Developments in Mathematics

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Preface

This book is devoted to a study of the oscillation theory of nonautonomous linear Hamiltonian differential systems and that of a spectral theory which is adapted to such systems. Systematic use will be made of basic facts concerning Lagrange subspaces of $\mathbb{R}^{2n}$ and argument functions on the set of symplectic matrices. We will also consistently apply some fundamental methods of topological dynamics and of ergodic theory, including Lyapunov exponents, exponential dichotomies, and rotation numbers. Further, we will show that our results concerning oscillation theory can be fruitfully applied to several basic issues in the theory of linear-quadratic control systems with time-varying coefficients.

Nonautonomous Oscillation Theory

In due course, we will give an outline of the specific problems, methods, and results to be discussed in the body of the book. Before doing that, it seems appropriate to collocate them in a priori way in the vast and nonhomogeneous area called oscillation theory of ordinary differential equations. In fact, the word "oscillation" has various meanings in this context. For example, it can refer to the study of the zeroes contained in some interval $\mathcal{I} \subseteq \mathbb{R}$ of a solution of an ordinary differential equation (ODE). In the case of a two-dimensional ODE, it can refer to the variation of the polar angle along a solution, i.e., to the "rotation" associated to that solution. Still again, it may indicate one of the many themes encountered in the study of the periodic solutions of an ordinary differential equation.

This book is about "rotation." Let us try to be a bit more precise. We will focus attention on various issues concerning the solutions of a linear Hamiltonian differential system

$$\mathbf{z}' = H(t)\,\mathbf{z}\,, \tag{1}$$

where $\mathbf{z} \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. The coefficient $H(\cdot)$ is a bounded measurable real $2n \times 2n$ matrix-valued function satisfying the symplectic condition $(JH)^T(t) = JH(t)$ for all $t \in \mathbb{R}$, where the "T" indicates the transpose and $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$ is the usual $2n \times 2n$ antisymmetric matrix: I_n is the $n \times n$ identity matrix and 0_n the $n \times n$ zero matrix. Generally speaking, we will be interested in the "rotation" of the solutions of (1). Of course, this notion is initially problematic because it is not immediately clear how to define it precisely, especially if $n \geq 2$. One of our main goals will be to do this. It will turn out that our concept of rotation is closely related to a more or less standard notion of a "point of verticality" of a solution of (1), namely, a focal point. It will also turn out that the concept of rotation considered here can be used to study some basic questions in spectral theory, which are formulated in terms of equation (1) and which will be discussed shortly.

Equation (1) is of course very significant. As a special case, one can set $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right]$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, and

$$H(t) = \begin{bmatrix} 0_n & I_n \\ G(t) & 0_n \end{bmatrix},$$

where $G^T = G$ is a real symmetric $n \times n$ matrix-valued function. Then (1) is equivalent to the second-order system

$$\mathbf{x}'' = G(t)\,\mathbf{x}\,, \tag{2}$$

which is often encountered in the study of mechanical systems near an equilibrium. Another special case is obtained by setting $n = 1$ and

$$H(t) = \begin{bmatrix} 0 & 1/p(t) \\ g(t) - \lambda\, d(t) & 0 \end{bmatrix}$$

for a real parameter λ; in this case (1) is equivalent to the classical Sturm–Liouville problem

$$-\,(p\,x')' + g(t)\,x = \lambda\, d(t)\,x\,. \tag{3}$$

Problem (3) has been studied with success from various points of view for over 150 years. The number and the location of the zeroes of a solution $x(\cdot)$ are a recurring theme. Information concerning these zeroes has implications for the spectral problem obtained by varying λ and by imposing boundary conditions, for example, of Dirichlet type: $x(a) = x(b) = 0$ where $a < b \in \mathbb{R}$. Then, as is well known, if p, g, and d satisfy certain general hypotheses, then the nth eigenfunction of (3) has $n - 1$ zeroes in (a, b), for $n = 1, 2, \ldots$

A more general spectral problem is obtained by using (1) as a point of departure. One introduces a parameter $\lambda \in \mathbb{R}$ and a positive semidefinite real weight function $\Gamma(t)$ in (1), so as to obtain

$$\mathbf{z}' = \big(H(t) + \lambda\, J^{-1}\, \Gamma(t)\big)\, \mathbf{z}\,. \tag{4}$$

This problem was studied systematically by Atkinson in [5]. It is noteworthy that if Γ is semidefinite but not everywhere definite, then the study of the boundary-value problem associated to (4) cannot be naturally carried out using standard functional-analytic techniques (due to the fact that one cannot multiply (4) by Γ^{-1}). However, in [5], one finds an "Atkinson condition" which, when imposed on (4), allows the development of a satisfactory spectral theory for (4).

Another of our goals is to show that our oscillation theory of (1) can be fruitfully applied to the spectral problem (4) especially when "the boundary conditions are imposed at $t = \pm\infty$," i.e., when (4) is considered on the whole line. Let us explain some of the issues involved in relating oscillation theory and spectral theory in the context of problem (4). Consider for a moment the version of (3) obtained by setting $p = d \equiv 1$:

$$-\,x'' + g(t)\, x = \lambda\, x\,. \tag{5}$$

This is the Schrödinger equation with potential $g(t)$ (a most important ordinary differential equation, due to its basic role in one-dimensional quantum mechanics). Fix $\lambda \in \mathbb{R}$, and consider a solution $x(t)$ of (5), say, that defined by the initial conditions $x(a) = 0$ and $x'(a) = 1$. This solution is called nonoscillatory in the interval (a, b) if it has no zeroes there; otherwise, it oscillates. There is a simple and fruitful way to study the presence/absence of zeroes of $x(\cdot)$ on (a, b), which is at the heart of the classical Sturm–Liouville theory. Namely, one introduces the polar angle $\theta(t)$ of the vector $\left[\begin{smallmatrix} x(t) \\ x'(t) \end{smallmatrix}\right]$ in the two-dimensional phase plane $\mathbb{R}^2$. It is clear that if $a < t < b$, then $x(t) = 0$ if and only if $\theta(t) = \pi/2 \bmod \pi$. Moreover, $\theta'(t) < 0$ at each zero t of $x(t)$, so we can determine the number of zeroes of $x(\cdot)$ in (a, b) by studying the evolution of $\theta(\cdot)$ there, that is, the "rotation" of $x(\cdot)$.

This simple observation does not generalize easily to the Hamiltonian system (1). It is rather straightforward to generalize the concept of zero of $x(\cdot)$: one sets $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right]$, requires that $\mathbf{x}(t) = \mathbf{0}$, and arrives at the concept of focal point, alias point of verticality. But it is not easy to extend the concept of polar angle in an appropriate way; in fact, it seems that this was only done in the 1950s and 1960s. One way is to introduce argument functions in the symplectic group, as done by Gel'fand, Lidskii, and Yakubovich. Another is to introduce the Maslov cycle and the corresponding Maslov index in the manifold of Lagrange subspaces of $\mathbb{R}^{2n}$. There is a corresponding angle, as was pointed out by Arnol'd (and by Conley in a little-known paper), which can be used to develop a Sturm–Liouville-type theory for (4). Still another method to generalize the Sturm–Liouville theory to Hamiltonian systems can be based on the polar coordinates of Barret and Reid.

A point which we will emphasize in this book is that one can study the argument functions, the index, and the polar coordinates from a dynamical point of view, more precisely, by using basic tools from topological dynamics and ergodic theory. One point of arrival in our theory is a quantity called the rotation number and its "complexification," the Floquet exponent for system (1). Using these quantities, we will connect the oscillation theory of (1) with the spectral theory of the Atkinson problem (4), much as the Sturm–Liouville theory connects the oscillation of solutions of (3) for each fixed λ to the spectral theory of (3).

Let us explain this matter in more detail. Let $\Gamma \geq 0$ be a real symmetric matrix-valued function. Consider the boundary-value problem

$$\begin{aligned} &\mathbf{z}' = \big(H(t) + \lambda\, J^{-1}\Gamma(t)\big)\,\mathbf{z}\,, \qquad \mathbf{z} = \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \mathbb{R}^{2n}\,, \\ &\mathbf{x}(a) = \mathbf{x}(b) = \mathbf{0}\,, \end{aligned} \tag{6}$$

where $a < b \in \mathbb{R}$. In [5] an analytic theory of the eigenvalues and eigenfunctions of (6) is worked out. Let us first try to extend that theory to the entire real axis: thus set $a = -\infty$ and $b = \infty$. One can expect that this will involve some analogue of the classical Weyl m-functions $m_{\pm}(\lambda)$ for (3), and in fact there is a rich literature concerning the "Weyl–Titchmarsh M-matrices" for (6). We will assume that $H(\cdot)$ and $\Gamma(\cdot)$ are uniformly bounded and will impose a natural "Atkinson condition" on the solutions of (5). It will then turn out that the dynamical concept of exponential dichotomy together with the above-mentioned notion of rotation number permits one to develop a satisfactory spectral theory for (6) with $a = -\infty$ and $b = \infty$. In particular, the introduction of the exponential dichotomy concept permits one to clarify the dynamical significance of the M-matrices.

To summarize what has been said so far, we will supplement the analytic methods which have been previously used to study the oscillation theory of (1) and the spectral theory of (4) with certain geometrical and dynamical techniques. The geometrical methods derive from the structure of the group of symplectic matrices and from that of the manifold of Lagrangian subspaces of $\mathbb{R}^{2n}$. Using dynamical methods, we define the rotation number and the Floquet exponent, which permit one to count the focal points of (1) and to develop the spectral theory of (4) using the exponential dichotomy concept.

The use of dynamical methods is made possible by carrying out a construction named after Bebutov, which we now explain. Begin with linear Hamiltonian differential system (1): we first view the coefficient function $H(\cdot)$ as an element of an appropriate functional space. This will often be the space of bounded continuous functions $\widetilde{H}$ from $\mathbb{R}$ to the Lie algebra of real infinitesimally symplectic matrices $\mathfrak{sp}(n,\mathbb{R}) = \{\widetilde{H} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \,|\, \widetilde{H}^TJ + J\widetilde{H} = 0_{2n}\}$. Next introduce the translation flow σ_t by setting $\sigma_t(\widetilde{H})(\cdot) = \widetilde{H}(\cdot + t)$ for all $t \in \mathbb{R}$. If the coefficient $H(\cdot)$ of (1) is uniformly continuous, then the closure $\mathrm{cls}\{\sigma_t(H) \mid t \in \mathbb{R}\}$ is compact (in the compact-open topology). Call the closure Ω: it is clearly invariant with respect to the translation flow. The idea now is to let H vary over Ω; to emphasize that we do not deal only with the "original" function $H(\cdot)$, we write ω to indicate a generic point of Ω. Note that each $\omega \in \Omega$ gives rise to a linear differential system of the form (1); call this system $(1)_\omega$.

At this point, one introduces the so-called cocycle obtained by considering the fundamental matrix solution of $(1)_\omega$ and letting ω run over Ω. One can now apply the Oseledets theory of the Lyapunov indices of solutions of $(1)_\omega$ ($\omega \in \Omega$). One can also apply the Sacker–Sell–Selgrade approach to the theory of exponential dichotomies. In addition, one can define the rotation number of the family of equations $(1)_\omega$. We will see that all these dynamical methods permit one to gain important insight into the oscillation theory of (1) and the spectral theory of (4).

In fact the main tool in the analysis consists in the systematic use of the rotation number, the Lyapunov index, the exponential dichotomy concept, and the Weyl matrices. These objects are also important in the discussion of two more notions which are of fundamental significance in the context of the linear Hamiltonian system (1): the property of disconjugacy, which is of basic significance in the calculus of variations, and the related property of existence of principal solutions, which in many interesting cases can be understood as a generalization to the nonuniformly hyperbolic case of the bundles provided by the existence of exponential dichotomy.

Applications to Control Theory

There are numerous applications of the oscillation theory of equation (1) to the theory of mechanical systems, to the calculus of variations, to control theory, and to other areas. We will not give an exhaustive account of these applications. But we will apply our results concerning equations (1) and (4) to certain problems in linear-quadratic (LQ) control theory. Among these are the linear-quadratic regulator problem, the Kalman–Bucy filter, the Yakubovich frequency theorem, and the question of Willems-type dissipativity in (linear) control systems. We now discuss in a bit more detail these applications to control theory.

First we recall the formulation of the LQ regulator problem. The point of departure consists of a linear control problem

$$\begin{aligned} &\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\,, \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m\,,\\ &\mathbf{x}(0) = \mathbf{x}\,. \end{aligned} \tag{7}$$

The matrices $A(\cdot)$, $B(\cdot)$ are taken to be bounded continuous functions; the time dependence is otherwise arbitrary. Let $\tau \in (0,\infty]$ be an extended positive real number. Introduce a quadratic functional

$$\mathcal{I}_{\mathbf{x}}(\mathbf{x},\mathbf{u}) = \langle \mathbf{x}(\tau), S\mathbf{x}(\tau)\rangle + \int_0^\tau \big(\langle \mathbf{x}(t), G(t)\,\mathbf{x}(t)\rangle + \langle \mathbf{u}(t), R(t)\,\mathbf{u}(t)\rangle\big)\,dt\,.$$

where S is a symmetric positive semidefinite matrix and $G(\cdot)$, $R(\cdot)$ are bounded continuous functions such that $G^T(t) = G(t) \geq 0$ and $R^T(t) = R(t) > 0$ for all $t \in \mathbb{R}$. If the upper limit τ is finite, one speaks of a finite-horizon problem, otherwise one has an infinite-horizon problem. If $\tau = \infty$ one sets $S = 0_n$. For each fixed initial condition $\mathbf{x} \in \mathbb{R}$, one seeks a control $\mathbf{u}\colon [0, \tau] \to \mathbb{R}^m$ which, when taken together with the corresponding solution of (7), minimizes $\mathcal{I}_{\mathbf{x}}(\mathbf{x}, \mathbf{u})$.

This basic problem has been studied in detail and has been solved both when $\tau < \infty$ and when $\tau = \infty$. Our contribution is to give a solution in the infinite-horizon case $\tau = \infty$ which uses the theory of exponential dichotomies and the rotation number as applied to an appropriate linear Hamiltonian system of the form (1). In this way one obtains, among other things, detailed information concerning the regular dependence of the optimal control on parameters.

The appropriate system (1) is obtained via a formal application of the Pontryagin maximum principle. According to this principle, a minimizing control $\mathbf{u}$ must maximize the Hamiltonian

$$\mathcal{H}(t, \mathbf{x}, \mathbf{y}, \mathbf{u}) = \langle \mathbf{y}, A(t)\,\mathbf{x} + B(t)\,\mathbf{u} \rangle - \frac{1}{2}\left(\langle \mathbf{x}, G(t)\,\mathbf{x} \rangle + \langle \mathbf{u}, R(t)\,\mathbf{u} \rangle \right),$$

for each $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, and an appropriate $\mathbf{y} \in \mathbb{R}^n$. Here $\mathbf{y}$ is interpreted as a variable dual to $\mathbf{x}$. This leads immediately to the "feedback rule"

$$\mathbf{u} = R^{-1}(t)\, B^T(t)\, \mathbf{y}\,.$$

Substituting for $\mathbf{u}$ in the Hamiltonian equations $\mathbf{x}' = \partial\mathcal{H}/\partial\mathbf{y}$, $\mathbf{y}' = -\partial\mathcal{H}/\partial\mathbf{x}$ leads to the differential system

$$\mathbf{z}' = \begin{bmatrix} A(t) & B(t)\,R^{-1}(t)\,B^T(t) \\ G(t) & -A^T(t) \end{bmatrix} \mathbf{z}\,. \tag{8}$$

Of course, (8) is a special case of (1).

We now arrive at the main point, which is that (under standard controllability and observability conditions on (7)) the system (8) admits exponential dichotomy. This is easily proved when one has available the basic facts concerning the rotation number of (8) and its relation to the existence of exponential dichotomy. Now, the existence of exponential dichotomy for (8) means that there is a linear projection $P = P^2\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that if $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right]$ is in the image of P, then the solution $\mathbf{z}(t)$ of (8) satisfying $\mathbf{z}(0) = \mathbf{z}$ decays exponentially as $t \to \infty$. It further turns out that $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{x}(t) \\ M(t)\,\mathbf{x}(t) \end{smallmatrix}\right]$ where $\mathbf{x}(0) = \mathbf{x}$ and $M(t)$ is a function taking values in the set of negative definite symmetric $n \times n$ matrices. Set $\mathbf{u}(t) = R^{-1}(t)\,B^T(t)\,M(t)\,\mathbf{x}(t)$ and note that $\mathbf{u}(t) \to \mathbf{0}$ exponentially as $t \to \infty$. So it is not so surprising that this $\mathbf{u}$ is in fact the unique control which minimizes $\mathcal{I}_{\mathbf{x}}(\mathbf{x}, \mathbf{u})$. If one varies $\mathbf{x}$, the dichotomy projection P and the symmetric matrix-valued function $M(t)$ do not change, so in fact we have solved the LQ regulator problem.

Let us note in passing that we have also solved the feedback stabilization problem for the control system (7). In fact, set $\mathbf{u}(t) = R^{-1}(t)\,B^T(t)\,M(t)\,\mathbf{x}(t)$ as above. Note that if $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$ is the solution of (8) mentioned above, then $\mathbf{x}(t)$ solves (7) with precisely this control $\mathbf{u}(t)$. Since $\mathbf{u}$ has the "feedback form" $\mathbf{u}(t) = K(t)\,\mathbf{x}(t)$ with $K(t) = R^{-1}(t)\,B^T(t)\,M(t)$, and since the linear system $\mathbf{x}' = \big(A(t) + B(t)\,R^{-1}(t)\,B^T(t)\,M(t)\big)\,\mathbf{x}$ is exponentially stable, we have "feedback stabilized" the system (7).

We can also study certain important properties of the Kalman–Bucy filter by applying our methods to an appropriate Hamiltonian system of the form (1). This is because, as Kalman and Bucy observed, the construction of their filter is closely tied to a "time-reversed" LQ regulator problem. We briefly describe the filter and the relevance of the theory of linear Hamiltonian systems in this context.

Let $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ $(t \geq 0)$ denote the state of a linear system which is disturbed by a d-dimensional white noise process: thus

$$d\boldsymbol{\xi}(t) = A(t)\,\boldsymbol{\xi}(t)\,dt + S(t)\,d\mathbf{w}(t)\,. \tag{9}$$

Here $\mathbf{w}(t)$ is a d-dimensional standard Brownian motion, and equation (9) is understood to be of Itô type. The state $\boldsymbol{\xi}(t)$ can only be partially observed; it is assumed that the observation process $\boldsymbol{\eta}(t)$ satisfies the Itô equation:

$$d\boldsymbol{\eta}(t) = B(t)\,\boldsymbol{\xi}(t)\,dt + S_1(t)\,d\mathbf{w}_1(t)\,.$$

where $\mathbf{w}_1(t)$ is a second, m-dimensional Brownian motion which is independent of $\mathbf{w}(t)$. The functions A, B, S, S_1 are assumed to be continuous and bounded and to have the appropriate dimensions. It is assumed that $\boldsymbol{\eta}(0) = \mathbf{0}$ and that $\boldsymbol{\xi}(0)$ is Gaussian, which implies that $\boldsymbol{\xi}(t)$ is Gaussian for all $t \geq 0$.

Let Σ_t be the σ-algebra generated by the set $\{\boldsymbol{\eta}(r) \mid 0 \leq r \leq t\}$ of measurements up to time t. The goal is to describe an estimate $\boldsymbol{\gamma}(t)$ for $\boldsymbol{\xi}(t)$, which minimizes the mean-square error $E\{(\mathbf{x}^T(\boldsymbol{\xi}(t) - \boldsymbol{\gamma}(t)))^2\}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$; here the expected value $E\{\cdot\}$ is taken over an appropriate probability space. It is well known that this best estimate is given by the conditional expectation

$$\boldsymbol{\gamma}(t) = \widehat{\boldsymbol{\xi}}(t) = E\,\{\boldsymbol{\xi}(t) \mid \Sigma_t\}\,.$$

To describe $\widehat{\boldsymbol{\xi}}(t)$, one introduces the error process $\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t) - \widehat{\boldsymbol{\xi}}(t)$. It turns out that $\tilde{\boldsymbol{\xi}}(t)$ is Gaussian with mean value zero and hence is determined by its $n \times n$ covariance matrix $M(t)$. Kalman and Bucy showed that $M(t)$ satisfies a Riccati equation

$$M' = -M\,B^T(t)\,(S_1S_1^T)^{-1}(t)\,B(t)\,M + M\,A^T(t) + A(t)\,M + (SS^T)(t)\,.$$

Now, this Riccati equation corresponds to the linear Hamiltonian system

$$\mathbf{z}' = \begin{bmatrix} -A^T(t) & B^T(t)\,(S_1S_1^T)^{-1}(t)\,B(t) \\ (SS^T)(t) & A(t) \end{bmatrix} \mathbf{z}\,, \tag{10}$$

via the matrix change of variables $M = YX^{-1}$. It turns out that, under standard controllability conditions, the system (10) admits exponential dichotomy. This leads to the conclusion that $M(t)$ tends exponentially fast to a "nonautonomous equilibrium" $M_\infty(t)$, which essentially describes the error process $\widetilde{\boldsymbol{\xi}}(t)$, and hence the signal $\boldsymbol{\xi}(t)$ if one takes the estimate $\widehat{\boldsymbol{\xi}}(t)$ to be known.

We will also apply our results concerning the oscillation theory of equation (1) and the spectral theory of the family (4) to the circle of ideas and results centered on the Yakubovich frequency theorem. This theorem was originally formulated and proved by Yakubovich for LQ control processes with periodic coefficients. We will state and prove a more general nonautonomous version of this theorem. We briefly sketch our results in this regard in the next paragraphs.

The point of departure is again the control system (7) combined with a quadratic functional

$$\widetilde{\mathcal{I}}_{\mathbf{x}}(\mathbf{x},\mathbf{u}) = \int_0^\infty \big(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\big)\,dt\,,$$

where the functions A, B, G, g, R are assumed to be bounded and continuous and to have the appropriate dimensions. The functional $\widetilde{\mathcal{I}}_{\mathbf{x}}(\mathbf{x},\mathbf{u})$ differs from the one encountered in the context of the LQ regulator in two respects. First of all, the cross-term $\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle$ is present in the integrand. Second and more importantly, though it is assumed that $G^T(t) = G(t)$ and that $R^T(t) = R(t) > 0$ for all t, it is not assumed that G is positive semidefinite for all t; indeed one is particularly interested in the case when $G(t) < 0$ ($t \in \mathbb{R}$).

We pose the problem of minimizing $\widetilde{\mathcal{I}}_{\mathbf{x}}(\mathbf{x},\mathbf{u})$ subject to (7). Since G is not assumed to be positive semidefinite, this problem need not have a solution. Nevertheless we proceed by applying the Pontryagin maximum principle in a formal way. Introduce the Hamiltonian

$$\widetilde{\mathcal{H}}(t,\mathbf{x},\mathbf{y},\mathbf{u}) = \langle \mathbf{y}, A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\rangle - \frac{1}{2}\big(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\big)\,.$$

A minimizing control $\mathbf{u}$ (if it exists) will maximize $\widetilde{\mathcal{H}}$ for each $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, and an appropriate $\mathbf{y} \in \mathbb{R}^n$. This leads to the feedback rule

$$\mathbf{u} = R^{-1}(t)\,B^T(t)\,\mathbf{y} - R^{-1}(t)\,g^T(t)\,\mathbf{x}\,,$$

and via the Hamiltonian equations $\mathbf{x}' = \partial\widetilde{\mathcal{H}}/\partial\mathbf{y}$, $\mathbf{y}' = -\partial\widetilde{\mathcal{H}}/\partial\mathbf{x}$, one is led to the differential system

$$\mathbf{z}' = H(t)\,\mathbf{z}\,, \qquad \text{with } H = \begin{bmatrix} A - B\,R^{-1}g^T & B\,R^{-1}B^T \\ G - g\,R^{-1}g^T & -A^T + g\,R^{-1}B^T \end{bmatrix}. \tag{11}$$

In the case when all the coefficients in (11) are T-periodic, Yakubovich showed that the minimization problem admits a solution if and only if (i) the system (11) has exponential dichotomy (frequency condition) and (ii) certain solutions of (11) have no focal points (nonoscillation condition). We will consider the case when A, B, G, g, R are bounded continuous functions of time and prove a satisfactory generalization of Yakubovich's theorem. It turns out that the frequency condition and the nonoscillation condition (which can be stated as above) imply that the optimal control problem can be solved for all $\mathbf{x} \in \mathbb{R}^n$. The converse statement is not quite true; as a matter of fact, and roughly speaking, the minimizing control must exhibit a uniform continuity condition in order to ensure that the frequency condition and the nonoscillation condition are valid.

The frequency theorem has many ramifications and applications, some of which will be considered in this book. Here we mention that the frequency theorem can be used to comment on the Willems concept of dissipativity in the context of control systems. This connection was pointed out and analyzed in the periodic case, by Yakubovich et al. [158]. We will discuss the connection between the frequency theorem and the Willems dissipativity concept when the relevant coefficients are aperiodic functions of time.

The main point here is to interpret the integrand of the functional $\widetilde{\mathcal{I}}_{\mathbf{x}}(\mathbf{x},\mathbf{u})$ as a power function. To explain this, set $\mathbf{x} = \mathbf{0}$ in equation (7). Let $\mathbf{u}\colon [t_1, t_2] \to \mathbb{R}^m$ be an integrable function, and let $\mathbf{x}(t)$ be the corresponding solution of (7) with $\mathbf{x}(t_1) = \mathbf{0}$. Let us write

$$\mathcal{Q}(t,\mathbf{x},\mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\right) .$$

Then the net energy entering the system due to the effect of $\mathbf{u}(\cdot)$ is obtained by integrating $\mathcal{Q}(t,\mathbf{x}(t),\mathbf{u}(t))$ in the interval $[t_1, t_2]$. Now one says that the system is dissipative if

$$\int_{t_1}^{t_2} \mathcal{Q}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \ge 0$$

whenever $t_1 < t_2 \in \mathbb{R}$. That is, "energy must be expended" to move the system from its equilibrium position $\mathbf{x} = \mathbf{0}$.

The basic result which we will prove is that, modulo details, the control system determined by (7) together with $\mathcal{Q}(t,\mathbf{x},\mathbf{u})$ is (strongly) dissipative if and only if the Hamiltonian system (11) satisfies the frequency condition and the nonoscillation condition. So the frequency theorem has deep consequences concerning the structure of LQ control processes.

Outline of the Contents

We end this introduction with a brief outline of the contents of the various chapters which will follow.

The long Chap. 1 contains a discussion of various tools from topological dynamics and from ergodic theory which will be systematically used throughout the book. We discuss the Birkhoff theorem and the Oseledets theorem, the Bebutov construction and some facts concerning flows, the Sacker–Sell–Selgrade theory of exponential dichotomies, and other matters as well.

Chapters 2 and 3 contain the basic theory of the oscillation of the solutions of (1), respectively, as well as a dynamical approach to the spectral theory of the Atkinson problem (4). In Chap. 2, we construct and discuss the rotation number for (1), which is roughly speaking "the average number of focal points" admitted by a so-called conjoined basis of solutions. This quantity can be defined in several ways, using the Gel'fand–Lidskii–Yakubovich argument functions, the Maslov index, and the Barrett–Reid polar angles. In Chap. 3 we complexify the rotation number so as to obtain the Floquet exponent, a quantity which is quite useful in the study of problem (4). We state and prove a basic result, namely, that if (4) satisfies an Atkinson condition, then the rotation number $\alpha = \alpha(\lambda)$ of (4) is constant for λ in an open subinterval $\mathcal{I} \subset \mathbb{R}$ if and only if (4) admits exponential dichotomy for all $\lambda \in \mathcal{I}$.

The Weyl M-matrices, or M-functions, arise in Chap. 3 as a tool used in the study of the spectral theory of (4) and especially in the proof of the theorem relating the constancy of the rotation number to the presence of exponential dichotomy. The M-functions are defined for nonreal values of the parameter λ. However, it is very important to understand their convergence properties in the limit as $\operatorname{Im}\lambda$ tends to zero, and Chap. 4 is dedicated to a study of this issue. In particular, we work out an extension to the Atkinson problem (4) of the classical Kotani theory, which is an important tool in the study of the refined spectral properties of the Schrödinger operator.

The notion of disconjugacy is very important in the context of the Hamiltonian linear differential system (1), because of its significance in the calculus of variations. Chapter 5 is devoted to a discussion of a generalization of the concept of disconjugacy, namely, weak disconjugacy. Under natural and mild auxiliary hypotheses, we prove the existence of a principal solution when (1) is weakly disconjugate. Our approach to the issue of (weak) disconjugacy relies on the systematic use of tools of topological dynamics; these allow a deep understanding of the conditions under which weak disconjugacy holds and also of the properties of the principal solutions.

The book concludes with Chap. 6 (the LQ regulator problem and the Kalman–Bucy filter), Chap. 7 (the nonautonomous version of the Yakubovich frequency theorem), and Chap. 8 (Willems dissipativity for LQ control processes).

Note finally that, in this book, methods and results which have been developed in the course of 100 years in the context of linear Hamiltonian systems with constant or periodic coefficients are extended to systems whose coefficients can exhibit a much more general time dependence. Indeed, techniques of topological dynamics and of ergodic theory which have been worked out in recent times permit us to apply new methods and adapt older ones to the study of a rich set of new scenarios which are not possible in the periodic case. In the end we obtain a coherent theory which has been successfully applied to a wide range of problems in the setting of nonautonomous linear Hamiltonian systems.

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Contents

Chapter 1
Nonautonomous Linear Hamiltonian Systems

This chapter is devoted to the general explanation of the framework of the analysis made in this book, and to stating the many foundational facts which will be required. With the aim of being relatively self-contained, precise references where the proofs of the stated properties can be found are included, and at the same time some proofs which the reader may consider elementary or well known, but for which it is not easy to find a completely appropriate reference in the literature, are given.

This long chapter is divided into four sections. The first presents the most fundamental notions and properties of topological dynamics and ergodic theory, including the concept and main characteristics of a skew-product flow, which are fundamental for the book.

The second section summarizes basic results concerning spaces of matrices, the Grassmannian and Lagrangian manifolds, and matrix-valued functions.

Section 1.3 is devoted to the description of the general framework of the book. Under mild conditions on the coefficient matrix, a nonautonomous linear system of ordinary differential equations defines continuous skew-product flows on the trivial and Grassmannian bundles above a compact metric space. Special attention is devoted to the Hamiltonian case, for which two special skew-product flows can be defined. For the first one, which is defined on the Lagrange bundle, the use of generalized polar coordinates simplifies the task of describing the dynamical behavior. The second one, which is closely related to the first, is defined on the bundle given by the set of symmetric matrices. It presents some interesting monotonicity properties.

The last section concerns one of the most fundamental concepts for the development of the analysis made in the book: that of exponential dichotomy, both in the general linear case and in the linear Hamiltonian case. Many of the properties ensured by its presence will be described in detail, and then applied later in the book. The closely related concept of Sacker–Sell spectrum is also discussed, and several aspects of the Sacker–Sell perturbation theory are explained. The section is

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completed with the less standard analysis of the behavior of the Grassmannian flows in the presence of exponential dichotomy.

1.1 Some Fundamental Notions

The concepts and properties summarized in this section will be used often throughout the book, many times without reference to these initial pages. Suitable references for all these notions include Nemytskii and Stepanov [110], Ellis [41], Sacker and Sell [133], Cornfeld et al. [35], Walters [148], Mañé [99], and Rudin [128, 129].

1.1.1 Basic Concepts and Properties of Topological Dynamics

Let Ω be a locally compact Hausdorff topological space. Let Σ_Ω and $\Sigma_\mathbb{R}$ represent the Borel sigma-algebras of Ω and $\mathbb{R}$, and let $\Sigma_* = \Sigma_\mathbb{R} \times \Sigma_\Omega$ be the product sigma-algebra; i.e. the intersection of all the sigma-algebras on $\mathbb{R} \times \Omega$ containing the sets $\mathcal{I} \times \mathcal{A}$ for $\mathcal{I} \in \Sigma_\mathbb{R}$ and $\mathcal{A} \in \Sigma_\Omega$. Mild conditions on Ω ensure that Σ_* agrees with the Borel sigma-algebra of $\mathbb{R} \times \Omega$: it is enough to assume that Ω admits a countable basis of open sets (see e.g. Proposition 7.6.2 of Cohn [30]).

It will be convenient to work under the hypothesis that Σ_* is indeed the Borel sigma-algebra of $\mathbb{R}\times\Omega$. So, throughout Sect. 1.1, Ω will represent a locally compact Hausdorff topological space which admits a countable basis of open sets. In fact, throughout the book, any flow will be defined on a set which satisfies, at a minimum, these conditions. Some of the results explained in this section require Ω to be a compact metric space, but this hypothesis will be specified whenever it is assumed.

A map $\sigma\colon \mathbb{R} \times \Omega \to \Omega$ is *Borel measurable* if $\sigma^{-1}(\mathcal{A}) \in \Sigma_*$ for all $\mathcal{A} \in \Sigma_\Omega$. A *global real Borel measurable flow* on Ω is a Borel measurable map $\sigma\colon \mathbb{R} \times \Omega \to \Omega$ such that $\sigma_0 = \mathrm{Id}_\Omega$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for all $s, t \in \mathbb{R}$, where $\sigma_t\colon \Omega \to \Omega,\ \omega \mapsto \sigma(t,\omega)$. The flow is *continuous* if σ satisfies the stronger condition of being a continuous map, in which case each map σ_t is a homeomorphism on Ω with inverse σ_{-t}. The notation (Ω,σ) will be frequently used to represent a real global flow on Ω, and the words *real* and *global* will be omitted when no confusion arises.

The *orbit* of a point $\omega \in \Omega$ is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$, and its *positive* (resp. *negative*) *semiorbit* is $\{\sigma_t(\omega) \mid t \in \mathbb{R}_+\}$, where $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \ge 0\}$ (resp. $\{\sigma_t(\omega) \mid t \in \mathbb{R}_-\}$, where $\mathbb{R}_- = \{t \in \mathbb{R} \mid t \le 0\}$).

Given a Borel measurable flow (Ω,σ), a Borel subset $\mathcal{A} \subseteq \Omega$ (i.e. an element $\mathcal{A}$ of Σ_Ω) is *σ-invariant* (resp. *positively* or *negatively σ-invariant*) if $\sigma_t(\mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{R}$ (resp. $t \in \mathbb{R}_+$ or $t \in \mathbb{R}_-$). Let $\mathbb{Y}$ be a topological space. If Σ is a sigma-algebra on Ω containing the Borel sets, a map $f\colon \Omega \to \mathbb{Y}$ is *Σ-measurable* if $f^{-1}(\mathcal{B}) \in \Sigma$ for every Borel subset $\mathcal{B} \subseteq \mathbb{Y}$; and f is *Borel measurable* when it is Σ_Ω-measurable. A Borel measurable function $f\colon \Omega \to \mathbb{Y}$ is *σ-invariant* if $f(\sigma_t(\omega)) =$

$f(\omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. It is obvious that a Borel subset $\mathcal{A}$ is σ-invariant if and only if its characteristic function $\chi_{\mathcal{A}}$ is σ-invariant.

If Σ is a sigma-algebra containing the Borel sets, the concepts of σ-invariant set $\mathcal{A} \in \Sigma$ and σ-invariant Σ-measurable map $f : \Omega \to \mathbb{Y}$ are defined analogously. Note that in fact this concept of invariance can be extended to any set or function, since it does not depend on measurability.

All these definitions of *σ-invariance* correspond to *strict σ-invariance*, although the word *strict* will be almost always omitted. A less restrictive definition of invariance, depending on a fixed measure, is given in Sect. 1.1.2.

The flow is *local* if the map σ is defined, Borel measurable, and satisfies the two initially required properties on an open subset $\mathcal{O} \subseteq \mathbb{R} \times \Omega$ containing $\{0\} \times \Omega$. Define $\mathcal{O}_\omega = \{t \in \mathbb{R} \mid (t, \omega) \in \mathcal{O}\}$ for $\omega \in \Omega$. The *orbit* of the point ω for a local flow (Ω, σ) is $\{\sigma_t(\omega) \mid t \in \mathcal{O}_\omega\}$, and it is *globally defined* if $\mathcal{O}_\omega = \mathbb{R}$. The *positive* (resp. *negative*) *semiorbit* of a point ω is the set $\{\sigma_t(\omega) \mid t \in \mathcal{O}_\omega \cap \mathbb{R}_+\}$ (resp. $\{\sigma_t(\omega) \mid t \in \mathcal{O}_\omega \cap \mathbb{R}_-\}$, and it is *globally defined* if $\mathcal{O}_\omega \cap \mathbb{R}_+ = \mathbb{R}_+$ (resp. $\mathcal{O}_\omega \cap \mathbb{R}_- = \mathbb{R}_-$). A (in general Borel) subset $\mathcal{A} \subseteq \Omega$ is *σ-invariant* (resp. *positively* or *negatively σ-invariant*) if it is composed of globally defined orbits (resp. globally defined positive or negative semiorbits).

Finally, replacing $\mathbb{R}$ by $\mathbb{R}_+$ (resp. by $\mathbb{R}_-$) provides the definition of a (global or local) real positive (resp. negative) *semiflow* on Ω. The definitions of positive (resp. negative) semiorbit and (strict) invariance are the obvious ones.

For the remaining definitions and properties discussed in this section, the flow σ is assumed to be continuous.

A compact σ-invariant subset $\mathcal{M} \subseteq \Omega$ is *minimal* if it does not contain properly any other such set; or, equivalently, if each of its positive or negative semiorbits is dense in it. The flow (Ω, σ) is *minimal* or *recurrent* if Ω itself is minimal, which obviously requires Ω to be compact. Note that Zorn's lemma ensures that, if Ω is compact, then it contains at least one minimal subset.

Suppose that the positive semiorbit of a point ω_0 for such a flow is relatively compact. Then the *omega-limit set* of the point (or of its positive semiorbit) is given by those points $\omega \in \Omega$ such that $\omega = \lim_{k\to\infty} \sigma(t_k, \omega_0)$ for some sequence $(t_k) \uparrow \infty$. The omega-limit set is nonempty, compact, connected, and σ-invariant. The concept of *alpha-limit set* is analogous, working now with a negative semiorbit and with sequences $(t_k) \downarrow -\infty$. Clearly, a minimal subset of Ω is the omega-limit set and the alpha-limit set of each of its elements.

Finally, assume in addition that Ω is a compact metric space, and let d_Ω represent the distance on Ω. The flow (Ω, σ) is *chain recurrent* if given $\varepsilon > 0$, $t_0 > 0$, and points $\omega, \tilde{\omega} \in \Omega$, there exist points $\omega = \omega_0, \omega_1, \ldots, \omega_m = \tilde{\omega}$ of Ω and real numbers $t_1 > t_0, \ldots, t_m > t_0$ such that $d_\Omega(\sigma_{t_i}(\omega_i), \omega_{i+1}) < \varepsilon$ for $i = 0, \ldots, m-1$. It is easy to check that minimality implies chain recurrence: just take $\omega_0 = \omega$ and $\omega_1 = \tilde{\omega}$ and keep in mind that the positive semiorbit of ω is dense in Ω. It is also easy to check that if (Ω, σ) is chain recurrent, then the set Ω is connected.

1.1.2 Basic Concepts and Properties of Measure Theory

Unless otherwise indicated, any measure appearing in the book is a positive normalized regular Borel measure. Given such a measure m, let Σ_m be the m-completion of the Borel sigma-algebra (see e.g. Theorem 1.36 of [128]), and represent with the same symbol m the extension of the initial measure to Σ_m. As usual, the notation "m-a.e." means *almost everywhere with respect to* m; "for m-a.e. $\omega \in \Omega$" means *for almost every* $\omega \in \Omega$; and $L^1(\Omega, m)$ represents the quotient set of Σ_m-measurable functions $f\colon \Omega \to \mathbb{R}$ with $\int_\Omega |f(\omega)|\, dm < \infty$ (so that two real functions represent the same class if they are m-a.e. equal, in which case they are the same element of $L^1(\Omega, m)$). See Sect. 1.2.4 for the general definitions of L^p spaces of matrix-valued functions on Ω.

Let m be a measure on Ω. Then m is *σ-invariant* if $m(\sigma_t(\mathcal{A})) = m(\mathcal{A})$ for every Borel subset $\mathcal{A} \subseteq \Omega$ and all $t \in \mathbb{R}$, which ensures the same property for every $\mathcal{A} \in \Sigma_m$. A Σ_m-measurable map $f\colon \Omega \to \mathbb{Y}$ (for a topological space $\mathbb{Y}$) is *σ-invariant with respect to m* if, for all $t \in \mathbb{R}$, $f(\sigma_t(\omega)) = f(\omega)$ m-a.e. And a subset $\mathcal{A} \in \Sigma_m$ is *σ-invariant with respect to m* if $\chi_\mathcal{A}$ has this property.

The expression "σ-invariant" (for sets, measures, or functions) will often be changed to "invariant" throughout the book, since in most cases no confusion arises.

Proposition 1.2 shows the relation between these concepts of σ-invariance with respect to m and the (strict) ones given in the previous section: it proves that, when moving for instance in the quotient space $L^1(\Omega, m)$, one can always consider that a "σ-invariant function" satisfies the "strict" definition. More information in this regard will be added in Proposition 1.5.

Remark 1.1 Recall that any Σ_m-measurable function $f\colon \Omega \to \mathbb{K}$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, agrees m-a.e. with a Borel measurable one (see [128], Lemma 1 of Theorem 8.12). In addition, if Σ is any sigma-algebra containing the Borel sets, and if a sequence $(f_n\colon \Omega \to \mathbb{K})$ of Σ-measurable functions converges everywhere to a function f, then f is Σ-measurable (see [128], Theorem 1.14). And, as a consequence of this last result, if $(f_n\colon \Omega \to \mathbb{K})$ is a sequence of Σ_m-measurable functions which converges m-a.e. to a function f, then f is Σ_m-measurable.

Proposition 1.2 *Let (Ω, σ) be a Borel measurable flow, and let m be a σ-invariant measure on Ω.*

(i) *Let the Σ_m-measurable function $f\colon \Omega \to \mathbb{K}$ be σ-invariant with respect to m. Then there exists a Σ_m-measurable function $f^*\colon \Omega \to \mathbb{K}$ which is (strictly) σ-invariant such that $f = f^*$ m-a.e.*
(ii) *Let the set $\mathcal{A} \in \Sigma_m$ be σ-invariant with respect to m. Then there exists a (strictly) σ-invariant set $\mathcal{A}^* \in \Sigma_m$ such that $\chi_\mathcal{A} = \chi_{\mathcal{A}^*}$ m-a.e.*

Proof

(i) The proof of this property is carried out in Lemma 1 of Chapter 1.2 of [35], and included here for the reader's convenience. It follows from Remark 1.1 that there is no loss of generality in assuming that f is Borel measurable. Define the sets $\mathcal{N} = \{(t,\omega) \in \mathbb{R}\times\Omega \mid f(\omega) \neq f(\sigma_t(\omega))\}$, and note that the hypotheses on σ ensure that this set belongs to $\Sigma_* = \Sigma_{\mathbb{R}} \times \Sigma_{\Omega}$, since the maps $\mathbb{R}\times\Omega \to \mathbb{R}$, $(t,\omega) \mapsto f(\omega)$ and $\mathbb{R}\times\Omega \to \mathbb{R}$, $(t,\omega)\mapsto f(\sigma_t(\omega))$ are Σ_*-measurable. Define now $\mathcal{N}_t = \{\omega \in \Omega \mid (t,\omega) \in \mathcal{N}\}$ for $t \in \mathbb{R}$, and $\mathcal{N}_\omega = \{t \in \mathbb{R} \mid (t,\omega) \in \mathcal{N}\}$ for $\omega \in \Omega$, and note that $\mathcal{N}_t \in \Sigma_\Omega$ for all $t \in \mathbb{R}$ and $\mathcal{N}_\omega \in \Sigma_{\mathbb{R}}$ for all $\omega \in \Omega$ (see Theorem 8.2 of [128]). By definition of σ-invariance with respect to m, $m(\mathcal{N}_t) = 0$ for all $t \in \mathbb{R}$. Define μ as the product measure of m and l on $\Omega \times \mathbb{R}$, where l is the Lebesgue measure on $\mathbb{R}$. Fubini's theorem (see Theorem 8.8 of [128]) ensures that the maps $\omega \mapsto l(\mathcal{N}_\omega)$ and $t \mapsto m(\mathcal{N}_t)$ are Borel, and that $\mu(\mathcal{N}) = \int_\Omega l(\mathcal{N}_\omega)\,dm = \int_{\mathbb{R}} m(\mathcal{N}_t)\,dl = 0$. Therefore the subset $\Omega_f \subseteq \Omega$ of points ω with $l(\mathcal{N}_\omega) = 0$ is Borel, and $m(\Omega_f) = 1$. Suppose that ω and $\sigma_t(\omega)$ belong to Ω_f for a pair $(t,\omega) \in \mathbb{R}\times\Omega$. Then $f(\omega) = f(\sigma_t(\omega))$. In order to prove this assertion, take $s \in \mathbb{R} - \mathcal{N}_{\sigma_t(\omega)}$ such that $s + t \in \mathbb{R} - \mathcal{N}_\omega$, and note that $f(\sigma_t(\omega)) = f(\sigma_s(\sigma_t(\omega))) = f(\sigma_{s+t}(\omega)) = f(\omega)$. Now define

$$f^*(\omega) = \begin{cases} f(\omega) & \text{if there exists } t \in \mathbb{R} \text{ with } \sigma_t(\omega) \in \Omega_f\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

which is Σ_m-measurable, since it agrees with f at least on Ω_f (and hence m-a.e.), and which is σ-invariant in the classical sense.

(ii) Let $g = \chi^*_{\mathcal{A}}$ be the σ-invariant function associated to $\chi_{\mathcal{A}}$ by (i). Then the set $\mathcal{B} = \{\omega \in \Omega \mid g(\omega) \in \{0,1\}\} = 1$ belongs to Σ_m, is σ-invariant, and has full measure for m: $m(\mathcal{B}) = 1$. The set $\mathcal{A}^* = \{\omega \in \Omega \mid g(\omega) = 1\} \subseteq \mathcal{B}$ also belongs to Σ_m and is σ-invariant. In addition, $g(\omega) = \chi_{\mathcal{A}^*}(\omega)$ for all $\omega \in \mathcal{B}$, so that $\chi_{\mathcal{A}} = \chi_{\mathcal{A}^*}$ m-a.e., as asserted.

One of the most fundamental results in measure theory is the Birkhoff ergodic theorem, one of whose simplest versions is now recalled.

Theorem 1.3 *Let (Ω,σ) and m be a Borel measurable flow and a σ-invariant measure on Ω. Given $f \in L^1(\Omega,m)$, there exists a (strictly) σ-invariant set $\Omega_f \in \Sigma_m$ with $m(\Omega_f) = 1$ such that, for all $\omega \in \Omega_f$, the limits*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\sigma_s(\omega))\,ds = \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^t f(\sigma_s(\omega))\,ds = \lim_{t\to-\infty}\frac{-1}{t}\int_t^0 f(\sigma_s(\omega))\,ds$$

exist, agree, and take on a real value $\tilde f(\omega)$. In addition, $\tilde f(\sigma_t(\omega)) = \tilde f(\omega)$ for all $\omega \in \Omega_f$ and $t \in \mathbb{R}$, $\tilde f$ belongs to $L^1(\Omega,m)$, and $\int_\Omega \tilde f(\omega)\,dm = \int_\Omega f(\omega)\,dm$.

Its proof in the case of a discrete flow (given by the iteration of an automorphism on Ω) can be found, for example, in Section II.1 of [99]. The procedure to deduce the result for a real flow from the discrete case is standard: define the automorphism $T(\omega) = \sigma(1,\omega)$ and, given $f \in L^1(\Omega,m)$, define $F(\omega) = \int_0^1 f(\sigma_s(\omega))\,ds$; then, Fubini's theorem ensures that $F \in L^1(\Omega,m)$, and the application of the discrete version of the theorem to this setting provides the sets Ω_f and the function $\tilde f$ satisfying the theses of the real version. The details are left to the reader.

Note that the function $\tilde{f}$ provided by the previous theorem can be considered to be σ-invariant in the strict sense: just define it to be 0 outside Ω_f. Note also that the set Ω_f contains a Borel subset with measure 1, which is clearly σ-invariant with respect to m. But in fact this Borel subset of Ω_f can be taken as a (strictly) σ-invariant set, as Proposition 1.5(i) below proves. Therefore, there is no loss of generality in assuming that the set Ω_f itself is Borel.

The following result, whose proof is included for completeness, will be required in Chap. 4. The notation $g\colon \Omega \to [0,\infty]$ is used for *extended-real* functions (which can take the value ∞), and the concept of Σ_m-measurability for such a function is clear.

Proposition 1.4 *Let (Ω,σ) be a Borel measurable flow, and let m be a σ-invariant measure on Ω. Let $f\colon \Omega \to [0,\infty)$ be a Σ_m-measurable function. Then, there exists a (strictly) σ-invariant set $\Omega_f \in \Sigma_m$ with $m(\Omega_f) = 1$ such that, for all $\omega \in \Omega_f$, the limits*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\sigma_s(\omega))\,ds = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^t f(\sigma_s(\omega))\,ds = \lim_{t\to-\infty} \frac{-1}{t}\int_t^0 f(\sigma_s(\omega))\,ds$$

exist, agree, and take a value $\tilde{f}(\omega) \in \mathbb{R}\cup\{\infty\}$. In addition, the extended-real function $\tilde{f}\colon \Omega \to [0,\infty]$ is Σ_m-measurable, and it satisfies $\tilde{f}(\sigma_t(\omega)) = \tilde{f}(\omega)$ for all $\omega \in \Omega_f$ and $t \in \mathbb{R}$, and $\int_\Omega \tilde{f}(\omega)\,dm = \int_\Omega f(\omega)\,dm$.

Proof Let $h\colon \Omega \to [0,\infty)$ be a Σ_m-measurable function. For each $k \in \mathbb{N}$, define $h_k = \min(h,k)$, which obviously belongs to $L^1(\Omega,m)$. Hence there exists a function $\tilde{h}_k \in L^1(\Omega,m)$ and a set $\Omega_{h_k} \in \Sigma_m$ with $m(\Omega_{h_k}) = 1$ satisfying the theses of Theorem 1.3. Define $\Omega_h^* = \cap_{k\in\mathbb{N}}\Omega_{h_k}$, which belongs to Σ_m, is σ-invariant, and has full measure for m. Note that the nondecreasing sequence $(h_k(\omega))$ converges to $h(\omega)$ for all $\omega \in \Omega_h^*$, and define $h^*(\omega) \in [0,\infty]$ as the limit of the nondecreasing sequence of σ-invariant functions $(\tilde{h}_k(\omega))$, also for $\omega \in \Omega_h^*$. Then, h^* is Σ_m-measurable (see Remark 1.1) and σ-invariant. In addition, if $h^* \in L^1(\Omega,m)$, then $h \in L^1(\Omega,m)$: apply the Lebesgue monotone convergence theorem and the Birkhoff Theorem 1.3 to get $0 \le \int_\Omega h(\omega)\,dm = \lim_{k\to\infty}\int_\Omega h_k(\omega)\,dm = \lim_{k\to\infty}\int_\Omega \tilde{h}_k(\omega)\,dm = \int_\Omega h^*(\omega)\,dm < \infty$.

Returning to the function f of the statement, note that if $f \in L^1(\Omega,m)$, the assertions follow from Theorem 1.3. Assume hence that $\int_\Omega f(\omega)\,dm = \infty$, and associate to it the sequences (f_k) and $(\tilde{f}_k)$, the set Ω_f^*, and the function f^*, as above. Therefore, $f^* \notin L^1(\Omega,m)$. Clearly, the sets

$$\mathcal{A} = \{\omega \in \Omega_f^* \mid f^*(\omega) = \infty\}\,,$$
$$\mathcal{A}_j = \{\omega \in \Omega_f^* \mid j \le f^*(\omega) < j+1\} \quad \text{for } j \ge 0$$

belong to Σ_m, are σ-invariant and disjoint, and satisfy $\Omega_f^* = \mathcal{A} \cup (\cup_{j=0}^{\infty} \mathcal{A}_j)$. Then, if $\omega \in \mathcal{A}$,

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(\sigma_s(\omega))\,ds &\geq \sup_{k\in\mathbb{N}} \lim_{t\to\infty} \frac{1}{t}\int_0^t f_k(\sigma_s(\omega))\,ds \\ &= \sup_{k\in\mathbb{N}} \tilde{f}_k(\omega)\,ds = f^*(\omega) = \infty\,,\end{aligned}$$

so that there exists $\lim_{t\to\infty}(1/t)\int_0^t f(\sigma_s(\omega))\,ds = f^*(\omega) = \infty$. The same property holds for the other two limits of the proposition. Now define

$$g = \sum_{j=0}^{\infty} \frac{1}{j+1}\,\chi_{\mathcal{A}_j} f$$

on Ω_f^*, note that it is Σ_m-measurable, and associate to it the sequences (g_k), $(\tilde{g}_k)$, and the set $\Omega_g^* \subseteq \Omega_f^*$, as at the beginning of the proof. Fix any $k \in \mathbb{N}$ and any $\omega \in \Omega_g^*$ outside $\mathcal{A}$, and take the unique $j \in \mathbb{N}$ such that $\omega \in \mathcal{A}_j \cap \Omega_g^*$. Then $g(\omega) = (1/j+1)f(\omega)$, and hence

$$g_k(\omega) = \frac{1}{j+1}\,\min(f(\omega), k(j+1)) = \frac{1}{j+1} f_{k(j+1)}(\omega)\,.$$

Since $\sigma_s(\omega) \in \mathcal{A}_j \cap \Omega_g^*$ for all $s \in \mathbb{R}$,

$$\begin{aligned}\tilde{g}_k(\omega) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t g_k(\sigma_s(\omega))\,ds = \frac{1}{j+1}\lim_{t\to\infty}\frac{1}{t}\int_0^t f_{k(j+1)}(\sigma_s(\omega))\,ds \\ &= \frac{1}{j+1}\tilde{f}_{k(j+1)}(\omega) \leq \frac{1}{j+1} f^*(\omega) \leq 1\end{aligned}$$

for all $k \in \mathbb{N}$. Note that g_k vanishes outside $\cup_{j=1}^{\infty}\mathcal{A}_j$. Hence $\int_\Omega g_k(\omega)\,dm = \int_\Omega \tilde{g}_k(\omega)\,dm \leq 1$, so that the Lebesgue dominated convergence theorem ensures that $g \in L^1(\Omega, m_0)$. Let $\tilde{g}$ and $\Omega_g \subseteq \Omega_f^*$ be the σ-invariant function and subset associated to g by Theorem 1.3, with $m(\Omega_g) = 1$. Then for all ω in the σ-invariant set $\mathcal{A}_j \cap \Omega_g, f(\omega) = (j+1)\,g(\omega)$ and hence

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\sigma_s(\omega))\,ds &= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^t f(\sigma_s(\omega))\,ds \\ &= \lim_{t\to-\infty}\frac{-1}{t}\int_t^0 f(\sigma_s(\omega))\,ds = (j+1)\,\tilde{g}(\omega) = (j+1)\,\chi_{\mathcal{A}_j}\tilde{g}(\omega)\,.\end{aligned}$$

Define $\Omega_f = \mathcal{A} \cup \Big((\cup_{j=0}^{\infty} \mathcal{A}_j) \cap \Omega_g \Big)$, and note that it belongs to Σ_m and satisfies $m(\Omega_f) = 1$. This σ-invariant set and the Σ_m-measurable and σ-invariant function

$$\tilde{f} = \begin{cases} f^*(\omega) & \text{if } \omega \in \mathcal{A} \\ \displaystyle\sum_{j=0}^{\infty} (j+1)\chi_{\mathcal{A}_j} \tilde{g} & \text{if } \omega \in (\cup_{j=0}^{\infty} \mathcal{A}_j) \cap \Omega_g \end{cases} \tag{1.1}$$

satisfy the statements regarding the limits. In addition, for all $\omega \in \Omega_f$,

$$\tilde{f}(\omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\sigma_s(\omega))\, ds \geq \lim_{t\to\infty} \frac{1}{t} \int_0^t f_n(\sigma_s(\omega))\, ds = \tilde{f}_n(\omega)\,,$$

so that $\tilde{f}(\omega) \geq f^*(\omega)$ on Ω_f. Hence, $\int_\Omega \tilde{f}(\omega)\, dm \geq \int_\Omega f^*(\omega)\, dm = \infty = \int_\Omega f(\omega)\, dm$, which completes the proof.

As in the case of Theorem 1.3, the function $\tilde{f}$ provided by Proposition 1.4 can be considered to be σ-invariant in the strict sense, and Proposition 1.5(i), which is proved immediately below, ensures that the set Ω_f contains a Borel subset with measure 1 which is σ-invariant with respect to m.

Proposition 1.5 *Let (Ω, σ) be a Borel measurable flow, and let m be a σ-invariant measure on Ω.*

(i) *Let $\mathcal{A} \in \Sigma_m$ be a (strictly) σ-invariant set with $m(\mathcal{A}) = 1$. Then $\mathcal{A}$ contains a (strictly) σ-invariant Borel set $\mathcal{B}$ with $m(\mathcal{B}) = 1$.*
(ii) *Let $f\colon \Omega \to \mathbb{R}$ be Σ_m-measurable and σ-invariant with respect to m_0. Then there exists $g\colon \Omega \to \mathbb{R}$ which is Borel and (strictly) σ-invariant such that $g = f$ m-a.e.*

Proof

(i) It suffices to prove that for all $n \in \mathbb{N}$ there exists a σ-invariant Borel set $\mathcal{B}_n \subseteq \mathcal{A}$ with $m(\mathcal{B}_n) \geq m(\mathcal{A}) - 1/n$, and then take $\mathcal{B} = \cup_{n\geq 1} \mathcal{B}_n$.

Fix $n \in \mathbb{N}$, and note that the regularity of the measure m implies the existence of a compact set $\mathcal{K}_n \subseteq \mathcal{A}$ with $m(\mathcal{A} - \mathcal{K}_n) \leq 1/n$. The Borel measurability of the flow ensures that the map $\mathbb{R} \times \Omega \to \mathbb{R}\,,\ (t, \omega) \mapsto \chi_{\mathcal{K}_n}(\sigma(t, \omega))$ is Borel measurable, and hence Fubini's theorem guarantees that the maps $h_n^j\colon \Omega \to \mathbb{R}$ given by

$$h_n^j(\omega) = \sum_{i=-j}^{j} \frac{1}{|i|^2 + 1} \int_j^{j+1} \chi_{\mathcal{K}_n}(\sigma_t(\omega))\, dt$$

are Borel measurable (see e.g. Theorem 8.8 of [128]). Clearly, $h_n^j \leq h_n^{j+1}$, so that the limit $h_n(\omega) = \lim_{j\to\infty} h_n^j(\omega)$ exists for all $\omega \in \Omega$, and the (bounded) function h_n is Borel measurable. Define

$$\mathcal{B}_n = \{\omega \in \Omega \mid \sigma_t(\omega) \in \mathcal{K}_n \text{ for all } t \text{ in a set of positive Lebesgue measure}\}\,,$$

which is contained in $\mathcal{A}$ and is Borel, since it agrees with $h_n^{-1}((0,\infty))$. Clearly, $\mathcal{B}_n$ is (strictly) σ-invariant. The Birkhoff Theorem 1.3 ensures that the limits $l_n(\omega) = \lim_{t\to\infty}(1/2t)\int_{-t}^{t}\chi_{\mathcal{K}_n}(\omega{\cdot}s)\,ds$ exist for all ω in a σ-invariant subset $\Omega_n \in \Sigma_m$ with $m(\Omega_n) = 1$, and that the function l_n is Σ_m-measurable and σ-invariant in Ω_n. Now write $\Omega_n = \Omega_n^0 \cup \Omega_n^+$, where $\Omega_n^0 = \{\omega \in \Omega_n \mid l_n(\omega) = 0\}$ and $\Omega_n^+ = \{\omega \in \Omega_n \mid l_n(\omega) > 0\}$, and note that these sets belong to Σ_m and are σ-invariant. Applying again the Birkhoff Theorem 1.3 to the function $\chi_{\mathcal{K}_n\cap\Omega_n^0} \le \chi_{\mathcal{K}_n}$ one proves that $m(\mathcal{K}_n \cap \Omega_n^0) = \int_\Omega \chi_{\mathcal{K}_n\cap\Omega_n^0}(\omega)\,dm = 0$. On the other hand, it is clear that $\Omega_n^+ \subseteq \mathcal{B}_n$. Since

$$m(\Omega_n^0) = m(\mathcal{K}_n \cap \Omega_n^0) + m((\Omega - \mathcal{K}_n)\cap\Omega_n^0) \le m(\Omega - \mathcal{K}_n) \le \frac{1}{n}$$

and

$$m(\mathcal{B}_n) \ge m(\Omega_n^+) = 1 - m(\Omega_n^0) \ge 1 - \frac{1}{n}\,,$$

the set $\mathcal{B}_n$ satisfies the required conditions.

(ii) Remark 1.1 and the definition of σ-invariance with respect to m show that there is no loss of generality in assuming that the function f is Borel measurable. Note also that $f = f^+ - f^-$ for $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$, which are Borel measurable, σ-invariant with respect to m, and nonnegative; hence it is enough to prove the result for $f \ge 0$. Now, on the one hand, repeating the argument of Proposition 1.2 one can check that the Borel set $\mathcal{N}_\omega = \{t \in \mathbb{R} \mid f(\sigma_t(\omega)) \ne f(\omega)\}$ has zero Lebesgue measure for all the points ω in a Borel set $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 1$. And, on the other hand, Proposition 1.4 provides a σ-invariant set $\Omega_f \in \Sigma_m$ with $m(\Omega_f) = 1$ and an extended-real Σ_m-measurable σ-invariant function $\tilde f$ such that $\tilde f(\omega) = \lim_{t\to\infty}(1/t)\int_0^t f(\sigma_s(\omega))\,ds$ exists for any $\omega \in \Omega_f$. Note that if $\omega \in \Omega_0 \cap \Omega_f$, then $\tilde f(\omega)$ exists and agrees with $f(\omega)$, so that the (σ-invariant) function $\tilde f$ takes real values in a σ-invariant and Σ_m-measurable set $\widetilde{\Omega}_f \subset \Omega_f$ with $m(\widetilde{\Omega}_f) = 1$. The already verified point (i) guarantees the existence of a Borel σ-invariant set $\mathcal{B} \subseteq \widetilde{\Omega}_f$ with $m(\mathcal{B}) = 1$. Define $g = \tilde f\chi_{\mathcal{B}}$, and note that $g(\omega) = \lim_{t\to\infty}(1/t)\int_0^t f(\sigma_s(\omega))\,\chi_{\mathcal{B}}(\sigma_s(\omega))\,ds$ for any $\omega \in \Omega$. A new application of Fubini's theorem ensures that the map $\omega \mapsto (1/t)\int_0^t f(\sigma_s(\omega))\,\chi_{\mathcal{B}}(\sigma_s(\omega))\,ds$ is Borel for any $t \in \mathbb{R}$, so that g is Borel (see Remark 1.1). Clearly, it is also σ-invariant. And it agrees with f on $\mathcal{B} \cap \Omega_0$, which completes the proof.

A (positive normalized regular Borel) measure m is *σ-ergodic* if it is invariant and, in addition, any σ-invariant set has measure 0 or 1. The following fundamental property will often be applied in combination with Theorem 1.3, which associates a σ-invariant function $\tilde{f} \in L^1(\Omega, m)$ to each $f \in L^1(\Omega, m)$.

Theorem 1.6 *Let (Ω, σ) and m be a Borel measurable flow and a σ-invariant measure on Ω. The measure m is σ-ergodic if and only if every σ-invariant function $f \in L^1(\Omega, m)$ is constant m-a.e. In other words, if and only if for every $f \in L^1(\Omega, m)$ there exists a (strictly) σ-invariant set $\Omega_f \in \Sigma_m$ with $m(\Omega_f) = 1$ such that $f(\omega_0) = \int_\Omega f(\omega)\, d\mu$ for every $\omega_0 \in \Omega_f$.*

The direct implication can be proved as (1)⇒(2) in Proposition II.2.1 of [99]. As for the converse implication: if $m(\mathcal{A}) \in (0, 1)$ for a σ-invariant subset $\mathcal{A} \subseteq \Omega$, then $\chi_\mathcal{A}$ is a nonconstant σ-invariant integrable function. Note once more that the sets Ω_f of the previous statement can be assumed to be Borel.

The following basic characterization of invariance will be useful in the proofs of several results.

Proposition 1.7 *Let (Ω, σ) be a Borel measurable flow, and let m be a measure on Ω. The following statements are equivalent:*

(1) *m is σ-invariant;*
(2) *$\int_\Omega f(\omega)\, dm = \int_\Omega f(\sigma_t(\omega))\, dm$ for all $f \in L^1(\Omega, m)$ and all $t \in \mathbb{R}$;*
(3) *$\int_\Omega f(\omega)\, dm = \int_\Omega f(\sigma_t(\omega))\, dm$ for all $f \in C(\Omega, \mathbb{R})$ and all $t \in \mathbb{R}$.*

Proof (1)⇒(2) If the measure is invariant, then $\int_\Omega s(\omega)\, dm = \int_\Omega s(\sigma_t(\omega))\, dm$ for every simple function s. Take a nonnegative function $f \in L^1(\Omega, m)$ and choose a nondecreasing sequence (s_k) of nonnegative simple functions such that $f(\omega) = \lim_{m\to\infty} s_k(\omega)$ for all $\omega \in \Omega$ (see [128], Theorem 1.17). Hence $f(\sigma_t(\omega)) = \lim_{m\to\infty} s_k(\sigma_t(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. Now apply the Lebesgue monotone convergence theorem in order to prove that

$$\int_\Omega f(\omega)\, dm = \lim_{k\to\infty} \int_\Omega s_k(\omega)\, dm = \lim_{k\to\infty} \int_\Omega s_k(\sigma_t(\omega))\, dm = \int_\Omega f(\sigma_t(\omega))\, dm\,.$$

Finally, any function $f \in L^1(\Omega, m)$ can be written as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are nonnegative elements of $L^1(\Omega, m)$. This proves (2).

(2)⇒(3) This property is obvious.

(3)⇒(1) Property (3) and the Lebesgue monotone convergence theorem yield $m(\mathcal{K}) = m(\sigma_t(\mathcal{K}))$ whenever $\mathcal{K} \subseteq \Omega$ is compact and $t \in \mathbb{R}$: just take a decreasing sequence of positive and continuous functions (f_k) with pointwise limit $\chi_\mathcal{K}$ and with bound 1. (For instance, $f_k(\omega) = 1/(1 + k\, d_\Omega(\omega, \mathcal{K}))$, where d_Ω is the distance in Ω and $d_\Omega(\omega, \mathcal{K}) = \inf_{\tilde{\omega}\in\mathcal{K}} d_\Omega(\omega, \tilde{\omega})$.) Hence, the regularity of the measure m ensures the same property for every Borel set $\mathcal{A} \subseteq \Omega$.

The occurrence or lack of invariant and ergodic measures is a fundamental question in measure theory. There are examples of noncontinuous flows on compact metric spaces (see [99], Exercise I.8.6) as well as more basic examples of continuous flows on noncompact spaces which do not admit any normalized invariant measure.

However, the situation is better when dealing with a continuous flow on a compact metric space, as stated in Theorem 1.8. This will be the setting from now on: until the end of this section, (Ω, σ) will represent a continuous flow on a compact metric space. A complete proof of Theorem 1.8 in the case of a discrete flow can be found in [99], Section I.8, and for a real flow in [110], Theorem 9.05 of Chapter VI. In fact the result was initially proved by Krylov and Bogoliubov [94].

Theorem 1.8 *Let σ be a continuous flow on a compact metric space Ω. Then there exists at least one σ-invariant measure on Ω.*

In order to deduce the existence of σ-ergodic measures from the above result, which is one of the assertions of the following theorem, consider the set $\mathfrak{M}(\Omega)$ of positive normalized regular Borel measures on Ω endowed with the weak* topology: the sequence of measures (m_k) converges to m if and only if $\lim_{k\to\infty} \int_\Omega f(\omega)\, dm_k = \int_\Omega f(\omega)\, dm$ for every continuous function $f\colon \Omega \to \mathbb{R}$. Then, $\mathfrak{M}(\Omega)$ is a metrizable compact space (see e.g. Theorems 6.4 and 6.5 of [148]), and it is clearly convex: any convex combination of measures $m_1, \ldots, m_n$ in $\mathfrak{M}(\Omega)$ (i.e. the sum $\lambda_1 m_1 + \ldots \lambda_n m_n$, where $\lambda_1, \ldots, \lambda_n \in [0, 1]$ and $\sum_{j=1}^n \lambda_j = 1$), belongs to $\mathfrak{M}(\Omega)$. Recall that given a convex subset $\mathfrak{M}$ of $\mathfrak{M}(\Omega)$, a point m is *extremal* if the equality $m = a\, m_1 + (1 - a)\, m_2$ for $a \in [0, 1]$ and $m_1, m_2 \in \mathfrak{M}$ ensures that $a \in \{0, 1\}$; and that the *closed convex hull* of a subset $\mathfrak{M}_1 \subseteq \mathfrak{M}$ is the closure of the set of convex combinations of points of $\mathfrak{M}_1$.

Theorem 1.9 *Let σ be a continuous flow on a compact metric space Ω.*

(i) *The nonempty set $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ of σ-invariant measures is a compact convex subset of $\mathfrak{M}(\Omega)$.*
(ii) *$\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ is the closed convex hull of the subset of its extremal points.*
(iii) *An element of $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ is an extremal point if and only if it is a σ-ergodic measure.*

In particular, there exist σ-ergodic measures, and every σ-invariant measure on Ω can be written as the limit in the weak topology of a sequence of convex combinations of σ-ergodic measures on Ω.*

Proof The proof of points (i) and (iii) can be easily carried out by adapting to the real case the arguments of Theorem 6.10 of [148] for the discrete case. To this end, use Theorem 1.8 and Proposition 1.7. Point (ii) is an immediate consequence of (i) and Krein–Milman theorem (see e.g. Theorem 3.23 of [129]), and the last assertions follow from the previous ones.

Another classical way to deduce the existence of σ-ergodic measures from the existence of σ-invariant ones is to use the Choquet representation theorem.

Remark 1.10 Let σ be a continuous flow on a compact metric space Ω. The *ergodic component* of a σ-invariant measure m on Ω is defined as the set of points $\omega_0 \in \Omega$ such that

$$\int_\Omega f(\omega)\, dm = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\sigma_s(\omega_0))\, ds$$

for all $f \in C(\Omega, \mathbb{R})$. In other words, it is the intersection of all the σ-invariant sets Ω_f associated by Theorem 1.3 to the continuous functions f. It is not hard to check that Ω_f is a Borel set if f is continuous. The separability of $C(\Omega, \mathbb{R})$ for the topology given by the norm $\|f\|_\Omega = \max_{\omega\in\Omega} |f(\omega)|$ implies that the ergodic component is also a Borel set, and that it has measure 1 in the case that m is ergodic: the ergodicity and Theorem 1.6 ensures that $m(\Omega_f) = 1$ for every continuous function f.

Theorem 1.9 ensures that, if the flow σ is continuous (and Ω is not necessarily compact), any minimal subset $\mathcal{K} \subseteq \Omega$ *concentrates* at least one σ-ergodic measure; that is, there exists a σ-ergodic measure m on Ω such that $m(\mathcal{K}) = 1$. In general, one says that a measure m on Ω *is concentrated on a subset* if this subset has measure 1. Recall that every measure is normalized unless otherwise indicated.

Let m be a measure on a compact metric space. The *topological support* of m, $\operatorname{Supp} m$, is the set $\Omega - \mathcal{O}$, where $\mathcal{O} \subset \Omega$ is the largest open subset with $m(\mathcal{O}) = 0$. Obviously, $\operatorname{Supp} m$ is a compact subset of Ω, with $m(\operatorname{Supp} m) = 1$: the measure is concentrated on its support. In addition,

Proposition 1.11 *Let (Ω, σ) be a continuous flow on a compact metric space and let* $\operatorname{Supp} m$ *be the topological support of a σ-invariant measure m.*

(i) *Suppose that* $\operatorname{Supp} m = \Omega$*, and let $\Omega_0 \subseteq \Omega$ satisfy $m(\Omega_0) = 1$. Then Ω_0 is dense in Ω.*
(ii) *If m is σ-invariant, so is* $\operatorname{Supp} m$.
(iii) *If m is σ-invariant and Ω is minimal, then* $\operatorname{Supp} m = \Omega$*. In fact Ω is minimal if and only if any σ-ergodic measure has full support.*

Proof

(i) Suppose for contradiction the existence of a nonempty open subset $\mathcal{O} \subset \Omega$ with $\Omega_0 \cap \mathcal{O}$ empty. Then $m(\mathcal{O}) = 0$, so that $\mathcal{O}$ is contained in the $\Omega - \operatorname{Supp} m$, which is empty. (Note that, in fact, the invariance of the measure is not required for this property.)
(ii) Let $\mathcal{O}$ be as in the definition of $\operatorname{Supp} m$. Then $\sigma_t(\mathcal{O})$ is open and $m(\sigma_t(\mathcal{O})) = m(\mathcal{O}) = 0$ for all $t \in \mathbb{R}$, so that $\sigma_t(\mathcal{O}) = \mathcal{O}$. That is, $\sigma_t(\operatorname{Supp} m) = \operatorname{Supp} m$ for all $t \in \mathbb{R}$.
(iii) $\operatorname{Supp} m$ is compact, since Ω is so. Hence, the first property in (iii) follows from (ii). The "if" assertion follows from Theorems 1.8 and 1.9: if $\mathcal{M} \subsetneq \Omega$ is a compact σ-invariant set, it concentrates a σ-ergodic measure m, and hence $\operatorname{Supp} m \subsetneq \Omega$.

The following property will be required several times in the book. Its proof is included here for the reader's convenience.

Proposition 1.12 *Let* (Ω, σ) *be a continuous flow on a compact metric space. Suppose that* $\Omega = \operatorname{Supp} m$ *for a* σ*-ergodic measure* m*. Then there exist subsets* $\Omega^{\pm} \subseteq \Omega$ *with* $m(\Omega^{\pm}) = 1$ *such that the positive* σ*-semiorbit of any* $\omega \in \Omega^{+}$ *and the negative* σ*-semiorbit of any* $\omega \in \Omega^{-}$ *are dense in* Ω*. In particular,* (Ω, σ) *is chain recurrent. In addition,* Ω *agrees with the omega-limit of any point* $\omega \in \Omega^{+}$ *and with the alpha-limit of any point* $\omega \in \Omega^{-}$*.*

Proof Let $\{\mathcal{O}_k \mid k \geq 1\}$ be a countable basis of open subsets of the compact set Ω. Since $\Omega = \operatorname{Supp} m$, then $m(\mathcal{O}_k) > 0$ for each $k \geq 1$. It follows from the Birkhoff Theorems 1.3 and 1.6 that the set

$$\Omega_k^{+} = \{\omega \in \Omega \mid \sigma_t(\omega) \in \mathcal{O}_k \text{ for some } t > 0\}$$

has measure 1, and hence also the countable intersection $\Omega^{+} = \bigcap_{k \geq 1} \Omega_k$ has full measure for m. Obviously any point in this intersection has dense positive semiorbit. The set Ω^{-} is defined from

$$\Omega_k^{-} = \{\omega \in \Omega \mid \sigma_t(\omega) \in \mathcal{O}_k \text{ for some } t < 0\}\,.$$

The chain recurrence follows easily from the fact that any point in the set $\Omega^{+} \cap \Omega^{-}$ has dense positive and negative semiorbits.

Take now $\omega \in \Omega^{+}$. If its σ-orbit is periodic, then its positive σ-semiorbit is finite and dense, and hence $\Omega = \mathcal{O}(\omega)$. Assume that this is not the case. Then the point $\omega{\cdot}1$ belongs to the closure of the positive semiorbit of ω (which agrees with Ω) but not to the orbit. Therefore $\omega{\cdot}1 \in \mathcal{O}(\omega)$, which ensures that $\{\sigma_t(\omega) \mid t \geq 0\} \subseteq \mathcal{O}(\omega)$. Since $\mathcal{O}(\omega)$ is closed, it follows that $\Omega = \text{closure}_{\Omega}\{\sigma_t(\omega) \mid t \geq 0\} \subseteq \mathcal{O}(\omega)$. This proves the last assertion in the case of Ω^{+}, and a similar argument proves it in the case of Ω^{-}.

Remarks 1.13

1. Note that in fact the last argument of the previous proof shows that if the positive (resp. negative) σ-semiorbit of a point $\omega \in \Omega$ is dense, then $\mathcal{O}(\omega) = \Omega$ (resp. $\mathcal{A}(\omega) = \Omega$).
2. It is easy to check that if Ω reduces to a point or is composed of just one periodic σ-orbit, then it admits a unique σ-invariant measure, which therefore is ergodic. In addition, it turns out that it has full topological support.

In most of the sections of the book, (Ω, σ) will indicate a fixed continuous flow on a compact metric space. The representation

$$\omega{\cdot}t = \sigma_t(\omega)$$

will be used from now on when no confusion may arise.

1.1.3 Skew-Product Flows

Let Ω and $\mathbb{Y}$ satisfy the conditions imposed on Ω in the previous section: they are locally compact Hausdorff topological spaces which admit countable bases of open sets. Hence, $\Omega \times \mathbb{Y}$ satisfies the same properties. In what follows, the product space $\Omega \times \mathbb{Y}$ is understood as a bundle over Ω: this is done throughout the book for several different spaces $\mathbb{Y}$. The sets Ω and $\mathbb{Y}$ will be referred to respectively as the *base* and the *fiber* of the bundle.

Let σ be a Borel measurable flow on Ω. A *skew-product flow on* $\Omega \times \mathbb{Y}$ *projecting onto* σ is a Borel measurable real flow

$$\tilde{\tau}\colon \mathbb{R} \times \Omega \times \mathbb{Y} \to \Omega \times \mathbb{Y}, \quad (\omega, y) \mapsto (\omega{\cdot}t, \tilde{\tau}_2(t, \omega, y))\,.$$

The flow (Ω, σ) is the *base flow* of $\tilde{\tau}$. Note that $\tilde{\tau}_2$ satisfies $\tilde{\tau}_2(s + t, \omega, y) = \tilde{\tau}_2(s, \omega{\cdot}t, \tau(t, \omega, y))$.

Some results concerning noncontinuous skew-product flows will be required in Chap. 4, and explained in the appropriate place. For the time being, let Ω and $\mathbb{Y}$ be compact metric spaces, and let $\tilde{\tau}$ be a continuous skew-product flow on $\Omega \times \mathbb{Y}$ with continuous base flow (Ω, σ). It is easy to check that, given a measure μ on $\Omega \times \mathbb{Y}$, the relation $m(\mathcal{A}) = \mu(\mathcal{A} \times \mathbb{Y})$ for every Borel set $\mathcal{A} \subseteq \Omega$ defines a measure m on Ω, which in addition is σ-invariant if μ is $\tilde{\tau}$-invariant. In this case, it is said that μ *projects onto m*.

Remark 1.14 In fact, μ projects onto m if and only if $\int_{\Omega\times\mathbb{Y}} f(\omega)\, d\mu = \int_{\Omega} f(\omega)\, dm$ for all $f \in C(\Omega, \mathbb{R})$. For the "if" assertion, keep in mind the regularity of the measures (see the proof of (3)⇒(1) in Proposition 1.7). The "only if" assertion is an easy consequence of the Lebesgue monotone convergence theorem (see the proof of (1)⇒(2) in Proposition 1.7).

The following result, whose proof is included for the reader's convenience, presents the well-known construction of a $\tilde{\tau}$-invariant measure projecting onto a fixed σ-ergodic measure m on Ω.

Proposition 1.15 *Let Ω and $\mathbb{Y}$ be compact metric spaces, and let $\tilde{\tau}$ be a continuous skew-product flow on $\Omega \times \mathbb{Y}$ with continuous base flow (Ω, σ). Let m be a σ-ergodic measure on Ω. Then,*

(i) *there exist $\tilde{\tau}$-invariant measures on $\Omega \times \mathbb{Y}$ projecting onto m.*
(ii) *The set $\mathfrak{M}_{\mathrm{inv},m}(\Omega \times \mathbb{Y}, \tilde{\tau})$ of the $\tilde{\tau}$- invariant measures projecting onto m is a convex compact set in the weak* topology.*
(iii) *There exist σ-ergodic measures on $\Omega \times \mathbb{Y}$ projecting onto m, and every $\tilde{\tau}$-invariant measure projecting onto m can be written as the limit in the weak* topology of a sequence of convex combinations of $\tilde{\tau}$-ergodic measures projecting onto m.*

Proof

(i) The Birkhoff Theorems 1.3 and 1.6 and the ergodicity of the measure m ensure that the set

$$\Omega_c = \left\{ \omega_0 \in \Omega \;\middle|\; \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(\omega_0{\cdot}s)\,ds = \int_\Omega f(\omega)\,dm \quad \forall f \in C(\Omega,\mathbb{R}) \right\}.$$

is σ-invariant and that $m(\Omega_c) = 1$: see Remark 1.10. Now fix $(\omega_0, y_0) \in \Omega_c \times \mathbb{Y}$. Let $C(\Omega\times\mathbb{Y},\mathbb{R})$ be the set of real continuous functions on the space $\Omega\times\mathbb{Y}$. Take also a sequence $(t_k) \uparrow \infty$. The Riesz representation theorem associates to the bounded linear functional defined by $C(\Omega\times\mathbb{Y},\mathbb{R}) \to \mathbb{R}$, $\tilde f \mapsto (1/(2t_k))\int_{-t_k}^{t_k} \tilde f(\tilde\tau(s,\omega_0,y_0))\,ds$, whose norm is 1, a (positive normalized regular Borel) measure μ_k, which satisfies

$$\int_{\Omega\times\mathbb{Y}} \tilde f(\omega,y)\,d\mu_k = \frac{1}{2t_k}\int_{-t_k}^{t_k} \tilde f(\tilde\tau(s,\omega_0,y_0))\,ds\,.$$

As stated in the previous section, the set of (positive normalized regular Borel) measures on $\Omega\times\mathbb{Y}$ is a metrizable compact set in the weak* topology. Therefore, the sequence (μ_k) admits a subsequence (μ_j) which converges weak* to a measure μ. That is,

$$\int_{\Omega\times\mathbb{Y}} \tilde f(\omega,y)\,d\mu = \lim_{j\to\infty}\frac{1}{2t_j}\int_{-t_j}^{t_j} \tilde f(\tilde\tau(s,\omega_0,y_0))\,ds$$

whenever $\tilde f \in C(\Omega\times\mathbb{Y},\mathbb{R})$. It follows easily from this fact, from the condition $\omega_0 \in \Omega_c$, and from Remark 1.14, that μ projects onto m. Note also that, if $l \in \mathbb{R}$ and $\tilde f \in C(\Omega\times\mathbb{Y},\mathbb{R})$, then

$$\begin{aligned}\int_{\Omega\times\mathbb{Y}} \tilde f\circ\tilde\tau_l(\omega,y)\,d\mu &= \lim_{j\to\infty}\frac{1}{2t_j}\int_{-t_j}^{t_j} \tilde f(\tilde\tau(s+l,\omega_0,y_0))\,ds\\ &= \lim_{j\to\infty}\frac{1}{2t_j}\int_{-t_j-l}^{t_j-l} \tilde f(\tilde\tau(s,\omega_0,y_0))\,ds = \int_{\Omega\times\mathbb{Y}} \tilde f(\omega,y)\,d\mu\,,\end{aligned}$$

as can be deduced from the boundedness of $\tilde f$. According to Proposition 1.7, this equality proves the $\tilde\tau$-invariance of μ, and completes the proof of (i).

(ii) Let $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$ be the set of the $\tilde\tau$-invariant measures on $\Omega\times\mathbb{Y}$ which project onto m. As in Theorem 1.9(i), an immediate application of the implication (3)⇒(1) of Proposition 1.7 proves that if a measure μ is the limit in the weak* topology of a sequence (μ_k) of elements of $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$, then $\mu \in \mathfrak{M}_{\mathrm{inv}}(\Omega\times\mathbb{Y},\tilde\tau)$. In order to check that μ projects onto m, keep in mind Remark 1.14, and note that if $f \in C(\Omega,\mathbb{R})$, then $\int_\Omega f(\omega)\,d\mu =$

$\lim_{k\to\infty}\int_\Omega f(\omega)\,d\mu_k = \lim_{k\to\infty}\int_\Omega f(\omega)\,dm = \int_\Omega f(\omega)\,dm$. The convexity of $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$ is clear.

(iii) As in the proof of Theorem 1.9(ii), properties (i) and (ii) allow one to apply the Krein–Milman theorem to prove that the nonempty convex compact set $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$ agrees with the closed convex hull of the subset of its extremal points. In addition, these extremal points are precisely the σ-ergodic measures projecting onto m. To prove this, assume first that the measure $\mu\in\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$ is ergodic, and apply Theorem 1.9(iii) to deduce that it is extremal in $\mu\in\mathfrak{M}_{\mathrm{inv}}(\Omega\times\mathbb{Y},\tilde\tau)$, which obviously ensures that it is extremal in $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$. Conversely, assume that a measure μ is extremal in $\mathfrak{M}_{\mathrm{inv},m}(\Omega\times\mathbb{Y},\tilde\tau)$ and that $\mu = a\,\mu_1 + (1-a)\,\mu_2$ for $a\in[0,1]$ and $\mu_1,\mu_2\in\mu\in\mathfrak{M}_{\mathrm{inv}}(\Omega\times\mathbb{Y},\tilde\tau)$. Then $m = a\,m_1+(1-a)\,m_2$, where m_1 and m_2 are the σ-invariant measures on Ω defined by the projections of μ_1 and μ_2. Since m is σ-ergodic, Theorem 1.9(iii) ensures that $a\in\{0,1\}$, so that μ is extremal in $\mathfrak{M}_{\mathrm{inv}}(\Omega\times\mathbb{Y},\tilde\tau)$ and hence $\tilde\tau$-ergodic. The assertions in (iii) are proved.

Proposition 1.16 *Let Ω and $\mathbb{Y}$ be compact metric spaces, and let $\tilde\tau$ be a continuous skew-product flow on $\Omega\times\mathbb{Y}$ with continuous base flow (Ω,σ). Let m be a σ-ergodic measure on Ω. Let $\Omega_0\in\Sigma_m$ be a σ-invariant set with $m(\Omega_0)=1$, and let $l\colon\Omega\to\mathbb{Y}$ be a Σ_m-measurable map with $\tilde\tau(t,\omega,l(\omega))=(\omega{\cdot}t,l(\omega{\cdot}t))$ for all $\omega\in\Omega_0$. Then,*

(i) *there exists a Borel set $\Omega_1\subseteq\Omega_0$ which is σ-invariant set and with $m(\Omega_1)=1$ such that the $\tilde\tau$-invariant set $\{(\omega,l(\omega))\mid\omega\in\Omega_1\}\subset\Omega\times\mathbb{Y}$ is Borel.*
(ii) *The graph of l concentrates a $\tilde\tau$-invariant measure μ_l which projects onto m, which is determined by $\int_{\Omega\times\mathbb{Y}}\tilde f(\omega,y)\,d\mu_l=\int_\Omega\tilde f(\omega,l(\omega))\,dm$ for all continuous functions $\tilde f\colon\Omega\times\mathbb{Y}\to\mathbb{R}$.*

Proof

(i) The regularity of m and Lusin's theorem guarantee the existence of a compact subset $\mathcal{M}\subseteq\Omega_0$ with $m(\mathcal{M})>0$ such that l is continuous at the points of $\mathcal{M}$. Define $\mathcal{M}_k=\{\omega{\cdot}t\mid\omega\in\mathcal{M}\,,t\in[-k,k]\}$ for $k=0,1,2,\ldots$, which is also a compact set, since σ is continuous. It is easy to check that $\Omega_1=\cup_{k\geq0}\mathcal{M}_k$ is a Borel σ-invariant set of positive measure; hence, by ergodicity, $m(\Omega_1)=1$. In addition, the map l is continuous at the points of all the sets $\mathcal{M}_k$, as one can easily deduce from the property of τ-invariance of l on Ω_0 and from the compactness of $\mathcal{M}$ and $[-k,k]$. Finally, one has that $\{(\omega,l(\omega))\mid\omega\in\Omega_1\}=\cup_{k\geq0}\{(\omega,l(\omega))\mid\omega\in\mathcal{M}_k\}$, and therefore it is Borel.
(ii) A $\tilde\tau$-invariant measure concentrated on $\mathcal{L}=\{(\omega,l(\omega))\mid\omega\in\Omega_1\}$, which is contained in the graph of l, is constructed in what follows. For all $\tilde f$ in the set $C(\Omega\times\mathbb{Y},\mathbb{R})$ of real continuous functions, the Σ_m-measurable map $\Omega\to\mathbb{R}\,,\ \omega\mapsto\tilde f(\omega,l(\omega))$ belongs to $L^1(\Omega,\mathbb{R})$, so that it is possible to define $L(\tilde f)=\int_\Omega\tilde f(\omega,l(\omega))\,dm$. Then L is a bounded linear functional with norm 1 on $C(\Omega\times\mathbb{Y},\mathbb{R})$, and the Riesz representation theorem provides a (positive normalized regular Borel) measure μ_l such that $L(\tilde f)=\int_{\Omega\times\mathbb{Y}}\tilde f(\omega,y)\,d\mu_l$.

Since $\int_{\Omega\times\mathbb{Y}} f(\omega)\,d\mu_l = L(f) = \int_\Omega f(\omega)\,dm$ for all $f \in C(\Omega,\mathbb{R})$, Remark 1.14 ensures that μ_l projects onto m. In addition, according to the equivalences established in Proposition 1.7, the measure μ_l is $\tilde\tau$-invariant: if $\tilde f \in C(\Omega\times\mathbb{Y},\mathbb{R})$ and $t\in\mathbb{R}$, then $\int_{\Omega\times\mathbb{Y}} \tilde f(\tilde\tau_t(\omega,y))\,d\mu_l = \int_\Omega \tilde f(\omega{\cdot}t, l(\omega{\cdot}t))\,dm = \int_\Omega \tilde f(\omega, l(\omega))\,dm = \int_{\Omega\times\mathbb{Y}} \tilde f(\omega,y)\,d\mu_l$, since m is σ-invariant. Finally, if $\mathcal{K} \subseteq (\Omega\times\mathbb{Y}) - \mathcal{L}$ is a compact set, the Lebesgue monotone convergence theorem ensures that $\mu_l(\mathcal{K}) = 0$ (see the proof of (3)$\Rightarrow$ (1) in Proposition 1.7), and hence the regularity of μ_l ensures that $\mu_l(\mathcal{L}) = 1$; that is, μ_l is concentrated on $\mathcal{L}$.

A skew-product flow may admit many types of compact invariant sets, whose complexity varies in an ample range. Among them are those described now, which are especially interesting from a dynamical point of view, and which will appear frequently in the following chapters. The first one represents an extension of the idea of an equilibrium point for an autonomous system, or of a T-periodic solution for a system with T-periodic coefficients.

Definition 1.17 Let Ω and $\mathbb{Y}$ be compact metric spaces, and let $\tilde\tau$ be a continuous skew-product flow on $\Omega\times\mathbb{Y}$ with continuous base flow (Ω,σ). A compact subset $\mathcal{K}\subset\Omega\times\mathbb{Y}$ is a *copy of the base (for the flow $\tilde\tau$)* if it is $\tilde\tau$-invariant and, in addition, $\mathcal{K}_\omega = \{y\in\mathbb{Y} \mid (\omega,y)\in\mathcal{K}\}$ reduces to a point for every $\omega\in\Omega$: in other words, if it agrees with the graph of a continuous map $c\colon\Omega\to\mathbb{Y}$ satisfying $c(\omega{\cdot}t) = \tilde\tau_2(t,\omega,c(\omega))$, so that $\mathcal{K} = \{(\omega,c(\omega)) \mid \omega\in\Omega\}$.

These invariant objects are the simplest ones from a dynamical point of view, since they reproduce homeomorphically the base Ω. The second type of set is a generalization of the first one.

Definition 1.18 Let Ω and $\mathbb{Y}$ be compact metric spaces, and let $\tilde\tau$ be a continuous skew-product flow on $\Omega\times\mathbb{Y}$ with continuous and minimal base flow (Ω,σ). A minimal subset $\mathcal{K}\subset\Omega\times\mathbb{Y}$ is an *almost automorphic extension of the base (for the flow $\tilde\tau$)* if it is $\tilde\tau$-invariant and, in addition, there exists $y\in\mathbb{Y}$ such that $\mathcal{K}_\omega = \{y\in\mathbb{Y} \mid (\omega,y)\in\mathcal{K}\}$ reduces to a point.

Clearly, a copy of the base provides the simplest example of an almost automorphic extension in the case of minimal base flow. However, there are examples of almost automorphic extensions which are not copies of the base. The most classical ones are those due to Millionščikov [104, 105] and Vinograd [147]. See Johnson [68] for a detailed dynamical description of these examples, and [67] for a later example of a scalar linear equation with this type of complicated invariant object, which can exhibit properties of high dynamical complexity (like sensitive dependence with respect to initial conditions). Example 8.44 contains a similar construction with most of the details explained, with an almost automorphic extension of the base whose fibers reduce to a singleton at a residual set of points of the base but not on a set of full measure.

1.2 Basic Properties of Matrices and Lagrange Planes

Throughout the book, $\mathbb{M}_{m\times d}(\mathbb{R})$ and $\mathbb{M}_{m\times d}(\mathbb{C})$ will represent the $(m \times d)$-dimensional vector spaces of real or complex $m \times d$ matrices; and, as in the previous pages, the symbol $\mathbb{K}$ will represent either $\mathbb{R}$ or $\mathbb{C}$. The cases $d = m = n$ and $d = m = 2n$ will be frequently considered. The symbols M^T and M^* denote respectively the transpose and conjugate transpose of M; and $\operatorname{Re} M$ and $\operatorname{Im} M$ represent the real and imaginary parts of a complex matrix M. The determinant and trace of a square matrix M will be represented by $\det M$ and $\operatorname{tr} M$. Recall that $\det M = \det M^T$, $\det(MN) = \det(NM)$, $\operatorname{tr} M = \operatorname{tr} M^T$, and $\operatorname{tr}(MN) = \operatorname{tr}(NM)$. If M is a *nonsingular* square matrix (i.e. if $\det M \neq 0$), then M^{-1} represents its inverse; I_d and 0_d are the identity and null matrices in $\mathbb{M}_{d\times d}(\mathbb{K})$ for all $d \in \mathbb{N}$; and $\mathbf{0}$ represents the null vector in $\mathbb{K}^d$ for all $d \in \mathbb{N}$.

A $d \times d$ matrix M is *symmetric* if $M^T = M$ and *hermitian* if $M^* = M$. A real symmetric matrix or a complex hermitian matrix is *selfadjoint*, in reference to the usual Euclidean inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{x}$ in $\mathbb{K}^d$: in both cases $\langle \mathbf{x}, M\mathbf{y}\rangle = \langle M\mathbf{x}, \mathbf{y}\rangle$ for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{K}^d$. The square matrix M is *unitary* when $M^*M = I_d$, and *orthogonal* if it is real and unitary. A real or complex $2n \times 2n$ matrix M is *symplectic* if $M^TJM = J$, where

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}.$$

Note that $J^2 = -I_{2n}$, and that any symplectic matrix is nonsingular: in fact, $\det M = 1$. This can be deduced, for instance, from the Iwasawa decomposition of M, described in Lemma 2.16. The simplest examples of symplectic matrices are I_{2n} and J.

The following notation will always be used:

- $\mathrm{GL}(m, \mathbb{K})$: set of nonsingular $m \times m$ matrices con coefficients in $\mathbb{K}$,
- $\mathrm{U}(m, \mathbb{C})$: set of (complex) unitary $m \times m$ matrices,
- $\mathrm{SU}(m, \mathbb{C})$: set of unitary $m \times m$ matrices with determinant 1,
- $\mathrm{O}(m, \mathbb{R})$: set of (real) orthogonal $m \times m$ matrices,
- $\mathrm{SO}(m, \mathbb{R})$: set of orthogonal $m \times m$ matrices with determinant 1,
- $\mathrm{Sp}(n, \mathbb{C})$: set of complex symplectic $2n \times 2n$ matrices,
- $\mathrm{Sp}(n, \mathbb{R})$: set of real symplectic $2n \times 2n$ matrices.

It is very easy to check that the first five sets are groups with respect to the matrix product. Proposition 1.23 below guarantees that also $\mathrm{Sp}(n, \mathbb{C})$ and $\mathrm{Sp}(n, \mathbb{R})$ are groups. In fact, all of them are Lie groups (see e.g. Sections 1 and 2 of Chapter II of Helgason [57]).

1.2.1 Symmetric, Hermitian, and Symplectic Matrices

Consider the sets of real and complex symmetric $d \times d$ matrices,

$$\mathbb{S}_d(\mathbb{R}) = \{M \in \mathbb{M}_{d\times d}(\mathbb{R}) \mid M = M^T\},$$
$$\mathbb{S}_d(\mathbb{C}) = \{M \in \mathbb{M}_{d\times d}(\mathbb{C}) \mid M = M^T\},$$

which constitute $(d \times (d+1))/2$-dimensional linear subspaces of $\mathbb{M}_{d\times d}(\mathbb{R})$ and $\mathbb{M}_{d\times d}(\mathbb{C})$. Let M belong to $\mathbb{S}_d(\mathbb{R})$. Then,

- M is a *positive definite matrix* ($M > 0$) if $\mathbf{x}^T M\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x} \neq \mathbf{0}$,
- M is a *positive semidefinite matrix* ($M \geq 0$) if $\mathbf{x}^T M\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$,
- M is a *negative definite matrix* ($M < 0$) if $-M > 0$,
- M is a *negative semidefinite matrix* ($M \leq 0$) if $-M \geq 0$.

The subsets

$$\mathbb{S}_d^+(\mathbb{R}) = \{M \in \mathbb{S}_d(\mathbb{R}) \mid M > 0\},$$
$$\mathbb{S}_d^+(\mathbb{C}) = \{M \in \mathbb{S}_d(\mathbb{C}) \mid \operatorname{Im} M > 0\}$$

will be frequently considered. Note that their closures on $\mathbb{S}_d(\mathbb{R})$ and $\mathbb{S}_d(\mathbb{C})$ are given by

$$\overline{\mathbb{S}_d^+}(\mathbb{R}) = \{M \in \mathbb{S}_d(\mathbb{R}) \mid M \geq 0\},$$
$$\overline{\mathbb{S}_d^+}(\mathbb{C}) = \{M \in \mathbb{S}_d(\mathbb{C}) \mid \operatorname{Im} M \geq 0\},$$

which can be easily deduced from the definitions of positive definite and semidefinite matrices.

Similarly, if M is a hermitian matrix, then

- M is *positive definite* (($M > 0$) if $\mathbf{x}^*M\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{C}^d$, $\mathbf{x} \neq \mathbf{0}$,
- M is *positive semidefinite* (($M \geq 0$) if $\mathbf{x}^*M\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^d$,
- M is *negative definite* (($M < 0$) if $-M > 0$,
- M is *negative semidefinite* (($M \geq 0$) if $-M \geq 0$.

The relations $M > N$, $M \geq N$, $M < N$ and $M \leq N$ for selfadjoint matrices have the obvious meaning: for instance, $M > N$ means that $M - N > 0$.

Well-known properties of positive (real or complex) matrices are: $M > 0$ (resp. $M \geq 0$) if and only if $\lambda > 0$ (resp. $\lambda \geq 0$) for all the eigenvalues λ of M (which are real); hence, $M > I_d$ (resp. $M \geq I_d$) if and only if $\lambda > 1$ (resp. $\lambda \geq 1$) for all the eigenvalues λ of M; and, if $M > 0$ (resp. $M \geq 0$) and P is nonsingular, then $P^*MP > 0$ (resp. $P^*MP \geq 0$).

Let $\mathbb{M}_{d\times d}(\mathbb{K})$ be a given positive definite (or semidefinite) matrix. Throughout the book, the expression "the unique positive definite (or semidefinite) square root of

M" (will be used very often. That this object exists for positive selfadjoint bounded operators is a well-known fact: see e.g. Theorem VI.9 of Reed and Simon [122]. The following result provides an easy and constructive proof in the matrix case.

Proposition 1.19 *Let M belong to $\mathbb{M}_{d\times d}(\mathbb{C})$.*

(i) *If $M \ge 0$, there exists a unique matrix $M^{1/2} \ge 0$ such that $(M^{1/2})^2 = M$. In addition, $M\,\mathbf{x} = \mathbf{0}$ if and only if $M^{1/2}\,\mathbf{x} = \mathbf{0}$, and $M^{1/2}$ is real if M is real.*
(ii) *If $M > 0$, then $M^{1/2} > 0$ and $(M^{1/2})^{-1} = (M^{-1})^{1/2}$.*
(iii) *The map defined from $\{M \in \mathbb{S}_d(\mathbb{C}) \mid M > 0\}$ to itself by sending M to $M^{1/2}$ is continuously differentiable.*

Proof If $M \ge 0$, all its eigenvalues are real and nonnegative, and it is a well-known fact that there exist a unitary matrix P (which is real if M is real) and a real diagonal matrix $D \ge 0$ such that $M = P^*DP$. It is obvious that there exists a unique diagonal matrix $\widetilde{D} \ge 0$ such that $D = \widetilde{D}^2$. Then the matrix $N = P^*\widetilde{D}P$ satisfies $N \ge 0$ and $N^2 = M$. This proves the existence, and the fact that N is real if M is real. Clearly, if $M > 0$, then $D > 0$ and hence $N > 0$. Now, if $N \ge 0$ satisfies $N^2 = M$, and $N\,\mathbf{x} = \lambda\,\mathbf{x}$, then $M\,\mathbf{x} = \lambda^2\,\mathbf{x}$. That is, the eigenvalues and the associated eigenvectors of N are uniquely determined, so that also the matrix is. In addition, if $M = P^*\widetilde{D}^2P > 0$, then $M^{-1} = P^*\widetilde{D}^{-2}P > 0$, so that $(M^{1/2})^{-1} = P^*\widetilde{D}^{-1}P = (M^{-1})^{1/2}$. Note also that $M\,\mathbf{x} = 0$ ensures that $\|M^{1/2}\,\mathbf{x}\| = 0$ for the Euclidean norm in $\mathbb{C}^d$, so that $M^{1/2}\,\mathbf{x} = \mathbf{0}$. These facts prove (i) and (ii). To prove (iii), one can apply the Inverse Function Theorem to the map $\mathbb{M}_{d\times d}(\mathbb{C}) \to \mathbb{M}_{d\times d}(\mathbb{C}),\ M \mapsto M^2$ at a point $M > 0$: it is continuously differentiable at M, and its differential, which sends $C \in \mathbb{M}_{d\times d}(\mathbb{C})$ to $MC + CM$, has no null eigenvalues. This last assertion is due to the positive definite character of M: assume that $MC + CM = 0_d$ in order to deduce that $DPCP^* + PCP^*D = 0_d$; and note that this implies that $PCP^* = 0_d$ and hence that $C = 0_d$.

Remark 1.20 Suppose that $0 < M \le N$ for two symmetric $d \times d$ matrix-valued functions. Then $I_d \le M^{-1/2}NM^{-1/2}$ and hence $I_d \le N^{1/2}M^{-1}N^{1/2}$, since both right-hand terms have the same eigenvalues. Therefore, $0 < N^{-1} \le M^{-1}$. Clearly, there is an analogous result if the inequality is strict.

Proposition 1.21

(i) *If $M \in \mathbb{S}_d^+(\mathbb{C})$, then it is nonsingular, and $-M^{-1} \in \mathbb{S}_d^+(\mathbb{C})$.*
(ii) *If $M \in \overline{\mathbb{S}_d^+}(\mathbb{C})$ is nonsingular, then $-M^{-1} \in \overline{\mathbb{S}_d^+}(\mathbb{C})$.*

Proof Write $M = A + iB$ for real symmetric matrices A and B, and assume that $B > 0$. Take $\mathbf{z} \in \mathbb{C}^n$ with $M\,\mathbf{z} = \mathbf{0}$ and note that $\mathbf{z}^*M^* = \mathbf{0}^*$. Then $2iB = M - \overline{M} = M - M^*$, and therefore $2i\,\mathbf{z}^*B\,\mathbf{z} = \mathbf{z}^*(M - M^*)\,\mathbf{z} = 0$, so that $\mathbf{z} = \mathbf{0}$. This proves the existence of M^{-1}. To check the second assertion in (i), as well as (ii), write $M^{-1} = C + iD$ for real matrices C and D. It follows from the identity $I_d = (A + iB)(C + iD)$ that $AC - BD = I_d$ and $BC + AD = 0_d$, so that also $C^TB + D^TA = 0_d$. These equalities ensure that $D^T + D^TBD = D^TAC = -C^TBC$, so that $D^T = -D^TBD - C^TBC$, which is obviously symmetric, and is negative semidefinite if $B \ge 0$. Finally, if $B > 0$ and

$\mathbf{z}^* D\mathbf{z} = 0$, then $C\mathbf{z} = \mathbf{0}$ and $D\mathbf{z} = \mathbf{0}$, so that $M^{-1}\mathbf{z} = \mathbf{0}$ and hence $\mathbf{z} = \mathbf{0}$; that is, $D < 0$.

The following basic properties refer to symplectic matrices.

Proposition 1.22 *If λ is an eigenvalue of $M \in \mathrm{Sp}(n, \mathbb{R})$, so is λ^{-1}.*

Proof Recall that any eigenvalue is different from zero. It can immediately be checked that $M^T J\mathbf{v} = \lambda^{-1} J\mathbf{v}$ if $M\mathbf{v} = \lambda\mathbf{v}$, so that the assertion follows from the coincidence of the set of eigenvalues of any matrix and that of its transpose.

Proposition 1.23 *Let $M = \left[\begin{smallmatrix} M_1 & M_3 \\ M_2 & M_4 \end{smallmatrix}\right]$ belong to $\mathrm{Sp}(n, \mathbb{C})$. Then M^T, M^* and M^{-1} are also symplectic matrices, and*

$$\begin{aligned}
M_1^T M_2 &= M_2^T M_1 \quad M_3^T M_4 = M_4^T M_3 \quad M_1^T M_4 - M_2^T M_3 = I_n \\
M_1 M_3^T &= M_3 M_1^T \quad M_2 M_4^T = M_4 M_2^T \quad M_4 M_1^T - M_2 M_3^T = I_n \\
M_4 M_2^T &= M_2 M_4^T \quad M_3 M_1^T = M_1 M_3^T \\
M_4^T M_3 &= M_3^T M_4 \quad M_2^T M_1 = M_1^T M_2
\end{aligned}$$

Proof If $M^T J M = J$, then $J M^T J = -M^{-1}$. This implies, on the one hand, that $M J M^T = J$ and $\overline{M} J M^* = J$, so that M^T and M^* are symplectic whenever M is. And, on the other hand, that $M^{-1} J (M^{-1})^T = J$, so that also $(M^{-1})^T$, and hence M^{-1}, are symplectic if M is.

One more consequence of the identity $M^{-1} = -J M^T J$ is that $M^{-1} = \left[\begin{smallmatrix} M_4^T & -M_3^T \\ -M_2^T & M_1^T \end{smallmatrix}\right]$. The remaining equalities are immediate consequences of the symplectic character of M, M^T, M^{-1}, and $(M^{-1})^T$.

Remarks 1.24

1. Unless otherwise indicated, $\|\cdot\| = \|\cdot\|_d$ will denote throughout the book some fixed norm on the vector space $\mathbb{K}^d$, and $\mathbb{M}_{m\times d}(\mathbb{K})$ will be provided with the associated operator norm, defined by $\|M\| = \max_{\|\mathbf{x}\|_d=1} \|M\mathbf{x}\|_m$. In general, no reference to the dimension will be made in the norm notation: the context will give the precise dimension d or $m \times d$. It can immediately be checked that, with this definition, $\|M\mathbf{x}\| \le \|M\|\,\|\mathbf{x}\|$, and hence that $\|MN\| \le \|M\|\,\|N\|$. Recall that all the norms are equivalent in the case of vector spaces of finite dimension. However, not every norm on $\mathbb{M}_{m\times d}(\mathbb{K})$ is associated as above to a vector norm.
2. The most frequently used norm will be the *Euclidean norm*, defined by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2} = (\mathbf{x}^*\mathbf{x})^{1/2}$ on $\mathbb{K}^d$ and by $\|M\| = \max_{\|\mathbf{x}\|=1} \|M\mathbf{x}\|$ on $\mathbb{M}_{d\times m}(\mathbb{K})$. It is the norm associated to the Euclidean inner product defined on $\mathbb{K}^d$ by $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{x}$. In this case, $\|M\| = \|M^*\|$: according to the Cauchy–Schwarz inequality, $\|M\mathbf{x}\|^2 = \langle \mathbf{x}, M^* M\mathbf{x}\rangle \le \|\mathbf{x}\|\,\|M^* M\mathbf{x}\| \le \|\mathbf{x}\|\,\|M^*\|\,\|M\mathbf{x}\|$, so that $\|M\mathbf{x}\| \le \|\mathbf{x}\|\,\|M^*\|$, which implies $\|M\| \le \|M^*\|$; and hence also $\|M^*\| \le \|(M^*)^*\| = \|M\|$. It is a well-known result that, if M is a square matrix, then $\|M\|^2$ agrees with the spectral radius $\rho(M^* M)$ of $M^* M$; i.e. with the maximum eigenvalue of the matrix

M^*M. In addition, $\|M\|^2 = \|M^*M\|$: since $N = M^*M$ is hermitian and positive semidefinite, $\|N\| = (\rho(N^*N))^{1/2} = (\rho(N^2))^{1/2} = \rho(N) = \|M\|^2$.

3. The choice of a particular norm on $\mathbb{M}_{d\times m}(\mathbb{R})$, which will now be defined, will be of importance in the proofs of some of the main results of the book. Given a real $d\times m$ matrix M, define $\|M\|_F = (\mathrm{tr}(M^TM))^{1/2}$. It is known (see e.g. Section 5.2 of Meyer [102]) that the continuous map $M \mapsto \|M\|_F$ defines a matrix norm (which does not come from a vector norm), called the *Frobenius norm*. The following properties will be useful:

   F1. $\|M\|_F = \|M^T\|_F$ if $M \in \mathbb{M}_{d\times m}(\mathbb{R})$;
   F2. $|\,\mathrm{tr}(MN)| \le \|M\|_F\|N\|_F$ if $M \in \mathbb{M}_{d\times m}(\mathbb{R})$ and $N \in \mathbb{M}_{m\times d}(\mathbb{R})$;
   F3. $\|MN\|_F \le \|M\|_F\|N\|_F$ if $M \in \mathbb{M}_{d\times m}(\mathbb{R})$ and $N \in \mathbb{M}_{m\times d}(\mathbb{R})$.

1.2.2 Grassmannian Manifolds

Let W be a m-dimensional linear subspace of $\mathbb{K}^d$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $k \in \{0, 1, \dots, m\}$, let $\mathcal{G}_k(W)$ represent the set of the k-dimensional subspaces of W. The set $\mathcal{G}_k(W)$ is diffeomorphic to the homogeneous space of left cosets $\mathrm{GL}(m,\mathbb{K})/\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{H}}$ being the closed Lie subgroup of $\mathrm{GL}(m,\mathbb{K})$ given by the matrices of the form $\left[\begin{smallmatrix} A & * \\ 0 & B\end{smallmatrix}\right]$, where $A \in \mathrm{GL}(k,\mathbb{K})$ and $B \in \mathrm{GL}(m-k,\mathbb{K})$. Here, $*$ represents any $k \times (m-k)$ matrix and 0 represents the zero $(m-k) \times k$ matrix. With this identification, which provides $\mathcal{G}_k(W)$ with a differentiable structure, $\mathcal{G}_k(W)$ is the *Grassmannian manifold of the k-dimensional linear subspaces of W*. In the real case, this manifold can be also identified with $\mathrm{SO}(m,\mathbb{R})/\mathcal{H}$, where $\mathcal{H}$ is the closed subgroup of $\mathrm{SO}(m,\mathbb{R})$ given by the matrices of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right]$ for $A \in \mathrm{O}(k,\mathbb{R})$, $B \in \mathrm{O}(m-k,\mathbb{R})$ and $\det A\cdot\det B = 1$. A similar identification is valid in the complex case, taking now $\mathcal{H}$ as the closed subgroup of $\mathrm{SU}(m,\mathbb{C})$ given by the matrices of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right]$ for $A \in \mathrm{U}(k,\mathbb{C})$, $B \in \mathrm{U}(m-k,\mathbb{C})$ and $\det A\cdot\det B = 1$. In both cases, $\mathcal{G}_k(W)$ is a compact and connected manifold of dimension $k(m-k)$, which agrees with the real or complex projective space on W if $k = 1$. The reader can find a discussion of these matters in Sections 17 and 18 of Chapter IV of Matsushima [101].

Let $\|\cdot\|$ represent the Euclidean norm in $\mathbb{K}^d$. It is possible to define a metric on $\mathcal{G}_k(\mathbb{K}^d)$ compatible with the topology: the distance between two k-dimensional linear subspaces g and h is defined by

$$d(g,h) = \max\left(\sup_{\mathbf{v}\in g,\,\|\mathbf{v}\|\le 1} d(\mathbf{v},h),\ \sup_{\mathbf{w}\in h,\,\|\mathbf{w}\|\le 1} d(\mathbf{w},g)\right),$$

where $d(\mathbf{v},h) = \inf_{\mathbf{w}\in h}\|\mathbf{v}-\mathbf{w}\|$. An equivalent and useful definition is given by Morris [107], Lemma 3.2:

$$d(g,h) = \sup_{\mathbf{v}\in g,\,\|\mathbf{v}\|=1} d(\mathbf{v},h) = \sup_{\mathbf{w}\in h,\,\|\mathbf{w}\|=1} d(\mathbf{w},g)\,. \tag{1.2}$$

In fact $d(g, h)$ agrees with the Hausdorff distance between $g \cap \mathbb{S}$ and $h \cap \mathbb{S}$, where $\mathbb{S}$ is the unit sphere in $\mathbb{K}^d$: see definition (1.25) below, and Section IV.2 of Kato [89].

Consider now the case $W = \mathbb{K}^d$, and take $M \in \mathrm{GL}(d, \mathbb{K})$. It is clear that the equivalence class $[M]$ of M in $\mathrm{GL}(d, \mathbb{K})/\widetilde{\mathcal{H}}$ is determined by the first k column vectors of M, and that the element of $\mathcal{G}_k(\mathbb{K}^d)$ corresponding to $[M]$ is the k-dimensional vector space generated by these column vectors. In other words, any $g \in \mathcal{G}_k(\mathbb{K}^d)$ can be represented by a $d \times k$ matrix of rank k, and two of these matrices G_1 and G_2 represent the same vector space g if and only if $G_1 = G_2 P$ for a matrix $P \in \mathrm{GL}(k, \mathbb{K})$. That is, $\mathcal{G}_k(\mathbb{K}^d)$ can be identified with the quotient space $\mathbb{M}^k_{d\times k}(\mathbb{K})/\mathrm{GL}(k, \mathbb{K})$, where $\mathbb{M}^k_{d\times k}(\mathbb{K}) \subset \mathbb{M}_{d\times k}(\mathbb{K})$ is the subset of matrices of rank k. Such a matrix G is called a *representation* of the element of $\mathcal{G}_k(\mathbb{K}^d)$ that it determines, and this fact is denoted by $g \equiv G$. Now represent by $\mathrm{U}^k_{d\times k}(\mathbb{K})$ the set $\{G \in \mathbb{M}^k_{d\times k}(\mathbb{K}) \mid G^*G = I_k\}$, and note that choosing an orthonormal basis of g provides a representation $G \in \mathrm{U}^k_{m\times k}(\mathbb{K})$. Clearly two matrices G_1 and G_2 in $\mathrm{U}^k_{d\times k}(\mathbb{K})$ represent the same vector space if and only if $G_1 = G_2 P$ for a unitary matrix P, which is orthogonal if $\mathbb{K} = \mathbb{R}$. That is, $\mathcal{G}_k(\mathbb{K}^d)$ can be identified with the quotient space $\mathrm{U}^k_{d\times k}(\mathbb{K})/\mathrm{U}(k, \mathbb{K})$ (where $\mathrm{U}(k, \mathbb{R}) = \mathrm{O}(k, \mathbb{R})$). A standard topological argument proves that the projections maps $\mathbb{M}^k_{d\times k}(\mathbb{K}) \to \mathbb{M}^k_{d\times k}(\mathbb{K})/\mathrm{GL}(k, \mathbb{K})$ and $\mathrm{U}^k_{d\times k}(\mathbb{K}) \to \mathrm{U}^k_{d\times k}(\mathbb{K})/\mathrm{U}(k, \mathbb{K})$ (which are continuous, by definition of the topology on the quotient spaces) are also open: see [101], Chapter IV.2. In particular, the set $\mathcal{G}_k(\mathbb{K}^d)$ is metrizable: see e.g. Lemma 1 of Stone [144]. The following result, whose proof is left to the reader, is another easy consequence of the open character of the projection maps.

Proposition 1.25 *Let (g_j) be a sequence of elements of $\mathcal{G}_k(\mathbb{K}^d)$, with $g_j \equiv G_j$ for $G_j \in \mathbb{M}^k_{d\times k}(\mathbb{K})$ (resp. $G_j \in \mathrm{U}^k_{d\times k}(\mathbb{K})$), and $g \in \mathcal{G}_k(\mathbb{K}^d)$, with $g \equiv G$ for $G \in \mathbb{M}^k_{d\times k}(\mathbb{K})$ (resp. $G \in \mathrm{U}^k_{d\times k}(\mathbb{K})$). Then $\lim_{j\to\infty} g_j = g$ in $\mathcal{G}_k(\mathbb{K}^d)$ if and only if, for $j \in \mathbb{N}$, there exists $P_j \in \mathrm{GL}(k, \mathbb{K})$ (resp. $P_j \in \mathrm{U}(k, \mathbb{K})$) such that $\lim_{j\to\infty} \|G_j P_j - G\| = 0$.*

This result has some consequences which will be useful in the proofs of several convergence results scattered throughout the book. For example, it is used to prove the next proposition.

Proposition 1.26 *Let (g_j) be a sequence of elements of $\mathcal{G}_k(\mathbb{K}^d)$, and $g \in \mathcal{G}_k(\mathbb{K}^d)$. Then,*

(i) $\lim_{j\to\infty} g_j = g$ *in $\mathcal{G}_k(\mathbb{K}^d)$ if and only if and if each vector $\mathbf{v} \in g$ is the limit in $\mathbb{K}^m$ of a sequence $(\mathbf{v}_j)$, with $\mathbf{v}_j \in g_j$.*
(ii) $\lim_{j\to\infty} g_j = g$ *in $\mathcal{G}_k(\mathbb{K}^d)$ if and only if and if the limit $\mathbf{v}$ of any convergent sequence $(\mathbf{v}_j)$ with $\mathbf{v}_j \in g_j$ belongs to g.*

Proof The proof is based on the information provided by Proposition 1.25.

(i) For the "if" assertion in (i), take a representation $G = [\mathbf{v}^1 \cdots \mathbf{v}^k]$ of g, for $i = 1, \ldots k$ write $\mathbf{v}^i = \lim_{j\to\infty} \mathbf{v}^i_j$ for $\mathbf{v}^i_j \in g_j$, note that $\lim_{j\to\infty} \|G_j - G\| = 0$ for $G_j = [\mathbf{v}^1_j \cdots \mathbf{v}^k_j]$, and deduce that the rank of the $m \times k$ matrix G_j is k for large enough j, so that G_j represents g_j. For the converse assertion, choose representations $g \equiv G$ and $g_j \equiv G_j$ with $\lim_{j\to\infty} G_j = G$, write $\mathbf{v} \in g$ as $\mathbf{v} = G\,\mathbf{c}$ for $\mathbf{c} \in \mathbb{K}^d$, and note that $\mathbf{v} = \lim_{j\to\infty} G_j\,\mathbf{c}$.

(ii) Note that the "only if" assertion is trivial if $\mathbf{v} = \mathbf{0}$, so assume that this is not the case and, without loss of generality, that $\|\mathbf{v}\| = 1$ and $\|\mathbf{v}_j\| = 1$. Take representations $g_j \equiv G_j$ for $G_j \in \mathrm{U}^k_{d\times k}(\mathbb{K})$ and $g \equiv G$ with $\lim_{j\to\infty} G_j = G$. Write $\mathbf{v}_j = G_j\,\mathbf{c}_j$ for $\mathbf{c}_j \in \mathbb{K}^d$, and note that, since $\|\mathbf{c}_j\| = \|\mathbf{v}_j\| = 1$ for the Euclidean norms, there exists a subsequence $(\mathbf{c}_m)$ with limit $\mathbf{c}$. Therefore, $\mathbf{v} = \lim_{m\to\infty} G_m\,\mathbf{c}_m = G\,\mathbf{c} \in g$, as asserted. For the converse assertion, choose any subsequence (g_m) of (g_j); write $g_m \equiv G_m$ for $G_m \in \mathrm{U}^k_{d\times k}(\mathbb{K})$; apply the compactness of the unit sphere in $\mathbb{K}^d$ to find a convergent subsequence (G_i) of (G_m) with limit G; deduce from the hypothesis that G represents g; use Proposition 1.25 to see that $\lim_{i\to\infty} g_i = g$; and conclude from the independence of the limit with respect to the choice of the subsequence that the limit of the initial sequence (g_j) is g.

1.2.3 Lagrangian Manifolds

The manifold $\mathcal{G}_n(\mathbb{K}^{2n})$ has a submanifold which will play a fundamental role throughout the book: the Lagrangian manifold, which is now described. Recall that two vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{K}^{2n}$ are *isotropic* if $\mathbf{z}^T J\mathbf{w} = 0$. Any linear subspace l of $\mathbb{K}^{2n}$ whose vectors are pairwise isotropic satisfies $\dim l \le n$, since l is contained in the Euclidean orthogonal subspace to $J{\cdot}l = \{J\,\mathbf{z} \mid \mathbf{z} \in l\}$. An n-dimensional linear subspace $l \subset \mathbb{R}^{2n}$ (or $l \subset \mathbb{C}^{2n}$) is a *real* (or *complex*) *Lagrange plane* if $\mathbf{z}^T J\mathbf{w} = 0$ for all $\mathbf{z}$ and $\mathbf{w}$ in l. Let $\mathcal{L}_\mathbb{R}$ and $\mathcal{L}_\mathbb{C}$ represent the sets of real and complex Lagrange planes. It is easy to check that the vector columns of a $2n{\times}n$ matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ form a basis of an element l of $\mathcal{L}_\mathbb{K}$ if and only if the rank of the matrix is n and $L_1^T L_2 = L_2^T L_1$. This situation will be represented as $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ throughout the book, and the matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ will be called a *representation* of l. Note that $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$ represent the same Lagrange plane if and only if there is a nonsingular $n \times n$ matrix P such that $L_1 = F_1 P$ and $L_2 = F_2 P$.

Remarks 1.27

1. There is a basic connection between symplectic matrices and Lagrange planes. Let V and l be a (real or complex) symplectic matrix and a (real or complex) Lagrange plane, and represent by $V{\cdot}l$ the vector space $\{V\,\mathbf{z} \mid \mathbf{z} \in l\}$. Then $V{\cdot}l$ is a new (real or complex) Lagrange plane: it has dimension n since the matrix V is nonsingular; and, if $\mathbf{z}$ and $\mathbf{w}$ belong to l, then $\mathbf{z}^T V^T J V\,\mathbf{w} = \mathbf{z}^T J\,\mathbf{w} = 0$. Note that if $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ and $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, then $V{\cdot}l \equiv V\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} V_1 L_1 + V_3 L_2 \\ V_2 L_1 + V_4 L_2 \end{smallmatrix}\right]$.
2. It follows from the previous remark that, if V is symplectic, then $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} V_3 \\ V_4 \end{smallmatrix}\right]$ represent Lagrange planes, since $\left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$ have this property. And the same

happens with $\left[\begin{smallmatrix} V_1^T \\ V_3^T \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} V_2^T \\ V_4^T \end{smallmatrix}\right]$, since according to Proposition 1.23, V^T is also symplectic.

3. If the real matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ represents the real Lagrange plane l, then the matrices $L_1 \pm iL_2$ are nonsingular. As a matter of fact, $(L_1 \pm iL_2)^*(L_1 \pm iL_2) = L_1^T L_1 + L_2^T L_2$, and this last matrix is positive definite, as easily deduced from the relation $\dim l = n$. In addition, it is possible to find a real representation $\left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$ of l with $\Phi_1 + i\Phi_2$ unitary: just take P to be the unique positive definite square root (see Proposition 1.19) of $L_1^T L_1 + L_2^T L_2$, where $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, and define $\left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] P^{-1}$. Finally, the real matrices $\left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \Psi_1 \\ \Psi_2 \end{smallmatrix}\right]$ represent the same plane with $\Phi_1 + i\Phi_2$ and $\Psi_1 + i\Psi_2$ unitary if and only if $\Psi_1 = \Phi_1 R$ and $\Psi_2 = \Phi_2 R$ with R orthogonal.

The spaces $\mathcal{L}_\mathbb{K}$ (for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$) are compact orientable manifolds of dimension $n(n+1)/2$: see Mishchenko et al. [106], Section 2.4. They can also be understood as submanifolds of $\mathcal{G}_n(\mathbb{K}^{2n})$, so that they are also metrizable. The following results clarify the meaning of convergence in $\mathcal{L}_\mathbb{K}$. Consider the subsets $\mathcal{L}_\mathbb{K}$ defined by

$$\mathcal{D}_{i_1,\dots,i_n}(\mathbb{K}) = \left\{ l \in \mathcal{L}_\mathbb{K} \mid l \equiv \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \text{ with } \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}_{i_1,\dots,i_n} \text{ nonsingular} \right\},$$

where $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]_{i_1,\dots,i_n}$ is the $n \times n$ submatrix of $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ whose jth row is i_jth row of the initial one, for $1 \le i_1 < \cdots < i_n \le 2n$. Note that the (nondisjoint) union of all these sets fills up the space $\mathcal{L}_\mathbb{K}$. In fact, these sets form the charts of the structure of a variety on $\mathcal{L}_\mathbb{K}$, as the following results imply. Direct proofs of them are included for the reader's convenience.

Proposition 1.28 *Each set $\mathcal{D}_{i_1,\dots,i_n}(\mathbb{K})$ is open in $\mathcal{L}_\mathbb{K}$.*

Proof To simplify the notation, the proof is carried out for the set $\mathcal{D}_\mathbb{K} = \mathcal{D}_{1,\dots,n}(\mathbb{K})$, which is the complement of the set given by $\mathcal{C}_\mathbb{K} = \{l \in \mathcal{L}_\mathbb{K} \mid l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \text{ with } \det L_1 = 0\}$. The ideas are the same in the remaining cases. In order to check that $\mathcal{C}_\mathbb{K}$ is closed, take a sequence (l_k) in $\mathcal{C}_\mathbb{K}$ with limit l, and apply Proposition 1.25 to find representations $l_k \equiv \left[\begin{smallmatrix} L_1^k \\ L_2^k \end{smallmatrix}\right]$ and $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\lim_{k\to\infty} \left[\begin{smallmatrix} L_1^k \\ L_2^k \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. Then $\det L_1 = \lim_{k\to\infty} \det L_1^k = 0$, so that $l \in \mathcal{C}_\mathbb{K}$.

Now define the map

$$d_{i_1,\dots,i_n} \colon \mathcal{D}_{i_1,\dots,i_n}(\mathbb{K}) \to \mathbb{M}_{n\times n}(\mathbb{K}), \quad l \mapsto M,$$

where $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ is the unique representation of l with $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]_{i_1,\dots,i_n} = I_n$ and M is the $n \times n$ submatrix composed of the remaining rows, preserving the relative order. It is obvious that $d_{i_1,\dots,i_n}$ is injective, and it is also obvious how to the inverse $d^{-1}_{i_1,\dots,i_n}$ on the subset $d_{i_1,\dots,i_n}(\mathcal{D}_{i_1,\dots,i_n}(\mathbb{K})) \subset \mathbb{M}_{n\times n}(\mathbb{K})$ must be defined.

Proposition 1.29 *The map $d_{i_1,\dots,i_n}$ is an embedding. More precisely,*

(i) *if the sequence (l_k) of elements of $\mathcal{L}_{\mathbb{K}}$ converges to the Lagrange plane l, and l belongs to a chart $\mathcal{D}_{i_1,\dots,i_n}(\mathbb{K})$, then $l_k \in \mathcal{D}_{i_1,\dots,i_n}(\mathbb{K})$ for k large enough, and* $\lim_{k\to\infty} d_{i_1,\dots,i_n}(l_k) = d_{i_1,\dots,i_n}(l)$.

(ii) *If* $\lim_{k\to\infty} d_{i_1,\dots,i_n}(l_k) = d_{i_1,\dots,i_n}(l)$, *then* $\lim_{k\to\infty} l_k = l$.

Proof The proof is carried out again for the case $\mathcal{D}_{\mathbb{K}} = \mathcal{D}_{1,\dots,n}(\mathbb{K})$. The other cases can be handled in an analogous manner. Let d represent $d_{1,\dots,n}$.

(i) The first assertion in (i) is a trivial consequence of Proposition 1.28. Now represent $l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$, with $M = d(l)$, and apply Proposition 1.25 to find representations $l_k \equiv \begin{bmatrix} L_1^k \\ L_2^k \end{bmatrix}$ with $\lim_{k\to\infty} \begin{bmatrix} L_1^k \\ L_2^k \end{bmatrix} = \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$, which obviously implies that $\lim_{k\to\infty} \left[\begin{smallmatrix} I_n \\ M_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ for $M_k = L_2^k (L_1^k)^{-1} = d(l_k)$. That is, $\lim_{k\to\infty} d(l_k) = d(l)$, as asserted.

(ii) This assertion is another easy consequence of Proposition 1.25.

Remark 1.30 The previous result ensures that the sets $\mathcal{D}_{1,\dots,n}(\mathbb{K})$ and $\mathcal{D}_{n+1,\dots,2n}(\mathbb{K})$ are homeomorphic to $\mathbb{S}_n(\mathbb{K})$.

The following result will be fundamental in Sect. 4.5. Its proof follows easily from Propositions 1.28 and 1.29(i), and the details are left to the reader.

Corollary 1.31 *If $l_k \colon \Omega \to \mathcal{L}_{\mathbb{K}}$ $(k = 1, 2, \dots)$ are continuous maps with $\lim l_k(\omega) = l(\omega)$ uniformly on Ω, then l is continuous. If, in addition, l takes values in a chart $\mathcal{D}_{i_1,\dots,i_n}(\mathbb{K})$, then $\lim_{k\to\infty} d_{i_1,\dots,i_n}(l_k(\omega)) = d_{i_1,\dots,i_n}(l(\omega))$ uniformly on Ω.*

1.2.4 Matrix-Valued Functions

Section 1.2 is completed by listing some more definitions and properties concerning real or complex matrix-valued functions. The scalar case is of course included. In what follows, $\|\cdot\|$ represents a fixed matrix norm, which can be the Euclidean norm or any equivalent one (see Remarks 1.24): the various concepts and properties to be discussed are independent of the particular choice of norm. Note that matrix-valued function $M \colon \Omega \to \mathbb{M}_{d\times m}(\mathbb{K})$ is measurable with respect to a fixed sigma-algebra on Ω containing the Borel sigma-algebra if each of its component functions has this property. The concepts of σ-invariance for matrix-valued functions are a particular case of the general ones given in Sects. 1.1.1 and 1.1.2.

Definition 1.32 Let m_0 be a σ-ergodic measure on Ω, and consider on the set of Σ_{m_0}-measurable functions taking values in $\mathbb{M}_{d\times m}(\mathbb{K})$ the equivalence relation which identifies functions which are equal m_0-a.e. Consider the quotient space $\mathcal{Q}$. For each $p \ge 1$, the space $L^p(\Omega, m_0)$ is the subset of $\mathcal{Q}$ consisting in functions satisfying

$\|M\|_p = \left(\int_\Omega \|M(\omega)\|^p \, dm_0\right)^{1/p} < \infty$; and $L^p(\Omega, m_0)$ is endowed with the norm-topology defined by $\|M\|_p$, which is the $L^p(\Omega, m_0)$*-norm*.

The notation $L^p(\Omega, m_0)$ makes no reference to the dimension of the matrix space, which will always be clearly determined by the context.

Remark 1.33 It is obvious that $M = \lim_{k\to\infty} M_k$ in $L^p(\Omega, m_0)$ if and only if $\lim_{k\to\infty} \|M_k - M\|_p = 0$. The spaces $L^p(\Omega, m_0)$ are hence independent of the initial choice of the matrix norm, due to the equivalence of any pair of them. Often the notation $L^1(\Omega, m_0)$ or $L^2(\Omega, m_0)$ will be used to refer to the spaces of integrable or square integrable matrix-valued functions, although the word "square" refers of course to the square of the norm.

Definition 1.34 Let (Ω, σ) be a continuous flow on a compact metric space Ω, and let Σ be a sigma-algebra on Ω which can be either the Borel one or its completion with respect to any fixed invariant measure. Let the function $M\colon \Omega \to \mathbb{M}_{d\times m}(\mathbb{K})$ be Σ-measurable and let $\Omega_0 \in \Sigma$ be a σ-invariant subset. Then M is *differentiable at* $\omega \in \Omega_0$ if there exists $(d/dt)M(\omega{\cdot}t)|_{t=0}$, in which case its value is represented by $M'(\omega)$. If $M'(\omega)$ exists for all $\omega \in \Omega_0$, then the function M is *differentiable along the flow* σ *on* Ω_0. The function M is a *solution along the flow on* Ω_0 of a differential equation $M' = h(\omega{\cdot}t, M)$ if $M'(\omega)$ exists for all $\omega \in \Omega_0$ and $M'(\omega{\cdot}t) = h(\omega{\cdot}t, M(\omega{\cdot}t))$ for all $\omega \in \Omega_0$ and all $t \in \mathbb{R}$. If $\Omega_0 = \Omega$, then M is said to be a *solution along the flow*.

Proposition 1.35 *Let* $M, N\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ *be differentiable at* ω*. Then,*

(i) *the functions* λM *(for* $\lambda \in \mathbb{K}$*),* $M + N$ *and* MN *are differentiable at* ω*, with* $(\lambda M)'(\omega) = \lambda M'(\omega)$*,* $(M+N)'(\omega) = M'(\omega) + N'(\omega)$ *and* $(MN)'(\omega) = M'(\omega)N(\omega) + M(\omega)N'(\omega)$*.*

(ii) *If* $M(\omega)$ *is nonsingular, then there exists*

$$(M^{-1})'(\omega) = -M^{-1}(\omega)M'(\omega)M^{-1}(\omega)\,.$$

(iii) *If* $M(\omega) > 0$*, there exists* $(M^{1/2})'(\omega)$*.*

Proof The functions $t \mapsto (\lambda M)(\omega{\cdot}t)$, $t \mapsto (M+N)(\omega{\cdot}t)$, $t \mapsto (MN)(\omega{\cdot}t)$, $t \mapsto M^{-1}(\omega{\cdot}t)$, and $t \mapsto M^{1/2}(\omega{\cdot}t)$ are well defined on a neighborhood of $t = 0$ and differentiable at 0. In the case of the square root, the assertion follows from Proposition 1.19(iii). The remaining properties are obvious.

Proposition 1.36 *Let* (Ω, σ) *be a continuous flow on a compact metric space* Ω*, and let* m_0 *be a* σ*-ergodic measure on* Ω*. Let the map* $f\colon \Omega \to \mathbb{R}$ *be* Σ_{m_0}*-measurable and differentiable along the flow on* Ω*. Suppose that the limit* $\lim_{t\to\infty}(1/t)\int_0^t f'(\omega{\cdot}s)\,ds = l(\omega) \in [-\infty, \infty]$ *exists for* m_0*-a.e.* $\omega \in \Omega$*. Then* $l(\omega) = 0$ *for a.e.* $\omega \in \Omega$*. In particular, if* $f' \in L^1(\Omega, m_0)$*, then* $\int_\Omega f'(\omega)\,dm_0 = 0$*. And this last property also holds for complex functions* $f\colon \Omega \to \mathbb{C}$*, as well as for real or complex matrix-valued functions.*

Proof Choose a constant $k > 0$ and a set $\widetilde{\Omega}_0 \in \Sigma_{m_0}$ with $m_0(\widetilde{\Omega}_0) > 0$ such that $|f(\omega)| \le k$ for all $\omega \in \widetilde{\Omega}_0$, and apply the Birkhoff Theorems 1.3 and 1.6 in order to find a σ-invariant set Ω_0 with $m_0(\Omega_0) = 1$ such that $\lim_{t\to\infty}(1/t)\int_0^t \chi_{\widetilde{\Omega}_0}(\omega{\cdot}s)\,ds = m_0(\widetilde{\Omega}_0)$ for all $\omega \in \Omega_0$. Then, for all $\omega \in \Omega_0$, there exists a sequence $(t_m) \uparrow \infty$ such that $\omega{\cdot}t_m \in \widetilde{\Omega}_0$ for all $m \in \mathbb{N}$, so that, if the limit $l(\omega)$ exists, then $l(\omega) = \lim_{m\to\infty}(1/t_m)(f(\omega{\cdot}t_m) - f(\omega)) = 0$. That is, $l \equiv 0$ m_0-a.e. The assertion for $f' \in L^1(\Omega, m_0)$ follows from the Birkhoff Theorem 1.3, and the last statements are obvious.

Recall finally that, given a topological space $\mathbb{Y}$, a function $M\colon \mathbb{Y} \to \mathbb{M}_{d\times m}(\mathbb{K})$ is *bounded* or *norm-bounded* on $\mathbb{Y}$ if $\sup_{y\in\mathbb{Y}} \|M(y)\| < \infty$. Observe that any continuous matrix-valued function M on a compact space $\mathbb{Y}$ is bounded.

1.3 Nonautonomous Linear Systems

From this point to the end of Chap. 1, (Ω, σ) denotes a real continuous flow on a compact metric space. Recall the notation $\omega{\cdot}t = \sigma(t, \omega)$ for $(t, \omega) \in \mathbb{R} \times \Omega$. Unless otherwise indicated, throughout the whole of Sect. 1.3, inclusive all subsections, $\|\cdot\|$ represents the Euclidean vector and matrix norms: see Remarks 1.24.1 and 1.24.2.

1.3.1 The Flows on the Trivial and Grassmannian Bundles

A continuous matrix-valued function $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, defines a family of nonautonomous $2n$-dimensional linear systems,

$$\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{1.3}$$

which, as will be explained in Sect. 1.3.2 derives frequently from a single nonautonomous linear system. Here, as in the rest of the book, the label (1.3) will be used to make reference both to the whole family and to the system corresponding to a given element ω, when the identity of this element is clear.

A *matrix solution* of the system (1.3) corresponding to $\omega \in \Omega$ is a matrix-valued function $V\colon \mathbb{R} \to \mathbb{M}_{d\times d}(\mathbb{K})$ such that $V'(t, \omega) = A(\omega{\cdot}t)\,V(t, \omega)$ for all $t \in \mathbb{R}$, where $V'(t, \omega) = (d/dt)\,V(t, \omega)$, and it is a *fundamental* matrix solution if $\det V(0, \omega) \ne 0$, which, by the Liouville formula, ensures that $V(t, \omega)$ is nonsingular for all t. Let $U_A(t, \omega)$ be the fundamental matrix solution of the system corresponding to ω with $U_A(0, \omega) = I_d$. Then, on the one hand, the uniqueness of solutions of (1.3) ensures that

$$U_A(t + s, \omega) = U_A(t, \omega{\cdot}s)\,U_A(s, \omega) \quad \text{for all } t, s \in \mathbb{R} \text{ and } \omega \in \Omega\,; \tag{1.4}$$

and, on the other hand, the continuity of A on Ω and the classical theory of ordinary differential equations ensure the continuity of $U_A\colon \mathbb{R}\times\Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$. (The continuity of U_A is proved under less restrictive conditions in Proposition 1.38 below.) These two properties have a fundamental consequence: the family (1.3) induces a global skew-product continuous flow

$$\tau_A\colon \mathbb{R}\times\Omega\times\mathbb{K}^d \to \Omega\times\mathbb{K}^d\,,\quad (t,\omega,\mathbf{z})\mapsto(\omega{\cdot}t,U_A(t,\omega)\,\mathbf{z})\,. \tag{1.5}$$

It is common to say that τ_A is a *linear* skew-product flow, due to the linearity of the second component of the homeomorphism

$$\{\omega\}\times\mathbb{K}^d \to \{\omega{\cdot}t\}\times\mathbb{K}^d\,,\quad (\omega,\mathbf{z})\mapsto(\omega{\cdot}t,U_A(t,\omega)\,\mathbf{z})\,.$$

Take now a linear subspace g of $\mathbb{K}^d$ and define

$$U_A(t,\omega){\cdot}g=\{U_A(t,\omega)\,\mathbf{z}\mid \mathbf{z}\in g\}\,,$$

and note that $\dim U_A(t,\omega){\cdot}g=\dim g$. Consequently, it is clear that the family (1.11) also defines a global skew-product continuous flow τ_A^k on the Grassmannian bundle $\Omega\times\mathcal{G}_k(\mathbb{K}^d)$,

$$\tau_A^k\colon \mathbb{R}\times\Omega\times\mathcal{G}_k(\mathbb{K}^d)\to\Omega\times\mathcal{G}_k(\mathbb{K}^d)\,,\quad (t,\omega,g)\mapsto(\omega{\cdot}t,U_A(t,\omega){\cdot}g)\,. \tag{1.6}$$

The results of Sect. 1.2.2 allow one to prove that these flows are also continuous:

Proposition 1.37 *The flows τ_A^k are continuous for $k=1,\dots,d$.*

Proof Fix $(t,\omega,g)\in\mathbb{R}\times\Omega\times\mathcal{G}_k(\mathbb{K}^d)$, and write it as $\lim_{j\to\infty}(t_j,\omega_j,g_j)$ in the same space. The goal is to prove that $\lim_{j\to\infty}U(t_j,\omega_j){\cdot}g_j=U(t,\omega){\cdot}g$. Take $\mathbf{w}\in U(t,\omega){\cdot}g$, so that $\mathbf{w}=U(t,\omega)\,\mathbf{v}$ for $\mathbf{v}\in g$; apply Proposition 1.26(i) to find a sequence $(\mathbf{v}_j)$ with $\mathbf{v}_j\in g_j$ and limit $\mathbf{v}$; note that $\mathbf{w}=\lim_{j\to\infty}U(t_j,\omega_j)\,\mathbf{v}_j$; and apply again Proposition 1.26(i) to get the desired conclusion.

Note that the previous proof only requires the continuity of the flow τ_A; i.e. the continuity of the base flow and the joint continuity of U_A on $\mathbb{R}\times\Omega$. In fact, the hypothesis of continuity of A is not necessary to ensure the continuity of τ_A, and hence that of the flows τ_A^k. The following result establishes much less restrictive hypotheses under which all these flows are continuous. A situation in which it is relevant is discussed at the end of Sect. 1.3.2. The hypotheses of Proposition 1.38 will be in force in Chap. 2, which is devoted to defining and analyzing the properties of the rotation number and the Lyapunov index. Given a Borel measurable matrix-valued function $M\colon\mathbb{R}\to\mathbb{M}_{d\times d}(\mathbb{K})$, let $\|M\|_\infty$ represent its *essential supremum*; i.e. the smallest $m\in[0,\infty]$ with $\|M(t)\|\le m$ for Lebesgue-a.e. $t\in\mathbb{R}$.

Proposition 1.38 *Let* $A: \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ *be a Borel measurable matrix-valued function satisfying the following regularity conditions: first,*

$$a = \sup_{\omega\in\Omega} \|A_\omega\|_\infty < \infty\,, \tag{1.7}$$

where $A_\omega(s) = A(\omega{\cdot}s)$*; and second, the map*

$$\Omega \to \mathbb{R}^d\,, \quad \omega \mapsto \int_{\mathbb{R}} A(\omega{\cdot}t)\,\mathbf{z}(t)\,dt \tag{1.8}$$

is continuous for all L^1*-functions* $\mathbf{z}: \mathbb{R} \to \mathbb{K}^d$*. Consider the family* (1.3) *given by* A*. Then the flows* τ_A *on* $\Omega \times \mathbb{K}^d$ *and* τ_A^k *on* $\Omega \times \mathcal{G}_k(\mathbb{K}^d)$ *for* $k = 1, \dots, d$*, respectively defined by* (1.5) *and* (1.6)*, are continuous.*

Proof For each $\omega \in \Omega$, the existence and uniqueness of a continuous matrix-valued function $t \mapsto U_A(t,\omega)$ satisfying

$$U_A(t,\omega) = I_d + \int_0^t A(\omega{\cdot}s)\,U_A(s,\omega)\,ds$$

is ensured by the standard theory of linear ordinary differential equations (see e.g. Problem 1 of Chapter 3 of [28]). As pointed out before, Proposition 1.37 ensures that all the statements follow from the continuity of the flow τ_A given by (1.5), which in turn follows from the continuity of $U_A: \mathbb{R} \times \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$. The continuity of U_A is now proved.

Take a sequence (t_m,ω_m) with limit (t_0,ω_0), and define $t^* = \sup_{m\in\mathbb{N}} |t_m|$. The goal is to prove that $\lim_{m\to\infty} \|U_A(t_m,\omega_m) - U_A(t_0,\omega_0)\| = 0$. Since $U_A(t,\omega) = I_d + \int_0^t A(\omega{\cdot}s)\,U_A(s,\omega)\,ds$, relation (1.7) ensures that $\|U_A(t,\omega)\| \le 1 + a\int_0^t \|U_A(s,\omega)\|\,ds$ for $t > 0$ and $\|U_A(t,\omega)\| \le 1 + a\int_t^0 \|U_A(s,\omega)\|\,ds$ for $t < 0$. Therefore, by the Gronwall lemma,

$$\|U_A(t,\omega)\| \le e^{a|t|} \le e^{at^*} \quad \text{for } (t,\omega) \in [-t^*, t^*] \times \Omega\,. \tag{1.9}$$

This property, the equality

$$U_A(t_m,\omega_m) - U_A(t_0,\omega_m) = \int_{t_0}^{t_m} A(\omega_m{\cdot}s)\,U_A(s,\omega_m)\,ds\,,$$

and again (1.7), yield

$$\|U_A(t_m,\omega_m) - U_A(t_0,\omega_m)\| \le a\,e^{at^*}\,|t_m - t_0|\,.$$

Hence, it suffices to prove that $\lim_{m\to\infty} U_A(t_0,\omega_m) = U_A(t_0,\omega_0)$. Note that

$$\begin{aligned} U_A(t_0,\omega_m) - U_A(t_0,\omega_0) &= \int_0^{t_0} \big(A(\omega_m{\cdot}s)\,U_A(s,\omega_m) - A(\omega_0{\cdot}s)\,U_A(s,\omega_0)\big)\,ds \\ &= \int_0^{t_0} \big(A(\omega_m{\cdot}s) - A(\omega_0{\cdot}s)\big)\,U_A(s,\omega_0)\,ds \\ &\quad + \int_0^{t_0} A(\omega_m{\cdot}s)\,\big(U_A(s,\omega_m) - U_A(s,\omega_0)\big)\,ds\,, \end{aligned}$$

and that $\beta_m = \big|\int_0^{t_0} (A(\omega_m{\cdot}s) - A(\omega_0{\cdot}s))\,U_A(s,\omega_0)\,ds\,\big|$ tends to zero as m tends to ∞, which follows easily from (1.9) and (1.8). Since

$$\|U_A(t_0,\omega_m) - U_A(t_0,\omega_0)\| \le \beta_m + \int_0^{t_0} a\,\|U_A(s,\omega_m) - U_A(s,\omega_0)\|\,ds\,,$$

if $t_0 \ge 0$ and

$$\|U_A(t_0,\omega_m) - U_A(t_0,\omega_0)\| \le \beta_m + \int_{t_0}^{0} a\,\|U_A(s,\omega_m) - U_A(s,\omega_0)\|\,ds\,,$$

if $t_0 < 0$, a new application of the Gronwall lemma ensures that

$$\|U_A(t_0,\omega_m) - U_A(t_0,\omega_0)\| \le \beta_m\, e^{a t_0}\,,$$

which yields the required property.

Note that conditions (1.7) and (1.8) are fulfilled if $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ is a continuous function.

Remark 1.39 Suppose that (1.7) holds, take any σ-invariant measure m_0 on the base Ω, and define $\Omega_a = \{\omega \in \Omega \mid \|A(\omega)\| \le a\}$. Then, for all $\omega \in \Omega$, $\int_{[0,1]} \chi_{\Omega_a}(\omega{\cdot}t)\,dt = 1$, so that Fubini's theorem ensures that $\int_\Omega \chi_{\Omega_a}(\omega{\cdot}t)\,dm_0 = 1$ for Lebesgue-a.e. $t \in [0,1]$, and the σ-invariance of m_0 yields $\int_\Omega \chi_{\Omega_a}(\omega)\,dm_0 = 1$. That is, $\|A(\omega)\| \le a$ for m_0-a.e. $\omega \in \Omega$. In particular, the matrix-valued function A belongs to $L^1(\Omega, m_0)$ for every σ-invariant measure m_0 on the base. (Another proof of this last property can be obtained by applying Proposition 1.4 to the function $\|A\|$.)

1.3.2 The Hull Construction

An important fact has already been mentioned, namely that the setup described in Sect. 1.3, which is associated to the family (1.3), can frequently be derived from a single $d \times d$ linear system, namely

$$\mathbf{z}' = A_0(t)\,\mathbf{z}\,, \tag{1.10}$$

where $A_0 : \mathbb{R} \to \mathbb{M}_{d\times d}(\mathbb{K})$ is bounded and uniformly continuous. To explain this assertion is the objective of this short section.

Represent the s-translation $t \mapsto A_0(s+t)$ by $A_s(t)$ for each $s \in \mathbb{R}$; and define Ω as the closure of $\{A_s \mid s \in \mathbb{R}\}$ on the set of bounded and uniformly continuous maps from $\mathbb{R}$ to $\mathbb{M}_{d\times d}(\mathbb{K})$ endowed with the compact-open topology. Then Ω is a compact metrizable space, called the *hull* of A_0, with a continuous flow defined by translation:

$$\sigma : \mathbb{R} \times \Omega \to \Omega\,, \quad (s,\omega) \mapsto \omega{\cdot}s\,,$$

with $\omega{\cdot}s(t) = \omega(s+t)$. Detailed proofs of these facts are given in Chapter III of Sell [140]. Now define the zero-evaluation map

$$A : \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})\,, \quad \omega \mapsto \omega(0)\,,$$

which is obviously continuous, and consider the corresponding family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ for $\omega \in \Omega$, which fits in the type (1.3). Represent $\omega_0 = A_0$, and note that

$$A(\omega_0{\cdot}t) = A(\sigma_t(\omega_0)) = \omega_0{\cdot}t(0) = A_t(0) = A_0(t)\,.$$

That is, the initial system (1.10) is included in the family.

It is well-known and easy to check that Ω can be identified with a point, a circle or a torus if the initial matrix-valued function A_0 is constant, periodic, or quasi-periodic, respectively.

This procedure is usually referred to as the *Bebutov hull construction*, in recognition of the contributions of M. V. Bebutov, who made a groundbreaking study of the dynamical system defined by the shift operator in the space of continuous functions on the real line: see [14]. Bebutov's promising career was cut off by the Second World War; he fell in July 1942 on the Voronezh front, at the age of 29. Thirty years later, the works of Miller and Sell [103] and Sell [140] showed the power of this tool in the analysis of nonautonomous differential equations. The advantage of the "collective" formulation is clear: unlike what happens with a single nonautonomous system, the family (1.11) defines a flow, as explained in the previous section; this fact allows one to use techniques coming from topological dynamics and ergodic theory for the analysis of the dynamical behavior; and in the applicability of these methods the compactness of Ω (which does not hold if one applies the standard method of adding the equation $t = 1$) is fundamental.

In the context of the Bebutov construction, a result proved to hold for all the systems in the hull obviously holds for the "initial" system (1.10). However, this cannot always be said of a result which is only proved to hold for almost all the systems in the hull Ω, where the words "almost all" refer to an ergodic measure fixed on Ω. In this situation, the fact that the orbit of ω_0 is dense in Ω sometimes has important dynamical consequences, as the reader will discover at various points of the book.

One of the most favorable situations arises when the flow σ on Ω is minimal and *uniquely ergodic* (i.e. a unique σ-invariant measure exists, which according to Theorem 1.9 is equivalent to the existence of a unique σ-ergodic measure). This happens, for instance, if the flow is *almost periodic* (i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_\Omega(\omega_1{\cdot}t, \omega_2{\cdot}t) < \varepsilon$ for all $t \in \mathbb{R}$ whenever $d_\Omega(\omega_1, \omega_2) < \delta$), and minimal: under these hypotheses it is easy to prove that, for all $f \in C(\Omega, \mathbb{R})$, the value of $\lim_{t\to\infty}(1/t)\int_0^t f(\omega{\cdot}s)\,ds$ is independent of the particular choice of ω; and hence the Birkhoff Theorems 1.3 and 1.6 and the regularity of the measures involved imply that a unique σ-ergodic measure exists. This is the situation which arises when Ω is constructed as the hull of a Bohr almost periodic matrix-valued function, as proved in Chapter VI of [140]. The fact that the set Ω_f provided by Theorem 1.3 agrees with Ω can be enough, in some situations, to ensure that a given property holds for every system of the family.

The setup described in Proposition 1.38 includes nonautonomous systems with a very wide class of coefficient functions. For instance, assume that the initial matrix-valued function $A_0\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ belongs to $L^\infty(\mathbb{R})$; i.e. there exists $a \in \mathbb{R}$ with $\|A_0(t)\| \le a$ for Lebesgue-a.e. $t \in \mathbb{R}$. Endow $L^\infty(\mathbb{R})$ with the weak* topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$, so that a given sequence (B_k) of $d \times d$ matrix-valued functions converges to B if and only if $\int_\mathbb{R} B_k(t)f(t)\,dt$ converges to $\int_\mathbb{R} B(t)f(t)\,dt$ for all L^1-functions $f\colon \mathbb{R} \to \mathbb{R}$. Since $L^1(\mathbb{R})$ is separable, the Banach–Alouglu theorem ensures that the closed ball $\mathcal{B}_a$ consisting of the matrix-valued functions B with $\|B(t)\| \le a$ for Lebesgue-a.e. $t \in \mathbb{R}$ is compact and metrizable. The set $C_c(\mathbb{R})$ of continuous functions with compact support is dense in $L^1(\mathbb{R})$, which can be used to check that a sequence (B_k) in $\mathcal{B}_a$ converges to B if and only if $\int_\mathbb{R} B_k(t)f(t)\,dt$ converges to $\int_\mathbb{R} B(t)f(t)\,dt$ for all $f \in C_c(\mathbb{R})$. Clearly, $\mathcal{B}_a$ contains the set $\{A_s \mid s \in \mathbb{R}\}$ of time-translated functions $A_s(t) = A_0(t+s)$. Let $\Omega \subset \mathcal{B}_a$ be the closure of this set, and define $\sigma\colon \mathbb{R}\times\Omega \to \Omega$ and $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ as before. The characterization of the convergence in $\mathcal{B}_a$ makes it easy to prove that σ is a continuous flow on Ω. And it is clear that A satisfies the conditions of Proposition 1.38.

See Johnson and Nerurkar [77] for more examples in which a family (1.11) defining a continuous skew-product flow on $\Omega \times \mathbb{K}^d$ arises from a single initial nonautonomous system.

1.3.3 The Hamiltonian Case: Flow on the Lagrangian Bundle

Represent

$$\mathfrak{sp}(n,\mathbb{R}) = \{H \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid H^T J + JH = 0_{2n}\}\,,$$

$$\mathfrak{sp}(n,\mathbb{C}) = \{H \in \mathbb{M}_{2n\times 2n}(\mathbb{C}) \mid H^T J + JH = 0_{2n}\}\,;$$

that is, $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{sp}(n,\mathbb{C})$ are the Lie algebras of real and complex infinitesimally symplectic matrices, whose corresponding Lie groups are $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$. Recall that J is the standard symplectic matrix $\left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$, that I_d and 0_d are the identity and null $d\times d$ matrices for all $d\in\mathbb{N}$, and that H^T is the transpose of H. That is, any element of $\mathfrak{sp}(n,\mathbb{K})$ is $H=\left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right]$ with H_2 and H_3 symmetric $n\times n$ matrix-valued functions on Ω. Equivalently, $H\in\mathfrak{sp}(n,\mathbb{K})$, with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, if and only if $JH\in\mathbb{S}_{2n}(\mathbb{K})$.

A continuous matrix-valued function $H\colon\Omega\to\mathfrak{sp}(n,\mathbb{K})$ defines a family of $2n$-dimensional linear Hamiltonian systems,

$$\mathbf{z}'=H(\omega{\cdot}t)\,\mathbf{z}\,,\quad \omega\in\Omega\,. \tag{1.11}$$

The family of n-dimensional Schrödinger equations

$$-\mathbf{x}''+G(\omega{\cdot}t)\,\mathbf{x}=\mathbf{0}\,,\quad \omega\in\Omega \tag{1.12}$$

determined by a real or complex symmetric $n\times n$ matrix-valued function G on Ω satisfying the two previous conditions, gives rise to a family (1.11) by taking $\mathbf{z}=\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{x}'\end{smallmatrix}\right]$ and $H=\left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n \end{smallmatrix}\right]$.

The labels (1.11) and (1.12) will refer both to the families of systems and to the particular system corresponding to a given element $\omega\in\Omega$. At any given moment the context will provide the exact meaning.

Since this type of system is the main object of analysis of the book, and since the matrix H will be almost always fixed, the general notation established in the previous section will be modified: the matrix $U(t,\omega)=\left[\begin{smallmatrix} U_1(t,\omega) & U_2(t,\omega) \\ U_3(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ (instead of $U_H(t,\omega)$) will denote the real fundamental matrix solution of equation (1.11) for $\omega\in\Omega$ with $U(0,\omega)=I_{2n}$. The global linear skew-product continuous flow induced by the family (1.11) on the linear bundle $\Omega\times\mathbb{K}^{2n}$, called now $\tau_{\mathbb{K}}$, is then

$$\tau_{\mathbb{K}}\colon\mathbb{R}\times\Omega\times\mathbb{K}^{2n}\to\Omega\times\mathbb{K}^{2n}\,,\quad (t,\omega,\mathbf{z})\mapsto(\omega{\cdot}t,U(t,\omega)\,\mathbf{z})\,. \tag{1.13}$$

Note that both flows $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are defined if $H\colon\Omega\to\mathfrak{sp}(n,\mathbb{R})$, but that just $\tau_{\mathbb{C}}$ is defined if $H\colon\Omega\to\mathfrak{sp}(n,\mathbb{C})$. Let $V(t,\omega)$ be a real matrix solution of (1.11). Since $H(\omega{\cdot}t)\in\mathfrak{sp}(n,\mathbb{K})$, it follows that

$$(V^T(t,\omega)JV(t,\omega))'=V^T(t,\omega)(H^T(\omega{\cdot}t)J+JH(\omega{\cdot}t))V(t,\omega)=0_{2n}\,.$$

Consequently, $V(t,\omega)$ belongs to the symplectic group

$$\mathrm{Sp}(n,\mathbb{K})=\{V\in\mathbb{M}_{2n\times 2n}(\mathbb{K})\mid V^TJV=J\}$$

if and only if $V(0,\omega)$ does. When this property holds for every $\omega\in\Omega$, V is called a *symplectic matrix solution*. Obviously, in this case, V is a fundamental matrix solution for every $\omega \in \Omega$. The main example is the fundamental matrix solution $U(t,\omega)$ defined above.

The fact that $V(t,\omega)$ is symplectic ensures that the family (1.11) defines new global real continuous skew-product flows on the real and complex Lagrange bundles. This assertion is now explained.

As a consequence of the symplectic character of $U(t,\omega)$, the vector space

$$U(t,\omega){\cdot}l = \{U(t,\omega)\,\mathbf{z} \mid \mathbf{z} \in l\}$$

is a Lagrange plane for all $t \in \Omega$ and $\omega \in \Omega$ in the case that l is: it has dimension n, since $U(t,\omega)$ defines an isomorphism on $\mathbb{K}^{2n}$; and, if $\mathbf{z}$ and $\mathbf{w}$ belong to l, then $\mathbf{z}^T U^T(t,\omega)\, JU(t,\omega)\,\mathbf{w} = \mathbf{z}^T J\,\mathbf{w} = 0$. This property implies that the map

$$\tau\colon \mathbb{R}\times\Omega\times\mathcal{L}_{\mathbb{K}} \to \Omega\times\mathcal{L}_{\mathbb{K}}\,, \quad (t,\omega,l) \mapsto (\omega{\cdot}t, U(t,\omega){\cdot}l) \tag{1.14}$$

defines a real *global* skew-product flow on $\mathcal{K}_{\mathbb{K}} = \Omega\times\mathcal{L}_{\mathbb{K}}$. In addition, if $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, then $U(t,\omega){\cdot}l \equiv U(t,\omega)\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_1(t,\omega)\,L_1 + U_3(t,\omega)\,L_2 \\ U_2(t,\omega)\,L_1 + U_4(t,\omega)\,L_2 \end{smallmatrix}\right]$. Note that $\mathcal{K}_{\mathbb{K}}$ is a compact metric space, since Ω and $\mathcal{L}_{\mathbb{K}}$ are compact. Note also that $\mathcal{K}_{\mathbb{K}}$ can be understood as a closed invariant subset of $\Omega\times\mathcal{G}_n(\mathbb{K}^{2n})$ for the corresponding flow, and that in fact the flow τ defined by (1.14) agrees with the restriction of this Grassmannian flow to $\mathcal{K}_{\mathbb{K}}$. In particular, it is a continuous flow.

Remark 1.40 In fact, as ensured by Proposition 1.38, the flows $\tau_{\mathbb{K}}$ and τ are continuous not only if H is continuous on Ω, but also if it satisfies conditions (1.7) and (1.8). It is also clear that, if $H_0\colon \mathbb{R} \to \mathfrak{sp}(n,\mathbb{K})$ is a bounded and uniformly continuous matrix-valued function, then the hull construction made in Sect. 1.3.2 provides a family of linear Hamiltonian systems over a continuous flow, and the flows $\tau_{\mathbb{K}}$ and τ are continuous.

1.3.4 The Hamiltonian Case: Generalized Polar Coordinates on $\mathcal{L}_{\mathbb{R}}$

As explained in Remark 1.27.3, the space $\mathcal{L}_{\mathbb{R}}$ can be identified with the homogeneous space of left cosets $\mathcal{G}/\mathcal{H}$, where

$$\mathcal{G} = \left\{ \begin{bmatrix} \Phi_1 & -\Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R})) \mid (\Phi_1 + i\Phi_2)^*(\Phi_1 + i\Phi_2) = I_n \right\} \simeq U(n,\mathbb{C})\,,$$

$$\mathcal{H} = \left\{ \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid R^T R = I_n \right\} \simeq \mathrm{O}(n,\mathbb{R})\,.$$

Recall that the symbol * represents the conjugate transpose, and that $U(n,\mathbb{C})$ and $\mathrm{O}(n,\mathbb{R})$ stand respectively for the groups of n-dimensional unitary complex and orthogonal real matrices.

Assume that the function $H\colon\Omega \to \mathfrak{sp}(n,\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. The above identification allows one to express the continuous flow τ on the real Lagrange bundle, defined by (1.14) from (1.11), in terms of the so-called *generalized polar coordinates*. This is explained in the following theorem, which follows from a more general result established in Theorem 9.1 of Chapter V of Reid [127]. As that author explains, the application of the polar transformation to the study of oscillation and comparison theorems for matrix differential equations was first presented by Barret in [12] and subsequently extended in Reid [123, 125]. See Remark 1.27.3 to understand the first sentence of the theorem.

Theorem 1.41 *Let* $l \equiv \left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right]$ *be a real Lagrange plane and let* Φ_1^0, Φ_2^0 *and* R^0 *be* $n\times n$ *real matrices such that* $\left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Phi_1^0 R^0 \\ \Phi_2^0 R^0 \end{smallmatrix}\right]$, *with* $\left[\begin{smallmatrix} \Phi_1^0 & -\Phi_2^0 \\ \Phi_2^0 & \Phi_1^0 \end{smallmatrix}\right] \in \mathcal{G}$ *and* R^0 *nonsingular. Then the* $2n\times n$ *solution of* (1.11) *corresponding to the initial datum* $\left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right]$ *is*

$$U(t,\omega)\begin{bmatrix} L_1^0 \\ L_2^0 \end{bmatrix} = \begin{bmatrix} L_1(t,\omega,L_1^0,L_2^0) \\ L_2(t,\omega,L_1^0,L_2^0) \end{bmatrix} = \begin{bmatrix} \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) \\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) \end{bmatrix},$$

where the $n\times n$ *matrix-valued functions*

$$t\mapsto \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\,, \quad t\mapsto \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\,, \quad \text{and} \quad t\mapsto R(t,\omega,\Phi_1^0,\Phi_2^0,R^0)$$

are the solutions of

$$\begin{aligned} \Phi_1' &= \Phi_2\, Q(\omega{\cdot}t,\Phi_1,\Phi_2)\,, \\ \Phi_2' &= -\Phi_1\, Q(\omega{\cdot}t,\Phi_1,\Phi_2)\,, \end{aligned} \tag{1.15}$$

and

$$R' = S(\omega{\cdot}t,\Phi_1,\Phi_2)\,R \tag{1.16}$$

given by the initial data Φ_1^0, Φ_2^0 *and* R^0 *respectively, with*

$$Q(\omega,\Phi_1,\Phi_2) = \begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} JH(\omega)\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \tag{1.17}$$

and

$$S(\omega,\Phi_1,\Phi_2) = \begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} H(\omega) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}. \tag{1.18}$$

Moreover,

$$\begin{aligned} R^T(t,\omega,\Phi_1^0,\Phi_2^0,R^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) &= L_1^T(t,\omega,L_1^0,L_2^0)\,L_1(t,\omega,L_1^0,L_2^0) \\ &\quad + L_2^T(t,\omega,L_1^0,L_2^0)\,L_2(t,\omega,L_1^0,L_2^0) \end{aligned}$$

and $\begin{bmatrix} \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0) & -\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0) \\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0) & \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0) \end{bmatrix} \in \mathcal{G}$ *for all* $t \in \mathbb{R}$.

Consequently, with these coordinates, the linear skew-product flow τ induced by (1.14) in $\mathcal{K}_{\mathbb{R}}$ can be expressed in the following way: given $l \in \mathcal{L}_{\mathbb{R}}$, represent it by $\begin{bmatrix} \Phi_1^0 \\ \Phi_2^0 \end{bmatrix} \in \mathcal{L}_{\mathbb{R}}$ with $\Phi_1^0 + i\Phi_2^0$ unitary (see Remark 1.27.3), and represent by $\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)$ and $\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)$ the matrix solutions of equations (1.15) with initial data Φ_1^0 and Φ_2^0. Then $U(t,\omega){\cdot}l \equiv \begin{bmatrix} \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0) \\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0) \end{bmatrix}$.

Remark 1.42 Assume that $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ and $\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$, with $\Phi_1 + i\Phi_2$ and $\Psi_1 + i\Psi_2$ unitary, represent the same real Lagrange plane, and look for R with $\Psi_1 = \Phi_1 R$ and $\Psi_2 = \Phi_2 R$ and $R^T R = I_n$. Then R is orthogonal (see again Remark 1.27.3), which implies that $\operatorname{tr} Q(\omega,\Phi_1,\Phi_2) = \operatorname{tr} Q(\omega,\Psi_1,\Psi_2)$ and $\operatorname{tr} S(\omega,\Phi_1,\Phi_2) = \operatorname{tr} S(\omega,\Psi_1,\Psi_2)$. In other words, despite the fact that the functions Q and S given by (1.17) and (1.18) are uniquely defined on $\Omega \times \mathcal{G}$ but not on $\mathcal{K}_{\mathbb{R}}$, the functions $\operatorname{tr} Q$ and $\operatorname{tr} S$ are actually functions on the quotient space $\mathcal{K}_{\mathbb{R}}$: given $\omega \in \Omega$ and $l \in \mathcal{L}_{\mathbb{R}}$, define

$$\operatorname{Tr} Q(\omega,l) = \operatorname{tr}\left(\begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} JH(\omega) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \right), \tag{1.19}$$

$$\operatorname{Tr} S(\omega,l) = \operatorname{tr}\left(\begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} H(\omega) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \right), \tag{1.20}$$

where $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ is any representation of l with $\Phi_1 + i\Phi_2$ unitary.

In addition, the functions $\operatorname{Tr} Q$ and $\operatorname{Tr} S$ are continuous on $\mathcal{K}_{\mathbb{R}}$ if H is continuous on Ω. To prove this assertion, note that Proposition 1.25 (or Proposition 1.29(i)) ensures that if $\lim_{k\to\infty} l_k = l$ in $\mathcal{K}_{\mathbb{R}}$, there exist suitable representations $l_k \equiv \begin{bmatrix} L_{1,k} \\ L_{2,k} \end{bmatrix}$ and $l \equiv \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ with $\lim_{k\to\infty} \begin{bmatrix} L_{1,k} \\ L_{2,k} \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ in $\mathbb{M}_{2n\times n}(\mathbb{R})$; hence, as explained in Remark 1.27.3, $l_k \equiv \begin{bmatrix} \Phi_{1,k} \\ \Phi_{2,k} \end{bmatrix} = \begin{bmatrix} L_{1,k}R_k^{-1} \\ L_{2,k}R_k^{-1} \end{bmatrix}$ and $l \equiv \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} L_1 R^{-1} \\ L_2 R^{-1} \end{bmatrix}$, where R_k and R are the unique positive square roots of $L_{1,k}^T L_{1,k} + L_{2,k}^T L_{2,k}$ and $L_1^T L_1 + L_2^T L_2$; and finally, $\Phi_{1,k} + i\Phi_{2,k}$ and $\Phi_1 + i\Phi_2$ are unitary, with $\lim_{k\to\infty} \Phi_{1,k} = \Phi_1$ and $\lim_{k\to\infty} \Phi_{2,k} = \Phi_2$. The continuity follows easily from these facts.

1.3.5 The Hamiltonian Case: The Riccati Equation

Consider the following open subsets of $\mathcal{L}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{R}}$:

$$\mathcal{D}_{\mathbb{C}} = \left\{ l \in \mathcal{L}_{\mathbb{C}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{D} = \mathcal{D}_{\mathbb{R}} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \right\}, \tag{1.21}$$

which agree with the sets $\mathcal{D}_{1,\dots,n}(\mathbb{C})$ and $\mathcal{D}_{1,\dots,n}(\mathbb{R})$ defined in Sect. 1.2.3: both sets $\mathcal{D}_{\mathbb{K}}$ and $\mathcal{D}_{1,\dots,n}(\mathbb{K})$ are composed of those Lagrange planes which admit a representation $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\det L_1 \neq 0$. As seen in Proposition 1.28, they are open. Obviously, each $l \in \mathcal{D}_{\mathbb{K}}$ admits a unique representation of the form $\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$, and the $n \times n$ matrix M has to be symmetric: $M \in \mathbb{S}_n(\mathbb{C})$ ($M \in \mathbb{S}_n(\mathbb{R})$ if $l \in \mathcal{D}$). In fact, if $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ belongs to $\mathcal{D}_{\mathbb{K}}$, then $M = L_2 L_1^{-1}$.

The set $\mathcal{D}$ is the complement in $\mathcal{L}_{\mathbb{R}}$ of the so-called *(vertical) Maslov cycle* $\mathcal{C}$, which is hence the set of real Lagrange planes represented by matrices $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\det L_1 = 0$. Both sets will play fundamental roles throughout the book.

It is assumed again throughout this section that $H\colon \Omega \to \mathfrak{sp}(n, \mathbb{K})$ is either a continuous function or satisfies the conditions described in Proposition 1.38. The joint continuity of U on $\mathbb{R} \times \Omega$ and the open character of $\mathcal{D}_{\mathbb{K}}$ ensure that, if $\omega \in \Omega$ and $l \in \mathcal{D}_{\mathbb{K}}$, then $U(t,\omega){\cdot}l \in \mathcal{D}_{\mathbb{K}}$ for t close enough to zero: if $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$ (with M_0 symmetric), then $U(t,\omega){\cdot}l$ can be represented by $\left[\begin{smallmatrix} I_n \\ M(t,\omega,M_0) \end{smallmatrix}\right]$ as long as $\det(U_1(t,\omega) + U_3(t,\omega)\, M_0) \neq 0$. Since $M(t,\omega,M_0) = L_2(t,\omega)\, L_1(t,\omega)^{-1}$, where $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] = U(t,\omega) \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$ is the $2n \times n$ matrix solution of (1.11) with initial datum $\left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$, the symmetric matrix-valued function $M(t,\omega,M_0)$ is the solution of the Riccati equation

$$\begin{aligned} M' &= -M\, H_3(\omega{\cdot}t)\, M - M\, H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\, M + H_2(\omega{\cdot}t) \\ &= h(\omega{\cdot}t, M) \end{aligned} \tag{1.22}$$

with $M(0,\omega,M_0) = M_0$. And

$$M(t,\omega,M_0) = (U_2(t,\omega) + U_4(t,\omega)\, M_0)(U_1(t,\omega) + U_3(t,\omega)\, M_0)^{-1}.$$

These facts imply that the family of equations (1.22) determines a local skew-product flow

$$\tau_s\colon \mathbb{R} \times \Omega \times \mathbb{S}_n(\mathbb{K}) \to \Omega \times \mathbb{S}_n(\mathbb{K}), \quad (t,\omega,M_0) \mapsto (\omega{\cdot}t, M(t,\omega,M_0)), \tag{1.23}$$

which is continuous on the open subset of $\mathbb{R} \times \Omega \times \mathbb{S}_n(\mathbb{K})$ on which it is defined.

By identifying $\mathcal{D}_{\mathbb{K}}$ with the vector space $\mathbb{S}_n(\mathbb{K})$ of the real $n \times n$ symmetric matrices, τ_s can be also considered as a local skew-product flow on $\Omega \times \mathcal{D}_{\mathbb{K}}$. Note that this flow is closely related to the restriction of the flow τ to the set

$\Omega \times \mathcal{D}_{\mathbb{K}} \subset \mathcal{K}_{\mathbb{K}}$: as explained above, if $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$, then $U(t,\omega){\cdot}l \equiv \left[\begin{smallmatrix} I_n \\ M(t,\omega,M_0) \end{smallmatrix}\right]$ as long as the solution $M(t,\omega,M_0)$ exists. It is important to emphasize the fact that, if $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ belongs to $\mathcal{D}_{\mathbb{K}}$, then $U(t,\omega){\cdot}l$ belongs to $\mathcal{D}_{\mathbb{K}}$ as long as $\det(U_1(t,\omega)\,L_1 + U_3(t,\omega)\,L_2) \neq 0$, which is independent of the representation chosen for l. This condition determines then the maximal interval of definition of any solution of (1.22).

Remark 1.43 Consider one of the Riccati equations (1.22). Let $M(t)$ solve it for t varying on an interval $\mathcal{I}$. The continuity of the flow τ (see Remark 1.40) ensures that, if $\mathcal{I}$ has finite right endpoint b and $\|M(t)\|$ is bounded on $\mathcal{I}$, then b is in the interior of the maximal interval of definition of $M(t)$. And a similar result holds for the left endpoint of $\mathcal{I}$. This property will be used often in the chapters to follow.

Fix now $\mathbb{K} = \mathbb{R}$. To study the monotonicity properties of the (continuous) flow τ_s on $\Omega \times \mathbb{S}_n(\mathbb{R})$ is the purpose of the rest of this section. This analysis reproduces that carried out by Johnson *at al.* in [85].

To begin, observe that the Banach space $\mathbb{S}_n(\mathbb{R})$ is *strongly ordered*. More precisely, it contains a closed convex solid cone, given by the positive semidefinite symmetric matrices $M \geq 0$. Its interior is given by the positive definite matrices $M > 0$. The (partial) *strong order relation* in $\mathbb{S}_n(\mathbb{R})$ is given by

$$\begin{aligned} M_1 \leq M_2 &\iff M_2 - M_1 \geq 0\,;\\ M_1 \lneqq M_2 &\iff M_1 \leq M_2 \text{ and } M_1 \neq M_2\,;\\ M_1 < M_2 &\iff M_2 - M_1 > 0\,. \end{aligned}$$

The relations $\geq$, $\gneqq$ and $>$ are defined in the obvious way.

Remarks 1.44

1. The norm $\|A\|_F = (\mathrm{tr}(A^TA))^{1/2}$ (see Remarks 1.24.1) is *monotone* on the set $\mathbb{S}_n(\mathbb{R})$, on which $\|A\|_F = (\mathrm{tr}(A^2))^{1/2}$. That is, if $0 \leq A \leq B$, then $\|A\|_F \leq \|B\|_F$: $\mathrm{tr}(A^2) \leq \mathrm{tr}(A^{1/2}BA^{1/2}) = \mathrm{tr}(B^{1/2}AB^{1/2}) \leq \mathrm{tr}(B^2)$. It follows easily that any other (equivalent) norm $\|\cdot\|$ is *semimonotone*: there exists $c > 0$ such that $0 \leq A \leq B$ implies $\|A\| \leq c\,\|B\|$.
2. Let $\|\cdot\|$ be a (semimonotone) matrix-norm, and let A, B and C be matrix-valued functions with $A \leq B \leq C$. Then, $\|B\| \leq \|B-A\| + \|A\| \leq c\,\|C-A\| + \|A\| \leq c\,\|C\| + (1+c)\|A\|$. In particular, if A and C are bounded, so is B.
3. Another easy consequence of these definitions is the existence of the limit of a decreasing sequence (A_m) of positive semidefinite matrices: the existence of a common bound for $\|A_m\|$ ensures the existence of a convergent subsequence, and it is very easy to check that this limit does not depend on the subsequence, and this proves the assertion.

The proof of Theorem 1.45 is given in Proposition 6 of Chapter 2 of Coppel [34], and is included here for the reader's convenience.

Theorem 1.45 *Suppose that the function $H\colon \Omega \to \mathfrak{sp}(n,\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. The local flow τ_s given by (1.23) is fiber-monotone on $\Omega \times \mathbb{S}_n(\mathbb{R})$. More precisely,*

(i) *if $\omega \in \Omega$ and $N_1 \le N_2$, then $M(t,\omega,N_1) \le M(t,\omega,N_2)$ in their common interval of definition;*
(ii) *if $\omega \in \Omega$ and $N_1 < N_2$, then $M(t,\omega,N_1) < M(t,\omega,N_2)$ in their common interval of definition.*

Proof The function $D(t) = M(t,\omega,N_1) - M(t,\omega,N_2)$ satisfies

$$D' = -D\,(H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\,S(t)) - \big(H_1^T(\omega{\cdot}t) + S(t)\,H_3(\omega{\cdot}t)\big)\,D\,,$$

where $S(t) = (M(t,\omega,N_1) + M(t,\omega,N_2))/2$. Hence, if $V(t)$ is the fundamental matrix solution of

$$V' = (H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\,S(t))\,V$$

with $V(0) = I_n$, then

$$D(t) = (V^T)^{-1}(t)\,D(0)\,V^{-1}(t)$$

when it exists, which proves the statements.

The point $\omega_0 \in \Omega$ and the matrix $N_0 \in \mathbb{S}_n(\mathbb{R})$ are fixed in the statement and proof of the following result, part of which is also given in Chapter 2 of [34].

Theorem 1.46 *Suppose that the function $H\colon \Omega \to \mathfrak{sp}(n,\mathbb{K})$ is continuous. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval containing 0 in its interior. Take $(\omega_0,N_0) \in \Omega \times \mathbb{S}_n(\mathbb{R})$, and let $\mathcal{J}$ be the maximal interval of definition of the solution $M(t,\omega_0,N_0)$ of the Riccati equation (1.22) associated to H. Represent by $N\colon \mathcal{I} \to \mathbb{S}_n(\mathbb{R})$ a C^1-map with $N(0) = N_0$.*

(i) *Suppose that $N'(t) \le h(\omega_0{\cdot}t, N(t))$ for $t \in \mathcal{I}$. Then,*

$$N(t) \le M(t,\omega_0,N_0) \quad \text{for } t \in \mathcal{I}\cap\mathcal{J} \ \text{ with } \ t \ge 0\,,$$
$$N(t) \ge M(t,\omega_0,N_0) \quad \text{for } t \in \mathcal{I}\cap\mathcal{J} \ \text{ with } \ t \le 0\,.$$

(ii) *Suppose that $N'(0) < h(\omega_0,N_0)$ and $N'(t) \le h(\omega_0{\cdot}t, N(t))$ for $t \in \mathcal{I}$. Then,*

$$N(t) < M(t,\omega_0,N_0) \quad \text{for } t \in \mathcal{I}\cap\mathcal{J} \ \text{ with } \ t > 0\,,$$
$$N(t) > M(t,\omega_0,N_0) \quad \text{for } t \in \mathcal{I}\cap\mathcal{J} \ \text{ with } \ t < 0\,.$$

(iii) *Suppose that* $N'(t) \geq h(\omega_0{\cdot}t, N(t))$ *for* $t \in \mathcal{I}$. *Then,*

$$N(t) \geq M(t,\omega_0,N_0) \quad \textit{for } t \in \mathcal{I} \cap \mathcal{J} \quad \textit{with} \quad t \geq 0\,,$$
$$N(t) \leq M(t,\omega_0,N_0) \quad \textit{for } t \in \mathcal{I} \cap \mathcal{J} \quad \textit{with} \quad t \leq 0\,.$$

(iv) *Suppose that* $N'(0) > h(\omega_0, N_0)$ *and* $N'(t) \geq h(\omega_0{\cdot}t, N(t))$ *for* $t \in \mathcal{I}$. *Then,*

$$N(t) > M(t,\omega_0,N_0) \quad \textit{for } t \in \mathcal{I} \cap \mathcal{J} \quad \textit{with} \quad t > 0\,,$$
$$N(t) < M(t,\omega_0,N_0) \quad \textit{for } t \in \mathcal{I} \cap \mathcal{J} \quad \textit{with} \quad t < 0\,.$$

Proof

(i) Let (ω_0, N_0) be a fixed element of $\Omega \times \mathbb{S}_n(\mathbb{R})$, and let $\mathcal{J}$ be the maximal interval of definition of the solution $M(t,\omega_0,N_0)$ of (1.22). Define $h_\varepsilon(\omega_0, M) = h(\omega_0, M) + \varepsilon I_n$ and represent by $M_\varepsilon(t,\omega,M_0)$ the solution of $M' = h_\varepsilon(\omega{\cdot}t, M)$ with $M_\varepsilon(0,\omega,M_0) = M_0$. Take $t_0 > 0$ in $\mathcal{I} \cap \mathcal{J}$ and $\varepsilon(t_0) > 0$ such that, if $0 < \varepsilon \leq \varepsilon(t_0)$, then the solution $M_\varepsilon(t,\omega_0,N_0)$ is defined for $t \in [0,t_0]$. The existence of $\varepsilon(t_0)$ is ensured by the joint continuity of $h_\varepsilon(\omega_0, M)$ on (ε, M). Fix $\varepsilon \in (0, \varepsilon(t_0)]$. Then,

$$N'(t) < h_\varepsilon(\omega_0{\cdot}t, N(t)) \quad \text{for } t \in [0,t_0]\,.$$

In particular, $N'(0) < h_\varepsilon(\omega_0, N_0) = h_\varepsilon(\omega_0, M_\varepsilon(0,\omega_0,N_0))$, and hence the continuity of N' and h ensure the existence of $t_\varepsilon \in (0,t_0]$ such that

$$N'(t) < h_\varepsilon(\omega_0{\cdot}t, M_\varepsilon(t,\omega_0,N_0)) \quad \text{for } t \in [0,t_\varepsilon]\,.$$

Integrating this inequality in $[0,t] \subseteq [0,t_\varepsilon]$ yields $N(t) < M_\varepsilon(t,\omega_0,N_0)$ for every $t \in [0,t_\varepsilon]$. The goal is to prove that this inequality holds on $[0,t_0]$. Assume for contradiction the existence of $t_\varepsilon^* \in [0,t_0]$ and $\mathbf{z} \in \mathbb{K}^n$ such that

$$N(t) < M_\varepsilon(t,\omega_0,N_0) \quad \text{for } t \in [0,t_\varepsilon^*] \quad \text{and} \quad N(t_\varepsilon^*)\,\mathbf{z} = M_\varepsilon(t_\varepsilon^*,\omega_0,N_0)\,\mathbf{z}\,. \tag{1.24}$$

It is easy to deduce from the last equality and the expression of h_ε that

$$\mathbf{z}^* h_\varepsilon(\omega_0{\cdot}t_\varepsilon^*, N(t_\varepsilon^*))\,\mathbf{z} = \mathbf{z}^* h_\varepsilon(\omega_0{\cdot}t_\varepsilon^*, M_\varepsilon(t_\varepsilon^*,\omega_0,N_0))\,\mathbf{z}\,.$$

Therefore the C^1-function $\varphi(t) = \mathbf{z}^*(N(t) - M_\varepsilon(t,\omega_0,N_0))\,\mathbf{z}$ satisfies $\varphi(t_\varepsilon^*) = 0$ and $\varphi'(t_\varepsilon^*) < \mathbf{z}^*(h_\varepsilon(\omega_0{\cdot}t_\varepsilon^*, N(t_\varepsilon^*)) - h_\varepsilon(t_\varepsilon^*,\omega_0, M_\varepsilon(t_\varepsilon^*,\omega_0,N_0)))\,\mathbf{z} = 0$. Consequently, $\varphi(t) > 0$ for $t < t_\varepsilon^*$ close enough to t_ε^*, which contradicts the first inequality in (1.24) and proves the inequality in the interval $[0,t_0]$. Taking now the limit as

$\varepsilon \to 0^+$ implies that $N(t) \le M(t, \omega_0, N_0)$ for $t \in [0, t_0]$. Since t_0 is any point in $\mathcal{I} \cap \mathcal{J} \cap (0, \infty)$, the first assertion in (i) is proved.

The proof of the second assertion is quite similar. Choose now a time $t_0 \in \mathcal{I} \cap \mathcal{J} \cap (-\infty, 0)$, and $\varepsilon(t_0) > 0$ such that, if $0 < \varepsilon \le \varepsilon(t_0)$, then $M_\varepsilon(t, \omega, N_0)$ exists for $t \in [t_0, 0]$. Fix such a value of ε and note that there exists $t_\varepsilon \in [t_0, 0]$ such that

$$N'(t) < h_\varepsilon(\omega_0{\cdot}t, M_\varepsilon(t, \omega_0, N_0)) \quad \text{for } t \in [t_\varepsilon, 0]\,.$$

Integrating this inequality in $[t, 0] \subseteq [t_\varepsilon, 0]$ yields

$$N(0) - N(t) < M_\varepsilon(0, \omega_0, N_0) - M_\varepsilon(t, \omega, N_0)\,,$$

i.e. $N(t) > M_\varepsilon(t, \omega_0, N_0)$ for every $t \in [t_\varepsilon, 0]$. From this point on the argument repeats that of point (i).

(ii) The hypothesis $N'(0) < h(\omega_0, N_0) = h(\omega_0, M(0, \omega_0, N_0))$ and the continuity of N' and h ensure the existence of $t_1 > 0$ in $\mathcal{I} \cap \mathcal{J}$ such that $N'(t) < h(\omega_0{\cdot}t, M(t, \omega_0, N_0))$ for $t \in [0, t_1]$. Integrating this inequality on $[0, t] \subseteq [0, t_1]$ yields $N(t) < M(t, \omega_0, N_0)$ for $t \in (0, t_1]$. In addition, again by hypothesis, $N'(t + t_1) \le h((\omega_0{\cdot}t_1){\cdot}t, N(t + t_1))$ for $t + t_1 \in \mathcal{I}$. As proved in (i), this condition ensures that, if $t \ge 0$ is such that $t + t_1 \in \mathcal{I}$ and such that t belongs to the interval of definition of $M(t, \omega_0{\cdot}t_1, N(t_1))$, then $N(t + t_1) \le M(t, \omega_0{\cdot}t_1, N(t_1))$. Therefore Theorem 1.45(ii) ensures that, for these values of t, $N(t + t_1) < M(t, \omega_0{\cdot}t_1, M(t_1, \omega_0, N_0)) = M(t + t_1, \omega_0, N_0)$. According to Remarks 1.44.2 and 1.43, these last inequalities ensure that $M(t, \omega_0{\cdot}t_1, M(t_1, \omega_0, N_0))$ is defined at least if $t \ge 0$ and $t + t_1 \in \mathcal{I} \cap \mathcal{J}$. Summarizing all this information, $N(t) < M(t, \omega_0, N_0)$ for $t > 0$ in $\mathcal{I} \cap \mathcal{J}$, which proves the first assertion in (ii).

Note now that if $N'(0) < h(\omega_0, N_0)$ and $N'(t) \le h(\omega_0{\cdot}t, N(t))$ for $t \in \mathcal{I}$, then there exists $t_1 < 0$ in $\mathcal{I} \cap \mathcal{J}$ such that $N(t) > M(t, \omega_0, N_0)$ for $t \in [t_1, 0)$. From here on the preceding argument can be modified in order to complete the proof of (ii).

(iii) & (iv) These proofs reproduce step by step the preceding ones.

The monotonicity properties of the dynamical system induced by (1.22) lead in a natural way to the idea of upper and lower solutions. Or, more precisely, to the generalization of these concepts appropriate to the nonautonomous case, which are the superequilibria and the subequilibria of the flow. These objects will play an important role in Chaps. 5 and 7.

Before defining them and analyzing their properties, and in order to avoid undue interruption of the discussion, another concept is introduced: the upper semicontinuity of a matrix-valued function $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$, which will also be fundamental in Chaps. 5 and 7. Thanks to the order structure of $\mathbb{S}_n(\mathbb{R})$, it is possible to give a direct definition (without considering a set-valued function) and to derive its main consequences.

Recall that, given a metric space $\mathcal{M}$ with distance d, the *Hausdorff distance* between two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{M}$ is

$$d_H(\mathcal{A},\mathcal{B}) = \max\left(\sup_{a\in\mathcal{A}}\inf_{b\in\mathcal{B}} d(a,b),\ \sup_{b\in\mathcal{B}}\inf_{a\in\mathcal{A}} d(a,b)\right), \tag{1.25}$$

and that it defines a metric on the set $\mathcal{P}_c(\mathcal{M})$ of nonempty compact subsets of $\mathcal{M}$: see Proposition 7.8 of Choquet [27] (and keep in mind that the distance between two bounded sets is finite). Recall also that some matrix norm $\|\cdot\|$ (equivalent to the Euclidean operator norm) is fixed.

Definition 1.47 A matrix-valued function $N\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ is said to be *upper semicontinuous* if $\sup_{\omega\in\Omega}\|N(\omega)\|<\infty$ and $N_0\le N(\omega_0)$ whenever $\omega_0=\lim_{m\to\infty}\omega_m$ and $N_0=\lim_{m\to\infty}N(\omega_m)$.

Note that any upper semicontinuous function is Borel measurable: the result for the maps $\omega\mapsto\mathbf{x}^T N(\omega)\,\mathbf{x}$ for any $\mathbf{x}\in\mathbb{R}^n$ fixed follows from the fact that any scalar semicontinuous function is the limit everywhere of a sequence of continuous functions and the information recalled in Remark 1.1; and hence the polarization formulas ensure that any component of $N(\omega)$ is a Borel measurable function.

Proposition 1.48

(i) *Any continuous function is upper semicontinuous.*
(ii) *Let $N\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ be upper semicontinuous. Then there exists a residual set $\Omega_N\subseteq\Omega$ of continuity points of N.*
(iii) *Let $(N_m\colon\Omega\to\mathbb{S}_n(\mathbb{R}))$ be a decreasing and uniformly bounded sequence of upper semicontinuous functions, and suppose that there exists $N(\omega)=\lim_{m\to\infty}N_m(\omega)$ for every $\omega\in\Omega$. Then $N\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ is upper semicontinuous.*

Proof

(i) This assertion is obvious.
(ii) Define $\mathcal{C}_N=\operatorname{closure}_{\mathbb{S}_n(\mathbb{R})}\{N(\omega)\mid\omega\in\Omega\}$ and, for all $\omega\in\Omega$, $n(\omega)=\{N\in\mathcal{C}_N\mid N\le N(\omega)\}$. Definition 1.47 ensures that $\mathcal{C}_N$ is compact and that $n(\omega)\in\mathcal{P}_c(\mathcal{C}_N)$, where $\mathcal{P}_c(\mathcal{C}_N)$ is the set of nonempty compact subsets of $\mathcal{C}_N$ endowed with the Hausdorff metric. This means that the map $n\colon\Omega\to\mathcal{P}_c(\mathcal{C}_N)$, $\omega\mapsto n(\omega)$ is well defined. The main step of the proof, whose argument is taken from Proposition 3.4 in Novo et al. [113], consists in proving that the map n is upper semicontinuous in the sense of Definition 7.7 of [27]. That is, for each open set $\mathcal{S}\subseteq\mathcal{C}_N$, the set $\mathcal{O}=\{\omega\in\Omega\mid n(\omega)\subseteq\mathcal{S}\}$ is open. Given a sequence (ω_m) in $\Omega-\mathcal{O}$ with limit ω_0, choose $N_m\in\mathcal{C}_N-\mathcal{S}$ with $N_m\le N(\omega_m)$ for all $m\in\mathbb{N}$, and take a suitable subsequence (ω_j) such that there exists $N_0=\lim_{j\to\infty}N_j$. Then $N_0\in\mathcal{C}_N-\mathcal{S}$ and, by hypothesis, $N_0\le N(\omega_0)$. Consequently, $n(\omega_0)\not\subseteq\mathcal{S}$, and hence $\omega_0\in\Omega-\mathcal{O}$, which is therefore a closed set.

According to Theorem 7.10 of [27] (whose proof also works in the upper semicontinuous case), the points of continuity of the map n form a residual subset $\Omega_N \subseteq \Omega$. It is easy to deduce from the definition of the Hausdorff metric that the map N is also continuous at these points.

(iii) Definition 1.47 is equivalent to: N is bounded and for all $N_0 \in \mathbb{S}_n(\mathbb{R})$ the set $\{\omega \in \Omega \mid N(\omega) \geq N_0\}$ is closed. Using this characterization, the proof of (iii) follows from the relation

$$\{\omega \in \Omega \mid N(\omega) \geq N_0\} = \bigcap_{m\in\mathbb{N}} \{\omega \in \Omega \mid N_m(\omega) \geq N_0\}\,.$$

The boundedness of N is obvious.

Definition 1.49 Let the Borel measurable map $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ have the property that the solution $M(t,\omega,N(\omega))$ of (1.22) is defined for every $t \geq 0$ (resp. $t \leq 0$) and $\omega \in \Omega$. Then N is

- a τ_s-*superequilibrium for* $t \geq 0$ (resp. *for* $t \leq 0$) if, for all $\omega \in \Omega$, $N(\omega{\cdot}t) \geq M(t,\omega,N(\omega))$ for $t \geq 0$ (resp. $N(\omega{\cdot}t) \leq M(t,\omega,N(\omega))$ for $t \leq 0$),
- a τ_s-*subequilibrium for* $t \geq 0$ (resp. *for* $t \leq 0$) if, for all $\omega \in \Omega$, $N(\omega{\cdot}t) \leq M(t,\omega,N(\omega))$ for $t \geq 0$ (resp. $N(\omega{\cdot}t) \geq M(t,\omega,N(\omega))$ for $t \leq 0$),
- a τ_s-*semiequilibrium for* $t \geq 0$ (resp. *for* $t \leq 0$) in each of the two previous cases,
- a τ_s-*equilibrium* if $M(t,\omega,N(\omega))$ exists for every $t \in \mathbb{R}$ and $\omega \in \Omega$ and it satisfies $N(\omega{\cdot}t) = M(t,\omega,N(\omega))$.

The prefix τ_s will be omitted when speaking of semiequilibria or equilibria, since in general no confusion arises.

Definition 1.50

- A superequilibrium N for $t \geq 0$ (resp. for $t \leq 0$) is *strong* if there exists a time $s_* > 0$ such that $N(\omega{\cdot}s_*) > M(s_*,\omega,N(\omega))$ (resp. $N(\omega{\cdot}(-s_*)) < M(-s_*,\omega,N(\omega))$) for every $\omega \in \Omega$.
- A subequilibrium N for $t \geq 0$ (resp. for $t \leq 0$) is *strong* if there exists a time $s_* > 0$ such that $N(\omega{\cdot}s_*) < M(s_*,\omega,N(\omega))$ (resp. $N(\omega{\cdot}(-s_*)) > M(-s_*,\omega,N(\omega))$) for every $\omega \in \Omega$.

The strong character of a superequilibrium N for $t \geq 0$ and Theorem 1.45 ensure that, for every $t \geq 0$ and $\omega \in \Omega$, one has $N(\omega{\cdot}(s_* + t)) = N((\omega{\cdot}t){\cdot}s_*) > M(s_*,\omega{\cdot}t,N(\omega{\cdot}t)) \geq M(s_*,\omega{\cdot}t,M(t,\omega,N(\omega))) = M(s_* + t,\omega,N(\omega))$, so that the strong inequality remains valid beyond s_*. Analogous properties hold in the three remaining cases.

Proposition 1.51 *Let $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ be a Borel measurable map such that the solution $M(t,\omega,N(\omega))$ of* (1.22) *is defined for every $t \in \mathbb{R}$ and $\omega \in \Omega$. Then,*

(i) *N is a (strong) superequilibrium for $t \geq 0$ if and only if it is a (strong) superequilibrium for $t \leq 0$.*
(ii) *N is a (strong) subequilibrium for $t \geq 0$ if and only if it is a (strong) subequilibrium for $t \leq 0$.*

Proof Assume that $t \in \mathbb{R}$ satisfies that

$$N(\omega{\cdot}t) \geq M(t, \omega, N(\omega)) \tag{1.26}$$

for all $\omega \in \Omega$. Then Theorem 1.45 ensures that

$$\begin{aligned} N(\omega{\cdot}(-t)) &= M(-t, \omega, M(t, \omega{\cdot}(-t), N(\omega{\cdot}(-t)))) \\ &\leq M(-t, \omega, N((\omega{\cdot}(-t)){\cdot}t)) = M(-t, \omega, N(\omega)) \end{aligned} \tag{1.27}$$

for all $\omega \in \Omega$. In addition, substituting $\geq$ by $>$, $\leq$ or $<$ in (1.26) changes $\leq$ by $<$, $\geq$ or $>$ in (1.27). From this the four assertions follow immediately.

As a consequence of the previous result, and under its hypotheses, it is possible to simply speak of semiequilibria and strong semiequilibria in $\mathbb{R}$, although in order to characterize them it is enough to consider just positive or negative values of time.

Recall that a metric on Ω is fixed from the beginning. For $\omega_0 \in \Omega$ and $\delta > 0$, let $\mathcal{B}_\Omega(\omega_0, \delta)$ represent the open neighborhood of ω_0 given by those points of Ω whose distance to ω_0 is less than δ. Recall also Definition 1.34 of differentiability along the flow. Recall that the function h determines the Riccati equation (1.22).

Proposition 1.52 *Let the map $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ be Borel measurable and differentiable along the flow on Ω. Suppose that it satisfies $N'(\omega) \geq h(\omega, N(\omega))$ for every $\omega \in \Omega$ and that $M(t, \omega, N(\omega))$ exists for every $t \geq 0$ (resp. $t \leq 0$). Then,*

(i) *$N(\omega)$ is a superequilibrium for $t \geq 0$ (resp. for $t \leq 0$).*
(ii) *If $N'(\omega) > h(\omega, N(\omega))$, then $N(\omega)$ is a strong superequilibrium for $t \geq 0$ (resp. for $t \leq 0$), with $N(\omega{\cdot}t) > M(t, \omega, N(\omega)$ for all $t > 0$.*
(iii) *If for every $\omega_0 \in \Omega$ there exist constants $\delta_{\omega_0} > 0$ and $s_{\omega_0} > 0$ (resp. $s_{\omega_0} < 0$) such that $N'(\omega{\cdot}s_{\omega_0}) > h(\omega{\cdot}s_{\omega_0}, N(\omega{\cdot}s_{\omega_0}))$ for all $\omega \in \mathcal{B}_\Omega(\omega_0, \delta_{\omega_0})$, then N is a strong superequilibrium for $t \geq 0$ (resp. for $t \leq 0$).*

The analogous statements hold in the case that $N'(\omega) \leq h(\omega, N(\omega))$ for every $\omega \in \Omega$.

Proof Assertions (i) and (ii) follow from Theorem 1.46(iii) and (iv). Under the hypotheses of point (iii) in the case $t \geq 0$, and according to Theorems 1.46(iv) and 1.45, one has

$$\begin{aligned} N(\omega{\cdot}t) &> M(t - s_{\omega_0}, \omega{\cdot}s_{\omega_0}, N(\omega{\cdot}s_{\omega_0})) \\ &\geq M(t - s_{\omega_0}, \omega{\cdot}s_{\omega_0}, M(s_{\omega_0}, \omega, N(\omega))) = M(t, \omega, N(\omega)) \end{aligned}$$

for every $\omega \in \mathcal{B}_\Omega(\omega_0, \delta_{\omega_0})$ and $t \geq s_{\omega_0}$. A standard compactness argument guarantees the strong character of the superequilibrium N for $t \geq 0$. An analogous argument completes the proof of (iii) for $t \geq 0$. The case $t \leq 0$ is proved in a similar way.

Note that a continuous equilibrium for the flow is exactly the same as a copy of the base for the flow τ_s: see Definition 1.17. The concept of equilibrium can hence be understood as a generalization of these interesting dynamical objects.

The following result, whose statement is based on Proposition 1.48(ii), gives more information about the dynamical consequences of the existence of a semicontinuous equilibrium.

Proposition 1.53 *Let* $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ *be a semicontinuous equilibrium, and let* $\Omega_N \subseteq \Omega$ *be the residual set of its continuity points. Then* Ω_N *is* σ*-invariant. Suppose that* Ω *is minimal and define* $l_N(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N(\omega) \end{smallmatrix}\right]$. *Then the set* $\mathcal{K} = \text{closure}_{\mathcal{K}_\mathbb{R}}\{(\omega, l_N(\omega)) \mid \omega \in \Omega_N\}$ *is an almost automorphic extension of the base* Ω *for the flow* τ *defined by* (1.14) *on* $\mathcal{K}_\mathbb{R}$.

Proof According to Proposition 1.29, Ω_N is also the set of continuity points of the map $l_N\colon \Omega \to \mathcal{L}_\mathbb{R}$. In addition N is an equilibrium, and hence $U(t,\omega)\left[\begin{smallmatrix} I_n \\ N(\omega) \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} I_n \\ N(\omega{\cdot}t) \end{smallmatrix}\right]$, as was explained at the beginning of the section. This means that $l(\omega{\cdot}t) = U(t,\omega){\cdot}l(\omega)$, which easily yields the σ-invariance of Ω_N and the τ-invariance of $\mathcal{K}$.

The continuity of N at the points of Ω_N ensures the equality of sets $\{l \in \mathcal{L}_\mathbb{R} \mid (\omega, l) \in \mathcal{K}\} = \{l_N(\omega)\}$ for every $\omega \in \Omega_N$. In addition, $\{(\omega, l_N(\omega)) \mid \omega \in \Omega_N\} \subseteq \mathcal{M}$ for every minimal subset $\mathcal{M} \subseteq \mathcal{K}$, and hence $\mathcal{K} \subseteq \mathcal{M}$. Therefore, the equality holds, which means that $\mathcal{K}$ is minimal and hence is an almost automorphic extension of the base, as asserted.

1.4 Exponential Dichotomy

The concept of exponential dichotomy (or hyperbolic splitting) is a fundamental tool in several fields, such as the study of the invertibility of selfadjoint operators in different spaces (Massera and Schaefer [100]), bifurcation theory (Chenciner and Iooss [26]), the study of invariant manifolds (Hirsch, Pugh and Shub [63]), the analysis of homoclinic orbits (Palmer [119]), the spectral theory of the Schrödinger operator (Johnson [71]), and control theory (Johnson and Nerurkar [75, 77]), among others.

The special characteristics of a nonautonomous dynamical system in the presence of exponential dichotomy play a fundamental role in all the chapters of this book. The aim of this long Sect. 1.4 is to summarize the different definitions and the many facts concerning the dichotomy property which will be used. The last parts of the section are devoted to the closely related notion of Sacker–Sell spectral decomposition, which is critical in several results of the following chapters. The robustness of the presence of exponential dichotomy, which implies the robustness of the spectral decomposition, will also be used often.

Throughout the whole of Sect. 1.4, including its subsections, $\|\cdot\|$ represents the Euclidean norm in $\mathbb{K}^d$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, as well as the corresponding operator norm in the set $\mathbb{M}_{d\times d}(\mathbb{K})$: see Remarks 1.24.1 and 1.24.2. The definitions and results of the section do not depend on this particular choice of the norm. Finally, whenever it appears, the symbol (Ω, σ) represents a real continuous flow on a compact metric space.

1.4.1 The General Linear Case: Definition in Terms of Projectors

Definition 1.54 Given a continuous function $A_0 : \mathbb{R} \to \mathbb{M}_{d\times d}(\mathbb{K})$, the linear system

$$\mathbf{z}' = A_0(t)\,\mathbf{z} \tag{1.28}$$

has an *exponential dichotomy on* $\mathbb{R}$ if there exist constants $\eta \geq 1$ and $\beta > 0$ together with a projection Q on $\mathbb{K}^d$ such that, for every $s, t \in \mathbb{R}$,

- $\|U_{A_0}(t)\, Q\, U_{A_0}^{-1}(s)\| \leq \eta\, e^{-\beta(t-s)}$ if $t \geq s$,
- $\|U_{A_0}(t)\, (I_d - Q)\, U_{A_0}^{-1}(s)\| \leq \eta\, e^{\beta(t-s)}$ if $t \leq s$,

where $U_{A_0}(t)$ is the fundamental matrix solution of (1.28) with $U_{A_0}(0) = I_d$.

Remark 1.55 As a matter of fact, the continuity of A_0 is not a necessary condition: the same definition applies to any system $\mathbf{z}' = A_0(t)\,\mathbf{z}$ if the matrix-valued function U_{A_0} is well defined and continuous on $\mathbb{R}$.

Let $\operatorname{Rg} Q$ and $\operatorname{Ker} Q$ represent the range and the kernel of the matrix $Q = Q^2$, so that $\mathbb{R}^d = \operatorname{Rg} Q \oplus \operatorname{Ker} Q$. The following properties, whose proofs are included for the reader's convenience, are basic in the theory of exponential dichotomies.

Proposition 1.56 *Suppose that* (1.28) *has an exponential dichotomy on* $\mathbb{R}$ *with projection* Q. *Then,*

(i) *the system* (1.28) *has no nonzero bounded solutions. More precisely,*

- *for all* $\mathbf{z}_1 \in \operatorname{Rg} Q$, $\|U_{A_0}(t)\,\mathbf{z}_1\| \geq (1/\eta)\, e^{-\beta t}\, \|\mathbf{z}_1\|$ *for* $t \leq 0$*;*
- *for all* $\mathbf{z}_2 \in \operatorname{Ker} Q$, $\|U_{A_0}(t)\,\mathbf{z}_2\| \geq (1/\eta)\, e^{\beta t}\, \|\mathbf{z}_2\|$ *for* $t \geq 0$*;*
- *for all* $\mathbf{z} \notin \operatorname{Rg} Q$, *there exists* $\lim_{t\to\infty} \|U_{A_0}(t)\,\mathbf{z}\| = \infty$*;*
- *for all* $\mathbf{z} \notin \operatorname{Ker} Q$, *there exists* $\lim_{t\to-\infty} \|U_{A_0}(t)\,\mathbf{z}\| = \infty$.

(ii)
$$\begin{aligned}\operatorname{Rg} Q &= \big\{\mathbf{z} \in \mathbb{K}^d \mid \lim_{t\to\infty} \|U_{A_0}(t)\,\mathbf{z}\| = 0\big\}\\ &= \big\{\mathbf{z} \in \mathbb{K}^d \mid \sup_{t\geq 0} \|U_{A_0}(t)\,\mathbf{z}\| < \infty\big\}.\end{aligned}$$

(iii)
$$\begin{aligned}\operatorname{Ker} Q &= \big\{\mathbf{z} \in \mathbb{K}^d \mid \lim_{t\to-\infty} \|U_{A_0}(t)\,\mathbf{z}\| = 0\big\}\\ &= \big\{\mathbf{z} \in \mathbb{K}^d \mid \sup_{t\leq 0} \|U_{A_0}(t)\,\mathbf{z}\| < \infty\big\}.\end{aligned}$$

In particular, "the exponential dichotomy is unique", that is, the projection Q is uniquely determined by the behavior of the system.

Proof (i) Assume that $\mathbf{z}_1 = Q\,\mathbf{z}_1$ (i.e. $\mathbf{z}_1 \in \operatorname{Rg} Q$) and $\mathbf{z}_2 = (I_d - Q)\,\mathbf{z}_2$ (i.e. $\mathbf{z}_2 \in \operatorname{Ker} Q$), and note that Definition 1.54 yields

$$\begin{aligned}\|U_{A_0}(t)\,\mathbf{z}_1\| &= \|U_{A_0}(t)\,Q\,\mathbf{z}_1\| \le \|U_{A_0}(t)\,Q\|\,\|\mathbf{z}_1\| \le \eta\,e^{-\beta t}\,\|\mathbf{z}_1\|\,,\\ \|\mathbf{z}_2\| &= \|(I_d - Q)\,\mathbf{z}_2\| \le \|(I_d - Q)\,U_{A_0}^{-1}(t)\|\,\|U_{A_0}(t)\,\mathbf{z}_2\|\\ &\le \eta\,e^{-\beta t}\,\|U_{A_0}(t)\,\mathbf{z}_2\|\end{aligned}$$

for $t \ge 0$ and

$$\begin{aligned}\|\mathbf{z}_1\| &= \|Q\,\mathbf{z}_1\| \le \|Q\,U_{A_0}^{-1}(t)\|\,\|U_{A_0}(t)\,\mathbf{z}_1\| \le \eta\,e^{\beta t}\,\|U_{A_0}(t)\,\mathbf{z}_1\|\,,\\ \|U_{A_0}(t)\,\mathbf{z}_2\| &= \|U_{A_0}(t)\,(I_d - Q)\,\mathbf{z}_2\| \le \|U_{A_0}(t)\,(I_d - Q)\|\,\|\mathbf{z}_2\| \le \eta\,e^{\beta t}\,\|\mathbf{z}_2\|\end{aligned}$$

for $t \le 0$. The two first assertions in (i) follow from these facts. Now, write any $\mathbf{z} \in \mathbb{K}^d$ as $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$ with $\mathbf{z}_1 = Q\,\mathbf{z} \in \operatorname{Rg} Q$ and $\mathbf{z}_2 = \mathbf{z} - \mathbf{z}_1 \in \operatorname{Ker} Q$. If $\mathbf{z}_2 \ne 0$, then $\|U_{A_0}(t)\,\mathbf{z}\| \ge \|U_{A_0}(t)\,\mathbf{z}_2\| - \|U_{A_0}(t)\,\mathbf{z}_1\| \ge (1/\eta)\,e^{\beta t}\,\|\mathbf{z}_2\| - \eta\,e^{-\beta t}\,\|\mathbf{z}_1\|$ for $t \ge 0$, which tends to ∞ as $t \to \infty$. Analogously, if $\mathbf{z}_1 \ne 0$, then $\|U_{A_0}(t)\,\mathbf{z}\| \ge \|U_{A_0}(t)\,\mathbf{z}_1\| - \|U_{A_0}(t)\,\mathbf{z}_2\| \ge (1/\eta)\,e^{-\beta t}\,\|\mathbf{z}_1\| - \eta\,e^{\beta t}\,\|\mathbf{z}_2\|$ for $t \le 0$, which tends to ∞ as $t \to -\infty$. The proof of (i) is complete.

(ii) & (iii) The contentions $\subseteq$ follow trivially from Definition 1.54. And the already verified properties stated in (i) show that

$$\{\mathbf{z} \in \mathbb{K}^d \mid \sup_{t\ge 0} \|U_{A_0}(t)\,\mathbf{z}\| < \infty\} \subseteq \operatorname{Rg} Q$$

and

$$\{\mathbf{z} \in \mathbb{K}^d \mid \sup_{t\le 0} \|U_{A_0}(t)\,\mathbf{z}\| < \infty\} \subseteq \operatorname{Ker} Q\,,$$

which completes the proof of (ii) and (iii).

The uniqueness of the exponential dichotomy on the whole line is clear: the range and kernel of Q are uniquely determined, and hence also the projection is unique.

Recall that (Ω, σ) is a continuous flow on a compact metric space. Let $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ represent either a continuous function or, more generally, a function satisfying the conditions described in Proposition 1.38. Consider the family of systems

$$\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,. \tag{1.29}$$

Let $U_A(t,\omega)$ be the fundamental matrix solution of the system corresponding to ω with $U_A(0,\omega) = I_d$. Recall that the flow τ_A given by (1.5) is continuous in both of the above situations: see Proposition 1.38.

Definition 1.57 A *projector* $\mathcal{Q} = \{Q(\omega)\}$ on $\Omega \times \mathbb{K}^d$ is a jointly continuous map

$$\mathcal{Q}\colon \Omega \times \mathbb{K}^d \to \Omega \times \mathbb{K}^d\,, \quad (\omega, \mathbf{z}) \mapsto (\omega, Q(\omega)\,\mathbf{z})$$

such that each $Q(\omega) \in \mathbb{M}_{d\times d}(\mathbb{K})$ defines a projection on $\mathbb{K}^d$.

Definition 1.58 The family (1.29) *has an exponential dichotomy over* Ω if there exist constants $\eta \geq 1$ and $\beta > 0$ together with a projector $\mathcal{Q} = \{Q(\omega)\}$ on $\Omega \times \mathbb{K}^d$ such that, for every $t, s \in \mathbb{R}$ and $\omega \in \Omega$,

- $U_A(t,\omega)\,Q(\omega) = Q(\omega{\cdot}t)\,U_A(t,\omega)$,
- $\|U_A(t,\omega)\,Q(\omega)\,U_A^{-1}(s,\omega)\| \leq \eta\, e^{-\beta(t-s)}$ if $t \geq s$,
- $\|U_A(t,\omega)\,(I_d - Q(\omega))\,U_A^{-1}(s,\omega)\| \leq \eta\, e^{\beta(t-s)}$ if $t \leq s$.

Remarks 1.59

1. In the situation described in Definition 1.58, it is usual to say that *the skew-product flow* τ_A *defined by* (1.29) *has an exponential dichotomy*. In fact, this concept can be defined for any continuous linear skew-product flow on a finite-dimensional vector bundle over a Hausdorff base, as in [133].
2. Definition 1.58 is clearly an extension to the nonautonomous setting of the concept of exponential dichotomy on $\mathbb{R}$ for a single linear system, given in Definition 1.54 and Remark 1.55. In particular, each of the systems of the family has an exponential dichotomy on the whole real line. (As a matter of fact, more can be said: see Theorem 1.60 below.) In particular, according to Proposition 1.56, the exponential dichotomy is unique: $Q(\omega)$ is uniquely determined for all $\omega \in \Omega$. The uniqueness of the projector justifies speaking of exponential dichotomy (on $\mathbb{R}$ or over Ω) instead of about *an* exponential dichotomy.
3. Proposition 1.56 and the previous remark ensure that the presence of exponential dichotomy over Ω of the family (1.29) implies the absence of nontrivial globally bounded solutions. This information will be completed in Theorems 1.61 and 1.78.
4. It is proved in [131] (Theorem 2 and Section 3) that if Ω is the hull of one of its elements (see Sect. 1.3.2), say ω_0, and the system $\mathbf{z}' = A(\omega_0{\cdot}t)\,\mathbf{z}$ has exponential dichotomy on $\mathbb{R}$, then the family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω. Recall also that if Ω is minimal then it is the hull of each of its elements.

Theorem 1.60 *The family* (1.29) *has exponential dichotomy over* Ω *if and only if all its systems have exponential dichotomy over* $\mathbb{R}$.

Proof The "only if" assertion is trivial: see Remark 1.59.2. The proof of the "if" assertion is based on the results of Sacker and Sell [131, 132]. So assume that each system of the family (1.29) has exponential dichotomy on $\mathbb{R}$, with projector Q_ω.

Proposition 1.56 proves that none of the systems of the family admits a nontrivial bounded solution. Define $\Omega_k = \{\omega \in \Omega \mid \dim \operatorname{Rg} Q_\omega = k\}$ for $k = 0, \ldots, d$. It is clear that $\Omega = \Omega_0 \cup \cdots \cup \Omega_d$, and that the sets $\Omega_0, \ldots \Omega_d$ are pairwise disjoint. Theorem 4 of [131] states that each Ω_k is a compact σ-invariant subset of Ω. Now consider the restriction of the flow τ_{A_0} (defined by (1.5)) to $\Omega_k \times \mathbb{K}^d$, and apply Theorem 2 of [132] in order to conclude that the family (1.29) has exponential dichotomy over Ω_k. Since there is a finite number of sets Ω_k, it is easy to deduce the exponential dichotomy of the family (1.29) over the whole base Ω.

As is explained in Remark 1.59.3, under the presence of exponential dichotomy over Ω, none of the systems (1.29) admits nontrivial globally bounded solutions. The converse result is not true in the general case, but it holds in some situations. The following result is due to Selgrade (see [139], Theorem 10.2) and characterizes the occurrence of exponential dichotomy in the chain recurrent case (see Sect. 1.1.1 for the concept of chain recurrence):

Theorem 1.61 *Suppose that (Ω, σ) is chain recurrent. Suppose also that the function $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. Then the family (1.29) has exponential dichotomy over Ω if and only if none of its systems has a nonzero globally bounded solution.*

Remarks 1.62

1. As explained at the end of Sect. 1.1.1, chain recurrence holds if the base flow is minimal. Consequently, the previous result ensures the occurrence of exponential dichotomy for the flow restricted to each of the minimal subsets of Ω if none of the systems (1.29) admits a nontrivial bounded solution.
2. It is well known that a constant linear system $\mathbf{z}' = A\,\mathbf{z}$ has exponential dichotomy on $\mathbb{R}$ if and only if A has no purely imaginary eigenvalues. And a periodic linear system $\mathbf{z}' = A(t)\,\mathbf{z}$ has exponential dichotomy (over $\mathbb{R}$ or over the hull Ω of A) if and only if $A(t)$ has no purely imaginary characteristic exponents; or, in other words, if and only if it has no zero Lyapunov exponents. In fact, both assertions are easy consequences of Theorem 1.61, since in both cases the hull of the initial system is minimal: see Sect. 1.3.2.

1.4.2 The General Linear Case: Definition in Terms of Subbundles

It is possible to give an alternative definition of the notion of exponential dichotomy, based on the idea of hyperbolic splitting, which is often very useful. The concepts of closed vector subbundle and Whitney sum are required for this. Note that, by continuity of the flow, any connected component of Ω is σ-invariant.

Definition 1.63 A *closed vector subbundle* of $\Omega\times\mathbb{K}^d$ is a closed subset F of $\Omega\times\mathbb{K}^d$ such that, for each $\omega \in \Omega$, the ω-fiber $F_\omega = \{\mathbf{z} \in \mathbb{K}^d\,, (\omega, \mathbf{z}) \in F\}$ is a linear subspace, and the function $\omega \mapsto \dim F_\omega$ is constant on each connected component of Ω. In the case that the function $\omega \mapsto \dim F_\omega$ is constant on Ω, its value $\dim F$ is the *dimension* of F.

The words "closed vector subbundle" will be substituted very often by "closed subbundle".

Remark 1.64 It can immediately be checked that a closed subbundle F is an invariant set for the skew-product flow τ_A defined by (1.5) on $\Omega \times \mathbb{K}^d$ if and only if $U_A(t,\omega)\cdot F_\omega = F_{\omega\cdot t}$ for all $(t,\omega) \in \mathbb{R} \times \Omega$.

Definition 1.65 The trivial vector bundle $\Omega \times \mathbb{K}^d$ is the *Whitney sum* of the closed vector subbundles $F_1, \ldots, F_m$ if $\mathbb{K}^d = (F_1)_\omega \oplus \cdots \oplus (F_m)_\omega$ for all $\omega \in \Omega$. This fact is represented by writing $\Omega \times \mathbb{K}^d = F_1 \oplus \cdots \oplus F_m$.

Definition 1.66 The family (1.29) has *exponential dichotomy over* Ω if there exist constants $\eta \geq 1$ and $\beta > 0$ and a splitting $\Omega \times \mathbb{K}^d = F^+ \oplus F^-$ of the bundle into the Whitney sum of two closed subbundles such that

- F^+ and F^- are invariant under the flow τ_A on $\Omega \times \mathbb{K}^d$,
- $\|U_A(t,\omega)\,\mathbf{z}\| \leq \eta\, e^{-\beta t}\|\mathbf{z}\|$ for every $t \geq 0$ and $(\omega,\mathbf{z}) \in F^+$,
- $\|U_A(t,\omega)\,\mathbf{z}\| \leq \eta\, e^{\beta t}\|\mathbf{z}\|$ for every $t \leq 0$ and $(\omega,\mathbf{z}) \in F^-$.

The proof of the equivalence of Definitions 1.58 and 1.66, which is carried out in Proposition 1.68, is based on the connection between the existence of a projector and of a decomposition of the bundle as a Whitney sum of two closed subbundles. A complete proof of the following result (see [133]), is included for the reader's convenience.

Proposition 1.67 *Given a projector $\mathcal{Q} \equiv \{Q(\omega)\}$ on $\Omega \times \mathbb{K}^d$, the sets*

$$\begin{aligned} \operatorname{Rg} \mathcal{Q} &= \{(\omega,\mathbf{z}) \mid Q(\omega)\,\mathbf{z} = \mathbf{z}\}\,, \\ \operatorname{Ker} \mathcal{Q} &= \{(\omega,\mathbf{z}) \mid Q(\omega)\,\mathbf{z} = \mathbf{0}\} \end{aligned}$$

are closed subbundles, with fibers $\operatorname{Rg} Q(\omega)$ *and* $\operatorname{Ker} Q(\omega)$*; and they satisfy* $\Omega \times \mathbb{K}^d = \operatorname{Rg}\mathcal{Q} \oplus \operatorname{Ker}\mathcal{Q}$ *as a Whitney sum.*

Conversely, given two closed subbundles F_1 and F_2 such that $\Omega \times \mathbb{K}^d = F_1 \oplus F_2$ as a Whitney sum, there exists a unique projector $\mathcal{Q}$ with $\operatorname{Rg}\mathcal{Q} = F_1$ *and* $\operatorname{Ker}\mathcal{Q} = F_2$.

Proof Assume first that $\mathcal{Q}$ is a projector. It is easy to deduce from the continuity required by Definition 1.57 that the sets $\operatorname{Rg}\mathcal{Q}$ and $\operatorname{Ker}\mathcal{Q}$ are closed. Obviously the fibers over each element of the base are the vector spaces $\operatorname{Rg} Q(\omega)$ and $\operatorname{Ker} Q(\omega)$. Now take a sequence (ω_j) with limit ω, and call $k = \dim \operatorname{Rg} Q(\omega)$. It follows from the continuity in Definition 1.57 that $\dim \operatorname{Rg} Q(\omega_j) \geq k$ and $\dim \operatorname{Ker} Q(\omega_j) = \dim \operatorname{Rg}(I_d - Q(\omega_j)) \geq d-k$ for large enough j, so that equality holds in both cases. Consequently, the maps $\dim \operatorname{Rg} Q\colon \Omega \to \{0,\ldots,d\}$ and $\dim \operatorname{Rg} Q\colon \Omega \to \{0,\ldots,d\}$

are continuous, and hence they are constant on each connected component of Ω. This completes the proof of the first assertion.

To prove the converse assertion, assume that $\Omega\times\mathbb{K}^d = F_1\oplus F_2$ as a Whitney sum and, for each $\omega\in\Omega$, define $Q(\omega)$ to be the projection on $\mathbb{K}^d$ with $\operatorname{Rg} Q(\omega) = (F_1)_\omega$ and $\operatorname{Ker} Q(\omega) = (F_2)_\omega$. This means that $Q(\omega)\,\mathbf{z} = \mathbf{z}^1$ if and only if $\mathbf{z} = \mathbf{z}^1 + \mathbf{z}^2$ with $\mathbf{z}^i \in F_i$ for $i = 1, 2$. The goal is to prove that $\Omega\times\mathbb{K}^d \to \mathbb{K}^d\,,\ (\omega,\mathbf{z}) \mapsto Q(\omega)\,\mathbf{z}$ is continuous. Take a sequence $((\omega_j,\mathbf{z}_j))$ with limit $(\omega,\mathbf{z})$, and write $\mathbf{z}_j = \mathbf{z}_j^1 + \mathbf{z}_j^2$ with $(\omega_j,\mathbf{z}_j^i)\in F^i$ for $i = 1, 2$; and, in the same way, write $\mathbf{z} = \mathbf{z}^1 + \mathbf{z}^2$. The property to prove is then $\lim_{j\to\infty}\mathbf{z}_j^1 = \mathbf{z}^1$. It will be proved below that the sequence $(\mathbf{z}_j^1,\mathbf{z}_j^2)$ is bounded in $\mathbb{K}^d\times\mathbb{K}^d$. Then, any subsequence admits a convergent subsequence, with limit $(\tilde{\mathbf{z}}^1,\tilde{\mathbf{z}}^2)$. Since the subbundles are closed, $(\omega,\tilde{\mathbf{z}}^1)\in F^1$ and $(\omega,\tilde{\mathbf{z}}^2)\in F^2$. And, since $\mathbf{z} = \tilde{\mathbf{z}}^1 + \tilde{\mathbf{z}}^2$, it follows that $\mathbf{z}^1 = \tilde{\mathbf{z}}^1$. The independence of the value of the limit with respect to the choice of the initial subsequence proves the assertion. Now, assume for contradiction that $(\mathbf{z}_j^1,\mathbf{z}_j^2)$ is not bounded, which since $(\mathbf{z}_j)$ is bounded is equivalent to the unboundedness of $(\mathbf{z}_j^1)$. Choose a suitable subsequence $(\mathbf{z}_m^1)$ with $\lim_{m\to\infty}\|\mathbf{z}_m^1\| = \infty$ and $\lim_{m\to\infty}\mathbf{z}_m^1/\|\mathbf{z}_m^1\| = \bar{\mathbf{z}}^1 \neq \mathbf{0}$. Then, $\lim_{m\to\infty}\mathbf{z}_m^2/\|\mathbf{z}_m^1\| = \lim_{m\to\infty}(\mathbf{z}_m - \mathbf{z}_m^1)/\|\mathbf{z}_m^1\| = -\bar{\mathbf{z}}^1$, which, using again the closed character of the subbundles, implies that $\bar{\mathbf{z}}^1 \in (F_1)_\omega \cap (F_2)_\omega = \{\mathbf{0}\}$ and provides the sought-for contradiction. This completes the proof.

Proposition 1.68 *Suppose that the function $A\colon\Omega\to\mathbb{M}_{d\times d}(\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. Then Definitions 1.58 and 1.66 are equivalent. For each $\omega\in\Omega$, let $Q(\omega)$ be the projection given by Definition 1.58, and let $F^\pm_\omega = \{\mathbf{z}\in\mathbb{K}^d \mid (\omega,\mathbf{z})\in F^\pm\}$ be the fibers of the subbundles F^+ and F^- of Definition 1.66. Then F^+_ω and F^-_ω are respectively the range and kernel of $Q(\omega)$.*

Proof The assertion follows easily from the information provided by Proposition 1.67, as will now be explained.

Suppose that the family (1.29) satisfies Definition 1.58 with projector $\mathcal{Q}$, and define $F^+ = \operatorname{Rg}\mathcal{Q}$ and $F^- = \operatorname{Ker}\mathcal{Q}$. These closed subbundles are invariant, since $(\omega{\cdot}t, U_A(t,\omega)\,Q(\omega)\,\mathbf{z}) = (\omega{\cdot}t, Q(\omega{\cdot}t)\,U_A(t,\omega)\,\mathbf{z})$. In addition, if $(\omega,\mathbf{z})\in F^+$, then $(\omega,\mathbf{z}) = (\omega, Q(\omega)\,\mathbf{z})$, so that $\|U_A(t,\omega)\,\mathbf{z}\| = \|U_A(t,\omega)\,Q(\omega)\,\mathbf{z}\| \le \eta\, e^{-\beta t}\|\mathbf{z}\|$ for all $t\ge 0$, and if $(\omega,\mathbf{z})\in F^-$, then $(\omega,\mathbf{z}) = (\omega,(I_d - Q(\omega))\,\mathbf{z})$, so that $\|U_A(t,\omega)\,\mathbf{z}\| = \|U_A(t,\omega)\,(I_d - Q(\omega))\,\mathbf{z}\| \le \eta\, e^{\beta t}\|\mathbf{z}\|$ for all $t\le 0$. That is, the family satisfies Definition 1.66 for the same η and β.

