-2n} \approx 0. \tag{6.3.40}$$

(iii) $\lambda \approx 0,\ \varepsilon < 0$. Eqs. (6.3.37) and (6.3.38) show that in this case no solution curves exist in the first real (λ, ψ)-quadrant if $\lambda \to 0,\ (\varepsilon < 0)$.

(iv) $\lambda \to \infty$. This subcase involves the behaviour of the wind/accretion flow near the central point mass M, where $r \approx 0$. Let us neglect at first the third term in Eq. (6.3.27), and insert the resulting rough approximation $\psi \approx 2\lambda,\ (\psi \gg 1)$ into the complete equation (6.3.27):

$$\psi = 2\varepsilon + 2\lambda - 2(n+1)(\lambda^2/\psi^{1/2})^{1/n} \approx 2\lambda[1 + \varepsilon/\lambda - 2^{-1/2n}(n+1)\lambda^{(3-2n)/2n}]. \tag{6.3.41}$$

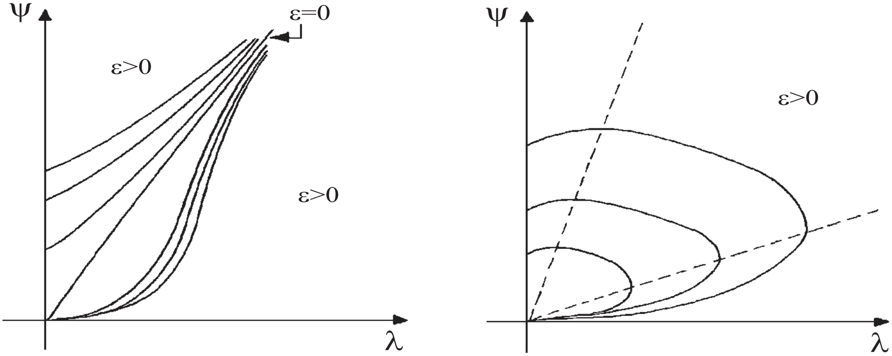

Fig. 6.3.2 Same as Fig. 6.3.1 for the polytropic index $n = 3/2$ if $C = 3/5 \times 2^{4/3} = 0.2381$ (on the left), and $C > 3/5 \times 2^{4/3}$ (on the right).

Let us now neglect the first term in Eq. (6.3.27), and insert the resulting rough approximation $\psi \approx (n+1)^{2n}\lambda^{4-2n}$, $(\psi \gg 1)$ into the complete Bernoulli equation (6.3.27):

$$\psi = \left[(n+1)/(\varepsilon + \lambda - \psi/2)\right]^{2n}\lambda^4 \approx (n+1)^{2n}\lambda^{4-2n}/\left[1 + \varepsilon/\lambda - (n+1)^{2n}\lambda^{3-2n}\right]^{2n}. \tag{6.3.42}$$

Obviously, if $0 < n < 3/2$, the terms associated with the minus sign in Eqs. (6.3.41) and (6.3.42) become the leading ones, and no integral curves exist in the real first (λ, ψ)-quadrant if $\lambda \to \infty$ (cf. Fig. 6.3.1 on the left).

Case 2, ($n = 3/2$). (i) $\lambda \approx 0$, $\varepsilon > 0$. We get similarly to Eqs. (6.3.37), (6.3.38), by neglecting the third and first term in Eq. (6.3.29), and inserting the resulting rough approximations $\psi \approx 2\varepsilon + 2\lambda$ and $\psi \approx [5C/2(\varepsilon + \lambda)]^3\lambda^4$ into the third and first term of Eq. (6.3.29), respectively:

$$\psi = 2\varepsilon + 2\lambda - 5C(\lambda^2/\psi^{1/2})^{2/3} \approx 2\varepsilon + 2\lambda - 2^{-1/3} \times 5C\lambda^{4/3}/(\varepsilon + \lambda)^{1/3} \approx 2\varepsilon + 2\lambda \approx 2\varepsilon > 0, \tag{6.3.43}$$

$$\psi = [5C/2(\varepsilon + \lambda - \psi/2)]^3\lambda^4 \approx (5C/2)^3\lambda^4/\left\{\varepsilon + \lambda - [5C/2(\varepsilon + \lambda)]^3\lambda^4/2]\right\}^3 \approx (5C/2\varepsilon)^3\lambda^4 \approx 0. \tag{6.3.44}$$

(ii) $\lambda \approx 0$, $\varepsilon = 0$. Exact solutions exist in this special case, valid for all λ if $C \leq 3/5 \times 2^{4/3} = 0.2381$. Dividing Eq. (6.3.29) by $\lambda^{4/3}/\psi^{1/3}$, we obtain

$$(\psi/\lambda)^{4/3} - 2(\psi/\lambda)^{1/3} + 5C = 0. \tag{6.3.45}$$

With the substitution $y = (\psi/\lambda)^{1/3}$ we get the fourth order algebraic equation

$$y^4 - 2y + 5C = 0. \tag{6.3.46}$$

The solutions of this equation (e.g. Bronstein and Semendjajew 1985, p. 133) are given by combinations $y_i = (\pm z_1^{1/2} \pm z_2^{1/2} \pm z_3^{1/2})/2$, $(i = 1, 2, 3, 4;\ z_1^{1/2}z_2^{1/2}z_3^{1/2} = -2)$ of the roots z_1, z_2, z_3 of the cubic resolvent of Eq. (6.3.46):

$$z^3 - 20Cz - 4 = 0. \tag{6.3.47}$$

The nature of the solutions in Eq. (6.3.46) depends on the sign of the discriminant of Eq. (6.3.47):

$$D = -(20C/3)^3 + 4. \tag{6.3.48}$$

If $D = 0$, i.e. $C = 3/5 \times 2^{4/3} = 0.2381$, Eq. (6.3.47) has the real solutions $z_1 = 2^{4/3},\ z_2 = z_3 = -2^{1/3}$, and Eq. (6.3.46) has a double positive real root $y_{1,2} = 2^{-1/3}$, the two other roots being complex conjugate. Thus, the exact solution valid for all λ is (Fig. 6.3.2 on the left):

$$\psi = \lambda/2, \qquad (C = 3/5 \times 2^{4/3} = 0.2381;\ n = 3/2;\ \varepsilon = 0). \tag{6.3.49}$$

Eq. (6.3.47) has at least one real positive root, since at $z = 0$ it takes the value -4. If $D > 0$, i.e. $C < 3/5 \times 2^{4/3}$, Eq. (6.3.47) has a positive and two complex conjugate roots, yielding two real and two complex conjugate roots of the fourth order equation (6.3.45). And we conclude from the considerations given below that the two real roots of Eq. (6.3.45) are always positive if C is in the range $0 < C < 3/5 \times 2^{4/3} = 0.2381$. The delimitation $C > 0$ results from Eq. (6.3.21). If $C \approx 3/5 \times 2^{4/3}$, $(C < 3/5 \times 2^{4/3})$, the two distinct real roots of Eq. (6.3.45) cannot differ too much from the root $\psi/\lambda = 1/2$ obtained for $C = 3/5 \times 2^{4/3}$, and are consequently positive. In the other limiting case $C \approx 0$, $(C > 0)$, the two real roots of Eq. (6.3.45) are positive again, taking the limiting forms $\psi/\lambda \approx 0$ and $\psi/\lambda \approx 2$. We conclude that for the considered small range of C-values, Eq. (6.3.45) has always two positive roots if $n = 3/2$ and $\varepsilon = 0$ (Fig. 6.3.1 on the right). If we denote by $\delta^{1/3}$ either one of the two positive roots of Eq. (6.3.45), the exact solution curve for all λ is

$$\psi = \delta\lambda, \qquad (0 < C < 3/5 \times 2^{4/3};\ n = 3/2;\ \varepsilon = 0). \tag{6.3.50}$$

If $D < 0$, i.e. $C > 3/5 \times 2^{4/3}$, the cubic (6.3.47) has three real roots, one positive and two negative, since the angle α of the "casus irreducibilis" is contained between 0 and $\pi/2$, $(0 \leq \cos\alpha = (3/5C)^{3/2}/4 < 1)$. The fourth order equation (6.3.45) or (6.3.46) possesses two pairs of complex conjugate roots (e.g. Bronstein and Semendjajew 1985), and no solution curves of Eq. (6.3.29) exist if $C > 3/5 \times 2^{4/3},\ \lambda \approx 0,\ \varepsilon = 0$ (Fig. 6.3.2 on the right).

(iii) $\lambda \approx 0,\ \varepsilon < 0$. No solution curves exist in the quadrant $\lambda, \psi > 0$, as seen from Eqs. (6.3.43) and (6.3.44).

(iv) $\lambda \to \infty$. Again, we divide Eq. (6.3.29) by $\lambda^{4/3}/\psi^{1/3}$, to obtain

$$(\psi/\lambda)^{4/3} - 2(1 + \varepsilon/\lambda)(\psi/\lambda)^{1/3} + 5C = 0. \tag{6.3.51}$$

If $\lambda \to \infty$, this equation becomes closely equal to Eq. (6.3.45), and the whole discussion parallels Case (ii), $\varepsilon = 0$. If for the moment we consider $2(1 + \varepsilon/\lambda)$ as a constant coefficient of $(\psi/\lambda)^{1/3}$, the discriminant of the resolvent of Eq. (6.3.51) would become $D = -(20C/3)^3 + 4(1 + \varepsilon/\lambda)^4$ [cf. Eq. (6.3.48)]. No asymptotic solutions exist if $D < 0$, i.e. if $C > 3(1 + \varepsilon/\lambda)^{4/3}/5 \times 2^{4/3} \approx 3/5 \times 2^{4/3}$, as already outlined under Case (ii), (see Fig. 6.3.2 on the right). The same is true if $C = 3/5 \times 2^{4/3} = 0.2381$ and $\varepsilon < 0$, because in this special case we have $D < 0$ too. Thus, asymptotic solutions are nonexistent if $C = 3/5 \times 2^{4/3}$, $(\varepsilon < 0)$, and $C > 3/5 \times 2^{4/3}$.

Case 3, ($3/2 < n < \infty$). Generally, two singular solutions exist, passing through the saddle point S_s, and the corresponding positive energy constant ε_s is given by Eq. (6.3.36).

(i) $\lambda \approx 0,\ \varepsilon > 0$. The relevant integral curves are those from Eqs. (6.3.37), (6.3.38), (see Figs. 6.3.3 and 6.3.4 on the left). Although the term associated with the minus sign in Eq. (6.3.37) becomes the leading one if $n > 2$, it ultimately approaches zero as $\lambda \to 0$. Eqs. (6.3.37) and (6.3.38) read

$$\psi \approx 2\varepsilon + O(\lambda, \lambda^{2/n}) \approx \text{const} > 0 \quad \text{and} \quad \psi \approx [(n+1)/\varepsilon]^{2n}\lambda^4 \approx 0. \tag{6.3.52}$$

(ii) $\lambda \approx 0,\ \varepsilon = 0$. If $n > 3/2$, we observe from Eqs. (6.3.39), (6.3.40) that the terms associated with the minus sign become the leading ones. No integral curves exist in the real first (λ, ψ)-quadrant if $\lambda \to 0$ and $\varepsilon = 0$.

(iii) $\lambda \approx 0,\ \varepsilon < 0$. Likewise, we observe from Eqs. (6.3.37), (6.3.38), (6.3.52) that no solution curves exist in the first quadrant if $\lambda \to 0$ and $\varepsilon < 0$.

(iv) $\lambda \to \infty$. The relevant equations are Eqs. (6.3.41) and (6.3.42). Since now $n > 3/2$, the negative terms become vanishingly small if $\lambda \to \infty$. Eq. (6.3.41) can be employed as it stands, while Eq. (6.3.42) reads after a first order expansion as (Parker 1963)

$$\psi \approx (n+1)^{2n}\lambda^{4-2n}[1 - 2n\varepsilon/\lambda + 2n(n+1)^{2n}\lambda^{3-2n}]. \tag{6.3.53}$$

If $3/2 < n < 2$, we observe from Eqs. (6.3.41) and (6.3.53) that there are two branches of ψ tending to infinity (Fig. 6.3.3 on the left), and if $2 < n < \infty$ the branch (6.3.53) becomes zero if $\lambda \to \infty$ (Fig. 6.3.3 on the right).

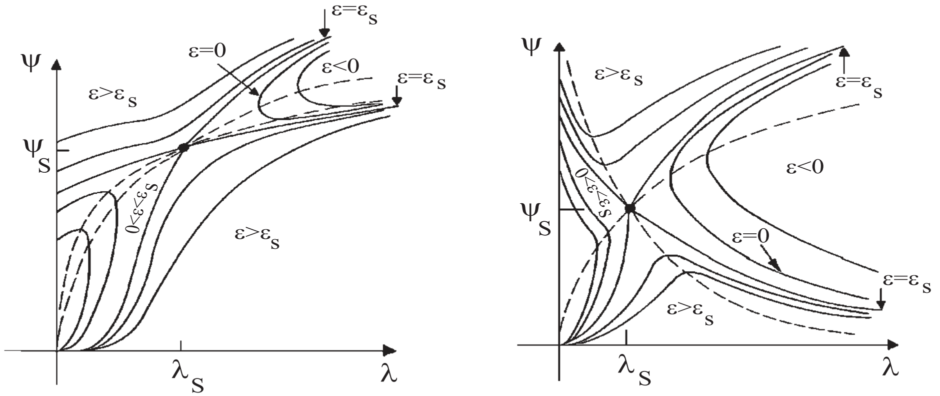

Fig. 6.3.3 Same as Fig. 6.3.1 if $3/2 < n < 2$ (on the left), and $2 < n < \infty$ (on the right). The energy constant of the two singular solutions passing through the singular saddle point S_s is denoted by ε_s.

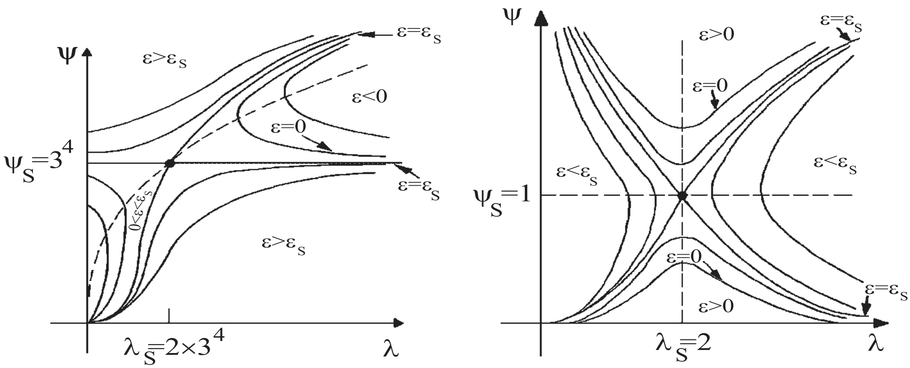

Fig. 6.3.4 Same as Fig. 6.3.1 if $n = 2$ (on the left), and $n = \pm\infty$ (on the right).

In the particular case $n = 2$ we observe from Eqs. (6.3.27), (6.3.32), (6.3.36), (6.3.53) that one singular solution is constant for all λ (Fig. 6.3.4 on the left):

$$\psi = \psi_s = 2\varepsilon_s = 3^4 = \text{const}, \qquad (n = 2). \tag{6.3.54}$$

Case 4, ($n = \pm\infty$). (i) $\lambda \approx 0$. The logarithmic term in Eq. (6.3.28) can be split into a sum, and it is seen that λ can safely be neglected with respect to $\ln \lambda^2$ if $\lambda \to 0$. Therefore, the two approximate solutions of Eq. (6.3.28) are obtained by neglecting $\ln \psi^{1/2}$ and ψ, respectively: $\psi \approx 2\varepsilon - \ln \lambda^4$ and $\psi \approx \lambda^4 \exp(-2\varepsilon)$. We insert these solutions into the neglected terms of Eq. (6.3.28):

$$\psi = 2[\varepsilon + \lambda - \ln(\lambda^2/\psi^{1/2})] \approx 2\varepsilon + 2\lambda + \ln \lambda^{-4} + \ln(2\varepsilon - \ln \lambda^4) \approx \ln \lambda^{-4} + O[\ln(\ln \lambda^{-4})] \to \infty, \tag{6.3.55}$$

$$\psi = \lambda^4 \exp(-2\varepsilon - 2\lambda + \psi) \approx \lambda^4 \exp[-2\varepsilon - 2\lambda + \lambda^4 \exp(-2\varepsilon)] \approx \lambda^4 \exp(-2\varepsilon) \approx 0. \tag{6.3.56}$$

(ii) $\lambda \to \infty$. In this case we can safely neglect $\ln \lambda^2$ with respect to λ, and the two approximate solutions of Eq. (6.3.28) are $\psi \approx 2\varepsilon + 2\lambda$ and $\psi \approx \exp(-2\varepsilon - 2\lambda)$. Eq. (6.3.28) becomes with these

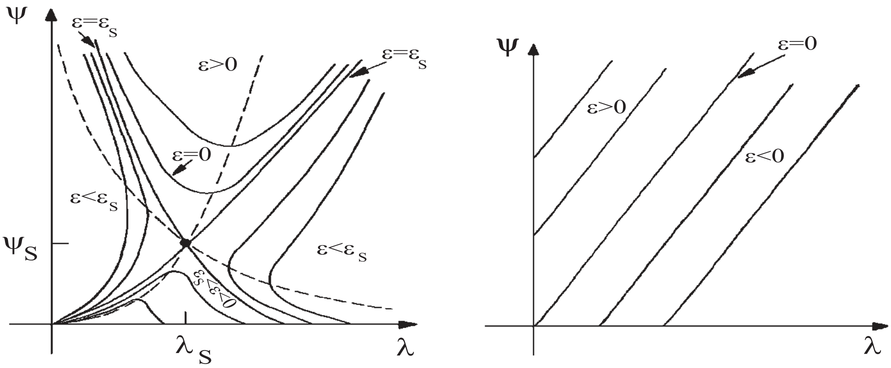

Fig. 6.3.5 Same as Fig. 6.3.1 if $-\infty < n < -1$ (on the left), and $n = -1$ (on the right).

approximations:

$$\psi = 2[\varepsilon + \lambda - \ln(\lambda^2/\psi^{1/2})] \approx 2\lambda - 3\ln\lambda \approx 2\lambda \to \infty, \tag{6.3.57}$$

$$\psi = \lambda^4 \exp(-2\varepsilon - 2\lambda + \psi) \approx \lambda^4 \exp[-2\varepsilon - 2\lambda + \exp(-2\varepsilon - 2\lambda)] \approx \lambda^4 \exp(-2\lambda) \approx 0. \tag{6.3.58}$$

Case 5, $(-\infty < n < -1)$. The singular saddle point S_s exists in this case with the negative energy constant ε_s from Eq. (6.3.36), (Fig. 6.3.5 on the left).

(i) $\lambda \approx 0,\ \varepsilon > 0$. Among the two approximate solutions $\psi \approx 2\varepsilon + 2\lambda$ and $\psi \approx [(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4$ obtained from Eq. (6.3.27) by neglecting the third and first term, respectively, only the first one subsists. But also this first solution $\psi \approx 2\varepsilon + 2\lambda$ must be discarded, as it would give after insertion into the right-hand side of Eq. (6.3.37) $\psi = 2\varepsilon + 2\lambda - 2(n+1)(\lambda^2/\psi^{1/2})^{1/n} \approx -2^{1-1/2n}(n+1)\lambda^{2/n}/\varepsilon^{1/2n} \to \infty$ if $\lambda \to 0$, contradicting the assumption that $\psi \approx 2\varepsilon + 2\lambda \neq \infty$.

The possibility remains that $\psi \gg 2\varepsilon + 2\lambda$, and with this assumption the exact parts of Eqs. (6.3.37), (6.3.38) yield the same consistent result:

$$\psi \approx [-2(n+1)]^{2n/(2n+1)}\lambda^{4/(2n+1)} \to \infty, \qquad (\lambda \to 0;\ \psi \gg 2\varepsilon). \tag{6.3.59}$$

(ii) $\lambda \approx 0,\ \varepsilon = 0$. Eq. (6.3.59) remains still valid, provided that $\psi \gg 0$.

(iii) $\lambda \approx 0,\ \varepsilon < 0$. The same is true also in this case, provided that $\psi \gg 2|\varepsilon|$, but we observe that also the approximate solution $\psi \approx [(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4 \to 0$ subsists, which is obtained by neglecting the first term in Eq. (6.3.27). This yields after substitution into the first term of Eq. (6.3.27) the consistent result $\psi = [(n+1)/(\varepsilon + \lambda - \psi/2)]^{2n}\lambda^4 \approx [(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4 \to 0$.

(iv) $\lambda \to \infty$. Among the two approximations $\psi \approx 2\lambda$ and $\psi \approx (n+1)^{2n}\lambda^{4-2n}$ from Eqs. (6.3.41), (6.3.42), only the first one holds good, and the relevant integral curve is $\psi \approx 2\lambda \to \infty$.

(v) $\lambda \approx -\varepsilon > 0,\ (\lambda < -\varepsilon;\ \varepsilon < 0)$. As suggested by the left-hand side of Fig. 6.3.5, the shape of the integral curves near the λ-axis can be obtained by letting $\lambda \to -\varepsilon$, provided that $\varepsilon < 0$. Since by assumption $\psi \approx 0$, we may safely neglect in Eq. (6.3.27) $\psi/2$ with respect to $\psi^{-1/2n}$, $(-\infty < n < -1)$, and we get approximately

$$\psi \approx [-(n+1)]^{2n}\varepsilon^4(-\lambda-\varepsilon)^{-2n} \approx 0, \qquad (\lambda \approx -\varepsilon > 0;\ \lambda < -\varepsilon). \tag{6.3.60}$$

Case 6, $(n = -1)$. No singular point S_s exists in this and in the following cases. The exact solution of the Bernoulli equation (6.3.27), valid for all λ, is simply (Fig. 6.3.5 on the right)

$$\psi = 2\lambda + 2\varepsilon, \qquad (n = -1;\ \lambda, \psi \geq 0). \tag{6.3.61}$$

Case 7, $(-1 < n < 0)$. (i) $\lambda \approx 0,\ \varepsilon > 0$. Again, the relevant equations are (6.3.37) and (6.3.38), but the exact part of Eq. (6.3.37) becomes after insertion of $\psi \approx 2\varepsilon + 2\lambda$ equal to $-\infty$ if $\lambda = 0$ and

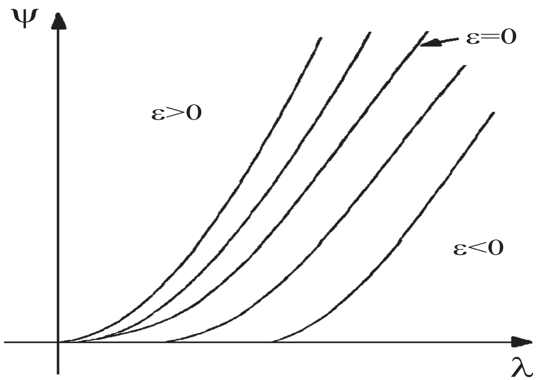

Fig. 6.3.6 Same as Fig. 6.3.1 if $-1 < n < 0$.

$-1 < n < 0$. However, the second possibility, viz. $\psi \approx [(n+1)/(\varepsilon+\lambda)]^{2n}\lambda^4 \to 0$, yields the consistent result (6.3.38), (cf. Fig. 6.3.6).

(ii) $\lambda \approx 0,\ \varepsilon = 0$. The exact equation (6.3.39) becomes by insertion of $\psi \approx 2\lambda$ equal to $-\infty$ if $\lambda = 0$ and $-1 < n < 0$. The sole consistent expansion is given by Eq. (6.3.40).

(iii) $\lambda \approx 0,\ \varepsilon < 0$. In the same way as for Case 1 (iii), we observe that no integral curves exist in the first real (λ, ψ)-quadrant if $\lambda \approx 0,\ (\varepsilon < 0)$.

(iv) $\lambda \to \infty$. The relevant equations are (6.3.41) and (6.3.42). It is seen at once that merely the integral curves (6.3.41) subsist if $-1 < n < 0:\ \psi \approx 2\lambda \to \infty$.

(v) $\lambda \approx -\varepsilon > 0,\ (\lambda > -\varepsilon;\ \varepsilon < 0)$. The behaviour of the integral curves near the λ-axis can be found similarly to Case 5 (v), (see Fig. 6.3.6). If $\psi \to 0$, we may neglect in Eq. (6.3.27) $\psi/2$ with respect to $\psi^{-1/2n}$ as long as $-1 < n < -1/2$. If $n = -1/2$, the two terms are of the same order of magnitude, and if $-1/2 < n < 0$ we may neglect $\psi^{-1/2n}$ with respect to $\psi/2$ as $\psi \to 0$. Our Eq. (6.3.27) becomes in these three cases, respectively (see Fig. 6.3.6):

$$\psi \approx (n+1)^{2n}\varepsilon^4(\lambda+\varepsilon)^{-2n}, \qquad (-1 < n < -1/2;\ \lambda \approx -\varepsilon > 0;\ \lambda > -\varepsilon), \tag{6.3.62}$$

$$\psi \approx 2(\lambda+\varepsilon)/(1+\varepsilon^{-4}), \qquad (n = -1/2;\ \lambda \approx -\varepsilon > 0;\ \lambda > -\varepsilon), \tag{6.3.63}$$

$$\psi \approx 2(\lambda+\varepsilon), \qquad (-1/2 < n < 0;\ \lambda \approx -\varepsilon > 0;\ \lambda > -\varepsilon). \tag{6.3.64}$$

Case 8, $(n = 0)$. There exists the single, normalized exact solution $\psi = \lambda^4$ from Eq. (6.3.18).

An examination of Figs. 6.3.1-6.3.6 shows an unexpected wide variety of solution types depending on the polytropic index n – even for the most simple wind/accretion model.

In the special case $n = 3/2$ three different subcases must be considered, namely if $C < 3/5 \times 2^{4/3}$, $C = 3/5 \times 2^{4/3}$, and $C > 3/5 \times 2^{4/3}$ (see Fig. 6.3.1 on the right, and Fig. 6.3.2). Solutions with negative energy constant $\varepsilon < 0$ are available in the first real quadrant only if $C < 3/5 \times 2^{4/3}$, and all integral curves approach asymptotically the two distinct exact solutions (6.3.50) obtained for $\varepsilon = 0$ (Fig. 6.3.1 on the right). If $C = 3/5 \times 2^{4/3}$, the velocity solutions consist of two distinct branches, asymptotic to the exact solution $\psi = \lambda/2$ from Eq. (6.3.49), (Fig. 6.3.2 on the left).

For polytropic indices $-\infty \le n < -1$ and $3/2 < n \le \infty$ all solution curves are of the saddle type, dominated by the two singular solutions passing through the saddle point S_s. At first sight a distinction between five different ranges of the polytropic index seems superfluous, but a more detailed inspection shows that these five subcases are quite distinct with respect to the boundary conditions, i.e. referring to their behaviour if $\lambda \to \infty,\ (r \to 0)$, and $\lambda \to 0,\ (r \to \infty)$. A singular solution exists only if $2 < n < \infty$, satisfying the boundary condition that v or ψ become zero if $r \to 0,\ (\lambda \to \infty)$, and remain finite at large distances from $M,\ (\lambda \to 0)$, (Fig. 6.3.3 on the right). Actually, it is just this solution that is very useful in stellar wind theory. An exact, analytical singular solution of constant velocity subsists if $n = 2$ (Eq. (6.3.54) and Fig. 6.3.4 on the left). And if $3/2 < n < 2$, the velocity approaches infinity near the point mass M, $(\lambda \to \infty;\ r \to 0)$, (Fig. 6.3.3 on the left). In the case $3/2 < n < \infty$, when $\varepsilon_s > 0$, all integral curves having $\varepsilon < 0$ are confined within a finite radius $r \propto 1/\lambda$ from the point mass M (Fig. 6.3.3 and Fig. 6.3.4 on the left).

A class of solutions existing exclusively if $-\infty < n < 0$ and $\varepsilon < 0$ involves a vanishing velocity $(\propto \psi^{1/2})$ at a finite nonzero distance r, $(1/r \propto \lambda \approx -\varepsilon)$, (see Figs. 6.3.5 and 6.3.6).

Finally, if $-1 \le n \le 0$ all solution curves are single-branched, monotonically increasing functions of λ (cf. right-hand side of Fig. 6.3.5 and Fig. 6.3.6).

To sum up, expansion flows (e.g. stellar winds) with constant n, satisfying the two obvious boundary conditions that v or $\psi = 0$ at $r = 0$, $(\lambda = \infty)$, and $v, \psi = \mathrm{const} < \infty$ if $r \to \infty$, $(\lambda \to 0)$ exist only for polytropic indices $2 < n < \infty$ (Fig. 6.3.3 on the right). Parker (1963, p. 62) has given arguments why the flow should "feel compelled" to adjust itself just to the continuous singular solution passing through the singular point S_s.

In order to assure the existence of an expanding polytropic wind $(2 < n < \infty)$, the reciprocal distance λ_0 at a certain reference level r_0 has to be confined within certain limits [Eq. (6.3.67)]. First, we observe from Eq. (6.3.36) that the energy constant ε_s of the singular solution is positive for a polytropic wind. Let us denote by $\psi_0 = \psi(\lambda_0) = \mu v^2(\lambda_0)/\mathcal{R}T_0$ the squared dimensionless velocity at the dimensionless reference level $\lambda_0 = \mu GM/\mathcal{R}T_0 r_0$, and let us express the energy constant (6.3.27) in reference level terms, taking into account that $(\lambda_0^2/\psi_0^{1/2})^{1/n} = \tau_0 = 1$, $(\tau = T/T_0)$ via Eq. (6.3.16):

$$\varepsilon_s = \psi_0/2 - \lambda_0 + (n+1)(\lambda_0^2/\psi_0^{1/2})^{1/n} = \psi_0/2 - \lambda_0 + n + 1 > 0, \qquad (2 < n < \infty). \tag{6.3.65}$$

Taking the reference level near the stellar surface, the condition that the wind velocity is vanishingly small near the point mass M implies $\psi_0 \approx 0$, and Eq. (6.3.65) gives $\lambda_0 < n + 1$.

A lower limit of λ_0 can be obtained from Eq. (6.3.30), written at reference level λ_0 under the form

$$\begin{aligned}(d\psi/d\lambda)_{\lambda=\lambda_0} &= 2\psi_0\big[(\psi_0^{1/2}/\lambda_0^2)^{1/n} - 2(1+1/n)/\lambda_0\big]\big/\big[\psi_0(\psi_0^{1/2}/\lambda_0^2)^{1/n} - 1 - 1/n\big]\\ &= (2\psi_0/\lambda_0)\big[\lambda_0 - 2(1+1/n)\big]\big/(\psi_0 - 1 - 1/n), \qquad (2 < n < \infty).\end{aligned} \tag{6.3.66}$$

Because ψ must be a decreasing function of λ, (v increases with r), the derivative (6.3.66) has to be negative. Since $\psi_0 \approx 0$, the denominator is already negative, and the numerator has to be positive, i.e. $\lambda_0 > 2(1 + 1/n)$. Thus, λ_0 is included between the limits (Parker 1963, Chap. V)

$$2(1 + 1/n) < \lambda_0 < n + 1, \qquad (2 < n < \infty). \tag{6.3.67}$$

In terms of the solar coronal temperature T_0 at reference level $r_0 = 10^6$ km this equation becomes by inserting for λ_0 from Eq. (6.3.11):

$$\begin{aligned}&\mu GM_\odot/(n+1)\mathcal{R}r_0 = 7.3\times10^5\ \mathrm{K} < T_0 < \mu GM_\odot n/2(n+1)\mathcal{R}r_0 = 3.6\times10^6\ \mathrm{K},\\ &(n = 10;\ \mu = 1/2),\end{aligned} \tag{6.3.68}$$

where the observationally suggested value of n at r_0 is about 10.

Actually, Eq. (6.3.67) already implies $n > 2$. If λ_0 is less than $2(1 + 1/n)$, gravitation is too weak, and the gas expands outwards in an explosive, uncontrolled manner. And if λ_0 is larger than $n + 1$, gravitation is too strong, there is no wind, and M possesses merely a static polytropic atmosphere.

We conclude the discussion of the basic equations (6.3.9), (6.3.10) of the wind/accretion problem with Yeh's (1970) analytic solution, which enables one to calculate explicitly distance $r = r(\varrho)$ and flow velocity $v = v(\varrho)$ in terms of the flow density ϱ, by evaluating the algebraic relationships (6.3.77), (6.3.78), (6.3.81), (6.3.86), (6.3.87). The analytic solutions (6.3.27)-(6.3.29) provide so far, only implicit functional relationships among v, ϱ, and r.

Let us integrate at first Eq. (6.3.9) with the polytropic law (6.3.3), to obtain the equivalent of the energy integrals (6.3.27)-(6.3.29):

$$v^2/2 + (n+1)K\varrho^{1/n} - GM/r = h = \mathrm{const}, \qquad (n \ne \pm\infty), \tag{6.3.69}$$

$$v^2/2 + K\ln\varrho - GM/r = h = \mathrm{const}, \qquad (n = \pm\infty). \tag{6.3.70}$$

The seeming singularity in Eq. (6.3.69) if $n = 0$, ($\varrho = \mathrm{const}$) can be removed at once by replacing $K\varrho^{1/n}$ with P/ϱ (cf. Sec. 1.2).

We insert for v from the mass conservation equation (6.3.10), getting a quartic equation in $1/r$:

$$1/r^4 - 2(GM\varrho^2/F^2)/r + 2(\varrho/F)^2\big[(n+1)K\varrho^{1/n} - h\big] = 0, \qquad (n \neq \pm\infty), \tag{6.3.71}$$

$$1/r^4 - 2(GM\varrho^2/F^2)/r + 2(\varrho/F)^2\big[K\ln\varrho - h\big] = 0, \qquad (n = \pm\infty). \tag{6.3.72}$$

The basic idea of Yeh (1970) is the factorization of Eqs. (6.3.71), (6.3.72) by introduction of the auxiliary variable s, satisfying the cubic equation (6.3.79) in s^2. Thus, we rewrite Eq. (6.3.71) as a difference of two squares, including the remaining terms in braces:

$$\Big[1/r^2 + 2^{1/3}(GM\varrho^2/F^2)^{2/3}s^2\Big]^2 - \Big[2^{2/3}(GM\varrho^2/F^2)^{1/3}s/r + 2^{-2/3}(GM\varrho^2/F^2)^{2/3}/s\Big]^2 +$$
$$(GM\varrho^2/F^2)^{4/3}\Big\{-2^{2/3}s^4 + 2^{-4/3}/s^2 + 2(G^2M^2\varrho/F)^{-2/3}\big[(n+1)K\varrho^{1/n} - h\big]\Big\} = 0, \quad (n \neq \pm\infty). \tag{6.3.73}$$

A quite similar equation holds if $n = \pm\infty$. Eq. (6.3.73) is split into two separate equations, by writing the first two terms as a product of their difference and sum. This product, as well as the last term, is equated to zero separately:

$$\Big[1/r^2 - 2^{2/3}(GM\varrho^2/F^2)^{1/3}s/r + (GM\varrho^2/F^2)^{2/3}(2^{1/3}s^2 - 2^{-2/3}/s)\Big]$$
$$\times\Big[1/r^2 + 2^{2/3}(GM\varrho^2/F^2)^{1/3}s/r + (GM\varrho^2/F^2)^{2/3}(2^{1/3}s^2 + 2^{-2/3}/s)\Big] = 0, \tag{6.3.74}$$

$$4s^6 - 2^{7/3}(G^2M^2\varrho/F)^{-2/3}\big[(n+1)K\varrho^{1/n} - h\big]s^2 - 1 = 0, \qquad (n \neq \pm\infty), \tag{6.3.75}$$

$$4s^6 - 2^{7/3}(G^2M^2\varrho/F)^{-2/3}\ (K\ln\varrho - h)s^2 - 1 = 0, \qquad (n = \pm\infty). \tag{6.3.76}$$

With the notations

$$q = (2^{7/3}/3)(G^2M^2\varrho/F)^{-2/3}\big[(n+1)K\varrho^{1/n} - h\big], \qquad (n \neq \pm\infty), \tag{6.3.77}$$

$$q = (2^{7/3}/3)(G^2M^2\varrho/F)^{-2/3}\ (K\ln\varrho - h), \qquad (n = \pm\infty), \tag{6.3.78}$$

Eqs. (6.3.75), (6.3.76) now take the simple form

$$s^6 - 3qs^2/4 - 1/4 = 0. \tag{6.3.79}$$

The extrema $s^2 = \pm q^{1/2}/2$ of the function $f(s^2) = s^6 - 3qs^2/4 - 1/4$ have different signs if $q > 0$, and are absent in the real domain if $q < 0$. If $q = 0$, an inflexion point exists at $s^2 = 0$. If $s = 0$, the function $f(s^2)$ takes the negative value $-1/4$, and therefore Eq. (6.3.79) has a single positive root s_1^2, the two other roots being either negative or complex conjugate. The discriminant $D = (1 - q^3)/64$ takes nonnegative values if $q \leq 1$, and the sole positive root of Eq. (6.3.79) is in this case given by Cardani's formula

$$s^2 = \big\{[1 + (1 - q^3)^{1/2}]^{1/3} + [1 - (1 - q^3)^{1/2}]^{1/3}\big\}/2, \qquad (q \leq 1), \tag{6.3.80}$$

with the two real roots

$$s_1 = -s_2 = 2^{-1/2}\big\{[1 + (1 - q^3)^{1/2}]^{1/3} + [1 - (1 - q^3)^{1/2}]^{1/3}\big\}^{1/2}, \qquad (q \leq 1). \tag{6.3.81}$$

It will be seen from Eqs. (6.3.83) and (6.3.85) that s_2 can be discarded, and that s_1 is further constrained by the delimitation $0 \leq s_1 \leq 1$, implying $-\infty \leq q \leq 1$. So, the relevant solution of Eq. (6.3.79) is given by s_1.

If on the other hand $D < 0,\ (q > 1)$, the sole positive root is not pertinent for our considerations, but is written down for completeness under trigonometric form

$$s^2 = q^{1/2}\cos[(1/3)\arccos(q^{-3/2})], \qquad (q > 1). \tag{6.3.82}$$

With the two roots (6.3.81), the two second order parentheses (6.3.74) can be solved at once, yielding four values of $1/r$:

$$1/r_{1,2,3,4} = (GM\varrho^2/2F^2)^{1/3}[s_{1,2} \pm (-s_{1,2}^2 + 1/s_{1,2})^{1/2}], \tag{6.3.83}$$

$$1/r_{1,2,3,4} = (GM\varrho^2/2F^2)^{1/3}[-s_{1,2} \pm (-s_{1,2}^2 - 1/s_{1,2})^{1/2}]. \tag{6.3.84}$$

Since $s_1 = -s_2$, we can dismiss the solutions (6.3.84), because they yield the same result as Eq. (6.3.83) if s_1 is replaced by $-s_2$, and vice versa. And the square root in Eq. (6.3.83) implies that we can also discard the nonpositive root s_2, and concentrate only on $s_1 \geq 0$, which is constrained further by the delimitation $0 \leq s_1 \leq 1$. Moreover, with this constraint, one of the two real solutions (6.3.83) is always nonnegative $(1/r_1 \geq 0)$, while the other solution $1/r_2$ is nonnegative only if $s_1 \geq (-s_1^2 + 1/s_1)^{1/2}$, i.e. if $1/2^{1/3} \leq s_1 \leq 1$.

