Control Engineering

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Linear Systems Theory

A Structural Decomposition Approach

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Contents

Preface

Structural properties play an important role in our understanding of linear systems in the state space representation. The structural canonical form representation of linear systems not only reveals the structural properties but also facilitates the design of feedback laws that meet various control objectives. In particular, it decomposes the system into various subsystems. These subsystems, along with the interconnections that exist among them, clearly show the structural properties of the system. The simplicity of the subsystems and their explicit interconnections with each other lead us to a deeper insight into how feedback control would take effect on the system, and thus to the explicit construction of feedback laws that meet our design specifications. The discovery of structural canonical forms and their applications in feedback design for various performance specifications has been an active area of research for a long time. The effectiveness of the structural decomposition approach has also been extensively explored in nonlinear systems and control theory in the recent past.

The aim of this book is to systematically present various canonical representations of the linear system, that explicitly reveal different structural properties of the system, and to report on some recent developments on its utilization in system analysis and design. The systems we will consider include the autonomous system, whose inherent properties are solely determined by a matrix that represents its dynamics; the unforced or unsensed system, whose inherent properties are dependent on a pair of matrices, the matrix that represents its internal dynamics and the measurement or control matrix; and the proper system whose inherent properties are determined by a matrix triple or a matrix quadruple. We will also consider linear descriptor systems whose structural properties are determined by a matrix quintuple. All the results will be presented in both continuous-time and discrete-time settings. The relationship between the structural properties of a continuous-time system and those of its discrete-time counterpart under a bilinear transformation will also be established.

The intended audience for this book includes graduate students, practicing control engineers and researchers in areas related to systems and control engineering. In writing this book, we have striven to make the presentation self-contained. A comprehensive review of various topics from matrix theory and linear systems theory is included in the beginning of the book. However, it is assumed that the reader has previous knowledge in both linear algebra and linear systems and control theory.

The first two authors would like to express their hearty thanks to late Professor Chin S. Hsu of Washington State University, for his kind help during their stay at Washington State University, and for his vivid and entertaining lectures on linear systems theory. A number of exercise problems in Chapter 3 are adopted from the homework assignments and examination questions of his course. They would also like to thank Professor Ali Saberi of Washington State University and Professor Pedda Sannuti of Rutgers University for their rigorous supervision during the PhD programs of the first two authors at Washington State University, and in particular, for their guidance to the theory of the special coordinate basis of linear systems, one of the key components presented in this manuscript. The third author would like to express his thanks to Professor Ali Saberi for introducing him to this research topic and his initial collaboration. The first author is particularly thankful to Professor Pedda Sannuti for his invaluable guidance on the preparation of scientific manuscripts. The second author would like to thank Professor Gongtian Yan of Beijing Institute of Control Engineering for his rigorous instruction on matrix theory and linear systems theory.

We are indebted to Mr. Xinmin Liu of the National University of Singapore for his assistance in developing a MATLAB software toolkit for this book, and to Professor Dazhong Zheng of Tsinghua University for his help in proofreading the manuscript. We are thankful to Dr. Delin Chu, Dr. Kemao Peng, Mr. Guoyang Cheng, Mr. Yingjie He and Dr. Minghua He, all of the National University of Singapore, for their many comments on the manuscript. We would also like to thank Professor Tong H. Lee of the National University of Singapore, Professor Frank L. Lewis of the University of Texas at Arlington, Professor Iven Mareels of the University of Melbourne, and Professor Mehrdad Saif of Simon Fraser University, for many beneficial discussions on the subject. The authors are grateful to their respective institutions, the National University of Singapore, the University

of Virginia, and the State University of New York at Stony Brook, for excellent environments for fundamental research.

We are indebted to Professor William S. Levine, the series editor, for his enthusiasm and encouragement of our effort in completing this book. We are also thankful to the editorial staff at Birkhäuser, in particular, Mr. Thomas Grasso and Mr. Seth Barnes, for their excellent editorial assistance.

Finally, we note that each of the computational algorithms included in the book has been implemented in the *Linear Systems Toolkit*, in the MATLAB environment. Access to the toolkit, a beta-version of which is available at the web site, `http://linearsystemskit.net`, will greatly facilitate understanding and application of the various analysis and design algorithms presented in the book. Interested readers who have our earlier versions of the software realization of the special coordinate basis, *i.e.*, those reported in Chen (1988) [17], Lin (1989) [84] and Lin *et al.* (1991) [90], are strongly encouraged to update to the new toolkit. The special coordinate basis, one of the structural decomposition techniques covered in this book, implemented in the new toolkit, is based on a numerically stable algorithm recently reported in Chu *et al.* (2002) [36] together with an enhanced procedure reported in this book.

This monograph was typeset by the authors using LaTeX. All simulations and numerical computations were carried out in MATLAB. Diagrams were generated using XFIG in LINUX and MATLAB with SIMULINK.

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Linear Systems Theory

Chapter 1

Introduction and Preview

1.1 Motivation

The state space representation of linear systems is fundamental to the analysis and design of dynamical systems. Modern control theory relies heavily on the state space representation of dynamical systems, which facilitates characterization of the inherent properties of dynamical systems. Since the introduction of the concept of a state, the study of linear systems in the state space representation itself has emerged as an ever active research area, covering a wide range of topics from the basic notions of stability, controllability, observability, redundancy and minimality to more intricate properties of finite and infinite zero structures, invertibility, and geometric subspaces. A deeper understanding of linear systems facilitates the development of modern control theory. The demanding expectations from modern control theory impose an ever increasing demand for the understanding and utilization of subtler properties of linear systems.

The importance of linear systems theory and the active research activities associated with it are also reflected in the continual publication of text books and monographs in linear systems theory, especially since publication of the classic works on state space approach and other associated topics such as controllability, observability and stability by Kalman and his co-workers [66,71–73], Gilbert [58], and Zadeh and Desoer [157]. A recent featured review of books on linear systems by Naidu [103] contains an extensive survey of books published in different periods of time. A few examples of earlier books published around the 1960s and 1970s are DeRusso *et al.* (1965) [47], Ogata (1967) [104], Brockett (1970) [14], Chen (1970) [32], Rosenbrock (1970) [112], and Blackman (1977) [13]. Examples of relatively recent books on linear systems that appeared in the 1980s and later are Kailath (1980) [70], McClamroch (1980) [98], Chen (1984) [33], DeCarlo (1989) [46], Sontag (1990) [132], Antoulas (1991) [3], Callier and Desoer (1991) [16], Rugh (1996) [114], Antsaklis and Michel (1997) [4], DeRusso (1998) [48], Sontag (1998) [133], Bay (1999) [10], Chen (1999) [34], Aplevich (2000) [5], and Trentelman *et al.* (2001) [141].

These books take different pedagogical approaches to presenting fundamental aspects of linear systems theory and to reporting on new advances in the field. This book takes a structural decomposition approach to the study of linear time-invariant systems. Structural decomposition is not a new concept, and several structural decompositions can be found in many existing text books on linear systems theory. Basic properties such as stability, controllability and observability of a linear time-invariant system in the state space representation can all be characterized in terms of coefficient matrices. For a controllable system, the inner working of how each control signal reaches different parts of the system can be characterized by the controllability index, a structural property that is invariant under state transformation. Appropriate state variables can be chosen such that the system is represented in a so-called controllable canonical form, from which the controllability indices can be readily identified and a stabilizing feedback law can be constructed in a straightforward way. The representation of controllable canonical form is a structural decomposition, which reveals the controllability indices of the system. Other examples of structural decomposition that can be found in linear systems theory text books include the observable canonical form (which reveals the observability indices), and the Kalman Decomposition (which characterizes the controllability and observability of the system modes).

The above structural decompositions demonstrate their power in our study of the problems of stabilization and state observation. However, the study of control problems beyond stabilization and state observation entails the understanding of structural properties more intricate than controllability and observability. For example, control problems such as H_2 and H_∞ optimal control are closely related to subtler structural properties such as finite and infinite zero structures and system invertibility properties. Naturally, there have always been efforts devoted to the study of various structural properties of linear systems and their utilization in the analysis and design of control systems. As such, it is appropriate to trace a short history of the development of structural decomposition techniques for linear systems. To the best of our knowledge, the concept of utilizing structural decomposition of a dynamical system in solving control problems beyond stabilization first arose while dealing with high gain and cheap control problems (see e.g., Sannuti [121]). By separating the finite and infinite zero structures of what are now known as uniform rank systems, Sannuti [121] showed the usefulness of utilizing the special coordinate basis, a structural decomposition, to discuss the important features of high gain and cheap control problems. Sannuti and Wason [123] later extended the concept of the special coordinate basis to general invertible systems and showed its significance in connection with multivariable root locus theory. By utilizing a modified structural algorithm of Silverman [131], Sannuti and Saberi [122] and Saberi and Sannuti [119] solidified the concept of the special coordinate basis of general linear multivariable systems, which is structure-wise fairly similar to the nine-fold canonical decomposition of Aling and Schumacher [2], and identified most of its important properties including those that are related to certain subspaces encountered in geometric control theory and the invariant indices of Morse [100]. However, all the properties of the special coordinate basis in the original work of [122,119] were reported without detailed proofs. The theory was recently completed by Chen [21], which includes rigorous proofs to all the aforementioned properties within the framework of the special coordinate basis for general strictly and nonstrictly proper systems. More recently, He and Chen [64] and He, Chen and Lin [65] further extended the technique to general linear singular or descriptor systems. Also, in the past several years, we, together with our co-workers, have been studying the structural properties and demonstrating the applications of structural decompositions in the solution of numerous control problems, in a systematic manner. A coherent approach to linear systems theory and control has emerged from our results, which are dispersed in the literature, many in relatively abstract forms. It is our intention to bring these results together with more detailed illustrations and interpretations, and put them under a single cover.

1.2 Preview of Each Chapter

Briefly, the book contains 12 chapters, which can be naturally divided into three parts. The first part, Chapters 1 to 3, deals with the needed background material and can serve as a comprehensive review of linear systems theory. In particular, Chapter 1 is the introduction to the book. It also introduces the notation to be used throughout the book, while Chapter 2 collects some basic facts from matrix theory. Chapter 3 summarizes essential elements of linear systems theory. Both Chapter 2 and Chapter 3 serve as a review of the background materials needed for the book.

The second part of the book, Chapters 4 to 7, presents various structural decompositions for linear systems, both in continuous-time and in discrete-time. Various intricate system properties are identified in the context of these structural decompositions. In particular, Chapter 4 presents structural decompositions for systems that are unforced and/or unsensed. For systems that are both unforced and unsensed, *i.e.*, autonomous systems, the structural properties center on system stability and include the stability structural decomposition (SSD), in which the system is decomposed into stable and unstable dynamics as well as dynamics that are associated with the imaginary axis poles, and the real Jordan decomposition (RJD). As in all other decompositions presented in this book, strong emphasis is placed on the numerical stability of the algorithms we develop for these decompositions. For unforced systems, we will present two structural decompositions, the observability structural decomposition (OSD) and the block diagonal observable structural decomposition (BDOSD). Dually, for unsensed systems, we will also present two structural decompositions, the controllability structural decomposition (CSD) and the block diagonal controllable structural decomposition (BDCSD). These structural decompositions for unforced and/or unsensed systems find many applications in control systems, including the sensor/actuator selection problem to be discussed in Chapter 9.

Chapters 5 and 6 present structural decompositions for proper linear systems and linear descriptor systems, respectively. Core to the structural decompositions for proper linear systems is the special coordinate basis (SCB) developed by Sannuti and Saberi [122] and the nine-fold canonical decomposition of Aling and Schumacher [2] for strictly proper systems. These structural decompositions display various structural properties of linear systems, including finite and infinite zero structures, system invertibility properties and geometric subspaces. The structural decomposition for regular systems have been instrumental in the solution of many control problems including the few control problems to be presented in Chapters 8 to 11 of this book. We expect the structural decomposition for descriptor systems will play similar roles in the solution of control problems for descriptor systems.

Chapter 7 studies the structural properties of a system under bilinear transformation. The bilinear and inverse bilinear transformations have widespread use in digital control and signal processing. It has also played a crucial role in solving the H_∞ control problem. In fact, the need to perform continuous-time to discrete-time model conversions arises in a range of engineering contexts, including sampled-data control system design and digital signal processing. As a consequence, numerous discretization procedures exist, including zero- and first-order hold input approximations, impulse invariant transformation, and bilinear transformation. In this chapter, we present a clear and comprehensive understanding of how the structures, *i.e.*, the finite and infinite zero structures, invertibility structures, as well as geometric subspaces of a general continuous-time (discrete-time) linear time-invariant system are mapped to those of its discrete-time (continuous-time) counterpart under the well-known bilinear (inverse bilinear) transformations.

The remaining chapters of this book contain several applications of the system structural decompositions presented in Chapters 4 to 7 in the analysis and design of linear control systems.

Chapter 8 presents algorithms for two system factorizations of general linear systems, the minimum phase and all-pass cascade factorization, which covers the well-known inner-outer factorization as a special case, and the generalized cascade factorization. These factorizations have been important algebraic problems in a variety of areas in electrical engineering, including systems and control analysis design. In particular, the minimum phase and all-pass cascade factorization factors a general nonminimum phase and non-left invertible system into a minimum phase and left invertible system cascaded with a stable all-pass system with unity gain. Our algorithm demonstrates how straightforward it is to obtain such a factorization, and consequently the inner-outer factorization, of a given system once it is displayed under the structural decomposition of Chapter 5.

Chapter 9 studies the flexibility in assigning structural properties to a given linear system, and introduces techniques for identifying sets of sensors which would yield desirable structural properties. It is well recognized that a major difficulty encountered in applying multivariable control synthesis techniques, such as the H_2 and H_∞ control techniques, to actual design is the inadequate study of the linkage between control performance and design implementation involving hardware selection, *e.g.*, appropriate sensors suitable for robustness and performance. This linkage provides a foundation upon which trade-offs can be considered at the preliminary design stage. Thus, one can introduce careful control design considerations into the overall engineering design process at an early stage.

Chapter 10 deals with the problem of asymptotic time-scale assignment. Based on the structural decomposition of a given system, a systematic procedure is developed for designing feedback laws that result in pre-specified eigenstructures of the closed-loop system. The essence of this time-scale assignment procedure is the utilization of subsystems which represent the finite and infinite as well as invertibility structures of the system, as revealed by the structural decomposition.

This time-scale assignment procedure has proven to be instrumental in the solution of many modern control problems including H_∞ control, H_2 control, loop transfer recovery and disturbance decoupling problems.

Chapter 11 addresses the problem of disturbance decoupling with or without internal stability by either state feedback or measurement feedback. The problem of disturbance decoupling has been extensively studied for the past three decades. It motivated the development of the geometric approach to linear systems and control theory, and has played a key role in a number of problems, including decentralized control, noninteracting control, model reference tracking control, and H_∞ optimal control. For the problem of disturbance decoupling with constant or static measurement feedback, there have been only a few results in the literature. With the aid of the structural decomposition, this chapter derives a set of structural conditions for the solvability of the disturbance decoupling problem with static measurement feedback and characterizes all the possible solutions for a class of systems which have a left invertible transfer function from the control input to the controlled output. For general systems, solutions can be derived by applying a similar procedure to an irreducible reduced-order system obtained from the given system using the structural decomposition technique of Chapter 5.

Chapter 12 includes the description of a MATLAB toolkit that implements all the analysis and design algorithms presented in the book. The toolkit itself is publicly available. The beta-version of the toolkit can be downloaded for free from the URL at `http://linearsystemskit.net`.

1.3 Notation

Throughout this book we shall adopt the following notation:

$$
\begin{aligned}
\mathbb{R} &:= \text{the set of real numbers,}\\
\mathbb{R}_+ &:= \text{the set of nonnegative real numbers,}\\
\mathbb{N} &:= \text{the set of all natural numbers, } i.e., 0, 1, 2, \ldots,\\
\mathbb{C} &:= \text{the entire complex plane,}\\
\mathbb{K} &:= \text{a scalar field associated with a vector space,}\\
\mathbb{C}^{\circ} &:= \text{the unit circle in the complex plane,}\\
\mathbb{C}^{\odot} &:= \text{the set of complex numbers inside the unit circle,}\\
\mathbb{C}^{\otimes} &:= \text{the set of complex numbers outside the unit circle,}\\
\mathbb{C}^{0} &:= \text{the imaginary axis in the complex plane,}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{C}^- &:= \text{the open left-half complex plane,} \\
\mathbb{C}^+ &:= \text{the open right-half complex plane,} \\
\operatorname{Re}\alpha &:= \text{the real part of a scalar } \alpha \in \mathbb{C}, \\
\operatorname{Im}\alpha &:= \text{the imaginary part of a scalar } \alpha \in \mathbb{C}, \\
\alpha^* &:= \text{the complex conjugate of a scalar } \alpha \in \mathbb{C}, \\
0 &:= \text{a scalar zero or a zero vector or a zero matrix,} \\
\emptyset &:= \text{an empty set,} \\
I &:= \text{an identity matrix of appropriate dimensions,} \\
I_k &:= k \times k \text{ identity matrix,} \\
\operatorname{diag}\{\cdots\} &:= \text{a diagonal matrix,} \\
\operatorname{blkdiag}\{\cdots\} &:= \text{a block diagonal matrix,} \\
X = [x_{ij}] &:= \text{a matrix } X \text{ with its entries being } x_{ij}, \\
X' &:= \text{the transpose of a matrix } X, \\
X^{\mathrm{H}} &:= \text{the conjugate transpose of a matrix } X, \\
\det(X) &:= \text{the determinant of a matrix } X, \\
\operatorname{rank}(X) &:= \text{the rank of a matrix } X, \\
\operatorname{normrank}(X) &:= \text{the normal rank of a rational matrix } X, \\
\operatorname{trace}(X) &:= \text{the trace of a matrix } X, \\
\operatorname{cond}(X) &:= \text{the condition number of a matrix } X, \\
X^\dagger &:= \text{the Moore–Penrose (pseudo) inverse of a matrix } X, \\
\lambda_i(X) &:= \text{the } i\text{-th eigenvalue of a matrix } X, \\
\lambda_{\min}(X) &:= \text{the minimum eigenvalues of a matrix } X \text{ whose } \lambda(X) \subset \mathbb{R}, \\
\lambda_{\max}(X) &:= \text{the maximum eigenvalues of a matrix } X \text{ whose } \lambda(X) \subset \mathbb{R}, \\
\lambda(X) &:= \text{the set of eigenvalues of a matrix } X, \\
\rho(X) &:= \text{the spectral radius of a matrix } X, \\
\sigma_i(X) &:= \text{the } i\text{-th singular value of a matrix } X, \\
\sigma_{\min}(X) &:= \text{the minimum singular value of a matrix } X, \\
\sigma_{\max}(X) &:= \text{the maximum singular value of a matrix } X, \\
\operatorname{im}(X) &:= \text{the image or range space of a matrix } X, \\
\ker(X) &:= \text{the kernel or null space of a matrix } X, \\
\mathcal{X} &:= \text{a vector space or subspace,}
\end{aligned}
$$

$$
\begin{aligned}
\dim(\mathcal{X}) &:= \text{the dimension of a subspace } \mathcal{X},\\
\mathcal{X}^{\perp} &:= \text{the orthogonal complement of a subspace } \mathcal{X},\\
C^{-1}\{\mathcal{X}\} &:= \text{the inverse image of subspace } \mathcal{X} \text{ associated with } C,\\
\mathcal{X}/\mathcal{V} &:= \text{the factor space of } \mathcal{X} \text{ modulo its subspace } \mathcal{V},\\
\oplus &:= \text{direct sum of vector subspaces},\\
\langle x, y\rangle &:= \text{the inner product of two vectors } x \text{ and } y,\\
\|\cdot\| &:= \text{a norm},\\
\|x\|_p &:= p\text{-norm of a vector } x,\ p \in [1,\infty],\\
|x| &:= \text{Euclidean norm of a vector } x,\\
\|X\|_p &:= p\text{-norm of a matrix } X,\ p \in [1,\infty],\\
\|X\|_{\mathrm{F}} &:= \text{Frobenius norm of a matrix } X,\\
\|g\|_p &:= \text{the } l_p\text{-norm of a signal, } g(t) \text{ or } g(k),\\
L_p &:= \text{the set of all functions whose } l_p\text{-norms are finite},\\
\|G\|_2 &:= \text{the } H_2\text{-norm of a stable system } G(s) \text{ or } G(z),\\
\|G\|_\infty &:= \text{the } H_\infty\text{-norm of a stable system } G(s) \text{ or } G(z),\\
\Sigma &:= \text{a continuous- or discrete-time linear system},\\
P_\Sigma(s) &:= \text{the system matrix of } \Sigma,\\
\mathbf{I}_i(\Sigma) &:= \text{the invariant lists of Morse, } i = 1,2,3,4,\\
S_\beta^\star(\Sigma) &:= \text{the finite zero structure of } \Sigma \text{ associated with a finite zero } \beta,\\
S_\infty^\star(\Sigma) &:= \text{the infinite zero structure of } \Sigma,\\
S_{\mathrm{R}}^\star(\Sigma) &:= \text{the right invertibility structure of } \Sigma,\\
S_{\mathrm{L}}^\star(\Sigma) &:= \text{the left invertibility structure of } \Sigma,\\
\mathcal{V}^{\mathrm{x}}(\Sigma) &:= \text{the weakly unobservable subspace of } \Sigma,\\
\mathcal{S}^{\mathrm{x}}(\Sigma) &:= \text{the strongly controllable subspace of } \Sigma,\\
\mathcal{R}^*(\Sigma) &:= \text{the controllable weakly unobservable subspace of } \Sigma,\\
\mathcal{N}^*(\Sigma) &:= \text{the distributionally weakly unobservable subspace of } \Sigma,
\end{aligned}
$$

and finally the symbol ■ is used to indicate the end of a proof.

Chapter 2

Mathematical Background

2.1 Introduction

In this chapter we recall some needed mathematical background materials. These include the fundamental facts and properties of vector spaces and matrix theory and the definitions and properties of various norms of vectors, matrices, signals and rational transfer functions. These materials are particularly useful for developing and understanding results in the following chapters. More detailed information on the subjects can be found in more specific mathematics textbooks, or other texts in linear systems and control theory (see, *e.g.*, Barnett [8], Chen [22], Chen [33], Desoer and Vidyasagar [49], Golub and Van Loan [59], Huang [68], Kailath [70], Kreyszig [78], Saberi *et al.* [120], Suda *et al.* [138], Trentelman *et al.* [141], Wielandt [150] and Wonham [154]).

2.2 Vector Spaces and Subspaces

We assume that the reader is familiar with the basic definitions of scalar fields and vector spaces.

Let $\mathcal{X}$ be a *vector space* over a certain scalar field $\mathbb{K}$. A subset of $\mathcal{X}$, say $\mathcal{S}$, is said to be a *subspace* of $\mathcal{X}$ if $\mathcal{S}$ itself is a vector space over $\mathbb{K}$. The *dimension* of a subspace $\mathcal{S}$, denoted by $\dim \mathcal{S}$, is defined as the maximal possible number of linearly independent vectors in $\mathcal{S}$.

We say that vectors $s_1, s_2, \ldots, s_k \in \mathcal{S}$, $k = \dim \mathcal{S}$, form a *basis* for $\mathcal{S}$ if they are *linearly independent*, *i.e.*, $\sum_{i=1}^{k} \alpha_i s_i = 0$ holds only if $\alpha_i = 0$. Two subspaces $\mathcal{V}$ and $\mathcal{W}$ are said to be independent if $\mathcal{V} \cap \mathcal{W} = \{0\}$.

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Throughout the book, we will focus our attention on two common vector spaces, *i.e.*, $\mathbb{R}^n$ (with a scalar field $\mathbb{K}=\mathbb{R}$) and $\mathbb{C}^n$ (with a scalar field $\mathbb{K}=\mathbb{C}$), and their subspaces. Thus, the *inner product* of two vectors, say x and y, in either $\mathbb{R}^n$ or $\mathbb{C}^n$, is given by

$$\langle x, y\rangle = x^{\mathrm{H}} y = x_1^* y_1 + x_2^* y_2 + \cdots + x_n^* y_n, \tag{2.2.1}$$

where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are respectively the entries of x and y, x^{H} is the conjugate transpose of x, and x_i^* is the complex conjugate of x_i. Vectors x and y are said to be *orthogonal* if $\langle x, y\rangle = 0$.

In what follows, we recall some frequently used concepts and properties of vector spaces and subspaces.

Definition 2.2.1 (Sums of subspaces). *Let $\mathcal{V}$ and $\mathcal{W}$ be the subspaces of a vector space $\mathcal{X}$. Then, the* sum *of subspaces $\mathcal{V}$ and $\mathcal{W}$ is defined as*

$$\mathcal{S} = \mathcal{V} + \mathcal{W} := \{v + w \mid v \in \mathcal{V},\ w \in \mathcal{W}\}. \tag{2.2.2}$$

If $\mathcal{V}$ and $\mathcal{W}$ are independent, then $\mathcal{S}$ is also called the direct sum *of $\mathcal{V}$ and $\mathcal{W}$ and is denoted by $\mathcal{S} = \mathcal{V} \oplus \mathcal{W}$. Obviously, in both cases, $\mathcal{S}$ is a subspace of $\mathcal{X}$.*

Definition 2.2.2 (Orthogonal complement subspace). *Let $\mathcal{V}$ be a subspace of a vector space $\mathcal{X}$. Then, the* orthogonal complement *of $\mathcal{V}$ is defined as*

$$\mathcal{V}^{\perp} := \{x \in \mathcal{X} \mid \langle x, v\rangle = 0,\ \forall v \in \mathcal{V}\}. \tag{2.2.3}$$

Again, $\mathcal{V}^{\perp}$ is a subspace of $\mathcal{X}$.

Definition 2.2.3 (Quotient space and codimension). *Let $\mathcal{V}$ be a subspace of a vector space $\mathcal{X}$. The* coset *of an element $x \in \mathcal{X}$ with respect to $\mathcal{V}$, denoted by $x + \mathcal{V}$, is defined as*

$$x + \mathcal{V} := \{w \mid w = x + v,\ v \in \mathcal{V}\}. \tag{2.2.4}$$

Under the algebraic operations defined by

$$(w + \mathcal{V}) + (x + \mathcal{V}) = (w + x) + \mathcal{V} \tag{2.2.5}$$

and

$$\alpha(w + \mathcal{V}) = \alpha w + \mathcal{V}, \tag{2.2.6}$$

it is straightforward to verify that all the cosets constitute the elements of a vector space. The resulting space is called the quotient space or factor space *of $\mathcal{X}$ by $\mathcal{V}$*

(or modulo $\mathcal{V}$) and is denoted by $\mathcal{X}/\mathcal{V}$. Its dimension is called the codimension *of $\mathcal{V}$ and is denoted by* codim $\mathcal{V}$,

$$\operatorname{codim}\mathcal{V} = \dim \mathcal{X}/\mathcal{V} = \dim \mathcal{X} - \dim \mathcal{V}. \tag{2.2.7}$$

Note that $\mathcal{X}/\mathcal{V}$ is not a subspace of $\mathcal{X}$ unless $\mathcal{V} = \{0\}$.

Definition 2.2.4 (Kernel and image of a matrix). *Given $A \in \mathbb{C}^{m\times n}$ (or $\mathbb{R}^{m\times n}$), a linear map from $\mathcal{X} = \mathbb{C}^n$ (or $\mathbb{R}^n$) to $\mathcal{Y} = \mathbb{C}^m$ (or $\mathbb{R}^m$), the* kernel or null space *of A is defined as*

$$\ker(A) := \{x \in \mathcal{X} \mid Ax = 0\}, \tag{2.2.8}$$

and the image *or* range space *of A is defined as*

$$\operatorname{im}(A) = A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}. \tag{2.2.9}$$

Obviously, $\ker(A)$ *is a subspace of $\mathcal{X}$, and* $\operatorname{im}(A)$ *is a subspace of $\mathcal{Y}$.*

Definition 2.2.5 (Inverse image of a subspace). *Given $A \in \mathbb{C}^{m\times n}$ (or $\mathbb{R}^{m\times n}$), a linear map from $\mathcal{X} = \mathbb{C}^n$ (or $\mathbb{R}^n$) to $\mathcal{Y} = \mathbb{C}^m$ (or $\mathbb{R}^m$), and $\mathcal{V}$, a subspace of $\mathcal{Y}$, the* inverse image *of $\mathcal{V}$ associated with the linear map is defined as*

$$A^{-1}\{\mathcal{V}\} := \{x \in \mathcal{X} \mid Ax \in \mathcal{V}\}, \tag{2.2.10}$$

which clearly is a subspace of $\mathcal{X}$.

Definition 2.2.6 (Invariant subspace). *Given $A \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$), a linear map from $\mathcal{X} = \mathbb{C}^n$ (or $\mathbb{R}^n$) to $\mathcal{X}$, a subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be A-*invariant *if*

$$A\mathcal{V} \subset \mathcal{V}. \tag{2.2.11}$$

Such a $\mathcal{V}$ is also called an invariant subspace *of A.*

The following are some useful properties of subspace manipulations: Let $\mathcal{S}$, $\mathcal{V}$ and $\mathcal{W}$ be subspaces of a vector space, we have

$$(\mathcal{S}^\perp)^\perp = \mathcal{S}, \tag{2.2.12}$$

$$(\mathcal{V} + \mathcal{W})^\perp = \mathcal{V}^\perp \cap \mathcal{W}^\perp, \tag{2.2.13}$$

$$(\mathcal{V} \cap \mathcal{W})^\perp = \mathcal{V}^\perp + \mathcal{W}^\perp, \tag{2.2.14}$$

$$\mathcal{S} + (\mathcal{V} \cap \mathcal{W}) \subset (\mathcal{S} + \mathcal{V}) \cap (\mathcal{S} + \mathcal{W}), \tag{2.2.15}$$

$$\mathcal{S} \cap (\mathcal{V} + \mathcal{W}) \supset (\mathcal{S} \cap \mathcal{V}) + (\mathcal{S} \cap \mathcal{W}). \tag{2.2.16}$$

If $\mathcal{V} \subset \mathcal{S}$, then we have

$$\mathcal{S} \cap (\mathcal{V} + \mathcal{W}) = \mathcal{V} + (\mathcal{S} \cap \mathcal{W}). \tag{2.2.17}$$

Given a linear map A and subspaces $\mathcal{V}$ and $\mathcal{W}$ of appropriate dimensions, we have

$$A(\mathcal{V} \cap \mathcal{W}) \subset A\mathcal{V} \cap A\mathcal{W}, \tag{2.2.18}$$

$$A(\mathcal{V} + \mathcal{W}) = A\mathcal{V} + A\mathcal{W}, \tag{2.2.19}$$

$$A^{-1}\{\mathcal{V} \cap \mathcal{W}\} = A^{-1}\{\mathcal{V}\} \cap A^{-1}\{\mathcal{W}\}, \tag{2.2.20}$$

$$A^{-1}\{\mathcal{V} + \mathcal{W}\} \supset A^{-1}\{\mathcal{V}\} + A^{-1}\{\mathcal{W}\}, \tag{2.2.21}$$

$$(A^{-1}\{\mathcal{V}\})^{\perp} = A^{\mathrm{H}}\mathcal{V}^{\perp}, \tag{2.2.22}$$

$$\ker(A^{\mathrm{H}}) = \{\mathrm{im}\,(A)\}^{\perp}, \tag{2.2.23}$$

where A^{H} is the conjugate transpose of A. Lastly, the following relations are equivalent:

$$A\mathcal{S} \subset \mathcal{V} \iff A^{\mathrm{H}}\mathcal{V}^{\perp} \subset \mathcal{S}^{\perp}. \tag{2.2.24}$$

2.3 Matrix Algebra and Properties

This section gives a brief review of basic matrix algebra and some useful matrix properties. For easy reference, we write an $m \times n$ matrix, say $A \in \mathbb{C}^{m\times n}$, as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}], \tag{2.3.1}$$

i.e., when a capital letter is used to denote a matrix, the corresponding lowercase letter with subscript ij refers to its (i, j) component. The *transpose* of A is defined as

$$A' := C = [c_{ij} := a_{ji}], \tag{2.3.2}$$

and its *conjugate transpose* is defined as

$$A^{\mathrm{H}} := C = [c_{ij} := a_{ji}^{*}], \tag{2.3.3}$$

with a_{ji}^{*} being the complex conjugate of a_{ji}.

2.3.1 Determinant, Inverse and Differentiation

For a square matrix $A \in \mathbb{C}^{n\times n}$, its *determinant*, det (A), is given by

$$\det(A) = \sum_{j=1}^{n}(-1)^{j+1}a_{ij}\det(A_{ij}), \quad \forall i = 1,2,\ldots,n, \tag{2.3.4}$$

where A_{ij} is an $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and j-th column of A. For $A(t) : \mathbb{R} \to \mathbb{C}^{m\times n}$, its *differentiation* is defined as

$$\frac{d}{dt}A(t) := \left[\frac{d}{dt}a_{ij}(t)\right] = \left[\dot{a}_{ij}(t)\right], \tag{2.3.5}$$

provided that all its entries are differentiable with respect to t.

If two square matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ satisfy $AB = BA = I$, then B is said to be the *inverse* of A and is denoted by A^{-1}. If the inverse of A exists, then A is said to be *nonsingular*; otherwise it is *singular*. We note that A is nonsingular if and only if det $(A) \neq 0$.

The following are some useful properties and identities of the determinant, inverse and differentiation: Given $A, B \in \mathbb{C}^{n\times n}$ and $\alpha \in \mathbb{C}$, we have

$$\det(AB) = \det(BA) = \det(A)\cdot\det(B), \tag{2.3.6}$$

$$\det(A') = \det(A), \tag{2.3.7}$$

$$\det(\alpha A) = \alpha^n \det(A). \tag{2.3.8}$$

Suppose A and B are square matrices of appropriate dimensions. Then,

$$\det\begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det(A)\cdot\det(B - CA^{-1}D) \tag{2.3.9}$$

if A is nonsingular, or

$$\det\begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det(B)\cdot\det(A - DB^{-1}C) \tag{2.3.10}$$

if B is nonsingular. Given $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, it follows from (2.3.9) and (2.3.10) that

$$\det(I_m + AB) = \det(I_n + BA). \tag{2.3.11}$$

Given a nonsingular $A \in \mathbb{C}^{n\times n}$, and $u, v \in \mathbb{C}^n$ satisfying $v^{\mathrm{H}}A^{-1}u \neq -1$, then

$$(A + uv^{\mathrm{H}})^{-1} = A^{-1} - \frac{A^{-1}uv^{\mathrm{H}}A^{-1}}{1 + v^{\mathrm{H}}A^{-1}u}, \tag{2.3.12}$$

which is known as the *Sherman–Morrison formula* (see Golub and Van Loan [59]) and its generalized versions are very handy in deriving many interesting results in systems and control theory. The *Sherman–Morrison–Woodbury formula* or simply the *Woodbury formula* below is one of its generalizations,

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}, \tag{2.3.13}$$

where $A \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{n\times n}$ are nonsingular, B and D are of appropriate dimensions, and $(DA^{-1}B + C^{-1})$ is nonsingular. The following identities are particularly useful:

$$(I+AB)^{-1}A = A(I+BA)^{-1}, \tag{2.3.14}$$

$$[I+C(sI-A)^{-1}B]^{-1} = I - C(sI-A+BC)^{-1}B, \tag{2.3.15}$$

and

$$(I-BD)^{-1} = I + B(I-DB)^{-1}D. \tag{2.3.16}$$

Identities below for the inverse of a block matrix (see, *e.g.*, [70]) are also handy: If A and B are nonsingular, then

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix} \tag{2.3.17}$$

and

$$\begin{bmatrix} A & D \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{bmatrix}. \tag{2.3.18}$$

If A is nonsingular, then

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}D\Delta^{-1}CA^{-1} & -A^{-1}D\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix}, \tag{2.3.19}$$

where $\Delta := B - CA^{-1}D$. Furthermore, if B is also nonsingular, then it follows from (2.3.13) that

$$A^{-1} + A^{-1}D(B-CA^{-1}D)^{-1}CA^{-1} = (A-DB^{-1}C)^{-1},$$

and thus (2.3.19) can be rewritten as

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} (A-DB^{-1}C)^{-1} & -A^{-1}D\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix}. \tag{2.3.20}$$

Similarly, if B is nonsingular, then

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} \nabla^{-1} & -\nabla^{-1}DB^{-1} \\ -B^{-1}C\nabla^{-1} & B^{-1}+B^{-1}C\nabla^{-1}DB^{-1} \end{bmatrix}, \tag{2.3.21}$$

where $\nabla = A - DB^{-1}C$. In addition, if A is also nonsingular,

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} \nabla^{-1} & -\nabla^{-1}DB^{-1} \\ -B^{-1}C\nabla^{-1} & (B - CA^{-1}D)^{-1} \end{bmatrix}. \tag{2.3.22}$$

Next, given $A(t) : \mathbb{R} \to \mathbb{C}^{m \times n}$ and $B(t) : \mathbb{R} \to \mathbb{C}^{n \times p}$ whose entries are differentiable with respect to t, we have

$$\frac{d}{dt}[A(t)B(t)] = \left[\frac{d}{dt}A(t)\right] B(t) + A(t)\left[\frac{d}{dt}B(t)\right]. \tag{2.3.23}$$

Given $A(t) : \mathbb{R} \to \mathbb{C}^{n \times n}$, if $A(t)$ is nonsingular for all t and its entries are differentiable with respect to t, then

$$\frac{d}{dt}\left[A(t)^{-1}\right] = -A(t)^{-1}\left[\frac{d}{dt}A(t)\right] A(t)^{-1}. \tag{2.3.24}$$

2.3.2 Rank, Eigenvalues and Jordan Form

Let us rewrite a matrix $A \in \mathbb{C}^{m \times n}$ as

$$A = [\,c_1 \quad c_2 \quad \cdots \quad c_n\,] = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}, \tag{2.3.25}$$

where c_i, $i = 1, 2, \ldots, n$, and r_i, $i = 1, 2, \ldots, m$, are respectively the columns and rows of A. The *rank* of A is defined as the maximum number of linearly independent vectors in $\{c_1, c_2, \ldots, c_n\}$, or equivalently, the maximum number of linearly independent vectors in $\{r_1, r_2, \ldots, r_m\}$, and is denoted by $\operatorname{rank}(A)$. Clearly,

$$\operatorname{rank}(A) = \dim\{\operatorname{im}(A)\} = n - \dim\{\ker(A)\}. \tag{2.3.26}$$

The following are some useful properties of matrix ranks. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$. Then,

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}, \tag{2.3.27}$$

which is known as *Sylvester's inequality*. If A is square and nonsingular, then

$$\operatorname{rank}(AB) = \operatorname{rank}(B). \tag{2.3.28}$$

In general, if A and B have the same dimensions, then

$$\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B). \tag{2.3.29}$$

The computation of matrix ranks plays a crucial role in obtaining the various structural decompositions of linear systems to be developed in this book. The computation of a matrix rank can be carried out efficiently by using the singular value decomposition to be reviewed in Section 2.3.4.

Next, given a square matrix $A \in \mathbb{C}^{n\times n}$, a scalar $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of A if

$$Ax = \lambda x \quad (\Longleftrightarrow \quad (\lambda I - A)x = 0), \tag{2.3.30}$$

for some nonzero vector $x \in \mathbb{C}^n$. Such an x is called a (right) *eigenvector* associated with the eigenvalue λ.