Conversely, assume that the family (1.29) satisfies Definition 1.66, and define $\mathcal{Q}$ to be the projector with $\operatorname{Rg}\mathcal{Q} = F^+$ and $\operatorname{Ker}\mathcal{Q} = F^-$. Due to the invariance of the subbundles, for all $\mathbf{z}\in\mathbb{K}^d$ one has

$$\begin{aligned} Q(\omega{\cdot}t)\,U_A(t,\omega)\,\mathbf{z} &= Q(\omega{\cdot}t)\,(U_A(t,\omega)\,Q(\omega)\,\mathbf{z} + U_A(t,\omega)\,(I_d - Q(\omega))\,\mathbf{z}) \\ &= U_A(t,\omega)\,Q(\omega)\,\mathbf{z}\,, \end{aligned}$$

since

$$U_A(t,\omega)\,Q(\omega)\,\mathbf{z} \in U_A(t,\omega){\cdot}F^+_\omega = F^+_{\omega\cdot t} = \operatorname{Rg} Q(\omega{\cdot}t)\,,$$
$$U_A(t,\omega)\,(I_d - Q(\omega))\,\mathbf{z} \in U_A(t,\omega){\cdot}F^-_\omega = F^-_{\omega\cdot t} = \operatorname{Ker} Q(\omega{\cdot}t)\,;$$

that is, $Q(\omega{\cdot}t)\,U_A(t,\omega) = U_A(t,\omega)\,Q(\omega)$. Moreover,

$$\begin{aligned} U_A(t,\omega)\,Q(\omega)\,U_A^{-1}(s,\omega) &= U_A(t,\omega)\,U_A^{-1}(s,\omega)\,Q(\omega{\cdot}s) \\ &= U_A(t-s,\omega{\cdot}s)\,Q(\omega{\cdot}s)\,, \end{aligned}$$

and hence, if $\mathbf{z} \in \mathbb{K}^d$ with $\|\mathbf{z}\| = 1$ and $t - s \geq 0$,

$$\|U_A(t,\omega)\,Q(\omega)\,U_A^{-1}(s,\omega)\,\mathbf{z}\| = \|U_A(t-s,\omega{\cdot}s)\,Q(\omega{\cdot}s)\,\mathbf{z}\| \leq \eta\,q\,e^{-\beta(t-s)}$$

where $q = \sup_{\omega\in\Omega}\|Q(\omega)\|$, since $(\omega{\cdot}s, Q(\omega{\cdot}s)\,\mathbf{z}) \in F^+$. And similarly,

$$U_A(t,\omega)\,(I_d - Q(\omega))\,U_A^{-1}(s,\omega) = U_A(t-s,\omega{\cdot}s)\,(I_d - Q(\omega{\cdot}s))\,,$$

so that if $\mathbf{z} \in \mathbb{K}^d$ with $\|\mathbf{z}\| = 1$ and $t - s \leq 0$,

$$\begin{aligned} \|U_A(t,\omega)\,(I_d - Q(\omega))\,U_A^{-1}(s,\omega)\,\mathbf{z}\| &= \|U_A(t-s,\omega{\cdot}s)\,(I_d - Q(\omega{\cdot}s))\,\mathbf{z}\| \\ &\leq \eta\,(1+q)\,e^{\beta(t-s)}\,, \end{aligned}$$

since $(\omega{\cdot}s, (I_d - Q(\omega{\cdot}s))\,\mathbf{z}) \in F^-$. Therefore, Definition 1.58 holds for a possibly larger constant η and the same β.

Remarks 1.69

1. It follows from Propositions 1.56 and 1.68 that

$$F^+ = \left\{(\omega,\mathbf{z}) \mid \lim_{t\to\infty}\|U_A(t,\omega)\,\mathbf{z}\| = 0\right\} = \left\{(\omega,\mathbf{z}) \mid \sup_{t\geq 0}\|U_A(t,\omega)\,\mathbf{z}\| < \infty\right\},$$

$$F^- = \left\{(\omega,\mathbf{z}) \mid \lim_{t\to-\infty}\|U_A(t,\omega)\,\mathbf{z}\| = 0\right\} = \left\{(\omega,\mathbf{z}) \mid \sup_{t\leq 0}\|U_A(t,\omega)\,\mathbf{z}\| < \infty\right\}.$$

   In particular, F^+_ω and F^-_ω can be referred to as the *vector spaces of the initial data giving rise to bounded solutions of* (1.29) *as $t \to \infty$ and as $t \to -\infty$*, respectively. And also as the *vector spaces of the initial data giving rise to solutions of* (1.29) *tending to* $\mathbf{0}$ *as $t \to \infty$ and as $t \to -\infty$*.
2. Any point $(\omega,\mathbf{z}) \in \Omega \times \mathbb{K}^d$ can be written as $(\omega, \mathbf{z}^- + \mathbf{z}^+)$ with $(\omega,\mathbf{z}^\pm) \in F^\pm$, and $\mathbf{z}^+$ and $\mathbf{z}^-$ are unique. Hence $U_A(t,\omega)\,\mathbf{z}$ behaves in the way determined by

$U_A(t,\omega)\,\mathbf{z}^{\mp}$ at $\pm\infty$. More precisely,

$$\lim_{t\to\pm\infty} \|U_A(t,\omega)\,\mathbf{z} - U_A(t,\omega)\,\mathbf{z}^{\mp}\| = 0\,. \tag{1.30}$$

For this reason it is also usual to refer to the closed subbundles F^+ and F^- as the *stable subbundle at* $-\infty$ and $+\infty$ respectively.

The following continuity property for closed subbundles will be repeatedly used. Recall the Definition 1.17 of the concept of copy of the base to understand the statement.

Proposition 1.70 *Let F be a closed subbundle of $\Omega\times\mathbb{K}^d$, and let $\Omega_c\subseteq\Omega$ be a connected component. Write $k=\dim F_\omega$ for all $\omega\in\Omega_c$. Then the map $\Omega_c\to\mathcal{G}_k(\mathbb{K}^d)$, $\omega\mapsto F_\omega$ is continuous.*

Consequently, if F is τ_A-invariant, the set $\{(\omega,F_\omega)\mid\omega\in\Omega\}$ is a copy of the base for the restriction of the flow τ_A^k defined by (1.6) to $\Omega_c\times\mathcal{G}_k(\mathbb{K}^d)$.

Proof Take a sequence $(\omega_j)\in\Omega_c$ with limit ω, and assume for contradiction that F_w is not the limit of F_{ω_j} in $\mathcal{G}_k(\mathbb{K}^d)$. According to Proposition 1.26(ii), there exists a sequence $((\omega_j,\mathbf{v}_j))$ with $(\omega_j,\mathbf{v}_j)\in F$ and with limit $(\omega,\mathbf{v})\notin F$. But this is impossible, since F is closed. This proves the continuity, and Remark 1.64 makes the second assertion trivial: $\tau_A^k(t,\omega,F_\omega)=(\omega{\cdot}t,U_A(t,\omega){\cdot}F_\omega)=(\omega{\cdot}t,F_{\omega\cdot t})$.

The section is completed with a discussion of another interesting fact concerning the exponential dichotomy concept, which will be useful in Chap. 5. Namely, the exponential dichotomy of a given family is equivalent to that of the adjoint family. This fact has a simple corollary which will be needed in Chaps. 6 and 7.

Given a linear subspace $g\subseteq\mathbb{K}^d$, let $g^\perp$ represent the orthogonal complement of g with respect to the Euclidean inner product, with $\dim g^\perp=d-\dim g$. And, given a closed subbundle F on $\Omega\times\mathbb{K}^d$, define

$$F^\perp=\{(\omega,\mathbf{z})\mid\mathbf{z}\in F_\omega^\perp\}\,.$$

It is easy to check that $F^\perp$ is a new closed subbundle, and it is obvious that $(F^\perp)^\perp=F$: $(g^\perp)^\perp=g$. See Remark 1.69.2 for the definition of the concept of stable subbundles at $\mp\infty$, which appears in the following discussion.

Proposition 1.71 *Suppose that the function $A\colon\Omega\to\mathbb{M}_{d\times d}(\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. The family of linear systems $\mathbf{z}'=A(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω with stable subbundles at $+\infty$ and $-\infty$ given by F^- and F^+ if and only if the adjoint family $\mathbf{w}'=-A^T(\omega{\cdot}t)\,\mathbf{w}$ has exponential dichotomy over Ω with stable subbundles at $+\infty$ and $-\infty$ given by $(F^-)^\perp$ and $(F^+)^\perp$.*

Proof Assume the exponential dichotomy of the family of systems $\mathbf{z}'=A(\omega{\cdot}t)\,\mathbf{z}$, and let $\Omega\times\mathbb{K}^d=F^+\oplus F^-$ be the corresponding Whitney sum. Let $\mathbf{w}(t)$ be a globally bounded solution of the system $\mathbf{w}'=-A^T(\omega{\cdot}t)\,\mathbf{w}$, and write $\mathbf{w}_0=\mathbf{w}(0)=$

$\mathbf{z}_0^+ + \mathbf{z}_0^-$ with $(\omega, \mathbf{z}_0^\pm) \in F^\pm$. Then, $\|\mathbf{w}_0\|^2 = \langle \mathbf{w}_0, \mathbf{z}_0^+\rangle + \langle \mathbf{w}_0, \mathbf{z}_0^-\rangle$ and $\langle \mathbf{w}_0, \mathbf{z}_0^\pm\rangle = \langle \mathbf{w}(t), \mathbf{z}^\pm(t)\rangle$ for all $t \in \mathbb{R}$, where $\mathbf{z}^\pm(t)$ represents the solution of the initial system (for the same ω as $\mathbf{w}(t)$) with initial datum $\mathbf{z}_0^\pm$. Taking the limit as $t \to \pm\infty$ yields $\langle \mathbf{w}_0, \mathbf{z}_0^\pm\rangle = 0$, so that $\mathbf{w}_0 = \mathbf{0}$ and $\mathbf{w}(t) \equiv \mathbf{0}$.

Therefore, the adjoint family has no nontrivial bounded solutions. Note that the presence of exponential dichotomy for the initial family ensures that Ω can be written as the following disjoint union: $\Omega = \Omega_0 \cup \cdots \cup \Omega_d$, with $\Omega_k = \{\omega \in \Omega \mid \dim F_\omega^+ = k\}$ for $k = 0, \ldots, d$. According to Lemma 10 of Sacker and Sell [132], each Ω_k is a compact invariant isolated subset of Ω. Fix one of these sets Ω_k. Theorem 1.61 and Remark 1.62 ensure that the adjoint family has exponential dichotomy over each minimal subset $\mathcal{M} \subseteq \Omega_k$. Let $\widetilde{F}_{\mathcal{M}}^+$ and $\widetilde{F}_{\mathcal{M}}^-$ be the corresponding closed subbundles. If $(\omega, \mathbf{w}_0) \in \widetilde{F}_{\mathcal{M}}^+$ and $(\omega, \mathbf{z}_0) \in F^+$, then $\langle \mathbf{w}_0, \mathbf{z}_0\rangle = \langle \mathbf{w}(t), \mathbf{z}(t)\rangle$, with limit 0 as $t \to \infty$. Here $\mathbf{z}(t)$ solves $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ with $\mathbf{z}(0) = \mathbf{z}_0$ and $\mathbf{w}(t)$ solves $\mathbf{w}' = -A^T(\omega{\cdot}t)\,\mathbf{w}$ with $\mathbf{w}(0) = \mathbf{w}_0$. Therefore, $(\widetilde{F}_{\mathcal{M}}^+)_\omega \subseteq (F_\omega^+)^\perp$, which ensures that $\dim(\widetilde{F}_{\mathcal{M}}^+)_\omega \le d - k$. Analogously, $\dim(\widetilde{F}_{\mathcal{M}}^-)_\omega \le k$. Consequently, both equalities hold, since $(\widetilde{F}_{\mathcal{M}}^+)_\omega \oplus (\widetilde{F}_{\mathcal{M}}^-)_\omega = \mathbb{K}^d$. One can now apply Theorem 2 of [132]: the fact that k does not depend on the choice of $\mathcal{M}$, together with the absence of nonzero bounded solutions, ensures the exponential dichotomy of the adjoint family of systems over the space Ω_k. And since the spaces Ω_k form a finite partition of Ω, it follows easily that the adjoint family has exponential dichotomy over the whole base: $\Omega \times \mathbb{K}^d = \widetilde{F}^+ \oplus \widetilde{F}^-$. Once this is known, repeating the above argument shows that $\widetilde{F}_\omega^+ = (F_\omega^+)^\perp$ and $\widetilde{F}_\omega^- = (F_\omega^-)^\perp$, which completes the proof.

Definition 1.72 The linear system $\mathbf{z}' = A_0(t)\,\mathbf{z}$ is of *Hurwitz type at* $+\infty$ if it has exponential dichotomy over $\mathbb{R}$ with projection $Q = I_d$, and *at* $-\infty$ if the projection is $Q = 0_d$.

The family of linear systems $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ for $\omega \in \Omega$ is of *uniform Hurwitz type at* $+\infty$ if it has exponential dichotomy over Ω with projector $\mathcal{Q} \equiv \{I_d\}$, and *at* $-\infty$ if the projector is $\mathcal{Q} \equiv \{0_d\}$.

Proposition 1.73 *Suppose that the function $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ is either continuous or satisfies the conditions described in Proposition 1.38. The family of linear systems $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ for $\omega \in \Omega$ is of uniform Hurwitz type at $+\infty$ if and only if the adjoint family $\mathbf{w}' = -A^T(\omega{\cdot}t)\,\mathbf{w}$ is of uniform Hurwitz type at $-\infty$.*

Proof The result is an immediate consequence of Proposition 1.71.

Proposition 1.74 *Suppose that any solution $\mathbf{z}(t)$ of any of the systems of the family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ for $\omega \in \Omega$ tends to $\mathbf{0}$ as $t \to \infty$ (resp. as $t \to -\infty$). Then the family is of uniform Hurwitz type at $+\infty$ (resp. at $-\infty$).*

Proof This result is proved in, for example, Lemma 4 of [133].

1.4.3 The Hamiltonian Case: Additional Properties

The exponential dichotomy property has some particular properties in the case of linear Hamiltonian systems. For the reader's convenience, the definition of the exponential dichotomy property is repeated and some of these properties are recalled. In this section, $H\colon\Omega \to \mathfrak{sp}(n,\mathbb{K})$ and $G\colon\Omega \to \mathbb{S}_n(\mathbb{K})$ represent either continuous functions or, more generally, functions satisfying the conditions imposed on A in Proposition 1.38. The families of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega\in\Omega \tag{1.31}$$

and of Schrödinger n-dimensional equations

$$-\,\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \mathbf{0}\,, \quad \omega\in\Omega \tag{1.32}$$

are considered, and the continuous map $U\colon\mathbb{R}\times\Omega \to \mathrm{Sp}(n,\mathbb{K})$ represents the fundamental matrix solution of the system (1.31) corresponding to ω with $U(0,\omega) = I_{2n}$. Recall that taking $H = \left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n\end{smallmatrix}\right]$ provides a system of type (1.31) equivalent to (1.32).

Definition 1.75 The family (1.31) has *exponential dichotomy over Ω* if there exist constants $\eta\ge 1$ and $\beta>0$ and a splitting $\Omega\times\mathbb{K}^{2n} = L^+\oplus L^-$ of the bundle into the Whitney sum of two closed subbundles such that

- L^+ and L^- are invariant under the flow $\tau_{\mathbb{K}}$ given by (1.13) on $\Omega\times\mathbb{K}^{2n}$,
- $\|U(t,\omega)\,\mathbf{z}\| \le \eta\, e^{-\beta t}\|\mathbf{z}\|$ for every $t\ge 0$ and $(\omega,\mathbf{z})\in L^+$,
- $\|U(t,\omega)\,\mathbf{z}\| \le \eta\, e^{\beta t}\|\mathbf{z}\|$ for every $t\le 0$ and $(\omega,\mathbf{z})\in L^-$.

The family of Schrödinger equations (1.32) has *exponential dichotomy over Ω* if this property holds for the associated family of linear Hamiltonian systems.

As proved in Proposition 1.68, this concept can also be formulated in terms of the existence of a projector $\mathcal{Q}$ with the properties required in Definition 1.58.

The analysis of the special facts concerning the Hamiltonian case starts with the following fundamental result. Recall the concept of a copy of the base, which is given in Definition 1.17.

Proposition 1.76 *Suppose that the function $H\colon\Omega\to\mathfrak{sp}(n,\mathbb{K})$ is either continuous or satisfies the conditions imposed on A in Proposition 1.38. Suppose also that the family (1.31) has exponential dichotomy over Ω, and let $\Omega\times\mathbb{K}^{2n} = L^+\oplus L^-$ be the corresponding decomposition. Then, for each $\omega\in\Omega$, the fibers*

$$l^\pm(\omega) = L^\pm_\omega = \{\mathbf{z}\in\mathbb{K}^{2n}\mid (\omega,\mathbf{z})\in L^\pm\} \tag{1.33}$$

are Lagrange planes, and they vary continuously with respect to ω. In particular, the closed subbundles $L^\pm$ are globally n-dimensional. In addition, $U(t,\omega){\cdot}l^\pm(\omega) =$

$l^{\pm}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$. That is, the sets $\{(\omega, l^{\pm}(\omega)) \mid \omega \in \Omega\} \subset \mathcal{K}_{\mathbb{K}}$ are copies of the base for the flow τ given by (1.14)*: $\tau(t, \omega, l(\omega)) = (\omega{\cdot}t, l^{\pm}(\omega{\cdot}t))$.*

Proof The symplectic character of the fundamental matrix $U(t, \omega)$ ensures that $\mathbf{w}^T J\mathbf{z} = \mathbf{w}^T U^T(t, \omega) J U(t, \omega)\,\mathbf{z}$ for all $t \in \mathbb{R}$ and for any pair of vectors $\mathbf{z}, \mathbf{w} \in \mathbb{K}^{2n}$, and hence the behavior of the solutions at $+\infty$ (resp. $-\infty$) described in Definition 1.75 yields $\mathbf{w}^T J\mathbf{z} = 0$ for any pair of vectors $\mathbf{z}$ and $\mathbf{w}$ in $l^+(\omega)$ (resp. $\mathbf{z}$ and $\mathbf{w}$ in $l^-(\omega)$). This fact and the impossibility of the existence of $n + 1$ linearly independent isotropic vectors ensure that $l^{\pm}(\omega)$ (resp. $L^{\pm}$) are n-dimensional vector spaces (resp. closed subbundles): recall that $\mathbb{K}^{2n} = l^+(\omega) \oplus l^-(\omega)$. The continuity of $l^{\pm}(\omega)$ follows from Proposition 1.70. Finally, the $\tau_{\mathbb{K}}$-invariance of $L^{\pm}$ means that $U(t, \omega){\cdot}l^{\pm}(\omega) = l^{\pm}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$, which proves the last assertions.

Remarks 1.77

1. The same argument proves that, in the case of a single Hamiltonian system $\mathbf{z}' = H_0(t)\,\mathbf{z}$ with exponential dichotomy, the vector spaces $l^+ = \operatorname{Rg} Q$ and $l^- = \operatorname{Ker} Q$ determined by the solutions bounded as $t \to \infty$ and $t \to -\infty$ (see Proposition 1.56) are Lagrange planes.
2. It follows from Propositions 1.68 and 1.56 that $(\omega, \mathbf{z}) \in L^+$ if and only if $\sup_{t\in[0,\infty)} \|U(t, \omega)\,\mathbf{z}\| < \infty$, in which case the solution $U(t, \omega)\,\mathbf{z}$ of (1.31) tends to zero exponentially fast as $t \to \infty$; and that $(\omega, \mathbf{z}) \in L^-$ if and only if $\sup_{t\in(-\infty,0]} \|U(t, \omega)\,\mathbf{z}\| < \infty$, in which case the solution $U(t, \omega)\,\mathbf{z}$ tends to zero exponentially fast as $t \to -\infty$.
3. As stated in Remark 1.69.1, $l^+(\omega)$ and $l^-(\omega)$ can be referred to as the Lagrange planes of the initial data of the solutions of (1.31) which give rise to bounded solutions as $t \to \infty$ and as $t \to -\infty$ respectively.

According to Proposition 1.76, in the linear Hamiltonian case the dimensions of the stable and unstable subbundles are the same for every minimal subset. As in the proof of Proposition 1.71, this property, Remark 1.62, and Theorem 2 of Sacker and Sell [132], yield the following fundamental result.

Theorem 1.78 *Suppose that the function $H\colon \Omega \to \mathfrak{sp}(n, \mathbb{K})$ is either continuous or satisfies the conditions imposed on A in Proposition 1.38. The family* (1.31) *has exponential dichotomy over Ω if and only if none of its systems admits a nonzero bounded solution.*

Remark 1.79 It is usual to refer to the presence of exponential dichotomy for the family (1.11) by saying that the corresponding dynamics is in the *uniformly hyperbolic case*. The case on nonuniform hyperbolicity, which is much more complex, occurs roughly speaking when exponentially increasing solutions coexist with nontrivial bounded ones for some of the systems of the family. An example of this situation is carefully described at the end of Chap. 8.

In some interesting cases in which exponential dichotomy is present, it happens that $l^+(\omega) \in \mathcal{D}_{\mathbb{K}}$ for all $\omega \in \Omega$, where $\mathcal{D}_{\mathbb{K}}$ is defined in (1.21). In other words, $l^+(\omega) \equiv$

$\left[\begin{smallmatrix} I_n \\ M^+(\omega) \end{smallmatrix}\right]$ for every $\omega \in \Omega$. In this case, it follows from Propositions 1.76 and 1.29(i) that the map $M^+ \colon \Omega \to \mathcal{D}_{\mathbb{K}}$, $\omega \mapsto M^+(\omega)$ is a continuous map. Similarly, it can be the case that $l^-(\omega) \equiv \left[\begin{smallmatrix} I_n \\ M^-(\omega) \end{smallmatrix}\right]$ for every $\omega \in \Omega$ for a continuous map $M^- \colon \Omega \to \mathcal{D}_{\mathbb{K}}$, $\omega \mapsto M^-(\omega)$.

Definition 1.80 Assume that the family (1.11) has exponential dichotomy over Ω. The $n \times n$ matrices $M_\Gamma^\pm(\omega) \in \mathbb{S}_n(\mathbb{K})$, as well as the continuous matrix-valued maps $M^\pm \colon \Omega \to \mathbb{S}_n(\mathbb{K})$, $\omega \mapsto M^\pm(\omega)$, if they exist, are called the *Weyl functions*, *Weyl matrices*, or *Weyl M-matrices* of the family (1.31).

It is usual to refer to M^+ (resp. M^-) as the Weyl function associated to the stable subbundle at $-\infty$ (resp. $+\infty$) of the family (1.31): see Remark 1.69.2. Note that the Weyl functions are continuous equilibria, according to Definition 1.49; and they define copies of the base $\{(\omega, M^\pm(\omega)) \ \omega \in \Omega\} \subset \Omega \times \mathbb{S}_n(\mathbb{R})$ for the flow τ_s given by (1.23), according to Definition 1.17.

Remarks 1.81

1. Assume that both M-functions exist. It follows easily that $\mathrm{Ker}(M^-(\omega) - M^+(\omega)) = \{\mathbf{0}\}$: otherwise there would exist a nonzero vector of the form $\left[\begin{smallmatrix} \mathbf{x} \\ M^-(\omega)\,\mathbf{x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{x} \\ M^+(\omega)\,\mathbf{x} \end{smallmatrix}\right]$ in the vector space $l^-(\omega) \cap l^-(\omega)$, which is impossible. Hence, the matrix-valued function $(M^- - M^+)^{-1}$ exists. The uniqueness of the projector $\mathcal{Q} = \{Q(\omega)\}$ and the fact that the range and kernel of $Q(\omega)$ are the Lagrange planes represented by $\left[\begin{smallmatrix} I_n \\ M^+(\omega) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I_n \\ M^-(\omega) \end{smallmatrix}\right]$ (see Propositions 1.68 and 1.76) ensure that

$$Q = \begin{bmatrix} (M^- - M^+)^{-1} M^- & -(M^- - M^+)^{-1} \\ M^+ (M^- - M^+)^{-1} M^- & -M^+ (M^- - M^+)^{-1} \end{bmatrix},$$

where all the matrices are evaluated in ω. Note that any regularity property that Q may have is inherited by $(M^- - M^+)^{-1}$, $(M^- - M^+)^{-1} M^-$ and $M^+(M^- - M^+)^{-1}$, and hence also by M^+ and M^-. On the other hand, it should be noted that if H is a C^r-function on a C^r-manifold Ω, then Q, M^+, and M^- need not be C^r.
2. Let $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right]$ solve (1.31). It can immediately be checked that

$$\begin{aligned} \langle \mathbf{x}(t_2), \mathbf{y}(t_2) \rangle - \langle \mathbf{x}(t_1), \mathbf{y}(t_1) \rangle &= \int_{t_1}^{t_2} \frac{d}{dt} \langle \mathbf{x}(t), \mathbf{y}(t) \rangle \, dt \\ &= \int_{t_1}^{t_2} \big(\langle H_2(\omega{\cdot}t)\, \mathbf{x}(t), \mathbf{x}(t) \rangle + \langle \mathbf{y}(t), H_3(\omega{\cdot}t)\, \mathbf{y}(t) \rangle \big) \, dt \,, \end{aligned} \tag{1.34}$$

so that, in the case that $H_2 \ge 0$ and $H_3 \ge 0$,

$$\begin{aligned}&\langle \mathbf{x}(t_2), \mathbf{y}(t_2)\rangle - \langle \mathbf{x}(t_1), \mathbf{y}(t_1)\rangle \\ &\qquad = \int_{t_1}^{t_2} \Big(\|H_2^{1/2}(\omega{\cdot}t)\,\mathbf{x}(t)\|^2 + \|H_3^{1/2}(\omega{\cdot}t)\,\mathbf{y}(t)\|^2\Big)\,dt\,.\end{aligned} \tag{1.35}$$

Assume now the global existence of the Weyl function M^+, and take a nonzero solution $U(t,\omega)\begin{bmatrix}\mathbf{x}_0\\ M^+(\omega)\,\mathbf{x}_0\end{bmatrix} = \begin{bmatrix}\mathbf{x}(t)\\ M^+(\omega{\cdot}t)\,\mathbf{x}(t)\end{bmatrix}$. Fixing $t_1 = 0$ and letting t_2 tend to ∞ in (1.35) shows that $-\mathbf{x}_0\,M^+(\omega)\,\mathbf{x}_0 \ge 0$, so that $M^+ \le 0$. An analogous argument proves that $M^- \ge 0$ if it is globally defined. In fact, M^+ and M^- are definite in certain situations: for instance, if $H_2 > 0$. Less restrictive conditions will be explained in Chap. 5: see Proposition 5.64.

1.4.4 Sacker–Sell Spectral Decomposition

In this section, $\mathcal{M} \subset C(\mathbb{R}, \mathbb{M}_{d\times d}(\mathbb{K}))$ represents any subset which satisfies the following two conditions. First, $\mathcal{M}$ is invariant under time translation; i.e. if $B \in \mathcal{M}$ then $B_t \in \mathcal{M}$ for all $t \in \mathbb{R}$, where $B_t(s) = B(t+s)$. And second, there exists a topology on $\mathcal{M}$ for which it is a compact metric space and for which the linear skew-product flow

$$\zeta\colon \mathbb{R}\times\mathcal{M}\times\mathbb{K}^d \to \mathcal{M}\times\mathbb{K}^d\,, \quad (t,B,\mathbf{z}) \mapsto (B_t, U(t,B)\,\mathbf{z}) \tag{1.36}$$

is continuous, where $U(t,B)$ is the fundamental matrix solution of $\mathbf{z}' = B(t)\,\mathbf{z}$ with $U(0,B) = I_d$.

In what follows, $\mathcal{M}_0 \subseteq \mathcal{M}$ is nonempty, compact, and (time-translation) invariant, so that the restriction of ζ to $\mathcal{M}_0\times\mathbb{K}^d$, denoted by the same symbol, is well defined. It is easy to adapt Definition 1.66 to express the notion of *exponential dichotomy of a family of systems over* $\mathcal{M}_0$; namely, *the family of systems* $\{\mathbf{z}' = B(t)\,\mathbf{z} \mid B \in \mathcal{M}_0\}$ *has exponential dichotomy* if there exist constants $\eta \ge 1$ and $\beta > 0$ together with a splitting $\mathcal{M}_0\times\mathbb{K}^d = F^+ \oplus F^-$ of the bundle into the Whitney sum of two closed subbundles such that

- F^+ and F^- are invariant under the flow ζ on $\mathcal{M}_0\times\mathbb{K}^d$,
- $\|U(t,B)\,\mathbf{z}\| \le \eta\, e^{-\beta t}\|\mathbf{z}\|$ for every $t \ge 0$ and $(\omega,\mathbf{z}) \in F^+$,
- $\|U(t,B)\,\mathbf{z}\| \le \eta\, e^{\beta t}\|\mathbf{z}\|$ for every $t \le 0$ and $(\omega,\mathbf{z}) \in F^-$.

The arguments of Proposition 1.68 imply that this definition is equivalent to the existence of constants $\eta \ge 1$ and $\beta > 0$ together with a projector $\mathcal{Q} = \{Q(B)\}$ on $\mathcal{M}_0\times\mathbb{K}^d$ such that, for every $t,s \in \mathbb{R}$ and $B \in \mathcal{M}_0$,

- $U(t,B)\,Q(B) = Q(B_t)\,U(t,B)$,
- $\|U(t,B)\,Q(B)\,U^{-1}(s,B)\| \le \eta\, e^{-\beta(t-s)}$ if $t \ge s$,
- $\|U(t,B)\,(I_d - Q(B))\,U^{-1}(s,B)\| \le \eta\, e^{\beta(t-s)}$ if $t \le s$.

Note also that, if $\mathcal{M}_1$ satisfies the conditions imposed on $\mathcal{M}_0$, with $\mathcal{M}_0 \subseteq \mathcal{M}_1$, and if the family $\{\mathbf{z}' = B(t)\,\mathbf{z} \mid B \in \mathcal{M}_1\}$ has exponential dichotomy, then also the family $\{\mathbf{z}' = B(t)\,\mathbf{z} \mid B \in \mathcal{M}_0\}$ has exponential dichotomy.

Definition 1.82 The *Sacker–Sell* or *dynamical spectrum* of $\mathcal{M}_0$, which will be denoted by $\Sigma(\mathcal{M}_0)$, is the set of $\lambda \in \mathbb{R}$ such that the family $\{\mathbf{z}' = (B(t) - \lambda I_d)\,\mathbf{z} \mid B \in \mathcal{M}_0\}$ does not have exponential dichotomy over $\mathcal{M}_0$.

Note that the existence of exponential dichotomy for the family of systems $\{\mathbf{z}' = B(t)\,\mathbf{z} \mid B \in \mathcal{M}_0\}$ is equivalent to the condition $0 \notin \Sigma(\mathcal{M}_0)$. The basic result given in Theorem 1.84, usually called the Sacker–Sell spectral theorem, appears in [133]. Its statement requires a preliminary definition.

Definition 1.83 The four *characteristic exponents* of the system $\mathbf{z}' = B(t)\,\mathbf{z}$ for the element $\mathbf{z}_0 \in \mathbb{K}^d$, $\mathbf{z}_0 \neq \mathbf{0}$, are the values of the limits

$$\limsup_{t\to\pm\infty} \frac{1}{t}\,\ln(\|U(t,B)\,\mathbf{z}_0\|)\,, \qquad \liminf_{t\to\pm\infty} \frac{1}{t}\,\ln(\|U(t,B)\,\mathbf{z}_0\|)\,.$$

In the case in which the four limits agree, their value is a *Lyapunov exponent* of the system.

It is clear that the values of the four limits in the previous definition are invariant along the orbits of the restriction of the flow ζ to $\mathcal{M}_0 \times \mathbb{K}^d$.

As stated in Lemma 1 of [133], if $\mathcal{M}_0$ is minimal, then the dynamical spectrum $\Sigma(\mathcal{M}_0)$ agrees with the dynamical spectrum of the system corresponding to each element $B \in \mathcal{M}_0$; i.e. with the set of points λ such that the system does not satisfy Definition 1.54. In the general case, $\lambda \in \Sigma(\mathcal{M}_0)$ if and only if there exists $B \in \mathcal{M}_0$ such that the corresponding system does not have exponential dichotomy on $\mathbb{R}$.

The following notation is important to understand the formulation of Theorem 1.84. Given $\lambda \in \mathbb{R} - \Sigma(\mathcal{M}_0)$, let $F^{\pm}_{\lambda}(\mathcal{M}_0)$ represent the stable and unstable subbundles at $\mp\infty$ for the family $\{\mathbf{z}' = (B(t) - \lambda\, I_d)\,\mathbf{z} \mid B \in \mathcal{M}_0\}$. In other words, the Whitney sum $\mathcal{M}_0 \times \mathbb{K}^d = F^{+}_{\lambda}(\mathcal{M}_0) \oplus F^{-}_{\lambda}(\mathcal{M}_0)$ satisfies the conditions of the previous definition of exponential dichotomy. The following assertions are part of [133], Lemmas 9 and 10, together with Theorems 2, 3 and 4.

Theorem 1.84 *The Sacker–Sell spectrum $\Sigma(\mathcal{M}_0)$ is compact and nonempty. If, in addition, $\mathcal{M}_0$ is connected, then $\Sigma(\mathcal{M}_0)$ is is the union of $m \le d$ non-overlapping closed intervals,*

$$\Sigma(\mathcal{M}_0) = [a_1, b_1] \cup \cdots \cup [a_m, b_m]\,, \tag{1.37}$$

with $a_1 \le b_1 < a_2 \cdots \le b_{m-1} < a_m \le b_m$. In this case, there exist closed subbundles $F^1_{\mathcal{M}_0}, \ldots, F^m_{\mathcal{M}_0}$ such that $\mathcal{M}_0 \times \mathbb{K}^d = F^1_{\mathcal{M}_0} \oplus \cdots \oplus F^m_{\mathcal{M}_0}$ as a Whitney sum, which are invariant for the flow ζ defined by (1.36) on $\mathcal{M}_0 \times \mathbb{K}^d$. A point $(\omega, \mathbf{z}_0)$ with $\mathbf{z}_0 \neq \mathbf{0}$ belongs to $F^j_{\mathcal{M}_0}$ if and only if its four characteristics exponents belong to $[a_j, b_j]$, for

$j = 1, \dots, m$. In addition, if

$$\lambda_0 < a_1 \le b_1 < \lambda_1 < a_2 \cdots \le b_{d-1} < \lambda_{m-1} < a_m \le b_m < \lambda_m,$$

then

$$F^j_{\mathcal{M}_0} = F^-_{\lambda_{j-1}}(\mathcal{M}_0) \cap F^+_{\lambda_j}(\mathcal{M}_0)$$

for $j = 1, \dots, m$. Finally, given $\mu_1,\ \mu_2 \notin \Sigma(\mathcal{M}_0)$ with $\mu_1 < \mu_2$, the following statements are equivalent:

(1) *there exists $\mu \in (\mu_1, \mu_2) \cap \Sigma(\mathcal{M}_0)$;*
(2) *$F^-_{\mu_1}(\mathcal{M}_0) \cap F^+_{\mu_2}(\mathcal{M}_0) \neq \mathcal{M}_0 \times \{\mathbf{0}\}$;*

and, in addition, $F^-_{\mu_1}(\mathcal{M}_0) \cap F^+_{\mu_2}(\mathcal{M}_0)$ is the sum of the closed subbundles $F^j_{\mathcal{M}_0}$ of $\mathcal{M}_0 \times \mathbb{K}^d$ associated to the intervals of $\Sigma(\mathcal{M}_0)$ contained in (μ_1, μ_2).

Remarks 1.85

1. Note that $(B, \mathbf{z}_0) \in F^j_{\mathcal{M}_0}$ for $j = 1, \dots, m$ if and only if, whenever $a < a_j \le b_j < b$, there exist constants c_1 and c_2 such that $c_1 e^{at} \le \|U(t,B)\,\mathbf{z}_0\| \le c_2 e^{bt}$ for $t \ge 0$ and $c_1 e^{bt} \le \|U(t,B)\,\mathbf{z}_0\| \le c_2 e^{at}$ for $t \le 0$. This property follows from the characterization of the elements of $F^j_{\mathcal{M}_0}$ in terms of their characteristic exponents given in the previous theorem.
2. Note also that the fundamental matrix solution of $\mathbf{z}' = (B(t) - \lambda I_d)\,\mathbf{z}$ which agrees with I_d at $t = 0$ is $e^{-\lambda t} U(t,B)$. In other words, the Sacker–Sell spectrum $\Sigma(\mathcal{M}_0)$ is determined by the properties of $\{U(t,B) \mid B \in \mathcal{M}_0\}$.

Definition 1.86 Suppose that $\mathcal{M}_0$ is connected. Then the sets $F^1_{\mathcal{M}_0}, \dots,\ F^m_{\mathcal{M}_0}$ are the *Sacker–Sell spectral subbundles over* $\mathcal{M}_0$.

Many more facts concerning the characteristic exponents and the Lyapunov exponents, as well as their relation with the Oseledets decomposition of the bundle $\Omega \times \mathbb{K}^d$, can be found in Sect. 2.5. In fact the formulation given there refers to the Hamiltonian case, but it is also valid in the general linear case. The analysis of the relation between the spectral decomposition, the Lyapunov exponents and the characteristic exponents is carried out by Johnson et al. in [86]. A more specific study in the linear Hamiltonian case can be found in Johnson [72] and Novo et al. [112].

Now let (Ω, σ) be as usual a real continuous flow on a compact metric space. Consider again the family of general linear systems (1.29). In this context, unless otherwise indicated, the *fixed* matrix-valued function A is assumed either to be continuous or to satisfy the conditions of Proposition 1.38, so that the flow τ_A defined by (1.5) on $\Omega \times \mathbb{K}^d$ is continuous. Define $\mathcal{M}_A = \{A_\omega : \mathbb{R} \to \mathbb{M}_{d\times d}(\mathbb{K}) \mid \omega \in \Omega\}$, where $A_\omega(t) = A(\omega{\cdot}t)$, and note that $\mathcal{M}_A$ and the flow ζ defined by (1.36) on $\mathcal{M}_A \times \mathbb{K}^d$ can be respectively identified with Ω and τ_A. It is usual to represent $\Sigma(A) = \Sigma(\mathcal{M}_A)$. In other words,

Definition 1.87 The *Sacker–Sell* (or *dynamical*) *spectrum* of (1.29), $\Sigma(A)$, is the set of $\lambda \in \mathbb{R}$ such that the family

$$\mathbf{z}' = (A(\omega{\cdot}t) - \lambda I_d)\,\mathbf{z}\,, \quad \omega \in \Omega \tag{1.38}$$

does not have exponential dichotomy over Ω. In addition, if Ω is connected, the Sacker–Sell subbundles for $\mathcal{M}_A$ provided by Theorem 1.84 are *Sacker–Sell spectral subbundles of the family* (1.29), and they are represented by $F_A^1, \dots, F_A^m$.

The Sacker–Sell spectral subbundles of (1.29) will play a fundamental role in Sect. 4.5, which is devoted to a perturbation analysis of their behavior.

The following result, regarding the relation between the spectral decomposition for a given family and its adjoint, will be used in Proposition 1.89, which describes the special shape of the Sacker–Sell spectral intervals and subbundles in the Hamiltonian case, due to the intrinsic symplectic structure of the dynamics.

Proposition 1.88 *There is a relationship between the Sacker–Sell spectra of* (1.29) *and of the adjoint family* $\mathbf{w}' = -A^T(\omega{\cdot}t)\,\mathbf{w}$, *namely* $\Sigma(-A^T) = -\Sigma(A) = [-b_m, -a_m] \cup \cdots \cup [-b_1, -a_1]$. *If, in addition, Ω is connected, then the Sacker–Sell spectral subbundle of the adjoint family corresponding to the interval* $[-b_j, -a_j]$ *is* $F_{-A^T}^{m+1-j} = \left(\bigoplus_{k\neq j} F_A^k\right)^{\perp}$ *for* $j = 1, \dots, m$, *where* F_A^k *is the Sacker–Sell spectral subbundle of the family* $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ *corresponding to the interval* $[a_k, b_k]$. *In particular,* $\dim F_{-A^T}^{m+1-j} = \dim F_A^j$ *for* $j = 1, \dots, m$.

Proof Proposition 1.71 establishes the equivalence between the exponential dichotomies over Ω of the family $\mathbf{z}' = (A(\omega{\cdot}t) - \lambda I_d)\,\mathbf{z}$ and its adjoint $\mathbf{w}' = (-A^T(\omega{\cdot}t) + \lambda I_d)\,\mathbf{w}$, from which the first assertion follows. Note that $F_{-A^T}^{m+1-j}$ is the spectral subbundle for the adjoint family corresponding to the spectral interval $[-b_j, -a_j]$. Take now $(\omega, \mathbf{w}_0) \in F_{-A^T}^{m+1-j}$ with $\mathbf{w}_0 \neq \mathbf{0}$, and take $(\omega, \mathbf{z}_0) \in F_A^k$ for $k \neq j$ with $\mathbf{z}_0 \neq \mathbf{0}$. Set $\mathbf{z}(t) = U_A(t, \omega)\,\mathbf{z}_0$ and $\mathbf{w}(t) = U_{-A^T}(t, \omega)\,\mathbf{w}_0$. Assume first that $k < j$, so that $b_k < a_j - 2\varepsilon$ for an $\varepsilon > 0$. According to Remark 1.85.1, there exists a constant c such that $|\langle \mathbf{z}_0, \mathbf{w}_0\rangle| = |\langle \mathbf{z}(t), \mathbf{w}(t)\rangle| \le c e^{(b_k+\varepsilon)t} e^{(-a_j+\varepsilon)t} = c e^{(b_k - a_j + 2\varepsilon)t}$ for $t \ge 0$. The limit of the last term as $t \to \infty$ is zero, so that $\mathbf{w} \in ((F_A^k)_\omega)^{\perp}$. In the case $k > j$, choose $\varepsilon > 0$ such that $b_j < a_k - 2\varepsilon$ and a new constant c with $|\langle \mathbf{z}_0, \mathbf{w}_0\rangle| = |\langle \mathbf{z}(t), \mathbf{w}(t)\rangle| \le c e^{(a_k-\varepsilon)t} e^{(-b_j-\varepsilon)t} = c e^{(a_k - b_j - 2\varepsilon)t}$ for $t \le 0$, and take the limit as $t \to -\infty$ to arrive at the same conclusion. Therefore, $F_{-A^T}^{m+1-j} \subseteq \left(\bigoplus_{k\neq j} F_A^k\right)^{\perp}$. Since the dimension of the last space is $\dim F_A^j$, and since this happens for all $j = 1, \dots, m$, an easy argument of dimension counting completes the proof.

Proposition 1.89 *The Sacker–Sell spectrum of the family* (1.31) *of linear Hamiltonian systems satisfies* $\Sigma(H) = -\Sigma(H)$. *That is,* $a_j = -b_{m+1-j}$ *for* $j = 1, \dots, m$. *If, in addition, Ω is connected, then the spectral subbundles corresponding to*

$[-b_j,-a_j]$ and $[a_j,b_j]$ have the same dimension for $j = 1,\dots,m$; and, if the family (1.31) has exponential dichotomy, then the closed subbundle L^+ (resp. L^-) of Definition 1.75 is given by the sum of the spectral subbundles corresponding to the spectral intervals contained in the positive (resp. negative) half-line.

Proof Given any closed subbundle F of $\Omega \times \mathbb{K}^d$ and a constant nonsingular $d \times d$ matrix C, denote

$$C{\cdot}F = \{(\omega, C\,\mathbf{z}) \mid (\omega, \mathbf{z}) \in F\}\,,$$

which is a new closed subbundle with the same dimension. Assume that the linear family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω and write $\Omega \times \mathbb{K}^d = F^+ \oplus F^-$. The change of variables $\tilde{\mathbf{z}} = C\,\mathbf{z}$ takes this family to $\tilde{\mathbf{z}}' = B(\omega{\cdot}t)\,\tilde{\mathbf{z}}$, with $B(\omega) = C\,A(\omega)\,C^{-1}$ and $U_B(t,\omega) = C\,U_A(t,\omega)\,C^{-1}$. Since $\|U_B(t,\omega)\,\tilde{\mathbf{z}}_0\| = \|C\,U_A(t,\omega)\,C^{-1}\tilde{\mathbf{z}}_0\|$, there exists a constant $c > 0$ such that

$$c^{-1}\|U_A(t,\omega)\,C^{-1}\tilde{\mathbf{z}}_0\| \le \|U_B(t,\omega)\,\tilde{\mathbf{z}}_0\| \le c\,\|U_A(t,\omega)\,C^{-1}\tilde{\mathbf{z}}_0\|$$

for all $(t,\omega) \in \mathbb{R} \times \Omega$. It follows from this fact and from Definition 1.66 that the family $\mathbf{z}' = B(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω, with $\Omega \times \mathbb{K}^d = C{\cdot}F^+ \oplus C{\cdot}F^-$. Consequently, since the same change of variables takes the family $\mathbf{z}' = (A(\omega{\cdot}t) - \lambda I_d)\,\mathbf{z}$ to $\tilde{\mathbf{z}}' = (B(\omega{\cdot}t) - \lambda I_d)\,\tilde{\mathbf{z}}$, Definition 1.82 yields

$$\Sigma(A) = \Sigma(B) = \Sigma(CA\,C^{-1})\,. \tag{1.39}$$

Assuming now that Ω is connected, the previous inequalities and the characterization of the spectral intervals given in Remark 1.85.1 yield an easy proof of the equalities

$$F^j_B = C{\cdot}F^j_A \quad \text{for } j = 1,\dots,m. \tag{1.40}$$

Returning to the Hamiltonian case, relation (1.39), it follows from the equality $H^TJ + JH = 0_{2n}$ and Proposition 1.88 that $\Sigma(H) = \Sigma(JHJ^{-1}) = \Sigma(-H^T) = -\Sigma(H)$. If Ω is connected, relation (1.40) and Proposition 1.88 guarantee that $\dim F_H^{m+1-j} = \dim F_{-H^T}^{m+1-j} = \dim F_H^j$, as asserted. Assume finally that the family (1.11) has exponential dichotomy over Ω, so that $0 \notin \Sigma(H)$. It follows easily from Remarks 1.77.2 and 1.85.1 that the spectral subbundles associated to the spectral intervals contained in $(0,\infty)$ (resp. in $(-\infty,0)$) are contained in L^+ (resp. in L^-). The already-established relation between spectral subbundles and a trivial analysis of dimensions prove the last assertion.

1.4.5 Perturbation Theory in the General Linear Case

One of the most important characteristics of the exponential dichotomy property is its *robustness*: it persists under small perturbations of the coefficient matrix of the initial family. Something similar can be asserted about the Sacker–Sell spectrum. To explain these assertions is the main goal of this section. Some consequences, required in the following chapters, will also be worked out.

All the results of this section are consequences of the following powerful theorem, due to Sacker and Sell: see [133], Theorem 6. The hypotheses on the sets $\mathcal{M}$ and the flow ζ described at the beginning of Sect. 1.4.4 are retained.

Theorem 1.90 *Let $\mathcal{M}_0$ be a nonempty, compact, and (time-translation) invariant subset of $\mathcal{M}$. Then, for every neighborhood $\mathcal{J}$ of $\Sigma(\mathcal{M}_0)$ in $\mathbb{R}$, there is a neighborhood $\mathcal{K}$ of $\mathcal{M}_0$ such that $\Sigma(\mathcal{M}_1) \subseteq \mathcal{J}$ for every compact and invariant subset $\mathcal{M}_1$ of $\mathcal{M}$ contained in $\mathcal{K}$. Moreover, if $\lambda \in \mathbb{R} - \Sigma(\mathcal{M}_0)$, then there exist $\rho > 0$ and a neighborhood $\mathcal{K}$ of $\mathcal{M}_0$ such that, if $\mathcal{M}_1$ is as above, then $(\lambda - \rho, \lambda + \rho) \subset \mathbb{R} - \Sigma(\mathcal{M}_1)$. Furthermore, if $\lambda \in \mathbb{R} - \Sigma(\mathcal{M}_0)$, then there exists a compact neighborhood $\mathcal{K}$ of $\mathcal{M}_0$ such that, if $\mathcal{M}_+$ is the largest compact and invariant subset of $\mathcal{M}$ contained in $\mathcal{K}$ and containing $\mathcal{M}_0$, then the family $\{\mathbf{z}' = (B(t) - \lambda I_d)\,\mathbf{z} \mid B \in \mathcal{M}_+\}$ has exponential dichotomy. In particular, the vector spaces $(F_\lambda^{\pm})_B$ of the initial data of the solutions of $\mathbf{z}' = (B(t) - \lambda I_d)\,\mathbf{z}$ which tend to zero as $t \to \pm\infty$ vary continuously with respect to $B \in \mathcal{M}_+$.*

Several applications of this theorem will be required in the book. The one most frequently used is described in what follows. (Others will be described in Sect. 4.5.1 and in the last examples of Sect. 7.2.) It refers to a parameterized perturbation of the family of systems $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$, namely

$$\mathbf{z}' = (A(\omega{\cdot}t) + K_\lambda(\omega{\cdot}t))\,\mathbf{z}\,, \tag{1.41}$$

where the following conditions are satisfied:

p1. (Ω, σ) is a continuous flow on a compact metric space, and $\sigma(t, \omega) = \omega{\cdot}t$;
p2. $A\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$ is continuous;
p3. $\Lambda \subset \mathbb{K}^m$ is a compact set containing $\mathbf{0}$;
p4. $\{K_\lambda \mid \lambda \in \Lambda\}$ is an m-parameter family of elements of $C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$, with $K_{\mathbf{0}} = 0_d$, such that the map $\Lambda \to C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$, $\lambda \mapsto K_\lambda$ is continuous for the topology of $C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$ given by the norm $\|B\|_\Omega = \max_{\omega\in\Omega} \|B(\omega)\|$.

Write $B_\omega(t) = B(\omega{\cdot}t)$ for any matrix-valued functions $B\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{K})$. Under the four conditions p1-4, for all $n \in \mathbb{N}$, the map

$$[0, n] \times \Omega \times \Lambda \to \mathbb{M}_{d\times d}(\mathbb{K})\,, \quad (t, \omega, \lambda) \mapsto (A + K_\lambda)_\omega(t)$$

is continuous, and hence uniformly continuous. Therefore, the function

$$\tilde{\varepsilon}\colon [0,\infty) \to \mathbb{R}\,, \qquad \delta \mapsto \sup_{\substack{|t|\le\delta \\ (\omega,\lambda)\in\Omega\times\Lambda}} \|(A+K_\lambda)_\omega(t) - (A+K_\lambda)_\omega(0)\|$$

is continuous, with $\tilde{\varepsilon}(0) = 0$, and moreover $\|(A+K_\lambda)_\omega(t) - (A+K_\lambda)_\omega(0)\| \le \tilde{\varepsilon}(|t|)$ for all $(\omega,\lambda) \in \Omega\times\Lambda$ and $|t| \le n$. Note that

$$\begin{aligned}\|(A+K_\lambda)_\omega(t) &- (A+K_\lambda)_\omega(s)\| \\ &= \|(A+K_\lambda)_{\omega\cdot s}(t-s) - (A+K_\lambda)_{\omega\cdot s}(0)\| \le \tilde{\varepsilon}(|t-s|)\,.\end{aligned}$$

In other words, $\tilde{\varepsilon}$ is a common continuity modulus for the set of continuous functions $\{(A+K_\lambda)_\omega\colon \mathbb{R}\to\mathbb{M}_{d\times d}\}$. Moreover, there exists $\kappa > 0$ such that $\|(A+K_\lambda)_\omega(t)\| \le \kappa$ for all $(t,\omega,\lambda) \in \mathbb{R}\times\Omega\times\Lambda$. Define the set

$$\mathcal{M} = \{C\colon \mathbb{R}\to\mathbb{M}_{d\times d} \mid \sup_{t\in\mathbb{R}} \|C(t)\| \le 2\kappa \text{ and } C \text{ has continuity modulus } 2\,\tilde{\varepsilon}\}\,,$$

and provide it with the topology of uniform convergence on compact subsets of $\mathbb{R}$. Consider also the subsets of $\mathcal{M}$ defined by $\mathcal{M}_0 = \{A_\omega \mid \omega\in\Omega\}$ and $\mathcal{M}_{\Lambda_*} = \{(A+K_\lambda)_\omega \mid \omega\in\Omega\,, \lambda\in\Lambda_*\}$, where $\Lambda_* \subseteq \Lambda$ is any compact neighborhood of $\mathbf{0}$ in Λ. It is easy to check that $\mathcal{M}$ is a compact Hausdorff topological space; that it is invariant by time-translation; and that the flow ζ given by (1.36) is continuous. Moreover, the subsets $\mathcal{M}_0$ and $\mathcal{M}_{\Lambda_*}$ are compact and time-invariant, and $\mathcal{M}_0 \subset \mathcal{M}_{\Lambda_*}$. In addition, the exponential dichotomy of the family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ over Ω is equivalent to the exponential dichotomy of the family $\{\mathbf{z}' = B(t)\,\mathbf{z} \mid B\in\mathcal{M}_0\}$, according to the definition given in Sect. 1.4.4. Applying Theorem 1.90 to this setting proves the following result. Recall that $\Sigma(A)$ represents the dynamical spectrum of the family $\{\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z} \mid \omega\in\Omega\}$: see Definition 1.87.

Theorem 1.91 *Suppose that conditions* p1–p4 *hold.*

(i) *Let $\mathcal{I} \subseteq \mathbb{R}$ be an open set containing $\Sigma(A)$. There exists a compact set $\Lambda_* \subseteq \Lambda$, which is a neighborhood of $\mathbf{0}$ in Λ, such that $\Sigma(A+K_\lambda) \subset \mathcal{I}$ whenever $\lambda\in\Lambda_*$. In addition, if $\lambda \in \mathbb{R}-\Sigma(A)$, then there exist $\varepsilon = \varepsilon(\lambda)$ and $\Lambda_* = \Lambda_*(\lambda)$ as above such that $(\lambda-\varepsilon, \lambda+\varepsilon)\cap\Sigma(A+K_\lambda)$ is empty whenever $\lambda\in\Delta^*$.*

(ii) *If, in particular, the unperturbed family $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω, then there exists a compact subset $\Lambda_* \subseteq \Lambda$, which determines a neighborhood of $\mathbf{0}$ in Λ, such that the family of systems $\{\mathbf{z}' = (A(\omega{\cdot}t) + K_\lambda(\omega{\cdot}t))\,\mathbf{z} \mid (\omega,\lambda)\in\Omega\times\Lambda_*\}$ has exponential dichotomy: it satisfies the conditions of Definition 1.58 for a projector $\widetilde{\mathcal{Q}} \equiv \{\bar{Q}(\omega,\lambda)\}$ and constants η and β. Consequently,*

  (ii.1) *the family (1.41) has exponential dichotomy over Ω for all $\lambda\in\Lambda_*$, with projector $\widetilde{\mathcal{Q}}_{K_\lambda} \equiv \{Q_{K_\lambda}(\omega)\}$ for $Q_{K_\lambda}(\omega) = \bar{Q}(\omega,\lambda)$ and common constants η and β.*

(ii.2) $\Omega \times \Lambda_* \times \mathbb{K}^d \to \Omega \times \Lambda_* \times \mathbb{K}^d$, $(\omega, \lambda, \mathbf{z}) \mapsto (\omega, \lambda, Q_{K_\lambda}(\omega)\,\mathbf{z})$ *is a jointly continuous map. In particular, the maps*

$$\Omega \times \Lambda_* \to \mathbb{M}_{d\times d}(\mathbb{K})\,, \quad (\omega, \lambda) \mapsto Q_{K_\lambda}(\omega)$$

and

$$\Lambda_* \to C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))\,, \quad \lambda \mapsto Q_{K_\lambda}$$

are continuous.

(ii.3) *Let* $\Omega \times \Lambda_* \times \mathbb{K}^d = \widetilde{F}^+ \oplus \widetilde{F}^-$ *be the decomposition provided by Definition 1.66. Let* $\Omega_c \subseteq \Omega$ *be a connected component, and assume that also* Λ_* *is connected. Then, there exists an integer* $k \geq 0$ *such that* $\dim \widetilde{F}^+_{\omega,\lambda} = k$ *and* $\dim \widetilde{F}^-_{\omega,\lambda} = d - k$ *for each* $(\omega, \lambda) \in \Omega_c \times \Lambda_*$, *and the maps*

$$\Omega_c \times \Lambda_* \to \mathcal{G}_k(\mathbb{K}^d)\,, \quad (\omega, \lambda) \mapsto \widetilde{F}^+_{\omega,\lambda}$$

and

$$\Omega_c \times \Lambda_* \to \mathcal{G}_{d-k}(\mathbb{K}^d)\,, \quad (\omega, \lambda) \mapsto \widetilde{F}^-_{\omega,\lambda}$$

are continuous.

Note that if $\Lambda = [0, 1] \subset \mathbb{R}$, then the set Λ_* contains an interval $[0, \mu]$ for some $\mu > 0$. Several consequences of Theorem 1.91, which are now explained, will be required in Chap. 3. Define

$$\mathcal{B}_\delta = \{K \in C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K})) \mid \|K\|_\Omega \leq \delta\}$$

for $\delta \geq 0$. Whenever the family $\mathbf{z}' = (A(\omega{\cdot}t) + K(\omega{\cdot}t))\,\mathbf{z}$ has exponential dichotomy over Ω, $\widetilde{Q}_K \equiv \{Q_K(\omega)\}$ and $\Omega \times \mathbb{K}^d = \widetilde{F}^+_K \oplus \widetilde{F}^-_K$ will represent the corresponding projector and hyperbolic splitting.

Theorem 1.92 *Suppose that conditions* p1 *and* p2 *hold, and that the family of linear systems* $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ *has exponential dichotomy over* Ω. *Then, there exists* $\delta > 0$ *such that*

(i) *the family* $\mathbf{z}' = (A(\omega{\cdot}t) + K(\omega{\cdot}t))\,\mathbf{z}$ *has exponential dichotomy over* Ω *for all* $K \in \mathcal{B}_\delta$.
(ii) *The map* $\mathcal{B}_\delta \to C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$, $K \mapsto Q_K$ *is continuous.*
(iii) *For each connected component* $\Omega_c \subseteq \Omega$, *there exists an integer* $k \geq 0$ *such that* $\dim(F^+_K)_\omega = k$ *for each* $\omega \in \Omega_c$ *and* $K \in \mathcal{B}_\delta$, *and the maps* $\Omega_c \to \mathcal{G}_k(\mathbb{K}^d)$, $\omega \mapsto (F^+_K)_\omega$ *and* $\Omega_c \to \mathcal{G}_{d-k}(\mathbb{K}^d)$, $\omega \mapsto (F^-_K)_\omega$ *are continuous.*
(iv) *There exists a real constant c with* $\|Q_K - Q_{0_d}\|_\Omega \leq c\,\|K\|_\Omega$ *whenever* $K \in \mathcal{B}_\delta$.

Proof (i), (ii) & (iii) Suppose for contradiction that (i) does not hold. Then there exists a sequence $(\widetilde{K}_j)$ in $C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$ with limit 0_d (in the uniform topology) such that the family $\mathbf{z}' = (A(\omega{\cdot}t) + \widetilde{K}_j(\omega{\cdot}t))\,\mathbf{z}$ does not have exponential dichotomy over Ω. But this contradicts Theorem 1.91(i) applied to a continuous mapping $[0,1] \to C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$, $\lambda \mapsto K_\lambda$ with $K_0 = 0_n$ and $K_{1/j} = \widetilde{K}_j$. A similar argument proves (ii) and (iii).

(iv) The proof is carried out for the most part by Coppel in Chapter 4 of [33]. Let $\eta \geq 1$ and $\beta > 0$ be the constants of Definition 1.58 for the exponential dichotomy of $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ over Ω, and choose $\delta < \beta/(8\eta^2)$. Recall that the projection $Q_K(\omega)$ is uniquely determined for all $\omega \in \Omega$, as Proposition 1.56 states. Following step by step Coppel's arguments, it follows that the continuous map $Q_K\colon \Omega \to \mathbb{M}_{d\times d}(\mathbb{C})$ is given for $K \in \mathcal{B}_\delta$ by

$$Q_K(\omega) = S_K(\omega)\, Q_{0_d}(\omega)\, S_K^{-1}(\omega) \tag{1.42}$$

for the matrix $S_K(\omega) = I_d + Q_K^1(\omega) - Q_K^2(\omega)$, where $Q_K^1(\omega) = Y_K^1(0,\omega)$, $Q_K^2(\omega) = I_d - Y_K^2(0,\omega)$, and the matrices Y_K^1 and Y_K^2 satisfy

$$\begin{aligned} Y_K^1(t,\omega) = {} & U_A(t,\omega)\, Q_{0_d}(\omega) \\ & + \int_0^t U_A(t,\omega)\, Q_{0_d}(\omega)\, U_A^{-1}(s,\omega)\, K(\omega{\cdot}s)\, Y_K^1(s,\omega)\, ds \\ & - \int_t^\infty U_A(t,\omega)\, (I_d - Q_{0_d}(\omega))\, U_A^{-1}(s,\omega)\, K(\omega{\cdot}s)\, Y_K^1(s,\omega)\, ds \end{aligned}$$

for $t \geq 0$ and

$$\begin{aligned} Y_K^2(t,\omega) = {} & U_A(t,\omega)\, (I_d - Q_{0_d}(\omega)) \\ & - \int_t^0 U_A(t,\omega)\, (I_d - Q_{0_d}(\omega))\, U_A^{-1}(s,\omega)\, K(\omega{\cdot}s)\, Y_K^2(s,\omega)\, ds \\ & + \int_{-\infty}^t U_A(t,\omega)\, Q_{0_d}(\omega)\, U_A^{-1}(s,\omega)\, K(\omega{\cdot}s)\, Y_K^2(s,\omega)\, ds \end{aligned}$$

for $t \leq 0$. Take $K \in \mathcal{B}_\delta$. It is not hard to deduce from Definition 1.58 that

$$\sup_{t\in[0,\infty)} \|Y_K^1(t,\omega)\| \leq 2\eta \quad \text{and} \quad \sup_{t\in(-\infty,0]} \|Y_K^2(t,\omega)\| \leq 2\eta$$

for all $\omega \in \Omega$, which in turn implies that

$$\sup_{\omega\in\Omega} \|Q_K^1(\omega) - Q_K^2(\omega)\| \leq \frac{4\eta^2}{\beta}\, \|K\|_\Omega \leq \frac{1}{2}\,. \tag{1.43}$$

Recall that $S_K(\omega) = I_d + Q_K^1(\omega) - Q_K^2(\omega)$. Therefore,

$$S_K^{-1}(\omega) = \sum_{k=0}^{\infty}(-1)^k (Q_K^1(\omega) - Q_K^2(\omega))^k\,,$$

$$S_K^{-1}(\omega) - I_d = (Q_K^1(\omega) - Q_K^2(\omega)) \sum_{k=0}^{\infty}(-1)^{k+1} (Q_K^1(\omega) - Q_K^2(\omega))^k\,.$$

These equalities and (1.43) ensure that

$$\sup_{\omega\in\Omega} \|S_K^{-1}(\omega)\| \le \sum_{k=0}^{\infty} \frac{1}{2^k} = 2\,,$$

$$\sup_{\omega\in\Omega} \|S_K(\omega) - I_d\| \le \frac{4\eta^2}{\beta}\,\|K\|_\Omega\,,$$

$$\sup_{\omega\in\Omega} \|S_K^{-1}(\omega) - I_d\| \le \frac{4\eta^2}{\beta}\,\|K\|_\Omega \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{8\eta^2}{\beta}\,\|K\|_\Omega\,.$$

These properties, the boundedness of Q_{0_d} on Ω, and the equality

$$Q_K(\omega) - Q_{0_d}(\omega) = (S_K(\omega) - I_d)\, Q_{0_d}(\omega)\, S_K^{-1}(\omega) + Q_{0_d}(\omega)\,(S_K^{-1}(\omega) - I_d)\,,$$

which follows from (1.42), all taken together, prove the assertion.

The following consequence of Theorem 1.91(i) will be required in Chap. 5. Assume that conditions p1–p4 hold, and that Ω and Λ are connected. Note that then the sets $\mathcal{M}_0 = \{A_\omega \mid \omega \in \Omega\}$ and $\mathcal{M}_\lambda = \{(A + K_\lambda)_\omega \mid \omega \in \Omega\}$ are connected, for all $\lambda \in \Lambda$. The notation established in Theorem 1.84 and Definition 1.38 is retained, adding now $d_j = \dim F_A^j$ for $j = 1, \ldots, m$. For $j = 1, \ldots, m$, let $\mathcal{I}_j$ be an open interval containing $[a_j, b_j]$ such that $\mathcal{I}_j \cap \mathcal{I}_{j+1}$ is empty for $j = 1, \ldots, m-1$. Under the conditions of Theorem 1.91(i), there exists a subset $\Lambda_* \subseteq \Lambda$ such that $\Sigma(A + K_\lambda)$ is contained in the disjoint union of the d intervals $\mathcal{I}_1, \ldots, \mathcal{I}_m$ if $\lambda \in \Lambda_*$. Assume also that Λ_* is connected. For each $\lambda \in \Lambda_*$ and $j = 1, \ldots, m$, let $F^j(\lambda)$ represent the Whitney sum of all the spectral subbundles of $\mathbf{z}' = (A(\omega{\cdot}t) + K_\lambda(\omega{\cdot}t))\,\mathbf{z}$ corresponding to spectral intervals contained in $\mathcal{I}_j$. Note that $F^j(0) = F_A^j$.

Corollary 1.93 *Suppose that Ω is connected. In the situation explained in the previous paragraph, the map $\Omega \times \Lambda_* \to \mathcal{G}_{d_j}(\mathbb{K}^d),\ (\omega, \lambda) \mapsto (F^j(\lambda))_\omega$ is well defined and continuous for $j = 1, \ldots, m$. In particular, $\dim F^j(\lambda) = d_j$ for $j = 1, \ldots, m$.*

Proof Choose real numbers $\mu_0, \ldots, \mu_m$ satisfying $\mu_0 < \inf \mathcal{I}_1$, $\sup \mathcal{I}_j < \mu_j < \inf \mathcal{I}_{j+1}$ for $j = 1, \ldots m-1$, and $\sup \mathcal{I}_m < \mu_m$. Take $\lambda \in \Lambda_*$, and note that the family

$$\mathbf{z}' = (A(\omega{\cdot}t) + K_\lambda(\omega{\cdot}t) - \mu_j I_d)\,\mathbf{z}$$

has exponential dichotomy for $j = 0, \ldots, m$. Write $\Omega \times \mathbb{K}^m = F^+(\lambda, \mu_j) \oplus F^-(\lambda, \mu_j)$ for the corresponding hyperbolic splitting, and recall that F_ω is the fiber over $\omega \in \Omega$ for any closed subbundle F. Fix $j = 0, \ldots, m$. Two properties hold: first, according to Theorem 1.91(ii.3), $\dim(F^\pm(\lambda, \mu_j))_\omega$ is constant with respect to $(\omega, \lambda) \in \Omega \times \Lambda_*$ (recall that Ω and Λ_* are connected), and the map $(\omega, \lambda) \mapsto (F^\pm(\lambda, \mu_j))_\omega$ from Ω to the suitable Grassmannian manifold is continuous; and second, according to the last assertion in Theorem 1.84,

$$(F^j(\lambda))_\omega = (F^-(\lambda, \mu_{j-1}))_\omega \cap (F^+(\lambda, \mu_j))_\omega$$

for $j = 1, \ldots, m$. The continuity of the map of the statement follows easily from these two facts, and in turn implies the last assertion about dimension.

The last theorem of this section will play a fundamental role in Sect. 3.1. It is an easy consequence of the more general result given by Johnson in Proposition 3.9 of [65] (see also its proof). As before, the set $C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$ is endowed with the topology given by the norm $\|B\|_\Omega = \max_{\omega\in\Omega} \|B(\omega)\|$.

Theorem 1.94 *Let* $\{\lambda \in \mathbb{C} \mid |\lambda| < r\} \to C(\Omega, \mathbb{M}_{d\times d}(\mathbb{K}))$, $\lambda \mapsto B_\lambda$ *be an analytic map for an* $r > 0$, *and let the families of systems* $\mathbf{z}' = B_\lambda(\omega{\cdot}t)\,\mathbf{z}$ *have exponential dichotomy over* Ω *whenever* $|\lambda| < r$. *Let* $\mathcal{Q}_\lambda = \{Q_\lambda(\omega)\}$ *be the projector provided by Definition 1.58. Then the map*

$$\{\lambda \in \mathbb{C} \mid |\lambda| < r\} \to \mathbb{M}_{d\times d}(\mathbb{K})\,, \quad \lambda \mapsto Q_\lambda(\omega)$$

is analytic for each $\omega \in \Omega$.

1.4.6 Perturbation Theory in the Linear Hamiltonian Case

The following theorem adds some information to that provided by Theorem 1.92 in the Hamiltonian case. In this context, let $Q_K(\omega)$ and $l_K^\pm(\omega)$ represent the projections of the exponential dichotomy over Ω and the corresponding Lagrange planes if this dichotomy property is present for the family $\mathbf{z}' = (H(\omega{\cdot}t) + K(\omega{\cdot}t))\,\mathbf{z}$ over Ω, and let $M_K^\pm(\omega)$ be the Weyl matrices if they exist: see Definition 1.80.

In what follows, $C(\Omega, \mathfrak{sp}(n, \mathbb{K}))$ is understood to be a subset of the space topological space $C(\Omega, \mathbb{M}_{2n\times 2n}(\mathbb{K}))$, on which the topology is given by the norm $\|B\|_\Omega = \max_{\omega\in\Omega} \|B(\omega)\|$.

Theorem 1.95 *Suppose that the family of linear Hamiltonian systems* $\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}$ *has exponential dichotomy over* Ω *for a matrix-valued function* $H \in C(\Omega, \mathfrak{sp}(n,\mathbb{K}))$. *Let* $\mathcal{B}_\delta \subset C(\Omega, \mathbb{M}_{2n\times 2n}(\mathbb{K}))$ *be the open neighborhood of* 0_{2n} *provided by Theorem 1.92, and represent* $\mathcal{B}_\delta^{\mathfrak{sp}} = \mathcal{B}_\delta \cap C(\Omega, \mathfrak{sp}(n,\mathbb{K}))$, *so that the family* $\mathbf{z}' = (H(\omega{\cdot}t) + K(\omega{\cdot}t))\,\mathbf{z}$ *is of Hamiltonian type and has exponential dichotomy over* Ω *for all* $K \in \mathcal{B}_\delta^{\mathfrak{sp}}$. *Then,*

(i) *the maps* $l^\pm\colon \Omega \times \mathcal{B}_\delta^{\mathfrak{sp}} \to \mathcal{L}_\mathbb{K}\,,\ (\omega, K) \mapsto l_K^\pm(\omega)$ *are continuous.*

(ii) *Suppose further that* $l^+_{0_{2n}}(\omega) \in \mathcal{D}_\mathbb{K}$ *for all* $\omega \in \Omega$, *so that the function* $M^+_{0_{2n}}\colon \Omega \to \mathbb{S}_n(\mathbb{K})$ *exists. Then* $\delta > 0$ *can be chosen in such a way that* $l_K^+(\omega) \in \mathcal{D}_\mathbb{K}$ *for all* $K \in \mathcal{B}_\delta^{\mathfrak{sp}}$ *and* $\omega \in \Omega$. *In addition, if* $l_K^+(\omega) \equiv \begin{bmatrix} I_n \\ M_K^+(\omega)\end{bmatrix}$, *the maps* $\Omega \times \mathcal{B}_\delta^{\mathfrak{sp}} \to \mathbb{S}_n(\mathbb{K}),\ (\omega, K) \mapsto M_K^+(\omega)$ *and* $M^+\colon \mathcal{B}_\delta^{\mathfrak{sp}} \to C(\Omega, \mathbb{S}_n(\mathbb{K})),\ K \mapsto M_K^+$, *are well defined and continuous. And the analogous statements hold for* $l^-_{0_{2n}}$.

Proof

(i) This assertion follows immediately from Theorem 1.92(iii).

(ii) The first assertion in (ii) follows from the open character of $\mathcal{D}_\mathbb{K}$, the compactness of Ω, and property (i). In addition, (i) and Proposition 1.29(i) imply the continuity of the map $\Omega \times \mathcal{B}_\delta^{\mathfrak{sp}} \to \mathbb{S}_n(\mathbb{K}),\ (\omega, K) \mapsto M_K^+(\omega)$. It is easy to deduce from this joint continuity property that the map M^+ is well-defined (i.e. that the Weyl function M_K^+ is continuous) and that it is continuous on $\mathcal{B}_\delta^{\mathfrak{sp}}$.

1.4.7 The Grassmannian Flows Under Exponential Dichotomy

Suppose that the family of linear Hamiltonian systems (1.31) has exponential dichotomy over Ω and, as before, represent by $l^\pm(\omega)$ the Lagrange planes of the initial data of the solutions of the system corresponding to ω which are bounded as $t \to \pm\infty$ (see Remark 1.69.1 and Proposition 1.76), so that the splitting $\Omega \times \mathbb{K}^{2n} = L^+ \oplus L^-$ with $L^\pm = \{(\omega, \mathbf{z}) \mid \mathbf{z} \in l^\pm(\omega)\}$ satisfies Definition 1.75. The following result is a consequence of the behavior of the solutions of (1.31) outside $L^\pm$ described in Remark 1.77.2 and Proposition 1.56(i). It shows how the topological behavior of the flow τ on $\mathcal{K}_\mathbb{K}$ inherits the complexity of the lower dimensional flows τ_H^k given by (1.6) on $\Omega \times \mathcal{G}_k(\mathbb{K}^{2n})$ for $k = 1, \dots, n$. In this section, the notation τ_k will substitute τ_H^k. As usual, $\langle \mathbf{z}_1, \dots, \mathbf{z}_k\rangle$ represents the vector space generated by $\mathbf{z}_1, \dots, \mathbf{z}_k$.

Proposition 1.96 *Suppose that the family* (1.31) *has exponential dichotomy over* Ω, *and write* $\mathbf{z} = \mathbf{z}^+(\omega) + \mathbf{z}^-(\omega)$ *for each* $(\omega, \mathbf{z}) \in \Omega \times \mathbb{K}^{2n}$, *with* $\mathbf{z}^\pm(\omega) \in l^\pm(\omega)$.

(i) *Take* $\omega_0 \in \Omega$, *k linearly independent vectors* $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{K}^{2n}$, *and a sequence of real numbers* $(t_m) \uparrow \infty$. *Suppose that there exist k linearly independent vectors* $\tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_k \in \mathbb{K}^{2n}$ *with*

$$\langle \tilde{\mathbf{z}}_j \rangle = \lim_{m \to \infty} \langle U(t_m, \omega_0)\, \mathbf{z}_j \rangle \quad \text{in } \mathcal{G}_1(\mathbb{K}^{2n})$$

for $j = 1, \ldots, k$. *Then,*

$$\langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_k \rangle = \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1, \ldots, \mathbf{z}_k \rangle \quad \text{in } \mathcal{G}_k(\mathbb{K}^{2n}) .$$

(ii) *Take* $(\omega_0, l_0) \in \mathcal{K}_{\mathbb{K}}$ *with* $\dim(l_0 \cap l^+(\omega_0)) = k \in \{0, \ldots, n\}$, *and write* $l_0 = \langle \mathbf{z}_1, \ldots, \mathbf{z}_n \rangle$ *with* $\mathbf{z}_j^-(\omega_0) = \mathbf{0}$ *for* $j = 1, \ldots, k$ *and* $\mathbf{z}_j^-(\omega_0) \neq \mathbf{0}$ *for* $j = k+1, \ldots, n$. *Take also* $(t_m) \uparrow \infty$, *and suppose that*

$$\begin{aligned} \tilde{\omega}_0 &= \lim_{m \to \infty} \omega_0 {\cdot} t_m , \\ \langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_k \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1, \ldots, \mathbf{z}_k \rangle \quad \text{in } \mathcal{G}_k(\mathbb{K}^{2n}) , \\ \langle \tilde{\mathbf{z}}_{k+1}, \ldots, \tilde{\mathbf{z}}_n \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_{k+1}, \ldots, \mathbf{z}_n \rangle \quad \text{in } \mathcal{G}_{n-k}(\mathbb{K}^{2n}) . \end{aligned} \tag{1.44}$$

Then,

$$\begin{aligned} \langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_k \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1^+(\omega_0), \ldots, \mathbf{z}_k^+(\omega_0) \rangle \in \mathcal{G}_k(l^+(\tilde{\omega}_0)) , \\ \langle \tilde{\mathbf{z}}_{k+1}, \ldots, \tilde{\mathbf{z}}_n \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_{k+1}^-(\omega_0), \ldots, \mathbf{z}_n^-(\omega_0) \rangle \in \mathcal{G}_{n-k}(l^-(\tilde{\omega}_0)) \\ !\langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_n \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1, \ldots, \mathbf{z}_n \rangle \in \mathcal{L}_{\mathbb{R}} . \end{aligned}$$

(iii) *Take* $(\omega_0, l_0) \in \mathcal{K}_{\mathbb{K}}$ *with* $\dim(l_0 \cap l^-(\omega_0)) = k \in \{0, \ldots, n\}$, *and write* $l_0 = \langle \mathbf{z}_1, \ldots, \mathbf{z}_n \rangle$ *with* $\mathbf{z}_j^+(\omega_0) = \mathbf{0}$ *for* $j = 1, \ldots, k$ *and* $\mathbf{z}_j^+(\omega_0) \neq \mathbf{0}$ *for* $j = k+1, \ldots, n$. *Take also* $(t_m) \downarrow -\infty$ *and suppose that conditions* (1.44) *hold. Then,*

$$\begin{aligned} \langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_k \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1^-(\omega_0), \ldots, \mathbf{z}_k^-(\omega_0) \rangle \in \mathcal{G}_k(l^-(\tilde{\omega}_0)) , \\ \langle \tilde{\mathbf{z}}_{k+1}, \ldots, \tilde{\mathbf{z}}_n \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_{k+1}^+(\omega_0), \ldots, \mathbf{z}_n^+(\omega_0) \rangle \in \mathcal{G}_{n-k}(l^+(\tilde{\omega}_0)) \\ \langle \tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_n \rangle &= \lim_{m \to \infty} U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1, \ldots, \mathbf{z}_n \rangle \in \mathcal{L}_{\mathbb{R}} . \end{aligned}$$

Proof Property (i) follows from Proposition 1.26(i).

The first assertion in (ii) is trivial when $k = 0$. For $k \geq 1$, it follows from the relations $U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1, \ldots, \mathbf{z}_k \rangle = U(t_m, \omega_0) {\cdot} \langle \mathbf{z}_1^+(\omega_0), \ldots, \mathbf{z}_k^+(\omega_0) \rangle \subseteq U(t_m, \omega_0) {\cdot} l^+(\omega_0) = l^+(\omega_0 {\cdot} t_m)$ and from the continuity of $l^+ \colon \Omega \to \mathcal{L}_{\mathbb{R}}$ (see Proposition 1.76). A similar argument proves the first assertion in (iii).

Since $\dim\langle\tilde{\mathbf{z}}_{k+1},\dots,\tilde{\mathbf{z}}_n\rangle = n-k$, the second equality in (ii) is an immediate consequence of the continuity of $l^-\colon\Omega\to\mathcal{L}_{\mathbb{K}}$ and the following property: given $\tilde{\mathbf{z}}\in\langle\tilde{\mathbf{z}}_{k+1},\dots,\tilde{\mathbf{z}}_n\rangle$ with $\|\tilde{\mathbf{z}}\|=1$, there exists a sequence $(\mathbf{w}_m)$ in $\langle\mathbf{z}_{k+1},\dots,\mathbf{z}_n\rangle$ with $\tilde{\mathbf{z}}=\lim_{m\to\infty}U(t_m,\omega_0)\,\mathbf{w}_m^-(\omega_0)$. To prove this last assertion, note that there exists a sequence $(\mathbf{y}_m)$ with $\mathbf{y}_m\in U(t_m,\omega_0)\cdot\langle\mathbf{z}_{k+1},\dots,\mathbf{z}_n\rangle$ and $\lim_{m\to\infty}\mathbf{y}_m=\tilde{\mathbf{z}}$ (see Proposition 1.26(i)), so that given $\varepsilon>0$ there exists m_1 such that $\|\tilde{\mathbf{z}}-\mathbf{y}_m\|\le\varepsilon/2$ for all $m\ge m_1$. Since $\tilde{\mathbf{z}}=\lim_{m\to\infty}\mathbf{y}_m/\|\mathbf{y}_m\|$, there is no loss of generality in assuming that $\|\mathbf{y}_m\|=1$. Assume also that $t_{m_1}>0$. Define the sequence $(\mathbf{w}_m)$ by $\mathbf{w}_m=(U^{-1}(t_m,\omega_0)\,\mathbf{y}_m)/\|U^{-1}(t_m,\omega_0)\,\mathbf{y}_m\|$, so that $\mathbf{w}_m\in\langle\mathbf{z}_{k+1},\dots,\mathbf{z}_n\rangle$ with $\|\mathbf{w}_m\|=1$ and

$$\mathbf{y}_m=\frac{U(t_m,\omega_0)\,\mathbf{w}_m}{\|U(t_m,\omega_0)\,\mathbf{w}_m\|}\,, \tag{1.45}$$

since $\|\mathbf{y}_m\|=1$. Two properties are now required:

(1) There exists $c_1>0$ such that $\|\mathbf{w}^+(\omega_0)\|\le c_1$ for all $\mathbf{w}\in l_0$ with $\|\mathbf{w}\|=1$, since $\mathbb{K}^{2n}\to l^+(\omega_0),\ \mathbf{w}\mapsto\mathbf{w}^+(\omega_0)$ is linear and continuous.
(2) There exist $m_2\ge m_1$ and $c_2>0$ with $\|U(t_m,\omega_0)\,\mathbf{w}_m\|^{-1}\le c_2$ for all $m\ge m_2$.

Suppose for the moment being that (2) is true. Since $\|\mathbf{w}_m\|=1$, properties (2) and (1), equality (1.45), and Definition 1.75, yield

$$\begin{aligned}\left\|\tilde{\mathbf{z}}-\frac{U(t_m,\omega_0)\,\mathbf{w}_m^-(\omega_0)}{\|U(t_m,\omega_0)\,\mathbf{w}_m\|}\right\| &\le \|\tilde{\mathbf{z}}-\mathbf{y}_m\|+\frac{\|U(t_m,\omega_0)\,\mathbf{w}_m^+(\omega_0)\|}{\|U(t_m,\omega_0)\,\mathbf{w}_m\|}\\ &\le\frac{\varepsilon}{2}+c_2\,c_1\,\eta\,e^{-\beta t_m}\end{aligned}$$

for all $m\ge m_2$, which is smaller than ε for large enough m. Hence the initial assertion holds for the sequence given by $\tilde{\mathbf{w}}_m=\|U(t_m,\omega_0)\,\mathbf{w}_m\|^{-1}\,\mathbf{w}_m$.

There remains to check property (2). As a preliminary step, the existence of $c_3>0$ such that $\|\mathbf{w}_m^-(\omega_0)\|\ge c_3$ is proved: assume for contradiction the existence of a subsequence of $(\mathbf{w}_m^-(\omega_0))$ with limit $\mathbf{0}$ and note that then the limit of a suitable subsequence of $(\mathbf{w}_m)$, with norm 1, is $\mathbf{0}$ if $k=0$ or, if $k\ge1$, it belongs to $l^+(\omega_0)\cap l_0=\langle\mathbf{z}_1,\dots,\mathbf{z}_k\rangle$ and at the same time to $\langle\mathbf{z}_{k+1},\dots,\mathbf{z}_n\rangle$, which is impossible. Therefore, $\|U(t_m,\omega_0)\,\mathbf{w}_m\|\ge\|U(t_m,\omega_0)\,\mathbf{w}_m^-(\omega_0)\|-\|U(t_m,\omega_0)\,\mathbf{w}_m^+(\omega_0)\|\ge c_3(1/\eta)\,e^{\beta t_m}-c_1\eta\,e^{-\beta t_m}$, with η and β satisfying Definition 1.75 and Proposition 1.56(i). This implies (2) and completes the proof of the second property in (ii). A similar argument works for second property in (iii).

The last assertions in (ii) and (iii) are consequences of two facts: the first one is that $\langle\tilde{\mathbf{z}}_1,\dots,\tilde{\mathbf{z}}_n\rangle=\lim_{m\to\infty}U(t_m,\omega_0)\cdot\langle\mathbf{z}_1,\dots,\mathbf{z}_n\rangle$ in $\mathcal{G}_n(\mathbb{K}^{2n})$, which is immediately deduced from the previous properties and from $l^+(\tilde{\omega})\cap l^-(\tilde{\omega}_0)=\{\mathbf{0}\}$; and the second one is that $U(t_m,\omega_0)\cdot\langle\mathbf{z}_1,\dots,\mathbf{z}_n\rangle$ belongs to the closed subspace $\mathcal{L}_{\mathbb{K}}$: see Sect. 1.3.3.

The preceding result is the fundamental tool in the proof of the following statement, which in turn is the key to the proof of Corollary 1.98. This result will be very useful in Chap. 7, where conditions are established which ensure the global existence of both Weyl functions for nonoscillatory Hamiltonian systems.

Proposition 1.97 *Suppose that the family* (1.11) *has exponential dichotomy over* Ω. *Given* $(\omega_0, l_0) \in \mathcal{K}_{\mathbb{K}}$, *let* $k \in \{0, \dots, n\}$ *be the dimension of* $l_0 \cap l^+(\omega_0)$ *and let* $\mathcal{O}(\omega_0, l_0)$ *be the omega-limit set of* (ω_0, l_0) *in the flow* τ *on* $\mathcal{K}_{\mathbb{K}}$. *Then,*

(i) $U(t, \omega_0){\cdot}(l_0 \cap l^+(\omega_0)) = U(t, \omega_0){\cdot}l_0 \cap l^+(\omega_0{\cdot}t)$ *and it has dimension* k *for all* $t \in \mathbb{R}$.
(ii) *For all* $(\omega, l) \in \mathcal{O}(\omega_0, l_0)$, $\dim\big(l \cap l^+(\omega)\big) = k$ *and* $\dim\,(l \cap l^-(\omega)) = n - k$. *In particular, if* $k = 0$, *then* $l = l^-(\omega)$ *for all* $(\omega, l) \in \mathcal{O}(\omega_0, l_0)$.
(iii) *The sets*

$$\mathcal{O}^+(\omega_0, l_0) = \{(\omega, l \cap l^+(\omega)) \mid (\omega, l) \in \mathcal{O}(\omega_0, l_0)\} \subset \Omega \times \mathcal{G}_k(\mathbb{K}^{2n}),$$

$$\mathcal{O}^-(\omega_0, l_0) = \{(\omega, l \cap l^-(\omega)) \mid (\omega, l) \in \mathcal{O}(\omega_0, l_0)\} \subset \Omega \times \mathcal{G}_{n-k}(\mathbb{K}^{2n}),$$

are respectively invariant for the flows τ_k *and* τ_{n-k} *on* $\Omega \times \mathcal{G}_k(\mathbb{K}^{2n})$ *and* $\Omega \times \mathcal{G}_{n-k}(\mathbb{K}^{2n})$.
(iv) *The set* $\mathcal{O}^+(\omega_0, l_0)$ *is the omega-limit set of* $(\omega_0, l_0 \cap l^+(\omega_0))$ *for the flow* τ_k *on* $\Omega \times \mathcal{G}_k(\mathbb{K}^{2n})$.

And the analogous results hold for the alpha-limit sets in the case that $\dim\,(l_0 \cap l^-(\omega_0)) = k$.

Proof (i) These properties follow trivially from the τ-invariance of the closed subbundle L^+, since $U(t, \omega_0)$ defines an automorphism on $\mathcal{L}_{\mathbb{R}}$ for all $t \in \mathbb{R}$.

(ii) Take a basis $\{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ of l_0 with $\mathbf{z}_j^-(\omega_0) = \mathbf{0}$ exactly for $j \in \{1, \dots, k\}$ (and for none of them if $k = 0$) and take a sequence $(t_m) \uparrow \infty$ with $\lim_{m\to\infty}(\omega_0{\cdot}t_m, U(t_m, \omega_0){\cdot}l_0) = (\omega, l)$ for which there exist the limits

$$\lim_{m\to\infty} U(t_m, \omega_0){\cdot}\langle \mathbf{z}_1, \dots, \mathbf{z}_k\rangle \quad \text{in } \mathcal{G}_k(\mathbb{K}^{2n}),$$

$$\lim_{m\to\infty} U(t_m, \omega_0){\cdot}\langle \mathbf{z}_{k+1}, \dots, \mathbf{z}_n\rangle \quad \text{in } \mathcal{G}_{n-k}(\mathbb{K}^{2n}).$$

Then conditions (1.44) hold, and Proposition 1.96(ii) implies the first assertion in (ii). The second one is obvious.

(iii) & (iv) These last properties are trivial consequences of (ii), (i), and the definition (1.6) of the flows τ_k and τ_{n-k}.

The last assertion is proved using Proposition 1.96(iii).

Corollary 1.98 *Suppose that the family* (1.11) *has exponential dichotomy over* Ω. *Let* $\mathcal{K} \subseteq \mathcal{K}_{\mathbb{K}}$ *be a* τ*-minimal set. Then,*

(i) *there exists an integer* $k \in \{0, \dots, n\}$ *such that* $\dim\big(l^+(\omega) \cap l\big) = k$ *and* $\dim\,(l^-(\omega) \cap l) = n - k$ *for every* $(\omega, l) \in \mathcal{K}$. *In particular, if* Ω *is minimal,*

$k = 0$ (resp. $k = n$) if and only if $\mathcal{K}$ is the copy of the base given by the graph of $l^-: \Omega \to \mathcal{L}_{\mathbb{K}}$ (resp. $l^+: \Omega \to \mathcal{L}_{\mathbb{K}}$).

(ii) *The sets $\mathcal{K}^+$ and $\mathcal{K}^-$ given by $\mathcal{K}^\pm = \{(\omega, l \cap l^\pm(\omega)) \mid (\omega, l) \in \mathcal{K}\}$ and $\mathcal{K}^- = \{(\omega, l \cap l^-(\omega)) \mid (\omega, l) \in \mathcal{K}\}$ are minimal for the corresponding flows τ_k on $\Omega \times \mathcal{G}_k(\mathbb{K}^{2n})$ and τ_{n-k} on $\Omega \times \mathcal{G}_{n-k}(\mathbb{K}^{2n})$.*

Proof Since any minimal set is the alpha-limit and the omega-limit of each of its points, these properties follow immediately from Proposition 1.97.

Fix $\omega_0 \in \Omega$ and let $\mathcal{O}(\omega_0) \subseteq \Omega$ be its omega-limit set in the base flow. Proposition 1.97(ii) ensures that, in the case that l_0 and $l^+(\omega_0)$ are supplementary, the omega-limit set $\mathcal{O}(\omega_0, l_0)$ for the flow τ is the graph of the continuous map $\mathcal{O}(\omega_0) \to \mathcal{L}_{\mathbb{K}}\,,\ \omega \mapsto l^-(\omega)$. Consequently, the dynamics on $\mathcal{O}(\omega_0, l_0)$ reproduces that of $\mathcal{O}(\omega_0)$. In particular, if Ω is σ-minimal, $\mathcal{O}(\omega_0, l_0)$ is a copy of the base for the flow τ on $\mathcal{K}_{\mathbb{K}}$: see Definition 1.17. But, as the following example shows, in general an omega-limit set can be “very large” and with highly complex dynamics, even in the case of a minimal base.

Example 1.99 Bjerklov and Johnson [17] give examples of two-dimensional systems $\tilde{\mathbf{x}}' = A(\omega{\cdot}t)\,\tilde{\mathbf{x}}$ with A continuous on an almost periodic base Ω for which $\Omega \times \mathcal{G}_1(\mathbb{R}^2)$ is minimal and chaotic in the sense of Li–Yorke. In fact their results are obtained for the real projective flow, and obviously $\mathcal{G}_1(\mathbb{R}^2)$ can be identified with the real projective line (and also with the set of Lagrange planes of $\mathbb{R}^2$). By taking λ large enough, the system

$$\mathbf{x}' = (A(\omega{\cdot}t) - \lambda I_2)\,\mathbf{x} \tag{1.46}$$

has exponential dichotomy: all its orbits tend exponentially to zero as $t \to \infty$. And the dynamics on $\Omega \times \mathcal{G}_1(\mathbb{R}^2)$ is the same: this new system comes from the initial one by taking $\mathbf{x} = e^{-\lambda t}\tilde{\mathbf{x}}$, so that the projective coordinate $m = x_2/x_1$ does not change. Note also that $\Omega \times \mathcal{G}_1(\mathbb{R}^2)$ is minimal and Li–Yorke chaotic for

$$\mathbf{y}' = (-A^T(\omega{\cdot}t) + \lambda I_2)\,\mathbf{y}\,, \tag{1.47}$$

since $(\omega, m) \to (\omega, -1/m)$ takes this projective flow to that for (1.46). Consider now the family of four-dimensional Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} A(\omega{\cdot}t) - \lambda I_2 & 0_2 \\ 0_2 & -A^T(\omega{\cdot}t) + \lambda I_2 \end{bmatrix} \mathbf{z}\,.$$

Clearly this system has exponential dichotomy over Ω, with $l^+(\omega) \equiv \left[\begin{smallmatrix} I_2 \\ 0_2 \end{smallmatrix}\right]$ and $l^-(\omega) \equiv \left[\begin{smallmatrix} 0_2 \\ I_2 \end{smallmatrix}\right]$ for all $\omega \in \Omega$. Take $\omega_0 \in \Omega$ and a Lagrange plane $l_0 \subset \mathbb{R}^4$ with basis $\left\{\left[\begin{smallmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{y}_2 \end{smallmatrix}\right]\right\}$. Note that once $\mathbf{x}_1$ is fixed, it must be the case that $\mathbf{x}_1^T\mathbf{y}_2 = 0$, so that $\mathbf{y}_2$ is unique up to a constant multiple. In other words, there is a unique Lagrange plane with this type of basis for each given direction $\mathbf{x}_1 \in \mathbb{R}^2$. For all $t \in \mathbb{R}$, $U(t, \omega_0){\cdot}l_0$ has the basis $\left\{\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{y}_2(t) \end{bmatrix}\right\}$, where $\mathbf{x}_1(t)$ and $\mathbf{y}_2(t)$ solve the systems (1.46) and (1.47) corresponding to ω_0 with initial data $\mathbf{x}_1$ and $\mathbf{y}_2$ respectively. Therefore, any element (ω, l) of $\mathcal{O}(\omega_0, l_0)$ has a basis of the form $\left\{\begin{bmatrix} \mathbf{x} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}\right\}$. Given such an element, let $m_+(\omega, l)$ and $m_-(\omega, l)$ represent the real projective lines corresponding to $\mathbf{x}$ and $\mathbf{y}$. Note also that the real line through the origin represented by $m_+(\omega, l)$ (resp. $m_-(\omega, l)$) is in fact the intersection $l \cap l^+(\omega)$ (resp. $l \cap l^-(\omega)$). In other words, $\mathcal{O}^{\pm}(\omega_0, l_0) \equiv \{(\omega, m_{\pm}(\omega, l)) \mid (\omega, l) \in \mathcal{O}(\omega_0, l_0)\}$, where $\mathcal{O}^{\pm}(\omega_0, l_0)$ are defined in Proposition 1.97(iii). One concludes that the maps

$$\mathcal{O}^{\pm}(\omega_0, l_0) \to \Omega \times \mathcal{G}_1(\mathbb{R}^2), \quad (\omega, l \cap l^{\pm}(\omega)) \mapsto (\omega, m_{\pm}(\omega, l))$$

are flow isomorphisms. Hence, $\mathcal{O}^{\pm}(\omega_0, l_0)$ are homeomorphic to $\Omega \times \mathcal{G}_1(\mathbb{R}^2)$, minimal for the flow τ_1, and the dynamics on them is Li–Yorke chaotic. And, since as explained above, $m_-(\omega, l)$ is uniquely determined by $m_+(\omega, l)$, it is also the case that $\mathcal{O}(\omega_0, l_0)$ is homeomorphic to $\Omega \times \mathcal{G}_1(\mathbb{R}^2)$. This fact indicates the complexity of the dynamics on the omega-limit set. Note also that $\mathcal{O}(\omega_0, l_0)$ is far away from being a copy of the base. In fact, for each $\omega \in \Omega$ and each $m_1 \in \mathcal{G}_1(\mathbb{R}^2)$ there is exactly one point $(\omega, m_1, m_2) \in \mathcal{O}(\omega_0, l_0)$.

Chapter 2
The Rotation Number and the Lyapunov Index for Real Nonautonomous Linear Hamiltonian Systems

Let (Ω, σ) be a real continuous flow on a compact metric space. The goal of this chapter is to introduce and analyze two objects, the rotation number and the Lyapunov index, associated to almost every linear Hamiltonian system of the family

$$\mathbf{z}' = H(\omega{\cdot}t)\, \mathbf{z}\,, \quad \omega \in \Omega\,. \tag{2.1}$$

The words "almost every" refer to an arbitrarily fixed σ-ergodic measure on Ω, and the matrix-valued function $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right] \in \mathfrak{sp}(n, \mathbb{R})$ is supposed to satisfy the conditions described in Proposition 1.38. That is, the following hypotheses will be in force throughout the chapter:

Hypotheses 2.1 The Borel measurable function $H: \Omega \to \mathfrak{sp}(n, \mathbb{R})$ satisfies:

- $\sup_{\omega\in\Omega} \|H_\omega\|_\infty < \infty$, where $H_\omega(s) = H(\omega{\cdot}s)$,
- the map $\Omega \to \mathbb{R}^{2n}$, $\omega \mapsto \int_{\mathbb{R}} H(\omega{\cdot}t)\, \mathbf{z}(t)\, dt$ is continuous for every L^1-function $\mathbf{z}: \mathbb{R} \to \mathbb{R}^{2n}$.

Recall that these conditions are fulfilled if H is a continuous function. Proposition 1.38 and Remark 1.40 ensure the continuity of the flows $\tau_{\mathbb{C}}$ and $\tau_{\mathbb{R}}$ induced by (2.1) on $\Omega \times \mathbb{C}^{2n}$ and $\Omega \times \mathbb{R}^{2n}$, which are defined by (1.13); and of the flow τ on $\mathcal{K}_{\mathbb{K}} = \Omega \times \mathcal{L}_{\mathbb{K}}$, which is defined by (1.14) (both for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$). This fact, as well as several of the properties stated in the previous chapter, will be fundamental in what follows. Recall that the family of n-dimensional Schrödinger linear equations

$$-\mathbf{x}'' + G(\omega{\cdot}t)\, \mathbf{x} = \mathbf{0}\,, \quad \omega \in \Omega\,, \tag{2.2}$$

is included in the above setting by taking $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n \end{smallmatrix}\right]$.

The following paragraphs contain a general overview of the results of this chapter and their relation with the various other topics taken up in the rest of the book.

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As explained in Sect. 1.3.3, due to the Hamiltonian character of the system (2.1), an initial symplectic matrix determines a symplectic matrix solution. It is known that the real symplectic group $\mathrm{Sp}(n,\mathbb{R})$ can be identified with a solid torus. In other words, it is homeomorphic to the topological product of a simply connected space and the unit circle $\mathbb{S}^1$. This important property was proved in [53] by Gel'fand and Lidskiĭ to characterize stability regions of linear periodic Hamiltonian systems. The position of the projection of a symplectic matrix over $\mathbb{S}^1$ determines an angle, called by them the argument of the matrix.

This concept is the starting point for Yakubovich's generalization of the Sturm theory for two-dimensional systems to linear periodic linear Hamiltonian systems of higher dimension

$$\mathbf{z}' = H_0(t)\,\mathbf{z}\,, \tag{2.3}$$

which is based on geometrical methods, in contrast to the analytical methods previously used by different authors. In [153, 154, 156], Yakubovich identified the oscillatory character of the periodic linear Hamiltonian system (2.3) with the property that the argument along the curve determined by a symplectic matrix solution of the system has an unbounded increment.

Somewhat later, V. Arnold [8] introduced his argument function on the manifold of Lagrange planes in $\mathbb{R}^{2n}$ and used it to study the Maslov index. This argument function can also be used to study oscillation problems for (2.3), as pointed out by Arnold himself in [9].

The argument functions of Yakubovich and Arnold can be put to use in a nonautonomous context, corresponding to the family of linear Hamiltonian systems (2.1). More precisely, let m_0 be a σ-ergodic measure on Ω. Johnson defines in [72] an m_0-dependent rotation number for the family (2.1) in terms of the time-average of the Arnold argument (see also Ruelle [130] for a related construction). This definition shows that the rotation number measures the average oscillation of the solutions of (2.1).

In the same paper, Johnson gives a definition of an analytic nature of the rotation number, which is based on the idea of average rotation due to the action of the Hamiltonian on the generalized unit disc. The analytic nature of this definition for real values of the parameter suggests a natural way to extend it to the complex plane, a question which will be central in Chap. 3.

Later, Novo et al. [112] defined the rotation number in a different way, based on the Yakubovich argument functions, which requires the polar coordinates on the symplectic group described by Barret [12] and Reid [123, 125] (see Sect. 1.3.4). These coordinates turn out to be an appropriate tool to study the flow induced by (2.1) in the Lagrangian bundles and to derive assertions concerning the ergodic limit which defines the rotation number, which in particular admits an ergodic representation in terms of these flows.

All these different ways to define the rotation number give rise to exactly the same object, which in addition constitutes a generalization of the well-known rotation number for two-dimensional systems: see Johnson and Moser [81].

In the description of the dynamics of the two-dimensional case, the Lyapunov exponent also plays a fundamental role: in fact the rotation number and the Lyapunov exponent are the main tools used in extending the Floquet theory for periodic systems to the nonautonomous setting. Also in the higher dimensional case it is interesting to find a single quantity which, roughly speaking, can play the role of the positive Lyapunov exponent in the analysis of the Lagrangian flow. This object is the Lyapunov index, defined as the sum of the positive Lyapunov exponents of the family (2.1) (always with respect to m_0). It is shown in [112] that the Lyapunov index also admits an ergodic representation in terms of the polar symplectic coordinates.

The rotation number and the Lyapunov index provide useful information about the behavior of (2.1). For example, it is a basic fact that, for certain one-parameter families of nonautonomous linear Hamiltonian systems, the constancy of the rotation number when the parameter varies in an open interval is equivalent to the occurrence of exponential dichotomy for the corresponding family of systems. These families are often referred to as *Atkinson spectral problems*, because their basic theory was worked out in Chapter 9 of Atkinson [5]. The details will be given in Chap. 3. The characteristics and differentiability properties of the rotation number, related to certain properties of the Lyapunov index, are also fundamental tools used to describe the limiting behavior on the real axis of the Weyl matrices associated to an Atkinson spectral problem. These results, which constitute a generalization of the classical Kotani theory, are written down and proved in Chap. 4. The Weyl functions can be used to analyze disconjugate linear Hamiltonian systems, as will be explained in Chap. 5.

The relation between the rotation number and the exponential dichotomy concept has been used to good effect in control theory, in the context of the nonautonomous linear regulator problem and the nonautonomous feedback control problem on the semi-infinite interval $[0,\infty)$. These matter will be described in Chap. 6. And it also turns out that the concepts of rotation number and exponential dichotomy permit a direct generalization of the Yakubovich Frequency Theorem from periodic control systems to general nonautonomous systems (2.1) with bounded measurable coefficients, as will be shown in Chap. 7.

Throughout this chapter, m_0 will represent a fixed σ-ergodic measure on Ω. The existence of such a measure is guaranteed by Theorem 1.9. The different approaches to the concept of the rotation number with respect to m_0 for the family (2.1) are worked out in Sect. 2.1.

A strong property of continuous variation of the rotation number for m_0 with respect to the coefficient matrix H is the main result of the second section. Of course, different choices of the ergodic measure m_0 may give rise to different values for the rotation number. In fact, in the case that the value is independent of the choice of the measure (which is of course the case if the base flow is uniquely ergodic), and the coefficient matrix H is continuous on the base, it will be shown that the rotation number can be obtained by taking as starting point any single system of the family. This result, which completes Sect. 2.2, is especially interesting in the case that the family (2.1) derives from a single nonautonomous linear Hamiltonian system via the Bebutov construction, as described in Sect. 1.3.2 (see also Remark 1.40): in general, there is no way to guarantee that the set of definition of the rotation number for m_0 includes this initial system; but this is the case if the Bebutov hull is uniquely ergodic. For instance, this is true if the starting point is a single linear Hamiltonian initial system given by a Bohr almost periodic matrix.

In Sect. 2.3, the Schwarzmann homomorphism defined by the flow on Ω is used to prove that the values of α are "quantized" in the set of coefficient matrices for which the family (2.1) has exponential dichotomy over Ω; and it is explained that, as a by-product of this fact, Yakubovich's discussion of stability zones for periodic Hamiltonian systems [157] can be extended to the general nonautonomous case. These first three sections reproduce basically the survey [45] of Fabbri et al. on the rotation number, which is in turn based on the previous works [72] and [112].

Section 2.4, first, establishes a condition on the coefficient matrix, namely $H_3 \geq 0$ (which is very common in the linear Hamiltonian systems appearing in the applications and is fulfilled always in the Schrödinger case), which suffices to ensure that the rotation number is nonnegative for all the ergodic measures on the base; second, it contains the proofs of some monotonicity properties of the rotation number; and third, it gives a new definition of the rotation number. These results are based on facts previously proved by Yakubovich [153, 154], Lidskiĭ [96], and Gel'fand and Lidskiĭ [53].

The fifth and last section concerns the definition and basic properties of the Lyapunov index of (2.1) with respect to m_0. Among these are the ergodic representation mentioned above, which is now extended to the measurable setting considered here. A brief reminder of the most basic facts of Oseledet's multiplicative ergodic theorem is given, and some particularities arising in the Hamiltonian case, which are fundamental for the proofs of the main results of Sect. 2.5, are carefully explained. The proof of the upper semicontinuity of the Lyapunov index with respect to the coefficient matrix of the nonautonomous Hamiltonian system completes the section.

2.1 Several Ways to Define the Rotation Number

The rotation number for the family of linear Hamiltonian systems of general dimension (2.1) admits different definitions, which extend those previously known in the two-dimensional case. In this section these approaches are explained, their equivalence is established, and an ergodic representation for the rotation number is provided. It will be seen later that each of these definitions is convenient for different purposes.

2.1.1 In Terms of an Argument on the Real Symplectic Group

The evolution of the argument of a symplectic matrix solution of the linear Hamiltonian system provides a definition for the rotation number.

The well-known definition of the rotation number for two-dimensional systems, together with Yakubovich's identification of the oscillation of a periodic linear Hamiltonian with the unbounded increment of the argument of a symplectic matrix solution, makes it natural to define a rotation number as the mean increment of the argument. This is done in [112] for the higher dimension case, and is explained in this section.

The definitions of an argument function on the group $\mathrm{Sp}(n,\mathbb{R})$ and of equivalence of arguments appear in Yakubovich and Starzhinskii [159]. They are based on a preceding definition, due to Gel'fand and Lidskiĭ [53], which is now explained. Let V be a given real symplectic matrix, and let $\mu_1,\ldots,\mu_n$ be the *eigenvalues of the first type* of the matrix V, repeated according to their multiplicities; i.e. those eigenvalues with modulus less than 1 or those with modulus 1 for which any corresponding eigenvector $\mathbf{v}$ satisfies $i\mathbf{v}^*J\mathbf{v}>0$ (see [159], Chapter III, Sections 1.2 and 2.7). Let arg stand for a fixed branch of the usual argument of a complex number. Define

$$\mathrm{Arg}_*\colon \mathrm{Sp}(n,\mathbb{R})\to\mathbb{R}\,,\qquad V\mapsto\sum_{j=1}^{n}\arg\mu_j\,. \tag{2.4}$$

It is known that $\mu_1,\ldots,\mu_n\colon\mathrm{Sp}(n,\mathbb{R})\to\mathbb{C}$ (when conveniently ordered) vary continuously (see [159], Chapter III, Section 2.10). Therefore Arg_* is a continuous multivalued function of V. Clearly, if $(\mathrm{Arg}_*V)_0$ is one of the values of Arg_*V, the other values are $(\mathrm{Arg}_*V)_m=(\mathrm{Arg}_*V)_0+2m\pi$, for $m\in\mathbb{Z}$, and each branch is a continuous function. In addition, if $V(t)$ is a continuous curve on $\mathrm{Sp}(n,\mathbb{R})$, and $\mu_1(t),\ldots,\mu_n(t)$ are the eigenvalues of the first type of $V(t)$, defined by continuity, then the argument increment $\varDelta\,\mathrm{Arg}_*V(t)|_{t=t_1}^{t=t_2}=\mathrm{Arg}_*V(t_2)-\mathrm{Arg}_*V(t_1)$ is independent of the choice of the branch: it is a continuous single-valued function. Finally, if $\mathbb{S}^1$ denotes the unit circle (understood in what follows as the interval $[0,2\pi]$ with endpoints identified), and $V\colon\mathbb{S}^1\to\mathrm{Sp}(n,\mathbb{R})$ is a continuous curve, then there exists $p\in\mathbb{Z}$ such that $\varDelta\,\mathrm{Arg}_*V(t)=\mathrm{Arg}_*V(2\pi)-\mathrm{Arg}_*V(0)=2p\,\pi$. This integer p is the *index* of the curve V.

As pointed out in [154], this definition of argument is difficult to manage; and it is not clear either that the associated concept of oscillation agrees with the usual one for the two-dimensional case, defined in terms of the number of zeros of the solutions, or that it agrees with the other still-to-be introduced definitions for higher dimension. However, according to the results of [153], it is possible to define several different arguments for a symplectic matrix, which are equivalent in a sense explained below, and which can be used clarify these points.

Definition 2.2 An *argument of symplectic matrices* is a countable-valued function $\mathrm{Arg}\colon\mathrm{Sp}(n,\mathbb{R})\to\mathbb{R}$ such that: if $(\mathrm{Arg}\,V)_0$ is any value of $\mathrm{Arg}\,V$, then the other ones are

$$(\operatorname{Arg} V)_m = (\operatorname{Arg} V)_0 + 2m\pi\,, \quad m \in \mathbb{Z}\,;$$

each of the different branches is a continuous function; and there exists a continuous curve $V: \mathbb{S}^1 \to \mathrm{Sp}(n, \mathbb{R})$ of index 1 with $\Delta \operatorname{Arg} V(t) = 2\pi$.

Note that any of the branches $(\operatorname{Arg} V)_m$ can be chosen to define the argument increment, due to the indicated relation between them.

Definition 2.3 Two argument functions Arg^i and Arg^{ii} are *equivalent* if there exists a uniform constant $c > 0$ such that, for any interval $[t_1, t_2] \subset \mathbb{R}$ and for any continuous curve $V: [t_1, t_2] \to \mathrm{Sp}(n, \mathbb{R})$, the inequality

$$\left|\Delta \operatorname{Arg}^i V(t)|_{t_1}^{t_2} - \Delta \operatorname{Arg}^{ii} V(t)|_{t_1}^{t_2}\right| < c$$

is satisfied. Here a continuous branch of each argument is taken along the curve.

The existence of several different argument functions which are equivalent to Arg_*, as well as the existence of non-equivalent arguments, is proved in [153]. Among these arguments, those listed below, which are equivalent to Arg_*, will play a role in proving different properties of the rotation number. Here, $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ is a real symplectic matrix, and from now on arg stands for the fixed branch of the argument satisfying $\arg(1) = 0$.

- The functions

$$\begin{aligned}
\operatorname{Arg}_1 V &= \arg\det(V_1 - iV_2)\,,\\
\operatorname{Arg}_2 V &= \arg\det(V_3 - iV_4)\,,\\
\operatorname{Arg}_3 V &= \arg\det(V_1 + iV_3)\,,\\
\operatorname{Arg}_4 V &= \arg\det(V_2 + iV_4)\,.
\end{aligned}$$

- The functions

$$\operatorname{Arg}^j_{T,S} V = \operatorname{Arg}_j(TVS)\,,$$

  for $T,\ S \in \mathrm{Sp}(n, \mathbb{R})$ and $j = 1, \ldots, 4$.

In the rest of this section, and unless otherwise indicated, Arg will represent any argument equivalent to Arg_*, for example one of those listed above. Given a real symplectic matrix solution of (2.1), $V(t, \omega) = \left[\begin{smallmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega) \end{smallmatrix}\right]$, define

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg} V(t, \omega)\,, \tag{2.5}$$

where a continuous branch of the argument is taken along the curve.

Recall that m_0 represents a fixed σ-ergodic measure on Ω. It turns out that α is well defined and depends only on the measure m_0. This is proved in the following theorem, one of the main results of [112], which in addition provides an ergodic representation for α in terms of the function $\operatorname{Tr} Q$, defined by (1.19) and closely related to the function Q given by (1.17). As seen in Theorem 1.41, the function Q determines the flow on $\mathcal{K}_\mathbb{R}$ in generalized polar coordinates. Recall that the set of (normalized) τ-invariant measures on $\mathcal{K}_\mathbb{R}$ projecting onto m_0 is nonempty: see Proposition 1.15(i).

Theorem 2.4 *The existence and the value of the limit* (2.5) *are independent of the choices of* Arg *and of* $V(t,\omega)$. *In addition, there is a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ such that the limit exists for every $\omega \in \Omega_0$ and takes the same constant value*

$$\alpha = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} Q(\omega, l)\, d\mu \tag{2.6}$$

for every normalized τ-invariant measure μ on $\mathcal{K}_\mathbb{R}$ projecting onto m_0, where the function $\operatorname{Tr} Q \colon \mathcal{K}_\mathbb{R} \to \mathbb{R}$ *is defined by* (1.19).

Proof The equivalence of two given argument functions Arg^i and Arg^{ii} implies

$$|\operatorname{Arg}^i V(t,\omega) - \operatorname{Arg}^{ii} V(t,\omega)| \le c + |\operatorname{Arg}^i V(0,\omega) - \operatorname{Arg}^{ii} V(0,\omega)|\,,$$

which ensures the independence of α with respect to the particular choice of Arg. On the other hand, according to Yakubovich's results [153], choosing a different symplectic matrix solution of (2.1) (which agrees with $V(t,\omega)\, C$ for a constant real symplectic matrix C) induces the substitution of Arg with an equivalent argument, and hence it affects neither the existence nor the value of α.

Choose now $\operatorname{Arg} = \operatorname{Arg}_1$ and write $\begin{bmatrix} V_1(0,\omega) \\ V_2(0,\omega) \end{bmatrix} = \begin{bmatrix} \Phi_1^0 R^0 \\ \Phi_2^0 R^0 \end{bmatrix}$, with $\Phi_1^0 + i\Phi_2^0$ unitary and $\det R^0 > 0$. Theorem 1.41 ensures that $\begin{bmatrix} V_1(t,\omega) \\ V_2(t,\omega) \end{bmatrix} = \begin{bmatrix} \Phi_1(t,\omega) R(t,\omega) \\ \Phi_2(t,\omega) R(t,\omega) \end{bmatrix}$, where $\begin{bmatrix} \Phi_1(t,\omega) \\ \Phi_2(t,\omega) \end{bmatrix}$ and $R(t,\omega)$ are the solutions of (1.15) and (1.16) with initial values $\begin{bmatrix} \Phi_1^0 \\ \Phi_2^0 \end{bmatrix}$ and R^0, and $\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega))$ has modulus 1. Clearly, $\det R(t,\omega) > 0$ for every $t \in \mathbb{R}$. These facts and the definition of Arg_1 imply

$$\operatorname{Arg}_1 V(t,\omega) = \arg\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega)) = -i \ln\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega))\,.$$

In addition,

$$(\Phi_1(t,\omega) - i\Phi_2(t,\omega))' = i\,(\Phi_1(t,\omega) - i\Phi_2(t,\omega))\, Q(\omega{\cdot}t, \Phi_1(t,\omega), \Phi_2(t,\omega))\,,$$

and hence, by the Liouville formula and the definition of $\operatorname{Tr} Q$,

$$\begin{aligned} \operatorname{Arg}_1 V(t,\omega) - \operatorname{Arg}_1 V(0,\omega) &= \int_0^t \operatorname{tr} Q(\omega{\cdot}s, \Phi_1(s,\omega), \Phi_2(s,\omega))\, ds \\ &= \int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\, ds \end{aligned} \tag{2.7}$$

for $l \equiv \begin{bmatrix} \Phi_1^0 \\ \Phi_2^0 \end{bmatrix}$. Consequently,

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_1 V(t,\omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\, ds\,. \tag{2.8}$$

Note that the hypotheses on H ensure that the scalar function $\operatorname{Tr} Q$ belongs to $L^1(\mathcal{K}_\mathbb{R},\mu)$ for all the τ-invariant normalized measures μ on $\mathcal{K}_\mathbb{R}$, as is easily deduced from Remark 1.39. Fix a τ-invariant measure μ projecting onto m_0. Birkhoff's ergodic theorem (see Theorem 1.3) provides a function $\tilde q_\mu \in L^1(\mathcal{K}_\mathbb{R},\mu)$ defined on a τ-invariant set $\mathcal{K}_\mu$ with $\mu(\mathcal{K}_\mu) = 1$, with $\tilde q_\mu(\tau(t,\omega,l)) = \tilde q_\mu(\omega,l)$ for all $(\omega,l) \in \mathcal{K}_\mu$ and $t \in \mathbb{R}$, with $\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} Q(\omega,l)\, d\mu = \int_{\mathcal{K}_\mathbb{R}} \tilde q_\mu(\omega,l)\, d\mu$, and such that the previous limit exists and agrees with $\tilde q_\mu(\omega,l)$; and the already known independence of the limit with respect to l ensures that $\tilde q_\mu$ only depends on the element ω of the base space: that is, $\mathcal{K}_\mu = \Omega_\mu \times \mathcal{L}_\mathbb{R}$, with $m_0(\Omega_\mu) = 1$, and there exists a function $q_\mu^*\colon \Omega \to \mathbb{R}$ with $\tilde q_\mu(\omega,l) = q_\mu^*(\omega)$ for all $\omega \in \Omega_\mu$ and $l \in \mathcal{L}_\mathbb{R}$. Summing up, one has

$$\begin{aligned} \exists \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_1 V(t,\omega) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\, ds \\ &= \tilde q_\mu(\omega,l) = q_\mu^*(\omega) \end{aligned} \tag{2.9}$$

for m_0-a.e. $\omega \in \Omega$ and every $l \in \mathcal{L}_\mathbb{R}$. In addition, $q_\mu^*(\omega) = q_\mu^*(\omega{\cdot}t)$ for all $\omega \in \Omega_\mu$ and $t \in \mathbb{R}$, so that according to Theorem 1.6, the ergodicity of m_0 guarantees that $q_\mu^*(\omega)$ takes on a constant value α_μ for m_0-a.e. $\omega \in \Omega$: that is,

$$\alpha_\mu = \int_\Omega q_\mu^*(\omega)\, dm_0 = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} Q(\omega,l)\, d\mu = q_\mu^*(\omega)\,, \tag{2.10}$$

where the last equality holds for m_0-a.e. $\omega \in \Omega$. Now take any other τ-invariant measure μ^* projecting onto m_0, and repeat the previous reasoning. It follows from (2.9) that $q_\mu^*(\omega) = q_{\mu^*}^*(\omega)$ for m_0-a.e. $\omega \in \Omega$, and hence (2.10) yields $\alpha_\mu = \alpha_{\mu^*}$. In other words, the quantity α_μ is independent of the choice of the measure. Therefore, (2.10), (2.9), and the already known independence of α_μ with respect to the choices of Arg_1 and V, are sufficient to prove the theorem.

Definition 2.5 The *rotation number* of the family of linear Hamiltonian systems (2.1) *with respect to* m_0 is the (m_0-a.e. constant) value of the limit (2.5).

Remark 2.6 The result is identical, and the quantity α is the same, by defining

$$\alpha = \lim_{t\to-\infty} \frac{1}{t} \operatorname{Arg} V(t,\omega)\,.$$

This fact allows one to derive similar relations for the several expressions of α obtained in the rest of the chapter.

Remark 2.7 Note that the definition (2.5) and the representation (2.6) extend the definition and the representation of the rotation number for two-dimensional systems, introduced for the almost periodic Schrödinger case $-x'' + g(\omega{\cdot}t)\,x = 0$ by Johnson and Moser in [73] and extended to the general case

$$\mathbf{z}' = \begin{bmatrix} h_1(\omega{\cdot}t) & h_3(\omega{\cdot}t) \\ h_2(\omega{\cdot}t) & -h_1(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega \tag{2.11}$$

by Giachetti and Johnson in [55]: the introduction of the real projective coordinate $\varphi = \arg(z_1 - iz_2)$ leads to the equation $\varphi' = f(\omega{\cdot}t, \varphi)$, with

$$\begin{aligned} f(\omega,\varphi) &= -h_2(\omega)\cos^2\varphi + h_3(\omega)\sin^2\varphi + 2h_1(\omega)\sin\varphi\cos\varphi \\ &= \begin{bmatrix} \cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_1(\omega{\cdot}t) & h_3(\omega{\cdot}t) \\ h_2(\omega{\cdot}t) & -h_1(\omega{\cdot}t) \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}; \end{aligned}$$

and for all the solutions $\varphi(t,\omega)$ of this equation (m_0-a.e.),

$$\alpha = \lim_{t\to\infty} \frac{1}{t}\,\varphi(t,\omega) = \int_{\Omega\times P^1(\mathbb{R})} f(\omega,\varphi)\,d\mu\,,$$

where μ is any invariant measure for the corresponding projective flow which projects onto m_0. Note that any nonzero solution $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ of (2.11) is the first column of a symplectic matrix solution $V(t,\omega) = \left[\begin{smallmatrix} z_1 & w_1 \\ z_2 & w_2 \end{smallmatrix}\right]$ (it is enough to take $w_1(0) = z_2(0)/(z_1^2(0) + z_2^2(0))$ and $w_2(0) = -z_1(0)/(z_1^2(0) + z_2^2(0))$), and that $\varphi(t,\omega) = \operatorname{Arg}_2(t,\omega)$.

Remark 2.8 Consider the Hamiltonian system $\mathbf{z}' = H_0\,\mathbf{z}$ given by a real constant matrix H_0. The eigenvalues of $e^{H_0 t}$ are $\mu(t) = e^{\beta t + i\alpha t}$ for the eigenvalues $\beta + i\alpha$ of H_0. It is easy to check that for $t > 0$, the fact that an eigenvalue is of the first type is independent of the value of t. This is in particular what happens if $\beta < 0$, in which case also $\bar{\mu}(t) = e^{\beta t - i\alpha t}$ is an eigenvalue of the first type. Now use Arg_* to define the rotation number of the system. Clearly, if $\beta < 0$, the sum of the arguments of the eigenvalues $\mu(t)$ and $\bar{\mu}(t)$ does not contribute to the value of the limit defining α: only the eigenvalues of the first type lying on the unit circle must be taken into account. Consequently, the choice of this argument to obtain α shows that the rotation number for the linear Hamiltonian system $\mathbf{z}' = H_0\,\mathbf{z}$ agrees with the sum of the imaginary parts $\alpha_1, \ldots, \alpha_s$ of those eigenvalues of H_0 which are purely imaginary (if they exist) and which give rise to eigenvalues of the first type of e^{H_0}. (In fact $\mathrm{Arg}_* \, e^{H_0 t}$ and $(\alpha_1 + \cdots + \alpha_s)\, t$ agree modulo 2π.) This is what one could reasonably expect (see Arnold and San Martin [7]). An analogous statement can be formulated in the periodic case, using now the characteristic exponents of the system.

2.1.2 Two Analytic Definitions

The idea of average rotation due to the action of the Hamiltonian on the generalized unit disc gives rise to a new definition for the rotation number. This definition, which was formulated and analyzed previous to the one described in the previous section, is given in [72]. In this paper, the Floquet coefficient for a one-parameter family of nonautonomous linear Hamiltonian systems is introduced and its relation with the Weyl matrices and with certain spectral problems is described. As already mentioned, this definition of the rotation number for real values of the parameter suggests a natural way to extend it to the complex plane, and this question will of fundamental significance in Chap. 3.

The framework of the problem considered in [72] is more general, including the linear Hamiltonian families (2.1) as a particular case: a rotation number is defined for nonautonomous linear systems whose coefficient matrices lie in the Lie algebra $\mathfrak{u}(p,q) = \{H \in \mathbb{M}_{(p+q)\times(p+q)}(\mathbb{C}) \mid H^* J_0 + J_0 H = 0_{p+q}\}$, where $J_0 = \left[\begin{smallmatrix} -I_p & 0 \\ 0 & I_q \end{smallmatrix}\right]$, with $p \geq 1$ and $q \geq 1$. The use of the Iwasawa decompositions of this Lie algebra and the corresponding Lie group $\mathrm{U}(p,q) = \{V \in \mathbb{M}_{(p+q)\times(p+q)}(\mathbb{C}) \mid V^* J_0 V = J_0\}$ allows one to prove that the rotation number is well defined, to work out some of its properties, and to explain its geometrical significance.

Returning to the linear Hamiltonian setting, the symplectic Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ can be mapped diffeomorphically onto $\mathfrak{u}(n,n) \cap \mathfrak{sp}(n,\mathbb{C}) \subset \mathfrak{u}(n,n)$ via the map $H \mapsto \widetilde{H} = K^{-1} H K$, where $K = \left[\begin{smallmatrix} iI_n & iI_n \\ -I_n & I_n \end{smallmatrix}\right]$: a direct computation proves that

$$\widetilde{H} = \frac{1}{2} \begin{bmatrix} H_1 - H_1^T - i(H_2 - H_3) & H_1 + H_1^T - i(H_2 + H_3) \\ H_1 + H_1^T + i(H_2 + H_3) & H_1 - H_1^T + i(H_2 - H_3) \end{bmatrix}, \tag{2.12}$$

and hence that $(J\widetilde{H})^T = J\widetilde{H}$, so that $\widetilde{H} \in \mathfrak{sp}(n,\mathbb{C})$, and also that $(J_0\widetilde{H})^* = -J_0\widetilde{H}$, so that $\widetilde{H} \in \mathfrak{u}(n,n)$; and a similar computation proves that the inverse map $\widetilde{H} \mapsto H = K\widetilde{H}K^{-1}$ takes any element of $\mathfrak{u}(n,n) \cap \mathfrak{sp}(n,\mathbb{C})$ to a real symplectic matrix. Therefore, it is possible to define a rotation number for the family (2.1) as the value of the one corresponding to the transform of the coefficient matrix in the new Lie algebra. This is the path followed in [72], which will be summarized below. But it is also possible and simpler to redefine the rotation number directly for the symplectic case using exactly the same construction, as is in fact done in what follows.

The following technical lemma summarizes some properties which will be used often from now on. The Euclidean norms $\|\mathbf{z}\| = (\mathbf{z}^*\mathbf{z})^{1/2}$ in any vector space $\mathbb{C}^m$

and $\|M\| = \max_{\|\mathbf{z}\|=1} \|M\,\mathbf{z}\|$ in any space $\mathbb{M}_{d\times m}(\mathbb{C})$ will be fixed until the end of Sect. 2.1.

Lemma 2.9 *Let $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ belong to* $\mathrm{Sp}(n,\mathbb{C})$ *and satisfy the following property: there is an open domain $\mathcal{O} \subset \mathbb{S}_n(\mathbb{C})$ such that* $\det(V_1+V_3M) \neq 0$ *whenever $M \in \mathcal{O}$. Then the map*

$$\widehat{V}\colon \mathcal{O} \to \mathbb{S}_n(\mathbb{C})\,, \quad M \mapsto \widehat{V}{\cdot}M = (V_2 + V_4M)(V_1 + V_3M)^{-1} \tag{2.13}$$

is well defined, and is differentiable. In addition, the Fréchet derivative of $\widehat{V}$ is given for all $M_0 \in \mathcal{O}$ by

$$d_{M_0}\widehat{V}{\cdot}M = (V_1^T + M_0V_3^T)^{-1}M(V_1 + V_3M_0)^{-1}\,, \tag{2.14}$$

and hence

$$\det d_{M_0}\widehat{V} = (\det(V_1 + V_3M_0))^{-2n}\,. \tag{2.15}$$

Proof According to the results of Sect. 1.2, $V\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] = \left[\begin{smallmatrix} V_1+V_3M \\ V_2+V_4M \end{smallmatrix}\right]$ represents a complex Lagrange plane, so that $(V_1 + V_3M)^T(V_2 + V_4M) = (V_2 + V_4M)^T(V_1 + V_3M)$. This implies that $\widehat{V}{\cdot}M = (V_2 + V_4M)(V_1 + V_3M)^{-1}$ is also symmetric and hence that $\widehat{V}$ is well defined. Clearly, it is continuous and differentiable.

Recall that the Fréchet derivative at M_0 is defined as the continuous linear operator $d_{M_0}\widehat{V}\colon \mathbb{S}_{\mathbb{C}}(n) \to \mathbb{S}_{\mathbb{C}}(n)$ such that

$$\lim_{\|M\|\to 0} \frac{1}{\|M\|}\,\|\widehat{V}{\cdot}(M_0 + M) - \widehat{V}{\cdot}M_0 - d_{M_0}\widehat{V}{\cdot}M\| = 0\,. \tag{2.16}$$

Since $(V_1 + V_3M_0)^{-1}(V_1 + V_3(M_0 + M)) = I_n + (V_1 + V_3M_0)^{-1}V_3M$ (which is a well-defined nonsingular matrix when $M_0 \in \mathcal{O}$ and $\|M\|$ is small enough), one has

$$\begin{aligned}&\widehat{V}{\cdot}(M_0 + M)\\ &\quad = \Big((V_2 + V_4(M_0 + M))\left(I_n + (V_1 + V_3M_0)^{-1}V_3M\right)^{-1}\Big)(V_1 + V_3M_0)^{-1}.\end{aligned}$$

Recall also that $(I_n + A)^{-1} = \sum_{k=0}^{\infty}(-1)^kA^k$ whenever $\|A\| < 1$. Condition (2.16) is then satisfied by

$$\begin{aligned} d_{M_0}\widehat{V}{\cdot}M &= \left(V_4 - (V_2 + V_4M_0)(V_1 + V_3M_0)^{-1}V_3\right)M(V_1 + V_3M_0)^{-1}\\ &= (V_1^T + M_0V_3^T)^{-1}M(V_1 + V_3M_0)^{-1}.\end{aligned}$$

The last equality follows from the symplectic character of V: Proposition 1.23 ensures that $(V_1^TV_4 - V_2^TV_3) + M_0(V_3^TV_4 - V_4^TV_3) = I_n + M_0\,0_n = I_n$ and

$(V_1^T + M_0 V_3^T)^{-1}(V_2^T + M_0 V_4^T) = (V_2 + V_4 M_0)(V_1 + V_3 M_0)^{-1}$. Equality (2.14) is hence proved, and (2.15) is an immediate consequence.

Recall that $\mathbb{S}_n^+(\mathbb{C})$ represents the open subset of the complex symmetric $n \times n$ matrices M such that $\operatorname{Im} M > 0$. The following lemma describes the action of the Lie group $\operatorname{Sp}(n, \mathbb{R})$ on $\mathbb{S}_n^+(\mathbb{C})$.

Lemma 2.10 *Let* $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ *belong to* $\operatorname{Sp}(n, \mathbb{R})$. *The map*

$$\widehat{V}\colon \mathbb{S}_n^+(\mathbb{C}) \to \mathbb{S}_n^+(\mathbb{C})\,, \quad M \mapsto \widehat{V}{\cdot}M = (V_2 + V_4 M)(V_1 + V_3 M)^{-1} \tag{2.17}$$

is a diffeomorphism.

Proof Define $N = (V_2^T + M^* V_4^T)(V_1 + V_3 M)$. The first step of the proof is to show that $\operatorname{Im} N < 0$. In fact, an easy computation yields

$$\begin{aligned} \operatorname{Im} N &= V_2^T V_3 \operatorname{Im} M - \operatorname{Im} M\, V_4^T V_1 - \operatorname{Im} M\, V_4^T V_3 \operatorname{Re} M + \operatorname{Re} M\, V_4^T V_3 \operatorname{Im} M \\ &= -\operatorname{Im} M + (V_1^T + \operatorname{Re} M\, V_3^T) V_4 \operatorname{Im} M - \operatorname{Im} M\, V_4^T (V_1 + V_3 \operatorname{Re} M)\,, \end{aligned}$$

since $V_2^T V_3 = V_1^T V_4 - I_n$ and $V_4^T V_3 = V_3^T V_4$ (see Proposition 1.23). In addition, if R is a real matrix, then $\mathbf{x}^T(R - R^T)\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Therefore, $\mathbf{x}^T \operatorname{Im} N\, \mathbf{x} = -\mathbf{x}^T \operatorname{Im} M\, \mathbf{x} < 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$, and hence $\operatorname{Im} N < 0$, as asserted.

According to Proposition 1.21, N is a nonsingular matrix. Then, in particular, there exists $(V_1 + V_3 M)^{-1}$. Lemma 2.9 ensures that $\widehat{V}{\cdot}M = (V_2 + V_4 M)(V_1 + V_3 M)^{-1}$ is also symmetric. In turn, this ensures that

$$\begin{aligned} \operatorname{Im}(\widehat{V}{\cdot}M) &= \frac{1}{2i}\,(\widehat{V}{\cdot}M - (\widehat{V}{\cdot}M)^*) \\ &= ((V_1 + V_3 M)^{-1})^* \frac{1}{2i}\,(N^* - N)(V_1 + V_3 M)^{-1} \\ &= -((V_1 + V_3 M)^{-1})^* \operatorname{Im} N (V_1 + V_3 M)^{-1} > 0\,. \end{aligned}$$

All these properties imply that $\widehat{V}$ is a well-defined map. It is easy to check that, if U is another real symplectic matrix, then $\widehat{UV} = \widehat{U} \circ \widehat{V}$. In particular there exists $\widehat{V}^{-1} = \widehat{V^{-1}}$, and this fact together with Lemma 2.9 completes the proof.

Recall that $U(t, \omega)$ represents the (real symplectic) fundamental matrix solution of (2.1) with $U(0, \omega) = I_{2n}$. Let $\widehat{U}(t, \omega)\colon \mathbb{S}_n^+(\mathbb{C}) \to \mathbb{S}_n^+(\mathbb{C})$ be the corresponding map defined by (2.17). For $t \in \mathbb{R}$, $\omega \in \Omega$, and $M_0 \in \mathbb{S}_n^+(\mathbb{C})$, let $d_{M_0}\widehat{U}(t, \omega)$ be the Fréchet derivative at the point M_0 of $\widehat{U}(t, \omega)$; i.e. the linear map on $\mathbb{S}_{\mathbb{C}}(n)$ defined by the corresponding expression (2.16). The rotation number of the family (2.1) with respect to m_0 can be defined as

$$\alpha = -\lim_{t\to\infty} \frac{1}{2n}\,\frac{1}{t}\,\arg\det d_{M_0}\widehat{U}(t, \omega)\,, \tag{2.18}$$

where a continuous branch of the argument is taken. Theorem 2.11 guarantees the coincidence of the limits (2.18) and (2.5) for all $M_0 \in \mathbb{S}_n^+(\mathbb{C})$. Consequently, it follows from Theorem 2.4 that this last limit exists m_0-a.e. in Ω and that its value is independent of the choice of $M_0 \in \mathbb{S}_n^+(\mathbb{C})$.

Theorem 2.11 *For all $M_0 \in \mathbb{S}_n^+(\mathbb{C})$, the limits* (2.18) *and* (2.5) *agree.*

Proof According to Lemma 2.9,

$$\det d_{M_0}\widehat{U}(t,\omega) = (\det(U_1(t,\omega) + U_3(t,\omega)M_0))^{-2n}\,,$$

and hence

$$-\frac{1}{2n}\arg\det d_{M_0}\widehat{U}(t,\omega) = \arg\det(U_1(t,\omega) + U_3(t,\omega)M_0)\,. \tag{2.19}$$

Let $\mathrm{Im}^{1/2}M_0$ be the unique positive definite square root of $\mathrm{Im}\,M_0$, and $\mathrm{Im}^{-1/2}M_0$ its inverse (see Proposition 1.19). Note that, since the matrix $C_{M_0} = \left[\begin{smallmatrix} \mathrm{Im}^{-1/2}M_0 & 0 \\ \mathrm{Re}\,M_0\,\mathrm{Im}^{-1/2}M_0 & \mathrm{Im}^{1/2}M_0 \end{smallmatrix}\right]$ is symplectic, Yakubovich's results summarized in Sect. 2.1.1 ensure the equivalence of Arg_3 and the new argument function defined by

$$\mathrm{Arg}^3_{I_{2n},C_{M_0}}\,V = \mathrm{Arg}_3(VC_{M_0})\,.$$

The fact that $\det\mathrm{Im}^{1/2}M_0 > 0$ implies that

$$\begin{aligned}
&\mathrm{Arg}^3_{I_{2n},C_{M_0}}\,U(t,\omega)\\
&\quad= \arg\det\big((U_1(t,\omega) + U_3(t,\omega)\,\mathrm{Re}\,M_0)\,\mathrm{Im}^{-1/2}M_0 + i\,U_3(t,\omega)\,\mathrm{Im}^{1/2}M_0\big)\\
&\quad= \arg\det(U_1(t,\omega) + U_3(t,\omega)M_0)\,,
\end{aligned}$$

which together with (2.19) ensures that the limits (2.18) and (2.5) agree.

Remark 2.12 Take $M_0 \in \mathbb{S}_n^+(\mathbb{C})$. If follows from Lemma 2.10 that the solution $M(t,\omega,M_0)$ of the Riccati equation (1.22) associated to (2.1) with initial datum $M(0,\omega,M_0) = M_0$ is defined for all $t \in \mathbb{R}$: it agrees with the map $\widehat{U}(t,\omega){\cdot}M_0$ defined by (2.17). In addition, considered as a function of t, the map $d_{M_0}\widehat{U}(t,\omega){\cdot}M$ is the solution with initial datum $d_{M_0}\widehat{U}(0,\omega){\cdot}M = M$ of the matrix differential equation

$$(\delta M)' = f(\omega{\cdot}t, M(t,\omega,M_0)){\cdot}\delta M \tag{2.20}$$

given by the variational equation associated to the solution $M(t,\omega,M_0)$ of the Riccati equation (1.22) for the Hamiltonian system (2.1). The expression of $f(M){\cdot}D$

is then given by

$$f(\omega,M)\cdot D = -D\,(H_1(\omega)+H_3(\omega)\,M) - \big(H_1^T(\omega)+M\,H_3(\omega)\big)\,D\,.$$

On the other hand, since $\arg\det V = \operatorname{Im}\ln\det V$, definition (2.18), Theorem 2.11, equation (2.20) and the Liouville formula lead to

$$\begin{aligned}\alpha &= -\lim_{t\to\infty}\frac{1}{2n}\,\frac{1}{t}\,\operatorname{Im}\int_0^t \operatorname{tr} f(\omega{\cdot}s, M(s,\omega,M_0))\,ds\\ &= \lim_{t\to\infty}\frac{1}{t}\,\operatorname{Im}\int_0^t \operatorname{tr}(H_1(\omega{\cdot}s)+H_3(\omega{\cdot}s)\,M(s,\omega,M_0))\,ds\end{aligned} \tag{2.21}$$

m_0-a.e. The last equality is due to the fact that the trace of the linear operator $D\mapsto -DA-A^TD$ is $-2n\operatorname{tr}A$.

Remark 2.13 The last expression in (2.21), which can be taken as a new way to define the rotation number, indicates that the analytic definition given in this section also generalizes one of those previously known for the two-dimensional case. That is, taking the complex projective coordinate $m = z_2/z_1$ associates to (2.11) the Riccati equation

$$m' = h_2(\omega{\cdot}t) - 2h_1(\omega{\cdot}t)\,m - h_3(\omega{\cdot}t)\,m^2\,,$$

whose variational equation associated to a solution $m(t,\omega,m_0)$ with $\operatorname{Im} m_0 > 0$ which is globally defined is $(\delta m)' = (-2h_1(\omega{\cdot}t) - 2h_3(\omega{\cdot}t)\,m(t,\omega,m_0))\,\delta m$. Then,

$$\alpha = \lim_{t\to\infty}\frac{1}{t}\,\operatorname{Im}\int_0^t (h_1(\omega{\cdot}s)+h_3(\omega{\cdot}s)\,m(s,\omega,m_0))\,ds\,.$$

As stated before, the fact that the rotation number is well defined is proved in [72] in a more general framework using a rather different approach, which indicates the geometrical significance of α. It is possible to apply this argument directly to the symplectic case, as explained in what follows, and this will provide the third definition for the rotation number.

The Lie group $\mathrm{Sp}(n,\mathbb{R})$ is embedded into $\mathrm{U}(n,n)\cap\mathrm{Sp}(n,\mathbb{C})\subset\mathrm{U}(n,n)$ via the map $V\mapsto\widetilde{V} = K^{-1}VK$, where $K = \left[\begin{smallmatrix} iI_n & iI_n\\ -I_n & I_n\end{smallmatrix}\right]$: the fact that $K^{-1}VK$ belongs to $\mathrm{U}(n,n)\cap\mathrm{Sp}(n,\mathbb{C})$ follows easily from the equalities $K^TJK = KJK^T = 2iJ$, $KJ_0K^* = -2iJ$ and $K^*JK = 2iJ_0$. Clearly, $\widetilde{U}(t,\omega) = K^{-1}U(t,\omega)K$ is the fundamental matrix solution with value I_{2n} at $t=0$ of the system

$$\tilde{\mathbf{z}}' = \widetilde{H}(\omega{\cdot}t)\,\tilde{\mathbf{z}}\,, \tag{2.22}$$

which is obtained from (2.1) by means of the linear change of variables $\tilde{\mathbf{z}} = K^{-1}\mathbf{z}$ and given by the corresponding matrix $\widetilde{H}(\omega) = K^{-1}H(\omega)K$ defined by (2.12), which, as seen before, belongs to the Lie algebra $\mathfrak{u}(n,n)\cap\mathfrak{sp}(n,\mathbb{C})$.

Now let $\mathbb{D}_{\mathbb{C}}$ represent the open set of the complex symmetric $n \times n$ matrices $\widetilde{M}$ with $I_n - \widetilde{M}^* \widetilde{M} > 0$.

Remark 2.14 Obviously, $\mathbb{D}_{\mathbb{C}}$ agrees with the unit open disk when $n = 1$. In fact $\mathbb{D}_{\mathbb{C}} = \{\widetilde{M} \in \mathbb{S}_n(\mathbb{C}) \mid \|\widetilde{M}\| < 1\}$ for the (Euclidean) norm chosen, as is easily deduced from the fact that $\|\widetilde{M}\|^2$ agrees with the spectral radius $\rho(\widetilde{M}^*\widetilde{M})$ of $\widetilde{M}^*\widetilde{M}$; i.e. with the maximum of the eigenvalues of the matrix $\widetilde{M}^*\widetilde{M}$ (which are all positive). This ensures the existence of $(I_n + \widetilde{M})^{-1}$ and $(I_n - \widetilde{M})^{-1}$ whenever $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$. In addition, the set $\mathbb{D}_{\mathbb{C}}$ is convex; i.e. $\lambda M_1 + (1-\lambda) M_2 \in \mathbb{D}_{\mathbb{C}}$ whenever $M_1, M_2 \in \mathbb{D}_{\mathbb{C}}$ and $\lambda \in [0, 1]$. Consequently, it is a connected and simply connected domain.

The Lie group $\mathrm{U}(n, n)$ acts on $\mathbb{D}_{\mathbb{C}}$, as the following lemma describes.

Lemma 2.15 *Let* $\widetilde{V} = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ *belong to* $\mathrm{U}(n, n)$. *The map*

$$\widehat{\widetilde{V}} \colon \mathbb{D}_{\mathbb{C}} \to \mathbb{D}_{\mathbb{C}}\,, \quad \widetilde{M} \mapsto \widehat{\widetilde{V}}\cdot\widetilde{M} = (V_2 + V_4\widetilde{M})(V_1 + V_3\widetilde{M})^{-1} \tag{2.23}$$

is a diffeomorphism. Moreover, this action can be extended to the closure of $\mathbb{D}_{\mathbb{C}}$ *in* $\mathbb{S}_n(\mathbb{C})$, *and the map* $\widehat{\widetilde{V}}$ *preserves the boundary of* $\mathbb{D}_{\mathbb{C}}$.

Proof Since $\widetilde{V} \in \mathrm{U}(n, n)$,

$$V_1^* V_1 - V_2^* V_2 = I_n\,, \quad V_3^* V_3 - V_4^* V_4 = -I_n \quad \text{and} \quad -V_1^* V_3 + V_2^* V_4 = 0_n\,.$$

It follows easily from these equalities that

$$(V_1 + V_3\widetilde{M})^*(V_1 + V_3\widetilde{M}) - (V_2 + V_4\widetilde{M})^*(V_2 + V_4\widetilde{M}) = I_n - \widetilde{M}^*\widetilde{M} \ge 0 \tag{2.24}$$

if $\widetilde{M}$ belongs to the closure of $\mathbb{D}_{\mathbb{C}}$. Consequently, $(V_1 + V_3\widetilde{M})$ is nonsingular: $\widetilde{V}\left[\begin{smallmatrix} I_n \\ \widetilde{M} \end{smallmatrix}\right] = \left[\begin{smallmatrix} V_1 + V_3\widetilde{M} \\ V_2 + V_4\widetilde{M} \end{smallmatrix}\right]$ has rank n, so that, if $(V_1 + V_3\widetilde{M})\,\mathbf{z} = \mathbf{0}$ for a vector $\mathbf{z} \ne \mathbf{0}$, then $(V_2 + V_4\widetilde{M})\,\mathbf{z} \ne \mathbf{0}$ and hence one has $0 = \|(V_1 + V_3\widetilde{M})\,\mathbf{z}\|^2 \ge \|(V_2 + V_2\widetilde{M})\,\mathbf{z}\|^2 > 0$, which is impossible. In addition, if $\widehat{\widetilde{V}}\cdot\widetilde{M}$ is given by (2.23), then (2.24) yields the equality

$$I_n - (\widehat{\widetilde{V}}\cdot\widetilde{M})^*(\widehat{\widetilde{V}}\cdot\widetilde{M}) = ((V_1 + V_3\widetilde{M})^{-1})^*(I_n - \widetilde{M}^*\widetilde{M})(V_1 + V_3\widetilde{M})^{-1}\,,$$

which proves that $\widehat{\widetilde{V}}$ maps $\mathbb{D}_{\mathbb{C}}$ into itself and its boundary into its boundary. The existence of the inverse map is proved as in Lemma 2.10, and the differentiability of $\widehat{\widetilde{V}}$ and its inverse map on $\mathbb{D}_{\mathbb{C}}$ is clear.

Consider now the map (2.23) given by $\widetilde{U}(t, \omega)$, which belongs to $\mathrm{U}(n, n)$. The previous lemma ensures that $\det(\widetilde{U}_1(t, \omega) + \widetilde{U}_3(t, \omega)\widetilde{M}_0) \ne 0$ whenever $\widetilde{M}_0 \in \mathrm{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$, so that the same property holds on a neighborhood of $\widetilde{M}_0$, in which the matrix $\widehat{\widetilde{U}}(t, \omega)\cdot\widetilde{M}$ given by (2.23) is therefore well defined. The rotation number is defined in [72] as

$$\alpha = -\lim_{t\to\infty} \frac{1}{2n}\frac{1}{t} \arg\det d_{\widetilde{M}_0}\widetilde{\widetilde{U}}(t,\omega) \tag{2.25}$$

for $\widetilde{M}_0 \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$ and $\omega \in \Omega$, where as before $d_{\widetilde{M}_0}$ represents the Fréchet derivative at $\widetilde{M}_0$, given by (2.16). The geometrical idea of this definition is that the rotation number must measure the average rotation due to the action of $\widetilde{U}(t,\omega)$ on the set $\mathbb{D}_{\mathbb{C}}$ and its boundary. (Note that an analogous extension to the closure is not possible for the action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathbb{S}_n^+(\mathbb{C})$ defined by (2.17).)

In order to prove that the limit (2.25) is independent of the choice of the element $\widetilde{M}_0 \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$ and that it agrees with the previously given ones (2.5) and (2.18) (and hence it takes the same value m_0-a.e.), some facts concerning the Iwasawa decomposition of the real symplectic group are needed.

Lemma 2.16 *Any matrix $V \in \mathrm{Sp}(n,\mathbb{R})$ can be written in a unique way as the product GS, where both G and S are real symplectic matrices and*

$$G \in \mathcal{G} = \left\{ \begin{bmatrix} \Phi_1 & -\Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid (\Phi_1 + i\Phi_2)^*(\Phi_1 + i\Phi_2) = I_n \right\},$$

$$S \in \mathcal{S} = \left\{ \begin{bmatrix} A & 0_n \\ B & (A^T)^{-1} \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \,\middle|\, \begin{matrix} A \text{ is lower triangular with} \\ \text{positive diagonal,} \\ A^T B \text{ is symmetric} \end{matrix} \right\}$$

In addition, S and G depend smoothly on V.

Proof Since $\left[\begin{smallmatrix} V_3 \\ V_4 \end{smallmatrix}\right]$ is a Lagrange plane, it follows that $V_4^T V_4 + V_3^T V_3 > 0$. It is easy to check that there exists a unique real lower triangular matrix A with positive diagonal such that $A^T A = (V_4^T V_4 + V_3^T V_3)^{-1}$. This matrix A is nonsingular. Define also $\Phi_1 = V_4 A^T$ and $\Phi_2 = -V_3 A^T$, so that $\Phi_1 + i\Phi_2$ is a unitary matrix. It is also easy to check that $-\Phi_2^T V_3 + \Phi_1^T V_4 = A(A^T A)^{-1} = (A^T)^{-1}$, $\Phi_1^T V_3 + \Phi_2^T V_4 = 0_n$, and that $\Phi_1^T V_1 + \Phi_2^T V_2 = A(V_4^T V_1 - V_3^T V_2) = A$. Finally, if $B = -\Phi_2^T V_1 + \Phi_2^T V_1$, then one obtains

$$\begin{bmatrix} \Phi_1 & -\Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix}^{-1} \begin{bmatrix} V_1 & V_3 \\ V_2 & V_4 \end{bmatrix} = \begin{bmatrix} \Phi_1^T & \Phi_2^T \\ -\Phi_2^T & \Phi_1^T \end{bmatrix} \begin{bmatrix} V_1 & V_3 \\ V_2 & V_4 \end{bmatrix} = \begin{bmatrix} A & 0_n \\ B & (A^T)^{-1} \end{bmatrix}.$$

Note that the last matrix is symplectic, so that $A^T B$ is symmetric (see Proposition 1.23). The smoothness of the decomposition is an easy consequence of the uniqueness.

The decomposition of the symplectic fundamental matrix solution of (2.1),

$$\begin{aligned} U(t,\omega) &= G(t,\omega)\,S(t,\omega) \\ &= \begin{bmatrix} \Phi_1(t,\omega) & -\Phi_2(t,\omega) \\ \Phi_2(t,\omega) & \Phi_1(t,\omega) \end{bmatrix} \begin{bmatrix} A(t,\omega) & 0_n \\ B(t,\omega) & (A^T)^{-1}(t,\omega) \end{bmatrix}, \end{aligned} \tag{2.26}$$

is continuous in t, and hence the corresponding decomposition for $\widetilde{U}(t,\omega)$,

$$\widetilde{U}(t,\omega) = \widetilde{G}(t,\omega)\,\widetilde{S}(t,\omega), \tag{2.27}$$

with

$$\begin{aligned} \widetilde{G}(t,\omega) &= K^{-1}G(t,\omega)K = \begin{bmatrix} \Phi_1(t,\omega) - i\Phi_2(t,\omega) & 0 \\ 0 & \Phi_1(t,\omega) + i\Phi_2(t,\omega) \end{bmatrix}, \\ \widetilde{S}(t,\omega) &= K^{-1}S(t,\omega)K\,, \end{aligned}$$

is also continuous in t. (As before, $K = \left[\begin{smallmatrix} iI_n & iI_n \\ -I_n & I_n \end{smallmatrix}\right]$.) The following technical lemma shows that $\widetilde{S}(t,\omega)$ does not contribute to the limit (2.25).

Lemma 2.17 *With the notation established above,*

$$\lim_{t\to\infty} \frac{1}{t}\, \arg\det d_{\widetilde{M}_0}\widehat{\widetilde{S}}(t,\omega) = 0 \tag{2.28}$$

for every $\widetilde{M}_0 \in \mathrm{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$ *and* $\omega \in \Omega$.

Proof The first step of the proof consists in checking that the functions

$$\begin{aligned} &\widehat{K}\colon \mathbb{D}_{\mathbb{C}} \to \mathbb{S}_n^+(\mathbb{C})\,, \qquad \widetilde{M} \mapsto i(I_n - \widetilde{M})(I_n + \widetilde{M})^{-1}\,, \\ &\widehat{S}\colon \mathbb{S}_n^+(\mathbb{C}) \to \mathbb{S}_n^+(\mathbb{C})\,, \; M \mapsto BA^{-1} + (A^T)^{-1}MA^{-1}\,, \\ &\widehat{K}^{-1}\colon \mathbb{S}_n^+(\mathbb{C}) \to \mathbb{D}_{\mathbb{C}}\,, \; M \mapsto (iI_n - M)(iI_n + M)^{-1} \end{aligned}$$

are well-defined diffeomorphisms. This property is proved using Lemma 2.10 for $\widehat{S}$, since $S \in \mathrm{Sp}(n,\mathbb{R})$. As explained in Remark 2.14, $(I_n - \widetilde{M})^{-1}$ exists for $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$. Clearly $(I_n + \widetilde{M})(I_n - \widetilde{M}) = I_n - \widetilde{M}^2$ is a symmetric matrix, and hence so is $(I_n - \widetilde{M})(I_n + \widetilde{M})^{-1}$. In addition,

$$\begin{aligned} \mathrm{Im}(\widehat{K}\cdot\widetilde{M}) &= \frac{1}{2i}\,((\widehat{K}\cdot\widetilde{M}) - (\widehat{K}\cdot\widetilde{M})^*) \\ &= \frac{1}{2}\,(I_n + \widetilde{M}^*)^{-1}\big((I_n + \widetilde{M}^*)(I_n - \widetilde{M}) + (I_n - \widetilde{M}^*)(I_n + \widetilde{M})\big)(I_n + \widetilde{M})^{-1} \\ &= (I_n + \widetilde{M}^*)^{-1}(I_n - \widetilde{M}^*\widetilde{M})(I_n + \widetilde{M})^{-1} > 0\,. \end{aligned}$$

These two properties show that the map $\widehat{K}$ is well defined. Moreover, if $M \in \mathbb{S}_n^+(\mathbb{C})$, then $\operatorname{Im}(iI_n + M) > 0$, so that $(iI_n + M)^{-1}$ exists (see Proposition 1.21). And it is not hard to check that

$$I_n - (\widehat{K}^{-1}{\cdot}M)^*(\widehat{K}^{-1}{\cdot}M) = 4\,(iI_n + M)^{-1})^* \operatorname{Im} M(iI_n + M)^{-1} > 0\,.$$

Lemma 2.9 asserts that $\widehat{K}$ and $\widehat{K}^{-1}$ are differentiable. In addition, $\widehat{K}^{-1} \circ \widehat{K}$ is the identity function on $\mathbb{D}_{\mathbb{C}}$, which completes the first step.

Lemma 2.9 also implies that

$$\begin{aligned}
&\det d_{\widetilde{M}}\widehat{K} = 2^{n^2} i^{-n^2} \det{}^{-2n}(I_n + \widetilde{M}) \quad \text{for every } \widetilde{M} \in \mathbb{D}_{\mathbb{C}}\,,\\
&\det d_M\widehat{S} = \det{}^{-2n} A \quad \text{for every } M \in \mathbb{S}_n^+(\mathbb{C})\,,\\
&\det d_M\widehat{K}^{-1} = 2^{-n^2} i^{n^2} \det{}^{-2n}(iI + M) \quad \text{for every } M \in \mathbb{S}_n^+(\mathbb{C})\,.
\end{aligned}$$

Note that, in the cases of K and K^{-1}, the equalities follow from the symplectic character of the matrices $(2i)^{-1/2}K$ and $(2i)^{1/2}K^{-1}$, which define the same maps as K and K^{-1}.

Fix now $\omega \in \Omega$. The equality $\widetilde{S}(t,\omega) = K^{-1}S(t,\omega)K$ ensures that $\widehat{\widetilde{S}}(t,\omega) = \widehat{K}^{-1} \circ \widehat{S}(t,\omega) \circ \widehat{K}$. Consequently, for all $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$,

$$\begin{aligned}
\det d_{\widetilde{M}}\widehat{\widetilde{S}}(t,\omega) &= \det d_{\widehat{S}(t,\omega)\cdot(\widehat{K}\cdot\widetilde{M})}\widehat{K}^{-1}\;\; \det d_{\widehat{K}\cdot\widetilde{M}}\widehat{S}(t,\omega)\; \det d_{\widetilde{M}}\widehat{K}\\
&= \Big(\det\Big(iI_n + \widehat{S}(t,\omega)\cdot(\widehat{K}\cdot\widetilde{M})\Big) \det A \det(I_n + \widetilde{M})\Big)^{-2n}.
\end{aligned} \tag{2.29}$$

It is possible to choose a continuous branch $\arg_1$ of the complex argument such that $|\arg_1 \det(I_n + \widetilde{M})| \le (n+1)\,\pi$ for all $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$. To check this assertion, note that all the eigenvalues of any $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$ lie in the unit disk, since $\widetilde{M}^*\widetilde{M} < I_n$, and consequently all the eigenvalues of $(I_n + \widetilde{M})$ lie in the right complex half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\} = \{z \in \mathbb{C} \mid -\pi/2 < \arg z < \pi/2\}$. Fix $\widetilde{M}_0 \in \mathbb{D}_{\mathbb{C}}$ and choose $\arg_1$ in order that $\arg_1 \det(I_n + \widetilde{M}_0) \in (0,\pi)$. Given any other $\widetilde{M}_1 \in \mathbb{D}_{\mathbb{C}}$, choose a continuous map $C\colon [0,1] \to \mathbb{D}_{\mathbb{C}}$ such that $C(0) = \widetilde{M}_0$ and $C(1) = \widetilde{M}_1$. At this point, it is possible to choose continuous functions $\rho_1, \dots, \rho_n\colon \mathbb{R} \to \mathbb{C}$ such that the set of eigenvalues of $I_n + C(t)$ coincides with the unordered n-tuple $\{\rho_1(t), \dots, \rho_n(t)\}$, which may have repeated elements (see e.g. Theorem II.5.2 of Kato [89]). Then $|\arg_1(\rho_j(1)) - \arg_1(\rho_j(0))| < \pi$ (since the graph of ρ_j does not cross the imaginary axis), and hence $|\arg_1 \det(I_n + \widetilde{M}_1) - \arg_1 \det(I_n + \widetilde{M}_0)| \le n\pi$, from which the assertion follows.

Note also that, for any $N \in \mathbb{S}_n^+(\mathbb{C})$, all the eigenvalues of $iI_n + N$ have positive definite imaginary part. Repeating the previous argument twice, one proves the existence of a continuous branch $\arg_2$ of the complex argument such that

$$\begin{aligned}&\left|\arg_2 \det\left(iI_n + \widehat{S}(t,\omega)\cdot(\widehat{K}\cdot\widetilde{M})\right)\right| \\ &\qquad \leq \left|\arg_2 \det\left(iI_n + \widehat{S}(t,\omega)\cdot(\widehat{K}\cdot\widetilde{M})\right) - \arg_2 \det\left(iI_n + \widehat{K}\cdot\widetilde{M}\right)\right| \\ &\qquad\quad + \left|\arg_2 \det\left(iI_n + \widehat{K}\cdot\widetilde{M}\right) - \arg_2 \det\left(iI_n + \widehat{K}\cdot\widetilde{M}_0\right)\right| \\ &\qquad\quad + \left|\arg_2 \det\left(iI_n + \widehat{K}\cdot\widetilde{M}_0\right)\right| \leq (2n+1)\,\pi\end{aligned}$$

for all $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$ and $t \geq 0$. It follows from (2.29) and from the real character of the matrix A that

$$\begin{aligned}&\arg\det d_{\widetilde{M}}\widehat{\widehat{S}}(t,\omega) \\ &\qquad = (-2n)\left(\arg_2 \det\left(iI_n + \widehat{S}(t,\omega)\cdot(\widehat{K}\cdot\widetilde{M})\right) + \arg_1 \det(I_n + \widetilde{M})\right)\end{aligned}$$

is a continuous branch of the argument and it is bounded in modulus by $2n\,(3n + 2)\,\pi$. Consequently, given any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\frac{1}{t}\left|\arg\det d_{\widetilde{M}}\widehat{\widehat{S}}(t,\omega)\right| \leq \varepsilon \quad \text{whenever } t \geq t_\varepsilon \text{ and } \widetilde{M} \in \mathbb{D}_{\mathbb{C}}\,. \tag{2.30}$$

This proves (2.28) in the case that $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$. In fact the argument can be simplified for a fixed $\widetilde{M} \in \mathbb{D}_{\mathbb{C}}$. But the advantage now is that (2.30) can immediately be extended to any $\widetilde{M} \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$, since, as can be deduced from (2.15), the determinant of $d_{\widetilde{M}}\widehat{\widehat{S}}(t,\omega)$ is continuous with respect to $\widetilde{M}$.

Theorem 2.18 *For every* $\widetilde{M}_0 \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$, *the limit* (2.25) *agrees with the limit* (2.5).

Proof The map induced on $\mathbb{D}_{\mathbb{C}}$ by $\widetilde{G}(t,\omega)$, namely

$$\widehat{\widehat{G}}(t,\omega)\cdot\widetilde{M} = (\Phi_1(t,\omega) + i\Phi_2(t,\omega))\widetilde{M}(\Phi_1(t,\omega) - i\Phi_2(t,\omega))^{-1}\,,$$

is linear, and hence it agrees with its Fréchet derivative at any point. From this fact, relation (2.27), and the definition of the group $\mathcal{G}$, it follows that

$$\begin{aligned}\det d_{\widetilde{M}_0}\widehat{\widehat{U}}(t,\omega) &= \det\widehat{\widehat{G}}(t,\omega)\det d_{\widetilde{M}_0}\widehat{\widehat{S}}(t,\omega) \\ &= \det^{-2n}(\Phi_1(t,\omega) - i\Phi_2(t,\omega))\det d_{\widetilde{M}_0}\widehat{\widehat{S}}(t,\omega)\,.\end{aligned}$$

This and Lemma 2.17 ensure that the limit (2.25) is equal to

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \arg\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega)) \tag{2.31}$$

and hence it is independent of the choice of $\widetilde{M}_0 \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$. But the expression (2.31) can be also obtained by choosing the argument Arg_2 in the first definition (2.5) and keeping in mind the equality $\begin{bmatrix} U_3(t,\omega) \\ U_4(t,\omega) \end{bmatrix} = \begin{bmatrix} -\Phi_2(t,\omega)(A^T)^{-1}(t,\omega) \\ \Phi_1(t,\omega)(A^T)^{-1}(t,\omega) \end{bmatrix}$ (see (2.26)), since $\det A(t,\omega) > 0$. This proves that (2.25) is well defined and coincides with the rotation number (2.5), and hence completes the proof.

Remark 2.19 Relation (2.31) shows that the limit (2.18) measures the index of rotation of the composition of the two maps

$$\mathbb{R}\times\Omega \to U(n,\mathbb{C})\,, \quad (t,\omega) \mapsto (\Phi_1 - i\Phi_2)(t,\omega)$$

and $U(n,\mathbb{C}) \to \mathbb{S}^1$, $\Phi \mapsto \det\Phi$, where $U(n,\mathbb{C})$ is the group of the unitary $n\times n$ matrices. This displays once more the geometrical significance of α. Compare (2.31) with the expression of the limit α in terms of the generalized polar coordinates appearing in the proof of Theorem 2.4.

Remark 2.20 Using the arguments applied in Remark 2.12, one can prove that

$$\begin{aligned} \alpha &= -\lim_{t\to\infty} \frac{1}{2n}\,\frac{1}{t}\,\text{Im}\int_0^t \text{tr}\tilde{f}(\omega{\cdot}s, \widetilde{M}(s,\omega,\widetilde{M}_0))\,ds \\ &= \lim_{t\to\infty}\frac{1}{2t}\,\text{Im}\int_0^t \text{tr}\Big(-i(H_2(\omega)-H_3(\omega)) \\ &\qquad + (H_1(\omega)+H_1^T(\omega) - i(H_2(\omega)+H_3(\omega)))\,\widetilde{M}(s,\omega,\widetilde{M}_0)\Big)\,ds \end{aligned} \tag{2.32}$$

for every $\widetilde{M}_0 \in \text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_{\mathbb{C}}$. It must be kept in mind that the solution $\widetilde{M}(t,\omega,\widetilde{M}_0)$ of the Riccati equation corresponding to the transformed system (2.22) with $\widetilde{M}(0,\omega,\widetilde{M}_0) = \widetilde{M}_0$ is defined for all $t\in\mathbb{R}$, as can be deduced from Lemma 2.15, since it agrees with $\widetilde{\widetilde{U}}(t,\omega){\cdot}\widetilde{M}_0$. This follows from definition (2.25), Theorem 2.18, the Liouville formula, and the fact that the map given by $t\mapsto d_{\widetilde{M}_0}\widetilde{\widetilde{U}}(t,\omega){\cdot}\widetilde{M}$ is the solution with initial datum $d_{\widetilde{M}_0}\widetilde{\widetilde{U}}(0,\omega){\cdot}\widetilde{M} = \widetilde{M}$ of the matrix differential equation

$$(\delta\widetilde{M})' = \tilde{f}(\omega{\cdot}t, \widetilde{M}(t,\omega,\widetilde{M}_0)){\cdot}\delta\widetilde{M} \tag{2.33}$$

given by

$$\begin{aligned}
&\tilde f(\omega,\widetilde M)\cdot D = \\
&\quad -\frac{1}{2}\,D\left(H_1 - H_1^T - i(H_2 - H_3) + (H_1 + H_1^T - i(H_2 + H_3))\,\widetilde M\right) \\
&\quad -\frac{1}{2}\left(H_1^T - H_1 - i(H_2 - H_3) + \widetilde M\,(H_1 + H_1^T - i(H_2 + H_3))\right)D,
\end{aligned} \tag{2.34}$$

which is obtained as the variational equation of the mentioned Riccati equation associated to its solution $\widetilde M(t,\omega,\widetilde M_0)$. The argument ω is omitted in $H_j(\omega)$ in the last equality.

2.1.3 In Terms of the Arnold–Maslov Index

Arnold's approach to the theory of the Maslov index suggests a new definition for the rotation number of the family (2.1). This index theory, which is related to certain asymptotic methods in perturbation theory, is also a fundamental tool in the generalization of the Sturm theory to linear Hamiltonian systems, as was shown by Arnold himself in [9]: for the higher-dimensional Schrödinger equation $-\mathbf{x}'' + G(t)\,\mathbf{x} = \mathbf{0}$, instead of zeros of solutions one can consider moments at which a Lagrange plane evolving under the action of the corresponding system is vertical, i.e. it is represented by $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\det L_1 = 0$. Roughly speaking, the Maslov index measures the number of these vertical moments, which are also known as *focal points* in much of the Sturm–Liouville literature.

Arnold [8] characterizes the Maslov index for a closed curve in the space of real symplectic planes (whose previous definition had been based on intersection index theory and hence was difficult to manage) in terms of the rotation index of certain maps on $\mathbb{S}^1$ (see also Bott [18]). This is the idea which suggests the new approach to α, which is described in [72]. To explain this definition and its connection with the preceding ones is the purpose of this section. To this end, the definition of the Maslov index for a closed curve in the set of real Lagrange planes $\mathcal{L}_\mathbb{R}$ is briefly recalled; the reader is referred to [8] for the details.

Let l_v be the *vertical* Lagrange plane, which generated by the n last coordinate vectors; that is, $l_v \equiv \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$. Define the *(vertical) Maslov cycle* by

$$\mathcal{C} = \{l \in \mathcal{L}_\mathbb{R} \mid \dim(l \cap l_v) \ge 1\}\,, \tag{2.35}$$

which is clearly the complement of the set $\mathcal{D}$ defined by (1.21), which is homeomorphic to $\mathbb{S}_n(\mathbb{R})$ and hence simply connected: see Remark 1.30. Obviously, $\mathcal{C} = \cup_{k=1}^n \mathcal{C}^k$, where

$$\mathcal{C}^k = \{l \in \mathcal{L}_\mathbb{R} \mid \dim(l \cap l_v) = k\}\,.$$

Each set $\mathcal{C}^k$ is an algebraic submanifold of $\mathcal{L}_\mathbb{R}$ of codimension $k(k+1)/2$. In particular, $\operatorname{codim} \mathcal{C}^1 = 1$. Moreover, $\mathcal{C}^1$ is two-sidedly embedded in $\mathcal{L}_\mathbb{R}$; i.e. there exists a continuous vector field tangent to $\mathcal{L}_\mathbb{R}$ which is transversal to $\mathcal{C}^1$, and hence one can refer to the positive and negative sides of $\mathcal{C}^1$. The vector field is given at each point $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \in \mathcal{L}_\mathbb{R}$ by the velocity vector of the curve $t \mapsto e^{it} \cdot l \equiv \left[\begin{smallmatrix} \cos t\, L_1 - \sin t\, L_2 \\ \sin t\, L_1 + \cos t\, L_2 \end{smallmatrix}\right]$, and the positive side is chosen as the one towards which these velocity vectors are directed.

Definition 2.21 Let $\lambda \colon \mathbb{S}^1 \to \mathcal{L}_\mathbb{R}$ be a smooth closed curve, and assume that λ only intersects $\mathcal{C}$ transversally, and hence only in $\mathcal{C}^1$. The *Maslov index* of λ is given by

$$c(\lambda) = d_+ - d_- \,,$$

where d_+ (resp. d_-) is the number of intersection points for which λ passes from the negative side of $\mathcal{C}^1$ to the positive side (resp. from the positive to the negative).

The results of [8] (see also Duistermaat [38]) show that the index map c is independent of the choice of l_v, so that it induces a group isomorphism $c \colon \pi_1(\mathcal{L}_\mathbb{R}) \to \mathbb{Z}$, where $\pi_1(\mathcal{L}_\mathbb{R})$ is the fundamental group of $\mathcal{L}_\mathbb{R}$. In particular the Maslov index is defined for any continuous loop in $\mathcal{L}_\mathbb{R}$.

The rotation number can be defined in a somewhat approximate way as follows: choose $l \in \mathcal{L}_\mathbb{R}$, and for each pair (t, ω) consider the curve $\lambda_{t,\omega,l} \colon [0, t] \to \mathcal{L}_\mathbb{R},\ s \mapsto U(s, \omega)\, l$; deform $\lambda_{t,\omega,l}$ to a closed curve $\tilde{\lambda}_{t,\omega,l}$ by sliding the final point $U(t, \omega)\, l$ to l through $\mathcal{L}_\mathbb{R} - \mathcal{C}$, which as recalled above is simply connected, and represent $d(t, \omega, l) = c(\tilde{\lambda}_{t,\omega,l})$; and then define

$$\alpha = - \lim_{t \to \infty} \frac{\pi}{t}\, d(t, \omega, l)\,. \tag{2.36}$$

The limit, when properly defined (see below), exists and is independent of the choices of l and ω (m_0-a.e.), as stated in the following theorem. Its proof is basically a consequence of Arnold's results, but a brief sketch is included here for the reader's convenience.

Theorem 2.22 *For every $l \in \mathcal{L}_\mathbb{R}$, the limit* (2.36) *agrees with the limit* (2.5)*.*

Proof Each real Lagrange plane $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ can be represented as $l \equiv \left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$ with $\Phi_1 - i\Phi_2$ unitary: it suffices to take $\Phi_j = L_j P^{-1}$, where P is the unique positive definite square root of $L_1^T L_1 + L_2^T L_2$ (see Remark 1.27.3 and Proposition 1.19). Consequently, the map

$$\operatorname{Det}^2 \colon \mathcal{L}_\mathbb{R} \to \mathbb{S}^1, \quad l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \mapsto \det{}^2(\Phi_1 - i\Phi_2) = \frac{\det^2(L_1 - iL_2)}{\det(L_1^T L_1 + L_2^T L_2)} \tag{2.37}$$

is well defined. In particular, the image of l does not depend on the representation chosen. It follows easily from Proposition 1.29(i) that it is a continuous function.

Let $\lambda\colon \mathbb{S}^1 \to \mathcal{L}_{\mathbb{R}}$ be a continuous loop. Define $\operatorname{Ind}\lambda$ as the rotation index of the composition $\operatorname{Det}^2\lambda\colon \mathbb{S}^1 \to \mathbb{S}^1$; i.e. $1/(2\pi)$ times the increment along the circumference of a continuous determination of $\arg\colon \mathbb{S}^1 \to \mathbb{R}$. It is possible to extend Ind to an isomorphism $\operatorname{Ind}\colon \pi_1(\mathcal{L}_{\mathbb{R}}) \to \mathbb{Z}$. As proved in [8], $-\operatorname{Ind}$ and c are in fact the same map (since they agree on a nonzero homotopy class), and this provides a simple characterization of the Maslov index in the symplectic case.

Now return to the limit (2.36). The independence of the choice of l follows from the invariance of c under homotopies. Choose $l \equiv \left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$, and note that $U(t,\omega)\cdot l \equiv \left[\begin{smallmatrix} U_1(t,\omega) \\ U_2(t,\omega) \end{smallmatrix}\right]$. This leads to

$$
\begin{aligned}
-\lim_{t\to\infty}\frac{\pi}{t}\,d(t,\omega,l) &= \lim_{t\to\infty}\frac{\pi}{t}\frac{1}{2\pi}\arg\frac{\det^2(U_1 - iU_2)(t,\omega)}{\det(U_1^TU_1 + U_2^TU_2)(t,\omega)} \\
&= \lim_{t\to\infty}\frac{1}{t}\arg\det(U_1(t,\omega) - iU_2(t,\omega)) \\
&= \lim_{t\to\infty}\frac{1}{t}\operatorname{Arg}_1 U(t,\omega)\,,
\end{aligned}
$$

which proves the result.

The arguments used in the proof of this result will be fundamental in Sect. 2.3, in which a relation between the properties of the rotation number and the presence of exponential dichotomy will be discussed.

Remark 2.23 Definition (2.36) shows that α/π measures the average number of oriented intersections with the vertical Maslov cycle $\mathcal{C}$ of the curve determined in $\mathcal{L}_{\mathbb{R}}$ by the evolution of a real Lagrange plane under the flow determined by (2.1). Therefore, it extends to the $2n$-dimensional case another of the usual ways to define α for the two-dimensional system (2.11):

$$
\alpha = \lim_{t\to\infty}\frac{\pi}{t}\,d(t,\omega)
$$

(m_0-a.e.), where $d(t,\omega)$ is the number of oriented zeros in $[0,t]$ of the first component of an arbitrarily chosen solution of the system (see [73]).

2.2 Continuous Variation of the Rotation Number

The ergodic representation of the rotation number obtained in Sect. 2.1.1 is the fundamental tool in the study of the continuity of the rotation number with respect to the $L^1(\Omega, m_0)$-topology in the set of potentials H defining linear Hamiltonian systems (2.1). The proof of this continuity property is the goal of this section. (See

Definition 1.32 for the definition of the above topology.) The analysis continues in Chap. 4, where the directional differentiability of the rotation number is established.

Recall that m_0 represents a fixed σ-ergodic measure on Ω; that any matrix-valued function H satisfying Hypotheses 2.1 belongs to $L^1(\Omega, m_0)$ (see Remark 1.39); and that, according to Theorem 2.4,

$$\alpha(H) = \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega, l)\, d\mu_H$$

for every τ^H-invariant measure μ_H projecting onto m_0. Here, τ^H represents the flow induced on $\mathcal{K}_{\mathbb{R}}$ by the family of linear systems determined by H,

$$\operatorname{Tr} Q_H(\omega, l) = \operatorname{tr}\left(\begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} JH(\omega) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \right) \tag{2.38}$$

for a representation $\left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$ of l with $\Phi_1 + i\Phi_2$ unitary (see Remark 1.42), and $\alpha(H)$ is the corresponding rotation number.

In order to define the $L^1(\Omega, m_0)$-topology on the set of matrix-valued functions taking values in $\mathfrak{sp}(n, \mathbb{R})$, the Euclidean norm $\|\cdot\|$ is chosen: see Definition 1.32 and Remarks 1.24.1, 1.24.2, and 1.33. Obviously, the continuity results are independent of the particular choice of this vector norm. The Frobenius norm $\|\cdot\|_F$, defined in Remark 1.24.3, will also be used in the proofs which follow.

Lemma 2.24 *Let $H, H_1, H_2 \colon \Omega \to \mathfrak{sp}(n, \mathbb{R})$ satisfy Hypotheses 2.1.*

(i) *There exists a real function $T_H \in L^1(\Omega, m_0)$ such that*

$$|\operatorname{Tr} Q_H(\omega, l)| \le T_H(\omega)$$

for all $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$, which in addition is continuous on Ω if the matrix-valued function H is continuous.

(ii) *There exists a constant c such that*

$$|\operatorname{Tr} Q_{H_1}(\omega, l) - \operatorname{Tr} Q_{H_2}(\omega, l)| \le c\, \|H_1(\omega) - H_2(\omega)\|$$

for all $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$.

Proof Equality (2.38), the continuity and properties of the map $\|\cdot\|_F$ (see Remark 1.24.3), the equality $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, the compactness of the set of unitary matrices, and the equivalence of the matrix norms $\|\cdot\|$ and $\|\cdot\|_F$, ensure that

$$|\operatorname{Tr} Q_H(\omega, l)| \le \|H(\omega)\|_F \left\| \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \begin{bmatrix} \Phi_1^T & \Phi_2^T \end{bmatrix} J \right\|_F \le c_0 \|H(\omega)\|_F \le c_1 \|H(\omega)\|$$

for a positive real constant c_1. Thus, (i) holds for $T_H(\omega) = c_1 \|H(\omega)\|$, which belongs to $L^1(\Omega, m_0)$ and is continuous if H is. And the value of $|\operatorname{Tr} Q_{H_1}(\omega, l) -$

$\mathrm{Tr}\, Q_{H_2}(\omega, l)|$ is obtained by substituting H by $H_1 - H_2$ on (2.38), so that assertion (ii) follows from the same argument.

Theorem 2.25 *Suppose that* $H = \lim_{m\to\infty} H_m$ *in the* $L^1(\Omega, m_0)$*-topology, where all the matrix-valued functions* $H, H_m\colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$ *satisfy Hypotheses 2.1. Then*

$$\alpha(H) = \lim_{m\to\infty} \alpha(H_m)\,.$$

Proof The argument used is standard in measure theory, and the proof is simpler if the limit matrix H is supposed to be continuous. For each $m \in \mathbb{N}$, take a τ^{H_m}-invariant normalized measure μ_{H_m} on $\mathcal{K}_\mathbb{R}$ projecting onto m_0. Then, according to Theorem 2.4, $\alpha(H_m) = \int_{\mathcal{K}_\mathbb{R}} \mathrm{Tr}\, Q_{H_m}(\omega, l)\, d\mu_{H_m}$. As usual, any measure μ_{H_m} defines a functional on the separable space of real continuous functions on $\mathcal{K}_\mathbb{R}$, and the norm of the functional is $\mu_{H_m}(\mathcal{K}_\mathbb{R}) = 1$. Proposition 1.15(ii) ensures that any subsequence of (μ_{H_m}) has a weak* convergent subsequence, say (μ_{H_k}). Its limit μ_H is τ^H-invariant, projects onto m_0, and satisfies $\lim_{k\to\infty} \int_{\mathcal{K}_\mathbb{R}} f(\omega, l)\, d\mu_{H_k} = \int_{\mathcal{K}_\mathbb{R}} f(\omega, l)\, d\mu_H$ for every continuous function f on $\mathcal{K}_\mathbb{R}$. In particular, again by Theorem 2.4, $\alpha(H) = \int_{\mathcal{K}_\mathbb{R}} \mathrm{Tr}\, Q_H(\omega, l)\, d\mu_H$. Therefore, in order to prove the result it suffices to check that

$$\begin{aligned} &\alpha(H_k) - \alpha(H) \\ &\quad = \int_{\mathcal{K}_\mathbb{R}} \mathrm{Tr}\, Q_{H_k}(\omega, l)\, d\mu_{H_k} - \int_{\mathcal{K}_\mathbb{R}} \mathrm{Tr}\, Q_H(\omega, l)\, d\mu_H \stackrel{k\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{2.39}$$

Note first that (ii) in Lemma 2.24 and the $L^1(\Omega, m_0)$-convergence imply that

$$\begin{aligned} &\left| \int_{\mathcal{K}_\mathbb{R}} (\mathrm{Tr}\, Q_{H_k} - \mathrm{Tr}\, Q_H)(\omega, l)\, d\mu_{H_k} \right| \\ &\qquad \le c \int_\Omega \|H_k(\omega) - H(\omega)\|\, dm_0 \stackrel{k\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{2.40}$$

Let T_H satisfy Lemma 2.24(i). Take $\varepsilon > 0$ and choose

- a constant $\delta > 0$ such that $\int_{\widetilde{\Omega}} T_H(\omega)\, dm_0 < \varepsilon$ if $\widetilde{\Omega} \subseteq \Omega$ and $m_0(\widetilde{\Omega}) < \delta$,
- a compact subset $\mathcal{K}^\varepsilon \subseteq \Omega$ with $m_0(\Omega - \mathcal{K}^\varepsilon) < \delta$ and a continuous symplectic matrix-valued function H^ε on Ω such that $H^\varepsilon|_{\mathcal{K}^\varepsilon} = H|_{\mathcal{K}^\varepsilon}$.

Consider the map $(\omega, l) \mapsto \mathrm{Tr}\, Q_{H^\varepsilon}(\omega, l) = \mathrm{tr}\big([\Phi_1^T, \Phi_2^T]\, J H^\varepsilon(\omega) \left[\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right]\big)$, which is continuous on $\mathcal{K}_\mathbb{R}$, and choose

- an open subset $\mathcal{O}^\varepsilon \subseteq \Omega$ with $\mathcal{K}^\varepsilon \subseteq \mathcal{O}^\varepsilon$ and

$$m_0(\mathcal{O}^\varepsilon - \mathcal{K}^\varepsilon) \sup_{(\omega,l)\in\mathcal{K}_\mathbb{R}} |\,\mathrm{Tr}\, Q_{H^\varepsilon}(\omega, l)| < \varepsilon\,,$$

- a continuous function r on Ω with $\chi_{\mathcal{K}^\varepsilon} \le r \le \chi_{\mathcal{O}^\varepsilon}$.

Let μ be any measure on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0. Then

$$\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu = \int_{\mathcal{K}_{\mathbb{R}}} r(\omega)\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)\,d\mu \\ - \int_{(\mathcal{O}^\varepsilon-\mathcal{K}^\varepsilon)\times\mathcal{L}_{\mathbb{R}}} r(\omega)\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)\,d\mu + \int_{(\Omega-\mathcal{K}^\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu\,.$$

Moreover, the definition of $\mathcal{O}^\varepsilon$ and (i) in Lemma 2.24 imply that

$$\left|\int_{(\mathcal{O}^\varepsilon-\mathcal{K}^\varepsilon)\times\mathcal{L}_{\mathbb{R}}} r(\omega)\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)\,d\mu\right| \le m_0(\mathcal{O}^\varepsilon-\mathcal{K}^\varepsilon)\sup_{(\omega,l)\in\mathcal{K}_{\mathbb{R}}}|\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)| \\ <\varepsilon\,,$$

$$\left|\int_{(\Omega-\mathcal{K}^\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu\right| \le \int_{\Omega-\mathcal{K}^\varepsilon} T_H(\omega)\,dm_0 < \varepsilon\,.$$

Consequently,

$$\left|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu_{H_k} - \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu_H\right| \\ < \left|\int_{\mathcal{K}_{\mathbb{R}}} r(\omega)\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)\,d\mu_{H_k} - \int_{\mathcal{K}_{\mathbb{R}}} r(\omega)\operatorname{Tr} Q_{H^\varepsilon}(\omega,l)\,d\mu_H\right| + 4\varepsilon\,.$$

The weak* convergence of the sequence of measures implies then that

$$\left|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu_{H_k} - \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q_H(\omega,l)\,d\mu_H\right| \stackrel{k\to\infty}{\longrightarrow} 0\,. \tag{2.41}$$

Relations (2.40) and (2.41) ensure that (2.39) holds, which proves the result.

The second result of this section analyzes the rotation number for a fixed and continuous potential H. As has already been pointed out, the rotation number depends on the choice of the measure m, so that it makes sense to represent it by $\alpha(m)$. The following theorem shows that, if it is the case that $\alpha(m)$ takes the same value for every σ-ergodic measure m on Ω, then any $\omega\in\Omega$ can be chosen for the definition (2.5) of the rotation number. Clearly, the required hypothesis holds when the base flow is uniquely ergodic, which is the case when Ω is constructed as the hull of a Bohr almost periodic function $H_0\colon\mathbb{R}\to\mathfrak{sp}(n,\mathbb{R})$: see Sect. 1.3.2 and Remark 1.40.

Theorem 2.26 *Suppose that the matrix-valued function H defining (2.1) is continuous, and that there exists a number $\alpha_*\in\mathbb{R}$ such that $\alpha(m)=\alpha_*$ for all σ-ergodic measures m on Ω. Then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\,ds = \alpha_*$$

uniformly in $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$. *Consequently the limits* (2.5), (2.18), (2.25) *and* (2.36) *take the value* α_* *irrespective of the point* ω *at which they are calculated.*

Proof The proof is carried out in the case $\alpha_* = 0$: the general case only requires substituting $\operatorname{Tr} Q$ with $\operatorname{Tr} Q - \alpha_*$. Note first that Theorem 2.4 ensures that

$$\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q(\omega, l)\, d\mu = 0 \tag{2.42}$$

for every τ-invariant measure μ on $\mathcal{K}_{\mathbb{R}}$: the result follows directly if μ is ergodic, since it projects onto a σ-ergodic measure on Ω; and this property together with the last assertion of Theorem 1.9 proves (2.42) in the general case, since $\operatorname{Tr} Q$ is continuous: see Remark 1.42.

Now suppose for contradiction that there exist $\varepsilon > 0$ and sequences $(t_k) \uparrow \infty$, (ω_k) in Ω, and (l_k) in $\mathcal{L}_{\mathbb{R}}$ such that

$$\left| \frac{1}{t_k} \int_0^{t_k} \operatorname{Tr} Q(\tau(s, \omega_k, l_k))\, ds \right| \geq \varepsilon \tag{2.43}$$

for $k \in \mathbb{N}$. The Riesz representation theorem associates to the bounded linear functional $C(\mathcal{K}_{\mathbb{R}}, \mathbb{R}) \to \mathbb{R},\ g \mapsto (1/t_k) \int_0^{t_k} g(\tau(s, \omega_k, l_k))\, ds$ (which has norm 1) a normalized measure μ_k. Theorem 1.9(i) ensures that the sequence (μ_k) admits a subsequence (μ_j) which converges weak* to a σ-invariant measure μ_0; that is,

$$\lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} g(\tau(s, \omega_j, l_j))\, ds = \int_{\mathcal{K}_{\mathbb{R}}} g(\omega, l)\, d\mu_0\,,$$

for every continuous function g. In particular, the inequality (2.43) implies that $|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q(\omega, l)\, d\mu_0| \geq \varepsilon$. This contradicts (2.42), which proves the first assertion of the theorem. The second assertion is a trivial consequence of, for example, (2.8), and of Theorems 2.11, 2.18, and 2.22.

2.3 The Rotation Number and the Schwarzmann Homomorphism

This section is devoted to establish a fact concerning the relation between the rotation number for the family of linear Hamiltonian systems (2.1), the presence of exponential dichotomy (see Definition 1.75), and the properties of the Schwarzmann homomorphism, whose definition will be given shortly. The result proved here can be used to obtain a gap labeling formula for the spectral problems corresponding to (2.1) and (2.2), as will be explained in detail in Chap. 3, Sect. 3.3.4.

Let $\mathcal{H}(\Omega, \mathbb{S}^1)$ be the set of the homotopy classes of the continuous maps $\phi\colon \Omega \to \mathbb{S}^1 \subset \mathbb{C}$. The class $[\phi]$ contains a map ϕ such that

$$\omega \mapsto \frac{d}{dt}\,\phi(\omega{\cdot}t)|_{t=0} = \phi'(\omega)$$

is a continuous function. Define

$$h\colon \mathcal{H}(\Omega, \mathbb{S}^1) \to \mathbb{R}\,, \quad [\phi] \mapsto \operatorname{Im} \int_\Omega \frac{\phi'(\omega)}{\phi(\omega)}\, dm_0\,.$$

It follows from Birkhoff's Theorems 1.3 and 1.6 that

$$h([\phi]) = \lim_{t\to\infty} \frac{1}{t}\, \arg \phi(\omega{\cdot}t) \qquad m_0\text{-a.e.} \tag{2.44}$$

Schwarzmann [138] proves that the map h is well defined and determines a homomorphism from the group $\mathcal{H}(\Omega, \mathbb{S}^1)$ to the additive group of real numbers. Consider now the group of real Čech one-cocycles with integer values: it can be viewed as the quotient space

$$\check{\mathcal{H}}^1(\Omega, \mathbb{Z}) = \frac{\mathcal{H}(\Omega, \mathbb{S}^1)}{\mathcal{E}}\,,$$

where $\mathcal{E}$ is the subgroup of $\mathcal{H}(\Omega, \mathbb{S}^1)$ given by the homotopy classes of the maps $\phi(\omega) = e^{2ir(\omega)}$, for continuous maps $r\colon \Omega \to \mathbb{R}$. This continuity and equality (2.44) imply that $h([\phi]) = 0$ for all $\phi \in \mathcal{E}$, and consequently the map h also induces a homomorphism from $\check{\mathcal{H}}^1(\Omega, \mathbb{Z})$ into $\mathbb{R}$.

Definition 2.27 The map $h\colon \check{\mathcal{H}}^1(\Omega, \mathbb{Z}) \to \mathbb{R}$ is the *Schwarzmann homomorphism* of the flow (Ω, σ).

Theorem 2.28 *Let m be a σ-ergodic measure on Ω, and let $\alpha(m)$ be the corresponding rotation number of the family* (2.1)*. If the family has exponential dichotomy over Ω, then $2\alpha(m) \in h\left(\check{\mathcal{H}}^1(\Omega, \mathbb{Z})\right)$.*

Proof Consider the decomposition $\Omega \times \mathbb{R}^{2n} = L^+ \oplus L^-$ which is determined by the exponential dichotomy, and recall that, according to Proposition 1.76, the sets $\{(\omega, l^\pm(\omega)) \mid \omega \in \Omega\} \subset \mathcal{K}_\mathbb{R}$ with $l^\pm(\omega) = \{\mathbf{z} \in \mathbb{R}^{2n} \mid (\omega, \mathbf{z}) \in \mathbb{R}^{2n}\}$ are copies of the base for the flow τ: see Definition 1.17.

Define ϕ_* as the composition of the continuous maps $\Omega \to \mathcal{L}_\mathbb{R}\,,\ \omega \mapsto l^+(\omega)$ and $\mathcal{L}_\mathbb{R} \to \mathbb{S}^1,\ l \mapsto \operatorname{Det}^2 l$, where this last (continuous) map is defined by (2.37). The map ϕ_* is well defined and continuous. In addition, according to equality (2.44) and the proof of Theorem 2.22, $2\alpha(m) = h([\phi_*])$, which completes the proof.

This section will be completed with an application of the preceding result: namely, a discussion of the concept of "instability zones" for linear nonautonomous Hamiltonian systems. The idea is taken from Yakubovich [157], who makes a similar analysis in the case of periodic coefficients.

To make this discussion clearer, consider first the periodic case. Represent by $\mathcal{H}_T$ the set of continuous T-periodic matrix-valued functions taking values in $\mathfrak{sp}(n, \mathbb{R})$. It is well known that the subset $\mathcal{H}_T^{\mathrm{un}}$ of $\mathcal{H}_T$ consisting of functions H_0 such that the system $\mathbf{z}' = H_0(t)\,\mathbf{z}$ is *totally unstable*, i.e. admits an exponential dichotomy, is divided into countably many connected, pairwise disjoint subsets $\mathcal{U}_k$: $\mathcal{H}_T^{\mathrm{un}} = \cup_{k\in\mathbb{Z}}\mathcal{U}_k$. There are various ways to label these regions, among them one stated in terms of the rotation number, which is now described. Let Ω be the circle obtained by identifying the endpoints of the interval $[0, T]$ and let σ be the translation flow: if $\omega \in \Omega$ and $t \in \mathbb{R}$, then $\omega{\cdot}t = \sigma(t, \omega) = \omega + t$ modulo T. It is a well-known result that the unique σ-ergodic measure m_0 on Ω is induced by the normalized Lebesgue measure on $[0, T]$. And each element of $\mathcal{H}_T$ can be uniquely identified with a continuous function $H\colon \Omega \to \mathfrak{sp}(n, \mathbb{R})$. It turns out that, for each $k \in \mathbb{Z}$, $\mathcal{U}_k$ consists of those coefficient matrices giving rise to a Hamiltonian family $\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}$ for which the rotation number takes the value $\pi k/T$: see [157], Theorem 2.

Returning to the general nonautonomous setting, a similar statement can be formulated. Theorem 2.28 guarantees that if the family (2.1) has an exponential dichotomy over Ω, then its rotation number α (with respect to a fixed σ-ergodic measure on Ω) takes values in an enumerable subgroup of the additive group of the real numbers, defined using the image of the Schwarzmann homomorphism: more precisely, $2\alpha \in \mathcal{S} = h(\check{H}^1(\Omega, \mathbb{Z}))$. In other words, one has a method to label the elements of the set $\mathcal{H}_\Omega^{\mathrm{un}}$ given by the set $\mathcal{H}$ of those maps $H\colon \Omega \to \mathfrak{sp}(n, \mathbb{R})$ satisfying Hypotheses 2.1 and for which the family (2.1) has an exponential dichotomy over Ω: namely, $\mathcal{H}_\Omega^{\mathrm{un}} = \cup_{s\in\mathcal{S}}\mathcal{U}_s$ where $\mathcal{U}_s$ is given by those matrices $H \in \mathcal{H}_\Omega^{\mathrm{un}}$ for which the rotation number is $s/2$.

2.4 Additional Properties in the Case $H_3 \geq 0$

The main results of this section are Theorem 2.31 and Theorem 2.36. The first one states that the rotation number of the linear Hamiltonian family (2.1) with respect to any σ-ergodic measure is nonnegative in the case that H_3 is continuous and positive semidefinite. Some preliminary results which have independent interest, used in its proof, are proved in Lemma 2.29 and Theorem 2.30.

Theorem 2.36, in which it is also assumed that $H_3 \geq 0$, provides a new definition of the rotation number in terms of the so-called proper focal points. It is closely related to that based on the Arnold–Maslov index, but easier to understand.

Lemma 2.29 *Let* $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ *be a symplectic matrix, and define* $W_V = (V_1 - iV_3)^{-1}(V_1 + iV_3)$.

(i) $W_V^T = W_V$, $W_V^* W_V = I_n$ *and* $\det W_V = r \det^2(V_1 + iV_3)$ *for some* $r > 0$. *In particular, W is diagonalizable, and all its eigenvalues lie in the unit circle of* $\mathbb{C}$.

(ii) *$W_V\mathbf{z} = \mathbf{z}$ if and only if $V_3\mathbf{z} = \mathbf{0}$. In particular, 1 is an eigenvalue of W_V if and only if $\det V_3 = 0$, and the eigenspace of W_V associated to 1 agrees with the kernel of V_3.*

Proof Remarks 1.27.2 and 1.27.3 ensure that $\begin{bmatrix} V_1^T \\ V_3^T \end{bmatrix}$ represents a Lagrange plane and that W_V is well defined. It can immediately be deduced from $V_3 V_1^T = V_1 V_3^T$ that $W_V^T = W_V$, and from this symmetry that $W_V^* W_V = I_n$. The symmetry also implies that

$$\det(V_1 - iV_3)^{-1} = \det(V_1^T - iV_3^T)^{-1} = \det(V_1 + iV_3)/\det(V_1 V_1^T + V_3 V_3^T)\,.$$

Theorem 8 of Chapter 8 [95] shows that W_V is diagonalizable, and this completes the proof of (i). The properties stated in (ii) are trivial consequences of the definition of W_V.

Consider now a single linear Hamiltonian system

$$\mathbf{z}' = H_0(t)\,\mathbf{z} = \begin{bmatrix} H_{01}(t) & H_{03}(t) \\ H_{02}(t) & -H_{01}^T(t) \end{bmatrix} \mathbf{z}\,, \tag{2.45}$$

where $H_0\colon \mathbb{R} \to M_{2n\times 2n}(\mathbb{R})$ is continuous. Represent by $V(t) = \begin{bmatrix} V_1(t) & V_3(t) \\ V_2(t) & V_4(t) \end{bmatrix}$ any real symplectic matrix solution of this system, and define

$$W_V(t) = (V_1(t) - iV_3(t))^{-1}(V_1(t) + iV_3(t))\,.$$

Theorem II.5.2 of [89] and Lemma 2.29.1 ensure the existence of continuous functions $\rho_1, \ldots, \rho_n\colon \mathbb{R} \to \mathbb{C}$ with $|\rho_j(t)| = 1$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$, such that the set of eigenvalues of $W_V(t)$, repeated according to their multiplicities, coincides with the unordered n-tuple $\{\rho_1(t), \ldots, \rho_n(t)\}$. Thus it is possible to take continuous argument functions $\varphi_1, \ldots, \varphi_n\colon \mathbb{R} \to \mathbb{R}$; i.e. $\rho_j(t) = e^{i\varphi_j(t)}$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$.

Theorem 2.30 *Suppose that $H_{03}(t) \geq 0$ for each $t \in \mathbb{R}$. With the above notation, the continuous function $\varphi_j\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing for $j = 1, \ldots, n$.*

Proof This result is stated by Yakubovich [154], and, as he notes, the proof is essentially due to Lidskiĭ [96]. The proof is included for the reader's convenience.

The first step in the proof is to check that

$$W_V' = i\,L_V(t)\,W_V\,, \tag{2.46}$$

where $L_V = 2\,(V_1 - iV_3)^{-1}\,H_{03}\,(V_1^T + iV_3^T)^{-1}$. Clearly, $L_V^* = L_V \geq 0$. Proposition 1.23 ensures that

$$V_1(t)V_3^T(t) = V_3(t)V_1^T(t) \quad \text{and} \quad V_4(t)V_1^T(t) - V_2(t)V_3^T(t) = I_n\,.$$

A straightforward computation starting from the relation $V'(t) = H_0(t)V(t)$ proves that (2.46) is equivalent to

$$2i\,H_{03}\,(V_1^T + iV_3^T)^{-1} = H_{03}\,((-V_2 + iV_4) + (V_2 + iV_4)(V_1 + iV_3)^{-1}(V_1 - iV_3))\,,$$

where the argument t of V_1, V_2, V_3, V_4, and H_{03} is omitted. It easy to check that

$$(V_1 + iV_3)^{-1}(V_1 - iV_3) = (V_1^T - iV_3^T)(V_1^T + iV_3^T)^{-1}\,,$$
$$(-V_2 + iV_4)(V_1^T + iV_3^T) + (V_2 + iV_4)(V_1^T - iV_3^T) = 2iI_n\,,$$

from which the previous equality follows. Therefore, (2.46) is proved.

An approximation method will be used to prove that each φ_j is nondecreasing. Fix any $t_1 \in \mathbb{R}$. Obviously, it suffices to prove the existence of a bounded open interval $\mathcal{I}$ centered in t_1 such that φ_j increases in $\mathcal{I}$ for $j = 1, \ldots, n$. For the time being, let $\mathcal{I}$ be an open interval centered in t_1. More restrictions on the interval $\mathcal{I}$ will be imposed later.

For $\varepsilon > 0$, there is a real-analytic function $L_V^\varepsilon : \mathcal{I} \to \mathbb{M}_{n\times n}(\mathbb{C})$ with $(L_V^\varepsilon)^* = L_V^\varepsilon$ such that $\sup_{t\in\mathcal{I}} \|L_V^\varepsilon(t) - L_V(t)\| \leq \varepsilon/2$, where the Euclidean matrix norm is used. Consider the differential system

$$W' = i\,(L_V^\varepsilon(t) + \varepsilon\, I_n)\, W\,, \tag{2.47}$$

and note that $L_V^\varepsilon(t) + \varepsilon\, I_n$ is real-analytic, strictly positive definite on I, and also selfadjoint. Let $W_V^\varepsilon(t)$ be the solution of (2.47) with $W_V^\varepsilon(t_1) = W_V(t_1)$, which hence takes values in the complex unitary group too: $(W_V^\varepsilon)^*(t) W_V^\varepsilon(t)$ is constant and, by Lemma 2.29(i), $(W_V^\varepsilon)^*(t_1) W_V^\varepsilon(t_1) = I_n$.

Fix $\varepsilon > 0$. Since $W_V^\varepsilon(t)$ varies analytically in t, Theorem II.1.10 of [89] yields real-analytic functions $\rho_1^\varepsilon, \ldots, \rho_n^\varepsilon : \mathcal{I} \to \mathbb{C}$ such that the unordered n-tuple $\{\rho_1^\varepsilon(t), \ldots, \rho_n^\varepsilon(t)\}$ is the set of (possibly repeated) eigenvalues of $W_V^\varepsilon(t)$. To apply this theorem requires a standard procedure of extension of $L_V^\varepsilon(t)$ to a complex-analytic selfadjoint matrix-valued function defined on an open and simply connected domain containing $\mathcal{I}$.

Take for each $j \in \{1, \ldots, n\}$ a branch of the argument of ρ_j^ε which is analytic in t in and with $\varphi_j^\varepsilon(t_1) \in [a - \pi, a + \pi)$. As explained in Section II.4.2 of [89], there are families $\{\mathbf{w}_1^\varepsilon(t), \ldots, \mathbf{w}_n^\varepsilon(t)\}$ of eigenvectors of $W_V^\varepsilon(t)$, with $W_V^\varepsilon(t)\,\mathbf{w}_j^\varepsilon(t) = \rho_j^\varepsilon(t)\,\mathbf{w}_j^\varepsilon(t)$, which also vary analytically in $t \in \mathcal{I}$. Note that, since $(W_V^\varepsilon)^* = (W_V^\varepsilon)^{-1}$ and $W_V^\varepsilon\,\mathbf{w}_j^\varepsilon = e^{i\varphi_j^\varepsilon}\mathbf{w}_j^\varepsilon$, then $(\mathbf{w}_j^\varepsilon)^* W_V^\varepsilon = e^{i\varphi_j^\varepsilon}(\mathbf{w}_j^\varepsilon)^*$. Here, and in what follows, the argument t is omitted. Now write $(\mathbf{w}_j^\varepsilon)^*\, W_V^\varepsilon\,\mathbf{w}_j^\varepsilon = e^{i\varphi_j^\varepsilon}(\mathbf{w}_j^\varepsilon)^*\,\mathbf{w}_j^\varepsilon$. Computing the derivative with respect to t and dividing by $e^{i\varphi_j^\varepsilon}$ gives

$$i\,(\mathbf{w}_j^\varepsilon)^*(L_V^\varepsilon + \varepsilon\, I_n)\,\mathbf{w}_j^\varepsilon + ((\mathbf{w}_j^\varepsilon)^*\,\mathbf{w}_j^\varepsilon)' = i\,(\varphi_j^\varepsilon)'(\mathbf{w}_j^\varepsilon)^*\,\mathbf{w}_j^\varepsilon + ((\mathbf{w}_j^\varepsilon)^*\,\mathbf{w}_j^\varepsilon)'\,,$$

which implies that $(\varphi_j^\varepsilon)' = ((\mathbf{w}_j^\varepsilon)^*(L_V^\varepsilon + \varepsilon\, I_n)\,\mathbf{w}_j^\varepsilon)/((\mathbf{w}_j^\varepsilon)^*\,\mathbf{w}_j^\varepsilon) > 0$. Hence $\varphi_j^\varepsilon(t)$ is strictly increasing on $\mathcal{I}$. This completes the second step of the proof.

Since the increasing character of φ_j is independent of the branch chosen for the argument, it is possible to assume from the beginning that $\varphi_{j_1}(t_1) = \varphi_{j_2}(t_1)$ if $\rho_{j_1}(t_1) = \rho_{j_2}(t_2)$. And clearly, for all $\varepsilon > 0$, there is loss of generality neither in reordering the functions ρ_j^ε in order to get $\rho_j^\varepsilon(t_1) = \rho_j(t_1)$ for $j = 1, \dots, n$, nor in choosing the continuous branch of φ_j^ε with $\varphi_j^\varepsilon(t_1) = \varphi_j(t_1)$ for $j = 1, \dots, n$.

To unify notation, call $W_V^0 = W_V$, $\rho_j^0 = \rho_j$ and $\varphi_j^0 = \varphi_j$. Since $W_V^\varepsilon(t)$ varies continuously in (t, ε), Theorem II.5.1 of [89] ensures that the unordered sets $\mathcal{E}^\varepsilon(t)$ of the eigenvalues of $W_V^\varepsilon(t)$ vary continuously in (t, ε), in the sense that the Hausdorff distance from $\mathcal{E}^{\varepsilon_1}(t_1)$ to $\mathcal{E}^{\varepsilon_2}(t_2)$ goes to zero as the sum $|t_1 - t_2| + |\varepsilon_1 - \varepsilon_2|$ goes to zero. It is easy to deduce from this and from the boundedness of $\mathcal{I}$ that the unordered sets $\mathcal{E}^\varepsilon(t)$ converge to the unordered set $\mathcal{E}^0(t)$ uniformly on $\mathcal{I}$ as $\varepsilon \to 0^+$.

Let $\tilde\rho_1, \dots, \tilde\rho_l$ (with $1 \le l \le n$) be the *distinct* eigenvalues of $W_V^0(t_1)$, and let the constant $\delta_1 > 0$ be smaller than the distance between any two of them. Let $\mathcal{B}_k$ be the intersection of the unit circle and the open ball of the complex plane centered in $\tilde\rho_k$ and of radius $\delta_1/3$. For all $k \in \{1, \dots, l\}$ choose $j_k \in \{1, \dots, n\}$ with $\rho_{j_k}^0(t_1) = \tilde\rho_k$. Note that if $\rho_j^0(t_1) = \tilde\rho_k$ for any other j, then $\varphi_{j_k}^0(t_1) = \varphi_j^0(t_1)$. And denote by $\arg_k$ the continuous determination of the complex argument such that $\varphi_{j_k}^0(t_1) = \arg_k \tilde\rho_k$.

As was previously announced, some further conditions will be imposed on the choice of $\mathcal{I}$. Since the set $\mathcal{E}^0(t)$ of eigenvalues of $W_V^0(t)$ varies continuously in t and is contained in the unit circle, it is possible to find a bounded open interval $\mathcal{I}$ centered in t_1 such that

$$\mathcal{E}^0(t) \subset \cup_{k=1}^l \mathcal{B}_k$$

for all $t \in \mathcal{I}$. The uniform convergence in $\mathcal{I}$ implies the existence of $\varepsilon_0 > 0$ such that $\mathcal{E}^\varepsilon(t) \subset \cup_{k=1}^l \mathcal{B}_k$ for all $t \in \mathcal{I}$ when $0 \le \varepsilon \le \varepsilon_0$. Note that the sets $\mathcal{B}_1, \dots \mathcal{B}_l$ are pairwise disjoint. Recall also that $\varphi_j^\varepsilon(t_1) = \varphi_j(t_1)$. Fix $\varepsilon \in [0, \varepsilon_0]$, and assume that there exists $t_2 \in \mathcal{I}$ with $\rho_j^\varepsilon(t_2) \in \mathcal{B}_k$. Then,

- $\rho_j^\varepsilon(t) \in \mathcal{B}_k$ for all $t \in \mathcal{I}$, as can be deduced easily from the continuity of ρ_j^ε with respect to t;
- $\rho_j^\varepsilon(t_1) = \tilde\rho_k$, since the only eigenvalue of $W_V^\varepsilon(t_1) = W_V^0(t_1)$ in $\mathcal{B}_k$ is $\tilde\rho_k$;
- $\varphi_j^\varepsilon(t) = \arg_k(\rho_j^\varepsilon(t))$ for all $t \in \mathcal{I}$, since $\varphi_j^\varepsilon(t_1) = \varphi_j^0(t_1) = \arg_k \tilde\rho_k$ and φ_j^ε is continuous in t.

Now fix $j \in \{1, \dots, n\}$, take $t_2 \in \mathcal{I}$, and choose $k \in \{1, \dots, l\}$ with $\rho_j^0(t_2) \in \mathcal{B}_k$, so that $\rho_j^0(t_1) = \tilde\rho_k$. Choose also sequences (j_m) in $\{1, \dots, n\}$ and $(\varepsilon_m) \downarrow 0$ with $\varepsilon_1 \le \varepsilon_0$ such that $\lim_{m\to\infty} \rho_{j_m}^{\varepsilon_m}(t_2) = \rho_j^0(t_2)$. Then there exists m_0 such that $\rho_{j_m}^{\varepsilon_m}(t_2) \in \mathcal{B}_k$ for all $m \ge m_0$. Moreover, as was seen above: first, $\rho_{j_m}^{\varepsilon_m}(t_1) = \tilde\rho_k$ for all $m \ge m_0$, so that $\lim_{m\to\infty} \rho_{j_m}^{\varepsilon_m}(t_1) = \tilde\rho_k = \rho_j^0(t_1)$; and second, $\varphi_{j_m}^{\varepsilon_m}(t_i) = \arg_k(\rho_{j_m}^{\varepsilon_m}(t_i))$ for $i = 1, 2$ and $m \ge m_0$. Consequently,

$$\begin{aligned}\varphi_j^0(t_2) - \varphi_j^0(t_1) &= \arg_k \rho_j^0(t_2) - \arg_k \rho_j^0(t_1) \\ &= \lim_{m\to\infty} (\arg_k \rho_{j_m}^{\varepsilon_m}(t_2) - \arg_k \rho_{j_m}^{\varepsilon_m}(t_1)) = \lim_{m\to\infty} (\varphi_{j_m}^{\varepsilon_m}(t_2) - \varphi_{j_m}^{\varepsilon_m}(t_1)) \,.\end{aligned}$$

The increasing character of the elements of the sequence $(\varphi_{j_m}^{\varepsilon_m})$ guarantees that $\varphi_j(t) = \varphi_j^0(t)$ is nondecreasing in $\mathcal{I}$. The proof is complete.

The main consequence of these properties, which concerns again the family of systems (2.1), is now stated and proved. Recall again that, given any ergodic measure m, $\alpha(m)$ represents the rotation number of the family with respect to m.

Theorem 2.31 *Suppose that $H_3 \colon \Omega \to \mathbb{S}_n(\mathbb{R})$ is continuous and takes positive semidefinite values. Then $\alpha(m) \geq 0$ for every σ-ergodic measure m on Ω.*

Proof According to Theorem 2.4,

$$\alpha(m) = \lim_{t\to\infty} \frac{1}{t}\ \operatorname{Arg}_3 U(t,\omega)$$

for m-a.e. $\omega \in \Omega$. Fix one of these points ω, define

$$W_U(t,\omega) = (U_1(t,\omega) - iU_3(t,\omega))^{-1}(U_1(t,\omega) + iU_3(t,\omega))$$

and choose continuous argument functions $\varphi_1(t,\omega),\dots,\varphi_n(t,\omega)$ for the eigenvalues of $W_U(t,\omega)$. Lemma 2.29(i) implies that $\varphi(t,\omega) = (1/2)\sum_{j=1}^n \varphi_j(t,\omega)$ is a continuous branch of $\operatorname{Arg}_3 U(t,\omega) = \arg\det(U_1(t,\omega) + iU_3(t,\omega))$, so that

$$\alpha(m) = \lim_{t\to\infty} \frac{1}{t}\,\varphi(t,\omega) = \lim_{t\to\infty} \frac{1}{t}\,(\varphi(t,\omega) - \varphi(0,\omega)) \tag{2.48}$$

m-a.e. Theorem 2.30 ensures that φ is nondecreasing in t, which proves the assertion.

Remark 2.32 Note that the same proof can be used to show that, in the case that H_3 is continuous and $H_3(\omega) \geq 0$ for all ω in a positively σ-invariant subset $\Omega_1 \subseteq \Omega$, and if m is a σ-ergodic measure on Ω with $m(\Omega_1) = 1$, then $\alpha(m) \geq 0$.

The result stated in Theorem 2.31 is complemented by the following one, which shows that a certain order in the coefficient matrices of the equation implies an order in the corresponding rotation numbers. The matrix-valued functions H^1 and H^2 are supposed to satisfy the conditions initially imposed on H: the matrices H^1 and H^2 satisfy Hypotheses 2.1, and JH^1 and JH^2 are symmetric.

Proposition 2.33 *Suppose that $JH^1 \leq JH^2$. Let m be a σ-ergodic measure on Ω, and let $\alpha_j(m)$ represent the rotation number of the family $\mathbf{z}' = H^j(\omega{\cdot}t)\,\mathbf{z}$ for $j = 1,2$. Then $\alpha_1(m) \leq \alpha_2(m)$.*

Proof Define $\operatorname{Tr} Q^j \colon \mathcal{K}_{\mathbb{R}} \to \mathbb{R}$ from H^j as $\operatorname{Tr} Q$ from H in (1.19), for $j = 1,2$. It is obvious $\operatorname{Tr} Q^1 \leq \operatorname{Tr} Q^2$, so that the result follows from (2.6).

As was stated previously, the second goal of this section is to give a new definition of the rotation number in the case $H_3 \geq 0$, which requires first to define the notion of proper focal points of a given matrix-valued solution. The following lemma is fundamental to this purpose, and will also be used in Chap. 5. Note that it refers to a single Hamiltonian system 2.45. A different approach to a similar result is given in Theorem 3 of [93].

Lemma 2.34 *Assume that* $H_{03} \geq 0$. *Given* $l \in \mathcal{L}_{\mathbb{R}}$, *represent* $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, *and* $\left[\begin{smallmatrix} L_1(t) \\ L_2(t) \end{smallmatrix}\right] = U(t)\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, *where* $U(t)$ *is the matrix-valued solution of* (2.45) *with* $U(0) = I_{2n}$. *Then,*

(i) *if the rank of* $L_1(t)$ *is constant on* (a, b) *with* $-\infty \leq a < b \leq \infty$, *then* $\operatorname{Ker} L_1(t)$ *is constant on* (a, b). *In addition, if* $a \in \mathbb{R}$ *(resp.* $b \in \mathbb{R}$*) and* $t \in (a, b)$, *then* $\operatorname{Ker} L_1(t) \subseteq \operatorname{Ker} L_1(a)$ *(resp.* $\operatorname{Ker} L_1(t) \subseteq \operatorname{Ker} L_1(b)$*).*

(ii) *Given any finite interval* $[a, b]$ *there exists a finite number of points* $t_0, \ldots, t_m$ *with* $a = t_0 < t_1 < \ldots < t_m = b$ *such that the rank of* $L_1(t)$ *is constant on* (t_{j-1}, t_j) *for* $j = 1, \ldots, m$. *Consequently, for* $j = 1, \ldots, m$, $\operatorname{Ker} L_1(t)$ *is constant on* (t_{j-1}, t_j) *and* $\operatorname{Ker} L_1(t) \subseteq \operatorname{Ker} L_1(t_{j-1}) \cap \operatorname{Ker} L_1(t_j)$ *for all* $t \in (t_{j-1}, t_j)$.

Proof Take $\left[\begin{smallmatrix} L_3 \\ L_4 \end{smallmatrix}\right] \equiv l_1 \in \mathcal{L}_{\mathbb{R}}$ such that $\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] \in \operatorname{Sp}(n, \mathbb{R})$; for instance, $L_3 = L_2 R^{-1}$ and $L_4 = -L_1 R^{-1}$ for $R = L_1^T L_1 + L_2^T L_2$. Then call $V(t) = U(t)\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_3(t) & L_1(t) \\ L_4(t) & L_2(t) \end{smallmatrix}\right]$, which is a symplectic matrix solution of (2.1); define

$$W_V(t) = (L_3(t) - iL_1(t))^{-1}(L_3(t) + iL_1(t))\,;$$

and choose continuous argument functions $\varphi_1(t), \ldots, \varphi_n(t)$ for the eigenvalues of $W_V(t)$, as in Theorem 2.30.

(i) Set $k(t) = \operatorname{Ker} L_1(t)$, and let d be the dimension of $\operatorname{Ker} L_1(t)$ for $t \in (a, b)$. Since the assertion is trivial for $d = 0$, assume that $d \geq 1$. The goal is hence to prove that $k(t)$ is constant on (a, b).

Take $s_1, s_2 \in (a, b)$ with $s_1 < s_2$. The first step of the proof consists in checking that, if $\mathbf{y}\colon (s_1, s_2) \to \mathbb{R}^n$ is C^1 and satisfies $\mathbf{y}(t) \in k(t)$ for all $t \in (s_1, s_2)$, then also $\mathbf{y}'(t) \in k(t)$ for $t \in (s_1, s_2)$. In turn, this proof is divided into three parts. First, since $L_1(t)\,\mathbf{y}(t) = \mathbf{0}$ and $V'(t) = H_0(t)\,V(t)$,

$$L_1(t)\,\mathbf{y}'(t) = -L_1'(t)\,\mathbf{y}(t) = -H_{03}(t)\,L_2(t)\,\mathbf{y}(t) \quad \text{if } t \in (s_1, s_2)\,. \tag{2.49}$$

Second, the symplectic character of $V(t)$ (see Proposition 1.23) ensures that $L_3^T L_2 - L_4^T L_1 = I_n$ and $L_1^T L_2 = L_2^T L_1$, so that $(L_3^T(t) + iL_1^T(t))\,L_2(t)\,\mathbf{y}(t) = \mathbf{y}(t) + L_4^T(t)\,L_1(t)\,\mathbf{y}(t) + iL_2^T(t)\,L_1(t)\,\mathbf{y}(t) = \mathbf{y}(t)$; consequently,

$$L_2(t)\,\mathbf{y}(t) = (L_3^T(t) + iL_1^T(t))^{-1}\mathbf{y}(t) \quad \text{if } t \in (s_1, s_2)\,. \tag{2.50}$$

And third, from $W_V(t)\,\mathbf{y}(t) = \mathbf{y}(t)$ (see Lemma 2.29(ii)) it follows that

$$i\,L_V(t)\,W_V(t)\,\mathbf{y}(t) + W_V(t)\,\mathbf{y}'(t) = \mathbf{y}'(t) \quad \text{if } t \in (s_1, s_2)\,,$$

where $L_V = 2\,(L_3 - iL_1)^{-1}\,H_{03}\,(L_3^T + iL_1^T)^{-1}$ (see the proof of Theorem 2.30), and hence that $i\,\mathbf{y}^T(t)\,L_V(t)\,\mathbf{y}(t) = (\mathbf{y}^T(t) - \mathbf{y}^T(t)\,W_V(t))\,\mathbf{y}'(t) = 0$. The definition of L_V and the existence of a unique positive semidefinite square root of H_{03} (see Proposition 1.19(i)) imply that

$$H_{03}(t)\,(L_3^T(t) + iL_1^T(t))^{-1}\,\mathbf{y}(t) = \mathbf{0} \quad \text{if } t \in (s_1, s_2)\,. \tag{2.51}$$

Equalities (2.49), (2.50), and (2.51) prove the assertion.

Now it is possible to deduce that the space $k(t)$ is constant on (a, b). To check this, use the fact that the rank of $L_1(t)$ is constant in order to find c_1 and c_2 in (a, b) and C^1 functions $\mathbf{y}_1, \ldots, \mathbf{y}_d\colon (c_1, c_2) \to \mathbb{R}^n$ such that $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_d(t)\}$ is a basis of $k(t)$ for each $t \in (c_1, c_2)$. (For instance: using the concepts given in [89], one has that the 0-group reduces to $\{0\}$ at an arbitrary point $c_0 \in (a, b)$: this is due to the fact that the multiplicity of the eigenvalue 0 is constant for the matrix $L_1(t)$ on (a, b). Therefore, Theorem II.5.4 of [89] guarantees that the (total) projection corresponding to 0 is C^1 at the point c_0, and this fact implies the assertion.) Then, since $\mathbf{y}_1'(t), \ldots, \mathbf{y}_d'(t) \in k(t)$, there exist continuous functions $c_{jl}\colon (t_1, \infty) \to \mathbb{R}$ for $j, l = 1, \ldots, d$ such that $\mathbf{y}_j'(t) = \sum_{l=1}^d c_{jl}(t)\,\mathbf{y}_l(t)$. Let $C(t)$ be the $d \times d$ matrix-valued function with element $c_{jl}(t)$ in the j-row and l-column, so that $[\mathbf{y}_1(t) \cdots \mathbf{y}_d(t)]' = [\mathbf{y}_1(t) \cdots \mathbf{y}_d(t)]\,C^T(t)$. Let $E(t) = \big[\,e_{jl}(t)\,\big]$ be the fundamental matrix solution of $\mathbf{x}' = C(t)\,\mathbf{x}$ with $E(t_0) = I_d$ for a fixed $t_0 \in (c_1, c_2)$. Then, by uniqueness of solutions, $[\mathbf{y}_1(t) \cdots \mathbf{y}_d(t)] = [\mathbf{y}_1(t_0) \cdots \mathbf{y}_d(t_0)]\,E^T(t)$; that is, $\mathbf{y}_j(t) = \sum_{l=1}^d e_{jl}(t)\,\mathbf{y}_l(t_0)$ for all $t \in (a, c)$, which implies $k(t) = k(t_0)$ for all $t \in (c_1, c_2)$. In order to deduce that the same holds for $t \in (a, b)$, assume first for contradiction that $c^* = \sup\{c \in (c_1, b) \mid k(t) \text{ is constant on } (c_1, c)\}$ is not b; i.e. that $c^* \in (c_1, b)$. The same argument as above guarantees the existence of $\varepsilon > 0$ such that $k(t)$ is constant in $(c^* - \varepsilon, c^* + \varepsilon)$, which contradicts the definition of c^*. And the same argument can be used again to obtain a contradiction if one assumes that $c_* = \inf\{c \in (a, c_2) \mid k(t) \text{ is constant on } (c, c_2)\}$ is not a.

The first assertion (i) is proved. The second one is an immediate consequence of the first one and the continuity of $L_1(t)$.

(ii) According to Theorem 2.30, the argument functions $\varphi_1(t), \ldots, \varphi_n(t)$ are nondecreasing on $\mathbb{R}$. If the dimension of $\operatorname{Ker} L_1(t)$ varies at a point $t_* \in (a, b)$, then at least one $\varphi_* \in \{\varphi_1, \ldots, \varphi_n\}$ has the following two properties: $\varphi_*(t_*)$ is an integer multiple of 2π; and either $\varphi_*(t_* - \varepsilon, \omega) < \varphi_*(t_*, \omega)$ for any $\varepsilon > 0$ or $\varphi_*(t_*, \omega) < \varphi_*(t_* + \varepsilon, \omega)$. Clearly, for each argument function, this happens at most at finitely many points of $[a, b]$, which proves the result.

As in the previous lemma, take $l \in \mathcal{L}_{\mathbb{R}}$, and represent $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} L_1(t) \\ L_2(t) \end{smallmatrix}\right] = U(t)\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. A point $t_0 \in \mathbb{R}$ is a *focal point* or *vertical point for* $\left[\begin{smallmatrix} L_1(t) \\ L_2(t) \end{smallmatrix}\right]$ if $\det L_1(t_0) = 0$, which means that this solution intersects the Maslov cycle $\mathcal{C}$ given by (2.35) at

t_0. That is to say, the *multiplicity of* t_0 *with respect to* $\begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix}$, which is by definition $\dim \operatorname{Ker} L_1(t_0)$, is positive.

Among these focal points, the so-called *proper* ones are fundamental in the analysis of the oscillatory properties of the Hamiltonian systems (2.1) when $H_3 \geq 0$. The equivalence stated in the next definition is a consequence of Lemma 2.34.

Definition 2.35 Assume that $H_{03} \geq 0$. A point $t_0 \in \mathbb{R}$ is a *proper focal point for* $\begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix} = U(t) \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, where $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \equiv l \in \mathcal{L}_{\mathbb{R}}$, if

$$\operatorname{Ker} L_1(t_0^-) \subsetneq \operatorname{Ker} L_1(t_0)\,,$$

where $\operatorname{Ker} L_1(t_0^-)$ denotes the left-hand limit of the kernel of $L_1(t)$ at the point t_0. Or equivalently, if

$$m(t_0) = \dim \operatorname{Ker} L_1(t_0) - \dim \operatorname{Ker} L_1(t_0^-) \geq 1\,.$$

In this case, $m(t_0)$ is the *multiplicity of the proper focal point* t_0 *with respect to* $\begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix}$.

Note that, when $H_{03} \geq 0$, Lemma 2.34 shows that $\begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix}$ has a finite number of proper focal points in each bounded subinterval of $\mathbb{R}$, although in a positive half-line it may have infinitely many proper focal points.

The alternative definition for the rotation number can now be stated. Consider again the whole family of linear Hamiltonian systems (2.1). Take $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$ and define the rotation number of the family (2.1) with respect to a fixed σ-ergodic measure m_0, with $H_3 \geq 0$, as

$$\alpha = \lim_{t\to\infty} \frac{\pi}{t} \sum_{t_0 \in \mathcal{F}_t(\omega,l)} m(t_0) \tag{2.52}$$

Here, $\mathcal{F}_t(\omega, l)$ is the set of its proper focal points contained in the interval $[0, t]$ of the $2n \times n$ matrix-valued solution $\begin{bmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{bmatrix} = U(t, \omega) \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ of (2.1) for $l \equiv \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. The next result shows that α is well defined and agrees indeed with the rotation number previously defined; in other words, that the limit is independent of the choice of $l \in \mathcal{L}_{\mathbb{R}}$ and that it agrees for every $\omega \in \Omega$ with the limit (2.5) defining the rotation number, so that in particular the value of the limit is constant m_0-a.e.

Theorem 2.36 *Suppose that* $H_3 \geq 0$. *For every* $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$, *the limit* (2.52) *agrees with the limit* (2.5).

Proof Use now a notation similar to that established at the beginning of the proof of Lemma 2.34: since the quantities occurring depend on ω, it is convenient to add ω to the notation. Note that the equivalent of argument functions explained in Sect. 2.1.1

ensures that the proof of Theorem 2.31 can be repeated in order to show that (2.48) holds for the choice of the argument given by $\varphi(t,\omega) = (1/2)\sum_{j=1}^{n}\varphi_j(t,\omega)$.

Observe that the dimension of $\operatorname{Ker} L_1(t,\omega)$ increases strictly at a point $t_* > 0$ (in other words, t_* is a proper focal point) if and only if there exists at least one argument $\varphi_* \in \{\varphi_1,\dots,\varphi_n\}$ which reaches at t_* an integer multiple of 2π "arriving from below"; i.e. $\varphi_*(t_*,\omega) \in 2\pi\mathbb{Z}$ and $\varphi_*(t_*-\varepsilon,\omega) < \varphi_*(t_*,\omega)$ for any $\varepsilon > 0$. In this case, φ_* contributes one unit to the quantity $m(t_*)$. In other words, $m(t_*)$ measures the number of those argument functions of the set $\{\varphi_1,\dots,\varphi_n\}$ which reach at t_* an integer multiple of 2π arriving from below.

The previous comment has the following consequences. First, if there is no proper focal point in $[0,\infty)$, or if there is just one, then the increment of each argument function $\varphi_j(t,\omega)$ is less than 4π, so that (2.48) ensures that $\alpha = 0$, and the equality of the theorem is trivial. Assume now that there exist at least two, and let $t_0 \geq 0$ be the smallest one. Fix any $t > 0$, and let $p_j(t)$ be the number of times that the argument function φ_j reaches an integer multiple of 2π arriving from below in the interval $(t_0, t]$, for $j = 1,\dots,n$. Then the number of proper focal points in $(t_0, t]$ is at least one for large enough t. And, if for these values of t, $\mathcal{F}_t - \{t_0\} = \{t_1,\dots,t_l\}$, then $m(t_1) + \cdots + m(t_l) = p_1(t) + \cdots + p_n(t)$. Take now $t > t_l$ which is less than the immediately next proper focal point (if it exists). Then

$$\begin{aligned}&\left|\sum_{j=1}^{n}\varphi_j(t,\omega) - \sum_{j=1}^{n}\varphi_j(t_0,\omega) - 2\pi\sum_{s=1}^{l} m(t_s)\right|\\&\qquad = \left|\sum_{j=1}^{n}\varphi_j(t,\omega) - \sum_{j=1}^{n}\varphi_j(t_0,\omega) - 2\pi\sum_{j=1}^{n} p_j(t)\right| \leq 2n\pi\,.\end{aligned}\tag{2.53}$$

The last inequality follows from $|\varphi_j(t,\omega) - \varphi_j(t_0,\omega) - 2\pi p_j(t)| \leq 2\pi$ for $j = 1,\dots,n$, which in turn follows easily from the definition of $p_j(t)$ and the nondecreasing character of φ_j.

The statement of the theorem is now an easy consequence of (2.48), (2.53), and the definition of $\mathcal{F}_t$.

Note the connections between the last result and the definition of the rotation number in terms of the Arnold–Maslov cycle which was discussed in Sect. 2.1.3.

2.5 The Lyapunov Index

This section is devoted to the definition and main properties of a new index β, which is closely related to the Lyapunov exponents of (2.1) and hence to the exponential growth of the solutions. It will be seen in Sect. 3.2 that, in the higher-dimensional case, this index plays a role similar to that of the positive Lyapunov exponent for two-dimensional systems with zero trace. Following Craig and Simon [36], it will be called the Lyapunov index of (2.1). As in the case of the rotation number (Theorem 2.4), this index admits an ergodic representation in terms of the polar symplectic coordinates introduced in Theorem 1.41. This representation is obtained in [112] for the case of a continuous coefficient matrix H and is adapted here to the more general setting under consideration, in which H is assumed to satisfy Hypotheses 2.1.

Throughout this section, $\|\mathbf{z}\|$ and $\|A\|$ represent the Euclidean norms of a vector and a matrix, respectively. The results which will be obtained are independent of these particular choices of the norm.

Recall that $U(t,\omega)$ represents the fundamental matrix solution of (2.1) with $U(0,\omega) = I_{2n}$ and that m_0 is a fixed σ-ergodic measure on Ω. Recall also (see Definition 1.83) that the four *characteristic exponents* of the system (2.1) corresponding to ω for the element $\mathbf{z}_0 \in \mathbb{R}^{2n}$, $\mathbf{z} \neq \mathbf{0}$, are the values of the limits

$$\limsup_{t\to\pm\infty} \frac{1}{t}\,\ln(\|U(t,\omega)\,\mathbf{z}_0\|)\,, \qquad \liminf_{t\to\pm\infty} \frac{1}{t}\,\ln(\|U(t,\omega)\,\mathbf{z}_0\|)\,,$$

which are invariant along the orbits of the flow $\tau_{\mathbb{R}}$ defined by (1.13) on the bundle $\Omega \times \mathbb{R}^{2n}$. In addition, if for a pair $(\omega, \mathbf{z}_0)$ the four limits agree, then their common value is a *Lyapunov exponent* of the system.