From Eq. (6.3.79) follows

$$q = (4s^6 - 1)/3s^2; \quad dq/ds = (16s^6 + 2)/3s^3, \tag{6.3.85}$$

and it is seen that q increases monotonically $(dq/ds > 0)$ from $-\infty$ to ∞, if s increases from 0 to ∞. As mentioned above, the delimitation $0 \leq s_1 \leq 1$ implies $-\infty \leq q \leq 1$. To sum up, the analytical solution of the wind/accretion problem (6.3.10), (6.3.69), (6.3.70) is given by

$$1/r_{1,2} = (GM\varrho^2/2F^2)^{1/3}[s_1 \pm (-s_1^2 + 1/s_1)^{1/2}], \tag{6.3.86}$$

where the solution r_1 exists only if $0 \leq s_1 \leq 1,\ (-\infty \leq q \leq 1)$, and the second solution r_2 only if $2^{-1/3} \leq s_1 \leq 1,\ (0 \leq q \leq 1)$. The velocity is found at once from the mass conservation equation (6.3.10):

$$v = F/\varrho r^2 = (G^2M^2\varrho/4F)^{1/3}\big[1/s_1 \pm 2(s_1 - s_1^4)^{1/2}\big]. \tag{6.3.87}$$

As already noted at the beginning of this subsection, the simple polytropic law (6.3.3) is a substitute for an energy equation. The heating of the solar corona, where the solar wind starts, is simulated by the amount by which the polytropic index n is larger than its value $n = 1/(\gamma - 1) = 3/2,\ (\gamma = 5/3)$ for an adiabatic completely ionized gas [cf. Eq. (1.7.60)]. In the solar corona it is observed that $n \approx 10$, i.e. there prevails near-isothermality. Radiation losses by free-free emission are of some importance only at the coronal base, and the major heat transport through the expanding corona occurs by thermal conduction, this being the salient energy source to the expanding corona. If thermal conduction is not fully adequate, then energy supply by conversion of magnetic energy into plasma energy, dissipation of hydrodynamic and hydromagnetic waves would be no longer confined to a relatively thin layer at the coronal base, but would extend to distances of several solar radii, throughout the corona (Parks 1991, Chap. 8).

In fact, for a perfectly isothermal corona $(n = \pm\infty)$ thermal conduction ceases, and the high coronal temperatures required for expansion can be maintained only by various dissipative processes (dissipation of magnetic energy, hydromagnetic and hydrodynamic wave dissipation), occurring throughout the corona. Far from the Sun we would expect that coronal heating by conduction and energy dissipation drops off, and n decreases towards the adiabatic value $n = 3/2$, mentioned above. Besides, as obvious from the right-hand side of Fig. 6.3.4, a strictly isothermal wind would yield a slowly, but continuously increasing wind velocity if $r \to \infty,\ (\lambda \to 0)$, and should therefore be discarded at large distances from M. Thus, for a more realistic description of stellar winds a changing polytropic index should be employed $(3/2 \leq n \leq \infty)$, where the upper limit applies near the star, and the lower limit at large distances. For instance, Pudovkin et al. (1997) quote $n = 1.1 - 2.5$ for the solar wind at the Earth's orbit and at the Earth's bow shock, while Totten et al. (1996) find $n \approx 2.17$ between 0.08 and 1.5 AU. On the other hand, on a smaller and more localized scale, the plasma in the vicinity of solar wind *stream interactions* possesses $n = 1 - 1.5$ (Newbury et al. 1997). And for solar flares (short-lived sudden bursts of light in the neighborhood of sunspots) the polytropic index is via Eq. (1.2.39) equal to $n \approx 1.5$ – close to the adiabatic exponent $\gamma = 1 + 1/n = 5/3$ of a nonrelativistic plasma (Garcia 2001).

The solar wind can not only be regarded as a hydrodynamic flow, but also equivalently as an evaporative phenomenon with the wind particles traveling upwards in the hot corona at speeds exceeding the escape velocity, and escaping freely without collisions from the solar exosphere. The principal merit of such exospheric models is the independent production of the solar wind, which is the result of the hot solar corona, and does not depend on the method of treatment (e.g. Brandt 1970).

Chiuderi and Ciamponi (1977, 1978) have considered the radio emission from the extended envelopes surrounding early type stars undergoing mass loss. They treat the dynamics of the expanding envelope in the polytropic approximation, without prescribing a priori density and temperature profiles. The polytropic atmosphere is described by a straightforward generalization of Parker's (1963) radial wind model with the inclusion of radiation pressure in the optically thick regime. The momentum equation (6.3.9) generalizes to

$$v\ dv/dr = -(1/\varrho)\ d(P_g + P_r)/dr - GM/r^2 = -(1+1/n)K\varrho^{1/n-1}\ d\varrho/dr + \kappa L/4\pi c r^2 - GM/r^2, \tag{6.3.88}$$

where the radial change of radiation pressure dP_r/dr has been taken from Eq. (6.1.1). Eddington's assumption (6.1.9) concerning the constancy of the ratio P_r/P writes

$$1-\beta = P_r/(P_g+P_r) = \kappa L/4\pi cGM = \text{const}, \qquad (\beta = P_g/P). \tag{6.3.89}$$

Eq. (6.3.88) becomes

$$v\ dv/dr = -(1+1/n)K\varrho^{1/n-1}\ d\varrho/dr - \beta GM/r^2. \tag{6.3.90}$$

Thus, with Eddington's assumption, Eq. (6.3.90) takes the same form as the original equation (6.3.9), βG playing the role of an effective gravitational constant. Eq. (6.3.90) can be integrated at once [cf. Eq. (6.3.69)]:

$$\begin{aligned} &v^2/2 + (n+1)K\varrho^{1/n} - \beta GM/r = v^2/2 + (n+1)P/\varrho - \beta GM/r \\ &= v^2/2 + (n+1)(P_0/\varrho_0)(v_0 r_0^2/vr^2)^{1/n} - \beta GM/r = \text{const}, \qquad (n \neq \pm\infty). \end{aligned} \tag{6.3.91}$$

Chiuderi and Ciamponi (1977, 1978) adopt a more mathematical description of this Bernoulli equation, and introduce, following Parker (1963), the dimensionless radius η and velocity u by

$$\begin{aligned} &\eta = (n+1)^{2n/(2n-3)}(\varrho_0 v_0^2/2P_0)^{1/(2n-3)}(\beta GM\varrho_0/P_0 r_0)^{-(2n+1)/(2n-3)}\ r/r_0; \\ &u = (n+1)^{-n/(2n-3)}(\varrho_0 v_0^2/2P_0)^{(n-2)/(2n-3)}(\beta GM\varrho_0/P_0 r_0)^{2/(2n-3)}\ v/v_0, \qquad (n \neq 3/2, \pm\infty). \end{aligned} \tag{6.3.92}$$

Eq. (6.3.91) takes after some algebra the form

$$\begin{aligned} &u^2 + 1/(u\eta^2)^{1/n} - 1/\eta = (n+1)^{-2n/(2n-3)}(\varrho_0 v_0^2/2P_0)^{-1/(2n-3)}(\beta GM\varrho_0/P_0 r_0)^{4/(2n-3)} \\ &\times\big(\varrho_0 v_0^2/2P_0 + n + 1 - \beta GM\varrho_0/P_0 r_0\big) = \text{const}, \qquad (n \neq 3/2, \pm\infty). \end{aligned} \tag{6.3.93}$$

Chiuderi and Ciamponi (1977) argue that the relevant values of r fall completely into the region where u, and consequently v practically has reached its asymptotic value $v_\infty = \text{const}$. In virtue of Eq. (6.3.10) the density changes in this case as $\varrho = \varrho_0(r_0/r)^2$, while the temperature varies according to the polytropic and perfect gas laws $T = T_0(P/P_0)/(\varrho/\varrho_0) = T_0(\varrho/\varrho_0)^{1/n} = T_0(r_0/r)^{2/n}$. With these density and temperature profiles the authors calculate the radio emission from the extended polytropic envelope of a radiostar.

6.3.2 Bipolytropic Winds

Bipolytropic models of the solar wind can be envisaged to consist of an electron-proton mixture with distinct electron and proton temperatures T_e and T_p, respectively. To be somewhat more general, we consider the steady outflow of matter along a radial flow tube, but with a cross-section proportional to r^b, where the divergence parameter ($b \geq 2$) measures the geometrical deviation from spherically symmetric expansion $b = 2$. The total pressure of the completely ionized plasma is given by the sum of electron and proton pressure $P = P_e + P_p$, satisfying the ideal gas law (1.2.5), and two separate polytropic laws (6.3.3) with distinct polytropic indices n_e, n_p, and distinct polytropic constants K_e, K_p :

$$P_e = n_d kT_e = k\varrho T_e/m_p = K_e\varrho^{1+1/n_e}; \quad P_p = n_d kT_p = k\varrho T_p/m_p = K_p\varrho^{1+1/n_p},$$
$$(\varrho = n_d(m_p + m_e) \approx n_d m_p). \tag{6.3.94}$$

k denotes the Boltzmann constant, and n_d is the number density of electrons or protons. The total mass density is $\varrho = n_d(m_e + m_p) \approx n_d m_p$, where m_e, m_p are the electron and proton masses, respectively.

The mass conservation equation (6.3.10) becomes

$$\varrho v r^b = \varrho_0 v_0 r_0^b, \tag{6.3.95}$$

and the equation of motion (6.3.9) now reads

$$(1/2)\ dv^2/dr = -(1/\varrho)\ d(P_e + P_p)/dr - GM/r^2. \tag{6.3.96}$$

Integration of this equation with the aid of Eq. (6.3.94) yields the Bernoulli integrals

$$v^2/2 + (n_e + 1)K_e\varrho^{1/n_e} + (n_p + 1)K_p\varrho^{1/n_p} - GM/r = \mathrm{const}, \qquad (n_e, n_p \neq \pm\infty), \tag{6.3.97}$$

$$v^2/2 + K_e \ln\varrho + (n_p + 1)K_p\varrho^{1/n_p} - GM/r = \mathrm{const}, \qquad (n_e = \pm\infty;\ n_p \neq \pm\infty), \tag{6.3.98}$$

$$v^2/2 + (K_e + K_p)\ln\varrho - GM/r = \mathrm{const}, \qquad (n_e, n_p = \pm\infty). \tag{6.3.99}$$

The density ϱ is eliminated with Eq. (6.3.95), taking into account that $K_e\varrho_0^{1/n_e} = kT_{e0}/m_p,\ K_p\varrho_0^{1/n_p} = kT_{p0}/m_p$ via Eq. (6.3.94):

$$v^2/2 + (k/m_p)\big[(n_e + 1)T_{e0}(v_0r_0^b/vr^b)^{1/n_e} + (n_p + 1)T_{p0}(v_0r_0^b/vr^b)^{1/n_p}\big] - GM/r = \mathrm{const},$$
$$(n_e, n_p \neq 0, \pm\infty), \tag{6.3.100}$$

$$v^2/2 + (k/m_p)\big[T_{e0}\ln(v_0r_0^b/vr^b) + (n_p + 1)T_{p0}(v_0r_0^b/vr^b)^{1/n_p}\big] - GM/r = \mathrm{const},$$
$$(n_e = \pm\infty;\ n_p \neq 0, \pm\infty), \tag{6.3.101}$$

$$v^2/2 + (k/m_p)(T_{e0} + T_{p0})\ln(v_0r_0^b/vr^b) - GM/r = \mathrm{const}, \qquad (n_e, n_p = \pm\infty). \tag{6.3.102}$$

Similarly to Eq. (6.3.11), Summers (1983a) introduces the dimensionless notations

$$\zeta = r/r_0; \quad \Psi = \Psi(\zeta) = m_p v^2(\zeta)/2kT_{e0}; \quad \tau_0 = T_{p0}/T_{e0}. \tag{6.3.103}$$

The three previous equations become with the constants expressed at the reference level $\zeta = 1$, equal to $[\Psi_0 = \Psi(1)]$

$$\Psi + (n_e + 1)(\Psi_0/\Psi)^{1/2n_e}\zeta^{-b/n_e} + \tau_0(n_p + 1)(\Psi_0/\Psi)^{1/2n_p}\zeta^{-b/n_p} - \nu/\zeta$$
$$= \Psi_0 + (n_e + 1) + \tau_0(n_p + 1) - \nu, \qquad (n_e, n_p \neq 0, \pm\infty), \tag{6.3.104}$$

$$\Psi + \ln[(\Psi_0/\Psi)^{1/2}\zeta^{-b}] + \tau_0(n_p + 1)(\Psi_0/\Psi)^{1/2n_p}\zeta^{-b/n_p} - \nu/\zeta$$
$$= \Psi_0 + \tau_0(n_p + 1) - \nu, \qquad (n_e = \pm\infty;\ n_p \neq 0, \pm\infty), \tag{6.3.105}$$

$$\Psi + (1 + \tau_0)\ln[(\Psi_0/\Psi)^{1/2}\zeta^{-b}] - \nu/\zeta = \Psi_0 - \nu, \qquad (n_e, n_p = \pm\infty), \tag{6.3.106}$$

where

$$\nu = m_p GM/kT_{e0}r_0 = \mathrm{const}. \tag{6.3.107}$$

The singular points of Eqs. (6.3.104)-(6.3.106) can be determined by equating to zero the numerator and denominator of their derivatives, which take the unified form

$$d\Psi/d\zeta = (\Psi/\zeta^2)\{b(1+1/n_e)(\Psi_0/\Psi)^{1/2n_e}\zeta^{1-b/n_e} + b\tau_0(1+1/n_p)(\Psi_0/\Psi)^{1/2n_p}\zeta^{1-b/n_p} - \nu\}$$
$$/\{\Psi - [(1+1/n_e)/2](\Psi_0/\Psi)^{1/2n_e}\zeta^{-b/n_e} - \tau_0[(1+1/n_p)/2](\Psi_0/\Psi)^{1/2n_p}\zeta^{-b/n_p}\}, \qquad (n_e, n_p \neq 0). \tag{6.3.108}$$

Additionally to the nodal singular point at the origin $\zeta_s, \Psi_s = 0$, the other singular saddle point is given by equating the braces separately to zero. These two equations can also be expressed under the alternative form (Summers 1983a):

$$2b\zeta_s\Psi_s = \nu, \qquad [\Psi_s = \Psi(\zeta_s)], \tag{6.3.109}$$

$$\Psi_s^{1+(1-2b)/2n_e} - [(1+1/n_e)/2]\Psi_0^{1/2n_e}(2b/\nu)^{b/n_e}$$
$$-\tau_0[(1+1/n_p)/2](2b/\nu)^{b/n_p}\Psi_0^{1/2n_p}\Psi_s^{(b-1/2)(1/n_p-1/n_e)} = 0, \qquad (n_e, n_p \neq 0). \tag{6.3.110}$$

In the special case $n_e, n_p = \pm\infty$ we find simply: $\Psi_s = (1+\tau_0)/2$. If $\zeta = \zeta_s$ and $\Psi = \Psi_s(\zeta_s)$ are inserted into Eqs. (6.3.104)-(6.3.106), the pair (Ψ_0, Ψ_s) can be determined with the aid of Eq. (6.3.110).

The branches of the Bernoulli equations (6.3.104)-(6.3.106) which are pertinent for an expanding wind are those satisfying the condition $v, \Psi \approx 0$ at $\zeta \approx 0$ (lower branch), and $v, \Psi \gg 1$ if $\zeta \to \infty$ (upper branch) for a previously determined value of $\Psi_0 = \Psi(1)$. If $n_e, n_p \neq 0, \pm\infty$, the wind speed (6.3.100) increases monotonically with r up to its terminal value $v = \mathrm{const.}$ If $n_e = \pm\infty$ and $n_p \neq 0, \pm\infty$, the gas velocity (6.3.101) increases without bound according to the asymptotic form $v^2 = (2bkT_{e0}/m_p)\ln r$, while in the isothermal case $n_e, n_p = \pm\infty$ we have $v^2 = [2bk(T_{e0}+T_{p0})/m_p]\ln r$ via Eq. (6.3.102). The wind speed at the orbit of the Earth is up to two times larger in a strongly diverging flow tube $b \approx 5$, as compared to spherically symmetric flow ($b = 2$), (Summers 1983a).

For the quiescent solar wind between about 0.3 and 10 AU Riley et al. (2001) quote $5/3 < n_p < 2.5$ and $5/3 < n_e < 10$, while a negative polytropic index for electrons ($n_e < 0$) in interplanetary magnetic clouds (interplanetary coronal mass ejections) is still a matter of debate (Osherovich et al. 1993, Newbury et al. 1997, Gosling 1999, Gosling et al. 2001, Riley et al. 2001, Garcia 2001).

Another bipolytropic model for strong mass outflow from quasiequilibrium states with excess energy has been proposed by Bisnovatyi-Kogan and Zeldovich (1966). The isentropic equation of state is assumed under the form

$$P = \begin{cases} K_1\varrho^{1+1/n_1} & \text{if} \quad \varrho > \varrho_a, \ \Gamma_1 = 1+1/n_1 \\ K_2\varrho^{1+1/n_2} & \phantom{\text{if}} \quad \varrho < \varrho_a, \ \Gamma_2 = 1+1/n_2 \end{cases} \tag{6.3.111}$$

Pressure continuity at $\varrho = \varrho_a$ demands $K_1\varrho_a^{1/n_1} = K_2\varrho_a^{1/n_2}$. The specific internal (thermal) energy of isentropic stellar matter is via Eqs. (5.12.12)-(5.12.14) equal to

$$\varepsilon^{(int)}/\varrho = [K/(\Gamma-1)]\varrho^{\Gamma-1} + \mathrm{const} = nK\varrho^{1/n} + \mathrm{const}, \qquad (\Gamma = 1+1/n). \tag{6.3.112}$$

In the the region $\varrho < \varrho_a$ the integration constant vanishes by virtue of the surface boundary condition $\varepsilon^{(int)}/\varrho = 0$ if $\varrho = 0$, and Eq. (6.3.112) becomes

$$\varepsilon^{(int)}/\varrho = n_2K_2\varrho^{1/n_2}, \qquad (\varrho < \varrho_a). \tag{6.3.113}$$

If $\varrho > \varrho_a$, the integration constant in Eq. (6.3.112) can be fixed via the continuity of internal energy at $\varrho = \varrho_a$: const $=\varepsilon^{(int)}/\varrho_a - n_1K_1\varrho_a^{1/n_1} = n_2K_2\varrho_a^{1/n_2} - n_1K_1\varrho_a^{1/n_1} = (n_2-n_1)K_1\varrho_a^{1/n_1}$. Thus:

$$\varepsilon^{(int)}/\varrho = n_1K_1\varrho^{1/n_1} + (n_2-n_1)K_1\varrho_a^{1/n_1}, \qquad (\varrho > \varrho_a). \tag{6.3.114}$$

Consider a star of core radius r_1 and core mass M_1 such that its mean density is $\varrho_m \gg \varrho_a$. Then, we may neglect the gravitational and internal energies of the envelope with respect to the same energies of the much more massive core. The total energy (5.12.11) of the whole star approximates via Eq. (6.3.114) to

$$E = -3GM_1^2/(5-n_1)r_1 + n_1K_1\int_0^{M_1} \varrho^{1/n_1}\, dM + (n_2-n_1)K_1\varrho_a^{1/n_1}M_1$$
$$= (n_1-3)GM_1^2/(5-n_1)r_1 + (n_2-n_1)K_1\varrho_a^{1/n_1}M_1, \qquad (6.3.115)$$

where the integral is just equal to the internal energy $U_1 = n_1GM_1^2/(5-n_1)r_1$ from Eq. (2.6.168) with $n \to n_1,\ \Gamma \to 1+1/n_1$.

The Bernoulli integral (6.3.69) subsists for the quasistationary polytropic outflow of the stellar envelope at $\varrho < \varrho_a$, having the energy constant $h = v_\infty^2/2 > 0$, as demanded by a steadily expanding wind with constant velocity at infinity v_∞. At the core surface r_1 we have $v \approx 0$, and Eq. (6.3.69) yields

$$h = (n_2+1)K_2\varrho_a^{1/n_2} - GM_1/r_1. \qquad (6.3.116)$$

If $n_2 \gg 1$, ($\Gamma_2 \approx 1$, as in ionization-dissociation zones, see Sec. 1.7), the condition $h > 0$ is equivalent to $\partial E/\partial M_1 > 0$, because in virtue of Eq. (2.6.21) we have $M_1r_1^{(3-n_1)/(n_1-1)} = \mathrm{const}$, $dr_1/dM_1 = (n_1-1)r_1/(n_1-3)M_1$, and the derivative of Eq. (6.3.115) becomes (Bisnovatyi-Kogan 2002, Sec. 8.3.1):

$$\partial E/\partial M_1 = -GM_1/r_1 + (n_2-n_1)K_1\varrho_a^{1/n_1} \approx -GM_1/r_1 + (n_2+1)K_2\varrho_a^{1/n_2} = h,$$
$$(n_2 \gg n_1;\ n_2 \gg 1;\ K_1\varrho_a^{1/n_1} = K_2\varrho_a^{1/n_2}). \qquad (6.3.117)$$

If $h = 0$, we have $\partial E/\partial M_1 \approx 0$, $(n_2-n_1)K_1\varrho_a^{1/n_1} \approx GM_1/r_1$, and $E \approx 2GM_1^2/(5-n_1)r_1 > 0$ via Eq. (6.3.115). Hence, the condition of adiabatic outflow $h > 0$ is more stringent than $E > 0$. If $E > 0$ and $h < 0$, the expansion of the star *as a whole* is energetically possible, but a quasistationary wind originating at density level ϱ_a cannot arise.

A three-polytropic model (three-zone polytrope) with $n_1 = 3$, $n_2 = -10/3$, $n_3 = 1.5$ has been adopted by Colpi et al. (1993) for the description of an exploding neutron star with mass slightly less than a minimum mass of about 0.2 $M_\odot$ – a mass located near the minimum of the unstable branch with $dM_{r1}/d\varrho_{r0} < 0$ in Fig. 5.12.1.

6.3.3 Magnetopolytropic Winds

Everywhere round the solar photosphere there appears to be a general background field of about 1-2 Gauss, while the fields of active regions may be several orders of magnitude higher. The solar wind gas can be approximated with a plasma of infinite conductivity, and the magnetic field is carried into space by the steady expansion ($\partial\ /\partial t = 0$) of coronal gas, with the field lines everywhere along the streamlines, but with the roots of the field lines fixed on the rotating Sun. The magnetic field is stretched out – the field lines being frozen-in. The interplanetary field lines connect all the plasma emitted from the same location on the rotating Sun and have the form of an Archimedes spiral (e.g. Brandt 1970). The spiral magnetic field pattern corotates with the rotating Sun, while the solar wind plasma moves nearly *radially.* This view of a corotating magnetic field is merely adopted to describe the *geometrical* features of the field, and should be carefully distinguished from physical corotation, which involves also significant, azimuthal nonradial motions of the plasma.

The magnetic field appears to act as a kind of safety valve on the corona, bottling up the corona with its enclosing lines of force until the temperature rises up to the level where coronal plasma bursts forth and expends the coronal heating energy in expansion. Parker (1963) has given arguments that the corona would first begin to push its way through the field along the equatorial plane of the Sun, and this prompted Weber and Davis (1967) to limit the expansion of coronal gases mainly to the near equatorial regions of the Sun, with negligible meridional components of velocity $\vec{v}$ and magnetic field induction vector $\vec{B}$. For the solar wind plasma, as for most astrophysical plasmas, we can safely set the magnetic permeability p equal to unity in the Gaussian unrationalized CGS-system, so the magnetic induction B is just equal to the magnetic field intensity H [cf. Eq. (3.10.5)].

Weber and Davis (1967) have examined a hydromagnetic, axially symmetric, stationary model of the solar wind, concentrating mainly on the equatorial plane, where the field is combed out by the solar wind. This model is essentially a one-dimensional one, and has been generalized among others by Sakurai (1985) and Lima et al. (2001) to a two-dimensional axisymmetric wind with frozen-in magnetic field. In an inertial frame centered on the Sun, the velocity $\vec{v}$ and the magnetic induction vector $\vec{B}$ of the wind have in the equatorial plane of the Sun the components

$$\vec{v} = v_r \vec{e}_r + v_\varphi \vec{e}_\varphi, \qquad (v_r = v_r(r);\ v_\varphi = v_\varphi(r);\ |v_r| \gg |v_\varphi|), \tag{6.3.118}$$

$$\vec{B} = B_r \vec{e}_r + B_\varphi \vec{e}_\varphi, \qquad [B_r = B_r(r);\ B_\varphi = B_\varphi(r)]. \tag{6.3.119}$$

Let us apply Ohm's law (3.10.10) in our inertial frame, where $\vec{J}/\sigma$ must be zero in the perfectly conducting solar wind plasma having conductivity $\sigma = \infty$. This condition writes via Eq. (3.10.12) as

$$\vec{E} = -\vec{v} \times \vec{B}/c. \tag{6.3.120}$$

The solar wind can be regarded as convecting with velocity $\vec{v}$ through the magnetic field $\vec{B}$ to produce the electric field (6.3.120), (Brandt 1970). Maxwell's equation (3.10.1) yields for steady-state conditions

$$(1/c)\ \partial\vec{B}/\partial t = -\nabla \times \vec{E} = 0, \tag{6.3.121}$$

and the φ-component of this equation becomes equal to

$$c(\nabla \times \vec{E})_\varphi = (c/r)\ d(rE_\lambda)/dr = (1/r)\ d[r(v_r B_\varphi - v_\varphi B_r)]/dr = 0, \tag{6.3.122}$$

by substituting for the electric field vector from Eq. (6.3.120) via Eq. (B.38). This integrates at once to

$$r(v_r B_\varphi - v_\varphi B_r) = \mathrm{const.} \tag{6.3.123}$$

To determine the integration constant, we make use of the above mentioned, nearly intuitive property that in a frame corotating with the angular rotation speed Ω of the Sun, the plasma velocity $\vec{v}_p(v_r, v_\varphi - \Omega r, 0)$ is always parallel to the magnetic field $\vec{B}(B_r, B_\varphi, 0)$, $(\Omega r \ll c)$. The parallelity condition in the corotating coordinate system writes $\vec{v}_p \times \vec{B} = 0$, or

$$v_r B_\varphi - v_\varphi B_r = -\Omega r B_r. \tag{6.3.124}$$

From Maxwell's equation $\nabla \cdot \vec{B} = 0$ we get $d(r^2 B_r)/dr = 0$ via Eq. (B.37), or

$$r^2 B_r = r_0^2 B_{r0} = \mathrm{const}, \tag{6.3.125}$$

where zero indexed values are taken at reference level $r = r_0$ near the solar surface. Combining Eqs. (6.3.123)-(6.3.125), the constant from Eq. (6.3.123) is found to be

$$r(v_r B_\varphi - v_\varphi B_r) = -\Omega r^2 B_r = -\Omega r_0^2 B_{r0}. \tag{6.3.126}$$

Returning to our inertial frame, the stationary φ-component of the equation of motion (2.1.1) takes via Eqs. (B.38), (B.42) the form

$$\begin{aligned}\varrho\ (D\vec{v}/Dt)_\varphi &= \varrho[(\vec{v}\cdot\nabla)\vec{v}]_\varphi = \varrho(v_r\ dv_\varphi/dr + v_r v_\varphi/r) = (\varrho v_r/r)\ d(rv_\varphi)/dr \\ &= (1/4\pi)[(\nabla \times \vec{H}) \times \vec{B}]_\varphi = (1/4\pi)[(\nabla \times \vec{B}) \times \vec{B}]_\varphi = (B_r/4\pi r)\ d(rB_\varphi)/dr.\end{aligned} \tag{6.3.127}$$

The mass loss rate of the Sun $dM/dt = 4\pi\varrho r^2\ dr/dt = 4\pi\varrho v_r r^2$ is constant under steady-state conditions, and therefore the ratio

$$B_r/4\pi\varrho v_r = r^2 B_r/4\pi\varrho v_r r^2 = \mathrm{const}, \tag{6.3.128}$$

is constant too. This finding allows us to integrate the azimuthal component (6.3.127) immediately:

$$L = r v_\varphi - r B_r B_\varphi/4\pi\varrho v_r = \mathrm{const.} \tag{6.3.129}$$

The first term is just the mechanical angular momentum per unit mass carried away by advection of the flow, and the second term represents the torque due to magnetic stresses (Lima et al. 2001). Their sum L is just the total angular momentum per unit mass carried away from the Sun.

To make further progress, we introduce the radial Alfvénic Mach number M_B, defined by

$$M_B^2 = v_r^2/v_B^2 = 4\pi p\varrho v_r^2/B_r^2 = 4\pi\varrho v_r^2/B_r^2, \qquad (p=1). \tag{6.3.130}$$

The Alfvén velocity v_B is given by Eq. (3.10.254) or (5.11.127), with the magnetic permeability $p = 1$ and $B = B_r$. We eliminate B_φ between Eqs. (6.3.126) and (6.3.129):

$$r\ d\varphi/dt = v_\varphi = \Omega r(M_B^2 L/\Omega r^2 - 1)/(M_B^2 - 1). \tag{6.3.131}$$

The critical (singular) point r_c of this equation is given by the condition that numerator and denominator both vanish simultaneously at this point, viz. if $M_B = 1$, we require that also $M_B^2 L/\Omega r_c^2 - 1 = 0$, or

$$L = \Omega r_c^2. \tag{6.3.132}$$

r_c denotes the location of the so-called Alvén critical point. It is seen from Eqs. (6.3.128) and (6.3.130) that $M_B^2/v_r r^2 = 4\pi\varrho v_r r^2/B_r^2 r^4$ is a constant, which may be evaluated at the critical point $r = r_c,\ v_r = v_{rc},\ M_B = 1:$

$$M_B^2/v_r r^2 = 1/v_{rc}r_c^2 \quad \text{or} \quad M_B^2 = v_r r^2/v_{rc}r_c^2 = \varrho_c/\varrho. \tag{6.3.133}$$

With the aid of Eqs. (6.3.132), (6.3.133) the azimuthal velocity (6.3.131) becomes

$$v_\varphi = \Omega r(v_r/v_{rc} - 1)/(M_B^2 - 1), \tag{6.3.134}$$

and the azimuthal magnetic field (6.3.124) is

$$B_\varphi = (\Omega B_r r/v_{rc})(1 - r^2/r_c^2)/(M_B^2 - 1). \tag{6.3.135}$$

From Eq. (6.3.133) we may obtain an estimate of the solar Alvén critical point r_c, expressed in terms of quantities r_E, v_{rE}, v_{BE} measured at Earth's orbit:

$$M_{BE} = v_{rE}^2/v_{BE}^2 = v_{rE}\ r_E^2/v_{rc}\ r_c^2 \quad \text{or} \quad r_c = r_E\ v_{BE}/(v_{rE}\ v_{rc})^{1/2}. \tag{6.3.136}$$

A lower limit to r_c can be obtained by setting $v_{rc} = v_{rE}:$

$$r_c > r_E\ v_{BE}/v_{rE} \approx 20\ r_\odot. \tag{6.3.137}$$

Hence, in virtue of Eqs. (6.3.132), (6.3.137) the total angular momentum of the solar wind, i.e. the angular momentum loss of the Sun can be evaluated as if there were solid body rotation up to a distance of at least 20 solar radii. To calculate the total angular momentum carried away from the mass M by the magnetized plasma of the wind, we assume that our evaluation (6.3.132) of the angular momentum per unit mass in the equatorial region applies to the entire stellar surface. The value of L at colatitude λ obtains if we replace r_c in Eq. (6.3.132) by the distance $\ell_c = r_c \sin\lambda$ from the solar rotation axis: $L(\lambda) = \Omega r_c^2 \sin^2\lambda$. With this value the total angular momentum loss of the mass M results by integration over the critical Alfvén surface $S_c = 4\pi r_c^2:$

$$dJ/dt = \varrho_c v_{rc}\int_{S_c} L(\lambda)\ dS_c = 2\pi\Omega\varrho_c v_{rc}r_c^4\int_0^\pi \sin^3\lambda\ d\lambda = 8\pi\Omega\varrho_c v_{rc}r_c^4/3 = (2\Omega r_c^2/3)\ dM/dt,$$
$$(dS_c = r_c^2\sin\lambda\ d\lambda\ d\varphi;\ dM/dt = 4\pi\varrho_c v_{rc}r_c^2 = 4\pi\varrho_0 v_{r0}r_0^2). \tag{6.3.138}$$

To determine the potential influence of solar wind plasma on the braking of solar rotation, we insert $\Omega = J/kMr^2$ for the angular velocity from Eq. (5.12.185), where the dimensionless gyration factor $k = 2I/3Mr^2$ takes the value $k = 0.4$ for a constant density polytrope $n = 0$, and $k = 0.0758$ for a $n = 3$ polytrope (Eq. (6.1.179) and Table 6.1.2). Eq. (6.3.138) becomes for the Sun

$$dJ_\odot/dt = (2J_\odot r_c^2/3kM_\odot r_\odot^2)\ dM_\odot/dt = J_\odot/\tau, \tag{6.3.139}$$

where $\tau = (3kM_\odot r_\odot^2/2J_\odot r_c^2)/(dM_\odot/dt) \approx (3-7)\times 10^9$ yr (Weber and Davis 1967, Brandt 1970). Thus, during an interval comparable to the age of the Sun, the magnetic field would be able to slow down substantially the solar rotation rate by the factor e, while a mass loss at the present rate produces a total solar mass loss of only $\approx 10^{-4}M_\odot$.

The loss of rotational kinetic energy from sufficiently rapidly rotating stars with strong magnetic fields affects the energetics of the wind, increasing by magnetic acceleration the wind velocity in comparison to the winds emanating from slowly rotating stars having weak field strength. The relevant magnetic acceleration is determined from the radial component of the equation of motion (2.1.1) via Eqs. (B.38), (B.42):

$$\varrho\ (D\vec{v}/Dt)_r = \varrho\ [\vec{v}\times\nabla)\vec{v}]_r = \varrho(v_r\ dv_r/dr - v_\varphi^2/r) = -dP/dr - GM\varrho/r^2$$
$$+(1/4\pi)[(\nabla\times\vec{B})\times\vec{B}]_r = -dP/dr - GM\varrho/r^2 - (B_\varphi/4\pi r)\ d(rB_\varphi)/dr. \tag{6.3.140}$$

The energy equation (6.3.142) can be most easily obtained by integrating Eq. (6.3.140). The radial terms integrate at once with the polytropic law (6.3.3), analogously to Eq. (6.3.69). The azimuthal quantities require some transformations. By virtue of Eqs. (6.3.124), (6.3.127) we have

$$v_\varphi^2\ dr/r - (B_\varphi/4\pi\varrho r)\ d(rB_\varphi) = (v_\varphi/r)\big[-r\ dv_\varphi + (B_r/4\pi\varrho v_r)\ d(rB_\varphi)\big]$$
$$+(\Omega rB_r - v_\varphi B_r)\ d(rB_\varphi)/4\pi\varrho rv_r = -v_\varphi\ dv_\varphi + (\Omega B_r/4\pi\varrho v_r)\ d(rB_\varphi). \tag{6.3.141}$$

The integration of the azimuthal terms in Eq. (6.3.141) is now immediate, since $B_r/4\pi\varrho v_r = \mathrm{const}$ via Eq. (6.3.128), and the integral of Eq. (6.3.140) finally becomes (Belcher and MacGregor 1976):

$$(v_r^2+v_\varphi^2)/2 + (n+1)K\varrho^{1/n} - GM/r - \Omega rB_rB_\varphi/4\pi\varrho v_r = \mathrm{const},\qquad (n\neq\pm\infty). \tag{6.3.142}$$

The first term $v_r^2/2$ is the kinetic energy per unit mass associated with radial motion, the second term $v_\varphi^2/2$ the rotational kinetic energy, the third term the enthalpy (3.8.82) $H = (n+1)P/\varrho = (n+1)K\varrho^{1/n}$, the fourth the potential gravitational energy, and the last term the magnetic energy term (r^2 times the radial component of the Poynting flux divided by the mass flux).

To determine $v_{r\infty}$ – the stellar wind velocity at infinity due to magnetic acceleration – we have to evaluate the asymptotic behaviour of $\vec{v}$ and $\vec{B}$. We assume – as in the nonmagnetic wind – that the radial velocity v_r approaches a constant value $v_{r\infty}$ [cf. Eq. (6.3.52)]. Then, we observe from Eq. (6.3.130) that the radial Alfvénic Mach number is asymptotically proportional to r, since $M_B^2 = 4\pi\varrho v_r^2r^4/B_r^2r^4 \propto r^2$, $(r\to\infty)$ via Eqs. (6.3.10), (6.3.125). From Eqs. (6.3.134), (6.3.135) results with this finding: $v_\varphi, B_\varphi \propto 1/r$, since $B_r \propto 1/r^2$. We are now in position to evaluate the asymptotic behaviour of the last term in the energy equation (6.3.142):

$$-\Omega rB_rB_\varphi/4\pi\varrho v_r = (\Omega rB_r/v_r)^2/4\pi\varrho = (\Omega^2r_0^2B_{r0}^2/4\pi\varrho_0v_{r0})/v_r = v_M^3/v_r = v_M^3/v_{r\infty},$$
$$(r\to\infty;\ (r^2B_r)^2/4\pi\varrho v_rr^2 = \mathrm{const};\ v_M = [(\Omega r_0B_{r0})^2/4\pi\varrho_0v_{r0}]^{1/3} = \mathrm{const}). \tag{6.3.143}$$

At infinity the sole subsisting terms in the energy equation (6.3.142) are the first and the last one:

$$v_{r\infty}^2/2 + v_M^3/v_{r\infty} = \mathrm{const},\qquad (r\to\infty;\ n\neq\pm\infty). \tag{6.3.144}$$

The topology of the hydromagnetic stellar wind can best be visualized from the ordinary differential equation (6.3.148). We derive Eq. (6.3.123), and get a differential equation connecting $d(rB_\varphi)/dr$ and $d(rv_\varphi)/dr$:

$$B_r\ d(rv_\varphi)/dr = v_r\ d(rB_\varphi)/dr + rB_\varphi\ dv_r/dr - rv_\varphi\ dB_r/dr. \tag{6.3.145}$$

Elimination of $d(rv_\varphi)/dr$ between Eqs. (6.3.127) and (6.3.145) yields

$$d(rB_\varphi)/dr = -(2v_rv_\varphi B_r + rv_rB_\varphi\ dv_r/dr)/(v_r^2 - B_r^2/4\pi\varrho),\qquad (dB_r/dr = -2B_r/r). \tag{6.3.146}$$

The pressure term in Eq. (6.3.140) can be transformed as follows:

$$(1/\varrho)\ dP/dr = (1+1/n)K\varrho^{1/n-1}\ d\varrho/dr = [(1+1/n)P/\varrho^2]\ d\varrho/dr$$
$$= [(1+1/n)P/\varrho][-(1/v_r)\ dv_r/dr - 2/r],\qquad (d\ln(\varrho v_rr^2)/dr = 0). \tag{6.3.147}$$

The radial momentum equation (6.3.140) becomes with the two last equations after some algebraic manipulations equal to

$$(r/v_r)\ dv_r/dr = \big\{\big[2(1+1/n)P/\varrho - GM/r + v_\varphi^2\big](v_r^2 - B_r^2/4\pi\varrho) + v_r v_\varphi B_r B_\varphi/2\pi\varrho\big\}$$
$$/\big\{\big[v_r^2 - (1+1/n)P/\varrho\big](v_r^2 - B_r^2/4\pi\varrho) - v_r^2 B_\varphi^2/4\pi\varrho\big\}, \qquad (n \neq 0, \pm\infty). \qquad (6.3.148)$$

The topological study of this equation shows that it possesses three critical (singular) points. The first critical point r_s (the so-called slow point) is a saddle point and becomes in absence of magnetic fields and rotation just equal to the sonic point (6.3.32) in Parker's radial stellar wind theory. At this point v_r equals the radial phase speed of slow magnetoacoustic waves. The second critical point is the Alfvén critical point r_c from Eq. (6.3.132), a higher order singularity, where the radial velocity v_r is just equal to the radial Alvén velocity $v_B = B_r/(4\pi\varrho)^{1/2}$, $(M_B = 1)$. And at the third critical saddle point r_f (the so-called fast point) v_r is equal to the radial phase speed of the fast magnetosonic waves. The relevant magnetohydrodynamic wind solution is that which passes through all three critical points (Weber and Davis 1967, Belcher and MacGregor 1976).