It then follows from (2.3.30) that, for an eigenvalue λ,

$$\text{rank}\,(\lambda I - A) < n \quad (\Longleftrightarrow \quad \det(\lambda I - A) = 0). \tag{2.3.31}$$

Thus, the eigenvalues of A are the roots of its *characteristic polynomial*,

$$\chi(\lambda) := \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \tag{2.3.32}$$

which has a total of n roots. The set of these roots or eigenvalues of A is denoted by $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The following property is the *Cayley–Hamilton theorem*,

$$\chi(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0. \tag{2.3.33}$$

The *spectral radius* of A is defined as

$$\rho(A) := \max\{\,|\lambda| \mid \lambda \in \lambda(A)\,\}, \tag{2.3.34}$$

and the *trace* of A, defined as

$$\text{trace}\,(A) := \sum_{i=1}^{n} a_{ii}, \tag{2.3.35}$$

is related to the eigenvalues of A as

$$\text{trace}\,(A) = \sum_{i=1}^{n} \lambda_i. \tag{2.3.36}$$

Suppose a square matrix $A \in \mathbb{C}^{n\times n}$ has n linearly independent eigenvectors, $x_1, x_2, \ldots, x_n$, respectively associated with eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$ (which need not be distinct). Let

$$T := [\, x_1 \quad x_2 \quad \cdots \quad x_n \,] \in \mathbb{C}^{n\times n}, \tag{2.3.37}$$

which is called the *eigenvector matrix* of A. Then, we have

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \tag{2.3.38}$$

In general, for a square matrix $A \in \mathbb{C}^{n\times n}$, there exists a nonsingular transformation $T \in \mathbb{C}^{n\times n}$ and an integer k such that

$$T^{-1}AT = J := \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}, \tag{2.3.39}$$

where J_i, $i = 1, 2, \ldots, k$, are some $n_i \times n_i$ *Jordan blocks* of the form:

$$J_i := \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}. \tag{2.3.40}$$

Obviously, $\lambda_i \in \lambda(A)$, $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} n_i = n$. The special structure of J in (2.3.39) is known as the *Jordan canonical form* of A. We have implemented an m-function, `jcf.m`, based on an algorithm modified from the result reported in Bingulac and Luse [12], for computing this Jordan canonical form. It is known that there are substantial numerical difficulties in computing the Jordan form (see, e.g., Kailath [70]). However, when it can be computed accurately, it will be very useful in displaying the finite zero structure and other properties of linear systems in the coming chapters. We would like to further note that although the Jordan canonical form is a powerful tool for analyzing system properties, it is seldom used in actual applications. The application of the Jordan canonical form in this book is mainly for theoretical development and technical analysis.

The following are some very handy inequalities on eigenvalues of general square matrices. More results for special matrices will be given in the next subsection. For an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, we have

$$\lambda_{\min}\left(\frac{A + A^{\mathrm{H}}}{2}\right) \le \operatorname{Re}\lambda_i(A) \le \lambda_{\max}\left(\frac{A + A^{\mathrm{H}}}{2}\right), \tag{2.3.41}$$

and

$$\lambda_{\min}\left(\frac{A - A^{\mathrm{H}}}{2j}\right) \le \operatorname{Im}\lambda_i(A) \le \lambda_{\max}\left(\frac{A - A^{\mathrm{H}}}{2j}\right), \tag{2.3.42}$$

where $j = \sqrt{-1}$.

For $A, B \in \mathbb{C}^{n\times n}$, we have the following inequalities on the eigenvalues of $A+B$,

$$\max_i |\lambda_i(A+B)| \leq \sqrt{\lambda_{\max}(A^{\mathrm{H}}A)} + \sqrt{\lambda_{\max}(B^{\mathrm{H}}B)}, \tag{2.3.43}$$

and

$$\begin{aligned}\lambda_{\min}\left(\frac{A^{\mathrm{H}}+A}{2}\right) + \lambda_{\min}\left(\frac{B^{\mathrm{H}}+B}{2}\right) &\leq \operatorname{Re}\lambda_i(A+B)\\ &\leq \lambda_{\max}\left(\frac{A^{\mathrm{H}}+A}{2}\right) + \lambda_{\max}\left(\frac{B^{\mathrm{H}}+B}{2}\right).\end{aligned} \tag{2.3.44}$$

For square matrices A and B with the same dimension, we have the following properties on the eigenvalues of AB,

$$\lambda(AB) = \lambda(BA), \tag{2.3.45}$$

$$\max_i |\lambda_i(AB)| \leq \max_i |\lambda_i(A)| \cdot \max_i |\lambda_i(B)|, \tag{2.3.46}$$

and

$$\max_i |\lambda_i(AB)| \leq \sqrt{\lambda_{\max}(A^{\mathrm{H}}A)} \cdot \sqrt{\lambda_{\max}(B^{\mathrm{H}}B)}. \tag{2.3.47}$$

For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$ with $n > m$, we have

$$\lambda(AB) = \lambda(BA) \cup \{\underbrace{0, 0, \ldots, 0}_{n-m}\} \quad \Rightarrow \quad \operatorname{trace}(AB) = \operatorname{trace}(BA). \tag{2.3.48}$$

In particular, for $x, z \in \mathbb{C}^n$,

$$\lambda(xz^{\mathrm{H}}) = \{z^{\mathrm{H}}x,\ \underbrace{0, 0, \ldots, 0}_{n-1}\} \quad \Rightarrow \quad z^{\mathrm{H}}x = \operatorname{trace}(xz^{\mathrm{H}}). \tag{2.3.49}$$

2.3.3 Special Matrices

In this section we discuss some commonly used special matrices. Special attention will be paid to positive and positive semi-definite matrices as they play a key role in solving many systems and control problems, especially in those problems related to system stability. Given an $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, we say that A is a *diagonal matrix* if $a_{ij} = 0$ whenever $i \neq j$. For a square diagonal matrix $A \in \mathbb{C}^{n\times n}$, we occasionally write

$$A = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \tag{2.3.50}$$

i.e., $a_{ii} = \alpha_i$, $i = 1, 2, \ldots, n$. Similarly, we write a *block diagonal matrix*

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix} = \text{blkdiag}\{A_1, A_2, \ldots, A_k\}. \tag{2.3.51}$$

The following are several important types of square matrices. We say that a matrix $A \in \mathbb{R}^{n \times n}$ is

1. *symmetric* if $A' = A$ (such a matrix has all eigenvalues on the real axis);
2. *skew-symmetric* if $A' = -A$ (such a matrix has all eigenvalues on the imaginary axis);
3. *orthogonal* if $A'A = AA' = I$ (such a matrix has all eigenvalues on the unit circle);
4. *nilpotent* if $A^k = 0$ for integer k (such a matrix has all eigenvalues at the origin);
5. *idempotent* if $A^2 = A$ (such a matrix has eigenvalues at either 1 or 0);
6. a *permutation matrix* if A is nonsingular and each one of its columns (or rows) has only one nonzero element, which is equal to 1.

We say that a matrix $A \in \mathbb{C}^{n \times n}$ is

1. *Hermitian* if $A^{\mathrm{H}} = A$ (such a matrix has all eigenvalues on the real axis);
2. *unitary* if $A^{\mathrm{H}}A = AA^{\mathrm{H}} = I$ (such a matrix has all eigenvalues on the unit circle);
3. *positive definite* if $x^{\mathrm{H}}Ax > 0$ for every nonzero vector $x \in \mathbb{C}^n$;
4. *positive semi-definite* if $x^{\mathrm{H}}Ax \geq 0$ for every vector $x \in \mathbb{C}^n$;
5. *negative definite* if $x^{\mathrm{H}}Ax < 0$ for every nonzero vector $x \in \mathbb{C}^n$;
6. *negative semi-definite* if $x^{\mathrm{H}}Ax \leq 0$ for every vector $x \in \mathbb{C}^n$;
7. *indefinite* if A is neither positive nor negative semi-definite.

If A is positive definite (positive semi-definite), we write $A > 0$ ($A \geq 0$), and if A is negative definite (negative semi-definite), we write $A < 0$ ($A \leq 0$). Given two Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$, we write $A \geq B$ if $A - B \geq 0$, and $A > B$ if $A - B > 0$.

In systems and control applications, we are particularly interested in results related to the positive definiteness and positive semi-definiteness of symmetric or Hermitian matrices. We present below some useful results on symmetric or Hermitian matrices and definite matrices.

Given a Hermitian matrix $A \in \mathbb{C}^{n\times n}$, which has all real eigenvalues, *i.e.*, $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{R}$, and letting

$$\lambda_{\min}(A) := \min\{\lambda(A)\}, \quad \lambda_{\max}(A) := \max\{\lambda(A)\}, \tag{2.3.52}$$

we have

$$\lambda_{\min}(A) = \min_{x\neq 0} \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}x}, \quad \lambda_{\max}(A) = \max_{x\neq 0} \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}x} \tag{2.3.53}$$

and

$$-\max_i |\lambda_i(A)| \leq \lambda_{\min}(A) \leq \lambda_i(A) \leq \lambda_{\max}(A) \leq \max_i |\lambda_i(A)|. \tag{2.3.54}$$

Suppose $A, B \in \mathbb{C}^{n\times n}$ are Hermitian. Then, we have the following inequalities concerning the eigenvalues of $A + B$,

$$\max_i |\lambda_i(A+B)| \leq \max_i |\lambda_i(A)| + \max_i |\lambda_i(B)|, \tag{2.3.55}$$

$$\lambda_{\min}(A) + \lambda_{\min}(B) \leq \lambda_i(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B), \tag{2.3.56}$$

$$\lambda_{\min}(A+B) \leq \min\{\lambda_{\max}(A) + \lambda_{\min}(B), \lambda_{\min}(A) + \lambda_{\max}(B)\}, \tag{2.3.57}$$

$$\lambda_{\max}(A+B) \geq \max\{\lambda_{\max}(A) + \lambda_{\min}(B), \lambda_{\min}(A) + \lambda_{\max}(B)\}. \tag{2.3.58}$$

We also have the following inequalities on the eigenvalues of AB,

$$\lambda_{\min}(B)\lambda_i(A^2) \leq \lambda_i(ABA) \leq \lambda_{\max}(B)\lambda_i(A^2), \quad i = 1, 2, \ldots, n, \tag{2.3.59}$$

where $\lambda_i(\cdot)$ is assumed to be arranged such that $\lambda_1(\cdot) \geq \lambda_2(\cdot) \geq \cdots \geq \lambda_n(\cdot)$.

It is known that a Hermitian matrix is positive definite (positive semi-definite) if and only if all its eigenvalues are positive (nonnegative), and it is negative definite (negative semi-definite) if and only if all its eigenvalues are negative (nonpositive). Let a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathrm{H}} & A_{22} \end{bmatrix}, \tag{2.3.60}$$

with A_{11} and A_{22} being square. Then, A is positive definite if and only if either one of the following conditions holds:

$$A_{11} > 0, \;\; A_{22} - A_{12}^{\mathrm{H}} A_{11}^{-1} A_{12} > 0; \tag{2.3.61}$$

or

$$A_{22} > 0, \quad A_{11} - A_{12}A_{22}^{-1}A_{12}^{\mathrm{H}} > 0. \tag{2.3.62}$$

Using this result repeatedly, one can show that A is positive definite if and only if all its leading *principal minors* are positive, and that A is positive semi-definite if and only if all its principal minors are nonnegative. We note that similar results can be obtained on negative definite and negative semi-definite matrices.

Given two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, we have

$$A \geq B > 0 \quad \Longleftrightarrow \quad B^{-1} \geq A^{-1} > 0. \tag{2.3.63}$$

If A and B commute, *i.e.*, $AB = BA$, then

$$A > B > 0 \quad \Rightarrow \quad A^k > B^k > 0. \tag{2.3.64}$$

Given Hermitian matrices $A, B, S \in \mathbb{C}^{n\times n}$ with $A > 0$, $B > 0$ and $S > 0$, then

$$ASA > BSB \quad \Rightarrow \quad A > B. \tag{2.3.65}$$

Unfortunately, the converse of (2.3.65) is generally not true. But, we have

$$A > B \quad \Rightarrow \quad \text{there exists an } S = S^{\mathrm{H}} > 0 \text{ such that } ASA > BSB. \tag{2.3.66}$$

The following properties of $A + B$ and AB are also very useful in the derivation of many results in linear systems and control theory. Given two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, and assuming that the eigenvalues of A and $A + B$ are respectively arranged as:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \tag{2.3.67}$$

and

$$\lambda_1(A+B) \geq \lambda_2(A+B) \geq \cdots \geq \lambda_n(A+B), \tag{2.3.68}$$

we have

$$\lambda_i(A+B) > \lambda_i(A), \quad i = 1, 2, \ldots, n, \tag{2.3.69}$$

if $B > 0$, and

$$\lambda_i(A+B) \geq \lambda_i(A), \quad i = 1, 2, \ldots, n, \tag{2.3.70}$$

if $B \geq 0$. Given two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ with $B > 0$, we have

$$\lambda(AB) \subset \mathbb{R} \quad \text{and} \quad \lambda(AB^{-1}) \subset \mathbb{R}, \tag{2.3.71}$$

i.e., the eigenvalues of both AB and AB^{-1} are real, in particular,

$$\lambda_{\min}(AB^{-1}) = \min_{x\neq 0} \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}Bx}, \tag{2.3.72}$$

and

$$\lambda_{\max}(AB^{-1}) = \max_{x\neq 0} \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}Bx}. \tag{2.3.73}$$

Obviously, if, in addition, A is also positive definite (positive semi-definite), then all the eigenvalues AB and AB^{-1} are positive (nonnegative).

Given two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ with $A \geq 0$ and $B \geq 0$, we have

$$\lambda_{\min}(B)\lambda_{\max}(A) \leq \lambda_{\min}(B)\,\mathrm{trace}\,(A) \leq \mathrm{trace}\,(AB) \leq \lambda_{\max}(B)\,\mathrm{trace}\,(A). \tag{2.3.74}$$

2.3.4 Singular Value Decomposition

Given a matrix $A \in \mathbb{C}^{m\times n}$, its *singular values* are defined as

$$\sigma_i(A) := \sqrt{\lambda_i(A^{\mathrm{H}}A)} = \sqrt{\lambda_i(AA^{\mathrm{H}})}, \quad i = 1, 2, \ldots, k, \tag{2.3.75}$$

where $k := \min\{m, n\}$, assuming that the eigenvalues of $A^{\mathrm{H}}A$ and AA^{H} are arranged in a descending order. Clearly, we have $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_k(A) \geq 0$. The *condition number* of A is defined as

$$\mathrm{cond}\,(A) = \sigma_1(A)/\sigma_k(A). \tag{2.3.76}$$

Let

$$\Delta_1 = \mathrm{diag}\,\{\sigma_1(A), \sigma_2(A), \ldots, \sigma_k(A)\}. \tag{2.3.77}$$

It can be shown that there exist two unitary matrices such that A can be decomposed as:

$$A = U\Delta V^{\mathrm{H}}, \tag{2.3.78}$$

where

$$\Delta = \begin{bmatrix} \Delta_1 \\ 0 \end{bmatrix}, \quad \text{if } m \geq n, \tag{2.3.79}$$

or

$$\Delta = [\,\Delta_1 \quad 0\,], \quad \text{if } m \leq n. \tag{2.3.80}$$

Decomposition (2.3.78) is called the *singular value decomposition* of A. We now recall some useful properties of singular values.

Proposition 2.3.1. *Given* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$, *we have*

$$\sigma_{\max}(AB) \le \sigma_{\max}(A)\sigma_{\max}(B). \tag{2.3.81}$$

If both A *and* B *are square matrices, we have*

$$\begin{aligned}\sigma_{\min}(A)\sigma_{\min}(B) &\le \sigma_{\min}(AB)\\ &\le \min\{\sigma_{\max}(A)\sigma_{\min}(B), \sigma_{\min}(A)\sigma_{\max}(B)\}\\ &\le \max\{\sigma_{\max}(A)\sigma_{\min}(B), \sigma_{\min}(A)\sigma_{\max}(B)\}\\ &\le \sigma_{\max}(AB)\\ &\le \sigma_{\max}(A)\sigma_{\max}(B).\end{aligned} \tag{2.3.82}$$

Proof. Observing that

$$\lambda_{\max}(BB^{\mathrm{H}})AA^{\mathrm{H}} = A[\lambda_{\max}(BB^{\mathrm{H}})I - BB^{\mathrm{H}}]A^{\mathrm{H}} + ABB^{\mathrm{H}}A^{\mathrm{H}}, \tag{2.3.83}$$

and noting that $\lambda_{\max}(BB^{\mathrm{H}})I - BB^{\mathrm{H}} \ge 0$, it follows from (2.3.70) that

$$\lambda_{\max}(BB^{\mathrm{H}})\lambda_i(AA^{\mathrm{H}}) \ge \lambda_i(ABB^{\mathrm{H}}A^{\mathrm{H}}), \tag{2.3.84}$$

and thus

$$\lambda_{\max}(BB^{\mathrm{H}})\lambda_{\max}(AA^{\mathrm{H}}) \ge \lambda_{\max}(ABB^{\mathrm{H}}A^{\mathrm{H}}),$$

or equivalently, $\sigma^2_{\max}(B)\sigma^2_{\max}(A) \ge \sigma^2_{\max}(AB)$. Hence, the result of (2.3.81) follows.

For square matrices $A, B \in \mathbb{C}^{n\times n}$, (2.3.84) also implies that

$$\sigma_{\max}(B)\sigma_i(A) \ge \sigma_i(AB), \quad i = 1, 2, \ldots, n. \tag{2.3.85}$$

Similarly, using

$$\lambda_{\max}(A^{\mathrm{H}}A)B^{\mathrm{H}}B = B^{\mathrm{H}}[\lambda_{\max}(A^{\mathrm{H}}A)I - A^{\mathrm{H}}A]B + B^{\mathrm{H}}A^{\mathrm{H}}AB, \tag{2.3.86}$$

and its properties, we can show that

$$\sigma_{\max}(A)\sigma_i(B) \ge \sigma_i(AB), \quad i = 1, 2, \ldots, n. \tag{2.3.87}$$

Next, noting that

$$-\lambda_{\min}(BB^{\mathrm{H}})AA^{\mathrm{H}} = A[BB^{\mathrm{H}} - \lambda_{\min}(BB^{\mathrm{H}})I]A^{\mathrm{H}} - ABB^{\mathrm{H}}A^{\mathrm{H}}$$

and

$$-\lambda_{\min}(A^{\mathrm{H}}A)B^{\mathrm{H}}B = B^{\mathrm{H}}[A^{\mathrm{H}}A - \lambda_{\min}(A^{\mathrm{H}}A)I]B - B^{\mathrm{H}}A^{\mathrm{H}}AB,$$

we can show that

$$\sigma_{\min}(B)\sigma_i(A) \leq \sigma_i(AB), \tag{2.3.88}$$

and

$$\sigma_{\min}(A)\sigma_i(B) \leq \sigma_i(AB), \tag{2.3.89}$$

$i = 1, 2, \ldots, n$. The result of (2.3.82) follows from (2.3.85), (2.3.87), and (2.3.88) or (2.3.89), by letting $i = 1$ and $i = n$, which by definition corresponds to the largest and the smallest singular value, respectively. ■

Using similar arguments as in the proof of Proposition 2.3.1, we can derive more interesting results for AB with arbitrary $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ by carefully considering the values of m, n and p. In particular, we have the following results:

1. If $n \leq p$, then
$$\sigma_{\max}(AB) \geq \sigma_{\max}(A)\sigma_{\min}(B). \tag{2.3.90}$$

2. If $n \leq m$, then
$$\sigma_{\max}(AB) \geq \sigma_{\min}(A)\sigma_{\max}(B). \tag{2.3.91}$$

3. If $\min\{m, n\} \leq \min\{m, p\}$, then
$$\sigma_{\min}(AB) \leq \sigma_{\min}(A)\sigma_{\max}(B). \tag{2.3.92}$$

4. If $\min\{n, p\} \leq \min\{m, p\}$, then
$$\sigma_{\min}(AB) \leq \sigma_{\max}(A)\sigma_{\min}(B). \tag{2.3.93}$$

5. If $p \leq n \leq m$ or $m \leq n \leq p$, then
$$\sigma_{\min}(AB) \geq \sigma_{\min}(A)\sigma_{\min}(B). \tag{2.3.94}$$

It is also straightforward to verify the following inequality concerning the maximal singular value of $A + B$,

$$\sigma_{\max}(A + B) \leq \sigma_{\max}(A) + \sigma_{\max}(B). \tag{2.3.95}$$

It is clear from the decomposition of (2.3.78) that the rank of A is given by the number of nonzero singular values of A. As the singular value decomposition only involves unitary transformations, which are numerically stable with a perfect condition number, *i.e.*, 1, it can be used to determine matrix ranks more accurately. The singular value decomposition can also be used to compute the inverse of non-singular matrices. For example, if A is square and nonsingular, then its inverse is given by

$$A^{-1} = V\Delta^{-1}U^{\mathrm{H}}. \tag{2.3.96}$$

Note that only the inverses of scalars are required in obtaining Δ^{-1}. Another application of the singular value decomposition is the computation of pseudo-inverses of a matrix. Given a matrix $A \in \mathbb{C}^{m\times n}$, its *pseudo-inverse* or *Moore–Penrose inverse* is defined to be the unique matrix $A^{\dagger} \in \mathbb{C}^{n\times m}$ such that

1. $AA^{\dagger}A = A$,
2. $A^{\dagger}AA^{\dagger} = A^{\dagger}$,
3. $AA^{\dagger} = (AA^{\dagger})^{\mathrm{H}}$, and
4. $A^{\dagger}A = (A^{\dagger}A)^{\mathrm{H}}$.

Let q be the number of nonzero singular values of A. Then, the singular value decomposition of A in (2.3.78) can be rewritten as

$$A = U\Delta V^{\mathrm{H}} = U\begin{bmatrix}\Delta_* & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{H}}, \tag{2.3.97}$$

where $\Delta_* = \mathrm{diag}\,\{\sigma_1(A), \sigma_2(A), \ldots, \sigma_q(A)\}$. It is straightforward to verify that the pseudo-inverse of A is given by

$$A^{\dagger} = V\begin{bmatrix}\Delta_*^{-1} & 0\\ 0 & 0\end{bmatrix}U^{\mathrm{H}}, \tag{2.3.98}$$

with its associated properties:

$$0_{m\times n}^{\dagger} = 0_{n\times m}, \quad (A^{\dagger})^{\dagger} = A, \quad (A^{\dagger})^{\mathrm{H}} = (A^{\mathrm{H}})^{\dagger}, \tag{2.3.99}$$

$$A^{\mathrm{H}} = A^{\mathrm{H}}AA^{\dagger} = A^{\dagger}AA^{\mathrm{H}}, \quad (A^{\mathrm{H}}A)^{\dagger} = A^{\dagger}(A^{\mathrm{H}})^{\dagger}, \tag{2.3.100}$$

$$A^{\dagger} = (A^{\mathrm{H}}A)^{\dagger}A^{\mathrm{H}} = A^{\mathrm{H}}(AA^{\mathrm{H}})^{\dagger}, \tag{2.3.101}$$

$$\mathrm{im}\,(A) = \mathrm{im}\,(AA^{\dagger}) = \mathrm{im}\,(AA^{\mathrm{H}}), \tag{2.3.102}$$

$$\mathrm{im}\,(A^{\dagger}) = \mathrm{im}\,(A^{\mathrm{H}}) = \mathrm{im}\,(A^{\dagger}A) = \mathrm{im}\,(A^{\mathrm{H}}A), \tag{2.3.103}$$

$$\mathrm{im}\,(I - A^{\dagger}A) = \ker\,(A^{\dagger}A) = \ker\,(A) = \mathrm{im}\,(A^{\mathrm{H}})^{\perp}, \tag{2.3.104}$$

$$\mathrm{im}\,(I - AA^{\dagger}) = \ker\,(AA^{\dagger}) = \ker\,(A^{\mathrm{H}}) = \ker\,(A^{\dagger}) = \mathrm{im}\,(A)^{\perp}. \tag{2.3.105}$$

2.4 Norms

Norms measure the length or size of a vector or a matrix. Norms are also defined for signals and rational transfer functions.

Given a linear space $\mathcal{X}$ over a scalar field $\mathbb{K}$, any real-valued scalar function of $x \in \mathcal{X}$ (usually denoted by $\|x\|$) is said to be a *norm* on $\mathcal{X}$ if it satisfies the following properties:

1. $\|x\| > 0$ if $x \neq 0$ and $\|x\| = 0$ if $x = 0$;
2. $\|\alpha x\| = |\alpha| \cdot \|x\|, \forall \alpha \in \mathbb{K}, \forall x \in \mathcal{X}$; and
3. $\|x + z\| \leq \|x\| + \|z\|, \forall x, z \in \mathcal{X}$.

2.4.1 Norms of Vectors

The following p-norms are the most commonly used norms on the vector space $\mathbb{C}^n$:

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad 1 \leq p < \infty, \tag{2.4.1}$$

and

$$\|x\|_\infty := \max_i |x_i|, \tag{2.4.2}$$

where $x_1, x_2, \ldots, x_n$ are the elements of $x \in \mathbb{C}^n$. In particular, $\|x\|_2$ is also called the *Euclidean norm* of x and is denoted by $|x|$ for simplicity.

2.4.2 Norms of Matrices

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{m \times n}$, its *Frobenius norm* is defined as

$$\|A\|_{\mathrm{F}} := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \left(\sum_{i=1}^{\min\{m,n\}} \sigma_i(A) \right)^{1/2}. \tag{2.4.3}$$

The *p-norm* of A is a norm induced from the vector p-norm, *i.e.*,

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p. \tag{2.4.4}$$

In particular, for $p = 1, 2, \infty$, we have

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \tag{2.4.5}$$

$$||A||_2 = \sqrt{\lambda_{\max}(A^{\mathrm{H}}A)} = \sigma_{\max}(A), \tag{2.4.6}$$

which is also called the *spectral norm* of A, and

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{2.4.7}$$

It can be shown that

$$||A|| \geq \rho(A), \tag{2.4.8}$$

where $||A||$ is any norm of A and $\rho(A)$ is the spectral radius of A. Also note that all these matrix norms are invariant under unitary transformations.

2.4.3 Norms of Continuous-time Signals

For any $p \in [1, \infty)$, let L_p^m denote the linear space formed by all measurable signals $g : \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$\int_0^\infty |g(t)|^p dt < \infty.$$

For any $g \in L_p^m$, $p \in [1, \infty)$, its *L_p-norm* is defined as

$$||g||_p := \left(\int_0^\infty |g(t)|^p dt \right)^{1/p}, \quad 1 \leq p < \infty. \tag{2.4.9}$$

Let L_∞^m denote the linear space formed by all signals $g : \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$|g(t)| < \infty, \quad \forall t \in \mathbb{R}_+.$$

The L_∞-norm of a $g \in L_\infty^m$ is defined as

$$||g||_\infty := \sup_{t \geq 0} |g(t)|. \tag{2.4.10}$$

The following *Hölder inequality* of signal norms is useful,

$$||fg||_1 \leq ||f||_p \cdot ||g||_q, \tag{2.4.11}$$

where $1 < p < \infty$ and $1/p + 1/q = 1$. It can also be shown that if $g(t) \in L_1 \cap L_\infty$, then $g(t) \in L_2$.

2.4.4 Norms of Discrete-time Signals

For any $p \in [1, \infty)$, let l_p^m denote the linear space formed by all discrete-time signals $g : \mathbb{N} \to \mathbb{R}^m$ such that

$$\sum_{k=0}^{\infty} |g(k)|^p < \infty.$$

For any $g \in l_p^m$, $p \in [1, \infty)$, its *l_p-norm* is defined as

$$\|g\|_p := \left(\sum_{k=0}^{\infty} |g(k)|^p \right)^{1/p}, \quad 1 \le p < \infty. \tag{2.4.12}$$

Let l_∞^m denote the linear space formed by all signals $g : \mathbb{N} \to \mathbb{R}^m$ such that

$$|g(k)| < \infty, \quad \forall k \ge 0.$$

The *l_∞-norm* of any $g \in l_\infty^m$ is defined as

$$\|g\|_\infty := \sup_{k \ge 0} \, |g(k)|. \tag{2.4.13}$$

It can be shown that, if $g(k) \in l_1^m$, then $\|g\|_p \le \|g\|_1 < \infty, p \in (1, \infty]$, which implies that $l_1 \subset l_p, p \in (1, \infty]$. In general, we have $l_1^m \subset l_p^m \subset l_\infty^m, p \in (1, \infty)$.

2.4.5 Norms of Continuous-time Systems

Given a stable and proper continuous-time system with a transfer matrix $G(s)$, its *H_2-norm* is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi} \text{trace} \left[\int_{-\infty}^{\infty} G(j\omega) G(j\omega)^{\mathrm{H}} d\omega \right] \right)^{1/2}, \tag{2.4.14}$$

and its H_∞-norm is defined as

$$\|G\|_\infty := \sup_{\omega \in [0, \infty)} \sigma_{\max}[G(j\omega)] = \sup_{\|w\|_2 = 1} \frac{\|h\|_2}{\|w\|_2}, \tag{2.4.15}$$

where $w(t)$ and $h(t)$ are respectively the input and output of $G(s)$.

Let (A, B, C, D) be a state space realization of the stable transfer matrix, $G(s)$, *i.e.*, $G(s) = C(sI - A)^{-1}B + D$. It is straightforward to verify that $\|G\|_2 < \infty$ if and only if $D = 0$. In the case of $D = 0$, $\|G\|_2$ can be exactly computed by solving either one of the following Lyapunov equations:

$$A'P + PA = -C'C, \quad AQ + QA' = -BB', \tag{2.4.16}$$

for unique solution $P > 0$ or $Q > 0$. More specifically,

$$\|G\|_2 = \sqrt{\text{trace}\,(B'PB)} = \sqrt{\text{trace}\,(CQC')}. \tag{2.4.17}$$

The computation of $\|G\|_\infty$ is tedious and can be done by searching for a scalar $\gamma > \sigma_{\max}(D)$ such that

$$M_\gamma = \begin{bmatrix} A + BR^{-1}D'C & \gamma^{-2}BR^{-1}B' \\ -C'(I + DR^{-1}D')C & -(A + BR^{-1}D'C)' \end{bmatrix}, \tag{2.4.18}$$

where $R := \gamma^2 I - D'D$, has at least one eigenvalue on the imaginary axis. If such a γ exists, say $\gamma = \gamma^*$, then $\|G\|_\infty = \gamma^*$; otherwise, $\|G\|_\infty = \sigma_{\max}(D)$.

2.4.6 Norms of Discrete-time Systems

Given a stable and proper discrete-time system with a transfer matrix $G(z)$, its H_2-*norm* is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi}\text{trace}\left[\int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})^{\mathrm{H}} d\omega\right]\right)^{1/2}, \tag{2.4.19}$$

and its H_∞-*norm* is defined as

$$\|G\|_\infty := \sup_{\omega\in[0,2\pi]} \sigma_{\max}[G(e^{j\omega})] = \sup_{\|w\|_2=1} \frac{\|h\|_2}{\|w\|_2}, \tag{2.4.20}$$

where $w(k)$ and $h(k)$ are respectively the input and output of $G(z)$.

Assume that (A, B, C, D) is a state space realization of $G(z)$. For the case when $D = 0$, $\|G\|_2$ can be computed by solving either one of the following Lyapunov equations:

$$A'PA - P = -C'C, \quad AQA' - Q = -BB', \tag{2.4.21}$$

for $P > 0$ or $Q > 0$. More specifically,

$$\|G\|_2 = \sqrt{\text{trace}\,(B'PB)} = \sqrt{\text{trace}\,(CQC')}. \tag{2.4.22}$$

The computation of $\|G\|_\infty$ is again tedious and can be done by transforming $G(z)$ into a continuous-time equivalence using a bilinear transformation. It can be shown that the H_∞-norm of $G(z)$ is equal to the H_∞-norm of its continuous-time counterpart under the bilinear transformation. We further note that, for both continuous- and discrete-time systems,

$$\|G_1G_2\|_\infty \le \|G_1\|_\infty \cdot \|G_2\|_\infty. \tag{2.4.23}$$

This property is a simple consequence of the inequality of (2.3.81).

Chapter 3

Review of Linear Systems Theory

3.1 Introduction

We review in this chapter some fundamental concepts of multivariable linear time-invariant systems. Many concepts, such as system responses and stability as well as controllability and observability, are widely discussed in the literature and can be found in most of the introductory text books on linear systems theory (see, *e.g.*, Antsaklis and Michel [4], Callier and Desoer [16], Chen [33], DeCarlo [46], Kailath [70], Rugh [114], and Zheng [158]). In particular, we will adopt some nice mathematical derivations of these results from Chen [33] and Zheng [158]. On the other hand, some issues, such as the invariant zero structure (also called finite zero structure) and infinite zero structure, invertibility structures as well as geometric subspaces of linear systems, might be somewhat abstract to general readers and new graduate students. These topics, which will be illustrated in detail in the coming sections, can be found in some advanced monographs and research theses in linear systems, such as Rosenbrock [112], Trentelman *et al.* [141], Verghese [146], Wonham [154], and some research articles (see, *e.g.*, Commault and Dion [41], Kouvaritakis and MacFarlane [77], MacFarlane and Karcanias [96], Moylan [102], Owens [105], Pugh and Ratcliffe [110], and Saberi, Chen and Sannuti [115]). In particular, the concepts of finite and infinite zero structures as well as that of invertibility structures will be introduced using the well-known Kronecker canonical form, and the materials on geometric subspaces will be adopted mostly from Trentelman *et al.* [141]. These basic concepts and results will be essential to

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the development of this book and helpful in understanding the results presented in the coming chapters.

We will focus primarily on continuous-time linear time-invariant systems characterized by the following state and output equations:

$$\Sigma : \begin{cases} \dot{x}(t) = A\ x(t) + B\ u(t), \\ y(t) = C\ x(t) + D\ u(t), \end{cases} \tag{3.1.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the system input, $y(t) \in \mathbb{R}^p$ is the system output, and A, B, C and D are constant matrices of appropriate dimensions. When it is clear in the context, we will drop the variable t in x, u and y in (3.1.1). Although (3.1.1) is the primary focus of our work in the book, we do consider, however, another type of linear systems in Chapter 6, *i.e.*, the so-called *singular systems* or *descriptor systems*, in which the state equation is given as $E\dot{x}(t) = Ax(t) + Bu(t)$ with E being a singular matrix. We will leave the detailed treatment of this type of systems to Chapter 6. Note that Σ has a *transfer function* (representation in the frequency domain):

$$H(s) = C(sI - A)^{-1}B + D. \tag{3.1.2}$$

Similarly, results for discrete-time systems characterized by

$$\Sigma : \begin{cases} x(k+1) = A\ x(k) + B\ u(k), \\ \quad y(k) \quad = C\ x(k) + D\ u(k), \end{cases} \tag{3.1.3}$$

where as usual $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the system input and $y(k) \in \mathbb{R}^p$ is the system output, will be presented simultaneously along with the development of continuous-time systems. The transfer function of the discrete-time system (3.1.3) is given by

$$H(z) = C(zI - A)^{-1}B + D. \tag{3.1.4}$$

The following topics on the basic concepts of linear systems are revisited in this chapter:

1. Dynamical responses of linear time-invariant systems;
2. System stability;
3. Controllability and observability;
4. System invertibilities and invertibility structures;
5. Finite and infinite zero structures;
6. Geometric subspaces; and
7. Properties of state feedback and output injection.

3.2 Dynamical Responses

In this section, we derive the solutions to the state and output responses of linear time-invariant systems. We first consider the continuous-time system Σ of (3.1.1). The solution of the state variable or the state response, $x(t)$, of Σ with an initial condition $x_0 = x(0)$ can be uniquely expressed as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq 0, \tag{3.2.1}$$

where the first term is the response due to the initial state, x_0, and the second term is the response excited by the external control force, $u(t)$. To introduce the definition of a matrix exponential function, we derive this result by separating it into the following two cases: i) the system is free of external input, *i.e.*, $u(t) = 0$; and ii) the system has a zero initial state, *i.e.*, $x_0 = 0$.

1. For the case when the external force $u(t) = 0$, the state equation of (3.1.1) reduces to

$$\dot{x} = Ax, \quad x(0) = x_0. \tag{3.2.2}$$

Let the solution to the above autonomous system be expressed as

$$x(t) = \bar{\alpha}_0 + \bar{\alpha}_1 t + \bar{\alpha}_2 t^2 + \cdots = \sum_{k=0}^{\infty} \bar{\alpha}_k t^k, \quad t \geq 0, \tag{3.2.3}$$

where $\bar{\alpha}_k \in \mathbb{R}^n$, $k = 0, 1, \ldots$, are parameters to be determined. Substituting (3.2.3) into (3.2.2), we obtain

$$\dot{x}(t) = \bar{\alpha}_1 + 2\bar{\alpha}_2 t + 3\bar{\alpha}_3 t^2 + \cdots = A\bar{\alpha}_0 + A\bar{\alpha}_1 t + A\bar{\alpha}_2 t^2 + \cdots. \tag{3.2.4}$$

Since the equality in (3.2.4) has to be true for all $t \geq 0$, we have

$$\bar{\alpha}_1 = A\bar{\alpha}_0, \quad \bar{\alpha}_2 = \frac{1}{2}A\bar{\alpha}_1 = \frac{1}{2!}A^2\bar{\alpha}_0, \quad \bar{\alpha}_3 = \frac{1}{3}A\bar{\alpha}_2 = \frac{1}{3!}A^3\bar{\alpha}_0,$$

and in general,

$$\bar{\alpha}_k = \frac{1}{k!}A^k\bar{\alpha}_0, \quad k = 0, 1, 2, \ldots, \tag{3.2.5}$$

which together with the given initial condition imply

$$x(t) = \left(\sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k \right) \bar{\alpha}_0 = e^{At}x_0, \quad t \geq 0, \tag{3.2.6}$$

where

$$\mathrm{e}^{At} := \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k. \tag{3.2.7}$$

It is straightforward to verify that

$$\frac{d}{dt}\mathrm{e}^{At} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} A^k t^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k+1} t^k = A\mathrm{e}^{At} = \mathrm{e}^{At}A. \tag{3.2.8}$$

2. For the case when the system (3.1.1) has a zero initial condition, *i.e.*, $x_0 = 0$, but with a nonzero external input, $u(t)$, we consider the following equality:

$$\frac{d}{dt}\left(\mathrm{e}^{-At}x\right) = \frac{d\mathrm{e}^{-At}}{dt}x + \mathrm{e}^{-At}\dot{x} = \mathrm{e}^{-At}(\dot{x} - Ax) = \mathrm{e}^{-At}Bu(t). \tag{3.2.9}$$

Integrating both sides of (3.2.9), we obtain

$$\mathrm{e}^{-At}x(t) - x_0 = \mathrm{e}^{-At}x(t) = \int_0^t \mathrm{e}^{-A\tau}Bu(\tau)d\tau, \tag{3.2.10}$$

which implies that

$$x(t) = \mathrm{e}^{At}\int_0^t \mathrm{e}^{-A\tau}Bu(\tau)d\tau = \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau. \tag{3.2.11}$$

It is straightforward to verify that the state response of (3.1.1) with an initial condition $x(0) = x_0$ is given by the sum of (3.2.6) and (3.2.11), *i.e.*, the solution given in (3.2.1). The uniqueness of the solution to (3.1.1) with an initial condition $x(0) = x_0$ can be shown as follows: Suppose x_1 and x_2 are the solutions to (3.1.1) with $x_1(0) = x_2(0) = x_0$. Let $\tilde{x}(t) = x_1(t) - x_2(t)$, and thus $\tilde{x}_0 = \tilde{x}(0) = 0$. We have

$$\dot{\tilde{x}} = \dot{x}_1 - \dot{x}_2 = Ax_1 + Bu - Ax_2 - Bu = A\tilde{x}. \tag{3.2.12}$$

It follows from (3.2.6) that $\tilde{x}(t) = \mathrm{e}^{At}\tilde{x}_0 \equiv 0$, *i.e.*, $x_1(t) \equiv x_2(t)$ for all $t \geq 0$. Lastly, it is simple to see that the corresponding output response of the system (3.1.1) is given as:

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t), \quad t \geq 0. \tag{3.2.13}$$

The term *zero-input response* refers to output response due to the initial state and in the absence of an input signal. The terms *unit step response* and the *impulse* *response*, for the continuous-time system (3.1.1) respectively refer to the output responses of (3.2.13) with zero initial conditions to the input signals,

$$u(t) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad u(t) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \delta(t), \tag{3.2.14}$$

where $\delta(t)$ is a unit *impulse function.*

The dynamical responses of the discrete-time system (3.1.3) can be computed by some simple manipulations. It is straightforward to show that the state response of (3.1.3) with an initial condition $x(0) = x_0$ can be expressed as

$$x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i), \quad k \geq 0, \tag{3.2.15}$$

and thus its corresponding output response is given as

$$y(k) = CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-i-1} Bu(i) + Du(k), \quad k \geq 0. \tag{3.2.16}$$

Similarly, the term zero-input response refers to output response due to the initial state and in the absence of input signal. The terms unit step response and the unit pulse response respectively refer to the output responses of (3.2.16) with zero initial conditions to the input signals,

$$u(k) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad u(k) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \delta(k), \tag{3.2.17}$$

where

$$\delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases} \tag{3.2.18}$$

3.3 System Stability

Stability, more specifically internal stability, is always a primary issue in designing a meaningful control system. For linear systems, either the continuous-time system (3.1.1) or the discrete-time system (3.1.3), the notion of internal stability of the system is related to the behavior of its state trajectory in the absence of the external input, u. Thus, the internal stability is related to the trajectory of

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{3.3.1}$$

for the corresponding continuous-time system (3.1.1), or

$$x(k+1) = Ax(k), \quad x(0) = x_0, \tag{3.3.2}$$

for the corresponding discrete-time system (3.1.3). Specifically, the continuous-time system (3.1.1) is said to be *marginally stable* or *stable in the sense of Lyapunov* or simply *stable* if the state trajectory corresponding to every bounded initial condition x_0 is bounded. It is said to be *asymptotically stable* if it is stable and, in addition, for any initial condition, the corresponding state trajectory $x(t)$ of (3.3.1) satisfies,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At}x_0 = 0. \tag{3.3.3}$$

It is straightforward to verify that the continuous-time linear system (3.1.1) or (3.3.1) is stable if and only if all the eigenvalues of A are in the closed left-half complex plane with those on the $j\omega$ axis having Jordan blocks of size 1. It is asymptotically stable if and only if all the eigenvalues of A are in the open left-half complex plane, *i.e.*, $\lambda(A) \subset \mathbb{C}^-$. This can be shown by first transforming A into a Jordan canonical form, say

$$J = P^{-1}AP = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix}, \tag{3.3.4}$$

where $P \in \mathbb{C}^{n\times n}$ is a nonsingular matrix, and

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i\times n_i}, \quad i = 1, 2, \ldots, q. \tag{3.3.5}$$

Then, we have

$$e^{At} = Pe^{Jt}P^{-1} = P \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_q t} \end{bmatrix} P^{-1}, \tag{3.3.6}$$

where

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & t^{n_i-1}e^{\lambda_i t}/(n_i-1)! \\ 0 & e^{\lambda_i t} & \cdots & t^{n_i-2}e^{\lambda_i t}/(n_i-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_i t} \end{bmatrix}, \tag{3.3.7}$$

$i = 1, 2, \ldots, q$. It is now clear that $e^{J_i t} \to 0$ as $t \to \infty$ if and only if $\lambda_i \in \mathbb{C}^-$, and thus

$$\lim_{t\to\infty} e^{At}x_0 = P \begin{bmatrix} \lim_{t\to\infty} e^{J_1 t} & & & \\ & \lim_{t\to\infty} e^{J_2 t} & & \\ & & \ddots & \\ & & & \lim_{t\to\infty} e^{J_q t} \end{bmatrix} P^{-1}x_0 = 0, \quad (3.3.8)$$

for any $x_0 \in \mathbb{R}^n$, if and only if $\lambda_i \in \mathbb{C}^-$, $i = 1, 2, \ldots, q$, or $\lambda(A) \subset \mathbb{C}^-$. On the other hand, the solutions remain bounded for all initial conditions if and only if $\lambda(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$ and $n_i = 1$ for $\lambda_i(A) \in \mathbb{C}^0$.