The following result establishes that, for m_0-a.e. system of the family (2.1), there exist $2n$ common Lyapunov exponents, which can be equal or distinct. The first statements are part of the Multiplicative Ergodic Theorem (see Oseledets [118] and Ruelle [130]). A straightforward proof can be found in Johnson *at al.* [86]. Note that these results do not require H to be continuous but only that it induce a continuous flow on $\Omega \times \mathbb{R}^{2n}$, a property which is guaranteed in the present case by Hypotheses 2.1, Proposition 1.38 and Remark 1.40.

The concept of a wedge product (see e.g. [22]) is fundamental for the understanding of Theorem 2.37. For $j = 1, \ldots, 2n$, let $\wedge^j\mathbb{R}^{2n}$ denote the vector space generated by all the wedge products $\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_j$, where $\mathbf{z}_1, \ldots, \mathbf{z}_j \in \mathbb{R}^{2n}$. Recall that the wedge product $\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_j$ is linear in each factor separately, and that interchanging two factors changes the sign of the product. The dimension of $\wedge^j\mathbb{R}^{2n}$ is $\binom{2n}{j}$, and its canonical basis is $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_j} \mid 1 \le i_1 < \cdots < i_j \le 2n\}$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2n}\}$ being the canonical basis of $\mathbb{R}^{2n}$. In addition, a $2n \times 2n$ matrix L induces a linear map $\wedge^j L\colon \wedge^j\mathbb{R}^{2n} \to \wedge^j\mathbb{R}^{2n}$ by the formula $\wedge^j L(\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_j) = L\mathbf{z}_1 \wedge \cdots \wedge L\mathbf{z}_j$, and its (Euclidean) norm $\|\wedge^j L\|$ is the (Euclidean) norm of its matrix in the canonical basis.

The ergodicity of the measure m_0 plays a fundamental role in the following statement. Recall that $\mathcal{G}_k(\mathbb{R}^{2n})$ represents the set of the k-dimensional subspaces of $\mathbb{R}^{2n}$, and that τ_k is the flow induced by (2.1) on $\Omega \times \mathcal{G}_k(\mathbb{R}^{2n})$: see Sects. 1.2.2 and 1.3.1.

Theorem 2.37 *There exist real numbers* $\rho_1 > \cdots > \rho_d$*, positive integers* $n_1, \ldots, n_d$ *with* $n_1 + \cdots + n_d = 2n$*, and a Borel* σ*-invariant subset* $\Omega_0 \subseteq \Omega$ *with* $m_0(\Omega_0) = 1$ *such that*

o.1) *for every* $j = 1, \ldots, d$ *and* $\omega \in \Omega_0$ *there exists a* n_j*-dimensional vector space* W^j_ω *with* $\lim_{|t|\to\infty}(1/t)\ln\|U(t,\omega)\,\mathbf{z}\| = \rho_j$ *for* $\mathbf{z} \in W^j_\omega - \{\mathbf{0}\}$.

o.2) *The map* $\Omega_0 \to \mathcal{G}_{n_j}(\mathbb{R}^{2n})\,,\ \omega \mapsto W^j_\omega$ *is Borel measurable for* $j = 1, \ldots, d$*, with* $\tau_{n_j}(t, \omega, W^j_\omega) = (\omega{\cdot}t, W^j_{\omega\cdot t})$ *for all* $\omega \in \Omega_0$ *and* $t \in \mathbb{R}$.

o.3) $\mathbb{R}^{2n} = W^1_\omega \oplus \cdots \oplus W^d_\omega$ *for all* $\omega \in \Omega_0$.

In addition, if $\beta_1 \geq \cdots \geq \beta_{2n}$ *represent the numbers* $\rho_1, \ldots, \rho_d$ *repeated according to their multiplicities* $n_1, \ldots, n_d$*, then*

$$\lim_{t\to\infty} \frac{1}{t} \ln\|\wedge^j U(t,\omega)\| = \beta_1 + \cdots + \beta_j \tag{2.54}$$

for $j = 1, \ldots, 2n$ *and* $\omega \in \Omega_0$.

Note that, for $j = 1, \ldots, d$, the set

$$W_j = \{(\omega, \mathbf{z}) \in \Omega \times \mathbb{R}^{2n} \mid \omega \in \Omega_0 \text{ and } \mathbf{z} \in W^j_\omega\} \subseteq \Omega \times \mathbb{R}^{2n}$$

has properties which recall those of the closed subbundles of Definition 1.63: it is $\tau_\mathbb{R}$-invariant (in the sense that it is composed of orbits) and, for a set of points ω of full measure m_0, its fibers W^j_ω are linear subspaces of $\mathbb{R}^{2n}$ of constant dimension. Of course, they don't need to be closed. It is usual to refer to these sets as the *Oseledets subbundles*.

Definition 2.38 The (possibly repeated) numbers $\beta_1, \ldots, \beta_{2n}$ are the *Lyapunov exponents* of the family of linear Hamiltonian systems (2.1) *with respect to* m_0.

The last point of the following result states a well-known property of the Lyapunov exponents in the Hamiltonian case, which justifies Definition 2.41. A proof is included for the reader's convenience. The set Ω_0 is that given in Theorem 2.37.

Lemma 2.39 *Let* V *be a real or complex symplectic matrix. Let* $\eta_1^2, \ldots, \eta_{2n}^2$ *be the eigenvalues of the matrix* V^*V*, with* $\eta_1 \geq \cdots \geq \eta_{2n} > 0$*. Then,*

(i) $\eta_{2n+1-j} = \eta_j^{-1}$ *for* $j = 1, \ldots, n$,
(ii) $\eta_1 \cdots \eta_n \geq 1$ *and* $\operatorname{tr}(V^*V) > n$,
(iii) $\|\wedge^j V\| = \eta_1 \cdots \eta_j$.

Proof It follows from Proposition 1.23 that V^*V is a symplectic matrix, which according to Proposition 1.22 ensures (i). Consequently, $\eta_j \geq 1$ for $j = 1, \ldots, n$, which proves (ii).

Identifying $\wedge^j\mathbb{C}^{2n}$ with $\mathbb{C}^d$ for $d = \binom{2n}{j}$ by taking coordinates in the canonical basis allows one to define an (Euclidean) inner product $\langle\cdot,\cdot\rangle$ on $\wedge^j\mathbb{C}^{2n}$. It can be checked that

$$\langle\, \mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_j, \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_j \,\rangle = \det \begin{bmatrix} \mathbf{w}_1^* \mathbf{z}_1 & \cdots & \mathbf{w}_j^* \mathbf{z}_1 \\ \vdots & \ddots & \vdots \\ \mathbf{w}_1^* \mathbf{z}_j & \cdots & \mathbf{w}_j^* \mathbf{z}_j \end{bmatrix}.$$

It follows easily that the maps $\wedge^j L$ and $\wedge^j L^*$ are adjoint for any $2n \times 2n$ matrix L, so that their matrices in the canonical basis have the property that one is the conjugate transpose of the other one. That is, if M is the matrix of $\wedge^j V$, then the matrix of $\wedge^j V^*$ is M^*. It is very easy to deduce that M^*M is the matrix of $\wedge^j (V^* V)$ and that its eigenvalues are the elements of the set $\{\eta_{i_1}^2 \cdots \eta_{i_j}^2 \mid 1 \le i_1 < \cdots < i_j \le 2n\}$. Hence, assertion (iii) follows from the fact that the Euclidean norm of $\wedge^j V$ is the positive square root of the spectral radius of M^*M (see Remark 1.24.2).

Proposition 2.40 *Let $\eta_1^2(t,\omega), \ldots, \eta_{2n}^2(t,\omega)$ be the eigenvalues of the matrix $U^T(t,\omega)\, U(t,\omega)$, with $\eta_1(t,\omega) \ge \cdots \ge \eta_{2n}(t,\omega) > 0$. Let Ω_0 be the set appearing in Theorem 2.37. Then,*

(i) $\|\wedge^j\, U(t,\omega)\| = \eta_1(t,\omega) \cdots \eta_j(t,\omega) \ge 1$,
(ii) $\beta_j = \lim_{t\to\infty} (1/t) \ln \eta_j(t,\omega)$ *for* $j = 1, \ldots, 2n$ *and* $\omega \in \Omega_0$,
(iii) *the Lyapunov exponents of* (2.1) *for the measure* m_0 *are* $\pm\beta_1, \ldots, \pm\beta_n$ *with* $\beta_1 \ge \cdots \beta_n \ge 0$.

Proof The assertions in (i) are proved in Lemma 2.39. Property (ii) follows from (i) and (2.54), and property (iii) follows (ii) and from $\eta_j(t,\omega) = \eta_{2n+1-j}^{-1}(t,\omega)$ (see Lemma 2.39(i)), which implies that $\beta_j = -\beta_{2n+j-1}$.

Definition 2.41 The *Lyapunov index* of the family of linear Hamiltonian systems (2.1) *with respect to* m_0 is

$$\beta = \beta_1 + \cdots + \beta_n = \lim_{t\to\infty} \frac{1}{t} \ln \|\wedge^n U(t,\omega)\| \tag{2.55}$$

for $\omega \in \Omega_0$, where $\wedge^n$ denotes the nth wedge product and $\pm\beta_1, \ldots, \pm\beta_n$ are the Lyapunov exponents of (2.1) with $\beta_1 \ge \cdots \beta_n \ge 0$.

Remarks 2.42

1. The existence of exponential dichotomy for the family (2.1) implies that 0 does not belong to its Sacker–Sell spectrum (see Definitions 1.82 and 1.87). According to Theorem 2.3 of [86], this ensures that 0 is not one of the Lyapunov exponents of the family, independently of the fixed measure m_0. In addition, in this case, the closed subbundles L^+ and L^- of Definition 1.75 are given by the sums of the Oseledets subbundles corresponding to negative and positive Lyapunov exponents, respectively.
2. All the results summarized here, as well as the definition of the Lyapunov index, have complete analogues for a complex linear Hamiltonian system.

Such a system is given by a map $H:\Omega \to \mathfrak{sp}(n,\mathbb{C})$, for which the corresponding fundamental matrix solution $U(t,\omega)$ belongs to $\mathrm{Sp}(n,\mathbb{C})$. Now, $\beta_j = \lim_{t\to\infty}(1/t)\ln\eta_j(t,\omega)$ where $\eta_1(t,\omega) \ge \cdots \ge \eta_{2n}(t,\omega)$ and $\eta_1^2(t,\omega),\ldots,\eta_{2n}^2(t,\omega)$ are the eigenvalues of the positive definite matrix $U^*(t,\omega)\,U(t,\omega)$; and, as before, $\beta_j = -\beta_{2n+j-1}$. The proof of the corresponding Proposition 2.40 is identical to that given above.

3. It is a well-known fact (and very easy to prove) that the Lyapunov exponents of a Hamiltonian system with constant coefficient matrix H are the real parts of the eigenvalues of H, and hence the Lyapunov index is the sum of those real parts which are positive. In addition, the classical Floquet theory proves that the Lyapunov exponents of a Hamiltonian system with periodic coefficients are the real parts of the characteristic exponents of the system, so that the sum of those which are positive provides the Lyapunov index. In both cases the ergodic measure m_0 is unique.

To get an ergodic representation of β for the real family (2.1), the following lemma is required. It is strongly based on Theorem 2.37, whose notation is maintained. As usual, the linear subspace of $\mathbb{R}^{2n}$ generated by the vectors $\mathbf{z}_1,\ldots,\mathbf{z}_k$ is denoted by $\langle\mathbf{z}_1,\ldots,\mathbf{z}_k\rangle$; and for a linear subspace V of $\mathbb{R}^{2n}$, $V^\perp$ represents the linear subspace (of dimension $2n-\dim V$) orthogonal to V for the Euclidean inner product.

Lemma 2.43 *Let $\Omega_0 \subseteq \Omega$ be the σ-invariant subset with $m_0(\Omega_0) = 1$ which is given in Theorem 2.37. For each $\omega \in \Omega_0$, the space $\mathbb{R}^{2n}$ admits a basis*

$$\{\mathbf{z}^-_{\omega,1},\ldots,\mathbf{z}^-_{\omega,n},\mathbf{z}^+_{\omega,1},\ldots,\mathbf{z}^+_{\omega,n}\}$$

satisfying

(i) $\lim_{|t|\to\infty}(1/t)\ln\|U(t,\omega)\,\mathbf{z}^-_{\omega,j}\| = \beta_j$ *for* $j = 1,\ldots,n$,
(ii) $\lim_{|t|\to\infty}(1/t)\ln\|U(t,\omega)\,\mathbf{z}^+_{\omega,j}\| = -\beta_j$ *for* $j = 1,\ldots,n$, *and*
(iii) *the subspaces* $l^-_\omega = \langle\mathbf{z}^-_{\omega,1},\ldots,\mathbf{z}^-_{\omega,n}\rangle$ *and* $l^+_\omega = \langle\mathbf{z}^+_{\omega,1},\ldots,\mathbf{z}^+_{\omega,n}\rangle$ *are real Lagrange planes.*

Proof Represent by $V_-(\omega)$, $V_0(\omega)$ and $V_+(\omega)$ the sum of the linear subspaces W^j_ω provided by Theorem 2.37 and corresponding to the strictly positive, null and strictly negative Lyapunov exponents, respectively. Property o.3) of Theorem 2.37 and Proposition 2.40(iii) guarantee the existence of an integer $k \in \{0,1,\ldots,n\}$ such that $\dim V_\pm(\omega) = n-k$ and $\dim V_0(\omega) = 2k$ and $\mathbb{R}^{2n} = V_-(\omega)\oplus V_0(\omega)\oplus V_+(\omega)$ for each $\omega\in\Omega_0$. Fix $\omega\in\Omega_0$ and note that

$$\mathbf{w}^TJ\mathbf{z} = 0 \quad \text{if} \begin{cases} \mathbf{z},\mathbf{w}\in V_-(\omega) \text{ or } \mathbf{z},\mathbf{w}\in V_+(\omega)\,, \\ \mathbf{z}\in V_-(\omega),\;\; \mathbf{w}\in V_0(\omega)\,, \\ \mathbf{z}\in V_+(\omega),\;\; \mathbf{w}\in V_0(\omega)\,. \end{cases} \tag{2.56}$$

These equalities follow from the relation $\mathbf{w}^T J\mathbf{z} = \mathbf{w}^T U^T(t,\omega) J U(t,\omega)\,\mathbf{z}$ for all $t \in \mathbb{R}$, since property o.1) implies that $U(t,\omega)\,\mathbf{z}$ tends to $\mathbf{0}$ exponentially fast as $t \to \infty$ for $\mathbf{z} \in V_+$, that $U(t,\omega)\,\mathbf{z}$ tends to $\mathbf{0}$ as $t \to -\infty$ exponentially fast for $\mathbf{z} \in V_-$, and that for each $\varepsilon > 0$ and each $\mathbf{z} \in V_0$, $\|U(t,\omega)\,\mathbf{z}\| \le e^{\varepsilon t}$ if $|t|$ is large enough.

In the case that $k = n$ (i.e. $V_0(\omega) = \mathbb{R}^{2n}$) the canonical basis of $\mathbb{R}^{2n}$ satisfies the required conditions. Assume that $k < n$. If one takes the union of bases of the Oseledets subspaces W^j_ω one obtains bases $\{\mathbf{z}^-_{\omega,k+1},\dots,\mathbf{z}^-_{\omega,n}\}$ and $\{\mathbf{z}^+_{\omega,k+1},\dots,\mathbf{z}^+_{\omega,n}\}$ of $V_-(\omega)$ and $V_+(\omega)$ respectively. Conditions (i) and (ii) follow directly from o.1) for $j = k+1,\dots,n$. In the case that $k = 0$, property (iii) is guaranteed by (2.56), and the proof is complete. In the remaining case, that is, $2 \le \dim V_0(\omega) = 2k < 2n$, Lemma 2.44 below provides a basis $\{\mathbf{z}^-_{\omega,1},\dots,\mathbf{z}^-_{\omega,k},\mathbf{z}^+_{\omega,1},\dots,\mathbf{z}^+_{\omega,k}\}$ of $V_0(\omega)$ such that (i) and (ii) are also satisfied for $j = 1,\dots,k$. A new application of (2.56) shows that also (iii) holds, and this completes the proof.

Lemma 2.44 *Let $V_0 \subseteq \mathbb{R}^{2n}$ be a linear subspace with $\dim V_0 = 2k$ for $k \ge 1$. Then V_0 has a basis $\{\mathbf{z}^-_1,\dots\mathbf{z}^-_k,\mathbf{z}^+_1,\dots,\mathbf{z}^+_k\}$ with $(\mathbf{z}^-_j)^T J\mathbf{z}^-_k = (\mathbf{z}^+_j)^T J\mathbf{z}^+_k = 0$ for $1 \le j \le k \le n$.*

Proof Any basis of V_0 satisfies the lemma if $k = 1$. Assume that the result is true for $s-1$. If $J\mathbf{z} \in V_0^\perp$ for all $\mathbf{z} \in V$, again any basis is suitable. Assume that this is not the case, and choose $\mathbf{z}^-_k \in V_0 - \{\mathbf{0}\}$ such that the orthogonal projection of $J\mathbf{z}^-_k$ on V_0 is $\mathbf{z}^+_k \ne \mathbf{0}$. Since $J\mathbf{z}^-_k - \mathbf{z}^+_k$ is orthogonal to V_0,

$$0 = (\mathbf{z}^-_k)^T(J\mathbf{z}^-_k - \mathbf{z}^+_k) = -(\mathbf{z}^-_k)^T\mathbf{z}^+_k \quad \text{and} \quad 0 = (\mathbf{z}^+_k)^T(J\mathbf{z}^-_k - \mathbf{z}^+_k)\,. \tag{2.57}$$

The first equality ensures that $\dim\langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle = \dim\langle \mathbf{z}^-_k,\mathbf{z}^+_k\rangle = 2$. Define $V_0^* = V_0 \cap \langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp$. It will be checked in the last paragraph of this proof that $\langle \mathbf{z}^-_k,\mathbf{z}^+_k\rangle \cap \langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp = \{\mathbf{0}\}$. This fact has two fundamental consequences: first, $\mathbb{R}^{2n} = \langle \mathbf{z}^-_k,\mathbf{z}^+_k\rangle \oplus \langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp = V_0 + \langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp$, and hence

$$\dim V_0^* = \dim V_0 + \dim\langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp - 2n = 2k - 2\,;$$

and second, $V_0 = V_0^* \oplus \langle \mathbf{z}^-_k,\mathbf{z}^+_k\rangle$, since $V_0^* \cap \langle \mathbf{z}^-_k,\mathbf{z}^+_k\rangle = \{\mathbf{0}\}$. Therefore, if one takes the union of the basis $\{\mathbf{z}^-_1,\dots\mathbf{z}^-_{k-1},\mathbf{z}^+_1,\dots,\mathbf{z}^+_{k-1}\}$ of V_0^* given by the induction hypothesis and $\{\mathbf{z}^-_k,\mathbf{z}^+_k\}$, one obtains a basis of V_0 with the required properties.

Finally, if $a\,\mathbf{z}^-_k + b\,\mathbf{z}^+_k \in \langle J\mathbf{z}^-_k, J\mathbf{z}^+_k\rangle^\perp$, then $0 = a\,(\mathbf{z}^-_k)^T J\mathbf{z}^-_k + b\,(\mathbf{z}^-_k)^T J\mathbf{z}^+_k = b\,(\mathbf{z}^-_k)^T J\mathbf{z}^+_k$ and $0 = a\,(\mathbf{z}^+_k)^T J\mathbf{z}^-_k + b\,(\mathbf{z}^+_k)^T J\mathbf{z}^+_k = a\,(\mathbf{z}^+_k)^T J\mathbf{z}^-_k$. The second equality in (2.57) ensures that $(\mathbf{z}^+_k)^T J\mathbf{z}^-_k = (\mathbf{z}^+_k)^T\mathbf{z}^+_k \ne 0$. This implies that $a = b = 0$ and completes the proof.

Remark 2.45 It can be assumed that the σ-invariant subset Ω_0 appearing in Theorem 2.37 and Lemma 2.43 is included in the ergodic component of the measure m_0 in Ω; that is, for each $\omega \in \Omega_0$, $\lim_{t\to\infty}(1/t)\int_0^t f(\omega{\cdot}s)\,ds = \int_\Omega f(\omega)\,dm_0$ for every $f \in C(\Omega,\mathbb{R})$: see Remark 1.10.

The following theorem provides the previously announced ergodic representation for the Lyapunov index β. The $n \times n$ matrix-valued function S was introduced in Theorem 1.41, and the function $\operatorname{Tr} S$ is defined by (1.20).

Theorem 2.46 *Let β be the Lyapunov index of the family* (2.1) *with respect to m_0. Then*

$$\beta = \sup_{\mu} \left\{ \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} S(\omega, l)\, d\mu \right\} = \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} S(\omega, l)\, d\mu_0\,,$$

where the supremum is taken over the set of τ-invariant measures on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0, and μ_0 is a certain τ-ergodic measure on this set.

Proof Let Ω_0 be the subset of Ω described in Theorem 2.37, with the additional property explained in Remark 2.45, and let $l^-_{\tilde\omega}$ and $l^+_{\tilde\omega}$ be the real Lagrange planes for $\tilde\omega \in \Omega_0$ provided by Lemma 2.43. The point $\tilde\omega$ will be conveniently chosen later. Write the $2n \times n$ real matrices $[\mathbf{z}^-_{\tilde\omega,1} \cdots \mathbf{z}^-_{\tilde\omega,n}] = \begin{bmatrix} L^-_1 \\ L^-_2 \end{bmatrix}$ and $[\mathbf{z}^+_{\tilde\omega,1} \cdots \mathbf{z}^+_{\tilde\omega,n}] = \begin{bmatrix} L^+_1 \\ L^+_2 \end{bmatrix}$ which represent these planes as $\begin{bmatrix} \Phi^-_1 R_- \\ \Phi^-_2 R_- \end{bmatrix}$ and $\begin{bmatrix} \Phi^+_1 R_+ \\ \Phi^+_2 R_+ \end{bmatrix}$, with $\Phi^\pm_1 + i\Phi^\pm_2$ unitary and $R_\pm$ nonsingular for $j = 1, 2$ (see Remark 1.27.3). In addition, the matrices $R_\pm$ can be chosen to have positive determinant. Note that the matrices $\Phi^\pm_1$, $\Phi^\pm_2$ and $R_\pm$ depend on the point $\tilde\omega$.

In order to simplify the notation, set $\mathbf{z}^\pm_j(t, \tilde\omega) = U(t, \tilde\omega)\, \mathbf{z}^\pm_{\tilde\omega,j}$ for $j = 1, \ldots, n$. Let $R_\pm(t, \tilde\omega) = R(t, \tilde\omega, \Phi^\pm_1, \Phi^\pm_2, R_\pm)$ and $\Phi^\pm_j(t, \tilde\omega) = \Phi_j(t, \tilde\omega, \Phi^\pm_1, \Phi^\pm_2)$ for $j = 1, 2$ represent the corresponding solutions of (1.16) and (1.15). According to Theorem 1.41,

$$\big[\, \mathbf{z}^\pm_1(t, \tilde\omega) \,\cdots\, \mathbf{z}^\pm_n(t, \tilde\omega) \,\big] = \begin{bmatrix} \Phi^\pm_1(t, \tilde\omega)\, R_\pm(t, \tilde\omega) \\ \Phi^\pm_1(t, \tilde\omega)\, R_\pm(t, \tilde\omega) \end{bmatrix},$$

and $(\Phi^\pm_1)^T(t, \tilde\omega)\, \Phi^\pm_1(t, \tilde\omega) + (\Phi^\pm_2)^T(t, \tilde\omega)\, \Phi^\pm_2(t, \tilde\omega) = I_n$. Therefore,

$$\begin{aligned} \det(R_-(t, \tilde\omega)^T R_-(t, \tilde\omega)) &= \det([\mathbf{z}^-_1(t, \tilde\omega) \cdots \mathbf{z}^-_n(t, \tilde\omega)]^T [\mathbf{z}^-_1(t, \tilde\omega) \cdots \mathbf{z}^-_n(t, \tilde\omega)]) \\ &= \|\mathbf{z}^-_1(t, \tilde\omega)\|^2 \cdots \|\mathbf{z}^-_n(t, \tilde\omega)\|^2 \det \widetilde{R}_-(t, \tilde\omega)\,, \end{aligned}$$

where the entry of the matrix $\widetilde{R}_-(t, \tilde\omega)$ corresponding to the jth row and kth column is $(\|\mathbf{z}^-_j(t, \tilde\omega)\|\, \|\mathbf{z}^-_k(t, \tilde\omega)\|)^{-1} (\mathbf{z}^-_j)^T(t, \tilde\omega)\, \mathbf{z}^-_k(t, \tilde\omega)$. It follows that $\det \widetilde{R}_-(t, \tilde\omega)$ (which is positive) is bounded from above. Consequently, Lemma 2.43 ensures that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \det R_-(t, \tilde\omega) \le \sum_{j=1}^{n} \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{z}^-_j(t, \tilde\omega)\| = \sum_{j=1}^{n} \beta_j = \beta\,. \tag{2.58}$$

Analogously,

$$\limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_+(t,\tilde\omega) \le -\beta\,.$$

On the other hand, since $\operatorname{tr} H(\tilde\omega{\cdot}t) = 0$, the fundamental matrix solution

$$V(t,\tilde\omega) = \begin{bmatrix} \Phi_1^-(t,\tilde\omega) & \Phi_1^+(t,\tilde\omega) \\ \Phi_2^-(t,\tilde\omega) & \Phi_2^+(t,\tilde\omega) \end{bmatrix} \begin{bmatrix} R_-(t,\tilde\omega) & 0 \\ 0 & R_+(t,\tilde\omega) \end{bmatrix}$$

of the system (2.1) for $\tilde\omega$ has constant determinant, and hence

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{1}{t}\, \ln\det V(t,\tilde\omega) \\ &\le \limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_-(t,\tilde\omega) + \limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_+(t,\tilde\omega) \\ &\le \beta - \beta = 0\,, \end{aligned}$$

from which it follows that

$$\limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_-(t,\tilde\omega) = -\limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_+(t,\tilde\omega) = \beta\,. \tag{2.59}$$

Under these conditions, the equation (1.16) satisfied by $R_-(t,\tilde\omega)$ and definitions (1.18) and (1.20) guarantee that

$$\beta = \limsup_{t\to\infty} \frac{1}{t}\, \ln\det R_-(t,\tilde\omega) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} S(\tau(s,\tilde\omega,l_{\tilde\omega}^-))\, ds\,. \tag{2.60}$$

The next goal is to prove the existence of a τ-invariant measure μ_0 on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0 such that

$$\beta = \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} S(\omega,l)\, d\mu_0\,. \tag{2.61}$$

First, choose an increasing sequence $(t_m)\uparrow\infty$ with

$$\beta = \lim_{m\to\infty} \frac{1}{t_m} \int_0^{t_m} \operatorname{Tr} S(\tau(s,\tilde\omega,l_{\tilde\omega}^-))\, ds \tag{2.62}$$

and apply the Riesz representation theorem in order to associate to the bounded linear functional $C(\mathcal{K}_{\mathbb{R}},\mathbb{R}) \to \mathbb{R}\,,\ g \mapsto (1/t_m)\int_0^{t_m} g(\tau(s,\tilde\omega,l_{\tilde\omega}^-))\, ds$ a normalized measure μ_m. According to Theorem 1.9(i), the sequence (μ_m) admits a subsequence (μ_j) which converges weak* to a τ-invariant measure μ_0^*; that is,

$$\lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} g(\tau(s,\tilde\omega,l_{\tilde\omega}^-))\, ds = \int g(\omega,l)\, d\mu_0^* \tag{2.63}$$

for every continuous function g. It is easy to deduce from Remarks 2.45 and 1.14 that μ_0^* projects onto m_0. This and (2.60) prove (2.61) in the case that H (and hence $\operatorname{Tr} S$) is continuous. The more general conditions imposed on H require some more work, which is now carried out.

Repeating the ideas of the proof of Lemma 2.24, one shows: that $\operatorname{Tr} S \in L^1(\mathcal{K}_{\mathbb{R}}, \mu)$ for any measure on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0, since $H \in L^1(\Omega, m_0)$; and that there exists a positive constant k such that

$$|\operatorname{Tr} S(\omega, l) - \operatorname{Tr} S_{H_*}(\omega, l)| \le k\, \|H(\omega) - H_*(\omega)\| \tag{2.64}$$

for all $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$, where $\operatorname{Tr} S_{H_*}$ represents the matrix-valued function defined by (1.20) from a matrix H_* satisfying Hypotheses 2.1. Now fix $\varepsilon > 0$. Since $H \in L^1(\Omega, m_0)$, there exists a continuous symplectic matrix-valued function H^ε such that

$$\int_\Omega \|H(\omega) - H^\varepsilon(\omega)\|\, dm_0 \le \frac{\varepsilon}{2k}\,. \tag{2.65}$$

In addition, since $H - H^\varepsilon \in L^1(\Omega, m_0)$, Birkhoff's ergodic theorem and the ergodicity of m_0 can be used to find a set $\Omega^\varepsilon \subseteq \Omega$ with $m_0(\Omega^\varepsilon) = 1$ such that, for each $\omega_\varepsilon \in \Omega^\varepsilon$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|H(\omega_\varepsilon{\cdot}s) - H^\varepsilon(\omega_\varepsilon{\cdot}s)\|\, ds = \int_\Omega \|H(\omega) - H^\varepsilon(\omega)\|\, dm_0 \tag{2.66}$$

(see Theorems 1.3 and 1.6). Now take $\varepsilon_m = 1/m$ for each $m \in \mathbb{N}$. The point $\tilde\omega \in \Omega_0$ used to obtain (2.60) will be chosen as an element of the intersection $\Omega_0 \cap (\cap_{m\in\mathbb{N}} \Omega^{1/m})$, which has full measure with respect to m_0. The continuity of $\operatorname{Tr} S_{H^{1/m}}$ and relations (2.63)–(2.66) imply that, for each $m \in \mathbb{N}$,

$$\begin{aligned}
&\left| \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} S(\omega, l)\, d\mu_0^* - \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \operatorname{Tr} S(\tau(s, \tilde\omega, l_{\tilde\omega}^-)\, ds \right| \\
&\qquad \le \left| \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr}\, (S - S_{H^{1/m}})(\omega, l)\, d\mu_0^* \right| \\
&\qquad\quad + \left| \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \operatorname{Tr}\, (S - S_{H^{1/m}})(\tau(s, \tilde\omega, l_{\tilde\omega}^-))\, ds \right| \\
&\qquad \le \int_\Omega k\, \|(H - H^{1/m})(\omega)\|\, dm_0 \\
&\qquad\quad + \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} k\, \|(H - H^{1/m})(\tilde\omega{\cdot}s)\|\, ds \le \frac{1}{m}\,,
\end{aligned}$$

which together with (2.62) proves (2.61).

The next step consists in showing that, for any τ-invariant measure μ on $\mathcal{K}_\mathbb{R}$ projecting onto m_0, one has $\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu \le \beta$. Given such a measure, one proves easily from the Birkhoff Theorem 1.3 that there exists a point $(\omega_0, l_0) \in \Omega_0 \times \mathcal{L}_\mathbb{R}$ such that

$$\begin{aligned} \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu &\le \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} S(\tau(s, \omega_0, l_0))\, ds \\ &= \lim_{t\to\infty} \frac{1}{t} \ln \det R_0(t, \omega_0)\,, \end{aligned} \tag{2.67}$$

where $R_0(t, \omega_0) = R(t, \omega_0, \Phi_1^0, \Phi_2^0, I_n)$ for $l_0 \equiv \begin{bmatrix} \Phi_1^0 \\ \Phi_2^0 \end{bmatrix}$ with $\Phi_1^0 + i\Phi_2^0$ unitary. Represent the n independent column vectors of the real Lagrange plane $\begin{bmatrix} \Phi_1^0 \\ \Phi_2^0 \end{bmatrix}$ by $\mathbf{z}_{\omega_0,1}, \ldots, \mathbf{z}_{\omega_0,n}$, and consider the basis $\{\mathbf{z}^-_{\omega_0,1}, \ldots, \mathbf{z}^-_{\omega_0,n}, \mathbf{z}^+_{\omega_0,1}, \ldots, \mathbf{z}^+_{\omega_0,n}\}$ obtained in Lemma 2.43. Then,

$$[\mathbf{z}_{\omega_0,1} \cdots \mathbf{z}_{\omega_0,n}] = [\mathbf{z}^-_{\omega_0,1} \cdots \mathbf{z}^-_{\omega_0,n}]\, A + [\mathbf{z}^+_{\omega_0,1} \cdots \mathbf{z}^+_{\omega_0,n}]\, B$$

for some $n \times n$ real matrices A and B. Let P be a nonsingular matrix such that AP is lower triangular, and represent by a_{jk} (resp. b_{jk}) the entry of the matrix AP (resp. BP) corresponding to the jth row and kth column. Then $[\tilde{\mathbf{z}}_{\omega_0,1} \cdots \tilde{\mathbf{z}}_{\omega_0,n}] = [\mathbf{z}_{\omega_0,1} \cdots \mathbf{z}_{\omega_0,n}]P$ defines a new basis $\{\tilde{\mathbf{z}}_{\omega_0,1} \cdots \tilde{\mathbf{z}}_{\omega_0,n}\}$ of the Lagrange plane with

$$\tilde{\mathbf{z}}_{\omega_0,j} = \sum_{k=j}^{n} a_{kj}\, \mathbf{z}^-_{\omega_0,k} + \sum_{k=1}^{n} b_{kj}\, \mathbf{z}^+_{\omega_0,k}$$

for $j = 1, \ldots, n$. This expression and Lemma 2.43 imply that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|U(t, \omega_0)\, \tilde{\mathbf{z}}_{\omega_0,j}\| \le \beta_j$$

for $j = 1, \ldots, n$. Under these conditions, it is easy to adapt the argument applied to $R_-(t, \omega)$ in the proof of (2.58) in order to check that $\lim_{t\to\infty}(1/t) \ln \det R_0(t, \omega_0) \le \beta$, which together with (2.67) proves that

$$\sup_\mu \left\{ \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu \right\} = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu_0^* = \beta\,, \tag{2.68}$$

as asserted.

The proof of Theorem 2.46 will be completed once one has checked the existence of a τ-ergodic measure μ_0 projecting onto m_0 and for which $\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu_0 = \beta$. Note first that that $\lim_{k\to\infty} \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu_k = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu$ if μ is the limit of the sequence (μ_k) in the weak* topology.

To prove this, recall that $H \in L^1(\Omega, m_0)$, so that given $\varepsilon > 0$ is possible to find a continuous matrix-valued function $H^*\colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$ with $\int_\Omega \|H(\omega) - H^*(\omega)\|\, dm_0 \le \varepsilon/(4k)$, where k satisfies the condition (2.64). In addition, since the function $\operatorname{Tr} S_{H^*}$ is continuous (see Remark 1.42), there exists j_0 such that $\Big|\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H^*}(\omega,l)\, d\mu_j - \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H^*}(\omega,l)\, d\mu\Big| < \varepsilon/2$ for $j \ge j_0$. Hence, using (2.64),

$$\begin{aligned}
&\left|\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\mu_j - \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\mu\right| \\
&\quad \le \int_{\mathcal{K}_\mathbb{R}} |\operatorname{Tr} S(\omega,l) - \operatorname{Tr} S_{H^*}(\omega,l)|\, d\mu_j + \int_{\mathcal{K}_\mathbb{R}} |\operatorname{Tr} S(\omega,l) - \operatorname{Tr} S_{H^*}(\omega,l)|\, d\mu \\
&\qquad + \left|\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H^*}(\omega,l)\, d\mu_j - \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H^*}(\omega,l)\, d\mu_j\right| \le \varepsilon
\end{aligned}$$

for $j \ge j_0$, which proves the assertion.

Define $\tilde\beta = \sup_\nu \Big\{\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\nu\Big\}$, where the supremum is taken over the set of τ-ergodic measures on $\mathcal{K}_\mathbb{R}$ projecting onto m_0. By (2.68), $\tilde\beta \le \beta$. Clearly, $\operatorname{Tr} S(\omega,l)\, d\mu \le \tilde\beta$ for any convex combination μ of σ-ergodic measures. Therefore, according to (2.68), Proposition 1.15(iii), and the property explained above, $\beta = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\mu_0^* \le \tilde\beta$, and hence $\tilde\beta = \beta$. Now take a sequence (ν_k) of τ-ergodic measures projecting onto m_0 with $\lim_{k\to\infty} \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\nu_k = \beta$. Proposition 1.15(ii) can be used to find a subsequence (ν_j) which converges to a τ-invariant measure μ_0 in the weak*-topology. The characterization given in Theorem 1.6 ensures that μ_0 is τ-ergodic: just keep in mind that the intersection of a countable number of sets with total measure has total measure. Therefore, $\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega,l)\, d\mu_0 = \beta$, and the proof of Theorem 2.46 is complete.

To formulate the next consequence of the previous theorem, let $\beta(H)$ represent the Lyapunov index of the family (2.1) with respect to the fixed σ-ergodic measure m_0.

Corollary 2.47 *Suppose that $H = \lim_{m\to\infty} H_m$ in the $L^1(\Omega, m_0)$-topology, where all the matrix-valued functions $H, H_m\colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$ satisfy Hypotheses 2.1. Then*

$$\beta(H) \ge \limsup_{m\to\infty} \beta(H_m)\,.$$

In other words, the Lyapunov exponent is a semicontinuous function with respect to the coefficient matrix.

Proof Let the map $H \mapsto \operatorname{Tr} S_H$ be defined by (1.20). Theorem 2.46 provides, for each $m \in \mathbb{N}$, a τ_m-ergodic measure μ_m projecting onto m_0 such that $\beta(H_m) = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H_m}(\omega,l)\, d\mu_m$. Let (H_j) be any subsequence of (H_m). The following steps repeat those of the proof of Theorem 2.25, where many more details are given: there exists a subsequence (μ_k) of (μ_j) which converges in the weak* topology to a measure μ which is τ^H-invariant and projects onto m_0; and

$$\lim_{k\to\infty}\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_{H_k}(\omega,l)\,d\mu_k = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_H(\omega,l)\,d\mu\,. \tag{2.69}$$

To prove this property one must use the analogue of Lemma 2.24 for $\operatorname{Tr} S_H$ instead of $\operatorname{Tr} Q_H$, which has already been required in the proof of Theorem 2.46. Finally, it follows from Theorem 2.46 and (2.69) that

$$\beta(H) \ge \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S_H(\omega,l)\,d\mu = \lim_{k\to\infty}\beta(H_k)\,,$$

and the assertion follows easily from this inequality and the definition of the superior limit of a sequence.

Remark 2.48 The result stated in Corollary 2.47 is optimal, in the following sense: the Lyapunov index is not a continuous function with respect to the coefficient matrix H, even when this matrix is two-dimensional, and even with respect to the uniform topology on the set of continuous matrix-valued functions on Ω. This assertion is proved, for instance, by the example constructed by Johnson in [70]. It consists of: a one-parameter scalar Schrödinger equation $-x'' + (g(\omega{\cdot}t) - \lambda)\,x = 0$ where g is determined as the uniform limit of a sequence of periodic functions; a point $\lambda_0 \in \mathbb{R}$; and a sequence (λ_m) such that $\lim_{m\to\infty}\lambda_m = \lambda_0$, $\lim_{m\to\infty}\beta(\lambda_m) = 0$, and $\beta(\lambda_0) > 0$. Here, of course, $\beta(\lambda)$ is the Lyapunov index of the equation given by λ.

Chapter 3
The Floquet Coefficient for Nonautonomous Linear Hamiltonian Systems: Atkinson Problems

Let (Ω, σ) be a real continuous flow on a compact metric space. In the previous chapter, the notions of rotation number and Lyapunov index for a family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{3.1}$$

were introduced and some of their properties were analyzed. The central objective of the present chapter is to study a deeper aspect of the relation between these objects and the concept of exponential dichotomy. To this end it is convenient to complexify the rotation number, in order to view it as the imaginary part of a complex number, called the Floquet coefficient of the family, whose real part equals the negative Lyapunov index.

More precisely, consider the $2n$-dimensional family of linear Hamiltonian systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda\, J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{3.2}$$

which can be understood as a perturbation of (3.1) (which corresponds to $\lambda = 0$) in the direction determined by Γ. Here, $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$ is the usual antisymmetric matrix; the matrix-valued function $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right]$ is continuous on Ω; $\Gamma = \left[\begin{smallmatrix} -\Gamma_2 & \Gamma_1^T \\ \Gamma_1 & \Gamma_3 \end{smallmatrix}\right]$ is a continuous real symmetric $2n \times 2n$ matrix-valued function on Ω (and hence $J^{-1}\Gamma = \left[\begin{smallmatrix} \Gamma_1 & \Gamma_3 \\ \Gamma_2 & \Gamma_1^T \end{smallmatrix}\right]$ takes values in the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$); and λ is a complex parameter. A non-degeneracy condition, to be described later, will be imposed on Γ. When this condition is satisfied, the family (3.2) is called an *Atkinson problem*, since the conditions of Chapter 9 of [5] are satisfied for each $\omega \in \Omega$. Note that, in order to include in the general formulation the perturbed n-dimensional linear

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Schrödinger equation

$$-\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \lambda\,\Delta(\omega{\cdot}t)\,\mathbf{x}\,, \quad \omega \in \Omega\,, \tag{3.3}$$

where Δ is a continuous real symmetric $n{\times}n$ matrix-valued function on Ω, it suffices to define $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{x}'\end{smallmatrix}\right]$, $H = \left[\begin{smallmatrix}0_n & I_n\\ G & 0_n\end{smallmatrix}\right]$ and $\Gamma = \left[\begin{smallmatrix}\Delta & 0_n\\ 0_n & 0_n\end{smallmatrix}\right]$.

Let m_0 be a fixed σ-ergodic measure on Ω. By letting the parameter λ of equations (3.2) take on real values, the corresponding rotation number and the Lyapunov index with respect to m_0 (see Definitions 2.5 and 2.41) can be considered as functions on $\mathbb{R}$ and represented by $\alpha_\Gamma(\lambda)$ and $\beta_\Gamma(\lambda)$. In fact, as explained in Remark 2.42.2, the same expression

$$\beta_\Gamma(\lambda) = \lim_{t\to\infty} \frac{1}{t}\, \ln \| \wedge^n U_{\Gamma,\lambda}(t,\omega)\|\,, \tag{3.4}$$

where $U_{\Gamma,\lambda}(t,\omega)$ is the fundamental matrix solution of (3.2) with $U_{\Gamma,\lambda}(0,\omega) = I_{2n}$, defines the Lyapunov index $\beta_\Gamma(\lambda)$ on the whole complex plane. The following fundamental result is proved by Craig and Simon in [36] and Kotani and Simon in [91]. Recall that a function $f\colon \mathbb{C} \to \mathbb{R}$ is *subharmonic* if it is *upper semicontinuous* (i.e. if $\limsup_{\lambda\to\lambda_0} f(\lambda) \le f(\lambda_0)$ for every $\lambda_0 \in \mathbb{C}$) and *submean* (i.e. if $f(\lambda_0) \le (1/2\pi)\int_0^{2\pi} f(\lambda_0 + \varepsilon e^{i\theta})\,d\theta$ for every $\lambda_0 \in \mathbb{C}$ and every $\varepsilon > 0$).

Theorem 3.1 *The Lyapunov index is a subharmonic function on* $\mathbb{C}$.

But recall that, in general, it is not a continuous function on the real axis, as pointed out in Remark 2.48.

On the other hand, the rotation number $\alpha_\Gamma(\lambda)$ is a continuous function on the real axis, as Theorem 2.25 proves. However, the extension of its definition to the complex plane is not immediate. In fact, it requires hypotheses on the perturbation Γ ensuring the existence of exponential dichotomy for the family (3.2) outside the real axis. Once this is established, the hyperbolic character of the flow ensures the existence of the Weyl functions (also called Weyl matrices); and in turn the Weyl functions allow the definition of a complex function, called the Floquet coefficient and denoted by $w_\Gamma(\lambda)$, which is holomorphic on the complex open upper half-plane, whose real part agrees with the negative Lyapunov index and whose imaginary part converges to the rotation number when the parameter λ approaches to the real axis.

These are basically the contents of the first two sections of this chapter. Naturally, this Floquet coefficient extends to the higher dimensional case a well-known object for two-dimensional systems (see Johnson and Moser [73]), which in turn generalizes the Floquet index for periodic systems (see Magnus and Winkler [98]). In fact, the function $w_\Gamma(\lambda)$ has properties analogous to those of the usual (periodic) Floquet exponents.

In particular, this holomorphic extension is very useful in studying the properties of the spectral problems given by (3.2) and (3.3). When $n = 1$, it was used to study the quasi-periodic Schrödinger operator by Johnson and Moser in [73] and the general two-dimensional AKNS system by Giachetti and Johnson in [55], among other papers. In the context of the Atkinson problem (3.2), it was used to derive a relation between the rotation number and the existence of exponential dichotomy for the linear Hamiltonian family by Johnson in [72] and by Johnson and Nerurkar in [75]. Modulo details, the relation can be stated as follows: if the rotation number is constant on an open interval $\mathcal{I} \subseteq \mathbb{R}$, then the system (3.2) has exponential dichotomy for each $\lambda \in \mathcal{I}$. The proof of this result requires the elements of Atkinson's spectral theory for (3.2) (see Atkinson [5], Chapter 9) together with a fundamental trace formula. The previously established properties of the Floquet coefficient are used in the proof of this trace formula, which also requires the relation between the constant character of the rotation number and the presence of exponential dichotomy. These two proofs are contained in the third and last section.

This chapter reproduces basically the scheme of the survey of the theory of the Floquet coefficient which was carried out by Fabbri et al. in [46], which in turn relies on the papers Johnson [72] and Johnson and Nerurkar [75, 77]. Here, substantially more details of the proofs, which are in fact nontrivial, are included.

The flow induced by the family (3.2) on $\mathcal{K}_\mathbb{C}$ and $\mathcal{K}_\mathbb{R}$ is denoted by $\tau_{\Gamma,\lambda}$, and, as said before, the fundamental matrix solution satisfying $U_{\Gamma,\lambda}(0,\omega) = I_{2n}$ is represented as $U_{\Gamma,\lambda}(t,\omega)$. Clearly, the matrix $U_{\Gamma,0}(t,\omega)$ agrees with $U(t,\omega)$, according to the notation established in the previous chapter, which will be maintained here. Finally, represent, as usual, $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0\}$ and $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda < 0\}$.

Remark 3.2 In order to unify the notation, the subscript Γ is also used to make reference to the Schrödinger case (3.3), although in this case using Δ would perhaps be more appropriate.

3.1 Exponential Dichotomy and the Weyl Functions

The results stated in this section are independent of the choice of the ergodic measure m_0. The definition of exponential dichotomy for the families (3.2) and (3.3) corresponding to a fixed value of the parameter $\lambda \in \mathbb{C}$ is given in Sect. 1.4.3. The corresponding (closed) stable subbundles at $\mp\infty$ will be represented by $L^{\pm}_{\Gamma,\lambda}$, and the corresponding continuous fibers will be represented by $l^{\pm}_{\Gamma,\lambda}(\omega)$.

As stated before, the definition of the Weyl functions and the Floquet coefficient outside the real axis requires the existence of exponential dichotomy for $\lambda \notin \mathbb{R}$, which in turn requires a non-degeneracy assumption on the unperturbed system (3.1). More precisely, as stated in Theorem 3.8, the exponential dichotomy outside the real axis is guaranteed by the following Atkinson type condition:

Hypotheses 3.3 The continuous matrix-valued function $\Gamma\colon \Omega \to \mathbb{S}_{2n}(\mathbb{R})$ is continuous, positive semidefinite (which, in the Schrödinger case, means that $\Delta\colon \Omega \to \mathbb{S}_{2n}(\mathbb{R})$ is positive semidefinite), and in addition each minimal subset of Ω contains at least one point ω_0 such that

$$\int_{-\infty}^{\infty} \|\Gamma(\omega_0{\cdot}t)\, U(t,\omega_0)\,\mathbf{z}\|^2\, dt > 0 \quad \text{whenever } \mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,. \tag{3.5}$$

Definition 3.4 A continuous symmetric matrix-valued function Γ satisfying Hypotheses 3.3 is an *Atkinson perturbation*.

Remarks 3.5

1. It can immediately be checked that, if $\Gamma > 0$ or, or if $\Delta > 0$ in the Schrödinger case, then Hypotheses 3.3 are valid.
2. Let $\Gamma^{1/2}$ be the unique positive semidefinite square root of Γ (see Proposition 1.19), and $\mathcal{I}$ any interval in $\mathbb{R}$. It is immediate to see that if the equality $\int_{\mathcal{I}} \|\Gamma^{1/2}(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\, dt = 0$ holds for a point $\omega \in \Omega$, then $\int_{\mathcal{I}} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\, dt = 0$.

The proof of the previous assertion concerning the occurrence of exponential dichotomy, which basically appears in [72] and [75], requires Theorem 1.78, which characterizes the occurrence of exponential dichotomy in terms of the absence of globally bounded solutions, as well as a technical lemma:

Lemma 3.6

(i) *Suppose that* (3.5) *holds for an element* $\omega_0 \in \Omega$. *Then, for each* $\lambda \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} \|\Gamma(\omega_0{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0)\,\mathbf{z}\|^2\, dt > 0 \quad \textit{whenever } \mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,.$$

(ii) *Let* $\mathcal{M} \subseteq \Omega$ *be a minimal subset and let* $\omega_0 \in \mathcal{M}$ *be an element such that* (3.5) *holds. Then, for each* $\lambda \in \mathbb{C}$, *there exist* $t_0 = t_0(\mathcal{M},\lambda) > 0$ *and* $\delta = \delta(\mathcal{M},\lambda) > 0$ *such that*

$$\int_0^{t_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\, dt > \delta\, \|\mathbf{z}\|^2 \quad \textit{whenever } \omega \in \mathcal{M} \textit{ and } \mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,.$$

(iii) *Suppose that* Γ *is an Atkinson perturbation. Then, for each* $\tilde\omega \in \Omega$ *and* $\lambda \in \mathbb{C}$ *there exist* $t_0 = t_0(\tilde\omega,\lambda) > 0$ *and* $\delta = \delta(\tilde\omega,\lambda) > 0$ *such that*

$$\int_0^{t_0} \|\Gamma(\tilde\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\tilde\omega)\,\mathbf{z}\|^2\, dt > \delta\, \|\mathbf{z}\|^2 \quad \textit{whenever } \mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,.$$

(iv) *Suppose that* Γ *is an Atkinson perturbation. Then, for each* $\lambda \in \mathbb{C}$ *there exist* $t_0 = t_0(\lambda) > 0$ *and* $\delta = \delta(\lambda) > 0$ *such that, for every* $\omega \in \Omega$,

$$\int_0^{t_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\, dt > \delta\, \|\mathbf{z}\|^2 \quad \textit{whenever } \mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,.$$

Proof

(i) Fix $\lambda \in \mathbb{C} - \{0\}$ and assume the existence of $\mathbf{z}_0 \in \mathbb{C}^{2n}$ such that $\int_{-\infty}^{\infty} \|\Gamma(\omega_0{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0)\, \mathbf{z}_0\|^2\, dt = 0$. It is easy to check that $\mathbf{z}(t) = U_{\Gamma,\lambda}(t,\omega_0)\, \mathbf{z}_0$ is a solution of the system (3.2) corresponding to ω_0 and to $\lambda = 0$, and hence $\mathbf{z}(t) = U(t,\omega_0)\, \mathbf{z}_0$. This and relation (3.5) for ω_0 imply $\mathbf{z}_0 = 0$.

(ii) Statement (i) applied to the point $\omega_0 \in \mathcal{M}$ ensures the existence of $s_0 = s_0(\omega_0) > 0$ and $\varepsilon > 0$ such that

$$\int_{-s_0}^{s_0} \|\Gamma(\omega_0{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0)\, \mathbf{z}\|^2\, dt > \varepsilon \quad \text{whenever } \|\mathbf{z}\| = 1\ :$$

otherwise, $\int_{-s_m}^{s_m} \|\Gamma(\omega_0{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0)\, \mathbf{z}_m\|^2\, dt < 1/m$ for sequences $(s_m) \uparrow \infty$ and $(\mathbf{z}_m)$ in the unit sphere of $\mathbb{C}^{2n}$; and hence, for the limit $\mathbf{z}_0$ of a convergent subsequence of $(\mathbf{z}_m)$, one has $\int_{-\infty}^{\infty} \|\Gamma(\omega_0{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0)\, \mathbf{z}_0\|^2\, dt = 0$.

Let $\mathcal{O} \subseteq \Omega$ be an open neighborhood of ω_0 such that

$$\int_{-s_0}^{s_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\, \mathbf{z}\|^2\, dt > \frac{\varepsilon}{2} \quad \text{whenever } \omega \in \mathcal{O} \text{ and } \|\mathbf{z}\| = 1\,. \tag{3.6}$$

Since $\mathcal{M}$ is minimal, there exist positive numbers $\tilde{t}_1, \ldots, \tilde{t}_p$ such that $\mathcal{M} \subseteq \mathcal{O}{\cdot}\tilde{t}_1 \cup \cdots \cup \mathcal{O}{\cdot}\tilde{t}_p$, as is easily deduced from the density of the σ-orbit of ω_0 in $\mathcal{M}$ and the compactness of the set $\mathcal{M}$. It follows easily that there exist a constant $l_0 > 0$ and a decreasing sequence $(t_m) \downarrow -\infty$ with $0 < t_m - t_{m+1} < l_0$ such that $\omega_m = \omega_0{\cdot}t_m \in \mathcal{O}$ for every $m \in \mathbb{N}$: just take $l_0 = \max\{\tilde{t}_1, \ldots, \tilde{t}_p\}$, choose $t_1 = -\tilde{t}_{i_1}$ with $\omega_0 \in \mathcal{O}{\cdot}\tilde{t}_{i_1}$, choose $t_2 = t_1 - \tilde{t}_{i_2}$ with $\omega_0{\cdot}t_1 \in \mathcal{O}{\cdot}\tilde{t}_{i_2}$, and so on. Now take any $s < t_1$ and write it as $s = t_m + l$ for an $m \in \mathbb{N}$ and an $l \in (-l_0, 0]$. Note that $\omega_0{\cdot}s = \omega_m{\cdot}l$ and that $l_0 + l > 0$, $-l_0 + l < 0$. Then, for all $\mathbf{z}$ in the unit sphere,

$$\begin{aligned}
&\int_{-s_0-l_0}^{s_0+l_0} \|\Gamma((\omega_0{\cdot}s){\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_0{\cdot}s)\, \mathbf{z}\|^2\, dt \\
&\qquad = \int_{-s_0-l_0+l}^{s_0+l_0+l} \|\Gamma(\omega_m{\cdot}t)\, U_{\Gamma,\lambda}(t-l,\omega_m{\cdot}l)\, \mathbf{z}\|^2\, dt \\
&\qquad \geq \int_{-s_0}^{s_0} \|\Gamma(\omega_m{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega_m)\, U_{\Gamma,\lambda}(-l,\omega_m{\cdot}l)\, \mathbf{z}\|^2\, dt > k\,\frac{\varepsilon}{2}\,,
\end{aligned}$$

where $k = \min_{\omega\in\Omega,\|z\|=1,\,l\in[-l_0,0]} \|U_{\Gamma,\lambda}(-l,\omega{\cdot}l)\, \mathbf{z}\|^2 > 0$. Relation (3.6) has been used in this reasoning.

Therefore, since the negative semiorbit $\{\omega_0{\cdot}s \mid s < t_1\}$ is dense in the compact set $\mathcal{M}$, there exists $\varepsilon^* > 0$ such that

$$\int_{-s_0-l_0}^{s_0+l_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt > \varepsilon^* \quad \text{whenever } \omega \in \mathcal{M} \text{ and } \|\mathbf{z}\| = 1\,.$$

Now it is easy to check that $\int_0^{2(s_0+l_0)} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt > 0$ whenever $\omega \in \mathcal{M}$ and $\mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}$. Write $t_0 = 2(s_0 + l_0)$. Note that the map $(\omega, \mathbf{z}) \mapsto \int_0^{t_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt$ is continuous and strictly positive, and hence its minimum value δ for $\omega \in \mathcal{M}$ and $\|\mathbf{z}\| = 1$ is strictly positive. Note that both t_0 and δ depend on the value of λ and on the minimal subset $\mathcal{M}$. This proves (ii).

(iii) Fix any element $\tilde{\omega} \in \Omega$ and let $\mathcal{M} \subseteq \Omega$ be a minimal subset contained in its omega-limit set. Let $t_0 = t_0(\mathcal{M},\lambda)$ and $\delta = \delta(\mathcal{M},\lambda)$ be the constants provided by Hypotheses 3.3 and statement (ii), and let $\mathcal{O} \subseteq \Omega$ be an open set containing $\mathcal{M}$ such that

$$\int_0^{t_0} \|\Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt > \frac{\delta}{2}\,\|\mathbf{z}\|^2$$

whenever $\omega \in \mathcal{O}$ and $\mathbf{z} \in \mathbb{C}^{2n} - \{\mathbf{0}\}$. Assume for contradiction that the assertion (iii) is false for every pair of positive constants t_0, δ. As in the beginning of the proof of (ii), an easy argument provides a point $\mathbf{z}_0$ belonging to the unit sphere of $\mathbb{C}^{2n}$ for which $\int_0^\infty \|\Gamma(\tilde{\omega}{\cdot}t)\, U_{\Gamma,\lambda}(t,\tilde{\omega})\,\mathbf{z}_0\|^2 = 0$. Now take $s > 0$ with $\tilde{\omega}{\cdot}s \in \mathcal{O}$. Then

$$\begin{aligned}\frac{\delta}{2}\,\|\tilde{\mathbf{z}}\|^2 &< \int_0^\infty \|\Gamma((\tilde{\omega}{\cdot}s){\cdot}t)\, U_{\Gamma,\lambda}(t,\tilde{\omega}{\cdot}s)\,\tilde{\mathbf{z}}\|^2\,dt \\ &= \int_s^\infty \|\Gamma(\tilde{\omega}{\cdot}t)\, U_{\Gamma,\lambda}(t-s,\tilde{\omega}{\cdot}s)\,\tilde{\mathbf{z}}\|^2\,dt = 0\end{aligned}$$

for $\tilde{\mathbf{z}} = U_{\Gamma,\lambda}^{-1}(-s,\tilde{\omega}{\cdot}s)\,\mathbf{z}_0$. This is impossible, and hence (iii) is proved.

(iv) The last assertion of the lemma follows from (iii) and a standard compactness argument.

Remark 3.7 Identical arguments prove the existence of $t_0 = t_0(\lambda) > 0$ and $\delta = \delta(\lambda) > 0$ such that property (iv) of the preceding lemma holds when integrating over $[-t_0, 0]$. In addition, if one keeps in mind Remark 3.5, it is very easy to deduce the existence of a number $\tilde{\delta} = \tilde{\delta}(\lambda) > 0$ such that property (iv) holds with Γ replaced by $\Gamma^{1/2}$ and δ replaced by $\tilde{\delta}$, for the same time $t_0 = t_0(\lambda)$.

Theorem 3.8 *Suppose that Γ is an Atkinson perturbation. Then,*

(i) *the family (3.2) has exponential dichotomy over Ω for* $\operatorname{Im}\lambda \neq 0$, *and*

(ii) *if* $\operatorname{Im}\lambda \neq 0$ *and* $\omega \in \Omega$, *the complex Lagrange planes* $l^{\pm}_{\Gamma,\lambda}(\omega)$ *given by the fibers of the closed subbundles* $L^{\pm}_{\Gamma,\lambda}$ *can be represented as* $\left[\begin{smallmatrix} I_n \\ M^{\pm}_{\Gamma}(\omega,\lambda)\end{smallmatrix}\right]$, *where the symmetric matrix-valued functions* $M^{\pm}_{\Gamma}$ *satisfy* $\pm \operatorname{Im}\lambda \, \operatorname{Im} M^{\pm}_{\Gamma}(\omega,\lambda) > 0$.

Proof

(i) Fix $\omega \in \Omega$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$. The main step of the proof consists in showing that the corresponding system (3.2) for ω does not admit a nonzero bounded solution. Defining

$$(\mathcal{L}^{\lambda}_{\omega}\mathbf{z})(t) = J\,\mathbf{z}'(t) - (JH(\omega{\cdot}t) + \lambda\,\Gamma(\omega{\cdot}t))\,\mathbf{z}(t)\,,$$

it turns out that, for every solution of the system (3.2),

$$\begin{aligned} 0 &= \int_a^b \big(\mathbf{z}^*(t)\,(\mathcal{L}^{\lambda}_{\omega}\mathbf{z})(t) - (\mathcal{L}^{\lambda}_{\omega}\mathbf{z})^*(t)\,\mathbf{z}(t)\big)\,dt \\ &= \mathbf{z}^*(t)J\mathbf{z}(t)\Big|_{t=a}^{t=b} - 2\,i\operatorname{Im}\lambda\int_a^b \mathbf{z}^*(t)\,\Gamma(\omega{\cdot}t)\,\mathbf{z}(t)\,dt \end{aligned} \tag{3.7}$$

whenever $a < b$. So, if there exists $\mathbf{z}_0 \neq \mathbf{0}$ such that $\mathbf{z}(t) = U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}_0$ is a bounded solution, then

$$\int_{-\infty}^{\infty} \|\Gamma^{1/2}(\omega{\cdot}t)\,U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}_0\|^2\,dt < \infty\,. \tag{3.8}$$

Let $t_0 = t_0(\lambda)$ be the constant provided by Lemma 3.6(iv). Condition (3.8) provides an increasing sequence $(t_m) \uparrow \infty$ such that

$$\begin{aligned} \frac{1}{m} &> \int_{t_m}^{t_m+t_0} \|\Gamma^{1/2}(\tilde\omega{\cdot}t)\,U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}_0\|^2\,dt \\ &= \int_0^{t_0} \|\Gamma^{1/2}((\omega{\cdot}t_m){\cdot}t)\,U_{\Gamma,\lambda}(t+t_m,\omega)\,\mathbf{z}_0\|^2\,dt \\ &= \int_0^{t_0} \|\Gamma^{1/2}((\omega{\cdot}t_m){\cdot}t)\,U_{\Gamma,\lambda}(t,\omega{\cdot}t_m)\,\mathbf{z}(t_m)\|^2\,dt\,, \end{aligned}$$

for every $m \in \mathbb{N}$. Since Ω is compact and $(\mathbf{z}(t_m))$ is bounded, it is possible to find a subsequence (t_j) and points $\tilde\omega \in \Omega$, $\tilde{\mathbf{z}}_0 \in \mathbb{C}^{2n}$ such that $\tilde\omega = \lim_{j\to\infty}\omega{\cdot}t_j$ and $\tilde{\mathbf{z}}_0 = \lim_{j\to\infty}\mathbf{z}(t_j)$. But then

$$\int_0^{t_0} \|\Gamma^{1/2}(\tilde\omega{\cdot}t)\,U_{\Gamma,\lambda}(t,\tilde\omega)\,\tilde{\mathbf{z}}_0\|^2\,dt = 0\,,$$

which together with Remark 3.5.2 and Lemma 3.6(iv) ensures that $\tilde{\mathbf{z}}_0 = \mathbf{0}$. Analogously, it is possible to find a decreasing sequence $(s_j) \downarrow -\infty$ such that

$\lim_{j\to\infty} \mathbf{z}(s_j) = \mathbf{0}$. Substituting a and b in (3.7) by s_j and t_j, it turns out that

$$\int_{-\infty}^{\infty} \|\Gamma^{1/2}(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega)\, \mathbf{z}_0\|^2\, dt = 0\,,$$

which contradicts, for instance, Lemma 3.6(i).

The absence of nonzero bounded solutions and Theorem 1.78 prove the existence of exponential dichotomy over Ω, which completes the proof of (i).

(ii) Let us fix $\omega \in \Omega$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, and let $t_0 = t_0(\lambda)$ and $\tilde{\delta} = \tilde{\delta}(\lambda)$ be the constants whose existence is proved in Lemma 3.6(iv) and Remark 3.7. Represent the Lagrange plane $l^{+}_{\Gamma,\lambda}(\omega)$ by $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. Assume for contradiction that L_1 is not invertible, choose $\mathbf{x} \in \mathbb{C}^n - \{0\}$ with $L_1\,\mathbf{x} = 0$, and note that then $L_2\,\mathbf{x} \neq \mathbf{0}$. Then, by formula (3.7), for all $b \geq t_0$,

$$\begin{aligned}
&\frac{1}{2\,i\operatorname{Im}\lambda}\,\mathbf{x}^*\left[\, L_1^*\; L_2^* \,\right] U^*_{\Gamma,\lambda}(t,\omega)\, J\, U_{\Gamma,\lambda}(t,\omega) \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \mathbf{x}\,\Bigg|_{t=0}^{t=b} \\
&\qquad = \int_0^b \mathbf{x}^*\left[\, L_1^*\; L_2^* \,\right] U^*_{\Gamma,\lambda}(t,\omega)\, \Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega) \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \mathbf{x}\, dt > \tilde{\delta}\,\|L_2\mathbf{x}\|^2\,.
\end{aligned}$$

Taking the limit as $b \uparrow \infty$ and keeping in mind that $\lim_{t\to\infty} U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z} = \mathbf{0}$ for all $\mathbf{z} \in l^{+}_{\Gamma,\lambda}(\omega)$ (see Definition 1.75), one has

$$-\frac{1}{2\,i\operatorname{Im}\lambda}\,\mathbf{x}^*\left[\, L_1^*\; L_2^* \,\right] J \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \mathbf{x} > 0\,,$$

which is impossible since $\mathbf{x}^*\left[\, L_1^*\; L_2^* \,\right] J \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \mathbf{x} = \left[\, \mathbf{0}\;\; \mathbf{x}^* L_2^* \,\right] J \left[\begin{smallmatrix} \mathbf{0} \\ L_2\,\mathbf{x} \end{smallmatrix}\right] = 0$. The same argument proves that L_2 is also an invertible matrix.

Thus, $l^{+}_{\Gamma,\lambda}(\omega)$ can be represented by $\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$, for an invertible symmetric matrix M. In addition, using again formula (3.7) for $b > t_0$, given any $\mathbf{x} \in \mathbb{C}^n - \{0\}$,

$$\begin{aligned}
&\frac{1}{2\,i\operatorname{Im}\lambda}\,\mathbf{x}^*\left[\, I_n\; M^* \,\right] U^*_{\Gamma,\lambda}(t,\omega)\, J\, U_{\Gamma,\lambda}(t,\omega) \begin{bmatrix} I_n \\ M \end{bmatrix} \mathbf{x}\,\Bigg|_{t=0}^{t=b} \\
&\quad = \int_0^b \mathbf{x}^*\left[\, I_n\; M^* \,\right] U^*_{\Gamma,\lambda}(t,\omega)\, \Gamma(\omega{\cdot}t)\, U_{\Gamma,\lambda}(t,\omega) \begin{bmatrix} I_n \\ M \end{bmatrix} \mathbf{x}\, dt > \delta\, \left\| \begin{bmatrix} \mathbf{x} \\ M\,\mathbf{x} \end{bmatrix} \right\|^2,
\end{aligned}$$

and hence, taking again the limit as $b \uparrow \infty$,

$$\begin{aligned}
\frac{1}{\operatorname{Im}\lambda}\,\mathbf{x}^* \operatorname{Im} M\,\mathbf{x} &= -\frac{1}{2\,i\operatorname{Im}\lambda}\,\mathbf{x}^*(M^* - M)\,\mathbf{x} \\
&= -\frac{1}{2\,i\operatorname{Im}\lambda}\,\mathbf{x}^*\left[\, I_n\; M^* \,\right] J \begin{bmatrix} I_n \\ M \end{bmatrix} \mathbf{x} > 0\,.
\end{aligned}$$

This implies that $\operatorname{Im}\lambda \operatorname{Im} M$ is a positive definite matrix and completes the proof of assertion (ii) for $L^+_{\Gamma,\lambda}$, with $M^+_\Gamma(\omega,\lambda) = M$. The proof is analogous for $L^-_{\Gamma,\lambda}$ (see Remark 3.7).

Let Γ be an Atkinson perturbation. According to Definition 1.80, the symmetric complex $n \times n$ matrices $M^\pm_\Gamma(\omega,\lambda)$ whose existence is ensured by Theorem 3.8 for $\omega \in \Omega$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, as well as the matrix-valued maps $M^\pm_\Gamma \colon \Omega \times (\mathbb{C} - \mathbb{R}) \to S_{\mathbb{C}}(n),\ (\omega,\lambda) \mapsto M^\pm_\Gamma(\omega,\lambda)$, are the *Weyl functions* or *Weyl matrices* (associated to Γ). An alternative and equivalent definition for the Weyl functions can be derived from the fact that Hypotheses 3.3 ensure that the systems (3.2) are in the limit-point case in $\pm\infty$. The reader is referred to Hinton and Shaw [61, 62] for the details of this alternative definition.

Theorem 3.9 *Let Γ be an Atkinson perturbation. Then the Weyl functions $M^\pm_\Gamma(\omega,\lambda)$ are jointly continuous in both variables and analytic on $\mathbb{C} - \mathbb{R}$ for each fixed $\omega \in \Omega$.*

Proof The continuity is a consequence of Theorem 1.95(ii). According to Theorem 1.94, $Q_\Gamma(\omega,\lambda)$ is analytic outside the real axis for $\omega \in \Omega$ fixed. As explained in Remark 1.81.1,

$$Q_\Gamma = \begin{bmatrix} (M^-_\Gamma - M^+_\Gamma)^{-1} M^-_\Gamma & -(M^-_\Gamma - M^+_\Gamma)^{-1} \\ M^+_\Gamma (M^-_\Gamma - M^+_\Gamma)^{-1} M^-_\Gamma & -M^+_\Gamma (M^-_\Gamma - M^+_\Gamma)^{-1} \end{bmatrix}$$

for $\lambda \in \mathbb{C} - \mathbb{R}$, where all the matrices are evaluated in (ω,λ). Therefore, the functions $(M^-_\Gamma - M^+_\Gamma)^{-1}$, $(M^-_\Gamma - M^+_\Gamma)^{-1} M^-_\Gamma$ and $M^+_\Gamma (M^-_\Gamma - M^+_\Gamma)^{-1}$ are analytic in λ, so that also $M^-_\Gamma - M^+_\Gamma$, M^-_Γ and M^+_Γ are.

In particular, the Weyl functions are symmetric Herglotz matrix-valued functions in the complex upper and lower half-planes for each fixed $\omega \in \Omega$. The definition of Herglotz matrix-valued function is now recalled:

Definition 3.10 A symmetric matrix-valued function M on $\mathbb{C}^+$ or $\mathbb{C}^-$ is *Herglotz* if it is analytic and $\operatorname{Im} M(\lambda)$ is either positive semidefinite or negative semidefinite on the whole half-plane.

Note also that $L^\pm_{\Gamma,\bar\lambda} = \overline{L^\pm_{\Gamma,\lambda}}$, as is easily deduced from $U_{\Gamma,\bar\lambda}(t,\omega) = \overline{U}_{\Gamma,\lambda}(t,\omega)$. Consequently, $M^\pm_\Gamma(\omega,\bar\lambda) = (M^\pm_\Gamma)^*(\omega,\lambda)$. The $\tau_{\Gamma,\lambda}$-invariance of the closed subbundles $L^\pm_{\Gamma,\lambda}$ ensures that, for every fixed non-real λ and $\omega \in \Omega$, the functions $t \mapsto M^\pm_\Gamma(\omega{\cdot}t,\lambda)$ are differentiable and satisfy the Riccati equation corresponding to the perturbed system (3.2) (see Sect. 1.3.5),

$$\begin{aligned} M' = &- M(H_3(\omega{\cdot}t) + \lambda\Gamma_3(\omega{\cdot}t))M - M(H_1(\omega{\cdot}t) + \lambda\Gamma_1(\omega{\cdot}t)) \\ &- (H_1^T(\omega{\cdot}t) + \lambda\Gamma_1^T(\omega{\cdot}t))M + H_2(\omega{\cdot}t) + \lambda\Gamma_2(\omega{\cdot}t)\,; \end{aligned} \tag{3.9}$$

i.e. $M_\Gamma^\pm(\omega,\lambda)$ are globally defined solutions along the flow of (3.9). Or, in other terms, they are continuous equilibria (see Definition 1.49).

Remark 3.11 Recall the notation $f'(\omega) = (d/dt)f(\omega{\cdot}t)|_{t=0}$, which was introduced for any measurable (scalar or matrix-valued) function f on Ω which is differentiable along the σ-orbits on the base, as well as the information provided by Proposition 1.36: $\int_\Omega f'(\omega)\,dm_0 = 0$ whenever $f\colon \Omega \to \mathbb{C}$ and $f' \in L^1(\Omega, m_0)$.

3.1.1 Symmetric Herglotz Matrix-Valued Functions

Some of the most important properties of the Weyl functions derive from their Herglotz character. The properties of the functions of this type are also fundamental in the analysis of the limiting behavior of the Floquet coefficient. To avoid further interruption in the discussion, some basic properties of symmetric Herglotz matrix-valued (or scalar) functions are recalled in this section.

Definition 3.12 Define, for each $\delta \in (0, \pi/2]$, the sector

$$\mathbb{C}_\delta^+ = \{z \in \mathbb{C}_+ \mid z = |z| \exp(i\theta) \text{ with } \theta \in [\delta, \pi - \delta]\}\,.$$

Let $G\colon \mathbb{C} \to \mathbb{S}_n(\mathbb{C})$ be a function, and take $\lambda_0 \in \mathbb{R}$. One says that G_{λ_0} *is the nontangential limit from the upper half-plane of* G *at* λ_0, and represented as $G_{\lambda_0} = \lim_{\lambda \searrow \lambda_0} G(\lambda)$, if $G_{\lambda_0} = \lim_{\lambda\to\lambda_0,\,\lambda\in\mathbb{C}_\delta^+} G(\lambda)$ for all $\delta \in (0,\pi)$. One says that G_{λ_0} is the nontangential limit from the lower half-plane of G at λ_0, and represented as $G_{\lambda_0} = \lim_{\lambda \nearrow \lambda_0} G(\lambda)$, if $G_{\lambda_0} = \lim_{\lambda\to\lambda_0,\,-\lambda\in\mathbb{C}_\delta^+} G(\lambda)$ for all $\delta \in (0,\pi)$.

The reader is referred to [90] and [54] for a more extensive description and for the proofs of the results contained in the following theorem. It can also be formulated for Herglotz functions with negative imaginary parts in $\mathbb{C}^+$, as well as for Herglotz functions defined on $\mathbb{C}^-$.

Theorem 3.13 *Let* $G\colon \mathbb{C}^+ \to \mathbb{S}_n(\mathbb{C})$ *be a Herglotz function, with* $\operatorname{Im} G \geq 0$. *Then,*

(i) *for Lebesgue a.e.* $\lambda_0 \in \mathbb{R}$ *there exists the nontangential limit from the upper half-plane* $\lim_{\lambda\searrow\lambda_0} G(\lambda)$.
(ii) *There exist real symmetric matrices* L *and* K *and a real matrix-valued function* $P(t)$ *defined for* $t \in \mathbb{R}$, *which is symmetric, nondecreasing and right-continuous, such that, for* $\lambda \in \mathbb{C}^+$, *the* Nevalinna–Riesz–Herglotz representation

$$G(\lambda) = L + K\,\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) dP(t) \tag{3.10}$$

holds, with

$$L = \operatorname{Re} G(i) \qquad \textit{and} \qquad K = \lim_{\eta\to\infty} \frac{1}{i\eta}\, G(i\eta) \geq 0\,.$$

(iii) *For* $\lambda \in \mathbb{R}$, *represent* $P\{\lambda\} = P(\lambda^+) - P(\lambda^-) = P(\lambda) - \lim_{\mu\to\lambda^-} P(\mu)$. *The* Stieltjes inversion formula

$$\frac{1}{2}\,(P\{\lambda_1\} + P\{\lambda_2\}) + \int_{(\lambda_1,\lambda_2)} dP(t) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} G(t+i\varepsilon)\, dt$$

holds. In addition,

$$P\{\lambda\} = \lim_{\varepsilon\to 0^+} \varepsilon \operatorname{Im} G(\lambda+i\varepsilon) = -i \lim_{\varepsilon\to 0^+} \varepsilon\, G(\lambda+i\varepsilon)\,,$$
$$0 = \lim_{\varepsilon\to 0^+} \varepsilon \operatorname{Re} G(\lambda+i\varepsilon)\,.$$

In particular, the matrix-valued measure dP in representation (3.10) *is uniquely determined.*

Remarks 3.14

1. Since $\operatorname{Im} G(i) = K + \int_{\mathbb{R}} (1/(t^2+1))\, dP(t)$ and $K \geq 0$, one has
$$0 \leq \int_{\mathbb{R}} \frac{1}{t^2+1}\, dP(t) \leq \operatorname{Im} G(i)\,.$$
2. Representation (3.10) is deduced from the *a priori* weaker equality
$$\frac{1}{\operatorname{Im}\lambda}\, \operatorname{Im} G(\lambda) = K + \int_{\mathbb{R}} \frac{1}{|t-\lambda|^2}\, dP(t)$$
for $\lambda \in \mathbb{C}^+$, K being a real symmetric matrix, since a symmetric matrix-valued analytic function is determined by its imaginary part up to an additive (symmetric) constant matrix.
3. Note that the existence of nontangential limits from the upper half-plane ensures that for Lebesgue a.e. $\lambda_0 \in \mathbb{R}$ there exist $\lim_{\varepsilon\to 0^+} G(\lambda_0 + \rho\,\varepsilon)$ for all $\rho \in \mathbb{C}^+$, and that all of them take the same value.
4. Let $G\colon \Omega \times \mathbb{C}^+ \to \mathbb{S}_n(\mathbb{C})$ be jointly continuous and Herglotz for each $\omega \in \Omega$ fixed. Then, for each $\omega \in \Omega$, there exist the limits $\lim_{\lambda\searrow\lambda_0} G(\omega,\lambda)$ for Lebesgue-a.e. $\lambda_0 \in \mathbb{R}$. Fubini's theorem guarantees the existence of a subset $\mathcal{R} \subseteq \mathbb{R}$ with full Lebesgue measure such that for all $\lambda_0 \in \mathcal{R}$ these limits exist for m_0-a.e. $\omega \in \Omega$. Therefore, the limit for $\lambda_0 \in \mathcal{R}$ is a Σ_{m_0}-measurable function: see Remark 1.1. Recall that Σ_{m_0} is the the m_0-completion of the Borel sigma-algebra of Ω. This fact is fundamental in this and in the following chapters, where many

matrix-valued functions on Ω defined in terms of the limits of Herglotz functions are integrated with respect to m_0.

The following theorem will play a fundamental role in Sect. 3.3.4.

Theorem 3.15 *Let (G_m) (for $m \in \mathbb{N}$) and G_* be symmetric Herglotz matrix-valued functions defined on $\mathbb{C}^+$ and with positive semidefinite imaginary parts. Suppose that $G_*(\lambda) = \lim_{m\to\infty} G_m(\lambda)$ uniformly on the compact subsets of $\mathbb{C}^+$, and write*

$$G_m(\lambda) = L_m + K_m\,\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) dP_m(t)\,,$$
$$G_*(\lambda) = L_* + K_*\,\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) dP_*(t)\,.$$

Then, $dP_* = \lim_{m\to\infty} dP_m$ in the weak* sense; that is,

$$\lim_{m\to\infty} \int_{\mathbb{R}} \mathbf{f}^*(t)\, dP_m(t)\, \mathbf{f}(t) = \int_{\mathbb{R}} \mathbf{f}^*(t)\, dP_*(t)\, \mathbf{f}(t) \tag{3.11}$$

for every $\mathbf{f}\colon \mathbb{R} \to \mathbb{C}^{2n}$ continuous and with compact support.

Proof The proof relies on the analogous result for the scalar case, which will be proved as a first step. So, assume that G_m and G_* are Herglotz scalar functions on $\mathbb{C}^+$ with positive imaginary parts and they have the above representation for nondecreasing right-continuous real functions P_m and P_* and real numbers L_m, K_m, L_*, and K_*. Note that, clearly, $L_* = \lim_{m\to\infty} L_m$, but it is not possible to ensure *a priori* that $K_* = \lim_{m\to\infty} K_m$. Define the sequence of positive measures (μ_m) by $d\mu_m(t) = (1/(t^2+1))\, dP_m(t)$. Note that the total variation of μ_m is

$$\mu_m(\mathbb{R}) = \int_{\mathbb{R}} \frac{1}{t^2+1}\, dP_m(t) \le C\,, \tag{3.12}$$

for a real constant C which does not depend on $m \in \mathbb{N}$: this is guaranteed by the convergence hypothesis and Remark 3.14.1.

Consider now the space $C_0(\mathbb{R}, \mathbb{C})$ of the bounded continuous functions on $\mathbb{R}$ which limit to 0 as t tends to ∞ and $-\infty$, endowed with the supremum norm. Any positive Borel measure μ of finite total variation defines a functional on $C_0(\mathbb{R}, \mathbb{C})$, sending f to $\int_{\mathbb{R}} f(t)\, d\mu(t)$, with norm given by $\mu(\mathbb{R})$ (see e.g. Theorem 6.19 of [128]). Hence (3.12) shows that (μ_m) can be understood as a bounded sequence contained in the dual of $C_0(\mathbb{R}, \mathbb{C})$. Since $C_0(\mathbb{R}, \mathbb{C})$ is separable, the Banach–Alaoglu theorem ensures the existence of a weak* convergent subsequence, say (μ_k), with limit given by a finite measure $\tilde{\mu}_\infty$. Call μ_∞ to the measure given by $d\mu_\infty(t) = (t^2+1)\, d\tilde{\mu}_\infty(t)$. Then,

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(t)\, \frac{1}{t^2+1}\, dP_k(t) = \int_{\mathbb{R}} f(t)\, d\tilde{\mu}_\infty(t) = \int_{\mathbb{R}} f(t)\, \frac{1}{t^2+1}\, d\mu_\infty(t) \tag{3.13}$$

for all $f \in C_0(\mathbb{R}, \mathbb{C})$. It follows immediately that

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(t)\, dP_k(t) = \int_{\mathbb{R}} f(t)\, d\mu_\infty(t)$$

for each continuous $f : \mathbb{R} \to \mathbb{C}$ with compact support: this is because the function g given by $g(t) = (t^2+1)\, f(t)$ belongs to $C_0(\mathbb{R}, \mathbb{C})$. In addition, due to (3.12), and by taking a suitable subsequence if necessary, it is possible to assume the existence of the limit $m_0 = \lim_{k\to\infty} \int_{\mathbb{R}} (1/(t^2+1))\, dP_k(t)$.

The proof in the scalar case is completed by checking that the measure dP_* agrees with μ_∞: hence dP_* is the weak* limit in the sense of (3.11) of any subsequence of the initial sequence (dP_m), and therefore the weak* limit of the sequence itself. Define

$$K_\infty = \operatorname{Im} G_*(i) - \tilde{\mu}_\infty(\mathbb{R}) = \operatorname{Im} G_*(i) - \int_{\mathbb{R}} \frac{1}{t^2+1}\, d\mu_\infty(t)\,.$$

Then, for every $\lambda \notin \mathbb{R}$,

$$\frac{1}{\operatorname{Im}\lambda}\, \operatorname{Im} G_*(\lambda) = K_\infty + \int_{\mathbb{R}} \frac{1}{|t-\lambda|^2}\, d\mu_\infty(t)\,. \tag{3.14}$$

In order to prove this, fix a value of λ and take the limits in the representation

$$\frac{1}{\operatorname{Im}\lambda}\, \operatorname{Im} G_k(\lambda) = K_k + \int_{\mathbb{R}} \frac{1}{|t-\lambda|^2}\, dP_k(t)\,:$$

first,

$$\begin{aligned}\lim_{k\to\infty} K_k &= \lim_{k\to\infty} \left(\operatorname{Im} G_k(i) - \int_{\mathbb{R}} \frac{1}{t^2+1}\, dP_k(t) \right)\\ &= \operatorname{Im} G_*(i) - m_0 = K_\infty + \tilde{\mu}_\infty(\mathbb{R}) - m_0\,;\end{aligned}$$

and second,

$$\begin{aligned}\lim_{k\to\infty} \int_{\mathbb{R}} \frac{1}{|t-\lambda|^2}\, dP_k(t) &= \lim_{k\to\infty} \int_{\mathbb{R}} (f(t)+1)\, \frac{1}{t^2+1}\, dP_k(t)\\ &= \int_{\mathbb{R}} f(t)\, d\tilde{\mu}_\infty(t) + m_0 = \int_{\mathbb{R}} (f(t)+1)\, d\tilde{\mu}_\infty - \tilde{\mu}_\infty(\mathbb{R}) + m_0\\ &= \int_{\mathbb{R}} \frac{1}{|t-\lambda|^2}\, d\mu_\infty - \tilde{\mu}_\infty(\mathbb{R}) + m_0\,,\end{aligned}$$

as can be deduced from the fact that the function $f(t) = -1 + (t^2+1)/|t-\lambda|^2$ belongs to $C_0(\mathbb{R}, \mathbb{C})$, since $\lambda \in \mathbb{C} - \mathbb{R}$. Hence (3.14) holds. Finally, such a representation for $(1/\operatorname{Im}\lambda)\, \operatorname{Im} G_*(\lambda)$ ensures the uniqueness of the measure (see Remark 3.14.2). Consequently, $d\mu_\infty = dP_*$, and the proof is complete in the scalar case.

It is now easy to deduce the result in the matrix case. Fix $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^{2n}$ and apply the scalar result to the sequence of scalar Herglotz functions $(\mathbf{z}^* G_m(\lambda)\,\mathbf{z})$ and its limit $\mathbf{z}^* G_*(\lambda)\,\mathbf{z}$. It follows that

$$\lim_{m\to\infty} \int_{\mathbb{R}} f(t)\,\mathbf{z}^* dP_m(t)\,\mathbf{z} = \int_{\mathbb{R}} f(t)\,\mathbf{z}^* dP_*(t)\,\mathbf{z}$$

for each continuous $f\colon \mathbb{R} \to \mathbb{C}$ of compact support. The polarization formula

$$\begin{aligned}\int_{\mathbb{R}} & f(t)\,\mathbf{z}^*\,dP(t)\,\mathbf{w} \\ &= \int_{\mathbb{R}} \frac{f(t)}{4}\Big((\mathbf{z}+\mathbf{w})^*\,dP(t)\,(\mathbf{z}+\mathbf{w}) - (\mathbf{z}-\mathbf{w})^*\,dP(t)\,(\mathbf{z}-\mathbf{w}) \\ &\qquad + i\,(\mathbf{z}+i\mathbf{w})^*\,dP(t)\,(\mathbf{z}+i\mathbf{w}) - i\,(\mathbf{z}-i\mathbf{w})^*\,dP(t)\,(\mathbf{z}-i\mathbf{w})\Big)\,,\end{aligned}$$

which holds for $P = P_m$ and $P = P_*$, guarantees that

$$\lim_{m\to\infty} \int_{\mathbb{R}} f(t)\,\mathbf{z}^* dP_m(t)\,\mathbf{w} = \int_{\mathbb{R}} f(t)\,\mathbf{z}^* dP_*(t)\,\mathbf{w}\,.$$

The statement follows immediately by writing any function $\mathbf{f}\colon \mathbb{R} \to \mathbb{C}^{2n}$ as $\mathbf{f}(t) = \sum_{j=1}^{2n} f_j(t)\,\mathbf{e}_j$, where $\{\mathbf{e}_1,\dots,\mathbf{e}_{2n}\}$ is the canonical basis of $\mathbb{C}^{2n}$.

3.2 The Floquet Coefficient in the Complex Plane

Throughout this section, Γ and m_0 will represent respectively an Atkinson perturbation (see Definition 3.4) and a fixed σ-ergodic measure on Ω (see Sect. 1.1.2).

3.2.1 *The Floquet Coefficient Outside the Real Axis*

The presence of exponential dichotomy and the properties of the corresponding Weyl functions allow one to define the Floquet coefficient for the families of systems (3.2) corresponding to values of the parameter λ outside the real axis. That is the goal of this section.

As in the case of rotation number and Lyapunov index, this coefficient depends on the fixed σ-ergodic measure m_0: set

$$w_\Gamma(\lambda) = \int_\Omega \operatorname{tr}\big(H_1(\omega) + \lambda\,\Gamma_1(\omega) + (H_3(\omega) + \lambda\,\Gamma_3(\omega))\,M_\Gamma^+(\omega,\lambda)\big)\,dm_0 \qquad (3.15)$$

for $\lambda \in \mathbb{C} - \mathbb{R}$.

Definition 3.16 The *Floquet coefficient with respect to* m_0 of the family of linear Hamiltonian systems (3.2) corresponding to $\lambda \in \mathbb{C}^+$ is given by (3.15).

Remark 3.17 It is usual to define the Floquet coefficient for $\lambda \in \mathbb{C}^-$ not as $w_\Gamma(\lambda)$ but as $w_\Gamma(\bar{\lambda})$ (which agrees with the conjugate of $w_\Gamma(\lambda)$). A possible reason for this alternative choice will be clear after having studied the limiting behavior on the real axis of the function w_Γ, in Sect. 3.2.4: in that way, the real part of the Floquet coefficient always agrees with the negative Lyapunov index $-\beta_\Gamma$ (see Theorem 3.30) and its imaginary part can be extended to a continuous function on the complex plane, coinciding with the rotation number α_Γ on the real axis (see Theorem 3.32).

As stated before, Definition 3.16 extends to the general case the concept of Floquet coefficient for nonautonomous two-dimensional linear Hamiltonian systems. The function $w_\Gamma(\lambda)$ was first defined in [72] following an alternative path explained below (Remark 3.34).

It follows from (3.15) that the Floquet coefficient is an analytic function on $\mathbb{C}^+$. An addition fact is that it can also be also defined in terms of M_Γ^-, as is shown in the following lemma.

Remark 3.18 Let $A\colon \mathbb{R} \to \mathbb{M}_{m\times m}(\mathbb{R})$ be a C^1 function, and assume that $A^{-1}(t)$ exists for all $t \in \mathbb{R}$. It follows from the Liouville formula that $(\ln\det A(t))' = \operatorname{tr}(A'(t)A^{-1}(t))$.

Lemma 3.19 *Fix* $\lambda \in \mathbb{C} - \mathbb{R}$. *Then,*

(i) *for every* $\omega \in \Omega$ *the matrix* $\begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega,\lambda) & M_\Gamma^-(\omega,\lambda) \end{bmatrix}$ *is nonsingular, and*

$$\lim_{t\to\infty} \frac{1}{t} \ln\det \begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega{\cdot}t,\lambda) & M_\Gamma^-(\omega{\cdot}t,\lambda) \end{bmatrix} = 0 \qquad m_0\text{-}a.e\,.$$

(ii) *The change of variables* $\mathbf{z} = \begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega{\cdot}t,\lambda) & M_\Gamma^-(\omega{\cdot}t,\lambda) \end{bmatrix} \mathbf{w}$ *takes* (3.2) *to*

$$\mathbf{w}' = \begin{bmatrix} H_{\Gamma,\lambda}^+(\omega{\cdot}t) & 0_n \\ 0_n & H_{\Gamma,\lambda}^-(\omega{\cdot}t) \end{bmatrix} \mathbf{w}\,, \tag{3.16}$$

where $H_{\Gamma,\lambda}^\pm(\omega) = H_1(\omega) + \lambda\Gamma_1(\omega) + (H_3(\omega) + \lambda\Gamma_3(\omega))M_\Gamma^\pm(\omega,\lambda)$.

(iii) *Finally,*

$$w_\Gamma(\lambda) = -\int_\Omega \operatorname{tr}\big(H_1(\omega) + \lambda\Gamma_1(\omega) + (H_3(\omega) + \lambda\Gamma_3(\omega))M_\Gamma^-(\omega,\lambda)\big)\,dm_0\,.$$

Proof

(i) Since the column vectors of the matrix $\begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega,\lambda) & M_\Gamma^-(\omega,\lambda) \end{bmatrix}$ determine a basis of $l_{\Gamma,\lambda}^+(\omega) \oplus l_{\Gamma,\lambda}^-(\omega) = \mathbb{C}^{2n}$, the matrix is nonsingular. In addition, it can be checked immediately that

$$\begin{bmatrix} (M_\Gamma^- - M_\Gamma^+)^{-1} & 0_n \\ 0_n & (M_\Gamma^- - M_\Gamma^+)^{-1} \end{bmatrix} \begin{bmatrix} M_\Gamma^- & -I_n \\ -M_\Gamma^+ & I_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ M_\Gamma^+ & M_\Gamma^- \end{bmatrix} = I_{2n} \,.$$

Note that the matrix $M_\Gamma^- - M_\Gamma^+$ is nonsingular, since it has negative (resp. positive) definite imaginary part when $\lambda \in \mathbb{C}^+$ (resp. $\lambda \in \mathbb{C}^-$): see Proposition 1.21(i). This equality permits one to determine the linear differential system satisfied by the matrix-valued function $t \mapsto \left[\begin{smallmatrix} I_n & I_n \\ M_\Gamma^+(\omega{\cdot}t,\lambda) & M_\Gamma^-(\omega{\cdot}t,\lambda) \end{smallmatrix}\right]$, which has coefficient matrix

$$\begin{aligned} &\begin{bmatrix} 0_n & 0_n \\ (M_\Gamma^+)' & (M_\Gamma^-)' \end{bmatrix} \begin{bmatrix} I_n & I_n \\ M_\Gamma^+ & M_\Gamma^- \end{bmatrix}^{-1} \\ &\qquad = \begin{bmatrix} 0_n & 0_n \\ (M_\Gamma^+)' & (M_\Gamma^-)' \end{bmatrix} \begin{bmatrix} (M_\Gamma^- - M_\Gamma^+)^{-1} & 0_n \\ 0_n & (M_\Gamma^- - M_\Gamma^+)^{-1} \end{bmatrix} \begin{bmatrix} M_\Gamma^- & -I_n \\ -M_\Gamma^+ & I_n \end{bmatrix}. \end{aligned}$$

The arguments $(\omega{\cdot}t, \lambda)$ are omitted. It is easy to check that the trace of the right-hand term is $\operatorname{tr}\big((M_\Gamma^+ - M_\Gamma^-)'(M_\Gamma^+ - M_\Gamma^-)^{-1}(\omega{\cdot}t,\lambda)\big)$. Note that the function $\operatorname{tr}\big((M_\Gamma^+ - M_\Gamma^-)'(M_\Gamma^+ - M_\Gamma^-)^{-1}(\omega,\lambda)\big)$ is continuous on Ω. Hence Birkhoff's ergodic theorem (see Theorems 1.3 and 1.6) and Remarks 3.18 and 3.11 ensure that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \ln\det &\begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega{\cdot}t,\lambda) & M_\Gamma^-(\omega{\cdot}t,\lambda) \end{bmatrix} \\ &= \int_\Omega \operatorname{tr}\big((M_\Gamma^+ - M_\Gamma^-)'(M_\Gamma^+ - M_\Gamma^-)^{-1}(\omega,\lambda)\big)\, dm_0 \\ &= \int_\Omega \big(\ln\det\big(M_\Gamma^+(\omega,\lambda) - M_\Gamma^-(\omega,\lambda)\big)\big)'\, dm_0 = 0 \end{aligned}$$

m_0-a.e. So (i) is proved.

(ii) Statement (ii) follows from a straightforward computation taking the Riccati equation (3.9) as the starting point.

(iii) The invariance of the closed Lagrange subbundles $L^\pm_{\Gamma,\lambda}$ which are determined by the exponential dichotomy, and the representation given by Theorem 3.8, make it possible to write

$$\begin{aligned} U_{\Gamma,\lambda}(t,\omega) &\begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega,\lambda) & M_\Gamma^-(\omega,\lambda) \end{bmatrix} \\ &= \begin{bmatrix} I_n & I_n \\ M_\Gamma^+(\omega{\cdot}t,\lambda) & M_\Gamma^-(\omega{\cdot}t,\lambda) \end{bmatrix} \begin{bmatrix} W^+_{\Gamma,\lambda}(t,\omega) & 0_n \\ 0_n & W^-_{\Gamma,\lambda}(t,\omega) \end{bmatrix}. \end{aligned} \tag{3.17}$$

Then $\begin{bmatrix} W^{+}_{\Gamma,\lambda}(t,\omega) & 0_n \\ 0_n & W^{-}_{\Gamma,\lambda}(t,\omega) \end{bmatrix}$ is the fundamental matrix solution of (3.16) which agrees with I_{2n} at $t = 0$. The fact that $\det U_{\Gamma,\lambda}(\omega, t) = 1$, statement (i), and again the Liouville formula lead us to

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{1}{t} \ln\det U_{\Gamma,\lambda}(t,\omega) \\ &= \lim_{t\to\infty} \frac{1}{t} \ln\det W^{+}_{\Gamma,\lambda}(t,\omega) + \lim_{t\to\infty} \frac{1}{t} \ln\det W^{-}_{\Gamma,\lambda}(t,\omega) \\ &= \int_\Omega \operatorname{tr} H^{+}_{\Gamma,\lambda}(\omega)\, dm_0 + \int_\Omega \operatorname{tr} H^{-}_{\Gamma,\lambda}(\omega)\, dm_0 \end{aligned} \tag{3.18}$$

m_0-a.e. This and definition (3.15) prove (iii).

Remark 3.20 Note that the last part of the previous proof ensures that, if $\lambda \in \mathbb{C} - \mathbb{R}$,

$$w_\Gamma(\lambda) = \pm \lim_{t\to\infty} \frac{1}{t} \ln\det W^{\pm}_{\Gamma,\lambda}(t,\omega) \tag{3.19}$$

m_0-a.e. That is, $w_\Gamma(\lambda)$ measures the exponential growth and the rotation of the matrices $W^{\pm}_{\Gamma,\lambda}(t,\omega)$, which are respectively induced by n linearly independent solutions of (3.2) with initial conditions in the complex Lagrange planes $l^{\pm}_{\Gamma,\lambda}(\omega)$. All these properties will be used in the following sections in order to analyze the Fréchet differentiability and the boundary behavior of the Floquet coefficient.

3.2.2 Fréchet Differentiability of the Floquet Coefficient

Throughout this section, $\|\cdot\|$ will represent the Euclidean norm of any vector in $\mathbb{R}^d$ or $\mathbb{C}^d$ for any dimension d, as well as the corresponding operator norm in any space of real or complex matrices. Nevertheless, the results are independent of the choice of an equivalent norm. Let $C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$ be the space of continuous matrix-valued functions $K\colon \Omega \to \mathfrak{sp}(n,\mathbb{C})$ endowed with the topology given by the norm $\|K\|_\Omega = \sup_{\omega\in\Omega} \|K(\omega)\|$. The same topology will be given to any space of continuous functions taking values in any vector space of matrices. And let λ_* be a fixed complex value of the parameter with $\operatorname{Im}\lambda_* \neq 0$. Theorems 1.92 and 1.95 guarantee the existence of an open neighborhood $\mathcal{B} \subset C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$ of 0_{2n} such that, for all $K = \begin{bmatrix} K_1 & K_3 \\ K_2 & -K_1^T \end{bmatrix} \in \mathcal{B}$, the family of systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda_* J^{-1}\Gamma(\omega{\cdot}t) + K(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega\,. \tag{3.20}$$

has exponential dichotomy over Ω, and such that there exist the Weyl functions representing the closed subbundles of the solutions bounded at $\pm\infty$. These are denoted $M^{\pm}(\omega, K)$. As in (3.15), define

$$w(K) = \int_{\Omega} \operatorname{tr}\big(H_1 + \lambda_* \Gamma_1 + K_1 + (H_3 + \lambda_* \Gamma_3 + K_3)\, M^+(\omega, K)\big)\, dm_0\,, \tag{3.21}$$

with $H_1, H_3, \Gamma_1, \Gamma_3, K_1$, and K_3 evaluated in ω; and represent the fundamental matrix solution with value I_{2n} at $t = 0$ by $U_K(t, \omega)$. Note that all of these quantities also depend on the fixed Γ and λ_*, although this dependence does not appear explicitly in the notation.

Let $Q(\omega, K)\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the projection associated to the exponential dichotomy of (3.20) (see Definition 1.58). Note that

$$Q = \begin{bmatrix} (M^- - M^+)^{-1} M^- & -(M^- - M^+)^{-1} \\ M^+ (M^- - M^+)^{-1} M^- & -M^+ (M^- - M^+)^{-1} \end{bmatrix}, \tag{3.22}$$

as seen in Remark 1.81.1. It follows from Theorems 1.92 and 1.95 that the two section maps defined by $\mathcal{B} \to C(\Omega, \mathrm{GL}(2n, \mathbb{C}))$, $K \mapsto Q(\cdot, K)$ and $\mathcal{B} \to C(\Omega, \mathbb{S}_n(\mathbb{C}))$, $K \mapsto M^{\pm}(\cdot, K)$ are continuous. By reducing $\mathcal{B}$ if needed, it can be arranged that $M^{\pm}(\cdot, K)$ and $Q(\cdot, K)$ are uniformly norm-bounded on Ω for $K \in \mathcal{B}$. In addition,

Lemma 3.21 $\sup_{\omega \in \Omega} \|M^{\pm}(\omega, K) - M^{\pm}(\omega, 0_{2n})\| = O(\|K\|_{\Omega})$ *for* $K \to 0_{2n}$.

Proof By Proposition 1.68 and Theorem 3.8, $(I_{2n} - Q(\omega, K)) \begin{bmatrix} I_n \\ M^+(\omega, K) \end{bmatrix} = \begin{bmatrix} 0_n \\ 0_n \end{bmatrix}$ for all $K \in \mathcal{B}$, which implies that

$$\begin{aligned} &(I_{2n} - Q(\omega, 0_{2n})) \begin{bmatrix} 0_n \\ M^+(\omega, K) - M^+(\omega, 0_{2n}) \end{bmatrix} \\ &\qquad = (Q(\omega, K) - Q(\omega, 0_{2n})) \begin{bmatrix} I_n \\ M^+(\omega, K) \end{bmatrix}. \end{aligned}$$

Call this matrix $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$. According to (3.22),

$$F_1 = (M^-(\omega, 0_{2n}) - M^+(\omega, 0_{2n}))^{-1} (M^+(\omega, K) - M^+(\omega, 0_{2n}))\,.$$

Therefore, since $\|A\| \le \|B\| \|B^{-1} A\|$ and $\|C\| \le \|\begin{bmatrix} C \\ D \end{bmatrix}\| \le \|C\| + \|D\|$ for square matrices A, B, C and D with B nonsingular, one has that

$$\begin{aligned} &\|M^+(\omega, K) - M^+(\omega, 0_{2n})\| \\ &\quad \le \|M^-(\omega, 0_{2n}) - M^+(\omega, 0_{2n})\| \|F_1\| \\ &\quad \le \|M^-(\omega, 0_{2n}) - M^+(\omega, 0_{2n})\| \left\| \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\| \\ &\quad \le \|M^-(\omega, 0_{2n}) - M^+(\omega, 0_{2n})\| \|Q(\omega, K) - Q(\omega, 0_{2n})\| (1 + \|M^+(\omega, K)\|)\,. \end{aligned}$$

According to Theorem 1.92(iv), $\sup_{\omega\in\Omega}\|Q(\omega,K)-Q(\omega,0_{2n})\| = O(\|K\|_\Omega)$ for $K\to 0_{2n}$, which together with the uniform bound of $M^\pm(\omega,K)$ for $(\omega,K)\in\Omega\times\mathcal{B}$ proves the assertion for M^+. The proof is analogous for M^-, keeping now in mind that

$$Q(\omega,0_{2n})\begin{bmatrix} 0_n \\ M^-(\omega,K)-M^-(\omega,0_{2n})\end{bmatrix} = (Q(\omega,0_{2n})-Q(\omega,K))\begin{bmatrix} I_n \\ M^-(\omega,K)\end{bmatrix}$$

for all $K\in\mathcal{B}$.

Define now

$$\widetilde{G}_K(\omega,t,s) = \begin{cases} U_K(t,\omega)\,Q(\omega,K)\,J^{-1}U_K^T(s,\omega) & \text{if } t\ge s\,, \\ -U_K(t,\omega)\,(I_{2n}-Q(\omega,K))\,J^{-1}U_K^T(s,\omega) & \text{if } s>t\,, \end{cases} \tag{3.23}$$

and note that the symmetric matrix-valued function given on $\Omega\times\mathcal{B}$ by

$$G(\omega,K) = \frac{1}{2}\left(\lim_{s\to 0-}\widetilde{G}_K(\omega,0,s) + \lim_{s\to 0+}\widetilde{G}_K(\omega,0,s)\right) \tag{3.24}$$

satisfies

$$G(\omega,K) = \left(Q(\omega,K)-\frac{1}{2}\,I_{2n}\right)J^{-1}\,. \tag{3.25}$$

Hence, by (3.22),

$$G = \begin{bmatrix} (M^- - M^+)^{-1} & \dfrac{1}{2}\,(M^- - M^+)^{-1}(M^- + M^+) \\ \dfrac{1}{2}\,(M^- + M^+)(M^- - M^+)^{-1} & M^+(M^- - M^+)^{-1}M^- \end{bmatrix}. \tag{3.26}$$

This section is basically devoted to proving the following *trace formula*:

Theorem 3.22 *The map* $w\colon\mathcal{B}\to\mathbb{C}\,,\ K\mapsto w(K)$ *is Fréchet differentiable at* 0_{2n}, *with*

$$d_{0_{2n}}w{\cdot}K = \int_\Omega \operatorname{tr}(G(\omega,0_{2n})JK(\omega))\,dm_0 = \int_\Omega \operatorname{tr}(Q(\omega,0_{2n})\,K(\omega))\,dm_0\,.$$

This result plays a fundamental role in the application of the properties of the rotation number to the study of the spectral problems associated to (3.2) and (3.3), as explained in Sect. 3.3. It appears in [72] in the more general setting considered there. The different proof presented here is motivated by an argument of Kotani and Simon [91]. The main step of the proof consists in checking the first equality, which follows easily once the following technical lemmas have been established.

Lemma 3.23 *The maps $M^{\pm}\colon \mathcal{B} \to C(\Omega, \mathbb{S}_{\mathbb{C}}(n))$, $K \mapsto M^{\pm}(\omega, K)$ are Fréchet differentiable with respect to K at 0_{2n} for all $\omega \in \Omega$. More precisely, there exist continuous maps $d_{0_{2n}}M^{\pm}\colon \Omega \times C(\Omega, \mathfrak{sp}(n,\mathbb{C})) \to C(\Omega, \mathbb{S}_{\mathbb{C}}(n))$ such that the section maps $d_{0_{2n}}M^{\pm}(\omega)\colon C(\Omega, \mathfrak{sp}(n,\mathbb{C})) \to C(\Omega, \mathbb{S}_{\mathbb{C}}(n))$ are bounded linear operators for all $\omega \in \Omega$, with*

$$M^{\pm}(\omega, K) - M^{\pm}(\omega, 0_{2n}) = d_{0_{2n}}M^{\pm}(\omega){\cdot}K + o(\|K\|_{\Omega}) \tag{3.27}$$

for $K \to 0_{2n}$. In addition, for all $K = \left[\begin{smallmatrix} K_1 & K_3 \\ K_2 & -K_1^T \end{smallmatrix}\right]$ in $C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$, the maps $t \mapsto d_{0_{2n}}M^{\pm}(\omega{\cdot}t){\cdot}K$ satisfy the matrix differential equations

$$(\delta M)' = f_{\Gamma,\lambda_*}(\omega{\cdot}t, M^{\pm}(\omega{\cdot}t, 0_{2n})){\cdot}\delta M + B_K^{\pm}(\omega{\cdot}t) \tag{3.28}$$

respectively, where

$$\begin{aligned} f_{\Gamma,\lambda}(\omega, M){\cdot}D = &- D\,(H_1(\omega) + \lambda\Gamma_1(\omega) + (H_3(\omega) + \lambda\Gamma_3(\omega))M)) \\ &- \big(H_1^T(\omega) + \lambda\Gamma_1^T(\omega) + M(H_3(\omega) + \lambda\Gamma_3(\omega))\big)\,D \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} B_K^{\pm}(\omega) = K_2(\omega) &- M^{\pm}(\omega, 0_{2n})K_1(\omega) - K_1^T(\omega)M^{\pm}(\omega, 0_{2n}) \\ &- M^{\pm}(\omega, 0_{2n})K_3(\omega)M^{\pm}(\omega, 0_{2n})\,. \end{aligned} \tag{3.30}$$

Proof As in Lemma 3.19(ii), the change of variables $\mathbf{z} = C(\omega{\cdot}t)\,\mathbf{w}$, with $C(\omega) = \left[\begin{smallmatrix} I_n & I_n \\ M_\Gamma^+(\omega,0_{2n}) & M_\Gamma^-(\omega,0_{2n}) \end{smallmatrix}\right]$, transforms (3.20) for $K = 0_{2n}$ into the system

$$\mathbf{w}' = \begin{bmatrix} H^+(\omega{\cdot}t) & 0_n \\ 0_n & H^-(\omega{\cdot}t) \end{bmatrix} \mathbf{w}\,,$$

where $H^{\pm}(\omega) = H_1(\omega) + \lambda_*\Gamma_1(\omega) + (H_3(\omega) + \lambda_*\Gamma_3(\omega))M^{\pm}(\omega, 0_{2n})$. The fundamental matrix solution of this system agreeing with I_{2n} at $t = 0$ is the matrix function $\left[\begin{smallmatrix} W^+(t,\omega) & 0_n \\ 0_n & W^-(t,\omega) \end{smallmatrix}\right]$, given by the equality analogous to (3.17). The boundedness of C and C^{-1} guarantees that the change of variables preserves the exponential dichotomy of the original system, and the associated projection for the transformed system is given by $\left[\begin{smallmatrix} I_n & 0_n \\ 0_n & 0_n \end{smallmatrix}\right]$ for any $\omega \in \Omega$ (see e.g. Proposition 1.56). Hence there exist positive constants η and β such that

$$\begin{aligned} &\|W^+(t,\omega)\,(W^+)^{-1}(s,\omega)\| \le \eta\, e^{-\beta(t-s)} \quad \text{for } t \ge s\,, \\ &\|W^-(t,\omega)\,(W^-)^{-1}(s,\omega)\| \le \eta\, e^{\beta(t-s)} \quad \text{for } t \le s\,. \end{aligned} \tag{3.31}$$

As a first consequence, $\lim_{t\to\infty} W^+(t,\omega) = \lim_{t\to-\infty} W^-(t,\omega) = 0_n$, and the homogeneous equations

$$(\delta M)' = f_{\Gamma,\lambda_*}(\omega{\cdot}t, M^\pm(\omega{\cdot}t, 0_{2n})){\cdot}\delta M\,,$$

with $f_{\Gamma,\lambda}$ given by (3.29), have no nonzero matrix solutions bounded at $\pm\infty$, respectively. This is due to the fact that any matrix solution M satisfies $M(0) = (W^\pm)^T(t,\omega)\,M(t)\,W^\pm(t,\omega)$.

A second consequence is that, for all $\omega \in \Omega$, the matrix $d_{0_{2n}}M^+(\omega){\cdot}K$ defined for $K \in C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$ by

$$d_{0_{2n}}M^+(\omega){\cdot}K = -\int_0^\infty (W^+)^T(s,\omega)\,B_K^+(\omega{\cdot}s)\,W^+(s,\omega)\,ds\,, \tag{3.32}$$

with B_K^+ given by (3.30), is well defined. Note that $B_K^+(\omega)$ is jointly continuous in $(\omega,K) \in \Omega \times C(\Omega,\mathfrak{sp}(n,\mathbb{C}))$, it is bounded on Ω for each $K \in C(\Omega,\mathfrak{sp}(n,\mathbb{C}))$ fixed, and it is linear in K for each $\omega \in \Omega$ fixed. A technical and standard argument allows one to deduce from this fact and (3.31) that the map $d_{0_{2n}}M^+\colon \Omega \times C(\Omega,\mathfrak{sp}(n,\mathbb{C})) \to C(\Omega,\mathbb{S}_\mathbb{C}(n))$ given by (3.32) is well defined and jointly continuous. In particular, for all $\omega \in \Omega$, the map $d_{0_{2n}}M^+(\omega)\colon C(\Omega,\mathfrak{sp}(n,\mathbb{C})) \to C(\Omega,\mathbb{S}_\mathbb{C}(n))$ is a bounded linear operator. In addition, it is easy to check that

$$\begin{aligned} &d_{0_{2n}}M^+(\omega{\cdot}t){\cdot}K \\ &\quad = -\int_t^\infty (W^+(s,\omega)(W^+)^{-1}(t,\omega))^T\,B_K^+(\omega{\cdot}s)\,W^+(s,\omega)(W^+)^{-1}(t,\omega))\,ds\,, \end{aligned}$$

which ensures that $t \mapsto d_{0_{2n}}M^+(\omega{\cdot}t){\cdot}K$ is the unique matrix solution of (3.28) which is bounded at ∞.

In order to prove (3.27), use (3.9) to obtain the matrix differential equation satisfied by $t \mapsto M^+(\omega{\cdot}t,K) - M^+(\omega{\cdot}t,0)$ with $K \in \mathcal{B}$:

$$\delta M' = f_{\Gamma,\lambda_*}(\omega{\cdot}t, M^+(\omega{\cdot}t,0)){\cdot}\delta M + \widetilde{B}_K^+(\omega{\cdot}t)\,,$$

where

$$\begin{aligned} \widetilde{B}_K^+(\omega) = K_2(\omega) &- M^+(\omega,K)\,K_1(\omega) - K_1^T(\omega)\,M^+(\omega,K) \\ &- M^+(\omega,K)\,K_3(\omega)\,M^+(\omega,K) \\ &- (M^+(\omega,K) - M^+(\omega,0_{2n}))\,H_3(\omega)\,(M^+(\omega,K) - M^+(\omega,0_{2n}))\,. \end{aligned}$$

Therefore, since $t \mapsto M^+(\omega{\cdot}t,K) - M^+(\omega{\cdot}t,0)$ and $t \mapsto \widetilde{B}_K^+(\omega{\cdot}t)$ are bounded on $\mathbb{R}$,

$$M^+(\omega,K) - M^+(\omega,0_{2n}) = -\int_0^\infty (W^+)^T(s,\omega)\,\widetilde{B}_K^+(\omega{\cdot}s)\,W^+(s,\omega)\,ds\,. \tag{3.33}$$

On the other hand, Lemma 3.21, (3.31), and the expressions of $\widetilde{B}_K^+(\omega)$ and $B_K^+(\omega)$ guarantee that

$$\sup_{\omega\in\Omega} \|\widetilde{B}_K^+(\omega) - B_K^+(\omega)\| = O(\|K\|_\infty^2) = o(\|K\|_\Omega)$$

for $K \to 0_{2n}$, which together with (3.32) and (3.33) gives (3.27).

The proof for M^- is analogous, taking the definition

$$d_{0_{2n}} M^-(\omega)\cdot K = \int_{-\infty}^0 (W^-)^T(s,\omega)\, B_K^-(\omega{\cdot}s)\, W^-(s,\omega)\, ds$$

as a starting point.

The following result is an immediate consequence of Lemma 3.23:

Lemma 3.24 *The map*

$$\begin{aligned} \tilde{\imath}\colon \Omega \times \mathcal{B} &\to \mathbb{C} \\ (\omega, K) &\mapsto \operatorname{tr}\big((H_3(\omega) + \lambda_* \Gamma_3(\omega) + K_3(\omega))(M^+(\omega,K) - M^-(\omega,K))\big) \end{aligned}$$

is Fréchet differentiable with respect to K at 0_{2n}*, and its derivative is given by*

$$\begin{aligned} d_{0_{2n}}\tilde{\imath}(\omega)\cdot K = \operatorname{tr}\Big(&(H_3(\omega) + \lambda_*\Gamma_3(\omega))(d_{0_{2n}}M^+(\omega)\cdot K - d_{0_{2n}}M^-(\omega)\cdot K) \\ &+ K_3(\omega)(M^+(\omega,0_{2n}) - M^-(\omega,0_{2n}))\Big) \end{aligned}$$

for $K \in C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$. *In particular,*

$$\tilde{\imath}(\omega,K) - \tilde{\imath}(\omega,0_{2n}) = d_{0_{2n}}\tilde{\imath}(\omega)\cdot K + o(\|K\|_\Omega) \quad \textit{for } K \to 0_{2n}\,.$$

Lemma 3.25 *For all* $K \in C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$,

$$\int_\Omega d_{0_{2n}}\tilde{\imath}(\omega)\cdot K\, dm_0 = 2\int_\Omega \operatorname{tr}(G(\omega,0_{2n})JK(\omega))\, dm_0\,.$$

Proof Define $D = M_0^+ - M_0^-$ and $S = M_0^+ + M_0^-$ where $M_0^\pm(\omega) = M^\pm(\omega,0_{2n})$. For a fixed element $K \in C(\Omega, \mathfrak{sp}(n,\mathbb{C}))$, write $\delta M^\pm(\omega) = d_{0_{2n}}M^\pm(\omega)\cdot K$ and $\delta S = \delta M^+ + \delta M^-$. Write also $H_j^{\lambda*} = H_j + \lambda_*\,\Gamma_j$ for $j = 1, 3$. The argument ω is omitted in which follows. It is not hard to check from (3.9) that

$$\operatorname{tr}\big((D^{-1})'\delta S\big) = 2\,\operatorname{tr}\big(H_1^{\lambda*}D^{-1}\delta S\big) + \operatorname{tr}\big(H_3^{\lambda*}SD^{-1}\delta S\big)\,,$$

and from the definition of $B_K^{\pm}$ that

$$\begin{aligned}&\operatorname{tr}(D^{-1}(B_K^+ + B_K^-))\\&\qquad = 2\operatorname{tr}\big(D^{-1}K_2 - D^{-1}SK_1 - D^{-1}M_0^+\,K_3\,M_0^-\big) - \operatorname{tr}(K_3D)\,.\end{aligned}$$

These equalities and (3.28) yield

$$\begin{aligned}\operatorname{tr}\big(D^{-1}\,\delta S\big)' &= 2\,\operatorname{tr}\big(H_1^{\lambda *}D^{-1}\delta S\big) + \operatorname{tr}\big(H_3^{\lambda *}SD^{-1}\delta S\big)\\&\quad + \operatorname{tr}\big(D^{-1}(f_{\Gamma,\lambda_*}(\,\cdot\,,M_0^+)\cdot\delta M^+) + D^{-1}(f_{\Gamma,\lambda_*}(\,\cdot\,,M_0^-)\cdot\delta M^-)\big)\\&\quad + 2\operatorname{tr}\big(D^{-1}K_2 - D^{-1}SK_1 - D^{-1}M_0^+K_3M_0^-\big) - \operatorname{tr}(K_3D)\,.\end{aligned}$$

In addition,

$$\operatorname{tr}(G_0JK) = \operatorname{tr}(D^{-1}K_2 - D^{-1}SK_1 - D^{-1}M_0^+K_3M_0^-)\,,$$

where $G_0(\omega) = G(\omega, 0_{2n})$, and

$$\begin{aligned}&\operatorname{tr}\big(D^{-1}(f_{\Gamma,\lambda_*}(\,\cdot\,,M_0^+)\cdot\delta M^+) + D^{-1}(f_{\Gamma,\lambda_*}(\,\cdot\,,M_0^-)\cdot\delta M^-)\big)\\&\qquad = -2\operatorname{tr}(H_1^{\lambda *}D^{-1}\delta S) - \operatorname{tr}\big(H_3^{\lambda *}SD^{-1}\delta S\big) - \operatorname{tr}\big(H_3^{\lambda *}(\delta M^+ - \delta M^-)\big)\,.\end{aligned}$$

It follows from the last three equalities that

$$\operatorname{tr}(D^{-1}\,\delta S)' = 2\operatorname{tr}(G_0JK) - \operatorname{tr}\big(K_3(M_0^+ - M_0^-) + H_3^{\lambda *}(\delta M^+ - \delta M^-)\big)\,,$$

and hence Lemma 3.24 guarantees that

$$\operatorname{tr}(D^{-1}(\omega)\,\delta S(\omega))' = 2\operatorname{tr}(G(\omega, 0_{2n})JK(\omega)) - d_{0_{2n}}\tilde{\imath}(\omega)\cdot K\,.$$

As explained in Remark 3.11, the $L^1(\Omega, m_0)$-integrability of these functions proves Lemma 3.25.

Proof of Theorem 3.22 Definition (3.21), Lemma 3.19 (which can immediately be adapted to the family (3.20)), and the formula for $\tilde{\imath}(\omega, K)$, yield

$$w(K) = \frac{1}{2}\int_\Omega \tilde{\imath}(\omega, K)\,dm_0.$$

Consequently, Lemmas 3.24 and 3.25 ensure that

$$\begin{aligned}w(K) - w(0_{2n}) &= \frac{1}{2}\int_\Omega d_{0_{2n}}\tilde{\imath}(\omega)\cdot K\,dm_0 + o(\|K\|_\Omega)\\&= \int_\Omega \operatorname{tr}(G(\omega, 0_{2n})JK(\omega))\,dm_0 + o(\|K\|_\Omega)\end{aligned}$$

for $K \to 0_{2n}$. Clearly, $C(\Omega, \mathfrak{sp}(n,\mathbb{C})) \to \mathbb{C}$, $K \mapsto \int_\Omega \operatorname{tr}(G(\omega, 0_{2n})JK(\omega))\, dm_0$ is a continuous map, which completes the proof of the first equality in the theorem. The second equality is an immediate consequence of the relation (3.25) between Q and G and the fact that $\operatorname{tr} K = 0$.

3.2.3 Derivative of the Floquet Coefficient with Respect to λ

Theorem 3.26, which is an easy consequence of Theorem 3.22, ensures the existence of the derivative of $w_\Gamma(\lambda)$, with respect to the argument λ when it lies outside the real axis, and provides the values of that derivative. This section is devoted to an analysis of the imaginary part of the derivative. A result will be obtained which will be required later to study the limiting behavior of the Floquet coefficient on the real axis.

As in the previous section, define, for $\operatorname{Im}\lambda \neq 0$,

$$\widetilde{G}_{\Gamma,\lambda}(\omega,t,s) = \begin{cases} U_{\Gamma,\lambda}(t,\omega)\, Q_{\Gamma,\lambda}(\omega)\, J^{-1} U^T_{\Gamma,\lambda}(s,\omega), & t \geq s\,, \\ -U_{\Gamma,\lambda}(t,\omega)\,(I_{2n} - Q_{\Gamma,\lambda}(\omega))\, J^{-1} U^T_{\Gamma,\lambda}(s,\omega)\,, & s > t, \end{cases} \tag{3.34}$$

where $Q_{\Gamma,\lambda}(\omega)$ is the projection determined by the exponential dichotomy of (3.2). According to Proposition 1.68, its range is $l^+(\omega)$ and its kernel is $l^-(\omega)$, so that Theorem 3.8 ensures that $Q_{\Gamma,\lambda}(\omega)$ is given by the expression (3.22) with $M^\pm = M^\pm_\Gamma(\omega,\lambda)$. Then, the symmetric matrix

$$G_\Gamma(\omega,\lambda) = \frac{1}{2}\left(\lim_{s\to 0-} \widetilde{G}_{\Gamma,\lambda}(\omega,0,s) + \lim_{s\to 0+} \widetilde{G}_{\Gamma,\lambda}(\omega,0,s)\right),$$

agrees with the matrix obtained by substituting $M^\pm$ by $M^\pm_\Gamma(\omega,\lambda)$ in (3.26). In particular, it is analytic with respect to $\lambda \in \mathbb{C} - \mathbb{R}$.

Theorem 3.26 *The derivative of the function $w_\Gamma(\lambda)$ given by (3.15) with respect to the parameter is given by*

$$w'_\Gamma(\lambda) = \int_\Omega \operatorname{tr}(G_\Gamma(\omega,\lambda)\, \Gamma(\omega))\, dm_0 \tag{3.35}$$

for $\lambda \in \mathbb{C} - \mathbb{R}$.

Proof The statement follows from Theorem 3.22: just fix $\lambda_* = \lambda$ with $\operatorname{Im}\lambda \neq 0$ and consider the perturbed systems (3.20) with $K = \varepsilon J^{-1}\Gamma$ for $|\varepsilon|$ small enough.

The properties of the function G_Γ, which are analyzed in what follows, will play a fundamental role in analyzing the limiting behavior of the Floquet coefficient, as well as in the relation between the rotation number and exponential dichotomy.

Theorem 3.27 $\operatorname{Im}\lambda\ \operatorname{Im} G_\Gamma(\omega,\lambda) \geq 0$ *for* $\lambda \in \mathbb{C}-\mathbb{R}$. *In particular, the matrix-valued function* $\lambda \mapsto G_\Gamma(\omega,\lambda)$ *is Herglotz for all* $\omega \in \Omega$.

Proof Fix $\lambda \in \mathbb{C}-\mathbb{R}$ and consider, for each $\omega \in \Omega$, the linear operator $\mathcal{L}^\lambda_\omega = J((d/dt)-H_\omega) - \lambda\,\Gamma_\omega$ (where $(H_\omega \mathbf{f})(t) = H(\omega{\cdot}t)\,\mathbf{f}(t)$ and $(\Gamma_\omega \mathbf{f})(t) = \Gamma(\omega{\cdot}t)\,\mathbf{f}(t)$) from $\mathcal{S}$ to $L^2(\mathbb{R},\mathbb{C}^{2n})$, where $\mathcal{S}$ is the dense subset of $L^2(\mathbb{R},\mathbb{C}^{2n})$ composed of the absolutely continuous functions with square integrable derivative. It follows from the exponential dichotomy of (3.2) that $\mathcal{L}^\lambda_\omega$ is invertible. To check this assertion, define

$$\mathbf{f}_\omega(t) = \int_{-\infty}^{\infty} \widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\mathbf{g}(s)\,ds\,, \tag{3.36}$$

for $\mathbf{g} \in L^2(\mathbb{R},\mathbb{C}^{2n})$, and note that its expression is given by (3.34). Then $\mathbf{f}_\omega \in L^2(\mathbb{R},\mathbb{C}^{2n})$. This fact follows from the Riesz–Thorin interpolation theorem: it is easy to deduce from Definition 1.58 that (3.36) defines a bounded operator from $L^\infty(\mathbb{R},\mathbb{C}^{2n})$ (the set of bounded measurable functions) to itself, while the equality

$$\begin{aligned}\int_{-\infty}^{\infty} \mathbf{f}_\omega(t)\,dt &= \int_{-\infty}^{\infty}\left(\int_s^{\infty} \widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,dt\right)\mathbf{g}(s)\,ds\\ &\quad + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{s} \widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,dt\right)\mathbf{g}(s)\,ds\,.\end{aligned}$$

guarantees that it defines a bounded operator from $L^1(\mathbb{R},\mathbb{C}^{2n})$ to itself. Therefore, (3.36) defines a bounded operator also from $L^2(\mathbb{R},\mathbb{C}^{2n})$ to itself. In addition, $\mathbf{f}_\omega$ is absolutely continuous on $\mathbb{R}$. Finally, it is easy to check that $\mathbf{f}'_\omega = J^{-1}\mathbf{g} + \big(H_\omega + \lambda J^{-1}\Gamma_\omega\big)\,\mathbf{f}_\omega$, which has two consequences: first, $\mathbf{f}'_\omega$ is square integrable, so that $\mathbf{f}_\omega \in \mathcal{S}$; and second, $\mathbf{g}$ coincides with $\mathcal{L}^\lambda_\omega\,\mathbf{f}_\omega$, so that the operator (3.36) is the inverse of the initial one.

It is easy to check that $\mathcal{L}^\lambda_\omega$ is a selfadjoint operator. Since its imaginary part is $-\operatorname{Im}\lambda\,\Gamma_\omega$, one can conclude that the imaginary part of its inverse is a positive (resp. negative) semidefinite operator in the case that $\lambda \in \mathbb{C}^+$ (resp. in the case that $\lambda \in \mathbb{C}^-$). A possible way to prove this assertion is to use the arguments of Proposition 1.21. And a consequence of it is that, if $\mathbf{g} \in L^2(\mathbb{R},\mathbb{R}^{2n})$,

$$\operatorname{Im}\lambda\ \operatorname{Im}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{g}^T(t)\,\widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\mathbf{g}(s)\,ds\,dt \geq 0$$

and, by choosing $\mathbf{g}(t,\mathbf{z}) = \psi(t)\,\mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^{2n}$, it turns out that the matrix

$$\operatorname{Im}\lambda\ \operatorname{Im}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\psi(t)\,\psi(s)\,ds\,dt$$

is positive semidefinite for every function $\psi \in L^2(\mathbb{R},\mathbb{R})$. Now let $\varphi \in C^1(\mathbb{R},\mathbb{R})$ satisfy $\varphi|_{(-\infty,-1]} \equiv -1/2$ and $\varphi|_{[1,\infty)} \equiv 1/2$. Then,

$$\begin{aligned}
\operatorname{Im}\lambda\, G_\Gamma(\omega,\lambda) &= \frac{\operatorname{Im}\lambda}{2}\left(\lim_{s\to 0^-}\widetilde{G}_{\Gamma,\lambda}(\omega,0,s) + \lim_{s\to 0^+}\widetilde{G}_{\Gamma,\lambda}(\omega,0,s)\right)\\
&= \lim_{k\to 0}\frac{\operatorname{Im}\lambda}{k^2}\int_{-k}^{k}\left(\int_{-k}^{t}\widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\varphi'(s/k)\,ds\right)\varphi'(t/k)\,dt\\
&\quad + \lim_{k\to 0}\frac{\operatorname{Im}\lambda}{k^2}\int_{-k}^{k}\left(\int_{t}^{k}\widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\varphi'(s/k)\,ds\right)\varphi'(t/k)\,dt\\
&= \lim_{k\to 0}\frac{\operatorname{Im}\lambda}{k^2}\int_{-k}^{k}\int_{-k}^{k}\widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\varphi'(t/k)\,\varphi'(s/k)\,ds\,dt\\
&= \lim_{k\to 0}\frac{\operatorname{Im}\lambda}{k^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{G}_{\Gamma,\lambda}(\omega,t,s)\,\varphi'(t/k)\,\varphi'(s/k)\,ds\,dt\,,
\end{aligned}$$

so that it has positive semidefinite imaginary part, as asserted in Theorem 3.27.

The following result is an immediate consequence of Theorems 3.26 and 3.27.

Corollary 3.28 *For* $\lambda \in \mathbb{C}-\mathbb{R}$,

$$\begin{aligned}
\operatorname{Im} w_\Gamma'(\lambda) &= \int_\Omega \operatorname{tr}(\Gamma^{1/2}(\omega)\operatorname{Im} G_\Gamma(\omega,\lambda)\,\Gamma^{1/2}(\omega))\,dm_0\\
&= \int_\Omega \operatorname{tr}(\Gamma(\omega)\operatorname{Im} G_\Gamma(\omega,\lambda))\,dm_0\,,
\end{aligned}$$

and hence $\operatorname{Im}\lambda\operatorname{Im} w_\Gamma'(\lambda) \ge 0$.

Remark 3.29 The maximum principle for harmonic functions and Corollary 3.28 ensure that $\operatorname{Im} w_\Gamma'$ is either identically zero or strictly positive (resp. negative) on the upper (resp. lower) half-plane. In fact, it will be proved (see representation (3.44)) that $\operatorname{Im} w_\Gamma' \equiv 0$ if and only if the rotation number α_Γ is a constant function on the entire real axis, which is not possible under mild conditions on the measure m_0 (see Theorem 3.50).

3.2.4 Limit of the Floquet Coefficient on the Real Axis

This section is devoted to the study of the relation between the limit of the Floquet coefficient on the real axis and the rotation number and the Lyapunov index of the (real) limit systems. Recall that $\alpha_\Gamma(\lambda)$ represents the rotation number of (3.2) for $\lambda \in \mathbb{R}$, and that $\beta_\Gamma(\lambda)$ represents the nonnegative Lyapunov index for $\lambda \in \mathbb{C}$.

The arguments used reproduce basically those appearing in [72]. Consider first the real part, $\operatorname{Re} w_\Gamma(\lambda)$.

Theorem 3.30 *For every* $\lambda \in \mathbb{C} - \mathbb{R}$,

$$\operatorname{Re} w_\Gamma(\lambda) = -\lim_{t\to\infty} \frac{1}{t} \ln \| \wedge^n U_{\Gamma,\lambda}(t,\omega)\| = -\beta_\Gamma(\lambda)$$

m_0*-a.e. Moreover, there exist the nontangential limits from the upper and lower complex half-planes*

$$\lim_{\lambda \searrow \lambda_0} \operatorname{Re} w_\Gamma(\lambda) = \lim_{\lambda \nearrow \lambda_0} \operatorname{Re} w_\Gamma(\lambda) = -\beta_\Gamma(\lambda_0)$$

for Lebesgue a.e. $\lambda_0 \in \mathbb{R}$.

Proof As explained at the beginning of Sect. 2.5, the existence of exponential dichotomy for the system (3.2) with $\operatorname{Im}\lambda \neq 0$ ensures the existence of $2n$ Lyapunov exponents $\mp\beta_1, \ldots, \mp\beta_n$ associated to the ergodic measure m_0, with $\beta_j > 0$ (see Remark 2.42.1). These exponents are respectively provided by solutions with initial data on the stable and unstable subbundles $L^\pm_{\Gamma,\lambda}$. In addition, according to the Oseledets theory, the sum of the positive Lyapunov exponents, i.e. the Lyapunov index $\beta_\Gamma(\lambda)$ of (3.2), agrees with the function $\lim_{t\to\infty}(1/t)\ln \| \wedge^n U_{\Gamma,\lambda}(t,\omega)\|$ for $\omega \in \Omega_0$, with $m_0(\Omega_0) = 1$ (see (3.4)). So, the first assertion will be proved once it is proved that $\operatorname{Re} w_\Gamma(\lambda) = -\beta_\Gamma(\lambda)$ if $\operatorname{Im}\lambda \neq 0$.

Fix $\omega \in \Omega_0$ and choose bases $\{\mathbf{z}^\pm_{\omega,1}, \ldots, \mathbf{z}^\pm_{\omega,n}\}$ of the subbundle fibers $l^\pm_{\Gamma,\lambda}(\omega)$ such that

$$\lim_{|t|\to\infty} \frac{1}{t} \ln \|\mathbf{z}^\pm_j(t,\omega)\| = \mp\beta_j \quad \text{for } j = 1, \ldots, n\,,$$

where $\mathbf{z}^\pm_j(t,\omega) = U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}^\pm_{\omega,j}$. Theorem 3.8(ii) guarantees the existence of nonsingular matrices $P^\pm$ such that $\begin{bmatrix} I_n \\ M^\pm_\Gamma(\omega) \end{bmatrix} = [\,\mathbf{z}^\pm_{\omega,1} \cdots \mathbf{z}^\pm_{\omega,n}]\,P^\pm$. Hence,

$$\begin{bmatrix} W^\pm_{\Gamma,\lambda}(t,\omega) \\ M^\pm_\Gamma(\omega{\cdot}t)\, W^\pm_{\Gamma,\lambda}(t,\omega) \end{bmatrix} = U_{\Gamma,\lambda}(t,\omega) \begin{bmatrix} I_n \\ M^\pm_\Gamma(\omega) \end{bmatrix} = [\,\mathbf{z}^\pm_1(t,\omega) \cdots \mathbf{z}^\pm_n(t,\omega)]\,P^\pm,$$

with $W^\pm_{\Gamma,\lambda}(t,\omega)$ defined by (3.17). Look at the n first rows of this matrix equality to conclude that

$$\begin{aligned} &\det(W^-_{\Gamma,\lambda}(t,\omega)^T\, W^-_{\Gamma,\lambda}(t,\omega)) \\ &\qquad = \|\mathbf{x}^-_1(t,\omega)\|^2 \cdots \|\mathbf{x}^-_n(t,\omega)\|^2 (\det P^-)^2 \det R(t,\omega)\,, \end{aligned}$$

where $\mathbf{x}_j^-$ represents the vector composed of the n first components of $\mathbf{z}_j^-$ and the entry ij of the matrix $R(t,\omega)$ is defined for every $i,j = 1,\dots,n$ by

$$(\|\mathbf{x}_i^-(t,\omega)\|\,\|\mathbf{x}_j^-(t,\omega)\|)^{-1}\mathbf{x}_i^-(t,\omega)^T\mathbf{x}_j^-(t,\omega)\,.$$

Therefore, the choice of $\mathbf{z}_j^-(t,\omega)$ and the boundedness of $\det R(t,\omega)$ lead to

$$\lim_{t\to\infty}\frac{1}{t}\,\ln|\det W^-_{\Gamma,\lambda}(t,\omega)| \le \sum_{j=1}^{n}\lim_{t\to\infty}\frac{1}{t}\,\ln\|\mathbf{z}_j^-(t,\omega)\| = \sum_{j=1}^{n}\beta_j = \beta_\Gamma(\lambda)\,.$$

Analogously, $\lim_{t\to\infty}(1/t)\ln|\det W^+_{\Gamma,\lambda}(t,\omega)| \le -\beta_\Gamma(\lambda)$. The argument leading to (3.18) also proves that

$$0 = \lim_{t\to\infty}\frac{1}{t}\,\ln|\det W^+_{\Gamma,\lambda}(t,\omega)| + \lim_{t\to\infty}\frac{1}{t}\,\ln|\det W^-_{\Gamma,\lambda}(t,\omega)| \le \beta_\Gamma(\lambda)-\beta_\Gamma(\lambda)\,,$$

so that $\lim_{t\to\infty}(1/t)\ln|\det W^\pm_{\Gamma,\lambda}(t,\omega)| = \pm\beta_\Gamma(\lambda)$. Hence, by (3.19),

$$\begin{aligned}\operatorname{Re} w_\Gamma(\lambda) &= -\lim_{t\to\infty}\frac{1}{t}\,\ln|\det W^-_{\Gamma,\lambda}(t,\omega)|\\ &= \lim_{t\to\infty}\frac{1}{t}\,\ln|\det W^+_{\Gamma,\lambda}(t,\omega)| = -\beta_\Gamma(\lambda)\,,\end{aligned}$$

which, as explained at the beginning of the proof, demonstrates the first assertion of the theorem.

To prove the second assertion, recall that the holomorphic function $-iw_\Gamma$ is Herglotz, since $\beta_\Gamma(\lambda) > 0$ for all $\lambda \notin \mathbb{R}$. One uses Theorem 3.13 to establish the existence of $\beta_\Gamma^+(\lambda_0) = -\lim_{\lambda\searrow\lambda_0}\operatorname{Re} w_\Gamma(\lambda) = \lim_{\lambda\searrow\lambda_0}\beta_\Gamma(\lambda)$ and $\beta_\Gamma^-(\lambda_0) = -\lim_{\lambda\nearrow\lambda_0}\operatorname{Re} w_\Gamma(\lambda) = \lim_{\lambda\nearrow\lambda_0}\beta_\Gamma(\lambda)$ at Lebesgue-a.e. $\lambda_0 \in \mathbb{R}$. Remark 3.14.3 ensures that $\tilde\beta_\Gamma(\lambda_0) = \lim_{\varepsilon\to0^+}\beta_\Gamma(\lambda_0+\rho\,\varepsilon)$ for every $\rho\in\mathbb{C}^+$. On the other hand, the function β_Γ is subharmonic on the entire complex plane (see Theorem 3.1). In particular, for any fixed $\lambda \in \mathbb{C}$: $\limsup_{\mu\to\lambda}\beta_\Gamma(\mu) \le \beta_\Gamma(\lambda)$; $\beta_\Gamma(\lambda) \le (1/2\pi)\int_0^{2\pi}\beta_\Gamma(\lambda+\varepsilon e^{i\theta})\,d\theta$ for every $\varepsilon > 0$; and there exists $\varepsilon_0 > 0$ such that $\beta_\Gamma(\lambda+\varepsilon e^{i\theta}) \le \beta_\Gamma(\lambda)+1$ for all $\varepsilon\le\varepsilon_0$ and $\theta\in[0,2\pi]$. Therefore, Fatou's lemma ensures that

$$\begin{aligned}\beta_\Gamma(\lambda) &\le \limsup_{\varepsilon\to0^+}\frac{1}{2\pi}\int_0^{2\pi}\beta_\Gamma(\lambda+\varepsilon e^{i\theta})\,d\theta\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\limsup_{\varepsilon\to0^+}\beta_\Gamma(\lambda+\varepsilon e^{i\theta})\,d\theta \le \beta_\Gamma(\lambda)\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Taking $\lambda_0 \in \mathbb{R}$ for which $\beta_\Gamma^+(\lambda_0)$ and $\beta_\Gamma^-(\lambda_0)$ exist, one sees that

$$\beta_\Gamma(\lambda_0) \leq \frac{1}{2\pi}\left(\int_0^\pi \beta_\Gamma^+(\lambda_0)\,d\theta + \int_\pi^{2\pi} \beta_\Gamma^-(\lambda_0)\,d\theta\right) \leq \beta_\Gamma(\lambda_0)\,,$$

which in turn guarantees that $\beta_\Gamma(\lambda_0) = (\beta_\Gamma^+(\lambda_0) + \beta_\Gamma^-(\lambda_0))/2$. This and the inequalities $0 \leq \beta_\Gamma^\pm(\lambda_0) \leq \beta_\Gamma(\lambda_0)$ prove that $\beta_\Gamma(\lambda_0) = \beta_\Gamma^+(\lambda_0) = \beta_\Gamma^-(\lambda_0)$, as asserted.

The next objective is to analyze the limit of the imaginary part of $w_\Gamma(\lambda)$.

Remarks 3.31

1. As explained in Remark 3.14.4, the Herglotz character of the Weyl functions ensures the existence of a subset $\mathcal{R} \subseteq \mathbb{R}$ with full Lebesgue measure such that for all $\lambda_0 \in \mathcal{R}$ there exist the limits $\lim_{\lambda \searrow \lambda_0} M_\Gamma^\pm(\omega,\lambda)$ for m_0-a.e. $\omega \in \Omega$. Throughout this chapter and the following one, the value of each limit will be represented as

$$M_\Gamma^\pm(\omega,\lambda_0) = \lim_{\lambda \searrow \lambda_0} M_\Gamma^\pm(\omega,\lambda)$$

   whenever it exists. Recall also that the functions $\omega \mapsto M_\Gamma^\pm(\omega,\lambda_0)$ are Σ_{m_0}-measurable. Clearly, $\pm\,\mathrm{Im}\, M_\Gamma^\pm(\omega,\lambda_0) \geq 0$.
2. Write $M_\lambda(t,\omega,M_0)$ for the solution of the Riccati equation (3.9), where $\lambda \in \mathbb{C}$ and $M_\lambda(0,\omega,M_0) = M_0$. Take $\lambda_0 \in \mathcal{R}$ and denote by Ω_{λ_0} the subset of points ω such that the limit $M_\Gamma^+(\omega,\lambda_0) = \lim_{\lambda \searrow \lambda_0} M_\Gamma^+(\omega,\lambda)$ exists. Assume that $M_\lambda(t,\omega,M_0)$ is globally defined. Then, for all $t \in \mathbb{R}$,

$$\lim_{\lambda \searrow \lambda_0} M_\Gamma^+(\omega{\cdot}t,\lambda) = \lim_{\lambda \searrow \lambda_0} M_\lambda(t,\omega,M_\Gamma^+(\omega,\lambda)) = M_{\lambda_0}(t,\omega,M_\Gamma^+(\omega,\lambda_0))\,,$$

   as can be deduced from the classical theorems on continuous dependence of solutions of ordinary differential equations with respect to initial conditions and parameters. Therefore, the limit $M_\Gamma^+(\omega{\cdot}t,\lambda_0)$ exists for all $t \in \mathbb{R}$ (that is, the σ-orbit of ω is contained in Ω_{λ_0}), and the map $t \mapsto M_\Gamma^+(\omega{\cdot}t,\lambda_0)$ solves the equation (3.9) for λ_0. Clearly, if $M_\lambda(t,\omega,M_0)$ is globally defined for all $\omega \in \Omega_{\lambda_0}$, then the set Ω_{λ_0} is σ-invariant. An analogous argument works for $M_\Gamma^-(\omega,\lambda_0)$.

These facts will be fundamental in the proof of the following result, which shows that the limit of the imaginary part of the Floquet coefficient determines the rotation number of the limit system. In turn, the properties of the Floquet coefficient will be used in a later analysis of the boundary behavior of the Weyl functions (see Sect. 4.3).

Theorem 3.32 *For all* $\lambda_0 \in \mathbb{R}$*,*

$$\lim_{\lambda\to\lambda_0,\,\mathrm{Im}\,\lambda>0} \mathrm{Im}\, w_\Gamma(\lambda) = \alpha_\Gamma(\lambda_0)\,. \tag{3.37}$$

In particular, $\mathrm{Im}\, w_\Gamma$ *is continuous on the closure of* $\mathbb{C}^+$.

Proof As stated in Remark 2.12, the rotation number of the family (3.2) for $\lambda_0 \in \mathbb{R}$ with respect to m_0 is given by

$$\begin{aligned}\alpha_\Gamma(\lambda_0) &= -\lim_{t\to\infty} \frac{1}{2n}\,\frac{1}{t}\ \mathrm{Im}\int_0^t \mathrm{tr} f_{\Gamma,\lambda_0}(\omega{\cdot}s, M_{\lambda_0}(s,\omega,M_0))\,ds \\ &= \lim_{t\to\infty} \frac{1}{t}\ \mathrm{Im}\int_0^t \mathrm{tr}\Big(H_1(\omega{\cdot}s) + \lambda_0\Gamma_1(\omega{\cdot}s) \\ &\qquad\qquad + (H_3(\omega{\cdot}s) + \lambda_0\Gamma_3(\omega{\cdot}s))\,M_{\lambda_0}(s,\omega,M_0)\Big)\,ds\end{aligned} \tag{3.38}$$

m_0-a.e. for all $M_0 \in \mathbb{S}_n^+(\mathbb{C})$ (i.e. with $\mathrm{Im}\, M_0 > 0$), where $M_{\lambda_0}(t,\omega,M_0)$ is the (globally defined) solution of the Riccati equation (3.9) with $M_{\lambda_0}(0,\omega,M_0) = M_0$. The operator $f_{\Gamma,\lambda}(\omega,M)$ is given by (3.29).

Fix λ_0 in the set $\mathcal{R}$ of Remark 3.31.1, so that the limit function $M_\Gamma^+(\omega,\lambda_0)$ exists for $\omega \in \Omega_{\lambda_0}$ with $m_0(\Omega_{\lambda_0}) = 1$.

To obtain an overview of the main arguments required in the following proof, assume that the function $\Omega_{\lambda_0} \to \mathbb{S}_n(\mathbb{C})\,,\ \omega \mapsto M_\Gamma^+(\omega,\lambda_0)$ satisfies the following conditions. First, it takes values on $\mathbb{S}_n^+(\mathbb{C})$, so that $M_{\lambda_0}(t,\omega,M^+(\omega,\lambda_0))$ is globally defined and agrees with $M_\Gamma^+(\omega{\cdot}t,\lambda_0)$, and Ω_{λ_0} is σ-invariant: see Lemma 2.10 and Remark 3.31.2. Second, it belongs to $L^1(\Omega,m_0)$: see Definition 1.32. Note that

$$\begin{aligned}&\lim_{\lambda\searrow\lambda_0} \mathrm{tr}(H_1(\omega) + \lambda\,\Gamma_1(\omega) + (H_3(\omega) + \lambda\,\Gamma_3(\omega))\,M_\Gamma^+(\omega,\lambda)) \\ &\qquad = \mathrm{tr}(H_1(\omega) + \lambda_0\Gamma_1(\omega) + (H_3(\omega) + \lambda_0\Gamma_3(\omega))\,M_\Gamma^+(\omega,\lambda_0))\end{aligned}$$

for $\omega \in \Omega_{\lambda_0}$. Third and finally, assume that the Lebesgue dominated convergence theorem can be applied to these directional limits. Then, on the one hand, definition (3.15) yields

$$\lim_{\lambda\searrow\lambda_0} w_\Gamma(\lambda) = \int_\Omega \mathrm{tr}(H_1(\omega) + \lambda_0\Gamma_1(\omega) + (H_3(\omega) + \lambda_0\Gamma_3(\omega))M_\Gamma^+(\omega,\lambda_0))\,dm_0\,;$$

and, on the other hand, relation (3.38) for $M_0 = M_\Gamma^+(\omega,\lambda_0)$ and Birkhoff's ergodic theorem (see Theorems 1.3 and 1.6) prove that

$$\alpha_\Gamma(\lambda_0) = \mathrm{Im}\int_\Omega \mathrm{tr}(H_1(\omega) + \lambda_0\Gamma_1(\omega) + (H_3(\omega) + \lambda_0\Gamma_3(\omega))M_\Gamma^+(\omega,\lambda_0))\,dm_0\,.$$

That is, $\lim_{\lambda\searrow\lambda_0} \operatorname{Im} w_\Gamma(\lambda) = \alpha_\Gamma(\lambda_0)$ for all $\lambda_0 \in \mathbb{R}$. As will be explained at the end of the proof, the convergence from the upper half-plane follows from this fact and the Herglotz character of w'_Γ.

Now, in general, it can be only asserted that the function $M^+_\Gamma(\omega, \lambda_0)$ is Σ_{m_0}-measurable and satisfies $\operatorname{Im} M^+_\Gamma(\omega, \lambda_0) \geq 0$, and hence this argument cannot be applied. The solution to this problem is quite technical: the framework considered here must be embedded in the more general one studied in [72], as was done in the last part of Sect. 2.1.2 of Chap. 2. That is, the family (3.2) is transformed for all $\lambda \in \mathbb{C}$ by means of the change of variables $\tilde{\mathbf{z}} = K^{-1}\mathbf{z}$; recall that $K = \left[\begin{smallmatrix} iI_n & iI_n \\ -I_n & I_n \end{smallmatrix}\right]$. The expression of the transformed system is given by the corresponding matrix (2.12), with H_j substituted by $H_j^\lambda = H_j + \lambda\Gamma_j$. It is clear that the exponential dichotomy is preserved, and that the fibers of the corresponding closed subbundles can be represented by $\left[\begin{smallmatrix} I_n \\ \widetilde{M}^\pm_\Gamma(\omega,\lambda) \end{smallmatrix}\right]$, where the new Weyl functions $\widetilde{M}^\pm_\Gamma(\omega, \lambda)$ are related to the "old" functions $M^\pm_\Gamma(\omega, \lambda)$ (for $\lambda \notin \mathbb{R}$) by the Cayley transform

$$\widetilde{M} = (iI_n - M)(iI_n + M)^{-1} \quad \text{and} \quad M = i(I_n - \widetilde{M})(I_n + \widetilde{M})^{-1}\,. \tag{3.39}$$

Consequently, $\widetilde{M}^\pm_\Gamma(\omega, \lambda) \in \mathbb{D}_\mathbb{C}$ for $\lambda \notin \mathbb{R}$, where $\mathbb{D}_\mathbb{C}$ is the set of the complex symmetric $n \times n$ matrices $\widetilde{M}$ with $I_n - \widetilde{M}^*\widetilde{M} > 0$: see the beginning of the proof of Lemma 2.17. Recall that $\lambda_0 \in \mathcal{R}$, and let $\widetilde{M}^+_\Gamma(\omega, \lambda_0)$ be the transform by (3.39) of the nontangential limit $M^+_\Gamma(\omega, \lambda_0)$, which belongs to the $\text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_\mathbb{C}$. Then the set $\Omega_{\lambda_0} \subseteq \Omega$ with $m_0(\Omega_{\lambda_0}) = 1$ such that $\widetilde{M}^+_\Gamma(\omega, \lambda_0)$ exists for $\omega \in \Omega_{\lambda_0}$ is σ-invariant, and the map $\omega \mapsto \widetilde{M}^+_\Gamma(\omega, \lambda_0)$ is a solution along the flow on Ω_{λ_0} of the Riccati equation (3.9) for λ_0: see Remark 3.31.2, and keep in mind that Lemma 2.15 ensures that the solution of the transformed Riccati equation corresponding to λ_0 is globally defined for every initial datum in the closure of $\mathbb{D}_\mathbb{C}$ and for all $\omega \in \Omega$. The boundedness of the $\text{closure}_{\mathbb{S}_n(\mathbb{C})}\mathbb{D}_\mathbb{C}$ ensures that the map is $L^1(\Omega, m_0)$-integrable. As before, these properties, Remarks 2.20 and the Birkhoff ergodic theorem lead to

$$\alpha_\Gamma(\lambda_0) = -\frac{1}{2n}\operatorname{Im}\int_\Omega \operatorname{tr}\tilde{f}_{\Gamma,\lambda_0}(\omega, \widetilde{M}^+_\Gamma(\omega, \lambda_0))\, dm_0\,, \tag{3.40}$$

where $\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M})$ represents the linear operator obtained as the variational equation of the Riccati equation corresponding to the transformed systems associated to its solution $\widetilde{M}$.

The definition of $\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M}){\cdot}D$ is obtained by the substitution of H_j for H_j^λ in (2.34). In turn, this expression yields

$$\begin{aligned} &\operatorname{tr}\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M}) \\ &\qquad = -n\operatorname{tr}\big(i(H_2^\lambda - H_3^\lambda) - (H_1^\lambda + (H_1^\lambda)^T - i(H_2^\lambda + H_3^\lambda))\widetilde{M}\big)\,, \end{aligned} \tag{3.41}$$

where H_j^λ represents $H_j^\lambda(\omega)$. A straightforward computation from the Riccati equation corresponding to the transformed systems proves that

$$\begin{aligned} i(H_2^\lambda - H_3^\lambda) - \widetilde{M}(H_1^\lambda + (H_1^\lambda)^T - i(H_2^\lambda + H_3^\lambda)) \\ = 2\,\widetilde{M}'(I_n + \widetilde{M})^{-1} + ((H_1^\lambda)^T - H_1^\lambda) \\ - 2i\,(I_n - \widetilde{M})H_3^\lambda(I_n + \widetilde{M})^{-1} - 2(I_n + \widetilde{M})(H_1^\lambda)^T(I_n + \widetilde{M})^{-1} \end{aligned}$$

for $\widetilde{M} = \widetilde{M}_\Gamma^+(\omega, \lambda)$. According to Remarks 3.18 and 3.11,

$$\int_\Omega \operatorname{tr}(\widetilde{M}'(I_n + \widetilde{M})^{-1})\, dm_0 = \int_\Omega (\ln\det(I_n + \widetilde{M}))' dm_0 = 0\,.$$

Therefore, by (3.39) and (3.19),

$$\begin{aligned} -\frac{1}{2n}\int_\Omega \operatorname{tr}\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M}_\Gamma^+(\omega,\lambda))\, dm_0 \\ = \int_\Omega \operatorname{tr} H_1^\lambda dm_0 + \int_\Omega \operatorname{tr}(H_3^\lambda(i(I_n - \widetilde{M})(I_n + \widetilde{M})^{-1}))\, dm_0 \qquad (3.42) \\ = \int_\Omega \operatorname{tr}(H_1^\lambda(\omega) + H_3^\lambda(\omega)M_\Gamma^+(\omega,\lambda))\, dm_0 = w_\Gamma(\lambda)\,. \end{aligned}$$

It follows from (3.41) that $\|\operatorname{tr}\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M})\|$ is uniformly bounded when $(\omega, \widetilde{M})$ varies on $\Omega \times \mathbb{D}_\mathbb{C}$. And, clearly,

$$\lim_{\lambda\searrow\lambda_0} \operatorname{tr}\tilde{f}_{\Gamma,\lambda}(\omega, \widetilde{M}_\Gamma^+(\omega,\lambda)) = \operatorname{tr}\tilde{f}_{\Gamma,\lambda_0}(\omega, \widetilde{M}_\Gamma^+(\omega,\lambda_0))$$

m_0-a.e. Consequently, relations (3.42) and (3.40) and the Lebesgue dominated convergence theorem ensure that

$$\lim_{\lambda\searrow\lambda_0} \operatorname{Im} w_\Gamma(\lambda) = \alpha_\Gamma(\lambda_0)\,. \qquad (3.43)$$

Recall again that, from the beginning, λ_0 is assumed to belong to $\mathcal{R}$: it is a point for which the Weyl functions converge nontangentially m_0-a.e. That is, so far it has been proved that $\operatorname{Im} w_\Gamma$ converges nontangentially to the rotation number α_Γ at Lebesgue-a.e. point of the real axis.

On the other hand, according to Corollary 3.28, $\operatorname{Im} w'_\Gamma(\lambda) \geq 0$ if $\operatorname{Im}\lambda > 0$. Theorem 3.13 and this Herglotz character ensure that, if $\lambda_0 \in \mathbb{R}$ and $\varepsilon > 0$, then

$$\operatorname{Im} w'_\Gamma(\lambda_0 + i\varepsilon) = k\,\varepsilon + \int_\mathbb{R} \frac{\varepsilon}{(t-\lambda_0)^2 + \varepsilon^2}\, d\mu(t)$$

for $k \geq 0$ and for a positive regular Borel measure $d\mu$ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} 1/(t^2+1)\, d\mu(t) < \infty$, which is induced by a nondecreasing nonnegative Borel measurable function $\mu \colon \mathbb{R} \to \mathbb{R}$. In addition, if λ_1 and λ_2 are continuity points of the distribution function μ, then

$$\begin{aligned}(\mu(\lambda_2) - \mu(\lambda_1))\,\pi &= \lim_{\varepsilon\to 0+} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} w'_\Gamma(t+i\varepsilon)\,dt \\ &= \lim_{\varepsilon\to 0+} \int_{\lambda_1}^{\lambda_2} \frac{\partial \operatorname{Im} w_\Gamma}{\partial t}(t+i\varepsilon)\,dt \\ &= \lim_{\varepsilon\to 0+} \operatorname{Im} w_\Gamma(\lambda_2+i\varepsilon) - \lim_{\varepsilon\to 0+} \operatorname{Im} w_\Gamma(\lambda_1+i\varepsilon)\,.\end{aligned}$$

This property and the nontangential limiting behavior (3.43) mean that $\pi\mu(\lambda)$ and α_Γ are the same function (up to an additive constant) Lebesgue-a.e. That is,

$$\operatorname{Im} w'_\Gamma(\lambda_0+i\varepsilon) = k\,\varepsilon + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\varepsilon}{(t-\lambda_0)^2+\varepsilon^2}\, d\alpha_\Gamma(t) \tag{3.44}$$

and hence the continuity of α_Γ on the real axis proved in Theorem 2.25 guarantees that

$$\alpha_\Gamma(\lambda_2) - \alpha_\Gamma(\lambda_1) = \lim_{\varepsilon\to 0+} \operatorname{Im} w_\Gamma(\lambda_2+i\varepsilon) - \lim_{\varepsilon\to 0+} \operatorname{Im} w_\Gamma(\lambda_1+i\varepsilon) \tag{3.45}$$

for all λ_1 and λ_2 in $\mathbb{R}$. Fixing $\lambda_1 \in \mathcal{R}$ and applying (3.43) ensures that $\alpha_\Gamma(\lambda_2) = \lim_{\varepsilon\to 0+} \operatorname{Im} w_\Gamma(\lambda_2+i\varepsilon)$ for every $\lambda_2 \in \mathbb{R}$. Take now sequences (λ_m) in $\mathbb{R}$ and (ε_m) in $\mathbb{R}_+$ with limits λ_0 and 0. The proof of the first and main assertion of the theorem will be complete once it has been checked that $\lim_{m\to\infty} \operatorname{Im} w_\Gamma(\lambda_m+i\varepsilon_m) = \alpha_\Gamma(\lambda_0)$, for which it suffices to check that $\lim_{m\to\infty}(\operatorname{Im} w_\Gamma(\lambda_m+i\varepsilon_m) - \operatorname{Im} w_\Gamma(\lambda_0+i\varepsilon_m) = 0$. Take any $\delta > 0$; find ρ such that if $\lambda \in \mathbb{R}$ and $|\lambda-\lambda_0| \leq \rho$ then $|\alpha_\Gamma(\lambda) - \alpha_\Gamma(\lambda_0)| < \delta$; and note that there exists m_1 such that $\lambda_m \in (\lambda_0-\rho, \lambda_0+\rho)$ whenever $m \geq m_1$. Reasoning as above and keeping in mind that $\operatorname{Im} w'_\Gamma(t+i\varepsilon_m) \geq 0$ for $t \in \mathbb{R}$, it follows that

$$\begin{aligned}&\lim_{m\to\infty} |\operatorname{Im} w_\Gamma(\lambda_m+i\varepsilon_m) - \operatorname{Im} w_\Gamma(\lambda_0+i\varepsilon_m)\,| \\ &\quad= \lim_{m\to\infty} \left| \int_{\lambda_0}^{\lambda_m} \operatorname{Im} w'_\Gamma(t+i\varepsilon_m)\,dt \right| \leq \lim_{m\to\infty} \left| \int_{\lambda_0-\rho}^{\lambda_0+\rho} \operatorname{Im} w'_\Gamma(t+i\varepsilon_m)\,dt \right| \\ &\quad= |\alpha_\Gamma(\lambda_0+\rho) - \alpha_\Gamma(\lambda_0-\rho)\,| \leq 2\delta\,.\end{aligned}$$

The second assertion of the theorem is a trivial consequence of the first one.

Remark 3.33 The previous proof implies in particular that the rotation number $\alpha_\Gamma(\lambda)$ is nondecreasing with respect to λ. This property also follows from the more general one stated in Proposition 2.33, since $\Gamma \geq 0$. In particular, α_Γ defines a positive Borel measure on $\mathbb{R}$.

Remark 3.34 The first definition for the Floquet coefficient outside the real axis appeared in [72]. It was suggested by relation (3.40) and given by

$$w_\Gamma(\lambda) = -\frac{1}{2n}\int_\Omega \operatorname{tr} f_{\Gamma,\lambda}(\omega, M_\Gamma^+(\omega,\lambda))\, dm_0\,,$$

which agrees with (3.15) for $\lambda \in \mathbb{C}^+$ as can be deduced from the definition (3.29) of $f_{\Gamma,\lambda}(\omega, M)$. Note that, according to Birkhoff's Theorems 1.3 and 1.6,

$$w_\Gamma(\lambda) = -\lim_{t\to\infty}\frac{1}{2n}\frac{1}{t}\int_0^t \operatorname{tr} f_{\Gamma,\lambda}(\omega{\cdot}s, M_\Gamma^+(\omega{\cdot}s,\lambda))\, ds$$

m_0-a.e. This equality also implies that $\operatorname{Re} w_\Gamma(\lambda)$ measures the average rate of change of volume determined by the motion of vectors tangent to $M_\Gamma^+(\omega,\lambda)$, whereas $\operatorname{Im} w_\Gamma(\lambda)$ measures the average rotation around $M_\Gamma^+(\omega,\lambda)$.

Remark 3.35 The arguments used to prove Theorem 3.32 can be easily adapted to check that, for all $\lambda_0 \in \mathbb{R}$,

$$\lim_{\lambda\to\lambda_0,\,\operatorname{Im}\lambda<0} \operatorname{Im} w_\Gamma(\lambda) = -\alpha_\Gamma(\lambda_0)\,.$$

The point now is to work with the nontangential limits of the Weyl functions from the lower half-plane and to use the equivalent definition of $w_\Gamma(\lambda)$ obtained in Lemma 3.19 (iii). Consequently, defining the Floquet coefficient as suggested in Remark 3.17 provides a function with the characteristics indicated there.

The last result of this section establishes the *trace formula* for α_Γ mentioned in the introduction to this chapter, which will be fundamental in the study of the relation between exponential dichotomy and the rotation number. It relates the positive measure determined by α_Γ (see Remark 3.33) with the measure $dP_{\Gamma,\omega}$ appearing in the representation

$$G_\Gamma(\omega,\lambda) = L_\Gamma(\omega) + K_\Gamma(\omega)\,\lambda + \int_{\mathbb{R}}\left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) dP_{\Gamma,\omega}(t) \tag{3.46}$$

for $\lambda \in \mathbb{C}^+$ (and for all $\omega \in \Omega$), which is provided by Theorems 3.27 and 3.13.

Theorem 3.36 *The trace formula*

$$\frac{1}{\pi}\, d\alpha_\Gamma = \int_\Omega \operatorname{tr}(\Gamma(\omega)\, dP_{\Gamma,\omega})\, dm_0$$

holds, where the equality is to be interpreted in the following weak sense: if* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *is a continuous function with compact support, then*

$$\frac{1}{\pi}\int_{\mathbb{R}} \varphi(t)\, d\alpha_\Gamma(t) = \int_\Omega \operatorname{tr}\Big(\Gamma(\omega)\int_{\mathbb{R}} \varphi(t)\, dP_{\Gamma,\omega}(t)\Big)\, dm_0\,.$$

Proof The Stieltjes inversion formula (see Theorem 3.13(iii)) ensures that

$$\begin{aligned}\frac{1}{2}\left(P_{\Gamma,\omega}\{\lambda_1\} + P_{\Gamma,\omega}\{\lambda_2\}\right) + \int_{(\lambda_1,\lambda_2)} dP_{\Gamma,\omega}(t)&\\ = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{\lambda_1}^{\lambda_2} \operatorname{Im} G_\Gamma(\omega, t+i\varepsilon)\, dt\,.&\end{aligned}$$

In addition, this function is bounded on Ω for each fixed finite interval $(\lambda_1,\lambda_2)\subset\mathbb{R}$. So, the continuity of $G_\Gamma(\,\cdot\,, i)$ on Ω and Remark 3.14.1 prove the existence of a positive matrix C such that

$$0 \le \int_{\mathbb{R}} \frac{1}{t^2+1}\, dP_{\Gamma,\omega}(t) \le \operatorname{Im} G_\Gamma(\omega, i) \le C$$

for all $\omega\in\Omega$, and hence, for $s\in\mathbb{R}$ such that $[\lambda_1,\lambda_2]\subseteq[-s,s]$, one has

$$\int_{[\lambda_1,\lambda_2]} dP_{\Gamma,\omega}(t) \le (s^2+1)\int_{[-s,s]} \frac{1}{t^2+1}\, dP_{\Gamma,\omega}(t) \le (s^2+1)\, C\,. \tag{3.47}$$

On the other hand, according to Theorems 3.32 and 3.26, if $(\lambda_1,\lambda_2)\subset\mathbb{R}$,

$$\begin{aligned}\frac{1}{\pi}\left(\alpha_\Gamma(\lambda_2) - \alpha_\Gamma(\lambda_1)\right) &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left(\operatorname{Im} w_\Gamma(\lambda_2+i\varepsilon) - \operatorname{Im} w_\Gamma(\lambda_1+i\varepsilon)\right)\\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{\lambda_1}^{\lambda_2}\int_\Omega \operatorname{tr}\left(\operatorname{Im} G_\Gamma(\omega, t+i\varepsilon)\,\Gamma(\omega)\right) dm_0\, dt\,.\end{aligned}$$

Hence Fubini's theorem, the Lebesgue dominated convergence theorem, and the previous inversion formula yield

$$\begin{aligned}&\frac{1}{\pi}\left(\alpha_\Gamma(\lambda_2) - \alpha_\Gamma(\lambda_1)\right)\\ &\quad= \operatorname{tr}\int_\Omega \Gamma^{1/2}(\omega)\left(\frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{\lambda_1}^{\lambda_2} \operatorname{Im} G_\Gamma(\omega, t+i\varepsilon)\, dt\right)\Gamma^{1/2}(\omega)\, dm_0\\ &\quad= \operatorname{tr}\int_\Omega \bigg(\Gamma^{1/2}(\omega)\,\frac{1}{2}\left(P_{\Gamma,\omega}\{\lambda_1\} + P_{\Gamma,\omega}\{\lambda_2\}\right)\Gamma^{1/2}(\omega)\\ &\qquad\qquad + \Gamma^{1/2}(\omega)\int_{(\lambda_1,\lambda_2)} dP_{\Gamma,\omega}(t)\;\Gamma^{1/2}(\omega)\bigg)\, dm_0\end{aligned}$$

for any finite interval $(\lambda_1, \lambda_2) \subset \mathbb{R}$. In particular,

$$0 \le \operatorname{tr} \int_\Omega \Gamma^{1/2}(\omega)\, P_{\Gamma,\omega}\{\lambda_1\}\, \Gamma^{1/2}(\omega)\, dm_0 \le \frac{2}{\pi}\,(\alpha_\Gamma(\lambda_2) - \alpha_\Gamma(\lambda_1))$$

for all $\lambda_1 \in \mathbb{R}$ and $\lambda_2 > \lambda_1$. The continuity of the rotation number implies that

$$\operatorname{tr} \int_\Omega \Gamma^{1/2}(\omega)\, P_{\Gamma,\omega}\{\lambda_1\}\, \Gamma^{1/2}(\omega)\, dm_0 = 0$$

for all $\lambda_1 \in \mathbb{R}$. This fact and the previous equality mean that

$$\frac{1}{\pi} \int_{(\lambda_1,\lambda_2)} d\alpha_\Gamma(t) = \int_\Omega \operatorname{tr}\left(\Gamma(\omega) \int_{(\lambda_1,\lambda_2)} dP_{\Gamma,\omega}(t)\right) dm_0\,, \tag{3.48}$$

and a standard measure-theoretic argument proves that the map sending each Borel subset $B \subset \mathbb{R}$ to $\int_\Omega \operatorname{tr}\big(\Gamma(\omega) \int_B dP_{\Gamma,\omega}(t)\big)\, dm_0$ defines a Borel measure which agrees with $(1/\pi)\, d\alpha_\Gamma$.

3.3 The Floquet Exponent and Atkinson Spectral Problems

The properties of the rotation number are related to the existence of exponential dichotomy for the perturbed systems (3.2) and (3.3) with $\lambda \in \mathbb{R}$, and consequently to the associated Atkinson spectral problems. This section contains a discussion of these interconnections. Throughout Sect. 3.3, the perturbation Γ of the family (3.2) is assumed to be of Atkinson type (see Definition 3.4).

A substantial amount of preliminary work is required for the proofs of the main results, which are finally carried out in Sects. 3.3.4 and 3.3.5. The first three subsections contain some basic facts concerning symmetric Herglotz matrix-valued functions, a certain one-parameter boundary value problem given by (3.2) on a finite interval and the associated spectral matrix-valued functions, and the limiting behavior of the characteristic and spectral functions as the interval increases to fill out the real line. And Sect. 3.3.3 recalls some results about the null controllability on a compact set of the systems considered. The results here presented are mainly due to Atkinson [5], Johnson [72], and Johnson and Nerurkar [75, 77].

As before, Γ represents an Atkinson perturbation (see Definition 3.4); for each value of the parameter $\lambda \in \mathbb{C} - \mathbb{R}$, $Q_{\Gamma,\lambda}(\omega)$ represents the projection associated to the exponential dichotomy of (3.2); $M^\pm_\Gamma(\omega, \lambda)$ are the Weyl functions (see Theorem 3.8); and $G_\Gamma(\omega, \lambda)$ is the symmetric Herglotz matrix-valued function $(Q_{\Gamma,\lambda}(\omega) - (1/2)I_{2n})J^{-1}$ (see Theorem 3.27). Recall that $Q_{\Gamma,\lambda}$ and G_Γ are given in terms of $M^\pm_\Gamma$ by the corresponding relations (3.22) and (3.26). Throughout this section, the Euclidean vector and matrix norms are used (see Remark 1.24.2).

3.3.1 A Boundary Value Problem in $[a, b]$

Let $[a, b]$ represent a real interval with $a < 0 < b$, and let A and B be real $(2n)\times(2n)$ matrices such that

$$A\,\mathbf{v} = B\,\mathbf{v} = \mathbf{0} \quad \text{implies} \quad \mathbf{v} = \mathbf{0}\,, \quad \text{and} \quad A^T JA = B^T JB\,. \tag{3.49}$$

It is known (see [5], Theorem 9.2.1) that the eigenvalues of the boundary value problem

$$\begin{cases} J\mathbf{z}' = (JH(\omega{\cdot}t) + \lambda\,\Gamma(\omega{\cdot}t))\,\mathbf{z}\,, \\ \exists\;\mathbf{v} \neq \mathbf{0} \quad \mathbb{C}^{2n}-\{\mathbf{0}\} \quad \mathbf{z}(a) = A\,\mathbf{v} \quad \text{and} \quad \mathbf{z}(b) = B\,\mathbf{v}\,, \end{cases} \tag{3.50}$$

are real and form a countable set. Note that λ is an eigenvalue if and only if there exists a vector $\mathbf{v} \neq \mathbf{0}$ such that $U^{-1}_{\Gamma,\lambda}(a,\omega)A\mathbf{v} = U^{-1}_{\Gamma,\lambda}(b,\omega)B\mathbf{v} = \mathbf{z}_0$, in which case $\mathbf{z}_0 \neq \mathbf{0}$ is the initial datum of a corresponding eigenfunction. It can be deduced immediately that the *characteristic* function

$$\begin{aligned} &F^{a,b}_{A,B}(\omega,\lambda) \\ &\quad = -\frac{1}{2}\big(U^{-1}_{\Gamma,\lambda}(a,\omega)\,A + U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\big)\big(U^{-1}_{\Gamma,\lambda}(a,\omega)\,A - U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\big)^{-1}J \end{aligned}$$

is well defined if and only if λ is not an eigenvalue of (3.50). In particular, it is well defined for all $\lambda \notin \mathbb{R}$. In addition, the equalities $(U^{-1}_{\Gamma,\lambda})^T J U^{-1}_{\Gamma,\lambda} = J$ (see Proposition 1.23) and $A^T JA = B^T JB$ imply that $F^{a,b}_{A,B}(\omega,\lambda)$ is symmetric. Clearly, the matrix-valued map $F^{a,b}_{A,B}$ is jointly continuous on $\Omega \times (\mathbb{C}-\mathbb{R})$ and analytic outside the real axis for each fixed $\omega \in \Omega$. The following result shows that $\lambda \mapsto F^{a,b}_{A,B}(\omega,\lambda)$ is a symmetric Herglotz matrix-valued function.

Proposition 3.37 *If* $\operatorname{Im}\lambda \neq 0$, *then* $\operatorname{Im}\lambda\ \operatorname{Im}F^{a,b}_{A,B}(\omega,\lambda) \leq 0$.

Proof This proof is basically taken from [5], Section 9.5. Fix $\omega \in \Omega$ and write $F_\lambda = F^{a,b}_{A,B}(\omega,\lambda)$ and $U_\lambda(t) = U_{\Gamma,\lambda}(t,\omega)$. It is easy to check that

$$\begin{aligned} &\big(U^{-1}_\lambda(a)\,A - U^{-1}_\lambda(b)\,B\big)^*\,J^*(F_\lambda - F^*_\lambda)J\,\big(U^{-1}_\lambda(a)\,A - U^{-1}_\lambda(b)\,B\big) \\ &\qquad = -A^T(U^{-1}_\lambda)^*(a)\,J\,U^{-1}_\lambda(a)\,A + B^T(U^{-1}_\lambda)^*(b)\,J\,U^{-1}_\lambda(b)\,B\,. \end{aligned} \tag{3.51}$$

On the other hand, $\big(U_\lambda^*(t)\,J\,U_\lambda(t)\big)' = 2i\,\mathrm{Im}\,\lambda\,U_\lambda^*(t)\,\Gamma(\omega{\cdot}t)\,U_\lambda(t)$, as can be deduced from (3.2). Hence,

$$J-(U_\lambda^{-1})^*(t)\,J\,U_\lambda^{-1}(t) = 2i\,\mathrm{Im}\,\lambda\,(U_\lambda^{-1})^*(t)\left(\int_0^t U_\lambda^*(s)\,\Gamma(\omega{\cdot}s)\,U_\lambda(s)\,ds\right)U_\lambda^{-1}(t)\,,$$

which together with $A^TJA = B^TJB$ yields

$$\begin{aligned}-A^T(U_\lambda^{-1})^*(a)\,J\,U_\lambda^{-1}(a)\,A + B^T(U_\lambda^{-1})^*(b)\,J\,U_\lambda^{-1}(b)\,B \\ = -2i\,\mathrm{Im}\,\lambda\Bigg(A^*(U_\lambda^{-1})^*(a)\left(\int_a^0 U_\lambda^*(s)\,\Gamma(\omega{\cdot}s)\,U_\lambda(s)\,ds\right)U_\lambda^{-1}(a)\,A \\ + B^*(U_\lambda^{-1})^*(b)\left(\int_0^b U_\lambda^*(s)\,\Gamma(\omega{\cdot}s)\,U_\lambda(s)\,ds\right)U_\lambda^{-1}(b)\,B\Bigg).\end{aligned} \tag{3.52}$$

This equality and (3.51) imply that $(W^*(F_\lambda - F_\lambda^*)\,W)/(2i\,\mathrm{Im}\,\lambda) \le 0$ for a nonsingular matrix W. The assertion of the proposition follows hence from the fact that $\mathrm{Im}\,F_\lambda = (F_\lambda - F_\lambda^*)/(2i)$, which is in turn a consequence of the symmetry of F_λ.

Remark 3.38 According to Lemma 3.6(iv) and Remark 3.7, given any $\lambda \in \mathbb{C}$ there exist constants $t_0 = t_0(\lambda) > 0$ and $\delta = \delta(\lambda) > 0$ such that, for every $\mathbf{z} \neq 0$,

$$\int_0^{t_0} \|\Gamma(\omega{\cdot}t)\,U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt > \delta\|\mathbf{z}\|^2\,,$$

$$\int_{-t_0}^{0} \|\Gamma(\omega{\cdot}t)\,U_{\Gamma,\lambda}(t,\omega)\,\mathbf{z}\|^2\,dt > \delta\|\mathbf{z}\|^2\,.$$

It follows from this fact, equalities (3.51) and (3.52), and condition (3.49), that $\mathrm{Im}\,\lambda\;\mathrm{Im}\,F_{A,B}^{a,b}(\omega,\lambda) < 0$ if $[-t_0(\lambda), t_0(\lambda)] \subseteq [a,b]$.

Let $\{\lambda_k \mid k \ge 1\}$ be the eigenvalues of (3.50), repeated according to their multiplicities and ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, and let $\{\boldsymbol{\theta}_k(t) \mid k \ge 1\}$ be a corresponding set of normalized eigenfunctions. Atkinson [5] defines in Section 9.3 a spectral matrix-valued function on $\mathbb{R}$, associated to (3.50) (and hence depending on ω, as do the eigenvalues and eigenfunctions):

$$P^{a,b}_{A,B}(t) = \begin{cases} -\sum\limits_{t \le \lambda_k \le 0} \boldsymbol{\theta}_k(0)\,\boldsymbol{\theta}_k^T(0) & \text{if } t < 0\,, \\ 0_{2n} & \text{if } t = 0\,, \\ \sum\limits_{0 < \lambda_k \le t} \boldsymbol{\theta}_k(0)\,\boldsymbol{\theta}_k^T(0) & \text{if } t > 0\,. \end{cases} \tag{3.53}$$

Note that $P^{a,b}_{A,B}(t)$ is a nondecreasing right-continuous step function: it is constant on each interval between successive eigenvalues, and the jump at the eigenvalue λ_j is $\sum_{\lambda_k=\lambda_j} \boldsymbol{\theta}_k(0)\,\boldsymbol{\theta}_k^T(0)$. Note also that $F^{a,b}_{A,B}(\omega,\lambda)$ agrees with $K(0,0,\lambda)$, where $K(s,t,\lambda)$ is the integral kernel for (3.50) defined in [5], Section 9.4. Consequently, according to [5], Theorem 9.7.5 and Problem 9.18, the representation for the Herglotz function $-F^{a,b}_{A,B}$ (see Theorem 3.13(ii) and Remark 3.14.2) takes the form

$$-F^{a,b}_{A,B}(\omega,\lambda) = L^{a,b}_{A,B}(\omega) + K^{a,b}_{A,B}(\omega)\,\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) dP^{a,b}_{A,B}(t) \tag{3.54}$$

for certain real symmetric matrices $L^{a,b}_{A,B}(\omega)$ and $K^{a,b}_{A,B}(\omega)$.

3.3.2 *Limiting Behavior as a → −∞ and b → ∞*

This section contains some important results concerning the limiting properties of the characteristic and spectral functions defined in Sect. 3.3.1 as the interval $[a,b]$ increases.

Theorem 3.39 *For every pair of matrices A and B satisfying* (3.49)*,*

$$\lim_{a\to-\infty,\, b\to\infty} F^{a,b}_{A,B}(\omega,\lambda) = -G_\Gamma(\omega,\lambda)\,,$$

on $\Omega \times (\mathbb{C}-\mathbb{R})$, and the convergence is uniform on compact subsets of the domain.

Proof The proof is based on the existence of exponential dichotomy for the system (3.2) for $\lambda \notin \mathbb{R}$, which was established in Theorem 3.8. Since

$$\begin{aligned} &\Big(2F^{a,b}_{A,B}(\omega,\lambda)\,J - (I_{2n} - 2Q_{\Gamma,\lambda}(\omega))\Big)\left(U^{-1}_{\Gamma,\lambda}(a,\omega))\,A - U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\right) \\ &\qquad = 2\left(Q_{\Gamma,\lambda}(\omega)\,U^{-1}_{\Gamma,\lambda}(a,\omega)\,A + (I - Q_{\Gamma,\lambda}(\omega))\,U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\right), \end{aligned}$$

one has that

$$\begin{aligned}&\Big\|2F^{a,b}_{A,B}(\omega,\lambda)\,J-(I_{2n}-2Q_{\Gamma,\lambda}(\omega))\Big\|\\ &\qquad\le 2\eta_\lambda\,\big(e^{a\beta_\lambda}\|A\|+e^{-b\beta_\lambda}\|B\|\big)\,\Big\|\big(U^{-1}_{\Gamma,\lambda}(a,\omega)\,A-U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\big)^{-1}\Big\|\,,\end{aligned}$$

where β_λ and η_λ are the constants associated to the exponential dichotomy of (3.2) (see Definition 1.58, and recall that $a<0<b$). Let $\mathcal{K}\subset\Omega\times(\mathbb{C}-\mathbb{R})$ be a compact subset. Then the constants β_λ and η_λ can be chosen to be the same for every $(\omega,\lambda)\in\mathcal{K}$, as can easily be deduced from Theorem 1.91(i). Call these common constants η and β. It is now easy to see that the statement of the theorem is a consequence of the relation $G_\Gamma=(Q_{\Gamma,\lambda}-(1/2)I_{2n})J^{-1}$, the previous bound, and the following assertion: there exist $c>0$ and $\rho>0$ such that

$$\Big\|\big(U^{-1}_{\Gamma,\lambda}(a,\omega)\,A-U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\big)^{-1}\Big\|\le\rho$$

for $a\le -c$, $b\ge c$ and $(\omega,\lambda)\in\mathcal{K}$. The next objective is to prove this assertion. Since $\|C^{-1}\|=\max_{\|\mathbf{v}\|=1}(1/\|C\mathbf{v}\|)$, the previous inequality follows from the existence of $\tilde\rho>0$ such that

$$\big\|\big(U^{-1}_{\Gamma,\lambda}(a,\omega)\,A-U^{-1}_{\Gamma,\lambda}(b,\omega)\,B\big)\mathbf{v}\big\|\ge\tilde\rho \tag{3.55}$$

for all $\mathbf{v}\in\mathbb{C}^{2n}$ with $\|\mathbf{v}\|=1$ whenever $a\le -c$, $b\ge c$ and $(\omega,\lambda)\in\mathcal{K}$. Assume for contradiction the existence of sequences $(a_k)\downarrow -\infty$, $(b_k)\uparrow\infty$, $\mathbf{v}_k\in\mathbb{C}^{2n}$ with $\|\mathbf{v}_k\|=1$ and $(\omega_k,\lambda_k)\in\mathcal{K}$ such that

$$\big\|\big(U^{-1}_{\Gamma,\lambda_k}(a_k,\omega_k)\,A-U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\,B\big)\mathbf{v}_k\big\|<\frac{1}{k}\,. \tag{3.56}$$

Note that there is no loss of generality in assuming that $a_k\le -k$ and $b_k\ge k$. Write $U^{-1}_{\Gamma,\lambda}(t,\omega)=Q_{\Gamma,\lambda}(\omega)U^{-1}_{\Gamma,\lambda}(t,\omega)+(I_{2n}-Q_{\Gamma,\lambda}(\omega))\,U^{-1}_{\Gamma,\lambda}(t,\omega)$. The inequality $\|\mathbf{w}_1-\mathbf{w}_2\|\ge\|\mathbf{w}_1\|-\|\mathbf{w}_2\|$ and the exponential dichotomy imply that

$$\begin{aligned}&\Big\|(I_{2n}-Q_{\Gamma,\lambda_k}(\omega_k))\,U^{-1}_{\Gamma,\lambda_k}(a_k,\omega_k)\,A\,\mathbf{v}_k\\ &\qquad\qquad-Q_{\Gamma,\lambda_k}(\omega_k)\,U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\,B\,\mathbf{v}_k\Big\|\\ &\qquad<\frac{1}{k}+\eta\,e^{\beta a_k}\|A\|+\eta\,e^{-\beta b_k}\|B\|\le\frac{2}{k}\end{aligned} \tag{3.57}$$

whenever $k\ge k_0$, for an index k_0 large enough. Choose now commonly indexed subsequences with $\mathbf{v}_k\to\tilde{\mathbf{v}}$, $\lambda_k\to\tilde\lambda$, $\omega_k{\cdot}a_k\to\bar\omega$ and $\omega_k{\cdot}b_k\to\tilde\omega$ and assume that $Q_{\Gamma,\tilde\lambda}(\tilde\omega)\,B\,\tilde{\mathbf{v}}\ne\mathbf{0}$. Then, there exist $\varepsilon_1>0$ and k_1 such that $\|Q_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\,B\,\mathbf{v}_k\|>\varepsilon_1$ whenever $k\ge k_1$. Bearing in mind this property together with the three equalities $Q^T_{\Gamma,\lambda} J\, Q_{\Gamma,\lambda} = 0_{2n}$ (deduced from (3.22)), $Q_{\Gamma,\lambda}(\omega{\cdot}t,\lambda)\, U_{\Gamma,\lambda}(t,\omega) = U_{\Gamma,\lambda}(t,\omega)\, Q_{\Gamma,\lambda}(\omega)$ (ensured by Definition 1.58), and $U^T_{\Gamma,\lambda} J\, U_{\Gamma,\lambda} = J$, one obtains

$$
\begin{aligned}
\varepsilon_1^2 &< \left|\mathbf{v}_k^T B^T Q^T_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\, JJ\, \overline{Q}_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\, B\, \overline{\mathbf{v}}_k\right| \\
&= \left|\mathbf{v}_k^T B^T Q^T_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\, J\, (I_{2n} - Q_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)) J\, \overline{Q}_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\, B\, \overline{\mathbf{v}}_k\right| \\
&= \Big|(Q_{\Gamma,\lambda_k}(\omega_k)\, U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\, B\, \mathbf{v}_k)^T J\cdot \\
&\qquad \cdot(I_{2n} - Q_{\Gamma,\lambda_k}(\omega_k)) U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\, J\, \overline{Q}_{\Gamma,\lambda_k}(\omega_k{\cdot}b_k)\, B\, \overline{\mathbf{v}}_k\Big| \\
&\le \left\|Q_{\Gamma,\lambda_k}(\omega_k)\, U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\, B\, \mathbf{v}_k\right\|\, \eta\, e^{-\beta b_k} \sup_{(\omega,\lambda)\in\mathcal{K}} \|Q_{\Gamma,\lambda}(\omega)\|\, \|B\|\,.
\end{aligned}
$$

Since $\mathcal{K}$ is compact, there exists a positive constant $\bar{\rho}$ such that

$$
\|Q_{\Gamma,\lambda_k}(\omega_k)\, U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\, B\, \mathbf{v}_k\| \ge \bar{\rho}\, e^{\beta b_k} \ge \bar{\rho} \tag{3.58}
$$

whenever $k \ge k_1$. Suppose now for contradiction that

$$
\|(I_{2n} - Q_{\Gamma,\lambda_k}(\omega_k))\, U^{-1}_{\Gamma,\lambda_k}(b_k,\omega_k)\, A\, \mathbf{v}_k\| < k
$$

for all $k \ge k_1$. Then it follows from (3.57) and (3.58) that $\bar{\rho}\, e^{\beta b_k} < k + 2/k$, which is impossible. Therefore

$$
\left\|(I_{2n} - Q_{\Gamma,\lambda_{k_j}}(\omega_{k_j}))\, U^{-1}_{\Gamma,\lambda_k}(b_{k_j},\omega_{k_j})\, A\, \mathbf{v}_{k_j}\right\| > k_j \tag{3.59}
$$

for a suitable subsequence (k_j).

Represent by $l^{\pm}_{\Gamma,\lambda}(\omega)$ the vector spaces of the initial data of solutions of (3.1) which are bounded as $t \to \pm\infty$, determined again by the exponential dichotomy if $(\omega,\lambda) \in \mathcal{K}$: see Remark 1.77.3. Let $\delta \in [0,1)$ satisfy $|\langle \mathbf{w}^+, \mathbf{w}^- \rangle| < \delta\, \|\mathbf{w}^+\|\|\mathbf{w}^-\|$ for all $(\omega,\lambda) \in \mathcal{K}$ and all pairs of nonzero vectors $\mathbf{w}^+ \in l^+_{\Gamma,\lambda}(\omega)$ and $\mathbf{w}^- \in l^-_{\Gamma,\lambda}(\omega)$. The existence of δ is checked below. It is easy to see that, if $\mathbf{w}^+ \in l^+_{\Gamma,\lambda}(\omega)$ and $\mathbf{w}^- \in l^-_{\Gamma,\lambda}(\omega)$, then $\|\mathbf{w}^+ - \mathbf{w}^-\|^2 \ge 2(1-\delta)\|\mathbf{w}^+\|\|\mathbf{w}^-\|$. Since

$$
(I_{2n} - Q_{\Gamma,\lambda_{k_j}}(\omega_{k_j}))\, U^{-1}_{\Gamma,\lambda_{k_j}}(a_{k_j},\omega_{k_j})\, A\, \mathbf{v}_{k_j} \in l^-_{\Gamma,\lambda_{k_j}}(\omega_{k_j})
$$

and

$$
Q_{\Gamma,\lambda_{k_j}}(\omega_{k_j})\, U^{-1}_{\Gamma,\lambda_{k_j}}(b_{k_j},\omega_{k_j})\, B\, \mathbf{v}_{k_j} \in l^+_{\Gamma,\lambda_{k_j}}(\omega_{k_j})\,,
$$

the inequalities (3.57), (3.58), and (3.59) yield

$$\frac{4}{k_j^2} \geq 2\,(1-\delta)\,\bar{\rho}\,k_j$$

for each index j. This is impossible, so that the condition (3.56) indeed leads to a contradiction.

In order to check the existence of the constant δ, assume the existence of a sequence $(\omega_m, \lambda_m, \mathbf{w}_m^+, \mathbf{w}_m^-)$ in $\mathcal{K} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with $\mathbf{w}_m^\pm \in l_{\Gamma,\lambda_m}^\pm(\omega_m)$ and $\|\mathbf{w}_m^\pm\| = 1$ such that $|\langle \mathbf{w}_m^+, \mathbf{w}_m^- \rangle| > 1 - (1/m)$ for all $m \in \mathbb{N}$. By choosing a suitable subsequence if needed, it can be assumed that the sequence $(\omega_m, \lambda_m, \mathbf{w}_m^+, \mathbf{w}_m^-)$ tends to $(\omega_0, \lambda_0, \mathbf{w}_0^-, \mathbf{w}_0^+) \in \mathcal{K} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Hence, $|\langle \mathbf{w}_0^+, \mathbf{w}_0^- \rangle| = 1$, so that $\mathbf{w}_0^+$ and $\mathbf{w}_0^-$ are linearly dependent. But this is impossible, since the continuous variation of $l_{\Gamma,\lambda}^\pm(\omega)$ in $\mathcal{K}$ implies that $\mathbf{w}_0^\pm \in l_{\Gamma,\lambda_0}^\pm(\omega_0)$.

This completes the proof of (3.55) (and hence of the theorem) in the case that $Q_{\Gamma,\tilde{\lambda}}(\tilde{\omega})\, B\,\tilde{\mathbf{v}} \neq \mathbf{0}$. But, if this inequality were not true, one would have $(I_{2n} - Q_{\Gamma,\tilde{\lambda}}(\bar{\omega}))\, A\,\tilde{\mathbf{v}} \neq \mathbf{0}$, as will be checked in what follows, and the argument would be analogous. Assume for contradiction that $Q_{\Gamma,\tilde{\lambda}}(\tilde{\omega})\, B\,\tilde{\mathbf{v}} = \mathbf{0}$ and $(I_{2n} - Q_{\Gamma,\tilde{\lambda}}(\bar{\omega}))\, A$ $\tilde{\mathbf{v}} = \mathbf{0}$. From the representation of $l_{\Gamma,\tilde{\lambda}}^+(\tilde{\omega})$ and $l_{\Gamma,\tilde{\lambda}}^-(\bar{\omega})$ proved in Theorem 3.8, one infers the existence of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^n$ such that $B\,\tilde{\mathbf{v}} = \begin{bmatrix} I_n \\ M_\Gamma^+(\tilde{\omega},\tilde{\lambda}) \end{bmatrix} \mathbf{x}$ and $A\,\tilde{\mathbf{v}} = \begin{bmatrix} I_n \\ M_\Gamma^-(\bar{\omega},\tilde{\lambda}) \end{bmatrix} \mathbf{y}$. The equality $B^T J B = A^T J A$ and the Herglotz character of the Weyl functions imply that $0 \leq \mathbf{x}^* \operatorname{Im} M_\Gamma^+(\tilde{\omega},\tilde{\lambda})\,\mathbf{x} = \mathbf{y}^* \operatorname{Im} M_\Gamma^-(\bar{\omega},\tilde{\lambda})\,\mathbf{y} \leq 0$, which can only occur in the case $\mathbf{x} = \mathbf{y} = \mathbf{0}$. Hence, $A\,\tilde{\mathbf{v}} = B\,\tilde{\mathbf{v}} = \mathbf{0}$, which, according to (3.49), implies $\tilde{\mathbf{v}} = \mathbf{0}$. But this is impossible, since $\|\tilde{\mathbf{v}}\| = 1$. The proof is complete.

Theorem 3.39 and the continuity of $G_\Gamma(\omega,\lambda)$ with respect to ω allow one to apply Theorem 3.15 in order to obtain the following conclusions. The matrix-valued functions $dP_{\Gamma,\omega}$ and $P_{A,B}^{a,b}(t)$ appear in the representations (3.46) and (3.54) respectively. Recall that $P_{A,B}^{a,b}(t)$ also depends on ω.

Theorem 3.40

(i) *For all $\omega \in \Omega$ and every pair of matrices A and B satisfying* (3.49)*,*

$$\lim_{a\to-\infty,\, b\to\infty} \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP_{A,B}^{a,b}(t)\, \boldsymbol{\psi}(t) = \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP_{\Gamma,\omega}(t)\, \boldsymbol{\psi}(t)$$

for every $\boldsymbol{\psi}\colon \mathbb{R} \to \mathbb{C}^{2n}$ which is continuous and has compact support.

(ii) *The map $\omega \mapsto dP_{\Gamma,\omega}$ is weak* continuous. In other words, if $\omega = \lim_{k\to\infty} \omega_k$, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP_{\Gamma,\omega_k}(t)\, \boldsymbol{\psi}(t) = \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP_{\Gamma,\omega}(t)\, \boldsymbol{\psi}(t)\,.$$

for every $\boldsymbol{\psi}\colon \mathbb{R} \to \mathbb{C}^{2n}$ which is continuous and has compact support.

Remark 3.41 It is possible to repeat the arguments used in Theorem 3.39 in order to check that, for any pair of sequences $(a_k) \downarrow -\infty$, $(b_k) \uparrow \infty$ and every pair of *bounded* sequences (A_k), (B_k) such that each pair A_k, B_k satisfies (3.49), one has

$$\lim_{k\to\infty} F^{a_k,b_k}_{A_k,B_k}(\omega,\lambda) = -G_\Gamma(\omega,\lambda)\,,$$

on $\Omega \times (\mathbb{C} - \mathbb{R})$, and the convergence is uniform on compact subsets. Consequently, according to Theorem 3.15,

$$\lim_{k\to\infty} \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP^{a_k,b_k}_{A_k,B_k}(t)\, \boldsymbol{\psi}(t) = \int_{\mathbb{R}} \boldsymbol{\psi}^*(t)\, dP_{\Gamma,\omega}(t)\, \boldsymbol{\psi}(t)$$

for every $\boldsymbol{\psi}\colon \mathbb{R} \to \mathbb{C}^{2n}$ which is continuous and has compact support.

3.3.3 Null Controllability on $\mathcal{B}_1 = \{\mathbf{y} \in \mathbb{R}^d \mid \|\mathbf{y}\| \le 1\}$

Consider a control system

$$\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u} \tag{3.60}$$

where $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{u} \in \mathbb{R}^m$, and A and B are continuous matrix-valued functions of the appropriate dimensions. Let $U_A(t)$ be the fundamental matrix solution of $\mathbf{x}' = A(t)\,\mathbf{x}$ satisfying $U_A(0) = I_d$. The following definition and result can be found in Barmish and Schmitendorf [11].

Definition 3.42 Write $\mathcal{B}_r = \{\mathbf{y} \in \mathbb{R}^d \mid \|\mathbf{y}\| \le r\}$. The system (3.60) is $\mathcal{B}_r$*-null controllable at* $(a, \mathbf{x}_1) \in \mathbb{R} \times \mathbb{R}^d$ in time $t_0 = t_0(a, \mathbf{x}_1) \ge 0$ if there exists a Borel measurable function $\mathbf{u}\colon [a,\infty) \to \mathcal{B}_r$ such that the solution of the corresponding equation (3.60) with $\mathbf{x}(a) = \mathbf{x}_1$ satisfies $\mathbf{x}(a + t_0) = \mathbf{0}$.

Theorem 3.43 *Suppose that there exist* $t_0 > 0$ *and* $\delta > 0$ *such that*

$$\int_a^{a+t_0} \|B^T(t)\,(U_A^T)^{-1}(t)\,U_A^T(a)\,\mathbf{x}\|\,dt \ge \delta\,\|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbb{R}^d$ *with* $\|\mathbf{x}\| = 1$. *Then the system* (3.60) *is* $\mathcal{B}_1$*-null controllable in time* t_0 *at* $(a, \mathbf{x}_1)$ *for all* $\mathbf{x}_1$ *with* $\|\mathbf{x}_1\| < \delta/4$.

Remark 3.44 The proof of the previous theorem can be carried out easily using Theorem 2.1 in [11] and its proof. In fact, the result stated there is much more general, but Theorem 3.43 is enough for the purposes of this chapter. The theorem is formulated in [11] for the integrand $H_{\mathcal{B}_1}(B^T(t)\,(U_A^T)^{-1}(t)\,U_A^T(a)\,\mathbf{x})$, with $H_{\mathcal{B}_1}(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{B}_1} \mathbf{y}^T\mathbf{x}$. But, as the authors point out, it is immediate to check that $H_{\mathcal{B}_1}(\mathbf{x}) = \|\mathbf{x}\|$.

Assume now that Γ satisfies the Atkinson Hypotheses (3.3), fix $\lambda_0 \in \mathbb{R}$, and consider the family of nonautonomous control systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda_0 J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} + J^{-1}\Gamma(\omega{\cdot}t)\,\mathbf{u}\,, \quad \omega \in \Omega\,. \tag{3.61}$$

The following result will play a fundamental role in the proof of Theorem 3.48.

Lemma 3.45 *There exists* $\delta = \delta(\lambda_0) > 0$ *and* $t_0 = t_0(\lambda_0) > 0$ *such that*

(i) *for every* $a \in \mathbb{R}$, $\omega \in \Omega$ *and* $\mathbf{z} \in \mathbb{R}^{2n}$,

$$\int_a^{a+t_0} \|\Gamma(\omega{\cdot}t)\,J\,(U^T_{\Gamma,\lambda_0})^{-1}(t,\omega)\,U^T_{\Gamma,\lambda_0}(a,\omega)\,\mathbf{z}\|\,dt \ge \delta\,\|\mathbf{z}\|\,.$$

(ii) *All the systems* (3.61) *corresponding to* λ_0 *and* $\omega \in \Omega$ *are* $\mathcal{B}_1$*-null controllable in time* t_0 *at every* $(a,\mathbf{z}_1) \in \mathbb{R}\times\{\mathbf{z}\in\mathbb{R}^{2n} \mid \|\mathbf{z}\| < \delta/4\}$.

(iii) *All the systems* (3.61) *corresponding to* λ_0 *and* $\omega \in \Omega$ *are* $\mathcal{B}_r$*-null controllable in time* t_0 *at every* $(a,\mathbf{z}_1) \in \mathbb{R}\times\{\mathbf{z}\in\mathbb{R}^{2n} \mid \|\mathbf{z}\| < c\}$ *for* $r = 4c/\delta$.

Proof

(i) Let $t_0 = t_0(\lambda_0) > 0$ be the constant provided by Lemma 3.6(iv). Suppose for contradiction the existence of sequences (t_k) in $\mathbb{R}$, (ω_k) in Ω and $(\mathbf{z}_k)$ in $\{\mathbf{z}\in\mathbb{R}^{2n} \mid \|\mathbf{z}\| = 1\}$ with

$$\int_{t_k}^{t_k+t_0} \|\Gamma(\omega_k{\cdot}t)\,J\,(U^T_{\Gamma,\lambda_0})^{-1}(t,\omega_k)\,U^T_{\Gamma,\lambda_0}(t_k,\omega_k)\,\mathbf{z}_k\|\,dt < \frac{1}{k}\,.$$

Then,

$$\begin{aligned}\frac{1}{k} &> \int_0^{t_0} \|\Gamma((\omega_k{\cdot}t_k){\cdot}t)\,J\,(U^T_{\Gamma,\lambda_0})^{-1}(t+t_k,\omega_k)\,U^T_{\Gamma,\lambda_0}(t_k,\omega_k)\,\mathbf{z}_k\|\,dt\\ &= \int_0^{t_0} \|\Gamma((\omega_k{\cdot}t_k){\cdot}t)\,U_{\Gamma,\lambda_0}(t,\omega_k{\cdot}t_k)\,J\,\mathbf{z}_k\|\,dt\,.\end{aligned}$$

Taking suitable subsequences with $\lim_{k\to\infty}\omega_k{\cdot}t_k = \tilde\omega$ and $\lim_{k\to\infty}\mathbf{z}_k = \tilde{\mathbf{z}}$, it follows that

$$\int_0^{t_0} \|\Gamma(\tilde\omega{\cdot}t)\,U_{\Gamma,\lambda_0}(t,\tilde\omega)\,J\,\tilde{\mathbf{z}}\|\,dt = 0\,,$$

and this contradicts the choice of t_0.

(ii) This is an immediate consequence of (i) and Theorem 3.43.

(iii) Take $\mathbf{x}_1$ with $\|\mathbf{x}_1\| < c = r\delta/4$ and let $\tilde{\mathbf{u}}\colon [a,\infty) \to \mathcal{B}_1$ be the control such that the solution $\tilde{\mathbf{x}}$ of the corresponding equation (3.60) with $\tilde{\mathbf{x}}(a) = (1/r)\,\mathbf{x}_1$ satisfies $\tilde{\mathbf{x}}(a+t_0) = \mathbf{0}$. Then the solution of (3.60) with control $r\,\tilde{\mathbf{u}}$ and initial datum $\mathbf{x}_1$ at a is given by $r\,\tilde{\mathbf{x}}$, and hence it takes the value $\mathbf{0}$ at time $a+t_0$.

Remark 3.46 By extending the definitions in a suitable way, it is possible to prove results analogous to those of Theorem 3.43 and Lemma 3.45 in "negative time". That is, for all $\lambda_0 \in \mathbb{R}$, all the systems (3.61) corresponding to λ_0 and $\omega \in \Omega$ are $\mathcal{B}_r$-null controllable *backwards* in time $-t_0$ at every $(a, \mathbf{z}_1) \in \mathbb{R} \times \{\mathbf{z} \in \mathbb{R}^{2n} \mid \|\mathbf{z}\| < c\}$, where t_0 only depends on λ_0, and r only depends on λ_0 and c.

3.3.4 Exponential Dichotomy and the Rotation Number

As in the preceding sections, Γ is assumed to be an Atkinson perturbation; i.e. to satisfy Hypotheses 3.3. The main results indicating the connections between the properties of the rotation number and the existence of exponential dichotomy for (3.2) are now stated and proved. All the results presented in the previous sections, as well as the following lemma, are used in the proof of Theorem 3.48. The definition and main properties of the topological support $\operatorname{Supp} m_0$ of the fixed ergodic measure are given in Sect. 1.1.2.

Lemma 3.47 *Suppose that* $\Omega = \operatorname{Supp} m_0$. *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be an open interval such that the rotation number* α_Γ *is constant on* $\mathcal{I}$. *Then*

$$\operatorname{tr}\left(\Gamma(\omega) \int_{\mathcal{I}_1} dP_{\Gamma,\omega}(t)\right) = 0$$

for any open subinterval $\mathcal{I}_1 \subseteq \mathcal{I}$ *and every* $\omega \in \Omega$.

Proof Fix the subinterval $\mathcal{I}_1$. The assertion of the lemma follows from (3.48) for all ω in a subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$. Take any $\omega \in \Omega = \operatorname{Supp} m_0$ and write it as $\omega = \lim_{k\to\infty} \omega_k$ for $\omega_k \in \Omega_0$ (see Proposition 1.11(i)). Let f be a scalar continuous function with $0 \le f \le \chi_{\mathcal{I}_1}$. Then,

$$0 \le \operatorname{tr}\left(\Gamma(\omega_k) \int_{\mathbb{R}} f(t)\, dP_{\Gamma,\omega_k}(t)\right) \le \operatorname{tr}\left(\Gamma(\omega_k) \int_{\mathcal{I}_1} dP_{\Gamma,\omega_k}(t)\right) = 0\,.$$

It is not difficult to deduce from the continuity of Γ, from the uniform bound (3.47), and from the weak* continuity established by Theorem 3.40(ii), that

$$\operatorname{tr}\left(\Gamma(\omega) \int_{\mathbb{R}} f(t)\, dP_{\Gamma,\omega}(t)\right) = \lim_{k\to\infty} \operatorname{tr}\left(\Gamma(\omega_k) \int_{\mathbb{R}} f(t)\, dP_{\Gamma,\omega_k}(t)\right) = 0\,.$$

Therefore, writing $\chi_{\mathcal{I}_1}(t) = \lim_{m\to\infty} f_m(t)$ for an increasing sequence (f_m) of non-negative continuous functions, and applying the Lebesgue dominated convergence theorem, one proves the assertion for the point ω.

Theorem 3.48 *Suppose that* $\Omega = \operatorname{Supp} m_0$. *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be an open interval such that the rotation number* α_Γ *is constant on* $\mathcal{I}$, *and fix* $\lambda_0 \in \mathcal{I}$. *Fix also* $\tilde{\omega} \in \Omega$. *Then* *the system*

$$\mathbf{z}' = \big(H(\tilde{\omega}{\cdot}t) + \lambda_0 J^{-1}\Gamma(\tilde{\omega}{\cdot}t)\big)\,\mathbf{z} \tag{3.62}$$

does not admit any nonzero bounded solution.

Proof Assume for contradiction the existence of a nonzero solution $\tilde{\mathbf{z}}(t)$ of the system (3.62) and of a constant c such that $\|\tilde{\mathbf{z}}(t)\| < c$ for all $t \in \mathbb{R}$. The contradiction is reached in two steps. In the first one, the possibility

$$\int_{-\infty}^{\infty} \tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt = \infty \tag{3.63}$$

is excluded, while in the second it is proved that

$$0 < \int_{-\infty}^{\infty} \tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt < \infty \tag{3.64}$$

is also impossible. This means that $\int_{-\infty}^{\infty} \tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt = 0$, which, according to Lemma 3.6, is inconsistent with the hypothesis that $\tilde{\mathbf{z}}(0) \neq \mathbf{0}$. This is the desired contradiction.

Let $t_0 = t_0(\lambda_0)$ be a positive constant satisfying Lemma 3.6(iv) and Remark 3.7, and consequently Lemma 3.45 and Remark 3.46. Let $[a_0, b_0]$ be a real interval containing 0: it may be any such interval for the time being, although later it will be more precisely chosen.

Lemma 3.45 applied to $(b_0, \tilde{\mathbf{z}}(b_0))$ and Remark 3.46 applied to $(a_0, \tilde{\mathbf{z}}(a_0))$ guarantee the existence of a constant r independent of a_0 and b_0 and of a Borel measurable control function $\tilde{\mathbf{u}}\colon \mathbb{R} \to \mathbb{R}^{2n}$, with $\|\tilde{\mathbf{u}}(t)\| \le r$ and with $\tilde{\mathbf{u}}(t) = \mathbf{0}$ for $t \notin [a_0 - t_0, a_0] \cup [b_0, b_0 + t_0]$, such that the solution $\mathbf{z}_0(t)$ of the system

$$\mathbf{z}' = \big(H(\tilde{\omega}{\cdot}t) + \lambda_0 J^{-1}\Gamma(\tilde{\omega}{\cdot}t)\big)\,\mathbf{z} + J^{-1}\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{u}}$$

with $\mathbf{z}_0(0) = \tilde{\mathbf{z}}(0)$ agrees with $\tilde{\mathbf{z}}$ on $[a_0, b_0]$ and vanishes outside $[a_0 - t_0, b_0 + t_0]$.

Take now sequences $(a_m) \downarrow -\infty$ and $(b_m) \uparrow \infty$ with $[a_0 - t_0, b_0 + t_0] \subseteq [a_1, b_1]$ and choose matrices A and B satisfying (3.49) such that λ_0 is not an eigenvalue for any of the (countable) family of boundary value problems

$$\begin{cases} J\mathbf{z}' = (JH(\tilde{\omega}{\cdot}t) + \lambda\,\Gamma(\tilde{\omega}{\cdot}t))\,\mathbf{z}\,, \\ \exists\; \mathbf{v} \neq \mathbf{0} \quad \text{such that} \quad \mathbf{z}(a_m) = A\,\mathbf{v} \quad \text{and} \quad \mathbf{z}(b_m) = B\,\mathbf{v}\,. \end{cases} \tag{3.65}$$

The fact that this choice is possible is proved in Lemma 3.49 below. As in Sect. 3.3.1, let $\{\lambda_k^m \mid k \ge 1\}$ be the ordered set of eigenvalues of (3.65), and let

$\{\boldsymbol{\theta}_k^m(t) \mid k \geq 1\}$ be a set of normalized eigenfunctions, so that

$$\int_{a_m}^{b_m} (\boldsymbol{\theta}_k^m)^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \boldsymbol{\theta}_l^m(t)\, dt = \delta_{kl}\,. \tag{3.66}$$

The associated spectral matrix $P_m(t)$ is defined by the corresponding expression (3.53), which implies that, for any interval $\mathcal{J} \subseteq \mathbb{R}$,

$$\int_{\mathcal{J}} \mathbf{f}^T(t)\, dP_m(t)\, \mathbf{g}(t) = \sum_{\lambda_k^m \in \mathcal{J}} \mathbf{f}^T(\lambda_k^m)\, \boldsymbol{\theta}_k^m(0)\, (\boldsymbol{\theta}_k^m)^T(0)\, \mathbf{g}(\lambda_k^m) \tag{3.67}$$

if the right-hand term is finite. Define also

$$\mathbf{w}_0(t) = \int_{a_1}^{b_1} U_{\Gamma,t}^T(l,\tilde{\omega})\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl$$

and note that, since $\mathbf{z}_0$ vanishes outside $[a_1, b_1]$,

$$\mathbf{w}_0(t) = \int_{a_m}^{b_m} U_{\Gamma,t}^T(l,\tilde{\omega})\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl$$

for all $m \geq 1$. It follows that, for a fixed value of m the following properties (a)–(d) hold.

(a) The eigenfunction expansion of $\mathbf{z}_0$ corresponding to (3.65), given by

$$s \mapsto \sum_{k=1}^{\infty} c_k^m\, \boldsymbol{\theta}_k^m(s) \quad \text{with} \quad c_k^m = \int_{a_m}^{b_m} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl\,,$$

defines a continuous function $\tilde{\boldsymbol{\theta}}_m(s)$, since the series converges absolutely and uniformly on $\mathbb{R}$. This last assertion is proved in Theorem 9.7.4 of [5].

(b) Statement (a), Theorem 9.6.3 of [5], and the orthonormality condition (3.66), ensure that

$$\begin{aligned}
0 &= \int_{a_m}^{b_m} \Big(\mathbf{z}_0^T(t) - \tilde{\boldsymbol{\theta}}_m^T(t)\Big)\, \Gamma(\tilde{\omega}{\cdot}t)\, \Big(\mathbf{z}_0(t) - \tilde{\boldsymbol{\theta}}_m(t)\Big)\, dt \\
&= \int_{a_m}^{b_m} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt \\
&\quad - 2\sum_{k=1}^{\infty} c_k^m \int_{a_m}^{b_m} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \boldsymbol{\theta}_k^m(t)\, dt + \sum_{k=1}^{\infty} (c_k^m)^2 \\
&= \int_{a_m}^{b_m} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt - \sum_{k=1}^{\infty} (c_k^m)^2\,;
\end{aligned}$$

that is,

$$\int_{a_m}^{b_m} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt = \sum_{k=1}^{\infty} \left| \int_{a_m}^{b_m} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl \right|^2 .$$

In other words, since $\mathbf{z}_0$ vanishes outside $[a_1, b_1]$,

$$\int_{-\infty}^{\infty} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt = \sum_{k=1}^{\infty} \left| \int_{-\infty}^{\infty} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl \right|^2 .$$

(c) The definitions of $\mathbf{w}_0(t)$ and of the spectral matrix $P_m(t)$ together with the last equality in (b) imply that

$$\begin{aligned}\int_{\mathbb{R}} &\mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) \\ &= \sum_{k=1}^{\infty} \int_{a_1}^{b_1} \mathbf{z}_0^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \boldsymbol{\theta}_k^m(l)\, dl \int_{a_1}^{b_1} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl \\ &= \int_{a_1}^{b_1} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt = \int_{-\infty}^{\infty} \mathbf{z}_0^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t)\, dt .\end{aligned}$$

(d) It follows from the differential equations satisfied by $\boldsymbol{\theta}_k^m$ and $\mathbf{z}_0$ that

$$\begin{aligned}\big((\boldsymbol{\theta}_k^m)^T&(t)\, J\, \mathbf{z}_0(t)\big)' \\ &= (\lambda_0 - \lambda_k^m)\, (\boldsymbol{\theta}_k^m)^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \mathbf{z}_0(t) + (\boldsymbol{\theta}_k^m)^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \tilde{\mathbf{u}}(t) .\end{aligned}$$

Therefore, the coefficients of the eigenfunction expansion of $\tilde{\mathbf{u}}$ corresponding to (3.65) are $(\lambda_k^m - \lambda_0) \int_{a_m}^{b_m} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl$. Consequently, (3.67), the definition of $\mathbf{w}_0$, and the corresponding Bessel inequality, yield

$$\begin{aligned}\int_{\mathbb{R}} &(t-\lambda_0)^2\, \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) \\ &= \sum_{k=1}^{\infty} (\lambda_k^m - \lambda_0)^2 \mathbf{w}_0^T(\lambda_k^m)\, \boldsymbol{\theta}_k^m(0)\, (\boldsymbol{\theta}_k^m)^T(0)\, \mathbf{w}_0(\lambda_k^m) \\ &= \sum_{k=1}^{\infty} (\lambda_k^m - \lambda_0)^2 \left| \int_{a_1}^{b_1} (\boldsymbol{\theta}_k^m)^T(l)\, \Gamma(\tilde{\omega}{\cdot}l)\, \mathbf{z}_0(l)\, dl \right|^2 \\ &\le \int_{a_1}^{b_1} \tilde{\mathbf{u}}^T(t)\, \Gamma(\tilde{\omega}{\cdot}t)\, \tilde{\mathbf{u}}(t)\, dt .\end{aligned}$$

Lemma 3.47 will play a fundamental role in proving the following assertion: there exists a constant $\varepsilon > 0$ such that, for large enough m,

$$\int_{\mathbb{R}} (t-\lambda_0)^2\,\mathbf{w}_0^T(t)\,dP_m(t)\,\mathbf{w}_0(t) \ge \varepsilon^2 \int_{\mathbb{R}} \mathbf{w}_0^T(t)\,dP_m(t)\,\mathbf{w}_0(t)\,. \tag{3.68}$$

Before checking (3.68), note that it allows one to exclude possibility (3.63): (3.68), (c), and (d) above, and the characteristics of the control function $\tilde{\mathbf{u}}$, ensure that

$$\begin{aligned}\varepsilon^2 \int_{a_0}^{b_0} \tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt &\le \varepsilon^2 \int_{-\infty}^{\infty} \mathbf{z}_0^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\mathbf{z}_0(t)\,dt\\ &\le \int_{a_1}^{b_1} \tilde{\mathbf{u}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{u}}(t)\,dt \le 2\,r^2\,t_0\,\|\Gamma\|_{\Omega}\,;\end{aligned}$$

but if (3.63) held it would be possible to choose the initial interval $[a_0, b_0]$ with $\int_{a_0}^{b_0} \tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt$ as large as desired, making this last inequality impossible.

In order to prove (3.68), note that, for all $l \in \mathbb{R}$, the spectral problem

$$\begin{cases} J\mathbf{z}' = (JH((\tilde{\omega}{\cdot}l){\cdot}t) + \lambda\,\Gamma((\tilde{\omega}{\cdot}l){\cdot}t))\,\mathbf{z}\,,\\ \exists\,\mathbf{v} \neq \mathbf{0} \quad \text{such that} \quad \mathbf{z}(a_m - l) = A\,\mathbf{v} \quad \text{and} \quad \mathbf{z}(b_m - l) = B\,\mathbf{v}\end{cases}$$

has the same eigenvalues as (3.65) and that $\boldsymbol{\theta}_k^m(t+l)$ is a normalized eigenfunction for λ_k^m. This means that the jump at the eigenvalue λ_k^m of the spectral matrix for this problem, denoted by $P_m^l(t)$, is $\sum_{\lambda_j^m=\lambda_k^m} \boldsymbol{\theta}_j^m(l)\,(\boldsymbol{\theta}_j^m)^T(l)$. The definition of $\mathbf{w}_0$ and the Schwarz inequality ensure that

$$\begin{aligned}\int_{\mathcal{I}} \mathbf{w}_0^T(t)\,dP_m(t)\,\mathbf{w}_0(t) &= \sum_{\lambda_j^m \in \mathcal{I}} \left| \int_{a_1}^{b_1} (\boldsymbol{\theta}_j^m)^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\mathbf{z}_0(l)\,dl \right|^2\\ &\le \int_{a_1}^{b_1} \mathbf{z}_0^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\mathbf{z}_0(l)\,dl \sum_{\lambda_j^m \in \mathcal{I}} \int_{a_1}^{b_1} (\boldsymbol{\theta}_j^m)^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\boldsymbol{\theta}_j^m(l)\,dl\\ &= \int_{a_1}^{b_1} \mathbf{z}_0^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\mathbf{z}_0(l)\,dl \int_{a_1}^{b_1} \operatorname{tr}\left(\Gamma(\tilde{\omega}{\cdot}l)\int_{\mathcal{I}} dP_m^l(t)\,dt\right)dl\,.\end{aligned}$$

The last equality follows from (3.67) and from

$$(\boldsymbol{\theta}_j^m)^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\boldsymbol{\theta}_j^m(l) = \operatorname{tr}(\Gamma(\tilde{\omega}{\cdot}l)\,\boldsymbol{\theta}_j^m(l)\,(\boldsymbol{\theta}_j^m)^T(l))\,.$$

Theorem 3.40 and Lemma 3.47 ensure that

$$\lim_{m\to\infty} \operatorname{tr}\left(\Gamma(\tilde{\omega}{\cdot}l)\int_{\mathcal{I}} dP_m^l(t)\,dt\right) = \operatorname{tr}\left(\Gamma(\tilde{\omega}{\cdot}l)\int_{\mathcal{I}} dP_{\Gamma,\tilde{\omega}{\cdot}l}(t)\,dt\right) = 0\,.$$

Therefore, using Lemma 3.47 and the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \int_{a_1}^{b_1} \operatorname{tr}\left(\Gamma(\tilde{\omega}{\cdot}l) \int_{\mathcal{I}} dP^l_m(t)\, dt \right) dl = 0\,,$$

from which

$$\lim_{m\to\infty} \int_{\mathcal{I}} \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) = 0\,. \tag{3.69}$$

On the other hand, taking ε equal to one-half of the distance of λ_0 to $\mathbb{R}-\mathcal{I}$,

$$\begin{aligned}
&\int_{\mathbb{R}} (t-\lambda_0)^2\, \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) \\
&\quad \geq \int_{\mathbb{R}-\mathcal{I}} (t-\lambda_0)^2\, \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) \geq 4\,\varepsilon^2 \int_{\mathbb{R}-\mathcal{I}} \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) \\
&\quad = 4\,\varepsilon^2 \int_{\mathbb{R}} \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t) - 4\,\varepsilon^2 \int_{\mathcal{I}} \mathbf{w}_0^T(t)\, dP_m(t)\, \mathbf{w}_0(t)\,,
\end{aligned}$$

and it follows from (3.69) and from property (c) that (3.68) holds for m large enough (since $\tilde{\mathbf{z}}$ and hence $\mathbf{z}_0$ do not vanish identically). This completes the first step of the proof.

As stated before, the second and last step of the proof gives rise to a contradiction under assumption (3.64). Consider a sequence of boundary value problems

$$\begin{cases} J\mathbf{z}' = (JH(\tilde{\omega}{\cdot}t) + \lambda\, \Gamma(\tilde{\omega}{\cdot}t))\,\mathbf{z}\,, \\ \exists\ \mathbf{v} \neq \mathbf{0} \quad \text{such that} \quad \mathbf{z}(-m) = A_m\mathbf{v} \quad \text{and} \quad \mathbf{z}(m) = B_m\mathbf{v}\,, \end{cases} \tag{3.70}$$

where now the matrices A_m and B_m are norm-bounded by 1, the pairs (A_m, B_m) satisfy (3.49), and the solution $\tilde{\mathbf{z}}$ is an eigenfunction (associated to the eigenvalue λ_0) for each $m \in \mathbb{N}$. A possible choice for these matrices is the following one: for a fixed A, define

$$\begin{aligned}
k_m &= \max(\|A\|, \max_{\omega\in\Omega} \|U_{\Gamma,\lambda_0}(m,\omega)\, U^{-1}_{\Gamma,\lambda_0}(-m,\omega)A\|)\,, \\
B_m &= (1/k_m)\, U_{\Gamma,\lambda_0}(m,\omega)\, U^{-1}_{\Gamma,\lambda_0}(-m,\omega)A\,, \\
A_m &= (1/k_m)A\,.
\end{aligned}$$

Let $\widetilde{P}^l_m(t)$ denote the *translated* spectral matrix (corresponding to the problem for $\tilde{\omega}{\cdot}l$ on the interval $[-m-l, m-l]$. Then its jump at the eigenvalue λ_0 is greater or equal than $\tilde{\mathbf{z}}(l)\,\tilde{\mathbf{z}}^T(l)/\int_{-m}^{m}\tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt$. Consequently,

$$\frac{\tilde{\mathbf{z}}^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\tilde{\mathbf{z}}(l)}{\int_{-m}^{m}\tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt}\le \operatorname{tr}\left(\Gamma(\tilde{\omega}{\cdot}l)\int_{\mathcal{I}_1} d\widetilde{P}^l_m(t)\right)$$

for any open subinterval $\mathcal{I}_1\subseteq\mathcal{I}$ containing λ_0. Fix l with $\tilde{\mathbf{z}}^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\tilde{\mathbf{z}}(l)>0$. Theorem 3.40 and Remark 3.41 ensure the existence of the limit as $m\to\infty$ of $(\widetilde{P}^l_m)$; call it $P_{\Gamma,\tilde{\omega}\cdot l}$. Therefore, Lemma 3.47 for $\tilde{\omega}{\cdot}l$ implies that

$$\lim_{m\to\infty}\frac{\tilde{\mathbf{z}}^T(l)\,\Gamma(\tilde{\omega}{\cdot}l)\,\tilde{\mathbf{z}}(l)}{\int_{-m}^{m}\tilde{\mathbf{z}}^T(t)\,\Gamma(\tilde{\omega}{\cdot}t)\,\tilde{\mathbf{z}}(t)\,dt}=0\,,$$

which contradicts (3.64). The proof is complete.

Lemma 3.49 *Given $(a_m)\downarrow-\infty$ and $(b_m)\uparrow\infty$ with $a_1<b_1$ and $A\in\mathrm{Sp}(n,\mathbb{R})$, there exists $B\in\mathrm{Sp}(n,\mathbb{R})$ satisfying (3.49) such that λ_0 is not an eigenvalue for any of boundary value problems (3.65).*

Proof Note that the second condition in (3.49) is automatically satisfied when A and B are symplectic. Define the real symplectic matrices $U_0^\lambda=I_{2n}$ and $U_m^\lambda=U_{\Gamma,\lambda}(a_m,\omega)\,U_{\Gamma,\lambda}^{-1}(b_m,\omega)$ if $m\ge 1$, for all $\lambda\in\mathbb{R}$. Then the remaining required conditions hold if and only if $\det(A-U_m^{\lambda_0}B)\ne 0$ for $m\ge 0$ (see Sect. 3.3.1). Fix $m\ge 0$ and define

$$\mathcal{O}_m=\{B\in\mathrm{Sp}(n,\mathbb{R})\mid\ \det(A-U_m^{\lambda_0}\,B)\ne 0\}\,.$$

It is clear that $\mathcal{O}_m$ is an open set in $\mathrm{Sp}(n,\mathbb{R})$. The main point of this proof is to check that it is dense. Fix $B\in\mathrm{Sp}(n,\mathbb{R})-\mathcal{O}_m$, so that λ_0 is an eigenvalue of (3.65), and approximate λ_0 by a sequence (λ_k) of real non-eigenvalues of (3.65). That is, $\det(A-U_m^{\lambda_k}B)\ne 0$. Write $A-U_m^{\lambda_k}B=A-U_m^{\lambda_0}B_k$ for $B_k=(U_m^{\lambda_0})^{-1}U_m^{\lambda_k}B$. Then $B_k\in\mathcal{O}_m$ for all k, and $\lim_{k\to\infty}B_k=B$. The asserted density is proved.

Since $\mathrm{Sp}(n,R)$ is a complete metric space, the Baire theorem ensures that the countable intersection of open dense sets is dense. Therefore there exists $B\in\bigcap_{m\ge 0}\mathcal{O}_m$, which proves the lemma.

The main result of this section follows easily from the previous considerations:

Theorem 3.50 *Suppose that $\Omega=\operatorname{Supp}m_0$. Let $\mathcal{I}\subseteq\mathbb{R}$ be an open interval. Then the families (3.2) have exponential dichotomy over Ω for all $\lambda\in\mathcal{I}$ if and only if the rotation number $\alpha_\Gamma(\lambda)$ (with respect to m_0) is constant on $\mathcal{I}$.*

Proof According to Theorem 2.28, in the presence of exponential dichotomy on an open interval $\mathcal{I}$, the quantity $2\alpha_\Gamma(\lambda)$ takes values in the discrete group determined by the image of the Schwarzmann homomorphism on the group of real Čech one-cocycles over Ω with integer values, for all $\lambda\in\mathcal{I}$. The continuity of α_Γ on the real axis, which is ensured by Theorem 2.25, implies then that α_Γ is constant on the interval. This proves the "only if" part of the theorem.

The "if" part follows from Theorems 3.48 and 1.78.

Remarks 3.51

1. Note that the proof of the "only if" assertion of the previous theorem does not require either the Atkinson condition described by Hypotheses 3.3 or the fact that $\operatorname{Supp} m_0 = \Omega$.
2. To complete the information provided by Theorem 3.50, note that the rotation number $\alpha_\Gamma\colon\mathbb{R}\to\mathbb{R}$ is a nondecreasing function, as Proposition 2.33 proves, and that it takes just nonnegative values in the case that $H_3 \ge 0$, as proved by Theorem 2.31. Note also that $H_3 = I_n$ in the Schrödinger case.

3.3.5 Exponential Dichotomy and Gap-Labeling

The chapter is completed with a brief analysis of the linear selfadjoint operators defined for $\omega \in \Omega$ by

$$\mathcal{L}_\omega = J\left(\frac{d}{dt} - H_\omega\right) \qquad \text{and} \qquad \mathcal{S}_\omega = -\frac{d^2}{dt^2} + G_\omega \tag{3.71}$$

on $L^2(\mathbb{R},\mathbb{C}^{2n})$ and $L^2(\mathbb{R},\mathbb{C}^n)$ respectively. As usual, $(H_\omega\mathbf{f})(t) = H(\omega{\cdot}t)\,\mathbf{f}(t)$ and $(G_\omega\mathbf{f})(t) = G(\omega{\cdot}t)\,\mathbf{f}(t)$. The domains of these operators are the sets of square integrable absolutely continuous functions with square integrable derivative. Note that the families of spectral problems $\mathcal{L}_\omega\mathbf{z} = \lambda\,\mathbf{z}$ and $\mathcal{S}_\omega\mathbf{x} = \lambda\,\mathbf{x}$ are given by the families of equations

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda\,J^{-1}\big)\,\mathbf{z}\,, \quad \omega \in \Omega \tag{3.72}$$

and

$$-\,\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \lambda\;\mathbf{x}\,, \quad \omega \in \Omega\,, \tag{3.73}$$

which agree with (3.2) and (3.3) with $\Gamma = I_{2n}$ and $\Delta = I_n$ respectively. These values of Γ and Δ are fixed for the rest of the section.

The case of the general linear Hamiltonian systems (3.72) will be considered first. The following result plays a fundamental role in the proof of the main theorem. Recall that $P_{I_{2n},\omega}(t)$ is the spectral matrix-valued function associated to the Herglotz matrix-valued function $G_{I_{2n}}(\omega,\lambda)$ by Theorem 3.13.