For small rotation rates and/or weak magnetic fields there is $v_M \approx 0$ in virtue of Eq. (6.3.143), and $v_{r\infty}$ in Eq. (6.3.144) is nearly equal to the Parker (1963) value for the nonmagnetic radial flow considered in Sec. 6.3.1. In contrast, the velocity at infinity departs markedly from the Parker value at sufficiently high rotation rates and/or magnetic field strengths. Below, we determine the value of v_M in this so-called fast rotator case for future reference in connection with the blow-off of circumstellar (protoplanetary) clouds by a T-Tauri like stellar wind (Sec. 6.3.4). The asymptotic value of $B_\varphi/(4\pi\varrho)^{1/2}$ occurring in Eq. (6.3.148) turns out to approach a constant value:

$$B_\varphi/(4\pi\varrho)^{1/2} = -\Omega r B_r/v_r(4\pi\varrho)^{1/2} = -\big[\Omega r_0 B_{r0}/(4\pi\varrho_0 v_{r0})^{1/2}\big]/v_r^{1/2}$$
$$= -v_M^{3/2}/v_r^{1/2} = -v_M^{3/2}/v_{r\infty}^{1/2}, \qquad (r \to \infty), \qquad (6.3.149)$$

where we have used Eqs. (6.3.128), (6.3.143), together with the previously noted asymptotic form $B_\varphi = -\Omega r B_r/v_r$ of Eq. (6.3.124). From the asymptotic behaviour of the relevant quantities follows that $|B_\varphi/(4\pi\varrho)^{1/2}|, v_r \gg v_\varphi, GM/r, B_r/(4\pi\varrho)^{1/2}, P/\varrho$, and Eq. (6.3.148) takes the asymptotic form

$$(r/v_r)\ dv_r/dr = \big[2(1+1/n)P/\varrho - GM/r + v_\varphi^2 + v_\varphi B_r B_\varphi/2\pi\varrho v_r\big]/(v_r^2 - B_\varphi^2/4\pi\varrho), \qquad (r \to \infty). \qquad (6.3.150)$$

We apply this equation to the fast critical point r_f, which for fast magnetic rotators is far outside the critical Alfvén point r_c, where $v_r = B_r/(4\pi\varrho)^{1/2}$, $(M_B = 1)$ via Eq. (6.3.130), (Belcher and MacGregor 1976). The vanishing of the denominator in Eq. (6.3.150) at $r = r_f$ requires according to Eq. (6.3.149) that $v_r = -B_\varphi/(4\pi\varrho)^{1/2} = v_M^{3/2}/v_{r\infty}^{1/2}$. Since r_f is far outside the star, the radial wind velocity v_r has essentially reached its terminal value $v_{r\infty}$. Thus, $v_r \approx v_{r\infty} \approx v_M^{3/2}/v_{r\infty}^{1/2}$ or $v_{r\infty} = v_M$. With this finding the energy equation (6.3.144) takes for fast magnetic rotators the simple form

$$v_{r\infty}^2/2 + v_M^3/v_{r\infty} = 3v_{r\infty}^2/2 = 3v_M^2/2 = \text{const}, \qquad (r \to \infty;\ n \neq \pm\infty), \qquad (6.3.151)$$

showing that at infinity two thirds of the energy flux is carried away by the electromagnetic field, and only one third by the particle flux.

In her study of small-amplitude hydromagnetic waves Abraham-Shrauner (1973) has employed a generalized bipolytropic law for the pressure components $P_\parallel$ and $P_\perp$, parallel and perpendicular to the magnetic field lines:

$$P_\parallel = K_\parallel \varrho^{1+1/n_\parallel} B^{m_\parallel}; \quad P_\perp = K_\perp \varrho^{1+1/n_\perp} B^{m_\perp}, \qquad (K_\parallel, n_\parallel, m_\parallel, K_\perp, n_\perp, m_\perp = \text{const}). \qquad (6.3.152)$$

Theoretical values of $\gamma_\parallel = 1 + 1/n_\parallel$, $\gamma_\perp = 1 + 1/n_\perp$ for collisionless plasmas have been obtained by Belmont and Mazelle (1992); these values depend on wave mode and on the plasma population (cf. Lin et al. (2001) for the Earth's magnetosheath).

6.3.4 Interaction of Polytropic Winds with Other Objects

Interactions of this kind involve complex magnetohydrodynamic processes, and the subsequent fragmentary presentation constitutes only a rough approach to real phenomena.

(i) Magnetized planets (Mercury, Earth, Jupiter, Saturn, Uranus, Neptune). Solar system bodies are immersed in the magnetized solar wind. The interaction between solar wind and planetary magnetic field introduces large scale currents that can almost confine the planetary magnetic field, forming a magnetosphere round the planet. The outer boundary of the magnetosphere – the magnetopause – separates the planetary magnetic field from the solar wind. The size of the magnetosphere is determined by the balance between solar wind energy and the planet's magnetic energy. The magnetospheric boundary is compressed on the side facing the solar wind and very elongated in the opposite direction, forming a magnetic tail. The outer, unperturbed, planetary magnetic field – as well as that of pulsars and some radio galaxies – can be approximated by a simple dipole field (3.10.25) of magnitude

$$H^2 = B^2 = B_r^2 + B_\lambda^2 = (a_m^2/r_p^6)(1 + 3\cos^2\lambda_m), \qquad (\vec{B} = p\vec{H};\ p = 1). \tag{6.3.153}$$

r_p is now the planetocentric distance, a_m the absolute value of the magnetic moment, and λ_m the magnetic zenith angle (magnetic colatitude).

If a magnetized polytropic plasma wind (the solar wind) encounters other plasmas or magnetic fields, there can develop at the interface more types of discontinuities than in ordinary fluids: Shocks, as well as tangential, contact, and rotational discontinuities (e.g. Landau and Lifschitz 1974, Parks 1991). In addition to the discontinuous fluid parameters of ordinary fluids (flow velocity $\vec{v}$, temperature T, density ϱ, and pressure tensor P_{ij}), the set of discontinuous variables in plasmas includes also electric current density $\vec{J}$, magnetic and electric fields $\vec{B}$ and $\vec{E}$. The solar wind can be regarded as a collisionless, supersonic superalfvénic plasma ($M_A, M_B \gg 1$) producing a bow shock wave as it encounters the planet's magnetic field (Fig. 6.3.7). Some boundary conditions concerning the magnetic and electric field have already been established in Eqs. (5.11.56), (5.11.58), namely the continuity of the normal component $B_n = \vec{B}\cdot\vec{n}$ of the magnetic field, and of the tangential component $E_\tau = \vec{E}\cdot\vec{\tau} = |\vec{E}\times\vec{n}|$ of the electric field. And these conditions have to hold also across a stationary shock front, such as a bow wave in the solar wind:

$$B_{n1} = \vec{B}_1\cdot\vec{n} = \vec{B}_2\cdot\vec{n} = B_{n2} = B_n, \tag{6.3.154}$$

$$E_{\tau1} = |\vec{E}_1\times\vec{n}| = |\vec{E}_2\times\vec{n}| = E_{\tau2} = E_\tau. \tag{6.3.155}$$

The subscripts 1 and 2 refer to the upstream and downstream side of the shock. It is always sufficient to consider plane shocks. Eq. (6.3.155) can be transformed into a more transparent form involving the magnetic field and the flow velocity, starting with Eq. (3.10.12), which takes for an ideal magnetohydrodynamic fluid (infinite conductivity σ) the simple form

$$\vec{E} = -(\vec{v}\times\vec{B})/c = (\vec{B}\times\vec{v})/c, \tag{6.3.156}$$

where $\vec{E}$ is the electric field measured in the laboratory frame, and $\vec{v}$ the bulk velocity with respect to this frame. The vectorial product $\vec{E}\times\vec{n}$ takes the form

$$\vec{E}\times\vec{n} = -\vec{n}\times\vec{E} = (1/c)[\vec{n}\times(\vec{v}\times\vec{B})] = (1/c)\big[(\vec{B}\cdot\vec{n})\vec{v} - (\vec{v}\cdot\vec{n})\vec{B}\big], \tag{6.3.157}$$

by using the vectorial identity $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$. We decompose $\vec{E}, \vec{B}$, and $\vec{v}$ into their normal and tangential components with respect to the shock front, and the sole surviving component of Eq. (6.3.157) is $E_\tau = (1/c)(B_n v_\tau - v_n B_\tau)$. The continuity condition (6.3.155) on both sides of the shock turns into

$$B_n(v_{\tau1} - v_{\tau2}) = v_{n1}B_{\tau1} - v_{n2}B_{\tau2}, \qquad (B_n = B_{n1} = B_{n2}). \tag{6.3.158}$$

Three other boundary conditions result – similarly to the Rankine-Hugoniot relations in ordinary fluids – from conservation of mass, momentum, and energy, as the magnetized fluid crosses the interface. The

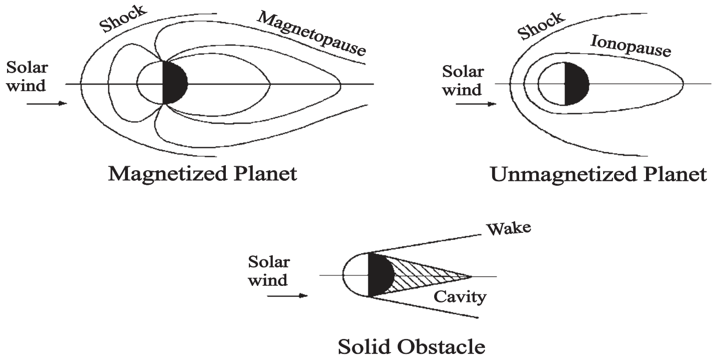

Fig. 6.3.7 Schematic figure illustrating the different types of boundaries created by the solar wind interacting with a magnetic planet, a nonmagnetic planet with atmosphere, and an atmosphereless nonmagnetic object, respectively (Parks 1991).

continuity equation (5.2.1) becomes in our stationary case $\nabla \cdot (\varrho \vec{v}) = 0$. Proceeding as in Eq. (5.11.55), the integration over the small volume V_ε of a surface shell element S_ε yields

$$\int_{V_e} \nabla \cdot (\varrho \vec{v})\ dV = \int_{S_\varepsilon} \varrho(\vec{v} \cdot \vec{n})\ dS = \varrho_2 v_{n2} - \varrho_1 v_{n1} = 0. \tag{6.3.159}$$

In fact, this conservation of vertical mass flow across the boundary could have been written down quite intuitively. The momentum equation (2.1.1) can be transformed into a more suitable form, by observing that its left-hand side becomes with summation over repeated indices equal to

$$\begin{aligned} &\varrho\ Dv_i/Dt = \varrho\ \partial v_i/\partial t + \varrho(\vec{v} \cdot \nabla)v_i = \partial(\varrho v_i)/\partial t - v_i\ \partial\varrho/\partial t + \varrho(\vec{v} \cdot \nabla)v_i \\ &= \partial(\varrho v_i)/\partial t + v_i\ [\nabla \cdot (\varrho \vec{v})] + \varrho(\vec{v} \cdot \nabla)v_i = \partial(\varrho v_i)/\partial t + \partial(\varrho v_i v_k)/\partial x_k, \\ &(\vec{v} = \vec{v}(v_1, v_2, v_3);\ i, k = 1, 2, 3). \end{aligned} \tag{6.3.160}$$

We transform the magnetic term according to Eqs. (2.6.52), (2.6.54), so that Eq. (2.1.1) turns into (e.g. Landau and Lifshitz 1959, 1974)

$$\begin{aligned} &\partial(\varrho v_i)/\partial t = \partial\big[- \varrho v_i v_k - \delta_{ik} P + (1/4\pi p)(B_i B_k - \delta_{ik} B^2/2)\big]\big/\partial x_k + \varrho\ \partial\Phi/\partial x_i \\ &= \partial\Pi_{ik}/\partial x_k + \varrho\ \partial\Phi/\partial x_i, \qquad (\vec{B} = \vec{B}(B_1, B_2, B_3);\ \vec{H} = \vec{B}/p), \end{aligned} \tag{6.3.161}$$

where δ_{ik} means the Kronecker delta. Ignoring the influence of gravitation in the thin shock front, we conclude that under steady-state conditions $(\partial\ /\partial t = 0)$ the divergence $\partial\Pi_{ik}/\partial x_k$ of the momentum flux tensor Π_{ik} vanishes. This implies – just as outlined in Eq. (6.3.159) – that its normal components $\Pi_{ik} n_i$ are continuous across the shock front. This reads in concise vectorial form as (e.g. Parks 1991)

$$\begin{aligned} &\varrho_1(\vec{v}_1 \cdot \vec{n})\vec{v}_1 + P_1 \cdot \vec{n} + (1/4\pi p)[B_1^2 \vec{n}/2 - (\vec{B}_1 \cdot \vec{n})\vec{B}_1] \\ &= \varrho_2(\vec{v}_2 \cdot \vec{n})\vec{v}_2 + P_2 \cdot \vec{n} + (1/4\pi p)[B_2^2 \vec{n}/2 - (\vec{B}_2 \cdot \vec{n})\vec{B}_2]. \end{aligned} \tag{6.3.162}$$

If velocity and magnetic induction are decomposed into their normal and tangent components, this equation splits into two scalar equations after scalar multiplication with $\vec{n}$ and $\vec{\tau}$, respectively:

$$\varrho_1 v_{n1}^2 + P_1 + B_{\tau 1}^2/8\pi p = \varrho_2 v_{n2}^2 + P_2 + B_{\tau 2}^2/8\pi p, \qquad (B_n = B_{n1} = B_{n2};\ B^2 = B_n^2 + B_\tau^2), \tag{6.3.163}$$

$$\varrho_1 v_{n1} v_{\tau 1} - B_n B_{\tau 1}/4\pi p = \varrho_2 v_{n2} v_{\tau 2} - B_n B_{\tau 2}/4\pi p. \tag{6.3.164}$$

Another boundary requirement is derived from the magnetohydrodynamic energy equation, which can be most easily obtained by taking the scalar product between $\vec{v}$ and the dissipationless Eq. (2.1.1):

$$\varrho\vec{v}\cdot(D\vec{v}/Dt) = -\vec{v}\cdot\nabla P + \varrho(\vec{v}\cdot\nabla\Phi) + (1/4\pi p)\ \vec{v}\cdot[(\nabla\times\vec{B})\times\vec{B}]. \qquad (6.3.165)$$

The left-hand side can be written with the continuity equation (5.2.1) as follows:

$$\varrho\vec{v}\cdot(D\vec{v}/Dt) = \varrho\vec{v}\cdot(\partial\vec{v}/\partial t) + \varrho\vec{v}\cdot[(\vec{v}\cdot\nabla)\vec{v}] = (\varrho/2)\ \partial v^2/\partial t + (\varrho/2)(\vec{v}\cdot\nabla v^2)$$
$$= \partial(\varrho v^2/2)/\partial t - (v^2/2)\ \partial\varrho/\partial t + (\varrho/2)(\vec{v}\cdot\nabla v^2) = \partial(\varrho v^2/2)/\partial t + (v^2/2)\ \nabla\cdot(\varrho\vec{v}) + (\varrho/2)(\vec{v}\cdot\nabla v^2)$$
$$= \partial(\varrho v^2/2)/\partial t + \nabla\cdot(\varrho v^2\vec{v}/2). \qquad (6.3.166)$$

The pressure term can be most easily transformed with the adiabatic equation (5.2.21) and the continuity equation (5.2.2):

$$DP/Dt = \partial P/\partial t + \vec{v}\cdot\nabla P = (\Gamma_1 P/\varrho)\ D\varrho/Dt = -\Gamma_1 P\ \nabla\cdot\vec{v} \quad \text{or}$$
$$\vec{v}\cdot\nabla P = [\Gamma_1/(\Gamma_1-1)]\ \nabla\cdot(P\vec{v}) + [1/(\Gamma_1-1)]\ \partial P/\partial t. \qquad (6.3.167)$$

The gravity term becomes with the continuity equation (5.2.1) equal to

$$\varrho(\vec{v}\cdot\nabla\Phi) = \nabla\cdot(\varrho\Phi\vec{v}) - \Phi\ \nabla\cdot(\varrho\vec{v}) = \nabla\cdot(\varrho\Phi\vec{v}) + \Phi\ \partial\varrho/\partial t = \nabla\cdot(\varrho\Phi\vec{v}) + \partial(\varrho\Phi)/\partial t,$$
$$(\partial\Phi/\partial t = 0). \qquad (6.3.168)$$

To transform the magnetic term in Eq. (6.3.165) we observe that

$$\vec{v}\cdot[(\nabla\times\vec{B})\times\vec{B}] = -(\nabla\times\vec{B})\cdot(\vec{v}\times\vec{B}) = c(\nabla\times\vec{B})\cdot\vec{E} = c(\nabla\times\vec{E})\cdot\vec{B} + c\ \nabla\cdot(\vec{B}\times\vec{E})$$
$$= -(\partial\vec{B}/\partial t)\cdot\vec{B} - \nabla\cdot[\vec{B}\times(\vec{v}\times\vec{B})] = -(1/2)\ \partial B^2/\partial t - \nabla\cdot[\vec{B}\times(\vec{v}\times\vec{B})], \qquad (6.3.169)$$

where we have used the vectorial relationships $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{b}\cdot(\vec{a}\times\vec{c})$, $\nabla\cdot(\vec{a}\times\vec{b}) = (\nabla\times\vec{a})\cdot\vec{b} - (\nabla\times\vec{b})\cdot\vec{a}$, as well as Eq. (6.3.156), and the Maxwell equation (3.10.1): $\nabla\times\vec{E} = -(1/c)\ \partial\vec{B}/\partial t$. The final form of the adiabatic energy equation is obtained by inserting Eqs. (6.3.166)-(6.3.169) into Eq. (6.3.165), (see Landau and Lifschitz (1974, §51) for a more complete form including thermal conduction, viscosity, and a finite electric conductivity σ) :

$$\partial\big[\varrho v^2/2 + P/(\Gamma_1-1) - \varrho\Phi + B^2/8\pi p\big]/\partial t$$
$$+\nabla\cdot\big[\varrho v^2\vec{v}/2 + \Gamma_1 P\vec{v}/(\Gamma_1-1) - \varrho\Phi\vec{v} + \vec{B}\times(\vec{v}\times\vec{B})/4\pi p\big] = 0, \qquad (\sigma=\infty). \qquad (6.3.170)$$

Under steady-state conditions the partial time derivative vanishes, and the divergence term must be zero. Therefore – by a quite analogous reasoning as effected in Eq. (6.3.159) – the normal component of the stationary energy flux, i.e. the scalar product between the term in the bracket of Eq. (6.3.170) and the normal vector $\vec{n}$ must be conserved across the adiabatic shock front:

$$\varrho v_n\big[v^2/2 + \Gamma_1 P/(\Gamma_1-1)\varrho\big] + \big[B^2 v_n - (\vec{v}\cdot\vec{B})B_n\big]/4\pi p = \text{const.} \qquad (6.3.171)$$

The normal component of the magnetic term has been evaluated as in Eq. (6.3.157). The gravity term has canceled out, because of the continuity of ϱv_n and Φ across the boundary surface.

The set of boundary (discontinuity) conditions that holds between quantities on the two sides of an adiabatic boundary (discontinuity) is then given by Eqs. (6.3.154), (6.3.155), (6.3.158), (6.3.159), (6.3.163), (6.3.164), (6.3.171). A major simplification of the solar plasma flow round the magnetosphere results, if ordinary gas dynamics is used instead of hydromagnetic theory. The magnetic field of the solar wind is neglected, and the solar wind pressure acting on a point of the magnetopause is given by the simple Newtonian formula

$$P = P_0\cos^2\psi = k\varrho v^2\cos^2\psi = B^2/8\pi p, \qquad (k = \text{const}). \qquad (6.3.172)$$

ϱ and v denote the undisturbed mass density and velocity of the upstreaming solar wind, and ψ is the angle between $\vec{v}$ and the inner normal to the magnetopause – the boundary surface of the magnetosphere. The so-called stagnation pressure $P_0 = k\varrho v^2$ is the solar wind pressure exerted on the nose of the magnetopause, where $\psi = 0$, and the *downstreaming* flow velocity vanishes. Like in the Chapman-Ferraro theory, the incident solar wind plasma is assumed free of magnetic field, and the magnetosphere free of plasma. Eq. (6.3.172) represents just the force balance between the Newtonian hydrodynamic pressure P of the solar wind and the hydrostatic magnetic pressure $B^2/8\pi p = H^2/8\pi$ of the planetary magnetic field at the magnetopause. The constant k equals 2 in the case of elastic (specular) reflection of solar wind particles on the magnetospheric boundary, and is unity if inelastic reflections are assumed. For, the total change of momentum of elastically reflected particles is $2m\vec{v} = 2\varrho v \cos\psi\ \vec{v}$, where $m = \varrho v \cos\psi$ is the mass of particles striking the magnetopause per unit area per unit time. The change of momentum in the normal direction to the magnetic obstacle – the solar wind pressure – is then just equal to $P = 2\varrho v \cos\psi\ (\vec{v}\cdot\vec{n}) = 2\varrho v^2 \cos^2\psi$, showing that indeed $k = 2$ for elastic reflections (Parks 1991, §8.3.1). For completely inelastic collisions with the magnetopause the total momentum change is only $\varrho v \cos\psi\ \vec{v}$, showing that in this case $k = 1$. A more sophisticated evaluation of the stagnation pressure P_0 is obtained from the hydrodynamics of a supersonic stream striking the nose of a blunt obstacle (e.g. Landau and Lifshitz 1959, §114). Let us denote by M_A the Mach number (6.3.2) of the upstreaming solar wind. The jump between the pressure P of the upstreaming solar wind and the downstream pressure P_2 behind the normal shock along the stagnation stream line $\psi = 0$ is given by

$$P_2 = P(2\gamma M_A^2 + 1 - \gamma)/(\gamma + 1), \qquad (M_A = v/a > 1;\ \gamma = c_P/c_V), \tag{6.3.173}$$

while the corresponding velocity and density jumps are

$$v_2 = v[2 + (\gamma - 1)M_A^2]/(\gamma + 1)M_A^2 = a[2 + (\gamma - 1)M_A^2]/(\gamma + 1)M_A, \tag{6.3.174}$$

$$\varrho_2 = \varrho(\gamma + 1)M_A^2/[2 + (\gamma - 1)M_A^2]. \tag{6.3.175}$$

The ratio between the pressure P_2 just behind the normal shock and the stagnation pressure P_0 is derived with $P_2/P_0 = (\varrho_2/\varrho_0)^\gamma$ from the Bernoulli equation $v_2^2/2 + \gamma P_2/(\gamma - 1)\varrho_2 = \gamma P_0/(\gamma - 1)\varrho_0$ [cf. Eq. (6.3.69) if $n = 1/(\gamma - 1)$] along the stagnation streamline *behind* the shock, where the flow velocity vanishes in the stagnation point at pressure P_0:

$$P_0 = \varrho_2(P_0/P_2)^{1/\gamma}[P_2/\varrho_2 + (\gamma - 1)v_2^2/2\gamma] = P_2[1 + (\gamma - 1)v_2^2/2a_2^2]^{\gamma/(\gamma-1)}. \tag{6.3.176}$$

The adiabatic sound velocity just behind the shock has been denoted by $a_2 = (\gamma P_2/\varrho_2)^{1/2}$. We now insert Eqs. (6.3.173), (6.3.174) into Eq. (6.3.176), to obtain after some algebra the well known formula

$$\begin{aligned} P_0 &= PM_A^2[(\gamma + 1)/2]^{(\gamma+1)/(\gamma-1)}/\big[\gamma - (\gamma - 1)/2M_A^2\big]^{1/(\gamma-1)} \\ &= \varrho v^2[(\gamma + 1)/2]^{(\gamma+1)/(\gamma-1)}/\gamma\big[\gamma - (\gamma - 1)/2M_A^2\big]^{1/(\gamma-1)} = k\varrho v^2. \end{aligned} \tag{6.3.177}$$

The ratio of adiabatic sound velocities is evaluated according to $a^2/a_2^2 = P\varrho_2/P_2\varrho$. Observing that in the present application $M_A \gg 1$, we get from Eq. (6.3.177)

$$k = [(\gamma + 1)/2]^{(\gamma+1)/(\gamma-1)}/\gamma^{\gamma/(\gamma-1)}, \qquad (M_A \gg 1), \tag{6.3.178}$$

showing that k is indeed of order unity: $k = 0.844$ and 0.881 if $\gamma = 2$ and $5/3$, respectively (Spreiter et al. 1966).

For the magnetized anisotropic plasma in the Earth's bow shock region Pudovkin et al. (1997) estimate the effective polytropic index to be $n = 1.1 - 2.5$, $(\gamma = 1 + 1/n = 1.4. - 1.9)$, while n may be negative in the magnetopause region.

It should be noted that during an adiabatic shock the entropy S increases, because the processes occurring in the shock front are irreversible (e.g. Sec. 1.1, Landau and Lifshitz 1959, §82). The ultimate cause of entropy increase are the dissipative processes occurring in the very thin layers, which actual shock fronts are. The amount of dissipation is entirely determined by the conservation laws of mass, momentum, and energy. However, the flow ahead and behind the shock front can be considered isentropic, with different, but constant entropies S_1 and S_2 on each side of the shock front.

The planetary magnetosphere thus acts as a real obstacle to the supersonic solar wind. However, the size and shape of the magnetopause are not known a priori, but must be determined as part of the gas dynamic solution. The solar wind modifies the outer parts of the planetary field by compression. For the Earth, available calculations indicate a compression of the planetary magnetic field B by a factor of about 2 on the sunward side. If the dipole field (6.3.153) – increased by the factor 2^2 – is inserted into Eq. (6.3.172), we obtain an approximate estimate for the distance of the nose of the magnetosphere from the centre of the magnetic planet (the stagnation distance):

$$r_{p0} = [a_m^2(1 + 3\cos^2\lambda_{m0})/2\pi k\varrho v^2]^{1/6}, \qquad (\psi = 0;\ p = 1). \tag{6.3.179}$$

λ_{m0} denotes the magnetic zenith angle of the stagnation point. Typically r_{p0} is about 10 Earth radii, while the bow shock is formed at about 14 Earth radii (stand-off distance) if $\psi = 0$ (e.g. Brandt 1970, Parks 1991). The region between bow shock and magnetopause is called the magnetosheath, and consists of shocked solar wind plasma and disordered interplanetary magnetic field, which generally do not penetrate into the planetary magnetosphere. A rough shape of the magnetopause can be obtained from Eq. (6.3.172):

$$\cos^2\psi = B^2/8\pi k\varrho v^2 = a_m^2(1 + 3\cos^2\lambda_m)/2\pi k\varrho v^2 r_p^6 = R, \tag{6.3.180}$$

allowing for a compression factor of 4 for the magnetic field B^2 from Eq. (6.3.153). We denote by α the latitude of a point on the magnetopause in a frame of polar (r_p, α)-coordinates with the origin in the planet's centre: $\alpha = 0$ if $\psi = 0$. The plane rectangular coordinates of a point on the magnetopause are $x = r_p\cos\alpha,\ y = r_p\sin\alpha$. Since $\cos^2\psi = dy^2/(dx^2 + dy^2) = (\sin\alpha\ dr_p + r_p\cos\alpha\ d\alpha)^2/(dr_p^2 + r_p^2\ d\alpha^2)$, we obtain from Eq. (6.3.180) an ordinary differential equation for the numerical determination of the magnetospheric boundary (Spreiter et al. 1970b):

$$(1/r_p)\ dr_p/d\alpha = [\sin 2\alpha \pm 2(R - R^2)^{1/2}]/2(R - \sin^2\alpha). \tag{6.3.181}$$

The proper choice of the sign in Eq. (6.3.181) is dictated by the consideration that as α increases from 0 to π, the planetocentric distance of the magnetopause r_p increases from r_{p0} to ∞, and hence R together with $\cos^2\psi$ diminishes from 1 to 0, neglecting the comparatively small variation of $\cos^2\lambda_m$ in Eq. (6.3.180). Thus, the denominator of Eq. (6.3.181) has to vanish at least for one critical value α_{cr}, and $dr_p/d\alpha$ would become infinite unless the numerator vanishes simultaneously. This yields the two simple equations

$$\sin 2\alpha_{cr} \pm 2(R_{cr} - R_{cr}^2)^{1/2} = 0; \quad \sin^2\alpha_{cr} = R_{cr}, \tag{6.3.182}$$

or

$$\sin 2\alpha_{cr} \pm 2(\sin^2\alpha_{cr} - \sin^4\alpha_{cr})^{1/2} = \sin 2\alpha_{cr} \pm |\sin 2\alpha_{cr}| = 0, \tag{6.3.183}$$

showing that the minus sign has to be taken in Eq. (6.3.181), as long as $0 \le \alpha_{cr} \le \pi/2$.

All these simple relationships are essentially confirmed by more recent evaluations (e.g. Petrinec and Russel 1997).

The dynamical properties of solar wind flow past a magnetosphere are thus represented by the numerical solution of the stationary magnetohydrodynamic equations of a dissipationless perfect plasma [Eqs. (6.3.156), (6.3.161), (6.3.170)] together with the Maxwell equations $\nabla\cdot\vec{B} = 0,\ \nabla\times\vec{E} = 0$, and the adiabatic equation of state $P = K\varrho^\gamma = K\varrho^{1+1/n}$. These equations have to be supplemented by the boundary conditions on both sides of the bow shock wave [Eqs. (6.3.154), (6.3.155), (6.3.158), (6.3.159), (6.3.163), (6.3.164), (6.3.171)].

(ii) Nonmagnetic (weakly magnetized) planets (Venus, Mars, and comets possessing an atmosphere). In this case there is no magnetosphere shielding the planetary (cometary) ionosphere from direct interaction with the solar wind. Since the solar wind with its frozen-in interplanetary magnetic field cannot penetrate another plasma, the solar wind is deflected round the ionosphere, preventing its penetration to lower levels in the atmosphere, where collision effects are dominant, or to the surface, where it would be absorbed as in the case of the Moon.

The pressure in the planetary ionosphere can be calculated approximately from the isothermal hydrostatic equation $dP_p/dr_p = (\mathcal{R}T_p/\mu)\ d\varrho_p/dr_p = -GM_p\varrho_p/r_p^2$, integrated over a massless ionosphere:

$$P_p = \mathcal{R}\varrho_p T_p/\mu = P_{p0}\exp\left[(GM_p\mu/\mathcal{R}T_p)(1/r_p - 1/r_{p0})\right], \qquad (T_p = \text{const}). \tag{6.3.184}$$

M_p is the planetary mass, and P_{p0} the stagnation pressure at the nose of the ionopause (the ionosphere boundary), located at stagnation distance r_{p0} from the planet's centre. A rough shape of the ionopause is obtained from Eq. (6.3.172), where the magnetostatic pressure is replaced by the gas pressure (6.3.184), and $P_0 = P_{p0}$, ($\cos\psi = 1$ if $r_p = r_{p0}$), (Spreiter et al. 1970b):

$$\cos^2\psi = \exp\left[(GM_p\mu/\mathcal{R}T_p)(1/r_p - 1/r_{p0})\right] = R. \qquad (6.3.185)$$

Eqs. (6.3.180) and (6.3.185) permit the rough calculation of the similar shapes of the magnetopause and ionopause, although the cavity carved into the solar wind is very different for the two applications, the stagnation distances r_{p0} being about 10 Earth radii for the Earth's magnetosphere, but only a few percent greater than the planetary radius for Venus or Mars, exceeding on the other side the cometary radius by many orders of magnitude (e.g. Biermann et al. 1967).

(iii) Nonmagnetic objects without atmosphere (most planetary satellites, asteroids). This topic has no relevance to polytropes and is mentioned merely for completeness. These objects act much like a dielectric obstacle ($\sigma = 0$) placed in the conducting solar wind. When the solar wind particles impact the surface of the Moon, they are neutralized and absorbed. A cavity and a wake are formed behind the Moon, but no bow shock wave or magnetosheath layer of thermalized shocks (Spreiter et al. 1970a). The interplanetary magnetic field is continuous from the solar wind into the Moon and into the lunar cavity, within which no particles and electric currents are flowing (Fig. 6.3.7).

(iv) Interstellar medium. Consider a solar wind with a mean supersonic velocity of 300-400 km/s and a mean number density of $n_{dE} = 10-20$ atoms/cm^3 at the Earth's orbit. In virtue of the solar wind model described by the first equation (6.3.52), the velocity $v \propto \psi^{1/2}$ assumes a nearly constant value at large distances from the Sun ($\lambda \to 0$). The mass conservation equation (6.3.10) yields

$$\varrho = \varrho_E(r_E/r)^2; \quad n_d = n_{dE}(r_E/r)^2, \qquad (6.3.186)$$

where E-indexed quantities denote values at the orbit of the Earth. The solar (stellar) wind continues its supersonic flow outward, until its pressure decreases to the level of the pressure P_i of the interstellar medium. A rough estimate of the stagnation pressure P_0 can be obtained from Eq. (6.3.172) for vertical incidence $\psi = 0$, if we replace the magnetic pressure by the pressure P_i of the interstellar medium:

$$P_0 = k\varrho v^2 = P_i. \qquad (6.3.187)$$

At a certain distance r_s (stand-off distance, shock radius) the solar wind is shocked, as it "feels" the pressure of interstellar gases and magnetic fields. We insert ϱ from Eq. (6.3.186) into Eq. (6.3.187), getting a rough estimate of the stagnation distance r_0 with respect to the Sun, where post-shocked solar wind comes to rest:

$$r = r_0 \approx r_E v(k\varrho_E/P_i)^{1/2}, \qquad (k \approx 1). \qquad (6.3.188)$$

A more refined expression of r_0 can be obtained in a similar way as effected in Eqs. (6.3.173)-(6.3.178). For the large Mach numbers involved, the Rankine-Hugoniot relations (6.3.173)-(6.3.175) reduce for the normal stationary shock between solar wind and interstellar medium to

$$P_2 = P(2\gamma v^2/a^2 + 1 - \gamma)/(\gamma+1) \approx 2\varrho v^2/(\gamma+1); \quad v_2 \approx v(\gamma-1)/(\gamma+1);$$
$$\varrho_2 \approx \varrho(\gamma+1)/(\gamma-1), \qquad (M_A^2 = v^2/a^2 = \varrho v^2/\gamma P \gg 1). \qquad (6.3.189)$$

Unindexed symbols denote quantities in the quiet solar wind ahead the shock, while post-shock values are indexed with 2. Although Fahr (1980) takes another view, it will be assumed that the difference between stand-off distance r_s (location of the shock wave) and stagnation distance r_0 (where the velocity of shocked solar wind becomes zero) is small as compared to r_s (Parker 1963, Talbot and Newman 1977). Therefore, the variation of solar gravity along the distance $r_0 - r_s$ will be negligible, and the Bernoulli equation between stand-off and stagnation distance assumes the simple form

$$v_2^2/2 + (n+1)K\varrho_2^{1/n} = v_2^2/2 + (n+1)K^{n/(n+1)}P_2^{1/(n+1)} = (n+1)K^{n/(n+1)}P_0^{1/(n+1)}, \qquad (6.3.190)$$

where the velocity at the stagnation point is zero, and n, ($n \neq -1, \pm\infty$) denotes the polytropic index of post-shocked solar wind. From Eq. (6.3.189) we get

$$v_2^2 = (\gamma-1)^2v^2/(\gamma+1)^2 = (\gamma-1)^2P_2/2\varrho(\gamma+1) = (\gamma-1)P_2/2\varrho_2$$
$$= (\gamma-1)K^{n/(n+1)}P_2^{1/(n+1)}/2, \qquad (6.3.191)$$

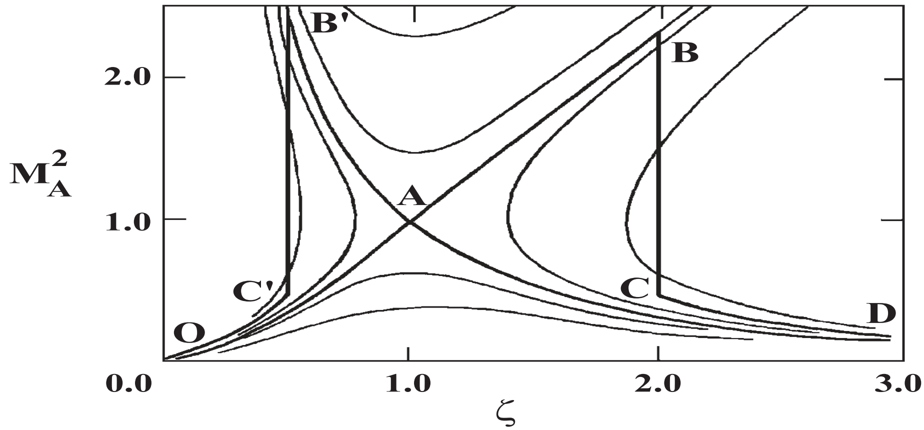

Fig. 6.3.8 Squared Mach number M_A^2 versus nondimensional distance ζ in units of the distance of the sonic point A from the central mass M. The flow is assumed adiabatic with $n = 3, (\gamma = 1 + 1/n = 4/3)$. The curve connecting the points $OABCD$ is the complete solution of a stellar wind expanding into the interstellar medium with pressure ratio $P_i/P_* = 0.0561$, where P_i is the pressure in the interstellar medium and P_* the surface pressure of the star. The contour $DAB'C'O$ is the complete solution of an accretion flow (see Sec. 6.4.2), having $P_0/P_*(r_0/r_*)^4 = 0.1797$, where r_* is the stellar radius, and zero indexed quantities denote pressure and distance at the reference level. The shocks of the two flows – depicted by the thicker streamlines – occur from B to C, and from B' to C', respectively (Holzer and Axford 1970).

and Eq. (6.3.190) becomes

$$P_2 = P_0[1 + (\gamma - 1)/4(n + 1)]^{-(n+1)}, \qquad (P_0 = P_i;\ M_A \gg 1). \tag{6.3.192}$$

We eliminate P_2 between Eqs. (6.3.189) and (6.3.192): $P_0 = P_i = [2\varrho v^2/(\gamma+1)][1+(\gamma-1)/4(n+1)]^{n+1}$. The stagnation distance of the solar wind is obtained after insertion of $\varrho = \varrho_E r_E^2/r_0^2$ via Eq. (6.3.186):

$$r_0^2 = \left[2\varrho_E r_E^2 v^2/P_i(\gamma + 1)\right]\left[1 + (\gamma - 1)/4(n + 1)\right]^{n+1} \approx r_s^2, \qquad (M_A \gg 1). \tag{6.3.193}$$

Comparison of this result with the approximate equation (6.3.188) if $\gamma = 1 + 1/n = 5/3$, for instance, shows indeed that $k = \left[2/(\gamma + 1)\right]\left[1 + (\gamma - 1)/4(n + 1)\right]^{n+1} = 0.88 \approx 1$. The interstellar pressure P_i is composed of the hydrostatic pressure of the interstellar gas, the hydrostatic pressure of the cosmic ray gas, and the pressure of the interstellar magnetic field, having a rough average value of $pH^2/24\pi$ [cf. Eq. (2.6.82)]. If $n_d = 1$ atom/cm^3 and $T = 100$ K, the gas pressure is only about 1.4×10^{-14} dyne/cm^2. Both, the cosmic ray pressure and the interstellar magnetic pressure are of order $(1 - 4) \times 10^{-12}$ dyne/cm^2, assuming for the interstellar magnetic induction a conventional figure of about $B = 10^{-5}$ Gauss, with the magnetic pressure ranging between the hydrostatic pressure $pH^2/8\pi$ and the mean pressure $pH^2/24\pi$. Only a fraction of the cosmic ray pressure acts on the solar wind, since a portion of this pressure penetrates through the inner solar system. Thus, the estimate for P_i is in the range $(1-4)\times 10^{-12}$ dyne/cm^2. With the previously mentioned values $n_{dE} = 10$ cm^{-3} and $v = 300$ km/s the equation (6.3.188) or (6.3.193) yields $r_0 \approx r_s = 60 - 120$ astronomical units – somewhere beyond the orbit of Pluto (Parker 1963, Chap. IX). The curve through the points $OABCD$ in Fig. 6.3.8 is the complete solution of a stellar wind expanding from the stellar surface O through the sonic point A, $(\zeta, M_A = 1)$ to the shock front with the interstellar medium at B, where the Mach number jumps down to point C, decreasing further through point D at $\zeta \approx 3$.