The following result is fundamental to the Lyapunov approach to stability analysis.

Theorem 3.3.1. *The continuous-time system of (3.3.1) is asymptotically stable if and only if for any given positive definite matrix $Q = Q' \in \mathbb{R}^{n\times n}$, the Lyapunov equation*

$$A'P + PA = -Q \quad (3.3.9)$$

has a unique and positive definite solution $P = P' \in \mathbb{R}^{n\times n}$.

Proof. The asymptotic stability of the system implies that all eigenvalues of A have negative real parts. Thus, the following matrix is well defined,

$$P = \int_0^\infty e^{A't}Qe^{At}dt. \quad (3.3.10)$$

In what follows, we will show that such a P is the unique solution to the Lyapunov equation (3.3.9) and is positive definite.

First, substitution of (3.3.10) in (3.3.9) yields

$$\begin{aligned} A'P + PA &= \int_0^\infty A'e^{At}Qe^{At}dt + \int_0^\infty e^{A't}Qe^{At}Adt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A't}Qe^{At}\right)dt \\ &= e^{A't}Qe^{At}\Big|_{t=0}^{\infty} \\ &= -Q, \end{aligned} \quad (3.3.11)$$

where we have used the fact that $e^{At} \to 0$ as $t \to \infty$. This shows that P as defined in (3.3.10) is indeed a solution to (3.3.9). To show that the solution (3.3.9) is unique, let P_1 and P_2 both be a solution, *i.e.*,

$$A'P_1 + P_1A = -Q, \quad (3.3.12)$$

and

$$A'P_2 + P_2A = -Q. \tag{3.3.13}$$

Subtracting (3.3.13) from (3.3.12) yields

$$A'(P_1 - P_2) + (P_1 - P_2)A = 0, \tag{3.3.14}$$

which implies that

$$\mathrm{e}^{A't}A'(P_1 - P_2)\mathrm{e}^{At} + \mathrm{e}^{A't}(P_1 - P_2)A\mathrm{e}^{At} = \frac{d}{dt}\mathrm{e}^{A't}(P_1 - P_2)\mathrm{e}^{At} = 0. \tag{3.3.15}$$

Integration of (3.3.15) from $t = 0$ to ∞ yields

$$\mathrm{e}^{A't}(P_1 - P_2)\mathrm{e}^{At}\Big|_{t=0}^{\infty} = P_1 - P_2 = 0. \tag{3.3.16}$$

This shows that P as defined in (3.3.10) is the unique solution to the Lyapunov equation (3.3.9).

It is clear that this P is symmetric since Q is. The positive definiteness of P follows from the fact that, for any nonzero $x \in \mathbb{R}^n$,

$$x'Px = \int_0^{\infty} x'\mathrm{e}^{A't}Q\mathrm{e}^{At}x\,dt > 0, \tag{3.3.17}$$

which in turn follows from the facts that Q is positive definite and that e^{At} is nonsingular for any t.

Conversely, if there are positive definite P and Q that satisfy the Lyapunov equation (3.3.9), then all eigenvalues of A have negative real parts. To show this, let λ be an eigenvalue of A with an associated eigenvector $v \neq 0$, *i.e.*,

$$Av = \lambda v,$$

which also implies that

$$v^*A' = \lambda^*v^*.$$

Pre-multiplying and post-multiplying (3.3.9) by v^* and v respectively yields

$$-v^*Qv = v^*A'Pv + v^*PAv = (\lambda^* + \lambda)v^*Pv = 2\mathrm{Re}(\lambda)v^*Pv,$$

which implies that $\mathrm{Re}(\lambda) < 0$, as both P and Q are positive definite. ■

Now let us get back to the system Σ of (3.1.1). The following characterization of the system inputs and outputs is due to Desoer and Vidyasagar ([49], p.59). It is valid for asymptotically stable systems.

Theorem 3.3.2. *Consider the continuous-time system* Σ *of (3.1.1) with* A *being asymptotically stable. For the case when* $D = 0$, *i.e.,* Σ *is strictly proper, we have*

1. *if* $u \in L_1^m$, *then* $y \in L_1^p \cap L_\infty^p$, $\dot{y} \in L_1^p$, *and* y *is absolutely continuous and* $y(t) \to 0$ *as* $t \to \infty$;
2. *if* $u \in L_2^m$, *then* $y \in L_2^p \cap L_\infty^p$, $\dot{y} \in L_2^p$, *and* y *is continuous and* $y(t) \to 0$ *as* $t \to \infty$;
3. *if* $u \in L_\infty^m$, *then* $y \in L_\infty^p$, $\dot{y} \in L_\infty^p$, *and* y *is uniformly continuous;*
4. *if* $u \in L_\infty^m$ *with* $u(t) \to u_\infty \in \mathbb{R}^m$ *as* $t \to \infty$, *then* $y(t) \to y_\infty \in \mathbb{R}^p$ *as* $t \to \infty$ *and the convergence is exponential;*
5. *if* $u \in L_q^m$, $1 < q < \infty$, *then* $y \in L_q^p$ *and* $\dot{y} \in L_q^p$.

For the case when $D \neq 0$, *i.e.,* Σ *is nonstrictly proper, we only have the following result: If* $u \in L_q^m$, $1 \le q \le \infty$, *then* $y \in L_q^p$.

Noting (3.3.6) and (3.3.7), it is straightforward to show that, for a stable system, there exists positive scalars $h_\mathrm{m} > 0$ and $\alpha > 0$ such that

$$\left\| \mathrm{e}^{At} \right\| \le h_\mathrm{m} \mathrm{e}^{-\alpha t}, \quad \forall t \ge 0. \tag{3.3.18}$$

The result of Theorem 3.3.2 can be verified through some direct manipulations.

Next, we proceed to address the stability issues for discrete-time systems. The discrete-time system of (3.3.2) is said to be *marginally stable* or *stable in the sense of Lyapunov* or simply *stable* if the state trajectory corresponding to every bounded initial condition x_0 is bounded. It is said to be *asymptotically stable* if it is stable and, in addition, for any initial condition x_0, the corresponding state trajectory $x(k)$ satisfies

$$\lim_{k \to \infty} x(k) = \lim_{k \to \infty} A^k x_0 = 0. \tag{3.3.19}$$

Similarly, we can show that the discrete-time system of (3.3.2) is stable if all the eigenvalues of A are inside or on the unit circle with those on the unit circle having Jordan blocks of size 1. It is asymptotically stable if and only if either one of the following conditions hold:

1. The eigenvalues of A are all inside the unit circle, *i.e.*, $\lambda(A) \subset \mathbb{C}^\odot$.
2. Given any positive definite matrix $Q = Q' \in \mathbb{R}^{n \times n}$, the discrete-time Lyapunov equation
$$A'PA - P = -Q \tag{3.3.20}$$
has a unique and positive definite solution $P = P' \in \mathbb{R}^{n \times n}$.

3.4 Controllability and Observability

Let us first focus on the issue of controllability. The concept of controllability is about controlling the state trajectory of a given system through its input. Simply stated, a system is said to be controllable if its state can be controlled in the state space from any point to any other point through an appropriate control input within a finite time interval. For a linear time-invariant system, it is equivalent to controlling the state trajectory from an arbitrary point to the origin of the state space. To be more precise, we consider the following continuous-time system:

$$\Sigma \, : \, \dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{3.4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Definition 3.4.1. *The given system Σ of (3.4.1) is said to be* controllable *if for any initial condition x_0 and any $x_1 \in \mathbb{R}^n$, there exist a time $t_1 > 0$ and a control signal $u(t)$, $t \in [0, t_1]$, such that the resulting state trajectory satisfies $x(t_1) = x_1$. Otherwise, Σ is said to be* uncontrollable.

We have the following results.

Theorem 3.4.1. *The given system Σ of (3.4.1) is controllable if and only if the matrix*

$$W_c(t) := \int_0^t e^{-A\tau} BB' e^{-A'\tau} d\tau \tag{3.4.2}$$

is nonsingular for all $t > 0$. $W_c(t)$ is called the controllability grammian *of* Σ.

Proof. If $W_c(t_1)$ is nonsingular for some $t_1 > 0$, we let

$$u(t) = -B' e^{-A't} W_c^{-1}(t_1)(x_0 - e^{-At_1} x_1), \quad t \in [0, t_1]. \tag{3.4.3}$$

Then, by (3.2.1), we have

$$\begin{aligned} x(t_1) &= e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-t)} Bu(t)dt \\ &= e^{At_1} x_0 - \left(\int_0^{t_1} e^{A(t_1-t)} BB' e^{-A't} dt\right) W_c^{-1}(t_1)(x_0 - e^{-At_1} x_1) \\ &= e^{At_1} x_0 - e^{At_1} \left(\int_0^{t_1} e^{-At} BB' e^{-A't} dt\right) W_c^{-1}(t_1)(x_0 - e^{-At_1} x_1) \\ &= e^{At_1} x_0 - e^{At_1} x_0 + x_1 \\ &= x_1. \end{aligned} \tag{3.4.4}$$

By definition, Σ is controllable.

We prove the converse by contradiction. Suppose Σ is controllable, but $W_c(t)$ is singular for some $t > 0$, which in fact can be shown implying that $W_c(t_1)$ is singular for all $t_1 > 0$. Then, there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0' W_c(t_1) x_0 = 0. \tag{3.4.5}$$

Thus, we have

$$\begin{aligned} 0 &= \int_0^{t_1} x_0' e^{-At} B B' e^{-A't} x_0 dt \\ &= \int_0^{t_1} \left(B' e^{-A't} x_0\right)' \left(B' e^{-A't} x_0\right) dt \\ &= \int_0^{t_1} \left|B' e^{-A't} x_0\right|^2 dt, \end{aligned} \tag{3.4.6}$$

which implies

$$B' e^{-A't} x_0 = 0, \quad \forall t \in [0, t_1]. \tag{3.4.7}$$

Since Σ is controllable, by definition, for any x_1, there exists a control $u(t)$ such that

$$x_1 = e^{At_1} x_0 + \int_0^{t_1} e^{At_1} e^{-At} B u(t) dt. \tag{3.4.8}$$

In particular, for $x_1 = 0$, we have

$$0 = e^{At_1} x_0 + e^{At_1} \int_0^{t_1} e^{-At} B u(t) dt, \tag{3.4.9}$$

or

$$x_0 = -\int_0^{t_1} e^{-At} B u(t) dt, \tag{3.4.10}$$

which together with (3.4.7) imply that

$$|x_0|^2 = x_0' x_0 = \left[-\int_0^{t_1} e^{-At} B u(t) dt\right]' x_0 = -\int_0^{t_1} u'(t) B' e^{-A't} x_0 dt = 0.$$

This is a contradiction as $x_0 \neq 0$. Hence, $W_c(t)$ is nonsingular for all $t > 0$. ∎

Theorem 3.4.2. *The given system Σ of (3.4.1) is controllable if and only if*

$$\text{rank}\,(Q_c) = n, \tag{3.4.11}$$

where

$$Q_c := [\,B \quad AB \quad \cdots \quad A^{n-1}B\,] \tag{3.4.12}$$

is called the controllability matrix *of Σ.*

Proof. We again prove this theorem by contradiction. Suppose rank $(Q_c) = n$, but Σ is uncontrollable. Then, it follows from Theorem 3.4.1 that

$$W_c(t) = \int_0^{t_1} e^{-At} BB' e^{-A't} dt, \quad \forall t_1 > 0 \tag{3.4.13}$$

is singular for some $t_1 > 0$. Also, it follows from the proof of Theorem 3.4.1, *i.e.*, equation (3.4.7), that there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0' e^{-At} B = 0, \quad \forall t \in [0, t_1]. \tag{3.4.14}$$

Differentiating (3.4.14) with respect to t and letting $t = 0$, we obtain

$$x_0' B = 0, \; x_0' AB = 0, \; \ldots, \; x_0' A^{n-1} B = 0, \tag{3.4.15}$$

or

$$x_0' [B \quad AB \quad \cdots \quad A^{n-1}B] = x_0' Q_c = 0, \tag{3.4.16}$$

which together with the fact that $x_0 \neq 0$ imply rank $(Q_c) < n$. Obviously, this is a contradiction, and hence, Σ is controllable.

Conversely, we will show that if Σ is controllable, then rank $(Q_c) = n$. If Σ is controllable, but rank $(Q_c) \neq n$, *i.e.*, rank $(Q_c) < n$, then, there exists a nonzero $x_0 \in \mathbb{R}^n$ such that $x_0' Q_c = 0$, *i.e.*,

$$x_0' B = 0, \; x_0' AB = 0, \; \ldots, \; x_0 A^{n-1} B = 0. \tag{3.4.17}$$

It follows from the Cayley–Hamilton Theorem, *i.e.*, (2.3.33), that

$$x_0' A^k B = 0, \quad k = n, n+1, \ldots \tag{3.4.18}$$

Thus, we have

$$x_0' e^{-At} B = 0 \tag{3.4.19}$$

and

$$x_0' \left(\int_0^t e^{-At} BB' e^{-A't} dt \right) x_0 = x_0' W_c(t) x_0 = 0, \tag{3.4.20}$$

which implies that $W_c(t)$ is singular for all $t > 0$, and hence, by Theorem 3.4.1, the given system Σ is uncontrollable. This is a contradiction. Thus, Q_c has to be of full rank. ■

Theorem 3.4.3. *The given system Σ of (3.4.1) is controllable if and only if, for every eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$,*

$$\text{rank}\, [\lambda_i I - A \quad B] = n. \tag{3.4.21}$$

This theorem is known as the PBH (Popov–Belevitch–Hautus) test, *developed by Popov* [109], *Belevitch* [11] *and Hautus* [63].

Proof. If Σ is controllable, we will show that (3.4.21) holds. Again, we prove this by contradiction. Suppose that (3.4.21) is not true for a controllable Σ, *i.e.*,

$$\text{rank}\,[\,\lambda_i I - A \quad B\,] < n, \tag{3.4.22}$$

for some λ_i. Then, there exists a nonzero $v \in \mathbb{C}^n$ such that

$$v'\,[\,\lambda_i I - A \quad B\,] = 0 \;\Rightarrow\; v'A = \lambda_i v', \;\; v'B = 0,$$

which implies

$$v'AB = \lambda_i v'B = 0, \;\; v'A^2B = \lambda_i v'AB = 0, \;\; \ldots, \;\; v'A^{n-1}B = 0.$$

Thus,

$$v'\,[\,B \quad AB \quad \cdots \quad A^{n-1}B\,] = v'Q_{\rm c} = 0,$$

or rank $(Q_{\rm c}) < n$, *i.e.*, Σ is uncontrollable. The contradiction implies that (3.4.21) is indeed true.

The proof of the converse part requires some state transformations that transform the given system into a certain special form. For example, the result is trivial once the given system is transformed into the so-called controllability structural decomposition (CSD) form given in Theorem 4.4.1 of Chapter 4. We leave the details to the interested readers. ■

We note that Theorem 3.4.3 builds an interconnection between the system controllability and the eigenstructure of the system matrix, *i.e.*, A. The system is controllable if all the eigenvalues of A satisfy the condition given in (3.4.21). On the other hand, the system is not controllable if one or more eigenvalues of A do not satisfy the condition given in (3.4.21). As such, we call an eigenvalue of A a *controllable mode* if it satisfies (3.4.21). Otherwise, it is said to be an *uncontrollable mode*. In many control system design methods, it is not necessary to require the given system to be controllable. The system can be properly controlled if all its uncontrollable modes are stable. Such a system is said to be *stabilizable* as it can still be made stable through a proper state feedback control. For easy reference, in what follows, we highlight the concept of stabilizability.

Definition 3.4.2. *The given system* Σ *of (3.4.1) is said to be* stabilizable *if all its uncontrollable modes are asymptotically stable. Otherwise,* Σ *is said to be* unstabilizable.

We have the following theorem.

Theorem 3.4.4. *For the given system Σ of (3.1.1), the following two statements are equivalent:*

1. *The pair (A, B) is stabilizable.*
2. *There exists an $F \in \mathbb{R}^{m\times n}$ such that, under the state feedback law*
$$u = Fx, \tag{3.4.23}$$
the resulting closed-loop system is asymptotically stable, i.e., $A + BF$ has all its eigenvalues in $\mathbb{C}^-$.

Proof. The result will be obvious once we have established Theorem 4.4.1 of Chapter 4. We thus omit the details here. ∎

Similarly, we can introduce the concept of observability and detectability for the following unforced system Σ:

$$\dot{x} = Ax, \quad y = Cx, \tag{3.4.24}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and A and C are constant matrices of appropriate dimensions. Basically, the system of (3.4.24) is said to be observable if we are able to reconstruct (or observe) the state variable, x, using only the measurement output y. More precisely, we have the following definition.

Definition 3.4.3. *The given system Σ of (3.4.24) is said to be* observable *if there exists a time, $t_1 > 0$, such that any initial state $x(0) = x_0$ can be uniquely determined from the measurement output $y(t)$, $t \in [0, t_1]$. Otherwise, Σ is said to be* unobservable.

We have the following results.

Theorem 3.4.5. *The given system Σ of (3.4.24) is observable if and only if the matrix*

$$W_o(t) := \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau \tag{3.4.25}$$

is nonsingular for all $t > 0$. $W_o(t)$ is called the observability grammian *of Σ.*

Proof. If $W_o(t_1)$ is nonsingular for some $t_1 > 0$, then the initial state x_0 can be computed using the measurement output $y(t)$ as follows,

$$W_o^{-1}(t_1) \int_0^{t_1} e^{A't} C' y(t) dt = W_o^{-1}(t_1) \left(\int_0^{t_1} e^{A't} C' C e^{At} dt \right) x_0 = x_0.$$

Hence, Σ is observable.

Conversely, if Σ is observable, we need to prove that $W_o(t)$ is nonsingular for all $t > 0$. We will show this by contradiction. Suppose Σ is observable, but $W_o(t)$ is singular for some $t > 0$, which in fact implies that $W_o(t_1)$ is singular for all $t_1 > 0$. Then, there exists a nonzero initial state $x_0 \in \mathbb{R}^n$ such that

$$0 = x_0' W_o(t_1) x_0 = \int_0^{t_1} x_0' e^{A't} C' C e^{At} x_0 dt = \int_0^{t_1} y'(t) y(t) dt = \int_0^{t_1} |y(t)|^2 dt,$$

which implies that $y(t) \equiv 0, t \in [0, t_1]$. It is impossible to determine the nonzero initial state x_0, and hence by definition, Σ is not observable. This contradiction shows that $W_o(t)$ is nonsingular for all $t > 0$. ∎

Remark 3.4.1. *In examining observability, we have assumed that* $u \equiv 0$ *and considered the unforced system (3.4.24). This is without loss of generality. In the situation when* $u \not\equiv 0$, *the proof of the theorem remains valid with* $y(t)$ *being replaced by*

$$\tilde{y}(t) := y(t) - C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau - D u(t) = C e^{At} x(0).$$

Next, examining the duality between $W_c(t)$ of (3.4.2) and $W_o(t)$ of (3.4.25) and the results of Theorems 3.4.1 and 3.4.5, it is clear that the given system Σ of (3.4.24) is observable if and only if the auxiliary (dual) system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} := -A'\tilde{x} + C'\tilde{u} \tag{3.4.26}$$

is controllable. Utilizing the results of Theorems 3.4.2 and 3.4.3, we can derive the following results.

Theorem 3.4.6. *The given system* Σ *of (3.4.24) is observable if and only if either one of the following statements is true:*

1. *The* observability matrix *of* Σ,

$$Q_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3.4.27}$$

is of full rank, i.e., rank $(Q_o) = n$.

2. *For every eigenvalue of* A, λ_i, $i = 1, 2, \ldots, n$,

$$\text{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n. \tag{3.4.28}$$

Similarly, the eigenvalues of A that satisfy (3.4.28) and those that do not satisfy (3.4.28) are called the *observable modes* and *unobservable modes* of Σ, respectively. The following is the definition of the system detectability.

Definition 3.4.4. *The given system Σ of (3.4.24) is said to be* detectable *if all its unobservable modes are asymptotically stable. Otherwise, Σ is said to be* undetectable.

Note that the concepts of stabilizability and detectability are important as they are necessary and sufficient for the existence of a measurement feedback control law that stabilizes the given system. We have the following results.

Theorem 3.4.7. *For the given system Σ of (3.1.1), the following two statements are equivalent:*

1. *The pair (A, C) is detectable.*
2. *There exists a $K \in \mathbb{R}^{n \times p}$ such that $A + KC$ has all its eigenvalues in $\mathbb{C}^-$.*

Furthermore, the following dynamical equation utilizing only the system output and control input is capable of asymptotically estimating the system state trajectory, $x(t)$, without knowing its initial value x_0:

$$\dot{\hat{x}} = A\hat{x} + Bu - K(y - C\hat{x} - Du), \quad \hat{x}_0 \in \mathbb{R}^n, \tag{3.4.29}$$

i.e., $e(t) := x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. The dynamical equation of (3.4.29) is commonly called the state observer or estimator *of Σ.*

Proof. Again, we will leave out details on the equivalence of the two statements. It can be easily verified using the observability structural decomposition of the pair (A, C) given in Theorem 4.3.1 of Chapter 4. The rest of the theorem can be shown in a straightforward way.

It follows from (3.1.1) and (3.4.29) that

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A + KC)e. \tag{3.4.30}$$

Clearly, $e(t) \to 0$ as $t \to \infty$ provided that $A + KC$ is asymptotically stable. ∎

By replacing x in the state feedback law (3.4.23) with $\hat{x}$ of (3.4.29), we obtain the so-called full order observer based output feedback control law:

$$\begin{cases} \dot{\hat{x}} = (A + BF + KC + KDF)\hat{x} - Ky, \\ u = F\hat{x}, \end{cases} \tag{3.4.31}$$

which has a dynamical order equal to n, the order of Σ. The closed-loop system comprising the given system Σ of (3.1.1) and the control law of (3.4.31) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{bmatrix} A & BF \\ -KC & A+BF+KC \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}. \tag{3.4.32}$$

Note that

$$\begin{bmatrix} A & BF \\ -KC & A+BF+KC \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A+BF & -BF \\ 0 & A+KC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}^{-1}.$$

It is then clear that the closed-loop system in (3.4.32) is asymptotically stable provided that both $A+BF$ and $A+KC$ have all their eigenvalues in $\mathbb{C}^-$. We note that the structure of the observer or state estimator is nonunique. The following is a more general state observer proposed by Luenberger [94]:

$$\dot{v} = Pv + My + Nu, \quad v \in \mathbb{R}^r, \tag{3.4.33}$$

which is an estimate of $Tx(t)$ for some constant matrix $T \in \mathbb{R}^{r \times n}$ provided that P is a stable matrix and

$$TA - PT = MC, \quad N + MD = TB. \tag{3.4.34}$$

Letting $e := v - Tx$, we have

$$\begin{aligned} \dot{e} = \dot{v} - T\dot{x} &= (MC - TA)x + Pv + (N + MD - TB)u \\ &= Pe + (MC + PT - TA)x + (N + MD - TB)u = Pe. \end{aligned}$$

Thus, $e(t) \to 0$ as $t \to \infty$.

Lastly, we conclude this section by noting that the concepts of controllability (stabilizability) and observability (detectability) for discrete-time systems, *i.e.*, (3.1.3), parallel those for continuous-time systems. In particular, the results of Theorems 3.4.2, 3.4.3 and 3.4.6 are directly applicable to discrete-time systems.

3.5 System Invertibilities

The topic of system invertibilities has been left out in many popular texts in linear systems, although it is important and crucial in almost every control problem. In fact, the concept of system invertibilities for a linear time-invariant system can be introduced naturally. Recall the general nonstrictly proper continuous-time system Σ of (3.1.1), which has a transfer function given by (3.1.2), *i.e.*,

$$H(s) = C(sI - A)^{-1}B + D. \tag{3.5.1}$$

Without loss of generality, we assume that both $[\,B' \;\; D'\,]$ and $[\,C \;\; D\,]$ are of full rank. We define the invertibility of Σ as follows.

Definition 3.5.1. *Consider the linear time-invariant system Σ of (3.1.1). Then,*

1. *Σ is said to be* left invertible *if there exists a rational matrix function of s, say $L(s)$, such that*
$$L(s)H(s) = I_m. \tag{3.5.2}$$

2. *Σ is said to be* right invertible *if there exists a rational matrix function of s, say $R(s)$, such that*
$$H(s)R(s) = I_p. \tag{3.5.3}$$

3. *Σ is said to be* invertible *if it is both left and right invertible.*

4. *Σ is said to be* degenerate *if it is neither left nor right invertible.*

By definition, it is clear that an invertible system has to be a square system, *i.e.*, the number of the system inputs, m, and the number of the system outputs, p, are identical. A square system is, however, not necessarily invertible. Unfortunately, confusion between invertibility and square systems is common in the literature. Many people take it for granted that a square system is invertible. We illustrate this in the following example.

Example 3.5.1. Consider a system Σ of (3.1.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.5.4}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5.5}$$

Note that both matrices B and C are of full rank. It is controllable and observable, and has a transfer function:

$$H(s) = \frac{1}{s^3 - 3s^2 + s} \begin{bmatrix} (s-1)^2 & s-1 \\ s-1 & 1 \end{bmatrix}. \tag{3.5.6}$$

Clearly, although square, it is a degenerate system as the determinant of $H(s)$ is identical to zero.

The system left and right invertibilities can be interpreted in the time domain as follows (see, e.g., [102] and [122]). For a left invertible system Σ, given any output y produced by Σ with an initial condition x_0, one is able to identify uniquely a control signal u that generates the given output y. For a right invertible system Σ, given any signal $y_{\mathrm{ref}} \in \mathbb{R}^p$, one is able to determine a control input u and an initial condition x_0 for Σ, which would produce an output $y = y_{\mathrm{ref}}$.

We further note that there are structures, *i.e.*, certain indices of integers, associated with the left and right invertibilities of linear systems. Unfortunately, these concepts cannot be easily introduced without the help of special structural forms of the system. We will leave these to the next section when we introduce the Kronecker canonical form of the system matrix of Σ.

Lastly, the concept of invertibilities of discrete-time systems follows identically from that of continuous-time systems.

3.6 Normal Rank, Finite Zeros and Infinite Zeros

The structures of finite zeros (also known as invariant zeros or transmission zeros) and infinite zeros (also known as the relative degrees) of linear systems play a dominant role in modern control theory. It is known that the locations of the closed-loop system poles primarily determine the performance, such as the transient response and settling time, of a control system. It is fortunate that these closed-loop poles can be freely assigned everywhere on the complex plane provided that the given open-loop system is controllable and observable. On the other hand, it is well understood now that the locations of the finite zeros have a significant influence on the overall performance of the closed-loop system as well. For example, a nonminimum phase zero (or unstable invariant zero) would impose a great limitation for many control performances. It is unfortunate that the finite zeros or invariant zeros are invariant under any feedback control, and thus any bad zeros would remain there in the closed-loop system. It is our belief that an unambiguous understanding of system zero structures is essential for the design of a control system.

The concepts of invariant zeros and infinite zeros (relative degrees) for single-input and single-output (SISO) systems are simple. For example, for a SISO system with a transfer function, say

$$H(s) = \frac{s(s+1)}{s^3 + 2s^2 + 3s + 4}, \tag{3.6.1}$$

it is simple to observe that the system possesses two finite zeros at $s = 0$ and $s = -1$, respectively, and a relative degree of 1. For general multi-input and multi-output (MIMO) systems, the concepts of finite and infinite zeros could be quite sophisticated. There have been mistakes in the definitions of these zeros in the literature. We need to introduce the notion of system normal rank before formally defining invariant zeros for MIMO systems.

Definition 3.6.1. *Consider the given system* Σ *of (3.1.1). The* normal rank *of its transfer function* $H(s) = C(sI - A)^{-1}B + D$, *or in short,* normrank$\{H(s)\}$, *is defined as*

$$\text{normrank}\{H(s)\} = \max\left\{\text{rank}\,[H(\lambda)] \,\middle|\, \lambda \in \mathbb{C}\right\}. \tag{3.6.2}$$

For example, the system in Example 3.5.1 has a 2×2 transfer matrix. But, it only has a normal rank of 1. It will be seen shortly that the computation of the system normal rank is trivial once we have developed the structural decomposition technique in the forthcoming chapters.

Next, we are ready to introduce the invariant zeros of the general system Σ of (3.1.1) characterized by a matrix quadruple (A, B, C, D), which can be defined via the Kronecker canonical form of the (Rosenbrock) *system matrix* [112] of Σ,

$$P_\Sigma(s) := \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}. \tag{3.6.3}$$

We first have the following definition for the invariant zeros, without associated multiplicity structure (see also [96]), and blocking zeros (see also [156]). The latter plays an important role in the strong stabilization of multivariable linear systems.

Definition 3.6.2. *Consider the given system* Σ *of (3.1.1). A scalar* $\beta \in \mathbb{C}$ *is said to be an* invariant zero *of* Σ *if*

$$\text{rank}\,\{P_\Sigma(\beta)\} < n + \text{normrank}\,\{H(s)\}. \tag{3.6.4}$$

A scalar $\beta \in \mathbb{C}$ *is said to be a* blocking zero *of* Σ *if* $H(\beta) \equiv 0$.

We note that the invariant zeros are equivalent to the so-called *transmission zeros* defined in the literature (see, e.g., [44,45]) when the given system is both controllable and observable.

Obviously, a blocking zero is an invariant zero, but an invariant zero is not necessarily a blocking zero for MIMO systems. For SISO systems, however, they are identical. It is interesting to note that there are other types of zeros defined in the literature, such as input decoupling zeros, *i.e.*, the uncontrollable modes of the pair (A, B), output decoupling zeros, *i.e.*, the unobservable modes of the pair (A, C), and input-output decoupling zeros, *i.e.*, the eigenvalues of A that are both uncontrollable and unobservable. The collection of all these zeros, including invariant zeros and blocking zeros, are called the system zeros of Σ.

Clearly, by definition, if β is an invariant zero of Σ, then there exist a nonzero vector $x_{\mathrm{R}} \in \mathbb{C}^n$ and a vector $w_{\mathrm{R}} \in \mathbb{C}^m$ such that

$$P_\Sigma(\beta)\begin{pmatrix} x_{\mathrm{R}} \\ w_{\mathrm{R}} \end{pmatrix} = \begin{bmatrix} \beta I - A & -B \\ C & D \end{bmatrix}\begin{pmatrix} x_{\mathrm{R}} \\ w_{\mathrm{R}} \end{pmatrix} = 0. \tag{3.6.5}$$

Here, x_{R} and w_{R} are respectively called the right *state zero direction* and right *input zero direction* associated with the invariant zero β of Σ. The following proposition gives a physical meaning to the invariant zero and its zero directions.

Proposition 3.6.1. *Let β be an invariant zero of Σ with a corresponding right state zero direction x_{R} and a right input zero direction w_{R}. Let the initial state of Σ be $x_0 = x_{\mathrm{R}}$ and the system input be*

$$u(t) = w_{\mathrm{R}} \mathrm{e}^{\beta t}, \quad t \geq 0. \tag{3.6.6}$$

Then, the output of Σ is identically zero, i.e., $y(t) = 0, t \geq 0$, and

$$x(t) = x_{\mathrm{R}} \mathrm{e}^{\beta t}, \quad t \geq 0. \tag{3.6.7}$$

This implies that with an appropriate initial state, the system input signal at an appropriate direction and frequency is totally blocked from the system output.

Proof. First, it is simple to verify that (3.6.5) implies that

$$A x_{\mathrm{R}} + B w_{\mathrm{R}} = \beta x_{\mathrm{R}}, \quad C x_{\mathrm{R}} + D w_{\mathrm{R}} = 0. \tag{3.6.8}$$

We next verify directly that (3.6.7) is a solution to the system Σ of (3.1.1) with the initial condition $x_0 = x_{\mathrm{R}}$ and the input $u(t)$ given in (3.6.6). Indeed, with $u(t)$ of (3.6.6) and $x(t)$ of (3.6.7), we have

$$Ax + Bu = A x_{\mathrm{R}} \mathrm{e}^{\beta t} + B w_{\mathrm{R}} \mathrm{e}^{\beta t} = (A x_{\mathrm{R}} + B w_{\mathrm{R}}) \mathrm{e}^{\beta t} = \beta x_{\mathrm{R}} \mathrm{e}^{\beta t} = \dot{x}. \tag{3.6.9}$$

Thus, $x(t)$ is indeed a solution to the state equation of Σ and it satisfies the initial condition $x(0) = x_{\mathrm{R}}$. In fact, $x(t)$ as given in (3.6.7) is the unique solution (see, e.g., Section 3.2). Next, we have

$$y(t) = Cx(t) + Du(t) = (C x_{\mathrm{R}} + D w_{\mathrm{R}}) \mathrm{e}^{\beta t} \equiv 0, \quad t \geq 0. \tag{3.6.10}$$

This concludes the proof of Proposition 3.6.1. ∎

The infinite zero structure of Σ can be either defined in association with the root locus theory (see, *e.g.*, [105]) or as the Smith–McMillan zeros of the transfer function at infinity (see [41], [110], [112] and [146]). To define the zero structure of $H(s)$ at infinity, one can use the Smith–McMillan description of the zero structure at finite frequencies of the transfer function matrix $H(s)$. Namely, a rational matrix $H(s)$ possesses an infinite zero of order k when $H(1/z)$ has a finite zero of precisely that order at $z = 0$. The number of zeros at infinity together with their orders define an infinite zero structure.

In what follows, however, we will introduce the well-known *Kronecker canonical form* for the system matrix $P_\Sigma(s)$, which is able to display the invariant zero structure, invertibility structures and infinite zero structure of Σ altogether. Although it is not a simple task (it is actually a pretty difficult task for systems with a high dynamical order), it can be shown (see Gantmacher [56]) that there exist nonsingular transformations U and V such that $P_\Sigma(s)$ can be transformed into the following form:

$$UP_\Sigma(s)V = \begin{bmatrix} \text{blkdiag}\{sI-J, L_{l_1},\ldots,L_{l_{p_\text{b}}}, R_{r_1},\ldots,R_{r_{m_\text{c}}}, I-sH, I_{m_0}\} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.6.11}$$

where 0 is a zero matrix corresponding to the redundant system inputs and outputs, if any; J is in Jordan canonical form, and $sI-J$ has the following $\sum_{i=1}^{\delta}\tau_i$ pencils as its diagonal blocks,

$$sI_{n_{\beta_i,j}} - J_{n_{\beta_i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \tag{3.6.12}$$

$j = 1,2,\ldots,\tau_i$ and $i = 1,2,\ldots,\delta$; and L_{l_i}, $i = 1,2,\ldots,p_\text{b}$, is an $(l_i+1)\times l_i$ bidiagonal pencil given by

$$L_{l_i} := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{3.6.13}$$

R_{r_i}, $i = 1,2,\ldots,m_\text{c}$, is an $r_i \times (r_i+1)$ bidiagonal pencil given by

$$R_{r_i} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \tag{3.6.14}$$

H is nilpotent and in Jordan form, and $I - sH$ has the following m_d pencils as its diagonal blocks,

$$I_{q_i+1} - sJ_{q_i+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \quad q_i > 0, \ i = 1, 2, \dots, m_\mathrm{d}, \tag{3.6.15}$$

and finally m_0 in I_{m_0} is the rank of D, i.e., $m_0 = \mathrm{rank}\,(D)$.

We have the following definitions.

Definition 3.6.3. *Consider the given system Σ of (3.1.1) whose system matrix $P_\Sigma(s)$ has a Kronecker form as in (3.6.11) to (3.6.15). Then,*

1. *β_i is said to be an invariant zero of Σ with a* geometric multiplicity *of τ_i and an* algebraic multiplicity *of $\sum_{j=1}^{\tau_i} n_{\beta_i,j}$. It has a zero structure*

$$S^\star_{\beta_i}(\Sigma) := \{n_{\beta_i,1}, n_{\beta_i,2}, \dots, n_{\beta_i,\tau_i}\}. \tag{3.6.16}$$

β_i is said to be a simple invariant zero *if $n_{\beta_i,1} = \cdots = n_{\beta_i,\tau_i} = 1$.*

2. *The* left invertibility structure *of Σ is defined as*

$$S^\star_\mathrm{L}(\Sigma) := \{l_1, l_2, \dots, l_{p_\mathrm{b}}\}. \tag{3.6.17}$$

3. *The* right invertibility structure *of Σ is defined as*

$$S^\star_\mathrm{R}(\Sigma) := \{r_1, r_2, \dots, r_{m_\mathrm{c}}\}. \tag{3.6.18}$$

4. *Finally, m_0 is the number of the infinite zeros of Σ of order 0. The* infinite zero structure *of Σ of order higher than 0 is defined as:*

$$S^\star_\infty(\Sigma) := \{q_1, q_2, \dots, q_{m_\mathrm{d}}\}. \tag{3.6.19}$$

We say that Σ has m_d infinite zeros of order $q_1, q_2, \dots, q_{m_\mathrm{d}}$, respectively. If $q_1 = \cdots = q_{m_d}$ and $m_0 = 0$, then Σ is said to be of uniform rank *q_1. On the other hand, if $m_0 > 0$ and $S^\star_\infty(\Sigma) = \emptyset$, then Σ is said to be of uniform rank 0.*

We note that all the invariant zeros, β_i, $i = 1, 2, \dots, \delta$, and their corresponding zero structures constitute the $\mathbf{I}_1$ *list of Morse* [100]. Furthermore, $S^\star_\mathrm{R}$, $S^\star_\mathrm{L}$ and $S^\star_\infty$ are corresponding to lists $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ of Morse, respectively. Also note that Σ is left invertible if $S_{\mathrm{R}}^\star = \emptyset$, it is right invertible if $S_{\mathrm{L}}^\star = \emptyset$, and invertible if both $S_{\mathrm{R}}^\star$ and $S_{\mathrm{L}}^\star$ are empty. The computation of these indices turns out to be quite simple using the techniques presented in the later chapters. We illustrate these structures in the following example.