Lemma 3.52 *The spectrum of the operator $\mathcal{L}_\omega$ agrees with the set of points of non-constancy of the nondecreasing matrix-valued function $P_{I_{2n},\omega}$. In particular, it is closed.*

Proof Although the proof is standard in spectral theory, a brief sketch is included. The statement of Theorem 3.40 and the ideas used in its proof allow to repeat the arguments of Coddington and Levinson [28], Chapter 9, Sections 3 and 5, in order to check that the mapping

$$\mathbf{f}(t) \mapsto \mathbf{g}(t) = \int_{\mathbb{R}} U^T_{I_{2n},t}(s,\omega)\,\mathbf{f}(s)\,ds$$

defines a unitary isomorphism Φ between the Lebesgue space $L^2(\mathbb{R},\mathbb{C}^{2n})$ and the space $L^2(\mathbb{R},\mathbb{C}^{2n},dP_{I_{2n},\omega})$ consisting of vector functions whose norm is square integrable with respect to $dP_{I_{2n},\omega}$, with inverse given by

$$\mathbf{g}(t) \mapsto \mathbf{f}(t) = \int_{\mathbb{R}} U_{I_{2n},s}(t,\omega)\,dP_{I_{2n},\omega}(s)\,\mathbf{g}(s).$$

In addition, the operator $\Phi \circ \mathcal{L}_\omega \circ \Phi^{-1}$ agrees with the usual multiplication operator densely defined on $L^2(\mathbb{R},\mathbb{C}^{2n},dP_{I_{2n},\omega})$, which maps any $\mathbf{g}(t)$ of its domain to $t\,\mathbf{g}(t)$. And it is well known that the spectrum of this operator (and hence that of $\mathcal{L}_\omega$) agrees with the set of non-constancy points of the matrix-valued function $P_{I_{2n},\omega}(t)$, as asserted. Since this set is closed, so is the spectrum.

The strong connection between the occurrence of exponential dichotomy and the spectrum of $\mathcal{L}_\omega$ is illustrated in the following theorem, which appears in [72].

Theorem 3.53 *Let $\tilde\omega \in \Omega$ have dense orbit. Then the complex number $\tilde\lambda$ belongs to the resolvent set of $\mathcal{L}_{\tilde\omega}$, which is open, if and only if the corresponding family* (3.72) *has exponential dichotomy over Ω.*

Proof The existence of exponential dichotomy for the equations corresponding to $\tilde\lambda$ provides integral kernels for the operators considered here. This kernel is given by the function $\widetilde{G}_{I_{2n},\tilde\lambda}(\omega,t,s)$ defined by the corresponding expression (3.34): as in the proof of Theorem 3.27, one checks that the mapping sending any $\mathbf{g} \in L^2(\mathbb{R},\mathbb{C}^{2n})$ to

$$\mathbf{f}_\omega(t) = \int_{\mathbb{R}} \widetilde{G}_{I_{2n},\tilde\lambda}(\omega,t,s)\,\mathbf{g}(s)\,ds$$

defines a bounded linear operator from $L^2(\mathbb{R},\mathbb{C}^{2n})$ to the domain of $\mathcal{L}_\omega$; and, in addition, $(\mathcal{L}_\omega - \tilde\lambda)\,\mathbf{f}_\omega = \mathbf{g}$. This means that $\tilde\lambda$ belongs to the resolvent of $\mathcal{L}_\omega$. Consequently, the “if” part of Theorem 3.53 is proved. Note that the density of the orbit of the element $\tilde\omega$ has not been required.

The proof of the converse implication is based on the fact that, in the case that the orbit of $\tilde\omega$ is dense and $\tilde\lambda$ belongs to the resolvent of the operator $\mathcal{L}_{\tilde\omega}$, none of the systems (3.72) corresponding to $\tilde\lambda$ admits a nonzero bounded solution. Once this assertion is proved, Theorem 1.78 ensures the exponential dichotomy over the whole of Ω.

Assume hence for contradiction the existence of such a bounded solution $\mathbf{z}_0$ for the equation corresponding to a point ω_0. The first goal is to prove that $\tilde\lambda$ belongs to the spectrum of $\mathcal{L}_{\omega_0}$. In the case that $\mathbf{z}_0$ is square integrable, $\tilde\lambda$ is an eigenvalue, and the assertion is true. Suppose therefore that $\lim_{k\to\infty} c_k = \infty$, where $c_k = (\int_{-k}^{k} \|\mathbf{z}_0(t)\|^2\, dt)^{1/2}$. Let $r_k\colon \mathbb{R} \to [0,1]$ be a C^2 real function equal to 1 on $[-k,k]$ and vanishing outside $[-k-1,k+1]$, with $|r_k'(t)| \le 2$. Define also $\mathbf{f}_k(t) = (r_k(t)/d_k)\,\mathbf{z}_0(t)$, with $d_k = (\int_{\mathbb{R}} \|r_k(t)\,\mathbf{z}_0(t)\|^2\, dt)^{1/2}$. It is easy to check that $d_k \ge c_k$ and to deduce two facts: first, that the sequence $((\mathcal{L}_{\omega_0} - \tilde\lambda)\,\mathbf{f}_k)$ tends to $\mathbf{0}$ in the $L^2(\mathbb{R},\mathbb{C}^{2n})$-topology; and second, that no subsequence of $(\mathbf{f}_k)$ converges in that topology, since the pointwise limit is $\mathbf{0}$ but the L^2-norm of all the functions is 1. According to the Weyl criterion (see e.g. Corollary 2 of Section XIII.7 of [39]), $\tilde\lambda$ belongs to the essential spectrum of the operator $\mathcal{L}_{\omega_0}$.

On the other hand, for all $s \in \mathbb{R}$, the operator $\mathcal{L}_{\tilde\omega\cdot s}$ is conjugate to $\mathcal{L}_{\tilde\omega}$ under translation by s: $\Phi_{-s} \circ \mathcal{L}_{\tilde\omega\cdot s} \circ \Phi_s = \mathcal{L}_{\tilde\omega}$ for $(\Phi_s\,\mathbf{f})(t) = \mathbf{f}(t+s)$; in particular, their spectra agree. Therefore, there exists an interval $(\tilde\lambda - \varepsilon, \tilde\lambda + \varepsilon)$ contained in the (open) resolvent of the operator $\mathcal{L}_{\tilde\omega\cdot s}$ for all $s \in \mathbb{R}$. According to Lemma 3.52, the matrix-valued functions $P_{I_{2n},\tilde\omega\cdot s}$ are constant on $(\tilde\lambda-\varepsilon, \tilde\lambda+\varepsilon)$. The density hypothesis provides a sequence (s_k) with $\omega_0 = \lim_{k\to\infty} \tilde\omega{\cdot}s_k$, and it is easy to deduce from the weak* convergence of the sequence of measures associated to $(P_{I_{2n},\tilde\omega\cdot s_k})$ guaranteed by Theorem 3.40(ii) that P_{I_{2n},ω_0} is constant on, for example, the subinterval $(\tilde\lambda - \varepsilon/2, \tilde\lambda + \varepsilon/2)$. Therefore, Lemma 3.52 implies that $\tilde\lambda$ cannot belong to the spectrum of $\mathcal{L}_{\omega_0}$, and this completes the proof of the equivalence.

The Schrödinger case can be analyzed in an analogous way. Set $K = \left[\begin{smallmatrix} I_n & 0_n \\ 0_n & 0_n \end{smallmatrix}\right]$, and let $G_K^1(\omega,\lambda)$ and $P_{K,\omega}^1(t)$ be the $n \times n$ submatrices of $G_K(\omega,\lambda)$ and $P_{K,\omega}(t)$ formed by the first n rows and n columns. Then, as in Lemma 3.52, it can be proved that $P_{K,\omega}^1$ defines a spectral measure for the operator $\mathcal{S}_\omega$, whose spectrum is closed. Repeating the proof of Theorem 3.53 one concludes:

Theorem 3.54 *Let $\tilde\omega \in \Omega$ have dense orbit. Then the complex number $\tilde\lambda$ belongs to the resolvent set of $\mathcal{S}_{\tilde\omega}$, which is open, if and only if the corresponding Schrödinger family* (3.73) *has exponential dichotomy over Ω.*

Corollary 3.55 *Suppose that there exists a σ-ergodic measure m such that $\Omega =$* Supp *m. Then the spectrum $\mathcal{S}$ of the operators $\mathcal{L}_\omega$ (or $\mathcal{S}_\omega$) is the same for m-a.e. $\omega \in \Omega$, and it agrees with the set of values of λ for which the corresponding family of linear Hamiltonian systems does not have exponential dichotomy over Ω. In addition, if $\alpha\colon \mathbb{R} \to \mathbb{R}$ is the rotation number of* (3.72) *(or* (3.73)*), then α is a continuous function which strictly increases on the set $\mathcal{S}$ and is constant on each (open) interval of $\mathbb{R} - \mathcal{S}$, where 2α takes values in the image of the Schwarzmann homomorphism.*

Proof Proposition 1.12 ensures that the orbit of m-a.e. $\omega \in \Omega$ is dense, so that Theorem 3.53 (or 3.54) proves that $\mathcal{S}$ is common for m-a.e. $\omega \in \Omega$, as well as the assertion concerning the exponential dichotomy. The remaining assertions follow from Theorems 2.25, 3.50 and 2.28, and Proposition 2.33.

Remark 3.56 The preceding result shows that, if Ω is the topological support of a σ-measure m, the corresponding rotation number can be used to label the different gaps of the spectrum of the operators $\mathcal{L}_{\tilde{\omega}}$ and $\mathcal{S}_{\tilde{\omega}}$, common for m-a.e. $\omega \in \Omega$. This *gap-labeling* was first obtained for the one-dimensional almost periodic Schrödinger operator in [73] and for the general two-dimensional case in [71].

Chapter 4
The Weyl Functions

Let (Ω, σ) be a real continuous flow on a compact metric space. In this chapter, the object of study is the family of linear Hamiltonian systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda\, J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{4.1}$$

where $\lambda \in \mathbb{C}$ is a complex parameter, $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right] : \Omega \to \mathfrak{sp}(n, \mathbb{R})$ and $\Gamma = \left[\begin{smallmatrix} -\Gamma_2 & \Gamma_1^T \\ \Gamma_1 & \Gamma_3 \end{smallmatrix}\right] : \Omega \to \mathbb{S}_{2n}(\mathbb{R})$ are continuous functions, and $\Gamma(\omega) \geq 0$ for all $\omega \in \Omega$. This one-parameter family will often be understood as a perturbation of the family of systems corresponding to $\lambda = 0$, namely

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,. \tag{4.2}$$

As was stated in Chap. 3, the family (4.1) defines an Atkinson spectral problem. Recall that the perturbed family of n-dimensional Schrödinger equations

$$-\,\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \lambda\,\Delta(\omega{\cdot}t)\,\mathbf{x}\,, \quad \omega \in \Omega\,, \tag{4.3}$$

where G and Δ are continuous symmetric $n \times n$ matrix-valued functions on Ω and $\Delta \geq 0$, is included in the general formulation (4.1) by taking $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix}\right]$, $H = \left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n \end{smallmatrix}\right]$ and $\Gamma = \left[\begin{smallmatrix} \Delta & 0_n \\ 0_n & 0_n \end{smallmatrix}\right]$.

The present chapter contains three different but closely related sets of results, which concern the limiting qualitative behavior as $\lambda \to 0$ of the flows determined by (4.1). Let m_0 be a fixed σ-ergodic measure on the base. In Sects. 4.1, 4.2, 4.3, and 4.5, the unperturbed family of linear Hamiltonian systems (4.2) is assumed to satisfy the Hypothesis 4.1, to be described shortly, and whose significance will be clarified in Sect. 4.4, once Kotani's theory has been summarized.

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Recall that $\mathbb{S}^+_{2n}(\mathbb{R})$ represents the set of real $2n\times 2n$ matrices which are symmetric and positive definite. Recall also that given any measurable function Z on Ω, the function Z' is defined by $Z'(\omega) = (d/dt)Z'(\omega{\cdot}t)|_{t=0}$ when this derivative exists. The notions of differentiability along the flow, and of solution along the flow of a differential equation, are set out in Definition 1.34. And the concept of integrable or square integrable matrix-valued function for the measure m_0 associated to a matrix operator norm is explained in Definition 1.32. In this chapter, unless otherwise indicated, the "measurable" sets and maps are always "Borel measurable", and the invariant sets and maps are (Borel) measurable.

Hypothesis 4.1 There exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable matrix-valued function $Z\colon \Omega \to \mathrm{Sp}(n,\mathbb{R}) \cap \mathbb{S}^+_{2n}(\mathbb{R})$ which belongs to $L^1(\Omega, m_0)$ and which is a solution along the flow on Ω_0 of the equation

$$Z' = -H^T(\omega{\cdot}t)Z - Z\,H(\omega{\cdot}t)\,. \tag{4.4}$$

In fact, this hypothesis is fulfilled in several interesting situations (see Theorem 4.15 and Sect. 4.4), and it leads to significant results, which are described in what follows.

It will be shown in Sect. 4.1.1 that the function Z of Hypothesis 4.1 provides a symplectic and square integrable change of variables taking the initial system (4.2) to skew-symmetric form, and which roughly speaking preserves the rotation number and the Lyapunov exponent of (4.1) with respect to m_0. But not any such a change of variables is useful for the main purposes of this chapter, in which the differentiability of the rotation number and the limit behavior of the Weyl functions are of interest. So, once certain basic results have been established, a fundamental consequence is derived: that a "suitable" change of variables can be associated to each suitable perturbation direction Γ; for instance to appropriate Atkinson perturbations, that is, those functions Γ satisfying Hypotheses 3.3. The construction of this suitable change of variables is the goal of Sect. 4.1.2, in which simultaneously two complex symmetric matrix-valued functions $N^{\pm}_{\Gamma}$ are described, which lie in $L^1(\Omega, m_0)$, satisfy $\pm\,\mathrm{Im}\,N^{\pm}_{\Gamma} > 0$, and solve along the flow the Riccati equation associated to (4.2). These functions will also play a fundamental role in the rest of the chapter. Theorem 4.15 completes the section and gives a characterization by means of several equivalent properties of those systems satisfying Hypothesis 4.1. In particular, they agree with the systems for which there exists the L^2-average of the solutions.

Let $\alpha_{\Gamma}(\lambda)$ be the rotation number of (4.1) with respect to m_0. The ergodic representation for the rotation number obtained in Theorem 2.4 applied to the systems obtained from (4.1) by means of the above-mentioned change of variables, together with the method of defining the change itself, is the main tool in proving the differentiability of $\alpha_{\Gamma}(\lambda)$ at $\lambda = 0$. This property is established in a set of directions Γ which includes those satisfying the Atkinson condition. The value of the derivative is explicitly obtained and can also be computed from Γ. These are the contents of Sect. 4.2.

Recall that, as seen in Sect. 3.1, the Atkinson Hypotheses 3.3 ensure the exponential dichotomy of the systems (4.1) for $\operatorname{Im}\lambda \neq 0$, together with the representation of the corresponding complex Lagrange planes $l^{\pm}_{\Gamma,\lambda}(\omega)$ by $\begin{bmatrix} I_n \\ M^{\pm}_{\Gamma}(\omega,\lambda) \end{bmatrix}$. The Weyl functions $M^{\pm}_{\Gamma}(\omega,\lambda)$ are jointly continuous in both variables and analytic outside the real axis for each fixed $\omega \in \Omega$: see Theorems 3.8 and 3.9. On the other hand, when $\lambda = 0$, the possibility of transforming (4.2) into skew-symmetric form preserving the Lyapunov index and the fact that all the solutions of this type of linear Hamiltonian system are bounded, ensure that the Lyapunov index of the non-perturbed system vanishes. Hence this system does not have exponential dichotomy: see Remark 2.42.1. The goal is to describe in detail the vertical limit of the Weyl functions from the upper and lower half-planes: it will be shown that $\lim_{\varepsilon\to\pm 0+} M^{\pm}_{\Gamma}(\omega, i\varepsilon) = \pm N^{\pm}(\omega)$ in measure. This result is proved in Sect. 4.3. In fact, the convergence occurs in the $L^1(\Omega, m_0)$-topology; this fact is established in the same section. The convergence is only proved for $\Gamma > 0$ in (4.1) and $\Delta > 0$ in (4.3); the question is still open in the general Atkinson case. In any case these results provide an extension of Kotani's theory, whose description is the goal of Sect. 4.4. This section also contains the generalization to the n-dimensional case of an inequality for the rotation number which is well-known in the scalar case, where it was obtained by Moser [108] and by Deift and Simon [37].

The last section of this chapter is devoted to establishing conditions both on the unperturbed family of systems and on the perturbation which ensure the existence of exponential dichotomy for small nonvanishing values of the parameter, as well as the uniform convergence of the Weyl functions. In fact, in the cases analyzed, the whole Sacker–Sell spectral decomposition varies uniformly as the parameter goes to 0. The limits of the spectral subbundles (and hence also of the Weyl functions) turn out again to be determined by Γ. A more detailed general description of the hypotheses and goals is given at the beginning of Sect. 4.5.

Most of the results contained in this chapter appear in Novo et al. [112], Johnson et al. [81] and Fabbri et al. [49]. These papers extend previous results for two-dimensional systems; see e.g. [114, 116, 117] and references therein.

The measure m_0 is fixed throughout the whole chapter except in Sect. 4.5, in the course of which the results are independent of any particular measure on the base. As in the previous chapters, $U(t,\omega)$ is the fundamental matrix solution satisfying $U(0,\omega) = I_{2n}$; τ represents the flow induced by the unperturbed systems (4.2) in $\mathcal{K}_{\mathbb{C}}$ and $\mathcal{K}_{\mathbb{R}}$, given by (1.14); and $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are defined by (1.13). The information provided by Remarks 1.24 and 1.33 will be used: the Euclidean norms $\|\mathbf{x}\| = (\mathbf{x}^T\mathbf{x})^{1/2} = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$ in any $\mathbb{R}^m$ and $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\,\mathbf{x}\|$ in any $\mathbb{M}_{d\times m}(\mathbb{R})$ will be fixed unless otherwise indicated, and used to define the spaces $L^p(\Omega, m_0)$ for $p = 1, 2$. And, as in the previous chapters, given any real matrix $A \geq 0$, $A^{1/2}$ will represent the unique positive semidefinite symmetric square root of A, and $A^{-1/2}$ its inverse in the case that it exists. If $A = \operatorname{Im} B$, the following notation will be used: $A^{1/2} = \operatorname{Im}^{1/2} B$ and $A^{-1/2} = \operatorname{Im}^{-1/2} B$.

4.1 A Suitable Symplectic Change of Variables

It will be assumed in this section that the unperturbed family (4.2) satisfies Hypothesis 4.1. As previously mentioned, it will be proved that this fact ensures the existence of a change of variables and of two complex Lagrange planes associated to each perturbation direction Γ of (4.1) in a wide set, which are suitable for later purposes. In particular, and roughly speaking, the change of variables preserves the rotation number and the Lyapunov index of (4.1) for each $\lambda \in \mathbb{R}$, and it takes the initial family (4.2) to skew-symmetric form. The construction of the suitable change of variables is carried out in two steps. In the first step, a change of variables is directly defined from the matrix Z provided by Hypothesis 4.1, which is hence independent of Γ. In the second step, the properties of the transformed flow obtained by means of the initial change of variables make it possible to obtain a new function A_Γ satisfying Hypothesis 4.1. In the definition of A_Γ, the matrix Γ plays a fundamental role. The change of variables associated to this function A_Γ is the "good one" for the purposes of the chapter.

4.1.1 A Symplectic Change of Variables from Hypothesis 4.1

Recall that

$$\mathbb{S}_d^+(\mathbb{C}) = \{A \in \mathbb{S}_d(\mathbb{C}) \mid \operatorname{Im} A > 0\} \quad \text{and} \quad \mathbb{S}_d^+(\mathbb{R}) = \{A \in \mathbb{S}_d(\mathbb{R}) \mid A > 0\}\,.$$

Recall also that the Riccati equation associated to (4.2) is

$$M' = -M\,H_3(\omega{\cdot}t)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t)\,. \tag{4.5}$$

The following technical lemma will be used in Proposition 4.3 and in the last two sections of the chapter.

Lemma 4.2

(i) *Given $N\colon \mathbb{R} \to \mathbb{S}_n^+(\mathbb{C})$, define*

$$C(t) = \begin{bmatrix} \operatorname{Im}^{1/2} N(t) & 0_n \\ -\operatorname{Im}^{-1/2} N(t)\operatorname{Re} N(t) & \operatorname{Im}^{-1/2} N(t) \end{bmatrix}. \tag{4.6}$$

Then the map N satisfies the equation (4.5) for a point $\omega \in \Omega$ if and only if the map $Z\colon \mathbb{R} \to \mathbb{S}_{2n}^+(\mathbb{R})$ defined by $Z = C^T C$ satisfies (4.4) for the same $\omega \in \Omega$. In addition, C and Z are symplectic, and $Z > 0$.

(ii) *Given* $Z\colon \mathbb{R} \to \mathrm{Sp}(n,\mathbb{R}) \cap \mathbb{S}^+_{2n}(\mathbb{R})$*, with* $Z = \left[\begin{smallmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{smallmatrix}\right]$*, define*

$$N(t) = Z_3^{-1}(t)(-Z_2(t) + iI_n)\,.$$

Then $\operatorname{Im} N > 0$ *and* $Z = C^T C$*, where* C *is defined by* (4.6)*.*

Proof

(i) Note that $\operatorname{Im}^{1/2} N$ and $\operatorname{Im}^{-1/2} N$ exist (see Proposition 1.19), so that also C is well defined. The proof of the first assertion in (i), which is a straightforward computation taking (4.5) as the starting point, does not present any complications, and therefore is omitted. It can immediately be checked that C is symplectic, which implies the same property for Z. Obviously, $Z > 0$.

(ii) The symplectic character of Z (see Proposition 1.23) ensures that

$$Z_2 Z_3 = Z_3 Z_2^T \quad \text{and} \quad I_n = Z_1 Z_3 + Z_2^T Z_2^T\,. \tag{4.7}$$

Note that $Z_3^{-1} = (Z_3^{-1})^T > 0$, since $Z = Z^T > 0$. Therefore, N is well defined, with $\operatorname{Im} N > 0$, and it is symmetric, since $Z_3^{-1} Z_2 = Z_2^T Z_3^{-1}$. In addition,

$$C^T C = \begin{bmatrix} Z_3^{-1} - Z_2^T Z_3^{-1} Z_2 & Z_2^T \\ Z_2 & Z_3 \end{bmatrix}.$$

The equalities (4.7) imply that $Z_1 = Z_3^{-1} - Z_2^T Z_2^T Z_3^{-1} = Z_3^{-1} - Z_2^T Z_3^{-1} Z_2$, which proves the equality $C^T C = Z$.

Proposition 4.3 *Suppose that there exist a* σ*-invariant subset* $\Omega_0 \subseteq \Omega$ *with* $m_0(\Omega_0) = 1$ *and a measurable function* $Z\colon \Omega \to \mathrm{Sp}(n,\mathbb{R}) \cap \mathbb{S}^+_{2n}(\mathbb{R})$ *which solves the equation* (4.4) *along the flow on* Ω_0*. Then,*

(i) *the complex* $n \times n$ *matrix-valued function*

$$N(\omega) = Z_3^{-1}(\omega)(-Z_2(\omega) + iI_n)\,, \tag{4.8}$$

is measurable, symmetric, and a solution along the flow on Ω_0 *of the Riccati equation* (4.5)*.*

(ii) *The real* $2n \times 2n$ *matrix-valued function*

$$C(\omega) = \begin{bmatrix} \operatorname{Im}^{1/2} N(\omega) & 0_n \\ -\operatorname{Im}^{-1/2} N(\omega) \operatorname{Re} N(\omega) & \operatorname{Im}^{-1/2} N(\omega) \end{bmatrix} \tag{4.9}$$

is measurable, satisfies $C^T C = Z$*, and is symplectic.*

If, in addition, $Z \in L^1(\Omega, m_0)$*, then* $N \in L^1(\Omega, m_0)$*,* $C \in L^2(\Omega, m_0)$*, and* $C^{-1} \in L^2(\Omega, m_0)$*.*

Proof Lemma 4.2 can be used to prove all the assertions in (i) and (ii) except for the measurability, which is obvious. To check the m_0-integrability and square integrability of N and C, note that $\|C^T\|^2 = \|C\|^2 = \|C^TC\| = \|Z\|$ (see Remark 1.24.2). Therefore, the integrability of Z ensures that C and $C^{-1} = -JC^TJ$ belong to $L^2(\Omega, m_0)$, which together with the equality $N = \operatorname{Im}^{1/2}N\,(\operatorname{Im}^{-1/2}N\operatorname{Re}N + i\operatorname{Im}^{1/2}N)$ implies that $N \in L^1(\Omega, m_0)$.

Theorem 4.4

(i) *Suppose that there exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable function $N: \Omega \to \mathbb{S}_n^+(\mathbb{C})$ which is a solution along the flow on Ω_0 of the Riccati equation (4.5). Define C by (4.9). The symplectic change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ transforms the system (4.2) for $\omega \in \Omega_0$ into skew-symmetric form,*

$$\widetilde{\mathbf{z}}' = \widetilde{H}(\omega{\cdot}t)\widetilde{\mathbf{z}} = \begin{bmatrix} \widetilde{H}_1(\omega{\cdot}t) & -\widetilde{H}_2(\omega{\cdot}t) \\ \widetilde{H}_2(\omega{\cdot}t) & \widetilde{H}_1(\omega{\cdot}t) \end{bmatrix} \widetilde{\mathbf{z}}, \tag{4.10}$$

where $\widetilde{H}_1(\omega) = -\widetilde{H}_1^T(\omega)$ and $\widetilde{H}_2(\omega) = \widetilde{H}_2^T(\omega)$.

(ii) *Conversely, suppose that there exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable function $C: \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$, with $C(\omega)$ nonsingular for $\omega \in \Omega_0$, which is differentiable along the flow on Ω_0, and such that the change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ transforms the system (4.2) for $\omega \in \Omega_0$ into skew-symmetric form (4.10). Then $Z = C^TC$ is a solution along the flow on Ω_0 of (4.4). If, in addition, C takes values in $\mathrm{Sp}(n, \mathbb{R})$, then so does Z, and the map N defined from Z by (4.8) is a solution along the flow on Ω_0 of (4.5).*

Proof

(i) The point ω is assumed to belong to Ω_0 and will be dropped from the notation. Denote $R = \operatorname{Re}N$ and $P = \operatorname{Im}^{1/2}N$. The Riccati equation (4.5) satisfied by $N = R + iP^2$ ensures that

$$\begin{aligned} R' &= -RH_3R + P^2H_3P^2 - RH_1 - H_1^TR + H_2\,, \\ P'P + PP' &= -RH_3P^2 - P^2H_3R - P^2H_1 - H_1^TP^2 \end{aligned} \tag{4.11}$$

Clearly, the transformed system is $\widetilde{\mathbf{z}}' = \widetilde{H}\widetilde{\mathbf{z}}$ with $\widetilde{H} = (C' + CH)C^{-1}$ on Ω_0. Write $\widetilde{H} = \left[\begin{smallmatrix} \widetilde{H}_1 & \widetilde{H}_3 \\ \widetilde{H}_2 & \widetilde{H}_4 \end{smallmatrix}\right]$. The expression of C yields

$$\begin{aligned} \widetilde{H}_1 &= P^{-1}(PP' + P^2H_3R + P^2H_1)P^{-1}\,, \\ \widetilde{H}_2 &= P^{-1}(-R' - RH_3R - RH_1 - H_1^TR + H_2)P^{-1} = \widetilde{H}_2^T\,, \\ \widetilde{H}_3 &= PH_3P = \operatorname{Im}^{1/2}N\,H_3\operatorname{Im}^{1/2}N\,, \\ \widetilde{H}_4 &= P^{-1}(-P'P - RH_3P^2 - H_1^TP^2)P^{-1} = -\widetilde{H}_1^T\,, \end{aligned} \tag{4.12}$$

and it follows easily from (4.11) that $\widetilde{H}_4 = \widetilde{H}_1$ and $\widetilde{H}_3 = -\widetilde{H}_2$ on Ω_0. This completes the proof of (i).

(ii) The hypotheses of (ii) say that $\widetilde{H} = (C' + CH)\,C^{-1}$ is skew-symmetric, which implies that $C^T(C' + CH) = -((C^T)' + H^T C^T)\,C$. That is, the matrix-valued function $Z = C^T C$ solves (4.4). The second assertion in (ii) follows easily from Lemma 4.2.

Remarks 4.5

1. If Hypothesis 4.1 holds, Proposition 4.3(i) shows that the relation (4.8) defines a matrix-valued function N satisfying the conditions of Theorem 4.4(i). Therefore a change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ taking (4.2) to skew-symmetric form (4.10) exists for $\omega \in \Omega_0$. Note also that Proposition 4.3 ensures that C is square integrable. The map C will be referred to as *the square integrable matrix-valued function associated to the function Z provided by Hypothesis* 4.1.
2. Under the conditions of Theorem 4.4(i), the transformed systems (4.10) are defined just for $\omega \in \Omega_0$. It follows from (4.12) and (4.11) that $\widetilde{H}_2, \widetilde{H}_3, \widetilde{H}_1 + \widetilde{H}_4$, and $\widetilde{H}_1 - \widetilde{H}_4$ are measurable on Ω_0, so that $\widetilde{H}$ is also measurable. Now Ω_0 is σ-invariant so, in order to have a globally defined measurable skew-symmetric matrix-valued function $\widetilde{H}$ as well as globally defined flows $\widetilde{\tau}_{\mathbb{R}}$ on $\Omega \times \mathbb{R}^{2n}$, $\widetilde{\tau}_{\mathbb{C}}$ on $\Omega \times \mathbb{C}^{2n}$, and $\widetilde{\tau}$ on $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{K}_{\mathbb{C}}$, it is enough to define $\widetilde{H}(\omega) = 0_{2n}$ for $\omega \notin \Omega_0$. The expressions $\widetilde{U}(t,\omega) = C(\omega{\cdot}t)\,U(t,\omega)\,C^{-1}(\omega)$ for $(t,\omega) \in \mathbb{R} \times \Omega_0$ and $\widetilde{U}(t,\omega) = I_{2n}$ for $(t,\omega) \in \mathbb{R} \times (\Omega - \Omega_0)$ define the fundamental matrix solution $\widetilde{U}(t,\omega)$ of $\widetilde{\mathbf{z}}' = \widetilde{H}(\omega{\cdot}t)\widetilde{\mathbf{z}}$ with $\widetilde{U}(t,\omega) = I_{2n}$, so that the flows are Borel measurable. Of course, nothing ensures that they are continuous on their corresponding phase spaces; but these flows are continuous on sets of the form $\mathcal{K} \times \mathbb{R}^{2n}$, $\mathcal{K} \times \mathbb{C}^{2n}$, $\mathcal{K} \times \mathcal{L}_{\mathbb{R}}$ or $\mathcal{K} \times \mathcal{L}_{\mathbb{C}}$ if C is continuous on $\mathcal{K} \subseteq \Omega_0$. Note also that, if C is defined from N as in (4.9), then the maps $t \mapsto \widetilde{H}(\omega{\cdot}t)$ are indeed continuous for all $\omega \in \Omega$: for $\omega \in \Omega_0$ this assertion follows from the continuity of the maps $t \mapsto C(\omega{\cdot}t)$, $t \mapsto C'(\omega{\cdot}t)$ and $t \mapsto C^{-1}(\omega{\cdot}t)$, in turn ensured by the hypotheses on N; and it is obvious for $\omega \notin \Omega_0$.
3. Note that all the solutions of any family of linear Hamiltonian systems $\widetilde{\mathbf{z}}' = \widetilde{H}(\omega{\cdot}t)\widetilde{\mathbf{z}}$ given by a skew-symmetric matrix-valued function $\widetilde{H}$ are bounded: it follows from $\widetilde{H}^T = -\widetilde{H}$ that the derivative of $\|\widetilde{\mathbf{z}}(t)\|^2$ is zero for every solution $\widetilde{\mathbf{z}}(t)$ of all these systems.

Assume now that Hypothesis 4.1 holds, and let C be the square integrable matrix-valued function associated to the function Z which it determines (see Remark 4.5.1). Of course, the change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ can be applied to any family of linear Hamiltonian systems different from (4.2), independently of the fact that $C^T C$ is no longer a solution of the corresponding equation (4.4): the symplectic character of C ensures that the transformed family of systems is also Hamiltonian, but it is not necessarily skew-symmetric. The goal of the following Propositions 4.6 and 4.7 is to show that the definitions (2.5) and (2.55) of the rotation number and the Lyapunov index can be directly extended to the transformed equations (in spite of the fact that the coefficient matrix is not necessarily continuous), and that they take the same values as for the initial family. As will be seen, these properties are due to the block-triangular expression of the matrix C, its square integrability, and the fact that $\mathrm{Im}^{1/2} N > 0$.

Proposition 4.6 *Suppose that Hypothesis 4.1 holds, let Ω_0 be the set appearing there, and let C be the square integrable matrix-valued function associated to the function Z that it provides (see Remark 4.5.1). Given a function $V: \mathbb{R} \times \Omega_0 \to \mathrm{Sp}(n, \mathbb{R})$, define $\widetilde{V}: \mathbb{R} \times \Omega_0 \to \mathrm{Sp}(n, \mathbb{R})$ by $\widetilde{V}(t, \omega) = C(\omega{\cdot}t)\, V(t, \omega)$. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_3 \widetilde{V}(t, \omega) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_3 V(t, \omega)$$

whenever one of the limits exists.

Proof Write $V(t, \omega) = \begin{bmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega) \end{bmatrix}$ and $\widetilde{V}(t, \omega) = \begin{bmatrix} \widetilde{V}_1(t,\omega) & \widetilde{V}_3(t,\omega) \\ \widetilde{V}_2(t,\omega) & \widetilde{V}_4(t,\omega) \end{bmatrix}$. The expression of $C(\omega)$ and the fact that $\mathrm{Im}^{1/2} N(\omega) > 0$ for $\omega \in \Omega_0$ ensure that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_3 \widetilde{V}(t, \omega) &= \lim_{t\to\infty} \frac{1}{t} \arg\det\big(\widetilde{V}_1(t, \omega) + i\widetilde{V}_3(t, \omega)\big) \\
&= \lim_{t\to\infty} \frac{1}{t} \arg\det\big(\mathrm{Im}^{1/2} N(\omega{\cdot}t)(V_1(t, \omega) + iV_3(t, \omega))\big) \\
&= \lim_{t\to\infty} \frac{1}{t} \arg\det(V_1(t, \omega) + iV_3(t, \omega)) \\
&= \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_3 V(t, \omega)
\end{aligned}$$

for all $\omega \in \Omega_0$, as asserted.

Proposition 4.7 *Suppose that Hypothesis 4.1 holds, let Ω_0 be the set appearing there, and let C be the square integrable matrix-valued function associated to the function Z that it provides (see Remark 4.5.1). Then,*

(i) $\lim_{t\to\infty} (1/t) \ln \| \wedge^n C(\omega{\cdot}t)\|$ *and* $\lim_{t\to\infty} (1/t) \ln \|\wedge^n C^{-1}(\omega{\cdot}t)\|$ *exist and are* 0 *for m_0-a.e. $\omega \in \Omega_0$.*
(ii) *Given $V: \mathbb{R} \times \Omega_0 \to \mathrm{Sp}(n, \mathbb{R})$, define $\widetilde{V}: \mathbb{R} \times \Omega_0 \to \mathrm{Sp}(n, \mathbb{R})$ by $\widetilde{V}(t, \omega) = C(\omega{\cdot}t)\, V(t, \omega)$. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \ln \| \wedge^n \widetilde{V}(t, \omega)\| = \lim_{t\to\infty} \frac{1}{t} \ln \| \wedge^n V(t, \omega)\|$$

for m_0-a.e. $\omega \in \Omega_0$ for which the second limit exists.

Proof

(i) Let $\eta_1^2, \dots, \eta_{2n}^2$ be the eigenvalues of the symplectic and positive definite matrix $C^T C$, with $\eta_1 \geq \cdots \geq \eta_{2n} > 0$. The results of Lemma 2.39 ensure that $\mathrm{tr}(C^T C) > n \geq 1$, $\| \wedge^n C\| \geq 1$, and $\| \wedge^n C\|^2 = \eta_1^2 \cdots \eta_n^2 \leq (\mathrm{tr}(C^T C))^n$. Consequently,

$$0 \leq \ln \|\wedge^n C\| \leq \frac{n}{2} \ln(\mathrm{tr}(C^T C)) \quad \text{and} \quad (\ln(\mathrm{tr}(C^T C)))' \leq \mathrm{tr}(C^T C)' . \tag{4.13}$$

Moreover, since $Z = C^T C$ is a solution along the flow on Ω_0 of the equation (4.4), one has that $\mathrm{tr}(C^T C)' = -2\,\mathrm{tr}(H C^T C)$. Since H is norm-bounded (it is continuous on Ω) and $Z \in L^1(\Omega, m_0)$, it follows that $\mathrm{tr}(C^T C)' \in L^1(\Omega, m_0)$. Hence (4.13), the Birkhoff Theorems 1.3 and 1.6, and Proposition 1.36 ensure that

$$0 \leq \lim_{t\to\infty} \frac{1}{t} \ln \|\wedge^n C(\omega{\cdot}t)\| \leq \frac{n}{2} \int_\Omega \mathrm{tr}(C^T(\omega)\, C(\omega))'\, dm_0 = 0$$

for m_0-a.e. $\omega \in \Omega_0$, which proves the assertion regarding the first limit in (i). To check it for the second limit, with the same argument, note that the symplectic and infinitesimally symplectic characters of C and H respectively ensure that C^{-1} is symplectic, and that $\widetilde{Z} = (C^{-1})^T C^{-1}$ agrees with $-J(Z^{-1})^T J$ and satisfies $\widetilde{Z}' = H^T(\omega{\cdot}t)\widetilde{Z} + \widetilde{Z} H(\omega{\cdot}t)$.

(ii) Since $\wedge^n \widetilde{V}(t,\omega) = \wedge^n C(\omega{\cdot}t) \wedge^n V(t,\omega) \wedge^n C^{-1}(\omega)$, assertion (ii) follows from (i).

Corollary 4.8 *If Hypothesis 4.1 holds, then all the Lyapunov exponents and the Lyapunov index of (4.2) with respect to m_0 vanish.*

Proof Let Z be the matrix-valued function provided by Hypothesis 4.1. Consider the family (4.10) obtained from (4.2) by the change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$, with C associated to Z by (4.8) and (4.9). According to Remark 4.5.3, the fundamental matrix solution $\widetilde{U}(t,\omega)$ of (4.10) which satisfies $\widetilde{U}(0,\omega) = I_{2n}$, is bounded on $\mathbb{R} \times \Omega_0$ (in fact, on $\mathbb{R} \times \Omega$). Therefore, the spectral radius of the matrix $\widetilde{U}^T(t,\omega)\,\widetilde{U}(t,\omega)$ is bounded on $\mathbb{R} \times \Omega_0$ (see Remark 1.24.2), and hence all of its eigenvalues are bounded. Consequently, points (iii) and (ii) of Lemma 2.39 prove that $\| \wedge^n \widetilde{U}(t,\omega)\|$ is bounded on $\mathbb{R} \times \Omega_0$, and that $\| \wedge^n \widetilde{U}(t,\omega)\| \geq 1$. These two properties yield $\lim_{t\to\infty}(1/t) \ln \|\wedge^n \widetilde{U}(t,\omega)\| = 0$. Proposition 4.7 implies that $\lim_{t\to\infty}(1/t) \ln \|\wedge^n U(t,\omega)\| = 0$ for m_0-a.e. $\omega \in \Omega_0$, and so the assertion holds by virtue of Definition 2.41.

4.1.2 A Symplectic Change of Variables Associated to Γ

Under Hypothesis 4.1, a change of variables which is associated to a continuous function $\Gamma\colon\Omega\to\mathbb{S}_{2n}(\mathbb{R})$ and which is suitable for later purposes will be found. That is the purpose of this section. Note that the condition $\Gamma\geq 0$ is not required in the first result.

Theorem 4.9 *Suppose that Hypothesis 4.1 holds, and let Ω_0 be the set that it provides. Let $\Gamma\colon\Omega\to\mathbb{S}_{2n}(\mathbb{R})$ be a continuous map. Then there is a σ-invariant subset $\widetilde{\Omega}_\Gamma\subseteq\Omega_0$ with $m_0(\widetilde{\Omega}_\Gamma)=1$ such that the limit*

$$A_\Gamma(\omega)=\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds \tag{4.14}$$

exists for every $\omega\in\widetilde{\Omega}_\Gamma$. In addition, the symmetric matrix-valued function A_Γ is measurable, belongs to $L^1(\Omega,m_0)$, and is a solution along the flow on $\widetilde{\Omega}_\Gamma$ of (4.4).

Proof The main idea is to reformulate (4.2) with respect to a new base in order to express (4.14) in terms of the mean value of an integrable function. This is done in several steps.

Let Z be the matrix-valued function provided by Hypothesis 4.1, let C be the square integrable matrix-valued map associated to Z, and let (4.10) be the transformed family of skew-symmetric linear Hamiltonian systems obtained from (4.2) via the change of variables $\widetilde{\mathbf{z}}=C(\omega{\cdot}t)\,\mathbf{z}$ for $\omega\in\Omega_0$ and defined by $\widetilde{H}(\omega)=0_{2n}$ outside Ω_0: see Remarks 4.5.1 and 4.5.2. The first step is to check that the compact subset of $\Omega\times\mathbb{M}_{2n\times n}(\mathbb{R})$ defined by

$$\Omega^1=\left\{\omega^1=\left(\omega,\begin{bmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{bmatrix}\right)\in\Omega\times\mathbb{M}_{2n\times n}(\mathbb{R})\;\middle|\;\big(\widetilde{\Phi}_1+i\widetilde{\Phi}_2\big)^*\big(\widetilde{\Phi}_1+i\widetilde{\Phi}_2\big)=I_n\right\}$$

is invariant under the Borel measurable flow $\widetilde{\tau}_{\mathbb{R}}$ induced on $\Omega\times\mathbb{M}_{2n\times n}(\mathbb{R})$ by the family (4.10) (see Remark 4.5.2). It is clear that Ω^1 is homeomorphic to the space $\Omega\times\mathcal{G}$, where $\mathcal{G}$ is defined in Sect. 1.3.4. Define $\Omega^1_0=\left\{\left(\omega,\left[\begin{smallmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{smallmatrix}\right]\right)\in\Omega^1\mid\omega\in\Omega_0\right\}$. According to Theorem 1.41 (which just requires that the maps $t\mapsto\widetilde{H}(\omega{\cdot}t)$ be continuous: see [127] and Remark 4.5.2), if $\omega^1\in\Omega^1_0$, then

$$\omega^1{\cdot}t=\widetilde{\tau}_{\mathbb{R}}(t,\omega^1)=\left(\omega,\begin{bmatrix}\widetilde{\Phi}^0_1\\ \widetilde{\Phi}^0_2\end{bmatrix}\right){\cdot}t=\left(\omega{\cdot}t,\begin{bmatrix}\widetilde{\Phi}_1(t)\,\widetilde{R}(t)\\ \widetilde{\Phi}_2(t)\,\widetilde{R}(t)\end{bmatrix}\right),$$

where the functions $\widetilde{\Phi}_1(t)=\widetilde{\Phi}_1(t,\omega,\widetilde{\Phi}^0_1,\widetilde{\Phi}^0_2)$, $\widetilde{\Phi}_2(t)=\widetilde{\Phi}_2(t,\omega,\widetilde{\Phi}^0_1,\widetilde{\Phi}^0_2)$, and $\widetilde{R}(t)=\widetilde{R}(t,\omega,\widetilde{\Phi}^0_1,\widetilde{\Phi}^0_2,I_n)$ are the solutions of equations (1.15) and (1.16) corresponding to the transformed systems (4.10) with initial data $\widetilde{\Phi}^0_1$, $\widetilde{\Phi}^0_2$ and I_n.

In addition,

$$\big(\widetilde{\varPhi}_1(t) + i\widetilde{\varPhi}_2(t)\big)^* \big(\widetilde{\varPhi}_1(t) + i\widetilde{\varPhi}_2(t)\big) = I_n\,. \tag{4.15}$$

It is easy to deduce from (1.18) and from the relation $\widetilde{H} = -\widetilde{H}^T$ the equality $(d/dt)\,(\widetilde{R}^T(t)\,\widetilde{R}(t)) = 0_n$. That is,

$$\widetilde{R}^T(t)\,\widetilde{R}(t) = \widetilde{R}^T(0)\,\widetilde{R}(0) = I_n \tag{4.16}$$

for every $t \in \mathbb{R}$, which together with (4.15) implies that $\widetilde{\tau}_{\mathbb{R}}(t,\omega^1) \in \Omega_0^1$. The result is obvious if $\omega^1 \notin \Omega_0^1$, since in this case $\widetilde{H}(\omega{\cdot}t) = 0_{2n}$. This completes the first step.

At this point the base flow is (Ω^1,σ^1), where $\sigma^1 = \widetilde{\tau}_{\mathbb{R}}|_{\Omega^1}$. In spite of the lack of continuity of this new base flow, the existence of σ^1-ergodic measures projecting onto m_0 by $\Pi\colon \Omega^1 \to \Omega,\ \omega^1 \mapsto \omega$ can be proved: this will be the second step of the proof. Use Lusin's theorem to find an increasing sequence $(\mathcal{K}_m)_{m\in\mathbb{N}}$ of compact subsets of Ω with $m_0(\mathcal{K}_m) \ge 1 - 1/m$ such that the restriction to K_m of the matrix-valued function C given by (4.9) is continuous. Then define

$$\Omega_m = \left\{\omega \in \Omega \;\middle|\; \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} \chi_{\mathcal{K}_m}(\omega{\cdot}s)\,ds = m_0(\mathcal{K}_m)\right\}$$

for each $m \ge 1$, and

$$\Omega_c = \left\{\omega \in \Omega \;\middle|\; \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(\omega{\cdot}s)\,ds = \int_\Omega f(\omega)\,dm_0 \quad \forall f \in C(\Omega,\mathbb{R})\right\}.$$

The ergodicity of the measure m_0, the Birkhoff Theorems 1.3 and 1.6, Proposition 1.5(i), and Remark 1.10, allow one to assume without loss of generality that the sets Ω_m for $m \ge 1$ and Ω_c are σ-invariant, with $m_0(\Omega_m) = m_0(\Omega_c) = 1$. Now fix a point $\omega_0^1 \in \Omega^1$ which projects onto $\omega_0 \in \Omega_c \cap (\cap_{m\ge 1}\Omega_m)$. Take also a sequence $(t_k) \uparrow \infty$. The Riesz representation theorem associates to the bounded linear operator $C(\Omega^1,\mathbb{R}) \to \mathbb{R}\,,\ f^1 \mapsto (1/(2t_k))\int_{-t_k}^{t_k} f^1(\omega_0^1{\cdot}s)\,ds$, with norm 1, a normalized measure μ_k^1. Theorem 1.9(i) ensures that the sequence (μ_k^1) admits a subsequence (μ_j^1) which converges weak* to a measure μ^1. That is,

$$\int_{\Omega^1} f^1(\omega^1)\,d\mu^1 = \lim_{j\to\infty} \frac{1}{2t_j}\int_{-t_j}^{t_j} f^1(\omega_0^1{\cdot}s)\,ds \tag{4.17}$$

whenever $f^1 \in C(\Omega^1,\mathbb{R})$. It follows easily from this fact, from the condition $\omega_0 \in \Omega_c$, and from Remark 1.14, that μ^1 projects onto m_0. Note also that, if $l \in \mathbb{R}$ and

$f^1 \in C(\Omega^1, \mathbb{R})$, then

$$\begin{aligned} &\lim_{j\to\infty} \frac{1}{2t_j} \int_{-t_j}^{t_j} f^1(\omega_0^1{\cdot}(s+l))\,ds \\ &\qquad = \lim_{j\to\infty} \frac{1}{2t_j} \int_{-t_j-l}^{t_j-l} f^1(\omega_0^1{\cdot}s)\,ds = \int_{\Omega^1} f^1(\omega^1)\,d\mu^1\,, \end{aligned} \tag{4.18}$$

as can be deduced from the boundedness of f^1. In order to verify that μ^1 is a σ^1-invariant measure, fix $l \in \mathbb{R}$, $f^1 \in C(\Omega^1, \mathbb{R})$, and $\delta > 0$, and choose an integer $m \in \mathbb{N}$ with $1/m \le \delta/(4\|f^1\|_{\Omega^1})$. Remark 4.5.2 ensures that the restriction of the map $\sigma_l^1 \colon \Omega^1 \to \Omega^1$, $\omega^1 \mapsto \omega^1{\cdot}l$ to the compact set $\mathcal{K}_m^1 = \{\omega^1 \in \Omega^1 \mid \omega \in \mathcal{K}_m\}$ is continuous. Consequently, the Tietze extension theorem ensures that the restriction of $f_l^1 = f^1 \circ \sigma_l^1$ to $\mathcal{K}_m^1$ admits a continuous extension to Ω^1, called $f_{l,m}^1$, which satisfies $\|f_{l,m}^1\|_{\Omega^1} = \left\| f_l^1|_{\mathcal{K}_m^1} \right\|_{\Omega^1} \le \|f^1\|_{\Omega^1}$. Since $f_l^1 - f_{l,m}^1 = (f_l^1 - f_{l,m}^1)\chi_{\Omega^1 - \mathcal{K}_m^1}$, the point $\omega_0^1{\cdot}s$ belongs to $\mathcal{K}_m^1$ if and only if $\omega_0{\cdot}s \in \mathcal{K}_m$, and $\omega_0 \in \Omega_m$, one has

$$\left| \lim_{j\to\infty} \frac{1}{2t_j} \int_{-t_j}^{t_j} \big(f_l^1(\omega_0^1{\cdot}s) - f_{l,m}^1(\omega_0^1{\cdot}s)\big)\,ds \right| \le 2\,\|f^1\|_{\Omega^1}(1 - m_0(\mathcal{K}_m)) \le \frac{\delta}{2}$$

and

$$\left| \int_{\Omega^1} \big(f_l^1(\omega^1) - f_{l,m}^1(\omega^1)\big)\,d\mu^1 \right| \le 2\,\|f^1\|_{\Omega^1}(1 - m_0(\mathcal{K}_m)) \le \frac{\delta}{2}\,.$$

Therefore, equality (4.17) for $f_{l,m}^1$ and (4.18) imply that

$$\begin{aligned} &\left| \int_{\Omega^1} f^1 \circ \sigma_l^1(\omega^1)\,d\mu^1 - \int_{\Omega^1} f^1(\omega^1)\,d\mu^1 \right| \\ &\qquad \le \left| \int_{\Omega^1} \big(f_l^1(\omega^1) - f_{l,m}^1(\omega^1)\big)\,d\mu^1 \right| \\ &\qquad\quad + \left| \lim_{j\to\infty} \frac{1}{2t_j} \int_{-t_j}^{t_j} \big(f_{l,m}^1(\omega_0^1{\cdot}s) - f_l^1(\omega_0^1{\cdot}s)\big)\,ds \right| \\ &\qquad\quad + \left| \lim_{j\to\infty} \frac{1}{2t_j} \int_{-t_j}^{t_j} f^1(\omega_0^1{\cdot}(s+l))\,ds - \int_{\Omega^1} f^1(\omega^1)\,d\mu^1 \right| \\ &\qquad \le \frac{\delta}{2} + \frac{\delta}{2} + 0 = \delta\,, \end{aligned}$$

which together with the arbitrary choice of δ ensures that

$$\int_{\Omega^1} f^1 \circ \sigma_l^1(\omega^1)\, d\mu^1 = \int_{\Omega^1} f^1(\omega^1)\, d\mu^1$$

whenever $f^1 \in C(\Omega^1, \mathbb{R})$ and $l \in \mathbb{R}$. Proposition 1.7 ensures that μ^1 is a σ^1-invariant measure. And, as checked previously, it projects onto m_0.

Now, to prove the existence of a σ^1-ergodic measure projecting onto m_0, note first that if a measure ν^1 is the limit in the weak* topology of a sequence (ν_k^1) of elements of the set $\mathfrak{M}_{\mathrm{inv},m_0}(\Omega^1, \sigma^1)$ of σ^1-invariant measures projecting onto m_0, then $\nu^1 \in \mathfrak{M}_{\mathrm{inv},m_0}(\Omega^1, \sigma^1)$. To see that ν^1 projects onto m_0, just repeat the argument of Proposition 1.15(ii). So, only the σ^1-invariance remains to be checked. Take $f^1 \in C(\Omega^1, \mathbb{R})$. It suffices to prove that $\int_{\Omega^1} f^1(\omega^1)\, d\nu^1 = \int_{\Omega^1} f_l^1(\omega^1)\, d\nu^1$, where $f_l^1 = f^1 \circ \sigma_l^1$. And, since

$$\int_{\Omega^1} f^1(\omega^1)\, d\nu^1 = \lim_{k\to\infty} \int_{\Omega^1} f^1(\omega^1)\, d\nu_k^1 = \lim_{k\to\infty} \int_{\Omega^1} f_l^1(\omega^1)\, d\nu_k^1\,,$$

it is enough to check that $\lim_{k\to\infty} \int_{\Omega^1} f_l^1(\omega^1)\, d\nu_k^1 = \int_{\Omega^1} f_l^1(\omega^1)\, d\nu^1$. The characterization provided by Proposition 1.7 has been used in the last assertions. Note that

$$\begin{aligned}\left| \int_{\Omega^1} \big(f_l^1(\omega^1) - f_{l,m}^1(\omega^1)\big)\, d\nu^1 \right| &= \left| \int_{\Omega^1 - \mathcal{K}_m^1} \big(f_l^1(\omega^1) - f_{l,m}^1(\omega^1)\big)\, d\nu^1 \right| \\ &\le 2\, \|f^1\|_{\Omega^1} (1 - m_0(\mathcal{K}_m))\,,\end{aligned}$$

since ν^1 projects onto m_0; and that the same relation holds for every ν_k^1. Hence,

$$\begin{aligned}&\left| \int_{\Omega^1} f_l^1(\omega^1)\, d\nu^1 - \int_{\Omega^1} f_l^1(\omega^1)\, d\nu_k^1 \right| \\ &\qquad \le \left| \int_{\Omega^1} \big(f_l^1(\omega^1) - f_{l,m}^1\big)\, d\nu^1 \right| + \left| \int_{\Omega^1} \big(f_l^1(\omega^1) - f_{l,m}^1\big)\, d\nu_k^1 \right| \\ &\qquad\quad + \left| \int_{\Omega^1} f_{l,m}^1(\omega^1)\, d\nu^1 - \int_{\Omega^1} f_{l,m}^1\, d\nu_k^1 \right| \\ &\qquad \le 4\|f^1\|_{\Omega^1}(1 - m_0(\mathcal{K}_m)) + \left| \int_{\Omega^1} f_{l,m}^1(\omega^1)\, d\nu^1 - \int_{\Omega^1} f_{l,m}^1\, d\nu_k^1 \right|,\end{aligned}$$

and a good choice of m guarantees that the initial value is as small as desired for large enough k. That is, $\nu^1 \in \mathfrak{M}_{\mathrm{inv},m_0}(\Omega^1, \sigma^1)$, as asserted.

It follows that $\mathfrak{M}_{\mathrm{inv},m_0}(\Omega^1, \sigma^1)$ is a closed subset of the set of measures on Ω^1 (which is a compact metrizable space in the weak* topology, since Ω^1 is a compact metric space: see Theorems 6.4 and 6.5 of [148]), and hence it is compact. Clearly it is also convex. Therefore, the Krein–Milman theorem (see e.g. Theorem 3.23

of [129]) ensures that $\mathfrak{M}_{\mathrm{inv},m_0}(\Omega^1,\sigma^1)$ has extremal points, and the argument used for instance in the proof of Theorem 6.10 of [148] proves that these extremal points are σ^1-ergodic measures. That is, there exist σ^1-ergodic measures projecting ont m_0, and the second step of the proof is complete.

From now on, m_0^1 will be a fixed σ^1-ergodic measure projecting onto m_0.

In the third step one looks for a convenient way to rewrite $U(t,\omega)$ for $\omega\in\Omega_0$. In what follows, $\omega^1=\left(\omega,\left[\begin{smallmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{smallmatrix}\right]\right)$ is assumed to belong to Ω_0^1. Define $\widetilde{H}^1(\omega^1)=\widetilde{H}(\omega)$, where $\widetilde{H}$ determines the systems (4.10), and consider the family

$$\widetilde{\mathbf{z}}'=\widetilde{H}^1(\omega^1{\cdot}t)\widetilde{\mathbf{z}}\,,\quad \omega^1\in\Omega_0^1\,. \tag{4.19}$$

Since $J\widetilde{H}^1=-(\widetilde{H}^1)^TJ=\widetilde{H}^1J$, Theorem 1.41 guarantees that the symplectic matrix-valued function $t\mapsto\widetilde{V}^1(\omega^1{\cdot}t)$ given by $\widetilde{V}^1(\omega^1)=\left[\begin{smallmatrix}\widetilde{\Phi}_1 & -\widetilde{\Phi}_2\\ \widetilde{\Phi}_2 & \widetilde{\Phi}_1\end{smallmatrix}\right]$ is a fundamental matrix solution of system (4.19), evaluated along the appropriate orbit of (Ω_0^1,σ^1). Now consider the family of systems

$$\mathbf{z}'=H^1(\omega^1{\cdot}t)\,\mathbf{z}\,,\quad \omega\in\Omega^1\,, \tag{4.20}$$

given by $H^1(\omega^1)=H(\omega)$, and note that when $\omega^1\in\Omega_0^1$, this system is obtained from (4.19) by means of the change of variables $\mathbf{z}=(C^1)^{-1}(\omega^1{\cdot}t)\widetilde{\mathbf{z}}$ with $C^1(\omega^1)=C(\omega)$. Then the matrix-valued function $t\mapsto V^1(\omega^1{\cdot}t)=(C^1)^{-1}(\omega^1{\cdot}t)\widetilde{V}^1(\omega^1{\cdot}t)$ is a fundamental matrix solution of (4.20) when $\omega^1\in\Omega_0^1$, where H^1 is evaluated along the appropriate orbit of (Ω_0^1,σ^1). Moreover, the boundedness of $\widetilde{V}^1$ on Ω^1 and the fact that $(C^1)^{-1}\in L^2(\Omega^1,m_0^1)$ (which is ensured by Proposition 4.3) imply that

$$V^1\in L^2(\Omega^1,m_0^1)\,. \tag{4.21}$$

Note also that the fundamental matrix solution $U^1(t,\omega^1)$ of (4.20) with initial value $U^1(0,\omega^1)=I_{2n}$ only depends on the first component ω of ω^1; that is, $U^1(t,\omega^1)=U(t,\omega)$ for every $t\in\mathbb{R}$ and $\omega^1\in\Omega^1$. Therefore, for $\omega\in\Omega_0$,

$$U(t,\omega)=U^1(t,\omega^1)=V^1(\omega^1{\cdot}t)\,(V^1)^{-1}(\omega^1)\,, \tag{4.22}$$

which completes the third step.

The proof of Theorem 4.9 can now be finished. The matrix-valued function Γ can be extended to the new base Ω^1 by defining $\Gamma^1(\omega^1)$ to be equal to $\Gamma(\omega)$. Clearly, if $\omega\in\Omega_0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}&U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds\\ &=\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}(U^1)^T(s,\omega^1)\,\Gamma^1(\omega^1{\cdot}s)\,U^1(s,\omega^1)\,ds\,.\end{aligned}$$

It follows from (4.22), (4.21), the Birkhoff Theorems 1.3 and 1.6, and Proposition 1.5, that there exists a (Borel) σ^1-invariant subset $\widetilde{\Omega}^1_\Gamma \subseteq \Omega^1$ with $m^1_0(\widetilde{\Omega}^1_\Gamma) = 1$ and a (Borel) measurable map A^1_Γ such that

$$A^1_\Gamma(\omega^1) = ((V^1)^T)^{-1}(\omega^1)\, V_\Gamma\, (V^1)^{-1}(\omega^1) \tag{4.23}$$

for all $\omega^1 \in \widetilde{\Omega}^1_\Gamma$, where

$$\begin{aligned} V_\Gamma &= \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} (V^1)^T(\omega^1{\cdot}s)\,\Gamma^1(\omega^1{\cdot}s)\,V^1(\omega^1{\cdot}s)\,ds \\ &= \int_{\Omega^1} (V^1)^T(\omega^1)\,\Gamma^1(\omega^1)\,V^1(\omega^1)\,dm^1_0\,. \end{aligned} \tag{4.24}$$

The projection of $\widetilde{\Omega}^1_\Gamma$ onto Ω yields a (Borel) σ-invariant subset $\widetilde{\Omega}_\Gamma$ with $m_0(\widetilde{\Omega}_\Gamma) = 1$ which consists of points of convergence for the limit (4.14) defining A_Γ. This proves the first assertion of Theorem 4.9. The condition (4.21) and the symplectic character of V^1 ensure that $(V^1)^{-1} \in L^2(\Omega^1, m^1_0)$, and hence that $A^1_\Gamma \in L^1(\Omega^1, m^1_0)$; therefore, $A_\Gamma \in L^1(\Omega, m_0)$. Finally, since $(V^1)'(\omega^1{\cdot}t) = H^1(\omega^1{\cdot}t)\,V^1(\omega^1{\cdot}t)$, the relation (4.23) implies

$$(A^1_\Gamma)'(\omega^1{\cdot}t) = -(H^1)^T(\omega^1{\cdot}t)\,A^1_\Gamma(\omega^1{\cdot}t) - A^1_\Gamma(\omega^1{\cdot}t)\,H^1(\omega^1{\cdot}t)\,,$$

from which the last statement follows.

Remark 4.10 The same arguments as above, which are based on the construction of the extended flow, ensure the existence of the correlation matrix

$$\begin{aligned} \mathbf{C}(r,\omega) &= \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} U^T(s+r,\omega)U(s,\omega)\,ds \\ &= ((V^1)^T)^{-1}(\omega^1)\left(\int_{\Omega^1} (V^1)^T(\omega^1{\cdot}r)V^1(\omega^1)\,dm_0\right)(V^1)^T(\omega^1) \end{aligned}$$

and the Fourier coefficients of the solutions of (4.2) for m_0-a.e. $\omega \in \Omega$. Conditions which permit the reconstruction of the solutions in terms of the Fourier series are given in Wiener and Wintner [149] and Scarpellini [135].

The matrix-valued function A_Γ satisfies some of the conditions assumed for Z in Hypothesis 4.1, but not all of them: nothing ensures that it is positive definite or symplectic. The following result proves that for the most usual perturbations Γ (i.e. for Atkinson perturbations), the limit (4.14) defines a positive definite matrix $A_\Gamma(\omega)$ for m_0-a.e. $\omega \in \Omega$; more precisely, for ω in the σ-invariant set $\widetilde{\Omega}_\Gamma$ of Theorem 4.9. This property will often be represented by writing $A_\Gamma > 0$. Theorem 4.13 guarantees that two conjugate complex Lagrange planes are determined for m_0-a.e. $\omega \in \Omega$ by A_Γ whenever $A_\Gamma > 0$. The symmetric representation of these planes provides the complex $n \times n$ matrix-valued functions from which the suitable symplectic change of variables associated to the perturbation Γ will finally be found.

Proposition 4.11 *Suppose that Hypothesis 4.1 holds. Let $\Gamma \geq 0$ be a continuous symmetric positive semidefinite $2n \times 2n$ matrix-valued function on Ω satisfying Hypotheses 3.3, and let $\widetilde{\Omega}_\Gamma$ be the subset provided by Theorem 4.9. Then $A_\Gamma(\omega) > 0$ for all $\omega \in \widetilde{\Omega}_\Gamma$.*

Proof The definitions and notation of the proof of Theorem 4.9 are used in what follows. According to Lemma 3.6(iv) for $\lambda = 0$,

$$\int_{-\infty}^{\infty} \|\Gamma(\omega{\cdot}t)\, U(t,\omega)\,\mathbf{z}\|^2\, dt > 0$$

whenever $\omega \in \Omega$ and $\mathbf{z} \neq \mathbf{0}$. Therefore,

$$\int_{-\infty}^{\infty} \mathbf{z}^T (V^1)^T(\omega^1{\cdot}t)\, (\Gamma^1)^2(\omega^1{\cdot}t)\, V^1(\omega^1{\cdot}t)\,\mathbf{z}\, dt > 0 \tag{4.25}$$

whenever $\omega^1 \in \Omega^1$ and $\mathbf{z} \neq \mathbf{0}$, as can be deduced from (4.22). According to (4.23), the positivity of $A_\Gamma(\omega)$ is equivalent to the positivity of the constant matrix V_Γ, defined by (4.24). Obviously, $V_\Gamma \geq 0$. Assume that $\mathbf{z}^T V_\Gamma\, \mathbf{z} = 0$ for a vector $\mathbf{z} \in \mathbb{R}^{2n}$. That is, $\int_{\Omega^1} f(\omega^1)\, dm_0^1 = 0$ for the function determined by $f(\omega^1) = \mathbf{z}^T (V^1)^T(\omega^1)\, \Gamma^1(\omega^1)\, V^1(\omega^1)\,\mathbf{z}$, which is defined for $\omega^1 \in \Omega_0^1$. Then, for every $t \in \mathbb{R}$, $\int_{\Omega^1} f(\omega^1{\cdot}t)\, dm_0^1 = 0$, so that $\mathbf{z}^T (V^1)^T(\omega^1{\cdot}t)\, \Gamma^1(\omega^1{\cdot}t)\, V^1(\omega^1{\cdot}t)\,\mathbf{z} = 0$ for m_0-a.e. $\omega^1 \in \Omega_0^1$; or, equivalently, $(\Gamma^1)^{1/2}(\omega^1{\cdot}t)\, V^1(\omega^1{\cdot}t)\,\mathbf{z} = \mathbf{0}$. Thus, for every $t \in \mathbb{R}$,

$$\mathbf{z}^T (V^1)^T(\omega^1{\cdot}t)\, (\Gamma^1)^2(\omega^1{\cdot}t)\, V^1(\omega^1{\cdot}t)\,\mathbf{z} = 0 \tag{4.26}$$

for m_0^1-a.e. $\omega^1 \in \Omega_0^1$. Fubini's theorem implies that for m_0^1-a.e. $\omega^1 \in \Omega_0^1$, (4.26) holds for Lebesgue-a.e. $t \in \mathbb{R}$, and hence (4.25) ensures that $\mathbf{z} = \mathbf{0}$. Therefore, $V_\Gamma > 0$ and the result is proved.

Part of point (i) in the following algebraic lemma is crucial in the remaining part of this section, while both points will be required in Sect. 4.5.

Lemma 4.12 *Let A be a $2n \times 2n$ real symmetric matrix. Then n of the eigenvalues of $J^{-1}A$ are the opposites of the other n of them. In addition,*

(i) *A is positive definite if and only if:* (a) *the matrix $J^{-1}A$ has purely imaginary eigenvalues and can be conjugated to a diagonal matrix; and* (b) *the sums of the eigenspaces of $J^{-1}A$ corresponding to eigenvalues with either positive or negative imaginary parts are complex Lagrange planes l^+ and l^- which can be respectively represented as $\left[\begin{smallmatrix} I_n \\ N^+ \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I_n \\ N^- \end{smallmatrix}\right]$, with $N^- = \overline{N^+}$ (complex conjugate) and $\pm\,\mathrm{Im}\, N^\pm > 0$.*

(ii) *Assume that $J^{-1}A$ is nonsingular, has only real eigenvalues and can be conjugated to a diagonal matrix. Then the sums of the eigenspaces of $J^{-1}A$ corresponding to either negative or positive eigenvalues are real Lagrange planes l^+ and l^-.*

Proof The eigenvalues of $J^{-1}A$ agree with those of $JJ^{-1}AJ^{-1} = AJ^{-1}$ and hence also with those of the transposed matrix $-J^{-1}A$, which implies the first assertion.

(i) Assume first that $A > 0$. Consider the inner product defined in $\mathbb{C}^{2n}$ by A, namely $\langle \mathbf{x}, \mathbf{y} \rangle_A = \mathbf{y}^* A\,\mathbf{x}$. The adjoint of the matrix $J^{-1}A$ with respect to this product is $-J^{-1}A$, since $\langle \mathbf{x}, J^{-1}A\,\mathbf{y} \rangle_A = -\mathbf{y}^* AJ^{-1}A\mathbf{x} = \langle -J^{-1}A\,\mathbf{x}, \mathbf{y} \rangle_A$. This property ensures (see e.g. Theorem 7 of Chapter 8 of [95]): (i) that the eigenvalues of $J^{-1}A$ are purely imaginary, which together with the previously proved property and the nonsingular character of A ensures that they are $\pm i\mu_1, \ldots, \pm i\mu_n$, with $\mu_j > 0$ for $j = 1, \ldots, n$; and (ii) the existence of a basis of $\mathbb{C}^{2n}$ composed of eigenvectors of $J^{-1}A$ which is orthonormal for the chosen inner product. In other words, there is a complex matrix P such that

$$P^* AP = I_{2n} \qquad \text{and} \qquad P^{-1}J^{-1}A\,P = D = \begin{bmatrix} i\Lambda & 0_n \\ 0_n & -i\Lambda \end{bmatrix},$$

with $\Lambda = \mathrm{Diag}(\mu_1, \ldots, \mu_n)$. Clearly, P can be chosen such that $P = \left[\begin{smallmatrix} P_1 & \overline{P_1} \\ P_2 & \overline{P_2} \end{smallmatrix}\right]$ and such that $J^{-1}A = PDP^{-1}$, and moreover l^+ and l^- can be respectively represented by $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \overline{P_1} \\ \overline{P_2} \end{smallmatrix}\right]$. Since $AP = JPD$, then $I_{2n} = P^*JPD$. Therefore, $P^*JP = D^{-1}$, that is,

$$\begin{bmatrix} P_2^* P_1 - P_1^* P_2 & P_2^* \overline{P}_1 - P_1^* \overline{P}_2 \\ \overline{P}_2^* P_1 - \overline{P}_1^* P_2 & P_2^* \overline{P}_1 - \overline{P}_1^* \overline{P}_2 \end{bmatrix} = \begin{bmatrix} -i\Lambda^{-1} & 0_n \\ 0_n & i\Lambda^{-1} \end{bmatrix}.$$

This ensures, first, that $P_2^T P_1 = P_1^T P_2$ and $\overline{P}_2^T \overline{P}_1 = \overline{P}_1^T \overline{P}_2$; in other words, that l^+ and l^- are Lagrange planes. And second, that $i(P_2^* P_1 - P_1^* P_2) = \Lambda > 0$, which in turn guarantees that P_1 is nonsingular. Consequently, $l^\pm$ are represented by $\left[\begin{smallmatrix} I_n \\ N^\pm \end{smallmatrix}\right]$, with $N^+ = P_2 P_1^{-1}$ and $N^- = \overline{P_2}\,\overline{P_1^{-1}} = \overline{N^+}$. Moreover, since N^+ is symmetric,

$$\mathrm{Im}\, N^+ = -\frac{i}{2}\,(N^+ - (N^+)^*) = -\frac{i}{2}\,(P_1^*)^{-1}(P_1^* P_2 - P_2^* P_1)P_1^{-1}\,.$$

Hence $\mathrm{Im}\, N^+ > 0$. This completes the proof of the asserted properties for the case $A > 0$.

Conversely, if A is real and symmetric, the assumptions of the last part of the statement concerning $J^{-1}A$ ensure the existence of matrices Λ and D as above, a nonsingular matrix P_1, and a symmetric matrix N^+ with $\mathrm{Im}\, N^+ > 0$

such that, if $P = \begin{bmatrix} P_1 & \overline{P_1} \\ N^+P_1 & \overline{N^+P_1} \end{bmatrix}$, then $J^{-1}A = PDP^{-1}$. It is easy to check that $P^*AP = -P^*J^{-1}PD$ and that

$$-P^*J^{-1}PD = \begin{bmatrix} 2P_1^* \operatorname{Im} NP_1\,\Lambda & 0_n \\ 0_n & 2P_1^T \operatorname{Im} N(P_1^T)^*\,\Lambda \end{bmatrix}$$

is positive definite: it is enough to prove that $\Lambda^{-1/2}P_1^* \operatorname{Im} NP_1\,\Lambda\Lambda^{-1/2}$ is positive definite, which is true since its eigenvalues agree with those of $P_1^* \operatorname{Im} NP_1$. Thus, $P^*AP > 0$ and hence $A > 0$, as asserted.

(ii) If $J^{-1}A\,\mathbf{z} = \lambda\,\mathbf{z}$ and $J^{-1}A\,\mathbf{w} = \mu\,\mathbf{w}$, with $\lambda\mu > 0$, then $\mathbf{z}^T J\,\mathbf{w} = (1/\mu)\,\mathbf{z}^T A\,\mathbf{w} = -(\lambda/\mu)\,\mathbf{z}^T J\,\mathbf{w}$, so that $\mathbf{z}^T J^{-1}\,\mathbf{w} = 0$ follows. Hence l^+ and l^- are real Lagrange planes.

See Definitions 1.17 and 1.63 to recall the concepts of copy of the base and closed subbundle: they appear in the last point of the following theorem, which will be required in Sect. 4.5. Recall also that, under Hypothesis 4.1, each continuous Γ provides a matrix A_Γ by way of (4.14), and that A_Γ satisfies the properties described in Theorem 4.9.

Theorem 4.13 *Suppose that Hypothesis 4.1 holds. Let Γ be a continuous $2n \times 2n$ matrix-valued function on Ω such that the m_0-integrable function A_Γ defined by (4.14) satisfies $A(\omega) > 0$ for all $\omega \in \widetilde{\Omega}_\Gamma$, where $\widetilde{\Omega}_\Gamma$ is the set determined by Theorem 4.9.*

(i) *There exists a σ-invariant subset $\Omega_\Gamma \subseteq \widetilde{\Omega}_\Gamma$ with $m_0(\Omega_\Gamma) = 1$ and real positive numbers $\mu_{\Gamma,1}, \dots, \mu_{\Gamma,n}$ such that, for every $\omega \in \Omega_\Gamma$, the eigenvalues of $J^{-1}A_\Gamma(\omega)$ are $-i\mu_{\Gamma,1}, \dots, -i\mu_{\Gamma,n}, i\mu_{\Gamma,1}, \dots, i\mu_{\Gamma,n}$.*

(ii) *If $\omega \in \Omega_\Gamma$, the n-dimensional linear subspaces $l_\Gamma^+(\omega)$ and $l_\Gamma^-(\omega)$ of $\mathbb{C}^{2n}$, respectively generated by the eigenvectors associated to the eigenvalues of $J^{-1}A_\Gamma$ with positive and negative imaginary part, are complex Lagrange planes, and the sets $\{(\omega, l_\Gamma^\pm(\omega)) \,|\, \omega \in \Omega_\Gamma\} \subset \mathcal{K}_\mathbb{C}$ are τ-invariant.*

(iii) *The planes $l_\Gamma^\pm(\omega)$ can be represented for $\omega \in \Omega_\Gamma$ by $\begin{bmatrix} I_n \\ N_\Gamma^\pm(\omega) \end{bmatrix}$, with $N_\Gamma^-(\omega) = \overline{N_\Gamma^+(\omega)}$ and $\pm \operatorname{Im} N_\Gamma^\pm(\omega) > 0$, and the functions $N_\Gamma^\pm$ are Borel measurable and solutions along the flow on Ω_Γ of the Riccati equation (4.5).*

(iv) *For $\omega \in \Omega_\Gamma$, the real matrix*

$$C_\Gamma(\omega) = \begin{bmatrix} \operatorname{Im}^{1/2} N_\Gamma^+(\omega) & 0_n \\ -\operatorname{Im}^{-1/2} N_\Gamma^+(\omega) \operatorname{Re} N_\Gamma^+(\omega) & \operatorname{Im}^{-1/2} N_\Gamma^+(\omega) \end{bmatrix} \tag{4.27}$$

is symplectic, and the function B_Γ given by $B_\Gamma(\omega) = C_\Gamma^T(\omega)\, C_\Gamma(\omega)$ is a Borel, real, positive definite, and symplectic $2n \times 2n$ matrix solution along the flow *on Ω_Γ of the equation (4.4). In addition, $B_\Gamma \in L^1(\Omega, m_0)$; C_Γ and $C_\Gamma^{-1} \in L^2(\Omega, m_0)$;*

$$B_\Gamma(\omega) = J \begin{bmatrix} I_n & I_n \\ N_\Gamma^+(\omega) & N_\Gamma^-(\omega) \end{bmatrix} \begin{bmatrix} iI_n & 0_n \\ 0_n & -iI_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ N_\Gamma^+(\omega) & N_\Gamma^-(\omega) \end{bmatrix}^{-1};$$

and $B_\Gamma(\omega)$ agrees with $A_\Gamma(\omega)$ for each point $\omega \in \Omega_\Gamma$ at which $A_\Gamma(\omega)$ is symplectic.

Proof (i), (ii) & (iii) Since A_Γ is a solution along the flow on $\widetilde{\Omega}_\Gamma$ of (4.4), one has that

$$A_\Gamma(\omega{\cdot}t) = (U^T)^{-1}(t,\omega)\, A_\Gamma(\omega)\, U^{-1}(t,\omega) \tag{4.28}$$

whenever $\omega \in \widetilde{\Omega}_\Gamma$ and $t \in \mathbb{R}$. Take $\omega \in \widetilde{\Omega}_\Gamma$. According to Lemma 4.12(i), the condition $A_\Gamma(\omega) > 0$ implies that $J^{-1}A_\Gamma(\omega)$ can be conjugated to a diagonal matrix and has eigenvalues $\pm i\mu_{\Gamma,1}(\omega), \dots, \pm i\mu_{\Gamma,n}(\omega)$. Let $\mathbf{z} \in \mathbb{C}^{2n}$ be an eigenvector of $J^{-1}A_\Gamma(\omega)$ associated to the eigenvalue $i\mu(\omega)$. It follows from (4.28) and from the symplectic character of $U(t,\omega)$ that

$$J^{-1}A_\Gamma(\omega{\cdot}t)\, U(t,\omega)\, \mathbf{z} = i\mu(\omega)\, U(t,\omega)\, \mathbf{z}\,; \tag{4.29}$$

that is, $U(t,\omega)\,\mathbf{z}$ is an eigenvector of $J^{-1}A_\Gamma(\omega{\cdot}t)$ associated to the eigenvalue $i\mu(\omega)$. Therefore, the eigenvalues of $J^{-1}A_\Gamma(\omega)$ are σ-invariant functions. Theorem 1.6 and Proposition 1.5(i) provide a Borel σ-invariant set $\Omega_\Gamma^* \subset \widetilde{\Omega}_\Gamma$ with $m_0(\Omega_\Gamma^*) = 1$ such that these eigenvalues take the constant values

$$-i\mu_{\Gamma,1}, \dots, -i\mu_{\Gamma,n}, i\mu_{\Gamma,1}, \dots, i\mu_{\Gamma,n}$$

for all $\omega \in \Omega_\Gamma^*$. Also, Lemma 4.12(i) ensures that $l_\Gamma^\pm(\omega)$ are complex Lagrange planes and that they admit the representation stated in (iii): $l^\pm(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N^\pm(\omega) \end{smallmatrix}\right]$. Another easy consequence of (4.29) is that $U(t,\omega){\cdot}l_\Gamma^\pm(\omega) = l_\Gamma^\pm(\omega{\cdot}t)$, which in turn ensures that the matrix-valued functions $N_\Gamma^\pm$ are solutions along the flow on Ω_Γ^* of the Riccati equation (4.5): see Sect. 1.3.5.

It remains to check that Ω_Γ^* can be reduced to a Borel σ-invariant set Ω_Γ with $m_0(\Omega_\Gamma) = 1$ such that the functions $l_\Gamma^\pm \colon \Omega_\Gamma \to \mathcal{L}_\mathbb{C}$ and $N_\Gamma^\pm \colon \Omega_\Gamma \to \mathbb{S}_n^+(\mathbb{C})$ are Borel measurable (which means that they admit Borel extensions to Ω), and such that the sets $\{(\omega, l_\Gamma^\pm(\omega)) \,|\, \omega \in \Omega_\Gamma\}$ are Borel. The ideas applied here are similar to those of the proof of Proposition 1.16(i). Let $i\mu$ be one of the eigenvalues, and note that (4.29) ensures that its multiplicity d is also invariant; i.e. the map $k_\mu \colon \Omega_\Gamma^* \to \mathcal{G}_d(\mathbb{C}^{2n})\,,\ \omega \mapsto \mathrm{Ker}(A_\Gamma(\omega) - i\mu I_{2n})$ is well defined. The measurability of A_Γ on $\widetilde{\Omega}_\Gamma$ established in Theorem 4.9, the regularity of m_0, and Lusin's theorem, taken together, yield a compact subset $\mathcal{M} \subseteq \Omega_\Gamma^*$ with $m(\mathcal{M}) > 0$ such that $A_\Gamma(\omega)$ is continuous at the points of $\mathcal{M}$. It follows easily from Proposition 1.26(ii) that k_μ is continuous on $\mathcal{M}$. Moreover, the same result ensures that the map $\omega \to k_\mu(\omega) \oplus k_\lambda(\omega)$ is continuous if $\lambda \neq \mu$, and hence that the maps $\mathcal{M} \to \mathcal{L}_{\mathbb{C}}\,,\ \omega \mapsto l^\pm(\omega)$ are continuous. Define $\mathcal{M}_j = \{\omega{\cdot}t \mid \omega \in \mathcal{M}\,, t \in [-j,j]\}$ for $j = 0,1,2,\ldots$, which is also a compact set. Then $\Omega_\Gamma = \cup_{j\ge 0}\mathcal{M}_j$ is a Borel σ-invariant set of positive measure and hence, by ergodicity, $m_0(\Omega_\Gamma) = 1$. In addition, the maps $l^\pm_\Gamma$ are continuous at the points of all the sets $\mathcal{M}_j$: this assertion can be deduced from the relation $U(t,\omega){\cdot}l^\pm_\Gamma(\omega) = l^\pm_\Gamma(\omega{\cdot}t)$ and from the compactness of $\mathcal{M}$ and $[-k,k]$. Therefore, the maps $l^\pm_\Gamma$ are Borel measurable on Ω_Γ, and the sets $\{(\omega, l^\pm_\Gamma(\omega)) \mid \omega \in \Omega_\Gamma\}$ are Borel. Finally, Remark 1.30 and Proposition 1.28 ensure that any Borel set $\mathcal{A} \subseteq \mathbb{S}^+_n(\mathbb{C})$ can be identified with a Borel set $\mathcal{B} \subset \mathcal{L}_{\mathbb{C}}$, in such a way that $\{\omega \in \Omega \mid N^\pm_\Gamma(\omega) \in \mathcal{A}\} = \{\omega \in \Omega \mid l^\pm_\Gamma(\omega) \in \mathcal{B}\}$, which ensures also the Borel measurability of $N^\pm$ on Ω_Γ.

(iv) It follows from Lemma 4.2(i) and from the properties of N^+_Γ stated in (iii) that $C_\Gamma(\omega)$ is symplectic for all $\omega \in \Omega_\Gamma$, and that the measurable matrix-valued function B_Γ is a real, definite positive, and symplectic solution along the flow on Ω_Γ of (4.4). Therefore, $B_\Gamma(\omega{\cdot}t) = (U^{-1})^T(t,\omega)\,B(\omega)\,U^{-1}(t,\omega)$ for $(t,\omega) \in \mathbb{R} \times \Omega_\Gamma$. Now consider the extended flow defined in the proof of Theorem 4.9. With the notation established there and setting $B^1_\Gamma(\omega^1) = B_\Gamma(\omega)$,

$$B^1_\Gamma(\omega^1{\cdot}t) = ((V^1)^T)^{-1}(\omega^1{\cdot}t)\,(V^1)^T(\omega^1)\,B^1_\Gamma(\omega^1)\,V^1(\omega^1)\,(V^1)^{-1}(\omega^1{\cdot}t)\,, \tag{4.30}$$

according to (4.22). The boundedness of $(\widetilde{V}^1)^{-1}$ and the square integrability of the matrix C^1 imply that $(V^1)^{-1} \in L^2(\Omega^1, m^1_0)$. Recall that the measure m^1_0 is ergodic. Apply Theorem 1.3 and Proposition 1.4 to the real functions $v(\omega^1) = \|(V^1)^{-1}(\omega^1)\|^2$ and $b_\Gamma(\omega^1) = \|B^1_\Gamma(\omega^1)\|$ and use Theorem 1.6 in order to prove the existence of constants $\tilde v \in \mathbb{R}$ and $\tilde b_\Gamma \in \mathbb{R} \cup \{\infty\}$ such that

$$\tilde v = \int_{\Omega^1} \|(V^1)^{-1}(\omega^1)\|^2\,dm^1_0 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \|(V^1)^{-1}(\omega^1{\cdot}s)\|^2\,ds\,,$$

$$\tilde b_\Gamma = \int_{\Omega^1} \|B^1_\Gamma(\omega^1)\|\,dm^1_0 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \|B^1_\Gamma(\omega^1{\cdot}s)\|\,ds$$

for all $\omega^1 \in \Omega^1_0$, where Ω^1_0 is a σ^1-invariant subset of Ω^1 with $m^1_0(\Omega^1_0) = 1$. Take $\omega^1 \in \Omega^1_0$. It follows from (4.30) that $\tilde b_\Gamma \le \|V^1(\omega^1)\|^2\|B^1_\Gamma(\omega^1)\|\,\tilde v < \infty$, so that $B^1_\Gamma \in L^1(\Omega^1, m^1_0)$; that is, $B_\Gamma \in L^1(\Omega, m_0)$. Consequently, making use of Proposition 4.3, one sees that C_Γ and C^{-1}_Γ belong to $L^2(\Omega, m_0)$.

On the other hand, since $N^-_\Gamma = \overline{N^+_\Gamma}$ and $\pm\,\mathrm{Im}\,N^\pm_\Gamma > 0$ (always in Ω_Γ), then $2\begin{bmatrix} I_n & I_n \\ N^+_\Gamma & N^-_\Gamma \end{bmatrix}^{-1} = \begin{bmatrix} i\,\mathrm{Im}^{-1} N^+_\Gamma & 0_n \\ 0_n & i\,\mathrm{Im}^{-1} N^+_\Gamma \end{bmatrix}\begin{bmatrix} N^-_\Gamma & -I_n \\ -N^+_\Gamma & I_n \end{bmatrix}$; and from here a straightforward computation proves the equality stated in (iv) for B_Γ. In turn, this equality shows that $J^{-1}B_\Gamma$ can be conjugated to the diagonal matrix $\left[\begin{smallmatrix} iI_n & 0_n \\ 0_n & -iI_n \end{smallmatrix}\right]$, and that the eigenspaces of $J^{-1}B_\Gamma(\omega)$ respectively associated to the eigenvalues i and $-i$ are the Lagrange planes $\left[\begin{smallmatrix} I_n \\ N^+_\Gamma(\omega) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I_n \\ N^-_\Gamma(\omega) \end{smallmatrix}\right]$. But this is exactly what happens with $J^{-1}A_\Gamma(\omega)$ if

$A_\Gamma(\omega)$ is symplectic, since in this case the eigenvalues of $J^{-1}A_\Gamma(\omega)$ have modulus 1 and are purely imaginary (see Lemma 4.12(i)). Therefore, in this case, $A_\Gamma(\omega) = B_\Gamma(\omega)$, which completes the proof.

Remark 4.14 Note that B_Γ can be considered as a normalized representation of A_Γ, in the sense that the eigenvalues of JB_Γ are $\pm i$ and the corresponding eigenvectors determine the same Lagrange planes as those of $J^{-1}A_\Gamma$. The function B_Γ satisfies all the conditions imposed on the matrix-valued function Z in Hypothesis 4.1. In addition, C_Γ is defined from B_Γ as C is from Z in Proposition 4.3(ii). Theorem 4.4 guarantees that the square integrable symplectic matrix-valued function C_Γ satisfies all the hypotheses necessary to define a change of variables that transforms the initial family of systems (4.2) into skew-symmetric form, with the properties established in Propositions 4.7 and 4.6.

The section is completed with a result which summarizes some of the preceding ones and which characterizes the Hamiltonian families satisfying Hypothesis 4.1. Special attention should be paid to points (6) and (7), which describe apparently much weaker conditions which, however, turn out to be equivalent to Hypothesis 4.1. Note that point (6) identifies the families for which Hypothesis 4.1 holds with those for which there exists the L^2-average of the solutions of the systems which correspond to a set of positive measure.

Theorem 4.15 *The following assertions are equivalent:*

(1) *Hypothesis 4.1 holds;*
(2) *there exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable function $A\colon \Omega \to \mathbb{S}^+_{2n}(\mathbb{R})$ which belongs to $L^1(\Omega, m_0)$ and is a solution along the flow on Ω_0 of the equation (4.4);*
(3) *there exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable function $N\colon \Omega \to \mathbb{S}_n(\mathbb{C})$ with $\operatorname{Im} N(\omega) > 0$ for $\omega \in \Omega_0$, which belongs to $L^1(\Omega, m_0)$ and is a solution along the flow on Ω_0 of (4.5);*
(4) *there exist a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ and a measurable function $C_1\colon \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ with $C_1(\omega)$ nonsingular for $\omega \in \Omega_0$, which belongs to $L^2(\Omega, m_0)$, which is differentiable along the flow on Ω_0, and which has the property that the change of variables $\widetilde{\mathbf{z}} = C_1(\omega{\cdot}t)\,\mathbf{z}$ takes the initial system into skew-symmetric form;*
(5) *there exist a continuous perturbation Γ, a σ-invariant subset $\Omega_\Gamma \subseteq \Omega$ with $m_0(\Omega_\Gamma) = 1$, and a measurable function $A_\Gamma\colon \Omega \to \mathbb{S}^+_{2n}(\mathbb{R})$ which belongs to $L^1(\Omega, m_0)$ and is a solution along the flow on Ω_Γ of (4.4), such that $A_\Gamma(\omega)$ is given by the expression (4.14) for all $\omega \in \Omega_\Gamma$, with $A_\Gamma(\omega) > 0$ at these points;*
(6) *the limit $A_{I_{2n}}$ given by the corresponding expression (4.14) for $\Gamma = I_{2n}$ exists on a Borel subset $\widetilde{\Omega} \subseteq \Omega$ with $m_0(\widetilde{\Omega}) > 0$;*
(7) *there exists a continuous perturbation $\Gamma > 0$ such that the limit A_Γ given by the corresponding expression (4.14) exists on a Borel subset $\widetilde{\Omega} \subseteq \Omega$ with $m_0(\widetilde{\Omega}) > 0$.*

Proof It is obvious that Hypothesis 4.1 ensures (2). Proposition 4.3, Theorem 4.4, Theorem 4.9 and Proposition 4.11, prove that it also ensures conditions (3), (4), (5), (6) and (7).

Assume now condition (2). In order to construct a function Z satisfying Hypothesis 4.1 from A, one needs only to repeat the method followed in the proof of Theorem 4.13 to construct B_Γ from A_Γ. Therefore, (1) holds.

It follows easily from points (i) and (ii) of Theorem 4.4 that (3) implies (4) and that (4) implies (2).

Obviously (6) ensures (7) and (5) ensures (2). In order to complete the proof, it suffices to check that (7) implies (5).

Assume henceforth that (7) is true. Then the set Ω_Γ, defined now as the set of points of convergence for the limit (4.14), has positive measure, since it contains $\widetilde{\Omega}$. In addition, it is σ-invariant. To check this assertion, take $\omega \in \Omega_\Gamma$ and $l \in \mathbb{R}$. It follows immediately from $\Gamma \geq 0$ that

$$\begin{aligned}\int_{-t+l}^{t-l} U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds &\leq \int_{-t+l}^{t+l} U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds\\ &\leq \int_{-t-l}^{t+l} U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds\,.\end{aligned}$$

It is easy to deduce that

$$A_\Gamma(\omega) = \lim_{t\to\infty}\frac{1}{2t}\int_{-t+l}^{t+l} U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds\,,$$

and then from this equality that $A_\Gamma(\omega{\cdot}l)$ exists and satisfies

$$A_\Gamma(\omega{\cdot}l) = (U^{-1})^T(l,\omega)\,A_\Gamma(\omega)\,U^{-1}(l,\omega)\,.$$

The asserted σ-invariance is hence proved. This fact and the ergodicity of m_0 imply that $m_0(\Omega_\Gamma) = 1$: the function A_Γ is defined m_0-almost everywhere, and hence is measurable (see e.g. Remark 1.1). And the previous equality guarantees that A_Γ is a solution along the flow on Ω_Γ of (4.4). It remains to prove that $A_\Gamma(\omega) > 0$ for all $\omega \in \Omega_\Gamma$ and that $A_\Gamma \in L^1(\Omega, m_0)$.

Assume first for contradiction that there exist $\omega \in \Omega_\Gamma$ and $\mathbf{z} \in \mathbb{R}^{2n}$ with $\mathbf{z}^T A_\Gamma(\omega)\,\mathbf{z} = 0$ and $\|\mathbf{z}\| = 1$. Then, since $U(s,\omega) \in \mathrm{Sp}(n,\mathbb{R})$ for all $s \in \mathbb{R}$ and $\|J\,\mathbf{w}\| = \|\mathbf{w}\|$ for all $\omega \in \mathbb{R}^{2n}$,

$$\begin{aligned}1 &= \frac{1}{2t}\int_{-t}^{t}\langle \mathbf{z},\mathbf{z}\rangle\,ds = \frac{1}{2t}\int_{-t}^{t}\langle U(\omega,s)\,\mathbf{z}, -J\,U(\omega,s)\,J\,\mathbf{z}\rangle\,ds\\ &\leq \left(\frac{1}{2t}\int_{-t}^{t}\|U(s,\omega)\,\mathbf{z}\|^2\,ds\right)^{1/2}\left(\frac{1}{2t}\int_{-t}^{t}\|U(s,\omega)\,J\,\mathbf{z}\|^2\,ds\right)^{1/2}.\end{aligned}$$

On the other hand, since $\Gamma > 0$, there exists $k > 0$ such that $\|\Gamma^{-1/2}(\omega{\cdot}s)\| \le k$ for all $s \in \mathbb{R}$, which means that $\|U(s,\omega)\,\mathbf{w}\| \le k\,\|\Gamma^{1/2}(\omega{\cdot}s)\,U(s,\omega)\,\mathbf{w}\|$ for all $s \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^{2n}$. In turn, this ensures that

$$\limsup_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} \|U(s,\omega)\,\mathbf{w}\|^2\,ds \le k\,\mathbf{w}^T A_\Gamma(\omega)\,\mathbf{w}\,,$$

and hence the previous inequality yields

$$1 \le k\,(\mathbf{z}^T A_\Gamma(\omega)\,\mathbf{z})^{1/2}((J\mathbf{z})^T A_\Gamma(\omega)\,J\mathbf{z})^{1/2} = 0\,.$$

This is the sought-for contradiction.

In order to prove that $A_\Gamma \in L^1(\Omega, m_0)$, note that

- the proof of Proposition 4.3 can be repeated to find N_Γ and C_Γ beginning with A_Γ, with the difference that it is not possible to assert that they belong to $L^1(\Omega, m_0)$ and $L^2(\Omega, m_0)$;
- the proof of Theorem 4.4(i) can be repeated to obtain a family of Hamiltonian systems (4.10) given by a skew-symmetric matrix $\widetilde{H}$;
- the three steps in the proof of Theorem 4.9 can be repeated until (4.22) has been obtained, with the sole exception of assertion (4.21), which cannot yet be ensured.

Now define $\Omega^1_\Gamma = \{\omega^1 \in \Omega^1 \mid \omega \in \Omega_\Gamma\}$, and note that, reasoning as in (4.23),

$$(V^1)^T(\omega^1)\,A_\Gamma(\omega)\,V^1(\omega^1) = \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}(V^1)^T(\omega^1{\cdot}s)\,\Gamma^1(\omega^1{\cdot}s)\,V^1(\omega^1{\cdot}s)\,ds$$

for all $\omega^1 \in \Omega^1_\Gamma$, where $\Gamma^1(\omega^1) = \Gamma(\omega)$. It follows easily from the definition of the Euclidean matrix norm that the function $(\Gamma^1)^{1/2}\,V^1 \in L^2(\Omega^1, m_0^1)$, which due to the continuity and strict positivity of Γ^1 ensures that $V^1 \in L^2(\Omega^1, m_0^1)$. That is, (4.21) holds, and the final part of the proof of Theorem 4.9 can be repeated in order to check that A_Γ belongs to $L^1(\Omega, m_0)$.

4.2 Directional Differentiability of the Rotation Number

Hypothesis 4.1 is in force throughout this section. Now, a perturbation Γ, given by a continuous real $2n \times 2n$ matrix-valued function on Ω, will be fixed and subjected to the hypothesis that it satisfies the condition (5) of Theorem 4.15: there exists a measurable function $A_\Gamma\colon \Omega \to \mathbb{S}^+_{2n}(\mathbb{R})$ which belongs to $L^1(\Omega, m_0)$ and which is a solution along the flow on Ω_Γ of (4.4), such that $A_\Gamma(\omega)$ is given by the expression (4.14) corresponding to Γ for all $\omega \in \Omega_\Gamma$, with $A_\Gamma(\omega) > 0$ at these points. Recall that this is equivalent to the remaining six conditions described in Theorem 4.15. Note also that there is no loss of generality in assuming that the conclusions of Theorem 4.13 hold in the points of the set Ω_Γ. This assumption will also be in force in this section.

Remark 4.16 As seen in Theorem 4.13, the eigenvalues of $J^{-1}A_\Gamma(\omega)$ are $-i\mu_{\Gamma,1},\ldots,-i\mu_{\Gamma,n},i\mu_{\Gamma,1},\ldots,i\mu_{\Gamma,n}$, and are common for every $\omega\in\Omega_\Gamma$.

Under these conditions, the square integrable matrix C_Γ given by (4.27) for $\omega\in\Omega_\Gamma$ provides a change of variables $\widetilde{\mathbf{z}}=C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$ which takes the systems (4.1) to the family

$$\widetilde{\mathbf{z}}'=(\widetilde{H}(\omega{\cdot}t)+\lambda\,J^{-1}\widetilde{\Gamma}(\omega{\cdot}t))\widetilde{\mathbf{z}},\quad \omega\in\Omega\,. \tag{4.31}$$

Here, $\widetilde{H}(\omega)=-\widetilde{H}^T(\omega)$ is obtained for $\omega\in\Omega_\Gamma$ as in Theorem 4.4(i), $\widetilde{\Gamma}(\omega)$ is given by

$$\widetilde{\Gamma}(\omega)=(C_\Gamma^T)^{-1}(\omega)\,\Gamma(\omega)\,C_\Gamma^{-1}(\omega) \tag{4.32}$$

for $\omega\in\Omega_\Gamma$, and both of them are defined to be 0_{2n} for $\omega\notin\Omega_\Gamma$ (see Remark 4.5.2). The symbols $\widetilde{\tau}_{\Gamma,\lambda}$ and $\widetilde{\tau}$ stand for the flows induced on $\mathcal{K}_\mathbb{R}$ by the family (4.31) for $\lambda\in\mathbb{R}$ and $\lambda=0$ respectively. Although these flows may not be continuous, the existence of ergodic measures projecting onto m_0 is checked as in the proof of Theorem 4.9. The rotation number of (4.1) for $\lambda\in\mathbb{R}$, which, according to Proposition 4.6, can also be directly determined from the family (4.31), will be represented as $\alpha_\Gamma(\lambda)$. Note that now λ represents a real parameter.

As mentioned before, this section is devoted to establishing the differentiability of the rotation number in the direction of the matrix Γ at the point $\lambda_0=0$. The precise statement of this property is given in Theorem 4.19, whose proof is based on the equations and properties of the transformed flows. The following auxiliary results, which are consequences of the close connection between the matrices A_Γ, B_Γ, and C_Γ, show the importance of the particular choice of the linear change of variables. The equality provided by the first result will play a fundamental role in Theorem 4.19.

Lemma 4.17 *Suppose that Hypothesis 4.1 holds, and that Γ and Ω_Γ satisfy the condition (5) of Theorem 4.15 together with Remark 4.16. If $\omega\in\Omega_\Gamma$ and $l\in\mathcal{L}_\mathbb{R}$, with $l\equiv\left[\begin{smallmatrix}L_1\\L_2\end{smallmatrix}\right]$, then*

$$\mathrm{tr}\left(\left(\left[L_1^T\,L_2^T\right]A_\Gamma(\omega)\begin{bmatrix}L_1\\L_2\end{bmatrix}\right)\left(\left[L_1^T\,L_2^T\right]B_\Gamma(\omega)\begin{bmatrix}L_1\\L_2\end{bmatrix}\right)^{-1}\right)=\mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n}\,.$$

Proof Some information obtained in the proofs of Lemma 4.12 and Theorem 4.13 will be used, and the notation established in Theorem 4.13 will be maintained. Write $\Lambda_\Gamma=\mathrm{Diag}(\mu_{\Gamma,1},\ldots,\mu_{\Gamma,n})$ and $D_\Gamma=\left[\begin{smallmatrix}i\Lambda_\Gamma & 0_n\\ 0_n & -i\Lambda_\Gamma\end{smallmatrix}\right]$. For $\omega\in\Omega_\Gamma$, choose Q_ω such that the column vectors of $\left[\begin{smallmatrix}Q_\omega\\ N_\Gamma^+(\omega)\,Q_\omega\end{smallmatrix}\right]$ are eigenvectors associated to $\mu_{\Gamma,1},\ldots,\mu_{\Gamma,n}$, respectively. Then the column vectors of $\left[\begin{smallmatrix} \overline{Q_\omega} \\ N_\Gamma^-(\omega)\,\overline{Q_\omega} \end{smallmatrix}\right]$ are eigenvectors associated to $-\mu_{\Gamma,1},\dots,-\mu_{\Gamma,n}$, respectively. Therefore, if $P_\omega = \left[\begin{smallmatrix} Q_\omega & \overline{Q_\omega} \\ N_\Gamma^+(\omega)\,Q_\omega & N_\Gamma^-(\omega)\,\overline{Q_\omega} \end{smallmatrix}\right]$, then $A_\Gamma(\omega) = JP_\omega D_\Gamma P_\omega^{-1}$. It follows easily from Theorem 4.13(iv) that $B_\Gamma(\omega) = JP_\omega S P_\omega^{-1}$, with $S = \left[\begin{smallmatrix} iI_n & 0_n \\ 0_n & -iI_n \end{smallmatrix}\right]$. Therefore,

$$\begin{aligned} \left[L_1^T\, L_2^T\right] A_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} &= \left[L_2^T\; -L_1^T\right] P_\omega D_\Gamma P_\omega^{-1}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \\ \left(\left[L_1^T\, L_2^T\right] B_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right)^{-1} &= \left(\left[L_2^T\; -L_1^T\right] P_\omega S P_\omega^{-1}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right)^{-1}. \end{aligned}$$

Now represent

$$\left[X\,\overline{X}\right] = \left[L_2^T\; -L_1^T\right] P_\omega \quad \text{and} \quad \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix} = P_\omega^{-1}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

and note that

$$0_n = \left[L_2^T\; -L_1^T\right]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \left[L_2^T\; -L_1^T\right] P_\omega P_\omega^{-1}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \left[X\,\overline{X}\right]\begin{bmatrix} Y \\ \overline{Y} \end{bmatrix};$$

that is, $XY = -\overline{XY}$. Consequently,

$$\begin{aligned} \left[L_1^T\, L_2^T\right] A_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} &= \left[X\,\overline{X}\right] D_\Gamma \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix} = -iX\Lambda_\Gamma Y + i\overline{X}\Lambda_\Gamma \overline{Y}, \\ \left(\left[L_1^T\, L_2^T\right] B_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right)^{-1} &= \left(\left[X\,\overline{X}\right] S \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix}\right)^{-1} \\ &= (-2iXY)^{-1} = \frac{i}{2}\, Y^{-1}X^{-1}. \end{aligned}$$

These equalities lead to

$$\begin{aligned} &\operatorname{tr}\left(\left(\left[L_1^T\, L_2^T\right] A_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right)\left(\left[L_1^T\, L_2^T\right] B_\Gamma(\omega)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right)^{-1}\right) \\ &\quad = \operatorname{tr}\left((-iX\Lambda_\Gamma Y + i\overline{X}\Lambda_\Gamma\overline{Y})\left(\frac{i}{2}\, Y^{-1}X^{-1}\right)\right) \\ &\quad = \frac{1}{2}\operatorname{tr}\left(X\Lambda_\Gamma X^{-1} + \overline{X}\Lambda_\Gamma\overline{X}^{-1}\right) = \operatorname{tr}\Lambda_\Gamma = \mu_{\Gamma,1} + \cdots + \mu_{\Gamma,n}, \end{aligned}$$

which completes the proof.

The following result expresses an important property of invariance with respect to the measure of integration, which will be the principal tool in the study of the differentiability of the rotation number. As in Remark 1.42, the function

$$T_\Gamma(\omega, l) = \operatorname{tr}\left(\begin{bmatrix}\Phi_1^T & \Phi_2^T\end{bmatrix} \widetilde{\Gamma}(\omega) \begin{bmatrix}\Phi_1 \\ \Phi_2\end{bmatrix}\right) \tag{4.33}$$

where $\widetilde{\Gamma}$ is defined by (4.32) and $\left[\begin{smallmatrix}\Phi_1 \\ \Phi_2\end{smallmatrix}\right]$ is any representation of l in $\mathcal{L}_\mathbb{R}$ with $\Phi_1 + i\Phi_2$ unitary, is well defined on $\Omega_\Gamma \times \mathcal{L}_\mathbb{R}$. In addition, given any τ-invariant measure $\widetilde{\mu}$ projecting onto m_0, T_Γ is measurable with respect to the $\widetilde{\mu}$-completion of the Borel sigma-algebra of $\Omega_\Gamma \times \mathcal{L}_\mathbb{R}$: the last assertion of Remark 1.42 ensures that T_Γ is continuous on any compact subset $\mathcal{M} \times \mathcal{L}_\mathbb{R} \subseteq \Omega_\Gamma \times \mathcal{L}_\mathbb{R}$ if $\widetilde{\Gamma}$ is continuous on $\mathcal{M}$, and the assertion follows from this fact and a standard application of Lusin's theorem to the measurable function $\widetilde{\Gamma}$.

Theorem 4.18 *Suppose that Hypothesis 4.1 holds, and that Γ and Ω_Γ satisfy the condition (5) of Theorem 4.15 and Remark 4.16. For every $\widetilde{\tau}$-invariant measure $\widetilde{\mu}$ on $\mathcal{K}_\mathbb{R}$ projecting onto m_0, one has*

$$\int_{\mathcal{K}_\mathbb{R}} T_\Gamma(\omega, l)\, d\widetilde{\mu} = \frac{1}{2}\int_\Omega \operatorname{tr}\widetilde{\Gamma}(\omega)\, dm_0 = \mu_{\Gamma,1} + \cdots + \mu_{\Gamma,n}\,.$$

Proof Take $\omega \in \Omega_\Gamma$ and $l \in \mathcal{L}_\mathbb{R}$, and write $l \equiv \left[\begin{smallmatrix}\widetilde{\Phi}_1^0 \\ \widetilde{\Phi}_2^0\end{smallmatrix}\right]$ with $\widetilde{\Phi}_1^0 + i\widetilde{\Phi}_2^0$ unitary. Let $\widetilde{F}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)$ be the $2n \times n$ matrix solution of (4.31) for $\lambda = 0$ with initial datum $\widetilde{F}(0, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0) = \left[\begin{smallmatrix}\widetilde{\Phi}_1^0 \\ \widetilde{\Phi}_2^0\end{smallmatrix}\right]$. Written in generalized polar coordinates (see Theorem 1.41),

$$\widetilde{F}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0) = \begin{bmatrix}\widetilde{\Phi}_1(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)\, \widetilde{R}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0, I_n) \\ \widetilde{\Phi}_2(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)\, \widetilde{R}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0, I_n)\end{bmatrix}.$$

Here, as usual, $\widetilde{\Phi}_1(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)$, $\widetilde{\Phi}_2(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)$, and $\widetilde{R}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0, I_n)$ are the solutions of the equations (1.15) and (1.16) corresponding to the transformed systems (4.10) with initial data $\widetilde{\Phi}_1^0$, $\widetilde{\Phi}_2^0$, and I_n. It is clear that $C_\Gamma^{-1}(\omega{\cdot}t)\, \widetilde{F}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)$ is a $2n \times n$ a matrix solution of the system (4.2). Write $C_\Gamma^{-1}(\omega)\, \widetilde{F}(0, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0) = \left[\begin{smallmatrix}\Phi_1^0 R \\ \Phi_2^0 R\end{smallmatrix}\right]$. Then $C_\Gamma^{-1}(\omega{\cdot}t)\, \widetilde{F}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0) = U(t, \omega)\left[\begin{smallmatrix}\Phi_1^0 R \\ \Phi_2^0 R\end{smallmatrix}\right]$, which implies that

$$C_\Gamma^{-1}(\omega{\cdot}t)\begin{bmatrix}\widetilde{\Phi}_1(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0) \\ \widetilde{\Phi}_2(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0)\end{bmatrix} = U(t, \omega)\begin{bmatrix}\Phi_1^0 \\ \Phi_2^0\end{bmatrix} R\widetilde{R}^{-1}(t, \omega, \widetilde{\Phi}_1^0, \widetilde{\Phi}_2^0, I_n)\,. \tag{4.34}$$

Relation (4.16) ensures that $\widetilde{R}^{-1}(t, \omega, \widetilde{\Phi}_1, \widetilde{\Phi}_2, I_n)$ and $(\widetilde{R}^T)^{-1}(t, \omega, \widetilde{\Phi}_1, \widetilde{\Phi}_2, I_n)$ are inverse matrices. This property, together with (4.34) and the expression of $\widetilde{\Gamma}$, ensures that

$$\begin{aligned}
&T_\Gamma(\widetilde{\tau}(t,\omega,l))\\
&\quad = \mathrm{tr}\Bigg(\big[\widetilde{\Phi}_1^T(t,\omega,\widetilde{\Phi}_1^0\,\widetilde{\Phi}_2^0)\ \ \widetilde{\Phi}_2^T(t,\omega,\widetilde{\Phi}_1^0,\widetilde{\Phi}_2^0)\big]\ \widetilde{\Gamma}(\omega{\cdot}t)\begin{bmatrix}\widetilde{\Phi}_1(t,\omega,\widetilde{\Phi}_1^0,\widetilde{\Phi}_2^0)\\ \widetilde{\Phi}_2(t,\omega,\widetilde{\Phi}_1^0,\widetilde{\Phi}_2^0)\end{bmatrix}\Bigg)\\
&\quad = \mathrm{tr}\Bigg(\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]\,U^T(t,\omega)\,\Gamma(\omega{\cdot}t)\,U(t,\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}RR^T\Bigg).
\end{aligned}$$

It is easy to deduce from the equality $C_\Gamma(\omega)\left[\begin{smallmatrix}\Phi_1^0\\ \Phi_2^0\end{smallmatrix}\right]R=\left[\begin{smallmatrix}\widetilde{\Phi}_1^0\\ \widetilde{\Phi}_2^0\end{smallmatrix}\right]$ that

$$(R^T)^{-1}\big[(\widetilde{\Phi}_1^0)^T\ \ (\widetilde{\Phi}_2^0)^T\big]=\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]\,C_\Gamma^T(\omega)\,,$$

which in turn implies

$$\begin{aligned}
RR^T &= \Bigg(\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]\,C_\Gamma^T(\omega)\,C_\Gamma(\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\Bigg)^{-1}\\
&= \Bigg(\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]\,B_\Gamma(\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\Bigg)^{-1}.
\end{aligned}$$

Therefore, Theorem 4.9 and Lemma 4.17 imply that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}T_\Gamma(\widetilde{\tau}(s,\omega,l))\,ds\\
&\quad = \mathrm{tr}\Bigg[\Bigg(\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]A_\Gamma(\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\Bigg)\Bigg(\big[(\Phi_1^0)^T\ \ (\Phi_2^0)^T\big]B_\Gamma(\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\Bigg)^{-1}\Bigg]\\
&\quad = \mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n}\,.
\end{aligned}$$

Since this happens for m_0-a.e. $\omega\in\Omega$ and every $l\in\mathcal{L}_\mathbb{R}$, the Birkhoff Theorem 1.3 ensures that

$$\int_{\mathcal{K}_\mathbb{R}}T_\Gamma(\omega,l)\,d\widetilde{\mu}=\mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n} \tag{4.35}$$

for every τ-invariant measure $\widetilde{\mu}$ projecting onto m_0.

On the other hand, for all $l\in\mathcal{L}_\mathbb{R}$, the vector space $J{\cdot}l=\{J\mathbf{z}\mid\mathbf{z}\in l\}$ is a real Lagrange plane, and $j\colon\mathcal{K}_\mathbb{R}\to\mathcal{K}_\mathbb{R}$, $(\omega,l)\mapsto(\omega,J{\cdot}l)$ is a homeomorphism which preserves the flow $\widetilde{\tau}$: $j(\widetilde{\tau}(t,\omega,l))=\widetilde{\tau}(t,j(\omega,l))$. This property is a consequence of the skew-symmetric character of $\widetilde{H}$, which implies that $\widetilde{U}^{-1}(t,\omega)J\,\widetilde{U}(t,\omega)=J$ for the fundamental matrix solution $\widetilde{U}(t,\omega)$ of (4.10) with $\widetilde{U}(0,\omega)=I_{2n}$, and hence that $\widetilde{U}(t,\omega){\cdot}(J{\cdot}l)=J\widetilde{U}(t,\omega){\cdot}l$. Given a τ-ergodic measure $\widetilde{\mu}$ projecting onto m_0, let a new measure $\widetilde{\nu}$ be defined by $\int_{\mathcal{K}_\mathbb{R}} f(\omega,l)\,d\widetilde{\nu} = \int_{\mathcal{K}_\mathbb{R}} (f\circ j)(\omega,l))\,d\widetilde{\mu}$. Then, if $\widetilde{\tau}_t(\omega,l) = \widetilde{\tau}(t,\omega,l)$, it is the case that $\int_{\mathcal{K}_\mathbb{R}} (f\circ\tau_t)(\omega,l)\,d\widetilde{\nu} = \int_{\mathcal{K}_\mathbb{R}} (f\circ\tau_t\circ j)(\omega,l)\,d\widetilde{\mu} = \int_{\mathcal{K}_\mathbb{R}} (f\circ j\circ\tau_t)(\omega,l)\,d\widetilde{\mu} = \int_{\mathcal{K}_\mathbb{R}} (f\circ j)(\omega,l)\,d\widetilde{\mu} = \int_{\mathcal{K}_\mathbb{R}} f(\omega,l)\,d\widetilde{\nu}$ for all $t\in\mathbb{R}$, which means that $\widetilde{\nu}$ is also $\widetilde{\tau}$-invariant. Therefore, (4.35) ensures that

$$\mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n} = \int_{\mathcal{K}_\mathbb{R}} T_\Gamma(\omega,l)\,d\widetilde{\nu} = \int_{\mathcal{K}_\mathbb{R}} (T_\Gamma\circ j)(\omega,l)\,d\widetilde{\mu}\,,$$

and hence that

$$\mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n} = \frac{1}{2}\int_{\mathcal{K}_\mathbb{R}} \big(T_\Gamma(\omega,l)+(T_\Gamma\circ j)(\omega,l)\big)\,d\widetilde{\mu}\,. \tag{4.36}$$

Finally, take again $(\omega,l)\in\Omega_\Gamma\times\mathcal{L}_\mathbb{R}$ and write $l\equiv\left[\begin{smallmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{smallmatrix}\right]$ with $\widetilde{\Phi}_1+i\widetilde{\Phi}_2$ unitary. Then $J{\cdot}l=\left[\begin{smallmatrix}-\widetilde{\Phi}_2\\ \widetilde{\Phi}_1\end{smallmatrix}\right]$ and $-\widetilde{\Phi}_2+i\widetilde{\Phi}_1$ is also unitary, so that

$$\begin{aligned}
&T_\Gamma(\omega,l)+(T_\Gamma\circ j)(\omega,l)\\
&\quad=\operatorname{tr}\left(\big[\,\widetilde{\Phi}_1^T\;\;\widetilde{\Phi}_2^T\big]\,\widetilde{\Gamma}(\omega)\begin{bmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{bmatrix}\right)+\operatorname{tr}\left(\big[-\widetilde{\Phi}_2^T\;\;\widetilde{\Phi}_1^T\big]\,\widetilde{\Gamma}(\omega)\begin{bmatrix}-\widetilde{\Phi}_2\\ \widetilde{\Phi}_1\end{bmatrix}\right)\\
&\quad=\operatorname{tr}\left(\begin{bmatrix}\widetilde{\Phi}_1^T & \widetilde{\Phi}_2^T\\ -\widetilde{\Phi}_2^T & \widetilde{\Phi}_1^T\end{bmatrix}\widetilde{\Gamma}(\omega)\begin{bmatrix}\widetilde{\Phi}_1 & -\widetilde{\Phi}_2\\ \widetilde{\Phi}_2 & \widetilde{\Phi}_1\end{bmatrix}\right)=\operatorname{tr}\widetilde{\Gamma}(\omega)\,,
\end{aligned}$$

since $\left[\begin{smallmatrix}\widetilde{\Phi}_1^T & \widetilde{\Phi}_2^T\\ -\widetilde{\Phi}_2^T & \widetilde{\Phi}_1^T\end{smallmatrix}\right]$ is the inverse of $\left[\begin{smallmatrix}\widetilde{\Phi}_1 & -\widetilde{\Phi}_2\\ \widetilde{\Phi}_2 & \widetilde{\Phi}_1\end{smallmatrix}\right]$. The statements of the theorem can now be proved using (4.35) together with this last equality and (4.36).

Theorem 4.19 *Suppose that Hypothesis 4.1 holds, and that Γ and Ω_Γ satisfy the condition (5) of Theorem 4.15 and Remark 4.16. Then there exists the derivative of the rotation number in the direction of the matrix Γ,*

$$\alpha'_\Gamma(0)=\frac{1}{2}\int_\Omega \operatorname{tr}\big((C_\Gamma^T)^{-1}(\omega)\,\Gamma(\omega)\,C_\Gamma^{-1}(\omega)\big)\,dm_0=\mu_{\Gamma,1}+\cdots+\mu_{\Gamma,n}\,. \tag{4.37}$$

Proof The fundamental result contained in Proposition 4.6 allows one to repeat step by step the arguments of Theorem 2.4 in order to prove that, for all $\lambda\in\mathbb{R}$,

$$\alpha_\Gamma(\lambda)=\int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} Q_{\Gamma,\lambda}(\omega,l)\,d\widetilde{\mu}_\lambda \tag{4.38}$$

for every $\widetilde{\tau}_{\Gamma,\lambda}$-invariant measure $\widetilde{\mu}_\lambda$ on $\mathcal{K}_\mathbb{R}$ projecting onto m_0, where the function $\operatorname{Tr} Q_{\Gamma,\lambda}$ is defined by (1.19) with H replaced by $\widetilde{H}+\lambda\,J^{-1}\widetilde{\Gamma}$. It can immediately be checked that

$$\operatorname{Tr} Q_{\Gamma,\lambda}(\omega, l) = \operatorname{Tr} Q_{\Gamma,0}(\omega, l) + \lambda T_\Gamma(\omega, l)\,,$$

where T_Γ is defined by (4.33). The argument proving the measurability of T_Γ with respect to the $\widetilde{\mu}_\lambda$-completion of the Borel sigma-algebra of $\Omega_\Gamma \times \mathcal{L}_\mathbb{R}$ proves the same property for $\operatorname{Tr} Q_{\Gamma,\lambda}$. Therefore,

$$\frac{\alpha_\Gamma(\lambda) - \alpha_\Gamma(0)}{\lambda} = \int_{\mathcal{K}_\mathbb{R}} T_\Gamma(\omega, l)\, d\widetilde{\mu}_\lambda\,.$$

Take now a sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ with limit 0 and, for each $n \in \mathbb{N}$, a normalized $\widetilde{\tau}_{\Gamma,\lambda_n}$-invariant measure $\widetilde{\mu}_{\lambda_n}$ on $\mathcal{K}_\mathbb{R}$ projecting onto m_0. There is no loss of generality in assuming that the sequence $(\widetilde{\mu}_{\lambda_n})_{n\in\mathbb{N}}$ converges in the weak* topology to a measure $\widetilde{\mu}$: otherwise an appropriate subsequence could be chosen. Again, although the transformed flows $\widetilde{\tau}_{\Gamma,\lambda_n}$ and $\widetilde{\tau}$ are not continuous, it is not hard to check that the limit measure $\widetilde{\mu}$ is invariant under the limit flow $\widetilde{\tau}$: in fact, Remark 4.5.2 allows one to adapt the arguments which were used to a similar end in the proof of Theorem 4.9. The approximation of the matrix-valued function $\widetilde{\Gamma} = (C_\Gamma^T)^{-1}\Gamma C_\Gamma^{-1}$ (which, according Theorem 4.13(iv), belongs to $L^1(\Omega, m_0)$) by a family of continuous matrix-valued functions on Ω, and Theorem 4.18, guarantee that

$$\begin{aligned}\lim_{n\to\infty} \frac{\alpha_\Gamma(\lambda_n) - \alpha_\Gamma(0)}{\lambda_n} &= \int_{\mathcal{K}_\mathbb{R}} T_\Gamma(\omega, l)\, d\widetilde{\mu} \\ &= \frac{1}{2}\int_\Omega \operatorname{tr} \widetilde{\Gamma}(\omega)\, dm_0 = \mu_{\Gamma,1} + \cdots + \mu_{\Gamma,n}\,.\end{aligned}$$

The invariance of the limit with respect to the chosen sequence proves Theorem 4.19.

In the final lines of this section, another representation of the rotation number of the unperturbed family (4.2) will be presented. The notation established in the previous proof is retained. Write $l \equiv \left[\begin{smallmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{smallmatrix}\right]$ with $\widetilde{\Phi}_1 + i\widetilde{\Phi}_2$ unitary. Then $\operatorname{Tr} Q_{\Gamma,0}(\omega, l) = \operatorname{tr}\big(-\widetilde{H}_2(\omega)\,(\widetilde{\Phi}_1\widetilde{\Phi}_1^T + \widetilde{\Phi}_2\widetilde{\Phi}_2^T)\big) = -\operatorname{tr}(\widetilde{H}_2(\omega)) = \operatorname{tr}(\operatorname{Im} N_\Gamma^+(\omega) H_3(\omega))$: the first equality follows easily from the definition of $\operatorname{Tr} Q_{0,\Gamma}$, the second uses the equality $\widetilde{\Phi}_1\widetilde{\Phi}_1^T + \widetilde{\Phi}_2\widetilde{\Phi}_2^T = I_n$, and the third comes from the third equality in (4.12). Therefore, according to (4.38), for every $\widetilde{\tau}$-invariant measure projecting onto m_0,

$$\alpha_\Gamma(0) = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} Q_{\Gamma,0}(\omega, l)\, d\widetilde{\mu} = \int_\Omega \operatorname{tr}\big(H_3(\omega) \operatorname{Im} N_\Gamma^+(\omega)\big)\, dm_0\,. \tag{4.39}$$

4.3 The Limits of the Weyl Functions on the Real Axis

The results of this section are obtained under the following fundamental conditions:

Hypotheses 4.20 Hypothesis 4.1 holds, and either $\Gamma > 0$ for the general Hamiltonian one-parameter family (4.1), or $\Delta > 0$ for the Schrodinger one-parameter family (4.3).

Remark 4.21 According to Remark 3.5.1, if Hypotheses 4.20 hold, then Hypotheses 3.3 hold as well. As was seen previously, this fact and Hypothesis 4.1 have two fundamental consequences: on the one hand, the existence of the Weyl functions $M_\Gamma^\pm(\omega, \lambda)$ for $\lambda \notin \mathbb{R}$: see Theorem 3.8; and, on the other hand, the existence of the functions $N_\Gamma^\pm$ on a σ-invariant set $\Omega_\Gamma \subseteq \Omega$ with $m_0(\Omega_\Gamma) = 1$, which provide the symplectic matrix-valued function C_Γ given by (4.27): see Proposition 4.11 and Theorem 4.13.

As stated in the introduction to this chapter, the goal of this section is to prove that the functions $N_\Gamma^\pm(\omega)$ are the vertical limits from the upper half-plane $\mathbb{C}_+$ of the Weyl functions $M_\Gamma^\pm(\omega, \lambda)$ in the $L^1(\Omega, m_0)$-topology. Section 4.5 sets out more restrictive conditions on Ω, on the unperturbed family of systems, and on the perturbation Γ, which ensure that functions $N_\Gamma^\pm$ are continuous and that the vertical convergence is in fact uniform on Ω.

As a first step, the vertical convergence in measure is proved in Theorem 4.26. This result is based on the existence of the directional derivative of the rotation number. The L^1-convergence, which requires some additional work in the Schrödinger case, is stated and proved in Theorems 4.28 and 4.31.

Remarks 4.22

1. For convenience in the proofs, the matrix norm $\|A\|_F = (\mathrm{tr}(A^TA))^{1/2}$ (see Remark 1.24.3) is fixed in this section. Therefore (see Definition 1.32), $\|A\|_p = \big(\int_\Omega (\mathrm{tr}(A^T(\omega)\,A(\omega)))^{p/2} dm_0\big)^{1/p}$ for $p = 1, 2$. Recall that the L^p-norms induced by two (equivalent) matrix norms are equivalent, so that the notion of convergence in the $L^p(\Omega, m_0)$-topology is independent of this particular choice of the norm.
2. Property F3 in Remark 1.24.3 can be applied to prove two facts, which will be repeatedly used in what follows: first, if the sequences (A_m) and (B_m) converge to A and B in the $L^2(\Omega, m_0)$-topology, then (A_mB_m) converges to AB in the $L^1(\Omega, m_0)$-topology; and second, if C is a continuous and positive definite matrix-valued function and the sequence (CA_m) converges to CA in the $L^p(\Omega, m_0)$-topology (for $p = 1, 2$), then (A_m) converges to A in the $L^p(\Omega, m_0)$-topology. Clearly, both properties can be formulated for one-parameter families (A_ε) and (B_ε) instead of for sequences.

Recall the information obtained in Sect. 3.1: the functions $M_\Gamma^\pm(\omega,\lambda)$, defined for $\operatorname{Im}\lambda \neq 0$ and $\omega \in \Omega$, are symmetric complex $n \times n$ matrix functions, which are jointly continuous in both variables and analytic outside the real axis for each $\omega \in \Omega$ fixed. In addition, $\pm \operatorname{Im}\lambda \operatorname{Im} M_\Gamma^\pm(\omega,\lambda) > 0$, $M_\Gamma^\pm(\omega,\overline{\lambda}) = (M_\Gamma^\pm)^*(\omega,\lambda)$ and, for all nonreal λ and all $\omega \in \Omega$, the functions $t \to M_\Gamma^\pm(\omega{\cdot}t,\lambda)$ are differentiable and satisfy the Riccati equation

$$\begin{aligned} M' &= -M(H_3(\omega{\cdot}t) + \lambda\Gamma_3(\omega{\cdot}t))M - M(H_1(\omega{\cdot}t) + \lambda\Gamma_1(\omega{\cdot}t)) \\ &\quad - (H_1^T(\omega{\cdot}t) + \lambda\Gamma_1^T(\omega{\cdot}t))M + H_2(\omega{\cdot}t) + \lambda\Gamma_2(\omega{\cdot}t)\,. \end{aligned} \tag{4.40}$$

As explained in Sect. 3.2.1, the Weyl functions determine the Floquet coefficient $w_\Gamma(\lambda)$ for the fixed ergodic measure m_0 on the upper half-plane, where it is an analytic function defined by the expressions

$$\begin{aligned} w_\Gamma(\lambda) = \pm \int_\Omega \operatorname{tr}\Big(&H_1(\omega) + \lambda\Gamma_1(\omega) \\ &+ (H_3(\omega) + \lambda\Gamma_3(\omega))M_\Gamma^\pm(\omega,\lambda)\Big)\, dm_0\,. \end{aligned} \tag{4.41}$$

In addition, as is proved in Sect. 3.2.4, the Floquet coefficient is extended to the real axis by the function $-\beta_\Gamma(\lambda) + i\alpha_\Gamma(\lambda)$, where $\beta_\Gamma(\lambda)$ and $\alpha_\Gamma(\lambda)$ denote the Lyapunov index and the rotation number of (4.1) respectively. Recall also that $\beta_\Gamma(\lambda)$ represents the Lyapunov index for λ inside or outside the real axis, as seen in Theorem 3.30.

The analysis carried out in this section requires one to transform again the families of systems (4.1) for $\lambda \in \mathbb{C}$ and $\omega \in \Omega_\Gamma$ (see Remark 4.21) in (4.31) by means of the symplectic change of variables $\widetilde{\mathbf{z}} = C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$. The Lagrange planes $l_{\Gamma,\lambda}^\pm(\omega) \equiv \left[\begin{smallmatrix} I_n \\ M_\Gamma^\pm(\omega,\lambda)\end{smallmatrix}\right]$ are transformed into the Lagrange planes

$$\begin{aligned} \tilde{l}_{\Gamma,\lambda}^\pm(\omega) &\equiv \begin{bmatrix} \operatorname{Im}^{1/2} N_\Gamma^+(\omega) & 0_n \\ -\operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\operatorname{Re} N_\Gamma^+(\omega) & \operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\end{bmatrix}\begin{bmatrix} I_n \\ M_\Gamma^\pm(\omega,\lambda)\end{bmatrix} \\ &\equiv \begin{bmatrix} I_n \\ \widetilde{M}_\Gamma^\pm(\omega,\lambda)\end{bmatrix} \end{aligned}$$

for $\omega \in \Omega_\Gamma$, where

$$\widetilde{M}_\Gamma^\pm(\omega,\lambda) = \operatorname{Im}^{-1/2} N_\Gamma^+(\omega)(M_\Gamma^\pm(\omega,\lambda) - \operatorname{Re} N_\Gamma^+(\omega))\operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\,, \tag{4.42}$$

and hence $\pm\operatorname{Im}\widetilde{M}_\Gamma^\pm(\omega,\lambda) > 0$. In particular, these matrices are nonsingular for $\omega \in \Omega_\Gamma$: see Proposition 1.21(i). In addition, as explained in Sect. 1.3.5, the measurable functions $\widetilde{M}_\Gamma^\pm(\omega,\lambda)$ are solutions along the flow on Ω_Γ of the corresponding Riccati equation (4.40) for the transformed perturbed systems (4.31), namely

$$
\begin{aligned}
M' = &-M(-\widetilde{H}_3(\omega{\cdot}t) + \lambda\widetilde{\Gamma}_3(\omega{\cdot}t))M - M(\widetilde{H}_1(\omega{\cdot}t) + \lambda\widetilde{\Gamma}_1(\omega{\cdot}t)) \\
&- (-\widetilde{H}_1(\omega{\cdot}t) + \lambda\widetilde{\Gamma}_1^T(\omega{\cdot}t))M + \widetilde{H}_2(\omega{\cdot}t) + \lambda\widetilde{\Gamma}_2(\omega{\cdot}t)\,,
\end{aligned}
\tag{4.43}
$$

where $\widetilde{H} = \begin{bmatrix} \widetilde{H}_1 & -\widetilde{H}_2 \\ \widetilde{H}_2 & \widetilde{H}_1 \end{bmatrix}$ and $J^{-1}\widetilde{\Gamma} = J^{-1}\,(C_\Gamma^T)^{-1}\Gamma C_\Gamma^{-1} = \begin{bmatrix} \widetilde{\Gamma}_1 & \widetilde{\Gamma}_3 \\ \widetilde{\Gamma}_2 & -\widetilde{\Gamma}_1^T \end{bmatrix}$.

The highly technical proof of the vertical convergence of the Weyl functions requires the algebraic results stated in the following lemma.

Lemma 4.23 *Suppose that Hypotheses 4.20 hold, and let Ω_Γ satisfy the conditions of Remark 4.21. Define the hermitian $n \times n$ matrices*

$$
\begin{aligned}
W(\omega,\lambda) &= I_n + (\widetilde{M}_\Gamma^+)^*(\omega,\lambda)\,\widetilde{M}_\Gamma^+(\omega,\lambda)\,, \\
T_1(\omega,\lambda) &= i\widetilde{M}_\Gamma^+(\omega,\lambda)\,W^{-1}(\omega,\lambda) - iW^{-1}(\omega,\lambda)\,(\widetilde{M}_\Gamma^+)^*(\omega,\lambda)\,, \\
T_2(\omega,\lambda) &= W^{-1}(\omega,\lambda) + \widetilde{M}_\Gamma^+(\omega,\lambda)\,W^{-1}(\omega,\lambda)\,(\widetilde{M}_\Gamma^+)^*(\omega,\lambda) - I_n
\end{aligned}
\tag{4.44}
$$

for $\omega \in \Omega_\Gamma$ and $\lambda \in \mathbb{C}_+$, with $\widetilde{M}_\Gamma^+(\omega,\lambda)$ defined by (4.42). Then,

(i) $\begin{bmatrix} T_1(\omega,\lambda) & iT_2(\omega,\lambda) \\ -iT_2(\omega,\lambda) & T_1(\omega,\lambda) \end{bmatrix}$ *is a negative definite matrix, and*

(ii) $\begin{bmatrix} T_1(\omega,\lambda) + I_n & iT_2(\omega,\lambda) \\ -iT_2(\omega,\lambda) & T_1(\omega,\lambda) + I_n \end{bmatrix}$ *is a positive semidefinite matrix.*

Proof Note first that $W^* = W > I_n$, and hence $0 < W^{-1} < I_n$.

(i) Since

$$
\begin{bmatrix} T_1 & iT_2 \\ -iT_2 & T_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I_n & -iI_n \\ -iI_n & I_n \end{bmatrix}\begin{bmatrix} T_1 + T_2 & 0_n \\ 0_n & T_1 - T_2 \end{bmatrix}\begin{bmatrix} I_n & iI_n \\ iI_n & I_n \end{bmatrix}, \tag{4.45}
$$

it suffices to verify that $T_1 + T_2 < 0$ and $T_1 - T_2 < 0$. To check the first relation, note that

$$
T_1 + T_2 = \big(\widetilde{M}_\Gamma^+ - iI_n\big)\,W^{-1}\,\big((\widetilde{M}_\Gamma^+)^* + iI_n\big) - I_n\,. \tag{4.46}
$$

It is well-known that the eigenvalues of AB and BA agree when A and B are $n \times n$-matrices, and easy to deduce from this fact that also those of $AB - I_n$ and $BA - I_n$ agree. Therefore, the eigenvalues of $T_1 + T_2$ agree with those of the matrix $((\widetilde{M}_\Gamma^+)^* + iI_n)(\widetilde{M}_\Gamma^+ - iI_n)W^{-1} - I_n$, which is equal to $-2\,\mathrm{Im}\,\widetilde{M}_\Gamma^+\,W^{-1}$; and the eigenvalues of this last matrix agree with those of $-2W^{-1/2}\,\mathrm{Im}\,\widetilde{M}_\Gamma^+\,W^{-1/2}$, which is negative definite. This proves that the eigenvalues of the hermitian matrix $T_1 + T_2$ are strictly negative and hence so is the matrix itself. As for the matrix $T_1 - T_2$, it is immediate to check that

$$T_1 - T_2 = -\big(\widetilde{M}_\Gamma^+ + iI_n\big)\, W^{-1}\, \big((\widetilde{M}_\Gamma^+)^* - iI_n\big) + I_n\,. \tag{4.47}$$

The equality

$$\big((\widetilde{M}_\Gamma^+)^* - iI_n\big)\,(T_1 - T_2)\,\big(\widetilde{M}_\Gamma^+ + iI_n\big) = -2\,\mathrm{Im}\,\widetilde{M}_\Gamma^+ - 4\,\mathrm{Im}\,\widetilde{M}_\Gamma^+\, W^{-1}\,\mathrm{Im}\,\widetilde{M}_\Gamma^+$$

follows easily, and completes the proof of (i): the right-hand matrix is negative definite and, as checked in Proposition 1.21(i), $\widetilde{M}_\Gamma^+ + iI_n$ is nonsingular, since it has positive definite imaginary part.

(ii) Substitute T_1 by $T_1 + I_n$ in (4.45): this provides an alternative expression for the matrix in (ii) which proves that this assertion is equivalent to checking that $T_1 + I_n + T_2$ and $T_1 + I_n - T_2$ are positive semidefinite matrices. The equality (4.46) yields

$$T_1 + I_n + T_2 = \big(\widetilde{M}_\Gamma^+ - iI_n\big)\, W^{-1}\, \big((\widetilde{M}_\Gamma^+)^* + iI_n\big)\,,$$

so that $T_1 + I_n + T_2$ is positive semidefinite. And (4.47) yields

$$\begin{aligned}
&(\widetilde{M}_\Gamma^+)^*\,(T_1 + I_n - T_2)\,\widetilde{M}_\Gamma^+\\
&\quad= 2(\widetilde{M}_\Gamma^+)^*\widetilde{M}_\Gamma^+ - (\widetilde{M}_\Gamma^+)^*\,\big(\widetilde{M}_\Gamma^+ + iI_n\big)\, W^{-1}\,\big((\widetilde{M}_\Gamma^+)^* - iI_n\big)\,\widetilde{M}_\Gamma^+\\
&\quad= 2(\widetilde{M}_\Gamma^+)^*\widetilde{M}_\Gamma^+ - \big(W + i\,\big((\widetilde{M}_\Gamma^+)^* + iI_n\big)\big)\, W^{-1}\,\big(W - i\,\big(\widetilde{M}_\Gamma^+ - iI_n\big)\big)\\
&\quad= (\widetilde{M}_\Gamma^+)^*\widetilde{M}_\Gamma^+ + i\widetilde{M}_\Gamma^+ - i(\widetilde{M}_\Gamma^+)^* + I_n\\
&\qquad - \big((\widetilde{M}_\Gamma^+)^* + iI_n\big)\, W^{-1}\,\big(\widetilde{M}_\Gamma^+ - iI_n\big)\\
&\quad= \big((\widetilde{M}_\Gamma^+)^* + iI_n\big)\,(I_n - W^{-1})\,\big(\widetilde{M}_\Gamma^+ - iI_n\big)\,,
\end{aligned}$$

which is a also positive semidefinite matrix.

Lemma 4.24 *Under the hypotheses of Lemma 4.23,*

(i) *if* $\varepsilon > 0$, *then*

$$\beta_\Gamma(i\varepsilon) = \pm \int_\Omega \mathrm{tr}\big(\widetilde{H}_2(\omega)\,\mathrm{Re}\,\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon) + \varepsilon\widetilde{\Gamma}_3(\omega)\,\mathrm{Im}\,\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon)\big)\, dm_0\,;$$

(ii) $\displaystyle\lim_{\varepsilon\to 0+} \frac{\beta_\Gamma(i\varepsilon)}{\varepsilon} = \alpha'_\Gamma(0) = \frac{1}{2}\int_\Omega \mathrm{tr}\,\widetilde{\Gamma}(\omega)\, dm_0\,.$

Proof

(i) It follows from Theorem 3.30 and (4.41) that the Lyapunov index of the family of systems corresponding to $\lambda = i\varepsilon$ with $\varepsilon > 0$ is given by

$$\beta_\Gamma(i\varepsilon) = \mp \int_\Omega \operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re} M_\Gamma^\pm - \varepsilon \Gamma_3(\omega)\operatorname{Im} M_\Gamma^\pm\big)\, dm_0\,, \tag{4.48}$$

where $M^\pm$ are evaluated in $(\omega, i\varepsilon)$. Write, as before, $\widetilde{H} = \left[\begin{smallmatrix}\widetilde{H}_1 & -\widetilde{H}_2\\ \widetilde{H}_2 & \widetilde{H}_1\end{smallmatrix}\right]$ and $J^{-1}\widetilde{\Gamma} = J^{-1}\,(C_\Gamma^T)^{-1}\Gamma C_\Gamma^{-1} = \left[\begin{smallmatrix}\widetilde{\Gamma}_1 & \widetilde{\Gamma}_3\\ \widetilde{\Gamma}_2 & -\widetilde{\Gamma}_1^T\end{smallmatrix}\right]$ for $\omega \in \Omega_\Gamma$. It is easy to check that

$$\begin{aligned} H_3(\omega) &= -\operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\, \widetilde{H}_2(\omega)\operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\,,\\ \Gamma_3(\omega) &= \operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\, \widetilde{\Gamma}_3(\omega)\operatorname{Im}^{-1/2} N_\Gamma^+(\omega)\,, \end{aligned}$$

and the equality (4.42) implies that

$$\begin{aligned} \operatorname{Re} M_\Gamma^\pm(\omega, i\varepsilon) &= \operatorname{Im}^{1/2} N_\Gamma^+(\omega)\operatorname{Re}\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon)\ \operatorname{Im}^{1/2} N_\Gamma^+(\omega) + \operatorname{Re} N_\Gamma^+(\omega)\,,\\ \operatorname{Im} M_\Gamma^\pm(\omega, i\varepsilon) &= \operatorname{Im}^{1/2} N_\Gamma^+(\omega)\operatorname{Im}\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon)\ \operatorname{Im}^{1/2} N_\Gamma^+(\omega)\,. \end{aligned}$$

Therefore, for $\omega \in \Omega_\Gamma$,

$$\begin{aligned} &\operatorname{tr}\big(\Gamma_3(\omega)\operatorname{Im} M_\Gamma^\pm(\omega, i\varepsilon)\big) = \operatorname{tr}\big(\widetilde{\Gamma}_3(\omega)\operatorname{Im}\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon)\big)\,,\\ &\operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re} M_\Gamma^\pm(\omega, i\varepsilon)\big)\\ &\qquad = \operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re} N_\Gamma^+(\omega)\big) - \operatorname{tr}\big(\widetilde{H}_2(\omega)\operatorname{Re}\widetilde{M}_\Gamma^\pm(\omega, i\varepsilon)\big)\,. \end{aligned} \tag{4.49}$$

Recall that the function N_Γ^+ is a solution along the flow on Ω_Γ of (4.5). In particular,

$$\begin{aligned} (\operatorname{Im} N_\Gamma^+(\omega))' = &-\operatorname{Im} N_\Gamma^+(\omega)\,\big(H_1(\omega) + H_3(\omega)\operatorname{Re} N_\Gamma^+(\omega)\big)\\ &- \big(\operatorname{Re} N_\Gamma^+(\omega) H_3(\omega) + H_1^T(\omega)\big)\operatorname{Im} N_\Gamma^+(\omega)\,, \end{aligned}$$

and thus

$$(\det\operatorname{Im} N_\Gamma(\omega))' = -2\det\operatorname{Im} N_\Gamma^+(\omega)\operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re} N_\Gamma^+(\omega)\big)\,;$$

that is,

$$(\ln\det\operatorname{Im} N_\Gamma(\omega))' = -2\operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re} N_\Gamma^+(\omega)\big)\,.$$

It follows from Theorem 4.13(iv) that $\operatorname{Re} N_\Gamma^+ \in L^1(\Omega, m_0)$. Therefore, as explained in Proposition 1.36,

$$\int_\Omega \operatorname{tr}\big(H_1(\omega) + H_3(\omega) \operatorname{Re} N_\Gamma^+(\omega)\big)\, dm_0 = 0\,. \tag{4.50}$$

The substitution of (4.49) and (4.50) in (4.48) proves (i).

(ii) The equality (3.44) and the nondecreasing character of α_Γ (see e.g. Remark 3.33) imply that $0 \le \int_\mathbb{R} 1/(t^2+1)\, d\alpha_\Gamma(t) \le \operatorname{Im} w'_\Gamma(i)$. Under these conditions, the existence of $\alpha'_\Gamma(0)$, which is guaranteed by Theorem 4.19, ensures that $\lim_{\varepsilon\to 0^+} \operatorname{Im} w'_\Gamma(i\varepsilon) = \alpha'_\Gamma(0)$: see e.g. Section VI.B of [90]. This limiting behavior, together with the equality $\beta_\Gamma = -\operatorname{Im} w_\Gamma$, the Cauchy–Riemann equations for w_Γ, and the fact that $\beta_\Gamma(0) = 0$ (see Corollary 4.8), ensures that

$$\alpha'_\Gamma(0) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon \operatorname{Im} w'_\Gamma(is)\, ds = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \frac{\partial \beta_\Gamma(is)}{\partial s}\, ds = \lim_{\varepsilon\to 0^+}\frac{\beta_\Gamma(i\varepsilon)}{\varepsilon}\,,$$

which proves the first equality in (ii). The second equality is a trivial consequence of (4.37) and the definition of $\widetilde{\Gamma}$.

Remarks 4.25

1. Recall that a family (A_ε) of measurable matrix-valued functions converges in measure to a measurable matrix-valued function A as $\varepsilon \to 0$ (with respect to the fixed measure m_0) if, for all $\delta > 0$,

$$\lim_{\varepsilon\to 0} m_0\left(\{\omega \in \Omega \mid \|A_\varepsilon(\omega) - A(\omega)\| \ge \delta\}\right) = 0\,.$$

   Obviously the definition is independent of the choice of the matrix norm and equivalent to componentwise convergence in measure. And it is also clear (and well known) that the convergence in the $L^1(\Omega, m_0)$-topology implies the convergence in measure.
2. To understand the proof of the following result it is important to keep in mind that the (m_0) convergence in measure of A_ε to A as $\varepsilon \to 0^+$ for functions defined on Ω holds if and only if every sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of positive numbers with limit 0 admits a subsequence $(\varepsilon_{m_j})_{m_j\in\mathbb{N}}$ such that $\lim_{j\to\infty} A_{\varepsilon_{m_j}}(\omega) = A(\omega)$ m_0-a.e. The result is well known in the scalar case (see e.g. Exercise 11.45 of [58]), and can immediately be extended to the case of matrix-valued functions.

Theorem 4.26 *Suppose that Hypotheses 4.20 hold. Then*

$$\lim_{\varepsilon\to 0^+} M_\Gamma^\pm(\omega, i\varepsilon) = N_\Gamma^\pm(\omega)$$

in measure.

Proof Let Ω_Γ be the set mentioned in Remark 4.21. It can immediately be checked that the change of variables $\widetilde{\mathbf{z}} = C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$ takes the functions $N^\pm_\Gamma(\omega)$ for $\omega \in \Omega_\Gamma$ to the constant matrices $\pm iI_n$. It will be proved that

$$\lim_{\varepsilon\to 0^+} \widetilde{M}^\pm_\Gamma(\omega, i\varepsilon) = \pm iI_n \qquad \text{in measure,} \tag{4.51}$$

which, since $m_0(\Omega_\Gamma) = 1$, is equivalent to the assertion of the theorem: see Remark 4.25.2. As in Lemma 4.23, define $W(\omega,\lambda) = I_n + (\widetilde{M}^+_\Gamma)^*(\omega,\lambda)\,\widetilde{M}^+_\Gamma(\omega,\lambda)$ and take $\lambda = i\varepsilon$ with $\varepsilon > 0$. A straightforward computation taking the Riccati equation (4.43) as the starting point guarantees that

$$\begin{aligned}
&\operatorname{tr}\big(W'(\omega,i\varepsilon)\,W^{-1}(\omega,i\varepsilon)\big)\\
&\qquad = 2\operatorname{tr}\big(\widetilde{H}_2(\omega)\operatorname{Re}\widetilde{M}^+_\Gamma(\omega,i\varepsilon) + \varepsilon\widetilde{\Gamma}_3(\omega)\operatorname{Im}\widetilde{M}^+_\Gamma(\omega,i\varepsilon)\big)\\
&\qquad\quad + \varepsilon\operatorname{tr}\left(\widetilde{\Gamma}(\omega)\begin{bmatrix} T_1(\omega,i\varepsilon) & iT_2(\omega,i\varepsilon)\\ -iT_2(\omega,i\varepsilon) & T_1(\omega,i\varepsilon)\end{bmatrix}\right),
\end{aligned} \tag{4.52}$$

where T_1 and T_2 are defined by (4.44). The interested reader can find in the following steps a possible way to prove this equality. The matrix $\widetilde{M}^+_\Gamma(\omega,i\varepsilon)$ is represented by M, and the argument ω is omitted:

$$\begin{aligned}
&\operatorname{tr}\big(\widetilde{H}_1M^*MW^{-1} - M^*M\widetilde{H}_1W^{-1}\big)\\
&\qquad = \operatorname{tr}\big(\widetilde{H}_1(W-I_n)W^{-1} - \widetilde{H}_1W^{-1}(W-I_n)\big) = 0\,,\\
&\operatorname{tr}\big(\widetilde{\Gamma}_1^TM^*MW^{-1} - M^*M\widetilde{\Gamma}_1W^{-1}\big)\\
&\qquad = \operatorname{tr}\big(\widetilde{\Gamma}_1^T(W-I_n)W^{-1} - \widetilde{\Gamma}_1W^{-1}(W-I_n)\big)\\
&\qquad = \operatorname{tr}\big(-\widetilde{\Gamma}_1^TW^{-1} + \widetilde{\Gamma}_1W^{-1}\big)\,,\\
&\operatorname{tr}\big(\widetilde{H}_2MW^{-1} - M^*\widetilde{H}_2W^{-1} + M^*\widetilde{H}_2M^*MW^{-1} + M^*M\widetilde{H}_2W^{-1}\big)\\
&\qquad = \operatorname{tr}\big(W\widetilde{H}_2MW^{-1} + M^*H_2WW^{-1}\big) = 2\operatorname{tr}\big(\widetilde{H}_2\operatorname{Re}M\big)\,,\\
&\operatorname{tr}\big(M^*\widetilde{\Gamma}_3M^*MW^{-1} - M^*M\widetilde{\Gamma}_3MW^{-1}\big)\\
&\qquad = \operatorname{tr}\big(M^*\widetilde{\Gamma}_3 - M^*\widetilde{\Gamma}_3W^{-1} - \widetilde{\Gamma}_3M + \widetilde{\Gamma}_3MW^{-1}\big)\\
&\qquad = -2i\operatorname{tr}\big(\widetilde{\Gamma}_3\operatorname{Im}M\big) + \operatorname{tr}\big(-\widetilde{\Gamma}_3W^{-1}M^* + \widetilde{\Gamma}_3MW^{-1}\big)\,,
\end{aligned}$$

and finally

$$
\begin{aligned}
&i\varepsilon\,\mathrm{tr}\Big(\widetilde{\Gamma}_1 W^{-1} + \widetilde{\Gamma}_1 M W^{-1} M^* - \widetilde{\Gamma}_1^T W^{-1} - \widetilde{\Gamma}_1^T M W^{-1} M^* \\
&\qquad\qquad - \widetilde{\Gamma}_2 M W^{-1} + \widetilde{\Gamma}_2 W^{-1} M^* + \widetilde{\Gamma}_3 M W^{-1} - \widetilde{\Gamma}_3 W^{-1} M^*\Big) \\
&\qquad = \varepsilon\,\mathrm{tr}\left(\begin{bmatrix} -\widetilde{\Gamma}_2 & \widetilde{\Gamma}_1^T \\ \widetilde{\Gamma}_1 & \widetilde{\Gamma}_3 \end{bmatrix}\begin{bmatrix} T_1(\omega, i\varepsilon) & iT_2(\omega, i\varepsilon) \\ -iT_2(\omega, i\varepsilon) & T_1(\omega, i\varepsilon) \end{bmatrix}\right).
\end{aligned}
$$

The next step in the proof of Theorem 4.26 consists in checking that

$$
0 = 2\beta_\Gamma(i\varepsilon) + \varepsilon \int_\Omega \mathrm{tr}\left(\widetilde{\Gamma}^{1/2}(\omega)\begin{bmatrix} T_1(\omega, i\varepsilon) & iT_2(\omega, i\varepsilon) \\ -iT_2(\omega, i\varepsilon) & T_1(\omega, i\varepsilon) \end{bmatrix}\widetilde{\Gamma}^{1/2}(\omega)\right) dm_0\,. \tag{4.53}
$$

To this end, note that $\mathrm{tr}(W' W^{-1}) = (\ln\det W)'$, and use the information provided by Proposition 1.36 to deduce that $\int_\Omega \mathrm{tr}(W' W^{-1})\, dm_0 = 0$: Lemma 4.24(i) ensures that

$$
\beta_\Gamma(i\varepsilon) = \int_\Omega \mathrm{tr}\big(\widetilde{H}_2(\omega)\,\mathrm{Re}\,\widetilde{M}_\Gamma^+(\omega, i\varepsilon) + \varepsilon\widetilde{\Gamma}_3(\omega)\,\mathrm{Im}\,\widetilde{M}_\Gamma^+(\omega, i\varepsilon)\big)\, dm_0\,,
$$

so that in particular the function in the integrand belongs to $L^1(\Omega, m_0)$; and, as stated in Lemma 4.23(i), the third and final function in (4.52) is strictly negative, so that Proposition 1.4 ensures that there exists

$$
\lim_{t\to\infty} \frac{\varepsilon}{t} \int_0^t \mathrm{tr}\left(\widetilde{\Gamma}(\omega{\cdot}s)\begin{bmatrix} T_1(\omega{\cdot}s, i\varepsilon) & iT_2(\omega{\cdot}s, i\varepsilon) \\ -iT_2(\omega{\cdot}s, i\varepsilon) & T_1(\omega{\cdot}s, i\varepsilon) \end{bmatrix}\right) ds
$$

for m_0-a.e. $\omega \in \Omega$, and the limit lies in $[-\infty, 0]$.

The equality (4.53) and Lemma 4.24(ii) imply that

$$
\lim_{\varepsilon\to 0+} \int_\Omega \mathrm{tr}\left(\widetilde{\Gamma}^{1/2}(\omega)\begin{bmatrix} T_1(\omega, i\varepsilon) + I_n & iT_2(\omega, i\varepsilon) \\ -iT_2(\omega, i\varepsilon) & T_1(\omega, i\varepsilon) + I_n \end{bmatrix}\widetilde{\Gamma}^{1/2}(\omega)\right) dm_0 = 0\,.
$$

Lemma 4.23(ii) ensures that the matrix inside the integral is positive semidefinite. This, the previous assertion and the definition of $\|\cdot\|_F$ (see Remark 4.22) imply that

$$
\lim_{\varepsilon\to 0+} \begin{bmatrix} T_1(\omega, i\varepsilon) + I_n & iT_2(\omega, i\varepsilon) \\ -iT_2(\omega, i\varepsilon) & T_1(\omega, i\varepsilon) + I_n \end{bmatrix}^{1/2} \widetilde{\Gamma}^{1/2}(\omega) = 0_{2n}
$$

in the $L^2(\Omega, m_0)$-topology, and hence in measure. In addition, the measurable matrix function $\widetilde{\Gamma}$ is positive definite when $\Gamma > 0$, and takes the form $\left[\begin{smallmatrix} \widetilde{\Delta} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\widetilde{\Delta} > 0$ when $\Gamma = \left[\begin{smallmatrix} \Delta & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\Delta > 0$. In both cases, Remark 4.22.2 and the characterization given in Remark 4.25.2 yield $\lim_{\varepsilon\to 0+}(T_1(\omega,i\varepsilon)+I_n) = 0_n$ in measure. In addition, it is easy to check that

$$2(T_1 + I_n) = W^{-1}\,\big(W - 2i(\widetilde{M}_\Gamma^+)^*\big)\big(W + 2i\widetilde{M}_\Gamma^+\big)\,W^{-1} + \big(2W^{-1} - I_n\big)^2\,,$$

which implies that $\lim_{\varepsilon\to 0+}(W(\omega,i\varepsilon) + 2i\widetilde{M}_\Gamma^+(\omega,i\varepsilon)) = 0_n$ in measure. Since $\operatorname{Re}(W + 2i\widetilde{M}_\Gamma^+) = \operatorname{Re}^2\widetilde{M}_\Gamma^+ + (\operatorname{Im}\widetilde{M}_\Gamma^+ - I_n)^2$, one has that $\lim_{\varepsilon\to 0}\widetilde{M}_\Gamma^+(\omega,i\varepsilon) = iI_n$ in measure, as asserted in (4.51) for $\widetilde{M}_\Gamma^+$. The analogous result for $\widetilde{M}_\Gamma^-$ can be proved in a similar way.

Remark 4.27 According to the characterization given in Remark 4.25.2, the convergence in measure implies that the functions $N_\Gamma^\pm$ coincide with the vertical pointwise limits from the upper half-plane of the Weyl functions if these limits exist for m_0-a.e. $\omega\in\Omega$. And it also ensures that $\lim_{\varepsilon\to 0+}\operatorname{Im}^{-1}M_\Gamma^\pm(\omega,i\varepsilon) = \operatorname{Im}^{-1}N_\Gamma^\pm(\omega)$ in measure.

The following result establishes the L^1-convergence in the case of a positive definite perturbation Γ. This choice of Γ excludes the Schrödinger case, which will be analyzed separately.

Theorem 4.28 *Consider the general Hamiltonian case* (4.1)*. Suppose that Hypothesis 4.1 holds and that* $\Gamma > 0$*. Then,*

$$\lim_{\varepsilon\to 0+} M_\Gamma^\pm(\omega,i\varepsilon) = N_\Gamma^\pm(\omega)$$

in the $L^1(\Omega,m_0)$*-topology.*

Proof The Weyl functions $M_\Gamma^\pm(\omega,i\varepsilon)$ satisfy the Riccati equation (4.40) for $\lambda = i\varepsilon$. It is not difficult to check that

$$\begin{aligned}
&\operatorname{tr}\big((\operatorname{Im}M_\Gamma^+)'(\omega,i\varepsilon)\,\operatorname{Im}^{-1}M_\Gamma^+(\omega,i\varepsilon)\big)\\
&\qquad = -2\operatorname{tr}\big(H_1(\omega) + H_3(\omega)\operatorname{Re}M_\Gamma^+(\omega,i\varepsilon) - \varepsilon\Gamma_3(\omega)\operatorname{Im}M_\Gamma^+(\omega,i\varepsilon)\big)\\
&\qquad\quad - \varepsilon\operatorname{tr}\big((C_{\Gamma,\varepsilon}^{-1})^T(\omega)\Gamma(\omega)C_{\Gamma,\varepsilon}^{-1}(\omega)\big)\,,
\end{aligned}$$

where $C_{\Gamma,\varepsilon}$ is obtained substituting $N_\Gamma^+(\omega)$ by $M_\Gamma^+(\omega,i\varepsilon)$ in (4.27). The left-hand term agrees with the derivative of $\ln\det\operatorname{Im}M_\Gamma^+(\omega,i\varepsilon)$, and the right-hand term is a continuous function, so that it belongs to $L^1(\Omega,m_0)$. Proposition 1.36 and the representation (4.48) guarantee that

$$\frac{\beta_\Gamma(i\varepsilon)}{\varepsilon} = \frac{1}{2}\int_\Omega \operatorname{tr}\big((C_{\Gamma,\varepsilon}^{-1})^T(\omega)\,\Gamma(\omega)\,C_{\Gamma,\varepsilon}^{-1}(\omega)\big)\,dm_0\,. \tag{4.54}$$

Moreover, Lemma 4.24(ii) yields

$$\lim_{\varepsilon\to 0^+} \frac{\beta_\Gamma(i\varepsilon)}{\varepsilon} = \alpha'_\Gamma(0) = \frac{1}{2}\int_\Omega \mathrm{tr}\big((C_\Gamma^{-1})^T(\omega)\,\Gamma(\omega)\,C_\Gamma^{-1}(\omega)\big)\,dm_0\,, \tag{4.55}$$

where $C_\Gamma(\omega)$ is given by (4.27). Thus,

$$\lim_{\varepsilon\to 0^+}\int_\Omega \mathrm{tr}\big((C_{\Gamma,\varepsilon}^{-1})^T(\omega)\,\Gamma(\omega)\,C_{\Gamma,\varepsilon}^{-1}(\omega)\big)\,dm_0 = \int_\Omega \mathrm{tr}\big((C_\Gamma^{-1})^T(\omega)\,\Gamma(\omega)\,C_\Gamma^{-1}(\omega)\big)\,dm_0\,.$$

The definition of the norm $\|\cdot\|_F$ which was used to define the L^2-norm ensures that $\lim_{\varepsilon\to 0^+}\|\Gamma^{1/2}\,C_{\Gamma,\varepsilon}^{-1}\|_2 = \|\Gamma^{1/2}\,C_\Gamma^{-1}\|_2$. In addition, the convergence in measure of $M_\Gamma^+(\omega,i\varepsilon)$ to $N_\Gamma^+(\omega)$ as $\varepsilon\to 0^+$ and the continuity of Γ imply the convergence in measure of $\Gamma^{1/2}\,C_{\Gamma,\varepsilon}^{-1}$ to $\Gamma^{1/2}\,C_\Gamma^{-1}$ as $\varepsilon\to 0^+$. These two facts (see Remark 4.29 below) guarantee that $\lim_{\varepsilon\to 0^+}\Gamma^{1/2}(\omega)\,C_{\Gamma,\varepsilon}^{-1}(\omega) = \Gamma^{1/2}(\omega)\,C_\Gamma^{-1}(\omega)$ in the $L^2(\Omega,m_0)$-topology. The continuity and positivity of $\Gamma^{1/2}$ ensure that also $C_{\Gamma,\varepsilon}^{-1}(\omega)$ converges to $C_\Gamma^{-1}(\omega)$ in $L^2(\Omega,m_0)$ (see Remark 4.22.2), which in turn means that

$$\lim_{\varepsilon\to 0^+}\mathrm{Im}^{1/2}M_\Gamma^+(\omega,i\varepsilon) = \mathrm{Im}^{1/2}N_\Gamma^+(\omega)\,,$$

$$\lim_{\varepsilon\to 0^+}\mathrm{Im}^{-1/2}M_\Gamma^+(\omega,i\varepsilon) = \mathrm{Im}^{-1/2}N_\Gamma^+(\omega)\,,$$

$$\lim_{\varepsilon\to 0^+}\mathrm{Re}\,M_\Gamma^+(\omega,i\varepsilon)\,\mathrm{Im}^{-1/2}M_\Gamma^+(\omega,i\varepsilon) = \mathrm{Re}\,N_\Gamma^+(\omega)\,\mathrm{Im}^{-1/2}N_\Gamma^+(\omega)$$

in $L^2(\Omega,m_0)$. Finally, since $A+iB = (AB^{-1/2}+iB^{1/2})B^{1/2}$, one has that

$$\lim_{\varepsilon\to 0^+} M_\Gamma^+(\omega,i\varepsilon) = N_\Gamma^+(\omega)$$

in the $L^1(\Omega,m_0)$-topology (see again Remark 4.22.2), as claimed. The analogous result for M_Γ^- and N_Γ^- can be proved in a similar way.

Remark 4.29 Assume that: A and all the elements of the sequence (A_m) are real matrix-valued functions on Ω belonging to $L^p(\Omega,m_0)$ (for $p=1,2$); $\lim_{m\to\infty}\|A_m\|_{L^p} = \|A\|_{L^p}$; and $\lim_{m\to\infty}A_m(\omega) = A(\omega)$ m_0-a.e. Under these conditions, $\lim_{m\to\infty}A_m = A$ in the L^p topology. In order to prove this, fix any $\varepsilon>0$. The L^p-integrability of A provides a number $\delta>0$ such that $\int_{\widetilde{\Omega}}\|A\|^p\,dm_0 < \varepsilon$ whenever $m_0(\widetilde{\Omega}) < \delta$, and the Egorov theorem implies that there exists Ω_ε with $m_0(\Omega-\Omega_\varepsilon) < \delta$ such that (A_m) converges to A uniformly on Ω_ε. Therefore,

$\int_{\Omega_\varepsilon} \|A_m - A\|^p \, dm_0$ converges to 0 and $\int_{\Omega-\Omega_\varepsilon} \|A\|^p \, dm_0 < \varepsilon$. It follows easily from these facts and from the convergence of $\|A_m\|_p$ to $\|A\|_p$ that

$$\begin{aligned}\limsup_{m\to\infty} \int_\Omega \|A - A_m\|^p \, dm_0 &\le \limsup_{m\to\infty} \int_{\Omega-\Omega_\varepsilon} \|A - A_m\|^p \, dm_0 \\ &\le \int_{\Omega-\Omega_\varepsilon} \|A\|^p \, dm_0 + \limsup_{m\to\infty} \int_{\Omega-\Omega_\varepsilon} \|A_m\|^p \, dm_0 \\ &\le 2 \int_{\Omega-\Omega_\varepsilon} \|A\|^p \, dm_0 < 2\varepsilon \,,\end{aligned}$$

which proves the assertion. It is clear that the hypothesis $\lim_{m\to\infty} A_m(\omega) = A(\omega)$ m_0-a.e. can be replaced by $\lim_{m\to\infty} A_m = A$ in measure: just use the characterization given in Remark 4.25.2. And it is also clear that this result can be formulated for a family (A_ε) instead of for a sequence (A_m).

The rest of the section deals with the Schrödinger family (4.3) with perturbation $\Delta > 0$. The notations α_Γ, N_Γ, and $M_\Gamma^\pm$ have the same meaning as in the previous sections.

Proposition 4.30 *Consider the Schrödinger case* (4.3)*. Suppose that Hypothesis 4.1 holds and that* $\Delta > 0$*. Then,*

$$\begin{aligned}\lim_{\varepsilon\to 0+} \operatorname{Im} M_\Gamma^\pm(\omega, i\varepsilon) &= \operatorname{Im} N_\Gamma^\pm(\omega) \,, \\ \lim_{\varepsilon\to 0+} \operatorname{Im}^{-1} M_\Gamma^\pm(\omega, i\varepsilon) &= \operatorname{Im}^{-1} N_\Gamma^\pm(\omega) \,, \\ \lim_{\varepsilon\to 0+} \operatorname{Re} M_\Gamma^\pm(\omega, i\varepsilon) \operatorname{Im}^{-1} M_\Gamma^\pm(\omega, i\varepsilon) &= \operatorname{Re} N_\Gamma^\pm(\omega) \operatorname{Im}^{-1} N_\Gamma^\pm(\omega)\end{aligned}$$

in the $L^1(\Omega, m_0)$*-topology.*

Proof According to (4.41), the imaginary part of the Floquet coefficient of the family of systems (4.3) corresponding to $\lambda = i\varepsilon$ with $\varepsilon > 0$ is given by

$$\operatorname{Im} w_\Gamma(i\varepsilon) = \int_\Omega \operatorname{tr}\big(\operatorname{Im} M_\Gamma^+(\omega, i\varepsilon)\big) \, dm_0 \,. \tag{4.56}$$

In addition, as was proved in (4.39),

$$\alpha_\Gamma(0) = \int_\Omega \operatorname{tr}\big(\operatorname{Im} N_\Gamma^+(\omega)\big) \, dm_0 \,.$$

Thus, Theorem 3.32 guarantees that

$$\lim_{\varepsilon\to 0+} \int_\Omega \operatorname{tr}\big(\operatorname{Im} M_\Gamma^+(\omega, i\varepsilon)\big) \, dm_0 = \int_\Omega \operatorname{tr}\big(\operatorname{Im} N_\Gamma^+(\omega)\big) \, dm_0 \,. \tag{4.57}$$

On the other hand, the relations (4.54) and (4.55) corresponding to this case imply that

$$\lim_{\varepsilon\to 0^+}\int_\Omega \operatorname{tr}\big(\Delta(\omega)\,\mathrm{Im}^{-1}M_\Gamma^+(\omega,i\varepsilon)\big)\,dm_0=\int_\Omega \operatorname{tr}\big(\Delta(\omega)\,\mathrm{Im}^{-1}N_\Gamma^+(\omega)\big)\,dm_0\,. \tag{4.58}$$

Relations (4.57) and (4.58), the convergence in measure established in Theorem 4.26, Remark 4.29, the positivity and continuity of Δ, and Remark 4.22.2, all taken together, prove the convergence of $\mathrm{Im}\,M_\Gamma^+(\omega,i\varepsilon)$ to $\mathrm{Im}\,N_\Gamma^+(\omega)$ and of $\mathrm{Im}^{-1}M_\Gamma^+(\omega,i\varepsilon)$ to $\mathrm{Im}^{-1}N_\Gamma^+(\omega)$ in the $L^1(\Omega,m_0)$-topology, as $\varepsilon\to 0^+$.

Let Ω_Γ be the set appearing in Remark 4.21. Repeating the computation of the proof of Theorem 4.4(i) for this particular case proves that the change of variables $\widetilde{\mathbf{z}}=C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$ takes (4.1) for $\omega\in\Omega_\Gamma$ to

$$\widetilde{\mathbf{z}}'=\begin{bmatrix}\widetilde{H}_1(\omega{\cdot}t) & \mathrm{Im}\,N_\Gamma^+(\omega{\cdot}t)\\ \big(-\mathrm{Im}\,N_\Gamma^+-i\varepsilon\,\mathrm{Im}^{-1/2}N_\Gamma^+\Delta\,\mathrm{Im}^{-1/2}N_\Gamma^+\big)(\omega{\cdot}t) & \widetilde{H}_1(\omega{\cdot}t)\end{bmatrix}\widetilde{\mathbf{z}}, \tag{4.59}$$

with $\widetilde{H}_1=-\widetilde{H}_1^T$. In addition, the nonsingular functions $\widetilde{M}_\Gamma^+(\omega,i\varepsilon)$ defined by (4.42) are solutions along the flow on Ω_Γ of the corresponding Riccati equation (4.43); i.e.

$$\begin{aligned}\widetilde{M}'=&-\widetilde{M}\,\mathrm{Im}\,N_\Gamma^+(\omega{\cdot}t)\widetilde{M}-\widetilde{M}\widetilde{H}_1(\omega{\cdot}t)+\widetilde{H}_1(\omega{\cdot}t)\widetilde{M}\\ &-\mathrm{Im}\,N_\Gamma^+(\omega{\cdot}t)-i\varepsilon\,\mathrm{Im}^{-1/2}N_\Gamma^+(\omega{\cdot}t)\,\Delta(\omega{\cdot}t)\,\mathrm{Im}^{-1/2}N_\Gamma^+(\omega{\cdot}t)\,.\end{aligned}$$

It is immediate to determine from this equation those which are satisfied by $\mathrm{Re}\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)$ and $\mathrm{Im}\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)$, and not difficult to check that, if $\omega\in\Omega_\Gamma$, then

$$\begin{aligned}&\Big(\operatorname{tr}\big(\big(I_n+\mathrm{Re}^2\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)+\mathrm{Im}^2\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)\big)\,\mathrm{Im}^{-1}\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)\big)\Big)'\\ &\quad=\varepsilon\operatorname{tr}\Big(\big(I_n+\mathrm{Re}^2\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)\big)\,\mathrm{Im}^{-1}\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)\,\mathrm{Im}^{-1/2}N_\Gamma^+(\omega)\,\Delta(\omega)\\ &\qquad\cdot\mathrm{Im}^{-1/2}N_\Gamma^+(\omega)\,\mathrm{Im}^{-1}\,\widetilde{M}_\Gamma^+(\omega,i\varepsilon)-\mathrm{Im}^{-1/2}N_\Gamma^+(\omega)\,\Delta(\omega)\,\mathrm{Im}^{-1/2}N_\Gamma^+(\omega)\Big)\,.\end{aligned}$$

It follows from the continuity of $\widetilde{M}_\Gamma^+$ and Δ and from the integrability of $\mathrm{Im}^{-1/2}N_\Gamma^+$, which is ensured by Theorem 4.13(iv), that the right-hand term belongs to $L^1(\Omega,m_0)$. Therefore, Proposition 1.36 ensures that

$$\begin{aligned}&\int_\Omega\operatorname{tr}\big(\Delta\,\mathrm{Im}^{-1}N_\Gamma^+\big)\,dm_0\\ &\quad=\int_\Omega\operatorname{tr}\big(\Delta\,\mathrm{Im}^{-1/2}N_\Gamma^+\,\mathrm{Im}^{-1}\widetilde{M}_\Gamma^+\,\big(I_n+\mathrm{Re}^2\widetilde{M}_\Gamma^+\big)\,\mathrm{Im}^{-1}\widetilde{M}_\Gamma^+\,\mathrm{Im}^{-1/2}N_\Gamma^+\big)\,dm_0\,,\end{aligned}$$

where N_Γ and Δ have argument ω, and $\widetilde{M}_\Gamma^+$ has arguments $(\omega,i\varepsilon)$. Using the relation (4.42), this equality can be also expressed in terms of the function

$M_\Gamma^+(\omega, i\varepsilon)$ as follows:

$$\int_\Omega \operatorname{tr}\big(\Delta \operatorname{Im}^{-1} N_\Gamma^+\big)\, dm_0 = \int_\Omega \operatorname{tr}\big(\Delta \operatorname{Im}^{-1} M_\Gamma^+ \left(\operatorname{Im} N_\Gamma^+ + P_\Gamma^+\right) \operatorname{Im}^{-1} M_\Gamma^+\big)\, dm_0$$

where $P_\Gamma^+(\omega, i\varepsilon)$ is defined by

$$P_\Gamma^+ = \left(\operatorname{Re} M_\Gamma^+ - \operatorname{Re} N_\Gamma^+\right) \operatorname{Im}^{-1} N_\Gamma^+ \left(\operatorname{Re} M_\Gamma^+ - \operatorname{Re} N_\Gamma^+\right).$$

This equality can be rewritten as

$$\int_\Omega \operatorname{tr}\big(F_\varepsilon^T(\omega)\, C_\Gamma^T(\omega)\, C_\Gamma(\omega)\, F_\varepsilon(\omega)\big)\, dm_0 = \int_\Omega \operatorname{tr}\big(F^T(\omega)\, C_\Gamma^T(\omega)\, C_\Gamma(\omega)\, F(\omega)\big)\, dm_0\,,$$

with

$$F_\varepsilon(\omega) = \begin{bmatrix} \operatorname{Im}^{-1} M_\Gamma^+(\omega, i\varepsilon)\, \Delta^{1/2}(\omega) \\ \operatorname{Re} M_\Gamma^+(\omega, i\varepsilon) \operatorname{Im}^{-1} M_\Gamma^+(\omega, i\varepsilon)\, \Delta^{1/2}(\omega) \end{bmatrix},$$

$$F(\omega) = \begin{bmatrix} \operatorname{Im}^{-1} N_\Gamma^+(\omega)\, \Delta^{1/2}(\omega) \\ \operatorname{Re} N_\Gamma^+(\omega) \operatorname{Im}^{-1} N_\Gamma^+(\omega)\, \Delta^{1/2}(\omega) \end{bmatrix}.$$

Thus, the $L^2(\Omega, m_0)$-norm of the functions $C_\Gamma F_\varepsilon$ is independent of ε. This fact, the convergence in measure of $C_\Gamma F_\varepsilon$ to $C_\Gamma F$, and Remark 4.29, imply that

$$\lim_{\varepsilon \to 0+} C_\Gamma(\omega)\, F_\varepsilon(\omega) = C_\Gamma(\omega)\, F(\omega)$$

in the $L^2(\Omega, m_0)$-topology. Finally, since $C_\Gamma^{-1} \in L^2(\Omega, m_0)$ it follows that $F_\varepsilon(\omega) \to F(\omega)$ as $\varepsilon \to 0^+$ in the $L^1(\Omega, m_0)$-topology (see Remark 4.22.2), which implies that

$$\lim_{\varepsilon \to 0+} \operatorname{Re} M_\Gamma^+(\omega, i\varepsilon) \operatorname{Im}^{-1} M_\Gamma^+(\omega, i\varepsilon) = \operatorname{Re} N_\Gamma^+(\omega) \operatorname{Im}^{-1} N_\Gamma^+(\omega)$$

in the same topology. The proof of Proposition 4.30 is complete.

Theorem 4.31 *Consider the Schrödinger case* (3.3)*. Suppose that Hypothesis 4.1 holds and that $\Delta > 0$. Then,*

$$\lim_{\varepsilon \to 0+} M_\Gamma^\pm(\omega, i\varepsilon) = N_\Gamma^\pm(\omega)$$

in the $L^1(\Omega, m_0)$-topology.

Proof In what follows, M_Γ^+ is evaluated in $(\omega, i\varepsilon)$, G, Δ and N_Γ^+ are evaluated in ω, and convergence always means as $\varepsilon \to 0^+$.

The Riccati equation (4.40) for M_Γ^+ is now

$$M' = -M^2 + G(\omega{\cdot}t) - i\varepsilon\,\Delta(\omega{\cdot}t)\,, \tag{4.60}$$

while N_Γ^+ satisfies the same equation for $\varepsilon = 0$. It is easy to check that

$$\begin{aligned}
&\big(\mathrm{tr}\big(\mathrm{Im}^{-1} M_\Gamma^+\big)\big)' = 2\,\mathrm{tr}\big(\mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}\,M_\Gamma^+\big) + \varepsilon\,\mathrm{tr}\big(\Delta\,\mathrm{Im}^{-2} M_\Gamma^+\big)\,,\\
&\big(\mathrm{tr}\big(\mathrm{Im}\,M_\Gamma^+ + \mathrm{Re}\,M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}\,M_\Gamma^+\big)\big)'\\
&\quad = 2\,\mathrm{tr}\big(\mathrm{Re}\,M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+ G\big) + \varepsilon\,\mathrm{tr}\big(\Delta\,\big(-I_n + \mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}^2 M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+\big)\big)\,.
\end{aligned}$$

All the functions in the right-hand terms are continuous. Therefore, according to Proposition 1.36,

$$\begin{aligned}
&\varepsilon \int_\Omega \mathrm{tr}\big(\Delta\,\mathrm{Im}^{-2} M_\Gamma^+\big)\,dm_0 = -2 \int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}\,M_\Gamma^+\big)\,dm_0\,,\\
&\varepsilon \int_\Omega \mathrm{tr}\big(\Delta(-I_n + \mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}^2 M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+)\big)\,dm_0\\
&\qquad\qquad = -2 \int_\Omega \mathrm{tr}\big(\mathrm{Re}\,M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+ G\big)\,dm_0\,.
\end{aligned}$$

The same computations for $N_\Gamma^+(\omega)$, now with $\varepsilon = 0$, show that

$$\begin{aligned}
&\int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1} N_\Gamma^+\,\mathrm{Re}\,N_\Gamma^+\big)\,dm_0 = 0\,,\\
&\int_\Omega \mathrm{tr}\big(\mathrm{Re}\,N_\Gamma^+\,\mathrm{Im}^{-1} N_\Gamma^+ G\big)\,dm_0 = 0\,.
\end{aligned}$$

Proposition 4.30 implies that $\mathrm{tr}\big(\mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}\,M_\Gamma^+\big)$ and $\mathrm{tr}\big(\mathrm{Re}\,M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+ G\big)$ converge to $\mathrm{tr}\big(\mathrm{Im}^{-1} N_\Gamma\,\mathrm{Re}\,N_\Gamma^+\big)$ and $\mathrm{tr}\big(\mathrm{Re}\,N_\Gamma^+\,\mathrm{Im}^{-1} N_\Gamma^+ G\big)$ in $L^1(\Omega, m_0)$. Therefore, the previous four equalities yield

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \varepsilon \int_\Omega \mathrm{tr}\big(\Delta\,\mathrm{Im}^{-2} M_\Gamma^+\big)\,dm_0 = 0\,,\\
&\lim_{\varepsilon\to 0^+} \varepsilon \int_\Omega \mathrm{tr}\big(\Delta\,\mathrm{Im}^{-1} M_\Gamma^+\,\mathrm{Re}^2 M_\Gamma^+\,\mathrm{Im}^{-1} M_\Gamma^+\big)\,dm_0 = 0\,;
\end{aligned}$$

or, in other words, keeping in mind that $\|A\|_2 = \|A^T\|_2$ (see property F1 in Remark 1.24.3),

$$\lim_{\varepsilon\to 0+} \varepsilon^{1/2} \left\|\mathrm{Im}^{-1} M_\Gamma^+ \Delta^{1/2}\right\|_2 = 0\,,$$
$$\lim_{\varepsilon\to 0+} \varepsilon^{1/2} \left\|\Delta^{1/2}\, \mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\right\|_2 = 0\,.$$

These equalities, Property F2 in Remark 1.24.3, and the Cauchy–Schwarz inequality ensure that

$$\begin{aligned} \lim_{\varepsilon\to 0+} &\left|\varepsilon \int \mathrm{tr}\big(\mathrm{Im}^{-1} M_\Gamma^+ \,\Delta\, \mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\big)\right| \\ &\le \lim_{\varepsilon\to 0+} \varepsilon \left\|\mathrm{Im}^{-1} M_\Gamma^+ \Delta^{1/2}\right\|_2 \left\|\Delta^{1/2}\, \mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\right\|_2 = 0\,. \end{aligned} \tag{4.61}$$

The Riccati equations (4.60) for $\varepsilon > 0$ and $\varepsilon = 0$ also yield

$$\begin{aligned} \big(\mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\big)' &= \mathrm{Re}\, M_\Gamma^+ \,\mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+ + \mathrm{Im}\, M_\Gamma^+ + \mathrm{Im}^{-1} M_\Gamma^+ G \\ &\quad + \varepsilon\, \mathrm{Im}^{-1} M_\Gamma^+ \Delta\, \mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\,, \\ \big(\mathrm{Im}^{-1} N_\Gamma^+ \,\mathrm{Re}\, N_\Gamma^+\big)' &= \mathrm{Re}\, N_\Gamma^+ \,\mathrm{Im}^{-1} N_\Gamma^+ \,\mathrm{Re}\, N_\Gamma^+ + \mathrm{Im}\, N_\Gamma^+ + \mathrm{Im}^{-1} N_\Gamma^+ G\,, \end{aligned}$$

which together with Proposition 1.36 ensures that

$$\begin{aligned} 0 &= \int_\Omega \mathrm{tr}\Big(\mathrm{Re}\, M_\Gamma^+ \,\mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+ + \mathrm{Im}\, M_\Gamma^+ + \mathrm{Im}^{-1} M_\Gamma^+ G \\ &\qquad + \varepsilon\, \mathrm{Im}^{-1} M_\Gamma^+ \Delta\, \mathrm{Im}^{-1} M_\Gamma^+ \,\mathrm{Re}\, M_\Gamma^+\Big)\, dm_0\,, \\ 0 &= \int_\Omega \mathrm{tr}\big(\mathrm{Re}\, N_\Gamma^+ \,\mathrm{Im}^{-1} N_\Gamma^+ \,\mathrm{Re}\, N_\Gamma^+ + \mathrm{Im}\, N_\Gamma^+ + \mathrm{Im}^{-1} N_\Gamma^+ G\big)\, dm_0\,, \end{aligned}$$

Property (4.61) and the L^1-convergence established in Proposition 4.30 together with the continuity of G yield

$$\begin{aligned} \lim_{\varepsilon\to 0+} &\int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1/2} M_\Gamma^+ \,\mathrm{Re}^2 M_\Gamma^+ \,\mathrm{Im}^{-1/2} M_\Gamma^+\big)\, dm_0 \\ &= \int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1/2} N_\Gamma^+ \,\mathrm{Re}^2 N_\Gamma^+ \,\mathrm{Im}^{-1/2} N_\Gamma^+\big)\, dm_0\,. \end{aligned}$$

According to Remark 4.29, this fact, together with the convergence in measure and the definition of $\|\cdot\|_F$, ensures that $\mathrm{Re}\, M_\Gamma^+ \,\mathrm{Im}^{-1/2} M_\Gamma^+$ converges to $\mathrm{Re}\, N_\Gamma^+ \,\mathrm{Im}^{-1/2} N_\Gamma^+$ in the $L^2(\Omega, m_0)$-topology. The proof of Theorem 4.31 is completed as was that of Theorem 4.28.

Note that these two results and equality (4.41) also provide an ergodic representation for the Lyapunov index $\beta_\Gamma(0)$ if $\Gamma > 0$ or $\Delta > 0$, namely

$$\beta_\Gamma(\lambda) = \mp \int_\Omega \operatorname{tr}\bigl(H_1(\omega) + H_3(\omega)\operatorname{Re} N_\Gamma^\pm(\omega)\bigr)\, dm_0\,. \tag{4.62}$$

The arguments used in this section can be repeated to prove a final result, which concerns the vertical convergence from the lower half-plane of the Weyl functions. Note the fundamental difference in the value of the limits.

Theorem 4.32 *Suppose that Hypotheses 4.20 hold. Then,*

$$\lim_{\varepsilon\to 0^-} M_\Gamma^\pm(\omega, i\varepsilon) = N_\Gamma^\mp(\omega)$$

in the $L^1(\Omega, m_0)$-topology.

Remark 4.33 Denote by μ both the volume form on $\mathcal{L}_\mathbb{R}$ and the induced measure on the σ-algebra of Borel sets, and by m_1 the complete product measure $m_0 \otimes \mu$ on the corresponding σ-algebra of $\mathcal{K}_\mathbb{R}$. The matrices C_Γ play a fundamental role in the proof of the occurrence of absolutely continuous dynamics for the systems (4.1) for $\Gamma > 0$ and (4.3) for $\Delta > 0$; i.e. of the existence of a τ-invariant measure on $\mathcal{K}_\mathbb{R}$ which is absolutely continuous with respect to m_1. The details of this assertion (which in fact involves a wider class of systems), as well as an explicit representation of the density function of such a measure in terms of C_Γ, can be found in Novo and Núñez in [111].

4.4 An Extension of the Kotani Theory

In this section, as was the case in the previous one, only positive definite perturbations $\Gamma > 0$ in (4.1) and $\Delta > 0$ in (4.3) will be considered. On the other hand, Hypothesis 4.1 is not initially imposed. Define

$$\mathcal{A}_\Gamma = \{\lambda \in \mathbb{R} \mid \beta_\Gamma(\lambda) = 0\}\,,$$

where $\beta_\Gamma(\lambda)$ represents the Lyapunov index of (4.1) with respect to the fixed σ-ergodic measure m_0. Kotani's theory for n-dimensional Schrödinger equations (with perturbation $\Delta = I_n$) and linear Hamiltonian systems (with perturbation $\Gamma = I_{2n}$) can be found in Kotani and Simon [91] and Sun [145] respectively. This theory allows one to identify $\mathcal{A}_\Gamma$ with the essential support of the absolutely continuous spectrum of multiplicity $2n$ of the associated operators. A straightforward generalization leads to the following result. Remarks 3.31.1 and 3.31.2 contain the definition and basic properties of $M_\Gamma^\pm(\omega, \lambda_0)$ for $\lambda_0 \in \mathbb{R}$. Recall also the information provided by Proposition 1.5(i).

Theorem 4.34 *If $\Gamma > 0$ or if, in the Schrödinger case, $\Delta > 0$, then there exists a subset $\mathcal{A}_{\Gamma,1} \subseteq \mathcal{A}_\Gamma$, with the same Lebesgue measure as $\mathcal{A}_\Gamma$, such that, for $\lambda_0 \in \mathcal{A}_{\Gamma,1}$,*

(i) *there exists a σ-invariant subset $\Omega_{\lambda_0} \subseteq \Omega$ with $m_0(\Omega_{\lambda_0}) = 1$ such that the limits $M_\Gamma^\pm(\omega, \lambda_0)$ exist and satisfy $\pm \operatorname{Im} M_\Gamma^\pm(\omega, \lambda_0) > 0$ for every $\omega \in \Omega_{\lambda_0}$, and the functions $\Omega_{\lambda_0} \to \mathbb{S}_n^+(\mathbb{C})$, $\omega \to \pm M_\Gamma^\pm(\omega, \lambda_0)$ can be extended to measurable functions on Ω.*

(ii) *The three matrix-valued functions $\operatorname{Im} M_\Gamma^\pm(\omega, \lambda_0)$, $\operatorname{Im}^{-1} M_\Gamma^\pm(\omega, \lambda_0)$, and $\operatorname{Re} M_\Gamma^\pm(\omega, \lambda_0) \operatorname{Im}^{-1} M_\Gamma^\pm(\omega, \lambda_0) \operatorname{Re} M_\Gamma^\pm(\omega, \lambda_0)$, belong to $L^1(\Omega, m_0)$.*

It is known that, given a general recurrent linear system with bounded solutions, there is a continuous change of variables taking it into skew-symmetric form (see e.g. Ellis and Johnson [42] and Cameron [24], and recall that the recurrence of the system means that the flow on its hull is minimal). In the case that the assumptions of recurrence and of boundedness of solutions do not hold, it is still possible to give the explicit expression of a measurable and symplectic change of variables taking the family of linear Hamiltonian systems (4.1) for $\lambda = \lambda_0 \in \mathcal{A}_{\Gamma,1}$ into skew-symmetric form and preserving its rotation number and Lyapunov index. The proof of this assertion is basically contained in that of Theorem 4.15. To verify it directly, define the real matrix-valued function

$$P_{\Gamma,\lambda_0}(\omega) = \begin{bmatrix} \operatorname{Im}^{1/2} M_\Gamma^+(\omega, \lambda_0) & 0 \\ -\operatorname{Im}^{-1/2} M_\Gamma^+(\omega, \lambda_0) \operatorname{Re} M_\Gamma^+(\omega, \lambda_0) & \operatorname{Im}^{-1/2} M_\Gamma^+(\omega, \lambda_0) \end{bmatrix},$$

which, as a consequence of Theorem 4.34, is nonsingular and belongs to $L^2(\Omega, m_0)$. It can immediately be checked that P_{Γ,λ_0} is symplectic. Theorem 4.4(i), Remark 4.5.2, and Propositions 4.6 and 4.7 prove the indicated properties for the change of variables defined by $\widetilde{\mathbf{z}} = P_{\Gamma,\lambda_0}(\omega{\cdot}t)\,\mathbf{z}$ for $\omega \in \Omega_{\lambda_0}$.

In addition, since $M_\Gamma^+(\omega, \lambda_0)$ is a solution along the flow with positive definite imaginary part of the Riccati equation (4.40) for $\lambda = \lambda_0$, Lemma 4.2 proves that $Z_{\Gamma,\lambda_0} = P_{\Gamma,\lambda_0}^T P_{\Gamma,\lambda_0}$ is a (positive and symplectic) solution along the flow on Ω_{λ_0} of the equation

$$Z' = -\big(H(\omega{\cdot}t) + \lambda_0 J^{-1}\Gamma(\omega{\cdot}t)\big)^T Z - Z\,\big(H(\omega{\cdot}t) + \lambda_0 J^{-1}\Gamma(\omega{\cdot}t)\big)\,.$$

These considerations together with the amplified Kotani theory provided by Theorem 4.34 show that, when the Lyapunov index of (4.1) vanishes for all λ in a set $\mathcal{A} \subseteq \mathbb{R}$ of positive Lebesgue measure, then the Hypothesis 4.1 imposed in the previous sections of the chapter holds for the systems (4.1) corresponding to Lebesgue-a.e. $\lambda \in \mathcal{A}$. Recall also that Corollary 4.8 shows the converse: Hypothesis 4.1 for a given family ensures that the Lyapunov index vanishes.

In particular, the results of Sect. 4.3 hold with H replaced by $H + \lambda_0 J^{-1}\Gamma$ for $\lambda_0 \in \mathcal{A}_{\Gamma,1}$. This fact has the following immediate consequence, which adds some information to that provided by Theorem 4.34.

Theorem 4.35 *Suppose that $\Gamma > 0$ or that, in the Schrödinger case, $\Delta > 0$. Suppose also that λ_0 belongs to the set $\mathcal{A}_{\Gamma,1}$ defined in Theorem 4.34. Then the matrix-valued function $\omega \mapsto M_\Gamma^\pm(\omega,\lambda_0)$ belong to $L^1(\Omega,m_0)$, and $\lim_{\varepsilon\to 0^+} M_\Gamma^\pm(\omega,\lambda_0+i\varepsilon) = M_\Gamma^\pm(\omega,\lambda_0) = \lim_{\varepsilon\to 0^-} M_\Gamma^\mp(\omega,\lambda_0+i\varepsilon)$ in the $L^1(\Omega,m_0)$-topology.*

Consider again the family

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \mu\, J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega\in\Omega \tag{4.63}$$

under the hypothesis Γ imposed on Γ in this section: $\Gamma > 0$ or $\Delta > 0$ if the family comes from a Schrödinger equation. Take μ_0 in the corresponding set $\mathcal{A}_{\Gamma,1}$ determined by Theorem 4.34, and note that the family can be written as

$$\mathbf{z}' = \big(H_{\mu_0}(\omega{\cdot}t) + \lambda\, J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega\in\Omega \tag{4.64}$$

for $H_{\mu_0} = H + \mu_0 J^{-1}\Gamma$ and $\lambda = \mu-\mu_0$. Theorem 4.35 states that the function $M_\Gamma^+(\omega,\mu_0)$ corresponding to (4.63) is the limit in the $L^1(\Omega,m_0)$-topology of the functions $M_\Gamma^+(\omega,\mu_0+i\varepsilon)$ as $\varepsilon\to 0^+$. As explained in Remark 4.27, this fact ensures that $M_\Gamma^+(\omega,\mu_0) = N_{\Gamma,\mu_0}^+(\omega)$ for m_0-a.e. $\omega\in\Omega$, where N_{Γ,μ_0}^+ is the function determined by Theorem 4.13 for the family (4.64). Consequently, the ergodic representations (4.62), (4.39), and (4.37) for the Lyapunov index $\beta_\Gamma(\mu)$ and the rotation number $\alpha_\Gamma(\mu)$ of (4.63) (which of course agree with those of (4.64)) obtained in the previous section and expressed in terms of $N_{\Gamma,\mu_0}^+(\omega)$ can be now rewritten in terms of $M_\Gamma^+(\omega,\mu_0)$.

This property has an interesting consequence in the particular case of the family of n-dimensional Schrödinger equations

$$-\,\mathbf{z}'' + G(\omega{\cdot}t)\,\mathbf{z} = \mu\,\mathbf{z}\,, \quad \omega\in\Omega\,, \tag{4.65}$$

which are perturbed in the direction of $\Delta = I_n$. Set $\Gamma_0 = \left[\begin{smallmatrix} I_n & 0\\ 0 & 0\end{smallmatrix}\right]$, and take $\mu_0\in\mathcal{A}_{\Gamma_0,1}$. Then relations (4.39) and (4.37) imply that $\alpha_{\Gamma_0}(\mu_0) = \int_\Omega \mathrm{tr}\big(\mathrm{Im}\, M_{\Gamma_0}^+(\omega,\mu_0)\big)\,dm_0$ and $\alpha'_{\Gamma_0}(\mu_0) = (1/2)\int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1} M_{\Gamma_0}^+(\omega,\mu_0)\big)\,dm_0$. Therefore,

$$\begin{aligned} 2\,\alpha_{\Gamma_0}(\mu_0)\,\alpha'_{\Gamma_0}(\mu_0) &= \int_\Omega \mathrm{tr}\big(\mathrm{Im}\, M_{\Gamma_0}^+(\omega,\mu_0)\big)\,dm_0 \int_\Omega \mathrm{tr}\big(\mathrm{Im}^{-1} M_{\Gamma_0}^+(\omega,\mu_0)\big)\,dm_0\\ &\ge \left(\int_\Omega \Big(\mathrm{tr}\big(\mathrm{Im}\, M_{\Gamma_0}^+(\omega,\mu_0)\big)\,\mathrm{tr}\big(\mathrm{Im}^{-1} M_{\Gamma_0}^+(\omega,\mu_0)\big)\Big)^{1/2} dm_0\right)^2 . \end{aligned}$$

Note that, if $\mu_1,\ldots,\mu_n$ are the eigenvalues of a positive definite real $n\times n$ matrix A, then

$$\begin{aligned} \mathrm{tr}(A)\,\mathrm{tr}(A^{-1}) &= (\mu_1+\cdots+\mu_n)(1/\mu_1+\cdots+1/\mu_n)\\ &= n + \sum_{i\ne j}(\mu_i/\mu_j+\mu_j/\mu_i) \ge n + 2n(n-1)/2 = n^2, \end{aligned}$$

since the number of pairs of elements of a set of n elements is $n(n-1)/2$ and $x + 1/x \geq 2$ if $x > 0$. Therefore,

Proposition 4.36 *With the preceding notation, if* $\mu_0 \in \mathcal{A}_{\Gamma_0,1}$*, then*

$$2\,\alpha_{\Gamma_0}(\mu_0)\,\alpha'_{\Gamma_0}(\mu_0) \geq n^2.$$

This extends the well-known inequality for the one-dimensional Schrödinger equation which states that $2\,\alpha(\mu)\,\alpha'(\mu) \geq 1$ at Lebesgue-a.e. $\mu \in \mathbb{R}$ with null Lyapunov index, which was established by Moser in [108] and by Deift and Simon in [37].

4.5 Uniform Convergence of the Weyl Functions in the Case of Bounded Solutions

This section is devoted to proving the uniform convergence of the Weyl functions of the systems (4.1) and to analyzing the variation of the corresponding Sacker–Sell spectral decomposition (see [133] and Sect. 1.4.4). This will be done under the following conditions:

Hypotheses 4.37 The base flow (Ω, σ) is minimal and all the solutions of the unperturbed linear Hamiltonian systems of the family (4.2) are bounded for every $\omega \in \Omega$.

Throughout the whole of Sect. 4.5, including its various subsections, $\|\cdot\|$ will represent the Euclidean vector and matrix norms: see Remark 1.24.2. The results are independent of this particular choice of norms.

Remark 4.38 The boundedness of all the solutions of the unperturbed family (4.2) guarantees the existence of constants c_1 and c_2 such that $0 < c_1 \leq \|U(t,\omega)\| \leq c_2$ and $c_1 \leq \|U^{-1}(t,\omega)\| \leq c_2$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. In fact, the existence of a constant c_2 such that $\|U(t,\omega)\| \leq c_2$ follows from the compactness of Ω and the linearity of the system, and this together with the symplectic character of U ensures that $\|U^{-1}(t,\omega)\| = \|J^{-1}U^T(t,\omega)J\| \leq \|U(t,\omega)\| \leq c_2$; moreover, the other inequalities are satisfied by $c_1 = 1/c_2$, since $1 \leq \|U\|\|U^{-1}\|$.

The following fundamental result has already been mentioned in the previous section. It is proved in Theorem 2.3 and Remark 2.4 of Ellis and Johnson [42] in the general case. The analogous property for the almost-periodic case had been previously proved by Cameron [24].

Theorem 4.39 *Suppose that Hypotheses 4.37 hold. Then, there exists a continuous map* $C\colon \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ *taking values on the set of nonsingular matrices, which is differentiable along the flow on* Ω*, and such that the change of variables* $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ *takes the family* (4.2) *into skew-symmetric form.*

Now, in spite of this simplification, the ergodic and topological structures of the corresponding flow on the Lagrange bundle are far from being completely classified in the higher dimension case, and numerous aspects of its dynamics remain unclear.

Nevertheless, in the two-dimensional case, a complete classification of recurrent linear systems is known and, taking it as the starting point, different properties of the solutions have been described (see Novo and Obaya [114, 115] and references therein). In particular, Núñez and Obaya [116] develop a one-parameter perturbation theory of these two-dimensional linear Hamiltonian systems, which is based on the connection between the ergodic and topological structures of the Lagrange (i.e. projective) bundle. This theory extends part of the classical perturbative results of Moser and Pöschel [109] (see also Eliasson [40]) for the quasi-periodic Schrödinger equation to a more general setting.

The section presents and completes the results of Fabbri et al. [49], which extend these two-dimensional properties to the higher dimension case, establishing conditions on the perturbation directions Γ of the initial (elliptic) family which suffice to ensure (a) the exponential dichotomy of the perturbed systems, and (b) the uniform convergence as the parameter goes to zero of the Sacker–Sell spectral decomposition. This *uniform convergence of closed subbundles* means convergence of the fibers (on the suitable Grassmannian manifold), with the additional requirement of the uniformity of the convergence with respect to the base space Ω. In addition, the limiting behavior of the corresponding closed subbundles as the perturbation parameter goes to zero is analyzed: it is proved that the limits are uniform over the base and can be determined *a priori* from the initial non-perturbed system. These results have a direct application to the study of the measurable and topological structures of the phase space of the systems under consideration: each perturbation direction determines a pair of maps from Ω to $\mathcal{L}_{\mathbb{C}}$ whose graphs determine invariant sets of the unperturbed flow. The point is that an understanding of these graphs contributes to an understanding of the global ergodic and topological structures of this flow, along the line of the results of Novo and Obaya [114] and Arnold et al. [6].

Remark 4.40 Hypotheses 4.37 are stronger than Hypothesis 4.1: the continuous matrix-valued function C of Theorem 4.39 satisfies the conditions of Theorem 4.4(ii), so that the continuous map $Z = C^T C$ satisfies the properties required in Hypothesis 4.1. In particular, Theorem 4.9 applies in this situation, so that definition (4.14) associates an $L^1(\Omega, m_0)$-function A_Γ to each continuous Γ. This information is crucial in what follows.

In fact, Hypotheses 4.37 will not be the only ones needed for the perturbation analysis. The results concerning the uniform variation with respect to λ of the Sacker–Sell spectral decomposition of (4.1) will also require hypotheses on the perturbation direction Γ stronger than those of the previous sections: more specifically, Γ will be required to belong to the set $\mathcal{C}$ now defined.

Definition 4.41 A continuous function $\Gamma\colon \Omega \to \mathbb{S}_{2n}(\mathbb{R})$ *belongs to the set* $\mathcal{C}$ if the function A_Γ determined by the limit (4.14) exists for every $\omega \in \Omega$.

The following result ensures that $\mathcal{C}$ is nonempty, which is not *a priori* obvious, and that it is closed in the set $C(\Omega, \mathbb{S}_{2n}(\mathbb{R}))$ of the continuous symmetric $2n \times 2n$ matrix-valued functions on Ω endowed with the uniform topology, given by the norm $\|\Gamma\|_\Omega = \sup_{\omega\in\Omega} \|\Gamma(\omega)\|$.

Proposition 4.42 *Suppose that there exists a continuous matrix-valued map $C\colon \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ taking values in the set of nonsingular matrices, which is differentiable along the flow on Ω, and such that the change of variables $\widetilde{\mathbf{z}} = C(\omega{\cdot}t)\,\mathbf{z}$ transforms the family of systems* (4.2) *into skew-symmetric form* (4.10). *Then,*

(i) *the matrix-valued function $\Gamma_C = C^T C$ belongs to $\mathcal{C}$, and $A_{\Gamma_C} = \Gamma_C > 0$. In particular, if Hypotheses 4.37 hold, then $\mathcal{C}$ is nonempty.*
(ii) *If all the solutions of all the unperturbed systems* (4.2) *are bounded, then the set $\mathcal{C}$ is a closed linear subspace of $C(\Omega, \mathbb{S}_{2n}(\mathbb{R}))$ endowed with the uniform topology.*

Proof

(i) Theorem 4.4(ii) implies that the continuous positive definite matrix-valued function $\Gamma_C = C^T C$ solves (4.4) along the flow on Ω. Consequently, $U^T(t,\omega)\,\Gamma_C(\omega{\cdot}t)\,U(t,\omega) = \Gamma_C(\omega)$, and hence the limit A_{Γ_C} agrees with Γ_C. In particular, A_{Γ_C} exists everywhere, so that $\Gamma_C \in \mathcal{C}$. (Note also that the continuous symplectic matrix-valued function $B_{\Gamma_C} = C^T_{\Gamma_C} C_{\Gamma_C}$, defined from $A_{\Gamma_C} = \Gamma_C = C^T C$ as in Theorem 4.13, also belongs to $\mathcal{C}$.) The last assertion in (i) follows from Theorem 4.39.
(ii) It is obvious that $\mathcal{C}$ is a linear subspace of $C(\Omega, \mathbb{S}_{2n}(\mathbb{R}))$. In order to check that it is a closed subspace, take a sequence (Γ_m) in $\mathcal{C}$ converging to a continuous matrix-valued function Γ. Remark 4.38 has two consequences: first, there exists $c > 0$ such that $\|U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\| \le c$ for all $(s,\omega) \in \mathbb{R}\times\Omega$; and second, given $\varepsilon > 0$, there exists m_0 such that $\|U^T(s,\omega)\,(\Gamma(\omega{\cdot}s) - \Gamma_m(\omega{\cdot}s))\,U(s,\omega)\| \le \varepsilon$ for all $(s,\omega) \in \mathbb{R}\times\Omega$ if $m \ge m_0$. Fix $\omega \in \Omega$ and choose two sequences $(t_k^1) \uparrow \infty$ and $(t_k^2) \uparrow \infty$ such that there exist

$$\widetilde{A}^j_\Gamma(\omega) = \lim_{k\to\infty} \frac{1}{2t_k^j} \int_{-t_k^j}^{t_k^j} U^T(s,\omega)\,\Gamma(\omega{\cdot}s)\,U(s,\omega)\,ds$$

for $j = 1, 2$. It can immediately be checked that

$$\widetilde{A}^j_\Gamma(\omega) = A_{\Gamma_m}(\omega) + \lim_{k\to\infty} \frac{1}{2t_k^j} \int_{-t_k^j}^{t_k^j} U^T(s,\omega)\,(\Gamma(\omega{\cdot}s) - \Gamma_m(\omega))\,U(s,\omega)\,ds$$

for $j = 1, 2$, and it is easy to deduce from this equality and the previous bound that $\|\widetilde{A}^1_\Gamma(\omega) - \widetilde{A}^2_\Gamma(\omega)\| \le 2\varepsilon$. This fact and an immediate argument by contradiction imply the existence of the limit $A_\Gamma(\omega)$, that is, the global existence of A_Γ.

The structure and properties of the set $\mathcal{C}$ are better known in the two-dimensional case (see [116]). In this situation, there coexist cases in which any continuous Γ belongs to $\mathcal{C}$ with other situations in which this is not true.

Proposition 4.43 *Suppose that Hypotheses 4.37 hold. Let a measurable map $Z: \Omega \to \mathbb{S}_{2n}(\mathbb{R})$ be a solution along the flow on Ω of (4.4). If Z has a continuity point, then Z is continuous on Ω. In particular, if $\Gamma \in \mathcal{C}$, then the function A_Γ defined by (4.14) is continuous on Ω.*

Proof Some ideas used in the proof of Theorem 3.1 of Furstenberg [52] will be used. It is possible to repeat step by step the proof of Theorem 4.9, whose notation is maintained here, taking as the starting point the continuous change of variables provided by Theorem 4.39. Note that now the function V^1 and the flow on Ω^1 are continuous. Given any $\omega \in \Omega$, represent $\omega^1 = \big(\omega, \big[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\big]\big) \in \Omega^1$. Take any pair $(t, \omega) \in \mathbb{R} \times \Omega$. According to (4.22), $U(t, \omega) = V^1(\omega^1{\cdot}t)\,(V^1)^{-1}(\omega^1)$; and (4.4) ensures that $Z(\omega) = U^T(t, \omega)\, Z(\omega{\cdot}t)\, U(t, \omega)$. These two equalities yield

$$\widetilde{Z}(\omega) = (V^1)^T(\omega^1{\cdot}t)\, Z(\omega{\cdot}t)\, V^1(\omega^1{\cdot}t) \tag{4.66}$$

for $\widetilde{Z}(\omega) = (V^1)^T(\omega^1)\, Z(\omega)\, V^1(\omega^1)$. Assume for contradiction the existence of a point $\widetilde{\omega} \in \Omega$ at which Z is not continuous, and note that this is equivalent to saying that $\widetilde{Z}$ is not continuous at $\widetilde{\omega}$. Then there exist $\delta > 0$ and a sequence $(\widetilde{\omega}_m)$ with limit $\widetilde{\omega}$ such that there exists $\lim_{m\to\infty} \|\widetilde{Z}(\widetilde{\omega}) - \widetilde{Z}(\widetilde{\omega}_m)\| \geq \delta$. Since, by (4.66),

$$\begin{aligned}
\widetilde{Z}(\widetilde{\omega}) - \widetilde{Z}(\widetilde{\omega}_m) &= (V^1)^T(\widetilde{\omega}_m^1{\cdot}t)\,(Z(\widetilde{\omega}{\cdot}t) - Z(\widetilde{\omega}_m{\cdot}t))\, V^1(\widetilde{\omega}_m^1{\cdot}t) \\
&\quad + ((V^1)^T(\widetilde{\omega}^1{\cdot}t) - (V^1)^T(\widetilde{\omega}_m^1{\cdot}t))\, Z(\widetilde{\omega}{\cdot}t)\, V^1(\widetilde{\omega}_m^1{\cdot}t) \\
&\quad + (V^1)^T(\widetilde{\omega}^1{\cdot}t)\, Z(\widetilde{\omega}{\cdot}t)\,(V^1(\widetilde{\omega}^1{\cdot}t) - V^1(\widetilde{\omega}_m^1{\cdot}t))\,,
\end{aligned}$$

the continuity of V^1 and of the flow on Ω^1 ensures that, for all $t \in \mathbb{R}$,

$$\delta \leq \lim_{m\to\infty} \|\widetilde{Z}(\widetilde{\omega}) - \widetilde{Z}(\widetilde{\omega}_m)\| \leq v^2 \limsup_{m\to\infty} \|Z(\widetilde{\omega}{\cdot}t) - Z(\widetilde{\omega}_m{\cdot}t)\|\,,$$

where $\|V^1\| \leq v$ on Ω^1. The conclusion is that for all $t \in \mathbb{R}$ there is a sequence (ω_m^t) with limit $\widetilde{\omega}{\cdot}t$ such that $\lim_{m\to\infty} \|Z(\widetilde{\omega}{\cdot}t) - Z(\omega_m^t)\| \geq \delta/v^2$.

The goal now is to deduce that Z is not continuous at any point of Ω, which will give the sought-for contradiction. Take $\omega \in \Omega$ and write it as $\omega = \lim_{k\to\infty} \widetilde{\omega}{\cdot}t_k$ for a sequence (t_k), which is possible since the base flow is minimal. For each $k \in \mathbb{N}$ choose $\omega_k \in \Omega$ which has distance less than $1/k$ from $\widetilde{\omega}{\cdot}t_k$, with $\|Z(\widetilde{\omega}{\cdot}t_k) - Z(\omega_k)\| \geq \delta/v^2$. This makes the continuity of Z at ω impossible, since $\lim_{k\to\infty} \omega_k = \omega$.

The proof of the first assertion is complete. To check the second one, note that the definition of A_Γ and the fact that $\Gamma \in \mathcal{C}$ ensure that A_Γ is the limit everywhere of a sequence of continuous functions, so that it is a function of the first Baire category, and hence it must have continuity points (see e.g. Theorem 7.5 of [27]).

Proposition 4.44 *Suppose that Hypotheses 4.37 hold, that $\Gamma \in \mathcal{C}$, that $A_\Gamma \geq 0$, and that there exists a point $\omega_0 \in \Omega$ with $A_\Gamma(\omega_0) > 0$. Then $A_\Gamma(\omega) > 0$ for all $\omega \in \Omega$.*

Proof Suppose for contradiction that there exist $\omega \in \Omega$ and $\mathbf{z}_0 \in \mathbb{R}^{2n}$ with $\mathbf{z}_0^T A_\Gamma(\omega)\,\mathbf{z}_0 = 0$. Then, if $\mathbf{w}(t) = U(t,\omega)\,\mathbf{z}_0/\|U(t,\omega)\,\mathbf{z}_0\|$, it follows from (4.28) that $\mathbf{w}^T(t)\,A_\Gamma(\omega{\cdot}t)\,\mathbf{w}(t) = 0$. Choosing a sequence (t_m) with $\omega_0 = \lim_{m\to\infty}\omega{\cdot}t_m$ such that there exists $\mathbf{w}_0 = \lim_{m\to\infty}\mathbf{w}(t_m)$, one gets $\mathbf{w}_0^T A_\Gamma(\omega_0)\,\mathbf{w}_0 = 0$, which contradicts the hypothesis.

The situation described in the previous corollary will often be represented by $A_\Gamma > 0$. The perturbation analysis will consider the families (4.1) in two cases: when $\lambda = i\varepsilon$ for small real values of ε in the case that $\Gamma \in \mathcal{C}$ and A_Γ is positive definite; and when $\lambda = \varepsilon$ for small real ε if $\Gamma \in \mathcal{C}$ and $J^{-1}A_\Gamma$ is nonsingular and can be conjugated to a real diagonal matrix. The advantages of working with perturbation directions in the set $\mathcal{C}$ are summarized in the following result. See Definitions 1.17 and 1.63 to understand its statements.

Proposition 4.45 *Suppose that Hypotheses 4.37 hold and take $\Gamma \in \mathcal{C}$. Then,*

(i) *if $A_\Gamma > 0$, then the corresponding sets $\{(\omega,\mathbf{z}) \mid \mathbf{z} \in l_\Gamma^\pm(\omega)\} \subset \Omega \times \mathbb{C}^{2n}$, with $l_\Gamma^\pm(\omega) \equiv \begin{bmatrix} I_n \\ N_\Gamma^\pm(\omega)\end{bmatrix}$, are closed invariant subbundles for the flow $\tau_\mathbb{C}$ defined by (1.13); the sets $\{(\omega, l_\Gamma^\pm(\omega)) \mid \omega \in \Omega\} \subset \mathcal{K}_\mathbb{C}$ are copies of the base for the flow τ; and the matrix-valued functions $N_\Gamma^\pm$ and C_Γ obtained in Theorem 4.13 are continuous. In particular, there exists a continuous and symplectic change of variables taking the initial system (4.2) to skew-symmetric form.*

(ii) *If $J^{-1}A_\Gamma(\omega)$ is nonsingular and can be conjugated to a real diagonal matrix for all $\omega \in \Omega$, then the sets $\{(\omega,\mathbf{z}) \mid \mathbf{z} \in l_\Gamma^\pm(\omega)\} \subset \Omega \times \mathbb{R}^{2n}$, with $l_\Gamma^\pm(\omega)$ provided by Lemma 4.12(ii), are closed $\tau_\mathbb{R}$-invariant subbundles of $\Omega \times \mathbb{R}^{2n}$.*

Proof

(i) Note that Proposition 4.43 guarantees that A_Γ is continuous. The relation $J^{-1}A_\Gamma(\omega{\cdot}t) = U(t,\omega)\,J^{-1}A_\Gamma(\omega)\,U^{-1}(t,\omega)$ for all $(t,\omega) \in \mathbb{R}\times\Omega$, which can be derived from (4.28) and from the symplectic character of $U(t,\omega)$, has two consequences. First, the eigenvalues $\pm i\mu_{\Gamma,1}(\omega),\ldots,\pm i\mu_{\Gamma,n}(\omega)$ of $J^{-1}A_\Gamma(\omega)$ are constant on the whole set Ω (and not just constant with respect to any ergodic measure): since

$$J^{-1}A_\Gamma(\omega{\cdot}t)\,\frac{U(t,\omega)\,\mathbf{z}}{\|U(t,\omega)\,\mathbf{z}\|} = \pm i\mu_{\Gamma,k}(\omega)\,\frac{U(t,\omega)\,\mathbf{z}}{\|U(t,\omega)\,\mathbf{z}\|} \tag{4.67}$$

whenever $J^{-1}A_\Gamma(\omega)\,\mathbf{z} = \pm i\mu_{\Gamma,k}(\omega)\,\mathbf{z}$, the assertion follows from the density of the orbit of an arbitrarily chosen $\omega \in \Omega$ and from the continuity of A_Γ. Second, if $F_{\pm i\mu_{\Gamma,k}}(\omega)$ is the corresponding eigenspace of $J^{-1}A_\Gamma(\omega)$, then $F_{\pm i\mu_{\Gamma,k}}(\omega{\cdot}t) = U(t,\omega){\cdot}F_{\pm i\mu_{\Gamma,k}}(\omega)$ for all $(t,\omega) \in \mathbb{R}\times\Omega$. This last property ensures that $k(\omega) = \dim F_{\pm i\mu_{\Gamma,k}}(\omega) = \dim \operatorname{Ker}(J^{-1}A_\Gamma(\omega) \mp i\mu_{\Gamma,k}I_{2n})$ is σ-invariant, which together with the minimality of the base flow and the continuity of A_Γ ensures that $k(\omega)$ is constant on Ω: note that if $\omega_1 = \lim_{m\to\infty} \omega_0{\cdot}t_m$, then $k(\omega_1) \le k(\omega_0)$, as can be deduced from an easy contradiction argument. An almost immediate consequence of these facts and of the continuity of A_Γ is that the sets $F_{\pm i\mu_k} = \{(\omega, \mathbf{z}) \mid J^{-1}A_\Gamma(\omega)\,\mathbf{z} = \pm i\mu_k\,\mathbf{z}\}$ are $\tau_{\mathbb{C}}$-invariant closed subbundles of $\Omega \times \mathbb{C}^{2n}$. Hence, the first assertion in (i) follows from the equality $\{(\omega, \mathbf{z}) \mid \mathbf{z} \in l_\Gamma^\pm(\omega)\} = F_{\pm i\mu_1} \oplus \cdots \oplus F_{\pm i\mu_n}$.

Now, the first assertion ensures that the maps $l^\pm\colon \Omega \to \mathcal{L}_{\mathbb{C}}$ are continuous and their graphs are copies of the base (see Proposition 1.70), so that the second assertion of (i) follows. And, according to Proposition 1.29(i), the representation $l_\Gamma^\pm(\omega) \equiv \begin{bmatrix} I_n \\ N_\Gamma^\pm \end{bmatrix}$ provided by Theorem 4.13(iii) implies the continuity of the functions $N_\Gamma^\pm$, which in turn yields the continuity of C_Γ and B_Γ, and proves the third assertion of (i). Once this is established, the last property follows from Theorem 4.13(iii) and Theorem 4.4(i).

(ii) Suppose that the hypotheses of (ii) are valid. Keeping in mind the information provided by Lemma 4.12(ii), it is possible to adapt the proof of (i) in order to deduce from (4.28) and from the symplectic character of $U(t,\omega)$ that the eigenvalues of $J^{-1}A_\Gamma(\omega)$ are σ-invariant and hence constant on Ω, that the sets $\{(\omega, \mathbf{z}) \mid \mathbf{z} \in l^\pm(\omega)\} \subset \Omega \times \mathbb{R}^{2n}$ are τ-invariant, and that they are closed subbundles.

Definition 4.46 Take $\Gamma \in \mathcal{C}$ with $A_\Gamma > 0$. The *subbundles associated to* Γ are the closed invariant subbundles $\{(\omega, \mathbf{z}) \in \Omega \times \mathbb{C}^{2n} \mid \mathbf{z} \in l_\Gamma^\pm(\omega)\}$ of $\Omega \times \mathbb{C}^{2n}$, where $l_\Gamma^+(\omega)$ and $l_\Gamma^-(\omega)$ are the complex Lagrange planes determined by the sums of the eigenspaces of $J^{-1}A_\Gamma(\omega)$ associated to the eigenvalues with positive and negative imaginary part.

If $\Gamma \in \mathcal{C}$, and if $J^{-1}A_\Gamma(\omega)$ is nonsingular and can be conjugated to a real diagonal matrix for all $\omega \in \Omega$, then the *subbundles associated to* Γ are the closed invariant subbundles $\{(\omega, \mathbf{z}) \in \Omega \times \mathbb{R}^{2n} \mid \mathbf{z} \in l_\Gamma^\pm(\omega)\}$ of $\Omega \times \mathbb{R}^{2n}$, where $l_\Gamma^+(\omega)$ and $l_\Gamma^-(\omega)$ are the real Lagrange planes determined by the sums of the eigenspaces of $J^{-1}A_\Gamma(\omega)$ associated to the positive and negative eigenvalues, respectively.

The following result is included in order to complete the analysis of the situations in which Hypotheses 4.1 are valid. In particular, point (ii) reveals once more the relevance of the set $\mathcal{C}$ in the description of the global dynamics induced by (4.2) on $\Omega \times \mathbb{R}^{2n}$.

Proposition 4.47

(i) *Suppose that Ω is minimal and that there exists a measurable map $N\colon \Omega \to \mathbb{S}_n^+(\mathbb{C})$ which has at least one continuity point and satisfies* (4.5) *along the flow on Ω. Then N is continuous.*

(ii) *The existence of a continuous map $N\colon \Omega \to \mathbb{S}_n^+(\mathbb{C})$ which satisfies* (4.5) *along the flow on Ω ensures the existence of a continuous change of variables taking* (4.2) *to skew-symmetric form* (4.10)*, which in addition is given by a* *symplectic matrix-valued function. Moreover, there exists* $\Gamma \in \mathcal{C}$ *such that* $N = N^+_\Gamma$.

(iii) *The existence of a continuous change of variables taking* (4.2) *to skew-symmetric form ensures the boundedness of all the solutions of all the systems of the family. In particular, when* Ω *is minimal, these two properties are actually equivalent.*

Proof

(i) Suppose that N satisfies the hypotheses of (i), and note that $\mathrm{Im}^{-1} N$ is well defined. The first step in the proof of the continuity of N is to check that N and $\mathrm{Im}^{-1} N$ are norm-bounded on Ω. Let ω_0 be a continuity point of N. Then there exist $\rho > 0$ and $\delta > 0$ such that, if $M \in \mathcal{B}_\rho = \{M \in \mathbb{S}_n(\mathbb{C}) \mid \|N(\omega_0) - M\| \le \rho\}$, then $\Im M > 0$; and, if $\omega \in \Omega_\delta = \{\omega \in \Omega\,,\ d_\Omega(\omega, \omega_0) < \delta\}$, then $N(\omega) \in \mathcal{B}_\rho$, where d_Ω represents the distance in the metric space Ω.

Since Ω is minimal and Ω_δ is open, there exists a time $t_0 > 0$ such that for all $\omega \in \Omega$ there exists $t_\omega \in [-t_0, t_0]$ with $\omega{\cdot}t_\omega \in \Omega_\delta$. Let $\tau_s(t, \omega, M) = (\omega{\cdot}t, M(t, \omega, M_0))$ be the local continuous flow induced by (4.2) on $\Omega \times \mathbb{S}_n(\mathbb{C})$ (see (1.23)), and note that $M(t, \omega, N(\omega)) = N(\omega{\cdot}t)$. As explained in Remark 2.12, it follows from Lemma 2.10 that the restriction of τ_s to $\Omega \times \mathbb{S}^+_n(\mathbb{C})$ is globally defined. Then the map

$$[-t_0, t_0] \times \Omega \times \mathcal{B}_\rho \to \mathbb{S}^+_n(\mathbb{C})\,, \quad (t, \omega, M_0) \mapsto M(t, \omega, M_0)$$

is globally defined and continuous, and hence it is bounded. In addition, $(-t_\omega, \omega{\cdot}t_\omega, N(\omega{\cdot}t_\omega))$ belongs to its domain for all $\omega \in \Omega$, and $N(\omega) = M(-t_\omega, \omega{\cdot}t_\omega, N(\omega{\cdot}t_\omega))$. This proves the boundedness of N on Ω, which in turns implies that of $\mathrm{Im}^{-1} N$.

Note next that ω_0 is also a continuity point of $\mathrm{Im}\, N$, and hence of $\mathrm{Im}^{-1/2} N$ (see Proposition 1.19). Hence, if C is defined from N by (4.9), then the matrix-valued function $B = C^T C$ is globally defined on Ω and continuous at ω_0. Lemma 4.2 proves that it is a solution along the flow on Ω of (4.4), so that Proposition 4.43 ensures that it is continuous. Since N can also be defined from B by the corresponding expression (4.8), the result in (i) is proved.

(ii) Theorem 4.4(i) implies the first assertion in (ii), where the change of variables C is defined from N by (4.9). Proposition 4.42(i) ensures that $A_{\Gamma_C} = \Gamma_C$ for $\Gamma_C = C^T C$. In particular, A_{Γ_C} is symplectic, so that it agrees with the function B_{Γ_C} defined in Theorem 4.13(iv). It is proved there that $N = N^+_{\Gamma_C}$, which completes the proof of (ii).

(iii) The first assertion of (iii) follows easily from Remark 4.5.3., and the second one from Theorem 4.39.

Finally, recall that Theorem 3.8 ensures the existence of the M-functions $M^\pm_\Gamma \colon \Omega \times (\mathbb{C} - \mathbb{R}) \to \mathbb{S}_n(\mathbb{C})$ for every Atkinson perturbation Γ, with the property that $\pm\,\mathrm{Im}\,\lambda\ \mathrm{Im}\, M^\pm_\Gamma(\omega, \lambda) > 0$. In the case that the limits $\lim_{\varepsilon\to 0^+} M^+_\Gamma(\omega, i\varepsilon) = N(\omega)$ exist and belong to $\mathbb{S}^+_n(\mathbb{C})$ for every $\omega \in \Omega$, the function N is a solution along the flow of the Riccati equation (4.5), as is easily deduced from classical results concerning the dependence of solutions of differential equations on parameters. Then point (i) of Proposition 4.47(i) ensures that N is continuous, since it has at least one continuity point (see again (see e.g. Theorem 7.5 of [27]), and points (ii) and (iii) ensure that all the solutions of all the systems of the unperturbed family (4.2) are bounded. That is, Hypotheses 4.37 hold if the base is minimal. The following subsection is devoted to analyzing the converse assertion. Before beginning the discussion, note that, since Hypothesis 4.1 holds (see Remark 4.40), it follows from Theorem 4.26 that $N(\omega) = N_\Gamma^+(\omega)$ for almost every $\omega \in \Omega$ with respect to any ergodic measure on Ω.

4.5.1 The Variation with Respect to a Complex Parameter

The proof of the main result of the section requires the following technical lemma, which analyzes several nontrivial consequences of the Sacker–Sell perturbation theorem as applied to the Sacker–Sell spectral decomposition. The framework of application of the results of [133] which is now required is similar to the one described in Sect. 1.4, but is not exactly the same. The results proved in Lemma 4.12(i) are implicit in the following statement. Recall that $\mathcal{G}_d(\mathbb{C}^{2n})$ represents the Grassmannian manifold of the d-dimensional linear subspaces of $\mathbb{C}^{2n}$, and that given a closed subbundle $F \subseteq \Omega \times \mathbb{C}^{2n}$, all the vector spaces F_ω given by the fibers over the points $\omega \in \Omega$ have the same dimension: see Definition 1.63 and remember that Ω is minimal, and hence connected.

Lemma 4.48 *Let D be a constant real positive definite symmetric $2n \times 2n$ matrix and $\varepsilon_1 > 0$. Let $T\colon \Omega \times [0, \varepsilon_1] \to \mathbb{S}_{2n}(\mathbb{C})$ be a jointly continuous map with $T(\omega, 0) = 0_n$ for all $\omega \in \Omega$. Consider the families of linear systems*

$$\mathbf{z}' = i\varepsilon J^{-1}\,(D + T(\omega{\cdot}t, \varepsilon))\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{4.68}$$

for $\varepsilon \in [0, \varepsilon_1]$. Let $\pm i\mu_1, \ldots, \pm i\mu_d$ be the different eigenvalues of $J^{-1}D$ with multiplicities $m_1, \ldots, m_d$ respectively, ordered so that $0 < \mu_1 < \cdots < \mu_d$, and set

$$\eta_0 = \frac{1}{2}\,\min\,(2\mu_1, \mu_2 - \mu_1, \ldots, \mu_d - \mu_{d-1})\,. \tag{4.69}$$

For each $\eta \in (0, \eta_0)$ there exists $\varepsilon(\eta) > 0$ such that, if $\varepsilon \in (0, \varepsilon(\eta))$, then the following statements are valid.

(i) *The Sacker–Sell spectrum of* (4.68) *is contained in the set*

$$\bigcup_{j=1}^{d}[\pm\varepsilon\mu_j-\varepsilon\eta,\pm\varepsilon\mu_j+\varepsilon\eta]\,,$$

and each of the 2*d (disjoint) intervals of this union contains at least one spectral interval.*

(ii) *For each* $\varepsilon\in(0,\varepsilon(\eta))$ *and* $j=1,\dots,d$, *the closed subbundle* $F_\varepsilon^{\pm j}$ *given by the sums of the spectral subbundles of* (4.68) *corresponding to the intervals contained in* $[\mp\varepsilon\mu_j-\varepsilon\eta,\mp\varepsilon\mu_j+\varepsilon\eta]$ *has dimension* m_j. *In addition, the maps* $\Omega\times(0,\varepsilon(\eta))\to\mathcal{G}_{m_j}(\mathbb{C}^{2n})$, $(\omega,\varepsilon)\to(F_\varepsilon^{\pm j})_\omega$ *are continuous, and* $\lim_{\varepsilon\to0^+}(F_\varepsilon^{\pm j})_\omega=F_0^{\pm j}$ *in* $\mathcal{G}_{m_j}(\mathbb{C}^{2n})$ *uniformly on* Ω, *where* $F_0^{\pm j}$ *are the eigenspaces of* $J^{-1}D$ *which are associated to* $\pm i\mu_j$, *respectively.*

(iii) *Let* m_0 *be any* σ*-ergodic measure on* Ω, *fix* $j=1,\dots,d$, *and let* $\widetilde{\beta}_j^{\pm}(\varepsilon)$ *be the sum of the Lyapunov exponents (equal or distinct) of* (4.68) *for* m_0 *which belong to the interval* $[\pm\varepsilon\mu_j-\varepsilon\eta,\pm\varepsilon\mu_j+\varepsilon\eta]$. *Then,*

$$\lim_{\varepsilon\to0^+}\frac{\widetilde{\beta}_j^{\pm}(\varepsilon)}{\varepsilon}=\pm m_j\mu_j\,.$$

In particular, the families of systems (4.68) *corresponding to these values of* ε *have exponential dichotomy. For* ε *small enough, the complex Lagrange planes given by the stable subbundles at* $\mp\infty$ *can be represented by* $\begin{bmatrix} I_n \\ M^{\pm}(\omega,\varepsilon)\end{bmatrix}$, *and the corresponding Weyl M-functions satisfy*

$$\lim_{\varepsilon\to0^+}M^{\pm}(\omega,\varepsilon)=N^{\pm}$$

uniformly on Ω, *where* $\begin{bmatrix} I_n \\ N^{\pm}\end{bmatrix}$ *represent respectively the complex Lagrange planes generated by the eigenvectors which are associated to the eigenvalues* $\pm i\mu_1,\dots,\pm i\mu_d$ *of* $J^{-1}D$.