The Sun (as many other stars) is expected to have encountered during its lifetime about 150 interstellar clouds, spending about 3×10^7 yr within these clouds. In most cases the encounter velocities (typically $v_i \approx 20$ km/s) will be much larger than the sound speed within the cloud (typically $a \leq 1$ km/s for cloud

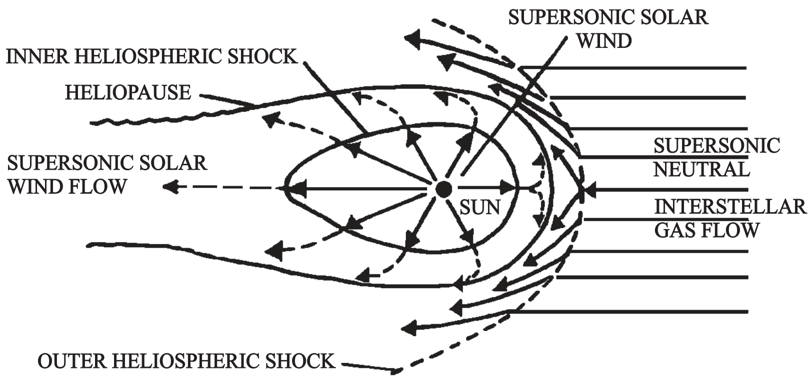

Fig. 6.3.9 Schematic impact configuration between the supersonic solar wind and the gas of an interstellar cloud encountering the Sun at supersonic speed v_i. The inner closed contour is the shock front of the solar wind (inner heliospheric shock), while the broken line represents the bowshock of the interstellar gas (outer heliospheric shock). The region between inner heliospheric shock and heliopause contains mainly subsonic solar wind flow, the region between heliopause and outer heliospheric shock mainly subsonic interstellar gases (Fahr 1980).

temperatures lower than 100 K), (Talbot and Newman 1977). Therefore, we are faced with the impact of two supersonic gas streams, as depicted schematically in Fig. 6.3.9. In the common stagnation point r_0 on the stagnation line of the two flows, the stagnation pressure P_0 of the solar wind equals the stagnation pressure P_{0i} of interstellar gas flow. We insert into Eq. (6.3.192) for P_2 via Eq. (6.3.189):

$$P_0 = \left[2\varrho v^2/(\gamma+1)\right]\left[1+(\gamma-1)/4(n+1)\right]^{n+1} = k\varrho v^2, \qquad (M_A \gg 1). \tag{6.3.194}$$

If $n = 1/(\gamma - 1)$, the constant k is identical to Eq. (6.3.178). Equating the stagnation pressure of the post-shocked solar wind and of the post-shocked interstellar gas, we get

$$P_0 = k\varrho v^2 = P_{0i} = k_i \varrho_i v_i^2. \tag{6.3.195}$$

The adiabatic exponent and polytropic index of solar wind and interstellar gas may be assumed nearly equal, and in this case Eq. (6.3.195) simplifies to (Parker 1963, Talbot and Newman 1977)

$$\varrho v^2 = \varrho_i v_i^2. \tag{6.3.196}$$

If we insert for the solar wind density $\varrho = \varrho_E r_E^2/r_0^2$ via Eq. (6.3.186), we obtain the stagnation distance of the solar wind (smallest extension of heliopause):

$$r = r_0 = (r_E v/v_i)(\varrho_E/\varrho_i)^{1/2}. \tag{6.3.197}$$

Talbot and Newman (1977) maintain that for interstellar densities ϱ_i larger than a certain value, the pressure P_i of the interstellar gas overwhelms the dynamic pressure $k\varrho v^2$ of the solar wind; no pressure equilibrium would establish between the two flows, and unhalted accretion of interstellar matter by the Sun would occur. This idea seem untenable for realistic interstellar densities $\varrho_i \lesssim 10^{-18}$ g cm^{-3} (e.g. Fahr 1980).

(v) Protoplanetary (circumstellar) cloud. Circumstellar clouds (disks) occur around a variety of stellar objects (e.g. Bjorkman 1997, and Sec. 6.4.3 for accretion disks). Early in the history of the solar system, the Sun – like other stars – may have gone through a T-Tauri like phase with an intense solar wind. Such a wind could have blown away the gas of a protoplanetary (circumstellar) cloud within which the early Sun and other stars were embedded. Magnetic acceleration associated with magnetic fields [Eqs. (6.3.144), (6.3.151)] could blow off considerably more mass than allowed by thermal processes alone.

Let us first derive the relevant equations of steady expansion or accretion flows possessing additional sources or sinks of mass, distributed in an arbitrary manner. The mass flow per unit time along the *exterior* normal of a surface element dS is $\varrho(\vec{v}\cdot\vec{n})\ dS$, where $\vec{v}\cdot\vec{n}$ is the fluid velocity along the exterior normal of dS. By the Gauss divergence theorem (5.11.55) we have

$$\int_S \varrho(\vec{v}\cdot\vec{n})\ dS = \int_V \nabla\cdot(\varrho\vec{v})\ dV. \tag{6.3.198}$$

Next, the variation per unit time of fluid mass inside volume V is given by the integral

$$\partial\Big[\int_V \varrho\ dV\Big]\Big/\partial t = \int_V (\partial\varrho/\partial t)\ dV. \tag{6.3.199}$$

If there exist additional mass sources or sinks connected with volume V, the mass variation (6.3.199) is just the difference between the mass accretion/loss rate dm/dt of volume V and the mass flow (6.3.198) along the exterior normal:

$$\int_V (\partial\varrho/\partial t)\ dV = dm/dt - \int_V \nabla\cdot(\varrho\vec{v})\ dV. \tag{6.3.200}$$

Diminishing the volume V up to the dimension of an arbitrary volume element dV, we obtain

$$A = (dm/dt)/dV = \partial\varrho/\partial t + \nabla\cdot(\varrho\vec{v}), \tag{6.3.201}$$

where A is the additional mass injection/loss rate per unit volume. For stationary conditions and spherical symmetry we have via Eq. (B.37):

$$(1/r^2)\ d(r^2\varrho v)/dr = A, \qquad [\vec{v} = \vec{v}(v,0,0);\ v = v(r)]. \tag{6.3.202}$$

The equation of conservation of momentum takes for a variable mass flow the form [Horedt 1978b, Eq. (13)]

$$D(m\vec{v})/Dt = -\nabla P\ dV + m\ \nabla\Phi + \vec{v}_A\ dm/dt = -\nabla P\ dV + (\varrho\ dV)\ \nabla\Phi + A\vec{v}_A\ dV. \tag{6.3.203}$$

dV is the volume occupied by the instantaneous mass element m. The pressure force acting on the volume element dV is $-\nabla P\ dV$, while the gravitational force acting on the mass inside dV is $m\ \nabla\Phi$. The last term is the momentum $\vec{v}_A\ dm/dt = A\vec{v}_A\ dV$ of the newly accreted/lost mass $dm/dt = A\ dV$, having velocity $\vec{v}_A$ in the considered inertial frame. Note, that in connection with variable mass flows we will use the notation $m = \varrho\ dV$ for the relationship between mass, density, and volume, because the symbol dm is reserved for the mass variation of the mass element m.

The momentum equation becomes after division by dV via Eqs. (6.3.160), (6.3.201) equal to

$$[D(m\vec{v})/Dt]/dV = \vec{v}\ (dm/dt)/dV + (m/dV)\ D\vec{v}/Dt = [\partial\varrho/\partial t + \nabla\cdot(\varrho\vec{v})]\vec{v} + \varrho\ \partial\vec{v}/\partial t$$
$$+\varrho(\vec{v}\cdot\nabla)\vec{v} = \partial(\varrho\vec{v})/\partial t + [\nabla\cdot(\varrho\vec{v})]\vec{v} + \varrho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \varrho\ \nabla\Phi + A\vec{v}_A, \qquad (m = \varrho\ dV). \tag{6.3.204}$$

For spherical symmetry and stationary motion this reads

$$(v/r^2)\ d(r^2\varrho v)/dr + \varrho v\ dv/dr = (1/r^2)\ d(r^2\varrho v^2)/dr = -dP/dr - GM\varrho/r^2 + Av_A. \tag{6.3.205}$$

And finally, the energy equation of an adiabatic variable mass flow is obtained analogously to Eq. (6.3.170) by scalar multiplication of the momentum equation (6.3.204) with $\vec{v}$. The left-hand side is transformed via Eqs. (6.3.166) and (6.3.201):

$$\vec{v}\cdot[D(m\vec{v})/Dt]/dV = v^2\ (dm/dt)/dV + (m/dV)\ \vec{v}\cdot(D\vec{v}/Dt) = Av^2 + \varrho\vec{v}\cdot(D\vec{v}/Dt)$$
$$= Av^2 + \partial(\varrho v^2/2)/\partial t - (v^2/2)\ \partial\varrho/\partial t + (\varrho/2)(\vec{v}\cdot\nabla v^2) = Av^2/2 + \partial(\varrho v^2/2)/\partial t + (v^2/2)\ \nabla\cdot(\varrho\vec{v})$$
$$+(\varrho/2)(\vec{v}\cdot\nabla v^2) = Av^2/2 + \partial(\varrho v^2/2)/\partial t + \nabla\cdot(\varrho v^2\vec{v}/2). \tag{6.3.206}$$

The right-hand side of Eq. (6.3.204) becomes by virtue of Eqs. (6.3.167), (6.3.168), (6.3.201) after scalar multiplication with $\vec{v}$ equal to

$$-\vec{v}\cdot\nabla P + \varrho(\vec{v}\cdot\nabla\Phi) + A\vec{v}\cdot\vec{v}_A = \partial\big[-P/(\Gamma_1-1) + \varrho\Phi\big]/\partial t + \nabla\cdot\big[-\Gamma_1 P\vec{v}/(\Gamma_1-1) + \varrho\Phi\vec{v}\big]$$
$$+A\big[\Gamma_1 P/(\Gamma_1-1)\varrho - \Phi + \vec{v}\cdot\vec{v}_A\big]. \tag{6.3.207}$$

The final form of the adiabatic energy equation of a variable mass flow results after equating Eqs. (6.3.206) and (6.3.207):

$$\partial\big[\varrho v^2/2 + P/(\Gamma_1-1) - \varrho\Phi\big]/\partial t + \nabla\cdot\big[\varrho v^2\vec v/2 + \Gamma_1 P\vec v/(\Gamma_1-1) - \varrho\Phi\vec v\big]$$
$$+A\big[\vec v\cdot(\vec v/2-\vec v_A) - \Gamma_1 P/(\Gamma_1-1)\varrho + \Phi\big] = 0. \qquad (6.3.208)$$

For stationary motion and spherical symmetry this reads [Holzer and Axford 1970, Eq. (4.3)]:

$$(1/r^2)\ d\big\{\varrho v r^2\big[v^2/2 + \Gamma_1 P/(\Gamma_1-1)\varrho\big]\big\}/dr + GM\varrho v/r^2 + A\big[v^2/2 - vv_A - \Gamma_1 P/(\Gamma_1-1)\varrho\big] = 0,$$
$$(v_A = v_A(r);\ \nabla\cdot(\varrho v\Phi) - A\Phi = (1/r^2)\ d(r^2\varrho v\Phi)/dr - (\Phi/r^2)\ d(r^2\varrho v)/dr = \varrho v\ d\Phi/dr). \quad (6.3.209)$$

In the case of constant mass flow ($A\propto d(\varrho v r^2)=0$) and $\Gamma_1 = 1+1/n$ this equation reduces to Eq. (6.3.69). For stationary variable mass flows with radial and azimuthal velocities, and with magnetic fields depending solely on radial distance r [as considered in Eqs. (6.3.118)-(6.3.151)], it is advisable to start the derivation of the radial momentum equation (6.3.205) ex novo. We are interested in the blow-off of a protoplanetary (circumstellar) cloud, initially at rest ($\vec v_A = 0$) with respect to M. The right-hand side of Eq. (6.3.140) remains unchanged, while the left-hand side writes via Eqs. (6.3.201), (B.42) as

$$[D(m\vec v)/Dt]_r = v_r\ dm/dt + m\ [(\vec v\cdot\nabla)\vec v]_r = Av_r\ dV + m(v_r\ dv_r/dr - v_\varphi^2/r). \qquad (6.3.210)$$

Dividing by dV, the analogue of Eq. (6.3.140) in the case of stationary winds with variable mass is

$$\varrho(v_r\ dv_r/dr - v_\varphi^2/r) + Av_r + dP/dr + GM\varrho/r^2 + (B_\varphi/4\pi r)\ d(rB_\varphi)/dr = 0. \qquad (6.3.211)$$

Making use of Eq. (6.3.141), this equation can be integrated in the same way as Eq. (6.3.140):

$$(v_r^2+v_\varphi^2)/2 + (n+1)K\varrho^{1/n} - GM/r - \Omega r B_r B_\varphi/4\pi\varrho v_r + \overline{v_r^2}\ln m = \text{const}, \qquad (n\neq\pm\infty), \qquad (6.3.212)$$

where $\overline{v_r^2}$ is a mean value of the squared radial velocity v_r. If $n=\pm\infty$, the term $(n+1)K\varrho^{1/n}$ turns into $K\ln\varrho$, while in the constant density case $K\varrho^{1/n}$ should be replaced by P/ϱ, $(n=0)$. The variable mass term in Eq. (6.3.211) has been transformed according to

$$Av_r\ dr/\varrho = v_r\ dm\ (dr/dt)\big/(\varrho\ dV) = v_r^2\ dm/m, \qquad (v_r = dr/dt;\ A = (dm/dt)\big/dV;\ m = \varrho\ dV). \qquad (6.3.213)$$

The gross details of the wind near the star are primarily determined by the thermal properties of the stellar corona; the magnetic term becomes via Eqs. (6.3.129), (6.3.132) equal to $-\Omega r B_r B_\varphi/4\pi\varrho v_r \approx \Omega L = \Omega^2 r_c^2$, $(v_\varphi\approx 0)$ and plays no role in driving the near wind (Belcher and MacGregor 1976, p. 503). Likewise, the squared azimuthal velocity of the wind near the stellar surface $v_\varphi^2 \approx \Omega^2 r_*^2$ is negligible. We also have $v_r = v_{r*}\approx 0$. The total mass m transported by the wind near the stellar surface is just equal to the stellar mass loss rate dM/dt.

At infinity the azimuthal velocity v_φ goes to zero like $1/r$, as already outlined subsequently to Eq. (6.3.142). And finally, the magnetic term is $v_M^3/v_{r\infty} = v_{r\infty}^2$, [$v_M = v_{r\infty}$ via Eqs. (6.3.143), (6.3.151)]. With these findings we are now able to write down Eq. (6.3.212) between the stellar radius $r=r_*$ and the infinity point:

$$(n+1)K\varrho_*^{1/n} - GM/r_* + \overline{v_r^2}\ln\big[(dM/dt)/m_\infty\big] = 3v_{r\infty}^2/2, \qquad (n\neq\pm\infty). \qquad (6.3.214)$$

The difference $m_\infty - dM/dt$ is just the mass that the stellar wind is able to pick up from the protoplanetary (circumstellar) cloud. An elementary average of $\overline{v_r^2}$ is obtained from

$$\overline{v_r^2} = \int_0^{v_{rc\infty}} v_r^2\ dv_r\Big/\int_0^{v_{rc\infty}} dv_r = v_{rc\infty}^2/3, \qquad (6.3.215)$$

where $v_{rc\infty}$ is the maximum velocity which a magnetic polytropic wind, transporting constant mass at all distances, can attain. If $dM/dt = m_\infty = \text{const}$, we get by virtue of Eq. (6.3.214)

$$(n+1)K\varrho_*^{1/n} - GM/r_* = 3v_{rc\infty}^2/2, \qquad (n\neq\pm\infty). \qquad (6.3.216)$$

Inserting Eqs. (6.3.215), (6.3.216) into Eq. (6.3.214), we get

$$v_{r\infty}^2 = v_{rc\infty}^2\big\{1 + (2/9)\ln\big[(dM/dt)/m_\infty\big]\big\}. \tag{6.3.217}$$

Since $v_{r\infty}^2 > 0$, there results $\ln\big[m_\infty/(dM/dt)\big] < 9/2$ or $m_\infty < 90\ dM/dt$, i.e. the magnetopolytropic wind can accrete and transport to infinity from the protoplanetary (circumstellar) cloud an up to 90 times larger mass than the original solar (stellar) mass loss rate dM/dt (Nerney 1980). If magnetic acceleration is not important, i.e. in the case of so-called slow magnetic rotators (e.g. the present solar case), the relevant equations (6.3.214), (6.3.216), (6.3.217) read for a thermally driven, mainly radial, polytropic wind as (Horedt 1978b, 1982a)

$$(n+1)K\varrho_*^{1/n} - GM/r_* + \overline{v_r^2}\ln\big[(dM/dt)/m_\infty\big] = v_{r\infty}^2/2, \qquad (n \neq \pm\infty), \tag{6.3.218}$$

$$(n+1)K\varrho_*^{1/n} - GM/r_* = v_{rc\infty}^2/2, \qquad (n \neq \pm\infty), \tag{6.3.219}$$

$$v_{r\infty}^2 = v_{rc\infty}^2\big\{1 + (2/3)\ln\big[(dM/dt)/m_\infty\big]\big\}, \tag{6.3.220}$$

i.e. such a wind could blow-off from the protoplanetary (circumstellar) cloud up to 4.5 times the original solar (stellar) mass loss: $\ln\big[m_\infty/(dM/dt)\big] < 3/2$ or $m_\infty < 4.5\ dM/dt$.

The solar (stellar) wind may not be able to efficiently peel off layers from the protoplanetary (circumstellar) nebula if shocks develop at the wind-nebula interface, possibly radiating away much of the available wind energy. In this case the net result may be even an infall of nebular parts into the star, when turbulence develops near the outflow at the wind-nebula interface (Nerney 1980, p. 732). The stability against radial perturbations of spherically symmetric polytropic winds and accretion flows will be touched at the end of Sec. 6.4.2.

6.4 Polytropic Accretion Flows, Accretion Disks and Tori

6.4.1 Line Accretion of Polytropic Flows

At first sight this topic is just the reverse side of the previously discussed polytropic expansion winds. But the origin of accretion flows and the reason for their occurrence are quite different. While expansion flows mostly arise due to particular conditions in the external layers of astronomical objects, the gas accretion onto stars, galaxies etc. takes place simply because of the gravitation of the central object onto the surrounding medium (e.g. interstellar, intergalactic, intracluster medium). The net effect of accretion flows is simply a mass increase of the central object, leaving aside secondary phenomena resulting from mass infall, such as shock waves and subsequent dissipation of kinetic energy. Relevant astrophysical applications of line accretion have already been mentioned at the beginning of Sec. 6.3.1.

The principal results obtained so far on supersonic Bondi-Hoyle-Lyttleton accretion theory have been pregnantly summarized by Font and Ibáñez (1998). Within our context, the salient feature of numerical 2-D and 3-D simulations is the fact that they often agree qualitatively and sometimes even quantitatively with the original oversimplified analytical theory (e.g. for the upstreaming flow ahead the bow or tail shock, the downstreaming flow near the axis, the mass accretion rate, and the stagnation point distance). Although Cowie (1977) and Soker (1990) have shown the linear accretion column to be unstable to linear short-wavelength perturbations, some numerical simulations exhibit structures resembling the linear high-density accretion column on the downstream axis of the axisymmetric flow – at least temporarily during quiescent stages (e.g. Fig. 6.4.6, Ruffert and Arnett 1994, Ruffert 1996, Font and Ibáñez 1998). The subsequent analytical approximations may serve therefore for more sophisticated numerical models as a first approximation to the downstreaming flow near the axis. The theory is mainly pertinent to supersonic flows, when the upstreaming Mach number at infinity is $M_{A\infty} \gg 1$. Generally, a bow shock develops in front of the accretor or – for some large accretors – a tail shock with stand-off distance on the downstreaming side. Moreover, as shown numerically by Koide et al. (1991), the analytic solution by Bisnovatyi-Kogan et al. (1979) for collisionless ballistic particle orbits is an acceptable approximation to hydrodynamic flow in front of the shock [cf. Eqs. (6.4.6)-(6.4.11)].

The accretion process is specified by a fairly small number of physical parameters: The central accreting mass M, the relative velocity at infinity v_∞ between M and accreting gases, the pressure and density of the gas being connected at infinity by a polytropic (adiabatic) law $P_\infty = K\varrho_\infty^{1+1/n}$, $(P_\infty = K\varrho_\infty^\gamma)$.

An approximate specific length is given by the radius of influence r_I, characterizing the domain of influence of the central gravitating mass. It is defined as the distance where the potential energy of the central mass is of the same order of magnitude as the gas energy at infinity. From the energy equation (6.3.91) it is obvious that for an isentropic flow the gas energy per unit mass at infinity is in absence of gravitation equal to the sum of kinetic energy $v_\infty^2/2$ and isentropic enthalpy $(n+1)P_\infty/\varrho_\infty = [\gamma/(\gamma-1)]P_\infty/\varrho_\infty$, $(\gamma = 1+1/n)$. If $n \not\approx 0, \infty$, the enthalpy may be approximately replaced by the squared sound velocity $a_\infty^2 = \gamma P_\infty/\varrho_\infty = (n+1)P_\infty/n\varrho_\infty$ from Eq. (2.1.49), and consequently the gas energy per unit mass at infinity is of the same order of magnitude as $v_\infty^2 + a_\infty^2$, $(n \not\approx 0, \infty)$. On the other side, allowing in Eq. (6.3.91) for the influence of radiation pressure in the optically thick case, the effective potential energy per unit mass is $\beta GM/r$, $(\beta = P_g/P)$. Equating the values of these two energies, we get up to the order of magnitude (Fahr 1980):

$$r_I = r = \beta GM/(v_\infty^2 + a_\infty^2), \qquad (0 \le \beta \le 1). \tag{6.4.1}$$

Introduction of the sound velocity a_∞ has been prompted by the finding of Bondi (1952) that in the pressure dominated case – when $v_\infty \approx 0$ – the sound velocity replaces the relative velocity at infinity (Sec. 6.4.2).

If the mean free length of path is larger than the radius of influence (6.4.1), the particle trajectories and the velocity distribution can be described in the single particle kinetic approximation with the Boltzmann equation. This case however, is of no interest in the present context, since the net mass accretion rate is small [cf. Eq. (6.4.5)]. We confine ourselves to the case when particle collisions are important near the accretion axis, i.e. if the mean free length of path $\Lambda_m = 1/s_m n_d$ is smaller than the radius of influence (6.4.1), where $s_m \approx 10^{-15}$ cm^2 is a mean effective collision cross-section of gas particles, and n_d their number density (e.g. Chapman and Cowling 1970).

We will pursue the chronological progress made on the subject, and determine at first the mass accretion rate under the crucial assumption that pressure gradients are neglected in the accreting flow. An accretor (e.g. star, galaxy) travelling through diffuse matter must sweep up particles in and near its track, and thereby accrete mass. As mentioned above, the accretion rates gained under the assumption that there accrete only particles whose initial collisionless orbits actually intersect the accretor surface, are discouragingly small (Eddington 1959, p. 391). The distance between $\vec{v}_\infty$ and the accretion axis – the target distance at infinity σ – is connected for grazing collisions to the radius r_s of an accretor through conservation of angular momentum in the hyperbolic two-body problem, taking into account that the relative velocity at infinity $\vec{v}_\infty$ between particle and accretor is perpendicular to σ, while at the grazing point (equal to the accretor radius r_s) the particle velocity (e.g. Stumpff 1959)

$$v_s = (2GM/r_s + v_\infty^2)^{1/2}, \tag{6.4.2}$$

is perpendicular to r_s. Thus, conservation of angular momentum in the orbital $(\vec{r}, \vec{v})$-plane between infinity and the grazing point reads

$$\sigma v_\infty = r_s v_s. \tag{6.4.3}$$

Elimination of v_s between Eqs. (6.4.2) and (6.4.3) yields for the target distance at infinity

$$\sigma = r_s(1 + 2GM/r_s v_\infty^2)^{1/2} \approx (2GMr_s)^{1/2}/v_\infty, \qquad (v_\infty^2 \ll 2GM/r_s). \tag{6.4.4}$$

The amount of matter swept up per second (the mass accretion rate) is

$$dM/dt = \pi\sigma^2 \varrho_\infty v_\infty = \pi r_s^2 \varrho_\infty v_\infty (1 + 2GM/r_s v_\infty^2) \approx 2\pi GM \varrho_\infty r_s/v_\infty. \tag{6.4.5}$$

This mass, colliding directly with the accretor, is generally negligibly small. To increase the mass accretion rate Hoyle and Lyttleton (1939a, b) assume that some kind of mass condensation is formed in the downstreaming flow. As before, the elements of gas are assumed to describe collisionless hyperbolic paths, which all converge to intersect on a line parallel to $\vec{v}_\infty$ and passing through M – the so-called accretion axis. Further, the motion is assumed to reach a steady state, all variables becoming independent of time.

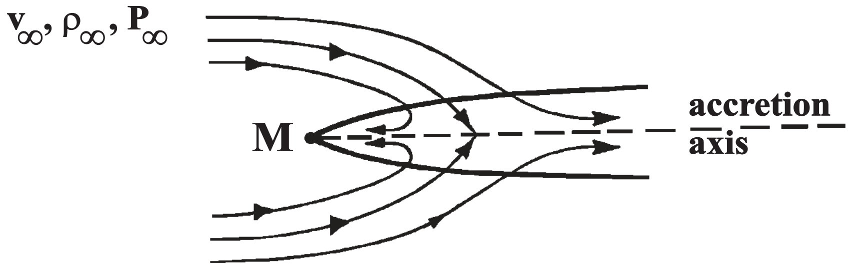

Fig. 6.4.1 Schematic view of line accretion as envisaged by Wolfson (1977). The heavy line centered on the central mass M is the bow shock front of initially supersonic gases, the broken line representing the accretion axis passing through M, parallel to the relative velocity at infinity $\vec{v}_\infty$. Three characteristic stream lines are shown, the inner one being specific for accreting gases, the middle streamline is centered on the stagnation point r_{a0} of the flow, and the outer one represents shocked gas leaving ultimately the mass M.

Particle collisions in the two opposing streams on both sides of the accretion axis in Fig. 6.4.1 will destroy the angular momentum of particles about M, yielding in this way an effective target distance much larger than σ from Eq. (6.4.4). If after collisions – occurring predominantly in the accretion column round the accretion axis – the surviving radial velocity component is insufficient to enable the particles to escape, they will eventually be swept into the accreting mass M. Let us suppose that the whole initial angular momentum σv_∞ from Eq. (6.4.3) is lost by collisions in the neighborhood of the accretion axis, i.e. in the accretion column. For the nearly one-dimensional flow in the thin accretion column the relevant distance between a gas particle and the accretor mass M is measured along the accretion axis, and denoted by $r = r_a$. The accretion column is viewed as a thin semi-infinite flux tube centered on the accretion axis, and having circular cross-section πs^2 of radius $s = s(r_a)$, $(s \ll r_a)$. With the very assumption that the effect of collisions on inflowing gases becomes appreciable only near the accretion axis, we determine subsequently the mass accretion rate A per unit length of accretion column. The collisionless, upstreaming hyperbolic motion of a gas element round M is given by the hyperbola (cf. Eq. (6.1.152) and Stumpff 1959)

$$r = a(e^2 - 1)/(1 + e\cos\varphi). \tag{6.4.6}$$

r is the radius vector with respect to M, the eccentricity is denoted by e, $(e > 1)$, and φ is the true anomaly of hyperbolic motion. The hyperbolic equivalent $a = GM/v_\infty^2$ of the semimajor axis in elliptic motion results from the energy equation of two-body motion

$$v^2 = GM(2/r + 1/a) = 2GM/r + v_\infty^2, \tag{6.4.7}$$

if we let $r \to \infty$ and $v \to v_\infty$. At infinity we have via Eq. (6.4.6)

$$1 + e\cos\varphi_\infty = 0. \tag{6.4.8}$$

At the point where the hyperbola intersects the accretion axis, the true anomaly has decreased just by $180^\circ : \varphi_a = \varphi_\infty - \pi$. Inserting this into Eq. (6.4.8), we get $\cos\varphi_\infty = \cos(\pi + \varphi_a) = -\cos\varphi_a = -1/e$. With this finding Eq. (6.4.6) yields the radial distance between M and the intersection point of the orbit with the accretion axis:

$$r_a = a(e^2 - 1)/(1 + e\cos\varphi_a) = a(e^2 - 1)/2. \tag{6.4.9}$$

The target distance σ can be expressed as a function of a and e, if we replace in Eqs. (6.4.2) and (6.4.3) the accretor radius for grazing collisions r_s by a general peridistance r_p. From Eq. (6.4.6) we find: $r_p = a(e - 1)$, $(\varphi = 0)$. The velocity at peridistance r_p is by virtue of Eq. (6.4.2) equal to

$$v_p = (2GM/r_p + v_\infty^2)^{1/2} = v_\infty[(e + 1)/(e - 1)]^{1/2}, \qquad [a = r_p/(e - 1) = GM/v_\infty^2]. \tag{6.4.10}$$

We introduce this into Eq. (6.4.3):

$$\sigma = r_p v_p/v_\infty = a(e^2 - 1)^{1/2} = (2ar_a)^{1/2} = (2GMr_a/v_\infty^2)^{1/2}. \tag{6.4.11}$$

The mass flux at infinity between target distance σ and $\sigma + d\sigma$ is

$$dm_\sigma/dt = 2\pi\varrho_\infty v_\infty \sigma\ d\sigma, \tag{6.4.12}$$

where it is always assumed that v_∞ is larger than the adiabatic sound velocity at infinity $a_\infty = (\gamma P_\infty/\varrho_\infty)^{1/2}$. Otherwise, the appropriate equations are those derived in Sec. 6.4.2 for the case of spherically symmetric accretion. Differentiating Eq. (6.4.11), we obtain $d\sigma = (GM/2v_\infty^2 r_a)^{1/2}\ dr_a$, and Eq. (6.4.12) reads

$$dm_\sigma/dt = 2\pi\varrho_\infty v_\infty(GM\ dr_a/v_\infty^2) = 2\pi GM\varrho_\infty\ dr_a/v_\infty, \tag{6.4.13}$$

showing the mass arriving on the accretion axis between the distances r_a and $r_a + dr_a$. Actually, Eq. (6.4.13) represents the mass added to the accretion column over the volume $\pi s^2\ dr_a$. The mass accretion rate per unit length of accretion column, viz. the mass accreted inside a volume of cross-section πs^2 and unit height is obtained from Eq. (6.4.13), (Bondi and Hoyle 1944):

$$A = (dm_\sigma/dt)/dr_a = dm/dt = 2\pi GM\varrho_\infty/v_\infty = \text{const.} \tag{6.4.14}$$

Let us denote by $v = v(r_a)$ the velocity of the mass $m = m(r_a)$ located in the accretion column within the unit of length. The symbol m means just the *linear* density in the accretion column, i.e. the mass contained in the volume πs^2, which is the volume of a circular cylinder with height equal to the unit of length in the accretion column. And $A = dm/dt$ is just the increase of linear density m per unit time.

The volume density ϱ in the accretion column is connected to linear density m by

$$\varrho = m/\pi s^2, \tag{6.4.15}$$

where πs^2 denotes the cross-section of the accretion column, and $s \ll r_a$.

To obtain the one-dimensional stationary mass conservation equation, we have to replace in Eq. (6.3.201) the mass per unit volume ϱ by the mass per unit length of accretion column m, and the infinitesimal volume element dV by the infinitesimal change of length dr_a. The mass accretion rate per unit volume (6.3.201) turns into the mass accretion rate per unit length of accretion column (6.4.14):

$$A = dm/dt = 2\pi GM\varrho_\infty/v_\infty = \nabla\cdot(m\vec{v}) = d(mv)/dr_a, \qquad (\partial m/\partial t = 0). \tag{6.4.16}$$

This integrates at once to

$$mv = Ar_a + \mathrm{const} = A(r_a - r_{a0}), \qquad [v(r_{a0}) = 0]. \tag{6.4.17}$$

The integration constant has been determined under the very condition that the flow velocity in the accretion column vanishes at some finite distance $r_a = r_{a0}$, being just the stagnation distance of the flow with respect to the accretor M. The gas flows towards M, $(v < 0)$ if $r_a < r_{a0}$. The motion is directed outwards $(v > 0)$ if $r_a > r_{a0}$.

The azimuthal velocities of newly accreting particles are dissipated by collisions near the accretion axis, so the sole surviving velocity is the radial one, which turns out from the hyperbolic two-body problem to be always constant and equal to v_∞ (e.g. Stumpff 1959, p. 107):

$$v_r = dr_a/dt = [GM/a(e^2-1)]^{1/2} e\sin\varphi_a = (GM/a)^{1/2} = v_\infty. \tag{6.4.18}$$

We have inserted into Eq. (6.4.18) from the trigonometric identity $\sin\varphi_a = (1-\cos^2\varphi_a)^{1/2} = (e^2-1)^{1/2}/e$, since $\cos\varphi_a = 1/e$.

Let us denote by

$$f = \pi s^2 P, \tag{6.4.19}$$

the cumulative linear pressure acting on the whole cross-section πs^2 of the accretion column. For the pressure P we use the polytropic law (6.3.3), which becomes via Eqs. (6.4.15), (6.4.19) in terms of linear pressure f and linear density m equal to

$$P = f/\pi s^2 = K\varrho^{1+1/n} = K(m/\pi s^2)^{1+1/n} \quad \text{or} \quad f = [K/(\pi s^2)^{1/n}]\, m^{1+1/n}. \tag{6.4.20}$$

We will restrict ourselves to polytropic indices inside the interval $0 < n \leq \infty$, which are the relevant ones in the present context.

We adopt a self-consistent picture of pressure effects in line accretion, and determine the pressure inside the accretion column from pressure equilibrium with the transverse pressure force exerted by the particles striking the accretion column from abroad. In virtue of Eqs. (6.4.7), (6.4.18) the transversal velocity v_φ of particles striking the accretion column from abroad at distance r_a is equal to $v_\varphi^2 = v^2 - v_r^2 = v^2 - v_\infty^2 = 2GM/r_a$. The mass striking the surface unit of the lateral surface of the accretion column per unit time is $A/2\pi s$, where $2\pi s$ is just the outer surface per unit length of accretion column. And the momentum of this newly impacting material [Bondi and Hoyle 1944, Eq. (7)]

$$Av_\varphi/2\pi s = (A/\pi s)(GM/2r_a)^{1/2} = P = f/\pi s^2, \tag{6.4.21}$$

should be equal to the pressure force P in the accretion column, in order to assure stationary equilibrium. From Eq. (6.4.21) we get for the cross-section radius of the accretion column

$$s = (A/\pi P)(GM/2r_a)^{1/2} \quad \text{or} \quad s = (f/A)(2r_a/GM)^{1/2}. \tag{6.4.22}$$

Inserting for s into Eq. (6.4.20), we get

$$f = K^{n/(n+2)}(A^2GM/2\pi)^{1/(n+2)} m^{(n+1)/(n+2)} r_a^{-1/(n+2)}. \tag{6.4.23}$$

We now write down the one-dimensional form of the stationary equation of motion (6.3.204). Like in Eq. (6.4.16), we have $\varrho \to \pi s^2 \varrho = m$, $dV \to dr_a$, and $P \to \pi s^2 P = f$. Analogously to the pressure force $\nabla P\ dV$ acting on the volume element dV, the linear pressure force exerted on the element of length dr_a is $\nabla f\ dr_a$. Thus, the pressure force ∇f acts on the unit of length $dr_a = 1$ in the accretion column, and ∇P in Eq. (6.3.204) has to be replaced by $\nabla f = df/dr_a$. According to Eq. (6.4.18) the velocity of accreting matter is now constant $\vec{v}_A = \vec{v}_\infty$, and the stationary form of Eq. (6.3.204) reads

$$[\nabla \cdot (m\vec{v})]\vec{v} + m(\vec{v} \cdot \nabla)\vec{v} = v\ d(mv)/dr_a + mv\ dv/dr_a = d(mv^2)/dr_a = Av + mv\ dv/dr_a$$
$$= -\nabla f + m\ \nabla\Phi + Av_\infty = -df/dr_a - GMm/r_a^2 + Av_\infty, \qquad (\varrho \to m;\ P \to f;\ \Phi = GM/r_a). \tag{6.4.24}$$

Eqs. (6.4.16) and (6.4.24) of mass and momentum conservation, together with the equation of state (6.4.23), are sufficient for the determination of the three unknowns of the flow, viz. velocity, density, and pressure.

The Bernoulli equation of the variable mass flow is obtained by integration of Eq. (6.4.24):

$$v^2/2 - GM/r_a + \int_{r_{a1}}^{r_a} [A(v - v_\infty)/m + (1/m)\ df/dr_a]\ dr_a = \text{const.} \tag{6.4.25}$$

This equation seems not suitable for further analytic evaluation, and reduces to the Bernoulli equation for constant mass flows [cf. Eq. (6.3.7)]

$$v^2/2 - GM/r_a + (n+1)f/m = v^2/2 - GM/r_a + (n+1)P/\varrho = \text{const}, \qquad (n \neq \infty), \tag{6.4.26}$$

only if $A = 0$, and $s = \text{const}$ in Eq. (6.4.20): $(1/m)\ df \propto (1 + 1/n)m^{1/n-1}\ dm$. The Bernoulli equation (6.4.26) adopted by Wolfson (1977) and Yabushita (1978a, b, 1979) is pertinent merely for constant mass flows $A = 0$.

Basically, we will confine ourselves to the analytic behaviour of accretion flows near the stagnation point ($r_a = r_{a0},\ v = 0$), and the infinity point $r_{a\infty} = \infty$. The so-called cut-off distance r_{ac} (the linear extension of the accretion column) is determined analytically from the pressure condition (6.4.21).