Example 3.6.1. Consider a system Σ of (3.1.1) characterized by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.6.20}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.6.21}$$

It can be shown (using the technique to be given later in Section 5.6 of Chapter 5) that with the following transformations

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & -1 \end{bmatrix},$$

the Kronecker canonical form of Σ is given as follows:

$$
UP_{\Sigma}(s)V = \left[\begin{array}{cc|cc|cc|ccccc}
s-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].
$$

Thus, we have $S_1^\star(\Sigma) = \{2\}$, $S_{\rm L}^\star(\Sigma) = \{2\}$, $S_{\rm R}^\star(\Sigma) = \{1\}$, $S_\infty^\star(\Sigma) = \{1,2\}$, *i.e.*, Σ has a nonsimple invariant zero at $s = 1$, and two infinite zeros of order 1 and 2, respectively. Σ is degenerate as both $S_{\rm L}^\star(\Sigma)$ and $S_{\rm R}^\star(\Sigma)$ are nonempty.

Again, we note that the aforementioned structural properties, such as the finite zero and infinite zero structures as well as invertibility structures, of continuous-time systems carry over to discrete-time systems without much effort.

3.7 Geometric Subspaces

The geometric approach to linear systems and control theory has attracted much attention over the past few decades. It was started in the 1970s and quickly matured in the 1980s when researchers attempted to solve disturbance decoupling and almost disturbance decoupling problems, which require the design of appropriate control laws to make the influence of the exogenous disturbances to the controlled outputs equal to zero or almost zero (see, *e.g.*, Basile and Marro [9], Schumacher [126], Willems [151,152], Wonham [154], and Wonham and Morse [155]). In fact, most of the concepts in linear systems can be tackled and studied nicely within the geometric framework (see, *e.g.*, the classical text by Wonham [154] and a recent text by Trentelman *et al.* [141]). The geometric approach is mathematically elegant in expressing abstract concepts in linear systems. It is, however, hard to compute explicitly various subspaces defined in the framework.

The purpose of this section is to introduce the basic concepts of some popular and useful geometric subspaces defined in the literature, such as the weakly unobservable subspaces and strongly controllable subspaces. These geometric subspaces play remarkable roles in solving many control problems, such as disturbance decoupling, H_2 and H_∞ control. We will show later that these subspaces can be easily obtained using the structural decomposition technique given in Chapter 5.

Let us consider the continuous-time system Σ of (3.1.1) and let us first focus on the weakly unobservable subspaces. Noting from Proposition 3.6.1, there are certain initial states of Σ, *i.e.*, the right state zero directions, for which there exist control signals that will make the system output identically zero. The set of all right state zero directions of the invariant zeros of Σ does not cover all such initial states. The weakly unobservable subspace of Σ does.

The following are the definition and properties of the weakly unobservable subspace adopted from Trentelman *et al.* [141].

Definition 3.7.1. *Consider the continuous-time system Σ of (3.1.1). An initial state of Σ, $x_0 \in \mathbb{R}^n$, is called* weakly unobservable *if there exists an input signal $u(t)$ such that the corresponding system output $y(t) = 0$ for all $t \geq 0$. The subspace formed by the set of all weakly unobservable points of Σ is called the* weakly unobservable subspace *of Σ and is denoted by $\mathcal{V}^*(\Sigma)$.*

The following lemma shows that any state trajectory of Σ starting from an initial condition in $\mathcal{V}^*(\Sigma)$ with a control input that produces an output $y(t) = 0$, $t \geq 0$, will always stay inside the weakly unobservable subspace, $\mathcal{V}^*(\Sigma)$.

Lemma 3.7.1. *Let x_0 be an initial state of Σ with $x_0 \in \mathcal{V}^*(\Sigma)$ and u be an input such that the corresponding system output $y(t) = 0$ for all $t \geq 0$. Then the resulting state trajectory $x(t) \in \mathcal{V}^*(\Sigma)$ for all $t \geq 0$.*

Proof. For an arbitrary $t_1 \geq 0$, we let $\tilde{x}_0 = x(t_1)$ be a new initial condition for Σ and define a new control input $\tilde{u}(t) = u(t + t_1)$, $t \geq 0$. It follows from (3.2.1) that

$$\begin{aligned}
\tilde{x}(t) &= e^{At}\tilde{x}_0 + \int_0^t e^{A(t-\tau)}B\tilde{u}(\tau)d\tau \\
&= e^{At}\left(e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau\right) + \int_0^t e^{A(t-\tau)}Bu(\tau + t_1)d\tau \\
&= e^{A(t+t_1)}x_0 + \int_0^{t_1} e^{A(t+t_1-\tau)}Bu(\tau)d\tau + \int_{t_1}^{t+t_1} e^{A(t+t_1-\tilde{\tau})}Bu(\tilde{\tau})d\tilde{\tau} \\
&= e^{A(t+t_1)}x_0 + \int_0^{t+t_1} e^{A(t+t_1-\tau)}Bu(\tau)d\tau \\
&= x(t + t_1), \quad t \geq 0,
\end{aligned} \tag{3.7.1}$$

and the corresponding system output

$$\tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t) = Cx(t + t_1) + Du(t + t_1) = y(t + t_1) = 0, \tag{3.7.2}$$

for all $t \geq 0$. By Definition 3.7.1, $\tilde{x}_0 = x(t_1) \in \mathcal{V}^*(\Sigma)$. The arbitrariness of t_1 implies that $x(t) \in \mathcal{V}^*(\Sigma)$ for all $t \geq 0$. ∎

The following theorem of [141] shows that the weakly unobservable subspace can be defined in an alternative way.

Theorem 3.7.1. *The weakly unobservable subspace of Σ, $\mathcal{V}^*(\Sigma)$, is equivalent to the largest subspace $\mathcal{V}$ that satisfies either one of the following conditions:*

1. $\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset (\mathcal{V} \times 0) + \mathrm{im}\left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\}.$
2. *There exists an $F \in \mathbb{R}^{m \times n}$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$ and $(C+DF)\mathcal{V} = 0$.*

Proof. First, let us prove that if $x_0 \in \mathcal{V}^*(\Sigma)$, then x_0 is in the largest subspace $\mathcal{V}$ that satisfies the condition in Item 1. It follows from Lemma 3.7.1 that for the given system Σ with an initial state x_0, there exists a u such that the resulting state $x(t) \in \mathcal{V}^*(\Sigma)$ and $y(t) = 0$ for all $t \geq 0$. Observing that

$$\dot{x}(0^+) := \lim_{t \to 0^+} \frac{1}{t} [x(t) - x_0] \in \mathcal{V}^*(\Sigma), \tag{3.7.3}$$

we have $\dot{x}(0^+) = Ax_0 + Bu_0 \in \mathcal{V}^*(\Sigma)$ and $Cx_0 + Du_0 = 0$, where $u_0 = u(0)$, or equivalently

$$\begin{bmatrix} A \\ C \end{bmatrix} x_0 + \begin{bmatrix} B \\ D \end{bmatrix} u_0 \in (\mathcal{V}^*(\Sigma) \times 0). \tag{3.7.4}$$

Thus, x_0 is in a subspace $\mathcal{V}$ that satisfies the condition in Item 1.

Next, we show that the condition in Item 1 implies the condition in Item 2. Let $\mathcal{V} \subset \mathbb{R}^n$ be any subspace that satisfies the condition in Item 1. Let us choose a basis $x_1, x_2, \ldots, x_r, \ldots, x_n$ for $\mathbb{R}^n$ such that $x_1, x_2, \ldots, x_r$ is a basis for $\mathcal{V}$. The condition in Item 1 implies that there exist $u_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, r$, such that

$$Ax_i + Bu_i \in \mathcal{V}, \quad Cx_i + Du_i = 0. \tag{3.7.5}$$

Let $F \in \mathbb{R}^{m \times n}$ be such that $Fx_i = u_i$, $i = 1, 2, \ldots, r$. We have

$$(A + BF)x_i \in \mathcal{V}, \quad (C + DF)x_i = 0. \tag{3.7.6}$$

Because $x_1, x_2, \ldots, x_r$ is a basis of $\mathcal{V}$, we have

$$(A + BF)\mathcal{V} \in \mathcal{V}, \quad (C + DF)\mathcal{V} = 0, \tag{3.7.7}$$

or equivalently, the condition in Item 1 implies that in Item 2.

Lastly, we show that any subspace $\mathcal{V}$ that satisfies the condition in Item 2, is a subspace of $\mathcal{V}^*(\Sigma)$. Let $x_0 \in \mathcal{V}$ and let us choose $u(t) = Fx(t)$. Then, the condition in Item 2 implies that the resulting state trajectory $x(t) \in \mathcal{V}$ and $y(t) = Cx(t) + Du(t) = (C + DF)x(t) = 0$ for all $t \geq 0$. Thus, by definition, $x_0 \in \mathcal{V}^*(\Sigma)$. Hence, $\mathcal{V} \subset \mathcal{V}^*(\Sigma)$. This concludes the proof of Theorem 3.7.1. ■

Using the result of Theorem 3.7.1, we can further define the stable and the unstable weakly unobservable subspaces of Σ.

Definition 3.7.2. *Consider a system Σ characterized by a quadruple (A, B, C, D). Then we define $\mathcal{V}^{\mathrm{x}}(\Sigma)$ to be the largest subspace $\mathcal{V}$ that satisfies*

$$(A + BF)\mathcal{V} \subset \mathcal{V}, \quad (C + DF)\mathcal{V} = 0, \tag{3.7.8}$$

and the eigenvalues of $(A+BF)|\mathcal{V}$ are contained in $\mathbb{C}^{\mathrm{x}} \subset \mathbb{C}$ for some $F \in \mathbb{R}^{n\times m}$. Obviously, $\mathcal{V}^{\mathrm{x}} = \mathcal{V}^*$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}$. We further define $\mathcal{V}^- := \mathcal{V}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^- \cup \mathbb{C}^0$, and $\mathcal{V}^+ := \mathcal{V}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^+$.

The following definition characterizes a subspace for which any state trajectory starting from within it vanishes in finite time and its corresponding system output can be made identically zero. Such a subspace is called the controllable weakly unobservable subspace of Σ, and is denoted by $\mathcal{R}^*(\Sigma)$.

Definition 3.7.3. *Consider the continuous-time system Σ of (3.1.1). An initial state of Σ, $x_0 \in \mathbb{R}^n$, is called* controllable weakly unobservable *if there exists an input signal $u(t)$ and $t_1 > 0$ such that the resulting system output $y(t) = 0$ for all $t \in [0, t_1]$ and the resulting state trajectory vanishes at $t = t_1$, i.e., $x(t_1) = 0$. The subspace formed by the set of all controllable weakly unobservable points of Σ is called the* controllable weakly unobservable subspace *of Σ and is denoted by $\mathcal{R}^*(\Sigma)$.*

Clearly, it follows from Definitions 3.7.1 and 3.7.3 that $\mathcal{R}^*(\Sigma) \subset \mathcal{V}^*(\Sigma)$. We next introduce the strongly controllable subspace of Σ, $\mathcal{S}(\Sigma)$. $\mathcal{S}$ and $\mathcal{V}$ are dual in the sense that $\mathcal{V}^{\mathrm{x}}(\Sigma^\star) = \mathcal{S}^{\mathrm{x}}(\Sigma)^\perp$, where $\Sigma^\star$ is characterized by the quadruple (A', C', B', D'). The physical interpretation of $\mathcal{S}$ is rather abstract and can be found in Trentelman *et al.* [141].

Definition 3.7.4. *Consider a system Σ characterized by a quadruple (A, B, C, D). Then we define the* strongly controllable subspace *of Σ, $\mathcal{S}^{\mathrm{x}}(\Sigma)$, to be the smallest subspace $\mathcal{S}$ that satisfies*

$$(A + KC)\mathcal{S} \subset \mathcal{S}, \quad \mathrm{im}\,(B + KD) \subset \mathcal{S}, \tag{3.7.9}$$

and the eigenvalues of the map that is induced by $A + KC$ on the factor space $\mathbb{R}/S$ are contained in $\mathbb{C}^\times \subset \mathbb{C}$ for some $K \in \mathbb{R}^{p\times n}$. We let $\mathcal{S}^* := \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}$, $\mathcal{S}^- := \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}^- \cup \mathbb{C}^0$, and $\mathcal{S}^+ := \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}^+$.

Intuitively, it is pretty clear from the definitions that the controllable weakly unobservable subspace is a subspace of the weakly unobservable subspace that is inside the strongly controllable subspace, *i.e.*,

$$\mathcal{R}^*(\Sigma) = \mathcal{V}^*(\Sigma) \cap \mathcal{S}^*(\Sigma). \tag{3.7.10}$$

This indeed turns out to be the case (see, *e.g.*, Trentelman *et al.* [141] for the detailed proof). Another popular subspace (paired with $\mathcal{R}^*$) is called the *distributionally weakly unobservable subspace* (denoted by $\mathcal{N}^*$) and is equivalent to the sum of the weakly unobservable subspace and the strongly controllable subspace, *i.e.*,

$$\mathcal{N}^*(\Sigma) = \mathcal{V}^*(\Sigma) + \mathcal{S}^*(\Sigma). \tag{3.7.11}$$

Finally, we define two more geometric subspaces of Σ, which were originally introduced by Scherer [124,125] for tackling H_∞ almost disturbance decoupling problems.

Definition 3.7.5. *For the given system Σ of (3.1.1) and for any $\lambda \in \mathbb{C}$, we define*

$$\mathcal{V}_\lambda(\Sigma) := \left\{ \zeta \in \mathbb{C}^n \;\middle|\; \exists\, \omega \in \mathbb{C}^m \; : \; 0 = \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right\}, \tag{3.7.12}$$

$$\mathcal{S}_\lambda(\Sigma) := \left\{ \zeta \in \mathbb{C}^n \;\middle|\; \exists\, \omega \in \mathbb{C}^{n+m} \; : \; \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \omega \right\}. \tag{3.7.13}$$

$\mathcal{V}_\lambda(\Sigma)$ and $\mathcal{S}_\lambda(\Sigma)$ are associated with the state zero directions of Σ if λ is an invariant zero of Σ. Clearly, $\mathcal{S}_\lambda(\Sigma) = \mathcal{V}_{\bar\lambda}(\Sigma^\star)^\perp$.

Once again, we note that all the aforementioned geometric subspaces can be explicitly computed using the structural decomposition technique to be developed in Chapter 5. In fact, the system given in Example 3.6.1 is already in the required form, and its geometric subspaces can thus be easily obtained. We now proceed to give all its geometric subspaces in the following example.

Example 3.7.1. Let us re-consider the system Σ with (A, B, C, D) being given in Example 3.6.1. It can be verified that the various geometric subspaces of Σ are given as:

$$\mathcal{V}^*(\Sigma) = \mathcal{V}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}, \quad \mathcal{V}^-(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\},$$

$$\mathcal{S}^*(\Sigma) = \mathcal{S}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

$$\mathcal{S}^-(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

$$\mathcal{R}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}, \quad \mathcal{N}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

and for $\lambda = 1$, which is the invariant zero of Σ,

$$\mathcal{V}_\lambda(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}, \quad \mathcal{S}_\lambda(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\}.$$

Note that in the computation of $\mathcal{V}$ and $\mathcal{S}$, we select F and K as follows:

$$F = -\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \star & \star & \star \\ 1 & 1 & 1 & 1 & 1 & \star & \star & \star \\ 1 & 1 & 1 & 1 & \star & 1 & 1 & 1 \end{bmatrix}, \quad K = -\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \star \\ 1 & 1 & \star \\ 1 & 1 & 1 \\ \star & \star & 1 \\ \star & \star & 0 \\ \star & \star & 1 \end{bmatrix},$$

where "$\star$" are appropriate scalars subject to the constraints on eigenvalues as in Definitions 3.7.2 and 3.7.4.

We conclude this section by noting that all the geometric subspaces defined for continuous-time systems can be used for discrete-time systems as well, except for $\mathcal{V}^-$, $\mathcal{S}^-$, $\mathcal{V}^+$ and $\mathcal{S}^+$, for which we need to modify their associated $\mathbb{C}^\times$ as $\mathbb{C}^\odot \cup \mathbb{C}^\circ$ (for the first two subspaces) and $\mathbb{C}^\otimes$ (for the last two subspaces).

3.8 Properties of State Feedback and Output Injection

It is straightforward to show that for a linear time-invariant system, its stability, controllability, observability, invertibility, and finite and infinite zero structures are all invariant under nonsingular state, input and output transformations. In fact, this property will enable us to develop the structural decomposition technique in the forthcoming chapters, which is to construct certain nonsingular state, input and output transformations for a given system such that all its structural properties can be explicitly displayed. In particular, the system transfer function remains unchanged under any nonsingular state transformation, which results in nonuniqueness for the realization of a system from the frequency domain (transfer function) to the time domain (state space representation).

In this section, we will study the behavior of the aforementioned system structural properties under two operations, the state feedback and the output injection, which are commonly used in systems and control theory. Given a continuous-time system Σ as characterized by (3.1.1), the *state feedback operation* is to introduce a control law

$$u = Fx + v, \tag{3.8.1}$$

and apply it to Σ. The resulting closed-loop system can then be written as

$$\Sigma_{\rm F} \; : \; \begin{cases} \dot{x} = (A + BF)\; x \; + \; B\; v, \\ y = (C + DF)\; x \; + \; D\; v, \end{cases} \tag{3.8.2}$$

i.e., it is characterized by a new quadruple $(A + BF, B, C + DF, D)$. The ideas of the output injection and state feedback are dual. But, it is hard to express the output injection in terms of an explicit expression as in (3.8.1). Instead, given a $K \in \mathbb{R}^{n \times p}$, we directly treat the *output injection* as an operation that generates a new system Σ_K characterized by a quadruple $(A + KC, B + KD, C, D)$.

The following two theorems show that the structural properties of Σ, including geometric subspaces, are invariant under state feedback and output injection.

Theorem 3.8.1. *Consider a given system Σ characterized by a matrix quadruple (A, B, C, D). Also, consider a state feedback gain matrix $F \in \mathbb{R}^{m \times n}$. Then, Σ_F as characterized by the quadruple $(A + BF, B, C + DF, D)$ has the following properties:*

1. *Σ_F is a controllable (stabilizable) system if and only if Σ is a controllable (stabilizable) system;*
2. *The normal rank of Σ_F is equal to that of Σ;*
3. *The invariant zero structure of Σ_F is the same as that of Σ;*
4. *The infinite zero structure of Σ_F is the same as that of Σ;*
5. *Σ_F is (left or right) invertible or degenerate if and only if Σ is (left or right) invertible or degenerate;*
6. *$\mathcal{V}^{\times}(\Sigma_F) = \mathcal{V}^{\times}(\Sigma)$ and $\mathcal{S}^{\times}(\Sigma_F) = \mathcal{S}^{\times}(\Sigma)$;*
7. *$\mathcal{R}^*(\Sigma_F) = \mathcal{R}^*(\Sigma)$ and $\mathcal{N}^*(\Sigma_F) = \mathcal{N}^*(\Sigma)$; and*
8. *$\mathcal{V}_\lambda(\Sigma_F) = \mathcal{V}_\lambda(\Sigma)$ and $\mathcal{S}_\lambda(\Sigma_F) = \mathcal{S}_\lambda(\Sigma)$.*

Proof. Item 1 is obvious. Items 3 and 4 follow directly from the following fact:

$$\begin{bmatrix} A + BF - sI & B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}. \tag{3.8.3}$$

Items 2 and 5 can be seen from the following simple manipulations:

$$\begin{aligned} H_F(s) &:= (C + DF)(sI - A - BF)^{-1}B + D \\ &= (C + DF)(sI - A)^{-1}[I - BF(sI - A)^{-1}]^{-1}B + D \\ &= (C + DF)(sI - A)^{-1}B[I - F(sI - A)^{-1}B]^{-1} + D \\ &= [C(sI - A)^{-1}B + D][I - F(sI - A)^{-1}B]^{-1} \\ &= H(s)[I - F(sI - A)^{-1}B]^{-1}. \end{aligned} \tag{3.8.4}$$

Since $[I - F(sI - A)^{-1}B]^{-1}$ is well defined almost everywhere on the complex plane, the results of Items 2 and 5 follow.

For Item 6, it is obvious from the definition that $\mathcal{V}^{\mathrm{x}}$ is invariant under any state feedback law. Next, for any subspace $\mathcal{S}$ that satisfies (3.7.9), we have

$$(A+KC+BF+KDF)\mathcal{S} = (A+KC)\mathcal{S}+(B+KD)F\mathcal{S} \subseteq \mathcal{S}. \tag{3.8.5}$$

Thus, $\mathcal{S}^{\mathrm{x}}$ is invariant under any state feedback laws as well.

Next, it follows from (3.7.10) and (3.7.11) that both $\mathcal{R}^*$ and $\mathcal{N}^*$ are invariant under state feedback. This proves Item 7.

Let us now prove Item 8. Recalling the definition of $\mathcal{V}_\lambda$, we have

$$\mathcal{V}_\lambda(\Sigma_{\mathrm{F}}) = \left\{ \zeta \in \mathbb{C}^n \,\middle|\, \exists\, \omega \in \mathbb{C}^m \ : \ 0 = \begin{bmatrix} A+BF-\lambda I & B \\ C+DF & D \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right\}.$$

Then, for any $\zeta \in \mathcal{V}_\lambda(\Sigma_{\mathrm{F}})$, there exists an $\omega \in \mathbb{C}^m$ such that

$$0 = \begin{bmatrix} A+BF-\lambda I & B \\ C+DF & D \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix},$$

or

$$0 = \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \begin{pmatrix} \zeta \\ \tilde{\omega} \end{pmatrix},$$

where $\tilde{\omega} = F\zeta + \omega$. Thus, $\zeta \in \mathcal{V}_\lambda(\Sigma)$ and hence $\mathcal{V}_\lambda(\Sigma_{\mathrm{F}}) \subseteq \mathcal{V}_\lambda(\Sigma)$. Similarly, one can show that $\mathcal{V}_\lambda(\Sigma) \subseteq \mathcal{V}_\lambda(\Sigma_{\mathrm{F}})$, and hence $\mathcal{V}_\lambda(\Sigma) = \mathcal{V}_\lambda(\Sigma_{\mathrm{F}})$. The result that $\mathcal{S}_\lambda(\Sigma_{\mathrm{F}}) = \mathcal{S}_\lambda(\Sigma)$ can be shown using similar arguments. ∎

Theorem 3.8.2. *Consider a given system Σ characterized by a matrix quadruple (A, B, C, D). Also, consider an output injection gain matrix $K \in \mathbb{R}^{n\times p}$. Then, Σ_{K} as characterized by the quadruple $(A+KC, B+KD, C, D)$ has the following properties:*

1. *Σ_{K} is an observable (detectable) system if and only if Σ is an observable (detectable) system;*
2. *The normal rank of Σ_{K} is equal to that of Σ;*
3. *The invariant zero structure of Σ_{K} is the same as that of Σ;*
4. *The infinite zero structure of Σ_{K} is the same as that of Σ;*
5. *Σ_{K} is (left or right) invertible or degenerate if and only if Σ is (left or right) invertible or degenerate;*

6. $\mathcal{V}^{\times}(\Sigma_{\rm K}) = \mathcal{V}^{\times}(\Sigma)$ *and* $\mathcal{S}^{\times}(\Sigma_{\rm K}) = \mathcal{S}^{\times}(\Sigma)$;

7. $\mathcal{R}^{*}(\Sigma_{\rm K}) = \mathcal{R}^{*}(\Sigma)$ *and* $\mathcal{N}^{*}(\Sigma_{\rm K}) = \mathcal{N}^{*}(\Sigma)$; *and*

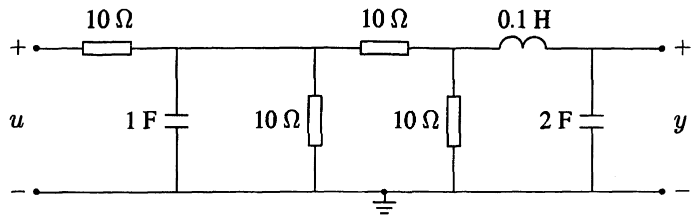
8. $\mathcal{V}_{\lambda}(\Sigma_{\rm K}) = \mathcal{V}_{\lambda}(\Sigma)$ *and* $\mathcal{S}_{\lambda}(\Sigma_{\rm K}) = \mathcal{S}_{\lambda}(\Sigma)$.

Proof. It is the dual version of Theorem 3.8.1. ∎

Note that Theorems 3.8.1 and 3.8.2 hold for discrete-time systems as well.

3.9 Exercises

3.1. Consider an electric network shown in the circuit below with its input, u, being a voltage source, and output, y, being the voltage across the 2 F capacitor. Assume that the initial voltages across the 1 F and 2 F capacitors are 1 V and 2 V, respectively, and that the inductor is initially uncharged.



Circuit for Exercise 3.1.

(a) Derive the state and output equations of the network.

(b) Find the unit step response of the network.

(c) Find the unit impulse response of the network.

(d) Determine the stability of the network.

(e) Determine the controllability and observability of the network.

(f) Determine the invertibility of the network.

(g) Determine the finite and infinite zero structures of the network.

3.2. Given

$$e^{At} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix},$$

determine the values of the scalars α and β, and the matrices A and A^{100}.

3.3. Given a linear system, $\dot{x} = Ax + Bu$, with $x(t_1) = x_1$ and $x(t_2) = x_2$ for some $t_1 > 0$ and $t_2 > 0$, show that

$$\int_{t_1}^{t_2} \mathrm{e}^{-A\tau} Bu(\tau) d\tau = \mathrm{e}^{-At_2} x_2 - \mathrm{e}^{-At_1} x_1.$$

3.4. Given a linear time-invariant system, $\dot{x} = Ax + Bu$, let

$$\tilde{A} := \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}.$$

(a) Verify that $\mathrm{e}^{\tilde{A}t}$ has the form

$$\mathrm{e}^{\tilde{A}t} = \begin{bmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{bmatrix}.$$

(b) Show that the controllability grammian of the system is given by

$$W_\mathrm{c}(t) = \int_0^t \mathrm{e}^{-A\tau} BB' \mathrm{e}^{-A'\tau} d\tau = E_3'(t) E_2(t).$$

(c) Compute $W_\mathrm{c}(t)$ for the system obtained in Exercise 3.1 with $t = 0.1$, 0.5 and 2 seconds.

3.5. Consider an uncontrollable system, $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that

$$\mathrm{rank}\,(Q_\mathrm{c}) = \mathrm{rank}\,([B \quad AB \quad \cdots \quad A^{n-1}B]) = r < n.$$

Let $\{q_1, q_2, \ldots, q_r\}$ be a basis for the range space of the controllability matrix, Q_c, and let $\{q_{r+1}, \ldots, q_n\}$ be any vectors such that

$$T = [\,q_1 \quad q_1 \quad \cdots \quad q_r \quad q_{r+1} \quad \cdots \quad q_n\,]$$

is nonsingular. Show that the state transformation

$$x = T\tilde{x} = T \begin{pmatrix} \tilde{x}_\mathrm{c} \\ \tilde{x}_{\bar{\mathrm{c}}} \end{pmatrix}, \quad \tilde{x}_\mathrm{c} \in \mathbb{R}^r, \quad \tilde{x}_{\bar{\mathrm{c}}} \in \mathbb{R}^{n-r},$$

transforms the given system into the form

$$\begin{pmatrix} \dot{\tilde{x}}_\mathrm{c} \\ \dot{\tilde{x}}_{\bar{\mathrm{c}}} \end{pmatrix} = \begin{bmatrix} A_\mathrm{cc} & A_{\mathrm{c}\bar{\mathrm{c}}} \\ 0 & A_{\bar{\mathrm{c}}\bar{\mathrm{c}}} \end{bmatrix} \begin{pmatrix} \tilde{x}_\mathrm{c} \\ \tilde{x}_{\bar{\mathrm{c}}} \end{pmatrix} + \begin{bmatrix} B_\mathrm{c} \\ 0 \end{bmatrix} u,$$

where $(A_\mathrm{cc}, B_\mathrm{c})$ is controllable. Show that the uncontrollable modes of the system are given by $\lambda(A_{\bar{\mathrm{c}}\bar{\mathrm{c}}})$.

3.6. Given an unobservable system characterized by

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

derive a state transformation matrix, T, and a new state variable, $\tilde{x}$, with

$$x = T\tilde{x} = T\begin{pmatrix} \tilde{x}_{\rm o} \\ \tilde{x}_{\bar{\rm o}} \end{pmatrix},$$

such that the given system can be transformed into the form

$$\begin{pmatrix} \dot{\tilde{x}}_{\rm o} \\ \dot{\tilde{x}}_{\bar{\rm o}} \end{pmatrix} = \begin{bmatrix} A_{\rm oo} & 0 \\ A_{\rm o\bar{o}} & A_{\bar{\rm o}\bar{\rm o}} \end{bmatrix} \begin{pmatrix} \tilde{x}_{\rm o} \\ \tilde{x}_{\bar{\rm o}} \end{pmatrix} + \begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix} u, \quad y = [C_{\rm o} \quad 0] \begin{pmatrix} \tilde{x}_{\rm o} \\ \tilde{x}_{\bar{\rm o}} \end{pmatrix},$$

where $(A_{\rm oo}, C_{\rm o})$ is observable. Moreover, the unobservable modes of the system are given by $\lambda(A_{\bar{\rm o}\bar{\rm o}})$.

3.7. Verify the result of Exercise 3.5 for the following systems:

$$\dot{x} = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 0 & 2 & -2 \\ -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u,$$

and

$$\dot{x} = \begin{bmatrix} -3 & -3 & 1 & 0 \\ 26 & 36 & -3 & -25 \\ 30 & 39 & -2 & -27 \\ 30 & 43 & -3 & -32 \end{bmatrix} x + \begin{bmatrix} 3 & 3 \\ -2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} u.$$

3.8. Verify the result of Exercise 3.6 for the following systems:

$$\dot{x} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 1 \quad 1]x,$$

and

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -2 & -2 \\ 0 & 0 & 0 & -1 \\ 1 & 2 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} u, \quad y = [1 \quad 1 \quad 1 \quad 1]x.$$

3.9. Show that if (A, B) is uncontrollable, then $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$.

3.10. It was shown in Theorem 3.8.1 that constant state feedback does not change the controllability of a linear system. Show by an example that a state feedback law may change the observability of the resulting system.

3.11. Similarly, it was shown in Theorem 3.8.2 that constant output injection does not change the observability of a linear system. Show by an example that a constant output injection may change the controllability of the resulting system.

3.12. Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x,$$

is left invertible. Given an output

$$y(t) = \begin{pmatrix} \cos\omega t + \omega \sin\omega t \\ \mathrm{e}^t - \cos\omega t \end{pmatrix}, \quad t \geq 0,$$

which is produced by the given system with an initial condition,

$$x(0) = \begin{pmatrix} 0 \\ 1 \\ \omega^2 \end{pmatrix},$$

determine the corresponding control input, $u(t)$, which generates the above output, $y(t)$. Also, show that such a control input is unique.

3.13. Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u, \quad y = [0 \quad 1 \quad 0]\, x,$$

is right invertible. Find an initial condition, $x(0)$, and a control input, $u(t)$, which together produce an output

$$y(t) = \alpha \cos\omega t, \quad t \geq 0.$$

Show that the solutions are nonunique.

3.14. Using the results of Theorems 3.8.1 and 3.8.2 with an appropriate state feedback gain matrix, F, and an appropriate output injection gain, K, show that both systems given in Exercise 3.12 and Exercise 3.13 have an infinite zero of order 2, and have no invariant zeros.

3.15. Show that for the system given in Exercise 3.12,

$$\mathcal{V}^* = \mathcal{R}^* = \{0\} \quad \text{and} \quad \mathcal{S}^* = \mathcal{N}^* = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

3.16. Show that for the system given in Exercise 3.13,

$$\mathcal{V}^* = \mathcal{R}^* = \operatorname{im}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{S}^* = \mathcal{N}^* = \mathbb{R}^3.$$

3.17. Given a linear system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x,$$

show that it is invertible, controllable and observable. Also, show that it has two infinite zeros of order 1 (and thus has a normal rank equal to 2), and has one invariant zero at $s = 1$ with a geometric multiplicity of 2 and an algebraic multiplicity of 2. Verify that such an invariant zero is also a blocking zero of the system.

3.18. Determine the geometric subspaces, $\mathcal{V}^*$, $\mathcal{R}^*$, $\mathcal{S}^*$ and $\mathcal{N}^*$, for the system given in Exercise 3.17.

3.19. Show that the geometric subspace, $\mathcal{S}_\lambda$, is invariant under any constant state feedback.

3.20. Show that the geometric subspaces, $\mathcal{V}^*$, $\mathcal{V}_\lambda$ and $\mathcal{S}_\lambda$, are invariant under any constant output injection.

Chapter 4

Decompositions of Unforced and/or Unsensed Systems

4.1 Introduction

In this chapter, we introduce the structural decomposition techniques for the following three types of linear time-invariant systems, which are relatively simple compared to general multivariable systems. The techniques presented in this chapter are very useful themselves and serve as an introduction to the more complete theory of structural decompositions of general systems discussed in the later chapters. The types of systems considered in this chapter are:

1. An autonomous system characterized by a constant matrix A, *i.e.*,

$$\dot{x} = Ax. \tag{4.1.1}$$

2. An unforced system characterized by a matrix pair (C, A), *i.e.*,

$$\dot{x} = Ax, \quad y = Cx. \tag{4.1.2}$$

3. An unsensed system characterized by a matrix pair (A, B), *i.e.*,

$$\dot{x} = Ax + Bu. \tag{4.1.3}$$

Note that the systems in (4.1.2) and (4.1.3) are dual to each other.

Specifically, we will introduce a *stability structural decomposition* (SSD) and the *real Jordan decomposition* (RJD) for the autonomous system of (4.1.1). We

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will then present two decompositions for the unforced system of (4.1.2), namely the *observability structural decomposition* (OSD) and the *block diagonal observable structural decomposition* (BDOSD). Dually, two structural decompositions, namely, the *controllability structural decomposition* (CSD) and the *block diagonal controllable structural decomposition* (BDCSD), are given for the unsensed system of (4.1.3). These decompositions are useful in deriving results for more complicated systems discussed in the later chapters. In fact, they are instrumental in solving many system and control problems such as sensor and actuator selection (see, *e.g.*, [31,92]) and almost disturbance decoupling (see, for example, [22,23,86]).

4.2 Autonomous Systems

Consider the linear time-invariant autonomous system Σ characterized by

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{4.2.1}$$

In this section, we present two structural decompositions for such an autonomous system, *i.e.*, the stability structural decomposition (SSD) and the real Jordan decomposition (RJD).

Theorem 4.2.1 (SSD). *Consider the autonomous system Σ of (4.2.1) characterized by a constant matrix A. There exists a nonsingular transformation $T \in \mathbb{R}^{n\times n}$ and nonnegative integers n_-, n_0 and n_+ such that*

$$T^{-1}AT = \tilde{A} = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix}, \tag{4.2.2}$$

where $A_- \in \mathbb{R}^{n_-\times n_-}$ with $\lambda(A_-) \subset \mathbb{C}^-$, $A_0 \in \mathbb{R}^{n_0\times n_0}$ with $\lambda(A_0) \subset \mathbb{C}^0$, and $A_+ \in \mathbb{R}^{n_+\times n_+}$ with $\lambda(A_+) \subset \mathbb{C}^+$. The SSD totally decouples the stable and unstable dynamics as well as those dynamics associated with the imaginary axis eigenvalues.

Note that the existence of the transformation T follows immediately from the real Jordan canonical decomposition given later in this section. In what follows, we proceed to present a constructive algorithm that realizes the above SSD. In fact, this SSD will be used later to improve numerical conditions in finding the Jordan and real Jordan canonical decompositions.

The key idea in the constructive procedure given below is motivated by the fact that, for a square constant matrix with distinct eigenvalues, there is a nonsingular transformation, the corresponding eigenvector matrix, which diagonalizes the given matrix. By treating all the stable eigenvalues, all the unstable eigenvalues and all those eigenvalues on the imaginary axis as single objects, which are obviously distinct, we can compute their corresponding eigenspaces and form a necessary transformation to block-diagonalize the given matrix into the structure of (4.2.2). The following constructive algorithm is adopted from Chen [19]:

STEP SSD.1.

Utilize the numerically stable real Schur decomposition (see, e.g., Golub and Van Loan [59]) to find an orthogonal matrix $P_1 \in \mathbb{R}^{n\times n}$ such that

$$P_1^{-1}AP_1 = M = \begin{bmatrix} M_1 & \star & \cdots & \star \\ 0 & M_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{bmatrix}, \tag{4.2.3}$$

for some integer k, where the symbol $\star$ represents a matrix of less interest. Moreover, for each $i = 1, 2, \ldots, k$, M_i is either a real scalar, say $M_i = \mu_i$, or a 2×2 matrix having a pair of complex eigenvalues at, say $\mu_i \pm j\omega_i$. Moreover, μ_i, $i = 1, 2, \ldots, k$, are arranged such that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$.

STEP SSD.2.

Let n_-, n_0 and n_+ be the numbers of the eigenvalues of A which belong to $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$, respectively. Also, let

$$T_{-0} = P_1 \begin{bmatrix} I_{n_-+n_0} \\ 0 \end{bmatrix}. \tag{4.2.4}$$

The columns of T_{-0} span the entire eigenvector space associated with the nonpositive eigenvalues of A.

STEP SSD.3.

Use the real Schur decomposition once again to find another orthogonal matrix $P_2 \in \mathbb{R}^{n\times n}$ such that

$$P_2^{-1}(-A)P_2 = Q = \begin{bmatrix} Q_- & \star & \star \\ 0 & Q_0 & \star \\ 0 & 0 & Q_+ \end{bmatrix}, \tag{4.2.5}$$

where $\lambda(Q_-) \subset \mathbb{C}^-$, $\lambda(Q_0) \subset \mathbb{C}^0$ and $\lambda(Q_+) \subset \mathbb{C}^+$. Let

$$T_+ = P_2 \begin{bmatrix} I_{n_+} \\ 0 \end{bmatrix}. \tag{4.2.6}$$

The columns of T_+ span the entire eigenvector space associated with the positive eigenvalues of A.

STEP SSD.4.

Let

$$T_1 = [\,T_{-0} \quad T_+\,]. \tag{4.2.7}$$

We have

$$T_1^{-1} A T_1 = \begin{bmatrix} A_{-0} & 0 \\ 0 & A_+ \end{bmatrix}, \tag{4.2.8}$$

where $\lambda(A_{-0}) \subset \mathbb{C}^- \cup \mathbb{C}^0$ and $\lambda(A_+) \subset \mathbb{C}^+$.

STEP SSD.5.

Again, apply the real Schur decomposition to matrix A_{-0} to find an orthogonal matrix $N_1 \in \mathbb{R}^{(n_- + n_0)\times(n_- + n_0)}$ such that

$$N_1^{-1} A_{-0} N_1 = \begin{bmatrix} R_- & \star \\ 0 & R_0 \end{bmatrix}, \tag{4.2.9}$$

where $\lambda(R_-) \subset \mathbb{C}^-$ and $\lambda(R_0) \subset \mathbb{C}^0$. Then, define

$$Z_- = N_1 \begin{bmatrix} I_{n_-} \\ 0 \end{bmatrix}. \tag{4.2.10}$$

STEP SSD.6.