Proof The proof of this result is based on the Sacker–Sell Theorem 1.90. Consider the metric space $BU=BU(\mathbb{R},\mathbb{M}_{2n\times2n}(\mathbb{C}))$ of all bounded and uniformly continuous complex $2n\times2n$ matrix-valued functions on $\mathbb{R}$, endowed with the compact-open topology. Let $\mathcal{M}$ be a connected compact subset of BU which is invariant by time translation; i.e. $B_t\in\mathcal{M}$ for all $t\in\mathbb{R}$ if $B\in\mathcal{M}$, where $B_t(s)=B(t+s)$. Let $U(t,B)$ be the fundamental matrix solution of

$$\mathbf{w}'=B(t)\,\mathbf{w} \tag{4.70}$$

with $U(0,B)=I_{2n}$. Then the real skew-symmetric flow

$$\zeta\colon\mathbb{R}\times\mathcal{M}\times\mathbb{C}^{2n}\to\mathcal{M}\times\mathbb{C}^{2n}\,,\quad(t,B,\mathbf{z})\mapsto(B_t,U(t,B)\,\mathbf{z})$$

is continuous. Now recall the definition of the Sacker–Sell spectrum of any nonempty, compact, and (time-translation) invariant subset $\mathcal{M}_0 \subseteq \mathcal{M}$: $\lambda \in \Sigma(\mathcal{M}_0)$ if $\lambda \in \mathbb{R}$ and the family $\{\mathbf{z}' = (B(t) - \lambda I_{2n})\,\mathbf{z} \mid B \in \mathcal{M}_0\}$ does not have exponential dichotomy over $\mathcal{M}_0$ (see Definition 1.82). That is, $\Sigma(\mathcal{M}_0) = \cup_{B\in\mathcal{M}} \Sigma(B)$, where $\Sigma(B)$ is the set of $\lambda \in \mathbb{R}$ such that the system $\mathbf{w}' = (B(t) - \lambda I_{2n})\,\mathbf{w}$ does not have exponential dichotomy on $\mathbb{R}$ (see Definition 1.54). Denote also $\rho(B) = \mathbb{R} - \Sigma(B)$ and $\rho(\mathcal{M}_0) = \mathbb{R} - \Sigma(\mathcal{M}_0) = \cap_{B\in\mathcal{M}_0}\rho(B)$. Now assume that $\mathcal{M}_0$ is connected and apply Theorem 1.84 in order to check that it makes sense to talk about the corresponding (ζ-invariant and closed) spectral subbundles of $\mathcal{M}_0 \times \mathbb{C}^{2n}$. More precisely, if $\lambda \in \rho(\mathcal{M}_0)$, the sets

$$F^+_\lambda(\mathcal{M}_0) = \{(B, \mathbf{w}) \in \mathcal{M} \times \mathbb{C}^{2n} \mid \|e^{-\lambda t} U(t, B)\,\mathbf{w}\| \to 0 \text{ as } t \to \infty\}\,,$$

$$F^-_\lambda(\mathcal{M}_0) = \{(B, \mathbf{w}) \in \mathcal{M} \times \mathbb{C}^{2n} \mid \|e^{-\lambda t} U(t, B)\,\mathbf{w}\| \to 0 \text{ as } t \to -\infty\}\,,$$

are $\bar{\tau}_{\mathbb{C}}$-invariant closed subbundles of $\mathcal{M}_0 \times \mathbb{C}^{2n}$, and their Whitney sum agrees with the whole space $\mathcal{M}_0 \times \mathbb{C}^{2n}$. In addition, given $\mu_1,\ \mu_2 \in \rho(\mathcal{M}_0)$ with $\mu_1 < \mu_2$, the following statements are equivalent:

(1) there exists $\mu \in (\mu_1, \mu_2) \cap \Sigma(\mathcal{M}_0)$;
(2) $F^-_{\mu_1}(\mathcal{M}_0) \cap F^+_{\mu_2}(\mathcal{M}_0) \neq \mathcal{M}_0 \times \{\mathbf{0}\}$.

In addition, if (1) or (2) holds, then

(3) $F^-_{\mu_1}(\mathcal{M}_0) \cap F^+_{\mu_2}(\mathcal{M}_0)$ is the sum of the spectral subbundles of $\mathcal{M}_0 \times \mathbb{C}^{2n}$ associated to the intervals of $\Sigma(\mathcal{M}_0)$ contained in (μ_1, μ_2).

Consider now the family of systems (4.68). For $\varepsilon \in (0, \varepsilon_1]$, the time rescaling $\mathbf{w}(t) = \mathbf{z}(t/\varepsilon)$ applied to (4.68) yields

$$\mathbf{w}' = iJ^{-1}\,(D + T(\omega{\cdot}(t/\varepsilon), \varepsilon))\,\mathbf{w}\,, \quad \omega \in \Omega\,. \tag{4.71}$$

For each pair $(\omega, \varepsilon) \in \Omega \times [0, \varepsilon_1]$, define $T_{\omega,\varepsilon}(t) = T(\omega{\cdot}(t/\varepsilon), \varepsilon)$ if $\varepsilon > 0$, and $T_{\omega,0}(t) = 0_n$. Then, if $0 \le \varepsilon \le \varepsilon_0 \le \varepsilon_1$, the sets

$$\mathcal{M}(\varepsilon) = \{iJ^{-1}(D + T_{\omega,\varepsilon}) \mid \omega \in \Omega\} \subset BU\,,$$

$$\mathcal{M}_{\varepsilon_0} = \cup_{\varepsilon\in[0,\varepsilon_0]}\mathcal{M}(\varepsilon) = \{iJ^{-1}(D + T_{\omega,\varepsilon}) \mid \omega \in \Omega, \varepsilon \in [0, \varepsilon_0]\} \subset BU$$

are connected compact invariant subsets of BU. These assertions follow from the connectivity of Ω, which is due to its minimality, and from the continuity of the map

$$\Omega \times [0, \varepsilon_1] \to BU\,, \quad (\omega, \varepsilon) \mapsto iJ^{-1}(D + T_{\omega,\varepsilon})\,, \tag{4.72}$$

which is deduced as follows from the hypotheses on T: let the sequence $(\omega_m, \varepsilon_m)$ of elements of $\Omega \times [0, \varepsilon_1]$ converge to $(\omega_0, \varepsilon_0)$; if $\varepsilon_0 > 0$, the continuity of the map $[a, b] \times [\varepsilon_0/2, \varepsilon_1] \times \Omega,\ \ (t, \varepsilon, \omega) \mapsto \omega{\cdot}(t/\varepsilon)$ allows one to conclude that $\lim_{m\to\infty} T_{\omega_n,\varepsilon_n}(t) = T_{\omega_0,\varepsilon_0}(t)$ uniformly on any interval $[a,b] \subset \mathbb{R}$; and, if $\varepsilon_0 = 0$, the assertion follows from the fact that $\sup_{\omega\in\Omega}(\sup_{t\in\mathbb{R}} \|T_{\omega,\varepsilon}(t)\|) \le \max_{\widetilde{\omega}\in\Omega} \|T(\widetilde{\omega},\varepsilon)\|$ is as small as desired if $\varepsilon > 0$ is small enough. This last continuity also implies that every neighborhood of $\mathcal{M}(0) = \{iJ^{-1}D\}$ in BU contains a set $\mathcal{M}_{\varepsilon_0}$ for $\varepsilon_0 > 0$ small enough. Note also that $\mathcal{M}(\varepsilon_*) \subseteq \mathcal{M}(\varepsilon^*)$ if $0 \le \varepsilon_* \le \varepsilon^* \le \varepsilon_1$. It is clear that

(4) if $\lambda \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon \in (0,\varepsilon_1]$, then the exponential dichotomy on $\mathbb{R}$ for the two systems

$$\mathbf{z}' = \big(i\varepsilon J^{-1}(D + T(\omega{\cdot}t,\varepsilon)) - \lambda I_{2n}\big)\,\mathbf{z}\,,$$

$$\mathbf{w}' = \left(iJ^{-1}(D + T_{\omega,\varepsilon}(t)) - \frac{\lambda}{\varepsilon}\, I_{2n}\right)\mathbf{w}\,,$$

occurs or not simultaneously. That is, the Sacker–Sell spectrum of the family of systems (4.68) (over Ω) coincides with $\varepsilon{\cdot}\Sigma(\mathcal{M}(\varepsilon))$.

(5) The time rescaling does not affect the spectral subbundles. More precisely, the fiber for the element $iJ^{-1}(D + T_{\widetilde{\omega},\varepsilon}) \in \mathcal{M}(\varepsilon)$ of the spectral subbundle of $\mathcal{M}(\varepsilon) \times \mathbb{C}^{2n}$ associated to the interval $[a,b]$ of the Sacker–Sell spectrum of the family $\{\mathbf{w}' = B(t)\,\mathbf{w} \mid B \in \mathcal{M}(\varepsilon)\}$, coincides with the fiber for $\widetilde{\omega}$ of the spectral subbundle of $\Omega \times \mathbb{C}^{2n}$ associated to the interval $[\varepsilon a, \varepsilon b]$ of the Sacker–Sell spectrum of the family $\{(4.68) \mid \omega \in \Omega\}$.

Consider now the constant coefficient system

$$\mathbf{w}' = iJ^{-1}D\,\mathbf{w}\,. \tag{4.73}$$

It is well known (and easy to check) that

$$\Sigma(\mathcal{M}(0)) = \Sigma(iJ^{-1}D) = \{-\mu_d, \ldots, -\mu_1, \mu_1, \ldots, \mu_d\}\,,$$

and that the eigenspaces $F_0^d, \ldots, F_0^1, F_0^{-1}, \ldots, F_0^{-d}$ of $J^{-1}D$ corresponding to the eigenvalues $i\mu_d, \ldots, i\mu_1, -i\mu_1, \ldots, -i\mu_d$, with $\dim F_0^{\pm j} = m_j$ for $j = 1, \ldots, d$, determine the corresponding spectral subbundles of $\mathcal{M}(0) \times \mathbb{C}^{2n}$.

Take $\eta \in (0, \eta_0)$ where η_0 is defined by (4.69), then consider the neighborhood of $\Sigma(iJ^{-1}D)$ in $\mathbb{R}$ given by

$$\mathcal{V}(\eta) = \cup_{j=1}^d [\pm\mu_j - \eta, \pm\mu_j + \eta]\,,$$

and choose $\lambda_0 = 0,\ \lambda_1, \ldots, \lambda_d$ with

$$\lambda_0 < \mu_1 - \eta\,, \qquad \mu_j + \eta < \lambda_j < \mu_{j+1} - \eta \ \text{ for } j = 1, \ldots, d-1\,, \qquad \mu_d + \eta < \lambda_d\,.$$

The Sacker–Sell perturbation theorem (see [133], Theorem 6) ensures the existence of a constant $\varepsilon(\eta) > 0$ such that the following properties hold:

(6) If $\varepsilon \in [0, \varepsilon(\eta)]$, then $\Sigma(\mathcal{M}(\varepsilon)) \subseteq \mathcal{V}(\eta)$ and hence $\Sigma(iJ^{-1}D) \subseteq \Sigma(\mathcal{M}_{\varepsilon(\eta)}) \subseteq \mathcal{V}(\eta)$. In particular, $\pm\lambda_j \in \rho(B)$ whenever $B \in \mathcal{M}_{\varepsilon(\eta)}$ and $j = 0, \dots, d$, since $\pm\lambda_j \in \rho(\mathcal{M}(\varepsilon))$.

(7) Let $F^{\pm}_{\lambda}(B)$ represent the complex vector spaces given by the fibers of the closed subbundles $F^{\pm}_{\lambda}(\mathcal{M}_{\varepsilon(\eta)})$ for $\lambda \in \rho(\mathcal{M}_{\varepsilon(\eta)})$ and $B \in \mathcal{M}_{\varepsilon(\eta)}$. Then, for $j = 0, \dots, d$, the dimensions $d^{\pm}_{\pm\lambda_j}$ of the spaces $F^{\pm}_{\pm\lambda_j}(B)$ are independent of $B \in \mathcal{M}_{\varepsilon(\eta)}$, and, in addition, the maps $\mathcal{M}_{\varepsilon(\eta)} \to \mathcal{G}_{d^{\pm}_{\pm\lambda_j}}(\mathbb{C}^{2n})\,,\; B \mapsto F^{\pm}_{\pm\lambda_j}(B)$ are continuous. Therefore the same properties hold for the vector spaces $F^{-}_{\lambda_{j-1}}(B) \cap F^{+}_{\lambda_j}(B)$ and $F^{-}_{-\lambda_j}(B) \cap F^{+}_{-\lambda_{j-1}}(B)$, and for the corresponding maps on $\mathcal{M}_{\varepsilon(\eta)}$.

On the other hand, for $j = 1, \dots, d$,

$$\begin{aligned} F^{-}_{-\lambda_j}(iJ^{-1}D) \cap F^{+}_{-\lambda_{j-1}}(iJ^{-1}D) &= F^{j}_{0}\,, \\ F^{-}_{\lambda_{j-1}}(iJ^{-1}D) \cap F^{+}_{\lambda_j}(iJ^{-1}D) &= F^{-j}_{0}\,. \end{aligned} \tag{4.74}$$

These equalities are easily deduced from (3) and from the spectral decomposition of (4.73), since

$$(\lambda_{j-1}, \lambda_j) \cap \Sigma(\mathcal{M}(0)) = \{\mu_j\} \quad \text{and} \quad (-\lambda_j, -\lambda_{j-1}) \cap \Sigma(\mathcal{M}(0)) = \{-\mu_j\}$$

for $j = 1, \dots, d$. Consequently, (7) ensures that

$$\dim\,(F^{-}_{-\lambda_j}(B) \cap F^{+}_{-\lambda_{j-1}}(B)) = \dim\,(F^{-}_{\lambda_{j-1}}(B) \cap F^{+}_{\lambda_j}(B)) = \dim F^{\pm j}_{0} = m_j$$

for every $B \in \mathcal{M}_{\varepsilon(\eta)}$. Note also that, for $j = 0, \dots, d$, the sets $F^{\pm}_{\pm\lambda_j}(B)$ (defined in (7)) agree with the fibers over the element B of the subbundles $F^{\pm}_{\pm\lambda_j}(\mathcal{M}(\varepsilon))$. In particular, the dimensions of the closed subbundles

$$\begin{aligned} F^{j}_{\varepsilon} &= F^{-}_{-\lambda_j}(\mathcal{M}(\varepsilon)) \cap F^{+}_{-\lambda_{j-1}}(\mathcal{M}(\varepsilon))\,, \\ F^{-j}_{\varepsilon} &= F^{-}_{\lambda_{j-1}}(\mathcal{M}(\varepsilon)) \cap F^{+}_{\lambda_j}(\mathcal{M}(\varepsilon)) \end{aligned}$$

are equal to $m_j > 0$. Once this fact has been established, the equivalence between the properties (1) and (2) ensures that each of the intervals

$$\begin{aligned} (-\lambda_j, -\lambda_{j-1}) \cap \mathcal{V}(\eta) &= [-\mu_j - \eta, -\mu_j + \eta\,]\,, \\ (\lambda_{j-1}, \lambda_j) \cap \mathcal{V}(\eta) &= [\mu_j - \eta, \mu_j + \eta\,] \end{aligned}$$

contains at least one spectral interval of $\Sigma(\mathcal{M}(\varepsilon))$. This assertion, together with property (6) and equality (4), proves (i) for $\varepsilon \in [0, \varepsilon(\eta)]$.

In order to prove (ii), note that (3) and (5) ensure that the sums of the spectral subbundles of (4.68) corresponding to the intervals contained in $[-\varepsilon\mu_j - \varepsilon\eta, -\varepsilon\mu_j + \varepsilon\eta]$ and $[\varepsilon\mu_j - \varepsilon\eta, \varepsilon\mu_j + \varepsilon\eta]$ are the sets F_ε^j and F_ε^{-j} defined in the previous paragraph, respectively. Therefore, they have dimension m_j, as asserted. In addition, also by (5), if $B_{\omega,\varepsilon} = iJ^{-1}(D + T_{\omega,\varepsilon})$, then

$$(F_\varepsilon^j)_\omega = F^-_{-\lambda_j}(B_{\omega,\varepsilon}) \cap F^+_{-\lambda_{j-1}}(B_{\omega,\varepsilon}) \quad \text{and} \quad (F_\varepsilon^{-j})_\omega = F^-_{\lambda_{j-1}}(B_{\omega,\varepsilon}) \cap F^+_{\lambda_j}(B_{\omega,\varepsilon})$$

for $\varepsilon \in (0, \varepsilon(\eta))$, and (4.74) states that

$$F_0^j = F^-_{-\lambda_j}(B_{\omega,0}) \cap F^+_{-\lambda_{j-1}}(B_{\omega,0}) \quad \text{and} \quad F_0^{-j} = F^-_{\lambda_{j-1}}(B_{\omega,0}) \cap F^+_{\lambda_j}(B_{\omega,0})\,.$$

On the other hand, the continuity stated in (7) and that of the map (4.72) ensure that also the maps

$$\Omega \times [0, \varepsilon(\eta)] \to \mathcal{G}_{d^\pm_{\lambda_j}}(\mathbb{C}^{2n})\,, \quad (\omega, \varepsilon) \mapsto F^\pm_{\pm\lambda_j}(B_{\omega,\varepsilon})\,, \tag{4.75}$$

for $j = 0, \ldots, n$, are continuous. These last properties ensure that the maps $\Omega \times (0, \varepsilon(\eta)) \to \mathcal{G}_{m_j}(\mathbb{C}^{2n}),\ (\omega, \varepsilon) \to (F_\varepsilon^{\pm j})_\omega$ are well defined and continuous; and that

$$\lim_{\varepsilon \to 0+} (F_\varepsilon^j)_\omega = F_0^j \quad \text{and} \quad \lim_{\varepsilon \to 0+} (F_\varepsilon^{-j})_\omega = F_0^{-j}$$

in $\mathcal{G}_{m_j}(\mathbb{C}^{2n})$ uniformly on Ω. The proof of (ii) is complete.

Now fix a σ-ergodic measure m_0 on Ω. For any $\eta \in (0, \eta_0)$ and $\varepsilon \in (0, \varepsilon(\eta))$, let $\beta^+_{j,1}(\varepsilon) \le \cdots \le \beta^+_{j,m_j}(\varepsilon)$ be the corresponding Lyapunov exponents of (4.68) which are contained in the interval $[\varepsilon\mu_j - \varepsilon\eta, \varepsilon\mu_j + \varepsilon\eta]$. The fact that they are exactly m_j in number follows from Theorem 2.3 of [86] and Theorem 2.37, since $\dim F_\varepsilon^j = m_j$: in fact F_ε^j decomposes as the sum of the Oseledets subbundles corresponding to those Lyapunov exponents contained in the fixed interval. Note also that the number m_j is common for all $\eta \in (0, \eta_0)$ if $\varepsilon \in (0, \varepsilon(\eta))$, as was seen before. Set $\beta_j^+(\varepsilon) = \beta^+_{j,1}(\varepsilon) + \cdots + \beta^+_{j,m_j}(\varepsilon)$, so that $\beta_j^+(\varepsilon)/(m_j\varepsilon) \in [\mu_j - \eta, \mu_j + \eta]$. Rewriting this information, for all $\eta \in (0, \eta_0)$ there exists $\varepsilon(\eta) > 0$ (the same as before) such that, if $\varepsilon \in (0, \varepsilon(\eta)$ then $|\beta_j^+(\varepsilon)/(m_j\varepsilon) - \mu_j| \le \eta$. This proves (iii) for $\beta_j^+(\varepsilon)$, and the proof is carried out in an analogous way for $\beta_j^-(\varepsilon)$.

The presence of exponential dichotomy for the systems (4.70) with $B \in \mathcal{M}_{\varepsilon(\eta)}$ follows from the fact that $\lambda_0 = 0 \notin \Sigma(\mathcal{M}_{\varepsilon(\eta)})$, since $0 \notin \mathcal{V}(\eta)$. With the previous notation, property (5) ensures that the sets

$$L_\varepsilon^\pm = \{(\omega, \mathbf{z}) \in \Omega \times \mathbb{C}^{2n} \mid \mathbf{z} \in F_0^\pm(B_{\omega,\varepsilon})\}$$

agree with the stable subbundles at $\mp\infty$ of (4.68). Note that they are n-dimensional (i.e. $d^{\pm}_{\lambda_0} = n$) and that their fibers $l^{\pm}_{\varepsilon}(\omega) = (L^{\pm}_{\varepsilon})_{\omega}$ over each element of the base are complex Lagrange planes, since the linear system is of Hamiltonian type: see Proposition 1.76. Similarly, (4.73) does not have exponential dichotomy (that is, $0 \notin \Sigma(iJ^{-1}D)$), and the Lagrange planes providing the stable subbundles at $+\infty$ and $-\infty$ are $l^{-}_0 = F^{-d}_0 \oplus \cdots \oplus F^{-1}_0 \equiv \left[\begin{smallmatrix} I_n \\ N^- \end{smallmatrix}\right]$ and $l^{+}_0 = F^{1}_0 \oplus \cdots \oplus F^{d}_0 \equiv \left[\begin{smallmatrix} I_n \\ N^+ \end{smallmatrix}\right]$: see Proposition 1.89. It is not hard to deduce from the continuity of the maps (4.75) and the independence of $B_{\omega,0}$ with respect to ω that $\lim_{\varepsilon\to 0^+} l^{\pm}_{\varepsilon}(\omega) = l^{\pm}_0$ in $\mathcal{G}_n(\mathbb{C}^{2n})$ uniformly on Ω. Hence, Proposition 1.29(i) and Corollary 1.31 imply that the fibers of the stable subbundles at $\mp\infty$ can be represented as $\left[\begin{smallmatrix} I_n \\ M^{\pm}(\omega,\varepsilon) \end{smallmatrix}\right]$ for ε small enough, and that $\lim_{\varepsilon\to 0^+} M^{\pm}(\omega,\varepsilon) = N^{\pm}$ uniformly on Ω. This completes the proof of the lemma.

As stated before, the main result of this section refers to the parametric variation of the perturbed family

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + i\varepsilon J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{4.76}$$

with real ε, in the case that Γ belongs to the set $\mathcal{C}$ and defines a positive definite limit A_Γ. This is true if, for instance, Γ belongs to $\mathcal{C}$ and satisfies the Atkinson Hypotheses 3.3: this can be checked using Propositions 4.11 and 4.44. What follows summarizes some of the results of Theorem 4.13 and 4.4(i); see also Remark 4.14. Let $\pm i\mu_{\Gamma,1}, \ldots, \pm i\mu_{\Gamma,d}$ be the different eigenvalues of $J^{-1}A_\Gamma$ with multiplicities $m_{\Gamma,1}, \ldots, m_{\Gamma,d}$ respectively, ordered to that $0 < \mu_{\Gamma,1} < \cdots < \mu_{\Gamma,d}$, and set

$$\eta_\Gamma = \frac{1}{2}\,\min\big(2\mu_{\Gamma,1}, \mu_{\Gamma,2} - \mu_{\Gamma,1}, \ldots, \mu_{\Gamma,d} - \mu_{\Gamma,d-1}\big)\,. \tag{4.77}$$

Let $l^{+}_{\Gamma}(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N^{+}_{\Gamma}(\omega) \end{smallmatrix}\right]$ and $l^{-}_{\Gamma}(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N^{-}_{\Gamma}(\omega) \end{smallmatrix}\right]$ be the complex Lagrange planes generated by the eigenvalues with positive and negative imaginary parts of $J^{-1}A_\Gamma(\omega)$ respectively. The symmetric $n \times n$ matrix-valued functions $N^{\pm}_{\Gamma}$ are solutions along the flow on Ω of the equation (4.5) and, according to Proposition 4.43, they are continuous on Ω. Moreover, the symplectic matrix-valued function C_Γ defined from these functions by (4.27) is continuous on Ω, and $\widetilde{\mathbf{z}} = C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$ determines a change of variables taking the initial systems (4.2) to a skew-symmetric family (4.10).

Remark 4.49 Note that if a family $\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}$ is taken to $\mathbf{w}' = \widetilde{H}(\omega{\cdot}t)\,\mathbf{w}$ by means of a change of variables $\mathbf{w} = C(\omega{\cdot}t)\,\mathbf{z}$ determined by a continuous map C, then the Sacker–Sell spectra of the two families coincide: the family $\mathbf{z}' = (H(\omega{\cdot}t) - \lambda I_{2n})\,\mathbf{z}$ is taken to $\mathbf{w}' = (\widetilde{H}(\omega{\cdot}t) - \lambda I_{2n})\,\mathbf{w}$. In addition, the spectral subbundles $F^{1}_{\widetilde{H}}, \ldots, F^{m}_{\widetilde{H}}$ and $F^{1}_{H}, \ldots, F^{m}_{H}$ of the two families are related by $F^{j}_{\widetilde{H}} = C{\cdot}F^{j}_{H} = \{(\omega, C(\omega)\,\mathbf{z}) \mid (\omega, \mathbf{z}) \in F^{j}_{H}\}$ for $j = 1, \ldots, m$. And clearly, the Lyapunov exponents of both families agree, as can be deduced from Definition 1.83.

The continuous variation of the Sacker–Sell spectral decomposition is stated in the following theorem. Note that Proposition 4.42(i) ensures the existence of perturbation directions Γ which satisfy the hypotheses. Clearly, a symmetric result can be stated for a negative definite limit A_Γ.

Theorem 4.50 *Suppose that Hypotheses 4.37 hold. Let $\Gamma \in \mathcal{C}$ give rise to a positive definite A_Γ, let $\pm i\mu_{\Gamma,1}, \dots, \pm i\mu_{\Gamma,d}$ be the different eigenvalues of $J^{-1}A_\Gamma$ with multiplicities $m_{\Gamma,1}, \dots, m_{\Gamma,d}$ respectively, ordered so that $0 < \mu_{\Gamma,1} < \dots < \mu_{\Gamma,d}$, and let η_Γ be defined by (4.77). For every $\eta \in (0, \eta_\Gamma)$ there exists $\varepsilon(\eta) > 0$ such that, if $\varepsilon \in (0, \varepsilon(\eta))$, then the following statements are valid.*

(i) *The Sacker–Sell spectrum of (4.76) is contained in the set*

$$\bigcup_{j=1}^{d} [\pm\varepsilon\mu_{\Gamma,j} - \varepsilon\eta, \pm\varepsilon\mu_{\Gamma,j} + \varepsilon\eta],$$

and each of the $2d$ (disjoint) intervals of this union contains at least one spectral interval.

(ii) *For each $\varepsilon \in (0, \varepsilon(\eta))$ and $j = 1, \dots, d$, the closed subbundles $F_\varepsilon^{\pm j}$ given by the sums of the spectral subbundles of (4.76) corresponding to the intervals contained in $[\mp\varepsilon\mu_j - \varepsilon\eta, \mp\varepsilon\mu_j + \varepsilon\eta]$ have dimension m_j. In addition, the maps $\Omega \times (0, \varepsilon(\eta)) \to \mathcal{G}_{m_j}(\mathbb{C}^{2n})$, $(\omega, \varepsilon) \to (F_\varepsilon^{\pm j})_\omega$ are continuous, and $\lim_{\varepsilon\to 0^+} (F_\varepsilon^{\pm j})_\omega = (F_0^{\pm j})_\omega$ in $\mathcal{G}_{m_j}(\mathbb{C}^{2n})$ uniformly on Ω, where $(F_0^{\pm j})_\omega$ are the eigenspaces of $J^{-1}A_\Gamma(\omega)$ associated to $\pm i\mu_{\Gamma,j}$, respectively.*

(iii) *Let m_0 be any σ-ergodic measure on Ω, fix $j = 1, \dots, d$, and let $\widetilde{\beta}^{\pm}_{\Gamma,j}(\varepsilon)$ be the sum of the Lyapunov exponents of (4.76) for m_0 which belong to the interval $[\pm\varepsilon\mu_{\Gamma,j} - \varepsilon\eta, \pm\varepsilon\mu_{\Gamma,j} + \varepsilon\eta]$. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{\beta}^{\pm}_{\Gamma,j}(\varepsilon)}{\varepsilon} = \pm m_{\Gamma,j}\mu_{\Gamma,j}\,.$$

In particular, the families of systems (4.76) corresponding to these values of ε have exponential dichotomy. In addition, for ε small enough, the stable subbundles at $\mp\infty$ can be represented by $\begin{bmatrix} I_n \\ M_\Gamma^{\pm}(\omega, i\varepsilon) \end{bmatrix}$, and the corresponding Weyl M-functions satisfy

$$\lim_{\varepsilon\to 0^+} M_\Gamma^{\pm}(\omega, i\varepsilon) = N_\Gamma^{\pm}(\omega)$$

uniformly on Ω, where $l_\Gamma^{\pm}(\omega) \equiv \begin{bmatrix} I_n \\ N_\Gamma^{\pm}(\omega) \end{bmatrix}$

Proof Let $U_\varepsilon(t, \omega)$ represent the fundamental matrix solution of (4.76) satisfying $U_\varepsilon(0, \omega) = I_{2n}$. As usual, the subindex is omitted for $\varepsilon = 0$.

The proof, which follows a scheme similar to that of the proof of Theorem 4.9, is divided into two steps. In the first one, (4.76) is reformulated with respect to a new base flow (Ω^1, σ^1), in order to find a continuous change of variables taking the unperturbed family of systems to $\mathbf{w}' = \mathbf{0}$. To this end, the unperturbed family (4.2) is transformed in the corresponding skew-symmetric family (4.10) by means of the continuous change of variables $\widetilde{\mathbf{z}} = C_\Gamma(\omega{\cdot}t)\,\mathbf{z}$. This matrix-valued function C_Γ will now play the role played by C in the proof of Theorem 4.9. It was explained there that the set $\Omega \times \mathcal{G}^1$, with $\mathcal{G}^1 = \left\{\begin{bmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{bmatrix} \,\middle|\, \begin{bmatrix}\widetilde{\Phi}_1 & -\widetilde{\Phi}_2\\ \widetilde{\Phi}_2 & \widetilde{\Phi}_1\end{bmatrix} \in \mathcal{G}\right\}$ for $\mathcal{G}$ defined in Sect. 1.3.4, is invariant under the flow $\widetilde{\tau}_{\mathbb{R}}$ induced on $\Omega \times \mathbb{M}_{2n\times n}(\mathbb{R})$ by the unperturbed transformed family (4.10). Let $\Omega^1 \subseteq \Omega \times \mathcal{G}^1$ be a minimal subset, and $\sigma^1 = \widetilde{\tau}_{\mathbb{R}}|_{\Omega^1}$. Also, represent by $\omega^1 = \left(\omega, \begin{bmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{bmatrix}\right)$ the elements of Ω^1, so that ω denotes the first component of ω^1.

Next, define $\widetilde{H}^1$, $\widetilde{V}^1$, H^1 and U^1 as in the proof of Theorem 4.9, set $C_\Gamma(\omega^1) = C_\Gamma(\omega)$ and $V^1_\Gamma = (C^1_\Gamma)^{-1}\widetilde{V}^1$, and note that: first, the change of variables $\widetilde{\mathbf{z}} = \widetilde{V}^1(\omega^1{\cdot}t)\,\mathbf{w}$ takes the family (4.19) to $\mathbf{w}' = \mathbf{0}$, so that $\mathbf{z} = V^1_\Gamma(\omega^1{\cdot}t)\,\mathbf{w}$ takes the unperturbed family (4.20) to $\mathbf{w}' = \mathbf{0}$; second, V^1_Γ is symplectic and continuous on Ω^1, and is a matrix solution along the flow on Ω^1 of equation (4.20); and third, $U^1(t, \omega^1) = V^1_\Gamma(\omega^1{\cdot}t)\,(V^1_\Gamma)^{-1}(\omega^1) = U(t, \omega)$ whenever ω is the first component of ω^1. Define also $\Gamma^1(\omega^1) = \Gamma(\omega)$, and consider the new extended family

$$\mathbf{z}' = \big(H^1(\omega^1{\cdot}t) + i\varepsilon J^{-1}\Gamma^1(\omega^1{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega^1 \in \Omega^1. \tag{4.78}$$

Let $U^1_\varepsilon(t, \omega^1)$ be the fundamental matrix solution of this system satisfying $U^1_\varepsilon(0, \omega^1) = I_{2n}$. It is obvious that $U^1_\varepsilon(t, \omega^1) = U_\varepsilon(t, \omega)$ for every $\omega^1 \in \Omega^1$. Consequently, the Sacker–Sell spectra of (4.76) and (4.78) coincide; the spectral decomposition of (4.78) can be obtained from that of (4.76) in a trivial way; and, if m^1_0 is a σ^1-ergodic measure on Ω^1 projecting onto m_0 (whose existence is guaranteed by the proof of Theorem 4.9), then the Lyapunov exponents of (4.78) with respect to m^1_0 agree with those of (4.76) with respect to m_0: see Remark 1.85.3 and Definition 1.83.

To complete the first step of the proof of Theorem 4.50, observe that the continuous and symplectic change of variables $\mathbf{z} = V^1_\Gamma(\omega^1{\cdot}t)\,\mathbf{w}$ takes the perturbed family (4.78) to

$$\mathbf{w}' = i\varepsilon J^{-1} W_{\Gamma^1}(\omega^1{\cdot}t)\,\mathbf{w}\,, \quad \omega^1 \in \Omega^1, \tag{4.79}$$

with $W_{\Gamma^1}(\omega^1) = (V^1_\Gamma)^T(\omega^1)\,\Gamma^1(\omega^1)\,V^1_\Gamma(\omega^1)$.

In the second step of the proof, a perturbative argument based on the ideas of [109] will provide a transformation of (4.79) into a new family of linear systems satisfying the hypotheses of Lemma 4.48.

Note that, since $U^1(t, \omega^1) = V^1_\Gamma(\omega^1{\cdot}t)\,(V^1_\Gamma)^{-1}(\omega^1)$, one has that

$$W_{\Gamma^1}(\omega^1{\cdot}t) = (V^1_\Gamma)^T(\omega^1)\,(U^1)^T(t, \omega)\,\Gamma^1(\omega^1{\cdot}t)\,U^1(t, \omega)\;V^1_\Gamma(\omega^1)\,.$$

Consequently, the limit

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} W_{\Gamma^1}(\omega^1{\cdot}s)\, ds = (V_\Gamma^1)^T(\omega^1)\, A_\Gamma^1(\omega^1)\, V_\Gamma^1(\omega^1) = D_\Gamma(\omega^1) \tag{4.80}$$

exists for every $\omega^1 \in \Omega^1$, and it determines a continuous and positive definite matrix-valued function D_Γ, since the hypotheses on Γ ensure these properties for the function

$$A_\Gamma^1(\omega^1) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} U^T(s,\omega^1)\Gamma^1(\omega^1{\cdot}s)U^1(s,\omega^1)\, ds = A_\Gamma(\omega)\,.$$

It is obvious that A_Γ^1 solves the equation $Z' = -(H^1)^T(\omega^1{\cdot}t)Z - ZH^1(\omega^1{\cdot}t)$ along the flow on Ω^1. This property, together with the last equality in (4.80) and the fact that V_Γ^1 solves (4.20) along the flow on Ω^1, implies that $D'_\Gamma(\omega^1)$ is σ^1-invariant, and hence it is constant on Ω^1: Theorem 1.6 ensures that it is almost everywhere constant for every σ^1-ergodic measure, and so the assertion follows from the continuity of D_Γ. Consequently, according to the Birkhoff Theorem 1.3,

$$\int_{\Omega^1} \big(W_{\Gamma^1}(\omega^1) - D_\Gamma\big)\, d\mu = 0 \tag{4.81}$$

for every σ^1-invariant measure μ in Ω^1. Note also that

$$\begin{aligned} J^{-1}D_\Gamma &= J^{-1}(V_\Gamma^1)^T(\omega^1)\, A_\Gamma^1(\omega^1)\, V_\Gamma^1(\omega^1) \\ &= (V_\Gamma^1)^{-1}(\omega^1)J^{-1}A_\Gamma^1(\omega^1)\, V_\Gamma^1(\omega^1)\,, \end{aligned} \tag{4.82}$$

since V_Γ^1 is symplectic. Consequently, the eigenvalues of $J^{-1}D_\Gamma$ agree with those of $J^{-1}A_\Gamma^1(\omega^1)$, and the respective eigenspaces of both matrices are related by means of the continuous matrix V_Γ^1.

Let the set $C(\Omega^1, \mathbb{M}_{2n\times 2n}(\mathbb{K}))$ of continuous matrix-valued functions on Ω^1 be endowed with the topology of the norm $\|B\|_{\Omega^1} = \max_{\omega^1\in\Omega^1} \|B(\omega^1)\|$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The results of Section 6 of Schwartzman [138] ensure the density of the set

$$\big\{f \in C(\Omega^1, \mathbb{M}_{2n\times 2n}(\mathbb{R}))\mid \text{ there exists } g\in C(\Omega^1) \text{ with } g'(\omega^1) = f(\omega^1)\big\}$$

(where, as usual, $g'(\omega^1)$ represents $(d/dt)g(\omega^1{\cdot}t)|_{t=0}$) in the set

$$\left\{f \in C(\Omega^1, \mathbb{M}_{2n\times 2n}(\mathbb{R}))\mid \int_{\Omega^1} f\, d\mu = 0 \;\text{ for every } \sigma^1\text{-invariant measure } \mu\right\}.$$

Now take a sequence of positive real numbers $(\kappa_j) \downarrow 0$ with $\kappa_1 < 1$. The previous property and (4.81) allow one to choose a sequence of matrix-valued functions (R_j)

in $C(\Omega^1, \mathbb{M}_{2n\times 2n}(\mathbb{R}))$ whose derivatives along the flow on Ω^1 exist, are continuous, and satisfy

$$\|R_j' - (W_{\Gamma^1} - D_\Gamma)\,\|_{\Omega^1} < \kappa_j \tag{4.83}$$

for every $j \in \mathbb{N}$. Denote $r_j = \|R_j\|_{\Omega^1}$ for every $j \in \mathbb{N}$. Take a strictly decreasing sequence of positive real numbers $(\varepsilon_j) \downarrow 0$ with $\varepsilon_j(r_j + r_{j+1}) \le \kappa_j$ for every $j \in \mathbb{N}$. Now define

$$R(\omega^1, \varepsilon) = \frac{\varepsilon - \varepsilon_{j+1}}{\varepsilon_j - \varepsilon_{j+1}}\, R_j(\omega^1) + \frac{\varepsilon_j - \varepsilon}{\varepsilon_j - \varepsilon_{j+1}}\, R_{j+1}(\omega^1) \quad \text{if } \varepsilon_{j+1} \le \varepsilon \le \varepsilon_j\,,$$

so that $\lim_{\varepsilon\to\varepsilon_j^+} R(\omega^1, \varepsilon) = R_j(\omega^1) = \lim_{\varepsilon\to\varepsilon_j^-} R(\omega^1, \varepsilon)$ and $\|\varepsilon R(\omega^1, \varepsilon)\|_{\Omega^1} \le \kappa_j < 1$ for all $\varepsilon \in (0, \varepsilon_1]$. Consequently, $R(\omega^1, \varepsilon)$ is continuous on $\Omega^1 \times (0, \varepsilon_1]$, and $\|\varepsilon J^{-1} R(\omega^1, \varepsilon)\|_{\Omega^1} < 1$ for all $\varepsilon \in (0, \varepsilon_1]$, which in turn ensures that $\det(I_{2n} + i\varepsilon J^{-1} R(\omega^1, \varepsilon)) \ne 0$ for every $(\omega^1, \varepsilon) \in \Omega^1 \times (0, \varepsilon_1]$. A straightforward computation allows one to deduce from (4.83) that the continuous linear change of variables $\mathbf{w} = (I_{2n} + i\varepsilon J^{-1} R(\omega^1{\cdot}t, \varepsilon))\,\widetilde{\mathbf{w}}$ takes (4.79) for $\varepsilon \in (0, \varepsilon_1]$ to

$$\widetilde{\mathbf{w}}' = i\varepsilon J^{-1}\left(D_\Gamma + \widetilde{W}_{\Gamma^1}(\omega^1{\cdot}t, \varepsilon)\right)\widetilde{\mathbf{w}}, \quad \omega^1 \in \Omega^1, \tag{4.84}$$

where $\|\widetilde{W}_{\Gamma^1}(\omega^1, \varepsilon)\|_{\Omega^1} \le c\kappa_j$ if $\varepsilon_{j+1} \le \varepsilon \le \varepsilon_j$, and where the constant c is independent of j. In particular, $\lim_{\varepsilon\to 0^+} \widetilde{W}_{\Gamma^1}(\omega^1, \varepsilon) = 0$ uniformly on Ω.

Now define $\widetilde{W}_{\Gamma^1}(\omega^1, 0) = 0$ and observe that the family (4.84) satisfies all the hypotheses of Lemma 4.48. Note also that the extended family (4.78) is taken to (4.84) by means of the continuous transformation

$$\mathbf{z} = V_\Gamma^1(\omega^1{\cdot}t)\,(I_{2n} + i\varepsilon J^{-1} R(\omega^1{\cdot}t, \varepsilon))\,\widetilde{\mathbf{w}}\,.$$

It follows from (4.82) that the eigenvalues $\pm\mu_{\Gamma,1}, \dots, \pm\mu_{\Gamma,d}$ of $J^{-1}A_\Gamma^1(\omega^1) = J^{-1}A_\Gamma(\omega)$ (which are independent of ω) agree with those of $J^{-1}D_\Gamma$, and that if $F(\omega^1)$ and $\widetilde{F}$ represent the eigenspaces of $J^{-1}A_\Gamma(\omega)$ and $J^{-1}D_\Gamma$ associated to the same eigenvalue, then $F(\omega^1) = V_\Gamma^1(\omega^1)\,\widetilde{F}$. Therefore, Lemma 4.48 and Remark 4.49 imply that the assertions in (i) and (ii) hold for the family (4.78), as well as the property stated in (iii) for the σ^1-ergodic measure m_0^1. The previous remark about the relation between the spectral decompositions and the Lyapunov exponents of (4.78) and (4.76) completes the proof of (i), (ii), and (iii) for the family (4.76).

In particular, the initial family of systems (4.76) has exponential dichotomy for $\varepsilon > 0$ small enough, and the corresponding closed subbundles converge uniformly on Ω as $\varepsilon \to 0$ to the Lagrange planes $V^1(\omega^1)\left[\begin{smallmatrix} I_n \\ N_\Gamma^\pm \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} I_n \\ N_\Gamma^\pm \end{smallmatrix}\right]$ are the complex Lagrange planes generated by the eigenvectors associated to the eigenvalues $\{\pm i\mu_{\Gamma,1}, \dots, \pm i\mu_{\Gamma,s}\}$ of $J^{-1}D_\Gamma$, respectively. The previously mentioned relation between the eigenspaces of $J^{-1}A_\Gamma(\omega)$ and $J^{-1}D_\Gamma$ implies that $\begin{bmatrix} I_n \\ N^{\pm}(\omega) \end{bmatrix}$ and $V^1(\omega^1)\begin{bmatrix} I_n \\ N_\Gamma^{\pm} \end{bmatrix}$ represent the same Lagrange plane $l_\Gamma^{\pm}(\omega^1)$. The last statement of the theorem follows from these facts: see Proposition 1.29(i) and Corollary 1.31.

4.5.2 The Variation with Respect to a Real Parameter

Consider now the perturbed families

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \varepsilon J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,,\quad \omega \in \Omega \tag{4.85}$$

for $\varepsilon \in \mathbb{R}$. In order to analyze the continuous variation of the spectral decomposition in this case, the following technical lemma will be required. The information provided by Lemma 4.12(ii) is required to understand its last statement.

Lemma 4.51 *Let D be a constant real positive definite symmetric $2n \times 2n$ matrix such that*

- *the different eigenvalues of $J^{-1}D$ are $\pm\mu_1, \dots, \pm\mu_d \in \mathbb{R}-\{0\}$ with multiplicities $m_1, \dots, m_d$ respectively, ordered so that $0 < \mu_1 < \cdots < \mu_d$,*
- *$J^{-1}D$ can be conjugated to a diagonal matrix,*

and define η_0 by (4.69). Let $T\colon [0,\varepsilon_1] \to C(\Omega, \mathbb{M}_{2n\times 2n}(\mathbb{C}))\,,\ \varepsilon \mapsto T(\omega,\varepsilon)$ be a continuous map with $T(\omega,0) = 0$ for an $\varepsilon_1 > 0$. Consider the families of linear systems

$$\mathbf{z}' = \varepsilon J^{-1}\,(D + T(\omega{\cdot}t,\varepsilon))\,\mathbf{z}\,,\quad \omega \in \Omega\,, \tag{4.86}$$

for $\varepsilon \in [0,\varepsilon_1]$. For each $\eta \in (0,\eta_0)$ there exists $\varepsilon(\eta) > 0$ such that, if $\varepsilon \in (0,\varepsilon(\eta))$, then the following statements are valid.

(i) *The Sacker–Sell spectrum of (4.86) is contained in the set*

$$\bigcup_{j=1}^{d} [\pm\varepsilon\mu_j - \varepsilon\eta, \pm\varepsilon\mu_j + \varepsilon\eta\,]\,,$$

and each of the $2d$ (disjoint) intervals of this union contains at least one spectral interval.

(ii) *For each $\varepsilon \in (0,\varepsilon(\eta))$ and $j = 1,\dots,d$, the closed subbundle $F_\varepsilon^{\pm j}$ given by the sums of the spectral subbundles of (4.86) corresponding to the intervals contained in $[\mp\varepsilon\mu_j - \varepsilon\eta, \mp\varepsilon\mu_j + \varepsilon\eta\,]$ has dimension m_j. In addition, the maps $\Omega \times (0,\varepsilon(\eta)) \to \mathcal{G}_{m_j}(\mathbb{C}^{2n}),\ \ (\omega,\varepsilon) \mapsto (F_\varepsilon^{\pm j})_\omega$ are continuous, and*

$\lim_{\varepsilon\to 0^+}(F_\varepsilon^{\pm j})_\omega = F_0^{\pm j}$ *in* $\mathcal{G}_{m_j}(\mathbb{C}^{2n})$ *uniformly on* Ω*, where* $F_0^{\pm j}$ *are the eigenspaces of* $J^{-1}D$ *which are associated to* $\mp\mu_j$*, respectively.*

(iii) *Let* m_0 *be any* σ*-ergodic measure on* Ω*, fix* $j = 1,\dots,d$*, and let* $\widetilde{\beta}_j^{\pm}(\varepsilon)$ *be the sum of the Lyapunov exponents (equal or distinct) of* (4.86) *for* m_0 *which belong to the interval* $[\pm\varepsilon\mu_j - \varepsilon\eta, \pm\varepsilon\mu_j + \varepsilon\eta]$*. Then,*

$$\lim_{\varepsilon\to 0^+}\frac{\widetilde{\beta}_j^{\pm}(\varepsilon)}{\varepsilon} = \pm m_j\mu_j\,.$$

In particular, the families of systems (4.86) *corresponding to these values of* ε *have exponential dichotomy. Finally, if* $l_\varepsilon^{\pm}(\omega)$ *represent the corresponding stable subbundles at* $\mp\infty$*, then* $l_\varepsilon^{\pm}(\omega)$ *converge uniformly on* Ω *as* $\varepsilon \to 0^+$ *to the real Lagrange planes* $l^{\pm}$ *generated by the eigenvectors of* $J^{-1}D$ *which are associated to the negative and positive eigenvalues, respectively.*

The proof of this result reproduces step by step that of Lemma 4.48; for this reason it is omitted. Theorem 4.52 will be treated in the same way, indeed it can be proved as was Theorem 4.50. The main point to keep in mind in order to understand its statement is that, under the following hypotheses on A_Γ, the linear invariant subbundles respectively generated by the eigenvectors which are associated to the negative or positive eigenvalues, $\{(\omega, l^{\pm}(\omega)) \mid \omega \in \Omega\}$, are closed and, according to Lemma 4.12(ii), $l^{\pm}(\omega)$ are real Lagrange planes.

Theorem 4.52 *Consider a perturbation* $\Gamma \in \mathcal{C}$ *such that*

– *the different eigenvalues of* $J^{-1}A_\Gamma(\omega)$ *are* $\pm\mu_{\Gamma,1},\dots,\pm\mu_{\Gamma,d} \in \mathbb{R}-\{0\}$*, with respective multiplicities* $m_{\Gamma,1},\dots,m_{\Gamma,d}$*;*
– $J^{-1}A_\Gamma(\omega)$ *can be conjugated to a diagonal matrix,*

and define η *by* (4.77)*. For every* $\eta \in (0,\eta_\Gamma)$ *there exists* $\varepsilon(\eta) > 0$ *such that, if* $\varepsilon \in (0,\varepsilon(\eta))$*, then the following statements are valid.*

(i) *The Sacker–Sell spectrum of* (4.85) *is contained in the set*

$$\bigcup_{j=1}^{d}[\pm\varepsilon\mu_{\Gamma,j} - \varepsilon\eta, \pm\varepsilon\mu_{\Gamma,j} + \varepsilon\eta]\,,$$

and each of the $2s$ *(disjoint) intervals of this union contains at least one spectral interval.*

(ii) *For each* $\varepsilon \in (0,\varepsilon(\eta))$ *and* $j = 1,\dots,d$*, the closed subbundles* $F_\varepsilon^{\pm j}$ *given by the sums of the spectral subbundles of* (4.85) *corresponding to the intervals contained in* $[\mp\varepsilon\mu_j - \varepsilon\eta, \mp\varepsilon\mu_j + \varepsilon\eta]$ *have dimension* m_j*. In addition, the maps* $\Omega\times(0,\varepsilon(\eta)) \to \mathcal{G}_{m_j}(\mathbb{C}^{2n})$*,* $(\omega,\varepsilon) \mapsto (F_\varepsilon^{\pm j})_\omega$ *are continuous, and* $\lim_{\varepsilon\to 0^+}(F_\varepsilon^{\pm j})_\omega = (F_0^{\pm j})_\omega$ *in* $\mathcal{G}_{m_j}(\mathbb{C}^{2n})$ *uniformly on* Ω*, where* $(F_0^{\pm j})_\omega$ *are the eigenspaces of* $J^{-1}A_\Gamma(\omega)$ *which are associated to* $\mp\mu_{\Gamma,j}$*, respectively.*

(iii) *Let* m_0 *be any* σ*-ergodic measure on* Ω*, fix* $j = 1, \ldots, d$*, and let* $\widetilde{\beta}^{\pm}_{\Gamma j}(\varepsilon)$ *be the sum of the Lyapunov exponents of* (4.85) *for* m_0 *which belong to the interval* $[\pm\varepsilon\mu_{\Gamma j} - \varepsilon\eta, \pm\varepsilon\mu_{\Gamma j} + \varepsilon\eta]$*. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{\beta}^{\pm}_{\Gamma j}(\varepsilon)}{\varepsilon} = \pm m_{\Gamma j}\mu_{\Gamma j}\,.$$

In particular, the families of systems corresponding to these values of ε *have exponential dichotomy. Finally, if* $l^{\pm}_{\Gamma,\varepsilon}(\omega)$ *represent the corresponding stable subbundles at* $\mp\infty$*, then* $\lim_{\varepsilon\to 0^+} l^{\pm}_{\Gamma,\varepsilon}(\omega) = l^{\pm}_{\Gamma}(\omega)$ *in* $\mathcal{K}_{\mathbb{R}}$ *uniformly on* Ω*.*

Note finally that Theorem 4.52 can be viewed as an extension to the nonautonomous linear Hamiltonian context of some of the results of Moser and Pöschel [109], which they proved in the case of a quasi-periodic two-dimensional linear Hamiltonian system.

Chapter 5
Weak Disconjugacy for Linear Hamiltonian Systems

The analysis of nonautonomous linear Hamiltonian systems with the disconjugacy property, which is closely related to their oscillation properties and which has applications in the calculus of variations, is an extended and classical branch of the study of linear differential systems. The texts of Hartman [56], Coppel [34], and Reid [127] contain the fundamental facts concerning this property. One of the most interesting properties of a disconjugate system is the existence of principal solutions: in many interesting situations they constitute an extension to the nonuniformly hyperbolic case of the Lagrange planes associated, in the case of exponential dichotomy, to the bounded solutions at $+\infty$ and $-\infty$: see Remarks 1.79 and 1.77.3.

More recently, Johnson et al. [81, 82] extended the classical analysis, using the methods of the modern theory of nonautonomous differential systems, many of which are drawn from the fields of topological dynamics and ergodic theory. These techniques allowed the authors to study the dynamical and ergodic properties of the principal solutions, and to go much deeper into the close relation between principal solutions, Lyapunov indices, and exponential dichotomy.

Later, Fabbri et al. [43, 48] introduced and analyzed a less restrictive condition called weak disconjugacy, often but not always equivalent to the classical disconjugacy property. The main advantage of weak disconjugacy as opposed to disconjugacy is that it holds under a much weakened version of the condition of identical normality, which is often imposed when studying the classical disconjugacy property. One of the reasons for the authors to introduce this concept was its clear relation with the oscillatory properties of the system analyzed (or, more precisely, with the absence of oscillation). It provided a suitable framework to optimize the hypotheses of certain results based on the properties of the rotation number.

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But in fact the interest of weak disconjugacy goes beyond this first analysis. As shown in Johnson et al. [78], under different additional conditions (still often providing a scenario less restrictive than the disconjugate one), the weak disconjugacy property also ensures the existence of principal solutions. The authors of [43] describe mild conditions under which the lack of oscillation of a linear Hamiltonian system is equivalent to its weak disconjugacy, as well as stronger conditions which ensure the existence of principal solutions for a given system. In [85], the close relation between principal solutions and exponential dichotomy is analyzed in detail and, as a consequence, it is shown that the Yakubovich frequency theorem (in its nonautonomous form as developed in [47]) can be applied to a wide range of optimization problems. This analysis relies on the strong connection between the uniform weak disconjugacy and the controllability properties of some systems constructed from the initial one; and clearly this relation has independent interest. Also in [84], which is dedicated to the analysis of dissipative linear quadratic control systems, the properties of weakly disconjugate systems allow the authors to relax the conditions ensuring the dissipativity. All these questions are explained in detail in the next chapters, in which the occurrence of weak disconjugacy for the Hamiltonian systems to be analyzed will play a fundamental role.

This chapter collects and unifies the results of all the mentioned papers, extending some of them. A somewhat more detailed description of its contents completes this introduction.

Under a very weak version of identical normality, the notion of weak disconjugacy can be characterized in terms of the nonoscillatory behavior of the system under study. To establish this connection is the goal of Sect. 5.1. The arguments are based on some of the properties explained in Sect. 2.4, which in turn are based on previous results of Yakubovich [153, 154], Lidskiĭ [96], and Gel'fand and Lidskiĭ [53].

In the rest of the chapter, a family of linear Hamiltonian systems defined over a continuous base flow is considered. If some additional conditions of uniformity hold, the weak disconjugacy property guarantees the presence of the so-called *uniform* principal solutions at $+\infty$ and $-\infty$ for each of the systems of the family. These matrix-valued solutions play a fundamental role in the dynamical description of the Lagrangian flow induced by the family. They define orbits of the Lagrangian flow, and always lie outside the vertical Maslov cycle. Section 5.2 is devoted to the proof of their existence in the uniform setting, which will be the scenario almost always considered in the rest of the chapter.

The results of Sect. 5.3, generally speaking, concern the connections between disconjugacy and weak disconjugacy. First, they describe several scenarios in which the uniform weak disconjugacy property studied in the previous section is equivalent to the true disconjugacy of all the systems, as well as others in which it is guaranteed by *a priori* less restrictive hypotheses. One of the conclusions of this analysis is that the main contribution of the theory of weak disconjugacy as opposed to the classical one concerns the situation where $H_3 \geq 0$ (where $H = \begin{bmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{bmatrix}$ is the coefficient matrix) but is not positive definite. Second, they establish conditions on a single nonautonomous linear Hamiltonian system which guarantee the uniform weak disconjugacy of the family constructed from that system by the usual hull procedure. This fact is especially significant, since most of the results of the section concern all the systems of the family, including hence the initial one. Third, they prove the existence of (not necessarily uniform) principal solutions in a setting much more general than that of disconjugacy. Some examples showing the optimality of the results complete the section.

The principal solutions always admit unique representations in $\mathcal{L}_{\mathbb{R}}$ determined by two real symmetric $n \times n$ matrix-valued functions $N^{\pm}$, which will be called principal functions. In the case of uniform weak disconjugacy, they are bounded solutions along the flow of the associated family of Riccati matrix equations, and they are semicontinuous on the base: semicontinuous equilibria, in the language of Sect. 1.3.5. As a consequence of their definition and of the monotonicity properties of the Riccati equation, the principal functions $N^{\pm}$ delimit a compact invariant zone in $\mathcal{K}_{\mathbb{R}}$ which concentrates any invariant measure on $\mathcal{K}_{\mathbb{R}}$, contains any minimal subset, and, frequently, contains also the graph of any continuous invariant function from the base to $\mathcal{L}_{\mathbb{R}}$. These are basically the contents of Sect. 5.4, which also include some properties derived from the semicontinuity of the principal functions: they determine almost automorphic extensions of the base flow (see Definition 1.18) in the case that the base flow is minimal. A fundamental comparison result completes the section.

Weak disconjugacy and principal solutions are closely related to many of the objects analyzed in the previous chapters. Section 5.5 details their relation with the Lyapunov index and the Oseledets subbundles. It is shown in Sect. 5.6 that, in the case of uniform weak disconjugacy, the principal solutions determine closed supplementary subbundles if and only if exponential dichotomy occurs, in which case the Weyl functions exist and agree with the principal functions. In addition, if $H_3 \geq 0$ and the family has exponential dichotomy, the uniform weak disconjugacy property is equivalent to the existence of both Weyl functions. This result will be of relevance in Chap. 7. And Sect. 5.7 presents an ergodic characterization of the presence of weak disconjugacy and, in some cases, disconjugacy, in terms of the rotation number of the family of linear Hamiltonian systems.

Sections 5.6 and 5.7 also contain perturbative results showing that a family of Hamiltonian systems with the properties analyzed in this chapter is always the limit of a one-parameter family of families possessing exponential dichotomy over Ω. These properties are used to establish a result of continuity of the principal solutions with respect to the parameter. It should be noted that, even in the context of linear Hamiltonian systems which are disconjugate in the classical sense, strong technical conditions are required to ensure the continuous dependence of the principal solutions with respect to the coefficient matrix: see Reid [126]. In Sect. 5.8, weak conditions are imposed on the perturbed families which guarantee that the weak convergence in measure of the principal solutions is equivalent to the convergence of the corresponding Lyapunov indices.

Finally, Sect. 5.9 presents an analysis which in a sense completes the previous one. The so-called *abnormal* linear Hamiltonian systems at $+\infty$ or $-\infty$ are those determining some positive or negative τ_r-semiorbits lying in $\Omega \times \mathcal{C}$, where $\mathcal{C}$ represents the vertical Maslov cycle. Such systems are clearly not disconjugate, and in fact define families (by the usual hull construction) which are not uniformly weakly disconjugate at $+\infty$ or $-\infty$. But still it is possible to combine the usual topological and measurable tools in order to describe interesting properties regarding the dynamical behavior of these abnormal systems.

As usual, the goals of the different sections are more precisely described at the beginning of each one of them.

5.1 Weak Disconjugacy and Nonoscillation

This section concerns a single linear Hamiltonian system

$$\mathbf{z}' = H_0(t)\,\mathbf{z}\,, \tag{5.1}$$

where $H_0 = \begin{bmatrix} H_{01} & H_{03} \\ H_{02} & -H_{01}^T \end{bmatrix}$ is continuous and bounded on $\mathbb{R}$ and takes values in $\mathfrak{sp}(n,\mathbb{R})$. The following concepts of weak disconjugacy and nonoscillation appear in Fabbri *at al.* [48] and Yakubovich [154], respectively. But first the classical concept of disconjugacy (see e.g. [34]) is recalled.

Definition 5.1 The linear Hamiltonian system (5.1) is *disconjugate on* $\mathbb{R}$ if, for every nonzero solution $\mathbf{z}(t) = \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix}$, the vector $\mathbf{z}_1(t)$ vanishes at most once on $\mathbb{R}$.

Definition 5.2 The linear Hamiltonian system (5.1) is *weakly disconjugate on* $[0,\infty)$ (resp. on $(-\infty,0]$) if there exists $t_0 \ge 0$ such that, for every nonzero solution $\mathbf{z}(t) = \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix}$ with $\mathbf{z}_1(0) = \mathbf{0}$, there holds $\mathbf{z}_1(t) \ne \mathbf{0}$ for all $t > t_0$ (resp. for all $t < -t_0$).

Clearly a disconjugate system satisfies this definition on both half-lines for $t_0 = 0$, which justifies the choice of the name for this less restrictive behavior.

In the following definition, Arg denotes any of the equivalent arguments for a real symplectic matrix defined in Sect. 2.1.1.

Definition 5.3 The linear Hamiltonian system (5.1) is said to be *nonoscillatory at* $+\infty$ (resp. *at* $-\infty$) if $\operatorname{Arg} V(t)$ is a bounded function on $[0,\infty)$ (resp. on $(-\infty,0]$), where $V(t)$ is any symplectic fundamental matrix solution and a continuous branch of the argument is taken along the curve.

Note that the above definition is independent of the choices of Arg and $V(t)$, as can be deduced from the Definition 2.3 of equivalence of arguments and the results of [153] summarized immediately below it.

Remark 5.4 In the case of a constant or periodic coefficient matrix H_0, the nonoscillation at $+\infty$ or at $-\infty$ is equivalent to the fact that the rotation number of the system (5.1) vanishes: Theorem 2.4 and Remark 2.6 prove the direct implication, and Remark 2.8 the converse one. More facts about the relation between nonoscillation and the vanishing of the rotation number are established in Sect. 5.7.

This section is devoted to an analysis of the connection between weak disconjugacy and oscillation: Proposition 5.7 shows that weak disconjugacy on a half-line implies nonoscillation at the corresponding limit point, and Proposition 5.9(ii) establishes conditions under which the converse is also true. The optimal assertion is given in Theorem 5.11.

An easy result will be required. It characterizes weak disconjugacy in terms of the symplectic fundamental matrix solution $U(t) = \left[\begin{smallmatrix} U_1(t) & U_3(t) \\ U_2(t) & U_4(t) \end{smallmatrix}\right]$ of (5.1) with $U(0) = I_{2n}$, which is fixed for the rest of the section, and which as seen in Sect. 1.3.3 is real and symplectic for all $t \in \mathbb{R}$.

Lemma 5.5 *The linear Hamiltonian system* (5.1) *satisfies Definition 5.2 of weak disconjugacy on* $[0, \infty)$ *(resp. on* $(-\infty, 0]$ *if and only* $\det U_3(t) \neq 0$ *for each* $t > t_0$ *(resp. for* $t < -t_0$*).*

Proof The assertion follows from the equality $U(t)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_3(t)\,\mathbf{z}_2 \\ U_4(t)\,\mathbf{z}_2 \end{smallmatrix}\right]$.

In the rest of the section, the matrix

$$W_U(t) = (U_1(t) - iU_3(t))^{-1}(U_1(t) + iU_3(t)) \tag{5.2}$$

is associated to the fixed fundamental matrix solution of (5.1). As stated in Sect. 2.4, it is possible to choose continuous functions $\rho_1, \ldots, \rho_n \colon \mathbb{R} \to \mathbb{C}$ with $|\rho_j(t)| = 1$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$, such that the set of eigenvalues of $W_U(t)$ coincides with the unordered n-tuple $\{\rho_1(t), \ldots, \rho_n(t)\}$. Let $\varphi_1, \ldots, \varphi_n \colon \mathbb{R} \to \mathbb{R}$ be continuous argument functions: $\rho_j(t) = e^{i\varphi_j(t)}$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$.

Lemma 5.6 *The sum* $(1/2)\sum_{j=1}^{n} \varphi_j(t)$ *is a continuous branch of the argument* $\operatorname{Arg}_3 U(t) = \arg\det(U_1(t) + iU_3(t))$.

Proof According to Lemma 2.29(i), $\det W_U(t) = r(t)\det^2(U_1(t) + iU_3(t))$, where the function r takes strictly positive values. All the functions involved are continuous, and the assertion follows easily.

Proposition 5.7 *If the system* (5.1) *is weakly disconjugate on* $[0, \infty)$ *(resp. on* $(-\infty, 0]$*), then it is nonoscillatory at* $+\infty$ *(resp. at* $-\infty$*).*

Proof Assume the weak disconjugacy on $[0, \infty)$. According to Lemmas 5.5 and 2.29(ii), there exists $t_0 \geq 0$ such that 1 is not an eigenvalue of $W_U(t)$ if $t > t_0$. The continuity of the angle functions φ_j ensures then that $\varphi_j(t) \in (2\pi m_j, 2\pi(m_j+1))$ for an $m_j \in \mathbb{Z}$ if $j = 1, \ldots, n$ and $t > t_0$. This fact and Lemma 5.6 prove the assertion. The other case can be proved similarly.

The arguments of the proof of Proposition 5.7 can be used to prove the following result, which is more general and which will be useful in Sect. 5.3.

Proposition 5.8 *If there exists a real Lagrange plane* $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ *such that the* $2n \times n$ *matrix-valued solution* $\left[\begin{smallmatrix} L_1(t) \\ L_2(t) \end{smallmatrix}\right] = U(t) \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ *of* (5.1) *satisfies* $\det L_1(t) \neq 0$ *for every* t *in a positive (resp. negative) half-line, then the system is nonoscillatory at* $+\infty$ *(resp. at* $-\infty$*).*

Proof As in the proof of Theorem 2.36, choose any matrix $\left[\begin{smallmatrix} L_3 \\ L_4 \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] \in Sp(n, \mathbb{R})$; for instance, $L_3 = L_2 R^{-1}$ and $L_4 = -L_1 R^{-1}$ for $R = L_1^T L_1 + L_2^T L_2$. Then $V(t) = U(t) \left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_3(t) & L_1(t) \\ L_4(t) & L_2(t) \end{smallmatrix}\right]$ takes values in $Sp(n, \mathbb{R})$. The arguments of Lemma 5.6 and Proposition 5.7 can be repeated in order to prove the existence of a continuous branch of $\operatorname{Arg}_3 V(t)$ bounded in a positive (resp. negative) half-line, which implies the assertion.

Note that the previous result can be rewritten as: if there exists a symmetric matrix-valued solution of the Riccati equation

$$M' = -M\, H_{03}(t)\, M - M\, H_{01}(t) - H_{01}^T(t)\, M + H_{02}(t)$$

defined on a positive (resp. negative) half-line, then the system is nonoscillatory at $+\infty$ (resp. at $-\infty$): see e.g. Sect. 1.3.5. Many examples illustrating this situation, from trivial to quite nontrivial, will be described in the book. In fact this is the case in most of the examples described in this chapter.

It is clear that the converse of Proposition 5.7 cannot be true in the general situation, even with $H_{03} \geq 0$. To see this, just think about the case $H_0 \equiv 0_{2n}$. However, more can be said in the case $H_{03} \geq 0$. Recall that the concept of proper focal point for a given $2n \times n$ matrix-valued solutions of the Hamiltonian system taking values in $\mathcal{L}_{\mathbb{R}}$ is given in Definition 2.35. The next result characterizes the nonoscillation at $\pm\infty$ in terms of the existence of a maximal or minimal proper focal point for any solution lying in $\mathcal{L}_{\mathbb{R}}$, and establishes an additional condition ensuring the converse wof Proposition 5.7.

Proposition 5.9 *Suppose that* $H_{03} \geq 0$*. Then,*

(i) *the linear Hamiltonian system* (5.1) *is nonoscillatory at* $+\infty$ *(resp. at* $-\infty$*) if and only if the number of positive (resp. negative) proper focal points is finite for the matrix-valued solution* $\left[\begin{smallmatrix} L_1(t) \\ L_2(t) \end{smallmatrix}\right] = U(t) \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ *where* $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ *is any element of* $\mathcal{L}_{\mathbb{R}}$*. In this case, if* t_m *is the largest (resp. if* $\tilde{t}_m$ *is the lowest) proper focal point, there exists* $s_l \geq t_m$ *(resp.* $\tilde{s}_l \leq \tilde{t}_m$*) such that* $\operatorname{Ker} L_1(t)$ *is constant on* (s_l, ∞) *(resp. on* $(-\infty, \tilde{s}_l)$*).*

(ii) *If, in addition, for every nonzero solution* $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ *of* (5.1) *with* $\mathbf{z}_1(0) = \mathbf{0}$ *the vector* $\mathbf{z}_1(t)$ *does not vanish identically on* $[t_1, \infty)$ *(resp. on* $(-\infty, t_1]$*) for all* $t_1 \in \mathbb{R}$*, then the system* (5.1) *is weakly disconjugate on* $[0, \infty)$ *(resp. on* $(-\infty, 0]$*).*

Proof

(i) The notation is now taken from the proof of Lemma 2.34. Suppose first the nonoscillation at $+\infty$. The characterization is an immediate consequence of: inequality (2.53), which ensures that $\varphi(t) = (1/2)\sum_{j=1}^{n}\varphi_j(t)$ is bounded if and only if the number of proper focal points is finite; the fact that $\varphi(t)$ is a continuous branch of $\operatorname{Arg}_3 V(t)$, guaranteed by Lemma 2.29(i); and Definition 5.3 of nonoscillation at $+\infty$. To prove the last assertion note that Definition 2.52 and Lemma 2.34 ensure that $\dim \operatorname{Ker} L_1(t)$ may only decrease on (t_m, ∞), so that there are at most $\dim \operatorname{Ker}(t_m)$ points at which it changes. The point s_l can be taken as the largest point at which $\dim \operatorname{Ker} L_1(t)$ changes, and the result follows from Lemma 2.34(i).

In the case of nonoscillation at $-\infty$, the argument is the same. It is necessary to repeat the reasoning of the proof of Theorem 2.36 in order to show the inequality analogous to (2.53) in $(-\infty, 0]$. Note also that to the left of the lowest focal point $\tilde{t}_m$ the dimension of $\operatorname{Ker} L_1(t)$ may only increase as t decreases.

(ii) Let s_l be the time associated by (i) to the initial Lagrange plane $l \equiv \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$, so that now $U_3(t)$ plays the role of $L_1(t)$. Suppose for contradiction that (5.1) is not weakly disconjugate, and observe that Lemma 5.5 ensures that there exists a sequence $(t_m) \uparrow \infty$ with $\det U_3(t_m) = 0$ for all $m \in \mathbb{N}$. Then there exists at least one argument function $\varphi_* \in \{\varphi_1, \ldots, \varphi_n\}$ (of those associated to the matrix function W_U) and a subsequence (t_j) with $\varphi_*(t_j) = 0$ modulus 2π. Since φ_* is bounded, continuous, and nondecreasing, this means that there exists a time t_j with $\varphi_*(t) = \varphi_*(t_j)$ for all $t \geq t_*$. Therefore, $\dim \operatorname{Ker} U_3(t) \geq 1$ for any $t \geq t_j$, and hence it is at least 1 for the constant vector space $k_U = \operatorname{Ker} U_3(t)$ for $t \in (s_l, \infty)$. Now take $\mathbf{z}_2 \in k_U$, $\mathbf{z}_2 \neq \mathbf{0}$ and consider the solution $\mathbf{z}(t) = U(t)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_3(t)\,\mathbf{z}_2 \\ U_4(t)\,\mathbf{z}_2 \end{smallmatrix}\right]$. Then $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ for $t > t_1$. This contradicts one of the hypotheses of (ii), which is hence proved.

Remark 5.10 Continuing with the notation of the previous result: it will be seen in Sect. 5.9 that, in the case of nonoscillation at $+\infty$, the constant value of $\dim \operatorname{Ker} L_1(t)$ on (s_l, ∞) agrees with the number of linearly independent solutions of (5.1) taking the form $U(t)\left[\begin{smallmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ on (s_l, ∞), with $\left[\begin{smallmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{smallmatrix}\right] \in l$; and that in the case of nonoscillation at $-\infty$, the constant value of $\dim \operatorname{Ker} L_1(t)$ on $(-\infty, \tilde{s}_l)$ agrees with the number of linearly independent solutions of (5.1) taking the form $U(t)\left[\begin{smallmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ on $(-\infty, \tilde{s}_l)$, with $\left[\begin{smallmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{smallmatrix}\right] \in l$.

According to Proposition 5.9(i), in the case $H_{03} \geq 0$, it can be stated that a system is nonoscillatory at $+\infty$ or at $-\infty$ if and only if it has a finite number of positive or negative proper focal points. In fact, this characterization is sometimes taken as the definition: see for instance [141].

Theorem 5.11 *Suppose that $H_{03}(t) \geq 0$ for each $t \in \mathbb{R}$. Then the system (5.1) is weakly disconjugate on $[0, \infty)$ (resp. on $(-\infty, 0]$) if and only if it is nonoscillatory at $+\infty$ (resp. at $-\infty$) and, for every nonzero solution $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ of (5.1) with $\mathbf{z}_1(0) = \mathbf{0}$, the vector $\mathbf{z}_1(t)$ does not vanish identically on $[t_1, \infty)$ (resp. on $(-\infty, t_1]$) for all $t_1 \in \mathbb{R}$.*

Proof The result follows from Proposition 5.7 and Proposition 5.9(ii), since the condition regarding the behavior on positive half-lines of nonzero solutions holds automatically if (5.1) is weakly disconjugate.

Corollary 5.12 *Suppose that* $H_{03}(t) > 0$ *for each* $t \in \mathbb{R}$. *Then the system* (5.1) *is weakly disconjugate on* $[0, \infty)$ *(resp. on* $(-\infty, 0]$ *if and only if it is nonoscillatory at* $+\infty$ *(resp. at* $-\infty$*).*

Proof If $\mathbf{z}_1(t) = \mathbf{0}$ for a nonzero solution, then $\mathbf{z}_1'(t) = H_{03}(t)\,\mathbf{z}_2(t) \neq \mathbf{0}$. This precludes the existence of solutions of (5.1) taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on any nondegenerate interval. Once this property is established, the result follows immediately from Theorem 5.11.

This section is completed with an example which illustrates the significance of the condition involving $\mathbf{z}_1(t)$ in Theorem 5.11 which must be added to nonoscillation in order to guarantee the existence of weak disconjugacy. In fact one might conjecture that if this condition holds and $0_n \not\equiv H_{03} \geq 0$, then the following stronger condition is valid: there is a sequence $(t_m) \uparrow \infty$ such that, if $l_v \equiv \left[\begin{smallmatrix}0_n\\ I_n\end{smallmatrix}\right]$ and $l(t_m) = U(t_m) \cdot l_v \equiv U(t_m)\left[\begin{smallmatrix}0_n\\ I_n\end{smallmatrix}\right]$, then $l(t_m) \cap l_v = \{\mathbf{0}\}$ for all $m \in \mathbb{N}$. Or, in geometrical terms, that $l(t_m)$ is outside the vertical Maslov cycle $\mathcal{C}$ defined by (2.35) for all $m \in \mathbb{N}$. This conjecture is indeed true if $n = 1$, but need not be true if $n \geq 2$, as the next example shows.

Example 5.13 Let $g\colon \mathbb{R} \to \mathbb{R}$ be the (continuous) π-periodic extension of the piecewise linear map satisfying $g(0) = g(\pi) = 0$ and $g(\pi/2) = 2$, define $G(t) = \int_0^t g(s)\,ds$, and note that $G(k\pi) = |k|\pi$ and $G(\pi/2 + k\pi) = \pi/2 + |k|\pi$ for all $k \in \mathbb{Z}$. Consider the 4-dimensional linear Hamiltonian system

$$\mathbf{z}' = H_0(t)\,\mathbf{z}\,, \tag{5.3}$$

where H_0 is the continuous 2π-periodic function given on $[2k\pi, (2k+2)\pi]$ by

$$H_0(t) = \begin{cases} \begin{bmatrix} 0 & 0 & g(t) & 0 \\ 0 & 0 & 0 & 0 \\ -g(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{for } 2k\pi \leq t \leq (2k+1)\pi\,, \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g(t) \\ 0 & 0 & 0 & 0 \\ 0 & -g(t) & 0 & 0 \end{bmatrix} & \text{for } (2k+1)\pi \leq t \leq (2k+2)\pi\,. \end{cases}$$

Note that $H_{03} \geq 0$. It is easy to obtain the matrix solution $U(t)$ of (5.3) with $U(0) = I_{2n}$, which is 4π-periodic. In fact, $U(t)$ is defined by

$$U(t) = \begin{cases} \begin{bmatrix} \cos G(t) & 0 & \sin G(t) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin G(t) & 0 & \cos G(t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{for } 0 \leq t \leq \pi\,, \\ \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\cos G(t) & 0 & -\sin G(t) \\ 0 & 0 & -1 & 0 \\ 0 & \sin G(t) & 0 & -\cos G(t) \end{bmatrix} & \text{for } \pi \leq t \leq 2\pi\,, \end{cases}$$

and the rule $U(t) = -U(t-2\pi)$ for $t \in \mathbb{R}$. From this expression it is immediate to check that for every nonzero solution $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ of (5.3) with $\mathbf{z}_1(0) = \mathbf{0}$, the vector $\mathbf{z}_1(t)$ does not vanish identically on any positive half-line, and that $\det U_3(t) = 0$ (i.e. $U(t){\cdot}l_v$ with $l_v = \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$ belongs to the vertical Maslov cycle $\mathcal{C}$) for all $t \in \mathbb{R}$. By, for example, Lemma 5.5, the system (5.3) is not weakly disconjugate on $[0,\infty)$; thus, by Proposition 5.9(ii), $\operatorname{Arg} U(t)$ must be unbounded as $t \to \infty$.

5.2 Uniform Weak Disconjugacy and Principal Solutions

In the rest of the chapter, (Ω,σ) will denote a real continuous flow on a compact metric space, and a family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{5.4}$$

with $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right] : \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ continuous and taking values in $\mathfrak{sp}(n,\mathbb{R})$, will be the object of study.

This section is devoted to establishing conditions ensuring the so-called uniform weak disconjugacy of the family (5.4), and to derive from this fact the existence and characteristics of the uniform principal solutions. The first point is hence to define these concepts.

As usual, $U(t,\omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U_3(t,\omega) \\ U_2(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ is the (symplectic) fundamental matrix solution of (5.4) with $U(0,\omega) = I_{2n}$. Recall that the flow τ induced by the family (5.4) on $\mathcal{K}_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}}$ is given by $\tau(t,\omega,l) = (\omega{\cdot}t, U(t,\omega){\cdot}l)$ (see (1.14)). Recall also that the set

$$\mathcal{D} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \right\} \subset \mathcal{L}_{\mathbb{R}}\,, \tag{5.5}$$

defined in (1.21) (which is the complement in $\mathcal{L}_\mathbb{R}$ of the vertical Maslov cycle $\mathcal{C}$ defined by (2.35)), is given by the Lagrange planes $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\det L_1 \neq 0$, in which case $l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ for the real symmetric matrix $M = L_2 L_1^{-1}$. This matrix M is the unique one *parameterizing* l *in* $\mathcal{D}$: see Remark 1.30. By a slight abuse of language, a real matrix $L = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ representing a Lagrange plane is said to *belong to* $\mathcal{L}_\mathbb{R}$; and a matrix-valued function with this property *takes values in* $\mathcal{L}_\mathbb{R}$. If L represents $l \in \mathcal{D}$, then L *belongs to* $\mathcal{D}$; and a matrix-valued function with this property *takes values in* $\mathcal{D}$. In this section, $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^m$ for all $m \in \mathbb{Z}$ (or any equivalent one), as well as the associated matrix norm (see Remarks 1.24).

Definition 5.14 The family (5.4) of linear Hamiltonian systems is *uniformly weakly disconjugate on* $[0,\infty)$ (resp. *on* $(-\infty,0]$) if there exists $t_0 \geq 0$ independent of ω such that for every nonzero solution $\mathbf{z}(t,\omega) = \left[\begin{smallmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right]$ with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, there holds $\mathbf{z}_1(t,\omega) \neq \mathbf{0}$ for all $t > t_0$ (resp. $\mathbf{z}_1(t,\omega) \neq \mathbf{0}$ for all $t < -t_0$).

Definition 5.15 A $2n \times n$ matrix solution $L(t,\omega) = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ of (5.4) is *principal on* $[t_1,\infty)$ (resp. *on* $(-\infty,t_1]$) if it takes values in $\mathcal{D}$ for all $t \geq t_1$ (resp. for $t \leq t_1$) and there exists

$$\lim_{t\to\infty} \left(\int_{t_1}^{t} L_1^{-1}(s,\omega)\, H_3(\omega{\cdot}s)\, (L_1^T)^{-1}(s,\omega)\, ds \right)^{-1} = 0_n \tag{5.6}$$

(resp. the same holds for the limit as $t \to -\infty$).

A principal solution on $[0,\infty)$ (resp. on $(-\infty,0]$) is *uniform principal at* ∞ (resp. *at* $-\infty$) if it takes values in $\mathcal{D}$ for all $t \in \mathbb{R}$.

The relation between disconjugacy on $\mathbb{R}$ of one of the systems of the family and the existence of (uniform) principal solutions at $\pm\infty$ for that system has been already mentioned in the introduction of the chapter (see also Proposition 5.29 below). Now the hypotheses on disconjugacy are relaxed to the weak version, which is compensated by the uniformity in the condition for all the systems.

Remarks 5.16

1. It is clear that the family (5.4) is uniformly weakly disconjugate on a half-line if and only if all its systems are weakly disconjugate on the same half-line and, in addition, the time t_0 of Definition 5.2 can be chosen to be the same for all $\omega \in \Omega$. In particular, if all the systems of the family are disconjugate, then the family is uniformly weakly disconjugate on both half-lines: just take $t_0 = 0$.
2. As in Lemma 5.5, the uniform weak disconjugacy on $[0,\infty)$ (resp. on $(-\infty,0]$) is equivalent to the existence of $t_0 \geq 0$ such that $\det U_3(t,\omega) \neq 0$ for all $\omega \in \Omega$ if $t > t_0$ (resp. $t < -t_0$), since $U(t,\omega) \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_3(t,\omega)\,\mathbf{z}_2 \\ U_4(t,\omega)\,\mathbf{z}_2 \end{smallmatrix}\right]$.

Some conditions which often appear when studying disconjugacy or weak disconjugacy will play a fundamental role throughout the chapter.

D1. The $n \times n$ matrix-valued function H_3 is positive semidefinite on Ω.

D2. For all $\omega \in \Omega$ and for any nonzero solution $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of the system (5.4) with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $[0,\infty)$.

D3. For any $\omega \in \Omega$ there exists a $2n \times n$ matrix solution $G(t,\omega) = \begin{bmatrix} G_1(t,\omega) \\ G_2(t,\omega) \end{bmatrix}$ of (5.4) taking values in $\mathcal{D}$ for all $t \in \mathbb{R}$. In other words, for all $\omega \in \Omega$ there exists $l_\omega \in \mathcal{L}_\mathbb{R}$ such that $U(t,\omega){\cdot}l_\omega \in \mathcal{D}$ for all $t \in \mathbb{R}$.

Note that condition D3 can be rewritten as: for any $\omega \in \Omega$ there exists a symmetric matrix-valued solution of the Riccati equation

$$M' = -M\,H_3(\omega{\cdot}t)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) \tag{5.7}$$

which is globally defined: see Sect. 1.3.5.

The main goal of this section is to prove the following characterization, whose scope will be analyzed in the rest of the chapter.

Theorem 5.17 *Suppose that D1 holds. The following properties are equivalent:*

(1) *the family (5.4) is uniformly weakly disconjugate on* $[0,\infty)$*;*
(2) *the family (5.4) is uniformly weakly disconjugate on* $(-\infty,0]$*;*
(3) *conditions D2 and D3 hold.*

In this case, each of the systems of the family admits uniform principal solutions at $+\infty$ *and* $-\infty$ *which are unique as matrix-valued functions taking values in* $\mathcal{L}_\mathbb{R}$ *and determine* τ*-invariant sets* $\{(\omega,\tilde{l}^{\pm}(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathcal{D}$.

The theorem is an immediate consequence of Theorems 5.25(i) and 5.26, stated below. One of its conclusions allows one to talk about uniform disconjugacy of the family, without particular mention of a precise half-line. This will be done beginning from the following section, once Theorem 5.17 has been proved.

Some previous work will simplify the proofs of the auxiliary theorems. Note that condition D1, which will be almost always assumed, is not required for the first results.

Proposition 5.18

(i) *Condition D2 holds if and only if there exist* $\delta > 0$ *and* $t_0 > 0$ *such that*

$$\int_0^{t_0} \|H_3(\omega{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{x}\|^2\,dt \ge \delta\|\mathbf{x}\|^2 \tag{5.8}$$

for all $\omega \in \Omega$ *and* $\mathbf{x} \in \mathbb{R}^n$*, where* $U_{H_1}(t,\omega)$ *is the fundamental matrix solution of* $\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x}$ *with* $U_{H_1}(0,\omega) = I_n$.

(ii) *Suppose that D2 holds, and let* t_0 *be the time provided by (i). Then none of the systems of the family (5.4) admits a solution taking the form* $\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{bmatrix}$ *on an interval of length* t_0.

(iii) *Condition D2 is equivalent to*

D2′. *For all* $\omega \in \Omega$ *and for any nonzero solution* $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ *of the system* (5.4) *with* $\mathbf{z}_1(0,\omega) = \mathbf{0}$*, the vector* $\mathbf{z}_1(t,\omega)$ *does not vanish identically on* $(-\infty, 0]$.

Proof

(i) Suppose that D2 does not hold and take a nonzero solution $\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ on $[0,\infty)$ of the system (5.4) corresponding to ω. Then $\mathbf{0} = H_3(\omega{\cdot}t)\,\mathbf{z}_2(t,\omega)$ and $\mathbf{z}_2'(t,\omega) = -H_1^T(\omega{\cdot}t)\,\mathbf{z}_2(t,\omega)$ for $t \ge 0$, so that $\mathbf{z}_2(t,\omega) = (U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{z}_2(0,\omega)$, and (5.8) does not hold for $\mathbf{x} = \mathbf{z}_2(0,\omega)$.

Conversely, if (5.8) does not hold, then the compactness of Ω and of the unit sphere in $\mathbb{R}^n$ ensure that

$$\int_0^m \|H_3(\omega_m{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_m)\,\mathbf{x}_m\|^2\,dt = 0$$

for each $m \in \mathbb{N}$, for a suitable point $(\omega_m, \mathbf{x}_m) \in \Omega \times \mathbb{R}^n$ with $\|\mathbf{x}_m\| = 1$. A convergent subsequence of $((\omega_m, \mathbf{x}_m))$ provides $(\omega_0, \mathbf{x}_0)$ with $\|\mathbf{x}_0\| = 1$ such that $\int_0^\infty \|H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0\|^2\,dt = 0$. Then the function $\mathbf{z}(t,\omega_0) = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega_0) \end{bmatrix}$ given by $\mathbf{z}_2(t,\omega_0) = (U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0$ is a nonzero solution of the system (5.4) corresponding to ω_0 on $[0,\infty)$, as can be deduced from the equality $H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0 = \mathbf{0}$ for each $t \ge 0$. This fact precludes D2.

(ii) Suppose for contradiction the existence of a solution of the system corresponding to ω taking the form $\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{bmatrix}$ for $t \in [a, a+t_0]$. Then, $\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t+a) \end{bmatrix}$ solves the system corresponding to $\omega{\cdot}a$ for $t \in [0,t_0]$. Hence, $\mathbf{0} = H_3(\omega{\cdot}a{\cdot}t)\,\mathbf{z}_2(t+a) = H_3((\omega{\cdot}a){\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega{\cdot}a)\,\mathbf{z}_2(a)$ for $t \in [0,t_0]$, and this yields

$$0 = \int_0^{t_0} \|H_3((\omega{\cdot}a){\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega{\cdot}a)\,\mathbf{z}_2(a)\|\,,$$

which contradicts (5.8).

(iii) It follows immediately from (ii) that D2 ensures D2′. Conversely, condition D2′ can be taken as the starting point to prove the analogue of (i), which will then ensure (ii) and hence D2.

Remark 5.19 Note that, if $H_3 > 0$, then (5.8) holds, so that D1 and D2 hold. Note also that this is the case when the family of Hamiltonian systems (5.4) comes from a family of Schrödinger equations $-\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \mathbf{0}$, since in this case $H_3 = I_n$.

The following notation will be used to indicate the following hypotheses:

D1$_\omega$. $H_3(\omega{\cdot}t) \ge 0$ for all $t \in \mathbb{R}$.

D2$_\omega$. For any nonzero solution $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of the system (5.4) corresponding to ω with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $[t_1,\infty)$ for all $t_1 \in \mathbb{R}$.

D2$'_\omega$. For any nonzero solution $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of the system (5.4) corresponding to ω with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $(-\infty, t_1]$ for all $t_1 \in \mathbb{R}$.

D3$_\omega$. There exists a $2n \times n$ matrix solution $G(t,\omega) = \begin{bmatrix} G_1(t,\omega) \\ G_2(t,\omega) \end{bmatrix}$ of the system (5.4) corresponding to ω taking values in $\mathcal{D}$ for all $t \in \mathbb{R}$. In other words, there exists a symmetric matrix solution of the Riccati equation (5.7) corresponding to ω which is globally defined.

Remarks 5.20

1. It is obvious that D1 and D3 hold if D1$_\omega$ and D3$_\omega$ hold for all $\omega \in \Omega$, respectively. The same holds for D2 and D2$'$: it is obvious that they hold if every system of the family satisfies D2$_\omega$ and D2$'_\omega$, respectively; and the converse assertion follows, for instance, from points (ii) and (iii) of Proposition 5.18.
2. Clearly, the weak disconjugacy on $[0,\infty)$ of the system given by ω ensures condition D2$_\omega$. Note that the weak disconjugacy on $[0,\infty)$ of all the systems precludes the existence of a nonzero solution taking the form $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of (5.4) on any positive half-line $[a,\infty)$, since $\mathbf{w}(t) = \mathbf{z}(t+a,\omega)$ is the solution of $\mathbf{z}' = H((\omega{\cdot}a){\cdot}t))\,\mathbf{z}$ with $\mathbf{w}(0) = \mathbf{z}(a,\omega)$. In particular, D2 holds. Similar relations hold for the weak disconjugacy on $(-\infty,0]$ and condition D2$'$.
3. Theorem 5.11 establishes that, under condition D1$_\omega$, the system (5.4) corresponding to ω is weakly disconjugate on $[0,\infty)$ (resp. on $(-\infty,0]$) if and only if D2$_\omega$ (resp. D2$'_\omega$) holds and it is nonoscillatory at $+\infty$ (resp. at $-\infty$). In other words, if D1$_\omega$ and D2$_\omega$ (or D2$'_\omega$) hold, the nonoscillatory character at $+\infty$ (or at $-\infty$) and the weak disconjugacy on $[0,\infty)$ (or on $(-\infty,0]$) of the corresponding system are equivalent properties.

The following result will not be required until the next section. However, it refers just to conditions D2 and D2$'$, so that it seems appropriate to include it at this point.

Lemma 5.21

(i) *Suppose that* D2$_{\omega_1}$ *holds for a point ω_1 in the omega-limit set of ω_0. Then* D2$_{\omega_0\cdot t}$ *holds for all $t \in \mathbb{R}$.*

(ii) *Suppose that* D2$'_\omega$ *holds for all ω in the alpha-limit set of ω_0. Then* D2$'_{\omega_0\cdot t}$ *holds for all $t \in \mathbb{R}$.*

(iii) *If Ω is minimal, then* D2$_{\omega_0}$ *(resp.* D2$'_{\omega_0}$*) holds for a point $\omega_0 \in \Omega$ if and only if* D2 *(resp.* D2$'$*) holds.*

(iv) *Condition* D2 *(resp.* D2$'$*) holds if and only if each minimal subset of Ω contains a point ω such that* D2$_\omega$ *(resp.* D2$'_\omega$*) holds.*

Proof In order to prove (i), note that the omega-limit sets of ω_0 and $\omega_0{\cdot}t$ agree for all $t \in \mathbb{R}$, so that it is enough to prove that D2$_{\omega_0}$ holds. Suppose for contradiction the existence of $\mathbf{z}_2 \neq \mathbf{0}$ such that $U(t,\omega_0)\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{bmatrix}$ on $[t_1,\infty)$. Take a sequence $(s_m) \uparrow \infty$ with $\omega_1 = \lim_{m\to\infty} \omega_0{\cdot}s_m$, and choose a subsequence (s_j) such that there exists the limit $\mathbf{w}_2 \neq \mathbf{0}$ of $(\mathbf{z}_2(s_j)/\|\mathbf{z}_2(s_j)\|)$. It follows that $\begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix} = \lim_{j\to\infty} U(s_j,\omega_0)\begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\| \end{bmatrix}$, and, consequently, $U(t,\omega_1)\begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix} =$

$\lim_{j\to\infty} U(t,\omega_0{\cdot}s_j)\,U(s_j,\omega_0)\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\|\end{bmatrix} = \lim_{j\to\infty} U(t+s_j,\omega_0)\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\|\end{bmatrix} = \lim_{j\to\infty}\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2(t+s_j)/\|\mathbf{z}_2(s_j)\|\end{bmatrix}$ for all $t\in\mathbb{R}$, which provides a solution taking the form $\begin{bmatrix}\mathbf{0}\\ \mathbf{w}_2(t)\end{bmatrix}$ on $\mathbb{R}$ for the system for ω_1. In particular, $D2_{\omega_1}$ does not hold, which yields the sought-for contradiction.

The proof of (ii) is identical, and the properties stated in (iii) are easy consequences of the previous ones and the minimality of Ω. In order to prove the "only if" implication in (iv), suppose for contradiction the existence of $\omega_0\in\Omega$ such that $D2_{\omega_0}$ does not hold, and choose a minimal subset $\mathcal{M}$ of its omega-limit set. By (i), $D2_\omega$ does not hold for all $\omega\in\mathcal{M}$, which is impossible by hypothesis. An analogous proof guarantees the result for $D2'$.

Remark 5.22 It will be explained in Chap. 6 that condition (5.8) (i.e. condition D2, according to Proposition 5.18(i)) is equivalent to the uniform null controllability of the family of control systems

$$\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u}\,,\quad \omega\in\Omega\,, \tag{5.9}$$

a condition which is in turn ensured by an *a priori* weaker one: each minimal subset of Ω contains a point ω such that the system (5.9) is null controllable (see the connection with point (iv) in the previous lemma). In turn, this last condition holds if $H_3\geq 0$ and each minimal subset of Ω contains a point ω with $H_3(\omega)>0$: see Remarks 6.8.1 and 6.2.1.

The following remark, which summarizes some of the results of Chapter 2 of Coppel [34], provides information which will be important in what follows.

Remark 5.23 Given a point $\omega\in\Omega$, let $G(t,\omega)=\begin{bmatrix}G_1(t,\omega)\\ G_2(t,\omega)\end{bmatrix}$ be a $2n\times n$ matrix solution of the corresponding system (5.4). Suppose that $G(t,\omega)$ takes values in $\mathcal{D}$ for every t in an interval $\mathcal{I}$. Take $a\in\mathcal{I}$ and define

$$I_G(a,t,\omega)=\int_a^t G_1^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(G_1^T)^{-1}(s,\omega)\,ds \tag{5.10}$$

for $t\in\mathcal{I}$. It is easy to check that

$$\begin{bmatrix} G_1(t,\omega) & G_1(t,\omega)\,I_G(a,t,\omega)\\ G_2(t,\omega) & G_2(t,\omega)\,I_G(a,t,\omega)+(G_1^T)^{-1}(t,\omega)\end{bmatrix} \tag{5.11}$$

is a fundamental matrix solution of (5.4). Consequently, a $2n\times n$ matrix-valued function $\widetilde{G}(t,\omega)$ solves (5.4) on $\mathcal{I}$ if and only if it takes the form

$$\begin{bmatrix}\widetilde{G}_1(t,\omega)\\ \widetilde{G}_2(t,\omega)\end{bmatrix}=\begin{bmatrix} G_1(t,\omega)\,(P(\omega)+I_G(a,t,\omega)\,Q(\omega))\\ G_2(t,\omega)\,(P(\omega)+I_G(a,t,\omega)\,Q(\omega))+(G_1^T)^{-1}(t,\omega)\,Q(\omega)\end{bmatrix} \tag{5.12}$$

for arbitrary real $n \times n$ matrices $P(\omega)$ and $Q(\omega)$. Moreover, as proved in [34] (Proposition 3 of Chapter 2), if $\widetilde{G}(t,\omega)$ belongs to $\mathcal{D}$ for $t \in \mathcal{I}$, then $P(\omega)$ is nonsingular and

$$I_{\widetilde{G}}(a,t,\omega) = (P(\omega) + I_G(a,t,\omega)\,Q(\omega))^{-1} I_G(a,t,\omega)\,(P^T)^{-1}(\omega) \tag{5.13}$$

for $t \in I$, with $I_{\widetilde{G}}$ defined from $\widetilde{G}_1$ as I_G from G_1 in (5.10). Finally, if $\mathcal{I}$ contains a half-line $[a,\infty)$, and there exists $\lim_{t\to\infty}(I_{\widetilde{G}}(a,t,\omega))^{-1} = 0_n$, then

$$\lim_{t\to\infty}(I_{\widetilde{G}}(a,t,\omega) + C)^{-1} = \lim_{t\to\infty}(I_{\widetilde{G}}(a,t,\omega))^{-1}(I_n + C(I_{\widetilde{G}}(a,t,\omega))^{-1})^{-1} = 0_n$$

for every constant matrix C, and hence there exists $\lim_{t\to\infty}(I_{\widetilde{G}}(b,t,\omega))^{-1} = 0_n$ whenever $[b,\infty) \subseteq \mathcal{I}$. The analogous result for the limits at $-\infty$ holds if $\mathcal{I}$ contains a half-line $(-\infty,b]$. These last properties are especially relevant when talking about uniform principal solutions on positive or negative half-lines: if this is the case, any t_1 in (5.6) provides the same limit, so that a uniform principal solution at $+\infty$ or at $-\infty$ is a principal solution on $[t_1,\infty)$ or on $(-\infty,t_1]$ for all $t_1 \in \mathbb{R}$.

Lemma 5.24 *Suppose that* D1 *and* D2 *hold, and let* t_0 *be the positive time of Proposition* 5.18*(i). Let* $G(t,\omega) = \begin{bmatrix} G_1(t,\omega) \\ G_2(t,\omega) \end{bmatrix}$ *be a* $2n \times n$ *matrix solution of* (5.4) *taking values in* $\mathcal{D}$ *for every* $t \geq t_1$ *and* $\omega \in \Omega$*. Then, for all* $\omega \in \Omega$*, the symmetric matrix* $I_G(a,t,\omega)$ *defined by* (5.10) *for* $t_1 \leq a < t$ *is positive definite if* $t - a \geq t_0$*.*

Proof It follows from (5.11) that

$$\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix} = \begin{bmatrix} G_1(t,\omega)\,I_G(a,t,\omega)\,\mathbf{x}_0 \\ G_2(t,\omega)\,I_G(a,t,\omega)\,\mathbf{x}_0 + (G_1^T)^{-1}(t,\omega)\,\mathbf{x}_0 \end{bmatrix}$$

solves (5.4) for all $\mathbf{x}_0 \in \mathbb{R}^n$. Take $t_2 \geq a + t_0$ and suppose for contradiction that there exists $(\omega_0,\mathbf{x}_0)$ with $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{x}_0^T I_G(a,t_2,\omega_0)\,\mathbf{x}_0 = 0$, which clearly implies that $\mathbf{x}_0^T I_G(a,t,\omega_0)\,\mathbf{x}_0 = 0$ for all $t \in [a,t_2]$. Hence $I_G(a,t,\omega_0)\,\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{z}_1(t,\omega_0) = \mathbf{0}$ for all $t \in [a,t_2]$, which contradicts Proposition 5.18(ii).

The results ensuring the properties stated in Theorem 5.17 can be now formulated and proved. The second statement of Theorem 5.25 will be required in Sect. 5.4.

Theorem 5.25 *Suppose that* D1*,* D2*, and* D3 *hold. Then,*

(i) *the family* (5.4) *is uniformly weakly disconjugate on* $[0,\infty)$ *and on* $(-\infty,0]$*.*
(ii) *For each* $\omega \in \Omega$ *and* $l \in \mathcal{L}_{\mathbb{R}}$*, there exists* $s_{\omega,l}$ *such that* $U(t,\omega){\cdot}l \in \mathcal{D}$ *whenever* $|t| > s_{\omega,l}$*.*

Proof

(i) Suppose without loss of generality that the matrix $\left[\begin{smallmatrix} G_1(t,\omega) \\ G_2(t,\omega) \end{smallmatrix}\right]$ of condition D3 is normalized to $G_1(0,\omega) = I_n$ for all $\omega \in \Omega$. Once this is done, it follows from (5.12) and from $\left[\begin{smallmatrix} U_3(0,\omega) \\ U_4(0,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$ that

$$U_3(t,\omega) = G_1(t,\omega)\, I_G(0,t,\omega)$$
$$U_4(t,\omega) = G_2(t,\omega)\, I_G(0,t,\omega) + (G_1^T)^{-1}(t,\omega)$$

for each $t \in \mathbb{R}$ and $\omega \in \Omega$, with $I_G(0,t,\omega)$ defined by (5.10). Lemma 5.24 ensures that $I_G(0,t,\omega)$ (and hence $U_3(t,\omega)$) is nonsingular whenever $|t| \geq t_0$, with t_0 provided by Proposition 5.18(i). As seen in Remark 5.16.2, this property is equivalent to the uniform weak disconjugacy at $+\infty$ and $-\infty$.

(ii) Fix $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$, represent $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, and choose any $\left[\begin{smallmatrix} L_3 \\ L_4 \end{smallmatrix}\right] \equiv l_1 \in \mathcal{L}_{\mathbb{R}}$ such that $\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] \in \mathrm{Sp}(n,\mathbb{R})$ (for instance, $L_3 = L_2 R^{-1}$ and $L_4 = -L_1 R^{-1}$ for $R = L_1^T L_1 + L_2^T L_2$). Then $V(t,\omega) = U(t,\omega) \left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_3(t,\omega) & L_1(t,\omega) \\ L_4(t,\omega) & L_2(t,\omega) \end{smallmatrix}\right]$ is a symplectic matrix solution of (5.4). According to Proposition 5.7, the (already established) uniform weak disconjugacy of the family on $[0,\infty)$ ensures that each of the systems of the family (5.4) is nonoscillatory at $+\infty$. Proposition 5.9(i) provides a time $s_{\omega,l}$ such that the vector space $k_V(\omega) = \operatorname{Ker} L_1(t,\omega)$ is constant on $(s_{\omega,l},\infty)$. Assume that $k_V(\omega)$ is nontrivial and take a nonzero $\mathbf{z} \in k_V(\omega)$. Then $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] \mathbf{z}$ is a nontrivial solution of (5.4) and takes the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right]$ for $t \geq s_{\omega,l}$, which is impossible according to Proposition 5.18(ii). The argument is analogous in the negative half-line.

Theorem 5.26 *Suppose that* D1 *holds. Then the family* (5.4) *is uniformly weakly disconjugate on* $[0,\infty)$ *if and only if it is uniformly weakly disconjugate on* $(-\infty,0]$. *If this is the case, then the system* (5.4) *possesses uniform principal solutions* $\left[\begin{smallmatrix} L_1^{\pm}(t,\omega) \\ L_2^{\pm}(t,\omega) \end{smallmatrix}\right]$ *at* $\pm\infty$ *for each* $\omega \in \Omega$, *and conditions* D2 *and* D3 *hold. In addition, the principal solutions are unique as matrix-valued functions taking values in* $\mathcal{L}_{\mathbb{R}}$. *Finally, if* $\tilde{l}^{\pm}(\omega)$ *are the real Lagrange planes represented by* $\left[\begin{smallmatrix} L_1^{\pm}(0,\omega) \\ L_2^{\pm}(0,\omega) \end{smallmatrix}\right]$, *then* $\tilde{l}^{\pm}(\omega{\cdot}t) = U(t,\omega)\cdot\tilde{l}^{\pm}(\omega)$ *for all* $t \in \mathbb{R}$ *and* $\omega \in \Omega$.

Proof The proof is carried out according to the following scheme. Assuming first the uniform weak disconjugacy on $[0,\infty)$, the existence of a principal solution at $+\infty$ for each $\omega \in \Omega$ with the stated properties is proved. Consequently, and according to Remark 5.20.2 and Definition 5.15, the family satisfies D2 and D3. Therefore, Theorem 5.25(i) ensures the uniform weak disconjugacy on $(-\infty,0]$. Some indications about how to adapt the first steps in order to ensure the existence of a principal solution at $-\infty$ complete the proof. This method of proof can be repeated taking the uniform weak disconjugacy on $(-\infty,0]$ as the starting point.

Assume the uniform weak disconjugacy on $[0,\infty)$. Recall that then D2 holds, as explained in Remark 5.20.2. Let $t_0 > 0$ satisfy $\det U_3(t,\omega) \neq 0$ for $t \geq t_0$ and $\omega \in \Omega$ (see Remark 5.16) and condition in Lemma 5.24. Consider the $2n\times n$ matrix-valued function $G(t,\omega) = \left[\begin{smallmatrix} U_3(t,\omega) \\ U_4(t,\omega)\end{smallmatrix}\right]$, which represents a real Lagrange plane for all $t\in\mathbb{R}$, solves (5.4), and takes values in $\mathcal{D}$ for $t \geq t_0$. Represent $I(t,\omega) = I_G(t_0,t,\omega)$, this last matrix being given by (5.10), and note that $I(t,\omega)$ is nondecreasing in t (since D1 holds). Lemma 5.24 ensures that $I(t,\omega)$ is positive definite for each $t \geq 2t_0$. Hence, $(I(t,\omega))^{-1}$ is positive definite for these values of t, and nonincreasing in t (see Remark 1.20). Therefore, there exists the limit

$$J_+(\omega) = \lim_{t\to\infty} (I(t,\omega))^{-1}$$

(see Remark 1.44.3). The goal now is to prove that $I_n - I(t,\omega)\,J_+(\omega)$ is nonsingular if $t \geq t_0$. Consider first the case $t \geq 2t_0$. By Lemma 5.24, $0 < I(t,\omega) < I(t+t_0,\omega)$, so that $J_+(\omega) < (I(t,\omega))^{-1}$ for $t \geq 2t_0$. Hence the matrix $I_n - I(t,\omega)\,J_+(\omega)$, whose eigenvalues agree with those of

$$I^{1/2}(t,\omega)\,\big((I(t,\omega))^{-1} - J_+(\omega)\big)\,I^{1/2}(t,\omega) > 0\,,$$

is nonsingular for each $t \geq 2t_0$. Now take $t \in [t_0, 2t_0]$ and $s \geq 2t_0$, and observe that the eigenvalues of the two matrices $I_n - I(t,\omega)\,I^{-1}(s,\omega)$ and $I_n - I^{-1/2}(s,\omega)\,I(t,\omega)\,I^{-1/2}(s,\omega)$ agree. Taking the limits as $s\to\infty$ one sees that the set of eigenvalues of the matrix $I_n - I(t,\omega)\,J_+(\omega)$ agrees with that of $I_n - J_+^{1/2}(\omega)\,I(t,\omega)\,J_+^{1/2}(\omega)$ (see e.g. Theorem II.5.1 of [89]). Thus, the assertion is proved once it has been checked that the eigenvalues of this last matrix are strictly positive if $t_0 \leq t \leq 2t_0$, which in turn follows from

$$I_n - J_+^{1/2}(\omega)\,I(t,\omega)\,J_+^{1/2}(\omega) \geq I_n - J_+^{1/2}(\omega)\,I(2t_0,\omega)\,J_+^{1/2}(\omega)\,:$$

the eigenvalues of the matrix in the right-hand term agree with those of the matrix $I_n - I(2t_0,\omega)\,J_+(\omega)$, which, as already seen, are strictly positive.

According to Remark 5.23, the $2n\times n$ matrix-valued function $L^+(t,\omega)$ given by

$$\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega)\end{bmatrix} = \begin{bmatrix} U_3(t,\omega)\,(I_n - I(t,\omega)\,J_+(\omega)) \\ U_4(t,\omega)\,(I_n - I(t,\omega)\,J_+(\omega)) - (U_3^T)^{-1}(t,\omega)\,J_+(\omega)\end{bmatrix} \tag{5.14}$$

solves (5.4) in $[t_0,\infty)$ and takes values in $\mathcal{L}_\mathbb{R}$. It has been just checked that in fact it takes values in $\mathcal{D}$ for $t \geq t_0$. Hence, by (5.13), if $I_{L^+}(t_0,t,\omega)$ is defined from L^+ by (5.10), then $(I_{L^+}(t_0,t,\omega))^{-1} = (I(t,\omega))^{-1} - J_+(\omega)$ if $t \geq 2t_0$, so that

$$\begin{aligned} 0_n &= \lim_{t\to\infty} (I_{L^+}(t_0,t,\omega))^{-1} \\ &= \lim_{t\to\infty}\left(\int_{t_0}^t (L_1^+)^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,((L_1^+)^T)^{-1}(s,\omega)\,ds\right)^{-1}. \end{aligned} \tag{5.15}$$

The same symbol $\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix}$ will denote the extension of the solution given on $[t_0,\infty)$ to the whole real line. Suppose now that $t \to \begin{bmatrix} \bar{L}_1(t,\omega) \\ \bar{L}_2(t,\omega) \end{bmatrix}$ is any $2n \times n$ matrix solution of (5.4) which takes values in $\mathcal{D}$ for $t \in [t_0,\infty)$ and satisfies $\lim_{t\to\infty}(I_{\bar{L}}(t_0,t,\omega))^{-1} = 0_n$. By Remark 5.23, $\begin{bmatrix} \bar{L}_1(t,\omega) \\ \bar{L}_2(t,\omega) \end{bmatrix}$ can be defined from $\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix}$ for $t \in [t_0,\infty)$ by expression (5.12) for suitable functions $\bar{P}(\omega)$ and $\bar{Q}(\omega)$. The matrix $\bar{P}(\omega)$ is invertible and, by (5.13),

$$\begin{aligned} 0_n &= \lim_{t\to\infty}(I_{\bar{L}}(t_0,t,\omega))^{-1} \\ &= \lim_{t\to\infty} \bar{P}^T(\omega)\,((I_{L^+}(t_0,t,\omega))^{-1}\bar{P}(\omega) + \bar{Q}(\omega)) = \bar{P}^T(\omega)\,\bar{Q}(\omega)\,, \end{aligned}$$

so that $\bar{Q}(\omega) = 0_n$. Hence $\begin{bmatrix} \bar{L}_1(t,\omega) \\ \bar{L}_2(t,\omega) \end{bmatrix} = \begin{bmatrix} L_1^+(t,\omega)\,\bar{P}(\omega) \\ L_2^+(t,\omega)\,\bar{P}(\omega) \end{bmatrix}$ for each $t \ge t_0$. By uniqueness of solutions of (5.4), the same equality holds for all $t \in \mathbb{R}$. That is, in terms of the matrix representation of Lagrange planes,

$$\begin{bmatrix} \bar{L}_1(t,\omega) \\ \bar{L}_2(t,\omega) \end{bmatrix} \equiv \begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix} \tag{5.16}$$

for all $t \in \mathbb{R}$.

The next goal is to check that

$$L^+(t+r,\omega) \equiv L^+(t,\omega{\cdot}r) \qquad \text{for all } \omega \in \Omega \text{ and } t,r \in \mathbb{R}\,; \tag{5.17}$$

i.e. they represent the same Lagrange plane. Note that $t \mapsto L^+(t+r,\omega)$ and $t \mapsto L^+(t,\omega{\cdot}r)$ solve the system corresponding to $\omega{\cdot}r$. Assume first that $r \ge 0$, so that $L^+(t+r,\omega)$ belongs to $\mathcal{D}$ for all $t \ge t_0$. And

$$\begin{aligned} &\lim_{t\to\infty}\left(\int_{t_0}^{t}(L_1^+)^{-1}(s+r,\omega)\,H_3((\omega{\cdot}r){\cdot}s)\,((L_1^+)^T)^{-1}(s+r,\omega)\,ds\right)^{-1} \\ &\qquad = \lim_{t\to\infty}\left(\int_{t_0+r}^{t+r}(L_1^+)^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,((L_1^+)^T)^{-1}(s,\omega)\,ds\right)^{-1} = 0_n\,, \end{aligned}$$

as can be deduced from (5.15) and the last assertion of Remark 5.23. Hence, (5.16) implies (5.17) for $r \ge 0$. This in turn implies that, if $r \ge 0$,

$$L^+(t,\omega{\cdot}(-r)) \equiv L^+(t-r+r,\omega{\cdot}(-r)) \equiv L^+(t-r,\omega),$$

which completes the proof of (5.17). Consequently, the Lagrange plane represented by $L^+(t,\omega) = L^+(t_0,\omega{\cdot}(t-t_0))$ belongs to $\mathcal{D}$ for all $(t,\omega) \in \mathbb{R}\times\Omega$.

The assertions concerning the uniform principal solution at $+\infty$ can now be proved. First, $L^+(t,\omega)$ always takes values in $\mathcal{D}$, so that relation (5.15) and a new application of the last assertion of Remark 5.23, ensure that $L^+(t,\omega)$ is a uniform principal solution at $+\infty$. Second, relation (5.16) implies that it is unique when considered as a function taking values in $\mathcal{L}_{\mathbb{R}}$. And third, (5.17) yields $U(r,\omega)L^+(0,\omega) = L^+(r,\omega) \equiv L^+(0,\omega{\cdot}r)$, so that if $l^+(\omega) \equiv \begin{bmatrix} L_1^+(0,\omega) \\ L_2^+(0,\omega) \end{bmatrix}$, then $U(r,\omega){\cdot}l^+(\omega) = l^+(\omega{\cdot}r)$.

As stated at the beginning of the proof, the uniform weak disconjugacy on $(-\infty,0]$ holds. To deal now with the existence, uniqueness, and invariance of the principal solution at $-\infty$, take $t_0 > 0$ satisfying Lemma 5.24 and $\det U_3(t,\omega) \neq 0$ for $t \leq -t_0$, write as before $G(t,\omega) = \begin{bmatrix} U_3(t,\omega) \\ U_4(t,\omega) \end{bmatrix}$, and define $\widetilde{I}(t,\omega) = I_G(-t_0,t,\omega)$ for $t \leq -t_0$. This last matrix is negative definite for $t \leq -2t_0$ and decreases as t decreases, so that $(\widetilde{I}(t,\omega))^{-1}$ is negative definite and increases as $t \to -\infty$. Hence, there exists $J_-(\omega) = \lim_{t\to-\infty}(\widetilde{I}(t,\omega))^{-1}$. Changing I to $\widetilde{I}$ and J_+ to J_- in (5.14) provides the definition of $\begin{bmatrix} L_1^-(t,\omega) \\ L_2^-(t,\omega) \end{bmatrix}$, which will now play the role played before by $\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix}$. The rest of the proof is identical with the previous one.

The proof of Theorem 5.17 is hence complete. Recall that, as stated in the introduction of this chapter, more information concerning the existence of (perhaps nonuniform) principal solutions for the systems of the family (5.4) under less restrictive hypotheses will be given at the end of the following section.

Among the most trivial examples of systems fitting the situation of Theorem 5.17, one can mention the autonomous cases $\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{z}$, with $L^+(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $L^-(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{z}' = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{z}$, with $L^+(0) = L^-(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The reader can find in this chapter several interesting nonautonomous examples of uniformly weakly disconjugate families: see e.g. Examples 5.38 and 5.47.

The section is completed with a result which presents sufficient conditions for the uniform weak disconjugacy of the family (5.4). In some cases, it allows one to identify this property very quickly: Remark 5.19 shows that the hypotheses of Proposition 5.27 are fulfilled if $H_3 > 0$ and $H_2 \geq 0$, and Remark 5.22 describes less restrictive conditions ensuring the same.

Proposition 5.27 *Suppose that* D1 *and* D2 *holds, and that $H_2 \geq 0$. Then the family* (5.4) *is uniformly weakly disconjugate.*

Proof Let t_0 satisfy (5.8). Clearly it suffices to check that the unique solution $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$ of (5.4) with $\mathbf{x}(0) = \mathbf{x}(t_0) = \mathbf{0}$ is the zero solution. Note that, for such a solution, $\int_0^{t_0} (\|H_2^{1/2}(\omega{\cdot}t)\,\mathbf{x}(t)\|^2 + \|H_3^{1/2}(\omega{\cdot}t)\,\mathbf{y}(t)\|^2)\,dt = 0$: see Remark 1.81.2. That is, on the one hand, $\int_0^{t_0} \|H_3^{1/2}(\omega{\cdot}t)\,\mathbf{y}(t)\|^2\,dt = 0$; and, on the other hand, for $t \in [0,t_0]$, $H_2(\omega{\cdot}t)\,\mathbf{x}(t) = \mathbf{0}$, so that $\mathbf{y}'(t) = -H_1^T(\omega{\cdot}t)\,\mathbf{y}(t)$ and hence $\mathbf{y}(t) = (U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{y}(0)$. These two equalities lead to $0 = \int_0^{t_0} \|H_3^{1/2}(\omega{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{y}(0)\|^2\,dt$, which, by (5.8), means that $\mathbf{y}(0) = \mathbf{0}$, so that $\mathbf{z}(0) = \mathbf{0}$ and hence $\mathbf{z} \equiv \mathbf{0}$.

5.3 Disconjugacy, Uniform Weak Disconjugacy, and Weak Disconjugacy

Consider the following three different possibilities for the family (5.4):

A. All the systems of the family are disconjugate.
B. The family is uniformly weakly disconjugate.
C. All the systems of the family are weakly disconjugate on $(-\infty, 0]$ or on $[0, \infty)$.

(Recall that Theorem 5.17 guarantees that the uniform weak disconjugacy of the family (5.4) holds simultaneously on both half-lines.) Then,

– A implies B and B implies C: see Remark 5.16.1;
– even when D1 holds, B does not imply A: see Example 5.38 below;
– even when $H_3 > 0$, C does not imply B (or A): see Example 5.39 below.

As stated in the introduction, to analyze the situations in which two of the conditions A, B, C (or the three of them) hold simultaneously is the first goal of this section. Some preliminary results concerning nonoscillation, which are of interest in their own right, will be used in the analysis, and the characterization of uniform weak disconjugacy provided by Theorem 5.17 will be of fundamental importance from now on. The properties of the rotation number of the family (5.4) associated to each σ-ergodic measure will provide in Sect. 5.7 more information about the relations holding between properties A, B and C when D1 and D2 hold.

It was mentioned in the introduction that the great advantage of weak disconjugacy, as compared to the classical disconjugacy, is that it holds under a much weaker version of the condition of identical normality, which is not required in order that B or C hold (see Example 5.38).

Definition 5.28 The system (5.4) corresponding to ω is *identically normal* on $\mathbb{R}$ if, for every nonzero solution $\left[\begin{smallmatrix}\mathbf{z}_1(t)\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$, the vector $\mathbf{z}_1(t)$ does not vanish identically on any nondegenerate interval.

So, it is clear that the weak disconjugacy on a half-line, as well as conditions $\mathrm{D2}_\omega$ and $\mathrm{D2}'_\omega$, are weaker than the identical normality on $\mathbb{R}$ of the corresponding system. Clearly, a disconjugate system is identically normal. Something more can be said in the case that D1 holds: the following result is proved in Sections 1 and 2 of Chapter 2 of [34]. Note that Theorem 5.17 can be understood as its extension to the less restrictive setting considered here.

Proposition 5.29 *Suppose that $H_3(\omega{\cdot}t) \geq 0$ for a point $\omega \in \Omega$ and every $t \in \mathbb{R}$. Then the corresponding system (5.4) is disconjugate on $\mathbb{R}$ if and only it is identically normal on $\mathbb{R}$ and it admits a $2n \times n$ matrix solution $\left[\begin{smallmatrix}G_1(t)\\ G_2(t)\end{smallmatrix}\right]$ taking values in $\mathcal{D}$ for all $t \in \mathbb{R}$. In this case, the corresponding system (5.4) possesses principal solutions at $+\infty$ and $-\infty$, which are unique as functions taking values in $\mathcal{L}_{\mathbb{R}}$.*

Remark 5.30 It is almost immediate that, if $H_3 > 0$, all the systems of the family are identically normal: if $\mathbf{z}_1(t) = \mathbf{0}$ for a nonzero solution, then $\mathbf{z}_1'(t) = H_3(\omega{\cdot}t)\,\mathbf{z}_2(t) \neq \mathbf{0}$. Thus, Remark 5.19, Theorem 5.17, and Proposition 5.29 ensure that, if $H_3 > 0$, properties A and B are equivalent, and that they are also equivalent to the fact that condition D3 holds. This conclusion was mentioned in the introduction: the main contribution of the theory of weak disconjugacy concerns the situations in which $H_3 \geq 0$ but it is not positive definite.

Under condition D1, Theorem 5.32 below describes a situation in which B and C hold or not simultaneously: this happens when the base flow has a dense semiorbit. To understand its scope, recall that the existence of positive and negative semiorbits which are dense in Ω holds in the case of existence of a σ-ergodic measure with total support Ω, as proved in Proposition 1.12. The proof of Theorem 5.32 is an immediate consequence of the following theorem, which is more general.

Theorem 5.31 *Let $\mathcal{O}$ and $\mathcal{A}$ be the omega-limit set and alpha-limit set of $\omega_0 \in \Omega$. Then,*

(i) *if the system (5.4) corresponding to ω_0 is nonoscillatory at $+\infty$, then all the systems corresponding to elements of $\{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathcal{O}$ are nonoscillatory at $+\infty$, and those corresponding to $\mathcal{O}$ are nonoscillatory at $-\infty$.*
(ii) *If the system (5.4) corresponding to ω_0 is nonoscillatory at $-\infty$, then all the systems corresponding to elements of $\{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathcal{A}$ are nonoscillatory at $-\infty$, and those corresponding to $\mathcal{A}$ are nonoscillatory at $+\infty$.*
(iii) *If $H_3(\omega_0{\cdot}t) \geq 0$ for all $t \geq 0$ and all the systems (5.4) corresponding to elements of $\{\omega_0\} \cup \mathcal{O}$ are weakly disconjugate on $[0,\infty)$, then the family restricted to $\mathcal{O}$ is uniformly weakly disconjugate.*
(iv) *If $H_3(\omega_0{\cdot}t) \geq 0$ for all $t \leq 0$ and all the systems (5.4) corresponding to elements of $\{\omega_0\} \cup \mathcal{A}$ are weakly disconjugate on $(-\infty,0]$, then the family restricted to $\mathcal{A}$ is uniformly weakly disconjugate.*

Proof

(i) As in Theorem 2.4 (see (2.7)), the definition of Arg_1 and Theorem 1.41 guarantee that

$$\int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\,ds = \mathrm{Arg}_1\,V(t,\omega) - \mathrm{Arg}_1\,V(0,\omega)\,,$$

where $V(t,\omega) = \begin{bmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega) \end{bmatrix}$ is a symplectic matrix solution of (5.4) with $l \equiv \begin{bmatrix} V_1(0,\omega) \\ V_2(0,\omega) \end{bmatrix}$. That is, the nonoscillation at $+\infty$ (resp. at $-\infty$) of the system corresponding to ω is equivalent to the existence of $l_\omega \in \mathcal{L}_{\mathbb{R}}$ and $c_{\omega,l_\omega} > 0$ such that $|\int_0^t \operatorname{Tr} Q(\tau(s,\omega,l_\omega))\,ds| \leq c_{\omega,l_\omega}$ for all $t \geq 0$ (resp. for all $t \leq 0$), in which case the same happens for all $l \in \mathcal{L}_{\mathbb{R}}$. Suppose that this is the case for the point

ω_0 and $t \ge 0$. Then

$$\left|\int_0^t \operatorname{Tr} Q(\tau(s+r,\omega_0,l_{\omega_0}))\,ds\right| = \left|\int_r^{t+r} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,ds\right|$$
$$\le \left|\int_0^r \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,ds\right| + \left|\int_0^{t+r} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,ds\right| \le 2c_{\omega_0,l_{\omega_0}}$$

for all $r \in \mathbb{R}$. This ensures the nonoscillation at $+\infty$ of the system corresponding to $\omega_0{\cdot}r$ for all $r \in \mathbb{R}$. Now, given $\omega_1 \in \mathcal{O}$, look for a sequence $(t_m) \uparrow \infty$ such that there exists $(\omega_1, l_1) = \lim_{m\to\infty} \tau(t_m, \omega_0, l_{\omega_0})$. Then, if $t \ge 0$,

$$\left|\int_0^t \operatorname{Tr} Q(\tau(s,\omega_1,l_1))\,ds\right| = \lim_{m\to\infty}\left|\int_0^t \operatorname{Tr} Q(\tau(s+t_m,\omega_0,l_{\omega_0}))\,ds\right|$$
$$= \lim_{m\to\infty}\left|\int_{t_m}^{t+t_m} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,ds\right| \le 2c_{\omega_0,l_{\omega_0}}\,,$$

and hence the system corresponding to ω_1 is nonoscillatory at $+\infty$. Analogously,

$$\left|\int_{-t}^0 \operatorname{Tr} Q(\tau(s,\omega_1,l_1))\,ds\right| = \lim_{m\to\infty}\left|\int_{t_m-t}^{t_m} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,ds\right| \le 2c_{\omega_0,l_{\omega_0}}$$

which ensures the nonoscillation at $-\infty$ and completes the proof of (i).

(iii) Note that the assumption $H_3(\omega_0{\cdot}t) \ge 0$ ensures condition D1 on $\widetilde{\mathcal{O}} = \{\omega_0{\cdot}t \mid t \ge 0\} \cup \mathcal{O}$. In addition, the weak disconjugacy hypothesis guarantees condition D2$_\omega$ for all $\omega \in \mathcal{O}$ (see Remark 5.20.2). Assertion (i) and Lemma 5.21(i) guarantee that all the systems corresponding to points $\omega \in \widetilde{\mathcal{O}}$ are nonoscillatory at $+\infty$ and satisfy D2$_\omega$, which ensure that all of them are weakly disconjugate: see Remark 5.20.3.

According to Lemma 5.5, the weak disconjugacy of the system (5.4) on $[0,\infty)$ for each $\omega \in \widetilde{\mathcal{O}}$ provides $t_\omega \ge 0$ with $\det U_3(t,\omega) \ne 0$ for each $t > t_\omega$. The time t_ω can be chosen as the smallest one with this property. In particular, $\det U_3(t_\omega,\omega) = 0$. Take $r > t_\omega$ and consider

$$\begin{bmatrix} Z_1(t,\omega) \\ Z_2(t,\omega) \end{bmatrix} = \begin{bmatrix} U_3(t-r,\omega{\cdot}r) \\ U_4(t-r,\omega{\cdot}r) \end{bmatrix},$$

which is a matrix solution of (5.4) taking values in $\mathcal{L}_{\mathbb{R}}$ and satisfying $\left[\begin{smallmatrix} Z_1(r,\omega) \\ Z_2(r,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$. Remark 5.23 yields

$$\begin{aligned} Z_1(t,\omega) &= U_3(t,\omega)\left(\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,ds\right)U_3^T(r,\omega)\,,\\ Z_2(t,\omega) &= U_4(t,\omega)\left(\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,ds\right)U_3^T(r,\omega)\\ &\quad + (U_3^T)^{-1}(t,\omega)\,U_3^T(r,\omega) \end{aligned}$$

for each $t \ge r$. Assume for now that there exists $t_0 > 0$, common for all $\omega \in \widetilde{\mathcal{O}}$, such that $\det Z_1(t,\omega) \ne 0$ for each $t \ge r + t_0$; or, equivalently, such that

$$\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,ds > 0 \tag{5.18}$$

for each $t \ge r + t_0$ and all $\omega \in \widetilde{\mathcal{O}}$. Then, $\det U_3(t,\omega{\cdot}r) = \det Z_1(t+r,\omega) \ne 0$ for each $t \ge t_0$, which implies that $t_{\omega\cdot r} < t_0$ if $r > t_\omega$. This property will be fundamental for the completion of the proof. In order to check the existence of this t_0, note that, since $\widetilde{\mathcal{O}}$ is compact, the arguments of Proposition 5.18(i) can be repeated to obtain $t_0 > 0$ and $\delta > 0$ such that (5.8) holds for all $\omega \in \widetilde{\mathcal{O}}$ and $\mathbf{x} \in \mathbb{R}^n$. Then, reasoning as in Proposition 5.18(ii), one proves that none of the systems (5.4) corresponding to elements of the positively σ-invariant set $\widetilde{\mathcal{O}}$ admits a solution taking the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ in $[r, r+t_0]$, since $r > 0$. And, in turn, this property allows one to repeat the proof of Lemma 5.24 in order to check (5.18).

Statement (iii) is equivalent to the boundedness from above of the set $\{t_\omega \mid \omega \in \mathcal{O}\}$. This will be checked now. Suppose for contradiction the existence of a sequence $(\omega_m)_{m\in\mathbb{N}}$ in $\mathcal{O}$ with $\lim_{m\to\infty} t_{\omega_m} = \infty$. Recall that $\det U_3(t_{\omega_m},\omega_m) = 0$ and note that there is no loss of generality in assuming that $t_{\omega_m} > t_0$ for every $m \in \mathbb{N}$. In addition, there exist m_0 and $t_1 \in (t_0, t_{\omega_{m_0}})$ with $\det U_3(t_1,\omega_{m_0}) \ne 0$: otherwise one would have $\det U_3(t,\omega_m) = 0$ for each $t \in (t_0, t_{\omega_m}]$, so that the continuity of $U_3(t,\omega)$ in ω would ensure that $\det U_3(t,\widetilde{\omega}) = 0$ for each $t > t_0$ for every accumulation point $\widetilde{\omega} \in \mathcal{O}$ of $(\omega_m)_{m\in\mathbb{N}}$ (and there exists at least one, since $\mathcal{O}$ is compact); but this is impossible by the weak disconjugacy of the system of the family (5.4) corresponding to $\widetilde{\omega}$ (see Remark 5.16.2).

According to Theorem II.5.2 of [89], it is possible to choose continuous functions $\rho_1,\ldots,\rho_n\colon\mathbb{R}\to\mathbb{C}$ such that the set of eigenvalues of $W_U(t,\omega_{m_0})$, with $W_U(t,\omega) = (U_1(t,\omega) - iU_3(t,\omega))^{-1}(U_1(t,\omega) + iU_3(t,\omega))$, coincides with the unordered n-tuple $\{\rho_1(t),\ldots,\rho_n(t)\}$, which may have repeated elements. In addition, according to Lemma 2.29(i), these functions have modulus 1. Let $\varphi_1\ldots,\varphi_n\colon\mathbb{R}\to\mathbb{R}$ be continuous branches of their arguments: $e^{i\varphi_j(t)} = \rho_j(t)$

for $j = 1, \dots, n$ and $t \in \mathbb{R}$. According to Theorem 2.30, φ_j is nondecreasing for $j = 1, \dots, n$. It follows from Lemma 2.29(ii) that $\det U_3(t, \omega_{m_0}) = 0$ if and only if there is $j \in \{1, \dots, n\}$ such that $\varphi_j(t) = 2m_j\pi$ for some $m_j \in \mathbb{Z}$. Since $\det U_3(t_1, \omega_{m_0}) \neq 0$, the arguments can be chosen so that $\varphi_j(t_1) \in (-2\pi, 0)$ for $j = 1, \dots, n$. Since $\det U_3(t_{\omega_{m_0}}, \omega_{m_0}) = 0$, there exist $l \in \{1, \dots, n\}$ and an integer $n_l \geq 0$ with $\varphi_l(t_{\omega_{m_0}}) = 2n_l\pi$. And since $\det U_3(t, \omega_{m_0}) \neq 0$ for all $t > t_{\omega_{m_0}}$, then $\varphi_l(t_2) \in (2n_l\pi, 2(n_l + 1)\pi)$ for all $t_2 > t_{\omega_{m_0}}$. Fix such a value t_2.
The definition of $\mathcal{O}$ provides a sequence $(s_k) \uparrow \infty$ with $\lim_{k\to\infty} \omega_0{\cdot}s_k = \omega_{m_0}$. The arguments of Theorem II.5.1 of [89] show that the unordered sets $\mathcal{E}(t, \omega)$ of the eigenvalues of the jointly continuous matrix-valued function $W_U(t, \omega)$ vary continuously in (t, ω), in the Hausdorff sense explained in the proof of Theorem 2.30. Therefore, by choosing k large enough, all the elements of $\mathcal{E}(t_1, \omega_0{\cdot}s_k)$ belong to $\{e^{i\varphi} \mid \varphi \in (-2\pi, 0)\}$, while at least one element of $\mathcal{E}(t_2, \omega_0{\cdot}s_k)$ belongs to $\{e^{i\varphi} \mid \varphi \in (2n_l\pi, 2(n_l + 1)\pi)\}$. For later purposes, choose such a value of k which in addition satisfies $s_k > t_{\omega_0}$. Recall that $n_l \geq 0$. A new application of Theorem II.5.2 of [89] provides continuous functions $\bar{\rho}_1, \dots, \bar{\rho}_n\colon \mathbb{R} \to \mathbb{C}$ such that $\mathcal{E}(t, \omega_0{\cdot}s_k) = \{\bar{\rho}_1(t), \dots, \bar{\rho}_n(t)\}$ for $t \in [t_1, t_2]$. It follows easily that there exists $\tilde{t} \in (t_1, t_2)$ such that 1 belongs to $\mathcal{E}(\tilde{t}, \omega_0{\cdot}s_k)$. Lemma 2.29(ii) implies that $\det U_3(\tilde{t}, \omega_0{\cdot}s_k) = 0$, so that $t_{\omega_0\cdot s_k} > t_1$. However, as checked at the beginning of the proof, $t_{\omega_0\cdot s_k} < t_0 < t_1$, since $s_k > t_{\omega_0}$. This is the sought-for contradiction, which completes the proof of (iii)
As an additional interesting fact, note that the time t_0, which is provided by Lemma 5.24, is in fact an upper bound for $\{t_\omega \mid \omega \in \mathcal{O}\}$. In order to check this, take $r > t_\omega$ for all $\omega \in \mathcal{O}$. The matrix-valued function $\begin{bmatrix} V_1(t,\omega) \\ V_2(t,\omega) \end{bmatrix} = \begin{bmatrix} U_3(t+r,\omega{\cdot}(-r)) \\ U_4(t+r,\omega{\cdot}(-r)) \end{bmatrix}$ solves (5.4) and satisfies $\det V_1(t, \omega) \neq 0$ for all $t \geq 0$. Since $U_3(0, \omega) = 0_n$, Remark 5.23 ensures that, for each $t \geq 0$,

$$U_3(t, \omega) = V_1(t, \omega) \left(\int_0^t V_1^{-1}(s, \omega)\, H_3(\omega{\cdot}s)\, (V_1^T)^{-1}(s, \omega)\, ds \right) Q(\omega)$$

for some nonsingular matrix $Q(\omega)$. Lemma 5.24 ensures that $\det U_3(t, \omega) \neq 0$ for each $t \geq t_0$, as asserted.

(ii) & (iv) These two proofs are analogous to those of points (i) and (iii).

Theorem 5.32 *Suppose that* D1 *holds and that there exists a positive (resp. negative) σ-semiorbit which is dense in Ω. Then, all the systems* (5.4) *are weakly disconjugate on* $[0, \infty)$ *(resp. on* $(-\infty, 0]$*) if and only if the family is uniformly weakly disconjugate.*

Example 5.39 shows the optimality of this result, in the sense that, even if $H_3 > 0$ (so that D1 and D2 hold: see Remark 5.19) and, at the same time, all the systems are simultaneously weakly disconjugate both on $(-\infty, 0]$ and on $[0, \infty)$, then the existence of a dense orbit (instead of semiorbit) does not suffice to guarantee the uniform weak disconjugacy of the family. And Example 5.40 shows that the nonoscillation and weak disconjugacy (on both half-lines) of all the systems in the omega-limit set of an initial point do not guarantee the same properties for the initial system.

The following result, which is an easy consequence of Theorems 5.32 and 5.17, and of Proposition 5.29, presents situations of equivalence of A, B, and C. Recall that the identical normality of all the systems of the family holds, for instance, if $H_3 > 0$.

Proposition 5.33 *Suppose that* D1 *holds, and that every system of the family* (5.4) *is identically normal. Then,*

(i) *the family* (5.4) *is uniformly weakly disconjugate if and only if all its systems are disconjugate.*
(ii) *If there exists a positive (resp. negative) σ-semiorbit which is dense in Ω, then all the systems of* (5.4) *are weakly disconjugate on* $(-\infty, 0]$ *(resp. on* $(0, \infty]$*) if and only if all of them are disconjugate.*

Proof Remark 5.16.1. proves the "if" statements of (i) and (ii). Under D1 and D2, the uniform weak disconjugacy of (5.4) ensures D3, according to Theorem 5.17. Therefore, Proposition 5.29 guarantees the disconjugacy of all the systems of the family: the proof of (i) is complete. In addition, the weak disconjugacy of all the systems ensures D2: see Remark 5.20.2. By Theorem 5.32, the family is uniformly weakly disconjugate if the hypothesis of (ii) holds. Thus, (i) completes the proof of (ii).

Much more can be said in the case of a minimal base. Theorem 5.32 and Lemma 5.21(iii) play a fundamental role in the proof of statement (ii) of the following result.

Proposition 5.34 *Suppose that* D1 *holds and that Ω is minimal. Then the family* (5.4) *is uniformly weakly disconjugate if and only if there exists a point ω_0 such that the corresponding system* (5.4) *is weakly disconjugate on* $[0, \infty)$ *or on* $(-\infty, 0]$.

Proof The "only if" assertion is trivial. Suppose that the system corresponding to a point $\omega_0 \in \Omega$ is weakly disconjugate on $[0, \infty)$. Proposition 5.7 ensures that it is nonoscillatory at $+\infty$, so that, by Theorem 5.31(i), all the systems of the family are. In addition, $D2_{\omega_0}$ holds (see Remark 5.20.2), so that Lemma 5.21 ensures that D2 holds. As explained in Remark 5.20.3, all the systems of the family are weakly disconjugate on $[0, \infty)$, and hence the assertion follows from Theorem 5.32. The proof is the same taking the weak disconjugacy of a system on $(-\infty, 0]$ as the starting point.

In particular, the last result shows that if a particular linear Hamiltonian system is weakly disconjugate on a half-line, and if it is determined by a recurrent coefficient matrix $H_0(t)$ with $H_{03} \geq 0$, then the family constructed on its hull (see Sect. 1.3.2) is uniformly weakly disconjugate. However, recurrence is a strong condition. The following result establishes hypotheses substituting it and providing the same conclusion. And Proposition 5.36 combines both results to optimize the information in the case of recurrence.

Proposition 5.35 *Suppose that the orbit of* ω_0 *is dense in* Ω*, and that*

1. $H_3(\omega_0{\cdot}t) \geq 0$ *for all* $t \in \mathbb{R}$ *(i.e.* D1$_{\omega_0}$ *holds),*
2. *for each nonzero vector* $\mathbf{z}_2 \in \mathbb{R}^n$ *there exist numbers* $t_0 > 0$ *and* $\delta_0 > 0$ *(depending on* $\mathbf{z}_2$*) such that, if* $s \in \mathbb{R}$ *and* $\left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right] = U(t, \omega_0{\cdot}s)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$*, then there is* $t_s \in [0, t_0]$ *with* $\|\mathbf{z}_1(t_s)\| \geq \delta_0$*,*
3. *there exists a* $2n \times n$ *matrix solution* $G(t, \omega_0) = \left[\begin{smallmatrix} G_1(t,\omega_0) \\ G_2(t,\omega_0) \end{smallmatrix}\right]$ *of the system* (5.4) *corresponding to* ω_0 *taking values in* $\mathcal{D}$*.*

Then the family (5.4) *is uniformly weakly disconjugate.*

Proof It is clear that D1 holds. In order to prove the same for D2, suppose for contradiction the existence of $\omega \in \Omega$ and $\mathbf{z}_2 \neq \mathbf{0}$ such that $\left[\begin{smallmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right] = U(t, \omega)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$ satisfies $\mathbf{z}_1(t, \omega) = \mathbf{0}$ for each $t \geq 0$. Let t_0 and δ_0 be the constants of hypothesis 2 for $\mathbf{z}_2$. Find a sequence (t_m) with $\omega = \lim \omega_0{\cdot}t_m$, and write $\left[\begin{smallmatrix} \mathbf{z}_{1,m}(t) \\ \mathbf{z}_{2,m}(t) \end{smallmatrix}\right] = U(t, \omega_0{\cdot}t_m)\left[\begin{smallmatrix} 0 \\ \mathbf{z}_2 \end{smallmatrix}\right]$. Then $\mathbf{z}_1(t, \omega) = \lim_{m\to\infty} \mathbf{z}_{1,m}(t)$ uniformly on $[0, t_0]$. However, for each m there is an $s_m \in [0, t_0]$ such that $\|\mathbf{z}_{1,m}(s_m)\| \geq \delta_0$, and a contradiction is easily established.

Now represent by $\mathcal{A}$ and $\mathcal{O}$ the alpha-limit and omega-limit sets of ω_0, and note that $\Omega = \mathcal{A} \cup \{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathcal{O}$. And recall that D3 holds globally if and only if D3$_\omega$ holds for all $\omega \in \Omega$.

According to Proposition 5.8, Hypothesis 3 ensures that the system corresponding to ω_0 is nonoscillatory at $+\infty$ and at $-\infty$. By Theorem 5.31(i), all the systems corresponding to points of $\mathcal{O}$ are nonoscillatory at $+\infty$, which according to Remark 5.20.3 ensures that all of them are weakly disconjugate on $[0, \infty)$. Hence, Theorem 5.31(iii) and Theorem 5.17 ensure that D3$_\omega$ holds for all $\omega \in \mathcal{O}$. Analogous arguments guarantee that it holds for all $\omega \in \mathcal{A}$. Finally, if $s \in \mathbb{R}$, hypothesis 3 yields the solution $G(t + s, \omega_0)$ taking values in $\mathcal{D}$ of the system $\mathbf{z}' = H((\omega_0{\cdot}s){\cdot}t)\,\mathbf{z}$, so that D3$_\omega$ also holds for all ω in the σ-orbit of ω_0. The proof is complete.

Proposition 5.36 *Suppose that* Ω *is minimal, and that there exists* $\omega_0 \in \Omega$ *such that* D1$_{\omega_0}$ *and* D2$_{\omega_0}$ *hold, and such that there exists a* $2n \times n$ *matrix solution* $G(t, \omega_0) = \left[\begin{smallmatrix} G_1(t,\omega_0) \\ G_2(t,\omega_0) \end{smallmatrix}\right]$ *of the system* (5.4) *corresponding to* ω_0 *taking values in* $\mathcal{D}$ *for all* t *in a positive or negative half-line. Then the family* (5.4) *is uniformly weakly disconjugate.*

Proof It is obvious that D1 holds, and Lemma 5.21(iii) ensures the same for D2. By Proposition 5.8, the system corresponding to ω_0 is nonoscillatory at $+\infty$ or at $-\infty$; Remark 5.20.3 yields its weak disconjugacy on $[0, \infty)$ or on $(-\infty, 0]$; and Proposition 5.34 completes the proof.

As stated in the introduction, one goal of this section is to establish conditions on a particular system ensuring the existence of principal solutions. Note that the second hypothesis of the following result is exactly that of the weak disconjugacy on $[0, \infty)$ (or on $(-\infty, 0]$, and that the third one, stronger than D2$_\omega$ (or than D2$'_\omega$) is rather weaker than the identical normality occurring in the case of disconjugacy (see Proposition 5.29). Examples 5.39 and 5.41 below show that the theorem is optimal, in two senses: the existence of a uniform principal solution cannot be ensured even in the identically normal case, and the weak disconjugacy of a particular system does not suffice to ensure that it has a principal solution on the corresponding half-line.

Theorem 5.37 *Suppose that the system corresponding to $\omega_0 \in \Omega$ satisfies D1$_{\omega_0}$ and $\det U_3(t,\omega_0) \neq 0$ for all $t \geq t_0$, and that it admits no solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on $[t_1,\infty)$ for all $t_1 \geq t_0$. Then it admits a principal solution on $[t_0,\infty)$, which is unique as a matrix-valued function taking values in $\mathcal{L}_{\mathbb{R}}$.*

Analogously, suppose that the system corresponding to $\omega_0 \in \Omega$ satisfies D1$_{\omega_0}$ and $\det U_3(t,\omega_0) \neq 0$ for all $t \leq t_0$, and that it admits no solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on $(-\infty,t_1]$ for all $t_1 \leq t_0$. Then it admits a principal solution on $(-\infty,t_0]$, which is unique as a matrix-valued function taking values in $\mathcal{L}_{\mathbb{R}}$.

Proof As usual, the proofs of these assertions are symmetric, so that just the first one will be explained. Fix any $t_1 \geq t_0$. The arguments of the proofs of points (i) and (ii) of Proposition 5.18 provide $s(t_1) > 0$ and $\delta(t_1) > 0$ such that

$$\int_{t_1}^{t_1+s(t_1)} \|H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}\|^2\,dt \geq \delta(t_1)\,\|\mathbf{x}\|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$, and show that there is no solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on $[t_1, t_1 + s(t_1)]$. The proof of Lemma 5.24 can be easily adapted to check that $I_G(t_1,t,\omega_0) > 0$ whenever $t \geq t_1 + s(t_1)$, where the maps $I_G(t_1,t,\omega_0)$ are defined for $t \geq t_1 \geq t_0$ from $G(t,\omega_0) = \left[\begin{smallmatrix}U_3(t,\omega_0)\\ U_4(t,\omega_0)\end{smallmatrix}\right]$ by (5.10). From here, the proof of Theorem 5.26 can be repeated step by step, taking as the starting point $I_G(t_0,t,\omega_0)$, until (5.16) is obtained, and this proves the statements of the first part of the theorem. The only point of difference is that the nonsingular character of $I_n - I_G(t_0,t,\omega_0)\,J_+(\omega_0)$ is proved first in the set $[t_0 + s(t_0),\infty)$ and then in $[t_0, t_0 + s(t_0)]$.

Note that D1, D2, and the weak disconjugacy on $[0,\infty)$ of the system corresponding to ω_0 guarantee the hypotheses of the previous theorem, and that under these conditions the family is uniformly weakly disconjugate if and only if the principal solution that it provides is uniform, as can be deduced from Theorem 5.17.

The last part of the section presents some examples which were announced above, and which demonstrate the optimality of the results given in this section. The main conclusion to be drawn from the first one has already been mentioned:

- Unless $H_3 > 0$, the uniform weak disconjugacy of the family is a condition less restrictive than the disconjugacy of all the systems, since it does not require the property of identical normality.

In the first three examples, Ω is the closure of the orbit of a particular one of its elements. In the second and third ones, $H_3 > 0$, so that all the systems are identically normal. In the second one, they are also weakly disconjugate both on $(-\infty,0]$ and on $[0, \infty)$. Some of the conclusions which can be inferred from this example have also been anticipated:

– The weak disconjugacy of all the systems of the family guarantees neither the uniform weak disconjugacy nor the existence of the uniform principal solutions, even in the case of identical normality.
– The additional conditions required in Theorem 5.32 and Proposition 5.33(ii) (existence of a dense *semi*orbit), and in Proposition 5.35 (properties 2 and 3), are not superfluous.

Just one of the systems of the third example is weakly disconjugate and nonoscillatory. The main conclusion to be drawn here is the following:

– The nonoscillation and/or weak disconjugacy of the systems corresponding to the points in the omega-limit set of a given one does not guarantee the same properties for the initial system, even in the case of identical normality.

And a conclusion corresponding to the fourth example has been mentioned before Theorem 5.37:

– Conditions $H_3 \geq 0$ and the weak disconjugacy of a particular system on $[0, \infty)$ (or on $(-\infty, 0]$) do not suffice to ensure the existence of principal solution on $[t_1, \infty)$ (or on $(-\infty, t_1]$) for that system, for all $t_1 \in \mathbb{R}$.

Example 5.38 Let $a\colon \mathbb{R} \to \mathbb{R}$ be the bounded and uniformly continuous function defined by

$$a(t) = \begin{cases} 0 & \text{if } |t| \leq 1\,, \\ |t| - 1 & \text{if } 1 \leq |t| \leq 2\,, \\ 1 & \text{if } |t| \geq 2\,. \end{cases}$$

Then $b(t) = \int_0^t a(s)\, ds$ takes the value 0 on $[-1, 1]$, and is strictly increasing outside that interval. Consider the two-dimensional Hamiltonian system

$$\mathbf{z}' = \begin{bmatrix} 0 & a(t) \\ 0 & 0 \end{bmatrix} \mathbf{z}\,.$$

It is easy to check that the hull Ω of the coefficient matrix is

$$\Omega = \left\{ \begin{bmatrix} 0 & a_s(t) \\ 0 & 0 \end{bmatrix} \mid s \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

with $a_s(t) = a(t+s)$. The solution of the system corresponding to $s \in \mathbb{R}$ with initial datum $\left[\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} x_s(t) \\ y_s(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \beta(b(t+s)-b(s)) \\ \beta \end{smallmatrix}\right]$. Therefore, $x_s(t) \neq 0$ for $|t| > 2$. For the limiting system, the solution with the same initial datum is $\left[\begin{smallmatrix} x_\infty(t) \\ y_\infty(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \beta\, t \\ \beta \end{smallmatrix}\right]$, and hence $x_\infty(t) \neq 0$ if $t \neq 0$. Therefore, the family is uniformly weakly disconjugate:

Definition 5.14 holds for $t_0 = 2$. However, the initial system (given by $s = 0$) is not disconjugate: in fact $x_0(t)$ vanishes on $[-1, 1]$, so that the system is not even identically normal. (And the same situation occurs for s small enough.) For further reference, note also that this family of Hamiltonian systems does not have exponential dichotomy over Ω, and that both principal solutions are given by the constant matrix $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

Example 5.39 Let $c\colon \mathbb{R} \to \mathbb{R}$ be a bounded and uniformly continuous function satisfying

$$c(t) = \begin{cases} 1 & \text{if } |t| \geq 3\pi\,, \\ -1 & \text{if } |t| \leq 2\pi\,. \end{cases}$$

Then the two-dimensional linear system

$$\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ c(t) & 0 \end{bmatrix} \mathbf{z}\,,$$

with $H_3(t) = 1 > 0$ for each $t \in \mathbb{R}$, is weakly disconjugate but not disconjugate: the first component of any solution takes the form $c_1 \cos t + c_2 \sin t$ for $t \in (-2\pi, 2\pi)$, so that it vanishes at least twice; and $c_3 e^t + c_4 e^{-t}$ for $|t| \geq 3\pi$, so that it does not vanish for large $|t|$. As in the previous example, the set $\Omega = \left\{\left[\begin{smallmatrix}0 & 1\\ c_s(t) & 0\end{smallmatrix}\right] \mid s \in \mathbb{R}\right\} \cup \left\{\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]\right\}$, with $c_s(t) = c(t+s)$, is the hull of the coefficient matrix. It is easy to check that all the systems of the corresponding family (5.4) are weakly disconjugate, but only the one given by $\omega_1 = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ is disconjugate. Proposition 5.33(i) ensures that the family of systems is not uniformly weakly disconjugate: as seen in Remark 5.30, condition D1 and identical normality hold for every system, since $H_3(\omega) = 1 > 0$. In fact, the absence of uniform weak disconjugacy can be also checked directly: as $s \to -\infty$, the "last" zero of the first component of the solution starting at $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ goes to $+\infty$. In addition, just the limiting system of the family admits uniform principal solutions, as can be deduced from Proposition 5.29, but all of the systems of the family admit principal solutions on suitable positive and negative half-lines. Note also that the principal solution on $[3\pi, \infty)$ of the initial system is given by $\left[\begin{smallmatrix}e^{-t}\\ -e^{-t}\end{smallmatrix}\right]$, which is (up to constant multiple) the unique solution bounded as $t \to \infty$; and that the principal solution on $(-\infty, -3\pi]$ is $\left[\begin{smallmatrix}e^{t}\\ e^{t}\end{smallmatrix}\right]$, the unique solution bounded as $t \to -\infty$. These properties add information to that obtained in Sect. 5.6 concerning the relation between the uniform principal solutions and the stable subbundles in the case of exponential dichotomy.

Example 5.40 Let $d\colon \mathbb{R} \to \mathbb{R}$ be a bounded and uniformly continuous function such that, for each $m \geq 1$,

$$d(t) = \frac{1}{m^2} \qquad \text{if } a_m \leq |t| \leq b_m\,,$$

where $(a_m) \uparrow \infty$, $(b_m) \uparrow \infty$, $a_{m+1} - b_m = 1$, and $b_m - a_m \geq 4\,\pi\, m$. Consider the two-dimensional linear Hamiltonian system

$$\mathbf{z}' = H_0(t)\,\mathbf{z} = \begin{bmatrix} 0 & 1 \\ -d(t) & 0 \end{bmatrix} \mathbf{z}\,, \quad t \in \mathbb{R}\,. \tag{5.19}$$

It is easy to deduce from the nature of any solution on the intervals on which d is constant, that the system is not weakly disconjugate: the first component of each solution vanishes at least once on any interval $[a_m, b_m]$. As in the previous examples, the hull of H_0 is $\Omega = \{H_{0,s} \mid s \in \mathbb{R}\} \cup \{\omega_1\}$ where $H_{0,s}(t) = H_0(t+s)$ and $\omega_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Again, the family satisfies D1 and all its systems are identically normal, since $H_3 > 0$, but the only one which is weakly disconjugate (on $(-\infty, 0]$ and on $[0, \infty)$) is the one corresponding to ω_1: the remaining ones behave as the initial system. According to Corollary 5.12, this means that the only nonoscillatory system (at $+\infty$ and $-\infty$) is the one corresponding to ω_1. Note that $\{\omega_1\}$ is the omega-limit and alpha-limit set of the initial system.

Example 5.41 Let $e\colon \mathbb{R} \to \mathbb{R}$ be a continuous function which is strictly positive on $(-1, 1)$ and zero outside this interval. The fundamental matrix solution $U(t)$ with $U(0) = I_2$ of the (single) two-dimensional linear Hamiltonian system

$$\mathbf{z}' = H_0(t)\,\mathbf{z} = \begin{bmatrix} 0 & e(t) \\ 0 & 0 \end{bmatrix} \mathbf{z}\,, \quad t \in \mathbb{R}$$

is $U(t) = \left[\begin{smallmatrix} 1 & E(t) \\ 0 & 1 \end{smallmatrix}\right]$ for $E(t) = \int_0^t e(s)\,ds$, so that the system is weakly disconjugate on $[0, \infty)$ and on $(-\infty, 0]$. It is easy to deduce that the condition (5.6) of Definition 5.15 is satisfied neither for a $t_1 > 0$ taking the limits as $t \to \infty$ nor for a $t_1 < 0$ taking the limit as $t \to -\infty$. That is, no principal solution exists. This is because $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is a solution of the system outside the interval $[-1, 1]$: one of the hypotheses of Theorem 5.37 does not hold.

5.4 General Properties of the Principal Functions

This section refers to the scenario described in Theorem 5.17: D1, D2, and D3 hold, or equivalently, D1 holds and the family is uniformly weakly disconjugate (on both half-lines); and there exist uniform principal solutions at $\pm\infty$ for each system (5.4), which are denoted by $\left[\begin{smallmatrix} L_1^{\pm}(t,\omega) \\ L_2^{\pm}(t,\omega) \end{smallmatrix}\right]$ and are unique as matrix-valued functions taking values in $\mathcal{L}_{\mathbb{R}}$ (in fact in $\mathcal{D}$). The symbols $\tilde{l}^{\pm}(\omega)$ will represent in the rest of the chapter the Lagrange planes represented by $\left[\begin{smallmatrix} L_1^{\pm}(0,\omega) \\ L_2^{\pm}(0,\omega) \end{smallmatrix}\right]$, which according to Definition 5.15 can be also represented by the real matrices $\left[\begin{smallmatrix} I_n \\ N^{\pm}(\omega) \end{smallmatrix}\right]$. It follows from the equality $\tilde{l}^{\pm}(\omega{\cdot}t) = U(t,\omega)\cdot\tilde{l}^{\pm}(\omega)$ established in Theorem 5.26 that

$$N^{\pm}(\omega{\cdot}t) = L_2^{\pm}(t,\omega)\,(L_1^{\pm})^{-1}(t,\omega)\,. \tag{5.20}$$

Or, in other words, that the maps $N^{\pm}(\omega)$ are globally defined symmetric solutions along the flow of the Riccati equation (5.7) (see Sect. 1.3.5). According to Definition 1.49, they are equilibria. Note that these functions are unique, as can be deduced from the uniqueness in $\mathcal{L}_{\mathbb{R}}$ of the uniform principal solutions.

Definition 5.42 The globally defined functions $N^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ parameterizing the principal solutions at $\pm\infty$ in $\mathcal{D}$ are the *principal functions*.

The analysis of the general properties of the principal functions, as well as of the dynamical and measurable consequences of these properties, is the object of this section. Recall that the concept of upper semicontinuous function $N\colon\Omega \to \mathbb{S}_n(\mathbb{R})$ is described in Definition 1.47, and that its main properties are described in Proposition 1.48.

Theorem 5.43 *Suppose that* D1, D2, *and* D3 *hold, and let t_0 satisfy the assertions of Remark 5.16 and Proposition 5.18(i). Then, the principal functions $N^{\pm}$ satisfy*

$$N^{\pm}(\omega) = \lim_{r\to\pm\infty} N_r(\omega) \tag{5.21}$$

for all $\omega \in \Omega$, where N_r is the continuous symmetric matrix-valued function given by

$$N_r(\omega) = -U_3^{-1}(r,\omega)\,U_1(r,\omega) \tag{5.22}$$

for $|r| > t_0$. In addition,

$$N_{r_1}(\omega) \le N_{r_2}(\omega) \le N_{-r_2}(\omega) \le N_{-r_1}(\omega) \quad \textit{for } \; t_0 < r_1 < r_2\,, \tag{5.23}$$

and hence

$$N_r(\omega) \le N^{+}(\omega) \le N^{-}(\omega) \le N_{-r}(\omega) \quad \textit{if } \; r > t_0\,. \tag{5.24}$$

In particular, $\mp N^{\pm}$ are (bounded) upper semicontinuous $n\times n$ matrix-valued functions on Ω, and the functions $\tilde{l}^{\pm}\colon \Omega \to \mathcal{L}_{\mathbb{R}}\,,\ \omega \to \tilde{l}^{\pm}(\omega)$ are Borel measurable.

Proof Fix $\omega\in\Omega$ and choose uniform principal solutions $L^{\pm}(t,\omega)$ at $\pm\infty$ normalized to $L_1^{\pm}(0,\omega) = I_n$ (so that $N^{\pm}(\omega) = L_2^{\pm}(0,\omega)$). According to Remark 5.23, for each fixed $r\in\mathbb{R}$ with $|r|\ge t_0$, the $2n\times n$ matrix-valued function

$$\begin{bmatrix} L_1^r(t,\omega) \\ L_2^r(t,\omega) \end{bmatrix} = \begin{bmatrix} L_1^{+}(t,\omega)\,(I_n - I(t,\omega)\,(I(r,\omega))^{-1}) \\ L_2^{+}(t,\omega)\,(I_n - I(t,\omega)\,(I(r,\omega))^{-1}) - ((L_1^{+})^T)^{-1}(t,\omega)\,(I(r,\omega))^{-1} \end{bmatrix},$$

with $I(t,\omega) = I_{L^+}(0,t,\omega)$ given by (5.10), solves (5.4). Note that $L_1^r(0,\omega) = L^+(0,\omega) = I_n$. By the definition of principal solution, $\lim_{r\to\infty}(I(r,\omega))^{-1} = 0_n$. Therefore, $\lim_{r\to\infty}\begin{bmatrix} L_1^r(t,\omega) \\ L_2^r(t,\omega)\end{bmatrix} = \begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega)\end{bmatrix}$ for all $t \in \mathbb{R}$. In particular, for $t = 0$, $\lim_{r\to\infty}\begin{bmatrix} L_1^r(0,\omega) \\ L_2^r(0,\omega)\end{bmatrix} = \begin{bmatrix} L_1^+(0,\omega) \\ L_2^+(0,\omega)\end{bmatrix}$; or, in other words,

$$N^+(\omega) = \lim_{r\to\infty} N_r(\omega) \quad \text{ for } N_r(\omega) = L_2^r(0,\omega)\,. \tag{5.25}$$

Note also that $L_1^r(r,\omega) = 0_n$. It follows from $\begin{bmatrix} 0_n \\ L_2^r(r,\omega)\end{bmatrix} = \begin{bmatrix} L_1^r(r,\omega) \\ L_2^r(r,\omega)\end{bmatrix} = U(r,\omega)\begin{bmatrix} L_1^r(0,\omega) \\ L_2^r(0,\omega)\end{bmatrix} = U(r,\omega)\begin{bmatrix} I_n \\ N_r(\omega)\end{bmatrix}$ that $0_n = U_1(r,\omega) + U_3(r,\omega)\,N_r(\omega)$. This implies (5.22) for $|r| > t_0$, which together with (5.25) completes the proof of (5.21) for N^+.

For later purposes, note that

$$N_r(\omega) = N^+(\omega) - (I(r,\omega))^{-1} \tag{5.26}$$

whenever $|r| \geq t_0$.

Now define $\widetilde{I}(t,\omega) = I_{L^-}(0,t,\omega)$ by (5.10). Repeating the above argument guarantees that $\begin{bmatrix} L_1^-(t,\omega) \\ L_2^-(t,\omega)\end{bmatrix} = \lim_{r\to-\infty}\begin{bmatrix} K_1^r(t,\omega) \\ K_2^r(t,\omega)\end{bmatrix}$ for each $t \in \mathbb{R}$, with

$$\begin{bmatrix} K_1^r(t,\omega) \\ K_2^r(t,\omega)\end{bmatrix} = \begin{bmatrix} L_1^-(t,\omega)\,(I_n - \widetilde{I}(t,\omega)\,(\widetilde{I}(r,\omega))^{-1}) \\ L_2^-(t,\omega)\,(I_n - \widetilde{I}(t,\omega)\,(\widetilde{I}(r,\omega))^{-1}) - ((L_1^-)^T)^{-1}(t,\omega)\,(\widetilde{I}(r,\omega))^{-1}\end{bmatrix}$$

for $|r| \geq t_0$. Then $N^-(\omega) = \lim_{r\to-\infty}\widetilde{N}_r(\omega)$, with $\widetilde{N}_r(\omega) = K_2^r(0,\omega)$. As before, $K_1^r(r,\omega) = 0_n$ yields $\widetilde{N}_r(\omega) = -U_3^{-1}(r,\omega)\,U_1(r,\omega)$ for $|r| > t_0$. This and relation (5.22), already established, prove that $\widetilde{N}_r(\omega) = N_r(\omega)$ for $|r| > t_0$, so that (5.21) also holds for N^-.

Relation (5.26) provides an almost immediate proof of (5.23) and (5.24): one just has to use that $-(I(r,\omega))^{-1}$ increases as $r \geq t_0$ increases, decreases as $r \leq -t_0$ decreases, and satisfies $I^{-1}(-r,\omega) < 0_n < (I(r,\omega))^{-1}$ for $r \geq t_0$.

Therefore the functions $\mp N^{\pm}(\omega)$ are the limits of two decreasing sequences of continuous functions which are uniformly bounded, as can be deduced from (5.23) for a fixed value of r_1 and Remark 1.44.2. Proposition 1.48(iii) ensures that they are upper semicontinuous. Of course, they are also Borel measurable: see e.g. Remark 1.1. Now, given a Borel set $\mathcal{B} \subseteq \mathcal{L}_{\mathbb{R}}$, note that $(\tilde{l}^{\pm})^{-1}(\mathcal{B}) = \{\omega \in \Omega \mid \tilde{l}^{\pm}(\omega) \in \mathcal{B}\} = \{\omega \in \Omega \mid \tilde{l}^{\pm}(\omega) \in \mathcal{B} \cap \mathcal{D}\}$; that $\mathcal{B} \cap \mathcal{D}$ can be identified with a Borel set $\mathcal{A} \subseteq \mathbb{S}_n(\mathbb{R})$ (see Remark 1.30); and that $(\tilde{l}^{\pm})^{-1}(\mathcal{B}) = \{\omega \in \Omega \mid N^{\pm}(\omega) \in \mathcal{A}\}$, which are Borel sets. The proof is complete.

Corollary 5.44 *Suppose that* D1, D2, *and* D3 *hold, and define* $\begin{bmatrix} L_1^{\pm}(t,\omega) \\ L_2^{\pm}(t,\omega) \end{bmatrix} = U(t,\omega)\begin{bmatrix} I_n \\ N^{\pm}(\omega) \end{bmatrix}$. *Then*

$$N^-(\omega) - N^+(\omega) = \lim_{t\to-\infty}\left(\int_t^0 (L_1^+)^{-1}(s,\omega)\,H_3(\omega{\cdot}s)((L_1^+)^T)^{-1}(s,\omega)\,ds\right)^{-1}$$
$$= \lim_{t\to\infty}\left(\int_0^t (L_1^-)^{-1}(s,\omega)\,H_3(\omega{\cdot}s)((L_1^-)^T)^{-1}(s,\omega)\,ds\right)^{-1}.$$

Proof The first equality follows from (5.26), and the second one from the analogous equality $N_r(\omega) = \widetilde{N}_r(\omega) = N^-(\omega) - (\widetilde{I}(r,\omega))^{-1}$ for $|r| \ge t_0$ (with the notation of the proof of the previous result).

Summing up the main results proved so far in this section: $N^{\pm}(\omega)$ are semicontinuous functions given by pointwise limits of continuous symmetric matrix-valued functions; they are bounded solutions along the flow of the Riccati equation (5.7); and they parameterize in $\mathcal{D}$ the uniform principal solutions at $\pm\infty$: $\tilde{l}^{\pm}(\omega) \equiv \begin{bmatrix} I_n \\ N^{\pm}(\omega) \end{bmatrix}$ for every $\omega \in \Omega$.

The following four results go more deeply into the dynamical and measure-theoretic properties of the functions $N^{\pm}$ and of the global flows associated to (5.4). Propositions 5.45 and 5.46 refer to the (residual) sets of continuity points of $N^{\pm}$. The first one states, in particular, that the principal functions provide minimal almost automorphic extensions of the base Ω if this one is minimal: see Definition 1.18. And the second one, Proposition 5.46, establishes that the dimension of the intersection of the vector spaces $\tilde{l}^+(\omega)$ and $\tilde{l}^-(\omega)$ reaches its minimum on the set of common continuity points, on which it is constant. Some examples follow the results and indicate their scope.

Proposition 5.45 *Suppose that* D1, D2, *and* D3 *hold.*

(i) *Let* m_0 *be a* σ*-ergodic measure on* Ω. *Then each one of the sets* $\widetilde{L}^{\pm} = \{(\omega, \tilde{l}^{\pm}(\omega)) \,|\, \omega \in \Omega\}$ *concentrates a* τ*-invariant measure* $\mu^{\pm}$ *projecting onto* m_0.
(ii) *The continuity points of* $N^{\pm}$ *form two residual invariant subsets* $\Omega^{\pm} \subseteq \Omega$, *which are* σ*-invariant.*
(iii) *If* Ω *is minimal, then the sets* $\mathcal{K}^{\pm} = \mathrm{closure}_{\mathcal{K}_{\mathbb{R}}}\{(\omega, \tilde{l}^{\pm}(\omega)) \,|\, \omega \in \Omega^{\pm}\}$ *are almost automorphic extensions of the base* Ω *for the flow* τ.

Proof

(i) Proposition 1.16(ii) states that the measures $\mu^{\pm}$ defined on $\mathcal{K}_{\mathbb{R}}$ by $\int_{\mathcal{K}_{\mathbb{R}}} f(\omega, l)\,d\mu^{\pm} = \int_{\Omega} f(\omega, \tilde{l}^{\pm}(\omega))\,dm_0$ for $f \in C(\mathcal{K}_{\mathbb{R}}, \mathbb{R})$ satisfy the assertion.
(ii) & (iii) Proposition 1.48(ii) establishes the residual character of the sets $\Omega^{\pm}$, and Proposition 1.53 contains the remaining assertions.

The σ-invariant sets $\Omega^\pm$ provided by Proposition 5.45(ii) play a role in the statement of the following result.

Proposition 5.46 *Suppose that* D1, D2, *and* D3 *hold and that* Ω *is minimal. Let* $\Omega_c = \Omega^+ \cap \Omega^-$ *be the* σ*-invariant set of common continuity points of* $N^\pm$*. Then, there is* $k \in \mathbb{N}$ *with* $0 \le k \le n$ *such that* $\dim\big(\tilde{l}^+(\omega) \cap \tilde{l}^-(\omega)\big) = k$ *for every* $\omega \in \Omega_c$*. In addition,* $\dim\big(\tilde{l}^+(\omega) \cap \tilde{l}^-(\omega)\big) \le k$ *for every* $\omega \in \Omega$*.*

Proof Note that

$$\dim\big(\tilde{l}^-(\omega) \cap \tilde{l}^+(\omega)\big) = \dim\big(\mathrm{Ker}\,\big(N^-(\omega) - N^+(\omega)\big)\big) \tag{5.27}$$

for all $\omega \in \Omega$: the vectors $\begin{bmatrix} I_n \\ N^-(\omega) \end{bmatrix} \mathbf{x}_0 \in \tilde{l}^-(\omega)$ and $\begin{bmatrix} I_n \\ N^+(\omega) \end{bmatrix} \mathbf{y}_0 \in \tilde{l}^+(\omega)$ coincide if and only if $\mathbf{x}_0 = \mathbf{y}_0$ and $N^-(\omega)\,\mathbf{x}_0 = N^+(\omega)\,\mathbf{y}_0$. Denote $\begin{bmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega) \end{bmatrix} = U(t,\omega) \begin{bmatrix} I_n \\ N^\pm(\omega) \end{bmatrix}$. Then, for all $\omega \in \Omega$ and $t \in \mathbb{R}$,

$$(L_2^+)^T(t,\omega)\, L_1^-(t,\omega) - (L_1^+)^T(t,\omega)\, L_2^-(t,\omega) = N^+(\omega) - N^-(\omega) \tag{5.28}$$

since equations (5.4) ensure that the left-hand term is independent of t. Hence, since

$$N^\pm(\omega{\cdot}t) = L_2^\pm(t,\omega)\,(L_1^\pm)^{-1}(t,\omega) = ((L_1^\pm)^{-1})^T(t,\omega)\,(L_2^\pm)^T(t,\omega)\,,$$

one has that

$$N^+(\omega) - N^-(\omega) = (L_1^+)^T(t,\omega)\,\big(N^+(\omega{\cdot}t) - N^-(\omega{\cdot}t)\big)\,L_1^-(t,\omega)\,. \tag{5.29}$$

Relations (5.27) and (5.29) ensure that the function

$$k(\omega) = \dim \mathrm{Ker}\,\big(N^-(\omega) - N^+(\omega)\big) = \dim\big(\tilde{l}^+(\omega) \cap \tilde{l}^-(\omega)\big)$$

is σ-invariant. Note also that $k(\omega)$ is the multiplicity of 0 as eigenvalue of the positive definite matrix $N^-(\omega) - N^+(\omega)$. That is, if $\mu_1(\omega) \le \cdots \le \mu_n(\omega)$ are these eigenvalues, and $\mu_0(\omega) = 0$, then $k(\omega) = \max\{k \in \{0,\dots,n\} \mid \mu_k(\omega) = 0\}$

Take $\omega_0 \in \Omega_c$, $\omega \in \Omega$, and a sequence (t_m) with $\lim_{m\to\infty} \omega{\cdot}t_m = \omega_0$. Let $\tilde{k} = k(\omega) = k(\omega{\cdot}t_m)$. Since the sequence $(N^-(\omega{\cdot}t_m) - N^+(\omega{\cdot}t_m))$ converges to $N^-(\omega) - N^+(\omega)$, the continuous variation of the set of eigenvalues with respect to the matrix ensures that $\mu_j(\omega_0) = \lim_{m\to\infty} \mu_j(\omega{\cdot}t_m)$ for $j = 0,\dots,n$: see e.g. Theorem II.5.1 of [89], and note that, in this case, the sets of eigenvalues are ordered subsets of $\mathbb{R}$. Therefore $\mu_j(\omega_0) = 0$ at least for $j = 0,\dots,\tilde{k}$, which ensures that $k(\omega_0) \ge \tilde{k} = k(\omega)$. Note that this holds for every $\omega \in \Omega$. In addition, if $\omega_1 \in \Omega_c$, then the same argument guarantees that $k(\omega_1) \ge k(\omega_0)$, so that both dimensions agree. The result is proved for $k = k(\omega_0)$, where ω_0 is any fixed point in the set Ω_c.

Examples 5.47 Note that in the case in which $N^{\pm}$ are continuous functions, the set Ω_c of the previous result agrees with Ω, so that the constant $\dim\big(\tilde{l}^+(\omega)\cap\tilde{l}^-(\omega)\big)$ takes the same value k for all $\omega \in \Omega$. It is possible to construct examples with $N^{\pm}$ continuous for which k takes any value, from 0 to n. The simplest ones are, of course, autonomous. For instance, with $n = 1$, the system $\mathbf{z}' = \left[\begin{smallmatrix}-1 & 1\\ -1 & 1\end{smallmatrix}\right]\mathbf{z}$ satisfies $N^+ = N^- = 1$, so that $k = 1$, while $\mathbf{z}' = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]\mathbf{z}$ satisfies $N^+ = -1$ and $N^- = 1$, so that $k = 0$. By combining these two systems one gets

$$\mathbf{z}' = \begin{bmatrix} -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{z}\,, \tag{5.30}$$

for which $N^+ = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ and $N^- = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, so that $k = 1$. This can be proved by direct computation: the solutions of (5.30) with initial data $\left[\begin{smallmatrix}I_2\\ N^+\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}I_2\\ N^-\end{smallmatrix}\right]$ are

$$\begin{bmatrix} 1 & 0\\ 0 & e^{-t}\\ 1 & 0\\ 0 & -e^{-t} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0\\ 0 & e^{t}\\ 1 & 0\\ 0 & e^{t} \end{bmatrix},$$

respectively, and they satisfy the conditions of Definition 5.15.

Of course, the two measures $\mu^{\pm}$ of Proposition 5.45(i) coincide in the case that $N^+ = N^-$ m_0-a.e. on Ω. Note also that $\mu^{\pm}$ are Dirac measures in the autonomous case. There are also simple nonautonomous cases in which the measures coincide, such as that given in Example 5.38 (with $N^+(\omega) = N^-(\omega) \equiv 0$ for all $\omega \in \Omega$, so that $k = 1$). Observe also that the functions N^+ and N^- are continuous if they agree everywhere in Ω, since according to Theorem 5.43 $-N^+$ and $N^+ = N^-$ are upper semicontinuous functions.

Another (very complicated) case with $n = 1$ will be considered in Example 8.44 of Chap. 8. In this example, which is of Millionščikov–Vinograd type with minimal base, the principal functions are noncontinuous maps which agree in the residual set of their continuity points and are different in full measure, so that the two τ-invariant measures $\mu^{\pm}$ are different. Therefore, in this case, the constant k of Proposition 5.46 equals 1, but $\dim(\tilde{l}^+(\omega)\cap\tilde{l}^-(\omega)) = 0$ m_0-a.e., and the sets $\mathcal{K}^{\pm}$ of Proposition 5.45(iii) are not copies of the base. As a matter of fact, they determine the unique minimal subset of the corresponding bundle $\mathcal{K}_{\mathbb{R}}$.

Observe finally that Proposition 5.46 ensures that if the two principal functions agree in at least one point of Ω (and hence $k = n$), then they agree at the σ-invariant set of their common continuity points. But this set can have zero measure, as in Example 8.44, or full measure, as in many of the examples given in this chapter.

The next two theorems explore the properties of the following subsets of $\mathcal{K}_{\mathbb{R}}$: assuming that hypotheses D1, D2, and D3 are valid, define

$$\begin{aligned}
\mathcal{J}^+ &= \left\{(\omega,l) \in \Omega \times \mathcal{D} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } N^+(\omega) \le M\right\}, \\
\mathcal{J}^- &= \left\{(\omega,l) \in \Omega \times \mathcal{D} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } M \le N^-(\omega)\right\}, \\
\mathcal{J} &= \left\{(\omega,l) \in \Omega \times \mathcal{D} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } N^+(\omega) \le M \le N^-(\omega)\right\},
\end{aligned} \tag{5.31}$$

so that $\mathcal{J} = \mathcal{J}^+ \cap \mathcal{J}^-$. These three sets posses some topological, dynamical and measurable properties which can be used to describe the global dynamics induced by the family (5.4) on $\mathcal{K}_{\mathbb{R}}$ and on $\Omega \times \mathbb{S}_n(\mathbb{R})$. A bit more precisely, recall that, if $l \in \mathcal{D}$ and $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$, then $U(t,\omega){\cdot}l \in \mathcal{D}$ as long as the solution $M(t,\omega,M_0)$ of the Riccati equation (5.7) with $M(t,\omega,M_0) = M_0$ is defined, and that these solutions define the flow τ_s on $\Omega \times \mathbb{S}_n(\mathbb{R})$ (see Sect. 1.3.5). Therefore, the properties of invariance and attractivity described by Theorem 5.48 show that the principal functions N^+ and N^- "delimit" the areas on which τ is globally defined as a flow, and as a positive or negative semiflow. And Theorem 5.49 proves that any τ-invariant measure on $\mathcal{K}_{\mathbb{R}}$ is concentrated on $\mathcal{J}$. The notion of copy of the base is given in Definition 1.17.

Theorem 5.48 *Suppose that D1, D2, and D3 hold. Then,*

(i) *The sets $\mathcal{J}^+$, $\mathcal{J}^-$, and $\mathcal{J}$, defined by (5.31), are positively τ-invariant, negatively τ-invariant, and τ-invariant, respectively.*
(ii) *The set $\mathcal{J}$ is compact. In addition, if a sequence $((\omega_j,l_j))$ of points of $\mathcal{J}^+$ (resp. of $\mathcal{J}^-$) converges to a point $(\omega_0,l_0) \in \Omega \times \mathcal{D}$, then $(\omega_0,l_0) \in \mathcal{J}^+$ (resp. $(\omega_0,l_0) \in \mathcal{J}^-$).*
(iii) *Take $(\omega,l) \in \mathcal{K}_{\mathbb{R}}$. Then $\tau(t,\omega,l) \in \Omega \times \mathcal{D}$ for all $t \ge 0$, $t \le 0$, and $t \in \mathbb{R}$, if and only if $(\omega,l) \in \mathcal{J}^+$, $(\omega,l) \in \mathcal{J}^-$, and $(\omega,l) \in \mathcal{J}$, respectively.*
(iv) *$\mathcal{J}$ is the maximal τ-invariant subset of $\Omega \times \mathcal{D}$. Moreover, the alpha-limit set and the omega-limit set of any τ-orbit in $\mathcal{K}_{\mathbb{R}}$ are contained in $\mathcal{J}$. In particular, $\mathcal{J}$ contains all the minimal subsets of $\mathcal{K}_{\mathbb{R}}$.*

Proof

(i) Consider the auxiliary linear equation

$$M' = -M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) = g(\omega{\cdot}t, M)\,,$$

whose solution with initial datum M_0, represented by $M_l(t,\omega,M_0)$, is globally defined for all $\omega \in \Omega$ and $M_0 \in \mathbb{S}_n(\mathbb{R})$. Take $(\omega,l) \in \mathcal{J}^+$ and represent l in the form $\left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. Let $\mathcal{I}$ be the maximal interval of definition of $M(t,\omega,M_0)$. The monotonicity properties established in Theorem 1.45 ensure that $N^+(\omega{\cdot}t) \le M(t,\omega,M_0)$ for all $t \in \mathcal{I}$. In addition, since $H_3 \ge 0$,

$$M'(t,\omega,M_0) = h(\omega{\cdot}t, M(t,\omega,M_0)) \le g(\omega{\cdot}t, M(t,\omega,M_0))$$

for $t \in \mathcal{I}$, so that Theorem 1.46(i) implies that $M(t,\omega,M_0) \le M_l(t,\omega,M_0)$ for $t \ge 0$, $t \in \mathcal{I}$. These inequalities imply that $\|M(s,\omega,M_0)\|$ is bounded in any interval $[0,t] \subset \mathcal{I}$: see Remark 1.44.2. It follows from Remark 1.43 that $M(t,\omega,M_0)$ is defined (at least) for $t \ge 0$, and hence (i) is proved for $\mathcal{J}^+$. The proof is analogous for $\mathcal{J}^-$. And both properties taken together imply that $\mathcal{J} = \mathcal{J}^+ \cap \mathcal{J}^-$ is τ-invariant.

(ii) Define $\mathcal{J}_r = \left\{(\omega,l) \in \Omega \times \mathcal{D} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ and } N_r(\omega) \le M \le N_{-r}(\omega)\right\}$ for N_r given by (5.22), and note that $\mathcal{J} = \cap_{r=1}^{\infty} \mathcal{J}_r$, as can be deduced from (5.21) and (5.24). The continuity of N_r and N_{-r} ensures that each set $\mathcal{J}_r$ is compact, so that also $\mathcal{J}$ is compact. Now take a sequence $((\omega_j,l_j))$ of points of $\mathcal{J}^+$ with limit $(\omega_0,l_0) \in \Omega \times \mathcal{D}$, and represent $l_j \equiv \left[\begin{smallmatrix} I_n \\ M_j \end{smallmatrix}\right]$ and $l_0 \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. Then, $\lim_{j\to\infty} M_j = M_0$. By hypothesis, $M_j \ge N^+(\omega_j)$. Since the function N^+ is norm-bounded on Ω, it is possible to take a subsequence $((\omega_k,l_k))$ such that there exists $\lim N^+(\omega_k) = N_0$. Hence, $M_0 \ge N_0$. The semicontinuity of $-N^+$ established in Theorem 5.43 ensures that $M_0 \ge N_0 \ge N^+(\omega_0)$, which proves the statement for $\mathcal{J}^+$. The proof is analogous for $\mathcal{J}^-$.

(iii) Just the "only if" assertions must be proved, since the "if" ones follow from (i). Suppose that $U(t,\omega){\cdot}l \in \mathcal{D}$ for all $t \ge 0$. Write $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$ and $L(t,\omega) = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. According to Remark 5.23, if $t \ge 0$, then

$$\tilde{l}^+(\omega{\cdot}t) \equiv \begin{bmatrix} L_1(t,\omega)\,(P(\omega) + I_L(0,t,\omega)\,Q(\omega)) \\ L_2(t,\omega)\,(P(\omega) + I_L(0,t,\omega)\,Q(\omega)) + (L_1^T)^{-1}(t,\omega)\,Q(\omega) \end{bmatrix},$$

where $P(\omega)$ is nonsingular. In particular, taking $t = 0$,

$$N^+(\omega) = M_0 + Q(\omega)\,P^{-1}(\omega)\,. \tag{5.32}$$

In addition, by (5.15) and (5.13),

$$0_n = \lim_{t\to\infty} I_{L^+}^{-1}(0,t,\omega) = P^T(\omega)\left(\lim_{t\to\infty} I_L^{-1}(0,t,\omega)\right)P(\omega) + P^T(\omega)\,Q(\omega)\,,$$

which implies that

$$Q(\omega)\,P^{-1}(\omega) = (P^{-1})^T(\omega)\,P^T(\omega)\,Q(\omega)\,P^{-1}(\omega) = -\lim_{t\to\infty} I_L^{-1}(0,t,\omega) \le 0\,.$$

This and (5.32) ensure that $N^+(\omega) \le M_0$. In other words, $(\omega,l) \in \mathcal{J}^+$, as asserted. The proof of the property for $\mathcal{J}^-$ is analogous, and both of them taken together imply the assertion for $\mathcal{J}$.

(iv) By (iii), the τ-orbit of a point $(\omega,l) \notin \mathcal{J}$ is not contained in $\Omega \times \mathcal{D}$. This proves the first assertion in (iv). Now take $(\omega_0,l_0) \in \mathcal{K}_{\mathbb{R}}$. By Theorem 5.25(ii), there exists $t_0 \ge 0$ such that $\tau(t,\omega_0,l_0) \in \Omega \times \mathcal{D}$ for $t \ge t_0$. Therefore, by (iii) and (i), $\tau(t,\omega_0,l_0) \in \mathcal{J}^+$ whenever $t \ge t_0$. Let the point (ω_1,l_1) belong to the omega-limit set of (ω_0, l_0), and write it as $(\omega_1, l_1) = \lim_{j\to\infty} \tau(t_j, \omega_0, l_0)$ for a sequence $(t_j) \uparrow \infty$. A new application of Theorem 5.25(ii) provides $t_1 \geq 0$ such that the point $\tau(t, \omega_1, l_1) = \lim_{j\to\infty} \tau(t + t_j, \omega_0, l_0)$ belongs to $\Omega \times \mathcal{D}$ whenever $t \leq -t_1$. Therefore, by (ii), $\tau(t, \omega_1, l_1) \in \mathcal{J}^+$ whenever $t \leq -t_1$, and hence (i) implies that $\tau(-t_1 + s, \omega_1, l_1) \in \mathcal{J}^+ \subset \Omega \times \mathcal{D}$ whenever $s \geq 0$. In other words, $\tau(t, \omega_1, l_1) \in \Omega \times \mathcal{D}$ for all $t \in \mathbb{R}$, and hence (iii) ensures that $(\omega_1, l_1) \in \mathcal{J}$, as asserted. The proof is analogous for the alpha-limit set. The last assertion of (iv) is now trivial: any minimal set is the omega-limit set of each of its orbits.

Recall that Σ_m represents the m-completion of the Borel sigma-algebra for a (positive normalized regular Borel) measure m.

Theorem 5.49 *Suppose that* D1, D2, *and* D3 *hold. Then,*

(i) *every τ-invariant measure μ on $\mathcal{K}_\mathbb{R}$ is concentrated on $\mathcal{J}$; that is, $\mu(\mathcal{J}) = 1$. In particular, let m be a σ-ergodic measure on Ω, let $\Omega_0 \in \Sigma_m$ be a σ-invariant set with $m(\Omega_0) = 1$, and let $l\colon \Omega \to \mathcal{L}_\mathbb{R}$ be a Σ_m-measurable map with $\tau(t, \omega, l(\omega)) = (\omega{\cdot}t, l(\omega{\cdot}t))$ for all $\omega \in \Omega_0$. Then the Σ_m-measurable set*

$$\Omega_1 = \{\omega \in \Omega_0 \,|\, (\omega, l(\omega)) \in \mathcal{J}\}$$

is σ-invariant with $m(\Omega_1) = 1$.

(ii) *Suppose further that there exists a subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) > 0$ for an ergodic measure m_0 on the base such that the σ-orbit of ω is dense in Ω for all $\omega \in \Omega_0$. Let $\mathcal{K} \subset \mathcal{K}_\mathbb{R}$ be a copy of the base. Then $\mathcal{K} \subseteq \mathcal{J}$.*

Proof

(i) Let μ be a τ-invariant measure on $\mathcal{K}_\mathbb{R}$. The Birkhoff Theorem 1.3 provides a τ-invariant set $\mathcal{K}_\mathcal{J}$ with $\mu(\mathcal{K}_\mathcal{J}) = 1$ and a function $\widetilde{\chi}_{\mathcal{J}^+} \in L^1(\mathcal{K}_\mathbb{R}, \mu)$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_{\mathcal{J}^+}(\tau(t, \omega, l))\, dt = \widetilde{\chi}_{\mathcal{J}^+}(\omega, l) \tag{5.33}$$

for every $(\omega, l) \in \mathcal{K}_\mathcal{J}$, with $\mu(\mathcal{J}^+) = \int_{\mathcal{K}_\mathbb{R}} \widetilde{\chi}_{\mathcal{J}^+}(\omega, l)\, d\mu$. Take $(\omega, l) \in \mathcal{K}_\mathcal{J}$. Theorem 5.25(ii) and points (iii) and (i) of Theorem 5.48 provide $t_0 \geq 0$ (depending on (ω, l)) such that $\tau(t, \omega, l) \in \mathcal{J}^+$ whenever $t \geq t_0$. Consequently, (5.33) ensures that $\widetilde{\chi}_{\mathcal{J}^+}(\omega, l) = 1$, which in turn ensures that $\mu(\mathcal{J}^+) = 1$. Analogously, $\mu(\mathcal{J}^-) = 1$, and therefore $\mu(\mathcal{J}) = \mu(\mathcal{J}^+ \cap \mathcal{J}^-) = 1$, as stated.

Assume now that m, Ω_0 and $l\colon \Omega \to \mathcal{L}_\mathbb{R}$ satisfy the conditions in the last assertion of (i). Applying what has already been proved to the τ-ergodic measure μ_l which is concentrated on the graph of l and which projects onto m (see Proposition 1.16(ii)) yields $1 = \mu_l(\mathcal{J}) = \int_\Omega \chi_\mathcal{J}(\omega, l(\omega))\, dm$, so that $(\omega, l(\omega)) \in \mathcal{J}$ for m-a.e. $\omega \in \Omega_0$. That is, $m(\Omega_1) = 1$. The τ-invariance of $\mathcal{J}$ guarantees the σ-invariance of Ω_1.

(ii) Write $\mathcal{K} = \{(\omega, l(\omega)) \mid \omega \in \Omega\}$ and note that the τ-orbit of $(\omega, l(\omega))$ is dense in $\mathcal{K}$ for all $\omega \in \Omega_0$. Let Ω_1 be the subset of Ω composed of the points ω with $(\omega, l(\omega)) \in \mathcal{J}$, which according to (i) is σ-invariant and satisfies $m_0(\Omega_1) = 1$. Points (i) and (ii) of Theorem 5.48 ensure that the (dense) τ-orbit of any point $\omega \in \Omega_0 \cap \Omega_1$ is contained in the compact set $\mathcal{J}$, and hence $\mathcal{K} \subseteq \mathcal{J}$.

Remark 5.50 The shape of the sets $\mathcal{J}^+$, $\mathcal{J}^-$, and $\mathcal{J}$ can vary from extremely simple, as in the autonomous examples described below the proof of Theorem 5.17 (before Proposition 5.27) to extremely complicated, as in the situation described in Example 8.44 and summarized in Examples 5.47. In this last case the set $\mathcal{J}^+$ is what in the literature is called a *pinched set*: the fiber over each point of the base reduces to a singleton just for a residual proper subset of Ω.

The section is completed with the following result, which is consequence of Theorem 5.26 and the comparison theorems for the Riccati equations of Sect. 1.3.5. It presents an extension to linear Hamiltonian systems of the Sturm comparison theorem, similar to the one obtained in [34] for disconjugate systems, as well as a comparison result for the corresponding Lagrange planes obtained from the uniform principal solutions. In the proof, $\mathcal{B}_\Omega(\omega_0, \delta_0)$ represents the open set of the points of Ω at a distance from ω_0 less than $\delta_0 > 0$.

Proposition 5.51 *Consider two families of linear Hamiltonian systems*

$$\mathbf{z}' = H^1(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{5.34}$$

$$\mathbf{z}' = H^2(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{5.35}$$

with $JH^1 \le JH^2$. Suppose that D1 and D2 hold for (5.34) and D1, D2, and D3 hold for (5.35). Then,

(i) *D3 also holds for (5.34). In addition, if $N_1^\pm(\omega)$ and $N_2^\pm(\omega)$ are the principal functions of (5.34) and (5.35) respectively, then*

$$N_1^+ \le N_2^+ \le N_2^- \le N_1^-\,.$$

(ii) *Suppose further that $H_1^1 \equiv H_1^2$ and that any minimal subset of Ω contains a point ω such that $H_2^1(\omega) > H_2^2(\omega)$, where $H^j = \begin{bmatrix} H_1^j & H_3^j \\ H_2^j & -(H_1^j)^T \end{bmatrix}$ for $j = 1, 2$. Then*

$$N_1^+ < N_2^+ \le N_2^- < N_1^-\,.$$

Proof

(i) Write the Riccati equations associated to $\mathbf{z}' = H^j(\omega{\cdot}t)\,\mathbf{z}$ as

$$\begin{aligned} M' &= -M\,H_3^j(\omega{\cdot}t)\,M - M\,H_1^j(\omega{\cdot}t) - (H_1^j)^T(\omega{\cdot}t)\,M + H_2^j(\omega{\cdot}t) \\ &= h_j(\omega{\cdot}t, M) \end{aligned}$$

for $j = 1, 2$, and consider also the linear equations

$$M' = -M\,H_1^j(\omega{\cdot}t) - (H_1^j)^T(\omega{\cdot}t)\,M + H_2^j(\omega{\cdot}t) = g_j(\omega{\cdot}t, M)$$

for $j = 1, 2$. Let $M_1^{\pm}(t,\omega)$ solve $M' = h_1(\omega{\cdot}t, M)$ in a maximal interval $\mathcal{I}_{\pm}$ containing 0 and let $M_{l,1}^{\pm}(t,\omega)$ solve $M' = g_1(\omega{\cdot}t, M)$ in $\mathbb{R}$, with $M_1^{\pm}(0,\omega) = M_{l,1}^{\pm}(0,\omega) = N_2^{\pm}(\omega)$. Then, for all $t \in \mathcal{I}_{\pm}$,

$$h_2(\omega{\cdot}t, M_1^{\pm}(t,\omega)) \le h_1(\omega{\cdot}t, M_1^{\pm}(t,\omega)) = (M_1^{\pm})'(t,\omega) \le g_1(t, M_1^{\pm}(t,\omega))\,,$$

so that, according to points (i) and (iii) of Theorem 1.46, $N_2^{\pm}(\omega{\cdot}t) \le M_1^{\pm}(t,\omega) \le M_{1,l}^{\pm}(t,\omega)$ for all points $t \in \mathcal{I}_{\pm} \cap [0,\infty)$ and $N_2^{\pm}(\omega{\cdot}t) \ge M_1^{\pm}(t,\omega) \ge M_{1,l}^{\pm}(t,\omega)$ for all $t \in \mathcal{I}_{\pm} \cap (-\infty, 0]$. These two inequalities and Remark 1.44.2 imply that $\|M_1^{\pm}(t,\omega)\|$ is bounded in any interval $[a,b] \subset \mathcal{I}$, and hence Remark 1.43 ensures that $M_1^{\pm}(t,\omega)$ is globally defined. Therefore: first, the $2n \times n$ matrix solutions of (5.34) with initial data $\left[\begin{smallmatrix} I_n \\ N_2^{\pm}(\omega)\end{smallmatrix}\right]$ take values in $\mathcal{D}$ for every $t \in \mathbb{R}$, so that D3 holds, and Theorem 5.17 ensures that the principal functions $N_1^{\pm}(\omega)$ exist; and second, Theorem 5.48(iii) ensures that $N_1^{+}(\omega) \le N_2^{\pm}(\omega) \le N_1^{-}(\omega)$. This property and the inequality $N_2^{+}(\omega) \le N_2^{-}(\omega)$ ensured by Theorem 5.43 are used to complete the proof of (i).

(ii) Note that $(N_1^{+})'(\omega) = h_1(\omega, N_1^{+}(\omega)) \ge h_2(\omega, N_1^{+}(\omega))$ for all $\omega \in \Omega$. According to Proposition 1.52(i), this ensures that N_1^{+} is a superequilibrium for the flow $\tau_{s,2}$ induced by $M' = h_2(\omega{\cdot}t, M)$ on $\Omega \times \mathbb{S}_n(\mathbb{R})$ (for $t \ge 0$ and $t \le 0$: see Proposition 1.51). Assume now the additional hypothesis in (ii). Given any $\omega_0 \in \Omega$, there exist a minimal set contained in the omega-limit set of $\{\omega_0{\cdot}t \mid t \ge 0\}$, a point $\widetilde{\omega}$ in this minimal set and a $\widetilde{\delta}_{\omega_0} > 0$ such that

$$\begin{aligned}(N_1^{+})'(\omega) - h_2(\omega, N_1^{+}(\omega)) &= -N_1^{+}(\omega)H_3^1(\omega)N_1^{+}(\omega) + N_1^{+}(\omega)H_3^2(\omega)N_1^{+}(\omega)\\ &\quad + H_2^1(\omega) - H_2^2(\omega) \ge H_2^1(\omega) - H_2^2(\omega) > 0\end{aligned}$$

for all $\omega \in \mathcal{B}_{\Omega}(\widetilde{\omega}, \widetilde{\delta}_{\omega_0})$. In addition, there exists a time $s_{\omega_0} > 0$ such that $\omega_0{\cdot}s_{\omega_0} \in \mathcal{B}_{\Omega}(\widetilde{\omega}, \widetilde{\delta}_{\omega_0})$. By continuity of the base flow, there exists δ_{ω_0} such that $\sigma_{s_{\omega_0}}(\mathcal{B}_{\Omega}(\omega_0, \delta_{\omega_0})) \subseteq \mathcal{B}_{\Omega}(\widetilde{\omega}, \widetilde{\delta}_{\omega_0})$. Hence $(N_1^{+})'(\omega{\cdot}s_{\omega_0}) > h_2(\omega{\cdot}s_{\omega_0}, N_1^{+}(\omega{\cdot}s_{\omega_0}))$ if $\omega \in \mathcal{B}_{\Omega}(\omega_0, \delta_{\omega_0})$, and therefore Proposition 1.52(iii) ensures that the superequilibrium N_1^{+} is strong. This implies the existence of $s^* > 0$ such that

$$\begin{aligned}N_1^{+}(\omega) = N_1^{+}((\omega{\cdot}s_*){\cdot}(-s_*)) &< M_2(-s_*, \omega{\cdot}s_*, N_1^{+}(\omega{\cdot}s^*))\\ &\le M_2(-s_*, \omega{\cdot}s_*, N_2^{+}(\omega{\cdot}s^*)) = N_2^{+}(\omega)\,.\end{aligned}$$

The first inequality in (i) and Theorem 1.45 have been used here. Consequently, $N_1^{+} < N_2^{+}$. The proof of $N_2^{-} < N_1^{-}$ is analogous.

This comparison property will be used in the proofs of some of the main results of the rest of the book. In particular, in combination with Theorem 5.58 below, it will be extremely useful in the determination of the existence of exponential dichotomy and/or uniform weak disconjugacy for several families of perturbed Hamiltonian systems.

5.5 Principal Solutions and Lyapunov Index

Throughout this section, m_0 represents a fixed σ-ergodic measure on Ω. According to Definition 2.41, the Lyapunov index of the family (5.4) with respect to m_0 is $\beta = \beta_1 + \cdots + \beta_n$, where $\beta_1 \geq \cdots \geq \beta_n \geq 0$ are the nonnegative Lyapunov exponents with respect to m_0 repeated according to their multiplicities. The remaining Lyapunov exponents are $-\beta_1 \leq \cdots \leq -\beta_n \leq 0$.

The following two propositions explain the behavior of certain solutions of the family of systems (5.4) which are important for the ergodic characterization given in Theorem 5.56. The set $\mathcal{D}$ is defined by (5.5). Recall once more that conditions D2 and D3 are equivalent to the uniform weak disconjugacy of the family (5.4) if D1 holds, as Theorem 5.17 guarantees.

Remark 5.52 Suppose that $\det G_1 \neq 0$. Then,

$$G_1^T G_1 + G_2^T G_2 = G_1^T (I_n + (G_1^T)^{-1} G_2^T G_2 G_1^{-1})\, G_1\,,$$

and, consequently,

$$\det(G_1^T G_1 + G_2^T G_2) = \det(G_1^T G_1)\det(I_n + (G_1^T)^{-1} G_2^T G_2 G_1^{-1}) \geq \det(G_1^T G_1)\,.$$

Proposition 5.53 *Suppose that* D1, D2, *and* D3 *hold. Then, there exists a* σ-*invariant subset* $\Omega_0 \subset \Omega$ *with* $m_0(\Omega_0) = 1$ *such that, for all* $\omega \in \Omega_0$, *there exist* $2n \times n$ *matrix solutions* $\begin{bmatrix} G_1^{\pm}(t,\omega) \\ G_2^{\pm}(t,\omega) \end{bmatrix}$ *of* (5.4) *taking values in* $\mathcal{D}$ *with*

$$\lim_{t\to\infty} \frac{1}{2t} \ln\det\big((G_1^{\pm})^T(t,\omega)\, G_1^{\pm}(t,\omega)\big) = \pm\beta\,. \tag{5.36}$$

Proof As proved in Theorem 2.46,

$$\beta = \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} S(\omega, l)\, d\mu_1$$

for a τ-ergodic measure μ_1 on $\mathcal{K}_\mathbb{R}$ projecting onto m_0, where $\operatorname{Tr} S(\omega, l)$ is defined by (1.20). Theorems 1.3 and 1.6 ensure then that

$$\beta = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} S(\tau(s,\omega,l))\, ds \tag{5.37}$$

for μ_1-a.e. $(\omega, l) \in \mathcal{K}_\mathbb{R}$. In addition, Theorem 5.49(i) ensures that μ_1 is concentrated on $\mathcal{J}$, which in turn implies that, for μ_1-a.e. $(\omega, l) \in \mathcal{K}_\mathbb{R}$ and any representation $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, the matrix-valued function $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ satisfies $\det L_1(t,\omega) \neq 0$ and $N^+(\omega{\cdot}t) \le L_2(t,\omega)\, L_1^{-1}(t,\omega) \le N^-(\omega{\cdot}t)$ for every $t \in \mathbb{R}$. It is easy to deduce the existence of a σ-invariant subset $\Omega_1 \subseteq \Omega$ with $m_0(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$ there exists (ω, l) such that this last condition and (5.37) hold. Fix one of these points (ω, l) with $\omega \in \Omega_1$. Equation (1.16) is satisfied by $R(t,\omega)$, with $R^T(t,\omega)\, R(t,\omega) = L_1^T(t,\omega)\, L_1(t,\omega) + L_2^T(t,\omega)\, L_2(t,\omega)$, and this together with the Liouville formula guarantees that

$$\beta = \lim_{t\to\infty} \frac{1}{2t} \ln\det\left(L_1^T(t,\omega)\, L_1(t,\omega) + L_2^T(t,\omega)\, L_2(t,\omega)\right).$$

Define $M(t,\omega) = L_2(t,\omega)\, L_1^{-1}(t,\omega)$ which is symmetric, and note that it is also bounded: see Remark 1.44.2 and recall that Theorem 5.43 guarantees that N^+ and N^- are bounded. Therefore,

$$\begin{aligned}\beta &= \lim_{t\to\infty} \frac{1}{2t} \ln\det\left(L_1^T(t,\omega)\left(I_n + M^2(t,\omega)\right) L_1(t,\omega)\right) \\ &= \lim_{t\to\infty} \frac{1}{2t} \ln\det\left(L_1^T(t,\omega)\, L_1(t,\omega)\right),\end{aligned}$$

so that $\left[\begin{smallmatrix} G_1^+(t,\omega) \\ G_2^+(t,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ satisfies (5.36). The proof of the assertion concerning $-\beta$ and a σ-invariant set Ω_2 is analogous: the starting point is the existence of a τ-ergodic measure μ_2 with $-\beta = \int_{\mathcal{K}_\mathbb{R}} \operatorname{Tr} S(\omega, l)\, d\mu_2$, which follows from the proof of Theorem 2.46, with a simple adaptation of the argument below (2.59). Finally, the set $\Omega_0 = \Omega_1 \cap \Omega_2$ satisfies the thesis.

Remark 5.54 For future purposes, the set Ω_0 of Proposition 5.53 is supposed from now on to be contained in the σ-invariant set of full measure for m_0 to which the Oseledets theorem 2.37 applies.

Proposition 5.55 *Suppose that* D1, D2, *and* D3 *hold. Let* Ω_0 *be the subset found in Proposition* 5.53 *and Remark* 5.54. *For* $\omega \in \Omega_0$, *let* $\left[\begin{smallmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega) \end{smallmatrix}\right]$ *be the principal solution of* (5.4) *at* $\pm\infty$, *and let* $\left[\begin{smallmatrix} F_1^\pm(t,\omega) \\ F_2^\pm(t,\omega) \end{smallmatrix}\right]$ *be any* $2n \times n$ *matrix solution of* (5.4) *with* $\det \begin{bmatrix} F_1^{\pm}(t,\omega) & L_1^{\pm}(t,\omega) \\ F_2^{\pm}(t,\omega) & L_2^{\pm}(t,\omega) \end{bmatrix} \neq 0$. *Then,*

$$\limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((F_1^{\pm})^T(t,\omega)\, F_1^{\pm}(t,\omega)\big) = \pm\beta\,.$$

Proof Fix $\omega \in \Omega_0$. According to Remark 5.23, there are matrices $P(\omega)$ and $Q(\omega)$ such that

$$\begin{bmatrix} F_1^+(t,\omega) \\ F_2^+(t,\omega) \end{bmatrix} = \begin{bmatrix} L_1^+(t,\omega) & L_1^+(t,\omega)\, I_{L^+}(0,t,\omega) \\ L_2^+(t,\omega) & L_2^+(t,\omega)\, I_{L^+}(0,t,\omega) + ((L_1^+)^T)^{-1}(t,\omega) \end{bmatrix} \begin{bmatrix} P(\omega) \\ Q(\omega) \end{bmatrix}.$$

The hypothesis on $\begin{bmatrix} F_1^+(t,\omega) \\ F_2^+(t,\omega) \end{bmatrix}$ ensures that the matrix $Q(\omega)$ is invertible: if $Q(\omega)\,\mathbf{x} = \mathbf{0}$, then $\begin{bmatrix} F_1^+(t,\omega) & L_1^+(t,\omega) \\ F_2^+(t,\omega) & L_2^+(t,\omega) \end{bmatrix} \begin{bmatrix} -\mathbf{x} \\ P(\omega)\,\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$. Since $I_{L^+}^{-1}(0,t,\omega)$ exists and tends to 0_n as $t \to \infty$ (see Definition 5.15), and since

$$P(\omega) + I_{L^+}(0,t,\omega)\, Q(\omega) = I_{L^+}(0,t,\omega)\, \big(I_{L^+}^{-1}(0,t,\omega)P(\omega) + Q(\omega)\big)\,,$$

there exists $s_{\omega,F}$ with $\det F_1^+(t,\omega) \neq 0$ for $t > s_{\omega,F}$. In addition, if $K(t,\omega) = (F_1^+)^{-1}(t,\omega)\, L_1^+(t,\omega)$ for $t > s_{\omega,F}$, then

$$K(t,\omega) = \big(I_{L^+}^{-1}(0,t,\omega)\, P(\omega) + Q(\omega)\big)^{-1}\, I_{L^+}^{-1}(0,t,\omega)\,,$$

and hence

$$\lim_{t\to\infty} K(t,\omega) = 0_n\,. \tag{5.38}$$

Let $\begin{bmatrix} F_1(t,\omega) \\ F_2(t,\omega) \end{bmatrix}$ be any $2n \times n$ matrix solution of (5.4). Since $\begin{bmatrix} F_1^+(t,\omega) & L_1^+(t,\omega) \\ F_2^+(t,\omega) & L_2^+(t,\omega) \end{bmatrix}$ is a fundamental matrix solution of (5.4), there are constant $n \times n$ matrices $C_1(\omega)$ and $C_2(\omega)$ such that $F_1(t,\omega) = F_1^+(t,\omega)\, C_1(\omega) + L_1^+(t,\omega)\, C_2(\omega)$. Hence, for $t > s_{\omega,F}$,

$$\begin{aligned} F_1^T(t,\omega)\, F_1(t,\omega) &= (C_1(\omega) + K(t,\omega)\, C_2(\omega))^T \\ &\quad \cdot (F_1^+)^T(t,\omega)\, F_1^+(t,\omega)\, (C_1(\omega) + K(t,\omega)\, C_2(\omega))\,, \end{aligned}$$

which together with (5.38) implies that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{2t} \ln\det\big(F_1^T(t,\omega)\, F_1(t,\omega)\big) \\ &\qquad \le \limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((F_1^+)^T(t,\omega)\, F_1^+(t,\omega)\big)\,. \end{aligned}$$

Remark 5.52 and the argument used to get relation (2.58) combined with the Oseledets theorem 2.37 prove that the right limit is $\le \beta$. In turn, Proposition 5.53 and the choice of ω provide a matrix solution $\left[\begin{smallmatrix} F_1(t,\omega) \\ F_2(t,\omega) \end{smallmatrix}\right]$ for which the left limit is exactly β. The proof is complete for β, and is analogous for $-\beta$. Note that $\left[\begin{smallmatrix} F_1^{\pm}(t,\omega) \\ F_2^{\pm}(t,\omega) \end{smallmatrix}\right]$ are not assumed to take values in $\mathcal{L}_{\mathbb{R}}$.

Recall that $\tilde{l}^{\pm}(\omega)$ represent the Lagrange planes associated to the uniform principal solutions at $\pm\infty$, that is, $\tilde{l}^{\pm}(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N^{\pm}(\omega) \end{smallmatrix}\right]$; and that the sets $\widetilde{L}^{\pm} = \{(\omega, \tilde{l}^{\pm}(\omega)) \mid \omega \in \Omega\} \subset \mathcal{K}_{\mathbb{R}}$ are τ-invariant: $\tau(t,\omega,\tilde{l}^{\pm}(\omega)) = (\omega{\cdot}t, \tilde{l}^{\pm}(\omega{\cdot}t))$. The following result provides an ergodic representation of the Lyapunov index with respect to a σ-ergodic measure m_0 on Ω, and describes $\tilde{l}^{\pm}(\omega)$ in terms of the Oseledets subbundles of the system.

Let $2k$ be the number of null Lyapunov exponents. The notation of Lemma 2.43 is now recalled, in order to represent

$$\begin{aligned} V_+(\omega) &= \langle \mathbf{z}^+_{\omega,1}, \dots, \mathbf{z}^+_{\omega,n-k} \rangle \,, \\ V_0(\omega) &= \langle \mathbf{z}^+_{\omega,n-k+1}, \dots, \mathbf{z}^+_{\omega,n}, \mathbf{z}^-_{\omega,n-k+1}, \dots \mathbf{z}^-_{\omega,n} \rangle \,, \\ V_-(\omega) &= \langle \mathbf{z}^-_{\omega,1}, \dots, \mathbf{z}^-_{\omega,n-k} \rangle \end{aligned}$$

for ω in the σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ appearing in Proposition 5.53 and Remark 5.54, where $\mathbf{z}^{\pm}_{\omega,1}, \dots, \mathbf{z}^{\pm}_{\omega,n}$ satisfy

$$\lim_{|t|\to\infty} \frac{1}{t} \ln \|U(t,\omega)\,\mathbf{z}^{\pm}_{\omega,j}\| = \mp\beta_j \,, \tag{5.39}$$

and the subspaces generated by $\{\mathbf{z}^-_{\omega,1}, \dots, \mathbf{z}^-_{\omega,n}\}$ and $\{\mathbf{z}^+_{\omega,1}, \dots, \mathbf{z}^+_{\omega,n}\}$ are real Lagrange planes. That is, $V_-(\omega)$ and $V_+(\omega)$ are the sum of the Oseledets subspaces corresponding to strictly positive and strictly negative Lyapunov exponents, respectively; and $V_0(\omega)$ is the Oseledets subspace associated to the null Lyapunov exponent. According to Theorem 2.37 and Proposition 2.40, the sets

$$\begin{aligned} V_+ &= \{(\omega, \mathbf{v}) \mid \omega \in \Omega_0,\ \mathbf{v} \in V_+(\omega)\} \\ V_0 &= \{(\omega, \mathbf{v}) \mid \omega \in \Omega_0,\ \mathbf{v} \in V_0(\omega)\} \\ V_- &= \{(\omega, \mathbf{v}) \mid \omega \in \Omega_0,\ \mathbf{v} \in V_-(\omega)\} \end{aligned}$$

are composed of τ_H-orbits, where τ_H is the flow induced by (5.4) on $\Omega_0 \times \mathbb{R}^{2n}$; and there exists $d \in \{1, \dots, n\}$ such that $\dim V_+(\omega) = \dim V_-(\omega) = d$ and $\dim V_0(\omega) = 2n - 2d$ for all $\omega \in \Omega_0$. Recall that $\mathcal{G}_d(\mathbb{R}^{2n})$ and $\mathcal{G}_{n-d}(\mathbb{R}^{2n})$ represent respectively the Grassmannian manifolds of the d-dimensional and $(n-d)$-dimensional linear subspaces of $\mathbb{R}^{2n}$, and that τ_d and τ_{n-d} are the corresponding flows induced by (5.4) on $\Omega \times \mathcal{G}_d(\mathbb{R}^{2n})$ and $\Omega \times \mathcal{G}_{n-d}(\mathbb{R}^{2n})$: see Sects. 1.2.2 and 1.3.1.

In the case $d = n$, the set $\mathcal{G}_0(\mathbb{R}^{2n})$ is made up of a unique element, which is the trivial linear subspace.

Theorem 5.56 *Suppose that D1, D2, and D3 hold, and let $\Omega_0 \subseteq \Omega$ be the σ-invariant subset with $m_0(\Omega_0) = 1$ found in Proposition 5.53 and Remark 5.54.*

(i) *For any $\omega \in \Omega_0$ there exist subspaces $W_0^{\pm}(\omega) \subset V_0(\omega)$ such that the Lagrange plane $\tilde{l}^+(\omega)$ coincides with $V_+(\omega) \oplus W_0^+(\omega)$ and the Lagrange plane $\tilde{l}^-(\omega)$ coincides with $V_-(\omega) \oplus W_0^-(\omega)$.*
(ii) *In particular, $\beta_1 \geq \ldots \geq \beta_n > 0$ if and only if $\tilde{l}^+(\omega)$ and $\tilde{l}^-(\omega)$ are supplementary subspaces m_0-a.e.*
(iii) *The maps*

$$\Omega_0 \to \mathcal{G}_d(\mathbb{R}^{2n}), \ \omega \mapsto V_\pm(\omega) \quad \textit{and} \quad \Omega_0 \to \mathcal{G}_{n-d}(\mathbb{R}^{2n}), \ \omega \mapsto W_0^{\pm}(\omega)$$

are Σ_{m_0}-measurable, and they satisfy $\tau_d(t, \omega, V_\pm(\omega)) = (\omega{\cdot}t, V_\pm(\omega{\cdot}t))$ and $\tau_{n-d}(t, \omega, W_0^{\pm}(\omega)) = (\omega{\cdot}t, W_0^{\pm}(\omega{\cdot}t))$ for all $\omega \in \Omega_0$ and $t \in \mathbb{R}$.
(iv) $\beta = \mp \displaystyle\int_\Omega \operatorname{tr}\big(H_1(\omega) + H_3(\omega)\, N^{\pm}(\omega)\big)\, dm_0.$
(v) *There exists a σ-invariant subset $\Omega_1 \subseteq \Omega_0$ with $m_0(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$, $W_0^+(\omega) = W_0^-(\omega)$ and $\dim\big(\tilde{l}^+(\omega) \cap \tilde{l}^-(\omega)\big) = k = \dim V_0(\omega)/2$. In particular, $\beta = 0$ if and only if $N^+(\omega) = N^-(\omega)$ (i.e. $\tilde{l}^+(\omega) = \tilde{l}^-(\omega)$) m_0-a.e.*

Proof

(i) Fix $\omega \in \Omega_0$. Suppose that $\mathbf{z}_{\omega,j}^+ \notin \tilde{l}^+(\omega)$ for an index $j \in \{1, \ldots, n-k\}$. Choose a subspace generated by n vectors $\mathbf{v}_{\omega,1}, \ldots, \mathbf{v}_{\omega,n} \in \{\mathbf{z}_{\omega,1}^-, \ldots, \mathbf{z}_{\omega,n}^-, \mathbf{z}_{\omega,1}^+, \ldots, \mathbf{z}_{\omega,n}^+\}$, with $\mathbf{v}_{\omega,1} = \mathbf{z}_{\omega,j}^+$, in such a way that it is a supplementary space of $\tilde{l}^+(\omega)$ (and not necessarily a Lagrange plane). Let $F^+(t,\omega) = \begin{bmatrix} F_1^+(t,\omega) \\ F_2^+(t,\omega) \end{bmatrix}$ be the $2n \times n$ matrix solution of (5.4) with initial datum $F(0,\omega) = [\mathbf{v}_{\omega,1}\, \mathbf{v}_{\omega,2} \cdots \mathbf{v}_{\omega,n}]$. Note that, by (5.39),

$$\sum_{m=1}^{n} \lim_{t\to\infty} \frac{1}{2t} \ln \|U(t,\omega)\, \mathbf{v}_{\omega,m}\| < \beta\,,$$

since $\lim_{t\to\infty}(1/2t) \ln \|U(t,\omega)\, \mathbf{v}_{\omega,1}\| = -\beta_j < 0$. Remark 5.52 and the arguments of the proof of Theorem 2.46 imply that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((F_1^+)^T(t,\omega)\, F_1^+(t,\omega)\big) \\ &\qquad \leq \limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((F_1^+)^T(t,\omega)\, F_1^+(t,\omega) + (F_2^+)^T(t,\omega)\, F_2^+(t,\omega)\big) \\ &\qquad \leq \sum_{m=1}^{n} \lim_{t\to\infty} \frac{1}{2t} \ln \|U(t,\omega)\, \mathbf{v}_{\omega,m}\| < \beta\,, \end{aligned}$$

which contradicts Proposition 5.55. Thus, $V_+(\omega) \subseteq \tilde{l}^+(\omega)$. Suppose now that $\mathbf{v} = \mathbf{v}_- + \mathbf{v}_+ + \mathbf{v}_0 \in \tilde{l}^+(\omega)$ with $\mathbf{v}_\pm \in V_\pm(\omega)$, $\mathbf{v}_0 \in V_0(\omega)$ and $\mathbf{v}_- \neq \mathbf{0}$. Since $V_+(\omega) \subseteq \tilde{l}^+(\omega)$, one has $\mathbf{v}_- + \mathbf{v}_0 = \mathbf{v} - \mathbf{v}_+ \in \tilde{l}^+(\omega)$, which in turn provides $(\mathbf{z}^+_{\omega,j})^T J\,(\mathbf{v}_- + \mathbf{v}_0) = 0$ for $j = 1,\dots,n-k$. The symplectic character of $U(t,\omega)$ and relation (5.39) imply that $(\mathbf{z}^+_{\omega,j})^T J\,\mathbf{v}_0 = (\mathbf{z}^+_{\omega,j})^T\, U^T(t,\omega) J U(t,\omega)\,\mathbf{v}_0 = 0$ for $j = 1,\dots,n-k$. The same argument shows that $(\mathbf{z}^+_{\omega,j})^T J\,\mathbf{v}_- = 0$ for $j = n-k+1,\dots,n$. Therefore, $(\mathbf{z}^+_{\omega,j})^T J\,\mathbf{v}_- = 0$ for $j = 1,\dots,n$, which implies the existence of $n+1$ isotropic linearly independent vectors, which, however, is impossible. Consequently, $\tilde{l}^+(\omega) = V_+(\omega) \oplus W_0^+(\omega)$ with $W_0^+(\omega) \subset V_0(\omega)$, as asserted. The proof of the assertion for $\tilde{l}^-(\omega)$ is analogous.

(ii) This property is a trivial consequence of (i).

(iii) According to Theorems 5.26 and 2.37, $U(t,\omega)\cdot\tilde{l}^\pm(\omega) = \tilde{l}^\pm(\omega{\cdot}t)$ and $U(t,\omega){\cdot}V_\pm(\omega) = V_\pm(\omega{\cdot}t)$. It is clear that $W_0^\pm(\omega) = \tilde{l}^\pm(\omega) \cap V_0(\omega)$ and that $\dim W_0^\pm(\omega) = \dim \tilde{l}^\pm(\omega) - \dim V_\pm(\omega) = n - d$ for all $\omega \in \Omega$. In particular, the maps $\Omega_0 \to \mathcal{G}_{n-d}(\mathbb{R}^{2n})\,,\ \omega \mapsto W_0^\pm(\omega)$ are well defined and satisfy

$$
\begin{aligned}
U(t,\omega){\cdot}W_0^+(\omega) &= U(t,\omega){\cdot}(\tilde{l}^+(\omega) \cap V_0(\omega)) \\
&= (U(t,\omega)\cdot\tilde{l}^+(\omega)) \cap (U(t,\omega){\cdot}V_0(\omega)) = \tilde{l}^+(\omega{\cdot}t) \cap V_0(\omega{\cdot}t) = W_0^+(\omega{\cdot}t)\,.
\end{aligned}
$$

There remains to check the Σ_m-measurability of the four maps. Let $g_1\colon \Omega_0 \to \mathcal{G}_{d_1}(\mathbb{R}^{2n})$ and $g_2\colon \Omega_0 \to \mathcal{G}_{d_2}(\mathbb{R}^{2n})$ be Σ_m-measurable maps such that $g_1(\omega) \cap g_2(\omega) = \{\mathbf{0}\}$ for all $\omega \in \Omega_0$. Then the "sum" map $\Omega_0 \to \mathcal{G}_{d_1+d_2}(\mathbb{R}^{2n})\,,\ \omega \mapsto g_1(\omega) \oplus g_2(\omega)$ is Σ_m measurable: Proposition 1.26(ii) proves that it is continuous on any compact subset $\mathcal{M} \subset \Omega_0$ if g_1 and g_2 are continuous on $\mathcal{M}$, and the assertion follows from this fact and a standard application of Lusin's theorem. Similarly, if g_1 and g_2 satisfy the condition $\dim(g_1(\omega) \cap g_2(\omega)) = d_3$ for all $\omega \in \Omega_0$, then the map $\Omega_0 \to \mathcal{G}_{d_3}(\mathbb{R}^{2n})\,,\ \omega \mapsto g_1(\omega) \cap g_2(\omega)$ is Σ_m-measurable. Keeping this in mind, the Σ_m-measurability of the maps follow easily from the Σ_m-measurability established in Theorem 2.37 and Theorem 5.43.

(iv) The arguments of the proof of Theorem 2.46 may be used to prove that

$$
\mp\beta = \limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((L_1^\pm)^T(t,\omega)\,L_1^\pm(t,\omega) + (L_2^\pm)^T(t,\omega)\,L_2^\pm(t,\omega)\big)
$$

for m_0-a.e. $\omega \in \Omega$. Remark 5.52 and the boundedness of $N^\pm(\omega{\cdot}t)$ proved in Theorem 5.43 imply then that

$$
\mp\,\beta = \limsup_{t\to\infty} \frac{1}{2t} \ln\det\big((L_1^\pm)^T(t,\omega)\,L_1^\pm(t,\omega)\big)\,. \tag{5.40}
$$

In addition,

$$
\begin{aligned}
(L_1^{\pm})'(t,\omega) &= H_1(\omega{\cdot}t)\, L_1^{\pm}(t,\omega) + H_3(\omega{\cdot}t)\, L_2^{\pm}(t,\omega) \\
&= \big(H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\, N^{\pm}(\omega{\cdot}t)\big)\, L_1^{\pm}(t,\omega)\,,
\end{aligned}
$$

which combined with (5.40) implies that

$$
\mp\beta = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{tr}\big(H_1(\omega{\cdot}s) + H_3(\omega{\cdot}s)\, N^{\pm}(\omega{\cdot}s)\big)\, ds\,.
$$

These equalities and the Birkhoff Theorems 1.3 and 1.6 prove the statements of (iv).

(v) With the notation of (i), define $l(\omega) = V_+(\omega) \oplus W_0^-(\omega)$ for $\omega \in \Omega_0$. The definitions of $V_+(\omega)$ and $V_0(\omega)$ together with the symplectic character of U and the behavior at ∞ described by (5.39) guarantee that $l(\omega)$ is a real Lagrange plane. It is easy to deduce from (iii) that the map $l\colon \Omega_0 \to \mathcal{L}_{\mathbb{R}}$ is Σ_m-measurable and that $\tau(t,\omega,l(\omega)) = (\omega{\cdot}t, l(\omega{\cdot}t))$. According to Theorem 5.49(i), there exists a σ-invariant set $\Omega_2 \subseteq \Omega_0$ with $m_0(\Omega_2) = 1$ such that $(\omega, l(\omega)) \in \mathcal{J}$ for every $\omega \in \Omega_2$. Representing $l(\omega)$ by $\left[\begin{smallmatrix} I_n \\ M(\omega)\end{smallmatrix}\right]$ and repeating the arguments of (iv) shows that $-\beta = \int_\Omega \operatorname{tr}\,(H_1(\omega) + H_3(\omega)\, M(\omega))\, dm_0$, so that

$$
\int_\Omega \operatorname{tr}\big(H_3(\omega)\,\big(M(\omega) - N^+(\omega)\big)\big)\, dm_0 = 0\,.
$$

Consequently, for all $t \in \mathbb{R}$ there exists $\Omega_t \subset \Omega_2$ with $m_0(\Omega_t) = 1$ such that $\operatorname{tr}\big(H_3(\omega{\cdot}t)(M(\omega{\cdot}t) - N^+(\omega{\cdot}t))\big) = 0$ for every $\omega \in \Omega_t$, which is equivalent to saying that $H_3(\omega{\cdot}t)\, N^+(\omega{\cdot}t) = H_3(\omega{\cdot}t)\, M(\omega{\cdot}t)$, since $N^+ \le M$ on Ω_2 and $H_3 \ge 0$. Define $\Omega_1 = \cap_{t\in\mathbb{Q}} \Omega_t \subset \Omega_2$, with $m_0(\Omega_1) = 1$. Then, for each $\omega \in \Omega_1$, $H_3(\omega{\cdot}t)\, N^+(\omega{\cdot}t) = H_3(\omega{\cdot}t)\, M(\omega{\cdot}t)$ for every $t \in \mathbb{R}$, since they agree on $\mathbb{Q}$ and are continuous in t. Therefore, if $L(t,\omega) = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega)\end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix} I_n \\ M(\omega)\end{smallmatrix}\right]$, both $L_1(t,\omega)$ and $L_1^+(t,\omega)$ solve

$$
L_1' = \big(H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\, N^+(\omega{\cdot}t)\big)\, L_1
$$

and take value I_n at $t = 0$, which yields $L_1(t,\omega) = L_1^+(t,\omega)$ for each $t \in \mathbb{R}$. Finally, D2 ensures that the $2n \times n$ matrix solution of (5.4) $L(t,\omega) - L^+(t,\omega)$ is the trivial one: $L(t,\omega) = L^+(t,\omega)$ for every $t \in \mathbb{R}$. This implies that $N^+(\omega) = M(\omega)$, $W_0^+(\omega) = W_0^-(\omega)$ and $\dim(\widetilde{L}^+(\omega) \cap \widetilde{L}^-(\omega)) = \dim W_0^+(\omega) = k = \dim V_0(\omega)/2$ for every $\omega \in \Omega_1$, as stated. The last assertion follows immediately from the equality $\dim V_0(\omega) = 2n$ for all $\omega \in \Omega_0$, which holds if $\beta = 0$.

Examples 5.57 Returning to Examples 5.47, note that for the autonomous system $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix}\right] \mathbf{z}$, where $N^+ = N^- = 1$, one must have $\beta = 0$ (which of course is trivial in this case): this illustrates the situation considered in Theorem 5.56(ii). The system $\mathbf{z}' = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \mathbf{z}$ satisfies $N^+ \neq N^-$, so that according to Theorem 5.56(v) it is the case that $\beta > 0$ (again, it is trivial to check that $\beta = 1$). The four-dimensional system (5.30), which is also autonomous, satisfies $0 < \dim(\tilde{l}^+ \cap l^-) = 1 < 2$, so that its Lyapunov exponents are 0 and $\pm\beta_2$ with $\beta_2 > 0$ (in fact $\beta_2 = 1$). And finally, in the nonautonomous Example 8.44, β must be positive, since $N^+(\omega) \neq N^-(\omega)$ for m_0-a.e. $\omega \in \Omega$.

5.6 Principal Solutions and Exponential Dichotomy

As in the previous two sections, the hypotheses imposed in this one are described in Theorem 5.17: conditions D1, D2, and D3 hold; or, in other words, systems (5.4) are uniformly weakly disconjugate (both on $[0, \infty)$ and $(-\infty, 0]$), and they possess uniform principal solutions at ∞ and $-\infty$. These principal solutions define the Lagrange planes $\tilde{l}^\pm(\omega) \in \mathcal{D}$ which are parameterized by the principal functions $N^\pm(\omega)$ for all $\omega \in \Omega$.

The results of this section are the analogues of those included in Johnson et al. [81, 82] for disconjugate systems, and concern the presence or absence of exponential dichotomy for (5.4) over Ω. This concept was defined in Sect. 1.4.3. The closed invariant subbundles provided by Definition 1.75 in the case of exponential dichotomy are represented by $L^\pm$. According to Proposition 1.76, $l^\pm(\omega) = L^\pm \cap \big(\{\omega\} \times \mathbb{R}^{2n}\big)$ are Lagrange planes for every $\omega \in \Omega$, and the maps $\Omega \to \mathcal{L}_\mathbb{R},\ \omega \mapsto l^\pm(\omega)$ are continuous. Recall also that the Lagrange planes $l^\pm(\omega)$ are composed of the initial data of the solutions of (5.4) which are bounded as $t \to \pm\infty$, respectively: see Remark 1.77.2.

In the dynamical situation described below by Theorems 5.58 and 5.59, the Lagrange planes $l^\pm(\omega)$ take values in $\mathcal{D}$, so that they can be represented in the form $\left[\begin{smallmatrix} I_n \\ M^\pm(\omega) \end{smallmatrix}\right]$ for continuous matrix-valued functions $M^\pm\colon \Omega \to \mathbb{S}_n(\mathbb{R})$. Recall that these matrix-valued functions are the *Weyl functions* of (5.4) (see Definition 1.80), and that they are continuous equilibria (see Definition 1.49) and define copies of the base $\{(\omega, M^\pm(\omega))\ \ \omega \in \Omega\} \subset \Omega \times \mathbb{S}_n(\mathbb{R})$ for the flow τ_s given by (1.23) (see Sect. 1.4.7).

Before stating the results of these theorems, it is convenient to recall some examples of families which are uniformly weakly disconjugate without having the exponential dichotomy property: this happens in the autonomous case $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix}\right] \mathbf{z}$ (with $N^+ = N^- \equiv 1$), as well as in the 4-dimensional system described in Examples 5.47 (with $N^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $N^- = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$); in the still simple but nonautonomous case given in as Example 5.38 (with $N^+(\omega) = N^-(\omega) \equiv 0$ for all $\omega \in \Omega$); and in the much more complicated case described in Example 8.44, for which the principal functions are two noncontinuous maps which agree in the residual set of their continuity points and are different in a set of full measure. Conversely, there are examples having exponential dichotomy without weak disconjugacy, as simple as $\mathbf{z}' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \mathbf{z}$; or of a more complicated nature: just take any two-dimensional Hamiltonian system with $H_3 \equiv 0$ (which precludes the existence of principal solutions) and with exponential dichotomy. Having this in mind makes it easier to understand the statements which follow.