The dimensionless distance x, velocity y, density z, and pressure w are introduced by

$$x = (v_\infty^2/GM)r_a; \quad x_0 = (v_\infty^2/GM)r_{a0}; \quad y = v/v_\infty; \quad z = (v_\infty^3/AGM)m; \quad w = (v_\infty/AGM)f. \tag{6.4.27}$$

Eqs. (6.4.17), (6.4.24), (6.4.23) become in dimensionless variables

$$yz = x - x_0, \qquad (y = 0 \ \text{ if } \ x = x_0), \tag{6.4.28}$$

$$yz\ dy/dx = 1 - y - z/x^2 - dw/dx, \tag{6.4.29}$$

$$w = C\ z^{(n+1)/(n+2)} x^{-1/(n+2)}, \tag{6.4.30}$$

$$C = K^{n/(n+2)}(A/2\pi GM)^{1/(n+2)} v_\infty^{(1-2n)/(n+2)}. \tag{6.4.31}$$

Eliminating z and w from these equations, we get after some algebra the basic ordinary differential equation

$$\{y^2 - [C(n+1)/(n+2)][y/x(x-x_0)]^{1/(n+2)}\}\ dy/dx = y^2(1-y)/(x-x_0) - y/x^2$$
$$+[Cn/(n+2)][y/x(x-x_0)]^{(n+3)/(n+2)}[-(1+1/n)x + (x-x_0)/n]. \tag{6.4.32}$$

In the isothermal case, when $P/\varrho = f/m = K = \mathcal{R}T/\mu = \text{const}$, the basic equation (6.4.32) reads

$$(y^2 - C)\ dy/dx = y^2(1-y)/(x-x_0) - y/x^2 - Cy/(x-x_0), \qquad (n = \infty). \tag{6.4.33}$$

The second special case $C = 0$ reduces to the well known pressure-free Bondi-Hoyle equation:

$$y\ dy/dx = y(1-y)/(x-x_0) - 1/x^2, \qquad (C = 0). \tag{6.4.34}$$

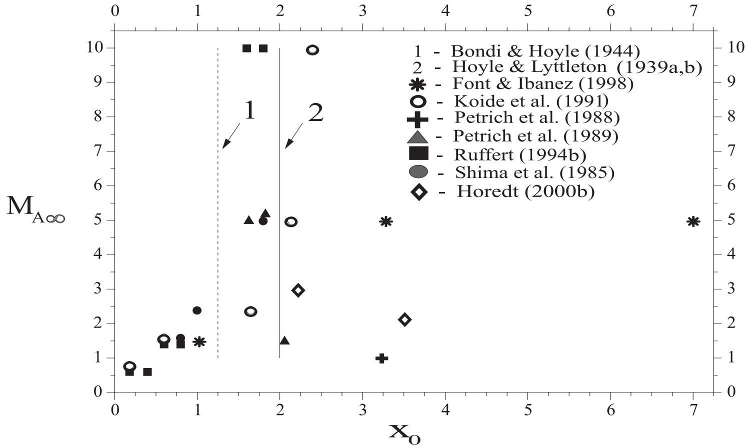

Fig. 6.4.2 Analytical and numerical values of the stagnation distance x_0 as a function of the upstreaming Mach number $M_{A\infty}$ in the unperturbed flow. The lines labeled 1 and 2 are given by Bondi and Hoyle (1944; $x_0 = 1.25$), and by Eq. (6.4.37), respectively (Horedt 2000b).

I. Behaviour near the Stagnation Point r_{a0}.

(i) $0 < n \le \infty$; $C \neq 0$. Since by virtue of Eq. (6.4.28) we have $y \approx 0$ in the vicinity of the stagnation point x_0, we assume the series expansion

$$y = b(x - x_0) + c(x - x_0)^2, \qquad (x \approx x_0;\ b, c = \mathrm{const}). \tag{6.4.35}$$

Inserting into Eq. (6.4.32), we observe that the zeroth order terms vanish, the first order terms in $x - x_0$ serving for the determination of c. Thus, the stagnation distance x_0 and the coefficient b, $(b > 0)$ remain undetermined, i.e. an infinity of solution curves pass through the stagnation point x_0. Indeed, it has been verified numerically that an infinitude of integral curves start from the stagnation point $(x > x_0)$ with quite different slopes, often exhibiting curious loops and converging to the thick curves in Figs. 6.4.3 and 6.4.4.

(ii) $C = 0$. The approximate solution of Eq. (6.4.34) becomes with the expansion (6.4.35) equal to (cf. Bondi and Hoyle 1944, p. 277)

$$y = [(x - x_0)/x_0^2][1 + (2/x_0)(1/x_0 - 1)(x - x_0)], \qquad (x \approx x_0), \tag{6.4.36}$$

where the stagnation distance x_0 is undetermined. An infinitude of solution curves pass through the stagnation point in this special case – all with the same slope $dy/dx = 1/x_0^2$ (Lyttleton 1972). Even the thick upper limiting curve in Fig. 6.4.5 starts with the slope $1/x_0^2 = 0.25$, $(x_0 = 2)$, but only up to $x - x_0 \approx 0.01$, when the slope increases suddenly.

In the pressure-free case $C = 0$ the dimensionless stagnation distance x_0 can be estimated by a heuristic argument devised by Hoyle and Lyttleton (1939a, b). The radial velocity (6.4.18) of gas particles reaching the accretion axis on collisionless hyperbolic paths is exactly v_∞. The transversal velocities v_φ are assumed to be annihilated by collisions in the accretion column, and if the remaining constant radial velocities v_∞ of newly accreting gas particles are smaller than the escape velocity $(2GM/r_a)^{1/2}$ at distance r_a from the accretor, they all will eventually be accreted: $v_\infty \le (2GM/r_a)^{1/2}$. In this ballistic picture the stagnation distance r_{a0} is just equal to

$$r_{a0} = 2GM/v_\infty^2 \quad \text{or} \quad x_0 = 2, \tag{6.4.37}$$

this being the distance at which the radial velocity of newly accreting gas v_∞ is just equal to the escape velocity $(2GM/r_{a0})^{1/2}$.

The precise knowledge of the stagnation point x_0 does not seem crucially, as suggested by the general aspect of the integral curves in Figs. 6.4.3-6.4.5: It may suffice to take $x_0 \approx 2$, (Eq. (6.4.37), Fig. 6.4.2).

The Mach number M_A in the accretion column is given by

$$M_A^2 = v^2/a^2 = v^2\varrho/\gamma P = v^2 m/\gamma f = y^2 z/\gamma w = (y^2 v_\infty^2 \varrho_\infty/\gamma P_\infty)(\varrho_\infty/\varrho)^{1/n} = y^2 M_{A\infty}^2(\varrho_\infty/\varrho)^{1/n},$$
$$[P = P_\infty(\varrho/\varrho_\infty)^{1+1/n}], \qquad (6.4.38)$$

where $M_{A\infty} = v_\infty/a_\infty = v_\infty(\varrho_\infty/\gamma P_\infty)^{1/2}$ denotes the upstreaming Mach number at infinity.

It will turn out that $M_{A\infty}$ can be expressed by the fundamental constant C of the problem. For the ratio ϱ/ϱ_∞ we get via Eqs. (6.4.14), (6.4.15), (6.4.22), (6.4.27):

$$\varrho = m/\pi s^2 = Az/2\pi v_\infty r_a w^2 = \varrho_\infty z/xw^2 \quad \text{or} \quad \varrho/\varrho_\infty = z/xw^2. \qquad (6.4.39)$$

The Mach number (6.4.38) becomes with Eqs. (6.4.30), (6.4.39) equal to

$$M_A^2 = y^2(\varrho_\infty/\varrho)^{1/n} M_{A\infty}^2 = y^2(xw^2/z)^{1/n} M_{A\infty}^2 = C^{2/n} y^2 (xz)^{1/(n+2)} M_{A\infty}^2. \qquad (6.4.40)$$

But Eq. (6.4.38) can also be written as

$$M_A^2 = y^2 z/\gamma w = (1/\gamma C) y^2 (xz)^{1/(n+2)}. \qquad (6.4.41)$$

Equating the last two equations we get the desired result

$$M_{A\infty}^2 = 1/\gamma C^{(n+2)/n}. \qquad (6.4.42)$$

If we put the condition that v is everywhere small (not only near the stagnation point x_0), we can neglect the left-hand side of Eq. (6.4.24). If further $C = 0$, $(w, f = 0)$, the right-hand side yields $m = Av_\infty r_a^2/GM$, $(v \approx 0;\ f = 0)$, and we get from Eq. (6.4.17) Lyttleton's [1972, Eq. (14)] *slow* solution, valid outside the stagnation point r_{a0} in the pressure-free case:

$$v \approx GM(r_a - r_{a0})/v_\infty r_a^2, \qquad (r_a \geq r_{a0};\ v \approx 0;\ C = 0). \qquad (6.4.43)$$

The whole mass, reaching the accretion axis inside the stagnation point r_{a0} at constant rate A, will be ultimately accreted. The mass accretion rate is therefore via Eqs. (6.4.14), (6.4.27) equal to [cf. Eq. (6.4.86)]

$$dM/dt = Ar_{a0} = 2\pi x_0 \varrho_\infty G^2 M^2/v_\infty^3. \qquad (6.4.44)$$

If $M_{A\infty} \gg 1$ and $x_0 \approx 2$, this equation agrees with some numerical simulations within a factor of about 2 (e.g. Koide et al. 1991, Ruffert and Arnett 1994, Ruffert 1996), while in the relativistic case numerically calculated values of dM/dt are sometimes more than an order of magnitude larger (Font and Ibáñez 1998, Table 3). As the dM/dt-relationships are considerably more involved and much more numerous (e.g. Hunt 1979, Shima et al. 1985, Koide et al. 1991, Ruffert 1994a, b, 1996, Foglizzo and Ruffert 1997, Font and Ibáñez 1998), no attempt has been made to collect all of them together.

II. Cut-off Distance r_{ac}. The linear accretion column may not extend to infinity, but merely up to the average interstellar distance. A more physical cut-off distance r_{ac} results from the condition that the pressure P_c in the accretion column is just equal to the pressure P_∞ in the unperturbed interstellar medium (Yabushita 1978a, b):

$$P_c = K\varrho_c^{1+1/n} = P_\infty = K\varrho_\infty^{1+1/n} \quad \text{or} \quad \varrho_c = \varrho_\infty. \qquad (6.4.45)$$

The cut-off density ϱ_c equals the density ϱ_∞ in the unperturbed interstellar cloud: The accretion column becomes nearly indistinguishable from the unperturbed cloud, excepting perhaps for velocity differences. The cut-off distance r_{ac} is meaningful if it is much larger than the stagnation distance r_{a0}, because otherwise no appreciable outer branch of the accretion column could develop: $x_c \gg x_0 \approx 1$. Thus, Eqs. (6.4.17) and (6.4.28) read at the cut-off distance

$$m_c v_c \approx Ar_{ac} \quad \text{or} \quad y_c z_c \approx x_c. \qquad (6.4.46)$$

Likewise, from Eq. (6.4.39) we get at the cut-off distance

$$\varrho_c/\varrho_\infty = z_c/x_c w_c^2 = 1. \qquad (6.4.47)$$

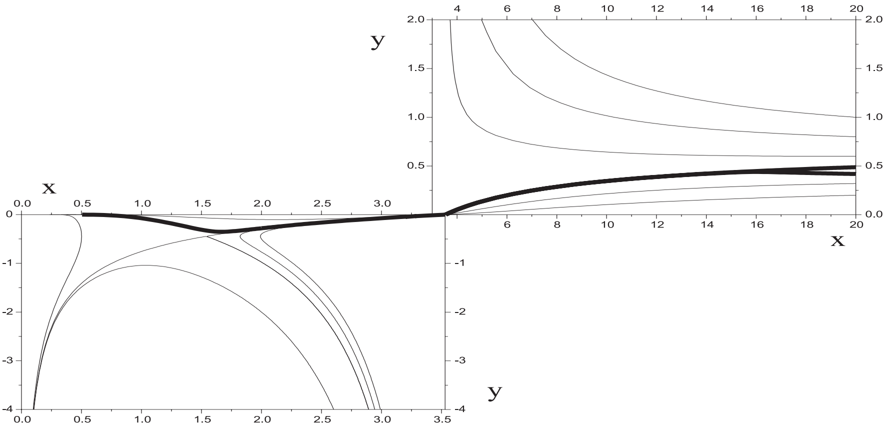

Fig. 6.4.3 Exterior ($x > x_0$) and interior ($x < x_0$) branch of Eq. (6.4.33) for a stagnation distance $x_0 = 3.53$ and $n = \infty,\ \gamma = 1 + 1/n = 1,\ M_{A\infty} = 2.25,\ (C = 0.1975)$, (Horedt 2000b).

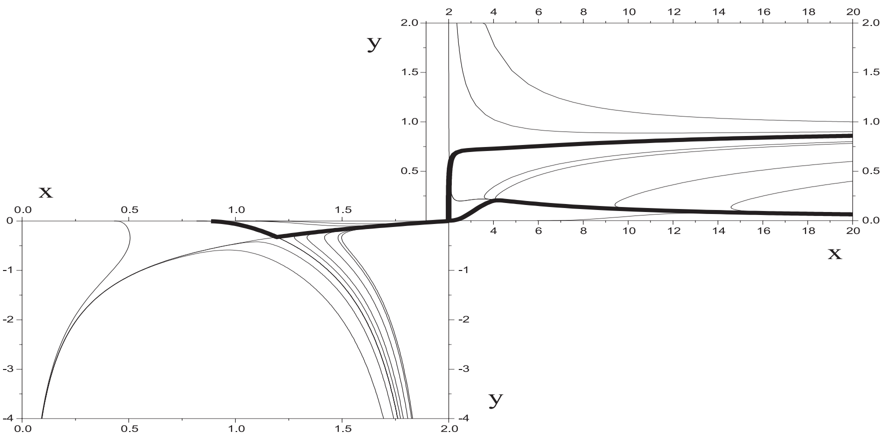

Fig. 6.4.4 Exterior ($x > x_0$) and interior ($x < x_0$) branch of Eq. (6.4.32) for a stagnation distance $x_0 = 2$ and $n = 1.5,\ \gamma = 1 + 1/n = 5/3,\ M_{A\infty} = 5,\ (C = 0.2022)$, (Horedt 2000b).

We combine the two previous equations into $y_c w_c^2 = 1$, and insert from Eqs. (6.4.30) and (6.4.46):

$$y_c w_c^2 = C^2 y_c z_c^{2(n+1)/(n+2)} x_c^{-2/(n+2)} = C^2 y_c^{-n/(n+2)} x_c^{2n/(n+2)} = 1. \tag{6.4.48}$$

The cut-off coordinate x_c can be expressed according to Eq. (6.4.42) in terms of the upstreaming Mach number at infinity:

$$x_c = y_c^{1/2}/C^{(n+2)/n} = y_c^{1/2}\gamma M_{A\infty}^2 \gg 1. \tag{6.4.49}$$

And finally, the cut-off distance is via Eq. (6.4.27) equal to

$$r_{ac} = GMx_c/v_\infty^2 = GM\gamma M_{A\infty}^2 y_c^{1/2}/v_\infty^2 = GM\varrho_\infty y_c^{1/2}/P_\infty = GM\mu y_c^{1/2}/\mathcal{R}T_\infty. \tag{6.4.50}$$

A basic assumption of line accretion is that throughout $s/r_a \ll 1$. With Eqs. (6.4.22), (6.4.27),

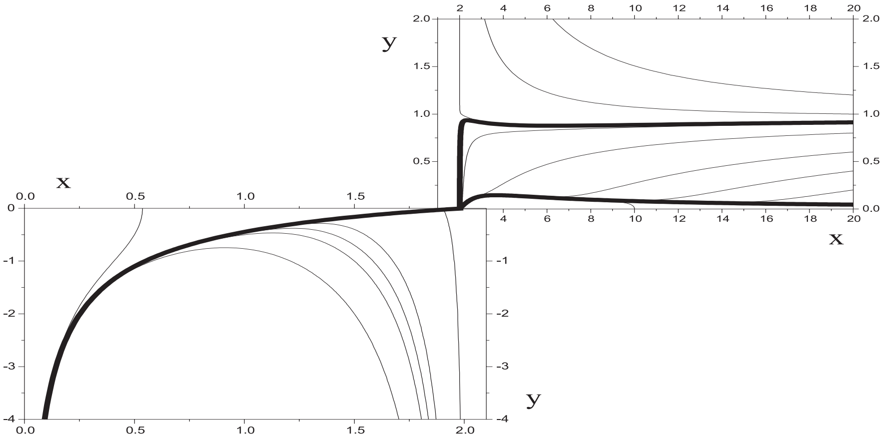

Fig. 6.4.5 Exterior ($x > x_0$) and interior ($x < x_0$) branch of Eq. (6.4.34) for a stagnation distance $x_0 = 2$ and $C = 0$ (Horedt 2000b; see also Lyttleton 1972).

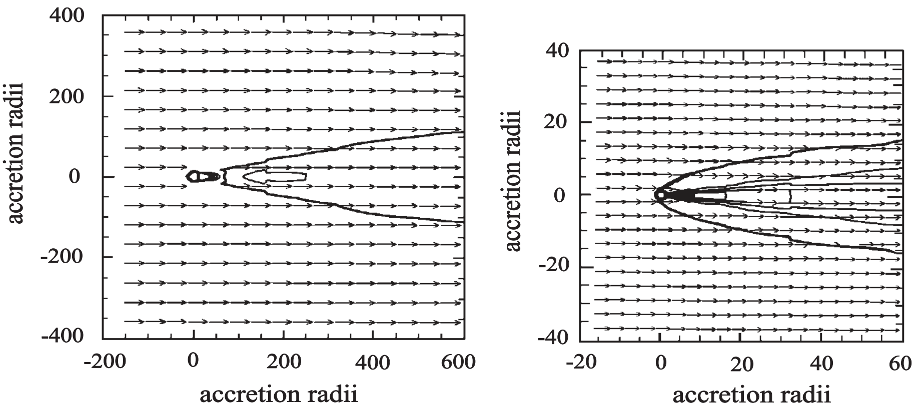

Fig. 6.4.6 Hydrodynamic 3-D calculations of a $M_{A\infty} = 10$ flow with $\gamma = c_P/c_V = 5/3$ (on the left, showing a tail shock), and $\gamma = 1.01$ (on the right), both resembling the accretion column near the downstreaming axis. Velocity patterns are designated by arrows, and "accretion radius" means $2GM/v_\infty^2$, ($x_0 = 2$) from Eq. (6.4.37), (Ruffert 1994b, 1996).

(6.4.48), (6.4.49) we find

$$s_c/r_{ac} = (f_c/A)(2/GMr_{ac})^{1/2} = 2^{1/2}w_c/x_c^{1/2} = (2/x_c y_c)^{1/2} = 2^{1/2}/\gamma^{1/2} y_c^{3/4} M_{A\infty}. \qquad (6.4.51)$$

As long as $y_c \not\approx 0$ and $M_{A\infty} \gg 1$, the value of s_c/r_{ac} is sufficiently small, and x_c is sufficiently large to be matched with ∞ : $y_c = y(x_c) \approx y(\infty) = y_\infty$. Thus

$$x_c \approx y_\infty^{1/2} \gamma M_{A\infty}^2 \gg 1; \quad s_c/r_{ac} \approx 2^{1/2}/\gamma^{1/2} y_\infty^{3/4} M_{A\infty} \ll 1, \qquad (y_\infty \not\approx 0;\ M_{A\infty} \gg 1). \qquad (6.4.52)$$

The case $y_\infty \not\approx 0$ will be termed *fast* solution, and $y_\infty = 1$ if $0 < n < \infty$ or $C = 0$.

Problems with the width s_c of the accretion column occur only for the *slow* solution, if $y_\infty \approx 0$ or even $y_\infty = 0$: The requirement $s_c/r_{ac} \ll 1$ is only marginally fitted – if at all (Horedt 2000b).

Numerical integration of Eqs. (6.4.32)-(6.4.34) indicates that an infinitude of exterior fast solutions exist ($y_\infty \neq 0$) if $n = \infty$, ($C \neq 0$, Fig. 6.4.3) or $C = 0$ (Fig. 6.4.5). The exterior slow solution ($y_\infty = 0$;

lower thick curve in Figs. 6.4.4 and 6.4.5) appears to be unique, like the fast exterior solution in Fig. 6.4.4 if $0 < n < \infty$, $(C \neq 0)$. If pressure is present $(C \neq 0)$, there exists an infinitude of interior accretion flows, all halted by pressure forces at a second interior stagnation point (Figs. 6.4.3, 6.4.4). If $C = 0$, the interior solution is unique with increasing absolute velocity (Fig. 6.4.5).

Analytical studies concerning the linear stability of the accretion line in the pressure-free Bondi-Hoyle case (6.4.34) have been undertaken by Cowie (1977) and Soker (1990): The three-dimensional flow along the accretion line – deviating slightly from the accretion axis – is unstable to radial perturbations along the accretion line. And the two-dimensional flow is found to be unstable against tangential modes, as well as against radial ones. These stability studies however, have no direct bearing on polytropes.

6.4.2 Spherically Symmetric Accretion of a Polytropic Gas

Like in Sec. 6.4.1 we confine ourselves to the case when the mean free path for collisions is much smaller than the characteristic length scale of motion, i.e. when collisions are important in the accretional flow. Consider then within Newtonian gravitation the steady, spherical accretion of an *isentropic* gas of adiabatic index γ obeying the equation of state (6.3.4):

$$P = P_\infty(\varrho/\varrho_\infty)^\gamma = P_\infty(\varrho/\varrho_\infty)^{1+1/n} = K\varrho^{1+1/n}, \qquad (n = 1/(\gamma-1);\ K = P_\infty/\varrho_\infty^{1+1/n}). \qquad (6.4.53)$$

P_∞ and ϱ_∞ are pressure and density at infinity, where the gas is at rest: $v_\infty = 0$. The Newtonian equations governing the problem are easily set up. The stationary continuity equation (5.2.1) becomes for spherical symmetry

$$\nabla \cdot (\varrho\vec{v}) = (1/r^2)\ d(r^2\varrho v)/dr = 0, \qquad (6.4.54)$$

which readily integrates to

$$A = dM/dt = 4\pi r^2\varrho v = \mathrm{const}, \qquad (6.4.55)$$

where v denotes the *inward* flow velocity ($v = |\vec{v}|$), and dM/dt is the mass accretion rate of the central mass M. The Eulerian equation of motion (5.2.10) reads

$$(\vec{v}\cdot\nabla)\vec{v} + (1/\varrho)\ \nabla P - \nabla\Phi = v\ dv/dr + (1/\varrho)\ dP/dr + GM/r^2 = 0. \qquad (6.4.56)$$

Integration via Eq. (6.4.53) yields with the boundary conditions at infinity $v = v_\infty = 0$ and $GM/r \to 0$ the Bernoulli equation

$$v^2/2 + [\gamma/(\gamma-1)](P_\infty/\varrho_\infty)[(\varrho/\varrho_\infty)^{\gamma-1} - 1] - GM/r = v^2/2 + (n+1)(P_\infty/\varrho_\infty)[(\varrho/\varrho_\infty)^{1/n} - 1]$$
$$-GM/r = v^2/2 + n(a^2 - a_\infty^2) - GM/r = 0, \qquad (n = 1/(\gamma-1);\ -\infty < n < -1;\ 0 < n < \infty). \qquad (6.4.57)$$

The local sound velocity is denoted by $a = (\gamma P/\varrho)^{1/2} = [(1+1/n)P/\varrho]^{1/2}$, and $a_\infty = (\gamma P_\infty/\varrho_\infty)^{1/2} = [(1+1/n)P_\infty/\varrho_\infty]^{1/2}$ is the sound velocity at infinity. It is seen at once that the sound velocity is not defined if $-1 \leq n \leq 0$.

Similarly to Eq. (6.4.27) we introduce the dimensionless variables

$$x = (a_\infty^2/GM)r; \quad y = v/a_\infty; \quad z = \varrho/\varrho_\infty, \qquad (-\infty < n < -1;\ 0 < n < \infty), \qquad (6.4.58)$$

and write Eqs. (6.4.55), (6.4.57) in the nondimensional form

$$\lambda = x^2yz = Aa_\infty^3/4\pi\varrho_\infty G^2M^2 = \mathrm{const}, \qquad (6.4.59)$$

$$y^2/2 + n(z^{1/n} - 1) = 1/x. \qquad (6.4.60)$$

The variables can be separated with the aid of the Mach number

$$M_A = v/a = v[\varrho/(1+1/n)P]^{1/2} = v[\varrho_\infty/(1+1/n)P_\infty]^{1/2}(\varrho/\varrho_\infty)^{-1/2n}$$
$$= (v/a_\infty)(\varrho/\varrho_\infty)^{-1/2n} = yz^{-1/2n}. \qquad (6.4.61)$$

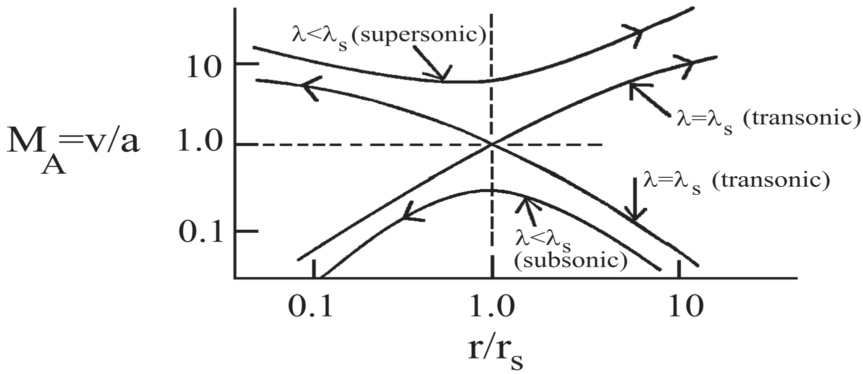

Fig. 6.4.7 Sketch of some spherically symmetric accretion and wind flows, as indicated by arrows (cf. Figs. 6.3.3, 6.3.4, 6.3.8). The Mach number M_A is plotted for three values of the dimensionless accretion parameter λ : subsonic accretion flow, supersonically expanding wind (both with $\lambda < \lambda_s$ if $3/2 \le n \le \infty$), and the two transonic wind and accretion flows occurring if $\lambda = \lambda_s$ (Shapiro and Teukolsky 1983).

With the mass conservation equation (6.4.59) we get

$$y = (\lambda/x^2)^{1/(2n+1)} M_A^{2n/(2n+1)}; \quad z = (\lambda/x^2 M_A)^{2n/(2n+1)}. \tag{6.4.62}$$

The Bernoulli equation (6.4.60) becomes via Eq. (6.4.62)

$$M_A^{4n/(2n+1)} (\lambda/x^2)^{2/(2n+1)}/2 + n(\lambda/x^2 M_A)^{2/(2n+1)} = 1/x + n. \tag{6.4.63}$$

Rearranging the terms we can write this equation as

$$f(M_A) = \lambda^{-2/(2n+1)} g(x), \qquad (-\infty < n < -1;\ 0 < n < \infty), \tag{6.4.64}$$

where

$$f(M_A) = M_A^{4n/(2n+1)} (1/2 + n/M_A^2), \tag{6.4.65}$$

$$g(x) = x^{4/(2n+1)} (1/x + n). \tag{6.4.66}$$

If $3/2 \le n < \infty$, the functions f and g are the sum of a positive and negative power – each of them has a minimum

$$f(M_{As}) = (2n+1)/2; \quad g(x_s) = [(2n+1)/4][(2n-3)/4n]^{(3-2n)/(2n+1)}, \tag{6.4.67}$$

that is obtained with $df/dM_A = 0$ and $dg/dx = 0$, i.e. for

$$M_A = M_{As} = 1; \quad x = x_s = (2n-3)/4n. \tag{6.4.68}$$

If $-\infty < n < -1$ and $3/2 \le n < \infty$, we can write:

$$\lambda_s = [g(x_s)/f(M_{As})]^{(2n+1)/2} = 2^{(2n-7)/2} [(2n-3)/n]^{(3-2n)/2}. \tag{6.4.69}$$

If $3/2 \le n < \infty$, the maximum value λ_s of the accretion parameter (6.4.59) occurs just for the transonic solution passing through the sonic point $M_{As} = 1$. This can be shown most easily by writing down Eq. (6.4.64) at distance x_s, and observing that $f(M_A) \ge f(M_{As})$:

$$\lambda^{-2/(2n+1)} g(x_s) = f(M_A) \ge f(M_{As}) \quad \text{or} \quad \lambda \le [g(x_s)/f(M_{As})]^{(2n+1)/2} = \lambda_s, \qquad (3/2 \le n < \infty). \tag{6.4.70}$$

The limiting value $\lambda_s = 1/4$ if $n = 3/2$ results from Eq. (6.4.69) by writing $\lim_{n\to 3/2}[(2n-3)/n]^{(3-2n)/2} = \lim_{n\to 3/2}\exp\{[(3-2n)/2]\ln[(2n-3)/n]\} = 1$. The value of λ_s is positive outside the interval [0, 3/2).

As seen from Fig. 6.3.1, no accretion flows are possible if $0 < n < 3/2$, because no solution curves exist connecting the infinity point $r = \infty,\ v_\infty = 0$ with the centre $r = 0$ of the point mass.

If $\lambda = 0$, the gas is at rest $(y, v = 0)$, and the dimensionless density changes according to Eq. (6.4.60) like in an extended atmosphere:

$$z^{1/n} = 1 + 1/nx, \qquad (\lambda = 0). \tag{6.4.71}$$

In the limiting isothermal case the integral of Eq. (6.4.56) becomes

$$v^2/2 + (P_\infty/\varrho_\infty)\ln(\varrho/\varrho_\infty) - GM/r = v^2/2 + a_\infty^2 \ln(\varrho/\varrho_\infty) - GM/r = 0,$$
$$(n = 1/(\gamma - 1) = \pm\infty), \tag{6.4.72}$$

and we have instead of Eq. (6.4.60):

$$y^2/2 + \ln z = 1/x, \qquad (n = \pm\infty). \tag{6.4.73}$$

There is $M_A = y$ if $n \to \pm\infty$ via Eq. (6.4.61), and z can be eliminated in Eq. (6.4.73) with $z = \lambda/x^2 M_A$ from Eq. (6.4.62):

$$M_A^2/2 - \ln M_A = -\ln\lambda + \ln x^2 + 1/x. \tag{6.4.74}$$

We write

$$\ln\lambda = g(x) - f(M_A), \qquad (f(M_A) = M_A^2/2 - \ln M_A;\ g(x) = \ln x^2 + 1/x). \tag{6.4.75}$$

The minima

$$f(M_{As}) = 1/2; \quad g(x_s) = 2 - \ln 4, \tag{6.4.76}$$

of f and g occur if

$$M_{As} = 1; \quad x_s = 1/2, \qquad (n = \pm\infty). \tag{6.4.77}$$

The maximum sonic value of the accretion parameter in the isothermal case is therefore

$$\lambda_s = \exp[g_s(x_s) - f_s(M_{As})] = e^{3/2}/4 = 1.120, \qquad (n = \pm\infty). \tag{6.4.78}$$

Via Eqs. (6.4.69) and (6.4.78) we get some values of λ_s : 1.376, 2.000, 0.250, 0.707, 0.872, 1.120 if $n = -5, -1.5, 1.5, 3, 5, \pm\infty$, respectively. Writing Eq. (6.4.69) under the form $\lambda_s = \lim_{n\to\pm\infty}\exp\{[(2n-7)/2]\ln 2 + [(3-2n)/2]\ln(2-3/n)\} = e^{3/2}/4$, we recover Eq. (6.4.78).

The transonic flow profiles are straightforward to deduce. The sonic distance r_s and the flow velocity v_s at the sonic point r_s are expressed via Eqs. (6.4.57), (6.4.58), (6.4.68):

$$r_s = (GM/a_\infty^2)x_s = (2n-3)GM/4na_\infty^2; \quad v_s^2 = a_s^2 = (GM/r_s + na_\infty^2)/(1/2 + n)$$
$$= 2na_\infty^2/(2n-3) = GM/2r_s, \qquad (r = r_s;\ -\infty \le n < -1;\ 3/2 \le n \le \infty). \tag{6.4.79}$$

At large distances $r \gg r_s$ the gravitational influence of the central mass is barely felt, and the Bernoulli equation (6.4.57) yields

$$\varrho = \varrho_\infty; \quad a = a_\infty; \quad T = T_\infty, \qquad (r/r_s \gg 1;\ v \approx 0). \tag{6.4.80}$$

The transonic flow velocity changes in virtue of Eqs. (6.4.55), (6.4.59) as

$$v \approx A/4\pi\varrho_\infty r^2 = \lambda_s G^2M^2/a_\infty^3 r^2, \qquad (r/r_s \gg 1). \tag{6.4.81}$$

Well inside the sonic radius r_s, the flow is significantly influenced by the gravitation of the central point mass M, and the term GM/r dominates over na^2 in Eq. (6.4.57):

$$v \approx (2GM/r)^{1/2}; \quad \varrho = A/4\pi r^2 v \approx (\lambda_s\varrho_\infty/2^{1/2})(GM/a_\infty^2 r)^{3/2} = \varrho_\infty(T/T_\infty)^n,$$
$$(r/r_s \ll 1;\ -\infty \le n < -1;\ 3/2 < n \le \infty). \tag{6.4.82}$$

Eqs. (6.4.79)-(6.4.82) subsist in the limiting case $n = \pm\infty$, $(T = T_\infty)$ too, excepting for the last equality in Eq. (6.4.82). In the other limiting case $n = 1/(\gamma - 1) = 3/2$, the sonic point is just in the origin, and the evaluation of flow profiles near the point mass M amounts to their study near the sonic point $r \approx r_s = 0$. Velocity and density run result via Eqs. (6.4.55), (6.4.57), (6.4.59):

$$v \approx a \approx (GM/2r)^{1/2}; \quad \varrho \approx A/4\pi r^2 v = (2^{1/2}\varrho_\infty/4)(GM/a_\infty^2 r)^{3/2} = \varrho_\infty(T/T_\infty)^{3/2},$$
$$(r \approx r_s = 0;\ n = 3/2;\ a_\infty^2 \ll GM/r;\ \lambda_s = 1/4). \tag{6.4.83}$$

In addition to the transonic solution passing through $M_A = 1$, there exists also a class of solutions which are subsonic throughout, satisfying at the same time the condition at infinity $v_\infty = 0$ with $0 \le \lambda < \lambda_s$ (cf. Figs. 6.3.3, 6.3.4, 6.4.7). The flow profiles if $r/r_s \gg 1$ are the same as in the transonic case from Eqs. (6.4.80), (6.4.81), but if $r \approx 0$, the pressure term $na^2 = (n+1)P/\varrho$ in Eq. (6.4.57) dominates over the kinetic energy per unit mass $v^2/2$, and the velocity no longer approaches the free-fall value (6.4.82). Thus, we get from Eqs. (6.4.53), (6.4.55), (6.4.57), (6.4.59):

$$\varrho = \varrho_\infty(a/a_\infty)^{2n} \approx \varrho_\infty(GM/na_\infty^2 r)^n; \quad v = A/4\pi r^2\varrho \approx n^n\lambda a_\infty(GM/a_\infty^2 r)^{2-n},$$
$$(r, v \approx 0;\ M_A < 1;\ 3/2 \le n < \infty). \tag{6.4.84}$$

In the limiting isothermal case we have via Eqs. (6.4.55), (6.4.59), (6.4.72):

$$\varrho \approx \varrho_\infty \exp(GM/a_\infty^2 r); \quad v = A/4\pi r^2\varrho \approx (\lambda G^2M^2/a_\infty^3 r^2)\exp(-GM/a_\infty^2 r),$$
$$(r, v \approx 0;\ M_A < 1;\ n = \pm\infty). \tag{6.4.85}$$

The mass accretion rate $dM/dt = 4\pi\lambda\varrho_\infty G^2M^2/a_\infty^3$, $(v_\infty = 0)$ for spherically symmetric accretion from Eq. (6.4.59) may be combined with the mass accretion rate $dM/dt = 2\pi x_0\varrho_\infty G^2M^2/v_\infty^3$, $(v_\infty \gg a_\infty)$ from Eq. (6.4.44):

$$dM/dt \approx 4\pi\Lambda\varrho_\infty G^2M^2/(a_\infty^2 + v_\infty^2)^{3/2}. \tag{6.4.86}$$

Indeed, in the two limits $a_\infty \gg v_\infty$ and $v_\infty \gg a_\infty$ this empirical equation converges to the forms (6.4.59) and (6.4.44) if Λ is replaced by λ and $x_0/2$, respectively (Bondi 1952). An attempt to include a finite size of the central point mass M has been made by Ruffert (1994a).