Apply the real Schur decomposition one more time but to the matrix $-A_{-0}$ to find an orthogonal matrix $N_2 \in \mathbb{R}^{(n_- + n_0)\times(n_- + n_0)}$ such that

$$N_2^{-1}(-A_{-0}) N_2 = \begin{bmatrix} S_0 & \star \\ 0 & S_+ \end{bmatrix}, \tag{4.2.11}$$

where $\lambda(S_0) \subset \mathbb{C}^0$ and $\lambda(S_+) \subset \mathbb{C}^+$, and define

$$Z_0 = N_2 \begin{bmatrix} I_{n_0} \\ 0 \end{bmatrix}. \tag{4.2.12}$$

STEP SSD.7.

Finally, let

$$Z_1 = [\,Z_- \quad Z_0\,] \quad \text{and} \quad T = T_1 \begin{bmatrix} Z_1 & 0 \\ 0 & I_{n_+} \end{bmatrix}. \tag{4.2.13}$$

It is straightforward to verify that

$$T^{-1}AT = \tilde{A} = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix}, \tag{4.2.14}$$

where $\lambda(A_-) \subset \mathbb{C}^-$, $\lambda(A_0) \subset \mathbb{C}^0$ and $\lambda(A_+) \subset \mathbb{C}^+$. This concludes the algorithm for the stability structural decomposition.

The above algorithm has been implemented in an m-function `ssd.m`, in [87]. In principle, one can modify the above procedure to deal with discrete-time systems by re-arranging the order of eigenvalues to obtain a required transformation that separates the given matrix A into three parts with their eigenvalues being respectively in $\mathbb{C}^{\odot}$ (the set of complex scalars inside the unit circle), $\mathbb{C}^{\circ}$ (the set of complex scalars on the unit circle) and $\mathbb{C}^{\otimes}$ (the set of complex scalars outside the unit circle). This will, however, require a re-programming of the real Schur decomposition. Following the result of Chen [19], we can utilize the above algorithm to develop a simple procedure that constructs the required transformation T for such a decomposition: Let α be a scalar on the unit circle of the complex plane but not an eigenvalue of A. We define a new matrix,

$$\bar{A} = \frac{1}{2}\Big[(A+\alpha I)^{-1}(A-\alpha I) + (A+\alpha^* I)^{-1}(A-\alpha^* I)\Big], \tag{4.2.15}$$

where α^* is the complex conjugate of α. It is easy to show that $\bar{A}$ is a real-valued matrix. Next, apply Steps SSD.1–SSD.7 to $\bar{A}$ to obtain a transformation T such that

$$T^{-1}\bar{A}T = \begin{bmatrix} \bar{A}_- & 0 & 0 \\ 0 & \bar{A}_0 & 0 \\ 0 & 0 & \bar{A}_+ \end{bmatrix}, \tag{4.2.16}$$

where $\lambda(\bar{A}_-) \subset \mathbb{C}^-$, $\lambda(\bar{A}_0) \subset \mathbb{C}^0$ and $\lambda(\bar{A}_+) \subset \mathbb{C}^+$. Then, it can be readily shown (see e.g., Chen [19]) that this same T yields

$$T^{-1}AT = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix}, \tag{4.2.17}$$

where $\lambda(A_-) \subset \mathbb{C}^{\odot}$, $\lambda(A_0) \subset \mathbb{C}^{\circ}$ and $\lambda(A_+) \subset \mathbb{C}^{\otimes}$. The discrete-time version of stability structural decomposition has also been implemented in an m-function, `dssd.m`, in [87].

We illustrate these techniques in the following example.

Example 4.2.1. Consider an autonomous system Σ of (4.2.1) characterized by

$$A = \begin{bmatrix} -1 & -1 & -3 & -1 & -1 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & -2 & -2 & -3 & -3 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}, \tag{4.2.18}$$

which has eigenvalues at 0, -1, 1, $-2j$ and $2j$. Following the SSD algorithm of Theorem 4.2.1, which has been implemented with an m-function, `ssd.m`, in [87], we obtain

$$T_1 = \begin{bmatrix} 0.57735 & 0.47385 & 0.66493 & 0 & 0.57735 \\ 0 & -0.81277 & 0.07790 & 0 & 0 \\ 0 & 0.33892 & -0.74283 & 0 & -0.57735 \\ -0.57735 & 0 & 0 & 0.70711 & 0 \\ 0.57735 & 0 & 0 & -0.70711 & 0.57735 \end{bmatrix},$$

which gives the following stability structural decomposition of A,

$$T_1^{-1} A T_1 = \left[\begin{array}{c|ccc|c} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0.21932 & 3.44308 & 0 & 0 \\ 0 & -1.17572 & -0.21932 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Using the m-function `dssd.m` of [87], we obtain

$$T_2 = \begin{bmatrix} 0 & -0.57735 & 0.57735 & 0.78072 & -0.23906 \\ 0 & 0 & 0 & -0.59739 & -0.55659 \\ 0 & 0 & -0.57735 & -0.18332 & 0.79565 \\ 0.70711 & 0.57735 & 0 & 0 & 0 \\ -0.70711 & -0.57735 & 0.57735 & 0 & 0 \end{bmatrix},$$

which gives the following stability structural decomposition of A,

$$T_2^{-1} A T_2 = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1.15182 & -2.39096 \\ 0 & 0 & 0 & 2.22785 & -1.15182 \end{array}\right].$$

Next, we introduce another decomposition for autonomous systems, *i.e.*, the real Jordan decomposition (RJD). The numerical difficulty associated with the transformations to the Jordan canonical forms is well understood in the literature. Yet, the Jordan forms have been proven to be convenient tools in dealing with linear systems.

Theorem 4.2.2 (RJD). *Consider the autonomous system Σ of (4.2.1), characterized by $A \in \mathbb{R}^{n\times n}$. There exists a nonsingular transformation $T \in \mathbb{R}^{n\times n}$ and an integer k such that*

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}, \tag{4.2.19}$$

where each block J_i, $i = 1, 2, \dots, k$, has the following form:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}, \tag{4.2.20}$$

if $\lambda_i \in \lambda(A)$ is real, or

$$J_i = \begin{bmatrix} \Lambda_i & I_2 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_2 \\ & & & \Lambda_i \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \mu_i & \omega_i \\ -\omega_i & \mu_i \end{bmatrix}, \tag{4.2.21}$$

if $\lambda_i = \mu_i + j\omega_i$, $\bar{\lambda}_i = \mu_i - j\omega_i \in \lambda(A)$ with $\omega_i > 0$.

The derivation of the above real Jordan canonical form can be found in many text books (see, *e.g.*, Wonham [154]). In what follows, we present a constructive algorithm for obtaining the transformation T. As pointed out earlier in Chapter 2, the numerical difficulties associated with the Jordan decomposition is well understood in the literature. However, when it can be computed accurately, it is very useful. The application of the real Jordan canonical form in this book is mainly for theoretical analysis.

We first repeatedly utilize the results of Theorem 4.2.1 to find a nonsingular and well-conditioned transformation $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}AP = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{bmatrix}, \tag{4.2.22}$$

where sub-matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, 2, \ldots, \ell$, have either a single or one repeated (if $n_i > 1$) real eigenvalue λ_i, or a single or a repeated (if $n_i > 2$) pair of complex eigenvalues λ_i and λ_i^*. Also, we have $\lambda_i \neq \lambda_j$, if $i \neq j$. This can be done using the following procedure:

1. Compute the eigenvalues of A. Let $\lambda_i = \mu_i + j\omega_i$, $\omega_i \geq 0$, $i = 1, 2, \ldots, \ell$, be all the distinct eigenvalues of A, i.e., $\lambda_i \neq \lambda_k$, $i \neq k$, with nonnegative imaginary parts. We also arrange λ_i such that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_\ell$. Furthermore, if $\mu_k = \mu_{k+1}$, we arrange λ_k and λ_{k+1} such that $\omega_k < \omega_{k+1}$.

2. Let k be an integer such that $\mu_1 = \mu_2 = \cdots = \mu_k < \mu_{k+1}$. We define a constant matrix
$$\bar{A} = A - \frac{\mu_k + \mu_{k+1}}{2} I, \tag{4.2.23}$$
which has k distinct eigenvalues in $\mathbb{C}^-$ with all their real parts equal to $(\mu_k - \mu_{k+1})/2$ and no eigenvalue on the imaginary axis. Utilizing the algorithm of the stability structural decomposition of Theorem 4.2.1, one can find a transformation T_1 such that
$$T_1^{-1} \bar{A} T_1 = \begin{bmatrix} \bar{A}_- & 0 \\ 0 & \bar{A}_+ \end{bmatrix}, \tag{4.2.24}$$
where $\lambda(\bar{A}_-) \subset \mathbb{C}^-$ and $\lambda(\bar{A}_+) \subset \mathbb{C}^+$. Such a transformation T_1 yields
$$T_1^{-1} A T_1 = \begin{bmatrix} A_{1,k} & 0 \\ 0 & A_{k+1,\ell} \end{bmatrix}, \tag{4.2.25}$$
where $A_{1,k}$ contains eigenvalues λ_i, $i = 1, 2, \ldots, k$, and $A_{k+1,\ell}$ contains eigenvalues λ_i, $i = k+1, k+2, \ldots, \ell$. Repeating the above procedure for $A_{k+1,\ell}$, we can block diagonalize A with each block containing eigenvalues with the same real part.

3. Next, for $A_{1,k}$, which contains the distinct eigenvalues $\lambda_i = \mu_1 + j\omega_i$, $i = 1, 2, \ldots, k$. These eigenvalues were arranged in Step 1 in such an order that $\omega_1 < \omega_2 < \cdots < \omega_k$. We define a constant matrix,
$$\hat{A} = (A + \beta I)^{-1} (A - \beta I), \tag{4.2.26}$$
where
$$\beta = \frac{\sqrt{\mu_1^2 + \omega_1^2} + \sqrt{\mu_1^2 + \omega_2^2}}{2}. \tag{4.2.27}$$

It can be verified that $\hat{A}$ has only one distinct eigenvalue in $\mathbb{C}^-$ and all the other eigenvalues are in $\mathbb{C}^+$. Utilizing the algorithm of the SSD of Theorem 4.2.1, one can find a transformation T_2 such that

$$T_2^{-1}\hat{A}T_2 = \begin{bmatrix} \hat{A}_- & 0 \\ 0 & \hat{A}_+ \end{bmatrix}, \tag{4.2.28}$$

where $\lambda(\hat{A}_-) \subset \mathbb{C}^-$ and $\lambda(\hat{A}_+) \subset \mathbb{C}^+$. Such a transformation T_2 yields

$$T_2^{-1}A_{1,k}T_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_{1,k}^* \end{bmatrix}, \tag{4.2.29}$$

where A_1 contains only one distinct eigenvalue λ_1, and $A_{1,k}^*$ contains eigenvalues λ_i, $i = 2, 3, \dots, k$. Repeating the above procedure for $A_{1,k}^*$, we can obtain a nonsingular transformation $T_{1,k}$ such that

$$T_{1,k}^{-1}A_{1,k}T_{1,k} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}. \tag{4.2.30}$$

Repeat the above procedure for all the blocks obtained in Step 2, which contain eigenvalues with the same real parts, to yield the desired block diagonalization as in (4.2.22).

Now, for each A_i with its corresponding λ_i being a real number, we use the result of (2.3.39) to obtain a nonsingular transformation $\tilde{S}_i = S_i \in \mathbb{R}^{n_i \times n_i}$ such that A_i is transformed into the Jordan canonical form. For each A_i which has eigenvalues $\lambda_i = \mu_i + j\omega_i$ and $\lambda_i^* = \mu_i - j\omega_i$, with $\omega_i > 0$, we follow Fama and Matthews [52] to define a new $(2n_i) \times (2n_i)$ matrix,

$$Z_i := \begin{bmatrix} A_i - \mu_i I_{n_i} & \omega_i I_{n_i} \\ -\omega_i I_{n_i} & A_i - \mu_i I_{n_i} \end{bmatrix}. \tag{4.2.31}$$

It is simple to show that Z_i has n_i real eigenvalues at 0 and n_i purely imaginary eigenvalues at $\pm j2\omega_i$. Define a constant matrix

$$\tilde{Z}_i = (A + \omega_i I)^{-1}(A - \omega_i I), \tag{4.2.32}$$

which has n_i eigenvalues at -1 and n_i unstable eigenvalues. Then, following the stability structural decomposition of Theorem 4.2.1, one can obtain a nonsingular transformation $S_i^0 \in \mathbb{R}^{(2n_i)\times(2n_i)}$ such that

$$(S_i^0)^{-1}\tilde{Z}_i S_i^0 = \begin{bmatrix} \tilde{Z}_{i0} & 0 \\ 0 & \tilde{Z}_{ix} \end{bmatrix}, \tag{4.2.33}$$

where $\tilde{Z}_{i0}$ has all its eigenvalues at -1 and $\tilde{Z}_{ix}$ has only unstable eigenvalues. It can be readily verified that

$$(S_i^0)^{-1} Z_i S_i^0 = \begin{bmatrix} Z_{i0} & 0 \\ 0 & Z_{ix} \end{bmatrix}, \tag{4.2.34}$$

where Z_{i0} has all its eigenvalues at 0 and Z_{ix} has no eigenvalue at 0. Next, we utilize the result of (2.3.39) to obtain a nonsingular transformation $S_i^1 \in \mathbb{R}^{n_i \times n_i}$ such that

$$(S_i^1)^{-1} Z_{i0} S_i^1 = \text{blkdiag}\Big\{ J_0^1, J_0^1, J_0^2, J_0^2, \dots, J_0^{\sigma_i}, J_0^{\sigma_i} \Big\}, \tag{4.2.35}$$

where J_0^m, $m = 1, 2, \dots, \sigma_i$, have the form,

$$J_0^m = \begin{bmatrix} 0 & I_{n_{im}-1} \\ 0 & 0 \end{bmatrix}. \tag{4.2.36}$$

Let us partition

$$S_i := S_i^0 \begin{bmatrix} S_i^1 & 0 \\ 0 & I_{n_i} \end{bmatrix} = \begin{bmatrix} S_{i,1}^{1,1} & \cdots & S_{i,1}^{1,n_{i1}} & X_{i,1}^{1,1} & \cdots & X_{i,1}^{1,n_{i1}} & \cdots\cdots \\ S_{i,1}^{2,1} & \cdots & S_{i,1}^{2,n_{i1}} & X_{i,1}^{2,1} & \cdots & X_{i,1}^{2,n_{i1}} & \cdots\cdots \end{bmatrix}$$

$$\begin{matrix} S_{i,\sigma_i}^{1,1} & \cdots & S_{i,\sigma_i}^{1,n_{i\sigma_i}} & X_{i,\sigma_i}^{1,1} & \cdots & X_{i,\sigma_i}^{1,n_{i\sigma_i}} & \star \\ S_{i,\sigma_i}^{2,1} & \cdots & S_{i,\sigma_i}^{2,n_{i\sigma_i}} & X_{i,\sigma_i}^{2,1} & \cdots & X_{i,\sigma_i}^{2,n_{i\sigma_i}} & \star \end{matrix} \Bigg], \tag{4.2.37}$$

where $S_{i,m}^{1,k}$, $S_{i,m}^{2,k}$, $X_{i,m}^{1,k}$ and $X_{i,m}^{2,k}$, $m = 1, 2, \dots, \sigma_i$ and $k = 1, 2, \dots, n_{im}$, are $n_i \times 1$ real-valued vectors. Next, define an $n_i \times n_i$ real-valued matrix,

$$\tilde{S}_i = \Big[S_{i,1}^{1,1} \;\; S_{i,1}^{2,1} \;\cdots\; S_{i,1}^{1,n_{i1}} \;\; S_{i,1}^{2,n_{i1}} \;\cdots\; S_{i,\sigma_i}^{1,1} \;\; S_{i,\sigma_i}^{2,1} \;\cdots\; S_{i,\sigma_i}^{1,n_{i\sigma_i}} \;\; S_{i,\sigma_i}^{2,n_{i\sigma_i}} \Big].$$

Finally, let

$$S = \begin{bmatrix} \tilde{S}_1 & & & \\ & \tilde{S}_2 & & \\ & & \ddots & \\ & & & \tilde{S}_\ell \end{bmatrix}, \tag{4.2.38}$$

and $T = PS \in \mathbb{R}^{n \times n}$. It is now straightforward to show that $T^{-1}AT$ is in the real Jordan canonical form as described in Theorem 4.2.2. The algorithm has been implemented in [87] with an m-function called `rjd.m`. We illustrate the above result in the following example.

Example 4.2.2. Consider an autonomous system Σ of (4.2.1) characterized by

$$A = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 2 & 3 & 2 & 2 \\ 4 & 0 & 2 & 2 & 3 & 2 \\ 4 & 0 & 2 & 1 & 4 & 2 \\ 4 & 0 & 3 & 0 & 4 & 2 \\ -20 & -4 & -12 & -9 & -16 & -11 \end{bmatrix}. \tag{4.2.39}$$

Using the m-function `rjd.m` of [87], we obtain a real Jordan canonical decomposition of A with

$$J = \left[\begin{array}{cc|cc|cc} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right], \tag{4.2.40}$$

and the required state transformation,

$$T = \begin{bmatrix} 0 & 0 & -0.12251 & 0.04504 & -0.07311 & -0.53231 \\ -0.63824 & -0.50617 & 0.26812 & 0.09233 & 0.24639 & -0.13302 \\ -0.13207 & -1.14440 & 0.22084 & 0.48295 & -0.15290 & 0.18649 \\ -0.13207 & -1.14440 & -0.41740 & -0.02322 & -0.15290 & 0.18649 \\ -0.13207 & -1.14440 & 0.08877 & -0.66145 & -0.15290 & 0.18649 \\ 1.03444 & 3.93938 & 0.00090 & -0.01943 & 0.58813 & 0.33547 \end{bmatrix}.$$

4.3 Unforced Systems

We consider an unforced system Σ characterized by

$$\dot{x} = Ax, \quad y = Cx, \tag{4.3.1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and A and C are constant matrices of appropriate dimensions. We note that there are quite a number of canonical forms associated with such a system, e.g., the observable canonical form and the observability canonical form (see, e.g., Chen [33] and Kailath [70]). These canonical forms are effective in studying the observability of the given system. However, they are not adequate to show the more intrinsic system structural properties. Two canonical forms are presented in this section for the unforced system (4.3.1), namely the observability structural decomposition (OSD) and the block diagonal observable structural decomposition (BDOSD). These canonical forms require both state and output transformations. The following theorem characterizes the properties of the OSD.

Theorem 4.3.1 (OSD). *Consider the unforced system of (4.3.1) with C being of full rank. Then, there exist nonsingular state transformation $T_s \in \mathbb{R}^{n\times n}$ and nonsingular output transformation $T_o \in \mathbb{R}^{p\times p}$ such that, in the transformed state and output,*

$$x = T_s\tilde{x}, \quad y = T_o\tilde{y}, \tag{4.3.2}$$

where

$$\tilde{x} = \begin{pmatrix} \hat{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{pmatrix}, \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,k_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p, \quad \tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_p \end{pmatrix}, \tag{4.3.3}$$

we have

$$\dot{\hat{x}}_0 = A_0\hat{x}_0 + L_0\tilde{y}, \tag{4.3.4}$$

and for $i = 1, 2, \ldots, p$,

$$\dot{\tilde{x}}_i = A_i\tilde{x}_i + L_i\tilde{y}, \quad \tilde{y}_i = [1 \quad 0]\,\tilde{x}_i, \tag{4.3.5}$$

where L_i, $i = 1, 2, \ldots, p$, are some constant matrices of appropriate dimensions and

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix}. \tag{4.3.6}$$

The matrix A_0 is of dimensions $n_0 \times n_0$, where $n_0 := n - \sum_{i=1}^{p} k_i$, and $\lambda(A_0)$ contains all the unobservable modes of the matrix pair, (C, A). Moreover, the set $\mathcal{O} := \{\, k_1, k_2, \ldots, k_p \,\}$ is the observability index *of (C, A).*

The result of Theorem 4.3.1 can be summarized in a more compact form as follows:

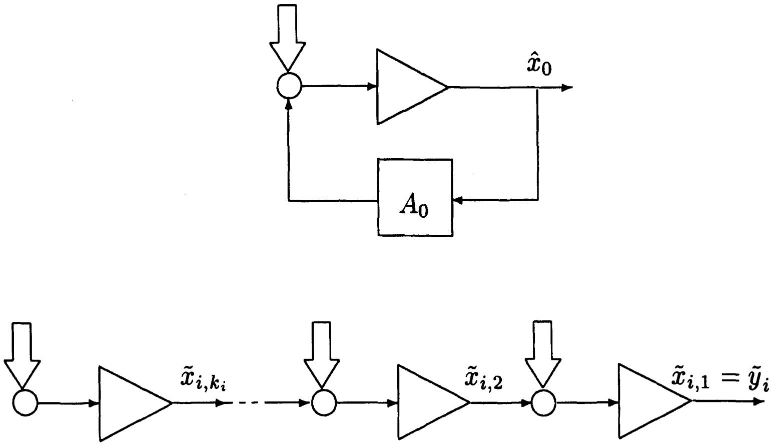
$$T_s^{-1}AT_s = \begin{bmatrix} A_0 & \star & 0 & \cdots & \star & 0 \\ 0 & \star & I_{k_1-1} & \cdots & \star & 0 \\ 0 & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{k_p-1} \\ 0 & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \tag{4.3.7}$$

and

$$T_o^{-1}CT_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{4.3.8}$$

where $\star$ represents a matrix of less interest.

The graphical form interpretation of the OSD is shown in Figure 4.3.1.



Note: the signals indicated by double-edged arrows are some linear combinations of $\tilde{y}_i$.

Figure 4.3.1: Interpretation of the observability structural decomposition.

Proof. We prove Theorem 4.3.1 by giving a step-by-step constructive algorithm that realizes the OSD. The key idea in the following proof is to identify the inherent chains of integrators. Noting the unforced system (4.3.1), we have

$$\dot{y} = C\dot{x} = CAx, \quad \ddot{y} = CA^2x, \quad \ldots, \quad y^{(k)} = CA^kx. \tag{4.3.9}$$

By repeatedly differentiating the system output y, we are able to identify the inherent system structure in terms of chains of integrators.

STEP OSD.1. *Initialization.*

Noting that matrix C is of full rank, we partition it as

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}, \tag{4.3.10}$$

where C_i, $i = 1, 2, \ldots, p$, are independent row vectors. For each C_i, $i = 1, 2, \ldots, p$, we assign a corresponding transformation matrix Z_i to it, which is initially set as:

$$Z_i := C_i. \tag{4.3.11}$$

We also define a flag vector f as

$$f := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{4.3.12}$$

which will be used as a flag in the iterative procedure in STEP OSD.2. Note that the elements of f will be replaced by zero and it will eventually become a zero vector. On the other hand, Z_i will be amended with additional rows and form parts of the required state transformation. We also initiate

$$Z := C, \tag{4.3.13}$$

and an empty matrix $\tilde{Z}$, which will be used to form a state transformation. These matrices are variables, *i.e.*, they might be amended with new components as we progress. Finally, we let $w := 0$.

STEP OSD.2. *Repetitive differentiation of the system output.*

This step will be repeated until f becomes a zero vector. We let

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix}. \tag{4.3.14}$$

For each nonzero element f_i, $i = 1, 2, \ldots, p$, we rewrite its corresponding transformation matrix,

$$Z_i = \begin{bmatrix} C_{i,1} \\ C_{i,2} \\ \vdots \\ C_{i,\alpha_i} \end{bmatrix}, \tag{4.3.15}$$

where $\alpha_i = \text{rank}\,(Z_i)$. Let $x_{i,\alpha_i} := C_{i,\alpha_i} x$, then we have

$$\dot{x}_{i,\alpha_i} = C_{i,\alpha_i} \dot{x} = C_{i,\alpha_i} A x. \tag{4.3.16}$$

The following tests are to be carried out for all Z_i, with nonzero flag f_i:

Case 1. If

$$\text{rank}\left(\begin{bmatrix} Z \\ C_{i,\alpha_i} A \end{bmatrix} \right) > \text{rank}(Z), \tag{4.3.17}$$

it implies that there are more integrators in the chain associated with the i-th output, which must be further identified. We then replace Z and Z_i with

$$Z := \begin{bmatrix} Z \\ C_{i,\alpha_i}A \end{bmatrix}, \quad Z_i := \begin{bmatrix} Z_i \\ C_{i,\alpha_i}A \end{bmatrix}, \tag{4.3.18}$$

and test the next Z_i whose corresponding flag $f_i \neq 0$.

Case 2. If

$$\text{rank}\left(\begin{bmatrix} Z \\ C_{i,\alpha_i}A \end{bmatrix} \right) = \text{rank}(Z), \tag{4.3.19}$$

there is no more inherent integration in the chain associated with this i-th output. For this case, we replace the corresponding flag f_i in the flag vector f with a scalar 0, which stops this output variable from further differentiation, and amend $\tilde{Z}$ as follows:

$$\tilde{Z} := \begin{bmatrix} \tilde{Z} \\ Z_i \end{bmatrix}, \tag{4.3.20}$$

which will be used to define new state variables. We also let

$$w := w + 1 \quad \text{and} \quad k_w := \alpha_i. \tag{4.3.21}$$

For future reference, we rewrite

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ \vdots \\ \tilde{Z}_w \end{bmatrix} \tag{4.3.22}$$

with $j = 1, 2, \ldots, w$,

$$\tilde{Z}_j = \begin{bmatrix} \tilde{C}_{j,1} \\ \tilde{C}_{j,2} \\ \vdots \\ \tilde{C}_{j,k_j} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{j,1} \\ \tilde{C}_{j,1}A \\ \vdots \\ \tilde{C}_{j,1}A^{k_j-1} \end{bmatrix}. \tag{4.3.23}$$

The above tests have to be carried out for all Z_i with flag $f_i \neq 0$. Note that in Case 2, there is an element in the flag vector f being replaced by a scalar 0. As such, f will eventually become a zero vector.

If $f = 0$, we move on to STEP OSD.3. Otherwise, we go back to repeat STEP OSD.2.

STEP OSD.3. *Preliminary transformation.*

Obviously, in STEP OSD.2 we have obtained a set of integers, $k_1, k_2, \ldots, k_p$, with $k_1 \le k_2 \le \cdots \le k_p$. Let $n_0 = n - \sum_{i=1}^{p} k_i$ and S_0 be an $n_0 \times n$ constant matrix such that

$$S := \begin{bmatrix} S_0 \\ \tilde{Z} \end{bmatrix} \tag{4.3.24}$$

is nonsingular. Generally, we can choose an S_0 whose rows constitute a basis of the null space of $\tilde{Z}$. Next, we define a new state variable

$$\bar{x} = \begin{pmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{pmatrix} = Sx, \tag{4.3.25}$$

where

$$\bar{x}_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,k_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p. \tag{4.3.26}$$

Noting that, for $i = 1, 2, \ldots, p$,

$$x_{i,1} = \tilde{C}_{i,1} x = \hat{y}_i, \tag{4.3.27}$$

$$x_{i,2} = \tilde{C}_{i,1} A x, \tag{4.3.28}$$

$$\vdots$$

$$x_{i,k_i} = \tilde{C}_{i,1} A^{k_i - 1} x, \tag{4.3.29}$$

we have

$$\dot{x}_{i,1} = \tilde{C}_{i,1} \dot{x} = \tilde{C}_{i,1} A x = x_{i,2}, \tag{4.3.30}$$

$$\dot{x}_{i,2} = \tilde{C}_{i,1} A \dot{x} = \tilde{C}_{i,1} A^2 x = x_{i,3}, \tag{4.3.31}$$

$$\vdots$$

$$\dot{x}_{i,k_i} = \tilde{C}_{i,1} A^{k_i - 1} \dot{x} = \tilde{C}_{i,1} A^{k_i} x = \sum_{s=1}^{p} \sum_{j=1}^{\min\{k_i+1, k_s\}} \alpha_{i,s,j} x_{s,j}, \tag{4.3.32}$$

for some appropriate constants $\alpha_{i,s,j}$. This last equation follows from the construction of $\tilde{Z}$ in the previous step. Also, we have

$$\dot{\bar{x}}_0 = A_0 \bar{x}_0 + \sum_{s=1}^{p} \sum_{j=1}^{k_s} A_{0,s,j} x_{s,j}, \tag{4.3.33}$$

for some appropriate constant vectors $A_{0,s,j}$.

STEP OSD.4. *Further simplification in (4.3.32).*

Let us define a new state variable

$$\tilde{x} = \begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{pmatrix} = W\bar{x}, \tag{4.3.34}$$

where $\tilde{x}_0 = \bar{x}_0$, and for $i = 1, 2, \ldots, p$,

$$\tilde{x}_i := \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,k_i} \end{pmatrix} := \begin{pmatrix} x_{i,1} - \sum_{s=1}^{p} \sum_{j=k_i+1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i} \\ x_{i,2} - \sum_{s=1}^{p} \sum_{j=k_i}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i+1} \\ \vdots \\ x_{i,k_i} - \sum_{s=1}^{p} \sum_{j=2}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-1} \end{pmatrix}, \tag{4.3.35}$$

and a new output variable,

$$\begin{aligned} \tilde{y}_i := \tilde{x}_{i,1} &= x_{i,1} - \sum_{s=1}^{p} \sum_{j=k_i+1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i} \\ &= x_{i,1} - \sum_{s=1}^{p} \alpha_{i,s,k_i+1} x_{s,1} = \hat{y}_i - \sum_{s=1}^{p} \alpha_{i,s,k_i+1} \hat{y}_s, \end{aligned} \tag{4.3.36}$$

which shows that $\tilde{y}_i = \tilde{x}_{i,1}$ is a linear combination of $\hat{y}_i$, $i = 1, 2, \ldots, p$. Here we note that the coefficient, $\alpha_{i,s,k}$, when $k > k_s$, is set to 0 in the definitions of (4.3.35) and (4.3.36). Then, we have

$$\begin{aligned} \dot{\tilde{x}}_{i,k_i} &= \dot{x}_{i,k_i} - \sum_{s=1}^{p} \sum_{j=2}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} \dot{x}_{s,j-1} \\ &= \sum_{s=1}^{p} \sum_{j=1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j} - \sum_{s=1}^{p} \sum_{j=2}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j} \\ &= \sum_{s=1}^{p} \alpha_{i,s,1} x_{s,1} = \sum_{s=1}^{p} \tilde{\alpha}_{i,s,1} \tilde{x}_{s,1}, \end{aligned} \tag{4.3.37}$$

$$\dot{\tilde{x}}_{i,k_i-1} = \dot{x}_{i,k_i-1} - \sum_{s=1}^{p} \sum_{j=3}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} \dot{x}_{s,j-2}$$

$$
\begin{aligned}
&= x_{i,k_i} - \sum_{s=1}^{p} \sum_{j=3}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-1} \\
&= \tilde{x}_{i,k_i} + \sum_{s=1}^{p} \sum_{j=2}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-1} - \sum_{s=1}^{p} \sum_{j=3}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-1} \\
&= \tilde{x}_{i,k_i} + \sum_{s=1}^{p} \alpha_{i,s,2} x_{s,1} = \tilde{x}_{i,k_i} + \sum_{s=1}^{p} \tilde{\alpha}_{i,s,2} \tilde{x}_{s,1}, \qquad (4.3.38)
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{\tilde{x}}_{i,1} &= \dot{x}_{i,1} - \sum_{s=1}^{p} \sum_{j=k_i+1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} \dot{x}_{s,j-k_i} \\
&= x_{i,2} - \sum_{s=1}^{p} \sum_{j=k_i+1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i+1} \\
&= \tilde{x}_{i,2} + \sum_{s=1}^{p} \sum_{j=k_i}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i+1} - \sum_{s=1}^{p} \sum_{j=k_i+1}^{\min\{k_i+1,k_s\}} \alpha_{i,s,j} x_{s,j-k_i+1} \\
&= \tilde{x}_{i,2} + \sum_{s=1}^{p} \alpha_{i,s,k_i} x_{s,1}, = \tilde{x}_{i,2} + \sum_{s=1}^{p} \tilde{\alpha}_{i,s,k_i} \tilde{x}_{s,1}. \qquad (4.3.39)
\end{aligned}
$$

We also have

$$\dot{\tilde{x}}_0 = A_0 \tilde{x}_0 + \sum_{s=1}^{p} \sum_{j=1}^{k_s} \tilde{A}_{0,s,j} \tilde{x}_{s,j}, \qquad (4.3.40)$$

for some constant vectors $\tilde{A}_{0,s,j}$.

STEP OSD.5. *Further simplification in (4.3.40).*

We now proceed to find a transformation such that the dynamics associated with $\tilde{x}_0$ is expressed only in terms of $\tilde{x}_{s1}$, $s = 1, 2, \ldots, p$.

If $\max\{k_1, k_2, \ldots, k_p\} = 1$, we will skip the following sub-steps and directly go to STEP OSD.6. Otherwise, we let $i := 0$, $\tilde{x}_{0,0} := \tilde{x}_0$, $\tilde{A}_{0,s,j,0} := \tilde{A}_{0,s,j}$, and carry out the following sub-steps:

Sub-step 5.1. First, we note that

$$\dot{\tilde{x}}_{0,i} = A_0 \tilde{x}_{0,i} + \sum_{s=1}^{p} \sum_{j=1}^{k_s-i} \tilde{A}_{0,s,j,i} \tilde{x}_{s,j}. \qquad (4.3.41)$$

We will eliminate $\tilde{x}_{s,k_s-i}$, $s = 1, 2, \ldots, p$, in the above expression. Let us define

$$\tilde{x}_{0,i+1} := \tilde{x}_{0,i} - \sum_{s=1}^{p} \tilde{A}_{0,s,k_s-i,i}\tilde{x}_{s,k_s-i-1}, \tag{4.3.42}$$

where we take $\tilde{A}_{0,s,k_s-i,i} = 0$ if $k_s - i - 1 \le 0$. We have

$$\begin{aligned}
\dot{\tilde{x}}_{0,i+1} &= \dot{\tilde{x}}_{0,i} - \sum_{s=1}^{p} \tilde{A}_{0,s,k_s-i,i}\dot{\tilde{x}}_{s,k_s-i-1} \\
&= A_0\tilde{x}_{0,i} + \sum_{s=1}^{p}\sum_{j=1}^{k_s-i} \tilde{A}_{0,s,j,i}\tilde{x}_{s,j} \\
&\quad - \sum_{s=1}^{p} \tilde{A}_{0,s,k_s-i,i}\left(\tilde{x}_{s,k_s-i} + \sum_{k=1}^{p} \alpha_{s,k,i+2}\tilde{x}_{k,1}\right) \\
&= A_0\tilde{x}_{0,i+1} + \sum_{s=1}^{p} A_0\tilde{A}_{0,s,k_s-i,i}\tilde{x}_{s,k_s-i-1} + \sum_{s=1}^{p}\sum_{j=1}^{k_s-i-1} \tilde{A}_{0,s,j,i}\tilde{x}_{s,j} \\
&\quad - \sum_{s=1}^{p}\sum_{k=1}^{p} \tilde{A}_{0,s,k_s-i,i}\alpha_{s,k,i+2}\tilde{x}_{k,1}.
\end{aligned} \tag{4.3.43}$$

Clearly, we have eliminated $\tilde{x}_{s,k_s-i}$, $s = 1, 2, \ldots, p$, in (4.3.43). Thus, we can rewrite (4.3.43) as

$$\dot{\tilde{x}}_{0,i+1} = A_0\tilde{x}_{0,i+1} + \sum_{s=1}^{p}\sum_{j=1}^{k_s-i-1} \tilde{A}_{0,s,j,i+1}\tilde{x}_{s,j}, \tag{4.3.44}$$

for some appropriate constant vectors $\tilde{A}_{0,s,j,i+1}$.

<u>Sub-step 5.2.</u> If $i = \max\{k_1, k_2, \ldots, k_p\} - 2$, then we will go to STEP OSD.6. Otherwise, let $i := i + 1$ and repeat Sub-step 5.1.

STEP OSD.6. *Finishing touch.*

Let $\hat{x}_0 := \tilde{x}_{0,i+1}$ and $\hat{A}_{0,s} := \tilde{A}_{0,i+1,s,1}$. We have

$$\dot{\hat{x}}_0 = A_0\hat{x}_0 + \sum_{s=1}^{p} \hat{A}_{0,s}\tilde{x}_{s,1}. \tag{4.3.45}$$

In view of (4.3.45) and (4.3.37) to (4.3.39), the transformed system is indeed in the OSD form. This completes the proof of Theorem 4.3.1. The software realization of the above algorithm has been implemented in [87] with a MATLAB function `osd.m`. ■

We illustrate the OSD in the following example.

Example 4.3.1. Consider an unforced system (4.3.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -2 & -1 & 4 & -2 & 3 & 0 \\ -2 & -1 & 3 & -1 & 3 & 0 \\ 1 & 1 & -2 & 3 & -2 & 0 \\ 2 & 1 & -2 & 2 & -3 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}, \tag{4.3.46}$$

and

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}. \tag{4.3.47}$$

We follow closely the step-by-step procedures of the OSD algorithm to construct necessary state and output transformations.

STEP OSD.1. *Initialization.*

We first partition

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} := \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix},$$

and set

$$Z_1 := C_1 = [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0],$$

$$Z_2 := C_2 = [-1 \quad 0 \quad 1 \quad -1 \quad 1 \quad 0],$$

$w := 0$, the flag

$$\boldsymbol{f} := \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$Z := C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix},$$

and $\tilde{Z} := [\,]$, an empty matrix.

STEP OSD.2. *Repetitive differentiation of the system output.*

Noting that

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $f_1 = 1 \neq 0$, we partition

$$Z_1 = [C_{1,1}] = [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0],$$

and compute

$$C_{1,1}A = [1 \quad 0 \quad 2 \quad 0 \quad -1 \quad 0].$$

It is easy to verify that

$$\text{rank}\left(\begin{bmatrix} Z \\ C_{1,1}A \end{bmatrix}\right) = 3 > \text{rank}(Z) = 2,$$

which satisfies the condition of Case 1, *i.e.*, (4.3.17). Thus, we set

$$Z := \begin{bmatrix} Z \\ C_{1,1}A \end{bmatrix} = \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ \hline 1 & 0 & 2 & 0 & -1 & 0 \end{array}\right],$$

and

$$Z_1 := \begin{bmatrix} Z_1 \\ C_{1,1}A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \end{bmatrix}.$$

Similarly, because $f_2 = 1 \neq 0$, we partition

$$Z_2 = [C_{2,1}] = [-1 \quad 0 \quad 1 \quad -1 \quad 1 \quad 0]$$

and compute

$$C_{2,1}A = [-2 \quad -1 \quad 3 \quad -2 \quad 3 \quad 0].$$

It can be readily verified that

$$\text{rank}\left(\begin{bmatrix} Z \\ C_{2,1}A \end{bmatrix}\right) = 4 > \text{rank}(Z) = 3,$$

which again satisfies the condition of Case 1, (4.3.17). Thus, we set

$$Z := \begin{bmatrix} Z \\ C_{2,1}A \end{bmatrix} = \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ \hline -2 & -1 & 3 & -2 & 3 & 0 \end{array}\right],$$

and

$$Z_2 := \begin{bmatrix} Z_2 \\ C_{2,1}A \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \end{bmatrix}.$$

Next, since

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

remains unchanged and $f_1 = 1$, we partition

$$Z_1 = \begin{bmatrix} C_{1,1} \\ C_{1,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \end{bmatrix},$$

and compute

$$C_{1,2}A = [-5 \quad -3 \quad 8 \quad -4 \quad 8 \quad 0],$$

and verify that

$$\operatorname{rank}\left(\begin{bmatrix} Z \\ C_{1,2}A \end{bmatrix}\right) = 5 > \operatorname{rank}(Z) = 4,$$

which corresponds to Case 1. Thus, we set

$$Z := \begin{bmatrix} Z \\ C_{1,2}A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \\ \hline -5 & -3 & 8 & -4 & 8 & 0 \end{bmatrix},$$

$$Z_1 := \begin{bmatrix} Z_1 \\ C_{1,2}A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ \hline -5 & -3 & 8 & -4 & 8 & 0 \end{bmatrix}.$$

Similarly, since $f_2 = 1$, we partition

$$Z_2 = \begin{bmatrix} C_{2,1} \\ C_{2,2} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \end{bmatrix},$$

and compute

$$C_{2,2}A = [-2 \quad -1 \quad 3 \quad -1 \quad 3 \quad 0],$$

and check that

$$\operatorname{rank}\left(\begin{bmatrix} Z \\ C_{2,2}A \end{bmatrix}\right) = 5 = \operatorname{rank}(Z),$$

which satisfies the condition of Case 2, *i.e.*, (4.3.19). Hence, we set

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\tilde{Z} := \begin{bmatrix} \tilde{Z} \\ Z_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \end{bmatrix} =: [\tilde{Z}_1] = \begin{bmatrix} \tilde{C}_{1,1} \\ \tilde{C}_{1,2} \end{bmatrix},$$

and

$$w := w + 1 = 1, \quad k_1 := \operatorname{rank}(\tilde{Z}_1) = 2.$$

Again, noting that

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with $f_1 = 1$, we proceed to partition

$$Z_1 = \begin{bmatrix} C_{1,1} \\ C_{1,2} \\ C_{1,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ -5 & -3 & 8 & -4 & 8 & 0 \end{bmatrix},$$

and compute

$$C_{1,3}A = [-3 \quad -1 \quad 4 \quad 2 \quad 4 \quad 0],$$

and check that

$$\text{rank}\left(\begin{bmatrix} Z \\ C_{1,3}A \end{bmatrix}\right) = 5 = \text{rank}(Z),$$

which satisfies the condition in Case 2. Thus, we set

$$f := \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\tilde{Z} := \begin{bmatrix} \tilde{Z} \\ Z_1 \end{bmatrix} = \left[\begin{array}{cccccc} -1 & 0 & 1 & -1 & 1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ -5 & -3 & 8 & -4 & 8 & 0 \end{array}\right] =: \left[\begin{array}{c} \tilde{Z}_1 \\ \hline \tilde{Z}_2 \end{array}\right] = \left[\begin{array}{c} \tilde{C}_{1,1} \\ \tilde{C}_{1,2} \\ \hline \tilde{C}_{2,1} \\ \tilde{C}_{2,2} \\ \tilde{C}_{2,3} \end{array}\right],$$

$$w := w + 1 = 2, \quad k_2 = \text{rank}(\tilde{Z}_2) = 3.$$

Since the flag, f, is identically zero, we move on to STEP OSD.3.