Theorem 5.58 *Suppose that D1, D2, and D3 hold. Then, the family (5.4) has exponential dichotomy over Ω if and only if $\mathbb{R}^{2n} = \tilde{l}^+(\omega) \oplus \tilde{l}^-(\omega)$ for every $\omega \in \Omega$, in which case $\tilde{l}^\pm(\omega) = l^\pm(\omega)$ for every $\omega \in \Omega$. In other words, if and only if $N^- > N^+$, in which case the Weyl functions $M^\pm\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ globally exist, agree with $N^\pm$, and hence satisfy $M^- > M^+$.*

Proof Since $N^- \ge N^+$, it follows from (5.27) that $N^- > N^+$ if and only if $\tilde{l}^-(\omega) \cap \tilde{l}^+(\omega) = \{\mathbf{0}\}$ (i.e. if and only if $\mathbb{R}^{2n} = \tilde{l}^+(\omega) \oplus \tilde{l}^-(\omega)$) for all $\omega \in \Omega$. Assume that this is the case. Denote, as usual, $\left[\begin{smallmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega) \end{smallmatrix}\right] = U(t,\omega) \left[\begin{smallmatrix} I_n \\ N^\pm(\omega) \end{smallmatrix}\right]$. The argument used to obtain (5.38) proves that, for each $\omega \in \Omega$,

$$\lim_{t\to\infty} (L_1^-)^{-1}(t,\omega)\, L_1^+(t,\omega) = 0_n\,, \tag{5.41}$$

which together with (5.28) and (5.29) yields

$$\begin{aligned} 0_n &= \lim_{t\to\infty} \big((L_2^+)^T(t,\omega)\, L_1^-(t,\omega) - (L_1^+)^T(t,\omega)\, L_2^-(t,\omega)\big)\, (L_1^-)^{-1}(t,\omega)\, L_1^+(t,\omega) \\ &= \lim_{t\to\infty} (L_1^+)^T(t,\omega)\, \big(N^+(\omega{\cdot}t) - N^-(\omega{\cdot}t)\big)\, L_1^+(t,\omega)\,, \end{aligned}$$

In turn, this together with $N^- - N^+ \ge 0$ (which is ensured by Theorem 5.43) implies that

$$\lim_{t\to\infty} (L_1^+)^T(t,\omega)\, \big(N^-(\omega{\cdot}t) - N^+(\omega{\cdot}t)\big)^{1/2} = 0_n\,. \tag{5.42}$$

Using again (5.29),

$$(L_1^-)^{-1}(t,\omega) = \big(N^+(\omega) - N^-(\omega)\big)^{-1} (L_1^+)^T(t,\omega)\, \big(N^+(\omega{\cdot}t) - N^-(\omega{\cdot}t)\big)\,,$$

so that (5.42) and the boundedness of $N^\pm$ yield

$$\lim_{t\to\infty} (L_1^-)^{-1}(t,\omega) = 0_n\,. \tag{5.43}$$

Repeating the previous arguments one proves that

$$\lim_{t\to-\infty} (L_1^+)^{-1}(t,\omega)\, L_1^-(t,\omega) = 0_n \quad \text{and} \quad \lim_{t\to-\infty} (L_1^+)^{-1}(t,\omega) = 0_n\,. \tag{5.44}$$

In order to prove the existence of exponential dichotomy, as well as the equality $\tilde{l}^{\pm}(\omega) = l^{\pm}(\omega)$ for every $\omega \in \Omega$, take $\mathbf{z}^{\pm}$ such that the solutions $U(t,\omega)\,\mathbf{z}^{\pm}$ are bounded as $t \to \pm\infty$ respectively, and write them as $\mathbf{z}^{\pm} = \left[\begin{smallmatrix} I_n \\ N^{+}(\omega)\end{smallmatrix}\right]\mathbf{c}_1^{\pm}(\omega) + \left[\begin{smallmatrix} I_n \\ N^{-}(\omega)\end{smallmatrix}\right]\mathbf{c}_2^{\pm}(\omega)$. Then, if $\left[\begin{smallmatrix} \mathbf{z}_1^{\pm}(t,\omega) \\ \mathbf{z}_2^{\pm}(t,\omega)\end{smallmatrix}\right] = U(t,\omega)\,\mathbf{z}^{\pm}$,

$$
\begin{aligned}
(L_1^-)^{-1}(t,\omega)\,\mathbf{z}_1^+(t,\omega) &= (L_1^-)^{-1}(t,\omega)\,L_1^+(t,\omega)\,\mathbf{c}_1^+(\omega) + \mathbf{c}_2^+(\omega)\,,\\
(L_1^+)^{-1}(t,\omega)\,\mathbf{z}_1^-(t,\omega) &= \mathbf{c}_1^-(\omega) + (L_1^+)^{-1}(t,\omega)\,L_1^-(t,\omega)\,\mathbf{c}_2^-(\omega)\,,
\end{aligned}
$$

which together with (5.41), (5.43), (5.44), and the choices of $\mathbf{z}^{\pm}$ allow one to take the limit as $t \to \infty$ in the first equality and as $t \to -\infty$ in the second one in order to conclude that $\mathbf{c}_2^+(\omega) = \mathbf{c}_1^-(\omega) = \mathbf{0}$. So, $\mathbf{z}^+ \in \tilde{l}^+(\omega)$ and $\mathbf{z}^- \in \tilde{l}^-(\omega)$. This and the fact that $\tilde{l}^+(\omega) \cap \tilde{l}^-(\omega) = \{\mathbf{0}\}$ show the absence of nonzero bounded solutions, which according to Theorem 1.78 ensures that the family (5.4) has exponential dichotomy over Ω. Note that it has also been proved that $l^{\pm}(\omega) \subseteq \tilde{l}^{\pm}(\omega)$. Since these are n-dimensional vector spaces, they agree.

Suppose on the other hand that the family (5.4) has exponential dichotomy over Ω. To carry out the proof of the converse only requires to check that $\tilde{l}^{\pm}(\omega) = l^{\pm}(\omega)$ for all $\omega \in \Omega$. To this end, consider the auxiliary perturbed families of Hamiltonian systems

$$
\mathbf{z}' = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) \\ H_2(\omega{\cdot}t) - \lambda I_n & -H_1^T(\omega{\cdot}t)\end{bmatrix}\mathbf{z} = H^{\lambda}(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,. \tag{5.45}
$$

Obviously, all these families satisfy D1. It is easy to check that they also satisfy D2: if $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ satisfies (5.45) on an interval, it also solves (5.4) on the same interval. In addition, if $\lambda < 0$, then $JH^{\lambda} \le JH$ and $H_2^{\lambda} > H_2$. Proposition 5.51 ensures that (5.45) satisfies also condition D3 for $\lambda < 0$, as well as the chain of inequalities

$$
\begin{aligned}
N^+(\omega,\lambda_2) < N^+(\omega,\lambda_1) &< N^+(\omega)\\
&\le N^-(\omega) < N^-(\omega,\lambda_1) < N^-(\omega,\lambda_2)
\end{aligned} \tag{5.46}
$$

if $\lambda_2 < \lambda_1 < 0$, where $N^{\pm}(\omega,\lambda)$ are the principal functions of the perturbed system. Therefore, there exist the limits

$$
N_0^{\pm}(\omega) = \lim_{\lambda\to 0^-} N^{\pm}(\omega,\lambda)
$$

for every $\omega \in \Omega$, they are finite, and they satisfy

$$
N_0^+(\omega) \le N^+(\omega) \quad \text{and} \quad N_0^-(\omega) \ge N^-(\omega)\,.
$$

It is easy to check that $N_0^{\pm}$ are solutions along the flow of the Riccati equation (5.7). Since they are globally defined, Theorem 5.48(iii) ensures that $N^+(\omega) \le N_0^{\pm}(\omega) \le N^-(\omega)$. Therefore, $N^{\pm}(\omega) = N_0^{\pm}(\omega)$ for all $\omega \in \Omega$. In terms of Lagrange planes, and according to Proposition 1.29(ii), this proves that

$$\lim_{\lambda\to 0^-} \tilde{l}^{\pm}(\omega,\lambda) = \tilde{l}^{\pm}(\omega)\,. \tag{5.47}$$

On the other hand, Theorems 1.92 and 1.95 guarantee the existence of $\lambda_0 < 0$ such that (5.45) has exponential dichotomy over Ω for all $\lambda \in [\lambda_0, 0)$, with

$$\lim_{\lambda\to 0^-} l^{\pm}(\omega,\lambda) = l^{\pm}(\omega) \tag{5.48}$$

on $\mathcal{L}_{\mathbb{R}}$. Let λ vary only on this interval. By (5.46), $N^+(\omega,\lambda) < N^-(\omega,\lambda)$, so that the corresponding equality (5.27) ensures that $\tilde{l}^+(\omega,\lambda)$ and $\tilde{l}^-(\omega,\lambda)$ are supplementary. As seen in the first step of this proof, under these conditions $l^{\pm}(\omega,\lambda) = \tilde{l}^{\pm}(\omega,\lambda)$ for all $\omega \in \Omega$. This together with (5.47) and (5.48) ensures that $\tilde{l}^{\pm}(\omega) = l^{\pm}(\omega)$. In particular, $\tilde{l}^+(\omega) \oplus \tilde{l}^-(\omega) = \mathbb{R}^{2n}$, which completes the proof of the equivalence under consideration.

Theorem 5.59 *Suppose that the family of Hamiltonian systems (5.4) satisfies D1 and has exponential dichotomy over Ω. Then the following assertions are equivalent:*

(1) *the family (5.4) satisfies conditions D2 and D3, i.e. it is uniformly weakly disconjugate;*
(2) *there exist both Weyl functions $M^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$; i.e. the Lagrange planes $l^{\pm}(\omega)$ belong to $\mathcal{D}$ for all $\omega \in \Omega$.*

In this situation, the Weyl functions agree with the principal functions, and $M^+ < M^-$.

Proof The implication (1)⇒(2) and the last assertions follow immediately from Theorem 5.58. Conversely, the existence of M^+ ensures D3, so that only D2 has to be proved. The relation $\mathbf{w}(t) = \left[\begin{smallmatrix} I_n & I_n \\ M^+(\omega\cdot t) & M^-(\omega\cdot t)\end{smallmatrix}\right]^{-1} \mathbf{z}(t)$ defines a continuous change of variables. A straightforward computation taking the Riccati equation (5.7) as the starting point proves that the transformed family of systems takes the form

$$\mathbf{w}' = \begin{bmatrix} H_1(\omega\cdot t) + H_3(\omega\cdot t)\, M^+(\omega\cdot t) & 0_n \\ 0_n & H_1(\omega\cdot t) + H_3(\omega\cdot t)\, M^-(\omega\cdot t)\end{bmatrix} \mathbf{w} \tag{5.49}$$

for $\omega \in \Omega$. Note that, for each $\omega \in \Omega$, the system (5.49) has exponential dichotomy, since the change of variables is bounded; and that the initial data of the solutions which are bounded as $t \to \infty$ and as $t \to -\infty$ form respectively the Lagrange planes represented by $\left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$: these planes are the transformed ones corresponding to $l^+(\omega)$ and $l^-(\omega)$, respectively. This means that any solution which takes a value $\left[\begin{smallmatrix}\mathbf{w}_1\\ \mathbf{0}\end{smallmatrix}\right]$ at any time is bounded as $t\to\infty$, while if it takes a value $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2\end{smallmatrix}\right]$ at any time then is bounded as $t\to-\infty$. Note also that if $\left[\begin{smallmatrix}\mathbf{w}_1(t)\\ \mathbf{w}_2(t)\end{smallmatrix}\right]$ solves (5.49), so do $\left[\begin{smallmatrix}\mathbf{w}_1(t)\\ \mathbf{0}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2(t)\end{smallmatrix}\right]$.

Suppose that there exists a nontrivial solution $\left[\begin{smallmatrix}\mathbf{z}_1(t)\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ of the system (5.4) corresponding to a point ω with $\mathbf{z}_1\equiv\mathbf{0}$ on $[0,\infty)$, so that $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]=U(t,\omega)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(0)\end{smallmatrix}\right]$ for all $t\ge 0$. Then there exists a point $\bar\omega$ and a solution of the system (5.4) corresponding to $\bar\omega$ of the form $\left[\begin{smallmatrix}\mathbf{0}\\ \bar{\mathbf{z}}_2(t)\end{smallmatrix}\right]$ on $\mathbb{R}$. To prove this assertion, take a sequence $(t_m)\uparrow\infty$ such that there exists $\bar\omega=\lim_{m\to\infty}\omega{\cdot}t_m$, and assume without loss of generality the existence of $\bar{\mathbf{z}}_2^0=\lim_{m\to\infty}\mathbf{z}_2(t_m)/\|\mathbf{z}_2(t_m)\|$. Then the solution $\left[\begin{smallmatrix}\bar{\mathbf{z}}_1(t)\\ \bar{\mathbf{z}}_2(t)\end{smallmatrix}\right]=U(t,\bar\omega)\left[\begin{smallmatrix}\mathbf{0}\\ \bar{\mathbf{z}}_2^0\end{smallmatrix}\right]$ satisfies the stated property: given any $t\in\mathbb{R}$,

$$\begin{aligned}\begin{bmatrix}\bar{\mathbf{z}}_1(t)\\ \bar{\mathbf{z}}_2(t)\end{bmatrix}&=U(t,\bar\omega)\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{z}}_2^0\end{bmatrix}=\lim_{m\to\infty}\frac{1}{\|\mathbf{z}_2(t_m)\|}\,U(t,\omega{\cdot}t_m)\,U(t_m,\omega)\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2(0)\end{bmatrix}\\ &=\lim_{m\to\infty}\frac{1}{\|\mathbf{z}_2(t_m)\|}\,U(t+t_m,\omega)\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2(0)\end{bmatrix}=\lim_{m\to\infty}\frac{1}{\|\mathbf{z}_2(t_m)\|}\begin{bmatrix}\mathbf{0}\\ \mathbf{z}_2(t+t_m)\end{bmatrix},\end{aligned}$$

so that $\bar{\mathbf{z}}_1(t)=\mathbf{0}$.

Write $\left[\begin{smallmatrix}\mathbf{0}\\ \bar{\mathbf{z}}_2(t)\end{smallmatrix}\right]=\left[\begin{smallmatrix}I_n & I_n\\ M^+(\bar\omega{\cdot}t) & M^-(\bar\omega{\cdot}t)\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbf{w}_1(t)\\ \mathbf{w}_2(t)\end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix}\mathbf{w}_1(t)\\ \mathbf{0}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2(t)\end{smallmatrix}\right]$ solve (5.49) and, since $\mathbf{w}_1(t)=-\mathbf{w}_2(t)$, one has that $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ are globally bounded. As explained in Sect. 1.4.1 (see Proposition 1.56 and Remark 1.59.2), the exponential dichotomy of the transformed system ensures the absence of nontrivial bounded solutions, so that $\mathbf{w}_1=\mathbf{w}_2\equiv\mathbf{0}$, and hence also $\bar{\mathbf{z}}\equiv\mathbf{0}$. This proves that D2 holds and completes the proof.

Remark 5.60 Let m_0 be a σ-ergodic measure on Ω, and let $k=k(m_0)$ be the integer provided by Theorem 5.56(v). Note that if the unperturbed family (5.4) satisfies D1, D2, and D3 and if it has exponential dichotomy over Ω, then $k(m_0)=0$, as is implied by Theorem 5.58. The converse is not necessarily true. For instance, in dimension 2 (i.e. with $n=1$), the case $k=0$ for all the σ-ergodic measures can correspond to exponential dichotomy over Ω with two spectral intervals, as trivial examples show; or to the absence of exponential dichotomy, as shown by the examples of disconjugate linear two-dimensional systems given by Millionščikov [104] and Vinograd [147], for which the Sacker–Sell spectrum consists of just one interval. Example 8.44 contains a detailed description of one of these last cases. What Theorem 5.56(v) indeed guarantees is that the Lyapunov index for the measure m_0 is strictly positive whenever $k(m_0)<n$, as in the above-mentioned cases.

The analysis of the relation between the Weyl and principal functions is completed with the following result, which shows that any family of linear Hamiltonian systems satisfying D1, D2, and D3 is the limit of a one-parameter family of families of systems which also satisfy these conditions and in addition have exponential dichotomy over Ω, and that the principal functions are always the pointwise limits of the corresponding one-parameter families of Weyl functions. Part of Theorem 5.61 was proved in [82] under the assumption of the existence of a σ-ergodic measure with total support (see Theorem 5.72). This assumption can be removed by using Proposition 5.51, which in turn is based on the semiequilibria properties established in Sect. 1.3.5. Recall that each of the principal functions is continuous at the points of a residual σ-invariant subset of Ω: see Proposition 5.45(ii).

Theorem 5.61 *Let $\Gamma = \left[\begin{smallmatrix} \Delta & 0_n \\ 0_n & 0_n \end{smallmatrix}\right] \geq 0$ be a symmetric continuous $2n \times 2n$ matrix-valued function on Ω such that each minimal subset of Ω has a point ω with $\Delta(\omega) > 0$. Consider the family*

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega \tag{5.50}$$

for $\lambda \in \mathbb{R}$, and suppose that the set

$$\mathcal{I} = \{\lambda \in \mathbb{R} \,|\, (5.50) \text{ satisfies D1, D2, and D3}\} \tag{5.51}$$

is nonempty. Then,

(i) $\mathcal{I} = (-\infty, \lambda_0]$ *for a point $\lambda_0 \in \mathbb{R}$. In addition, for $\lambda < \lambda_0$, the family (5.50) has exponential dichotomy over Ω and the Weyl functions exist; but it does not have exponential dichotomy for $\lambda = \lambda_0$.*

Let $M^{\pm}(\omega, \lambda)$ be the Weyl functions of (5.50) for $\lambda < \lambda_0$, and let $N^{\pm}(\omega, \lambda)$ be the principal functions of (5.50) for $\lambda \leq \lambda_0$. Then,

(ii) $M^{\pm}(\omega, \lambda) = N^{\pm}(\omega, \lambda)$ *for $\lambda < \lambda_0$ and, if $\lambda_2 < \lambda_1 < \lambda_0$, $M^{+}(\omega, \lambda_2) < M^{+}(\omega, \lambda_1) < N^{+}(\omega, \lambda_0) \leq N^{-}(\omega, \lambda_0) < M^{-}(\omega, \lambda_1) < M^{-}(\omega, \lambda_2)$.*
(iii) *For every $\omega \in \Omega$,*

$$\lim_{\lambda \to \lambda_0^-} M^{\pm}(\omega, \lambda) = N^{\pm}(\omega, \lambda_0)\,.$$

(iv) *If Ω is minimal and $\Omega^{\pm}_{\lambda_0}$ are the σ-invariant sets of continuity points of $N^{\pm}(\cdot, \lambda_0)$, then the sets $\mathcal{K}^{\pm}_{\lambda_0} = \mathrm{closure}_{\mathcal{K}_{\mathbb{R}}}\{(\omega, \tilde{l}^{\pm}(\omega, \lambda_0)) \,|\, \omega \in \Omega^{\pm}_{\lambda_0}\}$ are almost automorphic extensions of the base Ω.*

Proof (i), (ii) & (iii) Let $M^{\pm}(\omega, \lambda)$ and $N^{\pm}(\omega, \lambda)$ represent the Weyl and principal functions of (5.50), if they exist. Recall that, according to Theorem 5.59, this is the case if D1, D2, D3, and the exponential dichotomy over Ω hold, in which case $M^{\pm}(\omega, \lambda) = N^{\pm}(\omega, \lambda)$.

Obviously, the perturbed family (5.50) satisfies D1 for all $\lambda \in \mathbb{R}$. Since $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ solves (5.4) in an interval if and only if it solves (5.50) in the same interval, also condition D2 holds or not simultaneously for all $\lambda \in \mathbb{R}$. Suppose that $\mathcal{I}$ is nonempty, fix $\lambda_1 \in \mathcal{I}$, and choose any $\lambda_2 < \lambda_1$. Since $JH + \lambda_2\Gamma \leq JH + \lambda_1\Gamma$, Proposition 5.51 ensures that D3 also holds for λ_2. This ensures that

$(-\infty, \lambda_1] \subset \mathcal{I}$ for all $\lambda_1 \in \mathcal{I}$: $\mathcal{I}$ is either a negative half-line or $\mathbb{R}$. Since $(H + \lambda J^{-1}\Gamma)_1 = H_1$ and $(H + \lambda J^{-1}\Gamma)_2 = H_2 - \lambda\Delta$, Proposition 5.51 also ensures that $N^+(\omega, \lambda_2) < N^+(\omega, \lambda_1) \le N^-(\omega, \lambda_1) < N^-(\omega, \lambda_2)$ if $\lambda_2 < \lambda_1$. Hence, (5.27) implies that $\tilde{l}^+(\omega, \lambda_2) \cap \tilde{l}^-(\omega, \lambda_2) = \{\mathbf{0}\}$ and Theorem 5.58 ensures that the family (5.50) corresponding to λ_2 has exponential dichotomy over Ω, with $N^\pm(\omega, \lambda_2) = M^\pm(\omega, \lambda_2)$.

It is obvious that the supremum of $\mathcal{I}$ agrees with that of $\mathcal{I}_1 = [\lambda_1, \infty) \cap \mathcal{I}$. Whenever λ belongs to the interior of $\mathcal{I}_1$, the functions $M^\pm(\omega, \lambda)$ are solutions along the flow of

$$M' = -M\,H_3(\omega{\cdot}t)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) - \lambda\,\Delta(\omega{\cdot}t),$$

and, as checked above, $M^+(\omega, \lambda_1) \le M^\pm(\omega, \lambda) \le M^-(\omega, \lambda_1)$. Let m be a σ-ergodic measure on Ω. Since $\operatorname{Supp} m$ is compact (see Sect. 1.1.2), it contains a minimal set, and hence a point where Δ is positive definite. In particular, $\int_\Omega \Delta(\omega)\,dm = \bar{\Delta}$ for a constant matrix $\bar{\Delta} > 0$. According to Proposition 1.36, $\int_\Omega (M^\pm)'(\omega, \lambda)\,dm = 0_n$. So, if $\lambda \in \mathcal{I}_1$, then $0_n = B_\lambda - \lambda\bar{\Delta}$, where

$$B_\lambda = \int_\Omega \big(-M^\pm H_3\,M^\pm - M^\pm H_1 - H_1^T M^\pm + H_2\big)\,dm\,,$$

and where H_j and $M^\pm$ have arguments ω and (ω, λ). The boundedness of all the involved functions for $\lambda \in \mathcal{I}_1$ implies the existence of a constant $c > 0$ such that $\|B_\lambda\| < c$ for $\lambda \in \mathcal{I}_1$, so that also $\|\lambda\bar{\Delta}\| < c$. This precludes the upper unboundedness of the interval. In other words, $\lambda_0 = \sup \mathcal{I}_1 = \sup \mathcal{I}$ is a finite number.

Following now the sketch of the proof of (5.47) in Theorem 5.58, one first proves the existence of $N_0^\pm(\omega, \lambda_0) = \lim_{\lambda \to \lambda_0^-} M^\pm(\omega, \lambda)$, which in particular implies property D3 for the family (5.50) corresponding to λ_0 (i.e. $\lambda_0 \in \mathcal{I}$) and hence the existence of $N^\pm(\omega, \lambda_0)$; and then one deduces that $N_0^\pm(\omega, \lambda_0) = N^\pm(\omega, \lambda_0)$. This proves (iii) as well as the chain of inequalities in (ii).

To complete the proof of (i) it is enough to check that the family (5.50) corresponding to λ_0 does not have exponential dichotomy over Ω. Suppose for contradiction that the exponential dichotomy occurs. Then the robustness properties of this property recalled in Theorems 1.92 and 1.95 provide $\varepsilon > 0$ such that Weyl functions exist for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$. Therefore Theorem 5.59 proves that $[\lambda_0, \lambda_0 + \varepsilon] \subset \mathcal{I}$, which contradicts the definition of λ_0.

(iv) This assertion is implied by Proposition 1.53.

Remarks 5.62

1. It follows from Theorem 5.61(i), Theorem 3.50 and Remark 3.51.1 that the rotation number with respect to any σ-ergodic measure m on Ω is constant on $(-\infty, \lambda_0)$. Proposition 5.65 will show that in fact this constant value is zero.

2. Suppose that the family (5.4) satisfies D1, D2, and D3 and does not have exponential dichotomy over Ω. Then, for all Γ satisfying the conditions of the previous theorem, the corresponding interval $\mathcal{I}$ is $(-\infty, 0]$.

The next result of this section is a consequence of Theorems 5.58 and 5.61: Theorem 5.63 gives information about the Sacker–Sell spectral decomposition of the family (5.4) when this family is uniformly weakly disconjugate and does not have exponential dichotomy over Ω. See Definitions 1.82 and 1.87 and Theorem 1.84 to recall the definitions and main properties of the Sacker–Sell spectrum and Sacker–Sell decomposition when the base Ω is connected, which last is the case when it is given by the hull of a particular system. And recall that Proposition 1.89 provides extra information in the Hamiltonian case.

Theorem 5.63 *Suppose that Ω is connected, that D1, D2, and D3 hold for (5.4), and that there exists $\omega_0 \in \Omega$ with $\dim\big(\tilde{l}^+(\omega_0) \cap \tilde{l}^-(\omega_0)\big) = k \geq 1$. Then the Sacker–Sell spectrum $\Sigma(H)$ consists of at most $2n-2k+1$ intervals, and $0 \in \Sigma(H)$.*

Proof Theorem 5.58 ensures that the family (5.4) does not have exponential dichotomy over Ω, so that, by definition, $0 \in \Sigma(H)$. Suppose for contradiction that $\Sigma(H)$ contains more than $2n - 2k + 1$ intervals. Then, according to Definition 1.87 and Proposition 1.89,

$$\Sigma(H) = [-b_d, -a_d] \cup \cdots \cup [-b_1, -a_1] \cup [-b_0, b_0] \cup [a_1, b_1] \cup \cdots \cup [a_d, b_d]$$

for $d > n - k$, where $0 \leq b_0 < a_1 \leq b_1 < \cdots < a_d \leq b_d$, and the corresponding spectral decomposition is

$$\Omega \times \mathbb{R}^{2n} = F_H^{-d} \oplus \cdots \oplus F_H^{-1} \oplus F_H^0 \oplus F_H^1 \oplus \cdots \oplus F_H^d\,. \tag{5.52}$$

Choose $\varepsilon > 0$ such that the three intervals $S^+ = (-b_d - \varepsilon, -a_1 + \varepsilon)$, $S^0 = (-b_0 - \varepsilon, b_0 + \varepsilon)$ and $S^- = (a_1 - \varepsilon, b_d + \varepsilon)$ are disjoint, and note that $\Sigma(H) \subset S^+ \cup S^0 \cup S^+$. Now take $\Gamma = \left[\begin{smallmatrix} I_n & 0_n \\ 0_n & 0_n \end{smallmatrix}\right]$, and consider the family of systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} = H_\lambda(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,. \tag{5.53}$$

Theorem 1.91(i) implies the existence of $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$, then $\Sigma(H_\lambda) \subset S_\varepsilon^+ \cup S_\varepsilon^0 \cup S_\varepsilon^-$. For each of these values of λ, let $F^+(\lambda)$, $F^0(\lambda)$, and $F^-(\lambda)$ be the Whitney sums of the spectral bundles of (5.53) corresponding to those spectral intervals contained in S^+, S^0, and S^-, respectively. Note that, in particular, $F^+(0) = F_H^{-d} \oplus \cdots \oplus F_H^{-1}$, $F^0(0) = F_H^0$, and $F^-(0) = F_H^1 \oplus \cdots \oplus F_H^d$, so that the contradiction hypothesis ensures that $\dim F^+(0) = \dim F^-(0) > n - k$.

According to Corollary 1.93, the maps from $\Omega \times [-\lambda_0, \lambda_0]$ to the corresponding Grassmannian manifolds $(\omega, \lambda) \mapsto (F^+(\lambda))_\omega$, $(\omega, \lambda) \mapsto (F^0(\lambda))_\omega$, and $(\omega, \lambda) \mapsto (F^-(\lambda))_\omega$, are continuous. Consequently, $\dim(F^\pm(\lambda))_\omega = \dim(F^\pm(0))_\omega > n - k$. These two facts will be necessary to complete the proof.

Obviously Γ satisfies the hypotheses of Theorem 5.61, whose point (i) ensures that, if $-\lambda_0 < \lambda < 0$, then $\mathbf{z}' = H_\lambda(\omega{\cdot}t)\,\mathbf{z}$ has exponential dichotomy over Ω and the Weyl functions $M^\pm(\omega,\lambda)$ exist. In particular, 0 does not belong to $\Sigma(H_\lambda)$. A new application of Theorem 1.84 provides the splitting $F^0(\lambda) = F^0_+(\lambda) \oplus F^0_-(\lambda)$, where all the semiorbits starting at $F^0_+(\lambda)$ and $F^0_-(\lambda)$ have negative and positive characteristic exponents, respectively. That is, $L^\pm(\lambda,0) = F^\pm(\lambda) \oplus F^0_\pm(\lambda)$.

Let ω_0 be the point appearing in the hypotheses of the theorem. From this point on the proof is divided in two steps:

1. To check that $g_0 = \tilde{l}^+(\omega_0) \cap \tilde{l}^-(\omega_0)$ is contained in $(F^0(0))_{\omega_0}$.
2. To deduce that there exists $\lambda < 0$, as small as desired, such that $(F^0_\pm(\lambda))_{\omega_0}$ has dimension at least k, so that $\dim(F^0(\lambda))_{\omega_0} \geq 2k$.

Once this is done, a contradiction is immediately reached: as said before, $\dim(F^\pm(\lambda))_{\omega_0} > n-k$, so that

$$\begin{aligned}\dim \mathbb{R}^{2n} &= (\dim F^-(\lambda))_{\omega_0} + (\dim F^0(\lambda))_{\omega_0} + (\dim F^+(\lambda))_{\omega_0} \\ &> n-k+2k+n-k = 2n\,.\end{aligned}$$

To begin with step 1, let $\{\mathbf{z}_1,\ldots \mathbf{z}_k\}$ be a basis of g_0; i.e. $\mathbf{z}_j = \begin{bmatrix} I_n \\ N^\pm(\omega_0)\end{bmatrix}\mathbf{x}_j$ for a basis $\{\mathbf{x}_1,\ldots \mathbf{x}_k\}$ of $\mathrm{Ker}(N^+(\omega_0) - N^-(\omega_0))$. Define the vectors $\mathbf{z}_j^\pm(\lambda) = \begin{bmatrix} I_n \\ M^\pm(\omega_0,\lambda)\end{bmatrix}\mathbf{x}_j \in l^\pm(\omega,\lambda)$, where $l^\pm(\omega,\lambda) = (L^\pm(\lambda,0))_\omega$. Then, the k vectors $\mathbf{z}_1^+(\lambda),\ldots,\mathbf{z}_k^+(\lambda)$ are linearly independent, and the same is the case for $\mathbf{z}_1^-(\lambda),\ldots,\mathbf{z}_k^-(\lambda)$. According to Theorem 5.61(iii),

$$\lim_{\lambda\to 0^-}\mathbf{z}_j^+(\lambda) = \begin{bmatrix} I_n \\ N^+(\omega_0)\end{bmatrix}\mathbf{x}_j = \mathbf{z}_j = \begin{bmatrix} I_n \\ N^-(\omega_0)\end{bmatrix}\mathbf{x}_j = \lim_{\lambda\to 0^-}\mathbf{z}_j^-(\lambda)\,. \tag{5.54}$$

In addition, the vector spaces

$$\begin{aligned} g^+(\lambda) &= \langle \mathbf{z}_1^+(\lambda),\ldots,\mathbf{z}_k^+(\lambda)\rangle \subseteq l^+(\omega_0,\lambda)\,,\\ g^-(\lambda) &= \langle \mathbf{z}_1^-(\lambda),\ldots,\mathbf{z}_k^-(\lambda)\rangle \subseteq l^-(\omega_0,\lambda)\end{aligned}$$

are k-dimensional; i.e. they belong to the (compact) Grassmann manifold $\mathcal{G}_k(\mathbb{R}^{2n})$. Choose a sequence $(\lambda_m) \uparrow 0$ such that there exist $g^\pm = \lim_{m\to\infty} g^\pm(\lambda_m)$ in $\mathcal{G}_k(\mathbb{R}^{2n})$. Since $\lim_{m\to\infty}\mathbf{z}_j^\pm(\lambda_m) = \mathbf{z}_j$ for $j = 1\ldots,k$ and $\dim g_0 = k$, it follows from Proposition 1.26(i) that $g^+ = g^- = g_0$. And, since $g^\pm(\lambda_m) \subset l^\pm(\omega_0,\lambda_m) = (F^\pm(\lambda_m))_{\omega_0} \oplus (F^0_\pm(\lambda_m))_{\omega_0}$, the previously explained property of continuous variation of subbundles ensures that

$$\begin{aligned} g_0 &\subset \big((F^+(0))_{\omega_0} \oplus (F^0(0))_{\omega_0}\big) \cap \big((F^-(0))_{\omega_0} \oplus (F^0(0))_{\omega_0}\big) \\ &= (F^0(0))_{\omega_0}\,. \end{aligned} \tag{5.55}$$

This completes the first step.

Write now $\mathbf{z}_j^+(\lambda_m) = \mathbf{w}_j(\lambda_m) + \mathbf{v}_j(\lambda_m) \in (F^+(\lambda_m))_{\omega_0} \oplus (F_+^0(\lambda_m))_{\omega_0}$ and $\mathbf{z}_j = \mathbf{w}_j + \mathbf{v}_j \in (F^+(0))_{\omega_0} \oplus (F^0(0))_{\omega_0}$. As was seen above, the map $\lambda \mapsto (F^+(\lambda))_{\omega_0} \oplus (F^0(\lambda))_{\omega_0}$ is continuous on $[-\lambda_0, 0]$, and this implies the continuity of the projections over each of the spaces: see Proposition 1.67. Therefore, the convergence of $(\mathbf{z}_j^+(\lambda_m))$ to $\mathbf{z}_j$ implies that $(\mathbf{w}_j(\lambda_m))$ and $(\mathbf{v}_j(\lambda_m))$ converge to $\mathbf{w}_j$ and $\mathbf{v}_j$ for $j = 1, \ldots, k$. Recall that, by (5.54) and (5.55), $\lim_{m\to\infty} \mathbf{z}_j^+(\lambda_m) = \mathbf{z}_j \in g_0 \subset F_{\omega_0}^0$. Hence, the vector $\mathbf{w}_j = \lim_{m\to\infty} \mathbf{w}_j(\lambda_m) = \lim_{m\to\infty}(\mathbf{z}_j^+(\lambda_m) - \mathbf{v}_j(\lambda_m))$ belongs at the same time to $(F^+(0))_{\omega_0}$ and to $(F^0(0))_{\omega_0}$, so that $\mathbf{w}_j = \mathbf{0}$. Consequently, for large enough m, the k vectors $\mathbf{v}_1(\lambda_m), \ldots, \mathbf{v}_k(\lambda_m)$, which belong to $(F_+^0(\lambda_m))_{\omega_0}$, are linearly independent. This proves the assertion for $(F_+^0(\lambda_m))_{\omega_0}$. The same argument proves it for $(F_-^0(\lambda_m))_{\omega_0}$ as well. The proofs of step 2 and of the theorem itself are complete.

The section is completed by analyzing the special situation in which both H_2 and H_3 are positive semidefinite. A condition formally similar to D2 will also be considered, namely

D2*. For all $\omega \in \Omega$ and for any nonzero solution $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of the system (5.4) with $\mathbf{z}_2(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_2(t,\omega)$ does not vanish identically on $[0,\infty)$.

Repeating step by step the proof of Proposition 5.18(i), one proves that condition D2* holds if and only if there exist $\delta > 0$ and $t_0 > 0$ such that

$$\int_0^{t_0} \|H_2(\omega{\cdot}t)\, U_{H_1}(t,\omega)\, \mathbf{x}\|^2\, dt \geq \delta\, \|\mathbf{x}\|^2 \tag{5.56}$$

for all $\omega \in \Omega$ and $\mathbf{x} \in \mathbb{R}^n$. And, as Remark 5.22 explains, D2* is ensured by the existence of a point ω in each minimal set of Ω having the property that the system $\mathbf{x}' = -H_1^T(\omega{\cdot}t)\,\mathbf{x} + H_2(\omega{\cdot}t)\,\mathbf{u}$ is null controllable, which in turn holds if $H_2 \geq 0$ and each minimal set of Ω contains a point ω with $H_2(\omega) > 0$: see Remark 6.2.1.

Proposition 5.64 *Suppose that $H_2 \geq 0$, that $H_3 \geq 0$ (i.e. that D1 holds), and that condition* D2* *holds. Then,*

(i) *if the family* (5.4) *has exponential dichotomy over Ω and the Weyl function M^+ (resp. M^-) is globally defined, then $M^+ < 0$ (resp. $M^- > 0$).*

Suppose that also condition D2 *holds. Then,*

(ii) *the family* (5.4) *is uniformly weakly disconjugate, it has exponential dichotomy over Ω, both Weyl functions are globally defined and agree with the principal functions, and they satisfy $M^+ < 0$ and $M^- > 0$.*

(iii) *For all $\omega \in \Omega$ and $M_0 \geq 0$, the solution $M(t,\omega,M_0)$ of the Riccati equation* (5.7) *is defined on $(0,\infty)$, and there exist constants $t_0 > 0$, $\beta > 0$ and $\eta > 0$, independent of (ω, M_0), such that, for $t \geq t_0$,*

$$\|M(t,\omega,M_0) - M^-(\omega{\cdot}t)\| \leq \eta\, e^{-\beta t}\,.$$

(iv) *For all* $\omega \in \Omega$ *and* $M_0 \le 0$*, the solution* $M(t,\omega,M_0)$ *of the Riccati equation* (5.7) *is defined on* $(-\infty,0)$*, and there exist constants* $t_0 > 0$*,* $\beta > 0$ *and* $\eta > 0$*, independent of* (ω, M_0)*, such that, for* $t \ge t_0$*,*

$$\|M(t,\omega,M_0) - M^+(\omega{\cdot}t)\| \le \eta\, e^{\beta t}\,.$$

Proof

(i) Assume that M^+ is globally defined and take $\mathbf{x}_0 \in \mathbb{R}^n$ different from $\mathbf{0}$. Reasoning as in Remark 1.81.2, if $\left[\begin{smallmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix}\mathbf{x}_0\\ M^+(\omega)\,\mathbf{x}_0\end{smallmatrix}\right]$, then

$$-\mathbf{x}_0\, M^+(\omega)\,\mathbf{x}_0 = \int_0^\infty \Big(\|H_2^{1/2}(\omega{\cdot}t)\,\mathbf{x}(t)\|^2 + \|H_3^{1/2}(\omega{\cdot}t)\,\mathbf{y}(t)\|^2\Big)\,dt \ge 0\,,$$

so that $M^+ \le 0$. Suppose now that $\mathbf{x}_0\, M^+(\omega)\,\mathbf{x}_0 = 0$; i.e. $\mathbf{y}_0 = M^+(\omega)\,\mathbf{x}_0 = \mathbf{0}$. Then the previous equality ensures that $\mathbf{x}'(t) = H_1(\omega{\cdot}t)\,\mathbf{x}(t)$ and $\mathbf{y}'(t) = -H_1^T(\omega{\cdot}t)\,\mathbf{y}(t)$ for $t \ge 0$, so that $\left[\begin{smallmatrix}U_{H_1}(t,\omega)\,\mathbf{x}_0\\ \mathbf{0}\end{smallmatrix}\right]$ solves (5.4) on $[0,\infty)$, which by D2* means that $\mathbf{x}_0 = \mathbf{0}$. That is, $M^+(\omega) < 0$, as asserted. The proof is similar in the case of M^-.

(ii) The uniform weak disconjugacy of the family (1.11) has been established by Proposition 5.27 under less restrictive conditions. Therefore, the main step of the present proof is to prove the existence of exponential dichotomy. The rest of the assertions follow from these properties, Theorem 5.59, and point (i).

For reasons both historical and of convenience of presentation, the proof of the exponential dichotomy is postponed to Chap. 6: see Remark 6.12. In fact, that proof reproduces the arguments used by Johnson and Nerurkar [74, 76, 77] in their resolution of the linear regulator problem, in which the existence of exponential dichotomy is the key point; and it is convenient to adapt the proof carried out there to the present more abstract setting.

(iii) & (iv) For similar reasons, these proofs are also postponed to Chap. 6: see Remark 6.30.

5.7 Weak Disconjugacy and Rotation Number

This section provides an ergodic-theoretic characterization of the weak disconjugacy of linear Hamiltonian systems (5.4) under conditions D1 and D2 in terms of its rotation number, analyzed in Chap. 2. Roughly speaking, all the systems are weakly disconjugate on $[0,\infty)$ if and only if the average rotation of their solutions is zero. If m is a σ-ergodic measure on Ω, $\alpha(m)$ will denote the rotation number with respect to m. See Sect. 1.1.2 to recall the definition and main properties of the topological support of m_0, $\operatorname{Supp} m_0$, which appears frequently in this section.

Recall that a weakly disconjugate system on $[0,\infty)$ is nonoscillatory at $+\infty$, as ensured by Proposition 5.7. The first connection between weak disconjugacy and rotation number is hence an immediate consequence of the following result.

Proposition 5.65 *Suppose that the system* (5.4) *is nonoscillatory at* $+\infty$ *(or at* $-\infty$*) for all* $\omega \in \Omega_0$*, where* $m(\Omega_0) > 0$ *for a* σ*-ergodic measure* m *on* Ω*. Then* $\alpha(m) = 0$.

Proof The nonoscillation of the systems corresponding to points $\omega \in \Omega_0$ means that the functions $t \mapsto \operatorname{Arg} U(t, \omega)$ are bounded as $t \to \infty$ (or as $t \to -\infty$) for any of the equivalent arguments Arg defined in Sect. 2.1.1 if a continuous branch of the argument is taken along a given curve. Since $m(\Omega_0) > 0$, the assertion follows from Definition 2.5 and Theorem 2.4 (or Remark 2.6).

Theorem 5.66 *Suppose that* D1 *holds and that there exists a* σ*-ergodic measure* m_0 *on* Ω *with* $\operatorname{Supp} m_0 = \Omega$ *and* $\alpha(m_0) = 0$*. Then,*

(i) *all the systems of the family* (5.4) *are nonoscillatory at* $\pm\infty$.
(ii) *Suppose that, in addition,* D2 *holds. Then the family of systems* (5.4) *is uniformly weakly disconjugate; and, if* $\mathcal{K} \subset \mathcal{K}_{\mathbb{R}}$ *is a* τ*-invariant compact subset with* $\mathcal{K} = \{(\omega, l(\omega)) \mid \omega \in \Omega\}$ *for a continuous function* $l\colon \Omega \to \mathcal{L}_{\mathbb{R}}$*, then* $l(\omega) \in \mathcal{D}$ *for all* $\omega \in \Omega$.
(iii) *Suppose that, in addition,* D2 *holds and the family* (5.4) *has exponential dichotomy. Then the Weyl functions* $M^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ *exist globally and satisfy* $M^- > M^+$.

Proof

(i) Theorem 2.4 guarantees that

$$\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{Tr} Q(\omega, l)\, d\mu = 0$$

for every normalized τ-invariant measure μ on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0, where the function $\operatorname{Tr} Q$ is defined by (1.19). Suppose μ to be τ-ergodic. The recurrence result given by Schneiberg in [137] implies that for μ-a.e. $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$ there is a sequence $(t_m) \uparrow \infty$ such that

$$\int_0^{t_m} \operatorname{Tr} Q(\tau(s, \omega, l))\, ds = 0 \tag{5.57}$$

for each $m \in \mathbb{N}$. Fix one of these points (ω, l) and represent $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. As in Theorem 2.4 (see (2.7)), the definition of Arg_1 and Theorem 1.41 imply that

$$\int_0^{t} \operatorname{Tr} Q(\tau(s, \omega, l))\, ds = \operatorname{Arg}_1 V(t, \omega) - \operatorname{Arg}_1 V(0, \omega)\,, \tag{5.58}$$

where $V(t, \omega) = \left[\begin{smallmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega) \end{smallmatrix}\right]$ is a symplectic matrix solution of (5.4) with $\left[\begin{smallmatrix} V_1(0,\omega) \\ V_2(0,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. The equivalence of $\operatorname{Arg}_1 V(t, \omega)$ and $\operatorname{Arg}_3(V(t, \omega)\, S)$ for a constant real symplectic matrix S and relation (5.58) provide a positive constant ρ such that

$$\left| \int_0^t \operatorname{Tr} Q(\tau(s,\omega,l)) - \operatorname{Arg}_3 U(t,\omega) \right| \le \rho\,. \tag{5.59}$$

As in the proof of Theorem 5.31(iii), the eigenvalues of the unitary matrix-valued function $W_U(t,\omega) = (U_1(t,\omega) - iU_3(t,\omega))^{-1}(U_1(t,\omega) + iU_3(t,\omega))$ can be written as $e^{i\varphi_1(t)}, \dots, e^{i\varphi_n(t)}$ for functions $\varphi_1, \dots \varphi_n : \mathbb{R} \to \mathbb{R}$ which are continuous and nondecreasing in t. Lemma 2.29(i) and the definition of Arg_3 ensure then that $\varphi(t) = (1/2)\sum_{j=1}^n \varphi_j(t)$ is a nondecreasing continuous branch of $\operatorname{Arg}_3 U(t,\omega) = \arg\det(U_1(t,\omega) + iU_3(t,\omega))$. Relations (5.57) and (5.59) and the nondecreasing character of $\varphi(t)$ ensure that $\varphi(t)$ is bounded, so that the system (5.4) corresponding to ω is nonoscillatory at $+\infty$.

The nonoscillation at $+\infty$ of the system (5.4) has so far been checked for m_0-a.e. $\omega \in \Omega$. Fix now one of these points ω_0 for which in addition $\{\omega_0{\cdot}t \mid t \ge 0\}$ is dense in Ω. Then, Theorem 5.31(i) ensures that all the systems of Ω are nonoscillatory at $+\infty$, which proves (i).

(ii) Since $\operatorname{Supp} m_0 = \Omega$, m_0-almost every positive σ-semiorbit is dense in Ω (see Proposition 1.12). Due to this property and Theorem 5.32, to prove the uniform weak disconjugacy it suffices to check that each of the systems of the family is weakly disconjugate on $[0,\infty)$, which in the present conditions is equivalent to saying that all of them are nonoscillatory at $+\infty$: see Remark 5.20.3. Therefore, the first assertion (ii) follows from (i).

The second property stated in (ii) follows immediately from Proposition 1.12 and Theorem 5.49(ii). The interested reader can find in [48] a direct proof based on the nonoscillatory properties of the systems (5.4) under the assumed hypotheses.

(iii) This assertion follows immediately from (ii) and Theorem 5.59.

The following theorem is an immediate consequence of the two previous results, Propositions 1.12 and 5.18(iii), and Theorems 5.32 and 5.17.

Theorem 5.67 *Suppose that D1 and D2 (or D2′) hold, and that there exists a σ-ergodic measure m_0 with* $\operatorname{Supp} m_0 = \Omega$. *Then the following assertions are equivalent:*

(1) *condition D3 holds;*
(2) *the family (5.4) is uniformly weakly disconjugate;*
(3) *all the systems of the family (5.4) are weakly disconjugate on* $[0,\infty)$*;*
(4) *all the systems of the family (5.4) are weakly disconjugate on* $(-\infty,0]$*;*
(5) $\alpha(m_0) = 0$*;*
(6) $\alpha(m) = 0$ *for each σ-ergodic measure m on* Ω.

Note that, in particular, if D1 and D2 hold, if the family (5.4) has exponential dichotomy over Ω, and if there exists a σ-ergodic measure with full support for which the rotation number is zero, then the Weyl functions M^+ and M^- are globally defined, with $M^- > M^+$. This is a consequence of Theorems 5.67 and 5.58.

Example 5.68 In order to look more deeply into the preceding characterizations, note that the simultaneous presence of conditions D1, D2, and the exponential dichotomy over Ω are compatible with strictly positive rotation number for a measure with full support. For instance, consider the scalar Schrödinger equation $-x'' + g(\omega{\cdot}t)\,x = 0$ with an almost periodic coefficient g, whose resolvent set is composed by those values of $\lambda \in \mathbb{R}$ such that the system $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]' = \left[\begin{smallmatrix} 0 & 1 \\ q(\omega\cdot t)-\lambda & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ (which satisfies D1 and D2: see Remark 5.19) does not have exponential dichotomy over the uniquely ergodic base Ω: see Sect. 1.3.2 and Corollary 3.55. It is well known, for instance, that there exist periodic continuous functions g for which the spectrum is given by a union $[a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_m, \infty)$ with $a_1 < b_1 < a_2 < b_2 < \cdots < a_m$. Corollary 3.55 ensures that the rotation number $\alpha(\lambda)$ is nonnegative, constant in the intervals of the resolvent and strictly increasing with respect to λ on the spectrum. So one example is given by the system corresponding to the equation $-x'' + g(\omega{\cdot}t)\,x = \lambda x$ for a λ taken, for instance, in the interval (b_1, a_2) of the resolvent set. One can multiply examples by noting that for a large set of almost periodic functions g the spectrum is a Cantor set: see e.g. Moser [108]. Therefore, the possibilities for the choice of a λ as above are practically limitless.

The characterizations provided by Theorem 5.67 together with the continuity of the rotation number ensure the weak disconjugacy property for the limit of a suitable sequence of families of weakly disconjugate systems. See Definition 1.32 for the description of the $L^1(\Omega, m_0)$ topology.

Proposition 5.69 *Let m_0 be a σ-ergodic measure on Ω with* $\operatorname{Supp} m_0 = \Omega$. *Let $(H^k \colon \Omega \to \mathfrak{sp}(n, \mathbb{R}))$ be a sequence of continuous matrix-valued functions converging to a continuous function H in the $L^1(\Omega, m_0)$ topology. Suppose that, for each $k \in \mathbb{N}$, the following two properties hold: the family*

$$\mathbf{z}' = H^k(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega \tag{5.60}$$

satisfies D1 *and* D2*; and m_0-almost every system* (5.60) *is weakly disconjugate. Then the limit family of Hamiltonian systems $\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}$ is uniformly weakly disconjugate in the case that* D2 *holds.*

Proof Let $\alpha_k(m_0)$ be the rotation number of the family (5.60). Proposition 5.65 ensures that $\alpha_k(m_0) = 0$, so that by Theorem 2.25,

$$\alpha(m_0) = \lim_{k\to\infty} \alpha_k(m_0) = 0\,.$$

Theorem 5.66(ii) can be used to complete the proof, since the limit family satisfies conditions D1 and D2.

The next objective is to extend the characterizations given by Theorem 5.67 to the disconjugate case. This extension is based on Proposition 5.29 and on the fact that identical normality for all the systems (5.4) ensures condition D2.

Theorem 5.70 *Suppose that D1 holds, that every system of the family (5.4) is identically normal, and that there exists a σ-ergodic measure m_0 on Ω with* $\operatorname{Supp} m_0 = \Omega$. *Then the disconjugacy of all the systems of the family is equivalent to any of the six situations described in Theorem 5.67.*

In order to check the optimality of this result, think again of Example 5.39: all the systems of the elements of the hull of the initial system provide weakly disconjugate and identically normal systems, but just one of them is disconjugate; and the family is not uniformly weakly disconjugate. What fails in order to apply Theorem 5.70 is that in the present case there is a unique σ-ergodic measure m_0, with $\operatorname{Supp} m_0 \neq \Omega$: it is the measure concentrated on the σ-invariant compact set $\{\omega_1\}$. In order to check that any σ-ergodic measure m is precisely this one, use the regularity of m to write $m(\{\omega_1\}) = \inf\{m(\mathcal{V}) \mid \omega_1 \in \mathcal{V} \text{ and } \mathcal{V} \text{ is open}\}$; apply the Birkhoff Theorems 1.3 and 1.6 to find ω_2 such that $m(\mathcal{V}) = \lim_{t\to\infty}(1/t)\int_0^t \chi_{\mathcal{V}}(\omega_2{\cdot}s)\,ds$; use the fact that $\lim_{s\to\infty} d(\omega_2{\cdot}s, \omega_1) = 0$ to conclude that $\omega_2{\cdot}s \in \mathcal{V}$ for large enough s; and deduce that $m(\mathcal{V}) = 1$. This proves that $m(\{\omega_1\}) = 1$, so that all these ergodic measures agree: they are all concentrated on the same set $\{\omega_1\}$. Note finally that conditions D1 and D2 hold, so that Proposition 5.65 ensures that $\alpha(m_0) = 0$.

The following result, which is not directly related to weak disconjugacy, establishes an interesting property in a situation similar to the one analyzed in Theorem 5.67: the main and fundamental difference is that the existence of a measure m_0 with $\operatorname{Supp} m_0 = \Omega$ is not assumed.

Proposition 5.71 *Suppose that D1 and D2 hold, and that $\alpha(m) = 0$ for each σ-ergodic measure m on Ω. Given any $(\omega, l) \in \mathcal{K}_{\mathbb{R}}$, there exists a sequence $(t_m) \uparrow \infty$ (depending on (ω, l)) such that $U(t_m, \omega){\cdot}l \in \mathcal{D}$ for all $m \in \mathbb{N}$.*

Proof Let $\mathcal{K}$ be a minimal subset of the omega-limit set (for σ) of ω. Take $\omega_1 \in \mathcal{K}$ and $l_1 \in \mathcal{K}_{\mathbb{R}}$ such that (ω_1, l_1) belongs to the omega-limit set (for τ) of (ω, l). Theorem 5.67 ensures that the family (5.4) is uniformly weakly disconjugate over $\mathcal{K}$, and hence Theorem 5.25(ii) provides t_1 such that $(\omega_2, l_2) = \tau(t_1, \omega_1, l_1) \in \Omega \times \mathcal{D}$. Since (ω_2, l_2) also belongs to the omega-limit set of (ω, l), it is the case that $(\omega_2, t_2) = \lim_{m\to\infty} \tau(t_m, \omega, l)$. The assertion follows hence from the fact that $\mathcal{D}$ is open in $\mathcal{L}_{\mathbb{R}}$: see Proposition 1.28.

The section is completed with two perturbation results. The first one is very similar to Theorem 5.61: quite similar conclusions are obtained but with different hypotheses. Recall once more that conditions D1, D2, and D3 for a family of linear Hamiltonian systems imply the uniform weak disconjugacy on $(-\infty, 0]$ and $(0, \infty]$ and hence the existence of uniform principal solutions at $\pm\infty$.

Theorem 5.72 *Consider the perturbed families of linear Hamiltonian systems (5.50) for $\lambda \in \mathbb{R}$, given by a symmetric $2n \times 2n$ matrix-valued function $\Gamma = \left[\begin{smallmatrix} \Delta & 0_n \\ 0_n & 0_n \end{smallmatrix}\right] \geq 0$ which is continuous on Ω. Suppose that D1, D2, and D3 hold for the unperturbed family (5.4), that Γ satisfies the Atkinson Hypotheses 3.3, and that there exists a σ-ergodic measure m_0 with* $\operatorname{Supp} m_0 = \Omega$. *Let $\mathcal{I}$ be defined* *by* (5.51). *Then all the conclusions of Theorem* 5.61 *hold, with the exception that the inequalities in point* (ii)*, except for* $M^+(\omega, \lambda_1) < M^-(\omega, \lambda_1)$*, are not necessarily strict.*

Proof It is obvious that the family (5.50) satisfies D1 for all λ, and it is easy to check that also D2 holds, since it holds for $\lambda = 0$. Since $JH + \lambda\Gamma \le JH$ if $\lambda \le 0$, Proposition 5.51 ensures that D3 also holds for these values of λ. This proves that $(-\infty, 0] \subseteq \mathcal{I}$. Proposition 5.65 implies that the rotation number of (5.50) with respect to m_0 vanishes for each $\lambda < 0$. Therefore, Theorem 3.50 ensures the occurrence of exponential dichotomy for $\lambda < 0$.

Now fix $\lambda_1 < \sup \mathcal{I}$ and repeat step by step the proof of Theorem 5.61, with $m = m_0$. There are only two differences. The first one is that now Proposition 5.51 only ensures that $N^+(\omega, \lambda_2) \le N^+(\omega, \lambda_1) < N^-(\omega, \lambda_1) \le N^-(\omega, \lambda_2)$ if $\lambda_2 < \lambda_1$. And the second one is that the constant average matrix $\bar{\Delta} = \int_\Omega \Delta(\omega)\, dm_0$ is positive semidefinite, and not definite. But $\bar{\Delta}$ is nonzero, since the Atkinson condition precludes $\Delta \equiv 0_n$, it is continuous, and $\operatorname{Supp} m_0 = \Omega$; and the fact that it does not vanish is enough to conclude that $\mathcal{I}$ is bounded above. The rest of the arguments are identical to those used in proving Proposition 5.51.

The last result of this section is another illustration of the way in which the properties of the rotation number and the exponential dichotomy concept, and the relation among them, determine certain dynamical properties of a family of Hamiltonian systems. More precisely, it establishes conditions under which, even if both Lagrange planes lie in the Maslov cycle $\mathcal{C}$ defined by (2.35) for some values of ω, they can be approximated by Lagrange planes of perturbed systems which globally lie outside $\mathcal{C}$. This result will be used in Chap. 8.

Theorem 5.73 *Suppose that* D1 *holds: that there is a* σ*-ergodic measure* m_0 *on* Ω *with* $\operatorname{Supp} m_0 = \Omega$*; that the family* (4.2) *admits an exponential dichotomy; and that its rotation number with respect to* m_0 *is* $\alpha(m_0) = 0$*. Then there is a* $\rho > 0$*, such that the family*

$$\mathbf{z}' = H_\varepsilon(\omega{\cdot}t)\, \mathbf{z} = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) + \varepsilon I_n \\ H_2(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \quad \omega \in \Omega\,, \tag{5.61}$$

has exponential dichotomy over Ω *for* $\varepsilon \in [0, \rho)$*. Moreover, the Weyl functions* $M_\varepsilon^\pm(\omega)$ *exist globally for* $\varepsilon \in (0, \rho)$*, and*

$$M_{\varepsilon_1}^+(\omega) \le M_{\varepsilon_2}^+(\omega) < M_{\varepsilon_2}^-(\omega) \le M_{\varepsilon_1}^-(\omega)$$

whenever $0 < \varepsilon_1 < \varepsilon_2 < \rho$*.*

Proof The robustness of the exponential dichotomy provides $\rho > 0$ such that the family (5.61), which is a perturbation of (5.4), admits an exponential dichotomy for $\varepsilon \in (0, \rho)$: see e.g. Theorem 1.95. Therefore, its rotation number with respect to m_0 is zero for all $\varepsilon \in (0, \rho)$: see Theorem 3.50 and Remark 3.51.1. Since

$H_3(\omega{\cdot}t)+\varepsilon I_n>0$ for $\varepsilon>0$, the perturbed family satisfy condition D2 for $\varepsilon\in[0,\rho]$ (see Remark 5.19), and hence Theorem 5.66(ii) ensures that it is uniformly weakly disconjugate for these values of ε. Theorem 5.59 ensures the existence of the Weyl functions $M^{\pm}_{\omega}$, which agree with the uniform principal functions $N^{\pm}_{\varepsilon}$ and satisfy $M^{+}_{\varepsilon}<M^{-}_{\varepsilon}$. The comparison result given in Proposition 5.51 completes the proof.

5.8 Convergence of Sequences of Principal Functions

Consider a sequence of families of linear Hamiltonian systems

$$\mathbf{z}'=\begin{bmatrix} H_1^k(\omega{\cdot}t) & H_3^k(\omega{\cdot}t)\\ H_2^k(\omega{\cdot}t) & -(H_1^k)^T(\omega{\cdot}t)\end{bmatrix}\mathbf{z}=H^k(\omega{\cdot}t)\,\mathbf{z}\,,\quad \omega\in\Omega\,, \tag{5.62}$$

where $k\in\mathbb{N}$ and each $H^k\colon\Omega\to\mathfrak{sp}(n,\mathbb{R})$ is continuous. Theorem 11 of Chapter 2 in Coppel [34] shows that, if all these systems are disconjugate on $\mathbb{R}$, if H^k converges to H uniformly on Ω, and if $H_3>0$, then the limit systems $\mathbf{z}'=H(\omega{\cdot}t)\,\mathbf{z}$ are also disconjugate on $\mathbb{R}$. Proposition 5.69 above establishes a similar result for the weak disconjugacy case if a σ-ergodic measure with total support exists (recall that $H_3>0$ guarantees D2: see Remark 5.19). The question to be analyzed here is that of the convergence of the sequences of principal functions $N_k^{\pm}$ of (5.62) to those of the limit system, $N^{\pm}$.

The situation is trivial if the limit family is not just uniformly weakly disconjugate (as are the families (5.62)) but in addition has exponential dichotomy over Ω. In this case, and by reason of the robustness of the exponential dichotomy property described in Theorems 1.92 and 1.95, if k is large enough the principal functions $N_k^{\pm}$ are the Weyl functions of (5.62) (see Theorem 5.58), and they converge uniformly on Ω to the Weyl functions $N^{\pm}$ of the limit family.

Throughout this section, m_0 will be a fixed σ-ergodic measure on Ω. Theorem 5.74 establishes a relation between the weak convergence with respect to this measure of the principal functions and the convergence of the sequence of the corresponding Lyapunov indices. In fact they turn out to be equivalent if $H_3>0$, in which case the convergence is stronger: it holds in the $L^2(\Omega,m_0)$ topology.

The matrices whose convergence is going to be analyzed will always have the dimensions $n{\times}n$. As in Sect. 4.3, to define the space $L^2(\Omega,m_0)$ of $n{\times}n$ matrix-valued functions, the norm $\|A\|_F=\big(\mathrm{tr}(A^TA)\big)^{1/2}$ is chosen in $\mathbb{M}_{n\times n}(\mathbb{R})$, so that $\|A\|_2=\big(\int_\Omega \mathrm{tr}(A^T(\omega)\,A(\omega))\,dm_0\big)^{1/2}$: see Remark 4.22. This convenient choice of this norm does not affect the statements concerning the L^2 convergence, which are true for any other (equivalent) norm on the set of matrix-valued functions: see Remark 1.33.

Recall that if A and the elements of the sequence (A_k) belong to $L^2(\Omega,m_0)$, then $A=\lim_{k\to\infty}A_k$ in the weak topology of $L^2(\Omega,m_0)$ if

$$\int_\Omega A(\omega)\,B(\omega)\,dm_0=\lim_{k\to\infty}\int_\Omega A_k(\omega)\,B(\omega)\,dm_0$$

for every $n \times n$ matrix-valued function $B \in L^2(\Omega, m_0)$. Once again, this convergence is equivalent to componentwise convergence in the analogous topology for scalar functions, and is independent of the matrix norm chosen to define the space $L^2(\Omega, m_0)$.

Theorem 5.74, which is formulated in terms of uniform weak disconjugacy, can immediately be reformulated in terms of disconjugacy if $H_3 > 0$: see Remark 5.30. To understand the scope of point (iii) it is convenient to keep in mind that $A = \lim_{k\to\infty} A_k$ in the weak topology of $L^2(\Omega, m_0)$ at least in the following three cases: first, when $A = \lim_{k\to\infty} A_k$ in the $L^2(\Omega, m_0)$ topology; and second and third, when (A_k) is an L^2-bounded sequence of matrix-valued functions such that either $A(\omega) = \lim_{k\to\infty} A_k(\omega)$ m_0-a.e. or $A = \lim_{k\to\infty} A_k$ in measure (see Remarks 4.25). These assertions can be found in, for example, Theorems 13.42 and 13.44 and Corollary 13.45 of [58]. Recall also that the principal functions are bounded, and hence they belong to $L^2(\Omega, m_0)$.

Theorem 5.74 *Suppose that the families (5.62) satisfy D1, D2, and D3 for all $k \in \mathbb{N}$, that the sequence (H^k) converges to H uniformly on Ω, and that the limit family (5.4) also satisfies D1, D2, and D3. Denote by $N_k^{\pm}(\omega)$ and $N^{\pm}(\omega)$ the corresponding principal functions, and by β_k and β the Lyapunov indices with respect to m_0 of (5.62) and (5.4), respectively. Then,*

(i) *there exists $c > 0$ such that $\|N_k^{\pm}(\omega)\| \le c$ for all $k \in \mathbb{N}$ and $\omega \in \Omega$.*
(ii) *There exist suitable subsequences $(N_{k_j}^{\pm})$ and matrix-valued-functions $N_0^{\pm} \in L^2(\Omega, m_0)$ such that $\lim_{j\to\infty} N_{k_j}^{\pm} = N_0^{\pm}$ in the weak topology of $L^2(\Omega, m_0)$, and*

$$N^+(\omega) \le N_0^+(\omega) \le N_0^-(\omega) \le N^-(\omega) \tag{5.63}$$

for m_0-a.e. $\omega \in \Omega$.
(iii) *If $\lim_{k\to\infty} N_k^{\pm} = N^{\pm}$ in the weak topology, then $\lim_{k\to\infty} \beta_k = \beta$.*
(iv) *If $H_3 > 0$ and $\lim_{k\to\infty} \beta_k = \beta$, then $\lim_{k\to\infty} N_k^{\pm} = N^{\pm}$ in the $L^2(\Omega, m_0)$ topology.*

Proof

(i) Define $\Gamma = \begin{bmatrix} I_n & 0_n \\ 0_n & 0_n \end{bmatrix}$ and consider, for $\lambda \in \mathbb{R}$, the perturbed families

$$\mathbf{z}' = \big(H^k(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \qquad \omega \in \Omega \tag{5.64}$$

for each $k \in \mathbb{N}$, as well as

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \qquad \omega \in \Omega\,. \tag{5.65}$$

It follows from Theorem 5.61 that for each $\lambda < 0$ all these families have exponential dichotomy over Ω, that $\lim_{\lambda\to 0^-} M_k^{\pm}(\omega,\lambda) = N_k^{\pm}(\omega)$ pointwise for all $k \in \mathbb{N}$, with $M_k^+(\omega,\lambda) < N_k^+(\omega) \le N_k^-(\omega) < M_k^-(\omega,\lambda)$ for all

$k \in \mathbb{N}$ and $\lambda < 0$, and that $\lim_{\lambda\to 0^-} M^{\pm}(\omega,\lambda) = N^{\pm}(\omega)$. As usual, $M_k^{\pm}(\omega,\lambda)$ and $M^{\pm}(\omega,\lambda)$ represent the Weyl functions for the families (5.64) and (5.65). In turn, Theorem 1.95(ii) ensures that $\lim_{k\to\infty} M_k^{\pm}(\omega,\lambda) = M^{\pm}(\omega,\lambda)$ uniformly on Ω for each $\lambda < 0$. Fix such a $\lambda < 0$ and note that given $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that

$$M^{+}(\omega,\lambda) - \varepsilon I_n < N_k^{+}(\omega) \le N_k^{-}(\omega) < M^{-}(\omega,\lambda) + \varepsilon I_n \tag{5.66}$$

for $k \ge k_0(\lambda)$ and all $\omega \in \Omega$. These inequalities, the boundedness of the Weyl functions on Ω (which is ensured by their continuity), Remark 1.44.2, and the boundedness of the finite set of functions $N_k^{\pm}$ for $k \le k_0$ (established in Theorem 5.43), all taken together, imply (i).

(ii) The uniform bound for $N_k^{\pm}$ established in (i) provides a common bound for $\|N_k^{\pm}\|_2$. Therefore, there are subsequences (which can be supposed to be associated to a common subsequence (j) of the sequence of indices (k)) which converge in the weak topology to certain matrix-valued functions $N_0^{\pm} \in L^2(\Omega, m_0)$: see [136], Corollary 4.3. In addition, the weak convergence preserves the order, as explained in Remark 5.75.1 below. Therefore it follows from (5.66) that $M^{+}(\omega,\lambda) - \varepsilon I_n \le N_0^{+}(\omega) \le N_0^{-}(\omega) \le M^{-}(\omega,\lambda) + \varepsilon I_n$ for all $\varepsilon > 0$, and hence that $M^{+}(\omega,\lambda) \le N_0^{+}(\omega) \le N_0^{-}(\omega) \le M^{-}(\omega,\lambda)$. Taking now the limit as $\lambda \to 0^-$ completes the proof of (ii).

(iii) Theorem 5.56(iv) implies that

$$\begin{aligned} \beta_k &= \mp \int_\Omega \operatorname{tr}\left(H_1^k(\omega) + H_3^k(\omega)\, N_k^{\pm}(\omega)\right) dm_0\,, \\ \beta &= \mp \int_\Omega \operatorname{tr}\left(H_1(\omega) + H_3(\omega)\, N^{\pm}(\omega)\right) dm_0\,. \end{aligned} \tag{5.67}$$

Therefore,

$$\beta_k - \beta = \int_\Omega \operatorname{tr}\left(H_1^k - H_1 + (H_3^k - H_3)\, N_k\right) dm_0 + \int_\Omega \operatorname{tr}\left(H_3\,(N_k - N)\right) dm_0\,,$$

and the result follows easily from the hypotheses of (iii) and from (i).

(iv) The first step is to prove that the limits $N_0^{\pm}$ of the subsequences of $(N_{k_j}^{\pm})$ obtained in (ii) coincide with the principal functions $N^{\pm}$ if $H_3 > 0$.

Relations (5.67) ensure that

$$\begin{aligned} \beta_{k_j} &= \frac{1}{2} \int_\Omega \operatorname{tr}\left(H_3^{k_j}(\omega)\left(N_{k_j}^{-}(\omega) - N_{k_j}^{+}(\omega)\right)\right) dm_0\,, \\ \beta &= \frac{1}{2} \int_\Omega \operatorname{tr}\left(H_3(\omega)\left(N^{-}(\omega) - N^{+}(\omega)\right)\right) dm_0\,, \end{aligned}$$

so that

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \operatorname{tr}\Big(H_3^{k_j}(\omega)\Big(N_{k_j}^-(\omega)-N_{k_j}^+(\omega)\Big)\Big)\,dm_0 \\ &= \int_\Omega \operatorname{tr}\big(H_3(\omega)\big(N^-(\omega)-N^+(\omega)\big)\big)\,dm_0 \\ &= \int_\Omega \operatorname{tr}\big(H_3(\omega)\big(N_0^-(\omega)-N_0^+(\omega)\big)\big)\,dm_0\ :\end{aligned}$$

the first equality follows from the hypothesis $\beta=\lim_{k\to\infty}\beta_k$; and the second one from the uniform boundedness of $(H_3^{k_j})$ (which is deduced from the general hypotheses of the theorem) and of (N_{k_j}) (proved in (i)), and from the weak convergence of $H_3N_{k_j}^\pm$ to $H_3N_0^\pm$ (established in (ii)). Therefore,

$$\begin{aligned}\int_\Omega \operatorname{tr}\big(H_3(\omega)\big(N^-(\omega)-N_0^-(\omega)\big)\big)\,dm_0 \\ &= \int_\Omega \operatorname{tr}\big(H_3(\omega)\big(N^+(\omega)-N_0^+(\omega)\big)\big)\,dm_0\,.\end{aligned}$$

The inequalities (5.63) and the positivity of $H_3(\omega)$ yield $N^+(\omega)=N_0^+(\omega)$ and $N^-(\omega)=N_0^-(\omega)$ m_0-a.e. and, consequently, $\lim_{j\to\infty}N_{k_j}^\pm=N^\pm$ in the weak topology. The result applies to any subsequence of the initial sequence, and the limit is common to all subsequences, so that

$$\lim_{k\to\infty}N_k^\pm(\omega)=N^\pm(\omega)\quad \text{in the weak topology of } L^2(\Omega,m_0)\,. \tag{5.68}$$

The functions $N_k^\pm(\omega)$ and $N^\pm(\omega)$ are bounded solutions along the flow of the Riccati equations

$$M'=-MH_3^k(\omega{\cdot}t)M-MH_1^k(\omega{\cdot}t)-(H_1^k)^T(\omega{\cdot}t)M+H_2^k(\omega{\cdot}t)\,,$$

and so (5.7), Proposition 1.36 and (5.68) imply, reasoning as before, that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \operatorname{tr}\big((N_k^\pm H_3^kN_k^\pm)(\omega)\big)\,dm_0 \\ &= \lim_{k\to\infty}\int_\Omega \operatorname{tr}\big((-N_k^\pm H_1^k-(H_1^k)^TN_k^\pm+H_2^k)(\omega)\big)\,dm_0 \\ &= \int_\Omega \operatorname{tr}\big((-N^\pm H_1-(H_1)^TN^\pm+H_2)(\omega)\big)\,dm_0 \\ &= \int_\Omega \operatorname{tr}\big((N^\pm H_3N^\pm)(\omega)\big)\,dm_0\,;\end{aligned}$$

or, in other words, that

$$\lim_{k\to\infty} \|(H_3^k)^{1/2} N_k^{\pm}\|_2 = \|H_3^{1/2} N^{\pm}\|_2 \,. \tag{5.69}$$

On the other hand, it follows from (5.68), from the uniform boundedness of $(N_k^{\pm})$ established in (i), and from the uniform convergence of (H_3^k) to H_3, that $((H_3^k)^{1/2} N_k^{\pm})$ converge to $H_3^{1/2} N^{\pm}$ in the weak topology. This fact together with (5.69) ensures that the convergence holds, in fact, in the $L^2(\Omega, m_0)$-topology: see Remark 5.75.2 below. Finally,

$$\begin{aligned}\|N_k^{\pm} - N^{\pm}\|_2 \le \|H_3^{-1/2}(H_3^{1/2} - (H_3^k)^{1/2})\, N_k^{\pm}\|_2 \\ + \|H_3^{-1/2}((H_3^k)^{1/2} N_k^{\pm} - H_3^{1/2} N^{\pm})\|_2 \,,\end{aligned}$$

which combined with the previous L^2-convergence and, again, the uniform convergence of (H_3^k) to H_3 and the uniform boundedness of $(N_k^{\pm})$ are sufficient to prove that the sequences $(N_k^{\pm})$ converge to $N^{\pm}$ in the $L^2(\Omega, m_0)$-topology.

Remarks 5.75

1. Suppose that $n = 1$ and $f = \lim_{k\to\infty} f_k$ in the weak topology. Then, if $f_k \ge 0$, so is f; that is, the set $\Omega_0 = \{\omega \in \Omega \mid f(\omega) \ge 0\} \subset \Omega$ has full measure for m_0:

$$0 \le \lim_{k\to\infty} \int_\Omega f_k(\omega)\, \chi_{\Omega-\Omega_0}(\omega)\, dm_0 = \int_\Omega f(\omega)\, \chi_{\Omega-\Omega_0}(\omega)\, dm_0 \le 0\,,$$

so that $m_0(\Omega - \Omega_0) = 0$. The analogous result for $n \times n$ matrix-valued functions follows from this statement, as is now explained. Suppose that $A = \lim_{k\to\infty} A_k$ in the weak topology, and assume that $A_k \ge 0$. Let $\mathbf{x} \in \mathbb{Q}^n$ be a vector with only rational components. It can immediately be checked that the sequence of scalar nonnegative functions $(\mathbf{x}^T A_k \mathbf{x})$ converges to $\mathbf{x}^T A\, \mathbf{x}$ in the weak topology. Therefore there exists a subset $\Omega_{\mathbf{x}} \subseteq \Omega$ of full m_0 measure such that $\mathbf{x}^T A(\omega)\, \mathbf{x} \ge 0$ for all $\omega \in \Omega_{\mathbf{x}}$. The assertion follows easily from the equality $m_0(\cap_{\mathbf{x}\in\mathbb{Q}^n} \Omega_{\mathbf{x}}) = 1$ (since $\mathbb{Q}^n$ is countable) and from the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$.
2. If $A = \lim_{k\to\infty} A_k$ in the weak topology and $\|A\|_2 = \lim_{k\to\infty} \|A_k\|_2$, then

$$\begin{aligned}\lim_{k\to\infty} \|A - A_k\|_2^2 &= \lim_{k\to\infty} \int_\Omega \operatorname{tr}(A - A_k)^T (A - A_k)\, dm_0 \\ &= \lim_{k\to\infty} \int_\Omega (\operatorname{tr}(A^T A) + \operatorname{tr}(A_k^T A_k) - \operatorname{tr}(A^T A_k) - \operatorname{tr}(A_k^T A))\, dm_0 \\ &= \int_\Omega (\operatorname{tr}(A^T A) + \operatorname{tr}(A^T A) - \operatorname{tr}(A^T A) - \operatorname{tr}(A^T A))\, dm_0 = 0\,,\end{aligned}$$

so that $\lim_{k\to\infty} A_k = A$ in the $L^2(\Omega, m_0)$ topology.

5.9 Abnormal Linear Hamiltonian Systems

The last section of this chapter is devoted to a study of the index or order of abnormality of linear Hamiltonian systems, and of some topological and ergodic properties of families of systems containing abnormal systems. Roughly speaking, one of the systems of the family (5.4) is abnormal on a half-line if it has solutions defining τ-semiorbits which lie in $\Omega \times \mathcal{C}$, where $\mathcal{C}$ is the vertical Maslov cycle defined as $\mathcal{L}_{\mathbb{R}} - \mathcal{D}$ for $\mathcal{D}$ given by (5.5). Note that this situation is not compatible with the notion of identical normality: see Definition 5.28. In fact it is not even compatible with the less restrictive notion of disconjugacy, as explained below.

The main results of this section are Theorems 5.80 and 5.85. The first one states, among other things, that the index of abnormality defines a semicontinuous σ-invariant function $d\colon \Omega \to \{0, 1, \ldots, n\}$, which is locally constant on the residual invariant set Ω_c of its continuity points. Also there is always at least one point in Ω_c at which d assumes its minimum value. Moreover, in the case that there exists a σ-ergodic measure whose support is all of Ω, one has that Ω_c actually agrees with the set on which d assumes its minimum value. In the general case, d is constant on each minimal subset of Ω; and, in the case that $d \not\equiv 0$ (i.e. in the case of existence of abnormal systems in the family), the maximum value is attained in certain minimal subsets of Ω.

Theorem 5.85 considers the case of $d \not\equiv 0$ when in addition the family admits an exponential dichotomy. Among other properties, it is shown that at least one of the Lagrange planes must intersect the Maslov cycle $\mathcal{C}$. In fact, for any minimal subset $\mathcal{M} \subset \Omega$, one of the τ-invariant subsets $\{(\omega, l^+(\omega)) \mid \omega \in \mathcal{M}\}$, $\{(\omega, l^-(\omega)) \mid \omega \in \mathcal{M}\}$ of $\Omega \times \mathcal{L}_{\mathbb{R}}$ is contained in $\Omega \times \mathcal{C}$.

A final consequence of the previous results consists of a list of properties formulated in terms of the different functions introduced in the previous analysis which turn out to be equivalent to property D2, which, as already mentioned, it is also equivalent to the uniform null controllability of the family of linear control systems $\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u}$: see Remark 5.22.

To begin with the analysis, define

$$\Lambda^+(\omega) = \left\{\mathbf{z}_0 \in \mathbb{R}^{2n} \mid U(t,\omega)\,\mathbf{z}_0 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right] \text{ for } t \text{ in a positive half-line}\right\},$$

$$\Lambda^-(\omega) = \left\{\mathbf{z}_0 \in \mathbb{R}^{2n} \mid U(t,\omega)\,\mathbf{z}_0 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right] \text{ for } t \text{ in a negative half-line}\right\},$$

$$\Lambda(\omega) = \left\{\mathbf{z}_0 \in \mathbb{R}^{2n} \mid U(t,\omega)\,\mathbf{z}_0 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right] \text{ for } t \in \mathbb{R}\right\}.$$

Proposition 5.76

(i) *The sets $\Lambda^+(\omega)$, $\Lambda^-(\omega)$ and $\Lambda(\omega)$ are vector subspaces of $\mathbb{R}^{2n}$.*

(ii) *Let $\omega \in \Omega$ be fixed and let $l_v \equiv \left[\begin{smallmatrix} 0_n \\ I_n\end{smallmatrix}\right]$ be the vertical Lagrange plane. Then $U(t,\omega)\cdot\Lambda(\omega) \subseteq l_v$ for any $t \in \mathbb{R}$, and there exist real values $a^+(\omega)$ and $a^-(\omega)$ such that $U(t,\omega)\cdot\Lambda^+(\omega) \subseteq l_v$ for all $t \geq a^+(\omega)$ and $U(t,\omega)\cdot\Lambda^-(\omega) \subseteq l_v$ for all $t \leq a^-(\omega)$. In particular, the dimension of each of the three vector spaces is at most n.*

(iii) *For all* $t \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$U(t,\omega)\cdot \Lambda^{\pm}(\omega) = \Lambda^{\pm}(\omega{\cdot}t) \quad \textit{and} \quad U(t,\omega)\cdot \Lambda(\omega) = \Lambda(\omega{\cdot}t)\,.$$

Proof

(i) This assertion is almost trivial.

(ii) The assertions are obvious for $\Lambda(\omega)$. To analyze the case of $\Lambda^+(\omega)$, let $\{\mathbf{z}_1,\dots,\mathbf{z}_m\}$ be one if its bases, and choose a time $a\in\mathbb{R}$ large enough to ensure that $U(t,\omega)\,\mathbf{z}_1 = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2^1(t)\end{smallmatrix}\right],\dots,U(t,\omega)\,\mathbf{z}_m = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2^m(t)\end{smallmatrix}\right]$ for all $t\ge a$. This proves that $m\le n$ and $U(t,\omega)\cdot\Lambda^+(\omega)\subseteq l_v$ for all $t\ge a$. The proof is analogous for $\Lambda^-(\omega)$.

(iii) If $U(t,\omega)\,\mathbf{z} = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ for all $t\in[a^+(\omega),\infty)$, then $U(s,\omega{\cdot}t)\,U(t,\omega)\,\mathbf{z} = U(s+t,\omega)\,\mathbf{z} = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(s+t)\end{smallmatrix}\right]$ for all $s\in[a^+(\omega)-t,\infty)$. This ensures that $U(t,\omega)\cdot\Lambda^+(\omega)\subseteq\Lambda^+(\omega{\cdot}t)$ for all $t\in\mathbb{R}$ and $\omega\in\mathbb{R}$. Consequently, $\Lambda^+(\omega{\cdot}t) = U(t,\omega)\,U(-t,\omega{\cdot}t)\cdot\Lambda^+(\omega{\cdot}t)\subseteq U(t,\omega)\cdot\Lambda^+(\omega)$, so that the equality is proved for the case of Λ^+. The other two cases are handled analogously.

Now define the functions

$$d^+(\omega) = \dim\Lambda^+(\omega)\,,\quad d^-(\omega) = \dim\Lambda^-(\omega)\,,\quad \text{and}\quad d(\omega) = \dim\Lambda(\omega)\,,$$

which take values in $\{0,\dots,n\}$ and satisfy

$$d(\omega)\le\min(d^+(\omega),d^-(\omega))\,. \tag{5.70}$$

Definition 5.77 The system (5.4) corresponding to $\omega\in\Omega$ is *abnormal at* $+\infty$ if $d^+(\omega)>0$. The integer $d^+(\omega)$ is the *index* or *order of abnormality* of the system *at* $+\infty$.

The system (5.4) is *abnormal at* $-\infty$ if $d^-(\omega)>0$. The integer $d^-(\omega)$ is the *index* or *order of abnormality* of the system *at* $-\infty$.

The system (5.4) is *abnormal* if $d(\omega)>0$. The integer $d(\omega)$ is the *index* or *order of abnormality* of the system.

Examples 5.78 The simplest example of an abnormal system is the autonomous one $\mathbf{z}' = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\mathbf{z}$, with $d^+=d^-=d=1$. In fact it is an easy exercise to construct autonomous $2n$-dimensional Hamiltonian systems with $d^+=d^-=d=k$ for any $k\in\{0,1,\dots,n\}$. By defining $a(t)$ as a nonincreasing continuous function taking the value 1 on $(-\infty,0]$ and 0 on $[1,\infty)$, one obtains the nonautonomous system $\mathbf{z}' = \left[\begin{smallmatrix}0&a(t)\\0&0\end{smallmatrix}\right]\mathbf{z}$, for which $d^+=d=0$ and $d^-=1$. Note that here only the initial system is considered, not the family of systems over the hull. In this regard, see Example 5.81 below.

Remark 5.79 Note that abnormality at $+\infty$ (resp. at $-\infty$) of the system corresponding to a point $\omega\in\Omega$ means that this system has at least one nontrivial solution of the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ in a positive half-line $[a,\infty)$ (resp. in a negative half-line $(-\infty,a]$). Hence, on the one hand, the system cannot be disconjugate: see

Definition 5.1. And, on the other hand, the system corresponding to the point $\omega{\cdot}a$ admits a solution of the form $\begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2(t) \end{bmatrix}$ on $[0,\infty)$ (resp. on $(-\infty,0]$), so that is not weakly disconjugate on $[0,\infty)$ (resp. on $(-\infty,0]$: see Definition 5.2. Therefore, if D1 holds and one of the systems of the family is abnormal at $+\infty$ or at $-\infty$, then the family is not uniformly weakly disconjugate: see Definition 5.14 and Theorem 5.17. However, it is still possible to develop a theory of principal solutions in this context: the reader is referred to Reid [127] and Kratz [92] for a detailed study of abnormal systems, and to Reid [124] and Šepitka and Šimon Hilscher [141, 142] for the definition and analysis of the corresponding principal solutions.

Consider now the whole family (5.4). Define

$$\begin{aligned} d_M^{\pm} &= \max_{\omega\in\Omega} d^{\pm}(\omega)\,, \quad & d_m^{\pm} &= \min_{\omega\in\Omega} d^{\pm}(\omega)\,, \\ d_M &= \max_{\omega\in\Omega} d(\omega)\,, \quad & d_m &= \min_{\omega\in\Omega} d(\omega)\,, \end{aligned} \tag{5.71}$$

and note that (5.70) yields

$$d_m \le \min(d_m^+, d_m^-) \quad \text{and} \quad d_M \le \min(d_M^+, d_M^-)\,. \tag{5.72}$$

The next result collects some fundamental properties of these functions and quantities. Recall that $\mathcal{O}(\omega)$ and $\mathcal{A}(\omega)$ represent the omega and alpha-limit sets of the point ω for the base flow (Ω,σ).

Theorem 5.80

(i) *The functions d^+, d^-, and d are σ-invariant on Ω.*

(ii) *If $\omega\in\mathcal{O}(\omega_0)$, then $d^+(\omega_0)\le d(\omega)$; and if $\omega\in\mathcal{A}(\omega_0)$, then $d^-(\omega_0)\le d(\omega)$.*

(iii) *If m_0 is a σ-ergodic measure on Ω, then the functions d^+, d^-, and d are constant and coincide for m_0-a.e. $\omega\in\Omega$.*

(iv) *If $\mathcal{M}\subseteq\Omega$ is a minimal set, then the functions d^+, d^- and d are constant and coincide on $\mathcal{M}$. Hence, if Ω is minimal, then $d_M^{\pm}=d_m^{\pm}=d_M=d_m$.*

(v) *$d_M^{\pm}=d_M$, and there exists a minimal subset $\mathcal{M}\subseteq\Omega$ such that $d(\omega)=d^{\pm}(\omega)=d_M$ for each $\omega\in\mathcal{M}$.*

(vi) *The function d is upper semicontinuous, and the set of its continuity points is an open residual invariant subset $\Omega_c\subseteq\Omega$ on which d is locally constant, with*

$$\{\omega\in\Omega\mid d(\omega)=d_m\}\subseteq\Omega_c\,.$$

(vii) *If there exists a point $\omega_0\in\Omega$ with dense positive and negative semiorbits, then $d_m=d_m^{\pm}=d(\omega_0)=d^{\pm}(\omega_0)$. In particular, $\omega_0\in\Omega_c$.*

(viii) *If m_0 is a σ-ergodic measure on Ω with* $\operatorname{Supp} m_0=\Omega$, *then $d_m^{\pm}=d_m$, and there exists a subset $\Omega_1\subseteq\Omega$ with $m(\Omega_1)=1$ such that $d(\omega)=d^{\pm}(\omega)=d_m$ for all $\omega\in\Omega_1$. In particular, $m_0(\Omega_c)=1$. In addition, in this case, $\Omega_c=\{\omega\in\Omega\mid d(\omega)=d_m\}$.*

Proof

(i) These properties follow immediately from Proposition 5.76(iii), since $U(t,\omega)$ is a homeomorphism for all $(t,\omega) \in \mathbb{R} \times \Omega$.

(ii) Only the property that $d^+(\omega_0) \le d(\omega)$ if $\omega \in \mathcal{O}(\omega_0)$ will be proved: the second inequality can be checked in a completely analogous way.

The inequality is obvious if $d^+(\omega_0) = 0$. So, assume that $d^+(\omega_0) = d_0 > 0$. Since $\omega \in \mathcal{O}(\omega_0)$, there is a sequence $(t_m) \uparrow \infty$ with $\omega = \lim_{m\to\infty} \omega_0{\cdot}t_m$. It follows from (i) that $(\Lambda^+(\omega_0{\cdot}t_m))$ is a sequence in the compact manifold $\mathcal{G}_{d_0}(\mathbb{R}^{2n})$, so that it has a suitable convergent subsequence, say $(\Lambda^+(\omega_0{\cdot}t_j))$, with limit Λ. Clearly, it is enough to prove that $\Lambda \subseteq \Lambda(\omega)$. Recall that $\Lambda^+(\omega_0{\cdot}t_j) = U(t_j,\omega_0) \cdot \Lambda^+(\omega_0)$. Now choose $\mathbf{z} \in \Lambda$ and write it as $\mathbf{z} = \lim_{j\to\infty} U(t_j,\omega_0) \cdot \mathbf{z}_j$ with $\mathbf{z}_j \in \Lambda^+(\omega_0)$ (see Proposition 1.26(i)). Then

$$U(t,\omega)\,\mathbf{z} = \lim_{j\to\infty} U(t,\omega_0{\cdot}t_j)\,U(t_j,\omega_0)\,\mathbf{z}_j = \lim_{j\to\infty} U(t+t_j,\omega_0)\,\mathbf{z}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{bmatrix},$$

since $t + t_j > a^+(\omega_0)$ for large enough j, with $a^+(\omega_0)$ provided by Proposition 5.76(ii). This means that $\mathbf{z} \in \Lambda(\omega)$ and completes the proof.

(iii) The constant character of d^+, d^-, and d with respect to the σ-ergodic measure m_0 follows from the σ-invariance proved in (i) (see Theorem 1.6). In addition, the Poincaré Recurrence Theorem (see e.g. Theorem 1 of Chapter 1 of [35]) provides a subset Ω_0 with $m(\Omega_0) = 1$ such that $\omega \in \mathcal{O}(\omega) \cap \mathcal{A}(\omega)$ for all $\omega \in \Omega_0$. It follows from (5.70) and point (ii) above that $d(\omega) \le d^+(\omega) \le d(\omega) \le d^-(\omega) \le d(\omega)$ if $\omega \in \Omega_0$, and this completes the proof of (iii).

(iv) Note that $\mathcal{M} = \mathcal{A}(\omega) = \mathcal{O}(\omega)$ for all $\omega \in \mathcal{M}$. Therefore, using again (5.70) and (ii), one has that $d(\omega_1) \le d^\pm(\omega_1) \le d(\omega_2) \le d^\pm(\omega_2) \le d(\omega_1)$ for ω_1 and ω_2 in $\mathcal{M}$, and hence these three functions agree on $\mathcal{M}$. The last statement of (iv) follows immediately from this fact and from the definitions (5.71).

(v) Let $\omega_0 \in \Omega$ be such that $d^+(\omega_0) = d_M^+$. It follows from (5.72) and (ii) that $d_M \le d_M^+ = d^+(\omega_0) \le d(\omega) \le d_M$ for all $\omega \in \mathcal{O}(\omega_0)$, and hence, $d_M = d_M^+$. Analogously, if $d^-(\omega_0) = d_M^-$, then $d_M \le d_M^- = d^-(\omega_0) \le d(\omega) \le d_M$ for all $\omega \in \mathcal{A}(\omega_0)$, and consequently $d_M^- = d_M$. That is, $d_M^\pm = d_M$, as asserted. Take now a minimal subset $\mathcal{M} \subseteq \mathcal{O}(\omega_0)$ in order to conclude from the previous inequalities and (iv) that $d(\omega) = d^\pm(\omega) = d_M$ for all $\omega \in \mathcal{M}$.

(vi) The first goal is to check that $\lim_{m\to\infty} d(\omega_m) \le d(\omega_0)$ whenever this limit exists for a sequence (ω_m) with limit ω_0. Since $0 \le d(\omega_m) \le n$ and d takes only integer values, there is no loss of generality in assuming that $d(\omega_m) = d_0$ for each $m \in \mathbb{N}$. Now take a subsequence (ω_j) such that the sequence $(\Lambda(\omega_j))$ converges to a vector space Λ_0 in the compact manifold $\mathcal{G}_{d_0}(\mathbb{R}^{2n})$. The continuity of the flow τ_{d_0} on $\Omega \times \mathcal{G}_{d_0}(\mathbb{R}^{2n})$ implies that $U(t,\omega_0) \cdot \Lambda_0 = \lim_{j\to\infty} U(t,\omega_j){\cdot}\Lambda(\omega_j)$ for any $t \in \mathbb{R}$. Since $U(t,\omega_j){\cdot}\Lambda(\omega_j) \subseteq l_v$ for any $t \in \mathbb{R}$ (see Proposition 5.76(ii)), then also the limit is contained in l_v. This means that $\Lambda_0 \subseteq \Lambda(\omega_0)$ and hence that $d_0 \le d(\omega_0)$. The upper semicontinuity of d is proved.

Let $\Omega_c \subseteq \Omega$ be the residual set of continuity points of the (upper semicontinuous) function d, and take $\omega \in \Omega_c$. Since d takes only integer values, there is an open neighborhood of ω in Ω_c on which d is constant. This implies that Ω_c is open and that d is locally constant on it. In addition, property (i) and the continuity of the base flow ensure that Ω_c is invariant. It remains to check that each $\omega \in \Omega$ with $d(\omega) = d_m$ is a continuity point. Let (ω_k) be a sequence with limit ω. It follows from the definition of d_m and the upper semicontinuity of d that

$$d_m \le \liminf_{k\to\infty} d(\omega_k) \le \limsup_{k\to\infty} d(\omega_k) \le d(\omega) = d_m \,,$$

and hence that there exists $\lim_{k\to\infty} d(\omega_k) = d(\omega)$. The proof of (vi) is complete.

(vii) Take such a point ω_0. Since $\mathcal{O}(\omega_0) = \mathcal{A}(\omega_0) = \Omega$, it follows from (5.70), (5.72), and (ii) that $d_m \le d_m^{\pm} \le d^{\pm}(\omega_0) \le d(\omega) \le d^{\pm}(\omega)$ for all $\omega \in \Omega$. Taking $\omega = \omega_0$ shows that $d(\omega_0) = d^{\pm}(\omega_0)$, and taking the infimum for $\omega \in \Omega$ in this chain of inequalities shows that $d_m = d_m^{\pm} = d^{\pm}(\omega_0)$. The last assertion follows from (vi).

(viii) If $\operatorname{Supp} m = \Omega$, then there exists a subset $\Omega_1 \subseteq \Omega$ with $m(\Omega_1) = 1$ such that the positive and negative semiorbit of each $\omega_1 \in \Omega_1$ is dense in Ω: see Proposition 1.12. Therefore, the first assertions in (viii) follow from (vii). Concerning the set Ω_c note, first, that according to (vi), $\Omega_1 \subseteq \Omega_c$, so that $m(\Omega_c) = 1$; and, second, that if $\omega_1 \in \Omega_1$ and $\omega \in \Omega_c$ then there exists $(t_m) \uparrow \infty$ with $\lim_{m\to\infty} \omega_1{\cdot}t_m = \omega$, and hence the invariance of d proved in (i) together with (vii) ensures that $d(\omega) = \lim_{m\to\infty} d(\omega_1{\cdot}t_m) = \lim_{m\to\infty} d(\omega_1) = d(\omega_1) = d_m$.

Example 5.81 Consider again the nonautonomous system described in Examples 5.78. The hull of the initial coefficient matrix is

$$\Omega = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 0 & a_s(t) \\ 0 & 0 \end{bmatrix} \mid s \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\},$$

where $a_s(t) = a(t+s)$ and $a\colon \mathbb{R} \to \mathbb{R}$ is a nonincreasing continuous function agreeing with 1 at $(-\infty, 0]$ and with 0 at $[1, \infty)$. Identify Ω with $[-\infty, \infty]$ by associating the indices $-\infty$ to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, s to $\left[\begin{smallmatrix} 0 & a_s(t) \\ 0 & 0 \end{smallmatrix}\right]$ and ∞ to $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $d(s) = 0$ for $s \in (-\infty, \infty]$, while $d(-\infty) = 1$. That is, d is a discontinuous function, and it reaches its maximum on the minimal set $\{-\infty\} \subseteq \Omega$. Note that $\{\infty\}$ is also minimal, which does not, however, imply that the maximum of d is reached on it. Note that the set Ω_c of continuity points agree with the set at which d attains its minimum. The computation of $d^{\pm}$ for the different values of s is also a very easy exercise.

Note that Theorem 5.80(iv) and Proposition 5.76(iii) mean that the sets

$$\Lambda = \{(\omega, \mathbf{z}) \mid \mathbf{z} \in \mathcal{M} \text{ and } \mathbf{z} \in \Lambda(\omega)\}\,,$$
$$\Lambda^{\pm} = \{(\omega, \mathbf{z}) \mid \mathbf{z} \in \mathcal{M} \text{ and } \mathbf{z} \in \Lambda^{\pm}(\omega)\}\,,$$

define τ-invariant closed vector subbundles over each minimal subset $\mathcal{M} \subseteq \Omega$: see Definitions 1.63 and Remark 1.64.

The next goal is to study the number of independent solutions of the system (5.4) corresponding to a given point $\omega \in \Omega$ which take the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right]$ in a positive or negative half-line or in the full line, and with initial data in a given subspace $l \in \mathcal{L}_{\mathbb{R}}$. To this end, define

$$d^{\pm}\colon \mathcal{K}_{\mathbb{R}} \to \{0, \dots, n\}\,, \quad (\omega, l) \mapsto \dim(\Lambda^{\pm}(\omega) \cap l)\,,$$
$$d\colon \mathcal{K}_{\mathbb{R}} \to \{0, \dots, n\}\,, \quad (\omega, l) \mapsto \dim(\Lambda(\omega) \cap l)\,.$$

The following lemma provides a relation between $d(\omega)$ and $d(\omega, l)$, as well as between $d^{\pm}(\omega)$ and $d^{\pm}(\omega, l)$.

Lemma 5.82 *Let $\omega \in \Omega$ be fixed.*

(i) *If $k \in \{0, \dots, d^{+}(\omega)\}$, then there exists $l \in \mathcal{L}_{\mathbb{R}}$ such that $d^{+}(\omega, l) = k$.*
(ii) *If $k \in \{0, \dots, d^{-}(\omega)\}$, then there exists $l \in \mathcal{L}_{\mathbb{R}}$ such that $d^{-}(\omega, l) = k$.*
(iii) *If $k \in \{0, \dots, d(\omega)\}$, then there exists $l \in \mathcal{L}_{\mathbb{R}}$ such that $d(\omega, l) = k$.*

Proof

(i) The statement is trivial if $d^{+}(\omega) = 0$, so assume that $d^{+}(\omega) > 0$. First, take $k \in \{0, \dots, d^{+}(\omega)\}$ with $0 < k < n$. According to Proposition 5.76(ii), $U(a^{+}(\omega), \omega) \cdot \Lambda^{+}(\omega) = \langle \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2^1 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2^{d^{+}(\omega)} \end{smallmatrix}\right] \rangle$ for linearly independent vectors $\mathbf{z}_2^1, \dots, \mathbf{z}_2^{d^{+}(\omega)} \in \mathbb{R}^n$. Let $\{\mathbf{w}_1, \dots, \mathbf{w}_{n-k}\}$ be a basis of the subspace orthogonal to$\langle \mathbf{z}_2^1, \dots, \mathbf{z}_2^k \rangle$ in $\mathbb{R}^n$. It can immediately be checked that $l^{+} \equiv \left[\begin{smallmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{w}_1 & \cdots & \mathbf{w}_{n-k} \\ \mathbf{z}_2^1 & \cdots & \mathbf{z}_2^k & \mathbf{0} & \cdots & \mathbf{0} \end{smallmatrix}\right]$ belongs to $\mathcal{L}_{\mathbb{R}}$. In addition, if $\mathbf{z} \in U(a^{+}(\omega), \omega) \cdot \Lambda^{+}(\omega)$, then $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$, which ensures that $l^{+} \cap (U(a^{+}(\omega), \omega) \cdot \Lambda^{+}(\omega)) = \langle \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2^1 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2^k \end{smallmatrix}\right] \rangle$. Define now $l = U^{-1}(a^{+}(\omega), \omega) \cdot l^{+}$ and note that $\dim(l \cap \Lambda^{+}(\omega)) = \dim(l^{+} \cap (U(a^{+}(\omega), \omega) \cdot \Lambda^{+}(\omega))) = k$. This proves the result when $k \neq 0, n$. For $k = 0$, take $l \equiv U^{-1}(a^{+}(\omega), \omega) \left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$; and for $k = n$ (which requires one to assume that $d^{+}(\omega) = n$), take $l \equiv U^{-1}(a^{+}(\omega), \omega) \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$.

(ii) & (iii) The proof of (ii) is completely analogous to that of (i), and the proof of (iii) is simpler: just substitute $a^{+}(\omega)$ by 0 in the previous argument.

In particular, the maxima of d and $d^{\pm}$ on $\mathcal{K}_{\mathbb{R}}$ coincide with the maxima of d and $d^{\pm}$ on Ω, as defined on (5.71). In other words,

$$d_M = \max_{(\omega,l) \in \mathcal{K}_{\mathbb{R}}} d(\omega, l) \quad \text{and} \quad d_M^{\pm} = \max_{(\omega,l) \in \mathcal{K}_{\mathbb{R}}} d^{\pm}(\omega, l)\,.$$

Note also that the minimum of d on $\mathcal{K}_{\mathbb{R}}$ is 0.

Proposition 5.83

(i) *The functions d^+, d^-, and d are τ-invariant on $\mathcal{K}_\mathbb{R}$.*
(ii) *If $(\omega, l) \in \mathcal{O}(\omega_0, l_0)$, then $d^+(\omega_0, l_0) \leq d(\omega, l)$; and if $(\omega, l) \in \mathcal{A}(\omega_0, l_0)$, then $d^-(\omega_0, l_0) \leq d(\omega, l)$.*
(iii) *If μ is a τ-ergodic measure on $\mathcal{K}_\mathbb{R}$, then the functions d^+, d^-, and d are constant and coincide for μ-a.e. $(\omega, l) \in \mathcal{K}_\mathbb{R}$.*
(iv) *If $\mathcal{K} \subseteq \mathcal{K}_\mathbb{R}$ is a minimal set, then the functions d^+, d^-, and d are constant and coincide on $\mathcal{K}$.*
(v) *$d_M^\pm = d_M$, and there exists a minimal subset $\mathcal{K} \subseteq \mathcal{K}_\mathbb{R}$ such that $d(\omega, l) = d^\pm(\omega, l) = d_M$ for each $(\omega, l) \in \mathcal{K}$.*
(vi) *The function d is upper semicontinuous on $\mathcal{K}_\mathbb{R}$. In addition, for each $k \in \{0, \ldots, d_M\}$, the sets*

$$\mathcal{L}_k = \{(\omega, l) \in \mathcal{K}_\mathbb{R} \mid d(\omega, l) \geq k\}$$

are closed, and $\mathcal{L}_k - \mathcal{L}_{k+1} = \{(\omega, l) \in \mathcal{K}_\mathbb{R} \mid d(\omega, l) = k\}$ is an open residual and dense set in $\mathcal{L}_k$, which coincides with the set of continuity points of d in $\mathcal{L}_k$.

Proof

(i) In the case of d^+, (i) is due to the equality $\Lambda^+(\omega{\cdot}t) \cap (U(t, \omega){\cdot}l) = U(t, \omega){\cdot}(\Lambda^+(\omega) \cap l)$, which in turn follows from Proposition 5.76(iii) since $U(t, \omega)$ defines a homeomorphism for all $(t, \omega) \in \mathbb{R} \times \Omega$. The other two cases are handled analogously.
(ii) Suppose that $\lim_{m\to\infty}(\omega_0{\cdot}t_m, U(t_m, \omega_0){\cdot}l_0) = (\omega, l)$ in $\mathcal{L}_\mathbb{R}$ and, in addition, $\lim_{m\to\infty} \Lambda^+(\omega_0{\cdot}t_m) \cap (U(t_m, \omega_0){\cdot}l_0) = \Lambda$ in $\mathcal{G}_{d^+(\omega_0,l)}(\mathbb{R}^{2n})$. Then $\Lambda \subseteq \lim_{m\to\infty} U(t_m, \omega_0){\cdot}l_0 = l$ and, as seen in the proof of Theorem 5.80(ii), $\Lambda \subseteq \lim_{m\to\infty} \Lambda^+(\omega_0{\cdot}t_m) \subseteq \Lambda^+(\omega)$. From here property (ii) follows in the first case, and the second case is treated analogously.
(iii), (iv) & (v) The proofs of these properties are identical to the corresponding ones of Theorem 5.80.
(vi) The proof of the upper semicontinuity of d on $\mathcal{K}_\mathbb{R}$ can be carried out by arguing as in the proof of Theorem 5.80(vi), using the idea explained in the above point (ii). This upper semicontinuity property ensures that $\mathcal{L}_k$ is closed, so that $\mathcal{L}_k - \mathcal{L}_{k+1}$ is open in $\mathcal{L}_k$.

The hardest point in this proof is to prove the density of $\mathcal{L}_k - \mathcal{L}_{k+1}$, which is postponed for now. For the time being, assume that the density holds, and note that hence the residual set of continuity points of d in $\mathcal{L}_k$ (which is τ-invariant due to (i)) is necessarily contained in $\mathcal{L}_k - \mathcal{L}_{k+1}$. It remains to check that each $(\omega, l) \in \mathcal{L}_k$ with $d(\omega, l) = k$ is a continuity point of d. Let $((\omega_m, l_m))$ be a sequence in $\mathcal{L}_k$ with limit

(ω, l). It follows from the definition of $\mathcal{L}_k$ and the upper semicontinuity of d that

$$k \le \liminf_{m\to\infty} d(\omega_m, l_m) \le \limsup_{m\to\infty} d(\omega_m, l_m) \le d(\omega, l) = k\,,$$

and hence that there exists $\lim_{j\to\infty} d(\omega, l_j) = k$, as asserted.

In order to check the density of $\mathcal{L}_k - \mathcal{L}_{k+1}$, note that the case $k = n$ is trivial and assume therefore that $0 \le k < n$. As a first case take a point $(\omega, l) \in \mathcal{K}_\mathbb{R}$ with $d(\omega, l) = k + 1$. Write $l \equiv \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{z}_1^{k+2} & \cdots & \mathbf{z}_1^d & \mathbf{z}_1^{d+1} & \cdots & \mathbf{z}_1^n \\ \mathbf{z}_2^1 & \cdots & \mathbf{z}_2^{k+1} & \mathbf{z}_2^{k+2} & \cdots & \mathbf{z}_2^d & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$, where $2 \le k + 2 \le d \le n$ and $\mathbf{z}_1^j \ne \mathbf{0}$ for each $j = k + 2, \ldots, n$, and where the vectors $\mathbf{z}_2^1, \ldots, \mathbf{z}_2^d$ are linearly independent. (Note that the number of $\mathbf{0}$'s is 1 in the upper $n \times n$ matrix if $k = 0$, and 0 in the lower one if $d = n$.) Choose a vector $\widetilde{\mathbf{z}}_1^{k+1} \ne \mathbf{0}$ satisfying the following two conditions: it belongs to the subspace orthogonal to that generated by $\{\mathbf{z}_2^1, \ldots, \mathbf{z}_2^d\} - \{\mathbf{z}_2^{k+1}\}$ (which has dimension $n - d + 1$); and it does not belong to the subspace generated by $\{\mathbf{z}_1^{d+1}, \ldots, \mathbf{z}_1^n\}$ (note that this means nothing if $d = n$).

Then $\widetilde{\mathbf{z}}_1^{k+1}$ cannot be a linear combination of the vectors $\mathbf{z}_1^{k+2}, \ldots, \mathbf{z}_1^n$. To prove this, note that obviously nothing must be checked if $k = n - 1$. Assume for contradiction that $\widetilde{\mathbf{z}}_1^{k+1} = \sum_{j=k+2}^n \lambda_j \mathbf{z}_1^j$, and suppose for simplicity that $\lambda_{k+2} \ne 0$. Then l can be represented by $\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \widetilde{\mathbf{z}}_1^{k+1} & \cdots & \mathbf{z}_1^d & \mathbf{z}_1^{d+1} & \cdots & \mathbf{z}_1^n \\ \mathbf{z}_2^1 & \cdots & \mathbf{z}_2^{k+1} & \widetilde{\mathbf{z}}_2^{k+1} & \cdots & \mathbf{z}_2^d & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$, so that $\widetilde{\mathbf{z}}_1^{k+1}$ is orthogonal to $\mathbf{z}_2^{k+1}$. That is, $\widetilde{\mathbf{z}}_1^{k+1}$ belongs to the orthogonal space to $\langle\, \mathbf{z}_2^1, \ldots, \mathbf{z}_2^d \,\rangle$, which is given by $\langle\, \mathbf{z}_1^{d+1}, \ldots, \mathbf{z}_1^n \,\rangle$ since l is a Lagrange plane. But this contradicts the choice of $\widetilde{\mathbf{z}}_1$, which proves the assertion.

Note that the previous property has a fundamental consequence: if one chooses $\varepsilon > 0$ such that the matrix $\begin{bmatrix} \mathbf{0} & \cdots \varepsilon\widetilde{\mathbf{z}}_1^{k+1} & \mathbf{z}_1^{k+2} & \cdots & \mathbf{z}_1^d & \mathbf{z}_1^{d+1} & \cdots & \mathbf{z}_1^n \\ \mathbf{z}_2^1 & \cdots \;\mathbf{z}_2^{k+1} & \mathbf{z}_2^{k+2} & \cdots & \mathbf{z}_2^d & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$ represents a Lagrange plane l_ε, then $d(\omega, l_\varepsilon) = k$; or, in other words, l_ε belongs to the set $\mathcal{L}_k - \mathcal{L}_{k+1}$, whose density is being analyzed.

Note now that the column vectors of the above matrix are isotropic one to another for any value of ε, so that it represents a Lagrange plane in the case that its rank is n. And note also that $\begin{bmatrix} \varepsilon_1 \widetilde{\mathbf{z}}_1^{k+1} \\ \mathbf{z}_2^{k+1} \end{bmatrix}$ and $\begin{bmatrix} \varepsilon_2 \widetilde{\mathbf{z}}_1^{k+1} \\ \mathbf{z}_2^{k+1} \end{bmatrix}$ are linearly independent if $\varepsilon_1 \ne \varepsilon_2$. This implies that there exist at most finitely many values of ε for which $\dim l_\varepsilon < n$. In other words: except for these values of ε, l_ε is indeed a Lagrange plane. Therefore, in the case $d(\omega, l) = k + 1$, it is possible to take a sequence $(\varepsilon_m) \downarrow 0$ avoiding those values, so that (l_{ε_m}) is a sequence in $\mathcal{L}_k - \mathcal{L}_{k+1}$ with limit l. This completes the analysis in the case when $d(\omega, l) = k + 1$.

Consider now the case $d(\omega, l) = k + 2$. The same changes as before can be carried out first for the column $k + 2$, in order to obtain a family of Lagrange planes

$$l_\varepsilon \equiv \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \varepsilon\widetilde{\mathbf{z}}_1^{k+2} & \mathbf{z}_1^{k+3} & \cdots & \mathbf{z}_1^d & \mathbf{z}_1^{d+1} & \cdots & \mathbf{z}_1^n \\ \mathbf{z}_2^1 & \cdots & \mathbf{z}_2^{k+1} & \mathbf{z}_2^{k+2} & \mathbf{z}_2^{k+3} & \cdots & \mathbf{z}_2^d & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

contained in $\mathcal{L}_{k+1} - \mathcal{L}_{k+2}$ except for at most finitely many values of ε. Now, for each one of the "good" values of ε, it is possible to construct a family

$$l_{\eta,\varepsilon} \equiv \begin{bmatrix} \mathbf{0} & \cdots & \eta\widetilde{\mathbf{z}}_1^{k+1} & \varepsilon\widetilde{\mathbf{z}}_1^{k+2} & \mathbf{z}_1^{k+3} & \cdots & \mathbf{z}_1^d & \mathbf{z}_1^{d+1} & \cdots & \mathbf{z}_1^n \\ \mathbf{z}_2^1 & \cdots & \mathbf{z}_2^{k+1} & \mathbf{z}_2^{k+2} & \mathbf{z}_2^{k+3} & \cdots & \mathbf{z}_2^d & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

of Lagrange planes in $\mathcal{L}_k - \mathcal{L}_{k+1}$. In fact all the values of $(\eta, \varepsilon) \in (0, 1] \times (0, 1]$ are valid (once $\widetilde{\mathbf{z}}_1^{k+2}$ and $\widetilde{\mathbf{z}}_1^{k+1}$ have been fixed) except those of the form (η, ε_j) or (η_i, ε) for finitely many indexes j and i. So once again it is possible to choose a sequence $(l_{\eta_m,\varepsilon_m})$ in $\mathcal{L}_k - \mathcal{L}_{k+1}$ with limit l. Clearly the argument can be extended to the cases $d(\omega, l) = k + 3, \ldots, n$, which proves the asserted density of the set $\mathcal{L}_k - \mathcal{L}_{k+1}$ in $\mathcal{L}_k$. The proof of point (vi) is complete.

Assume now that the family (5.4) has exponential dichotomy over Ω, and let $\Omega \times \mathbb{R}^{2n} = L^+ \oplus L^-$ be the corresponding decomposition, with associated Lagrange planes $l^\pm(\omega) = \{\mathbf{z} \,|\, (\omega, \mathbf{z}) \in L^\pm\}$. Define the functions

$$\tilde{d}^\pm \colon \Omega \to \{0, \ldots, n\}, \quad \omega \mapsto d(\omega, l^\pm(\omega)) = \dim(\Lambda(\omega) \cap l^\pm(\omega))$$

and note that

$$\begin{aligned} \tilde{d}^+(\omega) + \tilde{d}^-(\omega) &= \dim(\Lambda(\omega) \cap l^+(\omega)) + \dim(\Lambda(\omega) \cap l^-(\omega)) \\ &\le \dim(\Lambda(\omega) \cap (l^+(\omega) \oplus l^-(\omega))) = d(\omega)\,. \end{aligned} \tag{5.73}$$

Define also the quantities

$$\tilde{d}_M^\pm = \max_{\omega\in\Omega} \tilde{d}^\pm(\omega) \quad \text{and} \quad \tilde{d}_m^\pm = \min_{\omega\in\Omega} \tilde{d}^\pm(\omega)\,.$$

Proposition 5.84 *Suppose that the family* (5.4) *has exponential dichotomy over Ω, and define the functions $\tilde{d}^\pm$ as above.*

(i) *The functions $\tilde{d}^\pm$ are σ-invariant, and hence they are m_0-a.e. constant with respect to any σ-ergodic measure m_0 on Ω.*
(ii) *The functions $\tilde{d}^\pm$ are upper semicontinuous, and the sets of their continuity points are open residual invariant subsets $\widetilde{\Omega}_c^\pm \subseteq \Omega$ on which $d_M^\pm$ are locally constant, with*

$$\{\omega \in \Omega \,|\, \tilde{d}^\pm(\omega) = \tilde{d}_m^\pm\} \subseteq \Omega_c^\pm\,.$$

In particular, the functions $\tilde{d}^\pm$ are constant on any minimal set $\mathcal{M} \subseteq \Omega$.
(iii) *There exist minimal sets $\mathcal{M}^\pm$ such that $\tilde{d}^\pm(\omega) = \tilde{d}_M^\pm$ for all $\omega \in \mathcal{M}^\pm$.*
(iv) *If there exists a point $\omega_0 \in \Omega$ with dense orbit, then $\tilde{d}^\pm(\omega_0) = \tilde{d}_m^\pm$. In particular, ω_0 is a continuity point for $\tilde{d}^+$ and $\tilde{d}^-$.*
(v) *If m_0 is a σ-ergodic measure on Ω with* Supp $m_0 = \Omega$, *then there exist subsets $\Omega_1^\pm \subseteq \Omega$ with $m(\Omega_1^\pm) = 1$ such that $\tilde{d}^\pm(\omega) = d_m^\pm$ for all $\omega \in \Omega_1^\pm$.*

In particular, the sets $\Omega_c^{\pm}$ *of* (ii) *have full measure* m_0. *In addition,* $\Omega_c^{\pm} = \{\omega \in \Omega \mid \tilde{d}^{\pm}(\omega) = \tilde{d}_m^{\pm}\}$.

Proof

(i) This property follows immediately from Proposition 5.76(iii) and the equalities $l^{\pm}(\omega{\cdot}t) = U(t,\omega){\cdot}l^{\pm}(\omega)$ (see Proposition 1.76).
(ii) The upper semicontinuity of $\tilde{d}^{\pm}$ follows from the continuity of the map $l^{\pm}\colon \Omega \to \mathcal{K}_{\mathbb{R}}$ (see again Proposition 1.76) and the upper semicontinuity of d (see Proposition 5.83(vi)). With this first property in mind, the subsequent assertions can be checked with the argument used in the proof of Proposition 5.80(vi). The constant character of $\tilde{d}^{\pm}$ on any minimal set is a trivial consequence of the upper semicontinuity.
(iii) Choose ω_0 with $\tilde{d}_M^{+} = \tilde{d}^{+}(\omega_0)$, and let $\mathcal{M} \subseteq \mathcal{O}(\omega_0)$ be a minimal set. Take $\omega \in \mathcal{M}$. The σ-invariance and upper semicontinuity of $\tilde{d}^{+}$ established in (i) and (ii) ensure that $\tilde{d}_M^{+} = \tilde{d}^{+}(\omega_0) \le \tilde{d}^{+}(\omega) \le \tilde{d}_M^{+}$, which proves (iii) for $\tilde{d}^{+}$. An analogous proof can be given for $\tilde{d}^{-}$.
(iv) Take such a point ω_0. Then, as above, $\tilde{d}_m^{\pm} \le \tilde{d}^{\pm}(\omega_0) \le \tilde{d}^{\pm}(\omega)$ for all $\omega \in \Omega$. Taking the infimum for $\omega \in \Omega$ shows that $\tilde{d}_m^{\pm} = d^{\pm}(\omega_0)$.
(v) See the proof of Proposition 5.80(viii).

The next result relates $\tilde{d}^{\pm}$ with d. In particular, it shows that, if the point ω belongs to $\mathcal{O}(\omega) \cup \mathcal{A}(\omega)$, then the number of independent solutions of the system (5.4) taking the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right]$ can be calculated in terms of the number of independent solutions of this form which are bounded as $t \to \pm\infty$; or, equivalently, whose initial data lie in the subspaces $l^{+}(\omega)$ and $l^{-}(\omega)$.

Theorem 5.85 *Suppose that the family* (5.4) *has exponential dichotomy over* Ω.

(i) *If* $\omega \in \mathcal{O}(\omega) \cup \mathcal{A}(\omega)$, *then* $d(\omega) = \tilde{d}^{+}(\omega) + \tilde{d}^{-}(\omega)$.
(ii) *If* m_0 *is a* σ*-ergodic measure, then there are constants* $\tilde{d}_*^{+}$, $\tilde{d}_*^{-}$, *and* d_*, *with* $d_* = \tilde{d}_*^{+} + \tilde{d}_*^{-}$, *and such that* $\tilde{d}^{\pm}(\omega) = \tilde{d}_*^{\pm}$ *and* $d(\omega) = d_*$ *for* m_0*-a.e.* $\omega \in \Omega$.
(iii) *If* m_0 *is a* σ*-ergodic measure with* $\operatorname{Supp} m_0 = \Omega$, *then the equalities of* (ii) *hold in the open residual invariant set* $\{\omega \in \Omega \mid d(\omega) = d_m\}$, *which has full measure* m_0. *In addition,*

$$\{\omega \in \Omega \mid d(\omega) = d_m\} = \{\omega \in \Omega \mid \tilde{d}(\omega) = \tilde{d}_m^{+}\} = \{\omega \in \Omega \mid \tilde{d}(\omega) = \tilde{d}_m^{-}\},$$

and $d_* = d_m$, $\tilde{d}_*^{+} = \tilde{d}_m^{+}$, $\tilde{d}_*^{-} = \tilde{d}_m^{-}$.
(iv) *If* $\mathcal{M} \subseteq \Omega$ *is a minimal set, then the equalities of* (ii) *hold for all* $\omega \in \mathcal{M}$.

Proof

(i) The result is obviously true if $d(\omega) = 0$, since (5.73) holds. Assume that $d(\omega) = d > 0$, and set $k = \dim(\Lambda(\omega) \cap l^{+}(\omega)) \ge 0$. Take d linearly independent vectors $\mathbf{z}_1 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_{1,2} \end{smallmatrix}\right], \dots, \mathbf{z}_d = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_{d,2} \end{smallmatrix}\right]$ in $\Lambda(\omega)$ such that $\mathbf{z}_j \in l^{+}(\omega)$ for $j = 1, \dots, k$; decompose $\mathbf{z}_j = \mathbf{z}_j^{+} + \mathbf{z}_j^{-}$ with $\mathbf{z}_j^{\pm} \in l^{\pm}(\omega)$ for $j = 1, \dots, d$, and note that $\mathbf{z}_j^- = \mathbf{0}$ for $j = 1, \ldots, k$. Let $U(t,\omega)\,\mathbf{z}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_{2,j}(t,\omega) \end{bmatrix}$ be the corresponding solutions for $j = 1, \ldots, k$. Assume that $\omega \in \mathcal{O}(\omega)$, take a sequence $(t_m) \uparrow \infty$ with $\lim_{m\to\infty} \omega{\cdot}t_m = \omega$, and note that there is no loss of generality in assuming the existence of vectors $\widetilde{\mathbf{z}}_1, \ldots, \widetilde{\mathbf{z}}_d$ such that

$$\langle \widetilde{\mathbf{z}}_1, \ldots, \widetilde{\mathbf{z}}_k \rangle = \lim_{m\to\infty} U(t_m,\omega) \cdot \langle \mathbf{z}_1, \ldots, \mathbf{z}_k \rangle \quad \text{in } \mathcal{G}_k(\mathbb{R}^{2n}),$$
$$\langle \widetilde{\mathbf{z}}_{k+1}, \ldots, \widetilde{\mathbf{z}}_d \rangle = \lim_{m\to\infty} U(t_m,\omega) \cdot \langle \mathbf{z}_{k+1}, \ldots, \mathbf{z}_n \rangle \quad \text{in } \mathcal{G}_{d-k}(\mathbb{R}^{2n}).$$

Proposition 1.96(ii) of [85] ensures that

$$\langle \widetilde{\mathbf{z}}_1, \ldots, \widetilde{\mathbf{z}}_k \rangle \in \mathcal{G}_k(l^+(\omega)) \;\text{ and }\; \langle \widetilde{\mathbf{z}}_{k+1}, \ldots, \widetilde{\mathbf{z}}_d \rangle \in \mathcal{G}_{d-k}(l^-(\omega)).$$

Repeating now the argument of Proposition 5.80(ii) shows that $U(t,\omega)\widetilde{\mathbf{z}}_j$ is of the form $\begin{bmatrix} \mathbf{0} \\ \widetilde{\mathbf{z}}_{2,j}(t,\omega) \end{bmatrix}$ for each $j = 1, \ldots, d$, so that $\widetilde{\mathbf{z}}_j \in \Lambda(\omega)$ for $j = 1, \ldots, d$. Therefore, $\tilde{d}^+(\omega) = d(\omega, l^+(\omega)) \geq k$ and $\tilde{d}^-(\omega) = d(\omega, l^-(\omega)) \geq d - k$. This fact and relation (5.73) imply that $d(\omega) = \tilde{d}^+(\omega) + \tilde{d}^-(\omega)$, as asserted. The proof of the case $\omega \in \mathcal{A}(\omega)$ is analogous.

(ii) Theorem 5.80(iii) and Proposition 5.84(i) ensure that the functions d, $\tilde{d}^+$, and $\tilde{d}^-$ are constant for m_0-a.e. $\omega \in \Omega$. In addition, the Poincaré Recurrence Theorem (see again [35]) provides a subset Ω_0 with $m(\Omega_0) = 1$ such that $\omega \in \mathcal{O}(\omega) \cap \mathcal{A}(\omega)$ for all $\omega \in \Omega_0$. These facts and (i) prove (ii).

(iii) Points (vi) and (viii) of Theorem 5.80 and (ii) and (v) of Proposition 5.84 imply that the equalities in (ii) hold for the set $\Omega_c \cap \Omega_c^+ \cap \Omega_c^-$, that is, in the set

$$\{\omega \in \Omega \mid d(\omega) = d_m\} \cap \{\omega \in \Omega \mid \tilde{d}(\omega) = \tilde{d}_m^-\} \cap \{\omega \in \Omega \mid \tilde{d}(\omega) = \tilde{d}_m^-\},$$

which has full measure m_0. Note that, in particular, $d_* = d_m$, $\tilde{d}_*^+ = \tilde{d}_m^+$, and $\tilde{d}_*^- = \tilde{d}_m^-$. In addition, if $d(\omega) = d_m$, then it follows from (5.73) that $\tilde{d}^+(\omega)$ and $\tilde{d}^-(\omega)$ also attain their minima at ω, so that the three sets agree.

(iv) Theorem 5.80(iv) and Proposition 5.84(ii) show that the functions d, $\tilde{d}^+$, and $\tilde{d}^-$ are constant on $\mathcal{M}$. On the other hand, the minimality property implies that $\mathcal{M} = \mathcal{O}(\omega)$ for any $\omega \in \mathcal{M}$. These facts and (i) prove the statement of (iv).

Corollary 5.86 *The following assertions are equivalent:*

(1) *the family of linear Hamiltonian systems* (5.4) *satisfies condition* D2 *of Sect.* 5.2;
(2) $d^+(\omega) = 0$ *for all* $\omega \in \Omega$;
(3) $d^-(\omega) = 0$ *for all* $\omega \in \Omega$;
(4) $d^+(\omega) = 0$ *for each* ω *which belongs to any minimal subset of* Ω;
(5) $d^-(\omega) = 0$ *for each* ω *which belongs to any minimal subset of* Ω;
(6) $d(\omega) = 0$ *for each* ω *which belongs to any minimal subset of* Ω.

And, if the family (5.4) has exponential dichotomy over Ω, then each of the following assertions is equivalent to the previous ones:

(7) $\tilde{d}^+(\omega) = 0$ *for each ω which belongs to any minimal subset of Ω.*
(8) $\tilde{d}^-(\omega) = 0$ *for each ω which belongs to any minimal subset of Ω.*

Proof (1)⇔(2) Assume the existence of $\omega \in \Omega$ with $d^+(\omega) = 0$. Then the system corresponding to the point ω has at least one nontrivial solution of the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ in a positive half-line $[a, \infty)$, so that the the system corresponding to the point $\omega{\cdot}(-a)$ has at least one nontrivial solution of the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ in $[0, \infty)$. Therefore, D2 does not hold. This shows that (1) implies (2). The converse assertion is trivial.

(1)⇔(3) According to Proposition 5.18, conditions D2 and D2′ are equivalent. Therefore it is enough to repeat the previous argument but in any negative half-line.

(2)⇔(4) It is obvious that (2) implies (4). Conversely, assume for contradiction that (2) does not hold, so that $d_M^+ > 0$. Theorem 5.80(v) ensures the existence of a minimal set $\mathcal{M} \subseteq \Omega$ such that $d^+(\omega) = d_M^+ > 0$ for all $\omega \in \mathcal{M}$, which precludes (4).

(4)⇔(5)⇔(6) These equivalences follow from Theorem 5.80(iv).

(6)⇔(7)⇔(8) Theorem 5.85(iv) ensures that the (nonnegative) functions d, $\tilde{d}^+$ and $\tilde{d}^-$ are constant on any minimal set $\mathcal{M} \subseteq \Omega$, and that $d = \tilde{d}^+ + \tilde{d}^-$. The remaining equivalences follow easily from these facts.

Remarks 5.87

1. The previous corollary and Theorem 5.80(iv) prove that the non-validity of condition D2 is equivalent to the existence of at least one minimal subset $\mathcal{M} \subseteq \Omega$ such that $d(\omega) > 0$ for all $\omega \in \mathcal{M}$.
2. If, in addition, the family (5.4) has exponential dichotomy over Ω, then at least one of the restricted (constant) functions $\tilde{d}^+|_{\mathcal{M}}$ or $\tilde{d}^-|_{\mathcal{M}}$ is strictly positive; or in other words, at least one of the associated Lagrange planes $l^+(\omega)$ or $l^-(\omega)$ lies on the vertical Maslov cycle $\mathcal{C}$ for all $\omega \in \mathcal{M}$. This is proved by Theorem 5.85(i)&(iv).
3. Recall also that, as stated in Remark 5.22, the non-validity of condition D2 is equivalent to the absence of uniform null controllability for the family (5.9).

Example 5.88 Example 8.48, in Chap. 8, provides a case of a minimal base with exponential dichotomy and $d^+ = d^- = 1$, so that Corollary 5.86 precludes D2: that is, the family (5.4) is not uniformly weakly disconjugate.

Chapter 6
Nonautonomous Control Theory: Linear Regulator Problem and the Kalman–Bucy Filter

The remaining three chapters of the book consider certain problems concerning linear control systems with time-varying coefficients which give rise in a natural way to nonautonomous linear Hamiltonian differential systems. The methods developed in the preceding chapters will be systematically used to study these control problems.

Chapter 6 begins with a discussion of the feedback stabilization problem for a nonautonomous linear control system: the stabilizing feedback control will be determined by formulating and solving an infinite horizon linear regulator problem. The minimizing pairs for the corresponding functional will be in a one-to-one correspondence with certain solutions of a nonautonomous linear Hamiltonian system constructed from the minimization problem. Previous results concerning the occurrence of exponential dichotomy and the properties of the rotation number for nonautonomous linear Hamiltonian systems will be used. In proving the results regarding the feedback stabilization problem, only some basic elements of control theory will be required; these have been for the most part introduced in Chap. 3. The Pontryagin Maximum Principle will be just referred to for purposes of motivation; and also the Riccati equation associated to the linear Hamiltonian system, which enjoys an important role in many treatments of the linear regulator problem, will make just a brief appearance here.

The linear regulator on a finite time interval (i.e. in the case of a *finite horizon*) has been thoroughly studied and is treated in standard texts; e.g. [51, 143]. The case of an infinite horizon is not quite as standard, but a substantial theory is available in this situation, as well: see e.g. [25] and [13]. In this chapter the treatment of the linear regulator problem in the infinite horizon case differs from some others in its systematic application of the theory of exponential dichotomies and the theory of the rotation number, and in its relative deemphasis of the role of the Riccati equation. In fact it will be seen that the stable dichotomy projection gives rise to

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a negative definite solution of the Riccati equation, which in addition possesses important regularity properties.

The first section of this chapter presents a preliminary heuristic approach to the linear regulator problem and to the feedback stabilization problem. This approach motivates the formulation of the rigorous results proved in the second section. In Sect. 6.3, the regularity properties of the matrix-valued function solving the feedback stabilization problem are analyzed. These results reproduce basically those of Johnson and Nerurkar [76, 77], extending some of them.

The second goal of this chapter is to study the Kalman–Bucy filter in a nonautonomous setting. This filter is a standard method used in control engineering for measuring the mean-square error between the output signal of a linear plant subject to a stochastic disturbance, and the estimated output signal. The concepts of exponential dichotomy and rotation number for linear Hamiltonian systems can (and will) be used to produce direct proofs of some basic results concerning the Kalman–Bucy filter. This is not surprising, in view of the previous use of these concepts in the study of the linear regulator problem, and in view of the well-known fact that the Kalman–Bucy filter is "dual" to that problem. Thus the asymptotic limit and the stability properties of the error covariance matrix can be quickly deduced. Also the Hurwitz property at $+\infty$ of the error propagation system follows immediately from the corresponding fact for the feedback system which is determined by the linear regulator problem. This discussion was first carried out in Johnson and Núñez [83] and is the content of Sect. 6.4, which begins with a precise description of the problem to be dealt with.

During the present chapter $\langle\ ,\ \rangle$ and $\|\cdot\|$ denote the Euclidean inner product and the Euclidean norm on $\mathbb{R}^d$ for any value of d, and the same symbol $\|\cdot\|$ represents the usual operator norm associated to the Euclidean norm on any matrix space $\mathbb{M}_{d\times m}(\mathbb{R})$: see Remark 1.24.2. And, if $M \in \mathbb{S}_n(\mathbb{R})$ (the set of $n\times n$-symmetric matrices) is positive semidefinite, $M^{1/2}$ will represent its unique positive semidefinite square root: see Proposition 1.19. In particular, $M^{1/2} > 0$ if $M > 0$.

6.1 An Heuristic Approach

Let $\mathbf{x} \in \mathbb{R}^n$ be a state vector, $\mathbf{u} \in \mathbb{R}^m$ be a control vector, and let $A\colon \mathbb{R} \to \mathbb{M}_{n\times n}(\mathbb{R})$ and $B\colon \mathbb{R} \to \mathbb{M}_{n\times m}(\mathbb{R})$ be bounded and uniformly continuous functions. The *feedback stabilization problem* for the linear control system

$$\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u} \tag{6.1}$$

consists in determining a linear time-varying control $\mathbf{u} = K(t)\,\mathbf{x}$ such that $\mathbf{x} \equiv \mathbf{0}$ is an exponentially asymptotically stable solution for the feedback system

$$\mathbf{x}' = (A(t) + B(t)K(t))\,\mathbf{x}\,. \tag{6.2}$$

It will be seen that a feedback matrix $K\colon \mathbb{R} \to \mathbb{M}_{m\times n}(\mathbb{R})$ can be determined by solving an appropriate linear regulator problem, which is now described. Let $G\colon \mathbb{R} \to \mathbb{S}_n(\mathbb{R})$ and $R\colon \mathbb{R} \to \mathbb{S}_m(\mathbb{R})$ be bounded and uniformly continuous functions. Suppose that $G \geq 0$ and $R \geq \rho_R I_m$ for a fixed $\rho_R > 0$ which does not depend on $t \in \mathbb{R}$. An *infinite horizon linear regulator problem* is posed by introducing a quadratic functional $\mathcal{I}_{\mathbf{x}_0}$ of the following form for each $\mathbf{x}_0 \in \mathbb{R}^n$: given a locally integrable control function $\mathbf{u}\colon [0,\infty) \to \mathbb{R}^m$, if $\mathbf{x}\colon [0,\infty) \to \mathbb{R}^n$ is the solution of (6.1) for that $\mathbf{u}$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$, then

$$\mathcal{I}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u}) = \frac{1}{2}\int_0^\infty \left(\langle \mathbf{x}(t), G(t)\,\mathbf{x}(t)\rangle + \langle \mathbf{u}(t), R(t)\,\mathbf{u}(t)\rangle\right) dt\,. \tag{6.3}$$

The reader is referred to Sect. 8.1 of Chap. 8 for an explanation of the meaning of the quantity $\mathcal{I}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u})$ when the pair $(\mathbf{x},\mathbf{u})$ solves the control problem (6.1). In this case, one speaks of an *infinite horizon* because the upper limit in the integral defining $\mathcal{I}_{\mathbf{x}_0}$ is ∞ rather than a finite number t_0. Each pair $(\mathbf{x},\mathbf{u})$ as above gives rise to an extended nonnegative real number $\mathcal{I}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u}) \in [0,\infty]$. The problem is solved by establishing conditions on A, B, G, and R which ensure the existence of at least one control function $\bar{\mathbf{u}}$ for which the corresponding pair $(\bar{\mathbf{x}},\bar{\mathbf{u}})$ satisfies $\mathcal{I}_{\mathbf{x}_0}(\bar{\mathbf{x}},\bar{\mathbf{u}}) < \infty$, and for which

$$\mathcal{I}_{\mathbf{x}_0}(\bar{\mathbf{x}},\bar{\mathbf{u}}) \leq \mathcal{I}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u})$$

for every other choice of pair $(\mathbf{x},\mathbf{u})$ as above. Actually, the objective is to minimize $\mathcal{I}_{\mathbf{x}_0}$ for each $\mathbf{x}_0 \in \mathbb{R}^n$ and also to arrange that the minimizing control $\bar{\mathbf{u}} = \bar{\mathbf{u}}_{\mathbf{x}_0}$ depend linearly on $\mathbf{x}_0$.

There are two distinct approaches to determining a minimizing pair $(\bar{\mathbf{x}},\bar{\mathbf{u}})$ of the functional $\mathcal{I}_{\mathbf{x}_0}$ given by (6.3) subject to the control problem given by (6.1) and $\mathbf{x}(0) = \mathbf{x}_0$. One of them involves an appeal to the Pontryagin Maximum Principle. The other one makes uses of the Dynamic Programming Principle of Bellman [51, 143]. The approach which proceeds via the Pontryagin Maximum Principle will now be illustrated.

Let $\mathbf{y} \in \mathbb{R}^n$ be a new variable, which is viewed as conjugate to the state variable $\mathbf{x}$. Introduce the linear Hamiltonian function

$$\begin{aligned}\mathcal{H}(t,\mathbf{x},\mathbf{y},\mathbf{u}) &= \langle \mathbf{y},\mathbf{x}'\rangle - \frac{1}{2}\big(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\big)\\ &= \langle \mathbf{y}, A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\rangle - \frac{1}{2}\big(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\big)\,,\end{aligned} \tag{6.4}$$

and write the corresponding Hamilton equations

$$\begin{aligned}\mathbf{x}' &= \frac{\partial \mathcal{H}}{\partial \mathbf{y}}(t,\mathbf{x},\mathbf{y},\mathbf{u})\\ \mathbf{y}' &= -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(t,\mathbf{x},\mathbf{y},\mathbf{u})\end{aligned} \tag{6.5}$$

for each control $\mathbf{u}$. According to the Pontryagin Maximum Principle, if the control $\bar{\mathbf{u}}$ gives rise to a minimizing pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ for $\mathcal{I}_{\mathbf{x}_0}$, then there is a motion $\bar{\mathbf{y}}(t)$ such that, for each t,

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}}(t, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{u}}) = \mathbf{0} \tag{6.6}$$

(or more primitively, $\mathcal{H}(t, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{u}}) = \max_{\mathbf{u}\in\mathbb{R}^m} \mathcal{H}(t, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \mathbf{u})$), and such that $(\bar{\mathbf{x}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{u}}(t))$ solves (6.5). This principle can be formulated and proved in an ample context which includes the linear regulator problem as a very special case; see e.g. [51] and [25]. However, this fact will not be used: instead, (6.6) will be regarded as an Ansatz which leads to the solution of the linear regulator problem.

Proceeding with this in mind, one obtains easily from (6.4) and (6.6) that $(\bar{\mathbf{y}}, \bar{\mathbf{u}})$ must satisfy the so-called *feedback rule*

$$\mathbf{u} = R^{-1}(t)\, B^T(t)\, \mathbf{y}\,. \tag{6.7}$$

Substituting (6.7) into (6.4), equations (6.5) yield

$$\mathbf{z}' = \begin{bmatrix} A(t) & B(t)\, R^{-1}(t)\, B^T(t) \\ G(t) & -A^T(t) \end{bmatrix} \mathbf{z}\,, \tag{6.8}$$

where $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right] \in \mathbb{R}^{2n}$.

The problem now is to find a solution $\bar{\mathbf{z}}(t) = \left[\begin{smallmatrix} \bar{\mathbf{x}}(t) \\ \bar{\mathbf{y}}(t) \end{smallmatrix}\right]$ of (6.8) with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ such that, if $\bar{\mathbf{u}}$ is determined from $\bar{\mathbf{y}}$ by the feedback rule (6.7), then the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ minimizes $\mathcal{I}_{\mathbf{x}_0}$. It turns out that, when standard controllability hypotheses hold, such a solution can be determined; moreover, its existence is due to the fact that, when these controllability hypotheses hold, the system (6.8) has exponential dichotomy.

Such controllability conditions will be discussed in due course. For now, just assume their validity and take for granted that (6.8) has exponential dichotomy. Let Q be the projection corresponding to the exponential dichotomy of (6.8) described in Definition 1.54. Let $U(t)$ be the fundamental matrix solution of (6.8) with $U(0) = I_{2n}$, and set $Q(t) = U(t)\, Q\, U^{-1}(t)$ for each $t \in \mathbb{R}$. Then the range of $Q(t)$, is a real Lagrange plane (see the proof of Proposition 1.56 and Remark 1.77.1), and the controllability conditions will also guarantee that it belongs to the set $\mathcal{D}$ defined by (1.21): it can be represented by the matrix $\left[\begin{smallmatrix} I_n \\ M^+(t) \end{smallmatrix}\right]$ for all $t \in \mathbb{R}$. More precisely, the parameterizing $n \times n$ matrix-valued function $M^+(t)$ is a real symmetric negative definite matrix for each $t \in \mathbb{R}$.

Now, the conditions $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ and $\bar{\mathbf{y}}(0) = M^+(0)\, \mathbf{x}_0$ determine a solution $\bar{\mathbf{z}}(t) = \left[\begin{smallmatrix} \bar{\mathbf{x}}(t) \\ \bar{\mathbf{y}}(t) \end{smallmatrix}\right]$ of (6.8) with $\bar{\mathbf{y}}(t) = M^+(t)\, \bar{\mathbf{x}}(t)$. Since $\bar{\mathbf{z}}(0)$ lies in the image of Q, this solution decays exponentially as $t \to \infty$. Set

$$K(t) = R^{-1}(t)\, B^T(t)\, M(t) \tag{6.9}$$

and $\bar{\mathbf{u}}(t) = K(t)\,\bar{\mathbf{x}}(t)$. It will turn out that $\mathcal{I}_{\mathbf{x}_0}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) < \infty$. By applying a standard completing-the-square argument, it can be proved that $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is the unique pair which minimizes $\mathcal{I}_{\mathbf{x}_0}$.

The reasoning just carried out is valid for all $\mathbf{x}_0 \in \mathbb{R}^n$, and $\bar{\mathbf{u}}$ depends linearly on $\mathbf{x}_0$. Thus, the linear regulator problem is solved. And, in addition, the exponential decay as $t \to \infty$ of $\bar{\mathbf{z}}(t) = \begin{bmatrix} \bar{\mathbf{x}}(t) \\ M^+(t)\,\bar{\mathbf{x}}(t) \end{bmatrix}$ implies that $\mathbf{x} \equiv \mathbf{0}$ is an exponentially stable solution of (6.2). Thus the matrix-valued function K solves the feedback stabilization problem.

It is important to emphasize that the key step in this solution of the linear regulator problem and of the feedback stabilization problem is the proof that the linear Hamiltonian system (6.8) has exponential dichotomy, with some additional properties regarding the Lagrange plane of the initial data of the bounded solutions on $[0, \infty)$. The proof requires the above-mentioned controllability hypotheses. It is carried out using the methods of Chap. 3; in particular Theorem 3.50, which relates the presence of exponential dichotomy to the constancy of the rotation number in a parameter interval for an Atkinson problem. It is also useful to keep in mind that the (Weyl) matrix-valued function $M^+ < 0$ which provides the solution of the linear regulator problem is in fact the negative of the positive definite matrix-valued function obtained by applying the Bellman Dynamic Programming Principle (for instance, in the text [51]).

6.2 The Rigorous Proofs

This section contains the rigorous solution of the infinite horizon linear regulator problem and the feedback stabilization problem. As usual, (Ω, σ) represents a real continuous flow on a compact metric space, and $\omega{\cdot}t \equiv \sigma(t, \omega)$. This flow may exhibit all ranges of recurrent behavior, from almost periodic (in particular periodic), to uniformly recurrent, to topologically transitive with positive topological entropy. Of course, Ω may contain wandering orbits, as well.

Let the functions $A\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$, $B\colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$, $G\colon \Omega \to \mathbb{S}_n(\mathbb{R})$, and $R\colon \Omega \to \mathbb{S}_m(\mathbb{R})$ be continuous, with $G \ge 0$ and $R > 0$, so in particular there exists $\rho_{\mathbb{R}} > 0$ such that $R(\omega) \ge \rho_R I_n$ for all $\omega \in \Omega$. Introduce the family of control systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}\,, \qquad \omega \in \Omega\,. \tag{6.10}$$

Also, for each $\omega \in \Omega$ and each $\mathbf{x}_0 \in \mathbb{R}^n$, introduce the functional

$$\mathcal{I}_{\mathbf{x}_0,\omega}(\mathbf{x}, \mathbf{u}) = \frac{1}{2}\int_0^\infty \big(\langle \mathbf{x}(t), G(\omega{\cdot}t)\,\mathbf{x}(t)\rangle + \langle \mathbf{u}(t), R(\omega{\cdot}t)\,\mathbf{u}(t)\rangle\big)\,dt\,, \tag{6.11}$$

evaluated on those pairs $(\mathbf{x},\mathbf{u})\colon [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ such that: $\mathbf{u}$ is a locally integrable control function; and $\mathbf{x}$ is the solution of the system (6.10) corresponding to $\mathbf{u}$ and ω with $\mathbf{x}(0) = \mathbf{x}_0$. Note that $\mathcal{I}_{\mathbf{x}_0,\omega}$ takes values in $[0,\infty]$.

As explained in Sect. 1.3.2, if the coefficients A, B, G, and R of (6.1) and (6.3) are bounded and uniformly continuous matrix-valued functions on $\mathbb{R}$, then a construction of Bebutov type (taking as starting point (A,B,G,R)) gives rise to a compact metric space Ω and ω-dependent families of problems (6.10) and functionals (6.11), in which the initial ones are included: more precisely, there exists a point $\omega_0 \in \Omega$ for which (6.10) and (6.11) coincide with (6.1) and (6.3). Roughly speaking, the goal of this section is to solve the linear regulator problem *uniformly on* Ω. Remark 6.16 explains in what cases the solution works for the initial system and functional if Ω is defined as the hull of its coefficients.

Definition 6.1 System (6.10) is *null controllable with unconstrained controls* if for each $\mathbf{x}_0 \in \mathbb{R}^n$ there exist a time $t_0 > 0$ and an integrable control function $\mathbf{u}\colon [0,t_0] \to \mathbb{R}^m$ such that the solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\mathbf{x}(t_0) = \mathbf{0}$. In this case, *the control* $\mathbf{u}$ *steers* $\mathbf{x}_0$ *to* $\mathbf{0}$ *in time* t_0.

In the rest of this book, the term *null controllable* will be used as synonymous with null controllable with unconstrained controls.

Remarks 6.2

1. Fix $\omega \in \Omega$. Let $U_A(t,\omega)$ be the fundamental matrix solution of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x}$ with $U_A(0,\omega) = I_n$. It is well known (see Conti [32], Theorem 7.2.2) that the null controllability of (6.10) for a fixed ω is equivalent to the condition
$$\int_0^\infty U_A^{-1}(t,\omega)\,B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt > 0\,. \tag{6.12}$$
In turn, (6.12) is clearly equivalent to the following condition: the only solution $\mathbf{y}(t)$ of $\mathbf{y}' = -A^T(\omega{\cdot}t)\,\mathbf{y}$ with $B^T(\omega{\cdot}t)\,\mathbf{y}(t) = \mathbf{0}$ for all $t \ge 0$ is the trivial one.
2. Fix $\omega \in \Omega$, assume that (6.12) holds, and choose t_0 such that
$$Q(t_0,\omega) = \int_0^{t_0} U_A^{-1}(t,\omega)\,B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt > 0\,.$$
It is easy to check that the continuous control $\mathbf{u}_\omega\colon [0,t_0] \to \mathbb{R}^m$ given by
$$\mathbf{u}_\omega(t) = -B^T(\omega{\cdot}t)(U_A^{-1})^T(t,\omega)\,Q^{-1}(t_0,\omega)\,\mathbf{x}_0$$
steers $\mathbf{x}_0$ to $\mathbf{0}$ in time t_0. Note also that $[0,t_0] \times \mathcal{O} \to \mathbb{R}^m,\ (t,\omega) \mapsto \mathbf{u}_\omega(t)$ is a jointly continuous map if $Q(t_0,\omega) > 0$ for all $\omega \in \mathcal{O} \subseteq \Omega$.
3. It follows easily from the first remark and from the continuity of B that if $B(\omega{\cdot}t)$ is nonsingular for some $t \ge 0$, then the system (6.10) is null controllable.
4. Assume that $B(\omega{\cdot}t)$ is positive semidefinite for all $t \ge 0$. Then $B(\omega{\cdot}t)\,\mathbf{x} = \mathbf{0}$ if and only if $B^{1/2}(\omega{\cdot}t)\,\mathbf{x} = \mathbf{0}$: see Proposition 1.19(i). This fact and the characterization

given in the first remark guarantee that system (6.10) is null controllable if and only if $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B^{1/2}(\omega{\cdot}t)\,\mathbf{u}$ has this property.

5. The previous remarks 1 and 4 can be used to prove also the equivalence of the null controllability of the three families of systems $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}$, $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + (B(\omega{\cdot}t)\,B^T(\omega{\cdot}t))^{1/2}\,\mathbf{u}$, and $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,\mathbf{u}$.
6. It is obvious that the systems (6.10) is null controllable if and only if $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} - B(\omega{\cdot}t)\,\mathbf{u}$ is. This and the previous property imply that, if $B(\omega{\cdot}t)$ is negative semidefinite for all $t \geq 0$, then the null controllability of (6.10) is equivalent to that of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + (-B)^{1/2}(\omega{\cdot}t)\,\mathbf{u}$.

The controllability conditions which will be imposed in the following discussion are now described. Recall that $G \geq 0$, so that $G^{1/2}$ exists and is positive semidefinite.

C1. Each minimal subset of Ω contains at least one point ω_1 such that the system

$$\mathbf{x}' = A(\omega_1{\cdot}t)\,\mathbf{x} + B(\omega_1{\cdot}t)\,\mathbf{u}$$

is null controllable.

C2. Each minimal subset of Ω contains at least one point ω_2 such that the system

$$\mathbf{x}' = -A^T(\omega_2{\cdot}t)\,\mathbf{x} + G^{1/2}(\omega_2{\cdot}t)\,\mathbf{u}$$

is null controllable.

Note that, according to Remarks 6.2.3, 6.2.4 and 6.2.6, the matrix $G^{1/2}$ in condition C2 can be replaced by G and by $-G$, and that the condition is automatically satisfied if $G > 0$.

Theorem 6.4 below asserts that conditions like C1 or C2 suffice to ensure the so-called uniform null controllability of a family of control systems over Ω. This property was proved by Johnson and Nerurkar in [74, 75]. Theorem 6.4 and several results derived from it in this chapter and the following ones illustrate this general observation: the dynamical properties of the compact metric flow (Ω, σ) can be related to the control-theoretic properties of the various control systems (6.10) (see e.g. [29, 77]).

Definition 6.3 The family of control systems (6.10) is *uniformly null controllable* if there exist numbers $t_0 > 0$ and $\delta > 0$ such that for all $\omega \in \Omega$,

$$\int_0^{t_0} U_A^{-1}(t,\omega)\,B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt \geq \delta\,I_n\,. \tag{6.13}$$

Theorem 6.4 *Condition* C1 *holds if and only if the family* (6.10) *is uniformly null controllable.*

Proof The details of the proof are given in Lemma 3.6: just repeat its arguments step by step, with B, $(U_A^{-1})^T$, and 0 taking the roles there played by Γ, U, and λ respectively, and working on $[0,\infty)$ instead of the real line.

Remarks 6.5

1. According to Remark 6.2.2, the uniform null controllability implies in particular the existence of a common time t_0 such that, for all $\mathbf{x}_0 \in \mathbb{R}^n$ and $\omega \in \Omega$ there is a continuous control which steers $\mathbf{x}_0$ to $\mathbf{0}$ in time t_0, a control which in addition varies continuously with respect to $\omega \in \Omega$.
2. Theorem 6.4 shows that in fact, the uniform null controllability is equivalent to the apparently less restrictive condition of null controllability of each individual system of the family.

Consider now the family of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} A(\omega{\cdot}t) & B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t) \\ G(\omega{\cdot}t) & -A^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z} = H(\omega{\cdot}t)\, \mathbf{z}\,, \quad \omega \in \Omega\,, \tag{6.14}$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right] \in \mathbb{R}^{2n}$. As usual, $U(t,\omega)$ represents the fundamental matrix solution of (6.14) with $U(0,\omega) = I_{2n}$. Consider also the perturbed family

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{6.15}$$

for

$$\Gamma(\omega) = \begin{bmatrix} G(\omega) & 0_n \\ 0_n & B(\omega)\, R^{-1}(\omega)\, B^T(\omega) \end{bmatrix}. \tag{6.16}$$

The next goal is to check that the controllability conditions C1 and C2 imply that the perturbation Γ satisfies the Atkinson Hypotheses 3.3 with respect to the family (6.14). The following auxiliary result will also be needed in Chap. 7.

Lemma 6.6 *The system* (6.10) *is null controllable if and only if*

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{u} \tag{6.17}$$

is null controllable.

Proof Let $\mathbf{y}(t)$ solve $\mathbf{y}' = -A^T(\omega{\cdot}t)\,\mathbf{y}$. Assume that

$$B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{y}(t) = \mathbf{0}$$

for $t \ge 0$. Then $\mathbf{y}^T(t) B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{y}(t) = 0$ for $t \ge 0$, which implies that $R^{-1/2}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{y}(t) = \mathbf{0}$ and hence that $B^T(\omega{\cdot}t)\,\mathbf{y}(t) = \mathbf{0}$ for all $t \ge 0$. The converse assertion is trivial. The characterization of null controllability given in Remark 6.2.1 implies the asserted equivalence.

Proposition 6.7 *Suppose that conditions* C1 *and* *C2* *hold, and let* Γ *be defined by* (6.16). *Then there exist* $t_0 > 0$ *and* $\delta > 0$ *such that for all* $\omega \in \Omega$*,*

$$\int_0^{t_0} \| \Gamma(\omega{\cdot}t)\, U(t,\omega)\, \mathbf{z}\|^2 \, dt \geq \delta \, \|\mathbf{z}\|^2 \quad \textit{whenever } \mathbf{z} \in \mathbb{R}^{2n} \, . \tag{6.18}$$

In particular, the family (6.15) *satisfies the Atkinson Hypotheses 3.3.*

Proof Note that (6.18) is true if and only if

$$\int_0^{t_0} U^T(t,\omega)\, \Gamma^2(\omega{\cdot}t)\, U(t,\omega)\, dt \geq \delta \, I_{2n} \, ,$$

and that $(U^{-1})^T(t,\omega)$ is the fundamental matrix solution of the system $\mathbf{z}' = -H^T(\omega{\cdot}t)\,\mathbf{z}$ with initial value I_{2n}; that is, (6.18) is true if and only if the family of control systems

$$\mathbf{z}' = -H^T(\omega{\cdot}t)\,\mathbf{z} + \Gamma(\omega{\cdot}t)\,\mathbf{w} \tag{6.19}$$

for $\omega \in \Omega$, where $\mathbf{w} = \left[\begin{smallmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{smallmatrix}\right] \in \mathbb{R}^{2n}$, is uniformly null controllable: see Definition 6.3. According to Theorem 6.4, it is enough to check that (6.19) is null controllable for all $\omega \in \Omega$.

So, fix $\omega \in \Omega$ and $\mathbf{z}_0 = \left[\begin{smallmatrix}\mathbf{x}_0\\ \mathbf{y}_0\end{smallmatrix}\right] \in \mathbb{R}^{2n}$. Theorem 6.4 and Remark 6.2.4 guarantee that, since condition C2 holds, the system

$$\mathbf{x}' = -A^T(\omega{\cdot}t)\,\mathbf{x} + G(\omega{\cdot}t)\,\mathbf{u} \tag{6.20}$$

is null controllable. Therefore, there exist a time t_1 and an integrable control $\mathbf{u}_1 \colon [0,t_1] \to \mathbb{R}^n$ such that the solution $\bar{\mathbf{x}}(t)$ of (6.20) for $\mathbf{u} = \mathbf{u}_1$ with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ satisfies $\bar{\mathbf{x}}(t_1) = \mathbf{0}$. Condition C1 and Theorem 6.4 also provide a time $t_2 > 0$ and a integrable control $\mathbf{u}_2 \colon [0,t_2] \to \mathbb{R}^n$ such that the solution $\bar{\mathbf{y}}(t)$ of (6.17) for $\mathbf{u} = \mathbf{u}_2$ with $\bar{\mathbf{y}}(0) = \mathbf{y}_0$ satisfies $\bar{\mathbf{y}}(t_2) = \mathbf{0}$.

Take $t_0 = \max(t_1, t_2)$ and set $\mathbf{u}_1(t) = \mathbf{0}$ for $t \in [t_1, t_0]$ if $t_1 < t_0$ and $\mathbf{u}_2(t) = \mathbf{0}$ for $t \in [t_2, t_0]$ if $t_2 < t_0$. Write out the control system (6.19) as

$$\begin{aligned}
\mathbf{x}' &= -A^T(\omega{\cdot}t)\,\mathbf{x} + G(\omega{\cdot}t)\,(-\mathbf{y} + \mathbf{w}_1)\,, \\
\mathbf{y}' &= A(\omega{\cdot}t)\,\mathbf{y} + B(\omega{\cdot}t)\,R^{-1}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(-\mathbf{x} + \mathbf{w}_2)\,,
\end{aligned}$$

and set $\mathbf{w}_1(t) = \mathbf{u}_1(t) + \bar{\mathbf{y}}(t)$ and $\mathbf{w}_2(t) = \mathbf{u}_2(t) + \bar{\mathbf{x}}(t)$. Then, if $\mathbf{w} = \left[\begin{smallmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{smallmatrix}\right]$, $\bar{\mathbf{z}} = \left[\begin{smallmatrix}\bar{\mathbf{x}}\\ \bar{\mathbf{y}}\end{smallmatrix}\right]$ is the solution of (6.19) satisfying $\bar{\mathbf{z}}(0) = \mathbf{z}_0 = \left[\begin{smallmatrix}\mathbf{x}_0\\ \mathbf{y}_0\end{smallmatrix}\right]$ and $\bar{\mathbf{z}}(t_0) = \mathbf{0}$. Thus $\mathbf{w}$ steers $\mathbf{z}_0$ to zero in time t_0: (6.19) is null controllable. This completes the proof.

Remarks 6.8

1. Definition 6.3, Theorem 6.4, Proposition 5.18, and Remark 6.2.5, taken together, prove the equivalence of: (1) condition C1; (2) the uniform null controllability of the family (6.10); (3) the uniform null controllability of the family (6.17); and (4) the fact that the Hamiltonian family (6.14) satisfies condition D2 of Sect. 5.2. Corollary 5.86 and Remarks 5.87 describe more equivalent situations.
2. Similarly, condition C2 (which, as said before, can be formulated for G instead of for $G^{1/2}$) is equivalent to the fact that the family (6.14) satisfies condition D2* of Sect. 5.6: just use the characterizations provided by Theorem 6.4 and (5.56).

The results of Chap. 5 play an important role in the following auxiliary result concerning the Atkinson problem (6.15). The notion of uniform weak disconjugacy is given in Chap. 5, Sect. 5.2. According to Theorem 5.17, since $B R^{-1} B^T \geq 0$, it is equivalent to speak of uniform weak disconjugacy on $[0, \infty)$ and of uniform weak disconjugacy on $(-\infty, 0]$, which justifies mentioning neither half-line.

Lemma 6.9 *Suppose that condition* C1 *holds. Then,*

(i) *the families of Hamiltonian systems* (6.15) *corresponding to* $\lambda \in (-1, 1)$ *are uniformly weakly disconjugate.*
(ii) *Let m be a σ-ergodic measure on Ω, and let $\alpha_\Gamma(\lambda)$ be the rotation number of the family* (6.15) *with respect to m. Then $\alpha_\Gamma(\lambda) = 0$ if $\lambda \in (-1, 1)$.*

Proof

(i) Given $\omega \in \Omega$ and $t_1 > 0$, consider the boundary value problem

$$\begin{aligned} \mathbf{z}' &= \big(H(\omega{\cdot}t) + \lambda J^{-1} \Gamma(\omega{\cdot}t)\big)\, \mathbf{z}\,, \\ \mathbf{x}(0) &= \mathbf{x}(t_1) = \mathbf{0}\,, \end{aligned} \tag{6.21}$$

where $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right]$ and Γ is defined by (6.16). The first and main step consists in checking that this problem has only the null solution for any fixed $\lambda \in (-1, 1)$ if t_1 is sufficiently large.

To this end, let $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right]$ be a solution of (6.21). Then,

$$\begin{aligned} 0 = \langle \mathbf{x}(t_1), \mathbf{y}(t_1) \rangle - \langle \mathbf{x}(0), \mathbf{y}(0) \rangle &= \int_0^{t_1} \frac{d}{dt} \langle \mathbf{x}(t), \mathbf{y}(t) \rangle \, dt \\ &= \int_0^{t_1} \big(\langle A\mathbf{x} + (1-\lambda)\, B R^{-1} B^T \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{x}, (\lambda+1)\, G\,\mathbf{x} - A^T \mathbf{y} \rangle \big)\, dt\,. \end{aligned}$$

The arguments $\omega{\cdot}t$ of A, B, R and G and t of $\mathbf{x}$ and $\mathbf{y}$ have been suppressed. It follows that

$$(\lambda+1)\int_0^{t_1}\|G^{1/2}(\omega{\cdot}t)\,\mathbf{x}(t)\|^2\,dt \\ = (\lambda-1)\int_0^{t_1}\|R^{-1/2}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,\mathbf{y}(t)\|^2\,dt\,.$$

Since $\lambda-1<0<\lambda+1$ and $R>0$, one has $G(\omega{\cdot}t)\,\mathbf{x}(t)=\mathbf{0}=B^T(\omega{\cdot}t)\,\mathbf{y}(t)$ for all $t\in[0,t_1]$, and hence

$$\mathbf{y}'=-A^T(\omega{\cdot}t)\,\mathbf{y}(t)\,;$$

i.e. $\mathbf{y}(t)=(U_A^{-1})^T(t,\omega)\,\mathbf{y}(0)$. The null controllability of all the systems of the family (6.10) ensured by condition C1 and Theorem 6.4 provide $t_0>0$ and $\delta>0$ (independent of the choice of ω) such that, if $t_1\ge t_0$,

$$\int_0^{t_1} U_A^{-1}(t,\omega)\,B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt\ge\delta I_n\,.$$

Consequently $\mathbf{y}(0)=\mathbf{0}$. Since $\mathbf{x}(0)=\mathbf{0}$ and the system is linear, it follows that $\mathbf{z}(t)=\mathbf{0}$ for all $t\in\mathbb{R}$. Thus, (6.21) has only the trivial solution if $t_1\ge t_0$, as asserted.

Definition 5.14 ensures then that the family of Hamiltonian systems (6.15) is uniformly weakly disconjugate if $\lambda\in(-1,1)$, which proves (i).

(ii) Once (i) is proved, (ii) follows from Propositions 5.7 and 5.65.

Remark 6.10 The uniform weak disconjugacy of the unperturbed family (6.14) (which was proved in the previous result under condition C1), the fact that it satisfies property D1 of Sect. 5.2, and Theorem 5.17, taken together, ensure that the family (6.14) satisfies condition D2 (this is already known: see Remark 6.8.1) and condition D3. These facts will be used in the following theorem.

Recall Definition 1.80 of the continuous Weyl functions $M^\pm\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ associated to the stable subbundles at $\mp\infty$ in the case of exponential dichotomy of a given family of real linear Hamiltonian systems.

Theorem 6.11 *Suppose that conditions* C1 *and* C2 *hold. Then the family of linear Hamiltonian systems* (6.14) *has exponential dichotomy over Ω. In addition, both Weyl functions M^+ and M^- are globally defined, and they satisfy $M^+<0$ and $M^->0$.*

Proof If Ω is the topological support of a σ-ergodic measure m, then Lemma 6.9, Proposition 6.7, and Theorem 3.50 ensure that the family (6.14) has exponential dichotomy over Ω. In the general case, some additional reasoning is required.

First, fix a minimal subset $\mathcal{M}$ of Ω and note that it agrees with the topological support of any σ-ergodic measure on Ω concentrated on it: see Sect. 1.1.2. As seen above, the family (6.14) has exponential dichotomy over $\mathcal{M}$.

It follows from the previous property and Theorem 1.78 that if a point ω belongs to a minimal subset of Ω, the corresponding system (6.14) admits no nonzero bounded solution. The following step is to find conditions sufficient to guarantee that the same holds for all the elements of Ω, so that a new application of Theorem 1.78 leads to the desired conclusion. Fix $\omega \in \Omega$ and suppose for contradiction that $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$ is a nonzero bounded solution of (6.14). Let $-\infty < s_1 < t_1 < \infty$. As seen in Remark 1.81.2,

$$\langle \mathbf{x}(t_1), \mathbf{y}(t_1)\rangle - \langle \mathbf{x}(s_1), \mathbf{y}(s_1)\rangle = \int_{s_1}^{t_1} \left(\|G^{1/2}\,\mathbf{x}\|^2 + \|R^{-1/2}B^T\mathbf{y}\|^2\right) dt\,, \tag{6.22}$$

where the arguments $\omega{\cdot}t$ and t are omitted. Then there exist two sequences $(s_k) \downarrow -\infty$ and $(t_k) \uparrow \infty$ such that $\mathbf{z}(s_k) \to \mathbf{0}$ and $\mathbf{z}(t_k) \to \mathbf{0}$ as $k \to \infty$. For, suppose for contradiction that there is no such sequence (s_k). Consider the negative semiorbit in $\Omega \times \mathbb{R}^{2n}$ of the point $(\omega, \mathbf{z}(0))$ with respect to the linear skew-product flow defined by the family (6.14). The alpha-limit set $\mathcal{A}(\omega, \mathbf{z}(0))$ is compact and invariant in $\Omega \times \mathbb{R}^{2n}$, and does not intersect the zero section of $\Omega \times \mathbb{R}^{2n}$. Consequently, each ω_0 in the projection $\mathcal{A}(\omega)$ of $\mathcal{A}(\omega, \mathbf{z}(0))$ onto Ω has the property that equation (6.14) admits a nonzero bounded solution. However, the compact invariant set $\mathcal{A}(\omega) \subseteq \Omega$ contains a minimal set. A contradiction has been reached, and the conclusion is that the desired sequence (s_k) exists. The existence of the sequence (t_k) is proved in a similar way, working now with the omega-limit set of the initial data.

As a consequence of this property and (6.22),

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left(\langle \mathbf{x}(t_k), \mathbf{y}(t_k)\rangle - \langle \mathbf{x}(s_k), \mathbf{y}(s_k)\rangle\right) \\ &= \int_{-\infty}^{\infty} \left(\|G^{1/2}(\omega{\cdot}t)\,\mathbf{x}(t)\|^2 + \|R^{-1/2}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,\mathbf{y}(t)\|^2\right) dt\,. \end{aligned}$$

This means that $G(\omega{\cdot}t)\,\mathbf{x}(t) = \mathbf{0} = B^T(\omega{\cdot}t)\,\mathbf{y}(t)$ for all $t \in \mathbb{R}$, and hence that $\mathbf{x}'(t) = A(\omega{\cdot}t)\,\mathbf{x}(t)$ and $\mathbf{y}'(t) = -A^T(\omega{\cdot}t)\,\mathbf{y}(t)$. Conditions C2 and C1, Theorem 6.4, and the characterization of null controllability in Remark 6.2.1 ensure that $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{y}(t) = \mathbf{0}$ for all $t \in \mathbb{R}$. In other words, (6.14) admits no nonzero bounded solution for each $\omega \in \Omega$, as asserted. This completes the proof of the existence of exponential dichotomy over Ω.

Now, condition C1 ensures that family (6.14) satisfies condition D2 and D3 (see Remark 6.10), which allows one to apply Theorem 5.58 in order to check the global existence of the Weyl functions $M^{\pm}(\omega)$ representing the Lagrange planes $l^{\pm}(\omega)$ of the initial data of the solutions bounded as $t \to \pm\infty$. And condition C2 ensures that also D2* holds (see Remark 6.8.2), which according to Proposition 5.64(i) guarantees that $M^+ < 0$ and $M^- > 0$. The proof is complete.

Remark 6.12 The proof of Proposition 5.64(ii) can be carried out by repeating the previous steps. In more detail: use conditions D2 and D2* and their respective characterizations (5.8) and (5.56) to prove that the families $\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u}$ and $\mathbf{x}' = -H_1^T(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u}$ are uniformly null controllable; define $\Gamma = \begin{bmatrix} H_2(\omega) & 0_n \\ 0_n & H_3(\omega) \end{bmatrix}$; adapt the proof of Proposition 6.7 to check that $\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}$ satisfies the Atkinson Hypotheses 3.3; and reason as in Lemma 6.9 and Theorem 6.11 to obtain the conclusion. See also the proof of Theorem 6.24 for a similar line of reasoning.

The following arguments are devoted to showing how, under conditions C1 and C2, Theorem 6.11 and its proof allow one to construct a stabilizing feedback control matrix K with good properties, as well as to solve the linear regulator problem. Thus the main goals of this section will be achieved.

Theorem 6.13 *Suppose that conditions C1 and C2 hold. There exists a continuous matrix-valued function $M^+\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ with $M^+ < 0$ such that if $K\colon \Omega \to \mathbb{M}_{m\times n}(\mathbb{R})$ is the continuous matrix-valued function defined by*

$$K(\omega) = R^{-1}(\omega)\,B^T(\omega)\,M^+(\omega)\,, \tag{6.23}$$

then $\mathbf{x} \equiv \mathbf{0}$ is a uniformly exponentially asymptotically stable solution of each of the equations of the family

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K(\omega{\cdot}t))\,\mathbf{x}\,, \qquad \omega \in \Omega\,. \tag{6.24}$$

More precisely, the family (6.24) is of uniform Hurwitz type at $+\infty$: there exist constants $\tilde\eta$ and β, independent of ω, such that any solution $\mathbf{x}(t)$ of (6.24) satisfies

$$\|\mathbf{x}(t)\| \le \tilde\eta\, e^{-\beta t}\,\|\mathbf{x}(0)\| \quad \textit{for } t \ge 0\,. \tag{6.25}$$

Proof Conditions C1 and C2 and Theorem 6.11 ensure the exponential dichotomy of the family of Hamiltonian systems (6.14) over Ω, as well as the existence of the continuous Weyl function M^+ representing the Lagrange planes $l^+(\omega)$, with $M^+ < 0$. In particular, for all $\mathbf{x}_0 \in \mathbb{R}^n$, the solution $\mathbf{z}(t)$ of (6.14) with initial datum $\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}_0 \\ M^+(\omega)\,\mathbf{x}_0 \end{bmatrix} \in l^+(\omega)$ takes the form $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ M^+(\omega{\cdot}t)\,\mathbf{x}(t) \end{bmatrix}$ and satisfies $\|\mathbf{z}(t)\| \le \eta\, e^{-\beta t}\|\mathbf{z}_0\|$ for $t \ge 0$, where the constants $\eta > 0$ and $\beta > 0$ are independent of ω: see Definition 1.75.

Note also that, if $K(\omega)$ is defined by (6.23), then the first component $\mathbf{x}(t)$ of $\mathbf{z}(t)$ is the solution of (6.24) with $\mathbf{x}(0) = \mathbf{x}_0$. The exponential decay of $\|\mathbf{z}(t)\|$ to zero ensures the same property for $\mathbf{x}(t)$. In fact, for all $\omega \in \Omega$ and $t \ge 0$,

$$\|\mathbf{x}(t)\| \le \tilde\eta\, e^{-\beta t}\,\|\mathbf{x}_0\|$$

for $\tilde\eta = \eta\,(1 + \|M^+(\omega)\|^2)^{1/2}$.

Therefore, for each $\omega \in \Omega$, the matrix $K(\omega)$ defined by (6.23) solves the feedback stabilization problem for the linear control system (6.10) under hypotheses C1 and C2. Note that K is continuous in ω; this property is referred to as *conservation of recurrence*. More regularity properties of K will be discussed in Sect. 6.3.

As stated before, the next step consists in solving the linear regulator problem for each $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Given $\mathbf{x}_0 \in \mathbb{R}^n$ and $\omega \in \Omega$, let $\bar{\mathbf{x}}(t)$ be the solution of (6.24) with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$, where K is given by (6.23), and define

$$\bar{\mathbf{u}}(t) = R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, M^+(\omega{\cdot}t)\, \bar{\mathbf{x}}(t) = K(\omega{\cdot}t)\, \bar{\mathbf{x}}(t)\,, \tag{6.26}$$

which is continuous on $\mathbb{R}$. Then (6.25) and the boundedness of B, G, R, and K ensure that $\mathcal{I}_{\mathbf{x}_0,\omega}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) < \infty$. It will be checked below that $(\mathbf{x}, \mathbf{u})$ is the sought-for minimizing pair. A preliminary technical result is required.

Lemma 6.14 *Suppose that condition* C2 *holds. Assume that* $\mathcal{I}_{\mathbf{x}_0,\omega}(\mathbf{x}, \mathbf{u}) < \infty$ *for a pair* $(\mathbf{x}, \mathbf{u})\colon [0, \infty) \to \mathbb{R}^m \times \mathbb{R}^n$ *which solves* (6.10)*, where* $\mathbf{u}$ *is integrable and* $\mathbf{x}(0) = \mathbf{x}_0$*, and where* $\mathcal{I}_{\mathbf{x}_0,\omega}$ *is given by* (6.11)*. Then,*

$$\mathbf{u} \in L^2([0, \infty), \mathbb{R}^m) \qquad \textit{and} \qquad \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}\,. \tag{6.27}$$

Proof The first assertion is immediate, since $R \geq \rho_R I_n > 0$ and $G \geq 0$. To check the second one, suppose for contradiction that there exist $(t_k) \uparrow \infty$ and $\varepsilon > 0$ such that $\|\mathbf{x}(t_k)\| = \varepsilon_k \geq \varepsilon$. Define $\omega_k = \omega{\cdot}t_k$, $\mathbf{x}_k(t) = \mathbf{x}(t + t_k)/\varepsilon_k$, $\mathbf{u}(t) = \mathbf{u}(t + t_k)/\varepsilon_k$ and $\mathbf{x}_k = \mathbf{x}_k(0) = \mathbf{x}(t_k)/\varepsilon_k$, and note that $\mathbf{x}_k(t)$ solves the initial value problem

$$\begin{aligned} \mathbf{x}' &= A(\omega_k{\cdot}t)\, \mathbf{x} + B(\omega_k{\cdot}t)\, \mathbf{u}_k(t)\,, \\ \mathbf{x}(0) &= \mathbf{x}_k\,. \end{aligned}$$

Define also

$$\begin{aligned} \mathcal{I}_k &= \frac{1}{2}\int_0^\infty \big(\langle \mathbf{x}_k(t), G(\omega_k{\cdot}t)\, \mathbf{x}_k(t)\rangle + \langle \mathbf{u}_k(t), R(\omega_k{\cdot}t)\, \mathbf{u}_k(t)\rangle\big)\, dt \\ &= \frac{1}{2\varepsilon_k^2}\int_0^\infty \Big(\langle \mathbf{x}(t+t_k), G(\omega_k{\cdot}t)\, \mathbf{x}(t+t_k)\rangle \\ &\qquad\qquad + \langle \mathbf{u}(t+t_k), R(\omega_k{\cdot}t)\, \mathbf{u}(t+t_k)\rangle\Big)\, dt \\ &= \frac{1}{2\varepsilon_k^2}\int_{t_k}^\infty \big(\langle \mathbf{x}(t), G(\omega{\cdot}t)\, \mathbf{x}(t)\rangle + \langle \mathbf{u}(t), R(\omega{\cdot}t)\, \mathbf{u}(t)\rangle\big)\, dt\,. \end{aligned}$$

Then $\lim_{n\to\infty} \mathcal{I}_k = 0$, since $\mathcal{I}_{\omega,\mathbf{x}_0}(\mathbf{x}, \mathbf{u}) < \infty$ and $\varepsilon_k \geq \varepsilon > 0$. This implies that $\lim_{k\to\infty} \int_0^\infty \|\mathbf{u}_k(t)\|^2\, dt = 0$, since $R > 0$ and $G \geq 0$. Let $t_0 > 0$ satisfy

$$\int_0^{t_0} U_A^T(t, \omega)\, G(\omega{\cdot}t)\, U_A(t, \omega)\, dt > 0 \tag{6.28}$$

for all $\omega \in \Omega$. The existence of this time t_0 is ensured by the uniform null controllability of the family $\mathbf{x}' = -A^T(\omega{\cdot}t)\,\mathbf{x} + G^{1/2}(\omega{\cdot}t)\,\mathbf{u}$, which in turn is guaranteed by condition C2 and by Theorem 6.4. The condition $\lim_{k\to\infty}\int_0^{t_0}\|\mathbf{u}_k(t)\|^2\,dt = 0$ implies the existence of a suitable subsequence $(\mathbf{u}_j)$ such that $\lim_{j\to\infty}\mathbf{u}_j(t) = \mathbf{0}$ for Lebesgue-a.e. $t \in [0, t_0]$ (see e.g. Theorem 3.12 of [128]).

Assume without loss of generality that $\omega_j \to \omega_*$ and $\mathbf{x}_j \to \mathbf{x}_*$, with $\|\mathbf{x}_*\| = 1$. It is easy to check that, for $t \in [0, t_0]$, the limit $\lim_{j\to\infty}\mathbf{x}_j(t) = \mathbf{x}_*(t) = U_A(t, \omega_*)\,\mathbf{x}_*$ is the solution of

$$\begin{aligned}
&\mathbf{x}' = A(\omega_*{\cdot}t)\,\mathbf{x}\,,\\
&\mathbf{x}(0) = \mathbf{x}_*\,.
\end{aligned}$$

This means that

$$\begin{aligned}
\int_0^{t_0} \mathbf{x}_*^T U_A^T(t,\omega_*)\,G(\omega_*{\cdot}t)\,U_A(t,\omega_*)\,\mathbf{x}_*\,dt &= \int_0^{t_0} \langle \mathbf{x}_*(t), G(\omega_*{\cdot}t)\,\mathbf{x}_*(t)\rangle \\
&\le \lim_{k\to\infty}\int_0^{t_0} \big(\langle \mathbf{x}_k(t), G(\omega_k{\cdot}t)\,\mathbf{x}_k(t)\rangle + \langle \mathbf{u}_k(t), R(\omega_k{\cdot}t)\,\mathbf{u}_k(t)\rangle\big)\,dt \\
&\le \lim_{k\to\infty} 2\mathcal{I}_k = 0\,,
\end{aligned}$$

which contradicts (6.28) and hence completes the proof.

Consequently, the search for a minimizing pair can be limited to those pairs $(\mathbf{x}, \mathbf{u})$ satisfying (6.27).

It is known (see e.g. Sect. 1.3.5) that the function $M^+(\omega)$ of Theorem 6.13 is a solution along the flow of the Riccati equation

$$M' = -M\,B\,R^{-1}B^T\,M - M\,A - A^T\,M + G\,,$$

where A, B, G, and R have argument $\omega{\cdot}t$. The usual completing-the-square trick will be applied to this equation. Thus let $(\mathbf{x}, \mathbf{u})\colon [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^m$ solve (6.10), where $\mathbf{u}$ is a locally integrable control and $\mathbf{x}(0) = \mathbf{x}_0$. Some manipulation leads to

$$\begin{aligned}
\frac{d}{dt}\,\langle M^+(\omega{\cdot}t)\,\mathbf{x}(t), \mathbf{x}(t)\rangle &= -\,\|R^{1/2}\big(\mathbf{u}(t) - R^{-1}B^TM^+\mathbf{x}(t)\big)\|^2 \\
&\quad + \big(\langle \mathbf{x}(t), G\,\mathbf{x}(t)\rangle + \langle \mathbf{u}(t), R\,\mathbf{u}(t)\rangle\big)
\end{aligned}$$

at the points t at which $\mathbf{x}'(t)$ exist (i.e. at Lebesgue-a.e. $t \in [0,\infty)$), where B, G, R, and M^+ have argument $\omega{\cdot}t$. So, if a pair $(\mathbf{x}, \mathbf{u})$ with the preceding conditions satisfies (6.27), then integrating the above relation gives

$$2\mathcal{I}_{\mathbf{x}_0,\omega}(\mathbf{x}, \mathbf{u}) = \int_0^\infty \|R^{1/2}(\mathbf{u}(t) - R^{-1}B^TM^+\mathbf{x}(t))\|^2\,dt - \langle M^+(\omega)\,\mathbf{x}_0, \mathbf{x}_0\rangle\,.$$

Hence $\mathcal{I}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u}) \geq -(1/2)\langle M^+(\omega)\,\mathbf{x}_0,\mathbf{x}_0\rangle$, which is nonnegative, since $M^+ < 0$, and the minimum value is attained if and only if (6.26) holds. This leads to:

Theorem 6.15 *Suppose that conditions* C1 *and* C2 *hold, and let M^+ be the matrix-valued function of Theorem 6.13. For each $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, let $\bar{\mathbf{x}}(t)$ be the solution of* (6.24) *with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$, where K is given by* (6.23)*, and define $\bar{\mathbf{u}}(t) = K(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)$. Then $(\bar{\mathbf{x}},\bar{\mathbf{u}})$ is the unique pair which minimizes the functional $\mathcal{I}_{\mathbf{x}_0,\omega}$ given by* (6.11)*, and $\mathcal{I}_{\mathbf{x}_0,\omega}(\bar{\mathbf{x}},\bar{\mathbf{u}}) = -(1/2)\langle M^+(\omega)\,\mathbf{x}_0,\mathbf{x}_0\rangle$.*

This completes the solution of the nonautonomous linear regulator problem under conditions C1 and C2.

Remark 6.16 If the family (6.10) satisfies Definition 6.3 (or equivalently condition C1, according to Theorem 6.4), then, for all $\omega \in \Omega$, $\mathbf{x}_1 \in \mathbb{R}^n$ and $t_1 \in \mathbb{R}$, there exists an integrable (continuous, as a matter of fact) control $\mathbf{u}\colon [t_1, t_1+t_0] \to \mathbb{R}^m$ such that the solution of the system (6.10) with $\mathbf{x}(t_1) = \mathbf{x}_1$ satisfies $\mathbf{x}(t_1+t_0) = \mathbf{0}$: just take $\mathbf{u}(t) = \bar{\mathbf{u}}(t+t_1)$, where $\bar{\mathbf{u}}\colon [0,t_0] \to \mathbb{R}^m$ is the control which steers $\mathbf{x}_1$ to $\mathbf{0}$ in time t_0 for the system corresponding to $\bar{\omega} = \omega{\cdot}t_1$. In other words, each of these systems is *uniformly* null controllable. More precisely, in this situation,

$$U_A(t_1,\omega)\left(\int_{t_1}^{t_1+t_0} U_A^{-1}(t,\omega)\,B(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt\right)U_A^T(t_1,\omega) \geq \delta I_n\,,$$

for each $t_1 \in \mathbb{R}$, where t_0 and δ are independent of t_1 and ω, as can be deduced from (6.13) for $\omega{\cdot}t_1$.

Conversely, suppose that a single system (6.1) satisfies

$$U_A(t_1)\left(\int_{t_1}^{t_1+t_0} U_A^{-1}(t)\,B(t)\,B^T(t)\,(U_A^{-1})^T(t)\,dt\right)U_A^T(t_1) \geq \delta I_n \tag{6.29}$$

for all $t_1 \in \mathbb{R}$, with common values of t_0 and δ. Here, as usual, U_A satisfies $\mathbf{x}' = A(t)\,\mathbf{x}$ with $U_A(0) = I_n$. Then a simple continuity argument proves that the family constructed in the hull is uniformly null controllable. In fact, if ω_0 is the element of the hull corresponding to the initial system, then

$$\int_0^{t_0} U_A^{-1}(t,\omega_0{\cdot}t_1)\,B((\omega_0{\cdot}t_1){\cdot}t)\,B^T((\omega_0{\cdot}t_1){\cdot}t)\,(U_A^{-1})^T(t,\omega_0{\cdot}t_1)\,dt \geq \delta I_n\,,$$

for all $t_1 \in \mathbb{R}$, which makes the proof of the assertion trivial.

Assume now that the (less restrictive) condition satisfied by the initial system is

$$\int_0^{\infty} U_A^{-1}(t)\,B(t)\,B^T(t)\,(U_A^{-1})^T(t)\,dt > 0$$

(i.e. that the initial system is null controllable: see Remark 6.2.1), and that in addition the hull Ω obtained from the data (A, B, G, R) by the usual Bebutov procedure is minimal, which is in particular the case if the initial coefficients are Bohr almost periodic: see Sect. 1.3.2. Then Theorem 6.4 ensures that the corresponding family of control systems (6.10) is uniformly null controllable.

This remark indicates possible ways to reformulate the hypotheses assumed in the main results of this section, Theorems 6.13 and 6.15, in terms of a single initial control system (6.1) and a single quadratic functional (6.3). Since the results are valid for all $\omega \in \Omega$, the conclusions of these two theorems can also be rewritten without any mention of Ω: the function M^+ can be directly defined from the Lagrange plane of the solutions of the corresponding single Hamiltonian system (6.8) bounded at $+\infty$; $K(t) = R^{-1}(t)\, B^T(t)\, M^+(t)$; and the minimum value for $\mathcal{I}_{\mathbf{x}_0}$ is $-(1/2)\langle M^+(0)\, \mathbf{x}_0, \mathbf{x}_0\rangle$.

This section is completed by showing how to solve the nonautonomous feedback stabilization and linear regulator problems when hypothesis C2 is replaced by the existence of exponential dichotomy for the family (6.14). This last property is in fact ensured by conditions C1 and C2, as proved by Theorem 6.11, but it can hold in more general situations: just consider the scalar autonomous case $A \equiv -1$, $B = R \equiv 1$ and $G \equiv 0$.

Theorem 6.17 *Suppose that condition* C1 *holds and the family* (6.14) *has exponential dichotomy over Ω. Then all the conclusions of Theorem 6.13 hold, except that now the Weyl function M^+ satisfies $M^+ \leq 0$.*

Proof Lemma 6.9 and Theorem 5.58 ensure the existence of the matrix-valued functions $M^\pm$ representing the Lagrange planes of the solutions of (6.14) which are bounded as $t \to \pm\infty$. In addition, $\pm M^+ \leq 0$, as Remark 1.81.2 ensures. The proof of Theorem 6.13 can be repeated to obtain the desired conclusions.

Note that $M^+ \equiv 0$ in the autonomous example mentioned before the theorem; this shows that the result is optimal.

To prove the analogue of Theorem 6.15 in this situation requires to restrict the domain of the operator $\mathcal{I}_{\mathbf{x}_0,\omega}$ to square integrable pairs $(\mathbf{x}, \mathbf{u})$. It also requires the following auxiliary lemma, which substitutes Lemma 6.14. Once this is done, the proof is identical.

Lemma 6.18 *Suppose that the pair* $(\mathbf{x}, \mathbf{u}) \in L^2([0,\infty), \mathbb{R}^n) \times L^2([0,\infty), \mathbb{R}^n)$ *satisfies* (6.10) *for a point $\omega \in \Omega$ and $t \geq 0$. Then* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

Proof The hypotheses ensure that $\mathbf{x}$ and $\mathbf{x}'$ belong to $L^2([0,\infty), \mathbb{R}^n)$, so that $\langle \mathbf{x}, \mathbf{x}'\rangle \in L^1([0,\infty), \mathbb{R}^n)$. Since $\|\mathbf{x}(t)\|^2 = 2\int_0^t \langle \mathbf{x}(s), \mathbf{x}'(s)\rangle\, ds + \|\mathbf{x}(0)\|^2$, there exists the limit as $t \to \infty$ of $\|\mathbf{x}(t)\|^2$, and the L^2-integrability of $\mathbf{x}$ ensures that its value is 0.

Theorem 6.19 *Suppose that condition* C1 *holds and the family* (6.14) *has exponential dichotomy over Ω. Let M^+ be the matrix-valued function of Theorem 6.17. Let $\mathcal{I}^*_{\mathbf{x}_0,\omega}$ represent the restriction of the functional $\mathcal{I}_{\mathbf{x}_0,\omega}$ given by* (6.11) *to the set of pairs* $(\mathbf{x}, \mathbf{u}) \in L^2([0,\infty), \mathbb{R}^n) \times L^2([0,\infty), \mathbb{R}^n)$ *satisfying* (6.10) *with* $\mathbf{x}(0) = \mathbf{x}_0$. *For* *each* $\omega \in \Omega$ *and* $\mathbf{x}_0 \in \mathbb{R}^n$*, let* $\bar{\mathbf{x}}(t)$ *be the solution of* (6.24) *with* $\bar{\mathbf{x}}(0) = \mathbf{x}_0$*, where* K *is given by* (6.23)*, and define* $\bar{\mathbf{u}}(t) = K(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)$*. Then* $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *is the unique pair which minimizes* $\mathcal{I}^*_{\mathbf{x}_0,\omega}$*, and* $\mathcal{I}^*_{\mathbf{x}_0,\omega}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = -(1/2)\langle M^+(\omega)\,\mathbf{x}_0, \mathbf{x}_0\rangle$.

Remark 6.20 As stated in the proof of Theorem 6.17, if the family (6.14) has exponential dichotomy over Ω and if also condition C1 holds, then both Weyl functions $M^\pm$ exist. This fact will be useful in the following section.

6.3 Regularity Properties of the Stabilizing Control

This third section is devoted to showing that, if A and B depend in a C^r manner on parameters, then the feedback stabilizing matrix K depends also in a C^r way on those parameters. Moreover, if Ω is a C^{r+1} manifold, if the one-parameter group $\{\sigma_t \mid t \in \mathbb{R}\}$ is determined by a C^r-vector field on Ω, and if the functions A and B are C^r on Ω, then in certain circumstances K can be chosen so as to be C^r-dependent on ω. In general, the regularity properties of K will depend on the regularity properties of the coefficient functions A, B, G, and R as well as on those of the function M^+. However, it will be explained that the regularity of M^+ depends to a certain extent on properties of the base flow σ.

The first objective of this section is to carry out the parametric analysis. So, without assuming anything else on Ω, suppose that $A = A(\omega, e)$, $B = B(\omega, e)$, $G = G(\omega, e)$, and $R(\omega, e)$ depend on a parameter e lying in a Banach space $\mathbb{E}$, and suppose that A, B, G, and R are continuous on $\Omega \times \mathbb{E}$, with values in the appropriate sets of matrices. This condition implies that the map $\mathbb{E} \to C(\Omega, \mathbb{M}_{n\times n}(\mathbb{R}))$, $e \mapsto A(\cdot, e)$ is continuous for the norm-topology defined by $\|C\|_\Omega = \max_{\omega\in\Omega} \|C(\omega)\|$, so that if e_n converges to e^* in $\mathbb{E}$ then $A(\cdot, e_n)$ converges to $A(\cdot, e^*)$ uniformly on Ω; and the same holds for B, G, and R.

Suppose also that the controllability condition C1, as well as the exponential dichotomy property of the family of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} A(\omega{\cdot}t, e) & B(\omega{\cdot}t, e)\,R^{-1}(\omega{\cdot}t, e)\,B^T(\omega{\cdot}t, e) \\ G(\omega{\cdot}t, e) & -A^T(\omega{\cdot}t, e) \end{bmatrix} \mathbf{z} = H(\omega{\cdot}t, e)\,\mathbf{z} \tag{6.30}$$

over Ω hold for $e = e_0$, where e_0 is a given point of $\mathbb{E}$. (Note that, according to Theorem 6.11, this situation is less restrictive than assuming the controllability conditions C1 and C2 for e_0.) Then C1 and the exponential dichotomy over Ω hold for all e in some open neighborhood $\mathcal{O} \subseteq \mathbb{E}$ of e_0. The assertion concerning C1 follows from Theorem 6.4, as is easily deduced from Definition 6.3 and the description of the continuity of A and B with respect to e. And the assertion about the exponential dichotomy follows from Theorem 1.91(ii). The same result ensures that, if $\mathcal{Q}(e) = \{Q(\omega, e)\}$ is the dichotomy projector of (6.30) for $e \in \mathcal{O}$ (see Definition 1.58), then the map Q is continuous on $\Omega \times \mathcal{O}$. Moreover, the dichotomy constants η and β, which *a priori* depend on $e \in \mathcal{O}$, can be chosen to be positive numbers which do not depend on $(\omega, e) \in \Omega \times \mathcal{O}$. In addition, an application of Theorem 1.95 shows that the map $\Omega \times \mathcal{O} \to \mathbb{S}_n(\mathbb{K})$, $(\omega, e) \mapsto M^+(\omega, e)$ is continuous. Here, $M^+(\omega, e)$ is the Weyl function provided by Theorem 6.17 for $e \in \mathcal{O}$. Hence, the map $\Omega \times \mathcal{O} \to \mathbb{M}_{m\times n}(\mathbb{R})$, $(\omega, e) \mapsto K(\omega, e) = R^{-1}(\omega, e)\, B^T(\omega, e)\, M^+(\omega, e)$ is jointly continuous.

Suppose now that, for a given integer $k \geq 1$, and for each $j = 1, \ldots k$, the Fréchet derivatives $D^j_e A$, $D^j_e B$, $D^j_e G$, and $D^j_e R$ with respect to e are defined and continuous on $\Omega \times \mathbb{E}$. In order to study the consequences regarding higher regularity of the dichotomy projections $Q(\omega, e)$ of the family (6.30) for $\omega \in \Omega$ and $e \in \mathcal{O}$, one verifies the hypotheses of Theorem 3.1 of Yi [160]. First, as already noted, the dichotomy constants η and β of (6.30) can be chosen so as not to depend on $e \in \mathcal{O}$. Second, there is a uniform constant δ such that, for each $j = 1, \ldots, k$, there holds

$$\sup_{\omega\in\Omega,\, e\in\mathcal{O}} \|D^j_e H(\omega, e)\| \leq \delta\,.$$

Thus Theorem 3.1 of [160] can be applied (with $N = 0$ in the notation of that theorem): the conclusions are that the map $\mathcal{O} \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$, $e \mapsto Q(\omega, e)$ is of class C^k for each $\omega \in \Omega$, that the map $(\omega, e) \mapsto Q(\omega, e)$ is continuous on $\Omega \times \mathcal{O}$, and that the Fréchet derivatives $D^j_e Q(\omega, e)$ are continuous on $\Omega \times \mathcal{O}$ for $j = 1, \ldots, k$. In fact that theorem applies to a fixed $\omega_0 \in \Omega$ and it ensures the continuity of $D^j_e Q(\omega_0, e)$ with respect to e, but its proof can be generalized in a direct way to prove the asserted joint continuity.

Recall that the present goal is to derive the regularity properties of the feedback matrices $K(\omega, e)$ provided by Theorem 6.19 for $e \in \mathcal{O}$. But using Remarks 6.20 and 1.81.1, it can be seen that $M^+ \colon \Omega \times \mathcal{O} \to \mathbb{S}_n(\mathbb{R})$ exists and is of class C^r in $e \in \mathcal{O}$ for each $\omega \in \Omega$, and that the Fréchet derivatives $D^j_e M^+$ are continuous on $\Omega \times \mathcal{O}$ for $j = 1, \ldots, k$. The following result is therefore proved.

Proposition 6.21 *Let $\mathbb{E}$ be a Banach space, and let A, B, G, and R be continuous functions on $\Omega \times \mathbb{E}$ taking values in the appropriate sets of matrices. Suppose that there exists $k \geq 0$ such that the Fréchet derivatives $D^j_e A$, $D^j_e B$, $D^j_e G$, and $D^j_e R$ are defined and continuous on $\Omega \times \mathbb{E}$ for each $j = 0, 1, \ldots, k$. And suppose also that there exists $e_0 \in \mathbb{E}$ such that: either the controllability conditions C1 and C2 hold for the matrices $A(\cdot, e_0)$, $B(\cdot, e_0)$, and $G(\cdot, e_0)$; or C1 holds and the corresponding family (6.30) has exponential dichotomy over Ω. Then there is an open neighborhood $\mathcal{O} \subseteq \mathbb{E}$ of e_0 such that the feedback control matrix-valued function $K \colon \Omega \times \mathcal{O} \to \mathbb{M}_{m\times n}(\mathbb{R})$ given by*

$$K(\omega, e) = R^{-1}(\omega, e)\, B^T(\omega, e)\, M^+(\omega, e)$$

is well defined and continuous on $\Omega \times \mathcal{O}$. Also its Fréchet derivatives $D^j_e K$ are defined and continuous on $\Omega \times \mathcal{O}$ for each $j = 0, 1, \ldots, k$. And, in addition, there exist positive constants $\bar{\eta}$ and $\bar{\beta}$ which do not depend on $(\omega, e) \in \Omega \times \mathcal{O}$, such that, *if* $\omega \in \Omega$, $e \in \mathcal{O}$, *and* $\mathbf{x}_0 \in \mathbb{R}^n$, *then the solution* $\mathbf{x}(t)$ *of the equation*

$$\mathbf{x}' = (A(\omega{\cdot}t, e) + B(\omega{\cdot}t, e)\, K(\omega{\cdot}t, e))\,\mathbf{x}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies*

$$\|\mathbf{x}(t)\| \le \bar{\eta}\, e^{-\bar{\beta} t}\|\mathbf{x}_0\|$$

for all $t \ge 0$.

Turning now to the second objective of this section, consider the situation in which Ω is a smooth manifold of class C^{k+1} for $k \ge 1$. Let $\|\cdot\|$ denote the Finsler structure induced on the tangent bundle of Ω by a fixed Riemannian metric on the base. Let f be a C^k-vector field on Ω, and let $\{\sigma_t \mid t \in \mathbb{R}\}$ be the 1-parameter group of diffeomorphisms of Ω induced by f. Then the map $\sigma \colon \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t \equiv \sigma_t(\omega)$ is a real C^k flow on Ω, and the k-order derivative $D^k_\omega \sigma_t$ of σ_t is C^1 with respect to t.

Assume that A, B, G, and R are all C^k-functions on Ω, that condition C1 holds, and that the family of systems (6.14) has exponential dichotomy over Ω, so that according to Theorem 6.19 a feedback matrix-valued function $K \colon \Omega \to \mathbb{M}_{m\times n}(\mathbb{R})$ exists. The objective is to find conditions sufficient to guarantee that K is C^k as a function on Ω. This is clearly possible if the function M^+ is C^k on Ω, as derived from definition (6.23). For this, however, it is not enough that A, B, G, and R are C^k-smooth. To obtain a sufficient condition, and following again Yi [160] (see also Palmer [119]), one can impose conditions on σ and on the family (6.14) ensuring that the dichotomy constant β dominates the hyperbolicity of $\{\sigma_t \mid t \in \mathbb{R}\}$.

To begin the discussion, observe that the ω-derivative $D_\omega \sigma_t(\omega)$ satisfies the variational equation

$$\frac{d}{dt} D_\omega \sigma_t(\omega) = Df(\sigma_t(\omega))\, D_\omega \sigma_t(\omega)\,, \tag{6.31}$$

and that $D_\omega \sigma_0(\omega)$ is the identity map in the tangent bundle of Ω for any ω. Define the *Bohl exponent* of the family of systems (6.31) by

$$\beta_B = \limsup_{|t|\to\infty} \frac{1}{|t|} \ln\left(\sup_{\omega\in\Omega} \|D_\omega \sigma_t(\omega)\|\right). \tag{6.32}$$

It is not hard to deduce from (6.31) that $\beta_B \le \sup_{\omega\in\Omega} \|Df(\omega)\|$, but the strict inequality may hold. And, if $\beta_1 > \beta_B$ is any fixed number, then there is a constant $\eta_1 \ge 0$ with $\sup_{\omega\in\Omega} \|D_\omega \sigma_t(\omega)\| \le \eta_1\, e^{\beta_1 |t|}$.

Fix such a value of $\beta_1 > \beta_B$ and differentiate successively equation (6.31) with respect to ω. Note that $D^j_\omega \sigma_0(\omega) = 0$ for $j = 2, \ldots, k$. Using the variation of parameters formula, one finds constants $\eta_2, \ldots, \eta_k$ such that

$$\sup_{\omega\in\Omega} \|D^j_\omega \sigma_t(\omega)\| \le \eta_j\, e^{j\,\beta_1 |t|} \quad \text{for } j = 2, \ldots, k\,.$$

Theorem 3.1 of [160] can be applied to prove the following result.

Proposition 6.22 *Assume that A, B, G, and R are all C^k functions on Ω, and that either conditions C1 and C2 hold or C1 holds and the family (6.14) has exponential dichotomy over Ω. Let η and β be the dichotomy constants provided by Theorem 6.11 and Definition 1.75. And let β_B be defined by (6.32). If $(k+1)\,\beta_B < \beta$, then the dichotomy projection Q is of class C^k on Ω.*

By combining, as before, Remarks 6.20 and 1.81.1, it follows that the map $\omega \mapsto M^+(\omega)$ is of class C^k if $(k+1)\beta_B < \beta$. Hence the feedback map $\omega \mapsto K(\omega)$ given by (6.23) is a C^k function of ω. To illustrate this result, suppose that Ω is the d-torus $\mathbb{R}^d/\mathbb{Z}^d$ and that σ is a minimal Kronecker flow on Ω: $\sigma(t,\omega) = \omega + \gamma t$ where γ is a vector of rationally independent frequencies. Then $D_\omega\sigma_t(\omega)$ is the identity for all $t \in \mathbb{R}$ and $\omega \in \Omega$, and hence $\beta_B = 0$. So, in this case, the functions M^+ and K are of class C^k on Ω when this is true for A, B, G, and R.

The third and last part of this section consists of a discussion of the so-called *pole relocation property* in the nonautonomous setting. This means the following. Take $\gamma > 0$. The objective is to choose the stabilizing feedback control K_γ in such a way that there exists a number $\eta_\gamma \ge 0$ with the property that, if $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, and if $\mathbf{x}(t)$ is the solution of the problem

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K(\omega{\cdot}t))\,\mathbf{x}$$

with $\mathbf{x}(0) = \mathbf{x}_0$, then $\|\mathbf{x}(t)\| \le \eta_\gamma e^{-\gamma t}\|\mathbf{x}_0\|$ for all $t \ge 0$. It is required that η_γ be independent of ω and $\mathbf{x}_0$.

Such a feedback control can be found by adapting a well-known technique (see [2] and [1]). The following hypotheses are assumed: A, B, G, and R are all continuous matrix-valued functions on Ω, and the controllability conditions C1 and C2 hold. Consider the modified family of control systems

$$\mathbf{x}' = (A(\omega{\cdot}t) + \gamma I_n)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}\,, \qquad \omega \in \Omega\,. \tag{6.33}$$

Then the controllability condition C1 holds for (6.33). In order to check this assertion, note that $U_{A+\gamma I_n}(t,\omega) = e^{\gamma t} U_A(t,\omega)$. By Theorem 6.4, there exist $t_0 > 0$ and $\delta > 0$ such that (6.13) holds for all $\omega \in \Omega$. Therefore, there exists $\delta_\gamma > 0$ such that

$$\int_0^{t_0} e^{-2\gamma t}\, U_A^{-1}(t,\omega)\, B(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, (U_A^{-1})^T(t,\omega)\, dt \ge \delta_\gamma\, I_n$$

for all $\omega \in \Omega$; or, in other words, such that

$$\int_0^{t_0} U_{A+\gamma I_n}^{-1}(t,\omega)\, B(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, (U_{A+\gamma I_n}^{-1})^T(t,\omega)\, dt \geq \delta_\gamma\, I_n$$

for all $\omega \in \Omega$. Once this fact is established, one can use Theorem 6.4 to guarantee that C1 holds for (6.33). In a similar way, it can be checked that condition C2 is valid for the family of control systems

$$\mathbf{x}' = -(A(\omega{\cdot}t) + \gamma I_n)^T \mathbf{x} + G^{1/2}(\omega{\cdot}t)\, \mathbf{u}\,, \qquad \omega \in \Omega\,.$$

Apply now Theorem 6.13 to obtain a continuous function $M_\gamma^+ \colon \Omega \to \mathbb{S}_n(\mathbb{R})$ such that, if $K_\gamma(\omega) = R^{-1}(\omega)\, B^T(\omega)\, M_\gamma^+(\omega)$, then $\mathbf{x} \equiv \mathbf{0}$ is a uniformly exponentially stable solution of each system of the family

$$\mathbf{x}' = \big(A(\omega{\cdot}t) + \gamma I_n + B(\omega{\cdot}t) K_\gamma(\omega)\big)\, \mathbf{x}\,, \qquad \omega \in \Omega\,.$$

It is then clear from (6.25) and from $U_{A+\gamma I_n}(t,\omega) = e^{-\gamma t} U_{A+\gamma I_n + BK}(t,\omega)$ that the required number γ_η indeed exists.

Note finally that, if condition C2 is changed to the existence of exponential dichotomy of (6.14) over Ω, the same conclusion holds but just for values of $\gamma > 0$ small enough to guarantee the exponential dichotomy of the families of Hamiltonian systems obtained by replacing A by $A + \gamma I_n$ in (6.14). The existence of such values of γ is ensured, for instance, by Theorem 1.91(ii). And, in this case, the sought-for K_γ is provided by Theorem 6.17.

6.4 The Kalman–Bucy Filter

As stated in the introduction to this chapter, the Kalman–Bucy filter is a standard method to measure the mean-square error between the unknown output signal and the estimated output signal of a linear plant subject to a white noise disturbance. It was first studied by Kalman and Bucy in [88] and has of course stimulated much further research: see e.g. Anderson and Moore [2, 3], Benavoli and Chisci [16], Fagnani and Willems [50], and Bell et al. [15].

The simplest case is, as usual, that in which the structure coefficients of the plant do not depend on time. In this situation, if one assumes that the signals in question are Gaussian, and if one further assumes that the initial error signal has a known expected value and a known covariance matrix, then the time-evolved covariance matrix of the error signal can be shown to tend exponentially fast to a constant matrix, which is a solution of a stationary Riccati equation.

When the linear plant has time-varying structure coefficients, the Kalman–Bucy filter has similar properties, though the analysis which leads to them seems less known. Bougerol [19–21] has studied the case when the coefficients are determined by a stationary ergodic process. He has proved that, under certain controllability hypotheses, the error covariance matrix tends almost certainly to a time-varying “stationary state” which is a solution of a time-dependent Riccati equation. The convergence takes place with nonuniform exponential velocity.

This section analyzes the Kalman–Bucy filter assuming that the coefficients are bounded and uniformly continuous functions of time. As in the case of the linear regulator problem, the concepts of exponential dichotomy and rotation number for linear Hamiltonian differential systems will play a basic role in the present approach to the Kalman–Bucy filter. Indeed, it is not surprising that dichotomies and rotation number turn out to be significant here as well, due to a formal analogy between two Riccati equation arising in the Kalman–Bucy filter model and the Riccati equation that presents itself when studying the linear regulator problem.

The situation analyzed here is actually subsumed in that studied by Bougerol, who works in a general measure space setting. His results hold under more general hypotheses than those of this section. However, the statements included here, under more restrictive hypotheses, are considerably stronger than his. Roughly speaking, his almost everywhere exponential estimates are strengthened to uniform exponential estimates. Also, the dichotomy-based approach allows one to appeal to general theoretical facts when reasoning, and to avoid detailed manipulation of a Riccati equation. In addition, the filter depends regularly on parameters in a wide sense.

As stated in the introduction, the presentation that follows is based on that of Johnson and Núñez [83].

The discussion begins with a review of some basic facts concerning the Kalman–Bucy filter as they are presented in [51] (see especially pages 135–141). Some standard concepts of probability theory and the theory of Itô differential equations will be used. The reader is referred to [51] for the necessary definitions and results.

Let A, B, S, and S_1 be bounded and uniformly continuous matrix-valued functions of the respective dimensions $n\times n$, $m\times n$, $n\times d$, and $m\times m$ where n, m, d are positive integers. Assume that $S_1S_1^T$ is strictly positive definite: there exists $\rho > 0$ such that $(S_1S_1^T)(t) \geq \rho I_m$ for all $t\in\mathbb{R}$.

The data A, B, S, and S_1 determine a linear system which will be written down shortly. Let $\boldsymbol{\xi}(t)\in\mathbb{R}^n$ denote the state of that linear system at time $t\geq 0$. Assume that the state can only be partially observed, and let $\boldsymbol{\eta}(t)\in\mathbb{R}^m$ be the observation of the state at time t. Assume also that $\boldsymbol{\xi}(t)$ is subject to a random d-dimensional disturbance, and that $\boldsymbol{\eta}(t)$ is subject to a random m-dimensional noise. The state evolution is modeled by the Itô differential equation

$$d\boldsymbol{\xi}(t) = A(t)\,\boldsymbol{\xi}(t)\,dt + S(t)\,d\mathbf{w}(t)\,,$$

while the equation for the observation is

$$d\boldsymbol{\eta}(t) = B(t)\,\boldsymbol{\xi}(t)\,dt + S_1(t)\,d\mathbf{w}_1(t)\,.$$

Here $\mathbf{w}(t)$ and $\mathbf{w}_1(t)$ are independent Brownian motion processes, of dimensions d and m. Let there be given a initial time, which might as well be taken to be $t = 0$. Assume that $\boldsymbol{\eta}(0) = \mathbf{0}$. And assume finally that $\boldsymbol{\xi}(0)$ is Gaussian, which implies that $\boldsymbol{\xi}(t)$ is Gaussian for all $t > 0$: this is explained in Chapter V.9 of [51].

Let Σ_t denote the σ-algebra generated by the set $\{\boldsymbol{\eta}(r) \mid 0 \leq r \leq t\}$ of observations available at time t. The goal is to describe an estimate $\boldsymbol{\gamma}(t)$ for the state $\boldsymbol{\xi}(t)$, based on the observation up to time t, which is Σ_t-measurable, with $E\{|\boldsymbol{\gamma}(t)|^2\} < \infty$ for $t \geq 0$, and which minimizes the mean-square error $E\{(\mathbf{x}^T(\boldsymbol{\xi}(t) - \boldsymbol{\gamma}(t)))^2\}$ for all $\mathbf{x} \in \mathbb{R}^n$. Here $E\{\cdot\}$ denotes the expected value with respect to a probability measure defined on a probability space with respect to which all occurring stochastic processes are defined and measurable. The first step is to write down an expression for this minimizer: as proved in [51] (see page 136), the minimizer is given by the conditional expectation vector

$$\hat{\boldsymbol{\xi}}(t) = E\{\boldsymbol{\xi}(t) \mid \Sigma_t\},$$

and it turns out that it also minimizes $E\{\|\boldsymbol{\xi}(t) - \boldsymbol{\gamma}(t)\|^2\}$. In particular, if $\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t) - \hat{\boldsymbol{\xi}}(t)$ represents the error process, then $E\{\tilde{\boldsymbol{\xi}}(t)\} = 0$ for all $t \geq 0$.

The main results concerning the Kalman–Bucy filter, given in Theorem V.9.2 of [51], read as follows. As explained in Remark V.9.1 of [51], the first assertion of the following theorem ensures that $\tilde{\boldsymbol{\xi}}(t)$ is Gaussian for all $t \geq 0$. Therefore, the covariance matrix

$$M(t) = E\{(\tilde{\boldsymbol{\xi}}(t) - E\{\tilde{\boldsymbol{\xi}}(t)\})(\tilde{\boldsymbol{\xi}}(t) - E\{\tilde{\boldsymbol{\xi}}(t)\})^T\} = E\{\tilde{\boldsymbol{\xi}}(t)\,\tilde{\boldsymbol{\xi}}(t)^T\}$$

is well defined (and positive semidefinite) for $t \geq 0$, and it determines the law of $\tilde{\boldsymbol{\xi}}(t)$.

Theorem 6.23 *Suppose that A, B, S, and S_1 are C^1 functions in $(0, \infty)$. The estimate $\hat{\boldsymbol{\xi}}(t)$ satisfies the Itô differential equation*

$$d\hat{\boldsymbol{\xi}}(t) = A(t)\,\hat{\boldsymbol{\xi}}(t)\,dt + F(t)(d\boldsymbol{\eta}(t) - B(t)\,\hat{\boldsymbol{\xi}}(t)\,dt),$$
$$\hat{\boldsymbol{\xi}}(0) = E\,\boldsymbol{\xi}(0),$$

where $F(t) = M(t)\,B^T(t)\,(S_1S_1^T)^{-1}(t)$. The error $\tilde{\boldsymbol{\xi}}(t)$ is independent of Σ_t; the error covariance matrix $M(t)$, which is defined and positive semidefinite for all $t \geq 0$, satisfies the equation

$$M' = -M\,B^T(t)\,(S_1S_1^T)^{-1}(t)\,B(t)\,M + M\,A^T(t) + A(t)\,M + (SS^T)(t) \tag{6.34}$$

on $(0, \infty)$, with $M(0)$ given by the covariance matrix of $\boldsymbol{\xi}(0)$; and, finally,

$$\boldsymbol{\eta}(t) - \int_0^t B(s)\,\hat{\boldsymbol{\xi}}(s)\,ds = \int_0^t S_1(s)\,d\hat{\mathbf{w}}_1(s),$$

where $\hat{\mathbf{w}}_1$ is a standard m-dimensional Brownian motion, which has the property that $\hat{\mathbf{w}}_1(t)$ is Σ_t-measurable for each $t > 0$.

Using the fact that the error $\tilde{\boldsymbol{\xi}}(t)$ equals $\boldsymbol{\xi}(t) - \hat{\boldsymbol{\xi}}(t)$, and making a simple computation, one finds that

$$d\tilde{\boldsymbol{\xi}}(t) = (A(t) - F(t)\,B(t))\,\tilde{\boldsymbol{\xi}}(t)\,dt + S(t)\,d\mathbf{w}(t) - F(t)\,S_1(t)\,d\mathbf{w}_1(t)\,.$$

This relation explains why it is important that the homogeneous system $d\tilde{\boldsymbol{\xi}}(t) = (A(t) - F(t)\,B(t))\,\tilde{\boldsymbol{\xi}}(t)\,dt$ be of Hurwitz type (see Definition 1.72): in this case $\tilde{\boldsymbol{\xi}}(t)$ "sees" only a nonanticipative process which depends boundedly on $d\mathbf{w}(t)$ and $d\mathbf{w}_1(t)$.

The proof of Theorem 6.23 is given in [51] and is omitted here. Note that it requires that A, B, S, and S_1 be C^1 functions of t.

Theorem 6.23 will be taken as a starting point to discuss the following points:

(1) the asymptotic limit $M_\infty(t)$ of the error covariance matrix $M(t)$;
(2) the (exponential) rate of approach of $M(t)$ to $M_\infty(t)$;
(3) the Hurwitz nature at $+\infty$ of the linear system $\mathbf{y}' = (A(t) - F(t)\,B(t))\,\mathbf{y}$.

The treatment of points (1), (2), and (3) will not require differentiability assumptions on A, B, S, and S_1. It will begin with (6.34), which is the Riccati equation corresponding to the linear Hamiltonian system

$$\mathbf{z}' = \begin{bmatrix} -A^T(t) & B^T(t)\,(S_1S_1^T)^{-1}(t)\,B(t) \\ (SS^T)(t) & A(t) \end{bmatrix} \mathbf{z} = \tilde{H}(t)\,\mathbf{z}\,. \tag{6.35}$$

Note that the Bebutov hull $(\widetilde{\Omega}, \tilde{\sigma})$ of the matrix-valued function $\widetilde{H}$ can be defined, since $\widetilde{H}$ has no stochastic component. As was explained in Sect. 1.3.2, the functions A, B, S, and S_1 "extend continuously" to $\widetilde{\Omega}$; by abusing notation, these extended functions will be also represented by A, B, S, and S_1. This procedure provides then the families of Riccati equations

$$M' = -M\,B^T(S_1S_1^T)^{-1}B\,M + M\,A^T + A\,M + SS^T \tag{6.36}$$

(with A, B, S, and S_1 evaluated in $\omega{\cdot}t$) and of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} -A^T(\omega{\cdot}t) & B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)\,B(\omega{\cdot}t) \\ (SS^T)(\omega{\cdot}t) & A(\omega{\cdot}t) \end{bmatrix} \mathbf{z} = \widetilde{H}(\omega{\cdot}t)\,\mathbf{z} \tag{6.37}$$

for $\omega \in \widetilde{\Omega}$, which include the original equation (6.34) and system (6.35): they coincide for a point $\omega_0 \in \widetilde{\Omega}$.

The first key point in the analysis of (1), (2), and (3) is this: certain controllability conditions will ensure the occurrence of exponential dichotomy over $\widetilde{\Omega}$ for the family (6.37), as well as the existence of Weyl functions $\widetilde{M}^{\pm}$ satisfying $\mp\widetilde{M}^{\pm} > 0$;

the function $M_\infty = \widetilde{M}^-$ will be the one which appears in (1) and (2); and, at the same time, the existence of M^- allows one to prove (3).

So, the following step is to describe the analogues of the controllability conditions C1 and C2 in the context of the families of control systems which correspond to the systems (6.35); these analogues will be assumed in the following analysis.

C3. Each minimal subset of $\widetilde{\Omega}$ contains at least one point ω_1 such that the system

$$\mathbf{x}' = -A^T(\omega_1{\cdot}t)\,\mathbf{x} + B^T(\omega_1{\cdot}t)\,\mathbf{u}$$

is null controllable.

C4. Each minimal subset of $\widetilde{\Omega}$ contains at least one point ω_2 such that the system

$$\mathbf{x}' = A(\omega_2{\cdot}t)\,\mathbf{x} + S(\omega_2{\cdot}t)\,\mathbf{u}$$

is null controllable.

Remark 6.16 explains the situations in which these properties are guaranteed by conditions on the control systems corresponding to the initial coefficients $A(t)$, $B(t)$, $S(t)$, and $S_1(t)$.

The family (6.37) can be compared with the family (6.14). In fact by making the substitutions $A \mapsto -A^T$, $B \mapsto B^T$, $S_1S_1^T \mapsto R$, and $SS^T \mapsto G$, system (6.37) is transformed into (6.14). In fact the ideas appearing in Sect. 6.2 are the fundamental ones needed to prove the exponential dichotomy over $\widetilde{\Omega}$ of the family (6.37).

Note first that, if conditions C3 and C4 hold, then Theorem 6.4 implies the uniform null controllability of the families of control systems over $\widetilde{\Omega}$: that is, the existence of numbers $t_0 > 0$ and $\delta > 0$ such that for all $\omega \in \widetilde{\Omega}$,

$$\begin{aligned}
&\int_0^{t_0} U_A^T(t,\omega)\,B^T(\omega{\cdot}t)\,B(\omega{\cdot}t)\,U_A(t,\omega)\,dt \ge \delta\, I_n\,,\\
&\int_0^{t_0} U_A^{-1}(t,\omega)\,S(\omega{\cdot}t)\,S^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,dt \ge \delta\, I_n\,.
\end{aligned} \tag{6.38}$$

These numbers t_0 and δ will be of basic significance in the analysis of the nonautonomous Kalman–Bucy filter.

Theorem 6.24 *Suppose that conditions* C3 *and* C4 *hold. Then the family of Hamiltonian systems* (6.37) *has exponential dichotomy over* $\widetilde{\Omega}$. *In addition, both Weyl functions* $\widetilde{M}^+$ *and* $\widetilde{M}^-$ *are globally defined, and they satisfy* $\widetilde{M}^+ < 0$ *and* $\widetilde{M}^- > 0$.

Proof The way to adapt the arguments used in the proof of Theorem 6.11 to this situation will be described. (See also Remark 6.12.) Set $\widetilde{\Gamma}(\omega) = \begin{bmatrix} (SS^T)(\omega) & 0_n \\ 0_n & B^T(\omega)\,(S_1S_1^T)^{-1}(\omega)\,B(\omega) \end{bmatrix}$, and consider the perturbed family of linear

Hamiltonian systems

$$\mathbf{z}' = \big(\widetilde{H}(\omega{\cdot}t) + \lambda J^{-1}\widetilde{\Gamma}(\omega{\cdot}t)\big)\,\mathbf{z} \tag{6.39}$$

for $\omega \in \widetilde{\Omega}$ and $-1 < \lambda < 1$, with $\widetilde{H}$ given by (6.37). One checks that the controllability conditions C3 and C4 imply that the family (6.39) satisfies the following uniform Atkinson condition: there exist $t_0 > 0$ and $\delta > 0$ such that for all $\omega \in \widetilde{\Omega}$,

$$\int_0^{t_0} \|\,\widetilde{\Gamma}(\omega{\cdot}t)\,U(t,\omega)\,\mathbf{z}\|^2\,dt \ge \delta\,\|\mathbf{z}\|^2 \quad \text{whenever } \mathbf{z} \in \mathbb{R}^{2n}\,.$$

This is demonstrated in the same way as in the proof of Proposition 6.7. Keep in mind that the null controllability of $\mathbf{x}' = A(\omega_2{\cdot}t)\,\mathbf{x} + S(\omega_2{\cdot}t)\,\mathbf{u}$ is equivalent to that of $\mathbf{x}' = A(\omega_2{\cdot}t)\,\mathbf{x} + (SS^T)^{1/2}(\omega_2{\cdot}t)\,\mathbf{u}$, as is easily deduced from the characterization given in Remark 6.2.1.

Next, let m be a $\widetilde{\sigma}$-ergodic measure on $\widetilde{\Omega}$, and let $\alpha_{\widetilde{\Gamma}}(\lambda)$ be the rotation number of the family (6.39) with respect to m. Repeating the proof of Lemma 6.9, one checks that, if $-1 < \lambda < 1$, then the corresponding perturbed families are uniformly weakly disconjugate, with $\alpha_{\widetilde{\Gamma}}(\lambda) = 0$.

Now, follow step by step the arguments of Theorem 6.11 in order to prove the occurrence of exponential dichotomy: first, Theorem 3.50, the Atkinson character of $\widetilde{\Gamma}$, and the properties of the rotation number ensure the existence of exponential dichotomy over each minimal subset of $\widetilde{\Omega}$ for the perturbed families corresponding to $\lambda \in (-1, 1)$, so that in particular this holds for $\lambda = 0$; and second, this property precludes the existence of a globally bounded solution for any of the systems of the family (6.37), which according to Theorem 1.78 ensures the existence of exponential dichotomy over the whole base.

Once this is established, Theorem 5.58 ensures that the corresponding Lagrange planes $\widetilde{l}^{\pm}(\omega)$ (which also determine the principal solutions) belong to $\mathcal{D}$ for all $\omega \in \widetilde{\Omega}$. And finally, the second inequality in (6.38) allows one to repeat the argument used to prove Proposition 5.64(i) in order to ensure that $\mp\widetilde{M}^{\pm} > 0$.

Remark 6.25 As stated in the previous proof, the family of systems (6.37) is uniformly weakly disconjugate, and the principal functions agree with the Weyl functions. In particular, Theorem 5.48(i) ensures that any solution of the Riccati equation (6.36) with initial datum $M_0 \ge 0$ (and hence with $M_0 \ge \widetilde{M}^{+}(\omega)$) is defined for all $t \ge 0$. More properties of these solutions will be described later.

The second fundamental point in the treatment of points (1), (2), and (3) presented in this chapter consists of an interesting property of the action of the symplectic matrices on the set of positive definite symmetric matrices, which is now described. It is derived from considerations presented in Bougerol [21] and Wojtkowski [152]. This action is described in the following technical lemmas, whose significance will become clear beginning from Proposition 6.28 on. Introduce the quadratic form

$$q\colon \mathbb{R}^{2n} \to \mathbb{R}\,, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle\,, \tag{6.40}$$

and recall the definition of the open subset $\mathcal{D} \subset \mathcal{L}_{\mathbb{R}}$ (see Sect. 1.3.5),

$$\mathcal{D} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \right\} \subset \mathcal{L}_{\mathbb{R}}\,.$$

Note that Theorem 6.24 ensures that the Lagrange plane $\tilde{l}^-(\omega) \equiv \begin{bmatrix} I_n \\ M_\infty(\omega) \end{bmatrix}$ belongs to $\mathcal{D}$ for all $\omega \in \Omega$, where $M_\infty = \widetilde{M}^-$.

Lemma 6.26 *Define*

$$\mathcal{L}_+ = \{ l \in \mathcal{L}_{\mathbb{R}} \mid \textit{if } \mathbf{z} \in l \textit{ then } q(\mathbf{z}) \ge 0 \}\,.$$

Then,

(i) *$\mathcal{L}_+$ is compact, and $\mathcal{L}_+ \cap \mathcal{D} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \text{ and } M \ge 0 \right\}$.*
(ii) *The interior of $\mathcal{L}_+$ is*

$$\begin{aligned} \operatorname{int} \mathcal{L}_+ &= \{ l \in \mathcal{L}_{\mathbb{R}} \mid \textit{if } \mathbf{0} \ne \mathbf{z} \in l \textit{ then } q(\mathbf{z}) > 0 \} \\ &= \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \textit{ and } M > 0 \right\}. \end{aligned}$$

Proof

(i) The closed and hence compact character of $\mathcal{L}_+$ follows easily from the continuity of q and Proposition 1.26(i). In addition, if $l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix}$ and $\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}_0 \\ M\mathbf{x}_0 \end{bmatrix} \in l$, then $q(\mathbf{z}_0) = \mathbf{x}_0^T M \mathbf{x}_0$, so that l belongs to $\mathcal{L}_+ \cap \mathcal{D}$ if and only if $l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix}$ for $M \ge 0$.
(ii) Define $\mathcal{L}_+^* = \{ l \in \mathcal{L}_{\mathbb{R}} \mid \text{if } \mathbf{0} \ne \mathbf{z} \in l \text{ then } q(\mathbf{z}) > 0 \}$. It is clear that $\mathcal{L}_+^* \supseteq \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \text{ and } M > 0 \right\}$. To prove the "$\subseteq$" statement, take first $l \in \mathcal{L}_{\mathbb{R}}$ with $l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix}$ for a matrix M which is not positive definite, and take $\mathbf{x}_0 \in \mathbb{R}^n$ with $\mathbf{x}_0 \ne \mathbf{0}$ and $\mathbf{x}_0^T M \mathbf{x}_0 \le 0$. Then $\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}_0 \\ M\mathbf{x}_0 \end{bmatrix} \in l$ and $q(\mathbf{z}_0) = \mathbf{x}_0^T M \mathbf{x}_0 \le 0$. And if $l \equiv \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ with $\det L_1 = 0$, then l contains a nonzero vector $\begin{bmatrix} \mathbf{0} \\ \mathbf{y}_0 \end{bmatrix}$, at which q takes on the value 0. In both cases, $l \notin \mathcal{L}_+^*$, which proves the equality of both sets. Obviously, $\mathcal{L}_+^* \subseteq \operatorname{int} \mathcal{L}_+$. Take now $l \in \operatorname{int} \mathcal{L}_+$ and assume for contradiction that $l \notin \mathcal{L}_+^*$. One can immediately discard the possibility that $l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix}$ with $M \ge 0$ and $M \not> 0$. This leads to $l \equiv \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ with $\det L_1 = 0$. Take $\mathbf{x}_0 \ne \mathbf{0}$ with $L_1 \mathbf{x}_0 = \mathbf{0}$ and set $\mathbf{y}_0 = L_2 \mathbf{x}_0 \ne \mathbf{0}$. It is easy to check that $l_\varepsilon \equiv \begin{bmatrix} L_1 - \varepsilon L_2 \\ L_2 \end{bmatrix}$ is a Lagrange plane; it follows from Proposition 1.25 that it belongs to $\mathcal{L}_+$ for $\varepsilon > 0$ small enough; and it contains the vector $\begin{bmatrix} -\varepsilon \mathbf{y}_0 \\ \mathbf{y}_0 \end{bmatrix} = \begin{bmatrix} L_1 - \varepsilon L_2 \\ L_2 \end{bmatrix} \mathbf{x}_0$. But this is impossible, since $q\left(\begin{bmatrix} -\varepsilon \mathbf{y}_0 \\ \mathbf{y}_0 \end{bmatrix} \right) = -\varepsilon \|\mathbf{y}_0\|^2 < 0$.

Let $\mathbb{S}_n^+(\mathbb{R})$ be the set of positive definite $n \times n$ matrices. A distance function $\mathtt{d}$ can be defined on $\mathbb{S}_n^+(\mathbb{R})$ as follows: if M_1 and M_2 are positive definite, then

$$\mathtt{d}(M_1, M_2) = \left(\sum_{j=1}^{n} \ln^2 \lambda_j \right)^{1/2}, \tag{6.41}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $M_1 M_2^{-1}$ (which are real and positive, since they are also the eigenvalues of $M_2^{-1/2} M_1 M_2^{-1/2}$). In fact, this distance function $\mathtt{d}$ is that defined by the Riemannian metric $ds^2 = \operatorname{tr}((M^{-1}dM)^2)$ on $\mathbb{S}_n^+(\mathbb{R})$: see [97], Section 3. In particular, it induces the usual topology on $\mathbb{S}_n^+(\mathbb{R})$.

Lemma 6.27 *Let the matrix* $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ *be symplectic. Suppose that* $q(V\mathbf{z}) > 0$ *whenever* $q(\mathbf{z}) > 0$. *Then the action*

$$\widehat{V}: \mathbb{S}_n^+(\mathbb{R}) \to \mathbb{S}_n^+(\mathbb{R}), \quad M \mapsto \widehat{V}{\cdot}M = (V_2 + V_4 M)(V_1 + V_3 M)^{-1}$$

is well defined. Suppose further that $q(V\mathbf{z}) > 0$ *for all nonzero* $\mathbf{z} \in \mathbb{R}^{2n}$ *with* $q(\mathbf{z}) \geq 0$. *Then the action is a strict contraction: there exists a positive constant* $\delta_V < 1$ *such that*

$$\mathtt{d}(\widehat{V}{\cdot}M_1, \widehat{V}{\cdot}M_2) \leq \delta_V \, \mathtt{d}(M_1, M_2)$$

whenever M_1 *and* M_2 *are positive definite.*

Proof In order to prove that $\widehat{V}$ is well defined, use Lemma 6.26(ii) to see that $M \in \mathbb{S}_n^+(\mathbb{R})$ parameterizes a Lagrange plane $l \in \operatorname{int} \mathcal{L}_+$, and that the assumption on V ensures that $V{\cdot}l \in \operatorname{int} \mathcal{L}_+ \subset \mathcal{D}$, so that it is parameterized by a positive definite matrix. And this matrix is precisely $\widehat{V}{\cdot}M$.