In the general relativistic case Michel (1972) has considered spherically symmetric Bondi accretion with the Schwarzschild metric (4.1.5). The relativistic Euler equation of motion (momentum conservation equation) can be derived with Eqs. (4.1.11), (4.1.14), taking into account that covariant derivation of sums and products obeys the same rules as ordinary derivation, and that the covariant derivative $\nabla_m g_{jk}$ of the metric tensor vanishes (Landau and Lifschitz 1987, §§85, 86):

$$\nabla_k T_j^k = u_j\ \nabla_k[(P + \varepsilon_r)u^k] + (P + \varepsilon_r)u^k\ \nabla_k u_j - \partial P/\partial x^j = 0, \qquad (\nabla_j P = \partial P/\partial x^j). \tag{6.4.87}$$

To make further progress, we recall from Eq. (4.1.12) that $u_j u^j = 1$, and therefore, by interchanging the summation indices j and ℓ :

$$\nabla_k(u_j u^j) = g_{j\ell}\ \nabla_k(u^\ell u^j) = g_{j\ell}u^j\ \nabla_k u^\ell + g_{j\ell}u^\ell\ \nabla_k u^j = 2g_{j\ell}u^j\ \nabla_k u^\ell$$
$$= 2g_{j\ell}g^{\ell m}u^j\ \nabla_k u_m = 2u^j\ \nabla_k u_j = 0. \tag{6.4.88}$$

We project Eq. (6.4.87) onto the j-direction of the four-velocity by multiplying with u^j, and taking into account Eq. (6.4.88):

$$u_j u^j\ \nabla_k[(P + \varepsilon_r)u^k] + (P + \varepsilon_r)u^k u^j\ \nabla_k u_j - u^j\ \partial P/\partial x^j = \nabla_k[(P + \varepsilon_r)u^k] - u^k\ \partial P/\partial x^k = 0. \tag{6.4.89}$$

We insert the last equality into Eq. (6.4.87), to obtain the relativistic Euler equation (e.g. Landau and Lifshitz 1959, §126; Shapiro and Teukolsky 1983, App. G):

$$(P + \varepsilon_r)u^k\ \nabla_k u_j + u_j u^k\ \partial P/\partial x^k - \partial P/\partial x^j = 0. \tag{6.4.90}$$

This equation takes a simple form for radial motion in the Schwarzschild field (4.1.5). The component along $x^1 = r$ becomes

$$(P + \varepsilon_r)(u^0\ \nabla_0 u_r + u^r\ \nabla_r u_r) + u_r u^r\ dP/dr - dP/dr = 0, \tag{6.4.91}$$

where $u^r = dr/ds$ is the radial component of the four-velocity u^j, and $u^0 = dt/ds$ its temporal component ($dx^0 = dt;\ dx^1 = dr$). We have $g_{00}(u^0)^2 + g_{rr}(u^r)^2 = 1$ via Eq. (4.1.12), or by inserting the external components (4.1.20) of the metric tensor:

$$u_0 = (1 - 2GM_{r1}/c^2r)u^0; \quad u_r = -u^r/(1 - 2GM_{r1}/c^2r);$$
$$(u^0)^2 = [1 - 2GM_{r1}/c^2r + (u^r)^2]/(1 - 2GM_{r1}/c^2r)^2. \tag{6.4.92}$$

It remains to calculate the covariant derivatives according to $\nabla_k A_j = \partial A_j/\partial x^k - \Gamma^\ell_{jk} A_\ell$, with the Christoffel symbols taken from Eqs. (4.1.17), (4.1.21):

$$\nabla_0 u_r = -\Gamma^0_{r0} u_0 = -(GM_{r1}/c^2r^2)u_0/(1 - 2GM_{r1}/c^2r); \quad \nabla_r u_r = du_r/dr - \Gamma^r_{rr} u_r$$
$$= du_r/dr + (GM_{r1}/c^2r^2)u_r/(1 - 2GM_{r1}/c^2r), \qquad (\Gamma^0_{r0} = \Gamma^0_{10};\ \Gamma^r_{rr} = \Gamma^1_{11}). \tag{6.4.93}$$

The equation of motion (6.4.91) takes with Eqs. (6.4.92), (6.4.93) the final form in the exterior static Schwarzschild field:

$$u^r\ du^r/dr + GM_{r1}/c^2r^2 + [1/(P + \varepsilon_r)][1 - 2GM_{r1}/c^2r + (u^r)^2]\ dP/dr = 0. \tag{6.4.94}$$

To the relativistic stationary Euler equation (6.4.94) we have to add also the equation of continuity (rest energy conservation or baryon number conservation):

$$\nabla_i(\varepsilon u^i) = 0. \tag{6.4.95}$$

Observing that the rest mass density ϱ can be written as the product of mean baryon mass m and baryon number density n_d, and that $\varepsilon = c^2\varrho = c^2 m n_d$, we can write this equation also under the more familiar form $\nabla_i(n_d u^i) = 0$. The covariant divergence (6.4.95) can be expressed by ordinary derivatives (e.g. Landau and Lifschitz 1987, §86):

$$\nabla_i(\varepsilon u^i) = (-g)^{-1/2}\ \partial[(-g)^{1/2}\varepsilon u^i]/\partial x^i = (1/r^2)\ d(r^2\varepsilon u^r)/dr \propto (1/r^2)\ d(r^2\varrho u^r)/dr = 0. \tag{6.4.96}$$

The determinant g of the metric tensor g_{jk} is for the external Schwarzschild metric (4.1.20) equal to $g = -r^4 \sin^2\lambda$, and the sole surviving derivative is with respect to $x^1 = r$, since the motion is assumed stationary and radial. The relativistic Bernoulli equation

$$[(P + \varepsilon_r)/\varrho]^2[1 - 2GM_{r1}/c^2r + (u^r)^2] = [(P_\infty + \varepsilon_{r\infty})/\varrho_\infty]^2 = \text{const}, \qquad (u^r(\infty) = u^r_\infty = 0), \tag{6.4.97}$$

constitutes the relativistic counterpart of Eq. (6.4.57), for instance, and is obtained by integration of Eq. (6.4.94), written under the form

$$(1/2)[(P + \varepsilon_r)/\varrho]\ d[1 - 2GM_{r1}/c^2r + (u^r)^2] + [1 - 2GM_{r1}/c^2r + (u^r)^2]\ dP/\varrho = 0. \tag{6.4.98}$$

To obtain Eq. (6.4.97), we have also inserted for dP/ϱ from the isentropic, relativistic first law of thermodynamics (4.1.57):

$$d[(P + \varepsilon_r)/\varrho] = d(P + \varepsilon_r)/\varrho - (P + \varepsilon_r)\ d\varrho/\varrho^2 = dP/\varrho + (1/\varrho)[d\varepsilon_r - (P + \varepsilon_r)\ d\varrho/\varrho] = dP/\varrho. \tag{6.4.99}$$

In terms of the relativistic adiabatic sound speed (4.1.66) we can write Eq. (6.4.94) as

$$u^r\ du^r/dr + GM_{r1}/c^2r^2 + [1 - 2GM_{r1}/c^2r + (u^r)^2](a^2/c^2\varrho)\ d\varrho/dr = 0, \tag{6.4.100}$$

because $[1/(P + \varepsilon_r)]\ dP/dr = [1/(P + \varepsilon_r)](dP/d\varrho)\ d\varrho/dr = (1/\varrho)(dP/d\varepsilon_r)\ d\varrho/dr = (a^2/c^2\varrho)\ d\varrho/dr$. We have used Eq. (4.1.57), and the fact that in adiabatic motion $P = P(\varrho)$, hence $dP = (dP/d\varrho)\ d\varrho$.

Eliminating $d\varrho/dr$ between Eqs. (6.4.96) and (6.4.100), we get the ordinary differential equation

$$du^r/dr = (u^r/r)\{(2a^2/c^2)[1 - 2GM_{r1}/c^2r + (u^r)^2] - GM_{r1}/c^2r\}$$
$$/\{(u^r)^2 - (a^2/c^2)[1 - 2GM_{r1}/c^2r + (u^r)^2]\}, \tag{6.4.101}$$

for the run of the radial component of the four-velocity. If one or the other of the braced factors vanishes, one has a turn-around point, and the flow is double-valued in either r or u^r. Only solutions that pass – as in the Newtonian case – through a singular point (r_s, u^r_s), where du^r/dr takes the form 0/0, correspond to material falling into, or flowing out from the central mass M_{r1}. The singular point is obtained by equating both braced factors to zero:

$$(u^r_s)^2 = GM_{r1}/2c^2r_s = (a_s^2/c^2)/(1+3a_s^2/c^2). \tag{6.4.102}$$

And it has been shown by Begelman (1978) and Chang (1985) that only one physically meaningful singular point exists in the relativistic regime if $0.5 < n \leq \infty$.

The proper, radial inward velocity v of a fluid element, as measured by the local stationary observer, is given by

$$v = d\ell/d\tau = (-g_{rr}/g_{00})^{1/2}c\ dr/dx^0 = cu^r/u^0(1-2GM_{r1}/c^2r)$$
$$= cu^r/[1-2GM_{r1}/c^2r+(u^r)^2]^{1/2}, \tag{6.4.103}$$

because via Eqs. (4.1.12), (4.1.20), (5.12.94)-(5.12.96) we have: $u^0 = [1-g_{rr}(u^r)^2]^{1/2}/g_{00}^{1/2}$, $d\ell = (-g_{rr})^{1/2}\ dr$, $d\tau = (g_{00})^{1/2}\ dx^0/c$. Inserting the critical values (6.4.102) into Eq. (6.4.103), we observe indeed that $v = v_s = a_s \ll c$, $(M_A = 1)$ at the critical radial coordinate $r = r_s \gg r_g$. At the Schwarzschild gravitational radius $r = r_g = 2GM_{r1}/c^2$ we have $v = c \geq a$, and at large distances, when $u^r \ll 1$, there is $v/c \approx u^r$ by Eq. (5.12.98).

To calculate the explicit value of the rest mass accretion rate, we assume the polytropic equation of state (4.1.83) for the relativistic pressure P. The first law of thermodynamics (4.1.57) may be written under the form $d(\varepsilon_r/\varrho) + P\ d(1/\varrho) = 0$, which integrates with Eq. (4.1.83) to

$$\varepsilon_r = \varrho_r c^2 = nK\varrho^{1+1/n} + \mathrm{const}\ \varrho = nP + \varrho c^2. \tag{6.4.104}$$

The integration constant is determined by comparison with Eq. (4.1.84). Hence, the adiabatic sound velocity (4.1.66) is expressed via Eqs. (4.1.57), (4.1.83), (6.4.104) as

$$a^2 = (\partial P/\partial\varrho_r)_S = (dP/d\varrho)\ d\varrho/d\varrho_r = [(n+1)P/n\varrho]/[(n+1)P/c^2\varrho+1]. \tag{6.4.105}$$

We substitute into Eq. (6.4.97) for $(P+\varepsilon_r)/\varrho = (n+1)P/\varrho + c^2$ from Eq. (6.4.104), and express $(n+1)P/\varrho$ by the sound speed (6.4.105):

$$(1-na^2/c^2)^{-2}[1-2GM_{r1}/c^2r+(u^r)^2] = (1-na_\infty^2/c^2)^{-2}. \tag{6.4.106}$$

Evaluating this Bernoulli equation via Eq. (6.4.102) at the sonic point r_s, we get

$$(1-na_s^2/c^2)^2(1+3a_s^2/c^2) = (1-na_\infty^2/c^2)^2. \tag{6.4.107}$$

At infinity we have $a_\infty \ll c$, and we find, by expanding to the lowest order:

$$a_s^2 \approx 2na_\infty^2/(2n-3) \quad \text{if} \quad -\infty \leq n < -1,\ 3/2 < n \leq \infty; \quad a_s^2/c^2 \approx 2a_\infty/3c \quad \text{if} \quad n = 3/2. \tag{6.4.108}$$

Thus, if $a_\infty \ll c$, we also have $a_s \ll c$, or $(n+1)P/c^2\varrho = na^2/(c^2-na^2) \ll 1$. Eq. (6.4.105) yields in virtue of Eq. (4.1.83) simply $\varrho_s/\varrho_\infty \approx (a_s/a_\infty)^{2n}$, as in the nonrelativistic case (6.4.84). With this finding the rest mass conservation equation (6.4.96) becomes via Eqs. (6.4.102), (6.4.103), (6.4.108) up to the lowest order:

$$r_s^2\varrho_s u_s^r \approx G^2M_{r1}^2\varrho_\infty a_s^{2n-3}/4ca_\infty^{2n} = [2n/(2n-3)]^{(2n-3)/2}G^2M_{r1}^2\varrho_\infty/4ca_\infty^3$$
$$= \lambda_s G^2M_{r1}^2\varrho_\infty/ca_\infty^3 = \mathrm{const}, \qquad (u_s^r \approx a_s/c = v_s/c;\ r_s \approx GM_{r1}/2a_s^2). \tag{6.4.109}$$

The factor $\lambda_s = [2n/(2n-3)]^{(2n-3)/2}/4$ is identical to Eq. (6.4.69). From Eq. (6.4.109) follows that the relativistic Bondi rest mass accretion rate

$$dM/dt \approx 4\pi r_s^2\varrho_s v_s \approx 4\pi cr_s^2\varrho_s u_s^r = 4\pi cr^2\varrho u^r \approx 4\pi\lambda_s\varrho_\infty G^2M_{r1}^2/a_\infty^3,$$
$$(-\infty \leq n < -1;\ 3/2 \leq n \leq \infty), \tag{6.4.110}$$

is in a first approximation just equal to its Newtonian counterpart (6.4.59), because the accretion rate is determined in the relativistic and nonrelativistic case by conditions at the same sonic point r_s, which in virtue of Eq. (6.4.102) lies far outside of the event horizon located at the Schwarzschild radius $r_g: \ r_s \gg r_g = 2GM_{r1}/c^2$.

The right-hand side of Eq. (6.4.106) is ≈ 1, and since inside the sonic coordinate r_s there is $u^r > v/c > a/c \ll 1$ via Eq. (5.12.98), we may approximate Eq. (6.4.106) by

$$1 + 2na^2/c^2 - 2GM_{r1}/c^2r + (u^r)^2 \approx 1 - 2GM_{r1}/c^2r + (u^r)^2 \approx 1 \quad \text{or} \quad u^r \approx (2GM_{r1}/c^2r)^{1/2},$$
$$(-\infty \leq n < -1;\ 3/2 < n \leq \infty;\ r \ll r_s). \tag{6.4.111}$$

At the Schwarzschild radius r_g we have $u^r = u^r_g \approx 1$. For the rest mass density run we get with Eqs. (6.4.110), (6.4.111): $\varrho/\varrho_\infty = 2^{-1/2}\lambda_s(GM_{r1}/a^2_\infty r)^{3/2}$, $\varrho_g/\varrho_\infty \approx \lambda_s(c/a_\infty)^3/4$ if $n \neq 3/2,\ r \ll r_s$. In the particular case $n = 3/2$ we get with $u^r_g \approx 0.782$ and $\lambda_s = 1/4$ via Eq. (6.4.110): $\varrho_g/\varrho_\infty \approx (c/a_\infty)^3/(16 \times 0.782)$, (Shapiro and Teukolsky 1983).

A combination of polytropic, spherically symmetric accretion onto black holes with an accretion powered outflowing wind has been considered by Das (2000, 2001). As it is to be expected on general grounds, the essential features of spherically symmetric accretion of a polytropic gas onto a black hole are not altered by its slow rotation or small linear velocity (Beskin and Pidoprygora 1995). The properties of some thin, disk-like polytropic accretion flows round black holes have been studied by Chakrabarti and Das (2001).

In the Newtonian case Theuns and David (1992) have included the effect of radiation pressure, obtaining solutions in closed form, in a similar way as outlined in Eqs. (6.3.69)-(6.3.87), (Yeh 1970). The equations of spherically symmetric Newtonian in- or outflow are given by the continuity equation (5.2.1), the energy equation [the polytropic equation of state (2.1.6)], and the nonstationary variant of the equation of motion (6.3.88):

$$Dv(r,t)/Dt = \partial v/\partial t + v\ \partial v/\partial r = -(1/\varrho)\ \partial P/\partial r - (GM/r^2)\left(1 - L/L_E + \int_{r_0}^{r} 4\pi\varrho r'^2\ dr'\right). \tag{6.4.112}$$

The integral in Eq. (6.4.112) represents the mass (self-gravity) of the flow, located between the surface radius r_0 of M and the radius r. The enhancement of Bondi accretion due to self-gravity of the flow can be important for stars immersed in cold dense clouds, for black holes, galaxies, and galactic clusters under a wide variety of astrophysical conditions (Chia 1978, 1979). This term will be henceforth ignored. The maximum luminosity attainable by an object in radiative and hydrostatic equilibrium is given by the Eddington limit

$$L_E = 4\pi cGM/\kappa = 1.2 \times 10^{38} M/M_\odot\ [\text{erg s}^{-1}], \tag{6.4.113}$$

if the opacity $\kappa = 0.40$ cm^2 g^{-1} is due to Thomson scattering in completely ionized hydrogen (Shapiro and Teukolsky 1983). Eq. (6.4.113) can be deduced at once from Eq. (6.1.3) if gas pressure is negligible: $dP_r/dP = 1$.

For stationary flows Eq. (6.4.112) can be integrated at once, if small changes in the mass and luminosity of the central object are ignored [cf. Eqs. (6.3.69), (6.3.70), (6.3.91)]:

$$v^2/2 + na^2 - f/r = na_C^2 = \text{const}, \qquad (n = 1/(\gamma-1);\ -\infty < n < -1;\ 0 < n < \infty;$$
$$f = GM(1 - L/L_E);\ a^2 = (1+1/n)P/\varrho;\ a_C^2 = (1+1/n)P_C/\varrho_C;\ P_C, \varrho_C = \text{const}), \tag{6.4.114}$$

$$v^2/2 + a^2 \ln\varrho - f/r = a^2 \ln\varrho_C = \text{const}, \qquad (n = 1/(\gamma-1) = \pm\infty;\ f = GM(1 - L/L_E);$$
$$a^2 = P/\varrho = K = \text{const};\ \varrho_C = \text{const}). \tag{6.4.115}$$

For computational convenience the integration constants have been written under the forms (6.4.114), (6.4.115). In the case of accretion flows P_C, ϱ_C, a_C are the values of pressure, density, and sound speed at infinity, where $v = 0$. Dimensionless values are introduced by

$$x = ra_C^2/f; \quad y = v/a_C; \quad z = a/a_C = (\varrho/\varrho_C)^{1/2n} \quad \text{if} \quad -\infty < n < -1,\ 0 < n < \infty;$$
$$x = ra^2/f; \quad y = v/a; \quad z = \varrho/\varrho_C \quad \text{if} \quad n = \pm\infty. \tag{6.4.116}$$

Eqs. (6.4.114) and (6.4.115) become, respectively:

$$y^2/2 + n(z^2 - 1) - 1/x = 0, \qquad (-\infty < n < -1;\ 0 < n < \infty), \tag{6.4.117}$$

$$y^2/2 + \ln z - 1/x = 0, \qquad (n = \pm\infty). \tag{6.4.118}$$

These two equations are supplemented by the equation of mass conservation (6.4.55), which reads in dimensionless variables as

$$x^2yz^{2n} = Aa_C^3/4\pi\varrho_C f^2 = \lambda = \mathrm{const} \quad \text{if} \quad -\infty < n < -1,\ 0 < n < \infty;$$
$$x^2yz = Aa^3/4\pi\varrho_C f^2 = \lambda = \mathrm{const} \quad \text{if} \quad n = \pm\infty. \tag{6.4.119}$$

Elimination of the dimensionless velocity among Eqs. (6.4.117)-(6.4.119) yields the two fourth order algebraic equations

$$x^4 - x^3/n(z^2 - 1) + \lambda^2/2nz^{4n}(z^2 - 1) = 0, \qquad (-\infty < n < -1;\ 0 < n < \infty), \tag{6.4.120}$$

$$x^4 - x^3/\ln z + \lambda^2/2z^2\ln z = 0, \qquad (n = \pm\infty), \tag{6.4.121}$$

with cumbersome solutions in closed form (cf. Eqs. (6.3.46)-(6.3.48), Bronstein and Semendjajew 1985, p. 185, Theuns and David 1992). Practically, four types of inflow/outflow solutions occur [cf. Fig. 6.4.7 for (i)-(iii)]: (i) The transonic flow passing through a sonic point $M_A = 1$, which is relevant for wind and accretion, both. (ii) Completely subsonic flow pertinent to accretion flows. (iii) Completely supersonic winds. (iv) Super-Eddington flow if $L > L_E$ in Eq. (6.4.112), with the net force acting on the flow f/r^2 directed outwards.

The global stability against radial perturbations of wind and accretion flows has been studied among others by Aikawa (1979), and Theuns and David (1992). Nonradial perturbations of accretion flows have been considered by Garlick (1979). There is general agreement that the transonic flow is stable, while subsonic accretion flows may become unstable, depending on the boundary conditions at the accretor's surface. These findings are concerned with the free oscillations in time of the flow, rather than with the *spatial* instability of forced oscillations due to some external source; in the latter case Bondi accretion is nonradially unstable if $r \to 0$ (Kovalenko and Eremin 1998). The instabilities occurring in *self-similar* collapse solutions are slightly related to this topic (Sec. 6.2.3, Hanawa and Matsumoto 2000a, b, Lai and Goldreich 2000, Lai 2000).

We derive at first a second order equation for the Eulerian perturbation $\delta F(r,t) = r^2(v\ \delta\varrho + \varrho\ \delta v)$ of the mass flux $F = r^2\varrho v$ from Eq. (6.3.10). The gravitational potential $\Phi = GM/r$, depending in this particular case solely on distance r, is an extrinsic attribute of the flow, rather than an intrinsic one (e.g. Chandrasekhar 1969, p. 29). Therefore $\delta\Phi = 0$, and the Eulerian perturbation (5.2.29) of the Eulerian equation of motion can be written under the from

$$\delta(D\vec{v}/Dt) = \partial\delta\vec{v}/\partial t + (\delta\vec{v}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\delta\vec{v} = -\delta[(1/\varrho)\ \nabla P] = -\nabla(\delta P/\varrho) = -\nabla(a^2\ \delta\varrho/\varrho), \tag{6.4.122}$$

because for our isentropic flow we have $\Gamma_1 = 1 + 1/n,\ \vec{A} = 0$, and $\delta[(1/\varrho)\ \nabla P] = \nabla(\delta P/\varrho)$ in virtue of Eqs. (5.2.73)-(5.2.86). The adiabatic energy equation (5.9.39) simplifies to

$$\delta P = (\Gamma_1 P/\varrho)\ \delta\varrho = (1 + 1/n)(P/\varrho)\ \delta\varrho = a^2\ \delta\varrho, \qquad (S = \mathrm{const};\ \nabla P = (\Gamma_1 P/\varrho)\ \nabla\varrho), \tag{6.4.123}$$

where $P = K\varrho^{1+1/n} = K\varrho^{\Gamma_1}$, and $a = (\Gamma_1 P/\varrho)^{1/2}$ denotes the adiabatic sound velocity (2.1.49).

From the equation of continuity (5.2.24) we deduce at once $r^2\ \partial\delta\varrho/\partial t = -\partial\delta F/\partial r$, and from $\delta F(r,t) = r^2(v\ \delta\varrho + \varrho\ \delta v)$ we find $\partial\delta F/\partial t = r^2\varrho\ \partial\delta v/\partial t + r^2 v\ \partial\delta\varrho/\partial t = r^2\varrho\ \partial\delta v/\partial t - v\ \partial\delta F/\partial r$. Because the mass flux F is constant in the unperturbed stationary flow, we can write $\nabla(a^2\ \delta\varrho/\varrho) = (1/r^2\varrho v)\ \partial(a^2r^2v\ \delta\varrho)/\partial r$ in Eq. (6.4.122), and derive with respect to the time:

$$r^2\varrho\ \partial^2\delta v/\partial t^2 + r^2\varrho(dv/dr)\ \partial\delta v/\partial t + r^2\varrho v\ \partial^2\delta v/\partial t\partial r + (1/v)\ \partial(a^2r^2v\ \partial\delta\varrho/\partial t)/\partial r = 0. \tag{6.4.124}$$

Inserting the derivatives of δF we get eventually

$$\partial^2\delta F/\partial t^2 + 2\ \partial(v\ \partial\delta F/\partial t)/\partial r + (1/v)\ \partial[v(v^2 - a^2)\ \partial\delta F/\partial r]/\partial r = 0. \tag{6.4.125}$$

Theuns and David (1992) write the perturbation under the form $\delta F(r,t) = \delta F(r)\exp(i\sigma t)$, where $\delta F(r)$ is a complex function of the real variable r, and σ is the angular oscillation frequency. Insertion into Eq. (6.4.125) yields

$$\sigma^2\ \delta F - 2i\sigma\ d(v\ \delta F)/dr - (1/v)\ d[v(v^2-a^2)\ d\delta F/dr]/dr = 0. \qquad (6.4.126)$$

We multiply with the complex conjugate $v\ \delta F^*(r)$ of $v\ \delta F(r)$, and separate the real and imaginary parts:

$$\sigma^2 v|\delta F|^2 + 2\sigma v^2\ \mathrm{Im}(\delta F^*\ d\delta F/dr) + v(v^2-a^2)|d\delta F/dr|^2 - (1/2)\ d[v(v^2-a^2)\ d|\delta F|^2/dr]/dr = 0,$$
$$(\delta F\ \delta F^* = |\delta F|^2;\ \mathrm{Re}(\delta F^*\ d\delta F/dr) = (1/2)\ d|\delta F|^2/dr), \qquad (6.4.127)$$

$$\sigma\ d(v^2|\delta F|^2)/dr + d[v(v^2-a^2)\ \mathrm{Im}(\delta F^*\ d\delta F/dr)]/dr = 0 \quad \text{or}$$
$$\sigma v^2|\delta F|^2 + v(v^2-a^2)\ \mathrm{Im}(\delta F^*\ d\delta F/dr) = C = \text{const.} \qquad (6.4.128)$$

We insert $\mathrm{Im}(\delta F^*\ d\delta F/dr)$ from Eq. (6.4.128) into Eq. (6.4.127), and integrate over a distance without singular points or discontinuities (shock waves):

$$\sigma^2\int_{r_1}^{r_2} v[(v^2+a^2)/(v^2-a^2)]|\delta F|^2\ dr - 2\sigma C\int_{r_1}^{r_2} v\ dr/(v^2-a^2) + D = 0;$$
$$D = (1/2)[v(v^2-a^2)\ d|\delta F|^2/dr]\Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} v(v^2-a^2)|d\delta F/dr|^2\ dr. \qquad (6.4.129)$$

For real values of σ, the discriminant of this second order equation has to be nonnegative, and its *sign* is independent of the sign and magnitude of v. Only the term

$$Q = -(1/2)[v(v^2-a^2)\ d|\delta F|^2/dr]\Big|_{r_1}^{r_2}\int_{r_1}^{r_2} v[(v^2+a^2)/(v^2-a^2)]|\delta F|^2\ dr, \qquad (6.4.130)$$

could make a negative contribution to the discriminant, so that a necessary and sufficient condition for radial stability of the flow is $Q \geq 0$.

(i) Subsonic accretion flow. If the perturbation is due to a standing wave [Eq. (5.1.32)], there are two radii r_1 and r_2 for which the perturbation $\delta F(r)$ vanishes at all times, and Eq. (6.4.130) yields $Q = 0$: The flow is stable. The same conclusion holds in virtue of Eq. (6.4.130) also if $d|\delta F|/dr = 0$ at r_1, r_2. If the medium is infinite and perturbed by radially travelling waves, the stability criterion (6.4.130) becomes

$$Q = (1/2)[v(v^2-a^2)\ d|\delta F|^2/dr]_{r=r_1}\int_{r_1}^{\infty} v[(v^2+a^2)/(v^2-a^2)]|\delta F|^2\ dr \geq 0. \qquad (6.4.131)$$

The flow could become unstable ($Q < 0$) only if $(d|\delta F|/dr)_{r=r_1} < 0$, which would mean that the perturbation is actually enhanced at the boundary layer $r = r_1$.

(ii) Transonic flow. One of the boundaries, say r_1, has to be taken in the sonic point r_s, $(v = a)$, since this point is a singular point of the flow. It suffices to show the regularity (continuity of the derivative) of $\delta F(r)$ at $r = r_s$, in order to prove the continuity of Q. We substitute the expansion

$$\delta F(r) = r^2(v\ \delta\varrho + \varrho\ \delta v) = (r-r_s)^\alpha[h_0 + h_1(r-r_s) + ...], \qquad (r \approx r_s;\ h_0, h_1 = \text{const}), \quad (6.4.132)$$

into Eq. (6.4.126), using the expansions of v and a near the sonic point r_s [Theuns and David 1992, Eq. (2.27)]. Equating the coefficients of equal powers of $r - r_s$, we get $\alpha = 0$ and $\alpha = -i\sigma r_s$ as admissible values of α. The exponent $\alpha = 0$ yields $\delta F(r) \approx h_0 = \text{const}$, while the other value gives $\delta F(r) \approx h_0(r-r_s)^{-i\sigma r_s} = h_0\exp[-i\sigma r_s\ln(r-r_s)] = h_0\exp[-i\sigma r_s\ln|r-r_s| + \pi\sigma r_s\ H(r_s-r)]$, by using the principal value of the natural logarithm of a complex number. The Heaviside function is equal to $H(r_s-r) = 1$ if $r_s - r > 0$, and $H(r_s-r) = 0$ if $r_s - r < 0$. Thus, δF is always regular in the singular sonic point, and the contribution to Q from the sonic point $v = a$ vanishes. Since for spherically symmetric accretion flows we have $v_\infty = 0$, the value of Q vanishes in the subsonic region between infinity and the sonic point r_s, which implies stability against radial modes. Garlick (1979) proved stability against nonradial modes. The supersonic region of the accretion flow is stable too, because any perturbation in the supersonic region between r_s and the accretor surface is carried away with the supersonic flow and reaches the accretor surface in a finite time; moreover, no information from the inner supersonic region $r < r_s$ reaches the outer subsonic region $r \geq r_s$ (Aikawa 1979, Garlick 1979).

In the case of outflowing winds the radial stability of the inner subsonic region between the origin of the wind ($v = 0$) and r_s, ($v = a$) is assured by the fact that $Q = 0$. And in the outer supersonic region the contribution to Q from the infinity point vanishes too, as can be seen from the asymptotic solution $\delta F = \exp(-i\sigma r/v_\infty)$ of Eq. (6.4.126) if $v \approx v_\infty = \mathrm{const}$, and $a^2 = (1 + 1/n)K\varrho^{1/n} \approx 0$ as $r \to \infty$. We have $|\delta F| = 1 = \mathrm{const}$, and $d|\delta F|^2/dr = 0$ if $r \to \infty$. Thus $Q = 0$, and polytropic winds are throughout stable against radially symmetric disturbances.

In the general relativistic regime Moncrief (1980) has performed a nonradial mode analysis on the *transonic* accretion flow solution, without finding unstable modes either in the subsonic or supersonic regions, in accordance to the Newtonian limit.

6.4.3 Polytropic Accretion Disks and Tori

The relevant difference between disks and tori is their extension in the equatorial (ℓ, φ)-plane of the system, i.e. $\ell_i \ll \ell_e$ in the case of disks, and $\ell_i \approx \ell_e$ for tori. ℓ_i and ℓ_e designates respectively, the distance – measured in the midplane $z = 0$ of the disk (torus) – between the central mass M and the interior (inner) and exterior (outer) border of the disk (torus). This subsection has some bearing on Sec. 5.10, focussed mainly on polytropic galactic (stellar system) disks.

Accretion disks are found round protostars (T-Tauri stars), interacting binary stars, galactic nuclei (Seyfert galaxies), and more speculatively, round quasars. Outflow disks due to the existence of stellar winds – occurring round a variety of binary and single (rapidly rotating) stars throughout the Hertzsprung-Russel diagram – are the counterparts of accretion disks (Bjorkman 1997). The accretion disks (tori) can be classified into a variety of ways: Thin and thick disks (geometrically or optically), semitransparent disks, disks with negligible or dominant self-gravity, gas or radiation pressure dominated disks, and various combinations of these groups (e.g. Hoshi 1981). We will use the notion of thick and thin disks always geometrically.

The term *polytropic* accretion disk is somewhat misleading, since the polytropic (isothermal) equation of state is generally used merely for a better approximation of the *vertical* structure of the disk. In cylindrical (ℓ, φ, z)-coordinates the gravity components of a central point mass M are

$$-GM\vec{r}/r^3 = \nabla\Phi_M = \nabla\Phi_M\big[-GM\ell/(\ell^2 + z^2)^{3/2},\ 0,\ -GMz/(\ell^2 + z^2)^{3/2}\big]. \tag{6.4.133}$$

The vertical gravity component of a self-gravitating thin disk without central point mass can be approximated with Eqs. (5.10.89), (5.10.101) as

$$\partial\Phi/\partial z = \int_0^z (\partial^2\Phi/\partial z'^2)\ dz' = -4\pi G \int_0^z \varrho\ dz' = -2\pi G\Sigma,$$
$$(z \geq 0;\ \partial\Phi/\partial z \gg \partial\Phi/\partial x,\ \partial\Phi/\partial y;\ (\partial\Phi/\partial z)_{z=0} = 0). \tag{6.4.134}$$

The vertical component of the equation of hydrostatic equilibrium (2.1.3) for a self-gravitating disk with central point mass may be approximated by adding to the gravitation of the central mass (6.4.133) the disk's self-gravitation (6.4.134):

$$dP/dz = -\varrho(GMz/r^3 + 2\pi G\Sigma), \qquad (z \geq 0). \tag{6.4.135}$$

This equation can be solved analytically in the two limiting cases $\Phi_M \ll \Phi$ and $\Phi_M \gg \Phi$ (Paczyński 1978a, b).

(i) $\boldsymbol{\Phi_M \ll \Phi}$. This case of overwhelming disk self-gravity has already been solved for the zero thickness disk by Eqs. (5.10.89)-(5.10.97). The hydrostatic equation (6.4.135) becomes $dP/dz = -2\pi G\Sigma\varrho$, turning after division with $d\Sigma/dz = 2\varrho$ into

$$dP/d\Sigma = -\pi G\Sigma. \tag{6.4.136}$$

Integration yields with the polytropic law $P = K\varrho^{1+1/n}$ the pressure and density run along the vertical extension of the thin accretion disk:

$$P = P_0 - \pi G\Sigma^2/2 = P_0(1 - \Sigma^2/\Sigma_1^2),$$
$$\big(P(0) = P_0;\ P(z_1) = P_1 = 0;\ \Sigma(0) = 0;\ \Sigma(z_1) = \Sigma_1;\ P_0 = \pi G\Sigma_1^2/2\big), \tag{6.4.137}$$

$$\varrho = (\varrho_0^{1+1/n} - \pi G\Sigma^2/2K)^{n/(n+1)} = \varrho_0(1 - \Sigma^2/\Sigma_1^2)^{n/(n+1)},$$
$$\big(n \neq -1;\ \varrho(0) = \varrho_0;\ \varrho(z_1) = 0;\ \varrho_0 = (\pi G\Sigma_1^2/2K)^{n/(n+1)}\big), \tag{6.4.138}$$

$$\Sigma = 2\int_0^z \varrho\ dz';\quad \Sigma_1 = 2\int_0^{z_1} \varrho\ dz'. \tag{6.4.139}$$

Zero indexed quantities denote values in the equatorial symmetry plane, and the total surface density of the thin disk is Σ_1. Pressure and density are assumed to vanish on the upper and lower boundary of the disk at z_1 and $-z_1$, respectively. The total surface density $\Sigma_1 = (2K/\pi G)^{1/2}\varrho_0^{(n+1)/2n} = (2P_0/\pi G)^{1/2}$ from Eq. (6.4.138) is identical to the surface density (5.10.94) of the zero thickness disk, and the vertically integrated pressure P_Σ of the thin accretion disk is the same as the cumulative pressure (5.10.95) acting in the equatorial plane of the zero thickness disk. The half-thickness of the disk may be calculated with the beta function (5.10.96), and the gamma function (C.9), (C.11):

$$z_1 = \int_0^{z_1} dz = \int_0^{\Sigma_1} d\Sigma/2\varrho = (\Sigma_1/2\varrho_0)\int_0^{\Sigma_1} (1 - \Sigma^2/\Sigma_1^2)^{-n/(n+1)}\ d\Sigma/\Sigma_1$$
$$= (\Sigma_1/4\varrho_0)\int_0^1 t^{-1/2}(1-t)^{-n/(n+1)}\ dt = (\Sigma_1/4\varrho_0)\ B[1/2, 1/(n+1)] = (\pi^{1/2}\Sigma_1/4\varrho_0)\ \Gamma[1/(n+1)]$$
$$/\Gamma[(n+3)/2(n+1)],\qquad (-1 < n < \infty;\ t = (\Sigma/\Sigma_1)^2;\ \Gamma(1/2) = \pi^{1/2}). \tag{6.4.140}$$

The mean vertical density in the thin disk is obtained from Eq. (6.4.140):

$$\varrho_m = \Sigma_1/2z_1 = 2\varrho_0\ \Gamma[(n+3)/2(n+1)]/\pi^{1/2}\Gamma[1/(n+1)],\qquad (-1 < n < \infty). \tag{6.4.141}$$

As it will turn out from Eq. (6.4.143), the isothermal accretion disk extends to infinity, and Eqs. (6.4.136)-(6.4.139) are valid in the limit $z_1 \to \infty$, $n = \pm\infty$, when $P = K\varrho$. Eq. (6.4.135) becomes $K\ d\varrho/dz = -2\pi G\varrho\Sigma$, and combining with $\varrho = \varrho_0(1 - \Sigma^2/\Sigma_1^2)$, we get

$$\pi G\Sigma_1^2\ dz/K = d\Sigma\big/(1 - \Sigma^2/\Sigma_1^2), \tag{6.4.142}$$

which integrates to

$$\Sigma = \Sigma_1[\exp(2\pi G\Sigma_1 z/K) - 1]\big/[\exp(2\pi G\Sigma_1 z/K) + 1] = \Sigma_1\tanh(\pi G\Sigma_1 z/K),\qquad (n = \pm\infty). \tag{6.4.143}$$

The equatorial density is $\varrho_0 = \pi G\Sigma_1^2/2K$ via Eq. (6.4.138), and within the heights $z = \pm K/\pi G\Sigma_1 = \pm\Sigma_1/2\varrho_0$ there are included $\Sigma/\Sigma_1 = \tanh(1) = 0.76$ parts of the disk mass.

(ii) $\boldsymbol{\Phi_M \gg \Phi}$. The structure equation (6.4.135) reads

$$dP/dz = (1 + 1/n)K\varrho^{1/n}\ d\varrho/dz = -GM\varrho z/r^3,\qquad (z \ll \ell \approx r \approx \mathrm{const}), \tag{6.4.144}$$

wherefrom we get (Paczyński 1991)

$$\varrho = [GMz_1^2/2(n+1)Kr^3]^n(1 - z^2/z_1^2)^n = \varrho_0(1 - z^2/z_1^2)^n,\qquad (n \neq -1, \pm\infty). \tag{6.4.145}$$

The half-thickness of the disk is therefore $z_1 = [2(n+1)K\varrho_0^{1/n}r^3/GM]^{1/2}$. Okazaki and Kato (1985) study one-armed oscillations under the particular assumption $z_1 = \mathrm{const}$, implying $\varrho(r) \propto r^{-3n}$, $(r \approx \ell;\ z \approx 0)$. The total surface density due to the density distribution (6.4.145) is

$$\Sigma_1 = 2\int_0^{z_1} \varrho\ dz = 2\varrho_0 z_1\int_0^{z_1} (1 - z^2/z_1^2)^n\ dz/z_1 = \varrho_0 z_1\int_0^1 t^{-1/2}(1-t)^n\ dt$$
$$= \varrho_0 z_1\ B(1/2, n+1) = \pi^{1/2}\varrho_0 z_1\ \Gamma(n+1)/\Gamma(n+3/2),\qquad (-1 < n < \infty;\ t = z^2/z_1^2). \tag{6.4.146}$$

From Eq. (6.4.145) we get for the pressure run

$$P = K\varrho^{1+1/n} = P_0(1 - z^2/z_1^2)^{n+1}, \qquad \big(P_0 = K^{-n}[GMz_1^2/2(n+1)r^3]^{n+1}\big), \tag{6.4.147}$$

and the integrated pressure (5.10.95) becomes

$$P_\Sigma = \int_{-z_1}^{z_1} P\ dz = 2P_0 z_1 \int_0^{z_1} (1 - z^2/z_1^2)^{n+1}\ dz/z_1 = P_0 z_1 \int_0^1 t^{-1/2}(1-t)^{n+1}\ dt$$
$$P_0 z_1\ B(1/2, n+2) = \pi^{1/2} P_0 z_1\ \Gamma(n+2)/\Gamma(n+5/2), \qquad (-2 < n < \infty;\ t = z^2/z_1^2). \tag{6.4.148}$$

The mean vertical density in thin accretion disks of negligible mass is via Eq. (6.4.146) equal to

$$\varrho_m = \Sigma_1/2z_1 = \pi^{1/2}\varrho_0\ \Gamma(n+1)/2\Gamma(n+3/2), \qquad (-1 < n < \infty). \tag{6.4.149}$$

The ratio ϱ_0/ϱ_m from Eqs. (6.4.141) and (6.4.149) has been calculated by Paczyński (1978a). For instance: $\varrho_0/\varrho_m = 1.0000, 1.6977, 2.1875, 2.7070$ in disks with negligible self-gravitation, and $\varrho_0/\varrho_m = 1.0000, 1.8395, 2.6221, 3.6429$ for self-gravitating disks, if the polytropic index is $n = 0, 1.5, 3$, and 5, respectively.

In the case of isothermal disks the hydrostatic equation (6.4.144) writes $d\varrho/\varrho = -GMz\ dz/Kr^3$, which integrates to

$$\varrho = \varrho_0 \exp(-GMz^2/2Kr^3), \qquad (n = \pm\infty;\ r \approx \text{const}). \tag{6.4.150}$$

The surface density within height z can be expressed with the error integral

$$\Sigma = 2\int_0^z \varrho\ dz' = 2\varrho_0 \int_0^z \exp(-GMz'^2/2Kr^3)\ dz' = (8Kr^3/GM)^{1/2}\varrho_0$$
$$\times \int_0^{(GM/2Kr^3)^{1/2}z} \exp(-t^2)\ dt = (2\pi Kr^3/GM)^{1/2}\varrho_0\ \mathrm{erf}[(GM/2Kr^3)^{1/2}z],$$
$$(n = \pm\infty;\ t = (GM/2Kr^3)^{1/2}z;\ \mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x \exp(-t^2)\ dt). \tag{6.4.151}$$

The total surface density becomes with $\mathrm{erf}(\infty) = 1$ equal to

$$\Sigma_1 = 2\int_0^\infty \varrho\ dz = (2\pi Kr^3/GM)^{1/2}\varrho_0, \qquad (n = \pm\infty), \tag{6.4.152}$$

and from the tables of the error function it is seen that within the density scale height $z = (2Kr^3/GM)^{1/2}$ there are included about 84% of the disk mass.