STEP OSD.3. *Preliminary transformation.*

Obviously, $n_0 = 6 - 5 = 1$. We select

$$S_0 := [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$$

to obtain a preliminary transformation,

$$S := \begin{bmatrix} S_0 \\ \tilde{Z} \end{bmatrix} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 1 & -1 & 1 & 0 \\ -2 & -1 & 3 & -2 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ -5 & -3 & 8 & -4 & 8 & 0 \end{array}\right].$$

The resulting transformed system is given by

$$Sx = \bar{x} = \begin{pmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \quad \bar{x}_1 = \begin{pmatrix} x_{1,1} \\ x_{1,2} \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} x_{2,1} \\ x_{2,2} \\ x_{2,3} \end{pmatrix},$$

$$\dot{\bar{x}} = SAS^{-1}\bar{x} = \left[\begin{array}{c|cc|ccc} -1 & 0.3333 & -2 & 1 & -0.3333 & 0.6667 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 6 & -14 & 0 & 0 & 5 \end{array}\right] \bar{x},$$

and

$$\begin{pmatrix} \hat{y}_2 \\ \hat{y}_1 \end{pmatrix} = CS^{-1}\bar{x} = \left[\begin{array}{c|cc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \bar{x}.$$

STEP OSD.4.

Following the procedure of STEP OSD.4 given in the proof, we obtain another state transformation matrix W,

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & 1 & 0 \\ 0 & 14 & 0 & 0 & -5 & 1 \end{bmatrix},$$

such that under the following transformation,

$$W\bar{x} = \tilde{x} = \begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \quad \tilde{x}_1 = \begin{pmatrix} \tilde{x}_{1,1} \\ \tilde{x}_{1,2} \end{pmatrix}, \quad \tilde{x}_2 = \begin{pmatrix} \tilde{x}_{2,1} \\ \tilde{x}_{2,2} \\ \tilde{x}_{2,3} \end{pmatrix},$$

we have

$$\dot{\tilde{x}} = WSAS^{-1}W^{-1}\tilde{x} = \left[\begin{array}{c|cc|ccc} -1 & -5 & -2 & 1 & 1 & 0.6667 \\ \hline 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & -14 & 0 & -14 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \end{array}\right] \tilde{x},$$

and

$$\tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = W_o AS^{-1}W^{-1}\tilde{x} = \left[\begin{array}{c|cc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \tilde{x},$$

with

$$W_o = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

STEP OSD.5.

It will take two sub-steps to remove the unwanted terms associated with $\tilde{x}_0$. Following the iterative procedures given in the proof, we first obtain a transformation,

$$M_1 = \begin{bmatrix} 1 & 2 & 0 & 0 & -0.6667 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which gives

$$M_1 W S A S^{-1} W^{-1} M_1^{-1} = \left[\begin{array}{c|cc|ccc} -1 & 2.3333 & 0 & 6.3333 & 0.3333 & 0 \\ \hline 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & -14 & 0 & -14 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \end{array}\right],$$

and the second transformation matrix,

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & -0.3333 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

STEP OSD.6. *Finishing touch.*

The complete required state and output transformations are then given by the following matrices:

$$T_s = (M_2 M_1 W S)^{-1} = \begin{bmatrix} 0 & 2 & 2 & -1 & -0.6667 & -0.5556 \\ 0 & 0 & -2 & 2 & 0.3333 & 0.4444 \\ 0 & -2 & -1 & 3 & 1 & 0.3333 \\ 0 & -7 & -3 & 3 & 2 & 1 \\ 0 & -2 & 0 & 0 & 0.3333 & 0.1111 \\ 1 & -2 & 0 & 0.3333 & 0.6667 & 0 \end{bmatrix},$$

and

$$T_o = W_o^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and the resulting transformed system is characterized by

$$T_s^{-1}AT_s = \left[\begin{array}{c|cc|ccc} -1 & 2.3333 & 0 & 4.3333 & 0 & 0 \\ \hline 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & -0 & 0 & 5 & 1 & 0 \\ 0 & -14 & 0 & -14 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \end{array}\right],$$

and

$$T_o^{-1}CT_s = \left[\begin{array}{c|cc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Clearly, the transformed system is indeed in the form of the observability structural decomposition as given in (4.3.7) and (4.3.8).

We now present another decomposition of the unforced system or the matrix pair (C, A), *i.e.*, the so-called block diagonal observable structural decomposition (BDOSD).

Theorem 4.3.2 (BDOSD). *Consider the unforced system of (4.3.1) with* (C, A) *being observable. Then, there exist an integer* $k \le p$, *a set of* k *integers* κ_1, κ_2, ..., κ_k, *and nonsingular transformations* T_s *and* T_o *such that*

$$T_s^{-1}AT_s = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}, \tag{4.3.48}$$

and

$$T_o^{-1}CT_s = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ \star & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & C_k \\ \star & \star & \cdots & \star \end{bmatrix}, \tag{4.3.49}$$

where the symbols $\star$ *represent some matrices of less interest, and* A_i *and* C_i, $i = 1, 2, \ldots, k$, *are in the OSD form*

$$A_i = \begin{bmatrix} \star & I_{\kappa_i - 1} \\ \star & 0 \end{bmatrix}, \quad C_i = [1 \quad 0 \quad \cdots \quad 0]. \tag{4.3.50}$$

Obviously, $\sum_{i=1}^{k} \kappa_i = n$.

Proof. This theorem is the dual version of Theorem 4.4.2 of the next section. The detailed proof of the BDOSD follows from that given in Theorem 4.4.2. The software realization of this canonical form in MATLAB, `bdosd.m`, can be found in [87]. ∎

Example 4.3.2. Consider an unforced system (4.3.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -2 & -1 & 4 & -2 & 3 \\ -2 & -1 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & -2 \\ 2 & 1 & -2 & 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 \end{bmatrix}. \tag{4.3.51}$$

Using `bdosd.m` of [87], we obtain

$$T_s = \begin{bmatrix} 9.202258 & 9.202258 & 9.202258 & 11.985440 & 20.334987 \\ -18.404516 & 0 & 0 & -15.621333 & -44.080817 \\ -27.606773 & -9.202258 & 0 & 0 & -6.419075 \\ -27.606773 & -18.404516 & -9.202258 & 0 & 9.202258 \\ 0 & 0 & 0 & 2.783182 & 8.349547 \end{bmatrix},$$

$$T_o = \begin{bmatrix} -2.783182 & -0.302446 \\ 0 & 1 \end{bmatrix},$$

$$T_s^{-1} A T_s = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$T_o^{-1} C T_s = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \hline -9.202258 & 0 & 0 & -9.202258 & -27.606773 \end{array}\right],$$

which is indeed in the BDOSD form of Theorem 4.3.2. Clearly, we have $k = 1$ and $\kappa_1 = 5$.

4.4 Unsensed Systems

We now proceed to introduce the controllability structural decomposition (CSD) and the block diagonal controllable structural decomposition (BDCSD) for the unsensed system Σ characterized by

$$\dot{x} = Ax + Bu, \tag{4.4.1}$$

where as usual $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. As mentioned earlier, the CSD and BDCSD are actually the dual versions of the OSD and BDOSD, respectively. We note that the CSD is also commonly known as the Brunovsky canonical form (see Brunovsky [15]). But, the same result was reported by Luenberger [95] earlier in 1967. This CSD turns out to be a key tool in solving the problems related to sensor/actuator selection (see, e.g., [31,92]), while the BDCSD is instrumental in solving the problem of H_∞ almost disturbance decoupling (see, e.g., [22,23,86]) and in deriving the structural decomposition for singular systems given later in Chapter 6.

Theorem 4.4.1 (CSD). *Consider the unsensed system of (4.4.1) with B being of full rank. Then, there exist nonsingular state and input transformations $T_\mathrm{s} \in \mathbb{R}^{n\times n}$ and $T_\mathrm{i} \in \mathbb{R}^{m\times m}$ such that, in the transformed input and state,*

$$x = T_\mathrm{s}\tilde{x}, \quad u = T_\mathrm{i}\tilde{u}, \tag{4.4.2}$$

where

$$\tilde{x} = \begin{pmatrix} \hat{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{pmatrix}, \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,k_i} \end{pmatrix}, \quad i = 1, 2, \dots, m, \quad \tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_m \end{pmatrix}, \tag{4.4.3}$$

we have

$$\dot{\hat{x}}_0 = A_0 \hat{x}_0, \tag{4.4.4}$$

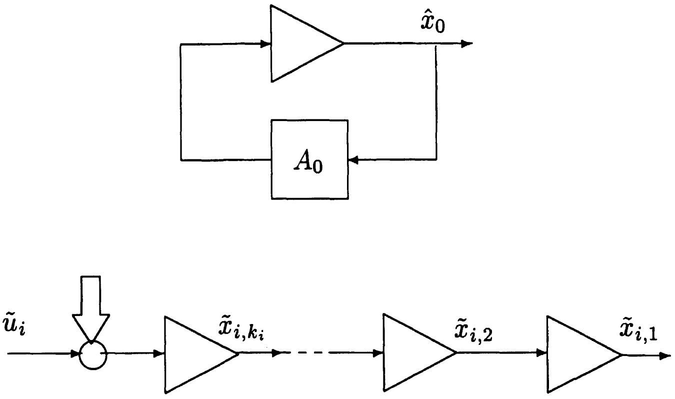
and for $i = 1, 2, \dots, m$,

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + B_i \Big(\tilde{u}_i + E_i \tilde{x} \Big), \tag{4.4.5}$$

where E_i, $i = 1, 2, \dots, m$, are some row vectors of appropriate dimensions, and

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.4.6}$$

The matrix A_0 is of dimensions $n_0 \times n_0$, where $n_0 = n - \sum_{i=1}^m k_i$, and $\lambda(A_0)$ contains all the uncontrollable modes of the matrix pair, (A, B). Moreover, the integer set, $\mathcal{C} := \{k_1, k_2, \dots, k_m\}$, is called the controllability index *of (A, B).*



Note: signals indicated by double-edged arrows are linear combinations of the states.

Figure 4.4.1: Interpretation of the controllability structural decomposition.

Theorem 4.4.1 follows dually from the result of Theorem 4.3.1. The CSD, *i.e.*, the controllability structural decomposition, can be summarized in a matrix form, $(\tilde{A}, \tilde{B}) := (T_s^{-1}AT_s, T_s^{-1}BT_i)$, with

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{4.4.7}$$

where $\star$ represents a matrix of less interest, or in a graphical form as in Figure 4.4.1. The software realization of such a decomposition in MATLAB can be found in [87] under an m-function `csd.m`. Readers are referred to Chapter 12 for a detailed help file on the usage of this m-function.

Example 4.4.1. Consider the unsensed system (4.4.1) characterized by a matrix pair, (A, B), with

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 6 & 3 & 2 & -1 & 0 \\ 5 & 4 & 2 & -1 & 0 \\ 10 & 8 & 5 & -2 & 0 \\ 11 & 7 & 4 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 1 \\ 2 & 0 \end{bmatrix}.$$

The m-function `csd.m` of [87] generates the following results:

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 2 & -5 & 1 & 1 & 3 \\ 0 & -3 & 0 & 1 & 2 \end{bmatrix}, \quad T_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.4.8}$$

and

$$T_s^{-1} A T_s = \left[\begin{array}{cc|c|cc} -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 5 & -1 & 1 & 1 & 2 \end{array}\right], \quad T_s^{-1} B T_i = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right].$$

The corresponding controllability index of (A, B) is given by $\mathcal{C} = \{1, 2\}$. The pair has two uncontrollable modes at -2 and 1, respectively.

The next theorem deals with the block diagonal controllable structural decomposition (BDCSD).

Theorem 4.4.2 (BDCSD). *Consider the unsensed system of (4.4.1) with (A, B) being controllable. Then, there exist an integer $k \leq m$, a set of k integers κ_1, $\kappa_2, \ldots, \kappa_k$, and nonsingular transformations T_s and T_i such that the transformed system, $(\tilde{A}, \tilde{B}) := (T_s^{-1} A T_s, T_s^{-1} B T_i)$, has the following form:*

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & \star & \cdots & \star & \star \\ 0 & B_2 & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_k & \star \end{bmatrix}, \tag{4.4.9}$$

where A_i and B_i, $i = 1, 2, \ldots, k$, are in the CSD form

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \star & \star & \cdots & \star \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{4.4.10}$$

and $\star$ represents a matrix of less interest. Obviously, $\sum_{i=1}^{k} \kappa_i = n$.

Proof. The existence of such a canonical form was shown in Wonham [154]. In what follows, we recall an explicit algorithm for the construction of the transformation matrices T_s and T_i derived earlier in Chen [22].

First, we follow the result of Theorem 4.2.2 to find a nonsingular transformation $Q \in \mathbb{R}^{n \times n}$ such that the matrix A is transformed into a real Jordan canonical form, *i.e.*,

$$\tilde{A} = Q^{-1}AQ = \begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_\ell \end{bmatrix}, \tag{4.4.11}$$

with

$$X_i = \begin{bmatrix} J_{\lambda_i}^1 & & & \\ & J_{\lambda_i}^2 & & \\ & & \ddots & \\ & & & J_{\lambda_i}^{\sigma_i} \end{bmatrix}, \quad i = 1, 2, \ldots, \ell, \tag{4.4.12}$$

where $\lambda_i = \mu_i + j\omega_i \in \lambda(A)$ with $\omega_i \geq 0$, and $\lambda_{i_1} \neq \lambda_{i_2}$, if $i_1 \neq i_2$. Moreover, for each $i \in \{1, 2, \ldots, \ell\}$ and $s = 1, 2, \ldots, \sigma_i$, $J_{\lambda_i}^s \in \mathbb{R}^{n_{is} \times n_{is}}$ has the real Jordan form

$$J_{\lambda_i}^s = \begin{bmatrix} \mu_i & 1 & & \\ & \ddots & \ddots & \\ & & \mu_i & 1 \\ & & & \mu_i \end{bmatrix}, \tag{4.4.13}$$

if $\omega_i = 0$, or

$$J_{\lambda_i}^s = \begin{bmatrix} \Lambda_i & I_2 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_2 \\ & & & \Lambda_i \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \mu_i & \omega_i \\ -\omega_i & \mu_i \end{bmatrix}, \tag{4.4.14}$$

if $\omega_i > 0$. For convenience in later presentation, we arrange the Jordan blocks in the way that $n_{i1} \geq n_{i2} \geq \cdots \geq n_{i\sigma_i}$. Next, compute

$$\tilde{B} = Q^{-1}B = \begin{bmatrix} B_{11}^1 & B_{11}^2 & \cdots & B_{11}^m \\ \vdots & \vdots & \ddots & \vdots \\ B_{1\sigma_1}^1 & B_{1\sigma_1}^2 & \cdots & B_{1\sigma_1}^m \\ B_{21}^1 & B_{21}^2 & \cdots & B_{21}^m \\ \vdots & \vdots & \ddots & \vdots \\ B_{2\sigma_2}^1 & B_{2\sigma_2}^2 & \cdots & B_{2\sigma_2}^m \\ \vdots & \vdots & \ddots & \vdots \\ B_{\ell 1}^1 & B_{\ell 1}^2 & \cdots & B_{\ell 1}^m \\ \vdots & \vdots & \ddots & \vdots \\ B_{\ell\sigma_\ell}^1 & B_{\ell\sigma_\ell}^2 & \cdots & B_{\ell\sigma_\ell}^m \end{bmatrix}. \tag{4.4.15}$$

It is straightforward to verify that the controllability of (A, B) implies: for each $i = 1, 2, \ldots, \ell$, there exists a B_{i1}^{ν} with $\nu \in \{1, 2, \ldots, m\}$ such that $(J_{\lambda_i}^1, B_{i1}^{\nu})$ is controllable, which is equivalent to the last row of B_{i1}^{ν} being nonzero if λ_i is real, or at least one of the last two rows of B_{i1}^{ν} being nonzero if λ_i is not real. Thus, it is simple to find a vector

$$T_1 = \begin{bmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{m1} \end{bmatrix}, \quad t_{11} \neq 0, \tag{4.4.16}$$

and partition

$$\tilde{B}_1 = \tilde{B}T_1 = \begin{bmatrix} \tilde{B}_{11}^1 \\ \vdots \\ \tilde{B}_{1\sigma_1}^1 \\ \tilde{B}_{21}^1 \\ \vdots \\ \tilde{B}_{2\sigma_2}^1 \\ \vdots \\ \tilde{B}_{\ell 1}^1 \\ \vdots \\ \tilde{B}_{\ell\sigma_\ell}^1 \end{bmatrix}, \tag{4.4.17}$$

such that $(J_{\lambda_i}^1, \tilde{B}_{i1}^1)$ is controllable. Because of the special structure of the real Jordan form and the fact that $n_{i1} \geq n_{i2} \geq \cdots \geq n_{i\sigma_i}$, the eigenstructures associated with $J_{\lambda_i}^s$, with $s > 1$, are uncontrollable by $\tilde{B}_1$. Thus, it is straightforward to show that there exist nonsingular transformations T_{s1}^i, $i = 1, 2, \ldots, \ell$, such that

$$(T_{s1}^i)^{-1} \begin{bmatrix} J_{\lambda_i}^1 & & & \\ & J_{\lambda_i}^2 & & \\ & & \ddots & \\ & & & J_{\lambda_i}^{\sigma_i} \end{bmatrix} T_{s1}^i = \begin{bmatrix} J_{\lambda_i}^1 & & & \\ & J_{\lambda_i}^2 & & \\ & & \ddots & \\ & & & J_{\lambda_i}^{\sigma_i} \end{bmatrix}, \tag{4.4.18}$$

and

$$(T_{s1}^i)^{-1} \begin{bmatrix} \tilde{B}_{i1}^1 \\ \tilde{B}_{i2}^1 \\ \vdots \\ \tilde{B}_{i\sigma_i}^1 \end{bmatrix} = \begin{bmatrix} \check{B}_{i1}^1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4.4.19}$$

with $(J_{\lambda_i}^1, \breve{B}_{i1}^1)$ being controllable. This can be done by utilizing the special structure of the CSD form (see Theorem 4.4.1). Next, perform a permutation transformation P_{s1} such that

$$(P_{s1})^{-1} \begin{bmatrix} T_{s1}^1 & & & \\ & T_{s1}^2 & & \\ & & \ddots & \\ & & & T_{s1}^\ell \end{bmatrix}^{-1} \tilde{A} \begin{bmatrix} T_{s1}^1 & & & \\ & T_{s1}^2 & & \\ & & \ddots & \\ & & & T_{s1}^\ell \end{bmatrix} P_{s1}$$
$$= \text{blkdiag}\left\{ J_{\lambda_1}^1, \ldots, J_{\lambda_\ell}^1, J_{\lambda_1}^2, \ldots, J_{\lambda_1}^{\sigma_1}, \ldots\ldots, J_{\lambda_\ell}^2, \ldots, J_{\lambda_\ell}^{\sigma_\ell} \right\},$$

and

$$(P_{s1})^{-1} \begin{bmatrix} T_{s1}^1 & & & \\ & T_{s1}^2 & & \\ & & \ddots & \\ & & & T_{s1}^\ell \end{bmatrix}^{-1} \tilde{B} \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ t_{21} & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ t_{m1} & 0 & \cdots & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \breve{B}_{11}^1 & \breve{B}_{11}^2 & \cdots & \breve{B}_{11}^m \\ \vdots & \vdots & \ddots & \vdots \\ \breve{B}_{\ell 1}^1 & \breve{B}_{\ell 1}^2 & \cdots & \breve{B}_{\ell 1}^m \\ 0 & \breve{B}_{12}^2 & \cdots & \breve{B}_{12}^m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \breve{B}_{1\sigma_1}^2 & \cdots & \breve{B}_{1\sigma_1}^m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \breve{B}_{\ell 2}^2 & \cdots & \breve{B}_{\ell 2}^m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \breve{B}_{\ell\sigma_\ell}^2 & \cdots & \breve{B}_{\ell\sigma_\ell}^m \end{bmatrix}.$$

Because λ_i, $i = 1, 2, \ldots, \ell$, are distinct, the controllability of $(J_{\lambda_i}^1, \breve{B}_{i1}^1)$ implies that

$$(\breve{A}_1, \breve{B}_1) := \left(\begin{bmatrix} J_{\lambda_1}^1 & & & \\ & J_{\lambda_2}^1 & & \\ & & \ddots & \\ & & & J_{\lambda_\ell}^1 \end{bmatrix}, \begin{bmatrix} \breve{B}_{11}^1 \\ \breve{B}_{21}^1 \\ \vdots \\ \breve{B}_{i1}^1 \end{bmatrix} \right), \tag{4.4.20}$$

is controllable. Hence, there exists a nonsingular transformation $X_1 \in \mathbb{R}^{k_1 \times k_1}$, where $k_1 = \sum_{i=1}^{\ell} n_{i1}$, such that

$$X_1^{-1}\breve{A}_1 X_1 = A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \star & \star & \star & \cdots & \star \end{bmatrix}, \tag{4.4.21}$$

and

$$X_1^{-1}\breve{B}_1 = B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4.4.22}$$

Next, repeating the above procedure for the pair

$$\left(\text{blkdiag}\left\{ J_{\lambda_1}^2, \dots, J_{\lambda_1}^{\sigma_1}, \dots\dots, J_{\lambda_\ell}^2, \dots, J_{\lambda_\ell}^{\sigma_\ell} \right\}, \begin{bmatrix} \breve{B}_{12}^2 & \cdots & \breve{B}_{12}^m \\ \vdots & \ddots & \vdots \\ \breve{B}_{1\sigma_1}^2 & \cdots & \breve{B}_{1\sigma_1}^m \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \breve{B}_{\ell 2}^2 & \cdots & \breve{B}_{\ell 2}^m \\ \vdots & \ddots & \vdots \\ \breve{B}_{\ell\sigma_\ell}^2 & \cdots & \breve{B}_{\ell\sigma_\ell}^m \end{bmatrix} \right),$$

one is able to separate (A_2, B_2). Repeating the same procedure for $k-2$ more times, where $k = \max\{\sigma_1, \sigma_2, \dots, \sigma_\ell\}$, one is able to obtain the required canonical form as in Theorem 4.4.2. This completes the proof of the theorem. The result has been implemented in [87] as an m-function `bdcsd.m`. ■

We conclude this chapter with the following example.

Example 4.4.2. Consider the unsensed system (4.4.1) characterized by matrices A and B with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \\ 4 & 3 \\ 5 & 2 \\ 6 & 1 \end{bmatrix}.$$

Using the MATLAB function bdcsd.m of [87], we obtain the following necessary transformations and transformed system:

$$T_s = \begin{bmatrix} -2.10371 & 0 & -4.20741 & 0 & -2.10371 & 0.78529 \\ -2.31866 & 0 & -4.63731 & 0 & -2.31866 & -0.71249 \\ -0.21495 & 9.10545 & -3.17845 & -3.17845 & -2.53360 & 0 \\ -5.71205 & 2.53360 & 3.82330 & 2.10371 & -2.74855 & 0 \\ 3.17845 & -0.21495 & 0.21495 & -0.21495 & -2.96350 & 0 \\ -2.96350 & 6.14195 & -6.14195 & 6.14195 & -3.17845 & 0 \end{bmatrix},$$

$$T_i = \begin{bmatrix} -0.48477 & 0 \\ -0.26982 & 0.97828 \end{bmatrix},$$

and

$$T_s^{-1} A T_s = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & -2 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad T_s^{-1} B T_i = \left[\begin{array}{c|c} 0 & -0.56147 \\ 0 & -0.29865 \\ 0 & -0.32323 \\ 0 & -0.76184 \\ 1 & -1.20895 \\ \hline 0 & 1 \end{array}\right].$$

This verifies the results of Theorem 4.4.2.

Note that although the results of this chapter are stated for continuous-time systems, they are valid for discrete-time systems as well.

4.5 Exercises

4.1. Show that if λ and v are respectively the eigenvalue and eigenvector of a matrix A, then $-\lambda$ and v are the eigenvalue and eigenvector of $-A$. Show that the result holds even if v is a generalized eigenvector.

4.2. Consider an upper triangular block-diagonal constant matrix,

$$A = \begin{bmatrix} A_1 & A_* \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ have no common eigenvalues. Let $T_2 \in \mathbb{R}^{n \times n_2}$ be a matrix whose columns span the eigenspace of A associated with $\lambda(A_2)$, *i.e.*, a vector space spanned by the eigenvectors and generalized eigenvectors, if any, of A associated with $\lambda(A_2)$, and let

$$T = [T_1 \quad T_2], \quad T_1 = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

Show that T is nonsingular, and

$$T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix},$$

where $\lambda(\tilde{A}_2) = \lambda(A_2)$.

4.3. Given a real-valued constant matrix, A, and a complex scalar, $\alpha \notin \lambda(A)$ with $|\alpha| = 1$, show that

$$\bar{A} = \frac{1}{2}\Big[(A + \alpha I)^{-1}(A - \alpha I) + (A + \alpha^* I)^{-1}(A - \alpha^* I)\Big],$$

is a real-valued matrix. Note that α^* is the complex conjugate of α. Show that if a nonsingular transformation, T, is such that

$$T^{-1}\bar{A}T = \begin{bmatrix} \bar{A}_- & 0 & 0 \\ 0 & \bar{A}_0 & 0 \\ 0 & 0 & \bar{A}_+ \end{bmatrix},$$

where $\lambda(\bar{A}_-) \subset \mathbb{C}^-$, $\lambda(\bar{A}_0) \subset \mathbb{C}^0$, and $\lambda(\bar{A}_+) \subset \mathbb{C}^+$, then

$$T^{-1}AT = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix},$$

where $\lambda(A_-) \subset \mathbb{C}^{\odot}$, $\lambda(A_0) \subset \mathbb{C}^{\circ}$, and $\lambda(A_+) \subset \mathbb{C}^{\otimes}$.

4.4. Verify the results of Exercise 4.1 to Exercise 4.3 with the matrix

$$A = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 2 \\ 1 & 2 & 1 & -1 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right].$$

4.5. Given a matrix $A \in \mathbb{R}^{n \times n}$ whose eigenvalues are given by $\lambda = \mu + j\omega$ and its complex conjugate $\lambda^* = \mu - j\omega$ with $\omega \neq 0$, define a new $(2n) \times (2n)$ matrix,

$$Z = \begin{bmatrix} A - \mu I & \omega I \\ -\omega I & A - \mu I \end{bmatrix}.$$

Show that matrix Z has n eigenvalues at 0, $n/2$ eigenvalues at $j2\omega$, and $n/2$ eigenvalues at $-j2\omega$. Verify the above result with the matrix A given in Example 4.2.2.

4.6. Compute an observability structural decomposition (OSD) for the unforced system

$$\dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x, \quad y = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} x.$$

4.7. Construct an unforced system, (A, C), which has the properties: (i) (A, C) is observable; (ii) the observability index of (A, C) is given by $\mathcal{O} = \{2, 2\}$; and (iii) A has eigenvalues at 0, 1, 2, and 3.

4.8. Construct an unsensed system, (A, B), which has the properties: (i) (A, B) is controllable; (ii) it has a controllability index of $\mathcal{C} = \{1, 1, 2\}$; and (iii) A has eigenvalues at 1, 2, 3, and 4.

4.9. Compute a block diagonal controllable structural decomposition (BDCSD) for the unsensed system obtained in Exercise 4.8.

4.10. Given an unforced system

$$\dot{x} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} x, \quad y = [\alpha \quad \star \quad \cdots \quad \star] \, x,$$

where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, show that the system is observable if and only if $\alpha \neq 0$.

4.11. Given an unsensed system

$$\dot{x} = \begin{bmatrix} \Lambda & I & & \\ & \ddots & \ddots & \\ & & \Lambda & I \\ & & & \Lambda \end{bmatrix} x + \begin{bmatrix} \star \\ \vdots \\ \star \\ \beta \end{bmatrix} u,$$

where

$$\Lambda = \begin{bmatrix} \mu & \omega \\ -\omega & \mu \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \omega \neq 0, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathbb{R}^2,$$

show that the system is controllable if and only if $\beta \neq 0$.

4.12. Given a controllable pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, show that if A has an eigenvalue with a geometric multiplicity of τ, *i.e.*, it has a total number of τ Jordan blocks associated with it, then $m \geq \tau$.

Chapter 5

Decompositions of Proper Systems

5.1 Introduction

We present in this chapter the structural decomposition of linear time-invariant systems as represented by a matrix triple or a matrix quadruple. In order to make the presentation easier to follow, we will start with the structural decomposition of single-input and single-output (SISO) systems. It will be followed by a detailed construction of the structural decomposition for a general strictly proper multi-input and multi-output (MIMO) system. The decomposition of a general nonstrictly proper MIMO system will then be given together with detailed proofs of system properties revealed under the decomposition. We will conclude the chapter with the structural decomposition of general discrete-time systems.

The development of the structural decomposition, or the special coordinate basis (SCB), for strictly proper systems given later in Section 5.3, follows from its development in Sannuti and Saberi [122]. However, the presentation of the proof and construction algorithm is very different, and is enhanced with many innovative results. In particular, we will replace quite a number of iterative steps in [122] with some single-step transformations. The algorithm is presented in a way that can be easily followed and implemented using software packages such as MATLAB. We will also completely resolve some issues left open in [122] by separating all transformed subsystems with proper structures, and by following the results of Chen [21] to give rigorous proofs to all the structural properties of general systems. The results of Chen *et al.* [24] on the interconnection of the

Kronecker canonical and Smith forms with the special coordinate basis of general multivariable systems are also presented.

5.2 SISO Systems

We consider a SISO system characterized by

$$\Sigma \,:\, \dot{x} = A\,x + B\,u, \quad y = C\,x, \tag{5.2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state, the input and the output. We assume that the transfer function of Σ is not identically zero. We have the following special coordinate basis (SCB) decomposition for Σ.

Theorem 5.2.1. *Consider the SISO system of (5.2.1). There exist nonsingular state, input and output transformations $\Gamma_\mathrm{s} \in \mathbb{R}^{n\times n}$, $\Gamma_\mathrm{i} \in \mathbb{R}$ and $\Gamma_\mathrm{o} \in \mathbb{R}$, which decompose the state space of Σ into two subspaces, x_a and x_d. These two subspaces correspond to the finite zero and infinite zero structures of Σ, respectively. The new state space, input and output space of the decomposed system are described by the following set of equations:*

$$x = \Gamma_\mathrm{s}\tilde{x}, \quad y = \Gamma_\mathrm{o}\tilde{y}, \quad u = \Gamma_\mathrm{i}\tilde{u}, \tag{5.2.2}$$

$$\tilde{x} = \begin{pmatrix} x_\mathrm{a} \\ x_\mathrm{d} \end{pmatrix}, \quad x_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a}}, \quad x_\mathrm{d} \in \mathbb{R}^{n_\mathrm{d}}, \quad x_\mathrm{d} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_\mathrm{d}} \end{pmatrix}, \tag{5.2.3}$$

and

$$\dot{x}_\mathrm{a} = A_\mathrm{aa}x_\mathrm{a} + L_\mathrm{ad}\tilde{y}, \tag{5.2.4}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.5}$$

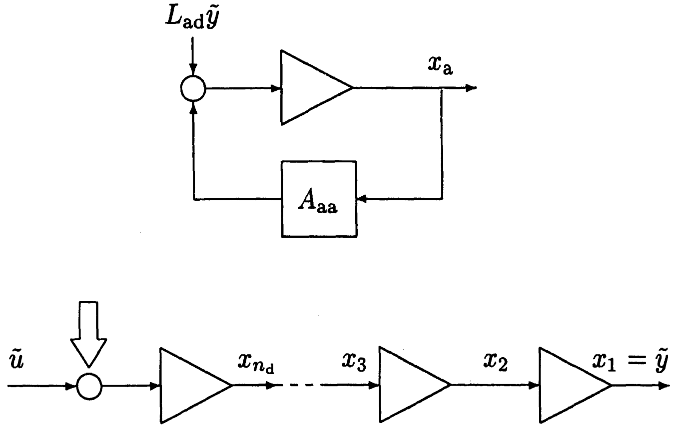
$$\dot{x}_2 = x_3, \tag{5.2.6}$$

$$\vdots$$

$$\dot{x}_{n_\mathrm{d}-1} = x_{n_\mathrm{d}}, \tag{5.2.7}$$

$$\dot{x}_{n_\mathrm{d}} = E_\mathrm{da}x_\mathrm{a} + E_1x_1 + E_2x_2 + \cdots + E_{n_\mathrm{d}}x_{n_\mathrm{d}} + \tilde{u}. \tag{5.2.8}$$

Furthermore, $\lambda(A_\mathrm{aa})$ contains all the system invariant zeros and n_d is the relative degree of Σ.



Note: the signal given by the double-edged arrow is a linear combination of the states.

Figure 5.2.1: Interpretation of structural decomposition of a SISO system.

Proof. The interpretation of Theorem 5.2.1 is given in Figure 5.2.1. In what follows, we present a step-by-step algorithm to construct the required Γ_s, Γ_i and Γ_o that realize the structural decomposition or the special coordinate basis of Σ. The proof of this theorem is considerably simpler than that for the general system given in Theorem 5.3.1. The idea behind is, however, quite similar.

STEP SISO-SCB.1. *Determination of the relative degree.*

The relative degree of Σ can be obtained by differentiating the output y. We let n_d be such that

$$CB = CAB = \cdots = CA^{n_d-2}B = 0, \tag{5.2.9}$$

and

$$\beta := CA^{n_d-1}B \neq 0. \tag{5.2.10}$$

Note that (5.2.9) and (5.2.10) imply that there are n_d inherent integrators between the input and the output of Σ. This can clearly be seen later in (5.2.13) to (5.2.16). Next, we let $n_a := n - n_d$.

STEP SISO-SCB.2. *Construction of a preliminary state transformation.*

Let Z_0 be an $n_\mathrm{a} \times n$ constant matrix such that

$$Z := \left[\frac{Z_0}{Z_\mathrm{d}}\right] := \begin{bmatrix} Z_0 \\ \hline C \\ CA \\ \vdots \\ CA^{n_\mathrm{d}-1} \end{bmatrix}, \tag{5.2.11}$$

is nonsingular. Note that we can choose a Z_0 whose rows form a basis of the null space of Z_d. Next, let

$$\bar{x} := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n_\mathrm{d}} \end{pmatrix} := Zx = \begin{bmatrix} Z_0 \\ C \\ CA \\ \vdots \\ CA^{n_\mathrm{d}-1} \end{bmatrix} x. \tag{5.2.12}$$

We have

$$\dot{x}_1 = C\dot{x} = CAx + CBu = CAx = x_2, \quad y = x_1, \tag{5.2.13}$$

$$\dot{x}_2 = CA\dot{x} = CA^2x + CABu = CA^2x = x_3, \tag{5.2.14}$$

$$\vdots$$

$$\dot{x}_{n_\mathrm{d}-1} = x_{n_\mathrm{d}}, \tag{5.2.15}$$

$$\dot{x}_{n_\mathrm{d}} = E_\mathrm{da}x_0 + \sum_{i=1}^{n_\mathrm{d}} \gamma_i x_i + \beta u, \tag{5.2.16}$$

for some appropriate E_da, and γ_i, $i = 1, 2, \ldots, n_\mathrm{d}$, and

$$\dot{x}_0 = A_{00}x_0 + \sum_{i=1}^{n_\mathrm{d}} \alpha_{0,i} x_i + \beta_0 u, \tag{5.2.17}$$

for some appropriate vectors A_{00}, $\alpha_{0,i}$, $i = 1, 2, \ldots, n_\mathrm{d}$, and β_0.

STEP SISO-SCB.3. *The elimination of u in the state equation of x_0.*

It follows from (5.2.16) that

$$u = \frac{1}{\beta}\left[\dot{x}_{n_\mathrm{d}} - E_\mathrm{da}x_0 - \sum_{i=1}^{n_\mathrm{d}} \gamma_i x_i\right], \tag{5.2.18}$$

which together with (5.2.17) imply that

$$\dot{x}_0 = A_{00}x_0 + \sum_{i=1}^{n_\mathrm{d}} \alpha_{0,i} x_i + \frac{\beta_0}{\beta}\left[\dot{x}_{n_\mathrm{d}} - E_\mathrm{da}x_0 - \sum_{i=1}^{n_\mathrm{d}} \gamma_i x_i\right]$$

$$= A_{\rm aa}x_0 + \sum_{i=1}^{n_{\rm d}} \bar{\alpha}_{0,i}x_i + \bar{\beta}\dot{x}_{n_{\rm d}}, \tag{5.2.19}$$

for some appropriate $A_{\rm aa}$, $\bar{\alpha}_{0,i}$, $i = 1, 2, \dots, n_{\rm d}$, and $\bar{\beta}$.

STEP SISO-SCB.4. *The elimination of $\dot{x}_{n_{\rm d}}$ in the state equation of x_0.*

We define a new state variable,

$$\tilde{x}_0 := x_0 - \bar{\beta}x_{n_{\rm d}}. \tag{5.2.20}$$

Then, we have

$$\begin{aligned}\dot{\tilde{x}}_0 = \dot{x}_0 - \bar{\beta}\dot{x}_{n_{\rm d}} &= A_{\rm aa}x_0 + \sum_{i=1}^{n_{\rm d}} \bar{\alpha}_{0,i}x_i + \bar{\beta}\dot{x}_{n_{\rm d}} - \bar{\beta}\dot{x}_{n_{\rm d}} \\ &= A_{\rm aa}(\tilde{x}_0 + \bar{\beta}x_{n_{\rm d}}) + \sum_{i=1}^{n_{\rm d}} \bar{\alpha}_{0,i}x_i \\ &= A_{\rm aa}\tilde{x}_0 + \sum_{i=1}^{n_{\rm d}} \tilde{\alpha}_{0,i}x_i, \end{aligned} \tag{5.2.21}$$

for some appropriate constant vectors $\tilde{\alpha}_{0,i}$, $i = 1, 2, \dots, n_{\rm d}$. Also, (5.2.16) can be re-written as

$$\begin{aligned}\dot{x}_{n_{\rm d}} &= E_{\rm da}x_0 + \sum_{i=1}^{n_{\rm d}} \gamma_i x_i + \beta u \\ &= E_{\rm da}\tilde{x}_0 + E_{\rm da}\bar{\beta}x_{n_{\rm d}} + \sum_{i=1}^{n_{\rm d}} \gamma_i x_i + \beta u \\ &= E_{\rm da}\tilde{x}_0 + \sum_{i=1}^{n_{\rm d}} \tilde{\gamma}_i x_i + \beta u, \end{aligned} \tag{5.2.22}$$

for some appropriate $\tilde{\gamma}_i$, $i = 1, 2, \dots, n_{\rm d}$.