The rest of the proof follows the arguments of the proof of Theorem 1.7 of [21], which can be adapted to the situation considered here. The first step is to write

$$V = \begin{bmatrix} I_n & V_3 V_4^{-1} \\ 0_n & I_n \end{bmatrix} \begin{bmatrix} (V_4^{-1})^T & 0_n \\ 0_n & V_4 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ V_4^{-1} V_2 & I_n \end{bmatrix}$$

and to ensure that $V_3 V_4^{-1}$ and $V_4^{-1} V_2$ are positive definite. This can be done as follows. Note first that, since $q(V\left[\begin{smallmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{smallmatrix}\right]) > 0$ and $q(V\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{y}_0 \end{smallmatrix}\right]) > 0$ whenever $\mathbf{x}_0 \neq \mathbf{0}$ and $\mathbf{y}_0 \neq \mathbf{0}$, it is the case that the four matrices V_1, V_2, V_3, and V_4 are nonsingular, as is easily checked by contradiction. On the other hand, according to Proposition 1.23, $V_4^T V_1 - V_2^T V_3 = I_n$, $V_2^T V_3 = V_3^T V_2$, and $V_4^T V_3 = V_3^T V_4$ which ensure that $V_1 = (V_4^T)^{-1} + V_3 V_4^{-1} V_2$. This proves that the above decomposition of V is valid. Also, since $V{\cdot}\left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} V_3 \\ V_4 \end{smallmatrix}\right]$ belongs to $\operatorname{int} \mathcal{L}_+$, one has $V_3 V_4^{-1} = (V_4 V_3^{-1})^{-1} > 0$. To check that $V_4^{-1} V_2$ is positive definite requires some more work. It is already known that it is nonsingular, and Proposition 1.23 ensures that it is symmetric. Assume for contradiction the existence of a negative eigenvalue λ and let $\mathbf{y}_0$ be an associated normalized eigenvector: $V_4^{-1}V_2\,\mathbf{y}_0 = \lambda\,\mathbf{y}_0$ and $\|\mathbf{y}_0\| = 1$. Then $V_2\,\mathbf{y}_0 = \lambda\,V_4\,\mathbf{y}_0$ and $V_1\,\mathbf{y}_0 = (V_4^T)^{-1}\,\mathbf{x}_0 + V_3\,V_4^{-1}V_2\,\mathbf{y}_0 = (V_4^T)^{-1}\,\mathbf{y}_0 + \lambda\,V_3\,\mathbf{y}_0$, so that, for all real numbers μ,

$$V\begin{bmatrix}\mathbf{y}_0\\ \mu\,\mathbf{y}_0\end{bmatrix} = \begin{bmatrix}(V_4^T)^{-1}\,\mathbf{y}_0 + (\lambda+\mu)\,V_3\,\mathbf{y}_0\\ (\lambda+\mu)\,V_4\,\mathbf{y}_0\end{bmatrix}.$$

Note that $V_4^TV_3 = V_3^TV_4 = V_3^T(V_4\,V_3^{-1})\,V_3 > 0$, so $\alpha = \mathbf{y}_0^TV_4^TV_3\,\mathbf{y}_0 > 0$. Hence,

$$\begin{aligned}q(V\left[\begin{smallmatrix}\mathbf{y}_0\\ \mu\mathbf{y}_0\end{smallmatrix}\right]) &= (\lambda+\mu)\,\mathbf{y}_0^TV_4^T(V_4^T)^{-1}\mathbf{y}_0 + (\lambda+\mu)^2\,\mathbf{y}_0^TV_4^TV_3\,\mathbf{y}_0\\ &= \lambda+\mu+(\lambda+\mu)^2\,\alpha = \alpha(\mu+\lambda)(\mu+\lambda+1/\alpha)\,.\end{aligned}$$

Choose now $\mu \in (-\lambda - 1/\alpha, -\lambda)$ with $\mu > 0$. Then $q(\left[\begin{smallmatrix}\mathbf{y}_0\\ \mu\mathbf{y}_0\end{smallmatrix}\right]) > 0$, whereas $q(V\left[\begin{smallmatrix}\mathbf{y}_0\\ \mu\mathbf{y}_0\end{smallmatrix}\right]) < 0$. This contradicts the assumptions on V.

It is not hard to prove that $\widehat{VW}{\cdot}M = \widehat{V}{\cdot}(\widehat{W}{\cdot}M)$ for two matrices V and W when all the terms are defined. Consequently, when acting on $\mathbb{S}_n^+(\mathbb{R})$ (and adapting the notation to that of [21]),

$$\widehat{V} = \sigma\circ\tau_2\circ\sigma\circ\gamma\circ\tau_1$$

where $\tau_1{\cdot}M = M + V_4^{-1}V_2$, $\gamma{\cdot}M = V_4MV_4^T$, $\sigma{\cdot}M = M^{-1}$, and $\tau_2{\cdot}M = M + V_3V_4^{-1}$. From here, the proof of the Bougerol theorem can be repeated. First, the definition (6.41) of the distance on $\mathbb{S}_n^+(\mathbb{R})$, implies that the maps γ and σ are isometries. Second, Proposition 1.6 of [21] proves that τ_1 is a contraction on $\mathbb{S}_n^+(\mathbb{R})$. Hence, $\sigma\circ\gamma\circ\tau_1$ is also a contraction. Now, it is easy to check that $(\gamma\circ\tau_1){\cdot}M \ge V_2V_4^T$ for all $M > 0$, and hence that $0 < (\sigma\circ\gamma\circ\tau_1){\cdot}M \le (V_2V_4^T)^{-1}$ for all $M > 0$. Therefore, Proposition 1.6 of [21] and the invertibility of $V_3V_4^{-1}$ provide a constant $\delta_V < 1$ such that

$$\begin{aligned}&\mathrm{d}((\tau_2\circ\sigma\circ\gamma\circ\tau_1){\cdot}M_1, (\tau_2\circ\sigma\circ\gamma\circ\tau_1){\cdot}M_2)\\ &\qquad\le \delta_V\,\mathrm{d}((\sigma\circ\gamma\circ\tau_1){\cdot}M_1, (\sigma\circ\gamma\circ\tau_1){\cdot}M_2) \le \delta_V\,\mathrm{d}(M_1, M_2)\end{aligned}$$

whenever $M_1 > 0$ and $M_2 > 0$, from which the assertion follows.

The importance of the previous properties in the forthcoming analysis is due to a fact which is proved in the following proposition: under conditions C3 and C4, if $U(t,\omega)$ is the fundamental matrix solution of (6.37) with $U(0,\omega) = I_{2n}$, then $U(t,\omega)$ satisfies the first hypothesis imposed on V on Lemma 6.27 for all $t > 0$, and the second one for $t \ge t_0$; so that it induces an action on $\mathbb{S}_n(\mathbb{R})$ which is a strict contraction from a certain time on; and this strict contraction will be the key in the asymptotic analysis carried out later.

Proposition 6.28 *Let q be defined by (6.40), and let $\widetilde{\tau}$ be the flow induced by (6.37) on $\widetilde{\Omega} \times \mathcal{L}_{\mathbb{R}}$. Then,*

(i) $q(U(t,\omega)\,\mathbf{z}) \geq q(\mathbf{z})$ *for all* $t \geq 0$, $\omega \in \widetilde{\Omega}$ *and* $\mathbf{z} \in \mathbb{R}^{2n}$.

In addition, if conditions C3 and C4 hold and if t_0 satisfies (6.38), then

(ii) $q(U(t,\omega)\,\mathbf{z}) > q(\mathbf{z})$ *for all* $t \geq t_0$, $\omega \in \widetilde{\Omega}$ *and* $\mathbf{z} \in \mathbb{R}^{2n} - \{\mathbf{0}\}$.
(iii) *The sets* $\widetilde{\Omega} \times \mathcal{L}_+$, $\widetilde{\Omega} \times (\mathcal{L}_+ \cap \mathcal{D})$ *and* $\widetilde{\Omega} \times \operatorname{int}\mathcal{L}_+$ *are positively* $\widetilde{\tau}$*-invariant, and* $\widetilde{\tau}_t(\widetilde{\Omega} \times \mathcal{L}_+) \subseteq \widetilde{\Omega} \times \operatorname{int}\mathcal{L}_+$ *if* $t \geq t_0$.
(iv) *The set* $\mathcal{K} = \widetilde{\tau}_{t_0}(\widetilde{\Omega} \times \mathcal{L}_+)$ *is compact, and it contains* $\widetilde{\tau}_t(\widetilde{\Omega} \times \mathcal{L}_+)$ *if* $t \geq t_0$.

Proof Write $\mathbf{z} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}$ and $U(t,\omega)\,\mathbf{z} = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$. According to (1.34),

$$\langle \mathbf{x}(t), \mathbf{y}(t)\rangle - \langle \mathbf{x}_0, \mathbf{y}_0\rangle = \int_0^t \left(\|S^T\mathbf{x}(s)\|^2 + \|S_1^{-1}B\,\mathbf{y}(s)\|^2\right) ds\,,$$

where S, S_1, and B have argument $\omega{\cdot}s$. This proves (i).

(ii) If the right-hand side of the previous equality is null for $t \geq t_0$, then $B(\omega{\cdot}t)\,\mathbf{y}(t) = \mathbf{0}$ and $S^T(\omega{\cdot}t)\,\mathbf{x}(t) = \mathbf{0}$ in $[0,t_0]$, which in turn ensure that $\mathbf{x}' = -A^T(\omega{\cdot}t)\,\mathbf{x}$ and $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$. That is, $B(\omega{\cdot}t)\,U_A(t,\omega)\,\mathbf{y}_0 = \mathbf{0}$ and $S^T(\omega{\cdot}t)\,(U_A^{-1})^T(t,\omega)\,\mathbf{x}_0 = \mathbf{0}$ in $[0,t_0]$, which under conditions C3 and C4 implies (by using (6.38)) that $\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{0}$ and proves (ii).
(iii) These properties follow immediately from (ii) and Remark 6.25.
(iv) The compactness of the set $\mathcal{K}$ is evident, and the last assertion follows from the equality $\widetilde{\tau}_{t_0+s}(\omega,l) = \widetilde{\tau}_{t_0}(\omega{\cdot}s, U(s,\omega){\cdot}l)$ and the positive $\widetilde{\tau}$-invariance of $\widetilde{\Omega} \times \mathcal{L}_+$.

With all these preliminaries out of the way, the behavior of the error covariance matrix of the Kalman–Bucy filter can be analyzed. In fact the family of Kalman–Bucy filters indexed by the points $\omega \in \widetilde{\Omega}$ will be studied: as explained before, the initial filter corresponds to one of the points $\omega_0 \in \widetilde{\Omega}$. Fix $\omega \in \widetilde{\Omega}$ and $M_0 \geq 0$, represent by $M(t,\omega,M_0)$ the solution of the corresponding Riccati equation (6.36) with $M(0,\omega,M_0) = M_0$, and deduce from Remark 6.25 and Proposition 6.28(iii) that $M(t,\omega,M_0)$ exists and is positive semidefinite for all $t \geq 0$. Theorem 6.23 ensures that $M_\omega(t) = M(t,\omega,M_\omega^0)$ is the error covariance matrix of the Kalman–Bucy filter given by the point $\omega \in \widetilde{\Omega}$ when M_ω^0 is the initial covariance matrix if the additional condition that $t \mapsto (A(\omega{\cdot}t), B(\omega{\cdot}t), S(\omega{\cdot}t), S_1(\omega{\cdot}t))$ is a C^1 function is satisfied. But in fact this last assumption is not imposed in what follows: it will be shown that, when evaluated along the positive $\widetilde{\sigma}$-orbit of a point ω, M_∞ attracts the solution $M(t,\omega,M_0)$ as $t \to \infty$; and that, in addition, the rate of convergence is exponential, and uniform in (ω, M_0). Note that this will be proved for all the solutions of the Riccati equation (6.36) given by a positive semidefinite initial datum. Actually, the result can be proved using certain general properties of the dichotomy projections. But a different approach will be used in the proof here given.

According to Proposition 6.28(iii) and (iv) and Lemma 6.26(ii), the compact set $\mathcal{K} = \widetilde{\tau}_{t_0}(\widetilde{\Omega} \times \mathcal{L}_+)$ is contained in $\widetilde{\Omega} \times \operatorname{int}\mathcal{L}_+$ and hence can be identified with a compact subset of $\widetilde{\Omega} \times \mathbb{S}_n^+(\mathbb{R})$, which will also be denoted by $\mathcal{K}$. In addition, for each $\omega \in \widetilde{\Omega}$ and each $t \geq t_0$, the symplectic matrix $U(t,\omega)$ satisfies the conditions imposed on V in Lemma 6.27, which hence ensures that $U(t,\omega)$ induces a strict contraction on $\mathbb{S}_n^+(\mathbb{R})$. This fact and the compactness of Ω ensure the existence of a common constant $\delta < 1$ such that $\mathtt{d}(\widehat{U}(t_0,\omega){\cdot}M_1, \widehat{U}(t_0,\omega){\cdot}M_2) \leq \delta\, \mathtt{d}(M_1, M_2)$ whenever (ω, M_1) and (ω, M_2) belong to $\mathcal{K}$.

Take now $s \geq 0$. Since $U(2t_0 + s, \omega) = U(t_0, \omega{\cdot}(t_0 + s))\, U(t_0 + s, \omega)$ and $(\omega{\cdot}(t_0 + s), \widehat{U}(t_0 + s, \omega){\cdot}M) \in \mathcal{K}$ whenever $M \geq 0$ (see Proposition 6.28(iv)), it follows that

$$\mathtt{d}(\widehat{U}(2t_0 + s, \omega){\cdot}M_1, \widehat{U}(2t_0 + s, \omega){\cdot}M_2) \leq \delta\, \mathtt{d}(\widehat{U}(t_0 + s, \omega){\cdot}M_1, \widehat{U}(t_0 + s, \omega){\cdot}M_2)$$

whenever $M_1 \geq 0$ and $M_2 \geq 0$. A recursive procedure proves that, if $t = (m + 1)\, t_0 + s$ for an integer $m \geq 1$ and $s \in [0, t_0)$, then

$$\mathtt{d}(\widehat{U}(t,\omega){\cdot}M_1, \widehat{U}(t,\omega){\cdot}M_2) \leq \delta^m \mathtt{d}(\widehat{U}(t_0 + s, \omega){\cdot}M_1, \widehat{U}(t_0 + s, \omega){\cdot}M_2)$$

whenever $M_1 \geq 0$ and $M_2 \geq 0$. In particular, for this value of t and all $\omega \in \widetilde{\Omega}$, and each initial datum $M_0 \geq 0$,

$$\mathtt{d}(M(t,\omega,M_0), M_\infty(\omega{\cdot}t)) \leq \delta^m \mathtt{d}(M_\omega(t_0 + s), M_\infty(\omega{\cdot}t))\,, \tag{6.42}$$

since $\widehat{U}(t,\omega){\cdot}M_0 = M(t,\omega,M_0)$, as explained in Sect. 1.3.5.

Recall the information provided by Theorem 6.24 in order to understand the following statement.

Theorem 6.29 *Suppose that conditions C3 and C4 hold, and let $M_\infty = \widetilde{M}^-$ be the Weyl function associated to the stable subbundle at $+\infty$. With the notation previously established, there exist constants $\beta > 0$ and $\eta > 0$ such that*

$$\|M(t,\omega,M_0) - M_\infty(\omega{\cdot}t)\| \leq \eta\, e^{-\beta t}$$

whenever $\omega \in \widetilde{\Omega}$, $M_0 \geq 0$, and $t \geq 2t_0$, where t_0 satisfies (6.38).

Proof Let δ satisfy (6.42) and set $\beta = -(\ln\delta)/(3t_0)$. It is easy to check that $\beta \leq -(m\ln\delta)/((m + 2)t_0)$ for all $m \geq 1$, and hence $\delta^m \leq e^{-ct}$ whenever $t \in [(m + 1)t_0, (m + 2)t_0)$. These facts and the inequality (6.42) ensure that

$$\mathtt{d}(M(t,\omega,M_0), M_\infty(\omega{\cdot}t)) \leq e^{-\beta t}\, \mathtt{d}(M(t_0 + s, \omega, M_0), M_\infty(\omega{\cdot}(t_0 + s))) \tag{6.43}$$

whenever $t \geq 2t_0$, $s \in [0, t_0)$, and $\omega \in \widetilde{\Omega}$. Now, in order to apply Proposition 6.28(iv), let $\mathcal{K}$ be, as before, the compact subset of $\widetilde{\Omega} \times \mathbb{S}_n^+(\mathbb{R})$ equivalent to the subset $\widetilde{\tau}_{t_0}(\widetilde{\Omega} \times \mathcal{L}_+)$ of $\Omega \times \mathcal{D}$. Then, for $t = (m + 1){\cdot}t_0 + s \geq 2t_0$, the four pairs $(\omega{\cdot}t, M(t,\omega,M_0))$, $(\omega{\cdot}t, M_\infty(\omega{\cdot}t))$, $(\omega{\cdot}(t_0 + s), M(t_0 + s, \omega, M_0))$ and

$(\omega{\cdot}(t_0+s), M_\infty(\omega{\cdot}(t_0+s)))$ belong to $\mathcal{K}$. It is obvious that the set $\mathcal{K}_2 = \{M \in \mathbb{S}_n^+(\mathbb{R}) \mid \text{there is } \omega \in \widetilde{\Omega} \text{ with } (\omega, M) \in \mathcal{K}\}$ is compact. Note that $M_\omega(t)$, $M_\infty(\omega{\cdot}t)$, $M(t_0+s, \omega, M_0)$ and $M_\infty(\omega{\cdot}(t_0+s))$ belong to this compact set. Of course, there exists a common bound η_1 for $\|M_1 - M_2\|$ if $M_1, M_2 \in \mathcal{K}_2$, and it is also clear that there exists η_2 such that $\mathrm{d}(M_1, M_2) \le \eta_2\|M_1 - M_2\|$ for all pairs if $M_1, M_2 \in \mathcal{K}_2$, since both norms induce the same topology. All these properties and (6.43) provide a number $\eta = \eta_1\,\eta_2$ such that the statement of the theorem holds.

Remark 6.30 The proof of Proposition 5.64(iii) repeats the arguments used in the previous theorem. The key point is to check that the results of Proposition 6.28 are also valid for the Hamiltonian system (5.4) of Chap. 5 when $H_2 \ge 0$, $H_3 \ge 0$, and conditions D2 and D2* hold, and this can be done following the same steps as in Proposition 6.28. Keep in mind that, according to (5.8) and (5.56), conditions D2 and D2* guarantee the existence of $t_0 > 0$ and $\delta > 0$ such that

$$\int_0^{t_0} \|H_3(\omega{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{x}\|^2\,dt \ge \delta\|\mathbf{x}\|^2\,,$$

$$\int_0^{t_0} \|H_2(\omega{\cdot}t)\,U_{H_1}(t,\omega)\,\mathbf{x}\|^2\,dt \ge \delta\|\mathbf{x}\|^2\,.$$

The proof of assertion (iv) requires some preliminary work, but the ideas are the same. Keep now in mind that conditions D2 and D2* ensure the existence of $t_0 > 0$ and $\delta > 0$ such that

$$\int_{-t_0}^{0} \|H_3(\omega{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{x}\|^2\,dt \ge \delta\|\mathbf{x}\|^2\,,$$

$$\int_{-t_0}^{0} \|H_2(\omega{\cdot}t)\,U_{H_1}(t,\omega)\,\mathbf{x}\|^2\,dt \ge \delta\|\mathbf{x}\|^2\,.$$

The first inequality is due to the equivalence between D2 and D2′ proved in Proposition 5.18(iii), and the second one can be checked with the same argument. The details are left to the reader.

Theorem 6.29 completes the analysis of questions (1) and (2). The treatment of the Kalman–Bucy filter is finished with a discussion of point (3): the Hurwitz character at $+\infty$ of the system $\mathbf{y}' = (A(\omega{\cdot}t) - F_\omega(t)\,B(\omega{\cdot}t))\,\mathbf{y}$, where $F_\omega(t) = M_\omega(t)\,B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)$ and $M_\omega(t) = M(t,\omega,M_\omega^0)$ is the error covariance matrix of the Kalman–Bucy filter given by the point $\omega \in \widetilde{\Omega}$; i.e. of the system

$$\mathbf{y}' = \big(A(\omega{\cdot}t) - M_\omega(t)\,B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)\,B(\omega{\cdot}t)\big)\,\mathbf{y} = A_\omega(t)\,\mathbf{y}\,, \tag{6.44}$$

when $\omega \in \widetilde{\Omega}$ is fixed. Actually, the exponential rate should be uniform in $\omega \in \widetilde{\Omega}$. In fact, a reasonable additional hypothesis is sufficient to provide the uniform Hurwitz character of the family (6.44).

Proposition 6.31 *Suppose that conditions C3 and C4 hold. There exist constants* $\eta_* > 0$ *and* $\beta_* > 0$ *such that, if* $\omega \in \widetilde{\Omega}$ *and* $\mathbf{y}_\omega(t)$ *is any solution of the equation* (6.44)*, then*

$$\|\mathbf{y}_\omega(t)\| \le \eta_*\, e^{-\beta_* t}\|\mathbf{y}_\omega(2t_0)\|$$

for $t \ge 2t_0$*, where* t_0 *satisfies* (6.38)*. In addition, if* $\widetilde{\Omega} \to \mathbb{S}_n(\mathbb{R}),\ \omega \mapsto M^0_\omega$ *is a continuous map, then there exists* $\bar{\eta} > 0$ *such that*

$$\|\mathbf{y}_\omega(t)\| \le \bar{\eta}\, e^{-\beta_* t}\|\mathbf{y}_\omega(0)\|$$

for all $t \ge 0$.

Proof Recall that the Hamiltonian system (6.37) has exponential dichotomy, as follows from Theorem 6.24, and that $\tilde{l}^-(\omega) \equiv \begin{bmatrix} I_n \\ M_\infty(\omega) \end{bmatrix}$ is the Lagrange plane of the solutions which tend to 0 as $t \to -\infty$. Of course, the function M_∞ is continuous on Ω, and hence bounded. Therefore, it follows easily from Definition 1.75 that there exist constants $\eta_1 > 0$ and $\beta_1 > 0$ such that, if $\mathbf{x}(t)$ is any solution of any one of the systems

$$\mathbf{x}' = \big(-A^T(\omega{\cdot}t) + B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)\,B(\omega{\cdot}t)\,M_\infty(\omega{\cdot}t)\big)\,\mathbf{x}\,, \tag{6.45}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ for arbitrary $\mathbf{x}_0 \in \mathbb{R}^n$, then $\|\mathbf{x}(t)\| \le \eta_1\, e^{-\beta_1 t}\,\|\mathbf{x}_0\|$ for all $t \le 0$. In other words, the family (6.45) is of uniform Hurwitz type at $-\infty$ (see Definition 1.72). Proposition 1.73 ensures that the family of adjoint systems

$$\bar{\mathbf{y}}' = \big(A(\omega{\cdot}t) - M_\infty(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)\,B(\omega{\cdot}t)\big)\,\bar{\mathbf{y}} = A_\infty(t)\,\bar{\mathbf{y}} \tag{6.46}$$

is uniformly Hurwitz at $+\infty$. According to Definition 1.58, there are constants $\eta_2 > 0$ and $\beta_2 > 0$ such that $\|U_{A_\infty}(t,\omega)\,U^{-1}_{A_\infty}(s,\omega)\| \le \eta_2\, e^{-\beta_2(t-s)}$ whenever $t \ge s$.

Now, $M_\omega(t)$ tends exponentially fast to $M_\infty(\omega{\cdot}t)$ in the sense of Theorem 6.29. Define

$$N_\omega(t) = (M_\infty(\omega{\cdot}t) - M_\omega(t))\,B^T(\omega{\cdot}t)\,(S_1S_1^T)^{-1}(\omega{\cdot}t)\,B(\omega{\cdot}t)$$

and take $\eta_3 > 0$ such that $\|N_\omega(t)\| \le \eta_3\, e^{-\beta t}$ for all $\omega \in \widetilde{\Omega}$ and all $t \ge 2t_0$. Let $\mathbf{y}_\omega(t)$ be a solution of equation (6.44) for $\omega \in \widetilde{\Omega}$. Then, since $A_\omega(t) = A_\infty(\omega{\cdot}t) + N_\omega(t)$, one has

$$\begin{aligned}\mathbf{y}_\omega(t) = {}& U_{A_\infty}(t,\omega)\,U^{-1}_{A_\infty}(2t_0,\omega)\,\mathbf{y}_\omega(2t_0)\\ &+ \int_{2t_0}^t U_{A_\infty}(t,\omega)\,U^{-1}_{A_\infty}(s,\omega)\,N_\omega(s)\,\mathbf{y}_\omega(s)\,ds\,,\end{aligned}$$

so that, if $t \geq 2t_0$,

$$\|\mathbf{y}_\omega(t)\| \leq \eta_2\, e^{-\beta_2(t-2t_0)}\|\mathbf{y}(2t_0)\| + \int_{2t_0}^{t} \eta_2\, \eta_3\, e^{-\beta_2(t-s)}\, e^{-\beta s}\, \|\mathbf{y}_\omega(s)\|\, ds$$

and hence

$$e^{\beta_2 t}\|\mathbf{y}_\omega(t)\| \leq \eta_2\, e^{2\beta_2 t_0}\|\mathbf{y}(2t_0)\| + \int_{2t_0}^{t} \eta_2\, \eta_3\, e^{-\beta s}\, e^{\beta_2 s}\|\mathbf{y}_\omega(s)\|\, ds\,.$$

The Gronwall inequality and the bound $\int_s^t e^{-\beta r} dr < 1/\beta$ for $s \geq 2t_0$, imply that, if $t \geq 2t_0$,

$$\|\mathbf{y}_\omega(t)\| \leq \eta_2\, e^{2\beta_2 t_0}\|\mathbf{y}(2t_0)\|\, e^{\eta_2\eta_3/\beta} e^{-\beta_2 t} = \eta_*\, e^{-\beta_* t}\|\mathbf{y}(2t_0)\|$$

for $\eta_* = \eta_2 e^{2\beta_2 t_2 + \eta_2\eta_3/\beta}$ and $\beta_* = \beta_2$, which are both independent of ω. This proves the first assertion.

Note finally that the last hypothesis of the proposition ensures that the map $[0, 2t_0] \times \widetilde{\Omega} \to \mathbb{S}_n(\mathbb{R}),\, ;(t,\omega) \mapsto M_\omega(t)$ is jointly continuous, since $M_\omega(t)$ solves (6.36). Therefore, also $A_\omega(t)$ is continuous on $[0, 2t_0] \times \widetilde{\Omega}$, and hence there exists $\kappa > 0$ such that $\|U_{A_\omega}(2t_0)\| \leq \kappa$ for all $\omega \in \Omega$. Consequently, for all $\omega \in \widetilde{\Omega}$, $\|\mathbf{y}_\omega(2t_0)\| = \|U_{A_\omega}(2t_0)\, \mathbf{y}_\omega(0)\| \leq \kappa\, \|\mathbf{y}_\omega(0)\|$, and the last assertion is true for the constant $\bar{\eta} = \kappa\, \eta_*$.

The continuity of the attracting matrix-valued function M_∞ can as usual be interpreted in terms of conservation of recurrence. So if, for example, the initial coefficients A, B, S, and S_1 are Bohr almost periodic functions and $\widetilde{\Omega}$ is their common hull, then for each $\omega \in \widetilde{\Omega}$, the function $t \mapsto M_\infty(\omega{\cdot}t)$ is Bohr almost periodic and has frequency module contained in the joint one of (A, B, S, S_1).

Note also that the regularity results of Sect. 6.3 can be applied to M_∞ whenever A, B, S, and S_1 satisfy the regularity hypotheses there assumed: M_∞ depends nicely on $\omega \in \widetilde{\Omega}$ and is a regular function of eventual parameters in the coefficients A, B, S, and S_1. In addition, it presents regularity properties when $\widetilde{\Omega}$ is a differentiable manifold.

Example 6.32 The section is completed with an example which illustrates how the Kalman–Bucy filter "works". Essentially the same example is treated in the original paper [88]. Consider the stochastic scalar differential equation

$$d\xi(t) = a\,\xi(t)\, dt + s\, dw(t)\,, \tag{6.47}$$

where a and s belong to $\mathbb{R}$, $s > 0$, and $w(t)$ is a standard one-dimensional Brownian motion (thus $\omega(t) \sim N(0,t)$ for each $t \geq 0$). The initial value $\xi(0)$ is taken to be Gaussian with mean μ_0 and variance m_0; that is, with the standard notation, $\xi(0) \sim N(\mu_0, m_0)$). When $a < 0$ the equation (6.47) gives rise to an Ornstein–Uhlenbeck process (see Baldi [10]). One wishes to estimate the state $\xi(t)$ of this process, based on an observation process $\eta(t)$.

Let the observation process satisfy

$$\begin{aligned} d\eta(t) &= b\,\xi(t)\,dt + s_1\,dw_1(t)\,,\\ \eta(0) &= 0\,, \end{aligned}$$

where b and b_1 belong to $\mathbb{R}$, $b_1 > 0$, and $w_1(t)$ is a standard Brownian motion process which is independent of $w(t)$. From the discussion carried out in this section, the optimal estimate $\hat{\xi}(t)$ satisfies

$$\begin{aligned} d\widehat{\xi}(t) &= a\,\widehat{\xi}(t)\,dt + f(t)(d\eta(t) - b\,\widehat{\xi}(t)\,dt)\,,\\ \widehat{\xi}(0) &= E\,\xi(0)\,, \end{aligned}$$

where $f(t) = m(t)\,b\,s_1^{-2}$ and $m(t)$ satisfies the Riccati equation

$$m' = -b^2 s_1^{-2} m^2 + 2\,a\,m + s^2 \tag{6.48}$$

with $m(0) = m_0$. The quantity $m(t)$ is the variance of the (Gaussian) error process $\widetilde{\xi}(t) = \xi(t) - \widehat{\xi}(t)$.

The next step is to study the behavior of the function $m(t)$ defining $f(t)$. One way to do this is to introduce the Hamiltonian differential system

$$\mathbf{z}' = \begin{bmatrix} -a & b^2 s_1^{-2} \\ s^2 & a \end{bmatrix} \mathbf{z} = H\,\mathbf{z} \tag{6.49}$$

and note that (6.48) is the Riccati equation associated to (6.49). The eigenvalues of H are $\pm\sqrt{a^2 + b^2 s^2 s_1^{-2}}$ with eigenvectors

$$\begin{bmatrix} 1 \\ b^{-2} s_1^2 \left(a \pm \sqrt{a^2 + b^2 s^2 s_1^{-2}} \right) \end{bmatrix},$$

from which one can give an explicit formula for the fundamental matrix solution of (6.49) and give and explicit formula for $m(t)$.

For a general (time-dependent) filter it is usually best to analyze the behavior of the error covariance matrix by introducing the corresponding linear Hamiltonian system (6.35): the presence of an exponential dichotomy will facilitate the determination of the stationary solution $M_\infty(t)$ of the Riccati equation (6.34), and the determination of the rate of exponential approach of a solution $M(t)$ of (6.34) with $M(0) \geq 0$ to $M_\infty(t)$. In the present example, however, it is easier to study the Riccati equation (6.48) directly. The zeros of the right-hand side of (6.48) are

$$m_{\mp} = b^{-2}s_1^2\left(a \pm \sqrt{a^2 + b^2 s^2 s_1^{-2}}\right).$$

(The inversion of signs is intentional, to maintain consistency with the previous notation.) Hence, (6.48) can be written as

$$m' = -b^2 s_1^{-2}(m - m_+)(m - m_-)\,. \tag{6.50}$$

Note also that $m^+ < 0 < m^-$. If $m(0) > m_+$ then one sees directly from (6.50) that $m(t)$ tends to m_- with an exponential rate which is initially $b^2 s_1^{-2}(m(0) - m_+)$, and which increases with t in such a way as to tend asymptotically to $b^2 s_1^{-2}(m_- - m_+) = 2\sqrt{a^2 + b^2 s^2 s_1^{-2}}$.

Chapter 7
Nonautonomous Control Theory: A General Version of the Yakubovich Frequency Theorem

The main purpose of this chapter is to state and prove a nonautonomous version of the well-known Yakubovich Frequency Theorem [156], which was originally formulated and proved for control systems $\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}$ with time-periodic coefficients. The extension of this theorem to the nonautonomous category is formulated in terms of a linear-quadratic optimization problem involving an indefinite quadratic function of $\mathbf{x}$ and $\mathbf{u}$ with nonperiodic coefficients. The nonperiodicity creates difficulties which can be overcome using methods previously discussed in this book. In particular, the Frequency Condition and the Nonoscillation Condition of the periodic case are rewritten in terms of the occurrence exponential dichotomy and of the properties of one of the Weyl functions.

The main results to follow appeared in the work of Fabbri *et al.* [47]. The narrative of these results given here presents many more details of the proofs; some of them appeared in Johnson and Núñez [84]. Two more equivalent conditions are added to the previous ones in the paper of Johnson et al. [80]. The paper [85] by Johnson et al. contains a supplementary analysis of the hypotheses under which the Frequency Theorem holds, which is also included here.

A more detailed description of the contents of the chapter completes this introduction. Consider the control system

$$\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\,, \tag{7.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$, together with the quadratic form

$$\widetilde{\mathcal{Q}}(t,\mathbf{x},\mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\right). \tag{7.2}$$

The functions A, B, G, g, and R are assumed to be bounded and uniformly continuous functions on $\mathbb{R}$, with values in the sets of real matrices of the appropriate dimensions. In addition, G and R are symmetric, and $R(t) \geq \rho_R I_m$ for a common

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$\rho_R > 0$ and all $t \in \mathbb{R}$. Note that now the condition $G \geq 0$, which was required in the previous chapter, is not imposed. The relation between the control problem (7.1) and the so-called *supply rate* $\widetilde{\mathcal{Q}}$ is explained in Sect. 8.1 of Chap. 8.

Fix $\mathbf{x}_0 \in \mathbb{R}^n$ and introduce the quadratic functional

$$\widetilde{\mathcal{I}}_{\mathbf{x}_0}(\mathbf{x}, \mathbf{u}) = \int_0^\infty \widetilde{\mathcal{Q}}(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \tag{7.3}$$

evaluated on the so-called *admissible pairs* $(\mathbf{x}, \mathbf{u})\colon [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^m$; i.e. those for which $\mathbf{u}$ belongs to $L^2((0,\infty), \mathbb{R}^m)$ and the solution $\mathbf{x}(t)$ of (7.1) for this control with $\mathbf{x}(0) = \mathbf{x}_0$ belongs to $L^2((0,\infty), \mathbb{R}^n)$. The problem of minimizing $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$ relative to the set of admissible pairs will be posed. In what follows, it will be assumed that (7.1) is L^2-stabilizable; that is, for each $\mathbf{x}_0 \in \mathbb{R}^n$, there exists at least one admissible pair $(\mathbf{x}, \mathbf{u})$. The problem is of a complex nature because now $\widetilde{\mathcal{Q}}$ is not assumed to be positive semidefinite. Indeed one can have $\inf \widetilde{\mathcal{I}}_{\mathbf{x}_0} = -\infty$.

Yakubovich [156, 157] presented a complete solution to the problem in the case when the coefficient matrices A, B, G, g, and R are all T-periodic functions. In particular, he showed that, in this case, the existence of a minimizing pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ for each $\mathbf{x}_0 \in \mathbb{R}$ is equivalent to the validity of a Frequency Condition and a Nonoscillation Condition. He also showed the equivalence of these with several other conditions of a classical nature; for example the existence of a Lyapunov-type function, and the existence of a stabilizing feedback control for (7.1).

In this chapter, Yakubovich's results will be reformulated and proved in the more general situation when A, B, G, g, and R are bounded and uniformly continuous. In this context, a fundamental role is played by the concepts of exponential dichotomy and rotation number for the family of linear Hamiltonian system which is naturally associated to (7.1) and $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$ via the Pontryagin Maximum Principle and the usual hull construction. More precisely, as described in the first section of this chapter: the Yakubovich Frequency Condition will be replaced by the condition that the Hamiltonian family admits exponential dichotomy, while his Nonoscillation Condition will be replaced by an assumption that in particular ensures that the rotation number of the family with respect to any ergodic measure vanishes. In fact these conditions are related, as is explained below.

In the second and main section the above-mentioned conditions are stated and their equivalence is proved. The basic results can be improved when some additional properties on the recurrence of the coefficients (which are always valid in the periodic case) are imposed. Some examples, which illustrate how these various equivalences can be applied, are given. The last part of Sect. 7.2 is devoted to relating the rotation number to the Frequency and Nonoscillation Conditions. More precisely, under the same additional hypothesis as above, the nonautonomous version of Yakubovich's condition of strong nonoscillation (see [157], p. 1030), which is reformulated in the nonautonomous setting in terms of the rotation number, can be weakened. And the presence of the Frequency and Nonoscillation Conditions can be characterized in terms of the instability zones for nonautonomous

Hamiltonian systems, which were labeled in Sect. 2.3 by means of the rotation number.

The third and last section is devoted to a description of certain scenarios in which the Frequency and Nonoscillation Conditions hold. Roughly speaking, there are two different ones, depending on the presence or absence of the uniform weak disconjugacy property discussed in Chap. 5.

The following notation will be in force throughout the chapter. As usual, $\langle\ ,\ \rangle$ and $\|\cdot\|$ are the Euclidean inner product and norm on $\mathbb{R}^d$ for any value of d; and $\|A\|$ represents the operator norm associated to the Euclidean norm for $A\in\mathbb{M}_{n\times m}(\mathbb{R})$. In addition, the Hilbert space $L^2([0,\infty),\mathbb{R}^d)$ (for $d\in\mathbb{N}$) will be endowed with the inner product $\langle\mathbf{u},\mathbf{v}\rangle_d=\int_0^\infty\mathbf{u}^T(t)\,\mathbf{v}(t)\,dt$ and the associated norm $\|\mathbf{u}\|_d=\langle\mathbf{u},\mathbf{u}\rangle_d^{1/2}$, and represented by L^2_d. And the norm $\|(\mathbf{x},\mathbf{u})\| = (\|\mathbf{x}\|_n^2+\|\mathbf{u}\|_m^2)^{1/2}$ will be considered in the product space $L^2_n\times L^2_m$.

7.1 The Frequency and Nonoscillation Conditions

This section is devoted to the definition of the Frequency and Nonoscillation Conditions in the general nonautonomous setting. The Pontryagin procedure is followed in order to minimize the functional (7.3) with respect to controls $\mathbf{u}\in L^2_m$ and solutions $\mathbf{x}\in L^2_n$ of (7.1).

Consider the Hamiltonian

$$\widetilde{\mathcal{H}}(t,\mathbf{x},\mathbf{y},\mathbf{u})=\langle\,\mathbf{y},\mathbf{x}'\rangle-\widetilde{\mathcal{Q}}(t,\mathbf{x},\mathbf{u})=\langle\,\mathbf{y},A(t)\,\mathbf{x}+B(t)\,\mathbf{u}\rangle-\widetilde{\mathcal{Q}}(t,\mathbf{x},\mathbf{u})\,,$$

with $\widetilde{\mathcal{Q}}$ defined by (7.2). Using a uniform stabilization condition which will be discussed in Sect. 7.2, the Pontryagin Maximum Principle can be proved to be valid. Namely, if $(\bar{\mathbf{x}},\bar{\mathbf{u}})\in L^2_n\times L^2_m$ is a minimizing pair for $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$, then there is a motion $\bar{\mathbf{y}}(t)$ such that $(\bar{\mathbf{x}}(t),\bar{\mathbf{y}}(t),\bar{\mathbf{u}}(t))$ simultaneously solves the corresponding Hamilton equations

$$\begin{aligned}\mathbf{x}'&=\frac{\partial\widetilde{\mathcal{H}}}{\partial\mathbf{y}}\,(t,\mathbf{x},\mathbf{y},\mathbf{u})\\ \mathbf{y}'&=-\frac{\partial\widetilde{\mathcal{H}}}{\partial\mathbf{x}}\,(t,\mathbf{x},\mathbf{y},\mathbf{u})\end{aligned}\tag{7.4}$$

and

$$\frac{\partial\widetilde{\mathcal{H}}}{\partial\mathbf{u}}\,(t,\bar{\mathbf{x}}(t),\bar{\mathbf{y}}(t),\bar{\mathbf{u}}(t))=\mathbf{0}\,.$$

This equality leads to the new *feedback rule*

$$\mathbf{u}=R^{-1}(t)\,B^T(t)\,\mathbf{y}-R^{-1}(t)\,g^T(t)\,\mathbf{x}\,,$$

which must be satisfied by $(\bar{\mathbf{x}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{u}}(t))$. Substituting this equality for $\mathbf{u}$ into Hamilton's equations (7.4) and writing $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ yields

$$\mathbf{z}' = H(t)\,\mathbf{z}\,, \qquad \text{with } H = \begin{bmatrix} A - B\,R^{-1}g^T & B\,R^{-1}B^T \\ G - g\,R^{-1}g^T & -A^T + g\,R^{-1}B^T \end{bmatrix}, \tag{7.5}$$

and the minimizing problem is hence rewritten as follows: to find a solution $\bar{\mathbf{z}}(t) = \left[\begin{smallmatrix}\bar{\mathbf{x}}(t)\\ \bar{\mathbf{y}}(t)\end{smallmatrix}\right]$ of (7.5) with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ such that, if $\bar{\mathbf{u}}$ is determined from $\bar{\mathbf{y}}$ by the feedback rule, then the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is admissible and minimizes $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$.

As explained in Sect. 1.3.2, if all the coefficient matrices of this problem are bounded and uniformly continuous functions on $\mathbb{R}$, then the Bebutov construction gives rise to the hull space Ω, and hence to families of Hamiltonian systems and of minimizing problems in which the initial ones are included. If the functions A, B, G, g, and R are all T-periodic functions with minimal period $T > 0$, then Ω is homeomorphic to a circle and the translation flow on it is equivalent to a one-parameter group of rigid motions on the circle. But the periodicity is of course not assumed here. And in fact there is no special reason to require Ω to be the hull of a particular system: the results will be obtained *uniformly* on Ω (and hence for each of its points) in the more general setting now described.

Let (Ω, σ) be a real continuous flow on a compact metric space, and let $A, G \colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$, $B, g \colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$, and $R \colon \Omega \to \mathbb{M}_{m\times m}(\mathbb{R})$ be continuous matrix-valued functions, with G and R symmetric and $R > 0$. Consider the family of control systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}\,, \qquad \omega \in \Omega\,, \tag{7.6}$$

define

$$\widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}, \mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}t)\,\mathbf{u}\rangle\right), \tag{7.7}$$

$$\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x}, \mathbf{u}) = \int_0^\infty \widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}(t), \mathbf{u}(t))\, dt\,, \tag{7.8}$$

for $\omega \in \Omega$, and consider the problem of minimizing this functional when the pair $(\mathbf{x}, \mathbf{u})$ is admissible; i.e. when it belongs to $L^2_n \times L^2_m$ and solves the problem (7.6) with $\mathbf{x}(0) = \mathbf{x}_0$. The following lemma states a consequence of the admissibility of a pair which will be used in the analysis. Its proof is identical to that of Lemma 6.18.

Lemma 7.1 *If the pair* $(\mathbf{x}, \mathbf{u})$ *is admissible for the functional* $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$*, then* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

As before, the uniform stabilization condition (which will be discussed in Sect. 7.2 and which in particular ensures the existence of at least one admissible pair $(\mathbf{x}, \mathbf{u})$ for each $\omega \in \Omega$), together with the Pontryagin Maximum Principle, relates the problem of minimizing $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ to the family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{7.9}$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and

$$H = \begin{bmatrix} A - B\,R^{-1} g^T & B\,R^{-1} B^T \\ G - g\,R^{-1} g^T & -A^T + g\,R^{-1} B^T \end{bmatrix}. \tag{7.10}$$

As usual, $U(t,\omega)$ will represent the fundamental matrix solution of this system with $U(0,\omega) = I_{2n}$, which is real and symplectic for all pairs (t,ω).

The reformulation of the Yakubovich Frequency and Nonoscillation Conditions is carried out making use of this family of Hamiltonian systems. In the case of the Frequency Condition, under the assumption of T-periodicity of the initial matrices, Yakubovich's definition is as follows: if $U(t)$ is the fundamental matrix solution of (7.5) with $U(0) = I_{2n}$, then

$$\det(U(T) - e^{i\beta} I_{2n}) \neq 0 \tag{7.11}$$

whenever $\beta \in [0, 2\pi)$. In other words, the initial periodic Hamiltonian system (7.5) has no null Lyapunov exponents, and therefore it has only one solution which is bounded on all of $\mathbb{R}$, namely the trivial one. This observation leads to the sought-for generalized formulation of the condition: the Frequency Condition will be the absence of nontrivial bounded solutions. Theorem 1.78 states that this hypothesis can be rewritten as

FC (Frequency Condition). The family (7.9) has exponential dichotomy over Ω.

Consider now the Nonoscillation Condition. Return to the case of periodic coefficients, and assume that the frequency condition (7.11) holds; or, in other words, that the system (7.5) has exponential dichotomy on $\mathbb{R}$. Let l^+ be the Lagrange plane of the initial data giving rise to solutions which are bounded as $t \to \infty$ (see Remark 1.77.1). Represent l^+ by $\left[\begin{smallmatrix}X_0\\ Y_0\end{smallmatrix}\right]$ and define $\left[\begin{smallmatrix}X(t)\\ Y(t)\end{smallmatrix}\right] = U(t)\left[\begin{smallmatrix}X_0\\ Y_0\end{smallmatrix}\right]$. Yakubovich's Nonoscillation Condition is then

$$\det X(t) \neq 0 \quad \text{for all } t \in \mathbb{R}\,. \tag{7.12}$$

As Yakubovich points out [157], this condition can be expressed geometrically in terms of the vertical Maslov cycle $\mathcal{C}$, which is the complement in $\mathcal{L}_{\mathbb{R}}$ of the set $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \left[\begin{smallmatrix}I_n\\ M\end{smallmatrix}\right] \right\} \subset \mathcal{L}_{\mathbb{R}} : \tag{7.13}$$

condition (7.12) holds if and only if $l^+(t) = U(t){\cdot}l^+$ belongs to $\mathcal{D}$ for all $t \in \mathbb{R}$. The extension to the present nonautonomous setting of the nonoscillation condition is hence clear. As usual, if the family (7.9) has exponential dichotomy over Ω, $l^+(\omega)$ represents the Lagrange plane of the vectors giving rise to solutions which are bounded as $t \to \infty$.

NC (Nonoscillation Condition). Condition FC holds and $l^+(\omega) \in \mathcal{D}$ for all $\omega \in \Omega$.

In other words, the Nonoscillation Condition is equivalent to the global existence of the Weyl function M^+: see Definition 1.80. Recall that the set $\mathcal{D}$ is open, a fact which will be used often in the chapter: see Proposition 1.28.

Remarks 7.2

1. If the Frequency and Nonoscillation Conditions are fulfilled, then all the systems of the family (7.9) satisfy Definition 5.3 of nonoscillation at $+\infty$ and at $-\infty$. This is proved by Proposition 5.8.
2. Note that the definition (2.36) of the rotation number in terms of the Maslov index and the subsequent Theorem 2.22 ensure that the rotation number of (7.9) with respect to any σ-ergodic measure on Ω vanishes when condition NC holds. (This is also proved by the previous remark and Proposition 5.65.)

7.2 The Extension of the Yakubovich Frequency Theorem

The main result of [156] (Theorem 2) asserts the logical equivalence of six conditions, when the coefficients A, B, G, g, and R are all T-periodic functions. One of these conditions is the solvability of the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ given by (7.8) subject to (7.6). Two more equivalent conditions are added to the list in Theorem 1 of [157].

The main goal of this section is to reformulate Yakubovich's six conditions (now called Y1, Y2, Y3, Y4, Y5, and Y6) in a way appropriate to the case of general nonautonomous control processes. Under an additional hypothesis (always valid in the periodic case), two more conditions, Y7 and Y8, will be added to the list. The logical equivalence of these conditions will be proved when the following *a priori* condition of exponential stabilizability at $+\infty$ of the family of control processes (7.6) is satisfied.

Hypothesis 7.3 There exists a continuous function $K_0 \colon \Omega \to \mathbb{M}_{m\times n}(\mathbb{R})$ such that the family of linear systems

$$\mathbf{x}' = \widehat{A}(\omega{\cdot}t)\,\mathbf{x} = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K_0(\omega{\cdot}t))\,\mathbf{x}\,, \qquad \omega \in \Omega\,, \tag{7.14}$$

is uniformly Hurwitz at $+\infty$; i.e. there exist constants $\eta > 0$ and $\beta > 0$ such that, for all $\omega \in \Omega$,

$$\|U_{\widehat{A}}(t,\omega)\, U_{\widehat{A}}^{-1}(s,\omega)\| \le \eta\, e^{-\beta(t-s)} \quad \text{if } t \ge s\,, \tag{7.15}$$

where $U_{\widehat{A}}(t,\omega)$ is the matrix-solution of (7.14) with $U_{\widehat{A}}(0,\omega) = I_n$.

It will be seen later on (in Proposition 7.33) that the controllability condition C1 of Sect. 6.2 implies Hypothesis 7.3. And it will seen now that, in the general case, Hypothesis 7.3 guarantees the L^2-stabilization condition required for Yakubovich's results. In fact they turn out to be equivalent in the periodic case: see Theorem 1 of [156]. A more restrictive condition is required in the nonautonomous case, due to the infinite-horizon nature of the optimization problem.

Proposition 7.4 *If Hypothesis 7.3 holds, then there exists at least one admissible pair for the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ given by (7.8) for each $\mathbf{x}_0 \in \mathbb{R}^n$.*

Proof Given $\mathbf{x}_0 \in \mathbb{R}^n$, let $\mathbf{x}(t)$ solve (7.14) with $\mathbf{x}(0) = \mathbf{x}_0$ and define $\mathbf{u}(t) = B(\omega{\cdot}t)\, K_0(\omega{\cdot}t)\, \mathbf{x}(t)$. Then $\mathbf{u}(t)$ and $\mathbf{x}(t)$ are square integrable in $[0,\infty)$, and $(\mathbf{x},\mathbf{u})\colon [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^m$ satisfies (7.6).

Suppose that Hypothesis 7.3 holds, and consider the family of control systems

$$\mathbf{x}' = \widehat{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widehat{\mathbf{u}}\,, \qquad \omega \in \Omega\,, \tag{7.16}$$

with $\widehat{A}$ defined by (7.14). There is a basic relation between the families (7.6) and (7.16): the equality

$$\mathbf{u}(t) = \widehat{\mathbf{u}}(t) + K_0(\omega{\cdot}t)\,\mathbf{x}(t) \tag{7.17}$$

establishes a correspondence between pairs $(\mathbf{x},\widehat{\mathbf{u}})$ which satisfy (7.16) and pairs $(\mathbf{x},\mathbf{u})$ solving (7.6). A systematic use of this correspondence and of the Hurwitz nature of the family (7.14) will be made in the rest of this section, especially in the proof of Theorem 7.10. Using the uniform boundedness of K_0, it is possible to show that there exist strictly positive constants c_1 and c_2 (independent of ω) such that, if (7.17) holds, then

$$c_1(\|\mathbf{x}\|_n^2 + \|\widehat{\mathbf{u}}\|_m^2) \le \|\mathbf{x}\|_n^2 + \|\mathbf{u}\|_m^2 \le c_2(\|\mathbf{x}\|_n^2 + \|\widehat{\mathbf{u}}\|_m^2)\,. \tag{7.18}$$

Lemma 7.5 *Suppose that Hypothesis 7.3 holds. For all $\omega \in \Omega$, $\mathbf{x}_0 \in \mathbb{R}^n$ and $\widehat{\mathbf{u}} \in L^2_m$, write the unique solution $\mathbf{x}(t)$ of (7.16) with $\mathbf{x}(0) = \mathbf{x}_0$ as*

$$\mathbf{x}(t) = \widehat{\mathbf{x}}_\omega(t) + \lambda_\omega(\widehat{\mathbf{u}})(t)\,, \tag{7.19}$$

where

$$\widehat{\mathbf{x}}_\omega(t) = U_{\widehat{A}}(t,\omega)\,\mathbf{x}_0\,, \tag{7.20}$$

$$\lambda_\omega(\widehat{\mathbf{u}})(t) = \int_0^t U_{\widehat{A}}(t,\omega)\,U_{\widehat{A}}^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\widehat{\mathbf{u}}(s)\,ds\,. \tag{7.21}$$

Then, there exist positive constants c_3 *and* c_4 *such that, for all* $\omega \in \Omega$, $\mathbf{x}_0 \in \mathbb{R}^n$ *and* $\widehat{\mathbf{u}} \in L^2_m$,

$$\|\widehat{\mathbf{x}}_\omega\|_n \le c_3\|\mathbf{x}_0\| \quad \textit{and} \quad \|\lambda_\omega(\widehat{\mathbf{u}})\|_n \le c_4\|\widehat{\mathbf{u}}\|_m\,. \tag{7.22}$$

In particular, $\mathbf{x} \in L^2_n$.

Proof The first inequality in (7.22) follows immediately from (7.15). The second one requires some more work. Define $b = \sup_{\omega\in\Omega}\|B(\omega)\|$. Note that, by Hölder's inequality,

$$\begin{aligned}\left(\int_0^t e^{-\beta(t-s)}\|\widehat{\mathbf{u}}(s)\|\,ds\right)^2 &= \left(\int_0^t (e^{-\beta(t-s)/2}\,\|\widehat{\mathbf{u}}(s)\|)\,e^{-\beta(t-s)/2}\,ds\right)^2\\ &\le \int_0^t e^{-\beta(t-s)}\,\|\widehat{\mathbf{u}}(s)\|^2\,ds\int_0^t e^{-\beta(t-s)}\,ds\\ &\le \frac{1}{\beta}\int_0^t e^{-\beta(t-s)}\,\|\widehat{\mathbf{u}}(s)\|^2\,ds\,.\end{aligned} \tag{7.23}$$

The second bound in (7.22) follows from this inequality and (7.15), since

$$\begin{aligned}\|\lambda_\omega(\widehat{\mathbf{u}})\|_n^2 &= \int_0^\infty \left\|\int_0^t U_{\widehat{A}}(t,\omega)\,U_{\widehat{A}}^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\widehat{\mathbf{u}}(s)\,ds\right\|^2 dt\\ &\le \frac{b^2\eta^2}{\beta}\int_0^\infty\left(\int_0^t e^{-\beta(t-s)}\,\|\widehat{\mathbf{u}}(s)\|^2\,ds\right)dt\\ &= \frac{b^2\eta^2}{\beta}\int_0^\infty\left(\|\widehat{\mathbf{u}}(s)\|^2\int_s^\infty e^{-\beta(t-s)}\,dt\right)ds = \frac{b^2\eta^2}{\beta^2}\,\|\widehat{\mathbf{u}}\|_m^2\,.\end{aligned}$$

This completes the proof.

Yakubovich's condition Y1 can be conveniently reformulated for the nonautonomous case, in a way which will be now described. Fix $\omega \in \Omega$ and suppose that the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ subject to (7.6) can be solved for each $\mathbf{x}_0 \in \mathbb{R}^n$. If it is assumed that Hypothesis 7.3 holds, then the Pontryagin Maximum Principle is valid: Yakubovich explains this in [156], pp. 619–621. The arguments used there rely on the abstract optimization theory given in [155], and can easily be adapted to the nonperiodic case thanks to the boundedness of G, g, and R. See also Carlson *et al.* [25]. Hence, as seen in Sect. 7.1, if $\mathbf{x}_0 \in \mathbb{R}^n$, then to each (admissible) minimizing pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ of $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ there corresponds a point $\mathbf{y}_0 \in \mathbb{R}^n$ such that the solution $\bar{\mathbf{z}}(t) = \begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{\mathbf{y}}(t) \end{bmatrix}$ of (7.9) with $\bar{\mathbf{z}}(0) = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}$ lies in L^2_{2n}, and $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{u}})$ satisfies

$$\mathbf{u}(t) = R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, \mathbf{y}(t) - R^{-1}(\omega{\cdot}t)\, g^T(\omega{\cdot}t)\, \mathbf{x}(t). \tag{7.24}$$

Note that such a solution $\bar{\mathbf{z}}(t)$ satisfies $\lim_{t\to\infty} \bar{\mathbf{z}}(t) = \mathbf{0}$. This is proved as Lemmas 6.18 and 7.1, since both $\bar{\mathbf{z}}$ and $\bar{\mathbf{z}}'$ lie in L^2_{2n}. In particular, for each $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, there exists at least one $\mathbf{y}_0 \in \mathbb{R}^n$ such that $\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}$ belongs to the vector space

$$l_b(\omega) = \left\{ \mathbf{z}_0 \in \mathbb{R}^{2n} \mid \sup_{t\in[0,\infty)} \|U(t,\omega)\, \mathbf{z}_0\| < \infty \right\} . \tag{7.25}$$

In fact by letting $\mathbf{x}_0$ vary in $\mathbb{R}^n$, one obtains at least n linearly independent solutions $\mathbf{z}_1(t), \dots, \mathbf{z}_n(t)$ of the system (7.9) corresponding to the fixed point ω which tend to $\mathbf{0}$ as $t \to +\infty$, and with $\mathbf{z}_1(0), \dots, \mathbf{z}_n(0) \in l_b(\omega)$. These n solutions play a role in the proof of the following lemma, which is required to formulate condition Y1.

Lemma 7.6 *Suppose that Hypothesis 7.3 holds and that the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ subject to (7.6) can be solved for each $\mathbf{x}_0 \in \mathbb{R}^n$ and each $\omega \in \Omega$. Then,*

(i) *for each $\omega \in \Omega$, the set $l_b(\omega)$ defined by (7.25) is a Lagrange plane. In addition,*

$$l_b(\omega) = \left\{ \mathbf{z}_0 \in \mathbb{R}^{2n} \mid \lim_{t\to\infty} \|U(t,\omega)\, \mathbf{z}_0\| = 0 \right\} . \tag{7.26}$$

(ii) *For each $\omega \in \Omega$ and each $\mathbf{x}_0 \in \mathbb{R}^n$ there exist a unique $\mathbf{y}_0 \in \mathbb{R}^n$ such that $\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}$ belongs to $l_b(\omega)$ and a unique pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ which minimizes $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$. In addition, if $\begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{\mathbf{y}}(t) \end{bmatrix} = U(t,\omega) \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}$, then $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{u}})$ solve (7.24).*

(iii) *$l_b(\omega) \in \mathcal{D}$ for all $\omega \in \Omega$.*

Proof

(i) Fix $\omega \in \Omega$ and define $l(\omega)$ to be the vector space generated by the previously found set of initial data $\{\mathbf{z}_1(0), \dots, \mathbf{z}_n(0)\} \subset \mathbb{R}^{2n}$, which has dimension n. Then $\lim_{t\to\infty} \|U(t,\omega)\, \mathbf{z}_0\| = 0$ for all $\mathbf{z}_0 \in l(\omega)$, which implies that $l(\omega)$ is a Lagrange plane: if $\mathbf{z}_0, \mathbf{w}_0 \in l(\omega)$, the symplectic character of $U(t,\omega)$ ensures that $\mathbf{z}_0^T J\, \mathbf{w}_0 = \mathbf{z}_0^T U^T(t,\omega) J U(t,\omega)\, \mathbf{w}_0$, which tends to zero as $t \to \infty$.

Clearly, $l(\omega) \subseteq l_b(\omega)$. Take now a nonzero vector $\mathbf{w}_0 \in l_b(\omega)$, and assume for contradiction that $\mathbf{w}_0 \notin l(\omega)$. Hence $\mathbf{w}_0 = \mathbf{w}_1 + \mathbf{w}_2$, with $\mathbf{w}_1 \in l(\omega)$ and $\mathbf{0} \neq \mathbf{w}_2 \in J{\cdot}l(\omega)$, the orthogonal complement of $l(\omega)$ in $\mathbb{R}^{2n}$. Then $U(t,\omega)\, \mathbf{w}_2$ is bounded as $t \to \infty$, since $U(t,\omega)\, \mathbf{w}_1$ tends to $\mathbf{0}$. On the other hand, $J\mathbf{w}_2 \in l(\omega)$, so that $U(t,\omega) J\mathbf{w}_2$ tends to $\mathbf{0}$ as $t \to \infty$. The equality

$\|\mathbf{w}_2\|^2 = -\mathbf{w}_2^T J J \mathbf{w}_2 = -(U(t,\omega)J\mathbf{w}_2)^T J(U(t,\omega)\,\mathbf{w}_2)$ leads to the sought-for contradiction. Hence (7.26) holds and $l_b(\omega) = l(\omega)$ is a Lagrange plane, as asserted.

(ii) The construction of $l_b(\omega)$ which was carried out before stating the lemma shows the existence of $\mathbf{y}_0$ for each $\mathbf{x}_0$ and described its relation with a minimizing pair. The uniqueness of $\mathbf{y}_0$ follows from the condition $\dim l_b(\omega) = n$, and this completes the proof of (ii).

(iii) This last property follows immediately from (ii): just let $\mathbf{x}_0$ vary in order to form a basis of $\mathbb{R}^n$.

Yakubovich's condition Y1 can finally be reformulated. Also the remaining seven conditions are described at this point: Theorem 7.10 states the already announced equivalence of these conditions. Recall that Hypothesis 7.3 is always assumed in this section.

Y1. The following two properties hold:

Y1$_1$. For each $\omega \in \Omega$, the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ given by (7.8) subject to (7.6) admits a solution for each $\mathbf{x}_0 \in \mathbb{R}^n$. That is, there exists a control function $\bar{\mathbf{u}} \in L^2_m$ such that the solution $\bar{\mathbf{x}}(t)$ of

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\bar{\mathbf{u}}$$

with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ belongs to L^2_n, and

$$\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\bar{\mathbf{x}},\bar{\mathbf{u}}) = \inf \widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u})\,,$$

where the infimum is taken over the set of admissible pairs.

Y1$_2$. The map $\Omega \to \mathcal{L}_{\mathbb{R}},\ \omega \mapsto l_b(\omega)$, with $l_b(\omega)$ defined by (7.25), is continuous.

Y2. The Frequency Condition FC and the Nonoscillation Condition NC hold for the family (7.9).

Y3. There exists a symmetric $n\times n$ matrix-valued function M^+ which is continuous on Ω and which is differentiable along the σ-orbits, with the following properties: first, M^+ is a solution along the flow of the Riccati equation

$$\begin{aligned} M' = &-MBR^{-1}B^T M - (A^T - gR^{-1}B^T)M \\ &- M(A - BR^{-1}g^T) + G - gR^{-1}g^T, \end{aligned} \tag{7.27}$$

where A, B, G, g, and R are evaluated in $\omega{\cdot}t$; and second, if

$$K = R^{-1}(-g^T + B^T M^+), \tag{7.28}$$

then the family of systems

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\, K(\omega{\cdot}t))\, \mathbf{x}\,, \qquad \omega \in \Omega \tag{7.29}$$

is of uniform Hurwitz type at $+\infty$.

Y4. There exists a symmetric $n\times n$ matrix-valued function M^+ which is continuous on Ω and which is differentiable along the σ-orbits, such that the form

$$V_\omega(t,\mathbf{x}) = \langle \mathbf{x}, M^+(\omega{\cdot}t)\,\mathbf{x}\rangle$$

satisfies

$$\begin{aligned} \frac{d}{dt}\, V_\omega(t,\mathbf{x}(t)) &= 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) \\ &\quad - \langle \mathbf{u}(t) - K(\omega{\cdot}t)\,\mathbf{x}(t), R(\omega{\cdot}t)(\mathbf{u}(t) - K(\omega{\cdot}t)\,\mathbf{x}(t))\rangle\,. \end{aligned} \tag{7.30}$$

Here $\mathbf{u}\colon [0,\infty) \to \mathbb{R}^m$ is an arbitrary continuous function, while $K\colon \Omega \to \mathbb{M}_{m\times n}(\mathbb{R})$ has the property that the family of systems (7.29) is of uniform Hurwitz type at $+\infty$. (It will turn out that K is defined from M^+ by (7.28).) In addition, $\mathbf{x}(t)$ is an arbitrary solution of the system (7.6) corresponding to ω and to the control $\mathbf{u}(t)$, and $\widetilde{\mathcal{Q}}_\omega$ is defined by (7.7).

Y5. There exist $\delta > 0$ independent of ω and a symmetric $n \times n$ matrix-valued function M_δ^+ which is continuous on Ω and which is differentiable along the σ-orbits, such that, for each $\omega \in \Omega$, the "Lyapunov function"

$$V_\omega^\delta(t,\mathbf{x}) = \langle \mathbf{x}, M_\delta^+(\omega{\cdot}t)\,\mathbf{x}\rangle$$

satisfies

$$\frac{d}{dt}\, V_\omega^\delta(t,\mathbf{x}(t)) \le 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \delta(\|\mathbf{x}(t)\|^2 + \|\mathbf{u}(t)\|^2) \tag{7.31}$$

for each continuous function $\mathbf{u}\colon [0,\infty) \to \mathbb{R}^m$. Here, $\mathbf{x}(t)$ is any solution of the system (7.6) corresponding to ω and to this control $\mathbf{u}(t)$, and $\widetilde{\mathcal{Q}}_\omega$ is defined by (7.7).

Y6. The functional $\widetilde{\mathcal{I}}_{\mathbf{0},\omega}$ is positive definite on the space of processes $(\mathbf{x},\mathbf{u}) \in L_n^2 \times L_m^2$ which satisfy (7.6) with $\mathbf{x}(0) = \mathbf{0}$. More precisely, there exists $\delta > 0$, independent of ω, such that

$$\int_0^\infty \widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t))\, dt \ge \delta \int_0^\infty (\|\mathbf{x}(t)\|^2 + \|\mathbf{u}(t)\|^2)\, dt \tag{7.32}$$

for all such pairs $(\mathbf{x},\mathbf{u})$. Here, $\widetilde{\mathcal{Q}}_\omega$ is defined by (7.7).

Assume now that there exists a σ-ergodic measure m_0 on Ω with $\operatorname{Supp} m_0 = \Omega$.

Y7. The Frequency Condition FC holds for the family (7.9), and the rotation number of the family (7.9) with respect to m_0 vanishes.

Y8. There exists $\delta > 0$ such that if the function $K\colon \Omega \to \mathfrak{sp}(n, \mathbb{R})$ is continuous and satisfies $\max_{\omega\in\Omega} \|K(\omega)\| < \delta$, then the rotation number of the family

$$\mathbf{z}' = (H(\omega{\cdot}t) + K(\omega{\cdot}t))\,\mathbf{z}\,, \quad \omega \in \Omega \tag{7.33}$$

with respect to m_0 vanishes.

Remark 7.7 Note that conditions Y3 and Y4 include Hypothesis 7.3. In addition, it will be seen in the proof of Y2⇒Y3 in the next theorem that also Y2 implies Hypothesis 7.3. That is, conditions Y2, Y3, or Y4 suffice by themselves to guarantee the solvability of the minimization problem.

Remarks 7.8

1. In the periodic case, there exists a unique σ-invariant (and hence σ-ergodic) measure m_0, which in addition satisfies $\operatorname{Supp} m_0 = \Omega$: see Remark 1.13.2. On the other hand, and again in the periodic case, the rotation number of the family (7.9) vanishes if and only if the systems are nonoscillatory, in the sense of Definition 5.3: see Remark 5.4. This means that Y7 agrees with the second equivalent condition of Theorem 1 of [157], which imposes the exponential dichotomy and the nonoscillation of the periodic Hamiltonian system.
2. In the same sense, Y8 is a nonautonomous version of the Yakubovich condition of strong nonoscillation. This condition, formulated as the third equivalent condition of Theorem 1 of [157], imposes that all the Hamiltonian systems in a neighborhood of the initial one are nonoscillatory. It is important to emphasize that Y8 gives a necessary and sufficient condition for the simultaneous validity of the Frequency Condition FC and the Nonoscillation Condition NC formulated exclusively in terms of the properties of the rotation number. In addition, Theorem 7.18 shows that this condition can be weakened in some cases.

A preliminary lemma will be useful at several points of the proofs of the main theorems of this section, as well as in Chap. 8. Recall once more (see Sect. 1.3.5) that the family of Riccati equations (7.27), which is associated to the family of linear Hamiltonian systems (7.9), defines a local skew-product flow τ_s on $\Omega \times \mathbb{S}_n(\mathbb{R})$: in time t it sends the pair (ω, M_0) to the pair $(\omega{\cdot}t, M(t, \omega, M_0))$, where $M(t, \omega, M_0)$ is the solution of the equation (7.27) with $M(0, \omega, M_0) = M_0$.

Lemma 7.9 *Take* $(\omega, M_0) \in \Omega \times \mathbb{S}_n(\mathbb{R})$ *and define*

$$V_{\omega,M_0}(t, \mathbf{x}) = \langle \mathbf{x}, M(t, \omega, M_0)\,\mathbf{x}\rangle$$

as long as $M(t,\omega,M_0)$ *exists. Then, for any pair* $(\mathbf{x}(t),\mathbf{u}(t))$ *solving* (7.6),

$$\begin{aligned}\frac{d}{dt}V_{\omega,M_0}(t,\mathbf{x}(t)) &= 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t))\\ &\quad - \langle \mathbf{u}(t)-K_{\omega,M_0}(t)\,\mathbf{x}(t), R(\omega{\cdot}t)(\mathbf{u}(t)-K_{\omega,M_0}(t)\,\mathbf{x}(t))\rangle\end{aligned} \tag{7.34}$$

for

$$K_{\omega,M_0}(t) = R^{-1}(\omega{\cdot}t)(-g^T(\omega{\cdot}t)+B^T(\omega{\cdot}t)\,M(t,\omega,M_0))\,.$$

Proof A straightforward computation from the Riccati equation (7.27) and the control system (7.6) proves the result.

Theorem 7.10 *Suppose that Hypothesis 7.3 holds. Then,*

(i) *all of the statements Y1, Y2, Y3, Y4, Y5, and Y6 are equivalent. In addition, if they hold, the minimizing pair* $(\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t))$ *for* $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ *is defined from the unique solution* $\left[\begin{smallmatrix}\bar{\mathbf{x}}(t)\\ \bar{\mathbf{y}}(t)\end{smallmatrix}\right]$ *of* (7.9) *satisfying* $\bar{\mathbf{x}}(0)=\mathbf{x}_0$ *which is bounded as* $t\to\infty$ *(which in turn satisfies* $\left[\begin{smallmatrix}\bar{\mathbf{x}}(0)\\ \bar{\mathbf{y}}(0)\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathbf{x}_0\\ M^+(\omega)\,\mathbf{x}_0\end{smallmatrix}\right]$*) by means of the feedback rule* (7.24)*; and* $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\bar{\mathbf{x}},\bar{\mathbf{u}}) = -(1/2)\langle\mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle$.

(ii) *If, in addition, there exists a* σ*-ergodic measure* m_0 *on* Ω *with full topological support, then they are all also equivalent to statements Y7 and Y8.*

Proof

(i) Following Yakubovich's strategy, the equivalence of the first six conditions will be proved by checking that Y1⇒Y2⇒Y3⇒Y4⇒Y1, and then that Y2⇒Y5⇒Y6⇒Y1. The steps which require new or extended arguments with respect to those given by Yakubovich in [156] are Y1⇒Y2 and Y6⇒Y1. Nevertheless, for the reader's convenience, most of the details of all the steps will be explained. The proof of the last assertions in (i) is implicit in the proof of the equivalences.

Y1⇒Y2. The main step consists in proving that, if Y1 holds, then the family (7.9) has exponential dichotomy over Ω. By Theorem 1.78, it is sufficient to prove that no equation (7.9) admits a nonzero solution which is bounded on all of $\mathbb{R}$. This is done in what follows.

As explained in Remark 1.27.3, for each $\omega\in\Omega$, the Lagrange plane $l_b(\omega)$ can be represented by a $2n\times n$ matrix $\left[\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right]$ with $\Phi_1(\omega)+i\Phi_2(\omega)\in U(n,\mathbb{R})$, this representation being unique up to multiplication by any matrix in $\mathrm{O}(n,\mathbb{R})$. Define

$$\widetilde{\Omega}_b = \left\{\left(\omega,\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\right) \mid \omega\in\Omega,\ l_b(\omega)\equiv\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix},\ \Phi_1^0+i\Phi_2^0\in U(n,\mathbb{R})\right\}\ ,$$

which due to the continuity of l_b and the compactness of $\mathrm{U}(n,\mathbb{R})$ is a compact subset of $\Omega \times \mathbb{M}_{2n\times n}(\mathbb{R})$. Theorem 1.41 ensures that

$$\widetilde{\sigma}_b\colon \mathbb{R}\times\widetilde{\Omega}_b \to \widetilde{\Omega}_b\,,\ \left(t,\omega,\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\right) \to \left(\omega{\cdot}t,\begin{bmatrix}\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\end{bmatrix}\right)$$

defines a continuous flow on $\widetilde{\Omega}_b$. Here, $\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)$ and $\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)$ are the solutions of (1.15) with initial data Φ_1^0 and Φ_2^0 respectively, and they satisfy $\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)+i\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\in U(n,\mathbb{R})$. Theorem 1.41 also ensures that, if $\left(\omega,\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\right)\in\widetilde{\Omega}_b$, then

$$U(t,\omega)\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}=\begin{bmatrix}\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,I_n)\\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,I_n)\end{bmatrix},$$

where $R(t,\omega,\Phi_1^0,\Phi_2^0,I_n)$ is the fundamental matrix solution with value I_n at $t=0$ of the system

$$\mathbf{x}'=S(\omega{\cdot}t,\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0),\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0))\,\mathbf{x}=S\left(\widetilde{\sigma}_b\left(t,\omega,\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\right)\right),\tag{7.35}$$

with S defined by (1.18). Take $\mathbf{x}_0\in\mathbb{R}^n$ and define $\mathbf{z}_0=\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\mathbf{x}_0\in l_b(\omega)$ and $\mathbf{z}(t)=U(t,\omega)\,\mathbf{z}_0$. It follows easily that $\|\mathbf{z}(t)\|=\|\mathbf{x}(t)\|$, where $\mathbf{x}(t)$ is the solution of (7.35) with $\mathbf{x}(0)=\mathbf{x}_0$. This and the alternative definition of $l_b(\omega)$ given by (7.26) ensure that every solution (7.35) tends to $\mathbf{0}$ as $t\to\infty$. Since this happens for all $\left(t,\omega,\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}\right)\in\widetilde{\Omega}_b$, the family (7.35) has exponential dichotomy over $\widetilde{\Omega}_b$ (see Proposition 1.74), which in turn implies that every nonzero solution of any of the systems (7.35) is unbounded as $t\to-\infty$ (see Proposition 1.56).

Since any solution $\mathbf{z}(t)$ of (7.9) which is bounded as $t\to\infty$ satisfies $\mathbf{z}(0)\in l_b(\omega)$, the above relation between norms ensures that $\mathbf{z}(t)$ is not bounded as $t\to-\infty$. This proves the exponential dichotomy of the family (7.9) over Ω, and hence the Frequency Condition. Note also that, with the usual notation for the Lagrange planes associated to the exponential dichotomy, $l_b(\omega)=l^+(\omega)$ under Hypothesis 7.3 when condition Y1 holds: see Remark 1.77.2. Hence the Nonoscillation Condition follows from Lemma 7.6(iii). The proof of Y1⇒Y2 is complete.

Y2⇒Y3. The Frequency and Nonoscillation Conditions show that $l^+(\omega)$ can be represented by $\begin{bmatrix}I_n\\ M^+(\omega)\end{bmatrix}$. As is explained in Sect. 1.3.5, the action of $U(t,\omega)$ takes the Lagrange plane $l^+(\omega)$ to $l^+(\omega{\cdot}t)$, and hence M^+ solves the Riccati equation (7.27) along the flow. Recall that there exist constants $\eta>0$ and $\beta>0$ such that for all $\omega\in\Omega$ and $\mathbf{z}_0\in l^+(\omega)$, the inequality $\|U(t,\omega)\,\mathbf{z}_0\|\le\eta\,e^{-\beta t}\|\mathbf{z}_0\|$ is valid: see Definition 1.75. Take any $\omega\in\Omega$ and any $\mathbf{x}_0\in\mathbb{R}^n$, define $\mathbf{z}_0=$

$\begin{bmatrix} I_n \\ M^+(\omega) \end{bmatrix} \mathbf{x}_0$, which belongs to $l^+(\omega)$, and represent $U(t,\omega)\,\mathbf{z}_0 = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$. Then $\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \in l^+(\omega{\cdot}t)$, and hence $\mathbf{y}(t) = M^+(\omega{\cdot}t)\,\mathbf{x}(t)$ for all $t \in \mathbb{R}$. This ensures that $\mathbf{x}(t)$ is the solution of (7.29) with $\mathbf{x}(0) = \mathbf{x}_0$, where K is defined by (7.28). Finally, $\|\mathbf{x}(t)\| \le \|\mathbf{z}(t)\|$, and the continuity of M^+ provides a $k > 0$ such that $\|\mathbf{z}_0\| \le k\|\mathbf{x}_0\|$. This shows that $\|\mathbf{x}(t)\| \le k\eta\, e^{-\beta t}\|\mathbf{x}_0\|$, and hence that Y3 holds (see Proposition 1.74).

Y3⇒Y4. This implication is a consequence of Lemma 7.9 applied to the matrix-valued function M^+ of Y3.

Y4⇒Y1. Assume that the condition stated in Y4 holds. Then, for all pairs $(\mathbf{x},\mathbf{u})$ with the properties required there,

$$\mathbf{x}^T(A^TM^+ + M^+A + (M^+)' - G + K^TRK)\,\mathbf{x} + 2\mathbf{u}^T(B^TM^+ - g^T - RK)\,\mathbf{x} = 0\,,$$

where $\mathbf{u}$ and $\mathbf{x}$ are evaluated in t and all the matrices are evaluated in $\omega{\cdot}t$. Take $t = 0$. For any $\mathbf{x}_0 \in \mathbb{R}^n$ it is possible to choose a pair such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{u}(0) = \mathbf{0}$, which ensures that $(M^+)' = -A^TM^+ - M^+A + G - K^TRK$ on Ω, and hence that $\mathbf{u}^T(B^TM^+ - g^T - RK)\,\mathbf{x} = 0$ for all pairs $(\mathbf{x},\mathbf{u})$ satisfying the properties required in Y4. And now, for any $\mathbf{x}_0 \in \mathbb{R}^n$, it is possible to choose a pair in such a way that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{u}(0) = (B^TM^+ - g^T - RK)\,\mathbf{x}_0$, which implies that $B^TM^+ - g^T - RK = 0$ on Ω. The last equality ensures that K and M^+ are related by (7.28), which together with the first inequality shows that M^+ is a solution along the flow of the Riccati equation (7.27). (Incidentally, note that Y4 implies Y3.)

The next step is to derive from equality (7.30) that

$$2\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0}(\mathbf{x},\mathbf{u}) = -\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle + \int_0^\infty \|R^{1/2}(\omega{\cdot}t)\,(\mathbf{u}(t) - K(\omega{\cdot}t)\,\mathbf{x}(t))\|^2 dt$$

for each admissible pair $(\mathbf{x},\mathbf{u}) \in L^2_n \times L^2_m$. Let K_0 satisfy Hypothesis 7.3 (for instance, $K_0 = K$). Given such a pair, consider the function $\widehat{\mathbf{u}}$ given by (7.17) (i.e. $\widehat{\mathbf{u}}(t) = \mathbf{u}(t) - K_0(\omega{\cdot}t)\,\mathbf{x}(t)$), which also belongs to L^2_m, and note that $\mathbf{x}$ solves the system $\mathbf{x}' = \widehat{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widehat{\mathbf{u}}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$, where $\widehat{A} = A + B\,K_0$. Take a sequence of continuous functions $(\widehat{\mathbf{u}}_k)$ in L^2_m with $\lim_{k\to\infty} \widehat{\mathbf{u}}_k = \widehat{\mathbf{u}}$ in L^2_m, and represent by $\mathbf{x}_k(t)$ the unique solution of the system $\mathbf{x}' = \widehat{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widehat{\mathbf{u}}_k(t)$ with $\mathbf{x}_k(0) = \mathbf{x}_0$. Then $\mathbf{x}(t) - \mathbf{x}_k(t)$ solves $\mathbf{x}' = \widehat{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,(\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_k)$ with initial datum $\mathbf{0}$. It follows from Lemma 7.5 that $\|\mathbf{x} - \mathbf{x}_k\|_n = \|\lambda_\omega(\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_k)\|_n \le c_4\|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_k\|_m$, so that $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}_k$ in L^2_n. In addition, if $\mathbf{u}_k$ is defined from $(\mathbf{x}_k, \widehat{\mathbf{u}}_k)$ by (7.17), then $\mathbf{u} = \lim_{k\to\infty} \mathbf{u}_k$ in L^2_m and $\mathbf{x}_k$ solves $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}_k(t)$. Equality (7.30) (which holds for continuous functions) and Lemma 7.1 applied to the admissible pair $(\mathbf{x}_k, \mathbf{u}_k)$ imply that

$$\begin{aligned} 2\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0}(\mathbf{x}_k,\mathbf{u}_k) &= -\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle \\ &\quad + \int_0^\infty \|R^{1/2}(\omega{\cdot}t)\,(\mathbf{u}_k(t) - K(\omega{\cdot}t)\,\mathbf{x}_k(t))\|^2\, dt\,, \end{aligned}$$

and this together with the already verified L^2-convergence proves the assertion.

Clearly $-(1/2)\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle$ is the minimum value of $\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0}$, and it is attained at the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$, where $\bar{\mathbf{u}}(t) = K(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{x}}$ solves (7.29) with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$. Note that the admissibility of this pair is guaranteed by the Hurwitz assumption on K. In addition, if $\bar{\mathbf{y}}(t) = M^+(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)$, then $\left[\begin{smallmatrix}\bar{\mathbf{x}}(t)\\ \bar{\mathbf{y}}(t)\end{smallmatrix}\right]$ solves (7.9), and the feedback rule (7.24) holds. These facts and Lemma 7.6(ii) ensure that the Lagrange plane $l_b(\omega)$ defined by (7.25) can be represented by $\left[\begin{smallmatrix}I_n\\ M^+(\omega)\end{smallmatrix}\right]$, which in turn implies the continuity of $\Omega \to \mathcal{L}_{\mathbb{R}},\ \omega \mapsto l_b(\omega)$.

Y2⇒Y5. Following the scheme given in [156], pp. 624–625, take $\delta > 0$ small enough to ensure that $R - \delta I_m > 0$, and consider the family of systems

$$\mathbf{z}' = H_\delta(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{7.36}$$

where H_δ is obtained by substituting G and R by $G - \delta I_n$ and $R - \delta I_m$ in (7.10). Define also

$$\begin{aligned}\widetilde{\mathcal{Q}}^\delta_\omega(t,\mathbf{x},\mathbf{u}) &= \widetilde{\mathcal{Q}}_\omega(t,\mathbf{x},\mathbf{u}) - \frac{\delta}{2}\,(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2)\\ &= \frac{1}{2}\Big(\langle \mathbf{x}, (G(\omega{\cdot}t) - \delta I_n)\,\mathbf{x}\rangle\\ &\qquad + 2\,\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, (R(\omega{\cdot}t) - \delta I_m)\,\mathbf{u}\rangle\Big)\end{aligned}$$

for $\omega \in \Omega$, and note that the family (7.36) is obtained from the old family of control problems (7.6) and the new family of quadratic forms $\{\widetilde{\mathcal{Q}}^\delta_\omega \mid \omega \in \Omega\}$ in the same way as (7.9) was constructed from (7.6) and $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$.

Theorems 1.92 and 1.95 ensure that, for $\delta > 0$ sufficiently small, the family (7.36) has exponential dichotomy over Ω, and that the corresponding Lagrange plane $l^+_\delta(\omega)$ belongs to $\mathcal{D}$ for all $\omega \in \Omega$: $l^+_\delta(\omega) = \left[\begin{smallmatrix}I_n\\ M^+_\delta(\omega)\end{smallmatrix}\right]$. In other words, condition Y2 is satisfied for these values of δ. Lemma 7.9 applied to the solution M^+_δ of the Riccati equation obtained from (7.36) ensures that, if $V^\delta_\omega(t,\mathbf{x}) = \langle \mathbf{x}, M^+_\delta(\omega{\cdot}t)\,\mathbf{x}\rangle$, then

$$\frac{d}{dt}\,V^\delta_\omega(t,\mathbf{x}(t)) \le 2\widetilde{\mathcal{Q}}^\delta_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) = 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \delta(\|\mathbf{x}(t)^2\| + \|\mathbf{u}(t)\|^2)$$

whenever $(\mathbf{x},\mathbf{u})\colon [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^m$ solves (7.6). That is, Y5 holds.

Y5⇒Y6. Take a pair $(\mathbf{x},\mathbf{u}) \in L^2_n \times L^2_m$ satisfying (7.6) with $\mathbf{x}(0) = \mathbf{0}$; i.e. an admissible pair for $\widetilde{\mathcal{I}}_{\mathbf{0},\omega}$. The arguments in the second step of the proof of Y4⇒Y1, based on Hypothesis 7.3, Lemmas 7.5 and 7.1, and the density of the set of continuous functions in the set L^2_m, can be repeated to derive from relation (7.31) (which is ensured by Y5) the following equality:

$$\begin{aligned} 0 &= \langle \mathbf{x}(0), M_\delta^+(\omega)\,\mathbf{x}(0)\rangle \\ &\le 2\int_0^\infty \big(\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \delta(\|\mathbf{x}(t)\|^2 + \|\mathbf{u}(t)\|^2)\big)\,dt\,, \end{aligned}$$

which implies (7.32) and hence Y6.

Y6⇒Y1. This assertion will be proved by amplifying some arguments used in [156]. These argument are sufficient to prove Y1_1. However, they are not in and of themselves sufficient to prove the continuity of the map $\omega \mapsto l_b(\omega)$ in Y1_2. Nevertheless it will be seen that the approach of [156] can be adapted to prove also the required continuity.

The relation between the families of control systems (7.6) and (7.16) established before Lemma 7.5, and the results proved there, will be systematically used in what follows. Define

$$\widehat{\mathcal{V}}_\omega(\mathbf{x}_0) = \big\{(\mathbf{x},\widehat{\mathbf{u}}) \in L_n^2 \times L_m^2 \mid \mathbf{x}(0) = \mathbf{x}_0 \text{ and (7.19) holds on } [0,\infty)\big\}\ .$$

It is easy to deduce from the inequality $\|\lambda_\omega(\widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2)\|_n \le c_4\|\widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2\|_m$ (see (7.22)) that $\widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$ is a closed subset of $L_n^2 \times L_m^2$. Set $\widehat{\mathcal{V}}_\omega = \widehat{\mathcal{V}}_\omega(\mathbf{0})$, and note that $\widehat{\mathcal{V}}_\omega$ is a closed linear subspace of the Banach space $L_n^2 \times L_m^2$. In fact $\widehat{\mathcal{V}}_\omega$ can be identified with the graph of the bounded linear transformation $\lambda_\omega\colon L_m^2 \to L_n^2,\ \widehat{\mathbf{u}} \mapsto \lambda_\omega(\widehat{\mathbf{u}})$ defined by (7.21): $\widehat{\mathcal{V}}_\omega = \{(\lambda_\omega(\widehat{\mathbf{u}}),\widehat{\mathbf{u}}) \mid \widehat{\mathbf{u}} \in L_m^2\}$. And $\widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$ is the affine space $(\widehat{\mathbf{x}}_\omega,\mathbf{0}) + \widehat{\mathcal{V}}_\omega$, with $\widehat{\mathbf{x}}_\omega$ defined by (7.20).

Associate in the same way the sets $\mathcal{V}_\omega(\mathbf{x}_0)$ and $\mathcal{V}_\omega$ to the family (7.6). Note that both $\widehat{\mathcal{V}}_\omega$ and $\mathcal{V}_\omega$ are Banach spaces in the norm inherited from $L_n^2 \times L_m^2$, and that the map $\widehat{\mathcal{V}}_\omega \to \mathcal{V}_\omega,\ (\mathbf{x},\widehat{\mathbf{u}}) \mapsto (\mathbf{x},\widehat{\mathbf{u}} - K_0(\omega{\cdot}t)\,\mathbf{x})$ defines a bijection between $\widehat{\mathcal{V}}_\omega$ and $\mathcal{V}_\omega$. This bijection is in fact bicontinuous. Recall also the definition (7.8) of $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u})$ for $(\mathbf{x},\mathbf{u}) \in \mathcal{V}_\omega(\mathbf{x}_0)$, given in terms of the map $\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x},\mathbf{u})$ defined by (7.7). Now, for $(\mathbf{x},\widehat{\mathbf{u}}) \in \widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$, define

$$\begin{aligned} \widehat{\mathcal{Q}}_\omega(t,\mathbf{x},\widehat{\mathbf{u}}) &= \widetilde{\mathcal{Q}}_\omega(t,\mathbf{x},\widehat{\mathbf{u}} + K_0(\omega{\cdot}t)\,\mathbf{x}) \\ &= \frac{1}{2}\Big(\langle \mathbf{x},\widehat{G}(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\langle \mathbf{x},\widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}\rangle + \langle \widehat{\mathbf{u}}, R(\omega{\cdot}t)\,\widehat{\mathbf{u}}\rangle\Big)\,, \end{aligned} \tag{7.37}$$

$$\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\widehat{\mathbf{u}}) = \int_0^\infty \widehat{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\widehat{\mathbf{u}}(t))\,dt\,, \tag{7.38}$$

where $\widehat{G} = G + g\,K_0 + K_0^T g^T + K_0^T R\,K_0$ and $\widehat{g} = g + K_0^T R$. The problem to be considered now is the minimization problem for the family of functionals $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$ subject to the family of control problems (7.16). Note that the corresponding set of admissible pairs is precisely $\widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$.

The assumed condition Y6 states the existence of a constant $\delta > 0$ independent of $\omega \in \Omega$ such that $\widetilde{\mathcal{I}}_{\mathbf{0},\omega}(\mathbf{x},\mathbf{u}) \geq \delta(\|\mathbf{x}\|_n^2 + \|\mathbf{u}\|_m^2)$ for all processes $(\mathbf{x},\mathbf{u}) \in \mathcal{V}_\omega$. It follows easily from (7.18) that there exists $\widehat{\delta} > 0$ independent of $\omega \in \Omega$ such that

$$\widehat{\mathcal{I}}_{\mathbf{0},\omega}(\mathbf{x},\widehat{\mathbf{u}}) \geq \widehat{\delta}(\|\mathbf{x}\|_n^2 + \|\widehat{\mathbf{u}}\|_m^2) \tag{7.39}$$

if $(\mathbf{x},\widehat{\mathbf{u}}) \in \widehat{\mathcal{V}}_\omega$. Next, let $j_\omega: L_m^2 \to \widehat{\mathcal{V}}_\omega$ be defined by $j_\omega(\widehat{\mathbf{u}}) = (\lambda_\omega(\widehat{\mathbf{u}}),\widehat{\mathbf{u}})$, which is a bounded linear transformation with bound independent of $\omega \in \Omega$, since, according to (7.22), λ_ω has these properties. It will be convenient to polarize the quadratic form $\widehat{\mathcal{I}}_{\mathbf{0},\omega} \circ j_\omega$. To this end, if $\widehat{\mathbf{u}}, \mathbf{v} \in L_m^2$, define

$$\begin{aligned}\widehat{q}_\omega(\widehat{\mathbf{u}},\mathbf{v}) = \frac{1}{2}\int_0^\infty \Big(&\langle \lambda_\omega(\widehat{\mathbf{u}})(t), \widehat{G}(\omega{\cdot}t)\,\lambda_\omega(\mathbf{v})(t)\rangle + \langle \lambda_\omega(\widehat{\mathbf{u}})(t), \widehat{g}(\omega{\cdot}t)\,\mathbf{v}(t)\rangle \\ &+ \langle \widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}(t), \lambda_\omega(\mathbf{v})(t)\rangle + \langle \widehat{\mathbf{u}}(t), R(\omega{\cdot}t)\,\mathbf{v}(t)\rangle\Big)\,dt\end{aligned} \tag{7.40}$$

and note that $\widehat{q}_\omega(\widehat{\mathbf{u}},\mathbf{v}) = \widehat{q}_\omega(\mathbf{v},\widehat{\mathbf{u}})$ and that $\widehat{q}_\omega(\widehat{\mathbf{u}},\widehat{\mathbf{u}}) = \widehat{\mathcal{I}}_{\mathbf{0},\omega} \circ j_\omega(\widehat{\mathbf{u}})$. The boundedness of λ_ω, $\widehat{G}$, $\widehat{g}$, and R, Hölder's inequality, and the lower bound (7.39), provide strictly positive constants c_5 and c_6 independent of $\omega \in \Omega$ such that

$$|\widehat{q}_\omega(\widehat{\mathbf{u}},\mathbf{v})| \leq c_5\|\widehat{\mathbf{u}}\|_m\|\mathbf{v}\|_m \quad \text{and} \quad \widehat{q}_\omega(\widehat{\mathbf{u}},\widehat{\mathbf{u}}) \geq c_6\|\widehat{\mathbf{u}}\|_m^2 \tag{7.41}$$

whenever $\widehat{\mathbf{u}}, \mathbf{v} \in L_m^2$. More properties of this map will be explained below.

For each $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$ defines a functional on L_m^2, sending $\widehat{\mathbf{u}}$ to

$$\begin{aligned}\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}(\widehat{\mathbf{x}}_\omega + \lambda_\omega(\widehat{\mathbf{u}}),\widehat{\mathbf{u}}) &= \int_0^\infty \widehat{\mathcal{Q}}_\omega(t,\widehat{\mathbf{x}}_\omega(t) + \lambda_\omega(\widehat{\mathbf{u}})(t),\widehat{\mathbf{u}}(t))\,dt \\ &= \widehat{q}_\omega(\widehat{\mathbf{u}},\widehat{\mathbf{u}}) + \int_0^\infty (\langle \lambda_\omega(\widehat{\mathbf{u}})(t), \widehat{G}(\omega{\cdot}t)\,\widehat{\mathbf{x}}_\omega(t)\rangle + \langle \widehat{\mathbf{x}}_\omega(t), \widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}(t)\rangle)\,dt \\ &\quad + \frac{1}{2}\int_0^\infty \langle \widehat{\mathbf{x}}_\omega(t), \widehat{G}(\omega{\cdot}t)\,\widehat{\mathbf{x}}_\omega(t)\rangle\,dt\,.\end{aligned}$$

It is clear that minimizing $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$ on $\widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$ is equivalent to minimizing the quantity $\widehat{q}_\omega(\widehat{\mathbf{u}},\widehat{\mathbf{u}}) - 2\varphi_\omega(\widehat{\mathbf{u}})$ on L_m^2, where φ_ω is the functional given on L_m^2 by

$$2\varphi_\omega(\widehat{\mathbf{u}}) = -\int_0^\infty (\langle \lambda_\omega(\widehat{\mathbf{u}})(t), \widehat{G}(\omega{\cdot}t)\,\widehat{\mathbf{x}}_\omega(t)\rangle + \langle \widehat{\mathbf{x}}_\omega(t), \widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}(t)\rangle)\,dt\,. \tag{7.42}$$

Note that this functional is again uniformly bounded in ω, due to (7.22) and the boundedness of $\widehat{G}$ and $\widehat{g}$. According to the Lax–Milgram theorem (see Corollary 5.8

of [23]), the inequalities (7.41) and the symmetry of $\widehat{q}_\omega$ ensure the existence of a unique $\widehat{\mathbf{u}}_\omega \in L^2_m$ such that

$$\widehat{q}_\omega(\widehat{\mathbf{u}}_\omega, \widehat{\mathbf{u}}_\omega) - 2\varphi_\omega(\widehat{\mathbf{u}}_\omega) = \min_{\widehat{\mathbf{u}} \in L^2_m} (\widehat{q}_\omega(\widehat{\mathbf{u}}, \widehat{\mathbf{u}}) - 2\varphi_\omega(\widehat{\mathbf{u}})) .$$

This means that there exists a unique process $(\mathbf{x}_\omega, \widehat{\mathbf{u}}_\omega)$, with $\mathbf{x}_\omega = \lambda_\omega(\widehat{\mathbf{u}}_\omega)$, at which $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$ attains its minimum value on the set of admissible pairs $\widehat{\mathcal{V}}_\omega(\mathbf{x}_0)$. In other words, condition Y1_1 holds for the optimization problem now considered. The goal now is to prove that the map $\Omega \to L^2_m,\ \omega \mapsto \widehat{\mathbf{u}}_\omega$ is continuous. This is the crucial point in the extension of Yakubovich's proof of the implication Y6⇒Y1 to the general nonautonomous case.

To this end, recall that the Lax–Milgram theorem can be viewed as a corollary of a fundamental result of Stampacchia ([23], Theorem 5.6), according to which the vector $\widehat{\mathbf{u}}_\omega$ is characterized as the unique $\widehat{\mathbf{u}} \in L^2_m$ such that

$$\widehat{q}_\omega(\widehat{\mathbf{u}}, \mathbf{v} - \widehat{\mathbf{u}}) \geq \varphi_\omega(\mathbf{v} - \widehat{\mathbf{u}}) \quad \text{for all } \mathbf{v} \in L^2_m . \tag{7.43}$$

The proof of Stampacchia's theorem must be analyzed in order to prove the asserted continuity. Let $\mathbf{w}_\omega$ be the unique element of L^2_m such that $\varphi_\omega(\widehat{\mathbf{u}}) = \langle \mathbf{w}_\omega, \widehat{\mathbf{u}} \rangle_m$ for all $\widehat{\mathbf{u}} \in L^2_m$. Further, let $\mu_\omega : L^2_m \to L^2_m$ be the unique bounded linear operator such that

$$\widehat{q}_\omega(\widehat{\mathbf{u}}, \mathbf{v}) = \langle \mu_\omega(\widehat{\mathbf{u}}), \mathbf{v} \rangle_m \quad \text{for all } \mathbf{v} \in L^2_m , \tag{7.44}$$

which according to (7.41) satisfies

$$\|\mu_\omega(\widehat{\mathbf{u}})\|_m = \sup_{\|\mathbf{v}\|_m = 1} |\langle \mu_\omega(\widehat{\mathbf{u}}), \mathbf{v} \rangle_m| = \sup_{\|\mathbf{v}\|_m = 1} |\widehat{q}_\omega(\widehat{\mathbf{u}}, \mathbf{v})| \leq c_5 \|\widehat{\mathbf{u}}\|_m$$

and $\langle \mu_\omega(\widehat{\mathbf{u}}), \widehat{\mathbf{u}} \rangle_m \geq c_6 \|\widehat{\mathbf{u}}\|^2$. According to (7.43), $\widehat{\mathbf{u}}_\omega$ is characterized as the unique $\widehat{\mathbf{u}} \in L^2_m$ such that

$$\langle \mu_\omega(\widehat{\mathbf{u}}), \mathbf{v} - \widehat{\mathbf{u}} \rangle_m \geq \langle \mathbf{w}_\omega, \mathbf{v} - \widehat{\mathbf{u}} \rangle_m \quad \text{for all } \mathbf{v} \in L^2_m .$$

This holds for a given $\widehat{\mathbf{u}} \in L^2_m$ if and only if there exists $\rho > 0$ such that

$$\langle \rho\, \mathbf{w}_\omega - \rho\, \mu_\omega(\widehat{\mathbf{u}}) + \widehat{\mathbf{u}} - \widehat{\mathbf{u}}, \mathbf{v} - \widehat{\mathbf{u}} \rangle_m \leq 0 \quad \text{for all } \mathbf{v} \in L^2_m ,$$

which in turn holds if and only if $\widehat{\mathbf{u}} = \rho\, \mathbf{w}_\omega - \rho\, \mu_\omega(\widehat{\mathbf{u}}) + \widehat{\mathbf{u}}$, i.e. if and only if $\widehat{\mathbf{u}}$ is a fixed point of the affine map

$$s_\omega : L^2_m \to L^2_m , \ \ \mathbf{v} \mapsto \rho\, \mathbf{w}_\omega - \rho\, \mu_\omega(\mathbf{v}) + \mathbf{v} . \tag{7.45}$$

If $\mathbf{v}_1, \mathbf{v}_2 \in L^2_m$, then

$$\begin{aligned}
&\|s_\omega(\mathbf{v}_1) - s_\omega(\mathbf{v}_2)\|^2_m \\
&\qquad = \|\mathbf{v}_1 - \mathbf{v}_2\|^2_m - 2\rho\langle \mu_\omega(\mathbf{v}_1 - \mathbf{v}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_m + \rho^2\|\mu_\omega(\mathbf{v}_1 - \mathbf{v}_2)\|^2_m \\
&\qquad \le \|\mathbf{v}_1 - \mathbf{v}_2\|^2_m(1 - 2\rho c_6 + \rho^2 c_5^2)\,.
\end{aligned}$$

If ρ is close enough to $2c_6/c_5^2$ and to the left of this value, then the constant $k = (1 - 2\rho c_6 + \rho^2 c_5^2)^{1/2}$ belongs to $(0, 1)$, so that the map s_ω is a uniform contraction on L^2_m. It is important to note that k is independent of $\omega \in \Omega$. The conclusion is that $\widehat{\mathbf{u}}_\omega$ is the unique fixed point of the uniform contraction s_ω on L^2_m, where s_ω is determined by the chosen value of ρ.

Lemma 7.11 below, which is fundamental for this proof, is a technical result which proves that the three maps

$$\begin{aligned}
\Omega \to L^2_n\,, &\quad \omega \mapsto \widehat{\mathbf{x}}_\omega\,, \\
\lambda\colon \Omega \times L^2_m \to L^2_n\,, &\quad (\omega, \widehat{\mathbf{u}}) \mapsto \lambda_\omega(\widehat{\mathbf{u}})\,, \\
s\colon \Omega \times L^2_m \to L^2_m\,, &\quad (\omega, \mathbf{v}) \mapsto s_\omega(\mathbf{v})
\end{aligned}$$

respectively defined by (7.20), (7.21), and (7.45) are continuous. Assume for the time being that Lemma 7.11 is valid. Since, for $\omega, \omega_1 \in \Omega$, one has

$$\begin{aligned}
\|\widehat{\mathbf{u}}_\omega - \widehat{\mathbf{u}}_{\omega_1}\|_m &= \|s_\omega(\widehat{\mathbf{u}}_\omega) - s_{\omega_1}(\widehat{\mathbf{u}}_{\omega_1})\|_m \\
&\le \|s_\omega(\widehat{\mathbf{u}}_\omega) - s_{\omega_1}(\widehat{\mathbf{u}}_\omega)\|_m + \|s_{\omega_1}(\widehat{\mathbf{u}}_\omega) - s_{\omega_1}(\widehat{\mathbf{u}}_{\omega_1})\|_m\,,
\end{aligned}$$

it follows that $(1 - k)\|\widehat{\mathbf{u}}_\omega - \widehat{\mathbf{u}}_{\omega_1}\|_m \le \|s_\omega(\widehat{\mathbf{u}}_\omega) - s_{\omega_1}(\widehat{\mathbf{u}}_\omega)\|_m$, which implies the continuity of $\Omega \to L^2_m\,, \omega \mapsto \widehat{\mathbf{u}}_\omega$. In turn, this implies the continuity of $\Omega \to L^2_m\,, \omega \mapsto \mathbf{x}_\omega$ with $\mathbf{x}_\omega = \widehat{\mathbf{x}}_\omega + \lambda_\omega(\widehat{\mathbf{u}}_\omega)$. That is, for each $\mathbf{x}_0 \in \mathbb{R}^n$ the map $\Omega \to L^2_n \times L^2_m$ sending ω to the unique minimizing pair for $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$ is continuous. This completes the fundamental part of this step of the proof.

The following step is to apply the Pontryagin Maximum Principle to the stabilized system (7.16). As stated before, that this can be done is proved in [156], pp. 619–621. Fix $\mathbf{x}_0 \in \mathbb{R}^n$, and let $(\mathbf{x}_\omega, \widehat{\mathbf{u}}_\omega)$ (with $\mathbf{x}_\omega = \lambda_\omega(\widehat{\mathbf{u}}_\omega)$) be the minimizing process for $\widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}$. Introduce the Hamiltonian

$$\widehat{\mathcal{H}}_\omega(t, \mathbf{x}, \mathbf{y}, \widehat{\mathbf{u}}) = \langle\, \mathbf{y}, \mathbf{x}'\rangle - \widehat{\mathcal{Q}}_\omega(t, \mathbf{x}, \widehat{\mathbf{u}}) = \langle\, \mathbf{y}, \widehat{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widehat{\mathbf{u}}\rangle - \widehat{\mathcal{Q}}_\omega(t, \mathbf{x}, \widehat{\mathbf{u}})\,.$$

According to the Maximum Principle, there is a solution $\mathbf{y}_\omega \in L^2_n$ of the adjoint equation

$$\mathbf{y}' = -\frac{\partial \widehat{\mathcal{H}}_\omega}{\partial \mathbf{x}}\,(t, \mathbf{x}_\omega, \mathbf{y}, \widehat{\mathbf{u}}_\omega) = -\widehat{A}^T(\omega{\cdot}t)\,\mathbf{y} + \widehat{G}(\omega{\cdot}t)\,\mathbf{x}_\omega + \widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}_\omega \tag{7.46}$$

such that $(\partial\widehat{\mathcal{H}}_\omega/\partial\widehat{\mathbf{u}})(t,\mathbf{x}_\omega(t),\mathbf{y}_\omega(t),\widehat{\mathbf{u}}_\omega(t)) = \mathbf{0}$. Hypothesis 7.3 and Proposition 1.73 ensure that the adjoint family $\mathbf{y}' = -\widehat{A}^T(\omega{\cdot}t)\,\mathbf{y}$ is of uniform Hurwitz type at $-\infty$, i.e. there exist positive constants $\bar\eta$ and $\bar\beta$ such that

$$\|(U^T_{\widehat{A}})^{-1}(t,\omega)\,U^T_{\widehat{A}}(s,\omega)\| \le \bar\eta\, e^{\bar\beta(t-s)} \tag{7.47}$$

whenever $\omega\in\Omega$ and $t\le s$. Since the map $t\mapsto \widehat{G}(\omega{\cdot}t)\,\mathbf{x}_\omega(t)+\widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}_\omega(t)$ belongs to L^2_n, it follows that

$$\mathbf{y}_\omega(t) = -\int_t^\infty (U^T_{\widehat{A}})^{-1}(t,\omega)\,U^T_{\widehat{A}}(s,\omega)\,(\widehat{G}(\omega{\cdot}s)\,\mathbf{x}_\omega(s)+\widehat{g}(\omega{\cdot}s)\,\widehat{\mathbf{u}}_\omega(s))\,ds$$

is the unique solution of (7.46) in L^2_n: on the one hand, it follows from (7.47) that it is square integrable (see the proof of Lemma 7.5); and, on the other hand, the existence of another square integrable solution of (7.46) would imply the existence of a solution in L^2_n of the homogeneous equation $\mathbf{y}' = -\widehat{A}^T(\omega{\cdot}t)\,\mathbf{y}$, and hence the existence of a globally bounded solution (see the proof of Lemma 6.18), contradicting Proposition 1.56.

The next objective is to check that the map $\Omega\to\mathbb{R}^n,\ \omega\mapsto\mathbf{y}_\omega(0)$ is continuous. This is a simple consequence of (7.47), of the continuity of $U_{\widehat{A}}$, and of the continuity of the map $\Omega\to L^2_n,\omega\mapsto\mathbf{g}_\omega$ defined by $\mathbf{g}_\omega(s) = \widehat{G}(\omega{\cdot}s)\,\mathbf{x}_\omega(s)+\widehat{g}(\omega{\cdot}s)\,\widehat{\mathbf{u}}_\omega(s)$, which in turn follows from the continuity and boundedness of $\widehat{G}$ and $\widehat{g}$ and from the continuity of the maps $\omega\mapsto\mathbf{x}_\omega$ and $\omega\mapsto\widehat{\mathbf{u}}_\omega$. The details are omitted: see the proof of Lemma 7.11 below for similar considerations.

The function $\mathbf{z}_\omega(t) = \left[\begin{smallmatrix}\mathbf{x}_\omega(t)\\ \mathbf{y}_\omega(t)\end{smallmatrix}\right]$ solves the linear Hamiltonian system $\mathbf{z}' = \widehat{H}(\omega{\cdot}t)\,\mathbf{z}$ given by

$$\widehat{H}(\omega) = \begin{bmatrix} \widehat{A}(\omega)-B(\omega)\,R^{-1}(\omega)\,\widehat{g}^T(\omega) & B(\omega)\,R^{-1}(\omega)\,B^T(\omega)\\ \widehat{G}(\omega)-\widehat{g}(\omega)\,R(\omega)\,\widehat{g}^T(\omega) & -\widehat{A}^T(\omega)+\widehat{g}(\omega)\,R^{-1}(\omega)\,B^T(\omega)\end{bmatrix}.$$

In addition, $\lim_{t\to\infty}\mathbf{z}_\omega(t) = \mathbf{0}$. This is proved again as was done in Lemmas 6.18 and 7.1, since $\mathbf{z}_\omega$ and $\mathbf{z}'_\omega$ lie in L^2_{2n}.

Recall that during the whole procedure, $\mathbf{x}_0$ is fixed. Represent the just-determined vector $\mathbf{y}_\omega(0)$ by $\mathbf{y}_{\mathbf{x}_0,\omega}$. The continuity of $\Omega\to\mathbb{R}^n,\ \omega\mapsto\mathbf{y}_{\mathbf{x}_0,\omega}$ has been established. As explained when describing condition Y1, the Lagrange plane $\widehat{l}_b(\omega)$ associated to the minimization problem studied above admits the representation $\left[\begin{smallmatrix}I_n\\ M_\omega\end{smallmatrix}\right]$, where the columns of M_ω are given by the vectors $\mathbf{y}_{\mathbf{e}_1,\omega},\dots,\mathbf{y}_{\mathbf{e}_n,\omega}$ associated to the coordinate vectors $\mathbf{e}_1,\dots,\mathbf{e}_n$ (see Lemma 7.6). This proves the continuity of the map $\Omega\to\mathcal{L}_\mathbb{R},\ \omega\mapsto\widehat{l}_b(\omega)$.

Putting all the above information together: the families of control systems (7.16) and functionals (7.38) satisfy condition Y1. This result must be now carried back to the original control systems (7.6) and functionals (7.8). It is obvious that the unique minimizing pair for $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u})$ in the set of admissible pairs $\mathcal{V}_\omega(\mathbf{x}_0)$ is $(\mathbf{x}_\omega,\mathbf{u}_\omega)$ with

$\mathbf{u}_\omega(t) = \widehat{\mathbf{u}}_\omega(t) + K_0(\omega{\cdot}t)\,\mathbf{x}_\omega(t)$, since defining $\widehat{\mathbf{u}}_\omega(t)$ from $\mathbf{u}$ by this equality yields $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u}) = \widehat{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\widehat{\mathbf{u}})$. In particular, condition Y1$_1$ holds. And, on the other hand, it is easy to check that the same function $\mathbf{z}_\omega(t) = \begin{bmatrix} \mathbf{x}_\omega(t) \\ \mathbf{y}_\omega(t) \end{bmatrix}$ satisfies the Hamiltonian system (7.9). In fact, $H(\omega) = \widehat{H}(\omega)$. This means that the Lagrange plane $l_b(\omega)$ given by (7.25) agrees with $\widehat{l_b}(\omega)$. Therefore, it is continuous, so that condition Y1$_2$ is satisfied. The proof of Y6⇒Y1 is finally complete (once Lemma 7.11 has been proved).

(ii) Suppose now that there exists a σ-ergodic measure m_0 with $\operatorname{Supp} m_0 = \Omega$. Note that, once the equivalences in (i) have been proved, it is enough to check that Y2⇒Y7⇒Y5, and Y7⇒Y8⇒Y7.

Y2⇒Y7. This implication has already been proved: see e.g. Remark 7.2.2. Note that it does not require the existence of m_0.

Y7⇒Y5. Repeating the ideas of the proof of Y2⇒Y5, take $\delta_0 > 0$ small enough to ensure that $R - \delta_0 I_m > 0$, define

$$\widetilde{\mathcal{Q}}^{\delta}_{\omega}(t,\mathbf{x},\mathbf{u}) = \widetilde{\mathcal{Q}}_{\omega}(t,\mathbf{x},\mathbf{u}) - \frac{\delta}{2}\,(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2)$$

for $\omega \in \Omega$ and $\delta \in (0,\delta_0]$, and consider the perturbed families (7.36), associated to (7.6) and the quadratic forms $\widetilde{\mathcal{Q}}^{\delta}_{\omega}$. By taking a smaller $\delta_0 > 0$, if needed, it is possible to guarantee that the families (7.36) have exponential dichotomy over Ω for $\delta \in [0,\delta_0]$ (see Theorem 1.92). Therefore, one can repeat the arguments of the proof of the "only if" assertion of Theorem 3.50 in order to deduce that also the rotation number of these families with respect to m_0 is zero (see also Remark 3.51.1).

Fix a value $\delta \in (0,\delta_0]$, and note that condition Y7 holds for the corresponding family (7.36). According to Theorem 5.73, this fact ensures the existence of $\varepsilon > 0$ such that

$$\mathbf{z}' = \left(H_\delta(\omega{\cdot}t) + \varepsilon \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} \right) \mathbf{z} \tag{7.48}$$

has exponential dichotomy over Ω, and such that the corresponding Weyl functions $M^{\pm}_{\delta,\varepsilon}$ are globally defined. (Note that the property $\operatorname{Supp} m_0 = \Omega$ is required at this point.)

Define $V^{\delta,\varepsilon}_{\omega}(t,\mathbf{x}) = \langle \mathbf{x}, M^{+}_{\delta,\varepsilon}(\omega{\cdot}t)\,\mathbf{x}\rangle$. The proof will be completed once it has been shown that for all $t \ge 0$, all $\omega \in \Omega$ and all pairs $(\mathbf{x}(t),\mathbf{u}(t))$ solving (7.6),

$$\frac{d}{dt}\,V^{\delta,\varepsilon}_{\omega}(t,\mathbf{x}(t)) \le 2\,\widetilde{\mathcal{Q}}^{\delta}_{\omega}(t,\mathbf{x}(t),\mathbf{u}(t)) = 2\,\widetilde{\mathcal{Q}}_{\omega}(t,\mathbf{x},\mathbf{u}) - \delta\,(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2)\,, \tag{7.49}$$

since this ensures that $M^+_{\delta,\varepsilon}$ satisfies Y5. Property (7.49) will be first established for $t = 0$. Let $M_\delta(t,\omega,M^+_{\delta,\varepsilon}(\omega))$ represent the solution of the Riccati equation corresponding to (7.36), namely

$$\begin{aligned} M' = &-MB(R-\delta I_m)^{-1}B^TM - (A^T - g(R-\delta I_m)^{-1}B^T)M \\ &- M(A - B(R-\delta I_m)^{-1}g^T) + (G-\delta I_n) - g(R-\delta I_m)^{-1}g^T, \end{aligned} \tag{7.50}$$

which satisfies $M_\delta(0,\omega,M^+_{\delta,\varepsilon}(\omega)) = M^+_{\delta,\varepsilon}(\omega)$, and which is defined for t in an open (bounded or unbounded) interval centered at 0. Here the coefficient functions A, B, G, g, and R have argument $\omega{\cdot}t$. Applying Lemma 7.9 to the family (7.36) and to the function $V^\delta_{\omega,M^+_{\delta,\varepsilon}(\omega)}(t,\mathbf{x}) = \langle \mathbf{x}, M_\delta(t,\omega,M^+_{\delta,\varepsilon}(\omega))\,\mathbf{x}\rangle$ yields

$$\frac{d}{dt}\,V^\delta_{\omega,M^+_{\delta,\varepsilon}(\omega)}(t,\mathbf{x}(t)) \le 2\,\widetilde{\mathcal{Q}}^\delta_\omega(t,\mathbf{x}(t),\mathbf{u}(t))\,.$$

Note that $M^+_{\delta,\varepsilon}(\omega)$ is a solution along the base flow of the Riccati equation associated to (7.48), which has the expression (7.50) with $B\,(R-\delta I_m)^{-1}\,B^T$ replaced by $B\,(R-\delta I_m)^{-1}\,B^T + \varepsilon I_n$. It is easy to deduce from this fact, from the Riccati equation (7.50) satisfied by $M_\delta(t,\omega,M^+_{\delta,\varepsilon}(\omega))$, and from $M_\delta(0,\omega,M^+_{\delta,\varepsilon}(\omega)) = M^+_{\delta,\varepsilon}(\omega)$, that

$$(M^+_{\delta,\varepsilon})'(\omega) = M_\delta'(t,\omega,M^+_{\delta,\varepsilon}(\omega))\,\Big|_{t=0} - \varepsilon\,(M^+_{\delta,\varepsilon})^2(\omega) \le M_\delta'(t,\omega,M^+_{\delta,\varepsilon}(\omega))\,\Big|_{t=0}$$

and hence, using again the equality $M_\delta(0,\omega,M^+_{\delta,\varepsilon}(\omega)) = M^+_{\delta,\varepsilon}(\omega)$, that

$$\frac{d}{dt}\,V^{\delta,\varepsilon}_\omega(t,\mathbf{x}(t))\,\Big|_{t=0} \le \frac{d}{dt}\,V^\delta_{\omega,M^+_{\delta,\varepsilon}(\omega)}(t,\mathbf{x}(t))\,\Big|_{t=0} \le 2\,\widetilde{\mathcal{Q}}^\delta_\omega(0,\mathbf{x}(0),\mathbf{u}(0))\,.$$

This proves (7.49) for $t = 0$, all $\omega \in \Omega$, and all pairs $(\mathbf{x},\mathbf{u})$ solving (7.6).

Now, given $s \in \mathbb{R}$, define $\mathbf{x}_s(t) = \mathbf{x}(s+t)$ and $\mathbf{u}_s(t) = \mathbf{u}(s+t)$ and note that the pair $(\mathbf{x}_s,\mathbf{u}_s)$ solves (7.6) for $\omega{\cdot}s$. It is easy to check that $(d/dt)V^{\delta,\varepsilon}_\omega(t,\mathbf{x}(t))|_{t=s} = (d/dt)V^{\delta,\varepsilon}_{\omega\cdot s}(t,\mathbf{x}_s(t))|_{t=0}$, which ensures that

$$\frac{d}{dt}\,V^{\delta,\varepsilon}_\omega(t,\mathbf{x}(t))\,\Big|_{t=s} \le 2\,\widetilde{\mathcal{Q}}^\delta_{\omega\cdot s}(0,\mathbf{x}_s(0),\mathbf{u}_s(0)) = 2\,\widetilde{\mathcal{Q}}^\delta_\omega(s,\mathbf{x}(s),\mathbf{u}(s))\,.$$

This completes the proof of (7.49) and of the implication Y7⇒Y5.

Y7⇒Y8. Since the family (7.9) has exponential dichotomy over Ω, Theorem 1.92 provides $\delta > 0$ such that the family (7.33) has exponential dichotomy over Ω if $\max_{\omega\in\Omega}\|K(\omega)\| < \delta$. The continuous variation in K of the rotation number with respect to m_0 (established in Theorem 2.25), the fact that its image lies in a discrete group when the exponential dichotomy property holds (ensured by Theorem 2.28), and the condition that it vanishes for (7.9) (included in Y7) prove that the rotation number of (7.33) is zero whenever $\max_{\omega\in\Omega}\|K(\omega)\| < \delta$, which means that Y8 holds.

Y8⇒Y7. Take a positive definite continuous function $\Gamma\colon\Omega \to \mathbb{S}_{2n}(\mathbb{R})$, and recall that Γ satisfies the conditions described in Hypotheses 3.3 with respect to (7.9) (see Remark 3.5.1). Consider the families of perturbed Hamiltonian systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda\, J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z}\,, \quad \omega\in\Omega\,. \tag{7.51}$$

Condition Y8 ensures that the rotation number with respect to m_0 is zero at least for the families corresponding to the values of λ which belong to an open interval centered at 0. Theorem 3.50 ensures that all these families have exponential dichotomy over Ω. That is, the statements of Y7 hold.

Lemma 7.11 *Suppose that Hypothesis 7.3 and condition Y6 hold, and fix $\mathbf{x}_0\in\mathbb{R}^n$. Then the maps*

$$\begin{aligned} \Omega &\to L^2_n\,, \quad \omega\mapsto\widehat{\mathbf{x}}_\omega\,,\\ \lambda\colon\Omega\times L^2_m &\to L^2_n\,, \quad (\omega,\widehat{\mathbf{u}})\mapsto\lambda_\omega(\widehat{\mathbf{u}})\,,\\ s\colon\Omega\times L^2_m &\to L^2_m\,, \quad (\omega,\mathbf{v})\mapsto s_\omega(\mathbf{v}) \end{aligned}$$

respectively defined by (7.20), (7.21) *and* (7.45) *are continuous.*

Proof The proof of the continuity of $\Omega\to L^2_n$, $\omega\mapsto\widehat{\mathbf{x}}_\omega$, which is somewhat simpler than the other ones, is omitted. Fix $(\omega,\widehat{\mathbf{u}})\in\Omega\times L^2_m$. Since $\lambda_\omega(\widehat{\mathbf{u}})-\lambda_{\omega_1}(\widehat{\mathbf{u}}_1)=\lambda_\omega(\widehat{\mathbf{u}})-\lambda_{\omega_1}(\widehat{\mathbf{u}})+\lambda_{\omega_1}(\widehat{\mathbf{u}}-\widehat{\mathbf{u}}_1)$, the second bound in (7.22) shows that to check the continuity of $\lambda\colon\Omega\times L^2_m\to L^2_n$ it suffices to check that, for each fixed $\omega_0\in\Omega$ and $\widehat{\mathbf{u}}\in L^2_m$, and for each sequence (ω_k) with limit ω_0, one has that $\lim_{k\to\infty}\|\lambda_{\omega_k}(\widehat{\mathbf{u}})-\lambda_{\omega_0}(\widehat{\mathbf{u}})\|_n = 0$. Write $V(t,s,\omega) = U_{\widehat{A}}(t,\omega)\,U^{-1}_{\widehat{A}}(s,\omega)\,B(\omega{\cdot}s)$ and $b=\sup_{\omega\in\Omega}\|B(\omega)\|$. The bound (7.15) ensures that, for all pairs $\omega,\bar\omega\in\Omega$, $\|V(t,s,\omega)-V(t,s,\bar\omega)\|\le 2\,b\,\eta\,e^{-\beta(t-s)}$ if $t\ge s$, and hence $\int_0^t\|V(t,s,\omega)-V(t,s,\bar\omega)\|\,ds\le 2\,b\,\eta/\beta$ for all $t\ge 0$ and $\int_s^\infty\|V(t,s,\omega)-V(t,s,\bar\omega)\|\,dt\le 2\,b\,\eta/\beta$ for all $s\ge 0$. Using Hölder's inequality as in (7.23),

$$\begin{aligned} \|\lambda_{\omega_k}(\widehat{\mathbf{u}})-\lambda_{\omega_0}(\widehat{\mathbf{u}})\|^2_n &= \int_0^\infty\|\lambda_{\omega_k}(\widehat{\mathbf{u}})(t)-\lambda_{\omega_0}(\widehat{\mathbf{u}})(t)\|^2\,dt\\ &\le\frac{2\,b\,\eta}{\beta}\int_0^\infty\left(\int_0^t\|V(t,s,\omega_k)-V(t,s,\omega_0)\|\,\|\widehat{\mathbf{u}}(s)\|^2\,ds\right)dt\\ &=\frac{2\,b\,\eta}{\beta}\int_0^\infty\|\widehat{\mathbf{u}}(s)\|^2\left(\int_s^\infty\|V(t,s,\omega_k)-V(t,s,\omega_0)\|\,dt\right)ds \end{aligned}$$

Take $\varepsilon > 0$ and write $\int_0^\infty = \int_0^{s_0} + \int_{s_0}^\infty$. Then,

$$\begin{aligned}\frac{2\,b\,\eta}{\beta}\int_{s_0}^\infty \|\widehat{\mathbf{u}}(s)\|^2 \left(\int_s^\infty \|V(t,s,\omega_k) - V(t,s,\omega_0)\|\,dt\right) ds \\ \le \frac{4\,b^2\eta^2}{\beta^2}\int_{s_0}^\infty \|\widehat{\mathbf{u}}(s)\|^2 \le \frac{\varepsilon^2}{2}\end{aligned}$$

if s_0 is large enough, for all $k \in \mathbb{N}$. Fix such a value of s_0, and note that to complete the proof of the continuity of λ it is enough to show that, given any $\bar{\varepsilon} > 0$,

$$\int_0^\infty \|V(t,s,\omega_k) - V(t,s,\omega_0)\|\,dt \le \bar{\varepsilon}$$

for all $s \in [0, s_0]$ if k is large enough, since then

$$\frac{2\,b\,\eta}{\beta}\int_0^{s_0} \|\widehat{\mathbf{u}}(s)\|^2 \left(\int_s^\infty \|V(t,s,\omega_k) - V(t,s,\omega_0)\|\,dt\right) ds \le \frac{2\,b\,\eta}{\beta}\,\bar{\varepsilon}\,\|\widehat{\mathbf{u}}\|_m^2\,.$$

And this is easy: first, for all $s \in [0, s_0]$,

$$\int_{t_0}^\infty \|V(t,s,\omega_k) - V(t,s,\omega_0)\|\,dt \le \frac{2\,b\,\eta}{\beta}\,e^{-\beta(t_0-s)} \le \frac{2\,b\,\eta}{\beta}\,e^{-\beta(t_0-s_0)} \le \frac{\bar{\varepsilon}}{2}$$

if t_0 is large enough, for all $k \in \mathbb{N}$; and second, fixing such a value of t_0, there exists k_0 such that $\|V(t,s,\omega_m) - V(t,s,\omega_0)\| \le \bar{\varepsilon}/(2t_0)$ if $t \in [0, t_0]$, $s \in [0, s_0]$, and $k \ge k_0$, so that $\int_0^{t_0} \|V(t,s,\omega_k) - V(t,s,\omega_0)\|\,dt < \bar{\varepsilon}/2$. Thus, λ is continuous.

The proof of the continuity of $s\colon \Omega \times L_m^2 \to L_m^2,\ (\omega, \mathbf{v}) \mapsto s_\omega(\mathbf{v})$ requires the proof of the continuity of the maps

$$\begin{aligned}\Omega \to L_m^2\,, &\quad \omega \mapsto \mathbf{w}_\omega\,,\\ \mu\colon \Omega \times L_m^2 \to L_m^2\,, &\quad (\omega, \widehat{\mathbf{u}}) \mapsto (\omega, \widehat{\mathbf{u}}) \mapsto \mu_\omega(\widehat{\mathbf{u}})\,.\end{aligned}$$

The first continuity property will now be analyzed. It is equivalent to the continuity of $\varphi\colon \Omega \to (L_m^2)^*,\ \omega \mapsto \varphi_\omega$ defined by (7.42), where $(L_m^2)^*$ is the dual of L_m^2 provided with the norm topology for $\|\varphi\|_{(L_m^2)^*} = \sup_{\|\mathbf{v}\|_m = 1} |\varphi(\mathbf{v})|$. This equivalence is due to the definition of $\mathbf{w}_\omega$, which satisfies $\varphi_\omega(\mathbf{v}) = \langle \mathbf{w}_\omega, \mathbf{v}\rangle$, and to the fact that

$$\|\mathbf{w}_{\omega_1} - \mathbf{w}_{\omega_2}\|_m = \sup_{\|\mathbf{v}\|_m = 1} \langle \mathbf{w}_{\omega_1} - \mathbf{w}_{\omega_2}, \mathbf{v}\rangle_m = \|\varphi_{\omega_1} - \varphi_{\omega_2}\|_{(L_m^2)^*}\,.$$

Define $\mathbf{f}\colon \Omega \times L_m^2 \to L_n^2\,, (\omega, \widehat{\mathbf{u}}) \mapsto \mathbf{f}_\omega(\widehat{\mathbf{u}})$ by

$$\mathbf{f}_\omega(\widehat{\mathbf{u}})(t) = \widehat{G}^T(\omega{\cdot}t)\,\lambda_\omega(\widehat{\mathbf{u}})(t) + \widehat{g}(\omega{\cdot}t)\,\widehat{\mathbf{u}}(t)\,.$$

The boundedness of $\widehat{G}$ and $\widehat{g}$ and the second bound in (7.22) ensure that the map $\mathbf{f}$ is well defined: in fact there exists $f_0 > 0$ such that $\|\mathbf{f}_\omega(\widehat{\mathbf{u}})\|_n \le f_0\,\|\widehat{\mathbf{u}}\|_m$ for all $\omega \in \Omega$ and every $\widehat{\mathbf{u}} \in L^2_m$. Now fix $\omega \in \Omega$ and $\widehat{\mathbf{u}} \in L^2_m$ with $\|\widehat{\mathbf{u}}\|_m = 1$, which will be used several times in what follows. Then,

$$2\,|\varphi_\omega(\widehat{\mathbf{u}}) - \varphi_{\omega_1}(\widehat{\mathbf{u}})| \le \left|\int_0^\infty (\widehat{\mathbf{x}}_\omega(t) - \widehat{\mathbf{x}}_{\omega_1}(t))^T \mathbf{f}_\omega(\widehat{\mathbf{u}})(t)\,dt\right| + \left|\int_0^\infty \widehat{\mathbf{x}}^T_{\omega_1}(t)\,(\mathbf{f}_\omega(\widehat{\mathbf{u}})(t) - \mathbf{f}_{\omega_1}(\widehat{\mathbf{u}})(t))\,dt\right|.$$

Hölder's inequality shows that

$$\left|\int_0^\infty (\widehat{\mathbf{x}}_\omega(t) - \widehat{\mathbf{x}}_{\omega_1}(t))^T \mathbf{f}_\omega(\widehat{\mathbf{u}})(t)\,dt\right| \le \|\widehat{\mathbf{x}}_\omega - \widehat{\mathbf{x}}_{\omega_1}\|_n\,\|\mathbf{f}_\omega(\widehat{\mathbf{u}})\|_n \le f_0\,\|\widehat{\mathbf{x}}_\omega - \widehat{\mathbf{x}}_{\omega_1}\|_n\,,$$

so that the continuity of $\Omega \to L^2_n,\ \omega \mapsto \widehat{\mathbf{x}}_\omega$ shows that this first term is as small as desired if ω_1 is close enough to ω. Write the second term as $\int_0^\infty = \int_0^{t_0} + \int_{t_0}^\infty$. Then, using the relation $\|\widehat{\mathbf{x}}_\omega(t)\| \le \eta\,e^{-\beta t}\,\|\mathbf{x}_0\|$ (which follows from (7.15)) and Hölder's inequality,

$$\left|\int_{t_0}^\infty \widehat{\mathbf{x}}^T_{\omega_1}(t)\,(\mathbf{f}_\omega(\widehat{\mathbf{u}})(t) - \mathbf{f}_{\omega_1}(\widehat{\mathbf{u}})(t))\,dt\right| \le 2\,\eta f_0\,\|\mathbf{x}_0\|\left(\int_{t_0}^\infty e^{-2\beta\,t}\right)^{1/2},$$

which is as small as desired if t_0 is large enough, independently of ω and ω_1. And, once such a value of t_0 has been fixed, one has

$$\begin{aligned}
&\left|\int_0^{t_0} \widehat{\mathbf{x}}^T_{\omega_1}(t)\,(\mathbf{f}_\omega(\widehat{\mathbf{u}})(t) - \mathbf{f}_{\omega_1}(\widehat{\mathbf{u}})(t))\,dt\right| \\
&\le \eta\,\|\mathbf{x}_0\|\int_0^{t_0}\Big(\|\widehat{G}(\omega{\cdot}t)\|\,\|\lambda_\omega(\widehat{\mathbf{u}})(t) - \lambda_{\omega_1}(\widehat{\mathbf{u}})(t)\| \\
&\quad + \|\widehat{G}(\omega{\cdot}t) - \widehat{G}(\omega_1{\cdot}t)\|\,\|\lambda_{\omega_1}(\widehat{\mathbf{u}})(t)\| + \|\widehat{g}(\omega{\cdot}t) - \widehat{g}(\omega_1{\cdot}t)\|\,\|\widehat{\mathbf{u}}(t)\|\Big)\,dt \\
&\le \eta\,\|\mathbf{x}_0\|\Bigg(g_0\int_0^{t_0}\|\lambda_\omega(\widehat{\mathbf{u}})(t) - \lambda_{\omega_1}(\widehat{\mathbf{u}})(t)\|\,dt \\
&\quad + c_4\left(\int_0^{t_0}\|\widehat{G}(\omega{\cdot}t) - \widehat{G}(\omega_1{\cdot}t)\|^2\,dt\right)^{1/2} + \left(\int_0^{t_0}\|\widehat{g}(\omega{\cdot}t) - \widehat{g}(\omega_1{\cdot}t)\|^2\,dt\right)^{1/2}\Bigg),
\end{aligned}$$

where $g_0 = \sup_{\omega\in\Omega}\|\widehat{G}(\omega)\|$. Hölder's inequality and the bounds (7.15) and (7.22) have been used. The continuity of $\widehat{G}$ and $\widehat{g}$ shows that the last two terms are as small as desired if ω_1 is close enough to ω, and the same property for the remaining term follows from

$$\int_0^{t_0} \|\lambda_\omega(\widehat{\mathbf{u}})(t) - \lambda_{\omega_1}(\widehat{\mathbf{u}})(t)\|\, dt \le \left(\int_0^{t_0}\int_0^{t_0} \|V(t,s,\omega) - V(t,s,\omega_1)\|^2\, ds\, dt\right)^{1/2}$$

and the uniform continuity of V on the compact set $[0,t_0]\times[0,t_0]\times\Omega$. This completes the proof of the continuity of the map $\Omega \to L^2_m,\ \omega \mapsto \mathbf{w}_\omega$.

The proof of the lemma will be completed once the continuity of the map $\mu\colon \Omega \times L^2_m \to L^2_m,\ (\omega, \widehat{\mathbf{u}}) \mapsto \mu_\omega(\widehat{\mathbf{u}})$ defined by (7.44) has been shown. As in the previous case, by definition of $\mu_\omega(\widehat{\mathbf{u}})$, this is equivalent to the continuity of the map $\widehat{q}\colon \Omega \times L^2_m \to (L^2_m)^*$ sending $(\omega, \widehat{\mathbf{u}})$ to the linear real map $\widehat{q}_{\omega,\widehat{\mathbf{u}}}$ given by $\widehat{q}_{\omega,\widehat{\mathbf{u}}}(\mathbf{v}) = \widehat{q}_\omega(\widehat{\mathbf{u}}, \mathbf{v})$, where this last expression is defined by (7.40). That is, given ω, $\widehat{\mathbf{u}}$, and $\varepsilon > 0$, it must be shown that $|\widehat{q}_\omega(\widehat{\mathbf{u}}, \mathbf{v}) - \widehat{q}_{\omega_1}(\widehat{\mathbf{u}}_1, \mathbf{v})| < \varepsilon$ whenever $\|\mathbf{v}\|_m = 1$ if $\|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_1\|_m$ and the distance of ω to ω_1 are small enough. The proof relies heavily on the properties of the function λ, including

$$\int_0^{t_0} \|\lambda_\omega(\mathbf{v})(t) - \lambda_{\omega_1}(\mathbf{v})(t)\|^2\, dt \\ \le \frac{2\,b\,\eta}{\beta}\int_0^{t_0}\int_0^{t_0} \|V(t,s,\omega) - V(t,s,\omega_1)\|\,\|\mathbf{v}(s)\|^2\, ds\, dt\,,$$

which is deduced by applying Hölder's inequality as in (7.23), and which is as small as desired for a fixed t_0 whenever $\|\mathbf{v}\|_m = 1$ if ω_1 is close enough to ω. The numerous details are omitted, since all the ideas involved have already been explained.

Remarks 7.12

1. The continuity of the functions M^+_δ and M^+ on Ω required in conditions Y2, Y4, and Y5 means that they have recurrence properties which are at least as strong as those of the data A, B, G, g, and R. This phenomenon of "conservation of recurrence" reduces to the T-periodicity of M^+_δ and M^+ in the case when the coefficient functions are all T-periodic.
2. The Weyl function M^+ provided by the Frequency and Nonoscillation Conditions is the same function M^+ which satisfies conditions Y3 and Y4.
3. As pointed out in the proof of Y1⇒Y3, the Lagrange planes $l_b(\omega)$ and $l^+(\omega)$ agree when Hypothesis 7.3 and condition Y1 hold.

Regarding condition Y1, the required continuity of the map $\omega \mapsto l_b(\omega)$ and of the minimizing pair may seem artificial. However, Example 7.13 shows that it is possible for $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}$ to admit a minimizing control for each $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, and nevertheless for the map $\omega \mapsto l_b(\omega)$ to be discontinuous. On the other hand, Theorem 7.14 states that if the base flow (Ω, σ) is minimal, then the first condition in Y1 ensures the second one. In this regard, recall that if A, B, G, g, and R are periodic, or more generally Bohr almost periodic, then the hull (Ω, σ) is minimal: see Sect. 1.3.2.

Example 7.13 Set $n = m = 1$ and define

$$G(t) = 1 + \gamma(t), \quad -A(t) = B(t) = R(t) = g(t) \equiv 1\,,$$

where γ is continuous and satisfies $\gamma(t) = 0$ for $|t| \geq 1$, as well as some further conditions to be specified below. It can immediately be checked that the common hull of $(A(t), B(t), G(t), g(t), R(t))$ is

$$\Omega = \{(-1, 1, 1 + \gamma_s(t), 1, 1) \mid s \in \mathbb{R}\} \cup \{(-1, 1, 1, 1, 1)\}\,,$$

where as usual $\gamma_s(t) = \gamma(t + s)$. It is possible to identify Ω with the set $\{s \in \mathbb{R}\} \cup \{\pm\infty\}$, where ∞ and $-\infty$ represent the same point. In this case, the functional to be minimized for each $s \in \mathbb{R}$ is

$$\widetilde{\mathcal{I}}_{s,x_0}(x, u) = \frac{1}{2}\int_0^\infty ((1 + \gamma_s(t))\, x^2(t) + 2\, x(t)\, u(t) + u^2(t))\, dt$$

associated to the two-dimensional Hamiltonian system

$$\mathbf{z}' = H_s(t)\, \mathbf{z} = \begin{bmatrix} -2 & 1 \\ \gamma_s(t) & 2 \end{bmatrix} \mathbf{z}\,; \tag{7.52}$$

and, for $\pm\infty$, the functional is

$$\widetilde{\mathcal{I}}_{\pm\infty,x_0}(x, u) = \frac{1}{2}\int_0^\infty (x^2(t) + 2\, x(t)\, u(t) + u^2(t))\, dt$$

with Hamiltonian system

$$\mathbf{z}' = H_{\pm\infty}\, \mathbf{z} = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{z}\,. \tag{7.53}$$

In addition, this constant system describes the behavior of any solution of each of the systems (7.52) for $|t|$ large enough. More precisely, the general solution of (7.52) for $t \geq 1 - s$ is $k_1 e^{-2t} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + k_2 e^{2t} \left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$. Hence each of the systems (including the limiting one) has a solution decaying exponentially as $t \to \infty$, which is unique up to a constant multiple: it agrees with $k_1 e^{-2t} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for t large enough. This implies that there exists at most one initial datum such that the corresponding solution of the (scalar) Riccati equation associated to (7.52),

$$m' = -m^2 + 4m + \gamma_s(t)\,, \tag{7.54}$$

is defined in $[0, \infty)$ and agrees with 0 for $t \geq 1 - s$. Clearly, independently of any additional property required on γ, this initial datum is 0 if $s \geq 1$, and the corresponding solution vanishes identically in $[0, \infty)$. This is also the situation for the Riccati equation $m' = -m^2 + 4m$ associated to (7.53).

Assume for the moment that such an initial datum indeed exists, and represent by $m_s^+(t)$ and $m_{\pm\infty}^+ \equiv 0$ the corresponding solutions of the Riccati equations corresponding to $s \in \mathbb{R}$ and to $\pm\infty$. Note that the functions $K_s(t) = R^{-1}(-g^T + B^T m_s^+(t)) = -1 + m_s^+(t)$ and $K_{\pm\infty} = R^{-1}(-g^T + B^T m_{\pm\infty}^+) \equiv -2$ determine the equations $x' = (A + BK_s(t))\,x = (-2 + m_s^+(t))\,x$ and $x' = (A + BK_{\pm\infty})\,x = -2x$, whose solutions decay exponentially as $t \to +\infty$. Then, a repetition of the arguments of the implications Y3⇒Y4⇒Y1 of Theorem 7.10 shows the existence of a unique minimizing pair for each of the functionals $\widetilde{\mathcal{I}}_{s,x_0}$, given by $(\bar{x}_s(t), m_s^+(t)\,\bar{x}_s(t))$ where $\bar{\mathbf{x}}_s$ solves $x' = (-2 + m_s^+(t))\,x$ with $\bar{x}(0) = x_0$; as well as for $\widetilde{\mathcal{I}}_{\pm\infty,x_0}$, given in all these cases by $(x_0 e^{-2t}, 0)$. In other words, condition $\mathrm{Y1}_1$ holds.

Now choose γ in such a way that the equation $m' = -m^2 + 4m + \gamma(t)$ has a bounded solution with $m^+(t) \equiv 4$ for $t \le -1$ and $m^+(t) \equiv 0$ for $t \ge 1$. (Just choose a C^1 function m^+ with these properties, and define $\gamma = (m^+)' + (m^+)^2 - 4m^+$.) Then the solution $m_s^+(t) = m^+(s+t)$ of (7.54) is bounded and agrees with 0 for $t \ge 1 - s$. In other words, the mentioned “special” initial datum for (7.54) exists: it is $m_s^+(0) = m^+(s)$, so that it agrees with 4 for $s \le -1$ and with 0 for $s \ge 1$ (which was already known). And recall that it is 0 also for $\pm\infty$.

In this way, the Lagrange plane $l_b(s)$ given by (7.26) is represented by $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ for $s \ge 1$ and $s = \pm\infty$ and by $\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]$ for $s \le -1$. In particular, $\lim_{s\to-\infty} l_b(s) \ne l_b(-\infty)$, and therefore condition $\mathrm{Y1}_2$ does not hold.

Theorem 7.14 *Suppose that (Ω, σ) is minimal and that Hypothesis 7.3 holds. Then condition $\mathrm{Y1}_1$ implies condition Y2. That is, the seven conditions $\mathrm{Y1}_1$, Y1, Y2, Y3, Y4, Y5, and Y6 are all equivalent. In addition, given any σ-ergodic measure m_0 on Ω, the seven conditions listed above are all equivalent to the properties Y7 and Y8 corresponding to m_0.*

Proof Under Hypotheses 7.3 and $\mathrm{Y1}_1$, Lemma 7.6 applies. If, in addition, the Frequency Condition holds, then the Lagrange plane $l_b(\omega)$ of the lemma, given by (7.25), agrees with the Lagrange plane $l^+(\omega)$ provided by the exponential dichotomy (see Remark 1.69.1). In addition, in the minimal case, any ergodic measure m_0 has full support (see Proposition 1.11(iii)). Hence, the goal is to prove that FC holds if $\mathrm{Y1}_1$ holds, and then apply Theorem 7.10 to complete the proof.

Define, for each integer $k \ge 1$, $\Omega_k = \{\omega \in \Omega \mid \|U(t,\omega)\,\mathbf{z}_0\| \le (1/2)\|\mathbf{z}_0\|$ for any $\mathbf{z}_0 \in l_b(\omega)$ and all $t \ge k\}$. The first step consists in using Lemma 7.6 to check that Ω_k is closed in Ω. So, take a sequence (ω_j) in Ω_k converging to a point $\omega_0 \in \Omega$, and assume by choosing a suitable subsequence if needed that $l_b(\omega_j)$ converges to a Lagrange plane $l(\omega_0)$. Then each $\mathbf{z}_0 \in l(\omega_0)$ is the limit of a sequence $(\mathbf{z}_j)$ with $\mathbf{z}_j \in l_b(\omega_j)$ (see Proposition 1.26(i)), so that $\|U(t,\omega_0)\,\mathbf{z}_0\| = \lim_{j\to\infty} \|U(t,\omega_j)\,\mathbf{z}_j\| \le \lim_{j\to\infty}(1/2)\|\mathbf{z}_j\| = (1/2)\|\mathbf{z}_0\|$ for each $t \ge k$. This and Lemma 7.6, taken together, show that $l(\omega_0) \subseteq l_b(\omega_0)$ (and hence that $l(\omega_0) = l_b(\omega_0)$, since both of them are n-dimensional), and that $\omega_0 \in \Omega_k$.

Lemma 7.6 also shows that $\Omega = \cup_{k\geq 1}\Omega_k$. By the Baire category theorem (see e.g. [27], Chap. 7), the compact metric space Ω is not a set of the first category in itself, so that some set Ω_k has non-empty interior. Fix an open set $\mathcal{O} \subset \Omega_k$ and take any point $\omega \in \Omega$. The minimal character of the flow provides a sequence $(t_j) \downarrow -\infty$ such that $\omega{\cdot}t_j \in \mathcal{O}$ for all $j \in \mathbb{N}$. There is no loss of generality in assuming that $t_j - t_{j+1} \geq k$ for all $j \in \mathbb{N}$. Take $\mathbf{z}_0 \in l_b(\omega)$, and set $\mathbf{z}_j = U(t_j,\omega)\,\mathbf{z}_0$; then $U(-t_1,\omega{\cdot}t_1)\,\mathbf{z}_1 = \mathbf{z}_0, U(t_1-t_2,\omega{\cdot}t_2)\,\mathbf{z}_2 = \mathbf{z}_1,\ldots,U(t_j-t_{j+1},\omega{\cdot}t_{j+1})\,\mathbf{z}_{j+1} = \mathbf{z}_j,\ldots$ It follows that $\|\mathbf{z}_j\| \leq (1/2)\|\mathbf{z}_{j+1}\|$ for $j \in \mathbb{N}$, and so $\mathbf{z}(t) = U(t,\omega)\,\mathbf{z}_0$ is unbounded as $t \to -\infty$. This fact and Theorem 1.78 ensure that the family (7.9) has exponential dichotomy over Ω. Hence the Frequency Condition FC holds, so that Y2 is satisfied. That is, condition $\mathrm{Y1}_1$ implies condition Y2.

The last statement follows immediately from Theorem 7.10.

Remark 7.15 Returning to the setting described at the beginning of the chapter, a natural question arises: how can the Frequency Theorem 7.10 be applied in order to ensure the solvability of the initial minimization problem, given by the single functional (7.3) and the single control system (7.1)? A first step could be to ensure Hypothesis 7.3: Proposition 7.33 below shows that a possible way to guarantee this hypothesis is to ensure the uniform controllability of the family of control systems (7.6) defined from (7.1); and Remark 6.16 explains that this is the case if the initial control system is uniformly null controllable, or if the initial system is simply controllable and the hull Ω of (A, B, G, g, R) is minimal. The second step should be to guarantee that one of the equivalent conditions Y2–Y6 holds. But recall (see Remark 7.7) that some of these hypotheses ensure by themselves Hypothesis 7.3: this is the case of Y2, Y3, and Y4. So it would be enough to ensure, for instance, that the family of linear Hamiltonian systems on the hull satisfies the Frequency and Nonoscillation Conditions. For the Frequency Condition, the exponential dichotomy over $\mathbb{R}$ of the initial Hamiltonian system is sufficient: see Remark 1.59.4. And the Nonoscillation Condition can be ensured if the Lagrange plane $l^+(t)$ composed of the solutions which are bounded as $t \to \infty$ can be represented as $l^+(t) \equiv \begin{bmatrix} I_n \\ M^+(t) \end{bmatrix}$ for all $t \in \mathbb{R}$ (i.e. if $l^+(t) \in \mathcal{D}$ for all $t \in \mathbb{R}$), where the symmetric matrix-valued function M^+ is bounded, since in this case the same happens for all the systems in the hull. This last condition holds automatically if the hull is minimal.

The section continues with two applications of Theorem 7.10, which are both based on the fact that the general robustness results for exponential dichotomies allow one to prove that the Frequency Condition and the Nonoscillation Condition are highly insensitive to small perturbations of the coefficient matrices A, B, G, g, and R.

Example 7.16 Suppose that A, B, G, g, and R are all continuous T-periodic functions. Suppose that the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$ given by (7.3) subject to (7.1) can be solved for each $\mathbf{x}_0 \in \mathbb{R}^n$. According to the Yakubovich theorem for the periodic case, the periodic family (7.9) (where $\omega{\cdot}t = \omega + t$ for $\omega \in [0,T)$ and $t \in \mathbb{R}$) satisfies the Frequency Condition and the Nonoscillation Condition. Hence it has exponential dichotomy over the circle $\Omega_0 = \mathbb{R}/[0,T]$, and the Lagrange plane $l^+(\omega)$ lies in $\mathcal{D}$ for each $\omega_0 \in \Omega_0$.

Next, let $\varepsilon > 0$, and let A_1, B_1, G_1, g_1, and R_1 be bounded and uniformly continuous matrix-valued functions of the appropriate dimensions, satisfying $\|A_1(t)\| \le \varepsilon, \dots, \|R_1(t)\| \le \varepsilon$ for all $t \in \mathbb{R}$. Of course, these functions need not be periodic. Then, as proved below, there exists $\varepsilon_* > 0$ such that, if $0 \le \varepsilon < \varepsilon_*$, the system $\mathbf{z}' = (H(t) + H_1(t))\,\mathbf{z}$ has exponential dichotomy on $\mathbb{R}$. Here, H and H_1 are respectively defined from (A, B, G, g, R) and $(A_1, B_1, G_1, g_1, R_1)$ as in (7.5). Remark 1.59.4 ensures that the family of systems defined over the hull Ω of $H + H_1$ have exponential dichotomy over Ω. And, in addition, the Lagrange plane $l^+(\omega)$ exists for every $\omega \in \Omega$ (see also below). Hence the family over $\widetilde{\Omega}$ satisfies the Frequency and Nonoscillation Conditions, so that Theorem 7.10 ensures that, for each $\widetilde{\omega} \in \widetilde{\Omega}$ (and in particular, for the point giving rise to the hull), the functional $\mathcal{I}_{\mathbf{x}_0,\widetilde{\omega}}$ admits a minimizing pair for each $\mathbf{x}_0 \in \mathbb{R}^n$.

The assertion concerning the exponential dichotomy can be proved by contradiction. Assume that there exists a sequence $(\widetilde{K}_m : \mathbb{R} \to \mathfrak{sp}(n, \mathbb{R}))$ of uniformly continuous functions with $\sup_{t\in\mathbb{R}} \|\widetilde{K}_m(t)\| \le 1/m$ such that $\mathbf{z}' = (H(t) + \widetilde{K}_m(t))\,\mathbf{z}$ does not have exponential dichotomy over $\mathbb{R}$, and look for a continuous mapping K from $[0, 1]$ to the set of uniformly continuous maps from $\mathbb{R}$ to $\mathfrak{sp}(n, \mathbb{R})$) with $\sup_{t\in\mathbb{R}} \|K(\lambda)(t)\| \le 2$ for all $\lambda \in [0, 1]$, $K(0) = 0_n$ and $K(1/j) = \widetilde{K}_j$. For all $\lambda \in [0, 1]$, consider the closure $\mathcal{M}_\lambda$ of the set $\{(H + K(\mu))_s \mid \mu \in [0, \lambda],\ s \in \mathbb{R}\}$ (where as usual $C_s(t) = C(s + t)$) on the set of bounded and uniformly continuous maps from $\mathbb{R}$ to $\mathfrak{sp}(n, \mathbb{R})$ endowed with the compact-open topology. It is easy to check that $\mathcal{M}_\lambda$ is a compact Hausdorff topological space; that it is invariant by time-translation; and that the flow ζ given by (1.36) is continuous. In addition, $\mathcal{M}_0 \subset \mathcal{M}_\lambda \subseteq \mathcal{M}_1$ for all $\lambda \in [0, 1]$. Theorem 1.91(i) provides the sought-for contradiction. And the same argument, together with the fact that $\mathcal{D}$ is open and an application of Theorem 1.91(ii.3), proves the assertion concerning the global existence of the Weyl function M^+ for the perturbed family of systems.

Example 7.17 For the second application, let $\mathbb{T}^k$ be the k-torus with angular variables $\theta_1, \dots, \theta_k$, identified with $(\mathbb{R}/[0, 2\pi])^k$. Let $\alpha_1, \dots, \alpha_k$ be real numbers. Write $\theta = (\theta_1, \dots, \theta_k)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$, and represent by $\theta + \alpha{\cdot}t$ the Kronecker flow on $\mathbb{T}^k$. Let A, B, G, g, and R be matrix-valued functions of the appropriate sizes, defined and continuous on $\mathbb{T}^k$, so that for each $\theta \in \mathbb{T}^k$, the functions $t \mapsto A(\theta + \alpha{\cdot}t), \dots, t \to R(\theta + \alpha{\cdot}t)$ are quasi-periodic functions. Consider the family of Hamiltonian systems

$$\mathbf{z}' = H(\theta + \alpha{\cdot}t)\,\mathbf{z} \tag{7.55}$$

for $\theta \in \mathbb{T}^k$, where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and H is defined from (A, B, G, g, R) as in (7.5). Suppose that, for some fixed frequency vector α, the family (7.55) has exponential dichotomy over $\mathbb{T}^k$, and that the Lagrange plane $l^+(\theta)$ belongs to $\mathcal{D}$ for all $\theta \in \mathbb{T}^k$. In other words, assume that the family (7.55) satisfies the Frequency and Nonoscillation Conditions.

An application of the Sacker–Sell perturbation theorem, to be explained below, now allows one to show that there exists $\varepsilon_0 > 0$ such that, if the distance on $\mathbb{T}^k$

between α and γ is $d_{\mathbb{T}^k}(\alpha,\gamma) \le \varepsilon_0$, then the family of Hamiltonian systems obtained by substituting α by γ in (7.55) has exponential dichotomy over $\mathbb{T}^k$, with $l^+(\theta) \in \mathcal{D}$ for all $\theta \in \mathbb{T}^k$. That is, for all frequency vectors γ near α, and for each $\theta \in \mathbb{T}^k$, the functional $\widetilde{\mathcal{I}}_{\theta,\mathbf{x}_0}$ admits a minimizing pair for each $x_0 \in \mathbb{R}^n$. Of course, varying α gives rise to a very "strong" perturbation of H. As already indicated, the point is that the Frequency Condition and the Nonoscillation Condition are insensitive even to such strong perturbations.

To guarantee the existence of the mentioned ε_0, define

$$\mathcal{M} = \{H_{\theta,\gamma} \mid \theta \in \mathbb{T}^k,\ \gamma \in \mathbb{T}^k\}\,,$$

where $H_{\theta,\gamma}\colon \mathbb{R} \to \mathfrak{sp}(n,\mathbb{R})$ is defined by $H_{\theta,\gamma}(t) = H(\theta + \gamma{\cdot}t)$, and consider it as a subset of the set of bounded and uniformly continuous maps from $\mathbb{R}$ to $\mathfrak{sp}(n,\mathbb{R})$ endowed with the compact-open topology. As in the previous example, it is easy to check that $\mathcal{M}$ is a compact Hausdorff topological space; that it is invariant by time-translation; and that the flow ζ given by (1.36) is continuous. Consider also its subsets

$$\mathcal{M}_\varepsilon = \{H_{\theta,\gamma} \mid \theta \in \mathbb{T}^k,\ \gamma \in \mathbb{R}^k \text{ with } d_{\mathbb{T}^k}(\alpha,\gamma) \le \varepsilon\}$$

for $\varepsilon > 0$, which are compact and invariant, and apply once more Theorem 1.91 to $\mathcal{M}_0 \subset \mathcal{M}_\varepsilon \subseteq \mathcal{M}$ to get the desired conclusion.

7.2.1 The Frequency Theorem and the Rotation Number

There are two goals to be achieved in this section. The first one is to see how condition Y8 (which, as is explained in Remark 7.8.2, is the reformulation of Yakubovich's condition of strong nonoscillation given in p. 1030 of [157] in a way appropriate for the theory of nonautonomous linear Hamiltonian systems) can be weakened in some cases. This will be done in Theorem 7.18. The second goal is to establish a connection between the families of linear Hamiltonian systems satisfying the Frequency and Nonoscillation Conditions and one of the instability zones labeled in terms of the rotation number (see Sect. 2.3): more precisely, these Hamiltonian systems are shown to belong to the zone corresponding to zero rotation number.

Theorem 7.18 gives a sufficient condition for the simultaneous validity of the Frequency Condition FC and the Nonoscillation Condition NC in terms of the properties of the rotation number for the perturbed families (7.51), where H is given by (7.9) and $\Gamma = \left[\begin{smallmatrix} -\Gamma_2 & \Gamma_1^T \\ \Gamma_1 & \Gamma_3 \end{smallmatrix}\right]$ is a suitable perturbation. This condition is weaker than Y8. This result will be complemented with Proposition 7.19, which provides a wide set of perturbations Γ for which the family (7.51) satisfies two of the hypotheses of

Theorem 7.18: namely, the Atkinson condition, and the existence of $\lambda_1 < 0$ (in fact $\lambda_1 = -1$) such that $B R^{-1} B^T + \lambda_1 \Gamma_3 \geq 0$.

Let $\alpha_\Gamma(m, \lambda)$ represent the rotation number of the family (7.51) corresponding to $\lambda \in \mathbb{R}$ with respect to the σ-ergodic measure m on Ω. Note that $\alpha_\Gamma(m, 0)$ is obviously independent of Γ.

Theorem 7.18 *Suppose that Hypothesis 7.3 holds and that there exists a σ-ergodic measure m_0 on Ω with $\Omega = \operatorname{Supp} m_0$. Take Γ satisfying the Atkinson Hypotheses 3.3 with respect to the unperturbed family (7.9) and such that there exist $\lambda_1 < 0$ with $B R^{-1} B^T + \lambda_1 \Gamma_3 \geq 0$. Suppose also that there exists $\lambda_2 > 0$ such that and $\alpha_\Gamma(m_0, \lambda_2) = 0$. Then the Frequency Condition FC and the Nonoscillation Condition NC are valid.*

Proof According to Theorem 2.31, if $B R^{-1} B^T + \lambda \Gamma_3 \geq 0$ then $\alpha_\Gamma(m_0, \lambda) \geq 0$. Hypotheses 3.3 guarantee that $\Gamma_3 \geq 0$, so that $\alpha_\Gamma(m_0, \lambda) \geq 0$ for all $\lambda \geq \lambda_1$. In addition, according to Proposition 2.33, the function $\lambda \mapsto \alpha_\Gamma(m_0, \lambda)$ is continuous and nondecreasing. These facts and the condition $\alpha_\Gamma(m_0, \lambda_2) = 0$ ensure that $\alpha_\Gamma(m_0, \lambda) = 0$ for all $\lambda \in (\lambda_1, \lambda_2)$. Consequently, since $\operatorname{Supp} m_0 = \Omega$, Theorem 3.50 guarantees the exponential dichotomy over Ω for the families corresponding to these values of λ. Taking $\lambda = 0$ shows that the Frequency Condition is valid, as well as that the rotation number of the unperturbed family (7.9) with respect to m_0 vanishes. Therefore, the equivalence of Y2 and Y7 under Hypothesis 7.3 completes the proof.

Proposition 7.19 *Suppose that condition C1 holds. Define*

$$\Gamma(\omega) = \begin{bmatrix} C(\omega) & 0_n \\ 0_n & B(\omega) R^{-1}(\omega) B^T(\omega) \end{bmatrix}$$

and set $\widetilde{A} = A - B R^{-1} g^T$. Suppose also that one of the following situations holds:

(i) *$C\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$ is a continuous function taking values in the set of positive definite symmetric matrices.*
(ii) *$C = \widetilde{G} = G - g R^{-1} g^T$ is positive semidefinite, and each minimal subset of Ω contains at least one element ω_2 such that $\mathbf{x}' = -\widetilde{A}^T(\omega_2{\cdot}t)\,\mathbf{x} + \widetilde{G}(\omega_2{\cdot}t)\,\mathbf{u}$ is null controllable.*
(iii) *$C = -\widetilde{G} = -G + g R^{-1} g^T$ is negative semidefinite, and each minimal subset of Ω contains at least one element ω_3 such that $\mathbf{x}' = -\widetilde{A}^T(\omega_3{\cdot}t)\,\mathbf{x} + \widetilde{G}(\omega_3{\cdot}t)\,\mathbf{u}$ is null controllable.*

Then there exist $t_0 > 0$ and $\delta > 0$ such that for all $\omega \in \Omega$,

$$\int_0^{t_0} \|\, \Gamma(\omega{\cdot}t)\, U(t,\omega)\, \mathbf{z}\|^2 \, dt \geq \delta\, \|\mathbf{z}\|^2 \quad \textit{whenever}\ \mathbf{z} \in \mathbb{R}^{2n}\,.$$

In particular, Γ satisfies the Atkinson Hypotheses 3.3 with respect to (7.9).

Proof The proof follows a scheme similar to that of the proof of Proposition 6.7. As in that proof, it is enough to show the null controllability for all $\omega \in \Omega$ of the corresponding system (6.19), which can be rewritten in the following form:

$$\begin{aligned} \mathbf{x}' &= -\widetilde{A}^T(\omega{\cdot}t)\,\mathbf{x} - \widetilde{G}(\omega{\cdot}t)\,\mathbf{y} + C(\omega{\cdot}t)\,\mathbf{w}_1\,, \\ \mathbf{y}' &= \widetilde{A}(\omega{\cdot}t)\,\mathbf{y} + B(\omega{\cdot}t)\,R^{-1}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,(-\mathbf{x} + \mathbf{w}_2)\,. \end{aligned} \tag{7.56}$$

Hence, fix $\omega \in \Omega$ and $\mathbf{z}_0 = \left[\begin{smallmatrix}\mathbf{x}_0\\ \mathbf{y}_0\end{smallmatrix}\right] \in \mathbb{R}^{2n}$. If (i) holds, Remark 6.2.3 ensures the null controllability of

$$\mathbf{x}' = -\widetilde{A}^T(\omega{\cdot}t)\,\mathbf{x} + C(\omega{\cdot}t)\,\mathbf{u}\,, \tag{7.57}$$

so that there is an integrable control $\mathbf{u}_1\colon [0, t_1] \to \mathbb{R}^n$ such that the corresponding solution of the system given by $\mathbf{u} = \mathbf{u}_1$ with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$ satisfies $\bar{\mathbf{x}}(t_1) = \mathbf{0}$. Analogously, under condition C1, Theorem 6.4 and Lemma 7.34 guarantee that $\mathbf{y}' = \widetilde{A}(\omega{\cdot}t)\,\mathbf{y} + B(\omega{\cdot}t)\,R^{-1}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,\mathbf{u}$ is null controllable, which in turn provides an integrable control $\mathbf{u}_2\colon [0, t_2] \to \mathbb{R}^n$ such that the solution $\bar{\mathbf{y}}(t)$ of this system with $\mathbf{u} = \mathbf{u}_2$ and $\bar{\mathbf{y}}(0) = \mathbf{y}_0$ satisfies $\bar{\mathbf{y}}(t_2) = \mathbf{0}$. Assume without loss of generality $t_1 = t_2 = t_0$, and define $\mathbf{w}_1(t) = \mathbf{u}_1(t) + C^{-1}(\omega{\cdot}t)\,\widetilde{G}(\omega{\cdot}t)\,\bar{\mathbf{y}}$ and $\mathbf{w}_2 = \mathbf{u}_2 + \bar{\mathbf{x}}$. Then $\left[\begin{smallmatrix}\bar{\mathbf{x}}\\ \bar{\mathbf{y}}\end{smallmatrix}\right]$ solves the corresponding system (7.56) with values $\left[\begin{smallmatrix}\mathbf{x}_0\\ \mathbf{y}_0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{0}\end{smallmatrix}\right]$ at times 0 and t_0, which proves the assertion in this case.

If the situation is described by (ii), the controllability of (7.57) is guaranteed by Theorem 6.4. Finally, in the case (iii), Remark 6.2.6 and Theorem 6.4 ensure again the null controllability of (7.57). From this point on the proof is carried out as above.

This completes the first point of this section. Now, to begin with the analysis connecting conditions FC and NC to the "instability zones", recall that the union of these zones is by definition formed by the set of Hamiltonian families which have exponential dichotomy over Ω. It has been seen at the end of Sect. 2.3 that these zones are countable in number and can be labeled in terms of the values of the rotation number.

Consider first the periodic case. Recall that $\mathcal{U}_0$ is the instability region determined by those T-periodic systems with exponential dichotomy for which the rotation number (with respect to the normalized Lebesgue measure on the circle Ω, which is ergodic for the translation flow) is zero. Under the assumption of existence of admissible pairs, Yakubovich proves in Theorem 3 of [157] that a T-periodic, linear Hamiltonian system of the form (7.9) (i.e. coming from an optimal control problem of the considered type) satisfies the periodic Frequency and Nonoscillation Conditions exactly when it lies in $\mathcal{U}_0$. Thus one of the results of [156] is equivalent to the statement that, if A, B, G, g, and R are T-periodic functions, then the problem of minimizing the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0}$ given by (7.3) subject to (7.1) is solvable for all $\mathbf{x}_0 \in \mathbb{R}^n$ if and only if the coefficient matrix of equation (7.5) lies in $\mathcal{U}_0$.

A similar statement can be formulated in the general nonautonomous setting. Let m_0 be a fixed σ-ergodic measure on Ω and let $\alpha(m_0)$ be the corresponding rotation number. As explained in Sect. 2.3, if the family (7.58) satisfies the Frequency Condition, then $2\alpha(m_0)$ belongs to the countable subgroup $\mathcal{S} = h(\check{H}^1(\Omega, \mathbb{Z}))$ given by the image of the Schwarzmann homomorphism h. If in addition it satisfies the Nonoscillation Condition, then its rotation number is zero (see Remark 7.2.2), so that the coefficient matrix belongs to the instability zone $\mathcal{U}_0$. The converse assertion can be formulated if in addition m_0 has the property that $\operatorname{Supp} m_0 = \Omega$ and Hypothesis 7.3 holds: if H belongs to the set $\mathcal{U}_0$ corresponding to this measure (i.e. if the family of systems (7.9) has exponential dichotomy over Ω and satisfies $\alpha(m_0) = 0$), then conditions FC and NC are satisfied. This is due to the equivalence between Y2 and Y7 proved in Theorem 7.10.

Summing up, the Schwarzmann homomorphism permits one to interpret the conditions FC and NC in terms of instability regions for linear nonautonomous Hamiltonian systems, at least when the corresponding flow (Ω, σ) admits a σ-ergodic measure m_0 whose support is all of Ω.

As an example, let Ω be the 3-sphere $\mathbb{S}^3$. It is well-known (see e.g. [4]) that there is a smooth vector field on $\mathbb{S}^3$ whose corresponding one-parameter group of diffeomorphisms admits an ergodic measure m_0 equivalent to the normalized Lebesgue measure on $\mathbb{S}^3$. In this case, $\check{H}^1(\Omega, \mathbb{Z}) = \{0\}$ and so $\mathcal{S} = \{0\}$. So if the family (7.58) has exponential dichotomy over Ω, then its rotation number is zero, and the Nonoscillation Condition is automatically satisfied. In this respect, the family (7.58) resembles a constant coefficient system when $\Omega = \mathbb{S}^3$. More generally, this holds whenever $\check{H}^1(\Omega, \mathbb{Z})$ is a finite group, since in this case the image of the Schwarzmann homomorphism is $\{0\}$.

7.3 Verification of the Frequency and Nonoscillation Conditions

As stated in the introduction to this chapter, the present section is devoted to the analysis of some scenarios in which the Frequency and Nonoscillation Conditions are fulfilled. This analysis will not be restricted to those Hamiltonian systems (7.9) arising from the minimization problem posed in the previous sections, but to a general family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{7.58}$$

where $H = \begin{bmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{bmatrix} : \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ is a continuous function taking values in $\mathfrak{sp}(n, \mathbb{R})$. Recall that in the case that FC and NC hold, the Lagrange plane $l^+(\omega)$ of the solutions of (7.58) which are bounded as $t \to \infty$ admits a unique representation $\begin{bmatrix} I_n \\ M^+(\omega) \end{bmatrix}$. Hence the $n \times n$ matrix-valued function M^+ is continuous on Ω, and the function $t \mapsto M^+(\omega{\cdot}t)$ solves the Riccati equation

$$M' = -MH_3(\omega{\cdot}t)M - MH_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)M + H_2(\omega{\cdot}t) = h(\omega{\cdot}t, M) \tag{7.59}$$

associated to (7.58). Or, in other words, the function M^+ is a solution along the flow of (7.59). In the language of Sects. 1.3.5 and 1.4.7, the function M^+ is a continuous equilibrium and the set $\{(\omega, M^+(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathbb{S}_n(\mathbb{R})$ is a copy of the base for the flow τ_s given by (1.23). The matrix $M^-(\omega)$ is associated in the same way to the Lagrange plane $l^-(\omega)$ if this plane also belongs to $\mathcal{D}$. The continuous functions $M^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$, when they exist, are the Weyl functions for (7.58): see Definition 1.80.

In order to analyze the different dynamical possibilities under which conditions FC and NC hold, conditions D1, D2, and D3 of Chap. 5 will play a fundamental role. They are now recalled for the reader's convenience.

D1. The $n \times n$ matrix-valued function H_3 is positive semidefinite on Ω.

D2. For all $\omega \in \Omega$ and every nonzero solution $\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix}$ of the system (5.4) with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $[0,\infty)$.

D3. For all $\omega \in \Omega$ there exists a $2n \times n$ matrix solution $G(t,\omega)$ of (7.58) taking values in the set $\mathcal{D}$ defined by (7.13) for all $t \in \mathbb{R}$.

Remarks 7.20

1. If D1 holds, condition D2 is equivalent to the uniform null controllability (see Definition 6.3) of the family $\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u}$, as Proposition 5.18 states; and both conditions D1 and D2 hold when $H_3 > 0$, as is explained in Remark 5.19.
2. Note also that D3 is equivalent to the following condition: for all $\omega \in \Omega$ there exists a solution of the Riccati equation (7.59) which is globally defined. In particular it is guaranteed by the Nonoscillation Condition. Therefore, if D1, FC, and NC hold, then *D2* is equivalent to the uniform weak disconjugacy of the family (7.9).

In most of the remaining results of the chapter, condition D1 will be assumed to hold. Note that this is the case if the family (7.58) is of the particular form (7.9), coming from a minimization problem. Since the Nonoscillation Condition ensures D3, the analysis of the families satisfying FC and NC will lead to two different possibilities: either D2 holds or it does not.

This section is divided into three parts. In the first one, the case of uniform weak disconjugacy is analyzed: this is the case in which D1, D2, and D3 hold. The occurrence of FC and NC will be characterized in terms of the properties of the principal solutions, and will imply that the sections of the two closed subbundles associated to the exponential dichotomy lie in $\mathcal{D}$. The required hypotheses can be substantially relaxed if the base flow is minimal. The second part considers the case of FC and NC in which D2 does not hold, so that the uniform weak disconjugacy is precluded, and is centered in the analysis of the robustness of this scenario. Finally, the third part presents a nontrivial example showing that the controllability condition C1, which is usually valid in the applications of the Frequency Theorem (and which will turn out to be equivalent to D2 for the family (7.9)), is in fact not necessary in order that its statements hold.

Note finally that all the results assuming the Frequency Condition FC and the Nonoscillation Condition NC admit a "symmetric" statement: they can be formulated taking as the starting point the exponential dichotomy of the family and the global existence of the Weyl function M^-.

7.3.1 The Case of Uniform Weak Disconjugacy

Theorem 5.25 proves that conditions D1, D2, and D3 ensure the uniform weak disconjugacy of the family (7.58), which in turn implies the existence of principal solutions at $\pm\infty$: see Definition 5.15. In fact, under condition D1, the uniform weak disconjugacy is equivalent to conditions D2 and D3, as stated in Theorem 5.17. Among other properties, the principal solutions are given by Lagrange planes $\widetilde{l}^{\pm}(\omega) \in \mathcal{D}$ for all $\omega \in \Omega$ with $U(t,\omega){\cdot}\tilde{l}^{\pm}(\omega) = \tilde{l}^{\pm}(\omega{\cdot}t)$, so that $\tilde{l}^{\pm}(\omega) = \left[\begin{smallmatrix} I_n \\ N^{\pm}(\omega)\end{smallmatrix}\right]$ for matrix-valued functions $N^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ which are solutions along the flow of the Riccati equation (7.59): see Sect. 1.3.5. In addition, the functions $\mp N^{\pm}$ are (upper) semicontinuous equilibria, and they satisfy $N^+ \le N^-$. Recall that the matrix-valued functions $N^{\pm}\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ are the so-called *principal functions* of (7.58).

The most valuable information for the purposes of this section has already been obtained in Theorems 5.58 and 5.59. Their statements are now rewritten for the reader's convenience. The examples described immediately before Theorem 5.58 help to understand the scope of these statements.

Theorem 7.21 *Suppose that* D1 *holds. Then,*

(i) *if* D2 *and* D3 *hold, then the family* (5.4) *satisfies the Frequency Condition* FC *if and only if* $N^- > N^+$; *i.e. if* $N^-(\omega) > N^+(\omega)$ *for all* $\omega \in \Omega$.
(ii) *If* FC *holds, then the family* (7.9) *satisfies conditions* D2 *and* D3 *if and only if there exist both Weyl functions* $M^{\pm}$, *in which case* NC *holds.*
(iii) *If* D2, D3, *and* FC *(and* NC*) hold, then the Weyl functions* $M^{\pm}$ *agree with the principal functions* $N^{\pm}$, *and hence they satisfy* $M^- > M^+$.
(iv) *If* FC *and* NC *hold, then* D2 *is equivalent to the uniform weak disconjugacy of the family* (7.9).

Theorems 5.48 and 5.49 describe the dynamical behavior and measurable behavior of the flow on $\mathcal{K}_{\mathbb{R}}$ under conditions D1, D2, and D3. The interesting situation for the purposes of this section is that in which exponential dichotomy is present: in this case the Weyl and principal functions agree, so that the set $\mathcal{J}$ of those theorems is defined in terms of the Weyl functions. And Theorem 5.61 completes the analysis of the relation between the Weyl and principal functions, showing that any family of linear Hamiltonian systems satisfying D1, D2, and D3 is the limit of a one-parameter family of families of systems which satisfy these conditions together with FC, and that the principal functions are always the pointwise limits of the corresponding one-parameter families of Weyl functions.

As anticipated above, the conditions ensuring the uniform weak disconjugacy of the initial family can be relaxed in the case of a minimal base. This is what Proposition 7.25 shows. A previous result is required, which is interesting in itself: of course, the existence of the Weyl functions does not require the existence of principal functions; but as Proposition 7.23 states, even in the absence of the principal functions, the Weyl functions play a similar role regarding the monotonicity behavior of the Lagrangian flow: see Theorem 5.48.

Consider now the new families of linear Hamiltonian systems, defined from (7.58),

$$\mathbf{z}' = \begin{bmatrix} H_1(\omega{\cdot}t) & 0_n \\ H_2(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z} = G(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega$$

and

$$\mathbf{z}' = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) + \varepsilon I_n \\ H_2(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z} = H_\varepsilon(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,. \tag{7.60}$$

Clearly, $JG \le JH \le JH_{\varepsilon_1} \le JH_{\varepsilon_2}$ if $0 \le \varepsilon_1 \le \varepsilon_2$ if D1 holds. The Riccati equations associated to these families are, respectively,

$$M' = -M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) = g(\omega{\cdot}t, M)\,, \tag{7.61}$$

$$\begin{aligned} M' &= -M\,(H_3(\omega{\cdot}t) + \varepsilon I_n)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) \\ &= h_\varepsilon(\omega{\cdot}t, M)\,. \end{aligned} \tag{7.62}$$

Note that $h_\varepsilon \le h \le g$ for all $\varepsilon \ge 0$ if D1 holds, with h given by (7.59). Recall that the solution of (7.59) with initial datum $M_0 \in \mathbb{S}_n(\mathbb{R})$ is represented by $M(t,\omega,M_0)$, and let $M_l(t,\omega,M_0)$ and $M_\varepsilon(t,\omega,M_0)$ represent the corresponding respective solutions of (7.61) and (7.62). Note also that $M_l(t,\omega,M_0)$ is globally defined, since $g(\omega,M)$ is a linear map.

Remark 7.22 Recall that if it is *a priori* possible to ensure that a solution $M(t)$ of one of these Riccati equations satisfies $M_1(t) \le M(t) \le M_2(t)$ on its domain for continuous matrix-valued functions $M_1(t)$ and $M_2(t)$, then $M(t)$ is defined at least where $M_1(t)$ and $M_2(t)$ are defined: see Remarks 1.44.2 and 1.43.

Proposition 7.23 *Suppose that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC. *Then,*

(i) *there exists* $\varepsilon_0 > 0$ *such that, if* $\varepsilon \in (0, \varepsilon_0]$*, then the family* (7.60) *has exponential dichotomy over* Ω *with* $l_\varepsilon^\pm(\omega) \equiv \left[\begin{smallmatrix} I_n \\ M_\varepsilon^\pm(\omega) \end{smallmatrix}\right]$*, and*

$$M^+(\omega) \le M_\varepsilon^+(\omega) \le M_{\varepsilon_0}^+(\omega) < M_{\varepsilon_0}^-(\omega) \le M_\varepsilon^-(\omega)\,. \tag{7.63}$$

(ii) *If* $M_0 \ge M^+(\omega)$*, then the function* $M(t, \omega, M_0)$ *is defined for every* $t \ge 0$*, and*

$$M^+(\omega{\cdot}t) \le M(t, \omega, M_0) \le M_l(t, \omega, M_0)\,.$$

(iii) *If* $M_0 \le M_\varepsilon^-(\omega)$ *for a given* $\varepsilon \in (0, \varepsilon_0]$*, then the function* $M(t, \omega, M_0)$ *is defined for every* $t \le 0$*, and*

$$M_l(t, \omega, M_0) \le M(t, \omega, M_0) \le M_\varepsilon^-(\omega{\cdot}t)\,.$$

Proof

(i) Theorems 1.92 and 1.95 ensure the existence of a number $\varepsilon_0 > 0$ such that the family (7.60) has exponential dichotomy over Ω with $l_\varepsilon^\pm(\omega) \equiv \left[\begin{smallmatrix} I_n \\ M_\varepsilon^\pm(\omega) \end{smallmatrix}\right]$ for all $\omega \in \Omega$. This and Remarks 7.20.1 and 7.20.2 show that this family satisfies D1, D2, and D3 if $0 < \varepsilon \le \varepsilon_0$. Theorem 7.21 ensures that the Weyl and principal functions agree for these values of ε, and that $M_\varepsilon^+ < M_\varepsilon^-$. Since JH_ε increases with ε, Proposition 5.51(i) ensures that $M_\varepsilon^+(\omega)$ increases and $M_\varepsilon^-(\omega)$ decreases with ε. To complete the proof of the chain of inequalities (7.63), recall that $M_0^+(\omega) = \lim_{\varepsilon\to 0+} M_\varepsilon^+(\omega)$, as Theorem 1.95 ensures.

(ii) Theorem 1.45 ensures that, if $M_0 \ge M^+(\omega)$, then $M^+(\omega{\cdot}t) \le M(t, \omega, M_0)$ wherever the second function is defined. In addition,

$$M'(t, \omega, M_0) = h(\omega{\cdot}t, M(t, \omega, M_0)) \le g(\omega{\cdot}t, M(t, \omega, M_0))\,,$$

so that Theorem 1.46(i) ensures that $M(t, \omega, M_0) \le M_l(t, \omega, M_0)$ for $t \ge 0$ where both solutions are defined. Since $M_l(t, \omega, M_0)$ is defined on $\mathbb{R}$, it follows that $M(t, \omega, M_0)$ is defined (at least) for $t \ge 0$: see Remark 7.22.

(iii) Assume that $M(t, \omega, M_0)$ is defined on $[t, 0] \subset (-\infty, 0]$. The first inequality in (iii) for this t follows, as above, from Theorem 1.46(i). In addition,

$$M'(t, \omega, M_0) = h(\omega{\cdot}t, M(t, \omega, M_0)) \ge h_\varepsilon(\omega{\cdot}t, M(t, \omega, M_0))\,,$$

and hence Theorems 1.46(iii) and Theorem 1.45 ensure that $M(t, \omega, M_0) \le M_\varepsilon(t, \omega, M_0) \le M_\varepsilon(t, \omega, M_\varepsilon^-(\omega)) = M_\varepsilon^-(\omega{\cdot}t)$. Again, Remark 7.22 ensures that $M(t, \omega, M_0)$ is defined (at least) for $t \le 0$.

Corollary 7.24 *Suppose that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC*.*

(i) *Let* ε_0 *be provided by Proposition 7.23. If* $M^+(\omega) \le M_0 \le M_\varepsilon^-(\omega)$ *for some* $\varepsilon \in (0, \varepsilon_0]$, *then the function* $M(t, \omega, M_0)$ *is defined for every* $t \in \mathbb{R}$, *and* $M^+(\omega{\cdot}t) \le M(t, \omega, M_0) \le M_\varepsilon^-(\omega{\cdot}t)$.
(ii) *If* $l^-(\omega) \in \mathcal{D}$ *for a point* $\omega \in \Omega$, *then* $M^+(\omega) < M^-(\omega)$.

Proof Assertion (i) follows immediately from the previous result. The inequality $M^+(\omega) \le M^-(\omega)$ is proved by taking the limit as $\varepsilon \to 0^+$ in $M^+(\omega) < M_\varepsilon^-(\omega)$, which is ensured by (7.63). Since

$$\dim\left(l^-(\omega) \cap l^+(\omega)\right) = \dim\left(\mathrm{Ker}\left(M^-(\omega) - M^+(\omega)\right)\right)$$

for all $\omega \in \Omega$ (which is proved as equality (5.27)), the inequality in (ii) is in fact strict.

Proposition 7.25 *Suppose that* Ω *is minimal, and that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC. *If there exists* $\omega_0 \in \Omega$ *with* $l^-(\omega_0) \in \mathcal{D}$, *then* $l^-(\omega) \in \mathcal{D}$ *for all* $\omega \in \Omega$. *In other words, under these conditions the family of Hamiltonian systems* (7.58) *is in the situation of uniform weak disconjugacy described by points (ii) and (iii) of Theorem 7.21.*

Proof Since $\mathcal{D}$ is open and $l^-\colon \Omega \to \mathcal{L}_\mathbb{R}$ is continuous, there exists an open neighborhood $\mathcal{O} \subset \Omega$ of ω_0 with $l^-(\omega) \in \mathcal{D}$ for all $\omega \in \mathcal{O}$, and Corollary 7.24(ii) ensures that $M^+(\omega) < M^-(\omega)$ for these points of the base. By the minimality of the base, there are positive times $t_1, \dots, t_k$ such that $\Omega = \sigma_{t_1}(\mathcal{O}) \cup \dots \cup \sigma_{t_k}(\mathcal{O})$. Take $\omega \in \Omega$ and $t_j \in \{t_1, \dots, t_k\}$ with $\omega = \widetilde{\omega}{\cdot}t_j$ for $\widetilde{\omega} \in \mathcal{O}$. Since $M^-(\widetilde{\omega}) \ge M^+(\widetilde{\omega})$, Proposition 7.23(ii) ensures that $M^-(\omega) = M^-(\widetilde{\omega}{\cdot}t_j) = M(t_j, \widetilde{\omega}, M^-(\widetilde{\omega}))$ exists, which together with Corollary 7.24(ii) proves the assertion.

The following immediate corollary is just a clearer way to rewrite the previous result.

Corollary 7.26 *Suppose that* Ω *is minimal and that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC. *Then,*

(i) D2 *holds if and only if* $l^-(\omega) \in \mathcal{D}$ *for every* $\omega \in \Omega$;
(ii) D2 *does not hold if and only if* $l^-(\omega) \notin \mathcal{D}$ *for every* $\omega \in \Omega$.

The last result of this subsection establishes more conditions ensuring the existence of both Weyl functions. It is important to note that condition D1 is not imposed, and that in the case that it holds, the conclusions of this proposition imply again the uniform weak disconjugacy of the Hamiltonian family (7.58).

Proposition 7.27 *Suppose that the family of Hamiltonian systems* (7.58) *satisfies the Frequency and Nonoscillation Conditions* FC *and* NC.

(i) *Let* $\{(\omega, l(\omega)) \mid \omega \in \Omega\} \subset \mathcal{D}$ *be a copy of the base such that the Lagrange planes* $l(\omega)$ *and* $l^+(\omega)$ *are supplementary for all* $\omega \in \Omega$. *Then* $l(\omega) = l^-(\omega)$ *for every* $\omega \in \Omega$, *so that the Weyl function* $M^-\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ *exists.*

(ii) *Suppose that* Ω *is* σ*-minimal, that* $\mathcal{K} \subset \mathcal{D}$ *is* τ*-minimal, and that the Lagrange planes* l *and* $l^+(\omega)$ *are supplementary for at least one point* $(\omega, l) \in \mathcal{K}$. *Then* $\mathcal{K} = \{(\omega, l^-(\omega)) \mid \omega \in \Omega\}$, *so that the Weyl function* $M^-\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ *exists.*

Proof

(i) Write $l(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N(\omega) \end{smallmatrix}\right]$ for all $\omega \in \Omega$. The hypotheses ensure that $\mathbf{w} = \left[\begin{smallmatrix} I_n & I_n \\ M^+(\omega) & N(\omega) \end{smallmatrix}\right]^{-1} \mathbf{z}$ defines a continuous change of variables. A straightforward computation from the Riccati equation (7.59) shows that the transformed family of linear Hamiltonian systems takes the form

$$\mathbf{w}' = \begin{bmatrix} H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\, M^+(\omega{\cdot}t) & 0_n \\ 0_n & H_1(\omega{\cdot}t) + H_3(\omega{\cdot}t)\, N(\omega{\cdot}t) \end{bmatrix} \mathbf{w}$$

for $\omega \in \Omega$. Obviously, each system of the transformed family has exponential dichotomy and the stable subbundles at $\pm\infty$ have dimension n. It is also obvious that the n-dimensional vector space of the solutions bounded as $t \to \infty$ is, for all $\omega \in \Omega$, the Lagrange plane represented by $\left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$, which is the transform of $\left[\begin{smallmatrix} I_n \\ M^+(\omega) \end{smallmatrix}\right]$. And if $\mathbf{w}(t) = \left[\begin{smallmatrix} \mathbf{w}_1(t) \\ \mathbf{w}_2(t) \end{smallmatrix}\right]$ is bounded as $t \to -\infty$ then $\mathbf{w}_1 = \mathbf{0}$: otherwise $\left[\begin{smallmatrix} \mathbf{w}_1(t) \\ \mathbf{0} \end{smallmatrix}\right]$ is a nontrivial bounded solution, which according to Proposition 1.56 is impossible. Hence the vector space of the initial data of solutions bounded as $t \to -\infty$ is the Lagrange plane represented by $\left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$, which is the transform of $\left[\begin{smallmatrix} I_n \\ N(\omega) \end{smallmatrix}\right]$. This proves that $N(\omega) = M^-(\omega)$ for all $\omega \in \Omega$, as asserted.

(ii) This assertion follows immediately from Corollary 1.98(i).

7.3.2 *The Absence of Uniform Weak Disconjugacy*

It will be assumed in what follows that the family (7.58) satisfies D1 and the Frequency and Nonoscillation Conditions FC and NC. According to Theorem 7.21, condition D2 holds if and only if $M^-(\omega)$ exists for all $\omega \in \Omega$. Proposition 7.28 says something more about this equivalence. Recall that the concept of abnormal system, which appears in its statement, is given in Definition 5.77.

Proposition 7.28 *Suppose that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC. *Then,*

(i) D2 *holds if and only if* $l^-(\omega) \in \mathcal{D}$ *for every* $\omega \in \Omega$ *and the family of Hamiltonian systems* (7.58) *is in the situation described by Theorem* 7.21*(iii).*

(ii) D2 *does not hold if and only if there is a* σ*-minimal subset* $\mathcal{M} \subseteq \Omega$ *such that* $l^-(\omega) \notin \mathcal{D}$ *for all* $\omega \in \mathcal{M}$. *In addition, in this case, all the systems of the family* (5.4) *corresponding to points* $\omega \in \mathcal{M}$ *are abnormal. More precisely, each system of the family* (7.58) *corresponding to a point* $\omega \in \mathcal{M}$ *has at least one nontrivial solution of the form* $\mathbf{z}^*(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_2^*(t) \end{bmatrix}$ *for* $t \in \mathbb{R}$, *with* $\mathbf{z}^*(t) \in l^-(\omega{\cdot}t)$.

Proof The equivalence stated in (i) follows from Theorem 7.21, and it proves the "if" implication in (ii). The "only if" assertion and the last statements in (ii) are explained in Remark 5.87.2.

Proposition 7.29 *Suppose that the family of Hamiltonian systems* (7.58) *satisfies* D1 *and the Frequency and Nonoscillation Conditions* FC *and* NC. *Suppose in addition that there exists a* σ*-ergodic measure* m *on* Ω *with* $\operatorname{Supp} m = \Omega$. *Then one of the following situations holds:*

1. *There is an open set* $\mathcal{O} \subseteq \Omega$ *such that* $l^-(\omega) \in \mathcal{D}$ *for every* $\omega \in \mathcal{O}$, *and* $\mathcal{O}$ *contains a* σ*-invariant set* Ω_0 *with* $m(\Omega_0) = 1$ *whose orbits are all dense;*
2. $l^-(\omega) \notin \mathcal{D}$ *for every* $\omega \in \Omega$.

Proof Suppose that condition 2 does not hold, so that the open set $\mathcal{O} = (l^-)^{-1}(\mathcal{D}) \subseteq \Omega$ is nonempty. If $\omega \in \mathcal{O}$, $M^-(\omega) < M^+(\omega)$ by Corollary 7.24(ii), and $l^-(\omega{\cdot}t) \in \mathcal{D}$ (i.e. $\omega{\cdot}t \in \mathcal{O}$) for all $t \geq 0$ by Proposition 7.23(ii). Since $\operatorname{Supp} m = \Omega$, the set $\Omega_0 = \{\omega \in \Omega \mid \{\omega{\cdot}t,\ t \leq 0\}$ is dense$\}$ has full measure for m (see Proposition 1.12). Finally, given $\omega \in \Omega_0$ there is $t \leq 0$ such that $\omega{\cdot}t \in \mathcal{O}$, and hence $\omega \in \mathcal{O}$. This completes the proof.

There are trivial examples of systems satisfying D1, D3, FC, and NC, but not D2. The simplest one is perhaps the two-dimensional constant system $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{z}$, for which $A \equiv -1$ and $B \equiv 0$. This trivial case also satisfies Hypothesis 7.3, which is required for the general Yakubovich Frequency Theorem 7.10.

The remaining results of this section analyze some dynamical consequences of the situation of absence of uniform weak disconjugacy. Both the Frequency and Nonoscillation Conditions are robust, but not D2. Theorem 7.31 analyzes two different types of one-parameter perturbations, one of which preserves D2 and the other of which makes it immediately disappear. And this result and Proposition 7.32 analyze the occurrence of almost automorphic dynamics in the endpoints of the intervals at which FC and NC hold. Example 7.37, in the following section, will illustrate the optimality of the results, in the sense that, for this example: the Frequency Theorem 7.10 applies; there exists an almost automorphic minimal set in the Riccati semiflow; and this minimal case does not reduce to a copy of the base.

In what follows it will be assumed that Ω is σ-minimal, so that Corollary 7.26 applies. Let $\Gamma = \begin{bmatrix} \Delta & 0_n \\ 0_n & 0_n \end{bmatrix} \geq 0$ be a continuous symmetric $2n \times 2n$ matrix-valued function on Ω, and suppose that $\Delta(\omega) > 0$ for a point $\omega \in \Omega$. Consider the perturbed families of Hamiltonian systems

$$\mathbf{z}' = \big(H(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) \\ H_2(\omega{\cdot}t) - \lambda\Delta(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix}\mathbf{z} \tag{7.64}$$

for $\omega \in \Omega$.

Remark 7.30 If D1, FC, and NC hold for (7.58) but D2 does not, then Γ does not satisfy the Atkinson condition given by Hypotheses 3.3: if $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ is a nontrivial solution on $[0,\infty)$ of the system (7.58) corresponding to ω, then $\Gamma(\omega{\cdot}t)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{0}\end{smallmatrix}\right]$; and this contradicts Lemma 3.6(iv). This fact precludes the possibility of using Theorem 3.50 in order to characterize the presence of exponential dichotomy by means of the properties of the rotation number.

The auxiliary families of systems

$$\begin{aligned}\mathbf{z}' &= \big(H_\varepsilon(\omega{\cdot}t) + \lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} \\ &= \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) + \varepsilon I_n \\ H_2(\omega{\cdot}t) - \lambda\Delta(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix}\mathbf{z}\end{aligned} \tag{7.65}$$

for $\omega \in \Omega$, will play a role in the statement and proof of the following result. The subindex ε and the argument λ will be used to make reference to these systems. Note that (7.64) agree with (7.65) for $\varepsilon = 0$ and with (7.58) for $\varepsilon = \lambda = 0$. Note also that all these families of systems share the submatrix $H_1(\omega)$. Note further that $JH_\varepsilon + \lambda\Gamma$ is increasing in ε and in λ, and that (7.65) always satisfies D1 and D2 (see Remark 7.20.1). Therefore, the comparison result of Proposition 5.51 can be applied. In addition, recall that each of the principal functions is continuous at the points of a residual subset of Ω: see Theorem 5.43 and Proposition 1.48(ii). And recall finally the information provided by Remark 7.22, which will be used in what follows without further reference.

Theorem 7.31 *Suppose that Ω is σ-minimal and that the family of Hamiltonian systems (7.58) satisfies D1 and the Frequency and Nonoscillation Conditions FC and NC, but not D2. Define*

$$\mathcal{I} = \{0\} \cup \{\lambda_0 \in \mathbb{R} \mid (7.64) \textit{ satisfies FC and NC } \textit{for } \lambda \in [0,\lambda_0) \textit{ or } \lambda \in (\lambda_0, 0]\}\,.$$

Then,

(i) *$\mathcal{I}$ is an open interval containing 0, and for $\lambda \in \mathcal{I}$, $l_0^+(\omega,\lambda) \in \mathcal{D}$ and $l_0^-(\omega,\lambda) \notin \mathcal{D}$ for all $\omega \in \Omega$. In addition, $M_0^+(\omega,\lambda_1) < M_0^+(\omega,\lambda_2)$ for every $\omega \in \Omega$ and for every pair of elements $\lambda_1 < \lambda_2$ of $\mathcal{I}$.*
(ii) *There exists a nonincreasing and lower semicontinuous extended-real function $\rho\colon \mathcal{I} \to (0,\infty]$ such that (7.65) satisfies D1, D2, D3, and FC for $\lambda \in \mathcal{I}$ if and*

only if $\varepsilon \in (0, \rho(\lambda))$. *In particular, for these values of* ε, *there exist the Weyl functions* $M_\varepsilon^\pm(\omega, \lambda)$.

(iii) *If* $\lambda \in \mathcal{I}$, *then*

$$M_0^+(\omega,\lambda) \le M_{\varepsilon_1}^+(\omega,\lambda) \le M_{\varepsilon_2}^+(\omega,\lambda) < M_{\varepsilon_2}^-(\omega,\lambda) \le M_{\varepsilon_1}^-(\omega,\lambda)$$

whenever $0 < \varepsilon_1 < \varepsilon_2 < \rho(\lambda)$ *and* $\omega \in \Omega$.

(iv) *If* λ_1 *and* λ_2 *belong to* $\mathcal{I}$ *and* $\lambda_1 < \lambda_2$, *then*

$$M_0^+(\omega,\lambda_1) \le M_\varepsilon^+(\omega,\lambda_1) < M_\varepsilon^+(\omega,\lambda_2) < M_\varepsilon^-(\omega,\lambda_2) < M_\varepsilon^-(\omega,\lambda_1)$$

whenever $\varepsilon \in (0, \rho(\lambda_2))$ *and* $\omega \in \Omega$.

(v) *If* $\lambda \in \mathcal{I}$ *satisfies* $\rho(\lambda) < \infty$, *then the family of systems* (7.65) *corresponding to* λ *and* $\varepsilon = \rho(\lambda)$ *satisfies* D1, D2, *and* D3 *(but not* FC*), and its principal functions are*

$$N_{\rho(\lambda)}^\pm(\omega,\lambda) = \lim_{\varepsilon\to\rho(\lambda)^-} M_\varepsilon^\pm(\omega,\lambda)\,.$$

In addition, if $\Omega_\lambda^\pm$ *are the* σ*-invariant residual sets of continuity points of* $N_{\rho(\lambda)}^\pm(\omega,\lambda)$, *then* $\tilde{l}_{\rho(\lambda)}^+(\omega,\lambda) \cap \tilde{l}_{\rho(\lambda)}^-(\omega,\lambda) \ne \{\mathbf{0}\}$ *for all* $\omega \in \Omega_\lambda^+ \cap \Omega_\lambda^-$. *Moreover, the sets* $\mathcal{K}_\lambda^\pm = \mathrm{closure}_{\mathcal{K}_\mathbb{R}}\{(\omega, \tilde{l}_{\rho(\lambda)}^\pm(\omega,\lambda)) \,|\, \omega \in \Omega_\lambda^\pm\}$ *are almost automorphic extensions of the base* Ω.

(vi) *If* $\lambda \in \mathcal{I}$ *satisfies* $\rho(\lambda) = \infty$, *then* $\lim_{\varepsilon\to\infty} M_\varepsilon^\pm(\omega,\lambda) = 0_n$.

(vii) *For each* $\lambda \in \mathcal{I}$ *fix* $k_\lambda \in \mathbb{R}$ *such that* $(1 + k_\lambda)\, I_n \le M_0^+(\omega,\lambda)$. *Then the limits* $D_{k_\lambda}^-(\omega,\lambda) = \lim_{\varepsilon\to0^+}(M_\varepsilon^-(\omega,\lambda) - k_\lambda I_n)^{-1}$ *exist, define continuous functions on* Ω, *and satisfy* $l_0^-(\omega,\lambda) \equiv \left[\begin{smallmatrix} D_{k_\lambda}^-(\omega,\lambda) \\ k_\lambda D_{k_\lambda}^-(\omega,\lambda)+I_n\end{smallmatrix}\right]$ *whenever* $\omega \in \Omega$ *and* $\lambda \in \mathcal{I}$.

Proof

(i) It obvious that $\mathcal{I}$ is an interval. Fix $\lambda_1 \in \mathcal{I}$. The robustness of the exponential dichotomy and of the existence of M^+ (see Theorems 1.92 and 1.95) guarantees the existence of an open interval $\mathcal{I}_1$ containing λ_1 such that (7.64) satisfies FC and NC. Consequently, $\mathcal{I}$ is open. Since NC holds, $l_0^+(\omega,\lambda) \in \mathcal{D}$ whenever $\omega \in \Omega$ and $\lambda \in \mathcal{I}$. In addition, $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ solves (7.58) if and only if it solves (7.64), so that this last family does not satisfy D2 for any value of λ. This together with Corollary 7.26 ensures that $l_0^-(\omega,\lambda) \notin \mathcal{D}$ whenever $\omega \in \Omega$ and $\lambda \in \mathcal{I}$. The inequality $M_0^+(\omega,\lambda_1) \le M_0^+(\omega,\lambda_2)$ for every $\omega \in \Omega$ and $\lambda_1 < \lambda_2$ in $\mathcal{I}$ follows by taking the limits as $\varepsilon \to 0^+$ in the second equality of (iii), which will be proved independently below. And the arguments used in the last part of the proof of Proposition 5.51 show that the inequality is strict.

(ii), (iii) & (iv) Fix $\lambda \in \mathcal{I}$ and define $\mathcal{I}(\lambda) = \{\varepsilon > 0 \,|\, (7.65)$ satisfies FC and NC$\}$. Theorems 1.92 and 1.95 ensure that $\mathcal{I}(\lambda)$ is nonempty and open. The following step is to show that $\mathcal{I}(\lambda)$ is an interval. The family (7.65) corresponding to a fixed $\varepsilon_2 \in \mathcal{I}(\lambda)$ satisfies D1, D2, and D3 (see Remark 7.20.1), so that Theorem 7.21

ensures that the Weyl functions $M^{\pm}_{\varepsilon_2}(\omega,\lambda)$ and the principal functions $N^{\pm}_{\varepsilon_2}(\omega,\lambda)$ exist and agree, with $M^{+}_{\varepsilon_2}(\omega,\lambda) < M^{-}_{\varepsilon_2}(\omega,\lambda)$ for all $\omega \in \Omega$. Now, if $\varepsilon_1 \in (0,\varepsilon_2)$, Proposition 5.51 ensures that the corresponding family (7.65) satisfies D1, D2, and D3, with $M^{+}_{0}(\omega,\lambda) \le N^{+}_{\varepsilon_1}(\omega,\lambda) \le M^{+}_{\varepsilon_2}(\omega,\lambda) < M^{-}_{\varepsilon_2}(\omega,\lambda) \le N^{-}_{\varepsilon_1}(\omega,\lambda)$, so that again Theorem 7.21 ensures that $M^{\pm}_{\varepsilon_1}(\omega,\lambda)$ exist and agree with $N^{\pm}_{\varepsilon_1}(\omega,\lambda)$. This means that $\varepsilon_1 \in \mathcal{I}(\lambda)$, and hence that this set is indeed an interval.

Define $\rho(\lambda) = \sup\mathcal{I}(\lambda)$ and note that $\rho(\lambda) \notin \mathcal{I}(\lambda)$. In order to show that the extended-real function $\rho\colon\mathcal{I} \to (0,\infty]$ is nonincreasing, take two elements $\lambda_1 < \lambda_2$ in $\mathcal{I}$. If $\varepsilon \in \mathcal{I}(\lambda_2)$ then, as was seen above, $N^{+}_{\varepsilon}(\omega,\lambda_2) = M^{+}_{\varepsilon}(\omega,\lambda_2) < M^{-}_{\varepsilon}(\omega,\lambda_2) = N^{-}_{\varepsilon}(\omega,\lambda_2)$ for all $\omega \in \Omega$. Proposition 5.51 shows that $N^{+}_{\varepsilon}(\omega,\lambda_1) < M^{+}_{\varepsilon}(\omega,\lambda_2) < M^{-}_{\varepsilon}(\omega,\lambda_2) < N^{-}_{\varepsilon}(\omega,\lambda_1)$, and hence Theorem 7.21 ensures the existence of the Weyl functions $M^{\pm}_{\varepsilon}(\omega,\lambda_1) = N^{\pm}_{\varepsilon}(\omega,\lambda_1)$. In particular, $\varepsilon \in \mathcal{I}(\lambda_1)$; that is, $\mathcal{I}(\lambda_2) \subseteq \mathcal{I}(\lambda_1)$, so that $\rho(\lambda_1) \ge \rho(\lambda_2)$. The existence of the Weyl functions and the nonincreasing character of ρ stated in (ii), and the inequalities in (iii) and (iv), have now been proved. It remains to check that ρ is a lower semicontinuous function; i.e. that $\rho(\lambda) \le \liminf_{m\to\infty}\rho(\lambda_m)$ for all sequences (λ_m) in $\mathcal{I}$ with limit $\lambda \in \mathcal{I}$. Take $\varepsilon \in (0,\rho(\lambda))$. Then the family (7.65) corresponding to ε and λ satisfies FC and NC, and hence there exists an integer $m_0 \ge 1$ such that the families corresponding to ε and λ_m also do so for all $m \ge m_0$. Therefore $\varepsilon \le \rho(\lambda_m)$ for all $m \ge m_0$, which proves the assertion.

(v) Fix $\lambda \in \mathcal{I}$ with $\rho(\lambda) < \infty$, and recall that the family (7.65) corresponding to this λ and to $\varepsilon = \rho(\lambda) > 0$ satisfies D1 and D2 (see Remark 7.20.1). In addition, (iii) ensures the existence of the limits

$$\widetilde{N}^{\pm}_{\rho(\lambda)}(\omega,\lambda) = \lim_{\varepsilon\to\rho(\lambda)^-} M^{\pm}_{\varepsilon}(\omega,\lambda)$$

(see Remark 1.44.3) as well as the inequalities $M^{+}_{\varepsilon}(\omega,\lambda) \le \widetilde{N}^{\pm}(\omega,\lambda) \le M^{-}_{\varepsilon}(\omega,\lambda)$ for all $\varepsilon \in (0,\rho(\lambda))$. This ensures that D3 holds as well. The equality $\widetilde{N}^{\pm}_{\rho(\lambda)}(\omega,\lambda) = N^{\pm}_{\rho(\lambda)}(\omega,\lambda)$ is checked as was (5.47) in the proof of Theorem 5.58. Note that it is not the case that $N^{+}_{\rho(\lambda)}(\omega,\lambda) < N^{-}_{\rho(\lambda)}(\omega,\lambda)$ for every $\omega \in \Omega$: if this property held, then Theorem 7.21 would ensure FC and NC for the pair $(\lambda,\rho(\lambda))$, and the robustness of these properties would contradict the definition of $\rho(\lambda)$. Consequently, there exists $\omega_0 \in \Omega$ with $\tilde{l}^{+}_{\rho(\lambda)}(\omega_0) \cap \tilde{l}^{-}_{\rho(\lambda)}(\omega_0) \ne \{\mathbf{0}\}$. Since $\dim\Big(\tilde{l}^{+}_{\rho(\lambda)}(\omega_0{\cdot}t) \cap \tilde{l}^{-}_{\rho(\lambda)}(\omega_0{\cdot}t)\Big)$ is constant for $t \in \mathbb{R}$ (as can be deduced from $U(t,\omega_0)\,\tilde{l}^{\pm}(\omega_0) = \tilde{l}^{\pm}(\omega_0{\cdot}t)$), the minimality of the base flow ensures that $\tilde{l}^{+}_{\rho(\lambda)}(\omega) \cap \tilde{l}^{-}_{\rho(\lambda)}(\omega) \ne \{\mathbf{0}\}$ for all the points ω at which both principal functions are continuous. The last assertion in (v) was proved in Proposition 1.53.

(vi) In order to simplify the notation, this proof will be carried out for the case $\lambda = 0$, and the corresponding index will be omitted: the general case admits an identical proof. So, assume that $\rho(0) = \infty$, and hence that the system (7.60) admits both Weyl functions for all $\varepsilon > 0$. Fix one of these values $\varepsilon > 0$, and take μ_ε such that $h_\varepsilon(\omega, \pm\mu I_n) < 0$ whenever $\mu > \mu_\varepsilon$ and $\omega \in \Omega$. Here h_ε is the function determining the Riccati equation (7.62) (corresponding to (7.65) for $\lambda = 0$), whose solution with initial datum M_0 is represented by $M_\varepsilon(t, \omega, M_0)$. Choose $\nu_\varepsilon > \mu_\varepsilon$ such that $-\nu_\varepsilon I_n \le M_\varepsilon^{\pm}(\omega) \le \nu_\varepsilon I_n$. Proposition 7.23(ii) ensures that the solution $M_\varepsilon(t, \omega, \nu_\varepsilon I_n)$ exists for $t \ge 0$ and $\omega \in \Omega$; and, since $h_\varepsilon(\omega, \nu_\varepsilon I_n) < 0$, Theorem 1.46(iv) ensures that $M_\varepsilon(t, \omega, \nu_\varepsilon I_n) < \nu_\varepsilon I_n$ for $t > 0$. Hence, ν_ε belongs to

$$\mathcal{I}_\varepsilon = \{\nu \in (\mu_\varepsilon, \nu_\varepsilon] \mid \text{there is } s_\nu \ge 0 \text{ such that} \\ M_\varepsilon(t, \omega, \nu_\varepsilon I_n) \le \nu I_n \text{ for } t \ge s_\nu \text{ and } \omega \in \Omega\}\,.$$

It is clear that $\mathcal{I}_\varepsilon$ is a nondegenerate interval. Note that $M_\varepsilon^{\pm}(\omega{\cdot}t) \le M_\varepsilon(t, \omega, \nu_\varepsilon I_n) \le \nu I_n$ for all $\nu \in \mathcal{I}_\varepsilon$, $t \ge s_\nu$ and $\omega \in \Omega$, so that the minimality of the base flow ensures that $M_\varepsilon^{\pm} \le \nu I_n$. In particular, by Proposition 7.23(ii), $M_\varepsilon(t, \omega, \nu I_n)$ exists for $\nu \in \mathcal{I}_\varepsilon$, $t \ge 0$ and $\omega \in \Omega$. The goal now is to prove that $\mathcal{I}_\varepsilon = (\mu_\varepsilon, \nu_\varepsilon]$. Let $\eta_\varepsilon = \inf \mathcal{I}_\varepsilon$ and assume for contradiction that $\eta_\varepsilon > \mu_\varepsilon$. Then, $h_\varepsilon(\omega, \eta_\varepsilon I_n) < 0$, so that Theorem 1.46(iv) ensures that $M_\varepsilon(t, \omega, \eta_\varepsilon I_n) < \eta_\varepsilon I_n$ for all $t > 0$. Fix $t_0 > 0$. The continuity of the Riccati flow ensures that there exists $\delta > 0$ such that $M_\varepsilon(t_0, \omega, \nu I_n) \le (\eta_\varepsilon - \delta) I_n$ for all $\omega \in \Omega$ if $\nu \in (\eta_\varepsilon, \nu_\varepsilon)$ is close enough to η_ε. Choose one of these values of ν, so that $M_\varepsilon(t, \omega, \nu_\varepsilon I_n) \le \nu I_n$ for $t \ge s_\nu$. Then, for $t \ge t_0 + s_\nu$, $M_\varepsilon(t, \omega, \nu_\varepsilon I_n) = M_\varepsilon(t_0, \omega{\cdot}(t - t_0), M_\varepsilon(t - t_0, \omega, \nu_\varepsilon I_n)) \le M_\varepsilon(t_0, \omega{\cdot}(t - t_0), \nu I_n) \le (\eta_\varepsilon - \delta) I_n$. But this implies that $\eta_\varepsilon - \delta \in \mathcal{I}_\varepsilon$ and contradicts the choice of η_ε.

Therefore, as was seen above, $M_\varepsilon^- \le (\inf \mathcal{I}_\varepsilon) I_n = \mu_\varepsilon I_n$. A symmetric argument (working now with $t \le 0$) shows that $-\mu_\varepsilon I_n \le M_\varepsilon^+$. These facts and Theorem 7.21 imply that

$$-\mu_\varepsilon I_n \le M_\varepsilon^+ < M_\varepsilon^- \le \mu_\varepsilon I_n\,. \tag{7.66}$$

In particular, $\mu_\varepsilon > 0$, and this conclusion is reached working only under the hypothesis that $h_\varepsilon(\omega, \pm\mu I_n) < 0$ whenever $\mu > \mu_\varepsilon$ and $\omega \in \Omega$.

Now, from the properties of the fixed number $\varepsilon > 0$ and the hypothesis $H_3 \ge 0$, one has $h_\varepsilon(\omega, \pm\mu I_n) \le -\mu^2 \varepsilon I_n \mp \mu(H_1(\omega) + H_1^T(\omega)) + H_2$, so that there exists

$$\mu_\varepsilon^* = \inf\{\mu_\varepsilon \mid h_\varepsilon(\omega, \pm\mu I_n) < 0 \text{ whenever } \mu > \mu_\varepsilon \text{ and } \omega \in \Omega\} \ge 0\,.$$

Then, $h_\varepsilon(\omega, \pm\mu I_n) < 0$ whenever $\mu > \mu_\varepsilon^*$ and $\omega \in \Omega$, so that, in fact, $\mu_\varepsilon^* > 0$. Now let ε vary in $(0, \infty)$. It will be proved below that μ_ε^* decreases as ε increases and that $\lim_{\varepsilon\to\infty} \mu_\varepsilon^* = 0$. Property (vi) will hence be proved by taking the limits as $\varepsilon \to \infty$ in (7.66) with μ_ε replaced by μ_ε^*.

If $0 < \varepsilon_1 < \varepsilon_2$, then $h_{\varepsilon_1}(\omega, \pm\mu I_n) > h_{\varepsilon_2}(\omega, \pm\mu I_n)$, so that $\mu_{\varepsilon_1}^* \ge \mu_{\varepsilon_2}^* > 0$, and hence the limit $\lim_{\varepsilon\to\infty} \mu_\varepsilon^* = \mu_\infty^* \ge 0$ exists. Suppose for contradiction that $\mu_\infty^* > 0$, and take $\mu_0 \in (0, \mu_\infty^*)$ and any $\mu_{\varepsilon_1}^*$. Then there exists $\varepsilon_2 \ge \varepsilon_1$

such that $h_{\varepsilon_2}(\omega, \pm\mu I_n) \leq -\mu^2\varepsilon_2 I_n \mp \mu(H_1(\omega) + H_1^T(\omega)) + H_2 < 0$ whenever $\mu \in (\mu_0, \mu^*_{\varepsilon_1}]$ and $\omega \in \Omega$. Therefore, since $h_{\varepsilon_2}(\omega, \pm\mu I_n) < h_{\varepsilon_1}(\omega, \pm\mu I_n)$, it is the case that $h_{\varepsilon_2}(\omega, \pm\mu I_n) < 0$ for all $\mu \geq \mu_0$ and $\omega \in \Omega$. This implies that $\mu^*_{\varepsilon_2} \leq \mu_0 < \mu^*_\infty$, which is impossible. This completes the proof of statement (vi).

(vii) Fix $\lambda \in \mathcal{I}$ and take $\varepsilon \in (0, \rho(\lambda))$. Then, by (iii),

$$(1 + k_\lambda)\, I_n \leq M_0^+(\omega, \lambda) \leq M_\varepsilon^+(\omega, \lambda) < M_\varepsilon^-(\omega, \lambda)$$

for every $\omega \in \Omega$. Therefore, $I_n < M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n$, and hence

$$0 < (M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1} < I_n$$

for all $\omega \in \Omega$ (see Remark 1.20). In addition, since $M_\varepsilon^-(\omega, \lambda)$ decreases with ε, it follows that $(M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1}$ increases with ε. This ensures that $D^-_{k_\lambda}(\omega, \lambda) = \lim_{\varepsilon \to 0^+} (M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1}$ exists (see Remark 1.44.3), with $0 \leq D_{k_\lambda}(\omega, \lambda) < I_n$. Moreover, the matrix $\begin{bmatrix} D^-_{k_\lambda}(\omega,\lambda) \\ k_\lambda D^-_{k_\lambda}(\omega,\lambda)+I_n \end{bmatrix}$ represents a Lagrange plane: obviously,

$$(D^-_{k_\lambda})^T (k_\lambda D^-_{k_\lambda} + I_n) = (k_\lambda (D^-_{k_\lambda})^T + I_n)\, D^-_{k_\lambda}\,;$$

and if $\begin{bmatrix} D^-_{k_\lambda}(\omega,\lambda) \\ k_\lambda D^-_{k_\lambda}(\omega,\lambda)+I_n \end{bmatrix} \mathbf{c} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ for a vector $\mathbf{c} \in \mathbb{R}^n$, then $\mathbf{c} = \mathbf{0}$, so that its range is n. Note also that

$$l_\varepsilon^-(\omega, \lambda) \equiv \begin{bmatrix} I_n \\ M_\varepsilon^-(\omega, \lambda) \end{bmatrix} \equiv \begin{bmatrix} (M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1} \\ k_\lambda (M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1} + I_n \end{bmatrix},$$

so that, since $l_0^-(\omega, \lambda) = \lim_{\varepsilon \to 0^+} l_\varepsilon^-(\omega, \lambda)$ (see e.g. Theorem 1.90),

$$l_0^-(\omega, \lambda) \equiv \begin{bmatrix} D^-_{k_\lambda}(\omega, \lambda) \\ k_\lambda D^-_{k_\lambda}(\omega, \lambda) + I_n \end{bmatrix}.$$

Finally, since k_λ is fixed, $D(\omega) = D^-_{k_\lambda}(\omega, \lambda)$ is the unique matrix such that $l_0^-(\omega, \lambda) \equiv \begin{bmatrix} D(\omega) \\ k_\lambda D(\omega)+I_n \end{bmatrix}$. It follows easily from Proposition 1.25 that $D^-_{k_\lambda}(\omega, \lambda)$ is continuous on Ω.

It is clear that the interval $\mathcal{I}$ of Theorem 7.31 is not necessarily bounded from above or below: to see this just consider the two-dimensional autonomous system $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ \lambda & 1 \end{bmatrix} \mathbf{z}$. The following result analyzes the dynamics of the systems (7.64) corresponding to $\lambda = \sup \mathcal{I}$ in the case that it is finite.

Proposition 7.32 *Under the conditions of Theorem 7.31 and with the notation there established, suppose that* $\lambda_0 = \sup \mathcal{I} < \infty$, *and let* $k_0 \in \mathbb{R}$ *satisfy* $(1 + k_0)I_n \le M_0^+(\omega, 0)$. *Then the limits*

$$D_{k_0}^+(\omega, \lambda_0) = \lim_{\lambda \to \lambda_0^-} (M_0^+(\omega, \lambda) - k_0 I_n)^{-1}$$

$$D_{k_0}^-(\omega, \lambda_0) = \lim_{\lambda \to \lambda_0^-} D_{k_0}^-(\omega, \lambda)$$

exist for every $\omega \in \Omega$ *and define upper semicontinuous functions on* Ω. *In addition, the matrices* $\begin{bmatrix} D_{k_0}^{\pm}(\omega,\lambda_0) \\ k_\lambda D_{k_0}^{\pm}(\omega,\lambda_0)+I_n \end{bmatrix}$ *represent Lagrange planes* $\bar{l}_{\lambda_0}^{\pm}(\omega)$, *and if* $\Omega_{\lambda_0}^{\pm}$ *represent the* σ*-invariant residual sets of continuity points of* $D_{k_0}^{\pm}(\omega, \lambda_0)$ *then the sets* $\mathcal{K}_{\lambda_0}^{\pm} = \text{closure}_{\mathcal{K}_\mathbb{R}} \left\{ \left(\omega, \bar{l}_{\lambda_0}^{\pm}(\omega)\right) \mid \omega \in \Omega_\lambda^{\pm} \right\}$ *are almost automorphic extensions of the base* Ω.

Proof Point (i) of the previous theorem implies that $M_0^+(\omega, \lambda)$ increases with λ, so that $(1 + k_0)\, I_n \le M_0^+(\omega, \lambda)$ for all $\lambda \in (0, \lambda_0)$, and hence (see Remark 1.20): $0 < (M_0^+(\omega, \lambda) - k_0 I_n)^{-1} \le I_n$ for all $\lambda \in (0, \lambda_0)$ and all $\omega \in \Omega$; and $(M_0^+(\omega, \lambda) - k_0 I_n)^{-1}$ decreases with λ. This ensures that $D_{k_0}^+(\omega, \lambda_0) = \lim_{\lambda \to \lambda_0^-} (M_0^+(\omega, \lambda) - k_0 I_n)^{-1}$ exists, and that $0 \le D_{k_0}^+(\omega, \lambda_0) \le I_n$. Also, the function $D_{k_0}^+(\omega, \lambda_0)$ is upper semicontinuous on Ω, as is ensured by Proposition 1.48(i).

Fix now $k_\lambda = k_0$ for all $\lambda \in (0, \lambda_0)$ and check the proof of Theorem 7.31(vii) in order to observe that $D_{k_0}^-(\omega, \lambda) < I_n$ increases as λ increases in $\mathcal{I}$, as $(M_\varepsilon^-(\omega, \lambda) - k_\lambda I_n)^{-1}$ does for a fixed and valid ε. Hence,

$$D_{k_0}^+(\omega, \lambda_0) = \lim_{\lambda \to \lambda_0^-} D_{k_0}^-(\omega, \lambda) \le I_n$$

exists for all $\omega \in \Omega$, and the function $D_{k_0}^-(\omega, \lambda_0)$ is upper semicontinuous on Ω, as follows from Proposition 1.48(i) since each $D_{k_0}^-(\omega, \lambda)$ is continuous.

The last two assertions are proved in the same way as the analogous ones in points (vi) and (v) of Theorem 7.31.

7.3.3 Presence and Absence of a Controllability Condition

When dealing with an optimization problem of the type described at the beginning of this chapter, the underlying Hamiltonian family takes the particular form (7.9), which in particular ensures condition D1. Frequently, in order to apply the Yakubovich Frequency Theorem, Condition C1 of Sect. 6.2 is assumed:

C1. Each minimal subset of Ω contains at least one point ω_1 such that the system

$$\mathbf{x}' = A(\omega_1{\cdot}t)\, \mathbf{x} + B(\omega_1{\cdot}t)\, \mathbf{u}$$

is null controllable.

Recall that this ensures that the family of control systems is uniformly null controllable, as Theorem 6.4 proves, and therefore it is equivalent to the null controllability of all the systems of the family. There are two basic reasons for making this assumption. The first one is that when C1 is fulfilled, Hypothesis 7.3, regarding exponential stabilization, holds:

Proposition 7.33 *The controllability condition* C1 *implies Hypothesis 7.3.*

Proof The assertion is proved in Sect. 6.2: just consider the auxiliary linear regulator problem given by (6.11) with a new $G > 0$, so that condition C2 there is automatically satisfied (see Remark 6.2.3). Theorem 6.13 provides a continuous function $K_0 \colon \Omega \to \mathcal{M}_{m\times n}(\mathbb{R})$ such that the family (7.14) is uniformly Hurwitz at $+\infty$.

The second reason is that, when dealing with Hamiltonian systems of the form (7.9), condition C1 turns out to be equivalent to condition D2 (see Remark 6.8.1 for a similar result), as will now be explained. That is, since condition D1 holds, under conditions FC and NC condition C1 is equivalent to the uniform weak disconjugacy of the family (7.9): see Remark 7.20.2. And this is a kind of "optimal" situation, which in particular implies the global existence of the Weyl functions (see Theorem 7.21(iii)). In addition, as Theorem 5.67 states (see also the remark below it), if there exists an ergodic measure with full support for which the rotation number is zero, and if condition FC holds, then condition C1 suffices to ensure the uniform weak disconjugacy of the Hamiltonian family.

Recall that Theorem 6.4 shows that condition C1 is equivalent to the uniform null controllability of the family (7.6). Recall also that, as explained in Remark 6.16, the uniform null controllability of the family (7.6) follows from the uniform null controllability of a single system corresponding to an element $\omega \in \Omega$ with dense σ-orbit. The following technical lemma is a generalization of Lemma 6.6 (for which the matrix-valued function g is identically zero).

Lemma 7.34 *Define*

$$\widetilde{A}(\omega{\cdot}t) = A(\omega{\cdot}t) - B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, g^T(\omega{\cdot}t)\,.$$

Then the system (7.6) *is null controllable if and only if*

$$\mathbf{x}' = \widetilde{A}(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{u} \tag{7.67}$$

is null controllable.

Proof It suffices to show that the null controllability of (7.67) is equivalent to that of

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\,\mathbf{u}\,,$$

and then apply Lemma 6.6. Take $\mathbf{x}_0 \in \mathbb{R}^n$. If $\mathbf{u} \colon [0, t_0] \to \mathbb{R}^m$ is an integrable control for this last system such that the solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\mathbf{x}(t_0) = \mathbf{0}$, then $\widehat{\mathbf{u}}(t) = \mathbf{u}(t) + R^{-1}(\omega{\cdot}t)\, g^T(\omega{\cdot}t)\, \mathbf{x}(t)$ is an integrable control for the system (7.67) which steers $\mathbf{x}_0$ to $\mathbf{0}$ in time t_0. The converse assertion is proved in the same way.

The previous lemma, Definition 6.3, Theorem 6.4 and Proposition 5.18 imply the following statement which was announced previously:

Corollary 7.35 *Condition* C1 *holds if and only if the Hamiltonian family* (7.9) *satisfies condition* D2.

Remark 7.36 Corollary 5.86 and Remarks 5.87 describe more equivalent situations. In addition, Theorem 7.21(ii) states that, under D2 (i.e. under the uniform null controllability hypothesis), conditions FC and NC ensure the global existence of M^-. And conversely, Theorem 7.21(ii) also states that FC and the global existence of M^+ and M^- ensure D2 and hence the uniform null controllability.

But in fact, to apply the Yakubovich Frequency Theorem does not require that C1 holds. The trivial autonomous example $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \mathbf{z}$ (for which $A \equiv -1$, $B = G = g \equiv 0$, and R is any positive real number), is probably the simplest case of this applicability in the absence of C1. The main purpose of this section is to give an example which is not trivial at all. In particular, it is nonautonomous.

Example 7.37 In this example, the Frequency and Nonoscillation Conditions are satisfied, the uniform null controllability condition on $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}$ is not fulfilled, the family of systems is not uniformly weakly disconjugate, and the exponential stabilization condition given by Hypothesis 7.3 is satisfied. In addition, the interval $\mathcal{I}$ defined in Theorem 7.31 is bounded from above, and the almost automorphic extensions of Proposition 7.32 are not copies of the base.

In the well-known example due to Vinograd [147] (based on the previous results of Millionščikov [104, 105]), a nonuniformly hyperbolic family of two-dimensional Hamiltonian systems is constructed: nontrivial bounded solutions coexist with exponentially increasing or decreasing ones. In this case, (Ω, σ) is an almost periodic minimal flow. A careful analysis of this problem can be found in [68] and is summarized in [87]. The interested reader can find in Example 8.44 all the details of a similar construction. By modifying their constructions, Johnson [69] writes down a nonuniformly hyperbolic family of Schrödinger equations $x'' - f(\omega{\cdot}t)\, x = 0$. His analysis shows that, for $\lambda < 0$ close to 0, the family of Hamiltonian systems constructed by taking $x_1 = x$ and $y_1 = x'$ in $x'' + (\lambda - f(\omega{\cdot}t))\, x = 0$, namely

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ f(\omega{\cdot}t) - \lambda & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

has exponential dichotomy over Ω, and that there exist the (scalar) Weyl functions $m^{\pm}(\omega, \lambda)$. This means that, for these values of λ, the systems satisfy D1, D3, FC, NC, and existence of m^-. It is clear (and ensured by Proposition 7.28) that D2 also holds. Theorem 5.61 applied to $\Gamma = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ ensures hence that this family has exponential dichotomy and that both Weyl functions exist whenever $\lambda \in (-\infty, 0)$.

Consider the family of linear Hamiltonian systems

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ f(\omega{\cdot}t) - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = H_\lambda(\omega{\cdot}t) \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \tag{7.68}$$

for $\omega \in \Omega$ and for $\lambda < 0$. As explained in Sect. 7.1, this Hamiltonian system is naturally associated to the problem of minimizing the functional

$$\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0,\lambda}(\mathbf{x},\mathbf{u}) = \int_0^\infty \frac{1}{2}\left(\langle \mathbf{x}(t), G(\omega{\cdot}t,\lambda)\,\mathbf{x}(t)\rangle + \langle \mathbf{u}(t), \mathbf{u}(t)\rangle\right) dt$$

where $G(\omega,\lambda) = \left[\begin{smallmatrix} f(\omega)-\lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]$, when it is evaluated on pairs $(\mathbf{x},\mathbf{u})\colon [0,\infty) \to \mathbb{R}^2 \times \mathbb{R}^2$ which are admissible: $\mathbf{u}$ belongs to $L^2([0,\infty),\mathbb{R}^2)$, and the solution $\mathbf{x}$ of the control system

$$\mathbf{x}' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{u}\,, \tag{7.69}$$

corresponding to this control $\mathbf{u}$ and with $\mathbf{x}(0) = \mathbf{x}_0$ also belongs to the space $L^2([0,\infty),\mathbb{R}^2)$. Note that by taking $K_2 \equiv -I_2$, the family of systems

$$\mathbf{x}' = \left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} K_2(\omega{\cdot}t)\right)\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x}$$

is uniformly Hurwitz at $+\infty$. In other words, Hypothesis 7.3 is fulfilled. Proposition 7.4 ensures that for each $(\mathbf{x}_0,\omega,\lambda)$ there is at least one admissible pair.

Note that (7.68) can be uncoupled to take the form

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ f(\omega{\cdot}t) - \lambda & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ -\lambda & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Therefore, for $\lambda < 0$, the Lagrange planes

$$l^+(\omega,\lambda) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ m^+(\omega,\lambda) & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad l^-(\omega,\lambda) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ m^-(\omega,\lambda) & 0 \\ 0 & 1 \end{bmatrix}$$

are composed of the initial data of the solutions bounded as $t \to \infty$ and $t \to -\infty$, respectively. It is obvious that the family (7.68) satisfies D1, D3, FC, and NC if $\lambda < 0$, which together with Hypothesis 7.3 ensures that the conclusions of the Yakubovich Frequency Theorem apply. That is, for all $\lambda < 0$, $\omega \in \Omega$, and

$\mathbf{x}_0 \in \mathbb{R}^n$, there exists a unique minimizing pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ for $\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0,\lambda}$, which is given by $\Big(\bar{\mathbf{x}}(t), \left[\begin{smallmatrix} m^+(\omega\cdot t,\lambda) & 0 \\ 0 & 0 \end{smallmatrix}\right] \bar{\mathbf{x}}(t)\Big)$ for the solution $\bar{\mathbf{x}}(t)$ of $\mathbf{x}' = \left[\begin{smallmatrix} m^+(\omega\cdot t,\lambda) & 0 \\ 0 & -1 \end{smallmatrix}\right] \mathbf{x}$ with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$; and the minimum value is

$$\widetilde{\mathcal{I}}_{\omega,\mathbf{x}_0,\lambda}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = -\frac{1}{2}\, \mathbf{x}_0^T \begin{bmatrix} m^+(\omega,\lambda) & 0 \\ 0 & \lambda/2 \end{bmatrix} \mathbf{x}_0 \,,$$

since $M^+(\omega,\lambda) = \left[\begin{smallmatrix} m^+(\omega,\lambda) & 0 \\ 0 & \lambda/2 \end{smallmatrix}\right]$.

The above considerations are valid for all $\lambda < 0$. Since $\left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ e^t \end{smallmatrix}\right]$ solves (7.68) for every value of λ, D2 never holds. (And of course, $l^-(\omega,\lambda) \notin \mathcal{D}$ for $\lambda < 0$.) In other words, the family of Hamiltonian systems (7.68) is not uniformly weakly disconjugate for any value of λ, and the family of control systems (7.69) (which is common for all λ) is not uniformly null controllable: condition C1 does not hold. Note that the Frequency and Nonoscillation Conditions do not hold for $\lambda = 0$.

Note finally that the family (7.68) can be reinterpreted as a perturbation of a family of the type (7.64) which satisfies the hypotheses of Theorem 7.31, with $H = H_{\lambda_0}$ for a fixed value $\lambda_0 < 0$, $\Delta \equiv I_2$, and the parameter λ substituted by $\lambda - \lambda_0$. With these identifications, the interval $\mathcal{I}$ defined in Theorem 7.31 is given by $\mathcal{I} = (-\infty, -\lambda_0)$. And the almost automorphic extensions of the base $\mathcal{K}^{\pm}_{-\lambda_0}$ provided by Proposition 7.32 (which are almost automorphic extensions for the system (7.68) corresponding to $\lambda = 0$) are not copies of the base. The details of this last assertion can be found in [68] and [87].

Chapter 8
Nonautonomous Control Theory: Linear-Quadratic Dissipative Control Processes

This chapter is devoted to the analysis of the dissipativity of linear control systems with time-varying coefficients and time-dependent quadratic supply rates: in other words, nonautonomous linear-quadratic control problems. These problems give rise in a natural way to linear nonautonomous linear Hamiltonian systems. The methods developed in the preceding chapters can and will be used in the analysis.

The concept of dissipativity as conceived by Willems [150, 151] (see also Trentelman and Willems [146]) is of great interest in systems theory and has been systematically developed by scientists working in that area (see for instance Hill and Moylan [60], Hill [59], Polishing [121], and Savin and Peterson [134]). As is stated in [151], the basic idea of Willems' theory is to take account in a systematic way of the energy transfer between a given dynamical system and its environment. The bookkeeping of energy transfer is carried out using a supply rate (or power function) together with a *storage function*. Generally speaking, a dissipative system exchanges energy with its environment, and this phenomenon is modeled by the supply rate: when this quantity is suitably integrated, it measures the flow of energy from the environment into the system; and the storage function measures the quantity of energy stored inside the system. The core concept is that a dissipative system with a storage function cannot store more energy than that received from the outside: the difference between the supplied and the internally stored energy is the dissipated energy.