The overall problems associated with polytropic accretion disks (e.g. radial structure, energy transport and emission, stability, evolution) would require a separate book (e.g. Shapiro and Teukolsky 1983, Camenzind et al. 1986, Korycansky and Pringle 1995, Bjorkman 1997); as these items have no direct bearing on polytropes, they are omitted.

The remainder of this section deals with accretion tori: $\ell_i \approx \ell_e$. It has been long suspected (e.g. Tassoul 1978, Sec. 7.3; Chandrasekhar 1981, §67) that the Rayleigh criterion (5.10.1) is a necessary and sufficient stability criterion only for *axisymmetric* motions, and is a necessary but not sufficient stability condition for *nonaxisymmetric* disturbances in differentially rotating systems. Indeed, Papaloizou and Pringle (1984, 1985, 1987) have shown that some polytropic differentially rotating tori are subject to a violent nonaxisymmetric mode of instability, even when the Rayleigh condition $d[\ell^2\Omega(\ell)]/d\ell > 0$ is satisfied. Below, we briefly sketch their demonstration.

Let us consider a differentially rotating, axisymmetric, polytropic and isentropic ($\Gamma_1 = 1 + 1/n$) torus of small extent and negligible self-gravity. The torus is under the influence of the external potential (6.4.133): $\Phi_M = GM/(\ell^2 + z^2)^{1/2}$. The angular velocity $\Omega(\ell)$ is in virtue of Eq. (3.1.11) a function of ℓ alone. To make the problem analytically tractable, we consider a power law of rotation

$$\Omega = \Omega(\ell) = \Omega_0(\ell_0/\ell)^q, \qquad (q, \Omega_0, \ell_0 = \text{const}). \tag{6.4.153}$$

The equation of motion of the differentially rotating torus can be written in the condensed form (3.1.8): $(1/\varrho)\ \nabla P - \nabla\Phi_M - \Omega^2\ell\vec{e}_\ell = 0$. This integrates with the polytropic equation of state (2.1.6) to

$$(n+1)P/\varrho - \Phi_M - \Phi_f = C_1 = \text{const}, \qquad (n \neq -1, \pm\infty), \tag{6.4.154}$$

where the centrifugal potential

$$\Phi_f = \Omega_0^2 \ell_0^{2q} \ell^{2-2q}/(2-2q), \tag{6.4.155}$$

results from the integration of $d\Phi_f/d\ell = \Omega^2 \ell = \Omega_0^2 \ell_0^{2q} \ell^{1-2q}$ under the condition of a vanishing integration constant at infinity ($q > 1$).

The pressure and density maximum of the torus occurs in the symmetry plane $z = 0$, and is taken just at distance ℓ_0 from the rotation axis. The structure of the considered slender polytropic tori may be approximated by expanding the potentials round the density maximum $\ell = \ell_0,\ z = 0:$

$$\Phi_M \approx (GM/\ell_0)\{1 - (\ell - \ell_0)/\ell_0 + [2(\ell - \ell_0)^2 - z^2]/2\ell_0^2\};$$
$$\Phi_f = \Omega_0^2 \ell_0^2/(2-2q) + \Omega_0^2 \ell_0 (\ell - \ell_0) + (1-2q)\Omega_0^2 (\ell - \ell_0)^2/2. \tag{6.4.156}$$

Because we have assumed the density maximum $\varrho = \varrho_0$ at $(\ell_0, 0)$, there is in this point $(1/\varrho_0)\ \nabla P_0 = K(1+1/n)\varrho_0^{1/n-1}\ \nabla \varrho_0 = \nabla(\Phi_M + \Phi_f)_{\ell=\ell_0, z=0} = 0$ or $\Omega_0^2 = GM/\ell_0^3$, and Eq. (6.4.154) writes

$$(n+1)K\varrho^{1/n} + (GM/2\ell_0^3)[(2q-3)(\ell - \ell_0)^2 + z^2] = C_1 + (2q-3)GM/\ell_0(2q-2) = C = \text{const.} \tag{6.4.157}$$

From this equation we get the maximum density $\varrho_0 = [C/(n+1)K]^n$, and the density run in the polytropic torus:

$$\varrho = \varrho_0[1 - (\ell - \ell_0)^2/a^2 - z^2/(2q-3)a^2]^n, \qquad (\ell - \ell_0, z \ll \ell_0). \tag{6.4.158}$$

The quantity

$$a = [2(n+1)K\varrho_0^{1/n}\ell_0^3/(2q-3)GM]^{1/2} \ll \ell_0, \qquad (n \neq -1, \pm\infty;\ K\varrho_0^{1/n} = P_0/\varrho_0), \tag{6.4.159}$$

is just the maximum half-thickness of the torus in the midplane – the torus extending from $\ell_i = \ell_0 - a$ to $\ell_e = \ell_0 + a$.

The Rayleigh (Solberg-Høiland) stability criterion (5.10.1) takes a simple form for the power law distribution (6.4.153) of angular velocity:

$$\kappa^2 = (2\Omega/\ell)\ d(\ell^2\Omega)/d\ell = 2(2-q)\Omega_0^2(\ell_0/\ell)^{2q} > 0. \tag{6.4.160}$$

Papaloizou and Pringle (1984) include also the particular case of neutral (marginal) stability $\kappa = 0$. In this particular case there is $q = 2$, and the specific angular momentum $j = \ell^2\Omega(\ell) = \ell_0^2\Omega_0$ is constant over the whole torus. If $q = 2$, the equidensity (isopycnic) surfaces (6.4.158) – which coincide for the barotropic fluid with the isobaric ones – are concentric circles, and the torus is unstable to low order nonaxisymmetric modes, growing on a dynamical time scale (Blaes 1985, p. 563, Goldreich et al. 1986).

The Rayleigh criterion (6.4.160) is not satisfied if $q > 2$. If $3/2 \leq q < 2$, the equidensity surfaces (6.4.158) are concentric ellipses elongated in the ℓ-direction, having eccentricity $e = [2(2-q)]^{1/2}$. The eccentricity tends to unity as $q \to 3/2$, and the rotation rate becomes Keplerian: $\Omega = \Omega_0(\ell_0/\ell)^{3/2} = (GM/\ell^3)^{1/2}$.

The linearized nonaxisymmetric perturbation equations in the inertial frame are derived by assuming the Eulerian perturbations of velocity $\vec{v} = \vec{v}[0, \ell\Omega(\ell), 0]$, density ϱ, and pressure P under the form

$$\delta\vec{v}(\ell, \varphi, z, t) = \delta\vec{v}(\ell, z)\exp[i(\sigma t + k\varphi)]; \quad \delta\varrho(\ell, \varphi, z, t) = \delta\varrho(\ell, z)\exp[i(\sigma t + k\varphi)];$$
$$\delta P(\ell, \varphi, z, t) = \delta P(\ell, z)\exp[i(\sigma t + k\varphi)]. \tag{6.4.161}$$

The Lagrangian displacement vector $\Delta\vec{r} = \Delta\vec{r}(\Delta\ell, \ell\Delta\varphi, \Delta z)$ is decomposed analogously:

$$\Delta\ell(\ell, \varphi, z, t) = \Delta\ell(\ell, z)\exp[i(\sigma t + k\varphi)]; \quad \Delta\varphi(\ell, \varphi, z, t) = \Delta\varphi(\ell, z)\exp[i(\sigma t + k\varphi)];$$
$$\Delta z(\ell, \varphi, z, t) = \Delta z(\ell, z)\exp[i(\sigma t + k\varphi)]. \tag{6.4.162}$$

We insert Eqs. (6.4.161), (6.4.162) into Eq. (5.9.42), performing the relevant derivatives:

$$\Delta\ell(\ell, z) = \delta v_\ell(\ell, z)/i(\sigma + k\Omega); \quad \ell\ \Delta\varphi(\ell, z) = \delta v_\varphi(\ell, z)/i(\sigma + k\Omega) - \ell(d\Omega/d\ell)\ \delta v_\ell(\ell, z)/(\sigma + k\Omega)^2;$$
$$\Delta z(\ell, z) = \delta v_z(\ell, z)/i(\sigma + k\Omega). \tag{6.4.163}$$

For our isentropic torus the perturbed energy equation (5.9.39) assumes the simple form

$$\delta P = (\Gamma_1 P/\varrho)\ \delta\varrho = [(1+1/n)P/\varrho]\ \delta\varrho, \qquad (n \neq -1), \tag{6.4.164}$$

because of the vanishing convective Schwarzschild discriminant (5.2.84). This implies that the Eulerian perturbations of the pressure term $(1/\varrho)\ \nabla P$ can be transformed via Eq. (6.4.164): $\delta[(1/\varrho)\ \nabla P] = -(\delta\varrho/\varrho^2)\ \nabla P + (1/\varrho)\ \nabla\delta P = -(\delta\varrho/\varrho^2)\ \nabla P + (\delta P/\varrho^2)\ \nabla\varrho + \nabla(\delta P/\varrho) = (\delta P/\varrho)[(1/\varrho)\ \nabla\varrho - (1/\Gamma_1 P)\ \nabla P] + \nabla(\delta P/\varrho) = \vec{A}\ \delta P/\varrho + \nabla(\delta P/\varrho) = \nabla(\delta P/\varrho)$. Also, the Eulerian perturbation $\delta\Phi_M$ of the gravitational potential – depending solely on location – vanishes, and the relevant equations of motion and continuity (5.9.35)-(5.9.38) become

$$\partial\delta v_\ell/\partial t + \Omega\ \partial\delta v_\ell/\partial\varphi - 2\Omega\ \delta v_\varphi = -\partial(\delta P/\varrho)/\partial\ell, \tag{6.4.165}$$

$$\partial\delta v_\varphi/\partial t + \Omega\ \partial\delta v_\varphi/\partial\varphi + (\delta v_\ell/\ell)\ d(\ell^2\Omega)/d\ell = -(1/\ell)\ \partial(\delta P/\varrho)/\partial\varphi, \tag{6.4.166}$$

$$\partial\delta v_z/\partial t + \Omega\ \partial\delta v_z/\partial\varphi = -\partial(\delta P/\varrho)/\partial z, \tag{6.4.167}$$

$$\partial\delta\varrho/\partial t + \Omega\ \partial\delta\varrho/\partial\varphi + \nabla\cdot(\varrho\ \delta\vec{v}) = 0. \tag{6.4.168}$$

We insert the perturbations (6.4.161), and introduce the new variable $W = \delta P/\varrho(\sigma + k\Omega)$:

$$i(\sigma + k\Omega)\ \delta v_\ell - 2\Omega\ \delta v_\varphi = -\partial(\delta P/\varrho)/\partial\ell = -\partial[(\sigma + k\Omega)W]/\partial\ell, \tag{6.4.169}$$

$$i(\sigma + k\Omega)\ \delta v_\varphi + (\delta v_\ell/\ell)\ d(\ell^2\Omega)/d\ell = -ik\ \delta P/\varrho\ell = -ik(\sigma + k\Omega)W/\ell, \tag{6.4.170}$$

$$i(\sigma + k\Omega)\ \delta v_z = -\partial(\delta P/\varrho)/\partial z = -\partial[(\sigma + k\Omega)W]/\partial z, \tag{6.4.171}$$

$$i(\sigma + k\Omega)\ \delta\varrho + (1/\ell)\ \partial(\varrho\ell\ \delta v_\ell)/\partial\ell + ik\varrho\ \delta v_\varphi/\ell + \partial(\varrho\ \delta v_z)/\partial z = 0. \tag{6.4.172}$$

After some algebra these equations can be expressed in terms of the Lagrangian displacements with the aid of the transformation formulas (5.9.43), (6.4.163):

$$[(\sigma + k\Omega)^2 - \kappa^2]\ \Delta\ell = \partial[(\sigma + k\Omega)W]/\partial\ell + 2k\Omega W/\ell, \tag{6.4.173}$$

$$[(\sigma + k\Omega)^2 - \kappa^2]\ell\ \Delta\varphi = 2i\Omega\ \partial W/\partial\ell + ik(\sigma + k\Omega)W/\ell, \tag{6.4.174}$$

$$(\sigma + k\Omega)^2\ \Delta z = \partial[(\sigma + k\Omega)W]/\partial z, \tag{6.4.175}$$

$$\delta\varrho + \nabla\cdot(\varrho\ \Delta\vec{r}) = 0. \tag{6.4.176}$$

We insert into Eq. (6.4.176) for $\delta\varrho = (\sigma + k\Omega)\varrho^2 W/(1+1/n)P$ from Eq. (6.4.164), and for $\Delta\vec{r}$ from Eqs. (6.4.173)-(6.4.175), to obtain the basic eigenvalue equation for the nonaxisymmetric stability of a differentially rotating, isentropic polytropic torus:

$$\begin{aligned}&(\sigma + k\Omega)^2\varrho^2 W/(1+1/n)P + (1/\ell)\ \partial\{\varrho\ell(\sigma + k\Omega)^2(\partial W/\partial\ell)/[(\sigma + k\Omega)^2 - \kappa^2]\}/\partial\ell\\ &-k^2(\sigma + k\Omega)^2\varrho W/\ell^2[(\sigma + k\Omega)^2 - \kappa^2] + [k(\sigma + k\Omega)W/2\ell]\ \partial\{\varrho\kappa^2/\Omega[(\sigma + k\Omega)^2 - \kappa^2]\}/\partial\ell\\ &+\partial(\varrho\ \partial W/\partial z)/\partial z = 0, \qquad (n \neq -1).\end{aligned} \tag{6.4.177}$$

The Rayleigh criterion (6.4.160) for nonaxisymmetric oscillations can be derived at once from this equation in the high wavenumber limit, by looking for solutions of the form

$$W(\ell, z) = W_0 \exp[i(j_\ell\ell + j_z z)], \qquad (W_0 = \mathrm{const};\ j_\ell, j_z \to \infty). \tag{6.4.178}$$

Preserving in Eq. (6.4.177) only the leading terms, we get the dispersion relation

$$\varrho W_0\{(\sigma+k\Omega)^2 j_\ell^2/[(\sigma+k\Omega)^2-\kappa^2]+j_z^2\}=0 \quad \text{or} \quad (\sigma+k\Omega)^2=\kappa^2 j_z^2/(j_\ell^2+j_z^2). \tag{6.4.179}$$

Stability requires $(\sigma+k\Omega)^2>0$, i.e. $\kappa^2>0$, with the eigenvalue spectrum in the range $-\kappa-k\Omega<\sigma<\kappa-k\Omega$, $(0<j_z^2/(j_\ell^2+j_z^2)<1)$.

The nonaxisymmetric instability is put best into evidence for modes which are even functions of the height z, although it will turn out that these modes are essentially independent of z. To make Eq. (6.4.177) analytically tractable, it is further assumed that $(\sigma+k\Omega)^2\ll\kappa^2$, so this equation transforms with the substitution $Q=(\sigma+k\Omega)W=\delta P/\varrho$ into

$$(\sigma+k\Omega)\varrho^2 Q/(1+1/n)P-(1/\ell)\ \partial\{[\varrho\ell(\sigma+k\Omega)^2/\kappa^2]\ \partial[Q/(\sigma+k\Omega)]/\partial\ell\}/\partial\ell$$
$$+k^2(\sigma+k\Omega)\varrho Q/\kappa^2\ell^2-(kQ/2\ell)\ \partial(\varrho/\Omega)/\partial\ell+[1/(\sigma+k\Omega)]\ \partial(\varrho\ \partial Q/\partial z)/\partial z=0. \tag{6.4.180}$$

Performing the derivative of $\sigma+k\Omega(\ell)$, we get with Eq. (6.4.160):

$$(\sigma+k\Omega)^2\big\{\varrho^2 Q/(1+1/n)P-(1/\ell)\ \partial[(\varrho\ell/\kappa^2)\ \partial Q/\partial\ell]/\partial\ell+k^2\varrho Q/\kappa^2\ell^2\big\}$$
$$-[2k(\sigma+k\Omega)Q/\ell]\ \partial(\Omega\varrho/\kappa^2)/\partial\ell+\partial(\varrho\ \partial Q/\partial z)/\partial z=0. \tag{6.4.181}$$

With the new variable $x=\ell-\ell_0\ll\ell_0$ this equation becomes approximately

$$(\sigma+k\Omega)^2\big[\varrho^2 Q/(1+1/n)P-(1/\kappa^2)\ \partial(\varrho\ \partial Q/\partial x)/\partial x+k^2\varrho Q/\kappa^2\ell_0^2\big]$$
$$-[2k\Omega_0(\sigma+k\Omega)Q/\kappa^2\ell_0]\ \partial\varrho/\partial x+\partial(\varrho\ \partial Q/\partial z)/\partial z=0, \qquad (n\neq-1), \tag{6.4.182}$$

where via Eq. (6.4.160) $\kappa^2\approx 2(2-q)\Omega_0^2=\text{const}$, $\ell\approx\ell_0=\text{const}$, and $\Omega\approx\Omega_0=\text{const}$, excepting in the term $\sigma+k\Omega$.

Solutions of this equation are sought under the form

$$Q(x,z)=\sum_{\alpha=0}^{\infty}U_\alpha(x)\ V_\alpha(x,z). \tag{6.4.183}$$

To get rid of $\partial Q/\partial z$, the functions $V_\alpha(x,z)$ are chosen to obey the differential equation

$$\partial(\varrho\ \partial V_\alpha/\partial z)/\partial z=-\lambda_\alpha\varrho^2 V_\alpha/P, \qquad (\lambda_\alpha=\text{const}). \tag{6.4.184}$$

They are also specified to be orthogonal functions of z with weight ϱ^2/P :

$$\int_{-z_1}^{z_1}(\varrho^2/P)\ V_\alpha(x,z)\ V_\beta(x,z)\ dz=\delta_{\alpha\beta}N_\alpha(x). \tag{6.4.185}$$

$N_\alpha(x)$ is referred to as a suitable normalization function, and $\delta_{\alpha\beta}$ is the Kronecker delta, while the integration is considered for constant x, $(-a\leq x\leq a)$ between the lower and upper height $z_1=\mp[(2q-3)(a^2-x^2)]^{1/2}$, $(q>3/2)$ of the torus, resulting from the condition $\varrho=0$ in Eq. (6.4.158).

With the new dimensionless variable $\zeta=z/z_1$, Eq. (6.4.158) becomes

$$\varrho=\varrho_0[(a^2-x^2)/a^2]^n(1-\zeta^2)^n, \qquad (n\neq-1,\pm\infty), \tag{6.4.186}$$

and Eq. (6.4.184) reads

$$\partial[(1-\zeta^2)^n\ \partial V_\alpha/\partial\zeta]/\partial\zeta+\lambda_\alpha a^2\varrho_0(2q-3)(1-\zeta^2)^{n-1}V_\alpha/P_0=0, \qquad (P_0=K\varrho_0^{1+1/n};\ x=\text{const}). \tag{6.4.187}$$

After simplification with $(1-\zeta^2)^{n-1}$ this equation turns into Eq. (3.10.156) for the Gegenbauer polynomials $G_\alpha^{n-1/2}(\zeta)\propto V_\alpha(x,z)$ of order α and index $n-1/2$ with the eigenvalue $\lambda_\alpha=\alpha(\alpha+2n-1)P_0/(2q-3)\varrho_0 a^2$.

We now multiply Eq. (6.4.182) by V_β and integrate with respect to z, taking into account Eqs. (6.4.183)-(6.4.185) with the summation convention over the repeated index α :

$$[(\sigma+k\Omega)^2/(1+1/n)-\lambda_\beta]N_\beta U_\beta+[(\sigma+k\Omega)^2/\kappa^2]\Big\{-U_\alpha\int_{-z_1}^{z_1}V_\beta[\partial(\varrho\ \partial V_\alpha/\partial x)/\partial x]\ dz$$
$$-\partial\Big[(\partial U_\alpha/\partial x)\int_{-z_1}^{z_1}\varrho V_\alpha V_\beta\ dz\Big]\Big/\partial x+(k^2/\ell_0^2)U_\alpha\int_{-z_1}^{z_1}\varrho V_\alpha V_\beta\ dz\Big\}$$
$$-[2k\Omega_0(\sigma+k\Omega)/\kappa^2\ell_0]U_\alpha\int_{-z_1}^{z_1}V_\alpha V_\beta(\partial\varrho/\partial x)\ dz=0,\qquad(\alpha,\beta=0,1,2,...).\tag{6.4.188}$$

In the zeroth approximation the basic function is independent of z :

$$Q=Q(x)=U_0(x)\ V_0,\qquad(V_0(x,z)\propto G_0^{n-1/2}(\zeta)=1;\ \lambda_0=0).\tag{6.4.189}$$

Let us consider Eq. (6.4.188) with $\alpha,\beta=0$:

$$[(\sigma+k\Omega)^2U_0/(1+1/n)]\int_{-z_1}^{z_1}\varrho^2\ dz/P+[(\sigma+k\Omega)^2/\kappa^2][-d(\Sigma\ dU_0/dx)/dx+k^2\Sigma U_0/\ell_0^2]$$
$$-[2k\Omega_0(\sigma+k\Omega)U_0/\kappa^2\ell_0]\ d\Sigma/dx=0.\tag{6.4.190}$$

The surface density is via Eqs. (5.10.96), (6.4.186) equal to

$$\Sigma(x)=\int_{-z_1}^{z_1}\varrho(x,z)\ dz=2\varrho_0a(2q-3)^{1/2}[(a^2-x^2)/a^2]^{n+1/2}\int_0^1(1-\zeta^2)^n\ d\zeta$$
$$=\varrho_0a(2q-3)^{1/2}[(a^2-x^2)/a^2]^{n+1/2}\int_0^1t^{-1/2}(1-t)^n\ dt$$
$$=\varrho_0a(2q-3)^{1/2}[(a^2-x^2)/a^2]^{n+1/2}B(1/2,n+1),\qquad(-1<n<\infty;\ q>3/2).\tag{6.4.191}$$

Similarly, we find with Eqs. (5.10.96), (6.4.159), (6.4.186), (C.11):

$$\int_{-z_1}^{z_1}\varrho^2\ dz/P=(2a\varrho_0^{1-1/n}/K)(2q-3)^{1/2}[(a^2-x^2)/a^2]^{n-1/2}\int_0^1(1-\zeta^2)^{n-1}\ d\zeta$$
$$=[2(n+1)\ell_0^3\varrho_0/(2q-3)GMa][(a^2-x^2)/a^2]^{n-1/2}\int_0^1t^{-1/2}(1-t)^{n-1}\ dt=[2(n+1)\Sigma$$
$$/(2q-3)\Omega_0^2(a^2-x^2)]\ B(1/2,n)/B(1/2,n+1)=(n+1)(2n+1)\Sigma/n(2q-3)\Omega_0^2(a^2-x^2),$$
$$(0<n<\infty;\ q>3/2;\ GM/\ell_0^3=\Omega_0^2;\ B(1/2;n)/B(1/2,n+1)=(2n+1)/2n).\tag{6.4.192}$$

Then, Eq. (6.4.190) writes as

$$d(\Sigma\ dU_0/dx)/dx+U_0\big[-(2n+1)\kappa^2\Sigma/(2q-3)\Omega_0^2(a^2-x^2)-k^2\Sigma/\ell_0^2$$
$$+2k\Omega_0(d\Sigma/dx)/(\sigma+k\Omega)\ell_0\big]=0,\qquad(0<n<\infty;\ q>3/2).\tag{6.4.193}$$

This equation possesses a regular singularity at the corotation surface $\sigma+k\Omega=\sigma+k\varphi/t=0$, when there occurs neutral stability of the perturbations (6.4.161) and (6.4.162), (cf. Sec. 5.9.3). This singularity can be avoided, if we have concomitantly $d\Sigma/dx=-2(n+1/2)\Sigma x/(a^2-x^2)=0$ in virtue of Eq. (6.4.191). This condition is nontrivially fulfilled at $x=0$.

We expand $\sigma+k\Omega=\sigma+k\Omega_0[(\ell_0+x)/\ell_0]^{-q}\approx\sigma+k\Omega_0-kq\Omega_0x/\ell_0$ near $x/\ell_0\approx0$, and observe that $\kappa^2\approx2(2-q)\Omega_0^2$ via Eq. (6.4.160). With these approximations Eq. (6.4.193) can be rewritten under the form

$$d(\Sigma\ dU_0/dx)/dx-[(2n+1)\Sigma U_0/(a^2-x^2)]\big\{2(2-q)/(2q-3)+k^2(a^2-x^2)/(2n+1)\ell_0^2$$
$$+2x/[(\sigma+k\Omega_0)\ell_0/k\Omega_0-qx]\big\}=0,\qquad(0<n<\infty;\ q>3/2).\tag{6.4.194}$$

If $q\geq2$, the Rayleigh criterion (6.4.160) is not satisfied, and the polytropic torus is unstable to axisymmetric perturbations. If $q\not\approx2$ and k sufficiently small, the second term in the braces $k^2(a^2-x^2)/(2n+1)\ell_0^2$ can be neglected with respect to the first one. If $x\approx0$, the condition of neutral (marginal) stability $\sigma + k\Omega = 0$ becomes $\sigma + k\Omega_0 \approx 0$. In this case Eq. (6.4.194) turns for the neutral modes $\sigma + k\Omega_0 \approx 0$ – by using the derivative of Eq. (6.4.191) – into the Gegenbauer equation

$$(a^2 - x^2)\ d^2U_0/dx^2 - (2n+1)x\ dU_0/dx - 2(2n+1)(3-q^2)U_0/q(2q-3) = 0. \tag{6.4.195}$$

Comparing with Eq. (3.10.156) we observe that solutions in terms of Gegenbauer polynomials $U_0(x) \propto G_\gamma^n(x/a)$ exist only if

$$\gamma(\gamma+2n) = 2(2n+1)(q^2-3)/q(2q-3), \qquad (0 < n < \infty;\ q > 3/2). \tag{6.4.196}$$

$\gamma = 0$ and $\gamma = 1$ are the relevant values of γ for which a value of q exists within the interval $(3/2, 2]$. If $\gamma = 1$, we get $q = 2$, and the second term in the brace of Eq. (6.4.194) cannot be neglected with respect to the first one. We are left with $\gamma = 0$, implying via Eq. (3.10.155) $U_0 \propto G_0^n = 1$, and by Eq. (6.4.196) $q = 3^{1/2}$. In the neighborhood of the neutral mode there occurs stability if $q < 3^{1/2}$, and instability if $q > 3^{1/2}$ (Papaloizou and Pringle 1985). Thus – although the Rayleigh criterion (6.4.160) is satisfied – polytropic tori with $3^{1/2} < q < 2$ are subject to nonaxisymmetric instabilities in the fundamental mode, driven by coupling of edge waves (Goldreich et al. 1986).

Further investigations on the modes of differentially rotating polytropic tori and disks have been effected for instance by Blaes (1985), Loska (1986), and Jaroszyński (1986). Christodoulou et al. (1997) study the intrinsic stability of zero-thickness galaxy rings (gravity g-modes, so-called I, J, L-modes), adopting an artificial external potential $\Phi_e \propto \ln \ell$, the polytropic pressure law (5.10.97) with $0.5 \le n_\Sigma \le 5$, and the power-law rotation profile (6.4.153) with $q = 1$.

Hachisu et al. (1988) have investigated the stability of rotating equilibrium tori with polytropic indices $n = 0, 0.5, 1, 1.5, 2.5, 3$, and initial specific angular momentum distribution similar to that of the contracting homogeneous sphere from Eq. (3.5.4). A dimensionless quantity

$$F = (K^n G^{3-2n} M^{10-4n} J^{2n-6})^{1/(n+1)}, \tag{6.4.197}$$

is introduced, which is composed of the polytropic constant K (specifying the entropy S), the gravitational constant G, the total mass M, and the total angular momentum $J = f_J MRv$, where v is the average rotation velocity over the total mass, R the total radius, and f_J a dimensionless numerical factor of order unity, depending on the polytropic structure.

In fact, an initial angular momentum distribution like that from Eq. (3.5.4) cannot be an equilibrium structure, because it belongs to a sphere, and a rotating cloud must evolve to a flattened configuration in order to acquire equilibrium. The internal energy U of a polytropic equilibrium configuration can be expressed by an integral like Eq. (2.6.95), which turns into $U = f_U \overline{P} M/\overline{\varrho} = f_U K \overline{\varrho}^{1/n} M = f_U' K \varrho_m^{1/n} M = f_U'' K M^{1+1/n} R^{-3/n}$, where $\overline{P} = K\overline{\varrho}^{1+1/n}$ and $\overline{\varrho}$ are average values of pressure and density over the total mass, $\varrho_m \propto M/R^3$ is the mean density of the configuration, and f_U, f_U', f_U'' are coefficients of order unity. Likewise, the rotational kinetic energy is $E_{kin} = f_E M v^2$, and the gravitational energy is according to Eq. (2.6.137) equal to $|W| = f_W G M^2/R$, where f_W is again a factor of order unity.

If the rotating polytrope contracts in a homologous fashion with local and global conservation of mass and angular momentum, the ratios

$$\alpha = U/|W| = f_U'' f_W^{-1} K G^{-1} M^{1/n-1} R^{-3/n+1} = f_U'' f_J^{3/n-1} f_W^{-1} K G^{-1} M^{4/n-2} J^{-3/n+1} v^{3/n-1};$$
$$\tau = E_{kin}/|W| = f_E f_W^{-1} G^{-1} M^{-1} R v^2 = f_E f_W^{-1} f_J^{-1} G^{-1} M^{-2} J v, \tag{6.4.198}$$

change in terms of the initial conditions, denoted by zero subscripts, as

$$\alpha = \alpha_0 (R/R_0)^{(n-3)/n}; \quad \tau = \tau_0 (v/v_0)^2 (R/R_0) = \tau_0 (R/R_0)^{-1}, \tag{6.4.199}$$

where $v/v_0 = R_0/R$ by conservation of angular momentum, and the f-coefficients are approximately constant during contraction. From Eq. (6.4.199) we observe that $\alpha\tau^{(n-3)/n} = \text{const}$ (Miyama 1992). This relationship is equivalent to $F = \text{const}$ from Eq. (6.4.197), if we insert for α, τ from Eq. (6.4.198). Since M, J are assumed constant, the quantity F must remain constant during an axisymmetric, adiabatic collapse.

If a collapsed equilibrium state is found with the same distribution of angular momentum as the initial state, it can be considered as a final state of cloud collapse, and if this state is unstable to fragmentation, it may be conjectured that the cloud fragments into pieces.

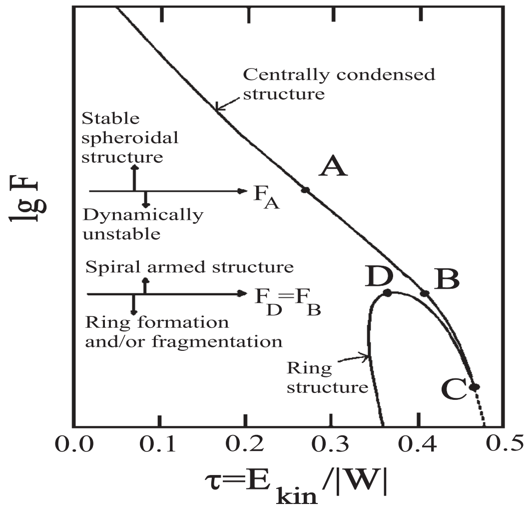

Fig. 6.4.8 Qualitative features of a polytropic model sequence if $0 \leq n \leq 3$. Dynamical instability towards nonaxisymmetric disturbances sets in at point A. Dynamical, axisymmetric, nonlinear ring mode instability sets in at point B in the spheroidal sequence – at the same F-value where the toroidal sequence attains its maximum at point D, ($F_B = F_D$). Point C is the bifurcation point from the spheroidal to the toroidal sequence (Hachisu et al. 1988).

In the well studied particular case of constant density polytropes $n = 0$, the Maclaurin spheroids become secularly unstable to nonaxisymmetric barlike perturbations (sectorial modes) at the first point of bifurcation, where the secularly and dynamically stable Jacobi sequence branches off and $\tau = E_{kin}/|W| = 0.1375$ (Secs. 5.7.4, 5.8.2, Christodoulou et al. 1995a). The first dynamical instability point towards nonaxisymmetric barlike distortions sets in at $\tau = 0.2738$. If strict axisymmetry is maintained, the Maclaurin spheroids become secularly unstable to axisymmetric ring deformations at $\tau = 0.3589$, where the Dyson-Wong toroids (Dyson 1892, 1893, Wong 1974) bifurcate from the Maclaurin sequence. Dynamical instability of Maclaurin spheroids to axisymmetric (ringlike) perturbations seems to set in $\tau = 0.4512$, rather than at $\tau = 0.4574$, as previously thought (Ostriker and Peebles 1973, p. 468, Hachisu et al. 1987, Table 4). And at $\tau = 0.4512$ the Maclaurin toroidal sequence branches off from the Maclaurin spheroidal sequence. The homogeneous Dyson-Wong toroids rotate throughout with constant angular velocity (rigid rotation), and do not have the same angular momentum distribution with mass as the Maclaurin toroids, which possess the same angular momentum distribution with mass as the original Maclaurin spheroids, i.e. the inner edge of Maclaurin toroids is nonrotating, for instance.

Fig. 6.4.8 schematically highlights the primary features of polytropic equilibrium sequences as calculated by Hachisu et al. (1988). The value of F from Eq. (6.4.197) determines the fate of the collapsing cloud. If $F > F_A$ (point A in Fig. 6.4.8), the cloud evolves to a dynamically (not necessarily secularly) stable state, and if $F < F_D$ (point D of toroidal structure in Fig. 6.4.8) the configuration is toroidal, rather than spheroidal, because τ for the toroidal sequence is smaller than for the spheroidal sequence ($F_B = F_D$). And finally, at F_C the linear bifurcation point appears from which the toroidal sequence starts. Hachisu et al. (1987, 1988) conjecture that $F < F_D$, ($\tau > 0.41$) is a condition for ring fragmentation, like $\kappa^2/\pi G\varrho_0 < 1$, where ϱ_0 is the equatorial density. For all considered polytropic indices $0 \leq n \leq 3$ with the angular momentum distribution (3.5.4) there bifurcates at $\tau \approx 0.45$ a toroidal sequence from the spheroidal sequence, like in the uniform density case $n = 0$. These results refer to axisymmetric disturbances, while Ostriker and Bodenheimer (1973) found a value of $\tau \approx 0.26$ if $n = 1.5, 3$ for the $k = 2$, $[\Delta r \propto P_j^k(\mu)\exp(ik\varphi)]$ nonaxisymmetric dynamical instability of spheroidal configurations (cf. Sec. 3.8.4, Fig. 3.8.2) – a result confirmed by the numerical work of Hachisu et al. (1988).

If the angular momentum is *constant* throughout the structure, the calculated sequences are always toroidal, and the nonaxisymmetric dynamical instability points F_A are estimated to occur at $\tau \gtrsim 0.14$, rather than at $\tau \approx 0.26$ (Hachisu et al. 1988). Indeed, for differential rotation laws of the form $\Omega \propto \ell^{-q}$, $(1.5 \leq q \leq 2)$ self-gravitating $n = 1.5$ tori are dynamically unstable to nonaxisymmetric (ellipsoidal) disturbances if $\tau > 0.16$, rather than if $\tau > 0.26$, as for the previously mentioned spheroidal models (Tohline and Hachisu 1990). But all $n = 1.5$ tori in the range $0.16 \leq \tau \leq 0.27$ do not exhibit fragmentation: A central ellipsoid is formed, surrounded by a disk of high specific angular momentum. The dynamical instability driven by the self-gravity of the torus is distinctly different from the previously considered Papaloizou-Pringle instability in disks with negligible self-gravity [Eqs. (6.4.153)-(6.4.196)].

Transport of mass and angular momentum has been investigated for a special self-gravitating disk with initial central mass to disk mass ratio of 1.5, and polytropic indices n, n_Σ=1.5 (Laughlin et al. 1998). The primary unstable nonaxisymmetric $k = 2$ mode saturates and evolves into a quasi-steady two-armed spiral due to nonlinear mode coupling.

Woodward et al. (1994) have put into evidence four different types of nonaxisymmetric eigenmodes (one of which is related to the Papaloizou-Pringle instability) in $n = 1.5$ disks with constant specific angular momentum, and central masses having 0.2–5 disk masses.

The tidal disruption of a spherical polytropic star due to a surrounding massive ring (disk) takes always place if $0.292 < n < 3$, provided that the star extends up to the point where the total potential has a local minimum (axisymmetric Roche problem). If $n < 0$ or $n > 3$, tidal stripping of the stellar surface never occurs, and in the intermediate region $0 < n < 0.292$ runaway excretion of the star depends on the mass ratio between star and ring (Woodward et al. 1992).

Doubtlessly, with the continuing computerization of science countless polytropic models will be calculated, but probably without reaching the generality, elegance, and deep insight of earlier analytical work. Obviously, polytropes are amply spread out over the whole astrophysical literature. If the author would have been aware of this fact, this book would not have been written.