STEP SISO-SCB.5. *The elimination of $x_2, \dots, x_{n_d}$ from the state equation of $\tilde{x}_0$.*

If $n_{\rm d} = 1$, no further transformation is required and we go to STEP SISO-SCB.6. Otherwise, we let $s := 0$, $\tilde{x}_{0,0} := \tilde{x}_0$, $\tilde{\alpha}_{0,0,i} := \tilde{\alpha}_{0,i}$ and $\tilde{\gamma}_{0,i} := \tilde{\gamma}_i$, $i = 1, 2, \dots, n_{\rm d}$. Then, we carry on the following iterative sub-steps.

Sub-step 5.1. First, note that we have

$$\dot{\tilde{x}}_{0,s} = A_{\rm aa}\tilde{x}_{0,s} + \sum_{i=1}^{n_{\rm d}-s} \tilde{\alpha}_{s,0,i}x_i, \tag{5.2.23}$$

and

$$\dot{x}_{n_\mathrm{d}} = E_\mathrm{da}\tilde{x}_{0,s} + \sum_{i=1}^{n_\mathrm{d}} \tilde{\gamma}_{s,i}x_i + \beta u. \tag{5.2.24}$$

We are now eliminating $x_{n_\mathrm{d}-s}$ from the above expression by defining

$$\tilde{x}_{0,s+1} := \tilde{x}_{0,s} - (\tilde{\alpha}_{s,0,n_\mathrm{d}-s})x_{n_\mathrm{d}-s-1}, \tag{5.2.25}$$

which together with $\dot{x}_{n_\mathrm{d}-s-1} = x_{n_\mathrm{d}-s}$ imply that

$$\begin{aligned}
\dot{\tilde{x}}_{0,s+1} &= \dot{\tilde{x}}_{0,s} - \tilde{\alpha}_{s,0,n_\mathrm{d}-s}\dot{x}_{n_\mathrm{d}-s-1} \\
&= A_\mathrm{aa}\left(\tilde{x}_{0,s+1} + \tilde{\alpha}_{s,0,n_\mathrm{d}-s}x_{n_\mathrm{d}-s-1}\right) \\
&\quad + \sum_{i=1}^{n_\mathrm{d}-s-1} \tilde{\alpha}_{s,0,i}x_i + \tilde{\alpha}_{s,0,n_\mathrm{d}-s}x_{n_\mathrm{d}-s} - \tilde{\alpha}_{s,0,n_\mathrm{d}-s}x_{n_\mathrm{d}-s} \\
&= A_\mathrm{aa}\tilde{x}_{0,s+1} + (A_\mathrm{aa}\tilde{\alpha}_{s,0,n_\mathrm{d}-s})x_{n_\mathrm{d}-s-1} + \sum_{i=1}^{n_\mathrm{d}-s-1} \tilde{\alpha}_{s,0,i}x_i. \quad (5.2.26)
\end{aligned}$$

Clearly, we have eliminated $x_{n_\mathrm{d}-s}$ in the above expression. Also, we have

$$\begin{aligned}
\dot{x}_{n_\mathrm{d}} &= E_\mathrm{da}\tilde{x}_{0,s} + \sum_{i=1}^{n_\mathrm{d}} \tilde{\gamma}_{s,i}x_i + \beta u \\
&= E_\mathrm{da}\left(\tilde{x}_{0,s+1} + \tilde{\alpha}_{s,0,n_\mathrm{d}-s}x_{n_\mathrm{d}-s-1}\right) + \sum_{i=1}^{n_\mathrm{d}} \tilde{\gamma}_{s,i}x_i + \beta u \\
&= E_\mathrm{da}\tilde{x}_{0,s+1} + E_\mathrm{da}\tilde{\alpha}_{s,0,n_\mathrm{d}-s}x_{n_\mathrm{d}-s-1} + \sum_{i=1}^{n_\mathrm{d}} \tilde{\gamma}_{s,i}x_i + \beta u. \quad (5.2.27)
\end{aligned}$$

For the next iteration, we re-write (5.2.26) as

$$\dot{\tilde{x}}_{0,s+1} = A_\mathrm{aa}\tilde{x}_{0,s+1} + \sum_{i=1}^{n_\mathrm{d}-s-1} \tilde{\alpha}_{s+1,0,i}x_i, \tag{5.2.28}$$

for some appropriate constant vectors $\tilde{\alpha}_{s+1,0,i}$, $i = 1, 2, \ldots, n_\mathrm{d} - s - 1$, and re-write (5.2.27) as

$$\dot{x}_{n_\mathrm{d}} = E_\mathrm{da}\tilde{x}_{0,s+1} + \sum_{i=1}^{n_\mathrm{d}} \tilde{\gamma}_{s+1,i}x_i + \beta u, \tag{5.2.29}$$

for some appropriate constant vectors $\tilde{\gamma}_{s+1,i}$, $i = 1, 2, \ldots, n_\mathrm{d} - s - 1$.

Sub-step 5.2. If $s = n_\mathrm{d} - 2$, we have obtained what we need and we go to STEP SISO-SCB.6. Otherwise, we let $s := s + 1$ and go back to Sub-step 5.1.

STEP SISO-SCB.6. *Finishing touch.*

Finally, we let

$$x_a := \tilde{x}_{0,s+1}, \quad y = \Gamma_o \tilde{y} = \tilde{y}, \quad u = \Gamma_i \tilde{u} = \frac{1}{\beta}\tilde{u} \tag{5.2.30}$$

and

$$L_{ad} := \tilde{\alpha}_{s+1,0,1}, \quad E_i := \tilde{\gamma}_{s+1,i}, \quad i = 1, 2, \ldots, n_d. \tag{5.2.31}$$

Then, (5.2.13) to (5.2.15) and (5.2.28) to (5.2.31) imply that

$$\dot{x}_a = A_{aa} x_a + L_{ad} \tilde{y}, \tag{5.2.32}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.33}$$

$$\dot{x}_2 = x_3, \tag{5.2.34}$$

$$\vdots$$

$$\dot{x}_{n_d} = E_{da} x_a + E_1 x_1 + \cdots + E_{n_d} x_{n_d} + \tilde{u}. \tag{5.2.35}$$

It is trivial to see that n_d is the relative degree of Σ. In Section 5.4, we will show that $\lambda(A_{aa})$ are the invariant zeros of Σ for general systems. ■

We note that the output transformation in Theorem 5.2.1 can be chosen to be equal to 1, i.e., $\Gamma_o = 1$. We illustrate the above procedure in the following example, which was given in Chen [22].

Example 5.2.1. Consider a SISO system Σ characterized by (5.2.1) with

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \tag{5.2.36}$$

and

$$C = [\,0 \quad 3 \quad -2 \quad 0\,]. \tag{5.2.37}$$

The structural decomposition of Σ proceeds as follows:

1. *Differentiating the system output.*

   It involves the following sub-steps.

   (a) First, we have

$$\dot{y} = C\dot{x} = CAx + CBu = [\,-2 \quad -1 \quad 0 \quad 1\,]\,x + 0 \cdot u.$$

(b) Since $CB = 0$, we compute

$$\ddot{y} = CA^2x + CABu = [1 \quad -1 \quad -3 \quad 1]\,x + 0 \cdot u.$$

(c) Since $CAB = 0$, we continue on computing

$$y^{(3)} = CA^3x + CA^2Bu = -[8 \quad 10 \quad 12 \quad 17]\,x - 6 \cdot u.$$

We move to the next step as $CA^2B \neq 0$.

2. *Constructing a preliminary state transformation.*

Let Z_0 be a vector such that

$$Z = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix}, \tag{5.2.38}$$

is nonsingular. Then, define a new set of state variables $\tilde{x}$,

$$\tilde{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} := Zx = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix} x = \begin{pmatrix} Z_0x \\ y \\ \dot{y} \\ \ddot{y} \end{pmatrix}. \tag{5.2.39}$$

It is simple to verify that Z with $Z_0 = [1 \quad 0 \quad 0 \quad 0]$ is a nonsingular matrix. Furthermore,

$$\dot{x}_0 = 8x_0 + x_1 + \frac{8}{3}x_2 - \frac{5}{3}x_3 + u, \tag{5.2.40}$$

$$\dot{x}_1 = x_2, \tag{5.2.41}$$

$$\dot{x}_2 = x_3, \tag{5.2.42}$$

$$\dot{x}_3 = -72x_0 - 9x_1 - 27x_2 + 10x_3 - 6u. \tag{5.2.43}$$

3. *Eliminating u in (5.2.40).*

Equation (5.2.43) implies that

$$u = -12x_0 - \frac{3}{2}x_1 - \frac{9}{2}x_2 + \frac{5}{3}x_3 - \frac{1}{6}\dot{x}_3. \tag{5.2.44}$$

Substituting this into (5.2.40), we obtain

$$\dot{x}_0 = -4x_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 - \frac{1}{6}\dot{x}_3. \tag{5.2.45}$$

We have eliminated u in $\dot{x}_0$. Unfortunately, we have also introduced an additional $\dot{x}_3$ in (5.2.45).

4. *Eliminating $\dot{x}_3$ in (5.2.45).*

   Define a new variable $\tilde{x}_0$ as

$$\tilde{x}_0 := x_0 + \frac{1}{6}x_3. \tag{5.2.46}$$

   We have

$$\dot{\tilde{x}}_0 = -4\tilde{x}_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 + \frac{2}{3}x_3, \tag{5.2.47}$$

   and

$$\dot{x}_3 = -72\tilde{x}_0 - 9x_1 - 27x_2 + 22x_3 - 6u. \tag{5.2.48}$$

5. *Eliminating x_2 and x_3 in (5.2.47).*

   This step involves two sub-steps.

   (a) Letting

$$\tilde{x}_{0,1} := \tilde{x}_0 - \frac{2}{3}x_2, \tag{5.2.49}$$

   we have

$$\dot{\tilde{x}}_{0,1} = -4\tilde{x}_{0,1} - \frac{1}{2}x_1 - \frac{9}{2}x_2, \tag{5.2.50}$$

   and

$$\dot{x}_3 = -72\tilde{x}_{0,1} - 9x_1 - 75x_2 + 22x_3 - 6u. \tag{5.2.51}$$

   (b) Letting

$$\tilde{x}_{0,2} := \tilde{x}_{0,1} + \frac{9}{2}x_1, \tag{5.2.52}$$

   we have

$$\dot{\tilde{x}}_{0,2} = -4\tilde{x}_{0,2} + \frac{35}{2}x_1, \tag{5.2.53}$$

   and

$$\dot{x}_3 = -72\tilde{x}_{0,2} + 315x_1 - 75x_2 + 22x_3 - 6u. \tag{5.2.54}$$

6. *Composing the nonsingular state, output and input transformations.*

   Let

$$x_{\mathrm{a}} := \tilde{x}_{0,2} \tag{5.2.55}$$

   or equivalently let

$$x = \Gamma_{\mathrm{s}}\tilde{x} = \Gamma_{\mathrm{s}} \begin{pmatrix} x_{\mathrm{a}} \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{5.2.56}$$

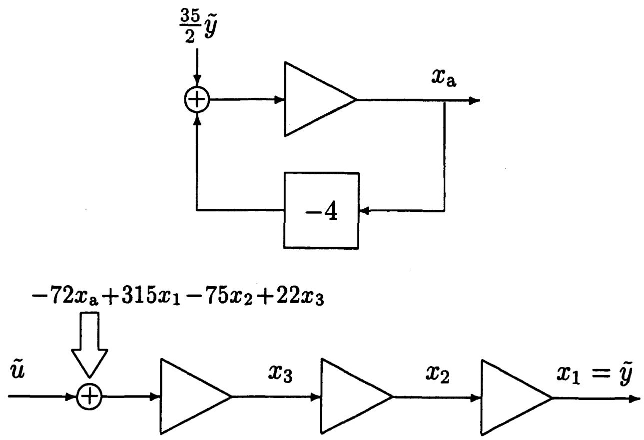


Figure 5.2.2: Interpretation of structural decomposition of Example 5.2.1.

with

$$\Gamma_s = \left\{ \begin{bmatrix} 1 & 9/2 & -2/3 & 1/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ -2 & -1 & 0 & 1 \\ 1 & -1 & -3 & 1 \end{bmatrix} \right\}^{-1}. \tag{5.2.57}$$

Also, let

$$u = \Gamma_i \tilde{u} = -\frac{1}{6}\tilde{u}, \quad y = \Gamma_o \tilde{y} = 1 \cdot \tilde{y}. \tag{5.2.58}$$

Finally, we obtain the dynamic equations of the transformed system,

$$\dot{x}_a = -4x_a + \frac{35}{2}x_1, \tag{5.2.59}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.60}$$

$$\dot{x}_2 = x_3, \tag{5.2.61}$$

$$\dot{x}_3 = -72x_a + 315x_1 - 75x_2 + 22x_3 + \tilde{u}, \tag{5.2.62}$$

which is now in the form given in Theorem 5.2.1. The graphical interpretation of the above decomposition is given in Figure 5.2.2. The given system has an invariant zero at -4 and a relative degree of 3 (equivalently an infinite zero of order 3).

5.3 Strictly Proper Systems

Next, we consider a general strictly proper linear system Σ characterized by

$$\begin{cases} \dot{x} = A\ x + B\ u, \\ y = C\ x, \end{cases} \tag{5.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output. Without loss of generality, we assume that both B and C are of full rank. We have the following structural or special coordinate basis decomposition of Σ.

Theorem 5.3.1. *Consider the strictly proper system Σ characterized by (5.3.1). There exist a nonsingular state transformation, $\Gamma_{\rm s} \in \mathbb{R}^{n\times n}$, a nonsingular output transformation, $\Gamma_{\rm o} \in \mathbb{R}^{p\times p}$, and a nonsingular input transformation, $\Gamma_{\rm i} \in \mathbb{R}^{m\times m}$, that will reveal all the structural properties of Σ. More specifically, we have*

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{5.3.2}$$

with the new state variables

$$\tilde{x} = \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm a} \in \mathbb{R}^{n_{\rm a}}, \quad x_{\rm b} \in \mathbb{R}^{n_{\rm b}}, \quad x_{\rm c} \in \mathbb{R}^{n_{\rm c}}, \quad x_{\rm d} \in \mathbb{R}^{n_{\rm d}}, \tag{5.3.3}$$

the new output variables

$$\tilde{y} = \begin{pmatrix} y_{\rm d} \\ y_{\rm b} \end{pmatrix}, \quad y_{\rm d} \in \mathbb{R}^{m_{\rm d}}, \quad y_{\rm b} \in \mathbb{R}^{p_{\rm b}}, \tag{5.3.4}$$

and the new input variables

$$\tilde{u} = \begin{pmatrix} u_{\rm d} \\ u_{\rm c} \end{pmatrix}, \quad u_{\rm d} \in \mathbb{R}^{m_{\rm d}}, \quad u_{\rm c} \in \mathbb{R}^{m_{\rm c}}. \tag{5.3.5}$$

Further, the state variable $x_{\rm d}$ can be decomposed as:

$$x_{\rm d} = \begin{pmatrix} x_{{\rm d},1} \\ x_{{\rm d},2} \\ \vdots \\ x_{{\rm d},m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{{\rm d},1} \\ y_{{\rm d},2} \\ \vdots \\ y_{{\rm d},m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{{\rm d},1} \\ u_{{\rm d},2} \\ \vdots \\ u_{{\rm d},m_{\rm d}} \end{pmatrix}, \tag{5.3.6}$$

$$x_{{\rm d},i} \in \mathbb{R}^{q_i}, \quad x_{{\rm d},i} = \begin{pmatrix} x_{{\rm d},i,1} \\ x_{{\rm d},i,1} \\ \vdots \\ x_{{\rm d},i,q_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_{\rm d}, \tag{5.3.7}$$

with $q_1 \le q_2 \le \cdots \le q_{m_\mathrm{d}}$. The state variable x_b can be decomposed as

$$x_\mathrm{b} = \begin{pmatrix} x_{\mathrm{b},1} \\ x_{\mathrm{b},2} \\ \vdots \\ x_{\mathrm{b},p_\mathrm{b}} \end{pmatrix}, \quad y_\mathrm{b} = \begin{pmatrix} y_{\mathrm{b},1} \\ y_{\mathrm{b},2} \\ \vdots \\ y_{\mathrm{b},p_\mathrm{b}} \end{pmatrix}, \tag{5.3.8}$$

$$x_{\mathrm{b},i} \in \mathbb{R}^{l_i}, \quad x_{\mathrm{b},i} = \begin{pmatrix} x_{\mathrm{b},i,1} \\ x_{\mathrm{b},i,2} \\ \vdots \\ x_{\mathrm{b},i,l_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p_\mathrm{b}, \tag{5.3.9}$$

with $l_1 \le l_2 \le \cdots \le l_{p_\mathrm{b}}$. And finally, the state variable x_c can be decomposed as

$$x_\mathrm{c} = \begin{pmatrix} x_{\mathrm{c},1} \\ x_{\mathrm{c},2} \\ \vdots \\ x_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \quad u_\mathrm{c} = \begin{pmatrix} u_{\mathrm{c},1} \\ u_{\mathrm{c},2} \\ \vdots \\ u_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \tag{5.3.10}$$

$$x_{\mathrm{c},i} \in \mathbb{R}^{r_i}, \quad x_{\mathrm{c},i} = \begin{pmatrix} x_{\mathrm{c},i,1} \\ x_{\mathrm{c},i,2} \\ \vdots \\ x_{\mathrm{c},i,r_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_\mathrm{c}, \tag{5.3.11}$$

with $r_1 \le r_2 \le \cdots \le r_{m_\mathrm{c}}$. The decomposed system can be expressed in the following dynamical equations:

$$\dot{x}_\mathrm{a} = A_\mathrm{aa} x_\mathrm{a} + L_\mathrm{ab} y_\mathrm{b} + L_\mathrm{ad} y_\mathrm{d}, \tag{5.3.12}$$

for each subsystem $x_{\mathrm{b},i}$, $i = 1, 2, \ldots, p_\mathrm{b}$,

$$\dot{x}_{\mathrm{b},i,1} = x_{\mathrm{b},i,2} + L_{\mathrm{bd},i,1} y_\mathrm{b} + L_{\mathrm{b},i,1} y_\mathrm{d}, \quad y_{\mathrm{b},i} = x_{\mathrm{b},i,1}, \tag{5.3.13}$$

$$\dot{x}_{\mathrm{b},i,2} = x_{\mathrm{b},i,3} + L_{\mathrm{bd},i,2} y_\mathrm{b} + L_{\mathrm{b},i,2} y_\mathrm{d}, \tag{5.3.14}$$

$$\vdots$$

$$\dot{x}_{\mathrm{b},i,l_i} = L_{\mathrm{bd},i,l_i} y_\mathrm{b} + L_{\mathrm{bd},i,l_i} y_\mathrm{d}, \tag{5.3.15}$$

for each subsystem $x_{\mathrm{c},i}$, $i = 1, 2, \ldots, m_\mathrm{c}$,

$$\dot{x}_{\mathrm{c},i,1} = x_{\mathrm{c},i,2} + L_{\mathrm{cb},i,1} y_\mathrm{b} + L_{\mathrm{cd},i,1} y_\mathrm{d}, \tag{5.3.16}$$

$$\vdots$$

$$\dot{x}_{\mathrm{c},i,r_i-1} = x_{\mathrm{c},i,r_i} + L_{\mathrm{cb},i,r_i-1} y_\mathrm{b} + L_{\mathrm{cd},i,r_i-1} y_\mathrm{d}, \tag{5.3.17}$$

$$\dot{x}_{\mathrm{c},i,r_i} = A_{\mathrm{c},i,\mathrm{a}} x_\mathrm{a} + A_{\mathrm{c},i,\mathrm{c}} x_\mathrm{c} + L_{\mathrm{cb},i,r_i} y_\mathrm{b} + L_{\mathrm{cd},i,r_i} y_\mathrm{d} + u_{\mathrm{c},i}, \tag{5.3.18}$$

and finally, for each subsystem $x_{\mathrm{d},i}$, $i = 1, 2, \ldots, m_{\mathrm{d}}$,

$$\dot{x}_{\mathrm{d},i,1} = x_{\mathrm{d},i,2} + L_{\mathrm{d},i,1} y_{\mathrm{d}}, \quad y_{\mathrm{d},i} = x_{\mathrm{d},i,1}, \tag{5.3.19}$$

$$\dot{x}_{\mathrm{d},i,2} = x_{\mathrm{d},i,3} + L_{\mathrm{d},i,2} y_{\mathrm{d}}, \tag{5.3.20}$$

$$\vdots$$

$$\dot{x}_{\mathrm{d},i,q_i} = A_{\mathrm{d},i,\mathrm{a}} x_{\mathrm{a}} + A_{\mathrm{d},i,\mathrm{c}} x_{\mathrm{c}} + A_{\mathrm{d},i,\mathrm{b}} x_{\mathrm{b}} + A_{\mathrm{d},i,\mathrm{d}} x_{\mathrm{d}} + u_{\mathrm{d},i}, \tag{5.3.21}$$

where $A_{\mathrm{aa}}, L_{\mathrm{ab}}, \ldots, A_{\mathrm{d},i,\mathrm{d}}$ *are constant matrices of appropriate dimensions.*

Proof. The basic idea in constructing the special coordinate basis decomposition of general strictly proper multivariable systems is pretty much the same as that in the proof of Theorem 5.2.1. Although the procedure for the decomposition of MIMO systems is more complicated, it still revolves around the identification of chains of integrators between the system input and output variables, which again will be done by repeatedly differentiating the system output variables. Nonetheless, we would like to note that for general MIMO systems, there might exist three different types of chains of integrators:

1. Chains that start from an input channel and end with an output. This type of chain gives the infinite zero structures of the given system and covers the subspace corresponding to x_{d};
2. Chains that start from an input channel but do not end with an output. This type of chain covers the subspace corresponding to x_{c}; and
3. Chains that do not start from an input but end with an output variable. This type of chain covers the subspace corresponding to x_{b}.

In general, these subspaces do not cover the whole state space of the given system. The remaining part forms a subspace corresponding to x_{a}, which is related to the invariant zeros of the system. These subsystems of x_{a}, x_{b}, x_{c} and x_{d} are illustrated in graphical form as given in Figure 5.3.1.

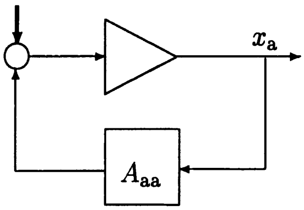
We proceed in the following with a step-by-step algorithm that decomposes the given system into the various subsystems.

STEP SCB.1. *Initialization.*

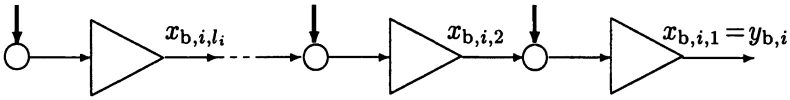
Noting that matrix C is of full rank, we partition it as

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}, \tag{5.3.22}$$

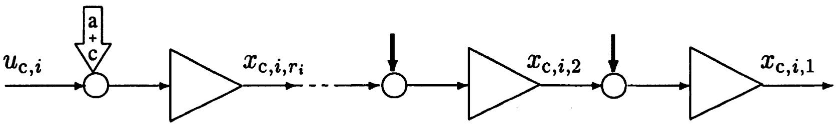
x_a – *the subsystem without direct input and output:*



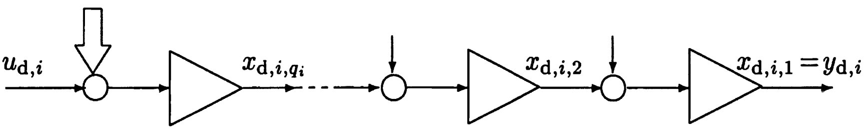
$x_{b,i}$ – *the chain of integrators without a direct input:*



$x_{c,i}$ – *the chain of integrators without a direct output:*



$x_{d,i}$ – *the chain of integrators with direct input and output:*



Note: the signal indicated by the double-edged arrow in $x_{d,i}$ is a linear combination of all the state variables; the signal indicated by the double-edged arrow marked with a + c *in $x_{c,i}$ is a linear combination of the state variables x_a and x_c; the signals indicated by the thick vertical arrows are some linear combinations of the output variables y_d and y_b; and the signals indicated by the thin vertical arrows are some linear combinations of the output variable y_d.*

Figure 5.3.1: Interpretation of structural decomposition of a MIMO system.

where C_i, $i = 1, 2, \ldots, p$, are independent row vectors. For each C_i, $i = 1, 2, \ldots, p$, we assign a corresponding transformation matrix Z_i to it, which is initially set as

$$Z_i := C_i. \tag{5.3.23}$$

We also define a flag vector f as

$$f := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.3.24}$$

which will be used as a flag in the iterative procedure in STEP SCB.2. Note that, as the algorithm is implemented, the elements of f will be replaced by zeros and it will eventually become a zero vector. On the other hand, Z_i will be amended with additional rows and form parts of the required state transformation. We also initialize

$$Z := C, \tag{5.3.25}$$

and three empty matrices $Z_{\rm d}$, $W_{\rm d}$ and $Z_{\rm b}$. Note that $Z_{\rm d}$, $W_{\rm d}$ and $Z_{\rm b}$ will be respectively used to form transformations associated with $x_{\rm d}$, $u_{\rm d}$ and $x_{\rm b}$. Again, these matrices are variable, *i.e.*, they might be amended with new components as we progress. Such a style of presentation is much easier to be implemented in software packages such as MATLAB. Finally, we let $v := 0$ and $w := 0$.

STEP SCB.2. *Repetitive differentiation of the system output.*

This step will be repeated until f becomes a zero vector. We let

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix}. \tag{5.3.26}$$

For each nonzero element f_i, $i = 1, 2, \ldots, p$, we rewrite its corresponding transformation matrix,

$$Z_i = \begin{bmatrix} C_{i,1} \\ C_{i,2} \\ \vdots \\ C_{i,\alpha_i} \end{bmatrix}, \tag{5.3.27}$$

where $\alpha_i = \text{rank}\,(Z_i)$. Let $x_{i,\alpha_i} := C_{i,\alpha_i} x$, then we have

$$\dot{x}_{i,\alpha_i} = C_{i,\alpha_i}\dot{x} = C_{i,\alpha_i}Ax + C_{i,\alpha_i}Bu. \tag{5.3.28}$$

The following tests are to be carried out for all Z_i, whose corresponding flag f_i is nonzero:

Case 1. If

$$\text{rank}\left(\begin{bmatrix} W_\text{d} \\ C_{i,\alpha_i}B \end{bmatrix}\right) > \text{rank}\,(W_\text{d}), \tag{5.3.29}$$

the chain of integrators associated with this i-th output reaches a system input. It is the end of this chain of integrators and it belongs to the subspace associated with x_d. This simply means that the given system has an infinite zero of order α_i. For this case, we replace the corresponding flag f_i in the flag vector f with a scalar 0, *i.e.*, to stop this output variable from further differentiation. Furthermore, we amend the matrices Z_d and W_d as follows:

$$Z_\text{d} := \begin{bmatrix} Z_\text{d} \\ Z_i \end{bmatrix}, \quad W_\text{d} := \begin{bmatrix} W_\text{d} \\ C_{i,\alpha_i}B \end{bmatrix}. \tag{5.3.30}$$

These matrices will be used to define new state and input variables related to x_d and u_d respectively. We also let

$$v := v + 1 \quad \text{and} \quad q_v := \alpha_i, \tag{5.3.31}$$

and test the next Z_i with a corresponding flag $f_i \neq 0$.

Case 2. If

$$\text{rank}\left(\begin{bmatrix} W_\text{d} \\ C_{i,\alpha_i}B \end{bmatrix}\right) = \text{rank}\,(W_\text{d}), \tag{5.3.32}$$

which implies that $C_{i,\alpha_i}B$ is either a zero vector or a linear combination of the rows of W_d. Note that we have so far identified a total of v infinite zeros and thus Z_d can be arranged as follows:

$$Z_\text{d} = \begin{bmatrix} Z_{\text{d},1} \\ Z_{\text{d},2} \\ \vdots \\ Z_{\text{d},v} \end{bmatrix}, \tag{5.3.33}$$

with $j = 1, 2, \dots, v$,

$$Z_{\text{d},j} = \begin{bmatrix} C_{\text{d},j,1} \\ C_{\text{d},j,2} \\ \vdots \\ C_{\text{d},j,q_j} \end{bmatrix}. \tag{5.3.34}$$

Similarly, we can rewrite

$$W_{\mathrm{d}} = \begin{bmatrix} C_{\mathrm{d},1,q_1} B \\ C_{\mathrm{d},2,q_2} B \\ \vdots \\ C_{\mathrm{d},v,q_v} B \end{bmatrix}. \tag{5.3.35}$$

Thus, the property of (5.3.32) implies that there exist a set of scalars, say $\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,v}$ such that

$$C_{i,\alpha_i} B = \sum_{j=1}^{v} \beta_{i,j} C_{\mathrm{d},j,q_j} B. \tag{5.3.36}$$

We then define

$$\left.\begin{aligned} \check{C}_{i,1} &= C_{i,1} - \sum_{j=1}^{v} \beta_{i,j} C_{\mathrm{d},j,q_j-\alpha_i+1} \\ \check{C}_{i,2} &= C_{i,2} - \sum_{j=1}^{v} \beta_{i,j} C_{\mathrm{d},j,q_j-\alpha_i+2} \\ &\vdots \\ \check{C}_{i,\alpha_i} &= C_{i,\alpha_i} - \sum_{j=1}^{v} \beta_{i,j} C_{\mathrm{d},j,q_j}. \end{aligned}\right\} \tag{5.3.37}$$

Here, we set $C_{\mathrm{d},j,s}$ to be a zero vector if $s \le 0$. It is simple to verify that

$$q_j - \alpha_i + 1 \le 1, \quad j = 1, 2, \ldots, v, \tag{5.3.38}$$

which implies that $\check{C}_{i,1}$ is a linear combination of the rows of the original output matrix C. Next, in view of (5.3.36) and (5.3.37), it is straightforward to verify that for $s = 1, 2, \ldots, \alpha_i$,

$$\check{C}_{i,s} B = 0, \tag{5.3.39}$$

and for $s = 1, 2, \ldots, \alpha_i - 1$,

$$\check{C}_{i,s+1} = \check{C}_{i,s} A - \sum_{j=1}^{v} \gamma_{i,s,j} C_{\mathrm{d},j,1}, \tag{5.3.40}$$

for some scalars $\gamma_{i,s,j}$. We next let

$$\check{C}_{i,\alpha_i+1} = \check{C}_{i,\alpha_i} A, \tag{5.3.41}$$

and test the following sub-cases:

Sub-case 2.1. If

$$\operatorname{rank}\left(\begin{bmatrix} Z \\ \check{C}_{i,\alpha_i+1} \end{bmatrix}\right) > \operatorname{rank}(Z), \tag{5.3.42}$$

it implies that there are more integrators in the chain associated with the i-th output, which must be further identified. We then update Z and Z_i as follows:

$$Z := \begin{bmatrix} Z \\ \check{C}_{i,\alpha_i+1} \end{bmatrix}, \quad Z_i := \begin{bmatrix} \check{C}_{i,1} \\ \check{C}_{i,2} \\ \vdots \\ \check{C}_{i,\alpha_i} \\ \check{C}_{i,\alpha_i+1} \end{bmatrix}, \tag{5.3.43}$$

and test the next Z_i with a corresponding flag $f_i \neq 0$.

Sub-case 2.2. If

$$\operatorname{rank}\left(\begin{bmatrix} Z \\ \check{C}_{i,\alpha_i+1} \end{bmatrix}\right) = \operatorname{rank}(Z), \tag{5.3.44}$$

there is no more inherent integrator in the chain associated with this i-th output. The chain of integrators ends up without an input and it belongs to the subspace associated with $x_{\rm b}$. For this case, we also replace the corresponding flag f_i in the flag vector f with a scalar 0, *i.e.*, to stop this output variable from further differentiation, and amend Z_i and $Z_{\rm b}$, respectively, as follows:

$$Z_i := \begin{bmatrix} \check{C}_{i,1} \\ \check{C}_{i,2} \\ \vdots \\ \check{C}_{i,\alpha_i} \end{bmatrix}, \quad Z_{\rm b} := \begin{bmatrix} Z_{\rm b} \\ Z_i \end{bmatrix}, \tag{5.3.45}$$

which will be used to define new state variables related to $x_{\rm b}$. We also let

$$w := w + 1 \quad \text{and} \quad l_w := \alpha_i. \tag{5.3.46}$$

For future reference, we rewrite

$$Z_{\rm b} = \begin{bmatrix} Z_{{\rm b},1} \\ Z_{{\rm b},2} \\ \vdots \\ Z_{{\rm b},w} \end{bmatrix}, \tag{5.3.47}$$

where

$$Z_{{\rm b},j} = \begin{bmatrix} C_{{\rm b},j,1} \\ C_{{\rm b},j,2} \\ \vdots \\ C_{{\rm b},j,l_j} \end{bmatrix}, \quad j = 1, 2, \ldots, w. \tag{5.3.48}$$

The above tests have to be carried out for all Z_i with its corresponding flag $f_i \neq 0$. Note that in either Case 1 or Sub-case 2.2, there is an element in the flag vector f being replaced by a scalar 0. As such, f will eventually become a zero vector.

Next, if $f = 0$, we move on to STEP SCB.3. Otherwise, we go back to repeat STEP SCB.2.

STEP SCB.3. *Interim transformations.*

We let $m_\mathrm{d} = v$ and rewrite

$$Z_\mathrm{d} = \begin{bmatrix} Z_{\mathrm{d},1} \\ Z_{\mathrm{d},2} \\ \vdots \\ Z_{\mathrm{d},m_\mathrm{d}} \end{bmatrix}, \tag{5.3.49}$$

where $\operatorname{rank}(Z_{\mathrm{d},i}) = q_i$, $i = 1, 2, \ldots, m_\mathrm{d}$. Obviously, $q_1 \leq q_2 \leq \cdots \leq q_{m_\mathrm{d}}$. Also, rewrite

$$W_\mathrm{d} = \begin{bmatrix} W_{\mathrm{d},1} \\ W_{\mathrm{d},2} \\ \vdots \\ W_{\mathrm{d},m_\mathrm{d}} \end{bmatrix}. \tag{5.3.50}$$

Next, let $p_\mathrm{b} = w$ and rewrite

$$Z_\mathrm{b} = \begin{bmatrix} Z_{\mathrm{b},1} \\ Z_{\mathrm{b},2} \\ \vdots \\ Z_{\mathrm{b},p_\mathrm{b}} \end{bmatrix}, \tag{5.3.51}$$

where $\operatorname{rank}(Z_{\mathrm{b},i}) = l_i$, $i = 1, 2, \ldots, p_\mathrm{b}$. Obviously, $l_1 \leq l_2 \leq \cdots \leq l_{p_\mathrm{b}}$. Also, let

$$n_\mathrm{d} = \sum_{i=1}^{m_\mathrm{d}} q_i, \quad n_\mathrm{b} = \sum_{i=1}^{p_\mathrm{b}} l_i, \quad n_0 = n - n_\mathrm{d} - n_\mathrm{b}, \tag{5.3.52}$$

and let Z_0 be an $n_0 \times n$ constant matrix such that

$$S := \begin{bmatrix} Z_0 \\ Z_\mathrm{b} \\ Z_\mathrm{d} \end{bmatrix} := \begin{bmatrix} Z_0 \\ Z_\mathrm{bd} \end{bmatrix} \tag{5.3.53}$$

is nonsingular. Generally, as in the previous decompositions, we can choose a Z_0 whose rows constitute a basis for the null space of $Z_{\rm bd}$. We then define a new state variable

$$\bar{x} = \begin{pmatrix} x_0 \\ \bar{x}_{\rm b} \\ x_{\rm d} \end{pmatrix} = Sx = \begin{bmatrix} Z_0 \\ Z_{\rm b} \\ Z_{\rm d} \end{bmatrix} x, \tag{5.3.54}$$

where

$$\bar{x}_{\rm b} = \begin{pmatrix} \bar{x}_{{\rm b},1} \\ \bar{x}_{{\rm b},2} \\ \vdots \\ \bar{x}_{{\rm b},p_{\rm b}} \end{pmatrix}, \quad \bar{x}_{{\rm b},i} = \begin{pmatrix} \bar{x}_{{\rm b},i,1} \\ \bar{x}_{{\rm b},i,2} \\ \vdots \\ \bar{x}_{{\rm b},i,l_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p_{\rm b}, \tag{5.3.55}$$

and

$$x_{\rm d} = \begin{pmatrix} x_{{\rm d},1} \\ x_{{\rm d},2} \\ \vdots \\ x_{{\rm d},m_{\rm d}} \end{pmatrix}, \quad x_{{\rm d},i} = \begin{pmatrix} x_{{\rm d},i,1} \\ x_{{\rm d},i,2} \\ \vdots \\ x_{{\rm d},i,q_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_{\rm d}. \tag{5.3.56}$$

Let $m_{\rm c} = m - m_{\rm d}$ and let $W_{\rm c}$ be an $m_{\rm c} \times m$ constant matrix such that

$$W = \begin{bmatrix} W_{\rm d} \\ W_{\rm c} \end{bmatrix} \tag{5.3.57}$$

is nonsingular. Again, $W_{\rm c}$ can be chosen such that its rows form a basis of the null space of $W_{\rm d}$. We then define a new input variable

$$\bar{u} = \begin{pmatrix} u_{\rm d} \\ \bar{u}_{\rm c} \end{pmatrix} = \begin{bmatrix} W_{\rm d} \\ W_{\rm c} \end{bmatrix} u \tag{5.3.58}$$

with

$$u_{\rm d} = \begin{pmatrix} u_{{\rm d},1} \\ u_{{\rm d},2} \\ \vdots \\ u_{{\rm d},m_{\rm d}} \end{pmatrix}, \tag{5.3.59}$$

and define a new output variable

$$\tilde{y} = \begin{pmatrix} y_{\rm d} \\ \bar{y}_{\rm b} \end{pmatrix} = My \tag{5.3.60}$$

with

$$y_{\rm d} = \begin{pmatrix} y_{{\rm d},1} \\ y_{{\rm d},2} \\ \vdots \\ y_{{\rm d},m_{\rm d}} \end{pmatrix} = \begin{pmatrix} x_{{\rm d},1,1} \\ x_{{\rm d},2,1} \\ \vdots \\ x_{{\rm d},m_{\rm d},1} \end{pmatrix} \tag{5.3.61}$$

and

$$\bar{y}_{\rm b} = \begin{pmatrix} \bar{y}_{\rm b,1} \\ \bar{y}_{\rm b,2} \\ \vdots \\ \bar{y}_{{\rm b},p_{\rm b}} \end{pmatrix} = \begin{pmatrix} \bar{x}_{\rm b,1,1} \\ \bar{x}_{\rm b,2,1} \\ \vdots \\ \bar{x}_{{\rm b},p_{\rm b},1} \end{pmatrix}. \tag{5.3.62}$$

STEP SCB.4. *Determination of* $x_{\rm d}$.