Among the classical fields of applications of the analysis of dissipativity, one can mention continuum mechanics, thermodynamics, viscoelasticity, and electricity.

These concepts are of interest in particular when the system under consideration is of control type. In this context, a particularly important case is that of a linear control system with a quadratic supply rate. In the study of dissipative systems, the focus is put on the construction of a storage function. But it seems that, despite the information contained in the preceding papers about this subject, it is still not known whether a dissipative linear-quadratic control problem always admits

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a storage function. Yakubovich et al. establish in [158] controllability conditions under which, if all the relevant coefficient matrices are periodic with a common period, then a strictly dissipative linear control system with a quadratic supply rate admits a quadratic strong storage function. The nonstrict case is not analyzed in that work, in which the authors make use of the Yakubovich Frequency Theorem. The generalization of this theorem given in Chap. 7, together with other methods described in the previous chapters, formed the basis for an extension of the result of [158] to the general nonautonomous (nonperiodic) case, when the coefficients of the linear-quadratic control problem are bounded and uniformly continuous functions of t. The coefficients might for example be all periodic, but two of them might have incommensurable periods. This extension appeared first in Fabbri et al. [44] and later, including also the case of nonstrict dissipativity, in Johnson and Núñez [84].

The discussion of the generalization contained in [44] and [84] is not the unique goal of Chap. 8. The strong conditions that the generalization requires (uniform null controllability, and that the Frequency and Nonoscillation Conditions of Chap. 7 hold) are weakened in order to include other scenarios: first, the existence of exponential dichotomy is relaxed to that of uniform weak disconjugacy; and second, the uniform null controllability is removed in order to describe situations of dissipativity and existence of storage function which have not heretofore been analyzed in the literature, even in the simplest cases of constant or periodic coefficients. In all the cases of dissipativity (normal or strict) here studied, a storage function (normal or strong) exists, and it turns out often to be the "optimal" one. Some of these results appeared for the first time in [84] and in Johnson et al. [79]. Putting all these results together, and hence analyzing in a systematic way the possible scenarios of dissipativity arising for linear-quadratic control problems, is the objective of this chapter.

The problem to be studied is formulated in Sect. 8.1, which contains the main definitions of dissipativity, strict dissipativity, storage function, and strong storage function. A general nonautonomous framework is imposed: the coefficients of the control system and of the supply rate are simply assumed to be bounded and uniformly continuous functions of t. Some properties deduced from the null controllability, which will be required later, will be stated and proved in Sect. 8.2.

Section 8.3 contains two results which generalize statements proved by Willems in [151] in the autonomous case. The first one states that the existence of a storage function for a given linear-quadratic control problem is equivalent to the fact that the so-called available storage is finite, in which case it (the available storage) is indeed a storage function: it is, to some extent, the "worst" possible one. This is the key point to prove this fundamental assertion: under a uniform null controllability property, the dissipativity of a linear-quadratic control problem is equivalent to the existence of a storage function. The second result shows that in fact, if the uniform null controllability holds, then the dissipativity is equivalent to the positivity of the so-called required supply, in which case this function is a new storage function: the "best" one. These results are included in [79].

Section 8.4 contains a proof of an easy but fundamental property, often used in the following sections: globally defined symmetric matrix-valued solutions of a Riccati equation constructed from the coefficients of the linear-quadratic problem provide storage functions (or strong storage functions) if they are positive semidefinite (or definite).

The main results about dissipativity are given in Sect. 8.5, under the fundamental hypothesis that the initial control problem is null controllable in a "uniform" way. It is divided into two parts. Some of the results of the first one, which also assumes the exponential dichotomy of the Hamiltonian system which arises from the linear-quadratic problem (among other conditions), generalize the statements proved in [158] to the nonautonomous case. Of course, the statements and proofs must be modified to take account of the nonperiodic nature of the coefficient matrices. The contents of the section reproduce and extend those of [44] and [84]. But whereas in those papers the results were based on those of Chap. 7 (following the ideas of [158]), the analysis presented here is independent of the Yakubovich Frequency Theorem. The main results establish equivalences between the normal or strict dissipativity and the properties of the vector space (Lagrange plane, as a matter of fact) determined by the solutions bounded at $-\infty$. In the second section the exponential dichotomy hypothesis is substituted by some conditions on weak disconjugacy, which are less restrictive, and hence the equivalence results there obtained are weaker, although of interest in certain applications.

The hypothesis of uniform null controllability is removed in Sect. 8.6, where the results of Sect. 8.4 are used to establish some dynamical conditions ensuring the dissipativity (normal or strict) of the linear-quadratic problem. A perturbative result shows that, even in the absence of such conditions, the dynamical methods described in the book may allow one to establish the dissipativity of a given problem.

Section 8.7 contains three nontrivial examples which demonstrate the optimality of the results of the previous sections: they describe interesting dynamical situations which are not included in the classical framework of the dissipativity analysis. And Sect. 8.8 shows that all the results can be easily adapted to the time-reversed problems, providing hence new scenarios of applicability of the nonautonomous techniques which constitute the main tools of all the book.

Each section begins with a more detailed description of the results contained therein and their scope.

Throughout this chapter, $\langle\ ,\ \rangle$ and $\|\cdot\|$ denote the Euclidean inner product and the Euclidean norm on $\mathbb{R}^d$ for all values of d; and given $A \in \mathbb{M}_{d\times m}(\mathbb{R})$, $\|A\|$ represents the usual operator norm associated to the Euclidean norm.

8.1 Statement of the Problem

This section describes the linear-quadratic control problems considered in the chapter, and includes the definitions of their dissipativity and strict dissipativity adopted in this book. In fact these definitions admit variations: Remark 8.3 gives alternative ones, and explains a possible reason for their coexistence in the literature.

As in the previous chapters, consider a time-varying linear control problem

$$\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\,, \tag{8.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a state vector, $\mathbf{u} \in \mathbb{R}^m$ is a control vector, and the coefficients A and B are bounded and uniformly continuous matrix-valued functions of the appropriate dimensions. Associate to (8.1) the quadratic functional

$$\widetilde{\mathcal{Q}}(t,\mathbf{x},\mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(t)\,\mathbf{x}\rangle + 2\,\langle \mathbf{x}, g(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(t)\,\mathbf{u}\rangle\right), \tag{8.2}$$

where G, g, and R are bounded and uniformly continuous matrix functions of the appropriate dimensions, with $G^T = G$, $R^T = R$, and $R(t) \ge \delta I_m$ for all $t \in \mathbb{R}$ and a common $\delta > 0$. Throughout the chapter, any control function $\mathbf{u}\colon [t_1, t_2] \to \mathbb{R}^m$ will always assumed to be square integrable.

The relation between $\widetilde{\mathcal{Q}}$ and the control system can be understood in this way: the quantity $\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}(t,\mathbf{x}(t),\mathbf{u}(t))\,dt$, when the pair $(\mathbf{x},\mathbf{u})$ solves the control system, represents the amount of "supply" (meaning for instance energy) which has to be delivered to the system in order to transfer it from its state in time t_1 to its state in time t_2. This is the reason for which $\widetilde{\mathcal{Q}}$ is called the *supply rate* or *power function*.

The pair given by the linear control system and the quadratic form will be called a *linear-quadratic* (or *LQ* for short) *control problem*. The classical problem of minimizing the quantity $\int_0^\infty \widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds$, when the functions $\mathbf{x}$ and $\mathbf{u}$ are square integrable functions on $[0,\infty)$ and solve the control system with the additional condition $\mathbf{x}(0) = \mathbf{x}_0$ for a fixed $\mathbf{x}_0$, has been considered in Chaps. 6 and 7.

Concerning the subject of this chapter, the rough idea is that a dissipative system is one which loses energy; or, in other terms, which requires energy coming from the environment to move from its equilibrium position to another one. The existence of this amount of energy is often guaranteed by the existence of a storage function, which roughly speaking bounds from below the energy that the system requires to pass from the state of minimum storage to a given state. These are the ideas formalized in the next definitions.

Definition 8.1 The control system (8.1) is *dissipative with supply rate* (8.2) if for each pair $t_1 < t_2 \in \mathbb{R}$ and for each control $\mathbf{u}\colon [t_1,t_2] \to \mathbb{R}^m$, the solution $\mathbf{x}(t)$ of (8.1) satisfying $\mathbf{x}(t_1) = \mathbf{0}$ has the property that

$$\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \ge 0\,.$$

The control system (8.1) is *strictly dissipative with supply rate* (8.2) if there exists $\delta > 0$ such that (8.1) is dissipative with the modified supply rate

$$\widetilde{\mathcal{Q}}_\delta(t,\mathbf{x},\mathbf{u}) = \widetilde{\mathcal{Q}}(t,\mathbf{x},\mathbf{u}) - \delta\,(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2)\,. \tag{8.3}$$

These concepts are also called *dissipativity* or *strict dissipativity of the LQ control problem given by* (8.1) *and* (8.2).

So, this definition responds to the idea that a system is dissipative if, when at rest at time $t = t_1$ and then “set into motion”, it cannot restore energy to the environment. Remark 8.3 gives an alternative formulation of dissipativity, which is very common in the literature, and which requires the next definition:

Definition 8.2 A function $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a *storage function* for the LQ control problem given by (8.1) and (8.2) if the following conditions hold. First, $V(t, \mathbf{0}) = 0$ and $V(t, \mathbf{x}) \geq 0$ for all $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$. Second, if $t_1 < t_2 \in \mathbb{R}$, if $\mathbf{u}: [t_1, t_2] \to \mathbb{R}^m$ is a control function, and if $\mathbf{x}(t)$ solves the corresponding system (8.1) (with arbitrary initial value $\mathbf{x}(t_1) \in \mathbb{R}^n$), then

$$\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \geq \frac{1}{2}\left(V(t_2, \mathbf{x}(t_2)) - V(t_1, \mathbf{x}(t_1))\right).$$

The function V is a *strong storage function* for the LQ control problem if it is a storage function and if, in addition, $V(t, \mathbf{x}) > 0$ for all $t \in \mathbb{R}$ and all nonzero $\mathbf{x} \in \mathbb{R}^n$.

The inequality in this definition formalizes the idea that the change of internal stored energy in a given time interval will never exceed the amount of energy that flows into the system in that interval. In other words: the control system cannot store more energy than is supplied to it from the outside. Note that the factor 1/2 is not included in the classical definitions: it appears here to make the notation consistent with that used in the preceding chapters. Note also that V is not even required to be continuous. However, the storage functions that the dynamical techniques provide in this chapter will be jointly continuous, and in fact quadratic in $\mathbf{x}$.

Remark 8.3 It is clear that the existence of a storage function ensures the dissipativity of the LQ system (the almost trivial details are given in the proof of Theorem 8.6). In fact, in some of the most cited references on dissipative systems (such as [151] or [146]), the definitions of dissipativity read as follows: the control system (8.1) is *dissipative with supply rate* (8.2) if there exists a storage function V for the LQ problem; and it is *strictly dissipative with supply rate* (8.2) if there exists $\delta > 0$ such that (8.1) is dissipative with the modified supply rate (8.3). Theorem 8.6 below shows what is possibly the main reason for the coexistence of these two different definitions: they turn out to be equivalent not only in the autonomous and periodic cases but also the general recurrent one if the system (8.1) is null controllable; or more generally, when all the systems of the family defined over the hull Ω of the data (A, B, G, g, R) are null controllable (see Sect. 1.3.2 and Remark 6.16 in this regard); and the null controllability is a common property in the main applied examples.

Definition 8.1 is chosen in this book, since it is less restrictive from a theoretical point of view, and it responds better to the rough idea of dissipativity explained above. But the reader may keep in mind what Willems explains in [151]: when talking about dissipative systems which arise in physical problems, the main question concerning dissipativity is not if a storage function exists (which is usually the case), but what this storage function might look like. In fact in all the situations described in this chapter in which (strict) dissipativity is guaranteed, a (strong) storage function exists. Thus the reader can adopt the second definition if it is more convenient for any reason.

As in the previous chapters, the analysis will be carried out for a family of LQ control systems defined over a compact metric space Ω with a continuous flow σ. In particular, this setting appears when Ω is the hull in the compact-open topology of the quintuple (A, B, G, g, R): see Sect. 1.3.2. If this is the case, the results regarding the dissipativity of the initial system (8.1) can derived from the results concerning the family (8.4) below by an obvious "restriction" process.

So, let A, B, G, g, and R be now given continuous matrix-valued functions on Ω of the appropriate dimensions, with $G = G^T$ and $R = R^T \geq \rho I_m$ for a $\rho > 0$, and consider the family of control systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}\,, \qquad \omega \in \Omega \tag{8.4}$$

together with the family of quadratic functionals

$$\widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}, \mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\,\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}t)\,\mathbf{u}\rangle\right). \tag{8.5}$$

From now on, the notation LQ_ω will be used to make reference to the linear-quadratic problem given by the system (8.4) and the functional (8.5) corresponding to a particular point $\omega \in \Omega$. The concepts of dissipativity or strict dissipativity of a particular LQ_ω pair are those given in Definition 8.1.

The results on dissipativity will be obtained in terms of the dynamical properties of the family of Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.6}$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and

$$H(\omega) = \begin{bmatrix} A(\omega) - B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) \\ G(\omega) - g(\omega)R^{-1}(\omega)g^T(\omega) & -A^T(\omega) + g(\omega)R^{-1}(\omega)B^T(\omega) \end{bmatrix}.$$

It is not the first time that this family has appeared in the book: the Hamiltonian family appears in Chap. 7 associated via the Pontryagin Maximum Principle to the minimizing problem for the functional

$$\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x}, \mathbf{u}) = \int_0^\infty \widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\,ds \tag{8.7}$$

for square integrable pairs $(\mathbf{x}, \mathbf{u})$ satisfying (8.4) with $\mathbf{x}(0) = \mathbf{x}_0$ (i.e. for admissible pairs), with $\widetilde{\mathcal{Q}}_\omega$ given by (8.5). This association requires the additional uniform stabilization condition described by Hypotheses 7.3. But note that, in fact, one can simply define (8.6) from (8.4) and (8.5) without assuming anything else.

8.2 Uniform Null Controllability and Time-Reversion

In order to avoid further interruption in the discussion, several facts concerning the controllability of the family (8.4) which will be needed later are discussed in this section. These properties, which require uniform null controllability, are based on the controllability that the time-reversed problems inherit from the initial ones. Some of these results appear in [77] and [44], but the proofs given here include more details.

To start with, note that the time-reversed map $\sigma^-(t,\omega) = \sigma(-t,\omega) = \omega{\cdot}(-t)$ also defines a real continuous flow on Ω. From now on, Ω^- will represent the same compact metric space Ω, but understood as the phase space of the flow σ^-.

Proposition 8.4 *The family* (8.4) *is uniformly null controllable if and only if the same holds for the family of time-reversed control systems*

$$\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x} - B(\omega{\cdot}(-t))\,\mathbf{u}\,, \qquad \omega \in \Omega^-\,; \tag{8.8}$$

in addition, in this case, a same time t_0 satisfies the property stated in Definition 6.3 for (8.4) *and for* (8.8).

Proof Obviously it suffices to proof the "only if" part. Assume hence the uniform null controllability of the family (8.4), and let t_0 and δ satisfy Definition 6.3. As usual, $U_A(t,\omega)$ represents the fundamental matrix solution of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x}$ with value I_n at $t = 0$ for each $\omega \in \Omega$. It is clear that $U_A(-t,\omega)$ is the fundamental matrix-solution of $\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x}$ with value I_n at $t = 0$ for each $\omega \in \Omega^-$. Then, if $\omega_0 \in \Omega$,

$$\begin{aligned}
\delta I_n &\le \int_0^{t_0} U_A^{-1}(s,\omega_0)\,B(\omega_0{\cdot}s)\,B^T(\omega_0{\cdot}s)\,(U_A^{-1})^T(s,\omega_0)\,ds \\
&= \int_0^{t_0} \Big(U_A^{-1}(t_0-s,\omega_0)\,B(\omega_0{\cdot}(t_0-s)) \\
&\qquad\qquad B^T(\omega_0{\cdot}(t_0-s))\,(U_A^{-1})^T(t_0-s,\omega_0)\Big)\,ds \\
&= \widetilde{U}\left(\int_0^{t_0} U_A^{-1}(-s,\omega)\,B(\omega{\cdot}(-s))\,B^T(\omega{\cdot}(-s))\,(U_A^{-1})^T(-s,\omega)\,ds\right)\widetilde{U}^T,
\end{aligned}$$

where $\omega = \omega_0{\cdot}t_0$ and $\widetilde{U} = U_A^{-1}(t_0, \omega_0)$. Obviously, any point $\omega \in \Omega^- = \Omega$ can be written as $\omega_0{\cdot}t_0$ for a point $\omega_0 \in \Omega$. Using the compactness of Ω and the continuity of $U_A(t_0, \omega{\cdot}(-t_0))$ with respect to ω, a $\delta^- > 0$ such that

$$\int_0^{t_0} U_A^{-1}(-s,\omega)\, B(\omega{\cdot}(-s))\, B^T(\omega{\cdot}(-s))\, (U_A^{-1})^T(-s,\omega)\, ds \geq \delta^- I_n$$

for all $\omega \in \Omega^-$ is obtained, and this proves the assertions.

The next result establishes consequences of the uniform null controllability which will be fundamental for the purposes of the chapter.

Proposition 8.5 *Suppose that the family (8.4) is uniformly null controllable, and let $t_0 > 0$ be the time of Definition 6.3.*

(i) *Given $\varepsilon > 0$ there exists $\delta_* > 0$ such that, if $\omega \in \Omega$ and if $\mathbf{x}_0 \in \mathbb{R}^n$ satisfies $\|\mathbf{x}_0\| \leq \delta_*$, then there is a continuous control function $\mathbf{u}_\omega\colon [0, t_0] \to \mathbb{R}^m$ with $\|\mathbf{u}_\omega(t)\| \leq \varepsilon$ for all $t \in [0, t_0]$ such that the solution $\mathbf{x}_\omega$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}_\omega(t)$ with $\mathbf{x}_\omega(0) = \mathbf{x}_0$ satisfies $\mathbf{x}_\omega(t_0) = \mathbf{0}$. In addition, the map $\Omega \times [0, t_0] \to \mathbb{R}^m,\ (\omega, t) \mapsto \mathbf{u}_\omega(t)$ is jointly continuous.*

(ii) *Given $\varepsilon > 0$ there exists $\delta_* > 0$ such that, if $\omega \in \Omega$ and if $\mathbf{x}_0 \in \mathbb{R}^n$ satisfies $\|\mathbf{x}_0\| \leq \delta_*$, then there is a continuous control function $\mathbf{v}_\omega\colon [-t_0, 0] \to \mathbb{R}^m$ with $\|\mathbf{v}_\omega(t)\| \leq \varepsilon$ for all $t \in [-t_0, 0]$ such that the solution $\mathbf{y}_\omega$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{v}_\omega(t)$ with $\mathbf{y}_\omega(0) = \mathbf{x}_0$ satisfies $\mathbf{y}_\omega(-t_0) = \mathbf{0}$. In addition, the map $\Omega \times [-t_0, 0] \to \mathbb{R}^m,\ (\omega, t) \mapsto \mathbf{v}_\omega(t)$ is jointly continuous.*

(iii) *For all $s \in \mathbb{R}$, $\mathbf{x}_0 \in \mathbb{R}^n$ and $\omega \in \Omega$ there exists a control function $\widetilde{\mathbf{u}}\colon [s-t_0, s] \to \mathbb{R}^n$ such that the solution $\widetilde{\mathbf{x}}\colon [s-t_0, s] \to \mathbb{R}^n$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widetilde{\mathbf{u}}(t)$ with $\widetilde{\mathbf{x}}(s-t_0) = \mathbf{0}$ satisfies $\widetilde{\mathbf{x}}(s) = \mathbf{x}_0$.*

Proof

(i) Let t_0 be the time appearing in Definition 6.3. The result follows very easily from Remark 6.2.2: the matrix

$$Q(t_0,\omega) = \int_0^{t_0} U_A^{-1}(s,\omega)\, B(\omega{\cdot}s)\, B^T(\omega{\cdot}s)\, (U_A^{-1})^T(s,\omega)\, ds$$

is continuous in ω and satisfies $Q(t_0,\omega)^{-1} \leq (1/\delta)I_n$ for δ satisfying Definition 6.3; hence, taking ρ such that $\|B(\omega{\cdot}t)(U_A^{-1})^T(t,\omega)\| \leq \rho$ for all $\omega \in \Omega$ and $t \in [0, t_0]$, the control function

$$\mathbf{u}_\omega(t) = -B^T(\omega{\cdot}t)(U_A^{-1})^T(t,\omega)\, Q^{-1}(t_0,\omega)\, \mathbf{x}_0$$

has the asserted properties with $\delta_* = \varepsilon\,\delta/\rho$.

(ii) Proposition 8.4 and point (i) provide $\delta^* > 0$ such that if $\|\mathbf{x}_0\| \leq \delta^*$ then there is a continuous control function $\mathbf{u}_\omega\colon [0, t_0] \to \mathbb{R}^m$ with $\|\mathbf{u}_\omega(t)\| \leq \varepsilon$ for all $t \in [0, t_0]$ such that the solution $\mathbf{x}_\omega$ of $\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x} - B(\omega{\cdot}(-t))\,\mathbf{u}_\omega(t)$

with $\mathbf{x}_\omega(0) = \mathbf{x}_0$ satisfies $\mathbf{x}_\omega(t_0) = \mathbf{0}$, and in such a way that the map $\Omega \times [0, t_0] \to \mathbb{R}^m,\ (t, \omega) \mapsto \mathbf{u}_\omega(t)$ is jointly continuous. It is easy to check that $\mathbf{v}_\omega: [-t_0, 0] \to \mathbb{R}^m,\ t \mapsto \mathbf{u}_\omega(-t)$ and $\mathbf{y}_\omega: [-t_0, 0] \to \mathbb{R}^m,\ t \mapsto \mathbf{x}_\omega(-t)$ satisfy the assertions in (ii).

(iii) Fix s, $\mathbf{x}$, and ω. Proposition 8.4, Definition 6.3 and Remark 6.2.2 provide a control $\bar{\mathbf{u}}: [0, t_0] \to \mathbb{R}^m$ such that the solution $\bar{\mathbf{x}}: [0, t_0] \to \mathbb{R}^n$ of

$$\mathbf{x}' = -A((\omega{\cdot}s){\cdot}(-t))\,\mathbf{x} - B((\omega{\cdot}s){\cdot}(-t))\,\bar{\mathbf{u}}(t)$$

with $\bar{\mathbf{x}}(0) = \mathbf{x}$ satisfies $\bar{\mathbf{x}}(t_0) = \mathbf{0}$. The assertion in (iii) is hence satisfied by $\widetilde{\mathbf{u}}: [s - t_0, s] \to \mathbb{R}^m,\ t \mapsto \bar{\mathbf{u}}(s - t)$ and $\widetilde{\mathbf{x}}: [s - t_0, s] \to \mathbb{R}^n,\ t \mapsto \bar{\mathbf{x}}(s - t)$.

8.3 Equivalence of Definitions Under Uniform Null Controllability: The Available Storage and Required Supply

The main results of this section are Theorems 8.6 and 8.15. The first establishes controllability conditions under which the dissipativity of the control system (8.1) with supply rate $\widetilde{Q}$ given by (8.2) is equivalent to the existence of a storage function (resp. strong storage function) for the same control system and supply rate (see Remark 8.3). The reader can find in [158] some information regarding previous results concerning this equivalence, as well as some recent results along the same lines. More precisely, in [158], the authors consider the case of T-periodic component functions, and prove that if the periodic control system is controllable and the LQ control problem is strictly dissipative, then a strong storage function exists. The information provided by Theorem 8.6 fills an important gap in this previous information, in the sense that neither periodicity nor strict dissipativity is assumed. (And much more will be said under additional hypotheses in Theorems 8.22, 8.23, and 8.34.)

Theorem 8.6 is now formulated, although it will be proved later, after the auxiliary result stated in Proposition 8.8, which in turn requires Definition 8.7. To understand the scope of the theorem, recall that the previous chapters explain several situations which guarantee the uniform null controllability: see for instance Theorem 6.4, Remarks 6.5.2, 6.8, 7.20, and 5.87, and Corollaries 7.35 and 5.86.

Theorem 8.6 *Suppose that the family* (8.4) *is uniformly null controllable. Then, for each $\omega \in \Omega$, the LQ$_\omega$ control problem is dissipative if and only if it admits a storage function. In other words, if the family* (8.4) *is uniformly null controllable, then, for each $\omega \in \Omega$, the Definitions 8.1 of dissipativity and strict dissipativity for the LQ$_\omega$ control system given by* (8.4) *and* (8.5) *are equivalent to those given in Remark 8.3.*

The next definition and result do not depend on any kind of properties of the family but just of a particular LQ problem. Therefore they are formulated for the initial LQ control system described in Sect. 8.1.

Definition 8.7 The *available storage* of the LQ control system given by (8.1) and (8.2) is the extended-real function V^a defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$V^a(t,\mathbf{x}) = \sup_{h\ge 0}\left\{ -2\int_t^{t+h} \widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \;\middle|\; \begin{array}{l} \mathbf{u}\colon[t,t+h]\to\mathbb{R}^m \text{ control} \\ \text{and } \mathbf{x} \text{ solves (8.1)} \\ \text{in } [t,t+h] \text{ with } \mathbf{x}(t)=\mathbf{x} \end{array} \right\}.$$

It is clear that $V^a(t,\mathbf{x}) \ge 0$: $-2\int_t^{t+h}\widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds = 0$ for $h=0$. The next result is a nonautonomous version of Theorem 1 of [151].

Proposition 8.8 *The following statements are equivalent:*

(1) *there exists a storage function V for the LQ control problem given by* (8.1) *and* (8.2)*; in other words, the control system* (8.1) *is dissipative with supply rate* (8.2) *in the sense of Remark 8.3;*
(2) *the available storage V^a for the LQ control problem satisfies $V^a(t,\mathbf{x}) < \infty$ for each $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^n$.*

In addition, under these conditions V^a is a storage function for the LQ control problem, and $V^a \le V$ for any other storage function V.

Proof (1)⇒(2) Let V be any storage function for the LQ control problem. Since V is nonnegative,

$$-2\int_t^{t+h}\widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \le V(t,\mathbf{x}(t)) - V(t+h,\mathbf{x}(t+h)) \le V(t,\mathbf{x}(t)) < \infty\,,$$

so that $V^a(t,\mathbf{x}) < \infty$, as asserted in (2). Note also that $V^a(t,\mathbf{x}) \le V(t,\mathbf{x})$ for each $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^n$, which proves the last assertion of the theorem.

(2)⇒(1) Assume that $V^a(t,\mathbf{x}) < \infty$ for each $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^n$. To prove this implication and complete the proof of the theorem it suffices to show that V^a satisfies the two conditions in Definition 8.2 of storage function for the LQ problem. The first one is that $V^a(t,\mathbf{0}) = 0$ for each $t\in\mathbb{R}$; or equivalently, that $V^a(t,\mathbf{0}) \le \varepsilon$ for each $\varepsilon > 0$ (since $V^a \ge 0$). Fix $t\in\mathbb{R}$ and $\varepsilon>0$ and note that the definition of (the *real* value) $V^a(t,\mathbf{0})$ ensures the existence of $h_\varepsilon \ge 0$ and a control $\mathbf{u}_\varepsilon\colon[t,t+h_\varepsilon]\to\mathbb{R}^m$ such that, if $\mathbf{x}_\varepsilon\colon[t,t+h_\varepsilon]\to\mathbb{R}^n$ is the solution of $\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}_\varepsilon(t)$ with $\mathbf{x}_\varepsilon(t) = \mathbf{0}$, then

$$V^a(t,\mathbf{0}) \le -2\int_t^{t+h_\varepsilon}\widetilde{\mathcal{Q}}(s,\mathbf{x}_\varepsilon(s),\mathbf{u}_\varepsilon(s))\,ds + \varepsilon\,.$$

Hence, since $\widetilde{\mathcal{Q}}$ is a quadratic form, for each $\lambda > 0$

$$V^a(t,\mathbf{0}) \le -\frac{2}{\lambda^2}\int_t^{t+h_\varepsilon} \widetilde{\mathcal{Q}}(s,\lambda\,\mathbf{x}_\varepsilon(s),\lambda\,\mathbf{u}_\varepsilon(s))\,ds + \varepsilon \le \frac{2}{\lambda^2}\,V^a(t,\mathbf{0}) + \varepsilon\,,$$

so that taking the limit as $\lambda \to \infty$ it follows that $V^a(t,\mathbf{0}) \le \varepsilon$, as asserted.

In order to prove the remaining statement, choose two times $t_1 < t_2$, a control $\mathbf{u}\colon [t_1,t_2] \to \mathbb{R}^m$ and any solution $\bar{\mathbf{x}}\colon [t_1,t_2] \to \mathbb{R}^n$ of the system $\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}(t)$. Take also any $h \ge 0$ and any control $\bar{\mathbf{u}}\colon [t_2,t_2+h] \to \mathbb{R}^m$, and let $\bar{\mathbf{x}}\colon [t_2,t_2+h] \to \mathbb{R}^n$ be the solution of $\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\bar{\mathbf{u}}(t)$ with $\bar{\mathbf{x}}(t_2) = \mathbf{x}(t_2)$. Now define the control $\widetilde{\mathbf{u}}\colon [t_1,t_2+h] \to \mathbb{R}^m$ by concatenating $\mathbf{u}$ on $[t_1,t_2)$ and $\bar{\mathbf{u}}$ on $[t_2,t_2+h]$, and note that the solution $\widetilde{\mathbf{x}}\colon [t_1,t_2+h] \to \mathbb{R}^n$ of $\mathbf{x}' = A(t)\,\mathbf{x}+B(t)\,\widetilde{\mathbf{u}}(t)$ with $\widetilde{\mathbf{x}}(t_1) = \mathbf{x}(t_1)$ agrees with $\mathbf{x}$ on $[t_1,t_2]$ and with $\bar{\mathbf{x}}$ on $[t_2,t_2+h]$. The definition of $V^a(t,\mathbf{x}(t_1))$ yields

$$\begin{aligned} V^a(t_1,\mathbf{x}(t_1)) &\ge -2\int_{t_1}^{t_2+h} \widetilde{\mathcal{Q}}(s,\widetilde{\mathbf{x}}(s),\widetilde{\mathbf{u}}(s))\,ds \\ &= -2\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds - 2\int_{t_2}^{t_2+h} \widetilde{\mathcal{Q}}(s,\bar{\mathbf{x}}(s),\bar{\mathbf{u}}(s))\,ds\,, \end{aligned}$$

so that taking the supremum over the set defining $V^a(t_2,\mathbf{x}(t_2))$ yields

$$V^a(t_1,\mathbf{x}(t_1)) \ge -2\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}(s,\mathbf{x}(s),\mathbf{u}(s))\,ds + V^a(t_2,\mathbf{x}(t_2))\,.$$

This is equivalent to the inequality in Definition 8.2. The proof is complete.

Proof of Theorem 8.6 Observe that it is enough to prove the equivalence of the definitions in the case of dissipativity, since once this fact is established, the definition of strict dissipativity is the same in both cases. (This is the reason why the last sentence of the theorem is equivalent to the previous one.)

It is simple to deduce that the existence of a storage function V_ω for the LQ$_\omega$ problem given by (8.4) and (8.5) guarantees its dissipativity. In fact, take $t_1 < t_2$, a control $\mathbf{u}\colon [t_1,t_2] \to \mathbb{R}^m$, and the solution $\mathbf{x}(t)$ of (8.4) with $\mathbf{x}(t_1) = 0$, and note that

$$\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_\omega(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \ge V_\omega(t_2,\mathbf{x}(t_2)) - V_\omega(t_1,\mathbf{x}(t_1)) = V_\omega(t_2,\mathbf{x}(t_2)) \ge 0\,,$$

which proves the assertion.

Assume now that the LQ$_\omega$ pair is dissipative. According to Proposition 8.8, it is enough to check that the available storage V^a_ω of the LQ$_\omega$ control problem (given by Definition 8.7 with $\widetilde{\mathcal{Q}}$ replaced by $\widetilde{\mathcal{Q}}_\omega$) satisfies $V^a_\omega(t,\mathbf{x}) < \infty$.

Fix $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^n$. According to Proposition 8.5(iii), there exist a time $t_1 < t$ and a control $\widetilde{\mathbf{u}}\colon [t_1,t] \to \mathbb{R}^n$ such that the solution $\widetilde{\mathbf{x}}\colon [t_1,t] \to \mathbb{R}^n$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\widetilde{\mathbf{u}}(t)$ with $\widetilde{\mathbf{x}}(t_1) = \mathbf{0}$ satisfies $\widetilde{\mathbf{x}}(t) = \mathbf{x}$. Take now any $h \ge 0$ and an arbitrary control $\bar{\mathbf{u}}\colon [t, t+h] \to \mathbb{R}^m$, define $\mathbf{u}^*\colon [t_1, t+h] \to \mathbb{R}^m$ by concatenating $\widetilde{\mathbf{u}}$ on $[t_1, t)$ and $\bar{\mathbf{u}}$ on $[t, t+h]$, and denote by $\mathbf{x}^*$ the solution of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}^*(t)$ with $\mathbf{x}^*(t_1) = \mathbf{0}$. It is clear that $\mathbf{x}^*$ agrees with $\widetilde{\mathbf{x}}$ on $[t_1, t]$ and with the solution $\bar{\mathbf{x}}$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\bar{\mathbf{u}}(t)$ with $\bar{\mathbf{x}}(t) = \mathbf{x}$ on $[t, t+h]$. Finally, the assumption in (ii) ensures that

$$\int_{t_1}^{t+h} \widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}^*(s), \mathbf{u}^*(s))\, ds \ge 0\,,$$

and consequently, by the definition of the available storage,

$$V_\omega^a(t, \mathbf{x}) \le -2\int_t^{t+h} \widetilde{\mathcal{Q}}_\omega(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds \le 2\int_{t_1}^{t} \widetilde{\mathcal{Q}}_\omega(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}(s))\, ds < \infty\,.$$

This completes the proof.

It is important to remark that, although the hypothesis of Theorem 8.6 refers to the whole family of control problems, the thesis is formulated for each LQ control problem of the family. In this regard it is convenient to bear in mind two questions. On the one hand, Remark 6.16 describes situations in which, if Ω is obtained as the common hull of the initial coefficients (A, B, G, g, R) of (8.1) and (8.2), the initial control problem satisfies conditions ensuring the uniform null controllability of the family. And, on the other hand, in the hull of an initial LQ control problem there may coexist elements for which a storage function exists together with others without this property. This can happen in particular in two situations: under the uniform controllability of the family systems if the hull is not minimal, as in Example 8.12; and when the hull is minimal but the family of control problems is not uniformly null controllable, as in Example 8.13. Those examples are postponed until Proposition 8.10, where it is proved that the coexistence is not possible in the case of minimality plus uniform null controllability.

It is also important to emphasize the fact that the controllability hypothesis is fundamental for the equivalence of the two classical definitions of dissipativity given in Theorem 8.6. That is the reason for the inclusion of the next simple example.

Example 8.9 The goal now is to construct an example of a family of LQ control problem which satisfies Definition 8.1 but for which a storage function does not exist, which according to Theorem 8.6 is only possible if the property of uniform null controllability does not hold. For instance, let (Ω, σ) be a minimal flow and let $A\colon \Omega \to \mathbb{R}$ be continuous. Then, taking $B \equiv 0$, one gets the family of "control problems" $x' = A(\omega{\cdot}t)\,x$, which obviously give rise to dissipative control problems no matter what the choice of G, g, and $R > 0$. However, taking $A \equiv 1, B \equiv 0, g \equiv 0$, $G \equiv -1$, and any $R > 0$ (not necessarily autonomous) one gets an LQ (dissipative) control problem which does not admit a storage function, since the available storage, independent of the point of the hull, is

$$V^a(t,x) = \sup_{h\geq 0}\left\{\int_t^{t+h} x^2\, e^{2(s-t)} ds\right\} = \infty$$

for every (t,x). Note that the supremum giving rise to $V^a(t,x)$ is actually obtained when $u \equiv 0$, since $B = 0$ (which ensures that the solution is independent of u), $g = 0$ and $R > 0$.

The next proposition, which was referred to above, contains an interesting result about the limiting behaviour of dissipativity. In particular, it ensures that the coexistence of dissipative and nondissipative LQ control problems is impossible if they correspond to the same minimal subset of Ω under the uniform null controllability assumption.

Proposition 8.10 *Suppose that there exists $\omega_0 \in \Omega$ such that the corresponding LQ_{ω_0} control problem is dissipative (resp. strictly dissipative). Then for each $\omega_1 \in closure_\Omega\{\omega_0{\cdot}t \mid t \in \mathbb{R}\}$ the LQ_{ω_1} control problem is dissipative (resp. strictly dissipative).*

Proof The proof is carried out in the case of dissipativity: the arguments are analogous in the strict situation. Take $\omega_1 \in \mathrm{closure}_\Omega\{\omega_0{\cdot}t \mid t \in \mathbb{R}\}$ and a sequence (s_n) with $\lim_{n\to\infty} \omega_0{\cdot}s_n = \omega_1$. Let $\bar{\mathbf{u}}\colon [t_1, t_2] \to \mathbb{R}^m$ be a control, let $\bar{\mathbf{x}}(t)$ be the solution of $\mathbf{x}' = A(\omega_1{\cdot}t)\,\mathbf{x} + B(\omega_1{\cdot}t)\,\bar{\mathbf{u}}(t)$ with $\bar{\mathbf{x}}(t_1) = \mathbf{0}$, and let $\mathbf{x}_n(t)$ be the solution of $\mathbf{x}' = A((\omega_0{\cdot}s_n){\cdot}t)\,\mathbf{x} + B((\omega_0{\cdot}s_n){\cdot}t)\,\bar{\mathbf{u}}(t)$ with $\mathbf{x}_n(t_1) = \mathbf{0}$ for each $n \in \mathbb{N}$. The definition (8.5) of $\widetilde{\mathcal{Q}}_\omega$, the classical results on continuous dependence of solutions with respect to the coefficients of the equations, and the Lebesgue dominated convergence theorem ensure that

$$\begin{aligned}
\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_{\omega_1}(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds &= \lim_{n\to\infty} \int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_{\omega_0\cdot s_n}(s, \mathbf{x}_n(s), \bar{\mathbf{u}}(s))\, ds \\
&= \lim_{n\to\infty} \int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_{\omega_0}(s + s_n, \mathbf{x}_n(s), \bar{\mathbf{u}}(s))\, ds \\
&= \lim_{n\to\infty} \int_{t_1+s_n}^{t_2+s_n} \widetilde{\mathcal{Q}}_{\omega_0}(s, \mathbf{x}_n(s - s_n), \bar{\mathbf{u}}(s - s_n))\, ds \geq 0\,.
\end{aligned}$$

The last inequality, which proves the result, follows from the assumed dissipativity, since $\bar{\mathbf{u}}_n\colon [t_1 + s_n, t_2 + s_n] \to \mathbb{R}^m,\ t \mapsto \bar{\mathbf{u}}(t - s_n)$ is a control, and $\mathbf{x}_n(t - s_n)$ is the solution of $\mathbf{x}' = A(\omega_0{\cdot}t)\,\mathbf{x} + B(\omega_0{\cdot}t)\,\bar{\mathbf{u}}_n(t)$ with $\mathbf{x}_n(t_1 + s_n - s_n) = \mathbf{0}$.

Remark 8.11 If one imposes the definition of dissipativity given in Remark 8.3 (i.e. the existence of a storage function), then the conclusions of Proposition 8.10 do in fact hold if and only if the uniform null controllability of the family (8.6) is assumed. The "if" assertion follows immediately from Theorem 8.6, and the "only if" assertion is proved by Example 8.13.

Example 8.12 This example shows that, even in the simple case of the autonomous control system $x' = x + u$, which is null controllable (since $B = 1 > 0$: see Remark 6.2.1), it is possible to have a nonautonomous supply rate giving rise to a hull Ω on which there coexist dissipative and nondissipative systems. Recall that Proposition 8.10 shows that this is impossible for problems corresponding to points in the same minimal subset of the hull, so that this hull will necessarily be nonminimal.

Take hence $A = B = R \equiv 1$, $g \equiv 0$, and let G be an increasing continuous function satisfying

$$G(t) = \begin{cases} -2 & \text{if } t < 0\,, \\ 0 & \text{if } t > 1\,. \end{cases}$$

The linear Hamiltonian system associated to the corresponding LQ problem is

$$\mathbf{z}' = \begin{bmatrix} 1 & 1 \\ G(t) & -1 \end{bmatrix} \mathbf{z}\,,$$

and it is easy to check that the hull Ω of the coefficient matrix is

$$\Omega = \left\{ \begin{bmatrix} 1 & 1 \\ G_s(t) & -1 \end{bmatrix} \mid s \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\},$$

with $G_s(t) = G(t+s)$. Consider first the right-limiting system $\mathbf{z}' = \left[\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right] \mathbf{z}$. Since the corresponding quadratic form is $\widetilde{\mathcal{Q}}_\infty(x,u) = u^2/2$, the corresponding LQ control problem satisfies the Definition 8.1 of dissipativity. However, the quadratic form associated to the left-limiting system $\mathbf{z}' = \left[\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right] \mathbf{z}$ is $\widetilde{\mathcal{Q}}_{-\infty}(x,u) = -xu + u^2/2$, and taking $u\colon [0,1] \to \mathbb{R}$, $t \mapsto 1$ and the solution $x(t) = e^t - 1$ of $x' = x + 1$ with $x(0) = 0$ one gets $\int_0^1 \widetilde{\mathcal{Q}}_2(x(s), u(s))\, ds = \int_0^1 (-e^s + 3/2)\, ds = -e + 5/2 < 0$. Therefore this last system corresponds to a nondissipative LQ control problem. Example 8.16 adds some more information about this nondissipative system.

Example 8.13 This example, which was announced in Remark 8.3, shows that Proposition 8.10 is not true for the definition of dissipativity given in Remark 8.3 unless the uniform null controllability is assumed: the coexistence of a point $\omega \in \Omega$ such that the corresponding LQ_ω control problem admits a storage function with other points for which this property does not hold is possible, even in the same minimal subset $\mathcal{M} \subseteq \Omega$. Proposition 8.10, Theorem 8.6 and Remark 6.5.2, taken together, state that this cannot happen when all the systems corresponding to points of $\mathcal{M}$ are null controllable.

As in Example 8.9, the problem will be scalar and with $B \equiv 0$, which precludes the null controllability of any one of the linear control systems $x' = A(\omega{\cdot}t)\, x + B(\omega{\cdot}t)\, u = A(\omega{\cdot}t)\, x$, and which gives families of dissipative LQ control problems irrespectively of the choices of G, g, and R. Take a $A\colon \Omega \to \mathbb{R}$ continuous and with additional properties to be described later, $G = -A$, $g \equiv 0$ and $R \equiv 1$. It is easy to check that

$$V^a_\omega(0, x) = \sup_{h \ge 0} \left\{ \frac{1}{2}\, x^2 \left(e^{2 \int_0^h A(\omega{\cdot}s)\, ds} - 1 \right) \right\} .$$

(As in Example 8.9, the supremum in the definition of $V^a_\omega(0, x)$ is actually attained when $u \equiv 0$.) The question hence is to choose A so that this supremum is finite for some values of ω and ∞ for other ones. And the existence of such functions A is a well-known fact, described by Poincaré in [120] (see also Johnson [64]). It is enough to take Ω as the (minimal) hull of a recurrent $\widetilde{A}\colon \mathbb{R} \to \mathbb{R}$ (and A as the time-zero evaluation operator: see Sect. 1.3.2) such that:

- First, $x' = \widetilde{A}(t)\, x$ does not have exponential dichotomy over $\mathbb{R}$ (which is equivalent to saying that the family $x' = A(\omega{\cdot}t)\, x$ does not have exponential dichotomy over Ω (see Remark 1.59.4).
- Second, $\sup_{t \in \ge 0} \left\{ \left| \int_0^t \widetilde{A}(s)\, ds \right| \right\} = \infty$ or $\sup_{t \in \le 0} \left\{ \left| \int_0^t \widetilde{A}(s)\, ds \right| \right\} = \infty$. Then there exists a residual subset $\mathcal{R} \subset \Omega$ such that, for any $\omega \in \mathcal{R}$, $\limsup_{t \to \infty} \int_0^t A(\omega{\cdot}s)\, ds = \infty$. This means that $V^a_\omega(0, x) = \infty$ for any $\omega \in \Omega$ and $x \in \mathbb{R}$, which according to Proposition 8.10 precludes the existence of storage function for the corresponding LQ problem.

There are particular examples given in the literature. The interested reader can find in [87], Theorem A.2, a short proof of the existence of the residual set $\mathcal{R}$, and of more interesting oscillatory properties of the solutions of the systems corresponding to the points of $\mathcal{R}$ (see also Example 8.44).

The second main result of this section, Theorem 8.15, establishes the equivalence between the dissipativity of a particular LQ_ω control problem and the existence of the *optimal* storage function, to be defined now, under the fundamental hypothesis of the uniform null controllability of the family (8.4).

Definition 8.14 Suppose that the family (8.4) is uniformly null controllable. The *required supply* of the LQ_ω control problem given by (8.4) and (8.5) is the extended-real function V^r_ω defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$V^r_\omega(t, \mathbf{x}) = \inf_{h \ge 0} \left\{ 2 \int_{t-h}^{t} \widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \;\middle|\; \begin{array}{l} \mathbf{u}\colon [t-h, t] \to \mathbb{R}^m \text{ control} \\ \text{and } \mathbf{x} \text{ solves (8.4) with} \\ \mathbf{x}(t) = \mathbf{x} \text{ and } \mathbf{x}(t-h) = \mathbf{0} \end{array} \right\} .$$

Proposition 8.5(iii) ensures that the set over which the infimum is taken is nonempty for all $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$. The next result provides a nonautonomous version of Theorem 2 of [151]: the equivalence between dissipativity and nonnegativity of the required supply. Note again that the thesis is formulated for each LQ control problem of the family: see the comment before Theorem 8.6, and recall that Example 8.12 displays a case of uniform null controllability for which dissipative and nondissipative problems coexist, so that positive and nonpositive required supplies coexist.

Theorem 8.15 *Suppose that the family (8.4) is uniformly null controllable. Fix $\omega \in \Omega$. Then the following statements are equivalent:*

(1) *the control system (8.4) is dissipative with supply rate (8.5);*
(2) *the required supply V_ω^r for the LQ$_\omega$ control problem satisfies $V_\omega^r(t, \mathbf{x}) \geq 0$ for each $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$.*

In addition, under these conditions V_ω^r is a storage function for the LQ$_\omega$ control problem, and $V_\omega \leq V_\omega^r$ for any other storage function V_ω.

Proof The equivalence of (1) and (2) is an easy consequence of Definition 8.1. Hence the first objective will be to check that if $V_\omega^r \geq 0$ then it is a storage function for the LQ$_\omega$ control problem.

It is clear from the definition that the (nonnegative) infimum of the set which appears in the definition of $V_\omega^r(t, \mathbf{0})$ is reached for $h = 0$, and is 0. Now take $t_1 \leq t_2$ and a pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ solving (8.4) in $[t_1, t_2]$. Take $h \geq 0$ and a control $\widetilde{\mathbf{u}}\colon [t_1 - h, t_1] \to \mathbb{R}^m$ such that the solution $\widetilde{\mathbf{x}}\colon [t_1 - h, t_1] \to \mathbb{R}^n$ with $\widetilde{\mathbf{x}}(t_1 - h) = \mathbf{0}$ satisfies $\widetilde{\mathbf{x}}(t_1) = \bar{\mathbf{x}}(t_1)$. Repeat the concatenating process already made twice before and note that the definition of V_ω^r ensures that

$$V_\omega^r(t_2, \bar{\mathbf{x}}(t_2)) \leq 2 \int_{t_1 - h}^{t_1} \widetilde{\mathcal{Q}}_\omega(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}(s))\, ds + 2 \int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_\omega(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds\,.$$

Therefore, taking the infimum of the set defining $V_\omega^r(t_1, \bar{\mathbf{x}}(t_1))$ yields

$$V_\omega^r(t_2, \bar{\mathbf{x}}(t_2)) \leq V_\omega^r(t_1, \bar{\mathbf{x}}(t_1)) + 2 \int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_\omega(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds\,,$$

as required.

The proof will be completed once it is shown that $V_\omega^r \geq V_\omega$ for any other storage function V_ω for the LQ$_\omega$ control problem. Fix $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, and choose $h \geq 0$ such that there exists a control $\bar{\mathbf{u}}\colon [t - h, t] \to \mathbb{R}^m$ for which the solution $\bar{\mathbf{x}}\colon [t - h, t] \to \mathbb{R}^n$ of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\bar{\mathbf{u}}(t)$ with $\mathbf{x}(t - h) = \mathbf{0}$ satisfies $\bar{\mathbf{x}}(t) = \mathbf{x}$. Then, since V_ω is a storage function,

$$\int_{t-h}^{t} \widetilde{\mathcal{Q}}_\omega(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds \geq V_\omega(t, \bar{\mathbf{x}}(t)) - V_\omega(t - h, \bar{\mathbf{x}}(t - h)) = V_\omega(t, \mathbf{x})\,,$$

so that the asserted inequality follows from the definition of $V_\omega^r(t, \mathbf{x})$.

Note that the last assertion of the preceding theorem states the optimality of the required supply mentioned above: it is the largest one among all the possible storage functions for the LQ$_\omega$ control problem. And, on the other hand, the available storage is the smallest one.

Example 8.16 Consider again the family of LQ control problems of Example 8.12. More precisely, consider the autonomous one given $x' = x + u$ and $\widetilde{\mathcal{Q}}_{-\infty}(x, u) = -xu + u^2/2$, given by $\mathbf{z}' = \left[\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right] \mathbf{z}$. Theorem 8.15 ensures that its required supply takes negative values at some pairs (t, x).

8.4 Riccati Equation and Storage Functions

Recall once more (see Sect. 1.3.5) that the family of Riccati equations defined from (8.6) by

$$M' = -M\,H_3(\omega{\cdot}t)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t)\,, \tag{8.9}$$

with $H_1 = A - B\,R^{-1}g^T$, $H_2 = G - g\,R^{-1}g^T$, and $H_3 = B\,R^{-1}B^T$, defines a local skew-product flow τ_s on $\Omega \times \mathbb{S}_n(\mathbb{R})$; in time t it sends the pair (ω, M_0) to the pair $(\omega{\cdot}t), M(t, \omega, M_0))$, where $M(t, \omega, M_0)$ is the solution of the equation (8.9) with $M(0, \omega, M_0) = M_0$.

Take one of these solutions, and define

$$V_{\omega,M_0}(t, \mathbf{x}) = \langle \mathbf{x}, M(t, \omega, M_0)\,\mathbf{x}\rangle$$

as long as it exists. Lemma 7.9 states that, for any pair $(\mathbf{x}(t), \mathbf{u}(t))$ solving (8.4),

$$\begin{aligned} \frac{d}{dt}\,V_{\omega,M_0}(t, \mathbf{x}(t)) = {} & 2\widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}(t), \mathbf{u}(t)) \\ & - \langle \mathbf{u}(t) - K_{\omega,M_0}(t)\,\mathbf{x}(t), R(\omega{\cdot}t)(\mathbf{u}(t) - K_{\omega,M_0}(t)\,\mathbf{x}(t))\rangle \end{aligned} \tag{8.10}$$

for

$$K_{\omega,M_0}(t) = R^{-1}(\omega{\cdot}t)(-g^T(\omega{\cdot}t) + B^T(\omega{\cdot}t)\,M(t, \omega, M_0))\,.$$

Relation (8.10) is the key point required to show the strong connection between the existence of globally defined nonnegative solutions of equations (8.9) and the dissipativity of the LQ control problems considered in this chapter. This relation is explained in the next result, which does not require extra controllability, dichotomy, or disconjugacy properties to be imposed on the family of Hamiltonian systems (8.6), and which is fundamental in the rest of the chapter.

Proposition 8.17

(i) *Suppose that there exist a point $\omega \in \Omega$ and a matrix $M_0 \geq 0$ (resp. $M_0 > 0$) such that $M(t, \omega, M_0)$ is a globally defined solution of the Riccati equation* (8.9)*, with $M(t, \omega, M_0) \geq 0$ (resp. $M(t, \omega, M_0) > 0$) for all $t \in \mathbb{R}$. Then the LQ_ω*

problem is dissipative and the function

$$V_{\omega,M_0}(t,\mathbf{x}) = \langle \mathbf{x}, M(t,\omega,M_0)\,\mathbf{x}\rangle$$

is a continuous storage function (resp. a strong storage function) for it.

(ii) *Suppose that there exist a point $\omega_0 \in \Omega$ with dense σ-orbit and a positive semidefinite matrix $M_0 \geq 0$ such that $M(t,\omega_0,M_0)$ is a globally defined and bounded solution of the corresponding Riccati equation, with $M(t,\omega_0,M_0) \geq 0$ for all $t \in \mathbb{R}$. Then each LQ_ω control problem of the family is dissipative and admits a continuous storage function.*

Proof

(i) The hypotheses on $M(t,\omega_0,M_0)$ ensure that $V_{\omega,M_0}(t,\mathbf{x})$ is continuous, globally defined and positive semidefinite (resp. definite), and it is obvious that $V_{\omega,M_0}(t,\mathbf{0}) = 0$. Therefore, since $R > 0$, integrating the relation (8.10) yields

$$2\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_\omega(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \geq V_{\omega,M_0}(t_1,\mathbf{x}(t_2)) - V_{\omega,M_0}(t_1,\mathbf{x}(t_1)) \tag{8.11}$$

if $t_1 \leq t_2$ and the pair $(\mathbf{x},\mathbf{u})$ solves (8.4) in $[t_1,t_2]$. This fact proves (i).

(ii) Choose any $\omega \in \Omega$ and write it as $\omega = \lim_{m\to\infty} \omega_0{\cdot}t_m$ for a suitable sequence (t_m). Since the sequence $(M(t_m,\omega_0,M_0))$ in $\mathbb{S}_n(\mathbb{R})$ is bounded, there exists a convergent subsequence, say $(M(t_j,\omega_0,M_0))$, with limit M_ω. It is obvious that $M_\omega \geq 0$. In addition, for any $t \in \mathbb{R}$, there exists $M(t,\omega,M_\omega)$: otherwise there would be a time s between 0 and t with $\|M(s,\omega,M_\omega)\|$ as large as desired (see Remark 1.43), but this is impossible since $M(s,\omega,M_\omega) = \lim_{j\to\infty} M(s+t_j,\omega_0,M_0)$ and the set $\{M(t,\omega_0,M_0) \mid t \in \mathbb{R}\}$ is, by hypothesis, bounded. It is also clear that $M(t,\omega,M_\omega) \geq 0$. This all means that $M(t,\omega,M_\omega)$ satisfies the conditions in (i), and this completes the proof of (ii).

8.5 The Optimal Situation: Uniform Null Controllability

The hypotheses for all the main results of this section include the uniform null controllability of the family (8.4) (see Definition 6.3). Recall that Theorem 6.4 proves the equivalence between this property and the apparently less restrictive condition C1 of Sects. 6.2 and 7.3.3, and hence with the null controllability of all the systems of the family: see Remark 6.5.2. The information provided by Theorem 6.4, Remarks 6.8 and 7.20, and Corollaries 7.35 and 5.86 contributes to give a better idea of the controllability scenario of this section.

The section is divided into three subsections, each one of which adds more fundamental hypotheses to that of controllability. In the first subsection, one assumes the exponential dichotomy of (8.6) and the global existence of the Weyl function M^-, which are proved to be equivalent to the Frequency and Nonoscillation

Conditions of Chap. 7. In other words, under these conditions, the Weyl function M^+ also globally exists. And in fact the main results state the equivalence between the uniform (strict) dissipativity of the family of LQ control problems and the fact that M^- is positive (definite) semidefinite. In addition, always working under these conditions, the optimal storage function, i.e. the required supply (see Theorem 8.15), is expressed in terms of M^-. Also, in this subsection the connection between the hypotheses here imposed and the Yakubovich Frequency Theorem (in its nonautonomous general version: see Theorem 7.10) is discussed. In fact, the Yakubovich Frequency Theorem was used in [44] and [84] to obtain most of the results proved here. The approach taken here permits one to simplify the proofs of [44] and [84].

As a historical comment, note that the analysis made in [44] and [84] was motivated by that of [158]. The results of this last paper are stated in the case of periodic coefficients, while those of [44] and [84] take up the situation of general time-varying coefficients. The second subsection is devoted to showing how the results proved so far can be formulated in terms of an initial system when Ω is given by its hull.

The third subsection requires the weak disconjugacy of all the systems (8.6), which together with the uniform null controllability ensures the uniform weak disconjugacy of the family, and hence the existence of the principal functions N^+ and N^-. The conditions imposed are less restrictive than in the previous section: this situation may be present in the absence of exponential dichotomy, as trivial examples show (see Example 8.35), while the hypotheses of the first subsection are considerably stronger than the occurrence of uniform weak disconjugacy. But the main result of this section also requires that a σ-ergodic measure with full support exist (which is not necessary to fulfill the conditions of the exponential dichotomy theorem of the first subsection: see Example 8.36), and that all the corresponding Lyapunov exponents are different from zero. In this case, an equivalence with the uniform dissipativity of the family is determined in terms of N^-, which under these conditions determines the required supply, but now just for m_0-a.a. systems of the family.

8.5.1 With Exponential Dichotomy and Global Existence of M^-

The main results of this section are Theorems 8.22 and 8.23. Under the fundamental hypothesis of uniform null controllability, they establish the equivalence between the (strict) uniform dissipativity of the family of LQ_ω control problems and some properties of the Weyl function M^-, whose global existence is also required; and they determine the optimal (strong) storage function in terms of M^-. The section also analyzes the relation between the framework here considered and that of the application of the Yakubovich Frequency Theorem.

As in the previous sections, the analysis will be carried out for a family (8.4) of control systems and a family (8.5) of supply rates varying in Ω. As a byproduct of the analysis, it will be shown that the optimal storage function (i.e. the required supply: see Theorem 8.15) varies continuously with respect to ω, a fact which has the fundamental consequence explained in Remark 8.25. However, in spite of the fact that the hypotheses and theses of the results of this section are formulated for the families of LQ control problems (and the linear Hamiltonian systems defined from them), they can be rewritten in terms of a single problem, without making reference to the whole family. The details of this comment are explained in Sect. 8.5.2.

Definition 8.18 The family of control systems (8.4) is *uniformly dissipative with family of supply rates* $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ given by (8.5) if for each $\omega \in \Omega$, for each pair $t_1 < t_2 \in \mathbb{R}$, and for each control $\mathbf{u}\colon [t_1, t_2] \to \mathbb{R}^m$, the solution $\mathbf{x}(t)$ of (8.4) satisfying $\mathbf{x}(t_1) = \mathbf{0}$ has the property that

$$\int_{t_1}^{t_2} \widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \geq 0\,;$$

i.e. if each LQ_ω control problem of the family is dissipative. The family (8.4) of control systems is *uniformly strictly dissipative with family of supply rates* $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ given by (8.5) if there exists $\delta > 0$ such that the family is uniformly dissipative with the modified supply rates

$$\widetilde{\mathcal{Q}}_{\omega,\delta}(t, \mathbf{x}, \mathbf{u}) = \widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}, \mathbf{u}) - \delta\,(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2)\,;$$

i.e. if each LQ_ω control problem of the family is strictly dissipative and the constant $\delta > 0$ of Definition 8.1 is common to the whole family.

It is clear that, in this context, the "uniformity" means nothing when referred to the dissipative case. However, it is meaningful in the case of strict dissipativity: see Proposition 8.28 and Remark 8.29.1, at the end of this section.

The hypotheses required in this section are now given. In fact there are several equivalent ways to formulate them, as Proposition 8.20 shows. The main results are formulated immediately after it.

Hypotheses 8.19 The family of control systems (8.4) is uniformly null controllable, the family (8.6) has exponential dichotomy over Ω, and the Weyl function M^- is globally defined.

Recall that the last condition means that the Lagrange plane $l^-(\omega)$ of the solutions which are bounded as $t \to -\infty$ (see Remark 1.77.3) admits the representation $\left[\begin{smallmatrix} I_n \\ M^-(\omega) \end{smallmatrix}\right]$ for all $\omega \in \Omega$; in other words, it lies outside the vertical Maslov cycle for all $\omega \in \Omega$. As seen in Sect. 7.3, even under the stabilization hypothesis assumed in the Yakubovich Frequency Theorem, the Lagrange plane $l^+(\omega)$ may be or may not have this property. But it turns out that Hypotheses 8.19 ensure also the global existence of M^+, as the next result recalls.

Proposition 8.20 *The following assertions are equivalent:*

(1) *Hypotheses 8.19 hold;*
(2) *the family of Hamiltonian systems* (8.6) *has exponential dichotomy over Ω and both Weyl functions $M^{\pm}$ are globally defined;*
(3) *the family of control systems* (8.4) *is uniformly null controllable, the family* (8.6) *has exponential dichotomy over Ω, and the Weyl function M^{+} globally exists.*

In addition, in this case, $M^{+} < M^{-}$, and the family of control systems (8.4) *satisfies Hypothesis 7.3.*

Proof Note that the family (8.6) satisfies condition D1 of Chap. 5 (which is a fundamental fact for this result). The proofs of the equivalences and the inequality $M^{+} < M^{-}$ follow easily from Corollary 7.35, which states that the uniform null controllability of the family (8.4) is equivalent to D2, and from Theorem 7.21, since the global existence of M^{-} or M^{+} ensures that D3 holds. Finally, Proposition 7.33 shows that Hypothesis 7.3 is satisfied.

Remark 8.21 It is implicit in the previous proof that Hypotheses 8.19 ensure the uniform weak disconjugacy of the family.

The main results of this section can now be stated.

Theorem 8.22 *Suppose that Hypotheses 8.19 hold. Then the following assertions are equivalent:*

(1) *the family of control systems* (8.4) *is uniformly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ given by* (8.5)*;*
(2) $M^{-} \geq 0$.

In addition, in this case the function $V_\omega^{-}(t,\mathbf{x}) = \langle \mathbf{x}, M^{-}(\omega{\cdot}t)\,\mathbf{x}\rangle$ is the required supply for the LQ_ω control problem, and is jointly continuous in the variables $(\omega, t, \mathbf{x})$.

Theorem 8.23 *Suppose that Hypotheses 8.19 hold. Then the following assertions are equivalent:*

(1) *the family of control systems* (8.4) *is uniformly strictly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ given by* (8.5)*;*
(2) $M^{-} > 0$.

In addition, in this case the function $V_\omega^{-}(t,\mathbf{x}) = \langle \mathbf{x}, M^{-}(\omega{\cdot}t)\,\mathbf{x}\rangle$ is the required supply for the LQ_ω control problem, it is strong, and it is jointly continuous in the variables $(\omega, t, \mathbf{x})$.

The next lemma reveals a strong connection between M^{-} and the required supply, which is the key point in the proofs of the main theorems of this section.

Lemma 8.24 *Suppose that Hypotheses 8.19 hold. With notation as above,*

$$V_\omega^{r}(t,\mathbf{x}) = \langle \mathbf{x}, M^{-}(\omega{\cdot}t)\,\mathbf{x}\rangle\,.$$

Proof Since $M(t,\omega,M^-(\omega)) = M^-(\omega{\cdot}t)$, relation (8.11) (which does not require any positivity in the solution of the Riccati equation), the definition of $V^r_\omega(t,\mathbf{x})$ and the trivial equality $V_{\omega,M^-(\omega)}(t,\mathbf{0}) = 0$, ensure that $V^r_\omega(t,\mathbf{x}) \ge V_{\omega,M^-(\omega)}(t,\mathbf{x}) = \langle \mathbf{x}, M^-(\omega{\cdot}t)\,\mathbf{x}\rangle$. So, the goal now is to prove the converse inequality.

As a first step, it will be proved that $V^r_\omega(0,\mathbf{x}_0) \le \langle \mathbf{x}_0, M^-(\omega)\,\mathbf{x}_0\rangle$ for all $\omega\in\Omega$ and $\mathbf{x}_0\in\mathbb{R}^n$. So, fix ω and $\mathbf{x}_0$. Applying (8.10) to $V_{\omega,M^-(\omega)}$ ensures that for any pair $(\mathbf{x}(t),\mathbf{u}(t))$ solving (8.4),

$$\begin{aligned}\frac{d}{dt}\,V_{\omega,M^-(\omega)}(t,\mathbf{x}(t)) &= 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t))\\ &\quad-\langle \mathbf{u}(t)-K^-(\omega{\cdot}t)\,\mathbf{x}(t), R(\omega{\cdot}t)(\mathbf{u}(t)-K^-(\omega{\cdot}t)\,\mathbf{x}(t))\rangle\end{aligned} \tag{8.12}$$

for

$$K^- = R^{-1}(-g^T + B^TM^-)\,. \tag{8.13}$$

Let $\mathbf{x}(t)$ be the solution of the equation

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K^-(\omega{\cdot}t))\,\mathbf{x} \tag{8.14}$$

with $\mathbf{x}(0) = \mathbf{x}_0$. Now fix $\varepsilon > 0$. Let $\delta^* > 0$ be provided by Proposition 8.5(ii) (with $\delta^* \le \varepsilon$ for later purposes). It is easy to check that $\left[\begin{smallmatrix}\mathbf{x}(t)\\ M^-(\omega\cdot t)\,\mathbf{x}(t)\end{smallmatrix}\right]$ solves the Hamiltonian system (8.6): the definition of K^- and the Riccati equation satisfied by M^- along the flow ensure that

$$\begin{aligned}\mathbf{x}'(t) &= \big(A - BR^{-1}\widetilde{g}\big)\,\mathbf{x} + BR^{-1}B^*\,n(t)\,\mathbf{x}(t)\,,\\ (M^-\,\mathbf{x}(t))' &= \big(G - gR^{-1}\widetilde{g}\big)\,\mathbf{x} + \big(g\,R^{-1}B^* - A^*\big)\,n(t)\,\mathbf{x}(t)\,,\end{aligned}$$

where all the coefficient matrices have argument $\omega{\cdot}t$. That is, $\left[\begin{smallmatrix}\mathbf{x}(t)\\ M^-(\omega\cdot t)\,\mathbf{x}(t)\end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix}\mathbf{x}_0\\ M^-(\omega)\,\mathbf{x}_0\end{smallmatrix}\right]$, and obviously $\left[\begin{smallmatrix}\mathbf{x}_0\\ M^-(\omega)\,\mathbf{x}_0\end{smallmatrix}\right]$ belongs to $l^-(\omega)$. This ensures that $\mathbf{x}(t)$ tends to zero exponentially as $t\to-\infty$: see Definition 1.75 and Proposition 1.76. Thus, there exists $h\ge 0$ such that $\|\mathbf{x}(t)\|\le\delta^*$ for all $t\le -h$. Such a value of h will be fixed in what follows.

Define a control function $\mathbf{u}(t)$ on $[-h,0]$ in the feedback form

$$\mathbf{u}(t) = K^-(\omega{\cdot}t)\,\mathbf{x}(t) \quad \text{for } t\in[-h,0]\,, \tag{8.15}$$

and apply Proposition 8.5(ii) in order to obtain a continuous function $\mathbf{u}_h\colon[-t_0,0]\to\mathbb{R}^m$ with $\|\mathbf{u}_h(t)\|\le\varepsilon$ for all $t\in[-t_0,0]$ such that the solution $\mathbf{x}_h(t)$ of $\mathbf{x}' = A((\omega{\cdot}h){\cdot}t)\,\mathbf{x} + B((\omega{\cdot}h){\cdot}t)\,\mathbf{u}_h(t)$ with $\mathbf{x}_h(0) = \mathbf{x}(-h)$ satisfies $\mathbf{x}_h(-t_0) = \mathbf{0}$.

Now consider the concatenated control

$$\bar{\mathbf{u}}(t) = \begin{cases} \mathbf{u}(t) & \text{for } t \in [-h, 0]\,, \\ \mathbf{u}_h(t+h) & \text{for } t \in [-t_0 - h, -h)\,. \end{cases}$$

Let $\bar{\mathbf{x}}(t)$ be the solution of $\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\bar{\mathbf{u}}(t)$ with $\bar{\mathbf{x}}(0) = \mathbf{x}_0$. It is clear that $\bar{\mathbf{x}}(t) = \mathbf{x}(t)$ for all $t \in [-h, 0]$ and $\bar{\mathbf{x}}(-t_0 - h) = \mathbf{x}_h(-t_0) = \mathbf{0}$. Integrating (8.12) for the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ in the interval $[-h - t_0, 0]$ and keeping in mind equality (8.15), one obtains

$$\begin{aligned} 2\int_{-t_0-h}^{0} \widetilde{\mathcal{Q}}_\omega(s, \bar{\mathbf{x}}(s), \bar{\mathbf{u}}(s))\, ds &= \langle \mathbf{x}_0, M^-(\omega)\,\mathbf{x}_0\rangle \\ &+ \int_{-t_0-h}^{-h} \langle \bar{\mathbf{u}}(s) - K^-(\omega{\cdot}s)\,\bar{\mathbf{x}}(s), R(\omega{\cdot}s)\,(\bar{\mathbf{u}}(s) - K^-(\omega{\cdot}s)\,\bar{\mathbf{x}}(s))\rangle\, ds\,. \end{aligned}$$

Hence, using the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ in the definition of $V^r_\omega(0, \mathbf{x}_0)$ yields

$$\begin{aligned} V^r_\omega(0, \mathbf{x}_0) &\le \langle \mathbf{x}_0, M^-(\omega)\,\mathbf{x}_0\rangle \\ &\quad + t_0\, r \sup\left\{\|\bar{\mathbf{u}}(t) - K^-(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)\|^2 \mid t \in [-t_0 - h, -h]\right\}, \end{aligned}$$

where $r \ge \|R(\omega)\|$ for all $\omega \in \Omega$. Now take $t \in [-t_0 - h, -h]$ and note that

$$\begin{aligned} \bar{\mathbf{x}}(t) &= U_A(t, \omega)\, U_A^{-1}(-t_0 - h, \omega)\,\mathbf{x}(-t_0 - h) \\ &\quad + \int_{-t_0-h}^{t} U_A(t, \omega)\, U_A^{-1}(s, \omega)\, B(\omega{\cdot}s)\,\bar{\mathbf{u}}(s)\, ds \\ &= U_A(t + h, \omega{\cdot}(-t_0 - h))\,\mathbf{x}(-t_0 - h) \\ &\quad + \int_{-t_0-h}^{t} U_A(t - s, \omega{\cdot}s)\, B(\omega{\cdot}s)\,\bar{\mathbf{u}}(s)\, ds\,. \end{aligned}$$

Also note that $t - s \in [0, t_0]$ for $s \in [-t_0 - h, t]$. So, if for all $t \in [0, t_0]$ and $\omega \in \Omega$, it is the case that $u \ge \|U_A(t, \omega)\|$, $b \ge \|B(\omega)\|$ and $k_0 \ge \|K^-(\omega)\|$ for all $\omega \in \Omega$, then

$$\|K^-(\omega{\cdot}t)\,\bar{\mathbf{x}}(t)\| \le k_0\,(u\,\delta_* + t_0\, u\, b\,\varepsilon) \le (1 + t_0\, b)\, k_0\, u\,\varepsilon$$

for $t \in [-t_0 - h, -t_0]$. Therefore, if $\rho = (1 + t_0\, b)\, k_0\, u$ (independent of ε), then $V^r_\omega(0, \mathbf{x}_0) \le \langle \mathbf{x}_0, M^-(\omega)\,\mathbf{x}_0\rangle + t_0\, r\,(1 + \rho)^2\,\varepsilon^2$. This shows that $V^r_\omega(0, \mathbf{x}_0) \le \langle \mathbf{x}_0, M^-(\omega)\,\mathbf{x}_0\rangle$ and completes the first step of the proof.

The second and last step of the proof is to show that $V^r_\omega(t, \mathbf{x}_0) = V^r_{\omega{\cdot}t}(0, \mathbf{x}_0)$ for all $t \in \mathbb{R}$. This follows easily from the definition of the required supply, from the equalities

$$\int_{-t-h}^{t} \widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\, ds = \int_{-h}^{0} \widetilde{\mathcal{Q}}_\omega(t+s, \mathbf{x}(t+s), \mathbf{u}(t+s))\, ds$$
$$= \int_{-h}^{0} \widetilde{\mathcal{Q}}_{\omega\cdot t}(s, \mathbf{x}(t+s), \mathbf{u}(t+s))\, ds\,,$$

and from the fact that the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ given by $\bar{\mathbf{x}}(s) = \mathbf{x}(t+s)$ and $\bar{\mathbf{u}}(s) = \mathbf{u}(t+s)$ solves the system (8.4) for $\omega{\cdot}t$ if the pair $(\mathbf{x}, \mathbf{u})$ solves it for ω. The proof is complete.

The proofs of the main results are easy consequences of the previous ones and the properties of the required supply as analyzed in Theorem 8.15:

Proof of Theorem 8.22 Note that the joint continuity with respect to all three arguments of the function $V_\omega^-(t, \mathbf{x}) = \langle \mathbf{x}, M^-(\omega{\cdot}t)\, \mathbf{x}\rangle$ follows from the continuity of the flow σ on $\mathbb{R} \times \Omega$ and of M^- on Ω (see e.g. Definition 1.80). Lemma 8.24 shows that the required supply is given by $V_\omega^-(t, \mathbf{x})$, so that Theorem 8.15 implies the equivalence between (1) and (2), as well as the assertion that V_ω^- is the required supply. The proof is complete.

Proof of Theorem 8.23 (1)⇒(2) Fix any $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{x}_0 \neq \mathbf{0}$. Take $h > 0$ and a pair $(\mathbf{x}, \mathbf{u})$ solving the system (8.4) with $\mathbf{u}$ square integrable, $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}(-h) = \mathbf{x}_0$. The assumed uniform strict dissipativity of the family of LQ control problems ensures the existence of a common $\delta > 0$ such that

$$2\int_{-h}^{0} \mathcal{Q}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \geq \delta \int_{-h}^{0} \big(\|\mathbf{x}(s)\|^2 + \|\mathbf{u}(s)\|^2\big)\, ds\,. \tag{8.16}$$

Now the proof follows the argument of the proof of Lemma 4 in [158]. The boundedness of A and B provides positive constants a and b such that

$$\int_{-h}^{0} \|\mathbf{x}'(s)\|^2\, ds \leq a \int_{-h}^{0} \|\mathbf{x}(s)\|^2\, ds + b \int_{-h}^{0} \|\mathbf{u}(s)\|^2\, ds\,.$$

Hence,

$$\|\mathbf{x}_0\|^2 = 2\int_{-h}^{0} \langle \mathbf{x}(s), \mathbf{x}'(s)\rangle\, ds \leq \int_{-h}^{0} \big(\|\mathbf{x}(s)\|^2 + \|\mathbf{x}'(s)\|^2\big)\, ds$$
$$\leq (a+1)\int_{-h}^{0} \|\mathbf{x}(s)\|^2\, ds + b\int_{-h}^{0} \|\mathbf{u}(s)\|^2\, ds$$
$$\leq (a+b+1)\int_{-h}^{0} \big(\|\mathbf{x}(s)\|^2 + \|\mathbf{u}(s)\|^2\big)\, ds\,,$$

and therefore (8.16) ensures that

$$2\int_{-h}^{0} \mathcal{Q}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \geq \delta\,(a+b+1)^{-1}\|\mathbf{x}_0\|^2\,.$$

This bound and the definition of the required supply V^r_ω yield $V^r_\omega(0,\mathbf{x}_0) \geq \delta\,(a+b+1)^{-1}\|\mathbf{x}_0\|^2 > 0$, and hence Lemma 8.24 shows that $M^-(\omega) > 0$ and that V^-_ω is the required supply, as asserted. It is obvious that V^-_ω is strong and that it is continuous in $(\omega, t, \mathbf{x})$.

(2)⇒(1) Following the proof of Y2⇒Y5 of Theorem 7.10, note that the quadratic functional obtained by substituting G and R by $G-\delta I_n$ and $R-\delta I_m$ in (8.5) is given by

$$\widetilde{\mathcal{Q}}^\delta_\omega(t,\mathbf{x},\mathbf{u}) = \widetilde{\mathcal{Q}}_\omega(t,\mathbf{x},\mathbf{u}) - \delta\left(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2\right). \tag{8.17}$$

Define also $H_\delta(\omega)$ by substituting G and R by $G-\delta I_n$ and $R-\delta I_m$ in the matrix H of (8.6). According to Theorems 1.92 and 1.95, it is possible to choose $\delta > 0$ small enough to guarantee: that $R - \delta I_m > 0$, that the Lagrange planes of the initial data of the solutions of $\mathbf{z}' = H_\delta(\omega{\cdot}t)\,\mathbf{z}$ which are bounded as $t \to \pm\infty$ are represented by $\begin{bmatrix} I_n \\ M^\pm_\delta(\omega) \end{bmatrix}$, and that $M^-_\delta(\omega) > 0$ for all $\omega \in \Omega$. It follows from the relation (8.12) corresponding to the value chosen for δ and from (8.17) that

$$\frac{d}{dt}\,V_{\omega, M^-_\delta(\omega)}(t,\mathbf{x}(t)) \leq 2\,\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \delta\left(\|\mathbf{x}(t)\|^2 + \|\mathbf{u}(t)\|^2\right)$$

for any pair solving (8.4). In turn, this ensures that

$$\begin{aligned}&\langle \mathbf{x}(t_2), M^-_\delta(\omega{\cdot}t)_2)\,\mathbf{x}(t_2)\rangle \\ &\qquad\leq 2\int_{t_1}^{t_2}\left(\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \delta(\|\mathbf{x}(t)\|^2 + \|\mathbf{u}(t)\|^2)\right) ds\end{aligned}$$

for every pair $(\mathbf{x},\mathbf{u})$ solving (8.4) with $\mathbf{u}$ square integrable and $\mathbf{x}(t_1) = \mathbf{0}$. This proves the asserted uniform strict dissipativity.

The last assertions were proved in the verification of (1)⇒(2).

Remark 8.25 As in the paper [158], in the situations described in Theorems 8.22 and 8.23, the optimal storage function turns out to be quadratic with respect to the state $\mathbf{x}$, and to have a t-dependence with recurrence properties which are at least as strong as those of the coefficients (in turn inherited from those of an initial LQ problem if the family is obtained via a Bebutov construction). Thus, for example, if A, B, G, g, R are all Bohr almost periodic functions with frequency module $\mathfrak{M}$, then V is almost periodic in t with frequency module contained in $\mathfrak{M}$.

Remark 8.26 It is easy to construct nonautonomous examples for which Theorems 8.22 and 8.23 imply the uniform dissipativity of the family over the hull. In

Example 8.36 the reader can find a concrete illustration of this, for which in addition the hull is not minimal.

The next result of the section concerns the behavior of the Weyl function M^+ (whose global existence is ensured by Proposition 8.20 under Hypotheses 8.19) in the case of uniform strict dissipativity. Define now $K^+ = R^{-1}(-g^T + B^T M^+)$ and let $\widetilde{U}(t,\omega) = U_{A+BK^+}(t,\omega)$ be the fundamental matrix solution of equation

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K^+(\omega{\cdot}t))\,\mathbf{x}$$

with $\widetilde{U}(0,\omega) = I_n$.

Theorem 8.27 *Suppose that Hypotheses 8.19 hold, and that the family of control systems* (8.4) *is uniformly strictly dissipative with family of supply rates* $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ *given by* (8.5)*. Let* $t_0 > 0$ *be the positive time satisfying the condition on uniform null controllability of Definition 6.3. Then, there exists* $\rho > 1$ *such that*

$$-\rho\, M^+(\omega) \le \left(\int_{-t_0}^{0} \widetilde{U}(t,\omega)^{-1} B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, (\widetilde{U}^{-1})^T(t,\omega)\, dt\right)^{-1}$$

for all $\omega \in \Omega$.

Proof The idea of the proof is taken from that of Theorem 1 of [158]. The control system (8.4) can be rewritten as

$$\mathbf{x}' = (A(\omega{\cdot}t) + B(\omega{\cdot}t)\,K^+(\omega{\cdot}t))\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{v} \tag{8.18}$$

for

$$\mathbf{v} = \mathbf{u} - K^+(\omega{\cdot}t)\,\mathbf{x}\,. \tag{8.19}$$

Hence the family of control systems (8.18) is uniformly null controllable. Fix now $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Proposition 8.5(iii) provides a square integrable control $\mathbf{u}\colon [-t_0, 0] \to \mathbb{R}^m$ such that the solution $\mathbf{x}\colon [-t_0, 0] \to \mathbb{R}^n$ of (8.4) with $\mathbf{x}(-t_0) = \mathbf{0}$ satisfies $\mathbf{x}(0) = \mathbf{x}_0$. Define $\mathbf{v}$ by (8.19). Note the relation (8.10) for $V_{\omega,M^+(\omega)}$ and this pair $(\mathbf{x},\mathbf{u})$ reads

$$\frac{d}{dt}\,\langle \mathbf{x}(t), M^+(\omega{\cdot}t)\,\mathbf{x}(t)\rangle = 2\widetilde{\mathcal{Q}}_\omega(t,\mathbf{x}(t),\mathbf{u}(t)) - \langle \mathbf{v}(t), R(\omega{\cdot}t)\mathbf{v}(t)\rangle,$$

and hence, by the uniform strict dissipativity,

$$\begin{aligned}\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle &+ \int_{-t_0}^{0} \|R^{1/2}(\omega{\cdot}s)\,\mathbf{v}(s)\|^2\,ds \\ &= \int_{-t_0}^{0} 2\,\widetilde{\mathcal{Q}}_\omega(s,\mathbf{x}(s),\mathbf{u}(s))\,ds \ge \delta \int_{-t_0}^{0} \big(\|\mathbf{x}(s)\|^2 + \|\mathbf{u}(s)\|^2\big)\,ds\,.\end{aligned}$$

It follows easily from the definition of $\mathbf{v}$ that there exists $\delta_1 \in (0,1)$ with

$$\int_{-t_0}^{0} (\|\mathbf{x}(s)\|^2 + \|\mathbf{u}(s)\|^2)\, ds \geq \delta_1 \int_{-t_0}^{0} \|R^{1/2}(\omega{\cdot}s)\,\mathbf{v}(s)\|^2\, ds\,.$$

Assume without loss of generality that $\delta\delta_1 < 1$, define $\rho = 1/(1-\delta\delta_1) \in (0,1)$, and deduce from the last inequalities that

$$-\rho\,\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle \leq \int_{-t_0}^{0} \|R^{1/2}(\omega{\cdot}s)\,\mathbf{v}(s)\|^2\, ds\,.$$

This means that

$$\begin{aligned} &-\rho\,\langle \mathbf{x}_0, M^+(\omega)\,\mathbf{x}_0\rangle \\ &\qquad \leq \inf\left\{ \int_{-t_0}^{0} \|R^{1/2}(\omega{\cdot}s)\,\mathbf{v}(s)\|^2\, ds \;\middle|\; \begin{array}{l} \mathbf{v}\colon[-t_0,0]\to\mathbb{R}^m \text{ control} \\ \text{and } \mathbf{x} \text{ solves (8.18) with} \\ \mathbf{x}(-t_0)=\mathbf{0} \text{ and } \mathbf{x}(0)=\mathbf{x}_0 \end{array} \right\}. \end{aligned}$$

It was noted previously that the set on the right is nonempty. Lemma 3 of [158] ensures that its infimum is precisely

$$\mathbf{x}_0^T \left(\int_{-t_0}^{0} \widetilde{U}^{-1}(t,\omega) B(\omega{\cdot}t)\, R^{-1}(\omega{\cdot}t)\, B^T(\omega{\cdot}t)\, (\widetilde{U}^{-1})^T(t,\omega)\, dt \right)^{-1} \mathbf{x}_0\,,$$

and this completes the proof.

The point of Theorem 8.27 is the following. According to Proposition 8.20, $M^-(\omega) > M^+(\omega)$ for all $\omega \in \Omega$. But even if $M^-(\omega)$ is positive definite for all $\omega \in \Omega$, the symmetric $n \times n$ matrix $M^+(\omega)$ need not be positive semidefinite. Thus Theorem 8.27 states in effect that $M^+(\omega)$ cannot be "too negative".

There is a clear connection between the techniques and results of this section and those of Sect. 7.2, which contains the proof of the general version of the Yakubovich Frequency Theorem. The point at which this connection is strongest is in the proof of Lemma 8.24. The next goal of this section is to show that in fact both frameworks are closely related.

Recall that the motivation of the Yakubovich Frequency Theorem is the minimization problem for the functional $\widetilde{\mathcal{I}}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u})$ defined by (8.7) when evaluated on the admissible pairs (i.e. square integrable pairs $(\mathbf{x},\mathbf{u})$ satisfying (8.4) with $\mathbf{x}(0) = \mathbf{x}_0$). Recall also that the Theorem establishes an uniform stabilization condition (Hypothesis 7.3) under which the solvability of the minimizing problem can be determined from the dynamical properties of the family (8.6). And recall further that the Frequency and Nonoscillation Conditions for (8.6) are equivalent to the occurrence of exponential dichotomy and the global existence of the Weyl function M^+.

Proposition 8.28 *Suppose that the family of control systems* (8.4) *satisfies Hypothesis 7.3, and that either (1): the family of LQ_ω control problems is uniformly strictly dissipative with family of supply rates given by* (8.5)*; or (2): it satisfies the Frequency and Nonoscillation Conditions. Then properties Y1, Y2, Y3, Y4, Y5, and Y6 of Sect. 7.2 hold for the families* (8.4)*,* (8.7)*,* (8.5)*, and* (8.6)*.*

Proof It is clear that, if (1) holds, the families (8.4) and (8.5) satisfy property Y6 of Sect. 7.2: an easy contradiction argument proves this assertion. And (2) is exactly condition Y2. Therefore, the result follows from Theorem 7.10.

Remarks 8.29

1. A natural question arise: under Hypothesis 7.3, does condition Y6 imply uniform strict dissipativity? The answer is no: Y6 does not imply dissipativity, even when $n = 1$. This is what Example 8.30 shows.
2. Proposition 8.20 shows that Hypotheses 8.19 (which include Hypothesis 7.3), imply the conditions Y2 (i.e. Frequency and Nonoscillation Conditions) of Sect. 7.2. Therefore, by Theorem 7.10, properties Y1, Y2, Y3, Y4, Y5, and Y6 also hold if Hypotheses 8.19 hold, as in Theorems 8.22 and 8.23. However, as seen in Sect. 7.3.3, the framework of the Yakubovich Frequency Theorem is less restrictive: it does not require the uniform null controllability. Example 7.37 illustrates a situation in which: the Frequency and Nonoscillation Conditions are satisfied and Hypothesis 7.3 holds, so that conditions Y1–Y6 are fulfilled; but the Weyl function M^- does not globally exist, which according to Proposition 8.20 implies the absence of uniform null controllability.

Example 8.30 In order to check that, even under Hypothesis 7.3, condition Y6 does not imply uniform strict dissipativity, take $n = 1$, and the autonomous coefficients given by $A \equiv 1$, $B \equiv 1$, $G \equiv 0$, $g \equiv -1$, and $R \equiv 1$. Then the corresponding control system $x' = x + u$ satisfies Hypothesis 7.3 (just take $K_0 \equiv -2$), and it is a trivial exercise to check that the Hamiltonian system $\mathbf{z}' = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{z}$ satisfies the Frequency and Nonoscillation Conditions. Hence, Theorem 7.10 ensures that the six conditions Y1–Y6 hold. However, the associated control problem is nondissipative. One way to check this assertion is to note that $x' = x + u$ is null controllable (see Remarks 6.2) and that the Weyl function (constant, as a matter of fact) n^- is $\sqrt{3} - 2 < 0$, and then use Lemma 8.24 and Theorem 8.15.

8.5.2 The Results for a Single System

As said before, the results of the preceding section can be formulated in terms of a single LQ problem. The idea is to ensure that the properties required for the whole family are inherited from those of one system.

To begin with, recall that Remark 6.16 describes two situations in which the family of control systems defined over the common hull Ω of (A, B, G, g, R) is uniformly null controllable. The simplest one corresponds to the case of a null controllable initial system and a minimal Ω. On the other hand, as seen in Sect. 1.4.1, when Ω is minimal, the exponential dichotomy of the initial Hamiltonian system ensures that of the whole family (8.6) over Ω; and it is easy to see that in this case the global existence and boundedness of M^+ (or M^-) for the initial system ensures the corresponding property for all the systems of the hull.

These results indicate a possible way to reformulate Proposition 8.20:

Proposition 8.31 *Consider the single LQ control problem defined by* (8.1) *and* (8.2)*. Suppose that A, B, G, g, and R are bounded and uniformly continuous functions, and that the hull Ω of (A, B, G, g, R) is minimal. Let $\mathbf{z}' = H(t)\,\mathbf{z}$ be the Hamiltonian system of type* (8.6) *obtained from the initial data, and let $U(t)$ be the fundamental matrix solution of this system with $U(0) = I_{2n}$. Then the following assertions are equivalent:*

(1) *the initial control system* (8.1) *is null controllable, the Hamiltonian system $\mathbf{z}' = H(t)\,\mathbf{z}$ has exponential dichotomy, and there exists a bounded function $M^-\colon \mathbb{R} \to \mathbb{S}_n(\mathbb{R})$ such that $\left[\begin{smallmatrix} I_n \\ M^-(t) \end{smallmatrix}\right] \equiv U(t)\cdot l^-$ in $\mathcal{L}_{\mathbb{R}}$, where l^- is the Lagrange space of the solutions which are bounded at $-\infty$;*
(2) *the Hamiltonian system $\mathbf{z}' = H(t)\,\mathbf{z}$ has exponential dichotomy, and there exist two bounded functions $M^\pm\colon \mathbb{R} \to \mathbb{S}_n(\mathbb{R})$ such that $\left[\begin{smallmatrix} I_n \\ M^\pm(t) \end{smallmatrix}\right] \equiv U(t)\cdot l^\pm$ in $\mathcal{L}_{\mathbb{R}}$, where $l^\pm$ are the Lagrange spaces of the solutions which are bounded at $\pm\infty$;*
(3) *the initial control system* (8.1) *is null controllable, the Hamiltonian system $\mathbf{z}' = H(t)\,\mathbf{z}$ has exponential dichotomy, and there exists a bounded function $M^+\colon \mathbb{R} \to \mathbb{S}_n(\mathbb{R})$ such that $\left[\begin{smallmatrix} I_n \\ M^+(t) \end{smallmatrix}\right] \equiv U(t)\cdot l^+$ in $\mathcal{L}_{\mathbb{R}}$, where l^+ is the Lagrange space of the solutions which are bounded at ∞.*

In addition, in this case, $M^-(t) - M^+(t) > \rho I_n$ for a common $\rho > 0$ and all $t \in \mathbb{R}$.

In the same line of ideas, an elementary continuity argument shows that if the initial LQ control problem is dissipative or strictly dissipative (according to Definition 8.1), then the family of all the LQ$_\omega$ problems over the hull is uniformly dissipative or strictly dissipative (according to Definition 8.18). That is, the dissipativity or strict dissipativity of the initial LQ problem is equivalent to the corresponding uniform property of the whole family.

Therefore, Theorem 8.22 can be reformulated as follows:

Theorem 8.32 *Consider the single LQ control problem defined by* (8.1) *and* (8.2)*. Suppose that A, B, G, g, and R are bounded and uniformly continuous functions, and that the hull Ω of (A, B, G, g, R) is minimal. Let $\mathbf{z}'(t) = H(t)\,\mathbf{z}$ be the Hamiltonian system of type* (8.6) *obtained from the initial data, and let $U(t)$ be the fundamental matrix solution of this system with $U(0) = I_{2n}$. And suppose also that the situation (1) of Proposition 8.31 holds. The following assertions are equivalent:*

(1) *the initial LQ control problem is dissipative;*
(2) *$M^-(t) \geq 0$ for all $t \in \mathbb{R}$.*

In addition, in this case the function $V^-(t,\mathbf{x}) = \langle \mathbf{x}, M^-(t)\,\mathbf{x}\rangle$ is the optimal storage function for the LQ control problem, and it is jointly continuous.

Theorem 8.23 can be reformulated in a very similar way (point (2) will read: $M^-(t) > \rho I_n$ for a common $\rho > 0$ and all $t \in \mathbb{R}$), and the same is the case with Theorem 8.27. Note that in all these results, the function M^- can be defined directly from the Lagrange plane of the solutions of the corresponding single Hamiltonian system bounded at $-\infty$, and $V^-(t,\mathbf{x}) = \langle \mathbf{x}, M^-(t)\,\mathbf{x}\rangle$ is the storage function.

The results summarized in this section should not be intended to be optimal: the minimality condition on Ω can be relaxed, in the line of what Remark 6.16 explains. The exponential dichotomy of $\mathbf{z}' = H(t)\,\mathbf{z}$ ensures that of (8.6) over the hull (see Remark 1.59.4); and there are situations less restrictive than minimality in which the global existence and properties of the Weyl functions are deduced and inherited from the corresponding ones of the initial system: for instance, if one can check that $M^-(t)$ and is norm-bounded on $\mathbb{R}$, and satisfies $M^-(t) \geq 0$ (or $M^-(t) \geq \rho I_n$ for a common $\rho > 0$) for all $t \in \mathbb{R}$, then the Weyl function M^- exists globally on the hull and it is positive semidefinite (or definite). This is the case in the nonautonomous Example 8.36.

8.5.3 The Uniformly Weakly Disconjugate Case

This section is focused on Theorem 8.34, which establishes conditions ensuring, on the one hand, the uniform weak disconjugacy of the family, and on the other hand, the equivalence between the positive semidefiniteness of the principal function N^- and the uniform dissipativity of the family of LQ control problems. In addition, the optimal storage function is determined up to zero measure.

Hypotheses 8.33 The family of control systems (8.4) is uniformly null controllable, and all the systems of the family (8.6) are weakly disconjugate simultaneously at $+\infty$ or at $-\infty$.

Theorem 8.34 *Suppose that Hypotheses 8.33 hold and that $\Omega = \operatorname{Supp} m_0$ for a σ-ergodic measure m_0. Then,*

(i) *the family (8.6) is uniformly weakly disconjugate, so that it admits principal functions N^+ and N^-.*

   Suppose also that all the Lyapunov exponents of the family of Hamiltonian systems (8.6) are different from zero. Then,

(ii) *the following assertions are equivalent:*

   (1) *the family of control systems (8.4) is uniformly dissipative with family of supply rates $\{\widetilde{Q}_\omega \mid \omega \in \Omega\}$ given by (8.5);*

   (2) $N^- \geq 0$.

   In addition, in this case, the function $\widetilde{V}^-_\omega(t,\mathbf{x}) = \langle \mathbf{x}, N^-(\omega{\cdot}t)\,\mathbf{x}\rangle$ is a storage function for the LQ$_\omega$ control problem for all $\omega \in \Omega$, and it is jointly continuous *in* $(t, \mathbf{x})$. *Moreover, there exists a* σ*-invariant subset* $\Omega_0 \subseteq \Omega$ *with* $m_0(\Omega_0) = 1$ *such that* $\widetilde{V}_\omega^-(t, \mathbf{x})$ *is the required supply for all* $\omega \in \Omega_0$.

Proof

(i) Proposition 1.12 ensures the existence of dense positive and negative σ-semiorbits in Ω, which is enough to deduce from Hypotheses 8.33 and Theorem 5.32 the uniform weak disconjugacy of the family.

(ii) Let Ω_0 be the σ-invariant set provided by Theorem 5.56, which has full measure m_0. That is, if $\omega \in \Omega_0$, then the Lagrange plane $\widetilde{l}^-(\omega) \equiv \left[\begin{smallmatrix} I_n \\ N^-(\omega) \end{smallmatrix}\right]$ determining the uniform principal solution at $-\infty$ agrees with the vector space of the initial data of the solutions of the Hamiltonian system (8.6) corresponding to ω with negative Lyapunov exponent. The proofs of the equivalence of (1) and (2), as well as of the last assertion of (ii), rely on Theorem 8.15, as was the case in the proof of Theorem 8.22. The crucial point is to establish the following result, which is analogous to that of Lemma 8.24:

$$V_\omega^r(t, \mathbf{x}) = \langle \mathbf{x}, N^-(\omega{\cdot}t)\,\mathbf{x}\rangle \quad \text{for } \omega \in \Omega_0\,, \quad t \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^n\,. \tag{8.20}$$

Note first that

$$V_\omega^r(t, \mathbf{x}) \geq \langle \mathbf{x}, N^-(\omega{\cdot}t)\,\mathbf{x}\rangle \quad \text{for } \omega \in \Omega\,, \quad t \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^n \;: \tag{8.21}$$

just reason as at the beginning of the proof of Lemma 8.24. Now fix $\omega \in \Omega_0$ and $\mathbf{x}_0 \in \mathbb{R}^n$, and note that one has to check that $V_\omega^r(t, \mathbf{x}_0) \leq \langle \mathbf{x}_0, N^-(\omega{\cdot}t)\,\mathbf{x}_0\rangle$. So, follow again the proof of Lemma 8.24, but now with the following changes:

– In relations (8.12) and (8.14) substitute M^- by N^- and K^- by $\widetilde{K}^- = R^{-1}(-g^T + B^T N^-)$.
– Deduce that the solution $\mathbf{x}(t)$ tends to $\mathbf{0}$ as $t \to -\infty$ from the fact that $\left[\begin{smallmatrix} \mathbf{x}(t) \\ N^-(\omega\cdot t)\,\mathbf{x}(t) \end{smallmatrix}\right]$ solves (8.6) with $\left[\begin{smallmatrix} \mathbf{x}_0 \\ N^-(\omega)\,\mathbf{x}_0 \end{smallmatrix}\right] \in \widetilde{l}^-(\omega)$, which ensures that its Lyapunov exponent is negative. (Note that now one does not have the *uniform* exponential convergence to $\mathbf{0}$ as $t \to -\infty$ of all the solutions of the all the corresponding equations (8.14), but this is not required.)

This completes the proof of (8.20). The next step is to prove the equivalence between (1) and (2).

So, assume (1), apply Theorem 8.15 in order to show that $N^+(\omega) \geq 0$ for all $\omega \in \Omega_0$, take $\omega \in \Omega_0$ with dense σ-orbit (which is possible, as Proposition 1.12 guarantees), and use the upper semicontinuity of N^- ensured by Theorem 5.43 to conclude that N^- is globally positive semidefinite.

The converse implication follows from (8.21) and Theorem 8.15.

To prove the last assertions of point (ii), recall that N^- solves the Riccati equation (8.9) along the base flow (see Sect. 5.4), and apply Proposition 8.17(i) to conclude that $\widetilde{V}_\omega(t, \mathbf{x})$ is a storage function for all $\omega \in \Omega$. Note finally that the continuity of $\widetilde{V}_\omega(t,\mathbf{x}) = \langle \mathbf{x}, N^-(\omega{\cdot}t)\,\mathbf{x}\rangle$ for each $\omega \in \Omega_0$ follows from that of $t \mapsto N^-(\omega{\cdot}t)$.

Observe that, due to the upper semicontinuity, the condition $N^- \geq 0$ is equivalent to the apparently less restrictive property of existence of a point $\omega \in \Omega$ with dense σ-orbit such that $N^-(\omega{\cdot}t) \geq 0$ for all $t \in \mathbb{R}$.

Example 8.35 Although in many situations the uniform weak disconjugacy and the exponential dichotomy of the family (8.6) hold or not simultaneously, this is of course not always the case. There are trivial examples for which the uniform disconjugacy can be proved using Theorem 8.34 and not Theorem 8.22. For instance, consider $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix}\right] \mathbf{z}$, which is determined from $x' = -x + u$ and $\widetilde{\mathcal{Q}}(x,t) = (-x^2 + u^2)/2$, which does not have exponential dichotomy (its only eigenvalue is 0), and for which the principal functions are the constants $n^+ = n^- = 1$: this constant function is the only globally defined solution of the Riccati equation $m' = -(m-1)^2$.

Example 8.36 In order to construct an example for which the information about dissipativity is provided by Theorem 8.22 and not by Theorem 8.34, it is possible to follow an idea similar to the one behind Example 8.12. Take $n = 1$, $A \equiv -2$, $B \equiv \sqrt{2}$, $R \equiv 1$, $g \equiv 0$, and let G be any increasing continuous function satisfying

$$G(t) = \begin{cases} -1 & \text{if } t < 0\,, \\ \ \ 0 & \text{if } t > 1\,. \end{cases}$$

The linear Hamiltonian system associated to the corresponding LQ problem is

$$\mathbf{z}' = \begin{bmatrix} -2 & 2 \\ G(t) & 2 \end{bmatrix} \mathbf{z}\,,$$

and the hull Ω of the coefficient matrix is now the set

$$\Omega = \left\{ \begin{bmatrix} -2 & 2 \\ G_s(t) & 2 \end{bmatrix} \,\Big|\, s \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix} \right\},$$

with $G_s(t) = G(t+s)$. Note that the autonomous linear problem is $x' = -2x + \sqrt{2}\,u$, which is null controllable.

All the systems of the family are weakly disconjugate and have exponential dichotomy. To prove this in the case of $\mathbf{z}' = H_{-\infty}\,\mathbf{z} = \left[\begin{smallmatrix} -2 & 2 \\ -1 & 2 \end{smallmatrix}\right] \mathbf{z}$, which is autonomous, note for instance that conditions D1 and D2 of Chap. 5 are fulfilled (see Remark 5.19). In addition the corresponding Riccati equation, which is $m' = -2m^2 + 4m - 1$, has globally defined solutions: the constant functions $1 \pm \sqrt{2}/2$. This ensures that also D3 holds. In fact this constant system has also exponential dichotomy, and hence the Weyl and principal functions agree, with $m^+(-\infty) = n^+(-\infty) = 1 - \sqrt{2}/2$ and $m^-(-\infty) = n^-(-\infty) = 1 + \sqrt{2}/2$:

see e.g. Theorem 5.58. The situation is the same for $\mathbf{z}' = H_\infty\,\mathbf{z} = \left[\begin{smallmatrix} -2 & 2 \\ 0 & 2 \end{smallmatrix}\right]\mathbf{z}$, where now $m^+(\infty) = n^+(\infty) = 0$ and $m^-(\infty) = n^-(\infty) = 4$. And, in the case of $\mathbf{z}' = H_s(t)\,\mathbf{z} = \left[\begin{smallmatrix} -2 & 2 \\ G_s(t) & 2 \end{smallmatrix}\right]\mathbf{z}$ for $s \in \mathbb{R}$, one can: firstly apply (twice) Proposition 5.51, since $JH_\infty \le JH_s(t) \le JH_{-\infty}$, in order to conclude that the principal functions $n^\pm(s)$ exist with $0 \le n^+(s) \le 1 - \sqrt{2}/2 < 1 + \sqrt{2}/2 \le n^-(s) \le 4$; and secondly deduce from Theorem 5.58 the presence of exponential dichotomy, with $m^\pm(s) = n^\pm(s)$. According to Theorem 1.60, the whole family of linear Hamiltonian systems has exponential dichotomy over Ω. (Incidentally, note that this last property ensures that $t \mapsto m^\pm(t)$ solve the Riccati equation $m' = -m^2 + 4m + G(t)$ associated to the initial Hamiltonian system, and that

$$\lim_{t\to-\infty} m^+(t) = 1 - \frac{\sqrt{2}}{2}\,, \qquad \lim_{t\to\infty} m^+(t) = 0\,,$$
$$\lim_{t\to-\infty} m^-(t) = 1 + \frac{\sqrt{2}}{2}\,, \qquad \lim_{t\to\infty} m^-(t) = 4\,;$$

moreover, Proposition 5.51 shows that $\mp m^\pm$ are nondecreasing functions.)

Therefore, Hypotheses 8.33 are fulfilled, as well as Hypotheses 8.19. However, the unique ergodic measures on Ω are those concentrated on its two proper minimal set: this follows from Birkhoff Theorem 1.3 and from the fact that these minimal sets are the alpha-limit and the omega-limit sets of any other element of Ω. Consequently, Ω does not admit an ergodic measure with full support, so that one cannot apply Theorem 8.34. On the other hand, nothing precludes an application of Theorem 8.23 to conclude that the family of LQ control problems is uniformly strictly dissipative, with strong storage function defined by

$$V_s(t,x) = \begin{cases} \dfrac{(2+\sqrt{2})\,x^2}{4} & \text{for } s = -\infty\,, \\[2ex] \dfrac{m^-(s+t)\,x^2}{2} & \text{for } s \in \mathbb{R}\,, \\[2ex] 2x^2 & \text{for } s = \infty\,, \end{cases}$$

which is jointly continuous with respect to (s,t,x). Of course, this means that every LQ systems of the family is strictly dissipative, including the initial one.

Remark 8.37 In contradistinction to the trivial situation described in Example 8.35, Example 8.44 presents a uniformly null controllable family of dissipative almost periodic LQ control problems over a minimal hull, for which the associated family of linear Hamiltonian systems does not have exponential dichotomy, and a stabilizing feedback control does not exist. However, the family is uniformly weakly disconjugate, and both principal functions are negative, so that it is again Theorem 8.34 (and not Theorem 8.22) which ensures the uniform dissipativity of the family. The example is of Millionščikov–Vinograd type ([104, 147]) and has a Lyapunov exponent with irregular behavior. Examples of this type have been mentioned several times in the previous chapters. Here, in Example 8.44, all the many details of the construction are given, in order to allow the reader to understand the idea behind this extremely complicated dynamical situation.

Remark 8.38 A natural question arises: is it possible to carry out an analysis similar to that of Sect. 8.5.2 in order to obtain the conclusions of Theorem 8.34 from hypotheses on a single system? The easiest situation corresponds, as usual, to the case of a minimal base flow. But even in this case one cannot ensure that if the Lyapunov exponents of the initial systems are different from zero, then the same happens on the hull. Example 8.44 is once more the tool to check whether or not this can happen: in fact, in this family there coexist systems with positive and null Lyapunov indexes.

8.6 In the Absence of Uniform Null Controllability

In this section a situation similar to that of Sect. 7.3.3 is considered: the uniform controllability condition is not required to obtain results on the dissipativity of all the systems of the family. However, these results are not as precise as in the previous section: they do not establish equivalences.

Recall that Proposition 8.17(i) shows that, in the case of existence, a globally defined solution of the Riccati equation (8.9) corresponding to a point $\omega \in \Omega$ which in addition is negative semidefinite provides a storage function for the LQ$_\omega$ problem (which therefore is dissipative), without extra controllability assumptions. This is the key point in the proofs of the next propositions.

Proposition 8.39 *Suppose that the family* (8.6) *admits exponential dichotomy and that the Weyl function* M^- *globally exists. Define*

$$V_\omega^-(t,\mathbf{x}) = \langle \mathbf{x}, M^-(\omega{\cdot}t)\,\mathbf{x}\rangle\,.$$

Then,

(i) *if* $M^-(\omega) \geq 0$ *for all* $\omega \in \Omega$*, then the family of control systems* (8.4) *is uniformly dissipative with family of supply rates* $\{\widetilde{Q}_\omega \mid \omega \in \Omega\}$ *given by* (8.5)*. In addition,* $V_\omega^-(t,\mathbf{x})$ *is a storage function for the LQ*$_\omega$ *control problem, and is jointly continuous in the variables* $(\omega, t, \mathbf{x})$.
(ii) *If* $M^-(\omega) > 0$ *for all* $\omega \in \Omega$*, then the family of control systems* (8.4) *is uniformly strictly dissipative with family of supply rates* $\{\widetilde{Q}_\omega \mid \omega \in \Omega\}$ *given by* (8.5)*. In addition, the storage function* $V_\omega^-(t,\mathbf{x})$ *is strong.*

Proof Since $M(t,\omega,M^-(\omega)) = M^-(\omega{\cdot}t)$, the assertions in (i) follow from Proposition 8.17(i) and Definition 8.18. In order to prove (ii), just repeat the arguments of the proof of (2)⇒(1) in the proof of Theorem 8.23.

Proposition 8.40 *Suppose that the family (8.6) is uniformly weakly disconjugate, and let N^- be its principal function at $-\infty$. Define*

$$\widetilde{V}_\omega^-(t,\mathbf{x}) = \langle \mathbf{x}, N^-(\omega{\cdot}t)\,\mathbf{x}\rangle\,.$$

Then, if $N^-(\omega) \ge 0$ for all $\omega \in \Omega$ (resp. $N^-(\omega) > 0$ for all $\omega \in \Omega$), then the family of control systems (8.4) is uniformly dissipative with family of supply rates $\{\widetilde{Q}_\omega \mid \omega \in \Omega\}$ given by (8.5). In addition, $\widetilde{V}_\omega^-(t,\mathbf{x})$ is a storage function (resp. strong storage function) for the LQ_ω control problem, and is continuous in the variables $(t,\mathbf{x})$.

Proof As in the previous result, the assertions follow from Proposition 8.17(i) and Definition 8.18.

Note that the hypotheses of the previous propositions do not require the uniform null controllability of the family (8.4). The trivial autonomous example $\mathbf{z}' = \left[\begin{smallmatrix} 1 & 0 \\ 2 & -1 \end{smallmatrix}\right]\mathbf{z}$ (which derives for instance from $A \equiv 1$, $B = g \equiv 0$, $G \equiv 2$, and any positive real number R) is a simple example which does not satisfy uniform null controllability (since the control system is $x' = x$), and for which $m^- = 1$ (and m^+ does not exist, since $l^+ \equiv \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$), so that it fits the hypotheses of Proposition 8.39. See Remark 7.35, Corollary 5.86, and Remarks 5.87 to recall once more the dynamical meaning of the absence of uniform null controllability.

The hypotheses of Proposition 8.40 also include the global existence of M^-, which cannot necessarily be asserted in the presence of exponential dichotomy without assuming the uniform null controllability: this is the situation in the autonomous example $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 0 \\ -2 & 1 \end{smallmatrix}\right]\mathbf{z}$ (coming for instance from $A \equiv -1$, $B = g \equiv 0$, $G \equiv -2$, and any $R > 0$), which satisfies the Frequency and Nonoscillation Conditions.

The following result, which is the last of the section, concerns a situation in which, in spite of the lack of the global existence of M^-, it is possible to establish conditions ensuring the uniform dissipativity (normal or strict) of the family. This results will be fundamental in the analysis of Examples 8.48 and 8.49. They illustrate a situation in which a uniformly dissipative family lacks "everything": uniform null controllability, exponential dichotomy, and uniform weak disconjugacy.

To formulate this result, consider the perturbed family

$$\mathbf{z}' = H_\varepsilon(\omega{\cdot}t)\,\mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.22}$$

where

$$H_\varepsilon(\omega) = \begin{bmatrix} A(\omega) - B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) + \varepsilon\Delta(\omega) \\ G(\omega) - g(\omega)R^{-1}(\omega)g^T(\omega) & -A^T(\omega) + g(\omega)R^{-1}(\omega)B^T(\omega) \end{bmatrix}$$

and where Δ is continuous and positive semidefinite on Ω.

Theorem 8.41 *Suppose that there exists an* $\varepsilon > 0$ *such that the Riccati equation* (8.9) *associated to the corresponding family* (8.22) *(obtained by replacing* H_3 *by* $H_3 + \varepsilon I_n$*) has a solution along the flow* M_ε *such that* $M_\varepsilon(\omega) \ge 0$ *for all* $\omega \in \Omega$*. Then the family of control systems* (8.4) *is uniformly dissipative with family of supply rates* $\{\widetilde{\mathcal{Q}}_\omega \mid \omega \in \Omega\}$ *given by* (8.5)*. In addition,* $V_\omega^\varepsilon(t, \mathbf{x}) = \langle \mathbf{x}, M_\varepsilon(\omega{\cdot}t)\,\mathbf{x}\rangle$ *is a storage function for the* LQ_ω *control problem, and it is continuous in* $(t, \mathbf{x})$.

Proof The main step of the proof is to check that, for all $t \ge 0$, all $\omega \in \Omega$ and all pairs $(\mathbf{x}(t), \mathbf{u}(t))$ solving (8.4),

$$\frac{d}{dt}\,V_\omega^\varepsilon(t, \mathbf{x}(t)) \le 2\,\widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}(t), \mathbf{u}(t))\,. \tag{8.23}$$

The proof of this fact reproduces step by step that of (7.49), although in this case the notation is simpler. Nevertheless, it is included here for the reader's convenience.

The inequality will be first established for $t = 0$. As usual, $M(t, \omega, M_\varepsilon(\omega))$ represents the solution of (8.9) with $\varepsilon = 0$ with initial data $M_\varepsilon(\omega)$. Relation (8.10) for $V_{\omega, M_\varepsilon(\omega)}(t, \mathbf{x}) = \langle \mathbf{x}, M(t, \omega, M_\varepsilon(\omega))\,\mathbf{x}\rangle$ yields

$$\frac{d}{dt}\,V_{\omega, M_\varepsilon(\omega)}(t, \mathbf{x}(t)) \le 2\widetilde{\mathcal{Q}}_\omega(t, \mathbf{x}(t), \mathbf{u}(t))\,.$$

It is easy to deduce from the fact that $M(0, \omega, M_\varepsilon(\omega)) = M_\varepsilon(\omega)$ and from the Riccati equations (8.9) and its analogue with H_3 replaced by $H_3 + \varepsilon I_n$, which $M_\varepsilon(\omega)$ satisfies along the base flow, that

$$M_\varepsilon'(\omega) = M'(t, \omega, M_\varepsilon(\omega))\,\big|_{t=0} - \varepsilon M_\varepsilon(\omega)\,\Delta(\omega)\,M_\varepsilon(\omega) \le M'(t, \omega, M_\varepsilon(\omega))\,\big|_{t=0}$$

and hence, using again the equality $M(0, \omega, M_\varepsilon(\omega)) = M_\varepsilon(\omega)$, that

$$\frac{d}{dt}\,V_\omega^\varepsilon(t, \mathbf{x}(t))\,\bigg|_{t=0} \le \frac{d}{dt}\,V_{\omega, M_\varepsilon(\omega)}(t, \mathbf{x}(t))\,\bigg|_{t=0} \le 2\widetilde{\mathcal{Q}}_\omega(0, \mathbf{x}(0), \mathbf{u}(0))\,.$$

This proves (8.23) for $t = 0$, and all $\omega \in \Omega$ and all pairs $(\mathbf{x}, \mathbf{u})$ solving (8.4).

Now, given $s \in \mathbb{R}$, define $\mathbf{x}_s(t) = \mathbf{x}(s + t)$ and $\mathbf{u}_s(t) = \mathbf{u}(s + t)$ and note that the pair $(\mathbf{x}_s, \mathbf{u}_s)$ solves (8.4) for $\omega{\cdot}s$. It is easy to check that $(d/dt)V_\omega(t, \mathbf{x}(t))|_{t=s} = (d/dt)V_{\omega\cdot s}(t, \mathbf{x}_s(t))|_{t=0}$, which ensures that

$$\frac{d}{dt}\,V_\omega^\varepsilon(t, \mathbf{x}(t))\,\bigg|_{t=s} \le 2\widetilde{\mathcal{Q}}_{\omega\cdot s}(0, \mathbf{x}_s(0), \mathbf{u}_s(0)) = 2\,\widetilde{\mathcal{Q}}_\omega(s, \mathbf{x}(s), \mathbf{u}(s))\,.$$

This completes the proof of (8.23).

The assertions of the theorem can be now proved: V_ω^ε is nonnegative, and integrating (8.23) at any interval $[t_1, t_2]$ shows that it is a storage function. This means that all the systems of the family are dissipative. And the continuity of

$(\omega, t, \mathbf{x}) \mapsto V^{\varepsilon}_{\omega}(t, \mathbf{x})$ with respect to (t, x) is an immediate consequence of that of $t \mapsto M_{\varepsilon}(\omega{\cdot}t)$.

Remark 8.42 Of course, there are two basic situations in which the Riccati equation associated to the perturbed family (8.22) admits a (not necessarily positive) solution along the flow, which is one of the hypotheses required in Theorem 8.41: when it has exponential dichotomy over Ω and at least one of the Weyl functions exists; and when, despite the lack of exponential dichotomy, the family is uniformly weakly disconjugate, so that the principal functions exist (see Theorem 5.58). But a fundamental question arises: is it possible to establish conditions on the unperturbed system of the family ensuring that one of these situations holds? The next three paragraphs give partial answers to this question.

Take first $\Delta \equiv I_n$. Suppose that there is a σ-ergodic measure m_0 on Ω with $\operatorname{Supp} m_0 = \Omega$; that the unperturbed family (8.6) (i.e. (8.22) with $\varepsilon = 0$) admits an exponential dichotomy; and that its rotation number with respect to m_0 (see Chap. 2 to review this concept) is $\alpha(m_0) = 0$. Theorem 5.73 provides $\rho > 0$ such that the family (8.22) has exponential dichotomy over Ω for $\varepsilon \in [0, \rho)$, and such that one has global existence of the Weyl functions $M^{\pm}_{\varepsilon}(\omega)$ for $\varepsilon \in (0, \rho)$ with

$$M^{+}_{\varepsilon_1}(\omega) \le M^{+}_{\varepsilon_2}(\omega) < M^{-}_{\varepsilon_2}(\omega) \le M^{-}_{\varepsilon_1}(\omega)$$

whenever $0 < \varepsilon_1 < \varepsilon_2 < \rho$. Thus if for a value of ε the Weyl function M^{-}_{ε} is positive semidefinite, then the hypotheses of Theorem 8.41 are fulfilled, and one of the theses of that theorem is improved: the storage function provided by M^{-}_{ε} is continuous in its three variables.

Something more can be said in this case: if $M^{-}_{\varepsilon}(\omega) > 0$ for all $\omega \in \Omega$, then the family of control systems (8.4) is uniformly strictly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_{\omega} \mid \omega \in \Omega\}$ given by (8.5); and the storage function $V^{\varepsilon}_{\omega}(t, \mathbf{x})$ is strong. In order to prove this, act as in Proposition 8.39(ii): repeat the arguments of (2)$\Rightarrow$(1) in the proof of Theorem 8.23, now combined with the ideas of the proof of Theorem 8.41.

The other "easy" case of applicability of Theorem 8.41 corresponds to the case in which: the perturbed family is uniformly weakly disconjugate, so that the principal functions exist; and the greater one, N^{-}_{ε}, is positive semidefinite. To this end, note that the family (8.22) satisfies $H_{\varepsilon,3} > 0$ if $\Delta > 0$, which suffices to guarantee conditions D1 and D2 (see Remark 5.19), so that there is at least a good chance to obtain a uniformly weakly disconjugate family. For instance, this is the case if $H_2 = G - gR^{-1}g^T$ is positive semidefinite (see Proposition 5.27). Of course, in general, the storage function now provided by N^{-}_{ε} will not be continuous with respect to ω.

8.7 Millionščikov–Vinograd Type Examples

Three examples which were previously announced are discussed in this section. The first one contains a detailed construction of Millionščikov–Vinograd type, and provides a scalar nonautonomous LQ control problem for which the following properties v1–v7 hold. The meaning of the term nonuniformly hyperbolic dynamics is explained in Remark 1.79, the definition of the Sacker–Sell spectrum is given in Sect. 1.4.4, and the concepts of uniform weak disconjugacy, and principal solutions and functions are explained in Chap. 5.

v1. The hull is minimal and uniquely ergodic.

v2. The Sacker–Sell spectrum of the associated family of Hamiltonian systems is $[-\beta_\infty, \beta_\infty]$ for a number $\beta_\infty > 0$. That is, the family does not have exponential dichotomy but its Lyapunov index (with respect to the unique ergodic measure) is positive: see Sect. 1.4.4. In other words, its dynamics is in the nonuniformly hyperbolic case. In addition, the rotation number of the family of linear Hamiltonian systems is zero.

v3. The family of Hamiltonian systems is uniformly weakly disconjugate. In addition, the principal functions are noncontinuous, and agree on the residual set of their continuity points but are distinct on a set of full measure. In particular, they determine an almost-automorphic extension of the base flow which is not a copy of the base.

v4. The family of LQ control problems on the hull is uniformly null controllable, and the uniform stabilization Hypothesis 7.3 is satisfied.

v5. All the LQ control problems of the family are dissipative and have a strong storage function.

v6. A stabilizing feedback control cannot be always constructed from the initial data by applying the method explained in Chap. 6.

v7. The infinite-horizon optimal control problems associated to the LQ control problems are not solvable for the points of a residual subset of the hull, but they are solvable for all the points in a subset of full measure of the hull.

The other two examples, which are based on the first one, illustrate how Theorem 8.41 can be applied to study the dissipativity of some nonautonomous systems without uniform null controllability.

Remark 8.43 The result to be explained now will be used twice in the next example. Consider a two-dimensional linear differential system $\mathbf{z}' = H(t)\,\mathbf{z}$ given by a T-periodic matrix-valued function H satisfying conditions which ensure the existence, uniqueness, and continuous variation with respect to initial data of the solutions of the system. Represent by $\mathbf{z}(t, \mathbf{z}_\theta)$ the solution with initial data $\mathbf{z}(0, \mathbf{z}_\theta) = \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$, and write it as $\mathbf{z}(t, \mathbf{z}_\theta) = r(t,\theta)\left[\begin{smallmatrix} \cos\varphi(t,\theta) \\ \sin\varphi(t,\theta) \end{smallmatrix}\right]$. It is simple to derive the scalar differential equations satisfied by $\varphi(t,\theta)$ and $r(t,\theta)$, although the particular expressions are not important for what follows. Write the first one as

$$\varphi' = h(t, \varphi)\,,$$

so that h is T-periodic in t. Assume that this equation has at least three distinct T-periodic solutions given by initial angles θ_1, θ_2, and θ_3 determining three different lines passing through $\mathbf{0}$ in the $\mathbf{z}$-plane. Then all its solutions are T-periodic. This is the property which will be required later.

To check it, note that $\varphi(T, \theta_i) = \theta_i$ for $i = 1, 2, 3$. Hence, if $U(t)$ is the fundamental matrix-valued solution of the Hamiltonian system with $U(0) = I_2$, then $U(T) \left[\begin{smallmatrix} \cos\theta_i \\ \sin\theta_i \end{smallmatrix}\right] = r(T, \theta_i) \left[\begin{smallmatrix} \cos\theta_i \\ \sin\theta_i \end{smallmatrix}\right]$ for $i = 1, 2, 3$. But this is only possible if two of the three numbers $r(T, \theta_i)$ take the same value λ, in which case λ is an eigenvalue for $U(T)$ with two linearly independent associated eigenvectors; that is, the only possibility is $U(T) = \lambda\, I_2$ for a constant λ. This means that $U(T) \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right] = \lambda \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$. In particular, for every $\theta \in [0, 2\pi)$ there exists an integer number k_θ such that $\varphi(T, \theta) = \theta + k_\theta \pi$. Now, given the point θ_1 previously used, define $\mathcal{I}$ as the largest neighborhood of θ_1 in $[0, 2\pi)$ composed by those values of θ for which $\varphi(T, \theta) = \theta$. The continuous variation with respect to the initial datum of the solutions of the angular equation ensures that $\mathcal{I}$ is open and closed in $[0, 2\pi)$, so that both intervals coincide. This proves the assertion.

Incidentally, note that if, in addition to the initial hypotheses, H defines a linear Hamiltonian system (that is, if $\operatorname{tr} H \equiv 0$), then $\det U \equiv 1$ (see Sect. 1.2), so that either $\lambda = 1$ or $\lambda = -1$. Since the equation $\varphi' = h(t, \varphi)$ has at least one T-periodic solution, the conclusion is that $U(T) = I_2$, which ensures that all the solutions of the linear Hamiltonian system are T-periodic.

Example 8.44 This scalar example consists of a family of LQ control problems defined over the Bebutov hull of an initial control problem, for which conditions v1–v7 are satisfied. In particular they are uniformly null controllable, and the associated family of Hamiltonian systems is uniformly weakly disconjugate, but it does not have exponential dichotomy over the hull. Therefore, Theorem 8.22 does not provide useful information, whereas Theorem 8.34 ensures the uniform dissipativity of the family. In fact, despite the uniform null controllability of the control problems, a stabilizing feedback control cannot be always constructed. The example, which is of Millionščikov–Vinograd type, presents a case of nonuniformly hyperbolic dynamics.

All the angles will be expressed in radians. To begin with, define

$$A \equiv -1\,, \quad B \equiv 1\,, \quad G(t) = g^2(t) - 1\,, \quad \text{and} \quad R \equiv 1$$

where the function $g(t)$ remains to be determined. These data give rise to the scalar control system $x' = -x + u$ and to the quadratic form

$$\mathcal{Q}(t, x, u) = \frac{1}{2}\,(G(t)\,x^2 + 2\,g(t)\,x\,u + u^2)\,.$$

The linear Hamiltonian system associated to this LQ control problem is

$$\mathbf{z}' = \begin{bmatrix} -1 - g(t) & 1 \\ -1 & 1 + g(t) \end{bmatrix} \mathbf{z}\,. \tag{8.24}$$

where $\mathbf{z} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2$. The construction to follow will produce a Bohr almost periodic function $g(t)$ and a corresponding (also Bohr almost periodic) function $G(t) = g^2(t) - 1$ with appropriate properties. In particular the family of linear Hamiltonian systems (8.6) over the (minimal) Bebutov hull will be uniformly weakly disconjugate but will not admit an exponential dichotomy. In fact, g will be the uniform limit on $\mathbb{R}$ of a sequence $\{g_k\}$ of T_k-periodic functions, where $T_k = j_k T_{k-1}$ for a positive integer j_k, for $k = 1, 2, \ldots$ Hence g will be a so-called limit-periodic function. And the corresponding system will display nonuniform hyperbolicity: it will have positive Lyapunov index in the absence of exponential dichotomy; or, in other words, its Sacker–Sell spectrum will be given by a nondegenerate interval centered at 0. The construction makes use of the well-known procedure of Millionščikov ([104]; see also Vinograd [147]) which has been applied by later authors in various contexts (e.g. [31, 69]).

Although the idea behind the construction of g is simple, to formalize it requires much work. This is the reason for which the construction is carried out in several steps.

Step 0. Let t_0 and T_0 be real numbers to be determined, with $0 < t_0 < T_0$. Set

$$\gamma_0(t) = \begin{cases} 0.1 & \text{if } 0 \le t < t_0\,, \\ -1 & \text{if } t_0 \le t < T_0\,, \end{cases}$$

and extend $\gamma_0(t)$ to be a T_0-periodic function on $\mathbb{R}$. Substituting g by γ_0 in (8.24), one obtains the T_0-periodic differential system which satisfies

$$\mathbf{z}' = \begin{cases} H^1\,\mathbf{z} = \begin{bmatrix} -1.1 & 1 \\ -1 & 1.1 \end{bmatrix}\mathbf{z} & \text{if } 0 \le t < t_0\,, \\ H^2\,\mathbf{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{z} & \text{if } t_0 \le t < T_0\,. \end{cases} \tag{8.25}$$

The first basic idea is that there exist two half-lines in the first quadrant of the $\mathbf{z}$-plane, determined by angles $\vartheta_0^+ < \vartheta_0^-$ in $(0, \pi/2)$, which delimit an open sector with the property that the orbit under (8.25) of any initial state in that sector satisfies the following properties: its argument strictly increases on $[0, t_0]$ and reaches a value as close to ϑ_0^- as desired if t_0 is large enough; its modulus is as large as desired if t_0 is large enough; and it moves along a circle centered at $\mathbf{0}$ in the clockwise sense and at angular speed 1 on $[t_0, T_0]$, so that it reaches ϑ_0^+ if $T_0 - t_0$ is large enough. To formalize this idea is the first purpose of this initial step.

Note that the matrix H^1 has eigenvalues $\pm\eta_0$ for

$$-\,\eta_0 = \sqrt{0.21} > 0.4 \tag{8.26}$$

(the anticorrespondence between signs is intentional). Therefore, the phase space structure of $\mathbf{z}' = H^1\,\mathbf{z}$ is that of a saddlepoint. In order to describe it a little better, let $\mathbf{z}_0^{\pm}$ be the normalized eigenvectors of H^1 associated to $\pm\eta_0$ which lie in the first quadrant of the $\mathbf{z}$-plane: explicitly,

$$\mathbf{z}_0^+ = r^+ \begin{bmatrix} 1 \\ 1.1 - \sqrt{0.21} \end{bmatrix} \quad \text{and} \quad \mathbf{z}_0^- = r^- \begin{bmatrix} 1 \\ 1.1 + \sqrt{0.21} \end{bmatrix}.$$

for $r^{\pm} = (2.42 \mp 2.2\sqrt{0.21})^{-1/2}$. Let $\vartheta_0^{\mp} \in [0, 2\pi]$ be the polar angles of $\mathbf{z}_0^{\mp}$ (in radians). It is obvious that $0 < \vartheta_0^+ < \vartheta_0^- < \pi/2$.

There is another way to determine the same angles ϑ_0^+ and ϑ_0^-. Denote $\mathbf{z}_\theta = \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$ (so that $\mathbf{z}_0^{\mp} = \mathbf{z}_{\vartheta_0^{\mp}}$) and write the solution of (8.25) starting at $\mathbf{z}_\theta$ in time 0 as

$$\mathbf{z}(t, \mathbf{z}_\theta) = r(t, \theta) \begin{bmatrix} \cos\varphi(t, \theta) \\ \sin\varphi(t, \theta) \end{bmatrix}.$$

It is a trivial exercise to check that $\varphi(t, \theta)$ is the solution of

$$\varphi' = (1 + \gamma_0(t))\sin(2\varphi) - 1 \tag{8.27}$$

with $\varphi(0, \theta) = \theta$. Note that the equation reduces to $\varphi' = -1$ in $(t_0, T_0]$, which ensures that

$$\varphi(t, \theta) = \varphi(t - t_0, \varphi(t_0, \theta)) = \varphi(t_0, \theta) - t + t_0 \qquad \text{for } t \in [t_0, T_0]\,; \tag{8.28}$$

and it reduces to $\varphi' = 1.1\sin(2\varphi) - 1$ in $(0, t_0]$, so that there exist two angles ϑ_0^+ and ϑ_0^- in $(0, \pi/2)$ with $\vartheta_0^+ < \pi/4 < \vartheta_0^-$ which determine constant solutions on $[0, t_0]$. The angles $\vartheta_0^{\pm}$ solve $1.1\sin(2\varphi) - 1 = 0$, that is, $\vartheta_0^{\pm} = (1/2)\arcsin(1/(1.1))$. For later reference, note that

$$\sin(2\,\vartheta_0^{\pm}) = \frac{1}{1.1} > 0.9. \tag{8.29}$$

Consequently,

$$0.5 < \vartheta_0^+ < 0.6 < \vartheta_0^- < 1.1 - \delta \quad \text{and} \quad \vartheta_0^- - \vartheta_0^+ > 0.4 + \delta\,, \tag{8.30}$$

where

$$\delta = 2^{-10}\,.$$

This constant δ is fixed for the rest of the example. Note that $\vartheta_0^+ + \delta < \vartheta_0^-$. Of course, $\vartheta_0^{\pm}$ are the same angles as before.

For later purposes, observe that $r(t,\theta)$ is the solution of the equation

$$r' = -(1 + \gamma_0(t))\cos(2\,\varphi(t,\theta))\,r$$

with $r(0,\theta) = 1$, where $\varphi(t,\theta)$ is as above. This equation reduces to $r' = 0$ in $[t_0, T_0]$, so that $r(t,\theta) = r(t_0,\theta)$ for $t \in [t_0, T_0]$: the phase space structure of $\mathbf{z}' = H^2\,\mathbf{z}$ is that of a center, with orbits given by circles centered at $\mathbf{0}$.

The next goal is to prove that a suitable choice of t_0 and T_0 provides two angles $\bar{\theta}_0^+$ and $\bar{\theta}_0^-$ which determine two T_0-periodic solutions of (8.27) with some additional characteristics. More precisely, the following properties will hold:

1. $\vartheta_0^+ < \bar{\theta}_0^+ < \bar{\theta}_0^- < \vartheta_0^+ + \delta$.
2. $\varphi(T_0, \bar{\theta}_0^\mp) = \bar{\theta}_0^\mp$.
3. $r(T_0, \bar{\theta}_0^\mp) = e^{\pm\widetilde{\beta}_0 T_0}$, with $\widetilde{\beta}_0 > 0.35$.
4. $\vartheta_0^+ < \varphi(t, \bar{\theta}_0^+) < \varphi(t, \bar{\theta}_0^-) < \vartheta_0^-$ for all $t \in \mathbb{R}$.

Note that points 2 and 3 can be written as

$$\mathbf{z}(T_0, \mathbf{z}_{\bar{\theta}_0^\mp}) = e^{\pm\widetilde{\beta}_0 T_0}\mathbf{z}_{\bar{\theta}_0^\mp} \quad \text{with } \widetilde{\beta}_0 > 0.35\,. \tag{8.31}$$

Let $\mathcal{V}_0$ be the closed sector in the $\mathbf{z}$-plane whose vertex is $\mathbf{0}$ and whose bounding rays are $\lambda_0^\mp = \{c\,\mathbf{z}_0^\mp \mid c \ge 0\}$. The set $\mathcal{V}_0$ is invariant under the fundamental matrix solution $\exp(tH^1)$ of equation (8.25) for $0 \le t < t_0$. Note that the vectors on the unit circle belonging to $\mathcal{V}_0$ are of the form $\mathbf{z}_\theta$ for $\theta \in [\vartheta_0^+, \vartheta_0^-]$, and that any θ in the interior of the interval defines a solution $\varphi(t,\theta)$ of (8.27) which strictly increases towards ϑ_0^-, and which has the property that $\varphi(t_0,\theta)$ is as close to ϑ_0^- as desired if t_0 is large enough. On the other hand, $r(t,\theta)$ decreases while $\varphi(t,\theta) \in (\vartheta_0^+, \pi/4)$ and increases for $\varphi(t,\theta) \in (\pi/4, \vartheta_0^-)$; and $r(t_0,\theta)$ is as large as desired if t_0 is large enough. Choose $\varepsilon > 0$ such that $-\eta_0 - \varepsilon > 0.4$, and a time $t_0 \ge 6$ such that

$$0 < \vartheta_0^- - \varphi(t_0,\theta) \le \delta/4 \quad \text{and} \quad r(t_0,\theta) \ge e^{(-\eta_0-\varepsilon)\,t_0} \tag{8.32}$$

for all $\theta \in [\vartheta_0^+ + \delta/8, \vartheta_0^-]$: it is enough to take t_0 such that (8.32) holds for $\theta = \vartheta_0^+ + \delta/8$. In particular, for all $\theta \in (\vartheta_0^+ + \delta/8, \vartheta_0^+ + \delta/4)$,

$$\varphi(t_0,\theta) - \theta > \vartheta_0^- - \frac{\delta}{4} - \vartheta_0^+ - \frac{\delta}{4} = \vartheta_0^- - \vartheta_0^+ - \frac{\delta}{2} > 0\,.$$

Define now

$$f\colon [\vartheta_0^+, \vartheta_0^-] \to \mathbb{R}\,, \quad \theta \mapsto \varphi(t_0,\theta) - \theta\,,$$

which is continuous and satisfies

$$f(\vartheta_0^\pm) = 0 \quad \text{and} \quad f(\theta) > \vartheta_0^- - \vartheta_0^+ - \frac{\delta}{2} \quad \text{for all } \theta \in \left(\vartheta_0^+ + \frac{\delta}{8},\ \vartheta_0^+ + \frac{\delta}{4}\right),$$

and choose $\bar{\theta}_0^+ \in (\vartheta_0^+, \vartheta_0^+ + \delta/8)$ and $\bar{\theta}_0^- \in (\vartheta_0^+ + \delta/4, \vartheta_0^-)$ to be the abscissae of the two intersection points of the graph of f and the line of height $\vartheta_0^- - \vartheta_0^+ - \delta$ (which are uniquely determined, as is explained below); i.e.

$$\varphi(t_0, \bar{\theta}_0^\pm) - \bar{\theta}_0^\pm = \vartheta_0^- - \vartheta_0^+ - \delta\,. \tag{8.33}$$

Note that

$$\bar{\theta}_0^- - \vartheta_0^+ = \varphi(t_0, \bar{\theta}_0^-) - \vartheta_0^- + \delta < \delta\,,$$

so that $\vartheta_0^+ < \bar{\theta}_0^+ < \bar{\theta}_0^- < \vartheta_0^+ + \delta$, which proves property 1. Then choose

$$T_0 = t_0 + \vartheta_0^- - \vartheta_0^+ - \delta\,,$$

and note for future reference that $T_0 < t_0 + 1.1 - 0.5 \leq 1.1\,t_0$ (since the first inequalities in (8.30) hold and $t_0 \geq 6$) and that $T_0 - t_0 > 0.4$ (due to the second assertion in (8.30)). The equalities (8.28) for $t = T_0$ and (8.33), together with the definition of T_0 yield

$$\varphi(T_0, \bar{\theta}_0^\mp) = \varphi(t_0, \bar{\theta}_0^\mp) - T_0 + t_0 = \bar{\theta}_0^\pm + \vartheta_0^- - \vartheta_0^+ - \delta - T_0 + t_0 = \bar{\theta}_0^\mp, \tag{8.34}$$

so that property 2 holds. As a matter of fact, relation (8.34) is satisfied for a point $\bar{\theta}$ if and only if $\varphi(t_0, \bar{\theta}) - \bar{\theta} = \vartheta_0^- - \vartheta_0^+ - \delta$, that is, if and only if $f(\bar{\theta}) = \vartheta_0^- - \vartheta_0^+ - \delta$; and this proves the uniqueness of $\bar{\theta}_0^\pm$ which was asserted before: if there were at least three points with this property, then all the solutions of the equation (8.27) would be T_0-periodic (see Remark 8.43), and hence the function f would take the constant value $\vartheta_0^- - \vartheta_0^+ - \delta$, which is not true.

Now observe that $\bar{\theta}_0^- > \vartheta_0^+ + \delta/4$, so that (8.32) implies that

$$r(T_0, \bar{\theta}_0^-) = r(t_0, \bar{\theta}_0^-) \geq e^{(-\eta_0 - \varepsilon)\,t_0} = e^{\widetilde{\beta}_0 T_0}$$

for $\widetilde{\beta}_0 = (-\eta_0 - \varepsilon)\,t_0/T_0 > 0.4\,t_0/(1.1\,t_0) > 0.35$. These properties prove point 3 for $\bar{\theta}_0^-$.

To verify property 3 also for $\bar{\theta}_0^+$ note that $\det U(T_0) = 1$, and that $e^{\widetilde{\beta}_0 T_0}$ is one of its eigenvalues (see (8.31), which is already known for $\bar{\theta}_0^-$), so that the other one is $e^{-\widetilde{\beta}_0 T_0}$. This fact together with $U(T_0)\begin{bmatrix}\cos\bar{\theta}_0^+\\ \sin\bar{\theta}_0^+\end{bmatrix} = r(T_0, \bar{\theta}_0^+)\begin{bmatrix}\cos\varphi(T_0,\bar{\theta}_0^+)\\ \sin\varphi(T_0,\bar{\theta}_0^+)\end{bmatrix} = r(T_0, \bar{\theta}_0^+)\begin{bmatrix}\cos\bar{\theta}_0^+\\ \sin\bar{\theta}_0^+\end{bmatrix}$ (here (8.34) is used) shows that $r(T_0, \bar{\theta}_0^+) = e^{-\widetilde{\beta}_0 T_0}$, as required. The proof of point 3 is hence complete.

To complete this part of the proof, note also that the orbits of the solutions of (8.25) starting at $\mathbf{z}_{\bar{\theta}_0^\mp}$ lie in the interior of the sector $\mathcal{V}_0$ for all $t \in [0, t_0]$, so that their argument decreases for $t \in [t_0, T_0]$. This fact together with (8.34) ensures property 4.

The initial step is completed by modifying $\gamma_0(t)$ in the following way: take $\varepsilon_0 > 0$ small, and define g_0 as the simplest piecewise linear continuous T_0-periodic function taking the values 0.1 on $[0, t_0]$ and -1 on $[t_0+\varepsilon_0, T_0-\varepsilon_0]$ (so that its graph on $[0, T_0]$ is formed by four segments). Hence,

5_{10}. $|g_0(t)| \le 1$ for all $t \in \mathbb{R}$.

The reason for the choice of this label and those to follow will be clear at the next step. In addition, it is possible to take ε_0 small enough to guarantee that the linear Hamiltonian system

$$\mathbf{z}' = \begin{bmatrix} -1 - g_0(t) & 1 \\ -1 & 1 + g_0(t) \end{bmatrix} \mathbf{z} \tag{8.35}$$

has the following properties:

6_{10}. If $U_0(t)$ is the fundamental matrix solution of (8.35) with $U_0(0) = I_2$, then there exist $\beta_0 > 0.35$ and two angles $\theta_0^\pm \in (0, \pi/2)$ such that

$$U_0(T_0)\,\mathbf{z}_{\theta_0^\mp} = e^{\pm\beta_0 T_0}\mathbf{z}_{\theta_0^\mp}\,.$$

7_{10}. $\vartheta_0^+ < \theta_0^+ < \theta_0^- < \vartheta_0^+ + \delta$, and hence $0 < \theta_0^- - \theta_0^+ < 2^{-10}$.

8_{10}. $0.5 < \vartheta_0^+ < \varphi_0(t, \theta_0^+) < \varphi_0(t, \theta_0^-) < 1.1$ for all $t \in \mathbb{R}$, where $\varphi_0(t, \theta)$ represents the solution of

$$\varphi' = (1 + g_0(t))\sin(2\varphi) - 1 \tag{8.36}$$

with $\varphi_0(t, \theta) = \theta$.

The main point in proving this assertion is to check that $\|U_0(t) - U(t)\|$ is as small as desired for all $t \in [0, T_0]$ if ε_0 is small enough, for which it may be convenient to fix the Euclidean norm: see Remark 1.24.2. This can be done by means of the Gronwall lemma. To be more precise, write the systems (8.25) and (8.35) as $\mathbf{z}' = H(t)\,\mathbf{z}$ and $\mathbf{z}' = H_0(t)\,\mathbf{z}$ and note that

$$U_0'(t) - U'(t) = H_0(t)\,(U_0(t) - U(t)) + (H_0(t) - H(t))\,U(t)\,,$$

so that $\|U(t) - U_0(t)\| = 0$ for $t \in [0, t_0]$ and

$$\|U_0(t) - U(t)\| \le \int_{t_0}^t \|H_0(s)\|\,\|U_0(s) - U(s)\|\,ds + \int_{t_0}^t \|H_0(s) - H(s)\|\,\|U(s)\|\,ds$$

for $t \in [t_0, T_0]$; and then apply the facts that $\|H_0(s)-H(s)\| = 0$ for $t \in [t_0+\varepsilon_0, T_0-\varepsilon_0]$ and

$$\int_{t_0}^{t_0+\varepsilon_0} \|H_0(s) - H(s)\| \, \|U(s)\| \, ds + \int_{T_0-\varepsilon_0}^{T_0} \|H(s) - H_0(s)\| \, \|U(s)\| \, ds \le c_1\, \varepsilon_0$$

for some constant $c_1 > 0$ which is independent of ε_0, together with the Gronwall lemma, in order to deduce that $\|U_0(t) - U(t)\| \le c_2\, \varepsilon_0$ for all $t \in [0, T_0]$, where c_2 is independent of ε_0. This is the announced property.

Now note that properties 1 and 4 provide $\rho > 0$ such that $\vartheta_0^+ < \bar{\theta}_0^+ - \rho$ and $\vartheta_0^+ \le \varphi(t,\theta) \le \vartheta_0^-$ for all $(t,\theta) \in [0, T_0] \times [\bar{\theta}_0^+ - \rho, \bar{\theta}_0^- + \rho]$. In addition, the bound of $\|U_0(t) - U(t)\|$ shows that, if ε_0 is small enough, then $|\varphi(t,\theta) - \varphi_0(t,\theta)| < \delta$ for $(t,\theta) \in [0, T_0] \times [\bar{\theta}_0^+ - \rho, \bar{\theta}_0^- + \rho]$. And it is easy to deduce directly from the bound $\|U(T_0) - U_0(T_0)\| \le c_2\varepsilon_0$ that the eigenvalues and the arguments $\theta_0^\pm$ of the lines of corresponding eigenvectors of the matrices $U(T_0)$ and $U_0(T_0)$ (where one chooses $\theta_0^+ < \theta_0^-$) are as close as desired by taking ε_0 small enough: use for instance the fact that $\det U_0(T_0) = \det U(T_0) = 1$. Therefore, there exists $\varepsilon_0 > 0$ small enough to ensure that properties 6_{10} and 7_{10} hold and, in addition, that $\theta_0^\pm \in [\bar{\theta}_0^+ - \rho, \bar{\theta}_0^- + \rho] \subset (\vartheta_0^+, \bar{\theta}_0^- + \rho]$. Consequently,

$$\vartheta_0^+ - \delta < \varphi_0(t, \theta_0^+) < \varphi_0(t, \theta_0^-) < \vartheta_0^- + \delta$$

for all $t \in [0, T_0]$. It is possible to say something more: it is already known that $\theta_0^+ > \vartheta_0^+$, and it is easy to deduce from the properties of (8.36) and from $\varphi_0(T_0, \theta_0^+) = \theta_0^+$ (which follows from 6_{10}) that $\varphi_0(t, \theta_0^+) > \theta_0^+$ for all $t \in (0, T_0)$. Therefore,

$$\vartheta_0^+ < \varphi_0(t, \theta_0^+) < \varphi_0(t, \theta_0^-) < \vartheta_0^- + \delta$$

for all $t \in [0, T_0]$, which together with (8.30) and the T_0-periodicity of $\varphi_0(t, \theta_0^\pm)$ (which is ensured by 6_{10}) prove 8_{10}. This completes the proof of the final assertion of the initial step.

Step k. The general step consists in modifying the initial system (8.35) by means of successive small perturbations in order to obtain systems with properties similar to those of the initial one, in such a way that: the directions giving the stable and unstable subbundles of the exponential dichotomy become closer at each step; but at the same time there exist solutions whose Lyapunov exponent is bounded from below by a common constant. The procedure now described is based on that of Section 5 of [69].

The goal will be achieved using an induction argument, whose hypotheses and thesis will be explained after fixing some notation.

Given a sequence of T_k-periodic continuous real functions (g_k), consider the systems

$$\mathbf{z}' = \begin{bmatrix} -1 - g_k(t) & 1 \\ -1 & 1 + g_k(t) \end{bmatrix} \mathbf{z} \tag{8.37}$$

and let $U_k(t)$ be the corresponding fundamental matrix solution with $U_k(0) = I_2$. Represent by $\mathbf{z}_k(t, \mathbf{z}_\theta)$ the solution of the preceding system with $\mathbf{z}_0(t, \mathbf{z}_\theta) = \mathbf{z}_\theta = \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$, and note that

$$\mathbf{z}_k(t, \mathbf{z}_\theta) = r_k(t, \theta) \begin{bmatrix} \cos\varphi_k(t, \theta) \\ \sin\varphi_k(t, \theta) \end{bmatrix},$$

where $\varphi_k(t, \theta)$ is the solution of the associated angular equation

$$\varphi' = (1 + g_k(t)) \sin(2\varphi) - 1 \tag{8.38}$$

with $\varphi_k(0, \theta) = \theta$, and $r_k(t, \theta)$ solves

$$r' = -(1 + g_k(t)) \cos(2\,\varphi_k(t, \theta))\, r \tag{8.39}$$

with $r_k(0, \theta) = 1$. These three equations will be often referred to as $(8.37)_k$, $(8.38)_k$ and $(8.39)_k$.

Recall that the value $\delta = 2^{-10}$ was fixed at the initial step, in which also the angle ϑ_0^+ was determined. It is important to keep in mind that $0.5 < \vartheta_0^+ < 0.6$ and that (8.29) holds. The induction hypothesis reads as follows. There exist an integer $k \geq 10$ together with a T_k-periodic function g_k such that all the following properties hold:

5_k. $|g_k(t)| \leq \sum_{j=0}^{k-10} 2^{-j} < 2$ for all $t \in \mathbb{R}$.

6_k. There exists $\beta_k > 0.35 - (1/5) \sum_{j=3}^{k-8} 2^{-j} > 0.3$ and two angles θ_k^+ and θ_k^- such that

$$U_k(T_k)\, \mathbf{z}_{\theta_k^\mp} = e^{\pm\beta_k T_k} \mathbf{z}_{\theta_k^\mp}\,;$$

in other words, $\varphi_k(t, \theta_k^\pm)$ are T_k-periodic functions, and $r_k(T_k, \theta_k^\mp) = e^{\pm\beta_k T_k}$. (In particular, the system $(8.37)_k$ has exponential dichotomy on $\mathbb{R}$, with Lyapunov index $\beta_k > 0.3$, and its rotation number is zero: see Remarks 1.62.2 and 2.8.)

7_k. $\vartheta_0^+ < \theta_k^+ < \theta_k^- < \vartheta_0^+ + \delta$ and $0 < \theta_k^- - \theta_k^+ < 2^{-k}$.

8_k. $0.5 < \vartheta_0^+ < \varphi_k(t, \theta_k^+) < \varphi_k(t, \theta_k^-) < 1.1$ for all $t \in \mathbb{R}$. In particular, there exist globally defined solutions $m_k^{\pm}$ of the Riccati equation

$$m' = -m^2 + 2\,(1 + g_k(t))\,m - 1\,,$$

and $0.5 < m_k^+(t) < m_k^-(t) < m_k^-(t) < 2$ for all $t \in \mathbb{R}$: they are given by $m_k^{\pm}(t) = \tan \varphi_k(t, \theta_k^{\mp})$, and so they are T_k-periodic functions.

Now define $T_{10} = T_0$, $g_{10}(t) = g_0(t)$, $\theta_{10}^{\pm} = \theta_0^{\pm}$, and $\beta_{10} = \beta_0$, where T_0, g_0, $\theta_0^{\pm}$, and β_0 have been obtained in the initial step, and note that they satisfy the conditions 5_{10}, 6_{10}, 7_{10}, and 8_{10}. Recall also that $T_{10} > t_0 \geq 6 > 2\,\delta$. The induction thesis is now stated: there exist a T_{k+1}-periodic function g_{k+1} and two angles θ_{k+1}^+ and θ_{k+1}^- such that all the properties 5_{k+1}, 6_{k+1}, 7_{k+1}, and 8_{k+1} hold, and such that, in addition,

9_{k+1}. T_{k+1} is an integer multiple of T_k .
10_{k+1}. $0 \leq g_{k+1}(t) - g_k(t) \leq 2^{9-k}$ for all $t \in \mathbb{R}$.
11_{k+1}. $\theta_k^+ < \theta_{k+1}^+ < \theta_{k+1}^- < \theta_k^-$.
12_{k+1}. $\varphi_k(t, \theta_k^+) \leq \varphi_{k+1}(t, \theta_{k+1}^+) < \varphi_{k+1}(t, \theta_{k+1}^-) \leq \varphi_k(t, \theta_k^-)$ and $m_k^+(t) \leq m_{k+1}^+(t) < m_{k+1}^-(t) \leq m_k^-(t)$ for all $t \in \mathbb{R}$.

The properties 5_{k+1}–12_{k+1} will be proved in the following order: 9_{k+1}, 10_{k+1}, 5_{k+1}, 11_{k+1}, part of 6_{k+1}, 7_{k+1}, 12_{k+1}, 8_{k+1}, and the rest of 6_{k+1}.

To begin with, take

$$\delta_k = \theta_k^- - \theta_k^+$$

and note that, according to 7_k,

$$0 < \delta_k < 2^{-k}\,. \tag{8.40}$$

Define $T_{k+1} = j_k\,T_k$ for a positive integer j_k, which will be determined later, and note that this ensures 9_{k+1}. Let γ_k be a continuous function supported on $[T_{k+1} - 2\,\delta, T_{k+1}]$ such that

$$0 \leq \gamma_k(t) \leq 2^9\,\delta_k \quad \text{for all } t \in \mathbb{R}\,, \tag{8.41}$$

and such that

$$\int_{T_{k+1}-2\,\delta}^{T_{k+1}} \gamma_k(t)\,dt = 0.85\,\delta_k\,. \tag{8.42}$$

By abusing notation slightly, let γ_k be also the T_{k+1}-periodic extension to $\mathbb{R}$ of the initial γ_k. Now define $g_{k+1} = g_k - \gamma_k$. Note that, irrespective of the value chosen for j_k, the inequality $0 \leq \gamma_k(t) \leq 2^{9-k}$ ensured by (8.40) and (8.41) guarantees 10_{k+1}, which in turn, together with 5_k, ensures also 5_{k+1}.

Take $\theta_0 \in [\theta_k^+, \theta_k^-]$, and recall that $\varphi_k(t, \theta_0)$ and $\varphi_{k+1}(t, \theta_0)$ are respectively the solutions of $(8.38)_k$ and $(8.38)_{k+1}$ with value θ_0 at $t = 0$. The coincidence of g_k and g_{k+1} on $[0, T_{k+1} - 2\,\delta]$ yields

$$\varphi_k(t, \theta) = \varphi_{k+1}(t, \theta) \quad \text{for all } t \in [0, T_{k+1} - 2\,\delta]\,. \tag{8.43}$$

In addition,

$$\begin{aligned} &\varphi_k'(t, \theta_0) - \varphi_{k+1}'(t, \theta_0) \\ &\qquad = (1 + g_{k+1}(t))\,(\sin(2\varphi_k(t, \theta_0)) - \sin(2\varphi_{k+1}(t, \theta_0))) \\ &\qquad\quad + (g_k(t) - g_{k+1}(t))\sin(2\varphi_k(t, \theta_0))\,. \end{aligned} \tag{8.44}$$

Now use the bound $|1 + g_{k+1}(t)| \le 3$ (which follows from the already proved property 5_{k+1}) and $|(g_{k+1}(t) - g_k(t))\sin(2\varphi_k(t, \theta_0)| \le \gamma_k(t)$ for all $t \in [0, T_{k+1}]$, together with the inequality $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$ and with the relations (8.43) and (8.42), in order to see that

$$|\varphi_k(t, \theta_0) - \varphi_{k+1}(t, \theta_0)| \le \int_{T_{k+1}-2\,\delta}^{t} 6\,|\varphi_k(s, \theta_0) - \varphi_{k+1}(s, \theta_0)|\,ds + 0.85\,\delta_k$$

for all $t \in [T_{k+1} - 2\delta, T_{k+1}]$. Therefore, (8.43), the Gronwall lemma, and the equality $\delta = 2^{-10}$, yield

$$|\varphi_k(t, \theta_0) - \varphi_{k+1}(t, \theta_0)| \le 0.85\,e^{12\,\delta}\,\delta_k < 0.87\,\delta_k \quad \text{if } \ t \in [0, T_{k+1}]\,. \tag{8.45}$$

This inequality will be used several times in the present step.

It follows from property 6_k that, if $\theta_0 \in (\theta_k^+, \theta_k^-]$, then

$$\lim_{j\to\infty} \varphi_k(j\,T_k, \theta_0) = \theta_k^- \quad \text{and} \quad \lim_{j\to\infty} \frac{1}{j\,T_k}\,\ln r_k(j\,T_k, \theta_0) = \beta_k\,.$$

These facts show that it is possible to choose a positive integer j_k large enough to ensure that, if $\vartheta_k = \theta_k^+ + 0.05\,\delta_k$ (which due to the definition of δ_k belongs to (θ_k^+, θ_k^-)), then

$$\varphi_k(T_{k+1}, \vartheta_k) > \theta_k^- - 0.05\,\delta_k \quad \text{and} \quad \ln r_k(T_{k+1}, \vartheta_k) > \widetilde{\beta}_{k+1}\,T_{k+1} \tag{8.46}$$

for $T_{k+1} = j_k\,T_k$ and $\widetilde{\beta}_{k+1} = \beta_k - 1/(5 \cdot 2^{k-8})$. This will be the value chosen for j_k, which will be fixed from now on. Note also that the inequalities (8.46) are valid for all $\theta_0 \in [\vartheta_k, \theta_k^-]$

In the next step, two functions will be defined and some of their properties will be described. The first one is

$$f_k\colon [\theta_k^+, \theta_k^-] \to \mathbb{R}\,, \quad \theta_0 \mapsto \varphi_k(T_{k+1}, \theta_0) - \theta_0\,.$$

Note that, since T_{k+1} is a multiple of T_k,

$$f_k(\theta_k^+) = f_k(\theta_k^-) = 0\,, \tag{8.47}$$

and it follows immediately from (8.46) and the definitions of δ_k and ϑ_k that

$$\max_{\theta_0\in[\theta_k^+,\theta_k^-]} f_k(\theta_0) \geq f_k(\vartheta_k) > \theta_k^- - 0.05\,\delta_k - (\theta_k^+ + 0.05\,\delta_k) = 0.9\,\delta_k\,. \tag{8.48}$$

The second function is

$$h_k\colon [\theta_k^+,\theta_k^-] \to \mathbb{R}\,, \quad \theta_0 \mapsto \varphi_k(T_{k+1},\theta_0) - \varphi_{k+1}(T_{k+1},\theta_0)\,.$$

It follows from 8_k and (8.45) that

$$\begin{aligned} \varphi_k(t,\theta_0) &\in (\vartheta_0^+, 1.1) \subset (0,\pi/2)\,, \\ \varphi_{k+1}(t,\theta_0) &\in [\varphi_k(t,\theta_0) - 0.87\,\delta_k, \varphi_k(t,\theta_0) + 0.87\,\delta_k] \\ &\subset (\vartheta_0^+ - 0.87\,\delta_k, 1.1 + 0.87\,\delta_k) \subset (0,\pi/2) \end{aligned} \tag{8.49}$$

for all $t \in [0, T_{k+1}]$ and all $\theta_0 \in [\theta_k^+,\theta_k^-]$. Here the inequalities (8.30) and $\delta_k \leq 0.001$ (which in turn follows from (8.40), since $k \geq 10$) have been used. In addition, $g_k(t) = g_{k+1}(t) + \gamma_k(t) \geq g_{k+1}(t)$, so that, if $\varphi \in (0,\pi/2)$ then

$$(1 + g_k(t))\sin(2\varphi) - 1 \geq (1 + g_{k+1}(t))\sin(2\varphi) - 1 \quad \text{for all } t \in \mathbb{R}\,, \tag{8.50}$$

and the inequality is strict if $\gamma_k(t) > 0$. Therefore, relations (8.49) and the standard results of comparison of solutions of scalar equations prove that

$$\varphi_k(t,\theta_0) \geq \varphi_{k+1}(t,\theta_0) \quad \text{for all } t \in [0, T_{k+1}] \tag{8.51}$$

whenever $\theta_0 \in [\theta_k^+,\theta_k^-]$, the inequality being strict at the end of the interval. Consequently, $h_k(\theta_0) > 0$ for all $\theta_0 \in [\theta_k^+,\theta_k^-]$; hence, according to (8.45),

$$0 < h_k(\theta_0) < 0.87\,\delta_k \tag{8.52}$$

for all $\theta_0 \in [\theta_k^+,\theta_k^-]$.

Now compare the graphs of the continuous functions f_k and h_k. It can immediately be deduced from (8.47), (8.48), and (8.52) that f_k and h_k coincide at (at least) two points, $\theta_{k+1}^\pm$, with

$$\theta_k^+ < \theta_{k+1}^+ < \vartheta_k < \theta_{k+1}^- < \theta_k^-\,. \tag{8.53}$$

These will be the points of the induction thesis; in particular, 11_{k+1} holds. Note that, at these points,

$$\varphi_{k+1}(T_{k+1}, \theta^{\pm}_{k+1}) = \theta^{\pm}_{k+1}\,, \tag{8.54}$$

so that they define two T_{k+1}-periodic solutions of $(8.38)_{k+1}$. This is one of the statements of 6_{k+1}; the other one will be proved at the end of this step. Note that the points θ^{+}_{k+1} are the only ones at which the graphs of f_k and h_k intersect: otherwise all the solutions of $(8.38)_{k+1}$ would be T_{k+1}-periodic (see Remark 8.43), and hence it would be the case that $f_k \equiv h_k$, which does not hold.

The next goal is to check the properties stated in 7_{k+1} for this choice of $\theta^{\pm}_{k+1}$. The first assertion is a trivial consequence of 11_{k+1} and 7_k. Next note that, since $f_k(\theta_0) > h_k(\theta_0)$ if and only if $\theta_0 \in (\theta^{+}_{k+1}, \theta^{-}_{k+1})$, it suffices to prove that

$$f_k(\theta^{+}_{k} + 0.5\,\delta_k) < h_k(\theta^{+}_{k} + 0.5\,\delta_k)\,,$$

or, equivalently, that

$$\varphi_{k+1}(T_{k+1}, \theta^{+}_{k} + 0.5\,\delta_k) < \theta^{+}_{k} + 0.5\,\delta_k\,,$$

since this and the property $\theta^{+}_{k} + 0.05\,\delta_k \in (\theta^{+}_{k+1}, \theta^{-}_{k+1})$ (see (8.53)) imply that $(\theta^{+}_{k+1}, \theta^{-}_{k+1}) \subset (\theta^{+}_{k}, \theta^{+}_{k} + 0.5\,\delta_k)$. To this end, go back to the relation (8.44) in order to deduce from (8.51), (8.45), $|\sin x - \sin y\,| \le |x - y|$, (8.49), $g_k - g_{k+1} = \gamma_k \ge 0$, property 8_k, and $\sin(2\,\vartheta^{+}_{0}) > 0.9$ (see (8.29)), that

$$\varphi'_k(t, \theta_0) - \varphi'_{k+1}(t, \theta_0) \ge -6 \cdot 0.87\,\delta_k + 0.9\,\gamma_k(t) \qquad \text{for } t \in [T_{k+1} - 2\,\delta, T_{k+1}]$$

for all $\theta_0 \in [\theta^{+}_{k}, \theta^{-}_{k}]$, which together with (8.42) yields

$$\varphi_k(T_{k+1}, \theta_0) - \varphi_{k+1}(T_{k+1}, \theta_0) \ge (-12\,\delta \cdot 0.87 + 0.9 \cdot 0.85)\,\delta_k > 0.5\,\delta_k$$

for all $\theta_0 \in [\theta^{+}_{k}, \theta^{-}_{k}]$. Therefore, bearing in mind that $\varphi_k(t, \theta_1) \le \varphi_k(t, \theta_2)$ if $\theta_1 \le \theta_2$ and that T_{k+1} is a multiple of T_k,

$$\begin{aligned}\varphi_{k+1}(T_{k+1}, \theta^{+}_{k} + 0.5\,\delta_k) &< \varphi_k(T_{k+1}, \theta^{+}_{k} + 0.5\,\delta_k) - 0.5\,\delta_k \\ &\le \varphi_k(T_{k+1}, \theta^{-}_{k}) - 0.5\,\delta_k = \theta^{-}_{k} - 0.5\,\delta_k = \theta^{+}_{k} + 0.5\,\delta_k\,.\end{aligned}$$

This completes the proof of 7_{k+1}. For future purposes, note that $\vartheta_k < \theta^{-}_{k+1} < \theta^{-}_{k}$ (see (8.53)), and hence it follows from the second inequality in (8.46) (which, as said above, is valid for all $\theta_0 \in [\vartheta_k, \theta^{-}_{k}]$) and the definition of $\widetilde{\beta}_{k+1}$ that

$$r_k(T_{k+1}, \theta^{-}_{k+1}) > \exp\left(\left(\beta_k - \frac{1}{5 \cdot 2^{k-8}}\right) T_{k+1}\right). \tag{8.55}$$

The next goal is to prove 12_{k+1}. It follows from the last inequality in 11_{k+1} and from (8.51) that $\varphi_k(t, \theta_k^-) > \varphi_k(t, \theta_{k+1}^-) \geq \varphi_{k+1}(t, \theta_{k+1}^-)$ for $t \in [0, T_{k+1}]$. On the other hand, since the functions $\varphi_k(t, \theta_{k+1}^\pm)$ and $\varphi_{k+1}(t, \theta_{k+1}^\pm)$ are T_{k+1}-periodic, relations (8.49) also hold for all $t \in [-T_{k+1}, 0]$ and all $\theta_0 \in [\theta_k^+, \theta_k^-]$. Therefore, the inequality (8.50) ensures that

$$\varphi_k(t, \theta_0) \leq \varphi_{k+1}(t, \theta_0) \quad \text{ for all } t \in [-T_{k+1}, 0]$$

whenever $\theta_0 \in [\theta_k^+, \theta_k^-]$. Hence, $\varphi_k(t, \theta_k^+) < \varphi_k(t, \theta_{k+1}^+) \leq \varphi_{k+1}(t, \theta_{k+1}^+)$ for $t \in [-T_{k+1}, 0]$. The first chain of inequalities in 12_{k+1} follows from these properties, from the T_{k+1}-periodicity of the functions $\varphi_k(t, \theta_k^\pm)$ and $\varphi_{k+1}(t, \theta_{k+1}^\pm)$, and from the second inequality in 11_{k+1}. And the second chain of inequalities in 12_{k+1} is a trivial consequence of the first one and 8_k, since $m_k^\pm(t) = \tan \varphi_k(t, \theta_k^\pm)$ and $m_{k+1}^\pm(t) = \tan \varphi_{k+1}(t, \theta_{k+1}^\pm)$.

Clearly, the inequalities of 12_{k+1} and property 8_k ensure 8_{k+1}.

To complete the proof of 6_{k+1}, one has to check that $r_{k+1}(T_{k+1}, \theta_{k+1}^\mp) = e^{\pm \beta_{k+1} T_k}$ for a number $\beta_{k+1} > 0.35 - (1/5) \sum_{j=3}^{(k+1)-8} 2^{-j}$. The first step is to prove that

$$0 \leq \varphi_{k+1}(t, \theta_{k+1}^-) < \frac{\pi}{4} \quad \text{ for all } \; t \in [T_{k+1} - 2\delta, T_{k+1}]\,. \tag{8.56}$$

The first inequality is included in 8_{k+1}. To prove the second one, apply the mean value theorem and the bound $|\varphi'_{k+1}| \leq 4$ (which follows from the equation $(8.38)_{k+1}$ and the bound 5_{k+1}) in order to check that

$$|\varphi_{k+1}(t, \theta_{k+1}^-) - \theta_{k+1}^-| = |\varphi_{k+1}(t, \theta_{k+1}^-) - \varphi_{k+1}(T_{k+1}, \theta_{k+1}^-)| \leq 8\,\delta$$

for all $t \in [T_{k+1} - 2\delta, T_{k+1}]$; hence, using now 7_{k+1}, (8.30) and $\delta = 1/2^{10}$,

$$\varphi_{k+1}(t, \theta_{k+1}^-) \leq \theta_{k+1}^- + 8\,\delta \leq \vartheta_0^+ + 9\,\delta \leq 0.6 + 9\,\delta < \frac{\pi}{4}$$

for all $t \in [T_{k+1} - 2\delta, T_{k+1}]$, which proves (8.56).

Property (8.56) has an immediate consequence: it ensures that

$$\gamma_k(t) \cos(2\varphi_{k+1}(t, \theta_{k+1}^-)) \geq 0 \quad \text{ for all } t \in [0, T_{k+1}]\,. \tag{8.57}$$

Consider now the functions $r_k(t, \theta_{k+1}^-)$ and $r_{k+1}(t, \theta_{k+1}^-)$, which respectively solve $(8.39)_k$ and $(8.39)_{k+1}$, with $r_k(0, \theta_{k+1}^-) = r_{k+1}(0, \theta_{k+1}^-) = 1$. The first one is given by

$$r_k(t, \theta_{k+1}^-) = \exp\left(- \int_0^t (1 + g_k(s)) \cos(2\varphi_k(s, \theta_{k+1}^-))\, ds \right),$$

and, due to (8.57), the second one satisfies

$$\begin{aligned} r_{k+1}(t,\theta_{k+1}^-) &= \exp\left(-\int_0^t (1+g_k(s)-\gamma_k(s))\cos(2\,\varphi_{k+1}(s,\theta_{k+1}^-))\,ds\right) \\ &\geq \exp\left(-\int_0^t (1+g_k(s))\cos(2\,\varphi_{k+1}(s,\theta_{k+1}^-))\,ds\right) \\ &= r_k(t,\theta_{k+1}^-)\exp\biggl(-\int_0^t (1+g_k(s))\Bigl(\cos(2\varphi_{k+1}(s,\theta_{k+1}^-)) \\ &\qquad\qquad - \cos(2\varphi_k(s,\theta_{k+1}^+))\Bigr)\,ds\biggr). \end{aligned}$$

The inequalities $|1+g_k(t)| \leq 3$, $|\cos x - \cos y| \leq |x-y|$ and (8.45), and the equality (8.43), yield

$$r_{k+1}(T_{k+1},\theta_{k+1}^-) \geq r_k(T_{k+1},\theta_{k+1}^-)\exp(-12\cdot 0.87\,\delta_k\,\delta)\,,$$

which, together with (8.55) and the bounds $T_{k+1} \geq t_0 \geq 6$ and $\delta_k \leq \delta = 1/2^{10}$ (which in turn follows from (8.40) since $k \geq 10$), ensures that

$$\begin{aligned} r(T_{k+1},\theta_{k+1}^-) &\geq \exp\left(\left(\beta_k - \frac{1}{5\cdot 2^{k-8}} - \frac{12\cdot 0.87\,\delta_k\,\delta}{T_{k+1}}\right)T_{k+1}\right) \\ &\geq \exp\left(\left(\beta_k - \frac{1}{5\cdot 2^{k-7}}\right)T_{k+1}\right). \end{aligned}$$

This fact and the bound of β_k provided by 6_k show that $r_{k+1}(T_{k+1},\theta_{k+1}^-) = e^{\beta_{k+1}T_k}$ for $\beta_{k+1} > \beta_k - 1/(5\cdot 2^{k-7}) > 0.35 - (1/5)\sum_{j=3}^{(k+1)-8} 2^{-j} > 0.3$. From this point on it is possible to reproduce the argument used at the initial step in order to deduce that $r_{k+1}(T_{k+1},\theta_{k+1}^+) = e^{-\beta_{k+1}T_k}$. Hence, 6_{k+1} is proved. This completes the induction step of the construction.

Final step and conclusions. Now define

$$g(t) = \lim_{k\to\infty} g_k(t) = g_{10}(t) - \sum_{k=10}^{\infty}\gamma_k(t) \tag{8.58}$$

for $t \in \mathbb{R}$. Properties 5_k and 9_{k+1} show that g is the uniform limit on $\mathbb{R}$ of a sequence of continuous T_k-periodic functions with $T_k = j_k T_{k-1}$, so $g(t)$ is limit-periodic. Applying the Bebutov construction (see Sect. 1.3.2) to the system (8.24) provides a compact metric space Ω equipped with a (time-translation) flow σ. Taking $\omega_0 = g \in \Omega$ and defining $\widetilde{g}\colon \Omega \to \mathbb{R}$ as the continuous operator of evaluation at $t = 0$, one gets $\widetilde{g}(\omega_0{\cdot}t) = g(t)$. One also obtains the family of equations

$$\mathbf{z}' = \begin{bmatrix} -1 - \widetilde{g}(\omega{\cdot}t) & 1 \\ -1 & 1 + \widetilde{g}(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.59}$$

which reduces to (8.24) for $\omega = \omega_0 = g$.

The description of this example will be completed by proving properties v1–v7. Before starting this task, it is convenient to explain the way in which the periodic systems (8.37) and the corresponding Weyl functions $m_k^{\pm}$ can be extended to the hull Ω, as well as some additional properties. This is done in the following paragraphs. A property stated in v1 must be anticipated: the flow on Ω is minimal (see below for a suitable reference).

Let $\mathcal{F}(f)$ represent the frequency module of an almost periodic function f; its definition can be found in, for example, [73]. It is also given in Lemma 8.45, at the end of the example, which shows that $\mathcal{F}(g) = \cup_{k \geq 10} \mathcal{F}(g_k)$. The following fundamental property is proved in Section 2 of [81]: for every almost periodic function f with $\mathcal{F}(f) \subseteq \mathcal{F}(g)$, there exists a continuous function $\widetilde{f}$ on Ω (the hull of g) with $\widetilde{f}(\omega_0{\cdot}t) = f(t)$ for all $t \in \mathbb{R}$ (where $\omega_0 = g \in \Omega$).

Fix now $k \geq 0$. According to 8_k, the functions $m_k^{\pm}(t)$ are T_k-periodic, as g_k is. Therefore, $\mathcal{F}(m_k^{\pm}) \subset \mathcal{F}(g)$. Let $\widetilde{g}_k \colon \Omega \to \mathbb{R}$ and $\widetilde{m}_k^{\pm} \colon \Omega \to \mathbb{R}$ be the continuous functions associated to $m_k^{\pm}$ and g_k by the procedure just explained, with

$$\widetilde{g}_k(\omega_0{\cdot}t) = g_k(t) \quad \text{and} \quad \widetilde{m}_k^{\pm}(\omega_0{\cdot}t) = m_k^{\pm}(t) \quad \text{for all } t \in \mathbb{R}\,. \tag{8.60}$$

It is easy to deduce from these equalities, the T_k-periodicity of g_k and of $m_k^{\pm}$, and the density of the orbit of ω_0 in Ω, that the functions $t \mapsto \widetilde{m}_k^{\pm}(\omega{\cdot}t)$ and $t \mapsto \widetilde{g}_k(\omega{\cdot}t)$ are T_k-periodic for every $\omega \in \Omega$.

Consider the family of periodic Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} -1 - \widetilde{g}_k(\omega{\cdot}t) & 1 \\ -1 & 1 + \widetilde{g}_k(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.61}$$

and note that the first equality in (8.60) ensures that the system corresponding to ω_0 coincides with (8.37). According to Remark 1.59.4, this property and the exponential dichotomy of the system (8.37) on $\mathbb{R}$ ensure that the family (8.61) has exponential dichotomy over Ω. And it follows from the minimality of the base flow, the continuity of the functions $\widetilde{m}_k^{\pm}$ on Ω, the second equality in (8.60), and the definition of the functions $m_k^{\pm}$ (see property 8_k), that $\widetilde{m}_k^{\pm}$ are the corresponding Weyl functions. In particular, these two functions are solutions along the flow of the Riccati equation associated to (8.61), namely

$$m' = -m^2 + 2\,(1 + \widetilde{g}_k(\omega{\cdot}t))\, m - 1\,. \tag{8.62}$$

Once this procedure has been performed for all $k \geq 0$, one can use (8.60) together with the minimality of the flow on Ω in order to deduce from the last chain of inequalities in 12_{k+1}) that

$$0.5 \leq \widetilde{m}_k^+(\omega) \leq \widetilde{m}_{k+1}^+(\omega) \leq \widetilde{m}_{k+1}^-(\omega) \leq \widetilde{m}_k^-(\omega) \leq 2 \tag{8.63}$$

for all $k \in \mathbb{N}$ and all $\omega \in \Omega$. Consequently, there exist the limits

$$\widetilde{n}^{\pm}(\omega) = \lim_{k\to\infty} \widetilde{m}_k^{\pm}(\omega)\,, \tag{8.64}$$

with

$$0.5 \leq \widetilde{n}^+(\omega) \leq \widetilde{n}^-(\omega) \leq 2 \tag{8.65}$$

for all $\omega \in \Omega$. Note also that the maps $\widetilde{n}^{\pm}(\omega)$ are globally defined solutions along the flow of the Riccati equation

$$m' = -m^2 + 2\,(1 + \widetilde{g}(\omega{\cdot}t))\,m - 1 \tag{8.66}$$

associated to (8.59). To prove this assertion: note that $\lim_{k\to\infty} \|\widetilde{g} - \widetilde{g}_k\|_\Omega = 0$, which in turn is a consequence of property 10_{k+1}, the equalities $\widetilde{g}(\omega_0{\cdot}t) = g(t)$ and $\widetilde{g}_k(\omega_0{\cdot}t) = g_k(t)$, and the minimality of the base flow; hence, if $\tau_{s,k}$ and τ_s represent the flows induced on $\Omega \times \mathbb{S}_1(\mathbb{R})$ by the Riccati equations (8.62) and (8.66), then

$$\begin{aligned}\tau_s(t,\omega,\widetilde{n}^{\pm}(\omega)) &= \lim_{k\to\infty} \tau_{s,k}(t,\omega,m_k^{\pm}(\omega))\\ &= \lim_{k\to\infty} (\omega{\cdot}t, m_k^{\pm}(\omega{\cdot}t)) = (\omega{\cdot}t, \widetilde{n}^{\pm}(\omega{\cdot}t))\,,\end{aligned}$$

as asserted.

The properties v1–v7 can now be proved.

v1: According to the results of Chapter VI of [140], the flow on Ω is minimal and admits a unique ergodic measure m_0, which has total support (see Proposition 1.11). In particular, v1 is satisfied.

v2: On the one hand, properties 6_k and 7_k ensure that the family (8.59) cannot admit exponential dichotomy: if it did, the perturbation theorem of Sacker and Sell (see e.g. Theorem 1.95) and point 7_k would imply that the limits of the two sequences of angles $(\theta_k^{\pm})$ determine the polar angles of the initial data of two different solutions of (8.59) for $\omega = \omega_0$ (those which are bounded at $\mp\infty$); and hence $\theta_k^- - \theta_k^+$ would be bounded apart from zero for large k, which is precluded by 8.

On the other hand, it follows from 6_k that the Lyapunov exponents of the system (8.61) corresponding to ω_0 (that is, of the system (8.37)) are $\pm\beta_k$, with $\beta_k > 0.3$: see Remark 2.42.3. Consequently, the Lyapunov index of the family (8.61) with respect to m_0 is β_k. A possible way to check this assertion is to apply for example Lemma 1 of [133], since Ω is minimal. It can also be deduced from 6_k and the T_k-periodicity of the extended function $\widetilde{g}_k$. The upper semicontinuity of the Lyapunov index guaranteed by Corollary 2.47 ensures that the Lyapunov index of the limit system (8.59) is a number $\beta_\infty \geq 0.3$. (As a matter of fact, $\beta_\infty = \lim_{k\to\infty} \beta_k$, as Lemma 8.46 proves.)

These two properties ensure that the Sacker–Sell spectrum of the family is $[-\beta_\infty, \beta_\infty] \supseteq [-0.3, 0.3]$: see Sect. 1.4.4; that is, the first assertion in v2 holds.

To complete the proof of v2, note that the family (8.59) has zero rotation number with respect to m_0. This property follows from the existence of the globally defined solutions $\widetilde{n}^{\pm}$ of the Riccati equation (8.66) and from Propositions 5.8 and 5.65.

v3: It is a well-known fact that the dynamical behavior of the flow τ determined by (8.59) on $\mathcal{K}_{\mathbb{R}} = \Omega \times \mathcal{L}_{\mathbb{R}}$ is highly complicated in the nonuniformly hyperbolic case. Part of this behavior is summarized below, in point 13. The reader can find an easy proof of its statements (as well as many more details) in Theorem 4.10 of [87], which is based on ideas which were previously considered in [64] and [66].

13. The set $\mathcal{K}_{\mathbb{R}} = \Omega \times \mathcal{L}_{\mathbb{R}}$ contains a unique minimal subset $\mathcal{M}$ for the flow τ. This set $\mathcal{M}$ is not uniquely ergodic: it supports two different τ-ergodic measures $\mu^{\pm}$, which are the unique τ-ergodic measures in $\mathcal{K}_{\mathbb{R}}$. These measures are associated to two nonclosed τ-invariant graphs $\{(\omega, \bar{l}^{\pm}(\omega)) \mid \omega \in \Omega\}$ by means of the relation $\int_{\mathcal{K}_{\mathbb{R}}} f(\omega, l)\, d\mu^{\pm} = \int_{\Omega} f(\omega, \bar{l}^{\pm}(\omega))\, dm$ for all continuous functions $f\colon \mathcal{K}_{\mathbb{R}} \to \mathbb{R}$.

In fact the maps $\bar{l}^{\pm}$ determine in full measure the principal solutions for (8.59), as is proved in what follows. Note first that the family (8.59) is uniformly weakly disconjugate; in fact all its systems are disconjugate. This is deduced, for instance, from Remark 5.30, since the existence of the functions $\widetilde{n}_\infty$ given by (8.64), which are globally defined solutions along the flow of the Riccati equation (8.66), guarantees property D3. This proves the first assertion in v3.

Consequently, Theorem 5.17 ensures the existence of uniform principal solutions at $\pm\infty$ which can be parameterized by the principal functions $n^{\pm}\colon \Omega \to \mathbb{R}$, which are defined by (5.20). Since the sets $\{(\omega, \widetilde{l}^{\pm}(\omega)) \mid \omega \in \Omega\}$ concentrate two ergodic measures (see Proposition 5.45(i)), it follows from the uniqueness established in point 13 that $\bar{l}^{\pm}(\omega) = \widetilde{l}^{\pm}(\omega)$ for m_0-almost every $\omega \in \Omega$, where $\widetilde{l}^{\pm}(\omega) \equiv \begin{bmatrix} 1 \\ n^{\pm}(\omega) \end{bmatrix}$.

In what follows, $\mathbf{z}^{\pm}(t, \omega)$ represent the solutions of (8.59) with $\mathbf{z}^{\pm}(0, \omega) = \begin{bmatrix} 1 \\ n^{\pm}(\omega) \end{bmatrix}$. It is proved in Theorem 5.56(i) that there exists a σ-invariant subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) = 1$ such that

$$\mp \beta_\infty = \lim_{|t|\to\infty} \frac{1}{t} \ln \|\mathbf{z}^{\pm}(t, \omega)\| \tag{8.67}$$

for every $\omega \in \Omega_0$. A first consequence of this fact is that $n^{+}(\omega) \neq n^{-}(\omega)$ for all $\omega \in \Omega_0$. However, these two functions cannot be different in the whole of Ω: if this were the case, Theorem 5.58 would ensure the exponential dichotomy over Ω of the family (8.59), which is precluded by v2. In fact, if Ω_c is the residual set of common continuity points for n^+ and n^- (see Proposition 5.45(ii)), then $n^+(\omega) = n^-(\omega)$ for all $\omega \in \Omega_c$. To prove this assertion, recall first that Theorem 5.17 also ensures that $n^\pm$ are globally defined solutions along the flow of the Riccati equation (8.66). Then, take a point $\widetilde{\omega}$ with $n^+(\widetilde{\omega}) = n^-(\widetilde{\omega})$, note that this implies that $n^+(\widetilde{\omega}{\cdot}t) = n^-(\widetilde{\omega}{\cdot}t)$ for all $t \in \mathbb{R}$, and use the fact that the σ-orbit of $\widetilde{\omega}$ is dense in the minimal set Ω. A consequence of this property is that n^+ and n^- are not continuous on Ω. This fact, together with Proposition (5.45) and the uniqueness of the minimal set $\mathcal{M}$ established in 13, shows that $\mathcal{M} = \text{closure}_{\mathcal{K}_\mathbb{R}}\{(\omega{\cdot}t, \widetilde{l}^+(\omega{\cdot}t)) \,|\, \omega \in \Omega^\pm\} = \text{closure}_{\mathcal{K}_\mathbb{R}}\{(\omega{\cdot}t, \widetilde{l}^-(\omega{\cdot}t)) \,|\, \omega \in \Omega^\pm\}$ (where ω is any element of Ω_c) is an almost automorphic extension of the base Ω for the flow τ, which does not reduce to a copy of the base. This completes the proof of the properties stated in v3.

Although the remaining properties of the example could be proved from the information collected so far, it is interesting to check that the principal functions $n^\pm$ agree everywhere with the limits $\widetilde{n}^\pm$ given by (8.64). This fact is proved in Lemma 8.46 below. Of course, its proof only requires already known properties. In particular, it follows from (8.65) that

$$0.5 \le n^+(\omega) \le n^-(\omega) \le 2 \tag{8.68}$$

for all $\omega \in \Omega$.

v4: The family of linear Hamiltonian systems (8.59) is associated to the family of LQ control problems given by

$$x' = -x + u \tag{8.69}$$

together with

$$\widetilde{\mathcal{Q}}_\omega(t,x,u) = \frac{1}{2}\,\big(\widetilde{G}(\omega{\cdot}t)\,x^2 + 2\,\widetilde{g}(\omega{\cdot}t)\,x\,u + u^2\big)\,,$$

where $\widetilde{G}(\omega) = \widetilde{g}^2(\omega) - 1$. Note that the (autonomous) control system (8.69) is (uniformly) null controllable, as is easily deduced for instance from Definition 6.3; and that the homogeneous linear system $x' = -x$ is of Hurwitz type at $+\infty$, i.e. Hypothesis 7.3 is satisfied. Hence, the properties stated in v4 hold.

v5: Proposition 8.17(i) and the inequalities (8.68) ensure that the family of LQ control problems corresponding to the family of linear Hamiltonian systems (8.59) via the relations $A = -1$, $B = 1$, $G = g^2 - 1$, and $R = 1$ is uniformly dissipative (that is, all the elements of the family are dissipative); and that each problem admits the strong storage function $V^-_\omega(t,x) = (1/2)\langle x, n^-(\omega))\,x\rangle$. These properties imply v5. Note also that, since $n^+(\omega) > 0$, the function $V^+_\omega(t,x) = (1/2)\langle x, n^+(\omega{\cdot}t)\,x\rangle$ is also a strong storage function, often "worse", in the sense that it is smaller for those points ω with $n^+(\omega) < n^-(\omega)$. In fact, Theorem 5.48(iv) yields

$$n^+(\omega{\cdot}t) \le n(t,\omega,n_0) \le n^-(\omega{\cdot}t)$$

for all $\omega \in \Omega$ and all $n_0 \in \mathbb{R}$ giving rise to a globally defined solution of (8.66). In other words, $V^-_\omega(t,x) = (1/2)\langle x, n^-(\omega{\cdot}t)\,x\rangle$ and $V^+_\omega(t,x) = (1/2)\langle x, n^+(\omega{\cdot}t)\,x\rangle$ are the "best" and "worst" storage functions defined from global solutions of the Riccati equation (8.66). Additional information is given in Theorem 8.34, which implies that V^-_ω is the required supply for m_0-a.e. $\omega \in \Omega$.

v6: A second consequence of (8.67) is that, if $\omega \in \Omega_0$, the feedback control

$$u = (n^+(\omega{\cdot}t) - \widetilde{g}(\omega{\cdot}t))\,x \tag{8.70}$$

exponentially stabilizes the LQ$_\omega$ control problem. To see this, one can repeat an argument which has appeared several times in the book, and which is included again for the reader's convenience. The goal is to prove that all the solutions of the linear equation $x' = (-1 + n^+(\omega{\cdot}t) - \widetilde{g}(\omega{\cdot}t))\,x$ decay exponentially to zero as $t \to \infty$ if $\omega \in \Omega_0$. Note that (8.67) provides constants $\eta > 0$ and $\beta > 0$ such that $\|\mathbf{z}^+(t,\omega)\| \le \eta\, e^{-\beta t}$ for all $t \ge 0$. Note also that $\mathbf{z}^+(t,\omega) = U(t,\omega)\,\mathbf{z}^+(0,\omega)$ belongs to $\widetilde{l}^+(\omega{\cdot}t) = U(t,\omega){\cdot}\widetilde{l}^+(\omega)$ (see Theorem 5.26), which ensures that it can be written as $\begin{bmatrix} x^+(t,\omega) \\ n^+(\omega{\cdot}t)\,x^+(t,\omega)\end{bmatrix}$. It follows easily that $x^+(t,\omega)$ is the solution of $x' = (-1 + n^+(\omega{\cdot}t) - \widetilde{g}(\omega{\cdot}t))\,x$ with $x^+(0,\omega) = 1$. Finally, $|x^+(t,\omega)| \le \|\mathbf{z}^+(t,\omega)\| \le \eta\, e^{-\beta t}$. The assertion follows from this fact, since all the remaining solutions of the scalar equation are multiples of $x^+(t,\omega)$.

On the other hand, there exists a residual set $\mathcal{R} \subset \Omega$ with $m_0(\mathcal{R}) = 0$ such that for all $\omega \in \mathcal{R}$ and all $\mathbf{z}_0 \in \mathbb{R}^2 - \{\mathbf{0}\}$, the solution $\mathbf{z}(t,\omega) = U(t,\omega)\,\mathbf{z}_0$ of (8.59) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\, \ln\|\mathbf{z}(t,\omega)\| = \beta_\infty \ge 0.3 > 0\,. \tag{8.71}$$

One way to prove this fact is to use the following fundamental property, which holds in the nonuniformly hyperbolic case. Its proof appears also in Theorem 4.10 of [87] (see also [64] and [66]):

14. Let $\mathcal{M} \subset \Omega \times \mathcal{K}_\mathbb{R}$ be the minimal set appearing in point 13. There exists a residual subset $\widetilde{\mathcal{R}} \subset \mathcal{M}$ such that, if $(\omega, l) \in \widetilde{\mathcal{R}}$ and $\widetilde{\mathbf{z}}(t,\omega)$ solves (8.59) with $\widetilde{\mathbf{z}}(0,\omega) \in l$, then

$$\limsup_{t\to\pm\infty} \frac{1}{t}\, \ln\|\widetilde{\mathbf{z}}(t,\omega)\| = \beta_\infty \quad \text{and} \quad \liminf_{t\to\pm\infty} \frac{1}{t}\, \ln\|\widetilde{\mathbf{z}}(t,\omega)\| = -\beta_\infty\,. \tag{8.72}$$

To deduce (8.71) from statement 14 requires some more work. The first point is to check that the projection of $\widetilde{\mathcal{R}}$ onto Ω is also a residual set $\mathcal{R}$. This is done in Lemma 8.47 below. Now take $(\omega, l) \in \widetilde{\mathcal{R}}$ and $\widetilde{\mathbf{z}} \in l$, so that (8.72) holds for $\widetilde{\mathbf{z}}(t,\omega) = U(t,\omega)\widetilde{\mathbf{z}}$; take any $\mathbf{z}_0 \in \mathbb{R}^2$ which is linearly independent of $\widetilde{\mathbf{z}}$ and define $\mathbf{z}(t,\omega) = U(t,\omega)\,\mathbf{z}_0$; recall that the determinant of the fundamental matrix solution $V(t,\omega) = [\widetilde{\mathbf{z}}(t,\omega)\;\; \mathbf{z}(t,\omega)\,]$ is a constant c (as is deduced from the Liouville formula), and that $|c| = |\det V(t,\omega)| \le \|\widetilde{\mathbf{z}}(t,\omega)\| \cdot \|\mathbf{z}(t,\omega)\|$; take any $\varepsilon > 0$; use the second equality in (8.72) to find a sequence $(t_k) \uparrow \infty$ with $\|\widetilde{\mathbf{z}}(t_k,\omega)\| < e^{(-\beta_\infty+\varepsilon)\,t_k}$ for all $k \in \mathbb{N}$; and deduce that $\|\widetilde{\mathbf{z}}(t_k,\omega)\| > |c|\,e^{(\beta_\infty-\varepsilon)\,t_k}$ and hence that $\limsup_{t\to\infty}(1/t)\,\ln\|\mathbf{z}(t,\omega)\| > \beta_\infty - \varepsilon$. This completes the proof of (8.71).

Relation (8.71) has the consequence that, if $\omega \in \Omega_0$, then the control system (8.69) cannot be stabilized by the control (8.70). Indeed, if $m(t,\omega)$ is any bounded solution of the Riccati equation (8.66) (as are $t \mapsto n^+(\omega{\cdot}t)$ and $t \mapsto n^-(\omega{\cdot}t)$), then $u = (m(t,\omega) - \widetilde{g}(\omega{\cdot}t))\,x$ does not stabilize (8.69): if this were the case, then any solution $x(t)$ of $x' = (-1 - \widetilde{g}(\omega{\cdot}t) - m(t,\omega))\,x$ would be bounded at $+\infty$, and hence it would provide the nontrivial bounded solution $\mathbf{z}(t,\omega) = \begin{bmatrix} x(t) \\ m(t,\omega)\,x(t) \end{bmatrix}$ of (8.59), contradicting (8.71). In this sense, "one cannot stabilize (8.69) using the data A, B, $\widetilde{G}$, $\widetilde{g}$, and R". This is what point v6 states.

v7: Finally, consider the infinite-horizon control problem of minimizing the functional $\widetilde{\mathcal{I}}_{x_0,\omega}(x,u) = \int_0^\infty \widetilde{\mathcal{Q}}_\omega(t,x(t),u(t))\,dt$ on the set of the admissible pairs; that is, on the set of pairs $(x(t),u(t)) \in L^2((0,\infty),\mathbb{R}) \times L^2((0,\infty),\mathbb{R})$ which solve (8.69) and satisfy $x(0) = x_0$. One can show that the problem is solvable for all $\omega \in \Omega_0$: the minimizing pair $(\bar{x}(t),\bar{u}(t))$ for $\widetilde{\mathcal{I}}_{x_0,\omega}$ is defined from the unique solution $\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix}$ of (8.59) satisfying $\bar{x}(0) = x_0$ which belongs to $L^2((0,\infty),\mathbb{R})$ by the relation $\bar{u}(t) = \bar{y}(t) - \widetilde{g}(\omega{\cdot}t)\,\bar{x}(t)$. (Note that (8.67) ensures the existence of $\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix}$, which is determined by the initial datum $\begin{bmatrix} \bar{x}(0) \\ \bar{y}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ n^+(\omega)\,x_0 \end{bmatrix}$.) To prove this assertion, take $\omega \in \Omega_0$, define $V_\omega^+(t,x) = n^+(\omega{\cdot}t)\,x^2$, and apply Lemma 7.9 to check that, if a pair $(x(t),u(t))$ is admissible, then

$$2\widetilde{\mathcal{I}}_{\omega,x_0}(x,u) = -n^+(\omega)\,x_0^2 + \int_0^\infty \|(u(t) - (-\widetilde{g}(\omega{\cdot}t) + n^+(\omega{\cdot}t))\,x(t))\|^2\,dt.$$

Consequently, $2\widetilde{\mathcal{I}}_{\omega,x_0}(x,u) \geq -n^+(\omega)\,x_0^2$ for all admissible pairs, and the equality holds for the pair $(\bar{x}(t),\bar{u}(t))$ described above. Conversely, if a pair $(x(t),u(t))$ is admissible and the equality holds, then $x(t)$ is square integrable and $u(t) = (-\widetilde{g}(\omega{\cdot}t) + n^+(\omega{\cdot}t))\,x(t)$, and it is easy to conclude from these facts that $(x(t),u(t)) = (\bar{x}(t),\bar{u}(t))$.

However, the optimization problem is not solvable for $\omega \in \mathcal{R}$ and $x_0 \neq 0$. To see this, suppose for contradiction the existence of $\omega \in \mathcal{R}$ and $x_0 \neq 0$ such that the problem admits a minimizing pair $(\bar{x}(t),\bar{u}(t)) \in L^2((0,\infty),\mathbb{R}) \times L^2((0,\infty),\mathbb{R})$. It has been seen in point v4 that the uniform stabilization Hypothesis 7.3 is satisfied. Consequently, as is explained in Chap. 7, if $\bar{y}(t) = \bar{u}(t) + \widetilde{g}(\omega{\cdot}t)\,\bar{x}(t)$, then $\bar{\mathbf{z}}(t) = \begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix}$ solves the system (8.59) corresponding to ω. And clearly $\bar{\mathbf{z}}(t) \in L^2((0,\infty),\mathbb{R}^2)$. In particular, there exists $\lim_{t\to\infty}\bar{\mathbf{z}}(t) = \mathbf{0}$: this is proved as in Lemmas 6.18 and 7.1, since both $\bar{\mathbf{z}}$ and $\bar{\mathbf{z}}'$ are square integrable on $(0,\infty)$. But the last property is precluded by (8.71).

These facts prove property v7. The description of the Millionščikov–Vinograd type example is hence complete (once the next lemmas, which were used in the final step, have been proved).

Lemma 8.45 *Let $\mathcal{F}(f)$ represent the frequency module of an almost periodic function f. In the situation described in Example 8.44, $\mathcal{F}(g) = \cup_{k\geq 10}\mathcal{F}(g_k)$.*

Proof Recall that $\mathcal{F}(f)$ is the additive group composed of the finite integer combinations of the so-called *frequencies* of the almost periodic function f, defined as those values of $\lambda \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\, e^{-i\lambda s}\, ds \neq 0\,.$$

It is simple to deduce from the convergence of the sequence (g_k) to g, which according to property 10_{k+1} is uniform on $\mathbb{R}$, that $\mathcal{F}(g) \subseteq \cup_{k\geq 10}\mathcal{F}(g_k)$. The converse inclusion requires some more work. Since $\mathcal{F}(g_k) = \{2\pi j/T_k \mid j \in \mathbb{Z}\}$ and T_{11} is a multiple of T_{10} (see 9_{k+1}), it is enough to see that $2\pi/T_{k+1} \in \mathcal{M}(g)$ for all $k \geq 10$. In other words (see (8.58)), that

$$c_{k+1} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(g_{10}(s) - \sum_{j=10}^{\infty} \gamma_j(s)\right) e^{-2\pi i s/T_{k+1}}\, ds \neq 0 \tag{8.73}$$

for all $k \geq 10$. Note that $2\pi/T_{k+1}$ does not belong to the frequency module of the T_{10}-periodic function g_{10}, and that the same happens with γ_j for $j = 10, \ldots, k-1$, since it is T_{j+1}-periodic. Therefore,

$$\begin{aligned} c_{k+1} &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(-\sum_{j=k}^{\infty} \gamma_j(s)\right) e^{-2\pi i s/T_{k+1}}\, ds \\ &= -\sum_{j=k}^{\infty} \lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma_j(s)\, e^{-2\pi i s/T_{k+1}}\, ds \\ &= -\sum_{j=k}^{\infty} \frac{1}{T_{j+1}}\int_0^{T_{j+1}} \gamma_j(s)\, e^{-2\pi i s/T_{k+1}}\, ds\,. \end{aligned}$$

Now recall that $\gamma_j \not\equiv 0$ is zero on the interval $[0, T_{j+1} - 2\delta]$ and nonnegative on the interval $[T_{j+1} - 2\delta, T_{j+1}]$, so that, if $j \geq k$,

$$\begin{aligned} &\mathrm{Re}\left(-\frac{1}{T_{j+1}}\int_0^{T_{j+1}} \gamma_j(s)\, e^{-2\pi i s/T_{k+1}}\, ds\right) \\ &\quad = -\frac{1}{T_{j+1}}\int_{T_{j+1}-2\delta}^{T_{j+1}} \gamma_j(s)\, \cos(2\pi s/T_{k+1})\, ds < 0\,. \end{aligned}$$

To check the last inequality, use property 9_{k+1} in order to write $T_{j+1} = m_j T_{k+1}$ for $m_j \in \mathbb{N}$, and note that $\cos(2\pi s/T_{k+1})$ is strictly positive on $((m_j - (1/4))\,T_{k+1}, m_j T_{k+1}] = (T_{j+1} - (1/4)\,T_{k+1}, T_{j+1}] \supset [T_{j+1} - 2\delta, T_{j+1}]$. The conclusion is that $\mathrm{Re}\, c_{k+1} < 0$ for all $k \geq 10$, which implies (8.73) and completes the proof.

Lemma 8.46 *In the situation described in Example 8.44, the functions $\widetilde{n}^{\pm}$ defined by the limits (8.64) agree with the principal functions $n^{\pm}$. In particular, $\beta_\infty = \lim_{k\to\infty} \beta_k$.*

Proof Consider again the Riccati equations

$$m' = -m^2 + 2\,(1 + \widetilde{g}_k(\omega{\cdot}t))\, m - 1 = h_k(\omega{\cdot}t, m) \tag{8.74}$$

and

$$m' = -m^2 + 2\,(1 + \widetilde{g}(\omega{\cdot}t))\, m - 1 = h(\omega{\cdot}t, m)\,, \tag{8.75}$$

and represent by $m_k(t,\omega,m_0)$ and $m(t,\omega,m_0)$ the respective solutions with initial data $m_0 \in \mathbb{R}$. Note that

$$h_k(\omega, m) \geq h(\omega, m) \qquad \text{whenever } m \geq 0$$

for all $\omega \in \Omega$, since $\widetilde{g}_k(\omega) \geq \widetilde{g}(\omega)$. In turn this property follows from the relation $g_k(t) > g(t)$ (which is due to the construction of g), from (8.60), and from the minimality of the base flow. This property will be used below twice to apply the comparison results given in Theorem 1.46. Recall also that $m_k(t,\omega,m_k^{\pm}(\omega)) = m_k^{\pm}(\omega{\cdot}t)$, $m(t,\omega,\widetilde{n}^{\pm}(\omega)) = \widetilde{n}^{\pm}(\omega{\cdot}t)$ and $m(t,\omega,n^{\pm}(\omega)) = n^{\pm}(\omega{\cdot}t)$ for all $k \in \mathbb{N}$, $t \in \mathbb{R}$ and $\omega \in \Omega$.

Theorems 5.48 and 5.58 ensure that a solution $m(t,\omega,m_0)$ of (8.75) is globally defined if and only if $n^+(\omega) \leq m_0 \leq n^-(\omega)$, and that a solution $m_k(t,\omega,m_0)$ of (8.74) is globally defined if and only if $m_k^+(\omega) \leq m_0 \leq m_k^-(\omega)$. The first of these properties, the equality $\widetilde{n}^{\pm}(\omega{\cdot}t) = m(t,\omega,\widetilde{n}^{\pm}(\omega))$, and (8.65) yield

$$n^+(\omega) \leq \widetilde{n}^+(\omega) \leq \widetilde{n}^-(\omega) \leq n^-(\omega)$$

for all $\omega \in \Omega$.

The next step is to prove that $n^-(\omega) \leq m_k^-(\omega)$ for all $\omega \in \Omega$ and all $k \in \mathbb{N}$, which obviously ensures that $n^-(\omega) \leq \widetilde{n}^-(\omega)$ and hence that they are equal. Assume for contradiction that $n^-(\omega) > m_k^-(\omega)$ for a certain point $\omega \in \Omega$ and a value of k. Then $m_k(t,\omega,n^-(\omega))$ is not globally defined. This fact, together with $m_k(t,\omega,n^-(\omega)) > m_k(t,\omega,m_k^-(\omega)) = m_k^-(\omega{\cdot}t)$ for all $t \in \mathbb{R}$ and the property $\lim_{t\to\infty} |m_k(t,\omega,n^-(\omega)) - m_k^-(\omega{\cdot}t)| = 0$ (which can be easily deduced from (1.30)), ensures that $m_k(t,\omega,n^-(\omega))$ tends to $+\infty$ at it approaches a negative value of t from the right. Since $n^-(\omega{\cdot}t) = m(t,\omega,n^-(\omega)) \geq m_k(t,\omega,n^-(\omega))$ for those values of

$t < 0$ at which the last solution is defined (see Theorem 1.46(i)), the function n^- cannot be globally defined, which is the sought-for contradiction.

The proof of the first assertion of the lemma will be complete once it has been checked that $n^+(\omega) \geq m_k^+(\omega)$ for all $\omega \in \Omega$ and all $k \in \mathbb{N}$, which yields $n^+(\omega) \geq \widetilde{n}^+(\omega)$. Let $\omega_1 \in \Omega_c$ be a continuity point of n^+ and n^-. As seen in the proof of v3, $n^+(\omega_1) = n^-(\omega_1)$, so that the four functions $n^\pm$ and $\widetilde{n}^\pm$ agree at this point. Choose now any point $\omega_2 \in \Omega$ and write $\omega_1 = \lim_{j\to\infty} \omega_2{\cdot}t_j$ for a suitable sequence $(t_j) \uparrow \infty$. Then $\lim_{j\to\infty} n^+(\omega_2{\cdot}t_j) = n^+(\omega_1) = \widetilde{n}^+(\omega_1)$. This fact ensures that $n^+(\omega_2{\cdot}t) > 0$ for all $t \in \mathbb{R}$: since $h(\omega, 0) = -1 < 0$ for all $\omega \in \Omega$, if $n^+(\omega_2{\cdot}t_0) \leq 0$ for a point $t_0 \in \mathbb{R}$, then $n^+(\omega_2{\cdot}t) \leq 0$ for all $t \geq t_0$; therefore $\widetilde{n}^+(\omega_1) \leq 0$, which is precluded by (8.65). Now assume for contradiction that $0 < n^+(\omega_2) < m_k^+(\omega_2)$ for a value of k. Then $m_k(t, \omega, n^+(\omega))$ is not globally defined, which together with $m_k(t, \omega, n^+(\omega)) < m_k^+(\omega{\cdot}t)$ and $\lim_{t\to-\infty} |m_k(t, \omega, n^+(\omega)) - m_k^+(\omega{\cdot}t)| = 0$ (see again (1.30)) ensures that $m_k(t, \omega, n^+(\omega))$ tends to $-\infty$ at it approaches a positive value of t from the left. But this is impossible, since $0 \leq n^+(\omega{\cdot}t) = m(t, \omega, n^+(\omega)) \leq m_k(t, \omega, n^+(\omega))$ for $t \geq 0$ if the last function is defined (see again Theorem 1.46(i)). This contradiction completes the proof of the equalities $n^\pm = \widetilde{n}^\pm$.

The last assertion of the lemma follows immediately from the first one, which shows that $n^\pm$ are the pointwise limits of $(m_k^\pm)$ on Ω, and from Theorems 5.58 and 5.74(iii).

Lemma 8.47 *Suppose that Ω is minimal. Let $\mathcal{M} \subset \mathcal{K}_\mathbb{R}$ be an almost automorphic extension of the base flow such that the fiber $\mathcal{M}_\omega = \{l \mid (\omega, l) \in \mathcal{M}\}$ reduces to a point for a residual set Ω_c of points of the base; i.e. $\mathcal{M}_\omega = \{l(\omega)\}$ for every $\omega \in \Omega_c$. Let $\widetilde{\mathcal{R}} \subset \mathcal{M}$ be a residual set in $\mathcal{M}$. Then the projection $\mathcal{R} = \{\omega \mid$ there exists $l \in \mathcal{L}_\mathbb{R}$ with $(\omega, l) \in \widetilde{\mathcal{R}}\}$ is a residual set in Ω.*

Proof The definition of residual set ensures the existence of a countable family $\{\widetilde{\mathcal{P}}_k \mid k \in \mathbb{N}\}$ of closed subsets of $\mathcal{M}$ with $(\mathrm{int}_{\mathcal{K}_\mathbb{R}} \widetilde{\mathcal{P}}_n) \cap \mathcal{M}$ empty such that $\mathcal{M} - \widetilde{\mathcal{R}} \subseteq \cup_{k\in\mathbb{N}} \widetilde{\mathcal{P}}_n$. Let $\Pi\colon \mathcal{K}_\mathbb{R} \to \Omega,\ (\omega, l) \mapsto \omega$ be the projection onto the base, so that $\mathcal{R} = \Pi(\widetilde{\mathcal{R}})$. Define $\mathcal{P}_k = \Pi(\widetilde{\mathcal{P}}_k)$ for $k \in \mathbb{N}$ and note that all these sets are closed in Ω. It is easy to deduce from the fact that $\Pi(\mathcal{M}) = \Omega$ (which is an easy consequence of the minimality of the base flow) that $\Omega - \mathcal{R} \subseteq \Pi(\mathcal{M} - \widetilde{\mathcal{R}}) \subseteq \cup_{k\in\mathbb{N}} \mathcal{P}_k$. The goal is hence to prove that $\mathrm{int}_\Omega\, \mathcal{P}_k$ is empty for all $k \in \mathbb{N}$.

Suppose for contradiction the existence of $k \in \mathbb{N}$, $\omega_1 \in \Omega$ and $\delta_0 > 0$ such that $\mathcal{B} \subseteq \mathcal{P}_k$, where $\mathcal{B}$ is the closed ball of points of Ω at a distance from ω_1 less than or equal to δ_0. It follows immediately that

$$\big(\Pi^{-1}(\mathcal{B} \cap \Omega_c)\big) \cap \mathcal{M} \subseteq \big(\Pi^{-1}(\mathcal{P}_k \cap \Omega_c)\big) \cap \mathcal{M}\,.$$

The property assumed on Ω_c has two consequences. The first one is that

$$\big(\Pi^{-1}(\mathcal{B} \cap \Omega_c)\big) \cap \mathcal{M} = (\mathcal{B} \times \mathcal{L}_\mathbb{R}) \cap (\Omega_c \times \mathcal{L}_\mathbb{R}) \cap \mathcal{M}\,,$$

since both sets agree with $\{(\omega, l(\omega)) \mid \omega \in \mathcal{B} \cap \Omega_c\}$. The second consequence is that

$$\big(\Pi^{-1}(\mathcal{P}_k \cap \Omega_c)\big) \cap \mathcal{M} \subseteq \widetilde{\mathcal{P}}_k\,.$$

This second consequence can be proved as follows: take $\omega \in \mathcal{P}_k \cap \Omega_c$ and $l \in \mathcal{L}_\mathbb{R}$ with $(\omega, l) \in \mathcal{M}$ (that is, take $(\omega, l) \in \big(\Pi^{-1}(\mathcal{P}_k \cap \Omega_c)\big) \cap \mathcal{M}$); note that, then, $l = l(\omega)$; and conclude that if it were the case that $(\omega, l) = (\omega, l(\omega)) \notin \widetilde{\mathcal{P}}_k \subseteq \mathcal{M}$, then $(\{\omega\} \times \mathcal{L}_\mathbb{R}) \cap \widetilde{\mathcal{P}}_k$ would be empty, so that $\omega \notin \Pi(\widetilde{\mathcal{P}}_k) = \mathcal{P}_k$ and a contradiction would be reached.

Therefore,

$$(\mathcal{B} \times \mathcal{L}_\mathbb{R}) \cap (\Omega_c \times \mathcal{L}_\mathbb{R}) \cap \mathcal{M} \subseteq \widetilde{\mathcal{P}}_k\,.$$

Clearly, the left-hand term is dense in $(\mathcal{B} \times \mathcal{L}_\mathbb{R}) \cap \mathcal{M}$. Therefore, taking closures in $\mathcal{K}_\mathbb{R}$ leads to $(\mathcal{B} \times \mathcal{L}_\mathbb{R}) \cap \mathcal{M} \subseteq \widetilde{\mathcal{P}}_k$: this is the sought-for contradiction. The lemma is proved.

Example 8.48 The example will consist of a three-dimensional family of nonautonomous LQ control problems which is uniformly strictly dissipative but which has the following properties: there is just one single autonomous control system in the family (but the quadratic functionals are time-dependent); and this autonomous control system is not controllable. The corresponding family of linear Hamiltonian systems will have exponential dichotomy, but the Weyl functions will not exist. The uniform strict dissipativity will hence be deduced from Theorem 8.41.

Consider again the family of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} -1 - \widetilde{g}(\omega{\cdot}t) & 1 \\ -1 & 1 + \widetilde{g}(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.76}$$

constructed in the previous example, where the following properties were proved: the family does not have exponential dichotomy, all the systems of the family (8.76) are disconjugate, and the uniform principal solutions determine the Lagrange planes $l^+(\omega) \equiv \Big[\begin{smallmatrix} 1 \\ n^+(\omega) \end{smallmatrix}\Big]$ and $l^-(\omega) \equiv \Big[\begin{smallmatrix} 1 \\ n^-(\omega) \end{smallmatrix}\Big]$, with $0.5 \le n^+(\omega) \le n^-(\omega) \le 2$ for each $\omega \in \Omega$: see (8.68).

According to Theorem 5.61(i), the family of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} -1 - \widetilde{g}(\omega{\cdot}t) & 1 \\ -1 + \lambda & 1 + \widetilde{g}(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,, \tag{8.77}$$

has exponential dichotomy over Ω for every $\lambda \in (0, \infty)$, with corresponding Lagrange planes $l^\pm_\lambda(\omega) \equiv \Big[\begin{smallmatrix} 1 \\ m^\pm(\omega,\lambda) \end{smallmatrix}\Big]$, and in addition $n^-(\omega) \le m^-(\omega, \lambda)$ for every $\lambda > 0$. In particular, $m^-(\omega, \lambda) > 0.5 \ge 0$ for all $\omega \in \Omega$ and $\lambda > 0$.

For the rest of the example, $\lambda > 0$ will be fixed. The robustness of the exponential dichotomy (see e.g. Theorem 1.95) ensures that, if $\varepsilon > 0$ is small enough, then the family of linear Hamiltonian systems

$$\mathbf{z}' = \begin{bmatrix} -1-\widetilde{g}(\omega{\cdot}t) & 1+\varepsilon \\ -1+\lambda & 1+\widetilde{g}(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,,$$

also has exponential dichotomy over Ω with corresponding Lagrange planes $\left[\begin{smallmatrix} 1 \\ m_\varepsilon^\pm(\omega,\lambda) \end{smallmatrix}\right]$ satisfying $\lim_{\varepsilon\to 0+} m_\varepsilon^\pm(\omega,\lambda) = m^\pm(\omega,\lambda)$. Therefore, for $\varepsilon > 0$ small enough, $m_\varepsilon^-(\omega,\lambda) > 0$ for every $\omega \in \Omega$. This fact will be of fundamental importance in the later discussion of this example.

Now it is possible to define the desired family of LQ control problems. Define, as in the previous example, $\widetilde{G}(\omega) = \widetilde{g}^2(\omega) - 1$. Consider the diagonal matrices

$$\widetilde{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\widetilde{G}_\lambda(\omega) = \widetilde{G}(\omega)\,\widetilde{B} + \lambda I_3$, and $\widetilde{R} = I_3$, and the (autonomous) linear control system

$$\mathbf{x}' = \widetilde{A}\,\mathbf{x} + \widetilde{B}\,\mathbf{u}\,, \tag{8.78}$$

with the (nonautonomous) family of quadratic functionals

$$\widetilde{\mathcal{Q}}_\omega^\lambda(t,\mathbf{x},\mathbf{u}) = \frac{1}{2}\,\big(\langle \mathbf{x}, \widetilde{G}_\lambda(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\,\langle \mathbf{x}, \widetilde{g}(\omega{\cdot}t)\,\widetilde{B}\,\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{u}\rangle\big)\,. \tag{8.79}$$

It can immediately be checked that condition (6.12) does not hold, so that the control problem (8.78) is not null controllable. The associated family of six-dimensional linear Hamiltonian systems is

$$\mathbf{z}' = \begin{bmatrix} \widetilde{A} - \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} & \widetilde{B} \\ -\widetilde{B} + \lambda I_3 & -\widetilde{A} + \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} \end{bmatrix} \mathbf{z}\,. \tag{8.80}$$

Note that (8.80) uncouples into three two-dimensional families: the family (8.77) and the constant coefficient ones

$$\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ \lambda & 1 \end{bmatrix} \mathbf{z} \quad \text{and} \quad \mathbf{z}' = \begin{bmatrix} 1 & 0 \\ \lambda & -1 \end{bmatrix} \mathbf{z}\,.$$

It is obvious that these constant systems have exponential dichotomy, and that the only (constant) Weyl functions are $m_2^+ = -\lambda/2$ for the first one, and $m_3^- = \lambda/2$ for last one. Therefore, the family (8.80) has exponential dichotomy over Ω, with

Lagrange planes

$$l^+(\omega,\lambda) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ m^+(\omega,\lambda) & 0 & 0 \\ 0 & -\lambda/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad l^-(\omega,\lambda) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ m^-(\omega,\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda/2 \end{bmatrix},$$

which obviously lie in the vertical Maslov cycle $\mathcal{C}$. (Incidentally, note that $\widetilde{d}^{\pm}(\omega) = 1$ for every $\omega \in \Omega$, where $\widetilde{d}^{\pm}$ are defined in Sect. 5.9. In fact the shape of $\widetilde{B}$ precludes condition D2 of Chap. 5, so that one cannot have $\widetilde{d}^+(\omega) = 0$ or $\widetilde{d}^-(\omega) = 0$ for any value of ω: see Corollary 5.86.) The same uncoupling procedure ensures that, for the values of $\varepsilon > 0$ previously associated to λ, the family of perturbed systems

$$\mathbf{z}' = \begin{bmatrix} \widetilde{A} - \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} & \widetilde{B} + \varepsilon I_3 \\ -\widetilde{B} + \lambda I_3 & -\widetilde{A} + \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega \tag{8.81}$$

has exponential dichotomy over Ω, with Lagrange planes $\left[\begin{smallmatrix} I_3 \\ M_\varepsilon^{\pm}(\omega,\lambda) \end{smallmatrix}\right]$, where

$$M_\varepsilon^+(\omega,\lambda) = \begin{bmatrix} m_\varepsilon^+(\omega,\lambda) & 0 & 0 \\ 0 & (1-\sqrt{1+\varepsilon\,\lambda})/\varepsilon & 0 \\ 0 & 0 & (-1-\sqrt{1+\varepsilon\,\lambda})/\varepsilon \end{bmatrix} \quad \text{and}$$

$$M_\varepsilon^-(\omega,\lambda) = \begin{bmatrix} m_\varepsilon^-(\omega,\lambda) & 0 & 0 \\ 0 & (1+\sqrt{1+\varepsilon\,\lambda})/\varepsilon & 0 \\ 0 & 0 & (-1+\sqrt{1+\varepsilon\,\lambda})/\varepsilon \end{bmatrix}.$$

Hence $M_\varepsilon^-(\omega,\lambda) > 0$ for $\varepsilon > 0$ small enough, and Theorem 8.41 and Remark 8.42 imply that the family of LQ control problems given by (8.78) and (8.79) is uniformly strictly dissipative. Of course, this is true for any value of $\lambda > 0$.

Note finally that the same construction could be carried out by taking as the starting point a system simpler than (8.76) but satisfying similar properties: absence of exponential dichotomy, presence of uniform weak disconjugacy, and with $n^- > 0$; for instance, $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix}\right] \mathbf{z}$, coming from $x' = -x+u$ and $\widetilde{Q}(x,t) = (-x^2+u^2)/2$, for which $n^+ = n^- = 1$: see Remark 8.35. And the same remarks apply to the next example.

Example 8.49 The last example will show that, in fact, the unperturbed family of LQ control problems of the previous example, given by (8.78) and (8.79) for $\lambda = 0$ (which is not null controllable, and for which the corresponding family of Hamiltonian system does not have exponential dichotomy and is not uniformly weakly disconjugate), is uniformly dissipative. Theorem 8.41 will again be the main tool used in the discussion.

So, consider a new perturbation of the family of Hamiltonian systems associated to (8.78) and (8.79), now given by

$$\mathbf{z}' = \begin{bmatrix} \widetilde{A} - \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} & \widetilde{B} + \varepsilon(I_3 - \widetilde{B}) \\ -\widetilde{B} & -\widetilde{A} + \widetilde{g}(\omega{\cdot}t)\,\widetilde{B} \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega \tag{8.82}$$

for $\varepsilon > 0$ and note that it uncouples into three two-dimensional families: the family (8.76) and the single constant coefficient systems

$$\mathbf{z}' = \begin{bmatrix} -1 & \varepsilon \\ 0 & 1 \end{bmatrix} \mathbf{z} \quad \text{and} \quad \mathbf{z}' = \begin{bmatrix} 1 & \varepsilon \\ 0 & -1 \end{bmatrix} \mathbf{z}\,.$$

It is obvious the last two systems have exponential dichotomy, and easy to compute their (constant) Weyl functions: $m_2^+ = 0$ and $m_2^- = 2/\varepsilon$ for that on the left, and $m_3^+ = -2/\varepsilon$ and $m_3^- = 0$ for that on the right. Therefore,

$$\widetilde{l}^+(\omega) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ n^+(\omega) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2/\varepsilon \end{bmatrix}, \qquad \widetilde{l}^-(\omega) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ n^-(\omega) & 0 & 0 \\ 0 & 2/\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(where $n^+(\omega)$ and $n^-(\omega)$ are the principal functions of (8.76)) are Lagrange planes satisfying $U(t,\omega)\cdot\widetilde{l}^\pm(\omega) = \widetilde{l}^\pm(\omega{\cdot}t)$. In particular, the family (8.82) satisfies condition D3 of Chap. 5. Since it also satisfies D1 and D2 (see e.g. Remark 5.19), Theorem 5.17 ensures that it is uniformly weakly disconjugate. There are several ways to prove that $\widetilde{l}^\pm$ define the uniform principal solutions at $\pm\infty$. For instance, it is very easy to check that they satisfy the conditions in Definition 5.6. Hence,

$$N_\varepsilon^-(\omega) = \begin{bmatrix} n^-(\omega) & 0 & 0 \\ 0 & 2/\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$$

is the principal function at $-\infty$. Since $N_\varepsilon^-(\omega) \geq 0$ for all $\omega \in \Omega$, Theorem 8.41 ensures that the family of LQ control problems given by (8.78) and (8.79) is uniformly strictly dissipative, and that $\widetilde{V}_\omega^\varepsilon(t,\mathbf{x}) = \langle \mathbf{x}, N_\varepsilon^-(\omega)\,\mathbf{x}\rangle$ defines a (nonstrong) storage function for every $\omega \in \Omega$. Note also that this storage function is not jointly continuous.

8.8 Back to the Time-Reversed Problem

As mentioned in the introduction to this chapter, the goal of this last section is to adapt the results of the previous ones to the time-reversed situation, providing hence new scenarios in which the dissipativity of LQ control problems can be deduced from the dynamical properties of certain families of linear Hamiltonian systems.

Consider, as in Sect. 8.2, the time-reversed flow σ^- on Ω^- and the family of control systems

$$\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x} - B(\omega{\cdot}(-t))\,\mathbf{u}\,, \qquad \omega \in \Omega^-\,. \tag{8.83}$$

Consider also the family of time-reversed quadratic forms

$$\begin{aligned}\widetilde{\mathcal{Q}}^-_\omega(t,\mathbf{x},\mathbf{u}) =& \frac{1}{2}\Big(\langle \mathbf{x}, G(\omega{\cdot}(-t))\,\mathbf{x}\rangle \\ &+ 2\langle \mathbf{x}, g(\omega{\cdot}(-t))\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}(-t))\,\mathbf{u}\rangle\Big)\end{aligned} \tag{8.84}$$

for $\omega \in \Omega^-$, as well as the family of linear Hamiltonian systems

$$\mathbf{z}' = H^-(\omega{\cdot}(-t))\,\mathbf{z}\,, \qquad \omega \in \Omega^-\,, \tag{8.85}$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and

$$H^-(\omega) = \begin{bmatrix} -A(\omega) + B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) \\ G(\omega) - g(\omega)R^{-1}(\omega)g^T(\omega) & A^T(\omega) - g(\omega)R^{-1}(\omega)B^T(\omega) \end{bmatrix}.$$

Note that, following the ideas of Chap. 7, this family of Hamiltonian systems would arise by applying the Pontryagin Principle to the family of problems obtained by trying to minimize

$$\widetilde{\mathcal{I}}^-_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u}) = \int_0^\infty \widetilde{\mathcal{Q}}^-_\omega(s,\mathbf{x}(s),\mathbf{u}(s))\,ds$$

for $\omega \in \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Here the admissible pairs $(\mathbf{x},\mathbf{u})$ are determined by referring to the control systems (8.83) with $\mathbf{x}(0) = \mathbf{x}_0$. Strictly speaking, the proof of the Pontryagin Principle requires a stabilization condition, but this condition is not required to construct the systems (8.85) from the data of (8.83) and (8.84).

It can immediately be checked the following simple but important properties.

Proposition 8.50 *The function* $\left[\begin{smallmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{smallmatrix}\right]$ *solves* (8.6) *if and only if* $\left[\begin{smallmatrix}\widetilde{\mathbf{x}}(t)\\ \widetilde{\mathbf{y}}(t)\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbf{x}(-t)\\ -\mathbf{y}(-t)\end{smallmatrix}\right]$ *solves* (8.85)*; and the function* $M(t,\omega,M_0)$ *solves* (8.9) *on* (a,b) *with* $M(t,\omega,M_0) = M_0$ *if and only if* $N(t,\omega,M_0) = -M(-t,\omega,M_0)$ *solves the Riccati equation defined* *from the time-reversed Hamiltonian system* (8.85) *on* $(-b,-a)$ *with* $N(0,\omega,M_0) = M_0$.

In particular, these facts imply that the symmetry $\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{x}(-t) \\ -\mathbf{y}(-t) \end{bmatrix}$ can be viewed as a map from the solution space of (8.6) onto the solution space of (8.85). They have a fundamental dynamical consequence regarding the existence of exponential dichotomy of the time-reversed systems and the structure of the corresponding Lagrange planes.

Proposition 8.51 *Suppose that the family of Hamiltonian systems* (8.6) *has exponential dichotomy over* Ω, *and that the Weyl function* M^+ *(resp.* M^-*) globally exists. Then the family* (8.85) *has exponential dichotomy over* Ω^- *and the Weyl function* $\widetilde{M}^-$ *globally exists, with* $\widetilde{M}^-(\omega) = -M^+(\omega)$ *for all* $\omega \in \Omega$ *(resp. the Weyl function* $\widetilde{M}^+$ *globally exists, with* $\widetilde{M}^+(\omega) = -M^-(\omega)$ *for all* $\omega \in \Omega$*).*

Proof Theorem 1.78 shows that the presence of exponential dichotomy for the family (8.85) over Ω^- is equivalent to the absence of nonzero bounded solutions, and hence the first assertion follows from Proposition 8.50. This last result shows that, for each $\omega \in \Omega$, a basis $\left\{ \begin{bmatrix} \mathbf{x}_1^\pm(t) \\ \mathbf{y}_1^\pm(t) \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{x}_n^\pm(t) \\ \mathbf{y}_n^\pm(t) \end{bmatrix} \right\}$ of $l^\pm(\omega{\cdot}t)$ provides a basis $\left\{ \begin{bmatrix} \mathbf{x}_1^\pm(-t) \\ -\mathbf{y}_1^\pm(-t) \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{x}_n^\pm(-t) \\ -\mathbf{y}_n^\pm(-t) \end{bmatrix} \right\}$ of $\widetilde{l}^\mp(\omega{\cdot}t)$. Here $l^\pm(\omega)$ and $\widetilde{l}^\pm(\omega)$ are the Lagrange planes of the initial data of the solutions bounded as $t \to \pm\infty$ of the families (8.6) and (8.85) respectively: see Remark 1.77.3. This implies the remaining assertions.

Corollary 8.52 *Hypotheses 8.19 hold for the families* (8.4) *and* (8.6) *over* Ω *if and only if they hold for the families* (8.83) *and* (8.85) *over* Ω^-.

Proof The assertion follows easily from Propositions 8.4, 8.20, and 8.51.

These basic facts are the keys to reformulating all the results of this chapter regarding dissipativity, but now for the time-reversed control family and supply rate. Note that the proofs do not need to be repeated: the "new" facts are consequences of those already known and those just summarized. The role previously played by M^-, in the main results, is hence now played by M^+ (more precisely by $-M^+$). The main results obtained by means of this reformulation are now given.

1. Suppose that there exist a point $\omega \in \Omega$ and a negative semidefinite matrix $M_0 \geq 0$ such that $M(t,\omega,M_0)$ is a globally defined solution of the time-reversed Riccati equation, with $M(t,\omega,M_0) \leq 0$ (resp. $M(t,\omega,M_0) < 0$) for all $t \in \mathbb{R}$. Then the time-reversed LQ_ω problem is dissipative (resp. strictly dissipative) and $V_{\omega,M_0}(t,\mathbf{x}) = \langle \mathbf{x}, M(t,\omega,M_0)\,\mathbf{x}\rangle$ is a storage function (resp. strong storage function) for it.
2. Suppose that there exists a point $\omega_0 \in \Omega$ with dense σ-orbit and a negative semidefinite matrix $M_0 \leq 0$ such that $M(t,\omega_0,M_0)$ is a globally defined and bounded solution of the corresponding Riccati equation, with $M(t,\omega_0,M_0) \leq 0$ for all $t \in \mathbb{R}$. Then each time-reversed LQ_ω control problem of the family is dissipative and admits a storage function.

3. Suppose that Hypotheses 8.19 hold. Then the following assertions are equivalent:

   (1) the family of control systems (8.83) is uniformly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84);
   (2) $M^+ \leq 0$.

   In addition, in this case the function $V_\omega^+(t, \mathbf{x}) = -\langle \mathbf{x}, M^+(\omega{\cdot}t)\,\mathbf{x}\rangle$ is the required supply for the time-reversed LQ$_\omega$ control problem, and is jointly continuous in the variables $(\omega, t, \mathbf{x})$.
4. Suppose that Hypotheses 8.19 hold. Then the following assertions are equivalent:

   (1) the family of control systems (8.83) is uniformly strictly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84);
   (2) $M^+ < 0$.

   In addition, in this case the function $V_\omega^+(t, \mathbf{x}) = -\langle \mathbf{x}, M^+(\omega{\cdot}t)\,\mathbf{x}\rangle$ is the required supply for the time-reversed LQ$_\omega$ control problem. It is strong, and is jointly continuous in the variables $(\omega, t, \mathbf{x})$.
5. Suppose that the family (8.6) admits exponential dichotomy and that the Weyl function M^+ globally exists. Define $V_\omega^+(t, \mathbf{x}) = -\langle \mathbf{x}, M^+(\omega{\cdot}t)\,\mathbf{x}\rangle$. Then,

   (i) if $M^+(\omega) \leq 0$ for all $\omega \in \Omega$, then the family of control systems (8.83) is uniformly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84). In addition, $V_\omega^+(t, \mathbf{x})$ is a storage function for the time-reversed LQ$_\omega$ control problem, and is jointly continuous in the variables $(\omega, t, \mathbf{x})$.
   (ii) If $M^+(\omega) < 0$ for all $\omega \in \Omega$, then the family of control systems (8.83) is uniformly strictly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84). In addition, the storage function $V_\omega^+(t, \mathbf{x})$ is strong.

6. With the hypotheses and notation established just before Theorem 8.41, define $W_\omega^\varepsilon(t, \mathbf{x}) = \langle \mathbf{x}, M_\varepsilon^+(\omega{\cdot}t)\,\mathbf{x}\rangle$ for each $\varepsilon \in (0, \rho)$. Then,

   (i) if there is an $\varepsilon \in (0, \rho)$ such that $M_\varepsilon^+(\omega) \leq 0$ for all $\omega \in \Omega$, then the family of control systems (8.83) is uniformly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84). In addition, $W_\omega^\varepsilon(t, \mathbf{x})$ is a storage function for the time-reversed LQ$_\omega$ control problem, and is jointly continuous in the variables $(\omega, t, \mathbf{x})$.
   (ii) If there is an $\varepsilon \in (0, \rho)$ such that $M_\varepsilon^+(\omega) < 0$ for all $\omega \in \Omega$, then the family of control systems (8.83) is uniformly strictly dissipative with family of supply rates $\{\widetilde{\mathcal{Q}}_\omega^- \mid \omega \in \Omega\}$ given by (8.84). In addition, the storage function $W_\omega^\varepsilon(t, \mathbf{x})$ is strong.

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