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Appendix A

Principal Constants Used in this Book (Cox and Giuli 1968, Gerthsen et al. 1977)

Atomic mass unit	$H = 1/N_A = m(^{12}\mathrm{C})/12 = 1.66055 \times 10^{-24}$ g
Avogadro number	$N_A = (6.02217 \pm 4) \times 10^{23}$ mole^{-1}
Boltzmann constant	$k = (1.38062 \pm 6) \times 10^{-16}$ erg K^{-1}
Gas constant	$\mathcal{R} = kN_A = (8.3143 \pm 3) \times 10^{7}$ erg K^{-1} mole^{-1}
Gravitational constant	$G = (6.673 \pm 3) \times 10^{-8}$ erg cm g^{-2}
Planck's constant	$h = (6.62620 \pm 5) \times 10^{-27}$ erg s
Radiation pressure constant	$a = 8\pi^5 k^4/15c^3h^3 = (7.5647 \pm 10) \times 10^{-15}$ erg cm^{-3} K^{-4}
Rest mass of electron	$m_e = (9.10956 \pm 5) \times 10^{-28}$ g
Rest mass of hydrogen atom	$^1\mathrm{H} = 1.6733 \times 10^{-24}$ g
Rest mass of neutron	$m_n = 1.67482 \times 10^{-24}$ g
Rest mass of proton	$m_p = (1.67261 \pm 1) \times 10^{-24}$ g
Solar effective temperature	$T_\odot = 5800$ K
Solar luminosity	$L_\odot = 3.90 \times 10^{33}$ erg s^{-1}
Solar mass	$M_\odot = 1.989 \times 10^{33}$ g
Solar radius	$r_\odot = 6.960 \times 10^{10}$ cm
Stefan-Boltzmann constant	$\sigma = ac/4 = (5.6696 \pm 10) \times 10^{-5}$ erg cm^{-2} K^{-4} s^{-1}
Velocity of light in vacuo	$c = (2.997925 \pm 1) \times 10^{10}$ cm s^{-1}

Appendix B

Some Vector Differentials in Orthogonal Curvilinear Coordinates

The square of the line element ds in N-dimensional curvilinear space is a scalar of the form (e.g. Landau and Lifschitz 1987)

$$ds^2 = \sum_{i=1}^{N}\sum_{k=1}^{N} g_{ik}\; dq_i\; dq_k, \qquad (g_{ik} = g_{ki}). \tag{B.1}$$

dq_i is the increment along the coordinate axis q_i, $(1 \le i \le N)$, and g_{ik} are the components of the so-called metric tensor. In orthogonal coordinates there is $g_{ik} = 0$ if $i \ne k$, and Eq. (B.1) becomes in three-dimensional space $N = 3$ equal to

$$ds^2 = g_{11}\; dq_1^2 + g_{22}\; dq_2^2 + g_{33}\; dq_3^2. \tag{B.2}$$

If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ denote the unit vectors along the curvilinear orthogonal coordinate axes q_1, q_2, q_3, we can write for the infinitesimal change $d\vec{r}$ of the position vector $\vec{r} = \vec{r}\,[r_1(q_1, q_2, q_3),\ r_2(q_1, q_2, q_3),\ r_3(q_1, q_2, q_3)]$:

$$d\vec{r} = (\partial\vec{r}/\partial q_1)\ dq_1 + (\partial\vec{r}/\partial q_2)\ dq_2 + (\partial\vec{r}/\partial q_3)\ dq_3 = |\partial\vec{r}/\partial q_1|\ dq_1\ \vec{e}_1$$
$$+|\partial\vec{r}/\partial q_2|\ dq_2\ \vec{e}_2 + |\partial\vec{r}/\partial q_3|\ dq_3\ \vec{e}_3 = h_1\ dq_1\ \vec{e}_1 + h_2\ dq_2\ \vec{e}_2 + h_3\ dq_3\ \vec{e}_3,$$
$$(\partial\vec{r}/\partial q_i = |\partial\vec{r}/\partial q_i|\vec{e}_i = h_i\vec{e}_i;\ |\partial\vec{r}/\partial q_i| = h_i;\ i = 1, 2, 3). \tag{B.3}$$

r_1, r_2, r_3 are the components of $\vec{r}$ along the coordinate axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and $\partial\vec{r}/\partial q_i$ is the tangent vector to the q_i-axis, being directed along the unit vector $\vec{e}_i$. The quantities h_i are called the scale factors. The line element (B.2) becomes

$$ds^2 = d\vec{r}\cdot d\vec{r} = \sum_{i=1}^{3} h_i^2\ dq_i^2 = \sum_{i=1}^{3} g_{ii}\ dq_i^2, \qquad (ds = |d\vec{r}|;\ h_i^2 = g_{ii}), \tag{B.4}$$

while the volume element is in our right-handed coordinate system equal to

$$dV = (h_1\ dq_1\ \vec{e}_1)\cdot[(h_2\ dq_2\ \vec{e}_2)\times(h_3\ dq_3\ \vec{e}_3)] = h_1h_2h_3\ dq_1dq_2dq_3, \qquad \big(\vec{e}_1\cdot(\vec{e}_2\times\vec{e}_3) = 1\big). \tag{B.5}$$

Since the curvilinear coordinate system (q_1, q_2, q_3) is orthogonal, the following three scalar products are zero:

$$(\partial\vec{r}/\partial q_1)\cdot(\partial\vec{r}/\partial q_2) = h_1h_2\ \vec{e}_1\cdot\vec{e}_2 = 0; \quad (\partial\vec{r}/\partial q_2)\cdot(\partial\vec{r}/\partial q_3) = h_2h_3\ \vec{e}_2\cdot\vec{e}_3 = 0;$$
$$(\partial\vec{r}/\partial q_3)\cdot(\partial\vec{r}/\partial q_1) = h_3h_1\ \vec{e}_3\cdot\vec{e}_1 = 0. \tag{B.6}$$

We derive these three equations with respect to q_3, q_1, and q_2, respectively (e.g. Batchelor 1967):

$$\partial[(\partial\vec{r}/\partial q_1)\cdot(\partial\vec{r}/\partial q_2)]/\partial q_3 = (\partial\vec{r}/\partial q_1)\cdot(\partial^2\vec{r}/\partial q_2\partial q_3) + (\partial\vec{r}/\partial q_2)\cdot(\partial^2\vec{r}/\partial q_3\partial q_1) = 0, \tag{B.7}$$

$$\partial[(\partial\vec{r}/\partial q_2)\cdot(\partial\vec{r}/\partial q_3)]/\partial q_1 = (\partial\vec{r}/\partial q_2)\cdot(\partial^2\vec{r}/\partial q_3\partial q_1) + (\partial\vec{r}/\partial q_3)\cdot(\partial^2\vec{r}/\partial q_1\partial q_2) = 0, \tag{B.8}$$

$$\partial[(\partial\vec{r}/\partial q_3)\cdot(\partial\vec{r}/\partial q_1)]/\partial q_2 = (\partial\vec{r}/\partial q_3)\cdot(\partial^2\vec{r}/\partial q_1\partial q_2) + (\partial\vec{r}/\partial q_1)\cdot(\partial^2\vec{r}/\partial q_2\partial q_3) = 0. \tag{B.9}$$

Inserting into Eq. (B.7) for the derivatives $\partial^2\vec{r}/\partial q_2\partial q_3,\ \partial^2\vec{r}/\partial q_3\partial q_1$ their values from Eqs. (B.8) and (B.9), we get

$$\partial[(\partial\vec{r}/\partial q_1)\cdot(\partial\vec{r}/\partial q_2)]/\partial q_3 = -2(\partial\vec{r}/\partial q_3)\cdot(\partial^2\vec{r}/\partial q_1\partial q_2) = -2h_3\ \vec{e}_3\cdot[\partial(h_1\vec{e}_1)/\partial q_2]$$
$$= -2h_3\ \vec{e}_3\cdot[(\partial h_1/\partial q_2)\ \vec{e}_1 + h_1(\partial\vec{e}_1/\partial q_2)] = -2h_1h_3\ \vec{e}_3\cdot(\partial\vec{e}_1/\partial q_2) = 0, \tag{B.10}$$

where we have used $\partial\vec{r}/\partial q_i = h_i\vec{e}_i,\ (i = 1, 2, 3)$ from Eq. (B.3), and $\vec{e}_1\cdot\vec{e}_3 = 0$. Because

$$\partial^2\vec{r}/\partial q_1\partial q_2 = \partial^2\vec{r}/\partial q_2\partial q_1 \quad \text{or} \quad \partial(h_1\vec{e}_1)/\partial q_2 = \partial(h_2\vec{e}_2)/\partial q_1, \tag{B.11}$$

Eq. (B.10) also yields $-2h_2h_3\ \vec{e}_3\cdot(\partial\vec{e}_2/\partial q_1) = 0$, showing that the vectors $\partial\vec{e}_1/\partial q_2$ and $\partial\vec{e}_2/\partial q_1$ are perpendicular to $\vec{e}_3$. On the other hand, the vector $\partial\vec{e}_1/\partial q_2$ is also perpendicular to $\vec{e}_1$, and the vector $\partial\vec{e}_2/\partial q_1$ perpendicular to $\vec{e}_2$. This can be shown by observing that $\vec{e}_1\cdot(\partial\vec{e}_1/\partial q_2) = (\partial\vec{e}_1^{\,2}/\partial q_2)/2 = 0$ and $\vec{e}_2\cdot(\partial\vec{e}_2/\partial q_1) = 0$, because $\vec{e}_i\cdot\vec{e}_i = |\vec{e}_i|^2 = 1 = \text{const}$, $(i = 1, 2, 3)$. Thus, the vector $\partial\vec{e}_1/\partial q_2$ must be directed along $\vec{e}_2$, and the vector $\partial\vec{e}_2/\partial q_1$ directed along $\vec{e}_1$. Therefore, we can split the second vectorial equality (B.11)

$$(\partial h_1/\partial q_2)\ \vec{e}_1 + h_1\ \partial\vec{e}_1/\partial q_2 = (\partial h_2/\partial q_1)\ \vec{e}_2 + h_2\ \partial\vec{e}_2/\partial q_1, \tag{B.12}$$

into two parts directed along the unit vectors $\vec{e}_2$ and $\vec{e}_1$, respectively:

$$\partial\vec{e}_1/\partial q_2 = (\partial h_2/\partial q_1)\ \vec{e}_2/h_1; \quad \partial\vec{e}_2/\partial q_1 = (\partial h_1/\partial q_2)\ \vec{e}_1/h_2. \tag{B.13}$$

By circular permutation of the indices we get four other similar equations, linking the unit vectors to their spatial derivatives:

$$\partial\vec{e}_1/\partial q_3 = (\partial h_3/\partial q_1)\ \vec{e}_3/h_1; \quad \partial\vec{e}_2/\partial q_3 = (\partial h_3/\partial q_2)\ \vec{e}_3/h_2;$$
$$\partial\vec{e}_3/\partial q_1 = (\partial h_1/\partial q_3)\ \vec{e}_1/h_3; \quad \partial\vec{e}_3/\partial q_2 = (\partial h_2/\partial q_3)\ \vec{e}_2/h_3. \tag{B.14}$$

The derivatives $\partial\vec{e}_i/\partial q_i$ are easily found (e.g. Batchelor 1967):

$$\partial\vec{e}_1/\partial q_1 = \partial(\vec{e}_2\times\vec{e}_3)/\partial q_1 = (\partial\vec{e}_2/\partial q_1)\times\vec{e}_3 + \vec{e}_2\times(\partial\vec{e}_3/\partial q_1)$$
$$= (\partial h_1/\partial q_2)(\vec{e}_1\times\vec{e}_3)/h_2 + (\partial h_1/\partial q_3)(\vec{e}_2\times\vec{e}_1)/h_3 = -(\partial h_1/\partial q_2)\ \vec{e}_2/h_2 - (\partial h_1/\partial q_3)\ \vec{e}_3/h_3;$$
$$\partial\vec{e}_2/\partial q_2 = -(\partial h_2/\partial q_3)\ \vec{e}_3/h_3 - (\partial h_2/\partial q_1)\ \vec{e}_1/h_1;$$
$$\partial\vec{e}_3/\partial q_3 = -(\partial h_3/\partial q_1)\ \vec{e}_1/h_1 - (\partial h_3/\partial q_2)\ \vec{e}_2/h_2. \tag{B.15}$$

The so-called "del" or "nabla" operator is a vector defining the infinitesimal change of a scalar or vector function along the orthogonal components $h_i\ dq_i,\ (i = 1,2,3)$ of the line element ds (e.g. Spiegel 1968):

$$\nabla = (\vec{e}_1/h_1)\ \partial\ /\partial q_1 + (\vec{e}_2/h_2)\ \partial\ /\partial q_2 + (\vec{e}_3/h_3)\ \partial\ /\partial q_3. \tag{B.16}$$

The ∇ operator acting on a scalar function $f = f(q_1, q_2, q_3)$ is a vector, called gradient of f

$$\nabla f = \text{grad}\ f = (\vec{e}_1/h_1)\ \partial f/\partial q_1 + (\vec{e}_2/h_2)\ \partial f/\partial q_2 + (\vec{e}_3/h_3)\ \partial f/\partial q_3. \tag{B.17}$$

When ∇ acts on the vector

$$\vec{F} = F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3, \qquad \big(\vec{F} = \vec{F}(F_1, F_2, F_3);\ F_i = F_i(q_1, q_2, q_3);\ i = 1,2,3\big), \tag{B.18}$$

we can form the scalar or the vectorial product between ∇ and $\vec{F}$. The scalar product between ∇ and $\vec{F}$ is a scalar, called divergence of $\vec{F}$. After some algebra we get by inserting Eqs. (B.13)-(B.16), (B.18):

$$\nabla\cdot\vec{F} = \text{div}\ \vec{F} = (\vec{e}_1/h_1)\cdot(\partial\vec{F}/\partial q_1) + (\vec{e}_2/h_2)\cdot(\partial\vec{F}/\partial q_2) + (\vec{e}_3/h_3)\cdot(\partial\vec{F}/\partial q_3)$$
$$= (1/h_1h_2h_3)\ [\partial(h_2h_3F_1)/\partial q_1 + \partial(h_3h_1F_2)/\partial q_2 + \partial(h_1h_2F_3)/\partial q_3]. \tag{B.19}$$

The vector product between ∇ and $\vec{F}$ is a vector called curl of $\vec{F}$

$$\nabla\times\vec{F} = \text{curl}\ \vec{F} = (\vec{e}_1/h_1)\times(\partial\vec{F}/\partial q_1) + (\vec{e}_2/h_2)\times(\partial\vec{F}/\partial q_2) + (\vec{e}_3/h_3)\times(\partial\vec{F}/\partial q_3)$$
$$= (\vec{e}_1/h_2h_3)[\partial(h_3F_3)/\partial q_2 - \partial(h_2F_2)/\partial q_3] + (\vec{e}_2/h_3h_1)[\partial(h_1F_1)/\partial q_3 - \partial(h_3F_3)/\partial q_1]$$
$$+(\vec{e}_3/h_1h_2)[\partial(h_2F_2)/\partial q_1 - \partial(h_1F_1)/\partial q_2], \tag{B.20}$$

where again we have used Eqs. (B.13)-(B.16), (B.18), and $\vec{e}_1\times\vec{e}_2 = \vec{e}_3,\ \vec{e}_2\times\vec{e}_3 = \vec{e}_1,\ \vec{e}_3\times\vec{e}_1 = \vec{e}_2$.

The divergence of the gradient of the scalar function f yields the Laplacian operator

$$\nabla\cdot(\nabla f) = \nabla^2 f = \text{div (grad}\ f) = (1/h_1h_2h_3)\big\{\partial[(h_2h_3/h_1)\ \partial f/\partial q_1]/\partial q_1$$
$$+\partial[(h_3h_1/h_2)\ \partial f/\partial q_2]/\partial q_2 + \partial[(h_1h_2/h_3)\ \partial f/\partial q_3]/\partial q_3\big\}. \tag{B.21}$$

The Laplacian of a vector $\vec{F}$ is defined through Eq. (B.21), (Hughes and Gaylord 1964):

$$\nabla^2\vec{F} = \nabla^2(F_1\vec{e}_1) + \nabla^2(F_2\vec{e}_2) + \nabla^2(F_3\vec{e}_3) = (1/h_1h_2h_3)\ \big\{\partial[(h_2h_3/h_1)$$
$$\times\partial(F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3)/\partial q_1]/\partial q_1 + \partial[(h_3h_1/h_2)\ \partial(F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3)/\partial q_2]/\partial q_2$$
$$+\partial[(h_1h_2/h_3)\ \partial(F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3)/\partial q_3]/\partial q_3\big\}. \tag{B.22}$$

The material (substantial or Stokes) derivative of a scalar or vector function Q is defined by

$$DQ/Dt = \partial Q/\partial t + (\vec{v}\cdot\nabla)Q = \partial Q/\partial t + (v_1/h_1)\ \partial Q/\partial q_1 + (v_2/h_2)\ \partial Q/\partial q_2 + (v_3/h_3)\ \partial Q/\partial q_3. \tag{B.23}$$

t denotes the time, and $\vec{v} = d\vec{r}/dt$ the velocity of a mass element. The symbol Q in Eq. (B.23) may be replaced by a scalar function $f = f[t, q_1(t), q_2(t), q_3(t)],\ (v_i = |\partial\vec{r}/\partial q_i|\ dq_i/dt = h_i\ dq_i/dt;\ i = 1,2,3)$

$$Df/Dt = \partial f/\partial t + (v_1/h_1)\ \partial f/\partial q_1 + (v_2/h_2)\ \partial f/\partial q_2 + (v_3/h_3)\ \partial f/\partial q_3, \tag{B.24}$$

or by a vector function $\vec{F} = F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3,\ F_i = F_i[t, q_1(t), q_2(t), q_3(t)]$, where the nabla operator acts on both, the vector components and the unit vectors $\vec{e}_i$. In particular, $\vec{F}$ may be replaced by the velocity vector $\vec{v} = \vec{v}\{v_1[t, q_1(t), q_2(t), q_3(t)],\ v_2[t, q_1(t), q_2(t), q_3(t)],\ v_3[t, q_1(t), q_2(t), q_3(t)]\}$:

$$\begin{aligned}
&D\vec{v}/Dt = \partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} = (\partial v_1/\partial t)\vec{e}_1 + (\partial v_2/\partial t)\vec{e}_2 + (\partial v_3/\partial t)\vec{e}_3\\
&+(v_1/h_1)\ \partial(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3)/\partial q_1 + (v_2/h_2)\ \partial(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3)/\partial q_2\\
&+(v_3/h_3)\ \partial(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3)/\partial q_3 = \vec{e}_1\ [\partial v_1/\partial t + (v_1/h_1)\ \partial v_1/\partial q_1 + (v_2/h_2)\ \partial v_1/\partial q_2\\
&+(v_3/h_3)\ \partial v_1/\partial q_3 + (v_2/h_1h_2)(v_1\ \partial h_1/\partial q_2 - v_2\ \partial h_2/\partial q_1) + (v_3/h_3h_1)(v_1\ \partial h_1/\partial q_3 - v_3\ \partial h_3/\partial q_1)]\\
&+\vec{e}_2\ [\partial v_2/\partial t + (v_1/h_1)\ \partial v_2/\partial q_1 + (v_2/h_2)\ \partial v_2/\partial q_2 + (v_3/h_3)\ \partial v_2/\partial q_3\\
&+(v_3/h_2h_3)(v_2\ \partial h_2/\partial q_3 - v_3\ \partial h_3/\partial q_2) + (v_1/h_1h_2)(v_2\ \partial h_2/\partial q_1 - v_1\ \partial h_1/\partial q_2)]\\
&+\vec{e}_3\ [\partial v_3/\partial t + (v_1/h_1)\ \partial v_3/\partial q_1 + (v_2/h_2)\ \partial v_3/\partial q_2 + (v_3/h_3)\ \partial v_3/\partial q_3\\
&+(v_1/h_3h_1)(v_3\ \partial h_3/\partial q_1 - v_1\ \partial h_1/\partial q_3) + (v_2/h_2h_3)(v_3\ \partial h_3/\partial q_2 - v_2\ \partial h_2/\partial q_3)]. \qquad \text{(B.25)}
\end{aligned}$$

Applications

(i) Cartesian coordinates $\boldsymbol{x_1, x_2, x_3}$. There is $q_1 = x_1,\ q_2 = x_2,\ q_3 = x_3$, and $\vec{F} = \vec{F}[F_1(x_1, x_2, x_3),\ F_2(x_1, x_2, x_3),\ F_3(x_1, x_2, x_3)]$, where F_1, F_2, F_3 are the components of $\vec{F}$ along the coordinate axes, and

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2; \quad dV = dx_1\ dx_2\ dx_3. \qquad \text{(B.26)}$$

All derivatives of the unit vectors are zero, since $h_1 = h_2 = h_3 = 1 = \text{const}$:

$$\nabla f = \vec{e}_1\ \partial f/\partial x_1 + \vec{e}_2\ \partial f/\partial x_2 + \vec{e}_3\ \partial f/\partial x_3, \qquad \text{(B.27)}$$

$$\nabla\cdot\vec{F} = \partial F_1/\partial x_1 + \partial F_2/\partial x_2 + \partial F_3/\partial x_3, \qquad \text{(B.28)}$$

$$\nabla\times\vec{F} = \vec{e}_1\ (\partial F_3/\partial x_2 - \partial F_2/\partial x_3) + \vec{e}_2\ (\partial F_1/\partial x_3 - \partial F_3/\partial x_1) + \vec{e}_3\ (\partial F_2/\partial x_1 - \partial F_1/\partial x_2), \qquad \text{(B.29)}$$

$$\nabla^2 f = \partial^2 f/\partial x_1^2 + \partial^2 f/\partial x_2^2 + \partial^2 f/\partial x_3^2. \qquad \text{(B.30)}$$

If the scalar f is replaced by the vector function $\vec{F} = F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3$, we find from Eq. (B.22):

$$\begin{aligned}
&\nabla^2\vec{F} = \nabla^2(F_1\vec{e}_1) + \nabla^2(F_2\vec{e}_2) + \nabla^2(F_3\vec{e}_3) = \vec{e}_1\ (\partial^2 F_1/\partial x_1^2 + \partial^2 F_1/\partial x_2^2 + \partial^2 F_1/\partial x_3^2)\\
&+\vec{e}_2\ (\partial^2 F_2/\partial x_1^2 + \partial^2 F_2/\partial x_2^2 + \partial^2 F_2/\partial x_3^2) + \vec{e}_3\ (\partial^2 F_3/\partial x_1^2 + \partial^2 F_3/\partial x_2^2 + \partial^2 F_3/\partial x_3^2). \qquad \text{(B.31)}
\end{aligned}$$

The material derivatives are

$$Df/Dt = \partial f/\partial t + v_1\ \partial f/\partial x_1 + v_2\ \partial f/\partial x_2 + v_3\ \partial f/\partial x_3, \qquad \text{(B.32)}$$

$$\begin{aligned}
&D\vec{v}/Dt = \vec{e}_1\ (\partial v_1/\partial t + v_1\ \partial v_1/\partial x_1 + v_2\ \partial v_1/\partial x_2 + v_3\ \partial v_1/\partial x_3)\\
&+\vec{e}_2\ (\partial v_2/\partial t + v_1\ \partial v_2/\partial x_1 + v_2\ \partial v_2/\partial x_2 + v_3\ \partial v_2/\partial x_3)\\
&+\vec{e}_3\ (\partial v_3/\partial t + v_1\ \partial v_3/\partial x_1 + v_2\ \partial v_3/\partial x_2 + v_3\ \partial v_3/\partial x_3). \qquad \text{(B.33)}
\end{aligned}$$

(ii) Spherical coordinates $\boldsymbol{r, \lambda, \varphi}$. There is $\vec{e}_r = \vec{e}_1,\ \vec{e}_\lambda = \vec{e}_2,\ \vec{e}_\varphi = \vec{e}_3;\quad q_1 = r,\ q_2 = \lambda,\ q_3 = \varphi;$ $\vec{F} = \vec{F}[F_r(r, \lambda, \varphi),\ F_\lambda(r, \lambda, \varphi),\ F_\varphi(r, \lambda, \varphi)]$, where $F_r, F_\lambda, F_\varphi$ are the components of the vector $\vec{F}$ along the spherical coordinate axes: The r-axis is directed along the radius vector $\vec{r}$, the zenith angle is λ, and φ is the azimuth angle. Note, that the coordinate system (r, λ, φ) is right-handed, whereas the (r, φ, λ)-system is left-handed:

$$\begin{aligned}
&ds^2 = dr^2 + r^2\ d\lambda^2 + r^2\sin^2\lambda\ d\varphi^2; \quad dV = r^2\sin\lambda\ dr\ d\lambda\ d\varphi;\\
&h_r = h_1 = 1; \quad h_\lambda = h_2 = r; \quad h_\varphi = h_3 = r\sin\lambda, \qquad \text{(B.34)}
\end{aligned}$$

$$\begin{aligned}
&\partial\vec{e}_r/\partial r = 0; \quad \partial\vec{e}_r/\partial\lambda = \vec{e}_\lambda; \quad \partial\vec{e}_r/\partial\varphi = \vec{e}_\varphi\sin\lambda; \quad \partial\vec{e}_\lambda/\partial r = 0; \quad \partial\vec{e}_\lambda/\partial\lambda = -\vec{e}_r;\\
&\partial\vec{e}_\lambda/\partial\varphi = \vec{e}_\varphi\cos\lambda; \quad \partial\vec{e}_\varphi/\partial r = 0; \quad \partial\vec{e}_\varphi/\partial\lambda = 0; \quad \partial\vec{e}_\varphi/\partial\varphi = -\vec{e}_r\sin\lambda - \vec{e}_\lambda\cos\lambda, \qquad \text{(B.35)}
\end{aligned}$$

$$\nabla f = \vec{e}_r \ \partial f/\partial r + (\vec{e}_\lambda/r) \ \partial f/\partial \lambda + (\vec{e}_\varphi/r \sin\lambda) \ \partial f/\partial \varphi, \tag{B.36}$$

$$\nabla \cdot \vec{F} = (1/r^2) \ \partial(r^2 F_r)/\partial r + (1/r \sin\lambda) \ \partial(F_\lambda \sin\lambda)/\partial\lambda + (1/r \sin\lambda) \ \partial F_\varphi/\partial\varphi, \tag{B.37}$$

$$\nabla \times \vec{F} = (\vec{e}_r/r \sin\lambda) \ [\partial(F_\varphi \sin\lambda)/\partial\lambda - \partial F_\lambda/\partial\varphi] + (\vec{e}_\lambda/r) \ [(1/\sin\lambda) \ \partial F_r/\partial\varphi - \partial(rF_\varphi)/\partial r] \\ +(\vec{e}_\varphi/r) \ [\partial(rF_\lambda)/\partial r - \partial F_r/\partial\lambda], \tag{B.38}$$

$$\nabla^2 f = (1/r^2) \ \partial(r^2 \ \partial f/\partial r)/\partial r + (1/r^2 \sin\lambda) \ \partial(\sin\lambda \ \partial f/\partial\lambda)/\partial\lambda + (1/r^2 \sin^2\lambda) \ \partial^2 f/\partial\varphi^2. \tag{B.39}$$

If we replace f by $\vec{F}$, and take into account Eqs. (B.22), (B.35), we get

$$\nabla^2 \vec{F} = \vec{e}_r \ [\nabla^2 F_r - 2F_r/r^2 - (2/r^2 \sin\lambda) \ \partial(F_\lambda \sin\lambda)/\partial\lambda - (2/r^2 \sin\lambda) \ \partial F_\varphi/\partial\varphi] \\ +\vec{e}_\lambda \ [(2/r^2) \ \partial F_r/\partial\lambda + \nabla^2 F_\lambda - F_\lambda/r^2 \sin^2\lambda - (2\cos\lambda/r^2 \sin^2\lambda) \ \partial F_\varphi/\partial\varphi] \\ +\vec{e}_\varphi \ [(2/r^2 \sin\lambda) \ \partial F_r/\partial\varphi + (2\cos\lambda/r^2 \sin^2\lambda) \ \partial F_\lambda/\partial\varphi + \nabla^2 F_\varphi - F_\varphi/r^2 \sin^2\lambda]. \tag{B.40}$$

The material derivatives are

$$Df/Dt = \partial f/\partial t + v_r \ \partial f/\partial r + (v_\lambda/r) \ \partial f/\partial\lambda + (v_\varphi/r \sin\lambda) \ \partial f/\partial\varphi, \tag{B.41}$$

$$D\vec{v}/Dt = \vec{e}_r \ [\partial v_r/\partial t + v_r \ \partial v_r/\partial r + (v_\lambda/r) \ \partial v_r/\partial\lambda + (v_\varphi/r \sin\lambda) \ \partial v_r/\partial\varphi - v_\lambda^2/r - v_\varphi^2/r] \\ +\vec{e}_\lambda \ [\partial v_\lambda/\partial t + v_r \ \partial v_\lambda/\partial r + (v_\lambda/r) \ \partial v_\lambda/\partial\lambda + (v_\varphi/r \sin\lambda) \ \partial v_\lambda/\partial\varphi + v_r \ v_\lambda/r - v_\varphi^2 \cot\lambda/r] \\ +\vec{e}_\varphi \ [\partial v_\varphi/\partial t + v_r \ \partial v_\varphi/\partial r + (v_\lambda/r) \ \partial v_\varphi/\partial\lambda + (v_\varphi/r \sin\lambda) \ \partial v_\varphi/\partial\varphi + v_r \ v_\varphi/r + v_\lambda \ v_\varphi \cot\lambda/r]. \tag{B.42}$$

(iii) Cylindrical coordinates $\boldsymbol{\ell, \varphi, z}$. There is $\vec{e}_\ell = \vec{e}_1$, $\vec{e}_\varphi = \vec{e}_2$, $\vec{e}_z = \vec{e}_3$; $q_1 = \ell$, $q_2 = \varphi$, $q_3 = z$; $\vec{F} = \vec{F}[F_\ell(\ell,\varphi,z), \ F_\varphi(\ell,\varphi,z), \ F_z(\ell,\varphi,z)]$, where F_ℓ, F_φ, F_z are the components of $\vec{F}$ along the distance ℓ from the z-axis, along the azimuth angle φ, and along the z-axis, respectively:

$$ds^2 = d\ell^2 + \ell^2 \ d\varphi^2 + dz^2; \quad dV = \ell \ d\ell \ d\varphi \ dz; \quad h_\ell = h_1 = 1; \quad h_\varphi = h_2 = \ell; \quad h_z = h_3 = 1, \tag{B.43}$$

$$\partial\vec{e}_\ell/\partial\varphi = \vec{e}_\varphi; \quad \partial\vec{e}_\varphi/\partial\varphi = -\vec{e}_\ell, \tag{B.44}$$

the remaining derivatives being zero.

$$\nabla f = \vec{e}_\ell \ \partial f/\partial\ell + (\vec{e}_\varphi/\ell) \ \partial f/\partial\varphi + \vec{e}_z \ \partial f/\partial z, \tag{B.45}$$

$$\nabla \cdot \vec{F} = (1/\ell) \ \partial(\ell F_\ell)/\partial\ell + (1/\ell) \ \partial F_\varphi/\partial\varphi + \partial F_z/\partial z, \tag{B.46}$$

$$\nabla \times \vec{F} = \vec{e}_\ell \ [-\partial F_\varphi/\partial z + (1/\ell) \ \partial F_z/\partial\varphi] + \vec{e}_\varphi \ (\partial F_\ell/\partial z - \partial F_z/\partial\ell) \\ +\vec{e}_z \ [-(1/\ell) \ \partial F_\ell/\partial\varphi + (1/\ell) \ \partial(\ell F_\varphi)/\partial\ell], \tag{B.47}$$

$$\nabla^2 f = (1/\ell) \ \partial(\ell \ \partial f/\partial\ell)/\partial\ell + (1/\ell^2) \ \partial^2 f/\partial\varphi^2 + \partial^2 f/\partial z^2. \tag{B.48}$$

If we replace f by $\vec{F}$, and use Eqs. (B.22), (B.44), we obtain

$$\nabla^2 \vec{F} = \vec{e}_\ell \ [\nabla^2 F_\ell - F_\ell/\ell^2 - (2/\ell^2) \ \partial F_\varphi/\partial\varphi] + \vec{e}_\varphi \ [(2/\ell^2) \ \partial F_\ell/\partial\varphi + \nabla^2 F_\varphi - F_\varphi/\ell^2] + \vec{e}_z \ (\nabla^2 F_z). \tag{B.49}$$

The material derivatives are

$$Df/Dt = \partial f/\partial t + v_\ell \ \partial f/\partial\ell + (v_\varphi/\ell) \ \partial f/\partial\varphi + v_z \ \partial f/\partial z, \tag{B.50}$$

$$D\vec{v}/Dt = \vec{e}_\ell \ [\partial v_\ell/\partial t + v_\ell \ \partial v_\ell/\partial\ell + (v_\varphi/\ell) \ \partial v_\ell/\partial\varphi + v_z \ \partial v_\ell/\partial z - v_\varphi^2/\ell] \\ +\vec{e}_\varphi \ [\partial v_\varphi/\partial t + v_\ell \ \partial v_\varphi/\partial\ell + (v_\varphi/\ell) \ \partial v_\varphi/\partial\varphi + v_z \ \partial v_\varphi/\partial z + v_\ell v_\varphi/\ell] \\ +\vec{e}_z \ [\partial v_z/\partial t + v_\ell \ \partial v_z/\partial\ell + (v_\varphi/\ell) \ \partial v_z/\partial\varphi + v_z \ \partial v_z/\partial z]. \tag{B.51}$$

(iv) Polar coordinates $\boldsymbol{\ell, \varphi}$. They are obtained from cylindrical coordinates if $z \equiv 0$.

Appendix C

Generalized N-dimensional Orthogonal Polar Coordinates

In analogy to polar and spherical coordinates we can define in N-dimensional space $(N = 1, 2, 3, ...)$ an orthogonal set of generalized polar coordinates $r, \varphi_1, \varphi_2, ...\varphi_{N-1}$, where r is the radial distance from the origin of a N-dimensional Cartesian coordinate frame $x_1, x_2, ...x_N$, and φ_i, $(i = 1, 2, 3, ...N-1)$ are the polar angles, where $0 \leq r \leq \infty$, $0 \leq \varphi_k \leq \pi$, $(k = 1, 2, 3, ...N-2)$, and $0 \leq \varphi_{N-1} \leq 2\pi$. We have (e.g. Madelung 1953)

$$x_1 = r\cos\varphi_1; \quad x_i = r\cos\varphi_i \prod_{k=1}^{i-1} \sin\varphi_k; \quad x_N = r\prod_{k=1}^{N-1} \sin\varphi_k; \quad r^2 = \sum_{i=1}^{N} x_i^2, \qquad (i = 2, 3, ...N-1). \tag{C.1}$$

By the same reasoning as for spherical coordinates we obtain for the line element

$$ds^2 = \sum_{i=1}^{N} dx_i^2 = dr^2 + r^2\ d\varphi_1^2 + r^2 \sum_{i=3}^{N} \prod_{k=1}^{i-2} \sin^2\varphi_k\ d\varphi_{i-1}^2, \tag{C.2}$$

with the scale factors [cf. Eq. (B.4)]

$$h_1 = 1; \quad h_2 = r; \quad h_i = r\prod_{k=1}^{i-2} \sin\varphi_k, \qquad (i = 3, 4, 5, ...N). \tag{C.3}$$

The volume element in N-dimensional polar coordinates is given by [cf. Eq. (B.5)]

$$dV = h_1\ dr \prod_{i=1}^{N-1} h_{i+1}\ d\varphi_i = r^{N-1}\ dr \prod_{i=1}^{N-1} (\sin\varphi_i)^{N-1-i}\ d\varphi_i. \tag{C.4}$$

The volume of a sphere of radius r is (e.g. Madelung 1953)

$$\begin{aligned} V &= \int_V dV = \int_0^r r^{N-1}\ dr \int_0^{2\pi} d\varphi_{N-1} \prod_{i=1}^{N-2} \int_0^{\pi} (\sin\varphi_i)^{N-1-i}\ d\varphi_i \\ &= (2\pi r^N/N) \prod_{i=1}^{N-2} \int_0^{\pi} (\sin\varphi_i)^{N-1-i}\ d\varphi_i = (2\pi r^N/N) \prod_{k=1}^{N-2} \int_0^{\pi} \sin^k\varphi\ d\varphi \\ &= \begin{cases} 2^{N/2}\pi^{N/2}r^N/N(N-2)(N-4)...6\times 4\times 2 & \text{if} \quad N = 2\nu;\ \nu = 2, 3, 4, ... \\ 2^{(N+1)/2}\pi^{(N-1)/2}r^N/N(N-2)(N-4)...5\times 3\times 1 & \quad\quad N = 2\nu+1;\ \nu = 1, 2, 3, ... \end{cases} \end{aligned} \tag{C.5}$$

Although the integration of Eq. (C.5) can be performed only if $N \geq 3$, the final result yields exact values also if $N = 1, 2$ [cf. Eq. (C.4)]:

$$V = \begin{cases} 2r & (N = 1,\ \text{volume of slab per unit surface}) \\ \pi r^2 & (N = 2,\ \text{cylindrical volume per unit height}) \\ 4\pi r^3/3 & (N = 3,\ \text{sphere}) \end{cases} \tag{C.6}$$

The surface of the N-dimensional sphere of radius r is simply

$$S = dV/dr = NV/r, \qquad (N = 1, 2, 3, ...). \tag{C.7}$$

For a radially symmetric distribution of matter the mass inside volume V is given by

$$M = \int_V \varrho \; dV = \int_0^r \varrho \; S \; dr, \tag{C.8}$$

where $\varrho = \varrho(r)$ denotes the density at radial distance r. With the aid of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) \; dt, \tag{C.9}$$

Eqs. (C.5), (C.7), (C.8) can be written as (cf. Kimura 1981a, Abramowicz 1983)

$$\begin{aligned} &V = r^N \; [\Gamma(1/2)]^N / \Gamma(N/2+1) = (2r^N/N) \; [\Gamma(1/2)]^N / \Gamma(N/2); \\ &S = dV/dr = 2r^{N-1} \; [\Gamma(1/2)]^N / \Gamma(N/2); \\ &M = \int_0^r \varrho \; S \; dr = \{2[\Gamma(1/2)]^N / \Gamma(N/2)\} \int_0^r \varrho r^{N-1} \; dr, \qquad (N = 1, 2, 3, ...). \end{aligned} \tag{C.10}$$

We have used the relationships (e.g. Madelung 1953, Smirnow 1967)

$$\begin{aligned} &\Gamma(x+j) = (x+j-1)(x+j-2)...(x+1)x \; \Gamma(x); \quad \Gamma(j+1) = j(j-1)...3 \times 2 \times 1 = j!; \\ &\Gamma(j+1/2) = (2j-1)(2j-3)...5 \times 3 \times 1 \; \Gamma(1/2)/2^j; \quad \Gamma(1) = 1; \quad \Gamma(1/2) = \pi^{1/2}, \\ &(j = 1, 2, 3, ...). \end{aligned} \tag{C.11}$$

If we denote by $\vec{e}_1, \vec{e}_2, ...\vec{e}_N$ the unit vectors along the coordinate axes $r, \varphi_1, \varphi_2, ...\varphi_{N-1}$, we obtain for the gradient of a scalar function $f = f(r, \varphi_1, \varphi_2, ...\varphi_{N-1})$ the equation [cf. Eqs. (B.17), (B.36), (B.45), (C.3)]:

$$\nabla f = \text{grad } f = (\partial f/\partial r) \; \vec{e}_1 + (1/r)(\partial f/\partial \varphi_1) \; \vec{e}_2 + \sum_{i=3}^N \left\{ \left[1 \Big/ \left(r \prod_{k=1}^{i-2} \sin \varphi_k \right) \right] \; (\partial f/\partial \varphi_{i-1}) \; \vec{e}_i \right\}. \tag{C.12}$$

The divergence of the vector $\vec{F} = \vec{F}(F_1, F_2, ...F_N)$ with the components F_i along the $\vec{e}_i$-axes is via Eqs. (B.19), (C.3) equal to

$$\begin{aligned} &\nabla \cdot \vec{F} = \text{div } \vec{F} = (1/r^{N-1}) \; \partial(r^{N-1} F_1)/\partial r + (1/r) \; [(N-2) F_2 \cot \varphi_1 + \partial F_2/\partial \varphi_1] \\ &+ \sum_{i=3}^N \left\{ \left[1 \Big/ \left(r \prod_{k=1}^{i-2} \sin \varphi_k \right) \right] \; [(N-i) F_i \cot \varphi_{i-1} + \partial F_i/\partial \varphi_{i-1}] \right\}, \end{aligned} \tag{C.13}$$

where

$$\prod_{j=1}^N h_j = r^{N-1} \prod_{k=1}^{N-2} (\sin \varphi_k)^{N-1-k}. \tag{C.14}$$

The Laplace operator of the scalar function f becomes in virtue of Eqs. (B.21), (C.3) equal to

$$\begin{aligned} &\nabla^2 f = \text{div (grad } f) = \sum_{i=1}^N \partial^2 f/\partial x_i^2 = (1/r^{N-1}) \; \partial(r^{N-1} \; \partial f/\partial r)/\partial r \\ &+ (1/r^2) \; \left[\partial^2 f/\partial \varphi_1^2 + (N-2) \cot \varphi_1 \; \partial f/\partial \varphi_1\right] \\ &+ \sum_{i=3}^N \left\{ \left[1 \Big/ \left(r^2 \prod_{k=1}^{i-2} \sin^2 \varphi_k \right) \right] \; \left[\partial^2 f/\partial \varphi_{i-1}^2 + (N-i) \cot \varphi_{i-1} \; \partial f/\partial \varphi_{i-1}\right] \right\}. \end{aligned} \tag{C.15}$$

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