With the properly defined state and input transformations as given in (5.3.54) and (5.3.58), respectively, we have

$$\dot{x}_0 = A_{00}x_0 + \breve{A}_{0\rm b}\bar{x}_{\rm b} + \breve{A}_{0\rm d}x_{\rm d} + B_{0\rm d}u_{\rm d} + B_{0\rm c}\bar{u}_{\rm c}, \tag{5.3.63}$$

where A_{00}, $\breve{A}_{0\rm b}$, $\breve{A}_{0\rm d}$, $B_{0\rm c}$ and $B_{0\rm d}$ are some appropriate constant matrices. For each subsystem $x_{{\rm d},i}$, $i = 1, 2, \dots, m_{\rm d}$, in view of (5.3.30), (5.3.33)–(5.3.35) and (5.3.40), we have

$$\dot{x}_{{\rm d},i,1} = x_{{\rm d},i,2} + L_{{\rm d},i,1}y_{\rm d}, \quad y_{{\rm d},i} = x_{{\rm d},i,1}, \tag{5.3.64}$$

$$\dot{x}_{{\rm d},i,2} = x_{{\rm d},i,3} + L_{{\rm d},i,2}y_{\rm d}, \tag{5.3.65}$$

$$\vdots$$

$$\dot{x}_{{\rm d},i,q_i} = A_{{\rm d},i,0}x_0 + \bar{A}_{{\rm d},i,\rm b}\bar{x}_{\rm b} + A_{{\rm d},i,\rm d}x_{\rm d} + u_{{\rm d},i}, \tag{5.3.66}$$

for some appropriate constant vectors $A_{{\rm d},i,0}$, $\bar{A}_{{\rm d},i,\rm b}$, $A_{{\rm d},i,\rm d}$ and $L_{{\rm d},i,j}$, $j = 1, 2, \dots, q_i - 1$. In fact, it is simple to verify from the procedure in STEP SCB.2 that $L_{i,j}$ has the form,

$$L_{{\rm d},i,j} = [\,\ell_{{\rm d},i,j,1} \ \ \cdots \ \ \ell_{{\rm d},i,j,i-1} \ \ 0 \ \ \cdots \ \ 0\,]. \tag{5.3.67}$$

For the subsystem $\bar{x}_{{\rm b},i}$, $i = 1, 2, \dots, p_{\rm b}$, in view of (5.3.39), (5.3.40) and (5.3.44), we have

$$\dot{\bar{x}}_{{\rm b},i,1} = \bar{x}_{{\rm b},i,2} + \bar{L}_{{\rm b},i,1}y_{\rm d}, \quad \bar{y}_{{\rm b},i} = \bar{x}_{{\rm b},i,1}, \tag{5.3.68}$$

$$\dot{\bar{x}}_{{\rm b},i,2} = \bar{x}_{{\rm b},i,3} + \bar{L}_{{\rm b},i,2}y_{\rm d}, \tag{5.3.69}$$

$$\vdots$$

$$\dot{\bar{x}}_{{\rm b},i,l_i} = \sum_{s=1}^{p_{\rm b}} \sum_{j=1}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j}\bar{x}_{{\rm b},s,j} + \sum_{s=1}^{m_{\rm d}} \sum_{j=1}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j}x_{{\rm d},s,j}, \tag{5.3.70}$$

for some constant row vectors $\alpha_{{\rm bb},i,s,j}$, $\alpha_{{\rm bd},i,s,j}$, and $\bar{L}_{{\rm b},i,j}$. Unfortunately, the dynamical equation in (5.3.70) is not in the desired form as specified in Theorem 5.3.1. We need to introduce further transformations that bring this part of the dynamics to depend on only the output variables $y_{\rm b}$ and $y_{\rm d}$.

STEP SCB.5. *Determination of* $x_{\rm b}$.

Define

$$x_{\rm b} := \begin{pmatrix} x_{{\rm b},1} \\ x_{{\rm b},2} \\ \vdots \\ x_{{\rm b},p_{\rm b}} \end{pmatrix}, \quad x_{{\rm b},i} := \begin{pmatrix} x_{{\rm b},i,1} \\ x_{{\rm b},i,2} \\ \vdots \\ x_{{\rm b},i,l_i} \end{pmatrix}, \quad y_{\rm b} := \begin{pmatrix} y_{{\rm b},1} \\ y_{{\rm b},2} \\ \vdots \\ y_{{\rm b},p_{\rm b}} \end{pmatrix}, \tag{5.3.71}$$

with

$$\begin{aligned} x_{{\rm b},i,1} := \bar{x}_{{\rm b},i,1} &- \sum_{s=1}^{p_{\rm b}} \sum_{j=l_i+1}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j} \bar{x}_{{\rm b},s,j-l_i} \\ &- \sum_{s=1}^{m_{\rm d}} \sum_{j=l_i+1}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j} x_{{\rm d},s,j-l_i}, \end{aligned} \tag{5.3.72}$$

$$\begin{aligned} x_{{\rm b},i,2} := \bar{x}_{{\rm b},i,2} &- \sum_{s=1}^{p_{\rm b}} \sum_{j=l_i}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j} \bar{x}_{{\rm b},s,j-l_i+1} \\ &- \sum_{s=1}^{m_{\rm d}} \sum_{j=l_i}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j} x_{{\rm d},s,j-l_i+1}, \end{aligned} \tag{5.3.73}$$

$$\vdots$$

$$\begin{aligned} x_{{\rm b},i,l_i-1} := \bar{x}_{{\rm b},i,l_i-1} &- \sum_{s=1}^{p_{\rm b}} \sum_{j=3}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j} \bar{x}_{{\rm b},s,j-2} \\ &- \sum_{s=1}^{m_{\rm d}} \sum_{j=3}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j} x_{{\rm d},s,j-2}, \end{aligned} \tag{5.3.74}$$

$$\begin{aligned} x_{{\rm b},i,l_i} := \bar{x}_{{\rm b},i,l_i} &- \sum_{s=1}^{p_{\rm b}} \sum_{j=2}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j} \bar{x}_{{\rm b},s,j-1} \\ &- \sum_{s=1}^{m_{\rm d}} \sum_{j=2}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j} x_{{\rm d},s,j-1}, \end{aligned} \tag{5.3.75}$$

$$\begin{aligned} y_{{\rm b},i} &:= x_{{\rm b},i,1} \\ &= \bar{x}_{{\rm b},i,1} - \sum_{s=1}^{p_{\rm b}} \sum_{j=l_i+1}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j} \bar{x}_{{\rm b},s,j-l_i} \\ &\quad - \sum_{s=1}^{m_{\rm d}} \sum_{j=l_i+1}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j} x_{{\rm d},s,j-l_i} \end{aligned}$$

$$= \bar{x}_{\mathrm{b},i,1} - \sum_{s=1}^{p_{\mathrm{b}}} \alpha_{\mathrm{bb},i,s,l_i+1}\bar{x}_{\mathrm{b},s,1} - \sum_{s=1}^{m_{\mathrm{d}}} \alpha_{\mathrm{bd},i,s,l_i+1}x_{\mathrm{d},s,1}$$

$$= \bar{x}_{\mathrm{b},i,1} - \sum_{s=1}^{p_{\mathrm{b}}} \alpha_{\mathrm{bb},i,s,l_i+1}\bar{y}_{\mathrm{b},s} - \sum_{s=1}^{m_{\mathrm{d}}} \alpha_{\mathrm{bd},i,s,l_i+1}y_{\mathrm{d},s} \qquad (5.3.76)$$

which shows that $y_{\mathrm{b},i}$ is a linear combination of y_{d} and $\bar{y}_{\mathrm{b}}$. Here we note that we set $\alpha_{\mathrm{bb},i,s,\ell} = 0$ when $\ell > l_s$ and $\alpha_{\mathrm{bd},i,s,\ell} = 0$ when $\ell > q_s$, in the definitions of (5.3.72)–(5.3.76). We then obtain

$$\begin{aligned}
\dot{x}_{\mathrm{b},i,l_i} &= \dot{\bar{x}}_{\mathrm{b},i,l_i} - \sum_{s=1}^{p_{\mathrm{b}}} \sum_{j=2}^{\min\{l_i+1,l_s\}} \alpha_{\mathrm{bb},i,s,j}\dot{\bar{x}}_{\mathrm{b},s,j-1} \\
&\quad - \sum_{s=1}^{m_{\mathrm{d}}} \sum_{j=2}^{\min\{l_i+1,q_s\}} \alpha_{\mathrm{bd},i,s,j}\dot{x}_{\mathrm{d},s,j-1} \\
&= \sum_{s=1}^{p_{\mathrm{b}}} \sum_{j=1}^{\min\{l_i+1,l_s\}} \alpha_{\mathrm{bb},i,s,j}\bar{x}_{\mathrm{b},s,j} + \sum_{s=1}^{m_{\mathrm{d}}} \sum_{j=1}^{\min\{l_i+1,q_s\}} \alpha_{\mathrm{bd},i,s,j}x_{\mathrm{d},s,j} \\
&\quad - \sum_{s=1}^{p_{\mathrm{b}}} \sum_{j=2}^{\min\{l_i+1,l_s\}} \alpha_{\mathrm{bb},i,s,j}\left(\bar{x}_{\mathrm{b},s,j} + \bar{L}_{\mathrm{b},s,j-1}y_{\mathrm{d}}\right) \\
&\quad - \sum_{s=1}^{m_{\mathrm{d}}} \sum_{j=2}^{\min\{l_i+1,q_s\}} \alpha_{\mathrm{bd},i,s,j}\left(x_{\mathrm{d},s,j} + L_{\mathrm{d},s,j-1}y_{\mathrm{d}}\right) \\
&= \sum_{s=1}^{p_{\mathrm{b}}} \alpha_{\mathrm{bb},i,s,1}\bar{x}_{\mathrm{b},s,1} + \sum_{s=1}^{m_{\mathrm{d}}} \alpha_{\mathrm{bd},i,s,1}x_{\mathrm{d},s,1} \\
&\quad - \sum_{s=1}^{p_{\mathrm{b}}} \sum_{j=2}^{\min\{l_i+1,l_s\}} \alpha_{\mathrm{bb},i,s,j}\bar{L}_{\mathrm{b},s,j-1}y_{\mathrm{d}} \\
&\quad - \sum_{s=1}^{m_{\mathrm{d}}} \sum_{j=2}^{\min\{l_i+1,q_s\}} \alpha_{\mathrm{bd},i,s,j}L_{\mathrm{d},s,j-1}y_{\mathrm{d}} \\
&= L_{\mathrm{bb},i,l_i}y_{\mathrm{b}} + L_{\mathrm{bd},i,l_i}y_{\mathrm{d}},
\end{aligned} \qquad (5.3.77)$$

where L_{bb,i,l_i} and L_{bd,i,l_i} are defined in a straightforward manner. We also have

$$\dot{x}_{\mathrm{b},i,l_i-1} = \dot{\bar{x}}_{\mathrm{b},i,l_i-1} - \sum_{s=1}^{p_{\mathrm{b}}} \sum_{j=3}^{\min\{l_i+1,l_s\}} \alpha_{\mathrm{bb},i,s,j}\dot{\bar{x}}_{\mathrm{b},s,j-2}$$

$$
\begin{aligned}
&-\sum_{s=1}^{m_{\rm d}} \sum_{j=3}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j}\dot{x}_{{\rm d},s,j-2} \\
&= \bar{x}_{{\rm b},i,l_i} + \bar{L}_{{\rm b},i,l_i-1}y_{\rm d} \\
&\quad -\sum_{s=1}^{p_{\rm b}} \sum_{j=3}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j}\Big(\bar{x}_{{\rm b},s,j-1} + \bar{L}_{{\rm b},s,j-2}y_{\rm d}\Big) \\
&\quad -\sum_{s=1}^{m_{\rm d}} \sum_{j=3}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j}\Big(x_{{\rm d},s,j-1} + L_{{\rm d},s,j-2}y_{\rm d}\Big) \\
&= \bar{x}_{{\rm b},i,l_i} - \sum_{s=1}^{p_{\rm b}} \sum_{j=2}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j}\bar{x}_{{\rm b},s,j-1} - \sum_{s=1}^{m_{\rm d}} \sum_{j=2}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j}x_{{\rm d},s,j-1} \\
&\quad + \sum_{s=1}^{p_{\rm b}} \alpha_{{\rm bb},i,s,2}\bar{x}_{{\rm b},s,1} + \sum_{s=1}^{m_{\rm d}} \alpha_{{\rm bd},i,s,2}x_{{\rm d},s,1} + \bar{L}_{{\rm b},i,l_i-1}y_{\rm d} \\
&\quad -\sum_{s=1}^{p_{\rm b}} \sum_{j=3}^{\min\{l_i+1,l_s\}} \alpha_{{\rm bb},i,s,j}\bar{L}_{{\rm b},s,j-2}y_{\rm d} - \sum_{s=1}^{m_{\rm d}} \sum_{j=3}^{\min\{l_i+1,q_s\}} \alpha_{{\rm bd},i,s,j}L_{{\rm d},s,j-2}y_{\rm d} \\
&= x_{{\rm b},i,l_i} + L_{{\rm bb},i,l_i-1}y_{\rm b} + L_{{\rm bd},i,l_i-1}y_{\rm d},
\end{aligned}
\tag{5.3.78}
$$

for some constant row vectors $L_{{\rm bb},i,l_i-1}$ and $L_{{\rm bd},i,l_i-1}$. Similarly, for $k = l_i - 2, l_i - 3, \ldots, 1$, we have

$$
\dot{x}_{{\rm b},i,k} = x_{{\rm b},i,k+1} + L_{{\rm bb},i,k}y_{\rm b} + L_{{\rm bd},i,k}y_{\rm d}, \tag{5.3.79}
$$

for some appropriately defined $L_{{\rm bb},i,k}$ and $L_{{\rm bd},i,k}$. The subsystem $x_{\rm b}$ is finally identified and is in the desired form.

Next, obviously, we can rewrite (5.3.63) as

$$
\dot{x}_0 = A_{00}x_0 + A_{0{\rm b}}x_{\rm b} + A_{0{\rm d}}x_{\rm d} + B_{0{\rm d}}u_{\rm d} + B_{0{\rm c}}\bar{u}_{\rm c}, \tag{5.3.80}
$$

and rewrite each subsystem $x_{{\rm d},i}$ in (5.3.64)–(5.3.66) as

$$
\dot{x}_{{\rm d},i,1} = x_{{\rm d},i,2} + L_{{\rm d},i,1}y_{\rm d}, \quad y_{{\rm d},i} = x_{{\rm d},i,1} \tag{5.3.81}
$$

$$
\dot{x}_{{\rm d},i,2} = x_{{\rm d},i,3} + L_{{\rm d},i,2}y_{\rm d}, \tag{5.3.82}
$$

$$
\vdots
$$

$$
\dot{x}_{{\rm d},i,q_i} = A_{{\rm d},i,0}x_0 + A_{{\rm d},i,{\rm b}}x_{\rm b} + A_{{\rm d},i,{\rm d}}x_{\rm d} + u_{{\rm d},i}, \tag{5.3.83}
$$

for some properly defined constant matrices $A_{{\rm d},i,0}$, $A_{{\rm d},i,{\rm b}}$ and $A_{{\rm d},i,{\rm d}}$.

We still need to do some additional work in (5.3.80) in order to decompose x_0 into subspaces related to x_a and x_c. In particular, we will have to make this part of the dynamics dependent only on the outputs y_d, y_b and the control $\bar{u}_c$.

STEP SCB.6. *Elimination of u_d from (5.3.80).*

In view of (5.3.83), we have

$$u_d = \begin{pmatrix} \dot{x}_{d,1,q_1} \\ \dot{x}_{d,2,q_1} \\ \vdots \\ \dot{x}_{d,m_d,q_{m_d}} \end{pmatrix} - \begin{bmatrix} A_{d,1,0} \\ A_{d,2,0} \\ \vdots \\ A_{d,m_d,0} \end{bmatrix} x_0 - \begin{bmatrix} A_{d,1,b} \\ A_{d,2,b} \\ \vdots \\ A_{md,d,b} \end{bmatrix} x_b - \begin{bmatrix} A_{d,1,d} \\ A_{d,2,d} \\ \vdots \\ A_{d,m_d,d} \end{bmatrix} x_d. \tag{5.3.84}$$

Substituting (5.3.84) into (5.3.80), we obtain

$$\dot{x}_0 = \bar{A}_{00}x_0 + \bar{A}_{0b}x_b + \bar{A}_{0d}x_d + B_{0d} \begin{pmatrix} \dot{x}_{d,1,q_1} \\ \dot{x}_{d,2,q_2} \\ \vdots \\ \dot{x}_{d,m_d,q_{m_d}} \end{pmatrix} + B_{0c}\bar{u}_c, \tag{5.3.85}$$

for some appropriate matrices $\bar{A}_{00}$, $\bar{A}_{0b}$ and $\bar{A}_{0d}$.

STEP SCB.7. *Elimination of $\dot{x}_{d,i,q_i}$, $i = 1, 2, \ldots, m_d$, from (5.3.85).*

We define a new state variable,

$$\tilde{x}_0 := x_0 - B_{0d} \begin{pmatrix} x_{d,1,q_1} \\ x_{d,2,q_2} \\ \vdots \\ x_{d,m_d,q_{m_d}} \end{pmatrix}. \tag{5.3.86}$$

Then, we have

$$\begin{aligned} \dot{\tilde{x}}_0 &= \bar{A}_{00}x_0 + \bar{A}_{0b}x_b + \bar{A}_{0d}x_d + B_{0c}\bar{u}_c \\ &= \bar{A}_{00} \left[x_0 - B_{0d} \begin{pmatrix} x_{d,1,q_1} \\ x_{d,2,q_2} \\ \vdots \\ x_{d,m_d,q_{m_d}} \end{pmatrix} \right] + \left[\bar{A}_{0d}x_d + \bar{A}_{00}B_{0d} \begin{pmatrix} x_{d,1,q_1} \\ x_{d,2,q_2} \\ \vdots \\ x_{d,m_d,q_{m_d}} \end{pmatrix} \right] \\ &\qquad + \bar{A}_{0b}x_b + B_{0c}\bar{u}_c \\ &= \bar{A}_{00}\tilde{x}_0 + \bar{A}_{0b}x_b + \tilde{A}_{0d}x_d + B_{0c}\bar{u}_c, \end{aligned} \tag{5.3.87}$$

where $\tilde{A}_{0\mathrm{d}}$ is defined in an obvious manner. Similarly, x_0 in (5.3.70) and (5.3.83) should be replaced by $\tilde{x}_0$ and its coefficients together with coefficients associated with x_{d} in (5.3.70) and (5.3.83) can be redefined in a straightforward manner. These coefficients are, however, irrelevant to the further decomposition. For use in the next step, we rewrite (5.3.87) as

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \sum_{i=1}^{p_{\mathrm{b}}}\sum_{j=1}^{l_i} A_{0\mathrm{b},i,j}x_{\mathrm{b},i,j} + \sum_{i=1}^{m_{\mathrm{d}}}\sum_{j=1}^{q_i} A_{0\mathrm{d},i,j}x_{\mathrm{d},i,j} + B_{0\mathrm{c}}\bar{u}_{\mathrm{c}}, \tag{5.3.88}$$

for some constant vectors $A_{0\mathrm{b},i,j}$ and $A_{0\mathrm{d},i,j}$.

STEP SCB.8. *Elimination of* $x_{\mathrm{b},i,j}$, $j = 2,3,\dots,l_i$, $i = 1,2,\dots,p_{\mathrm{b}}$, *and* $x_{\mathrm{d},i,j}$, $j = 2,3,\dots,q_i$, $i = 1,2,\dots,m_{\mathrm{d}}$, *from (5.3.88).*

We now proceed to find a transformation such that the dynamics associated with $\tilde{x}_0$ is expressed in terms of only $y_{\mathrm{d},s}$, $s = 1,2,\dots,m_{\mathrm{d}}$, and $y_{\mathrm{b},s}$, $s = 1,2,\dots,p_{\mathrm{b}}$.

If $\max\{q_1,q_2,\dots,q_{m_{\mathrm{d}}},l_1,l_2,\dots,l_{p_{\mathrm{b}}}\} = 1$, we will skip the following sub-steps and directly move on to STEP SCB.9. Otherwise, we initialize $i := 0$ and set $\tilde{x}_{0,0} := \tilde{x}_0$; $A_{0\mathrm{b},s,j,0} := A_{0\mathrm{b},s,j}$, $s = 1,2,\dots,p_{\mathrm{b}}$, $j = 1,2,\dots,l_s$, $A_{0\mathrm{d},s,j,0} := A_{0\mathrm{d},s,j}$, $s = 1,2,\dots,m_{\mathrm{d}}$, $j = 1,2,\dots,q_s$, and then carry out the following sub-steps:

Sub-step 8.1. First, we note that

$$\dot{\tilde{x}}_{0,i} = \bar{A}_{00}\tilde{x}_{0,i} + \sum_{s=1}^{p_{\mathrm{b}}}\sum_{j=1}^{l_s-i} A_{0\mathrm{b},s,j,i}x_{\mathrm{b},s,j} + \sum_{s=1}^{m_{\mathrm{d}}}\sum_{j=1}^{q_s-i} A_{0\mathrm{d},s,j,i}x_{\mathrm{d},s,j} + B_{0\mathrm{c}}\bar{u}_{\mathrm{c}}.$$

We now proceed to eliminate x_{d,s,q_s-i}, $s = 1,2,\dots,m_{\mathrm{d}}$, and x_{b,s,l_s-i}, $s = 1,2,\dots,p_{\mathrm{b}}$, in the above expression. Let us define

$$\tilde{x}_{0,i+1} := \tilde{x}_{0,i} - \sum_{s=1}^{p_{\mathrm{b}}} A_{0\mathrm{b},s,l_s-i,i}x_{\mathrm{b},s,l_s-i-1} - \sum_{s=1}^{m_{\mathrm{d}}} A_{0\mathrm{d},s,q_s-i,i}x_{\mathrm{d},s,q_s-i-1},$$

where we take $A_{0\mathrm{b},s,l_s-i,i} = 0$ if $l_s - i - 1 \le 0$, and $A_{0\mathrm{d},s,q_s-i,i} = 0$ if $q_s - i - 1 \le 0$. In view of (5.3.68)–(5.3.83), we have

$$\dot{\tilde{x}}_{0,i+1} = \dot{\tilde{x}}_{0,i} - \sum_{s=1}^{p_{\mathrm{b}}} A_{0\mathrm{b},s,l_s-i,i}\dot{x}_{\mathrm{b},s,l_s-i-1} - \sum_{s=1}^{m_{\mathrm{d}}} A_{0\mathrm{d},s,q_s-i,i}\dot{x}_{\mathrm{d},s,q_s-i-1}$$

$$
\begin{aligned}
&= \bar{A}_{00}\tilde{x}_{0,i} + \sum_{s=1}^{p_\mathrm{b}} \sum_{j=1}^{l_s-i} A_{0\mathrm{b},s,j,i} x_{\mathrm{b},s,j} + \sum_{s=1}^{m_\mathrm{d}} \sum_{j=1}^{q_s-i} A_{0\mathrm{d},s,j,i} x_{\mathrm{d},s,j} \\
&\quad - \sum_{s=1}^{p_\mathrm{b}} A_{0\mathrm{b},s,l_s-i,i} \Big(x_{\mathrm{b},s,l_s-i} + L_{\mathrm{b},s,l_s-i-1} y_\mathrm{d} \Big) \\
&\quad - \sum_{s=1}^{m_\mathrm{d}} A_{0\mathrm{d},s,q_s-i,i} \Big(x_{\mathrm{d},s,q_s-i} + L_{\mathrm{d},s,q_s-i-1} y_\mathrm{d} \Big) + B_{0\mathrm{c}}\bar{u}_\mathrm{c} \\
&= \bar{A}_{00}\tilde{x}_{0,i} + \sum_{s=1}^{p_\mathrm{b}} \sum_{j=1}^{l_s-i-1} A_{0\mathrm{b},s,j,i} x_{\mathrm{b},s,j} + \sum_{s=1}^{m_\mathrm{d}} \sum_{j=1}^{q_s-i-1} A_{0\mathrm{d},s,j,i} x_{\mathrm{d},s,j} \\
&\quad - \sum_{s=1}^{p_\mathrm{b}} A_{0\mathrm{b},s,l_s-i,i} L_{\mathrm{b},s,l_s-i-1} y_\mathrm{d} \\
&\quad - \sum_{s=1}^{m_\mathrm{d}} A_{0\mathrm{d},s,q_s-i,i} L_{\mathrm{d},s,q_s-i-1} y_\mathrm{d} + B_{0\mathrm{c}}\bar{u}_\mathrm{c}. \qquad (5.3.89)
\end{aligned}
$$

Clearly, we have eliminated in (5.3.89) all x_{d,s,q_s-i}, $s = 1, 2, \ldots, m_\mathrm{d}$, and all x_{b,s,l_s-i}, $s = 1, 2, \ldots, p_\mathrm{b}$. Thus, we can re-write (5.3.89) as

$$
\begin{aligned}
\dot{\tilde{x}}_{0,i+1} = \bar{A}_{00}\tilde{x}_{0,i+1} &+ \sum_{s=1}^{p_\mathrm{b}} \sum_{j=1}^{l_s-i-1} A_{0\mathrm{b},s,j,i+1} x_{\mathrm{b},s,j} \\
&+ \sum_{s=1}^{m_\mathrm{d}} \sum_{j=1}^{q_s-i-1} A_{0\mathrm{d},s,j,i+1} x_{\mathrm{d},s,j} + B_{0\mathrm{c}}\bar{u}_\mathrm{c}, \qquad (5.3.90)
\end{aligned}
$$

for some appropriate constant vectors $A_{0\mathrm{b},s,j,i+1}$ and $A_{0\mathrm{d},s,j,i+1}$.

Sub-step 8.2. If $i = \max\{q_1, q_2, \ldots, q_{m_\mathrm{d}}, l_1, l_2, \ldots, l_{p_\mathrm{b}}\} - 2$, then we will go to STEP SCB.9. Otherwise, we let $i := i + 1$ and go back to Sub-step 8.1 to repeat the process until there are only y_d and y_b left in the expression associated with $\tilde{x}_{0,i+1}$ dynamics.

STEP SCB.9. *Separation of x_a and x_c.*

Let $\hat{x}_0 := \tilde{x}_{0,i+1}$. We then have

$$
\dot{\hat{x}}_0 = \bar{A}_{00}\hat{x}_0 + L_{0\mathrm{b}} y_\mathrm{b} + L_{0\mathrm{d}} y_\mathrm{d} + B_{0\mathrm{c}}\bar{u}_\mathrm{c}. \qquad (5.3.91)
$$

Next, we apply the result of Theorem 4.4.1 of Chapter 4 to $(\bar{A}_{00}, B_{0\mathrm{c}})$. It follows from Theorem 4.4.1 that there exist an $n_0 \times n_0$ nonsingular transformation $T_{0\mathrm{s}}$ and an $m_\mathrm{c} \times m_\mathrm{c}$ nonsingular transformation $T_{0\mathrm{i}}$, which yield the following decomposition:

$$\hat{x}_0 = T_{0\mathrm{s}} \begin{pmatrix} x_\mathrm{a} \\ x_\mathrm{c} \end{pmatrix}, \quad \bar{u}_\mathrm{c} = T_{0\mathrm{i}} u_\mathrm{c}, \tag{5.3.92}$$

$$x_\mathrm{c} = \begin{pmatrix} x_{\mathrm{c},1} \\ x_{\mathrm{c},2} \\ \vdots \\ x_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \quad x_{\mathrm{c},i} = \begin{pmatrix} x_{\mathrm{c},i,1} \\ x_{\mathrm{c},i,2} \\ \vdots \\ x_{\mathrm{c},i,r_i} \end{pmatrix}, \quad u_\mathrm{c} = \begin{pmatrix} u_{\mathrm{c},1} \\ u_{\mathrm{c},2} \\ \vdots \\ u_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \tag{5.3.93}$$

where $\{r_1, r_2, \dots, r_{m_\mathrm{c}}\}$ is the controllability index of the pair $(\bar{A}_{00}, B_{0\mathrm{c}})$. Furthermore, we can write the dynamical equations of x_a and x_c as follows:

$$\dot{x}_\mathrm{a} = A_\mathrm{aa} x_\mathrm{a} + L_\mathrm{ab} y_\mathrm{b} + L_\mathrm{ad} y_\mathrm{d}, \tag{5.3.94}$$

and for each $x_{\mathrm{c},i}$, $i = 1, 2, \dots, m_\mathrm{c}$,

$$\dot{x}_{\mathrm{c},i,1} = x_{\mathrm{c},i,2} + L_{\mathrm{cb},i,1} y_\mathrm{b} + L_{\mathrm{cd},i,1} y_\mathrm{d}, \tag{5.3.95}$$

$$\vdots$$

$$\dot{x}_{\mathrm{c},i,r_i-1} = x_{\mathrm{c},i,r_i} + L_{\mathrm{cb},i,r_i-1} y_\mathrm{b} + L_{\mathrm{cd},i,r_i-1} y_\mathrm{d}, \tag{5.3.96}$$

$$\dot{x}_{\mathrm{c},i,r_i} = A_{\mathrm{c},i,\mathrm{a}} x_\mathrm{a} + A_{\mathrm{c},i,\mathrm{c}} x_\mathrm{c} + L_{\mathrm{cb},i,r_i} y_\mathrm{b} + L_{\mathrm{cd},i,r_i} y_\mathrm{d} + u_{\mathrm{c},i}, \tag{5.3.97}$$

for some constant matrices. Obviously, x_0 in the dynamical equations associated with x_d, i.e., (5.3.81) to (5.3.83), can be replaced by x_a and x_c in a simple manner. In particular, for each subsystem $x_{\mathrm{d},i}$, $i = 1, 2, \dots, m_\mathrm{d}$,

$$\dot{x}_{\mathrm{d},i,1} = x_{\mathrm{d},i,2} + L_{\mathrm{d},i,1} y_\mathrm{d}, \quad y_{\mathrm{d},i} = x_{\mathrm{d},i,1}, \tag{5.3.98}$$

$$\dot{x}_{\mathrm{d},i,2} = x_{\mathrm{d},i,3} + L_{\mathrm{d},i,2} y_\mathrm{d}, \tag{5.3.99}$$

$$\vdots$$

$$\dot{x}_{\mathrm{d},i,q_i} = A_{\mathrm{d},i,\mathrm{a}} x_\mathrm{a} + A_{\mathrm{d},i,\mathrm{b}} x_\mathrm{b} + A_{\mathrm{d},i,\mathrm{c}} x_\mathrm{c} + A_{\mathrm{d},i,\mathrm{d}} x_\mathrm{d} + u_{\mathrm{d},i}, \tag{5.3.100}$$

for some appropriate constant vectors $A_{\mathrm{d},i,\mathrm{a}}$, $A_{\mathrm{d},i,\mathrm{b}}$ $A_{\mathrm{d},i,\mathrm{c}}$ and $A_{\mathrm{d},i,\mathrm{d}}$. To complete the whole process, we recall the dynamical equations associated with x_b as derived in STEP SCB.5, i.e., for each $x_{\mathrm{b},i}$, $i = 1, 2, \dots, p_\mathrm{b}$,

$$\dot{x}_{\mathrm{b},i,1} = x_{\mathrm{b},i,2} + L_{\mathrm{bb},i,1} y_\mathrm{b} + L_{\mathrm{bd},i,1} y_\mathrm{d}, \quad y_{\mathrm{b},i} = x_{\mathrm{b},i,1}, \tag{5.3.101}$$

$$\dot{x}_{\mathrm{b},i,2} = x_{\mathrm{b},i,3} + L_{\mathrm{bb},i,2} y_\mathrm{b} + L_{\mathrm{bd},i,2} y_\mathrm{d}, \tag{5.3.102}$$

$$\vdots$$

$$\dot{x}_{\mathrm{b},i,l_i} = L_{\mathrm{bb},i,l_i} y_\mathrm{b} + L_{\mathrm{bd},i,l_i} y_\mathrm{d}. \tag{5.3.103}$$

Finally, we note that the invariant index lists $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ and $\mathbf{I}_4$ of Morse [100], or equivalently the invariant zero structure, the right invertibility structure $S_{\mathrm{R}}^\star$, the left invertibility structure $S_{\mathrm{L}}^\star$ and the infinite zero structure $S_\infty^\star$ (see Definition 3.6.3), of the given system Σ can be easily obtained from the above structural decomposition. In particular, we have

$$S_{\mathrm{R}}^\star(\Sigma) = \mathbf{I}_2 = \{r_1, r_2, \dots, r_{m_{\mathrm{c}}}\}, \quad S_{\mathrm{L}}^\star(\Sigma) = \mathbf{I}_3 = \{l_1, l_2, \dots, l_{p_{\mathrm{b}}}\}, \tag{5.3.104}$$

$$S_\infty^\star(\Sigma) = \mathbf{I}_4 = \{q_1, q_2, \dots, q_{m_{\mathrm{d}}}\}, \tag{5.3.105}$$

and $\mathbf{I}_1$ is related to the eigenstructure of A_{aa}, *i.e.*, the eigenvalues and the sizes of their associated Jordan blocks. We note that these properties will be further justified in detail in Section 5.6.

This completes the algorithm that realizes the structural decomposition or the special coordinate basis of general strictly proper multivariable systems. ∎

We illustrate the above algorithm in the following example.

Example 5.3.1. Consider a strictly proper system Σ characterized by (5.3.1) with

$$A = \begin{bmatrix} 0 & 0 & 2 & -1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 & 2 & 2 \\ 0 & 2 & 4 & -5 & 3 & 2 & -4 & 3 & 2 & -4 & 0 & 5 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 0 & 3 & -3 & 0 & 3 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & -2 & 2 & 1 & -3 & 2 & 1 & 1 & 1 & -2 \\ 0 & 3 & 3 & -1 & -2 & 3 & 0 & -2 & 3 & 0 & 2 & 2 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & -1 & 3 & 0 & 1 & 3 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & 0 & 3 & 0 & 1 & 4 & -3 \\ 0 & 2 & 2 & 1 & -1 & 2 & 0 & 0 & 2 & 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & 2 & -2 & 0 & 2 & 1 & -2 & 0 \\ 0 & -1 & -3 & 7 & -3 & -1 & 4 & -3 & -1 & 4 & 2 & -4 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

We proceed to construct the structural decomposition or the special coordinate basis of Σ by following the detailed procedures given in the constructive proof of Theorem 5.4.1.

STEP SCB.1. *Initialization.*

We first partition

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \end{bmatrix},$$

and set

$$Z_1 := C_1, \quad Z_2 := C_2, \quad Z_3 := C_3, \quad Z_4 := C_4,$$

$$\boldsymbol{f} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad Z := C.$$

Also set $Z_{\rm d} :=$ [], $W_{\rm d} :=$ [], and $Z_{\rm b} :=$ [], where [] stands for an empty matrix. Lastly, we set $v := 0$ and $w := 0$.

STEP SCB.2. *Repetitive differentiation of the system output.*

Noting that

$$\boldsymbol{f} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

with $f_1 = 1$, we partition

$$Z_1 = [\,C_{1,1}\,] = [0 \;\; 1 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; -1 \;\; 2 \;\; -1],$$

and compute

$$C_{1,1}B = [0 \;\; 0 \;\; 0 \;\; 0].$$

Thus, we have

$$\operatorname{rank}\left(\begin{bmatrix} W_{\rm d} \\ C_{1,1}B \end{bmatrix}\right) = 0 = \operatorname{rank}(W_{\rm d}),$$

which satisfies the condition of Case 2, *i.e.*, (5.3.32). Next, we let $\check{C}_{1,1} = C_{1,1}$ and calculate

$$\check{C}_{1,2} = \check{C}_{1,1}A = [0 \;\; 0 \;\; 0 \;\; 2 \;\; -1 \;\; 0 \;\; 2 \;\; -2 \;\; 0 \;\; 2 \;\; 1 \;\; -1 \;\; 0].$$

It is easy to check that

$$\text{rank}\left(\begin{bmatrix} Z \\ \check{C}_{1,2} \end{bmatrix}\right) = 5 > \text{rank}(Z) = 4,$$

which corresponds to Sub-case 2.1, *i.e.*, (5.3.42). We then update Z and Z_1 as follows:

$$Z := \begin{bmatrix} Z \\ \check{C}_{1,2} \end{bmatrix} = \left[\begin{array}{ccccccccccccc} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \end{array}\right],$$

and

$$Z_1 := \begin{bmatrix} \check{C}_{1,1} \\ \check{C}_{1,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \end{bmatrix}.$$

Similarly, because $f_2 = 1$, we partition

$$Z_2 = [\, C_{2,1} \,] = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

and compute

$$C_{2,1}B = [1 \;\; 1 \;\; 0 \;\; 0].$$

It is simple to see that

$$\text{rank}\left(\begin{bmatrix} W_\text{d} \\ C_{2,1}B \end{bmatrix}\right) = 1 > \text{rank}(W_\text{d}) = 0,$$

which satisfies the condition of Case 1, *i.e.*, (5.3.29), and thus implies that the chain of integrators associated with the 2nd output variable reaches a system input. We then stop this output from further differentiation by setting $f_2 := 0$ and thus

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Also, set

$$Z_\text{d} := \begin{bmatrix} Z_\text{d} \\ Z_2 \end{bmatrix} = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

$$W_\text{d} = \begin{bmatrix} W_\text{d} \\ C_{2,1}B \end{bmatrix} = [1 \;\; 1 \;\; 0 \;\; 0],$$

and

$$v := v + 1 = 1, \quad q_1 = \text{rank}(Z_2) = 1.$$

We now move on to the next output variable. Again, noting that $f_3 = 1$, we partition

$$Z_3 = [\, C_{3,1} \,] = [0 \;\; 1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; -1],$$

and compute

$$C_{3,1}B = [1 \;\; 1 \;\; 0 \;\; 0].$$

Then, check

$$\text{rank}\left(\begin{bmatrix} W_\text{d} \\ C_{2,1}B \end{bmatrix}\right) = 1 = \text{rank}\,(W_\text{d}),$$

which corresponds to Case 2,

$$Z_\text{d} = [\, Z_{\text{d},1} \,] = [\, C_{\text{d},1,q_1} \,] = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

and

$$W_\text{d} = [1 \;\; 1 \;\; 0 \;\; 0].$$

Clearly, we have $C_{3,1}B = W_\text{d}$. Thus, we calculate

$$\check{C}_{3,1} = C_{3,1} - C_{\text{d},1,q_1} = [0 \;\; 1 \;\; 1 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; -1]$$

and

$$\check{C}_{3,2} = \check{C}_{3,1}A = [1 \;\; 1 \;\; 1 \;\; -4 \;\; 2 \;\; 1 \;\; -3 \;\; 2 \;\; 1 \;\; -3 \;\; -1 \;\; 3 \;\; -1].$$

Next, check

$$\text{rank}\left(\begin{bmatrix} Z \\ \check{C}_{3,2} \end{bmatrix}\right) = 6 > \text{rank}\,(Z) = 5,$$

which satisfies the condition of Sub-case 2.1. We set

$$Z := \begin{bmatrix} Z \\ \check{C}_{3,2} \end{bmatrix} = \left[\begin{array}{ccccccccccccc} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \\ \hline 1 & 1 & 1 & -4 & 2 & 1 & -3 & 2 & 1 & -3 & -1 & 3 & -1 \end{array}\right],$$

and

$$Z_3 := \begin{bmatrix} \check{C}_{3,1} \\ \check{C}_{3,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -4 & 2 & 1 & -3 & 2 & 1 & -3 & -1 & 3 & -1 \end{bmatrix}.$$

As the last flag $f_4 = 1$, we proceed to partition

$$Z_4 = [\, C_{4,1} \,] = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; 1 \;\; -1 \;\; 0],$$

and compute

$$C_{4,1}B = [1 \;\; 1 \;\; 0 \;\; 0].$$

Obviously,

$$\text{rank}\left(\begin{bmatrix} W_{\rm d} \\ C_{4,1}B \end{bmatrix}\right) = 1 = \text{rank}\,(W_{\rm d}).$$

It belongs to Case 2. Note that we have so far obtained

$$Z_{\rm d} = [\,Z_{\rm d,1}\,] = [\,C_{{\rm d},1,q_1}\,] = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

and

$$W_{\rm d} = [1 \;\; 1 \;\; 0 \;\; 0].$$

Clearly, $C_{4,1}B = W_{\rm d}$. We then calculate

$$\check{C}_{4,1} = C_{4,1} - C_{{\rm d},1,q_1} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 1 \;\; -1 \;\; 0],$$

and

$$\check{C}_{4,2} = \check{C}_{4,1}A = [1 \;\; 0 \;\; 0 \;\; -3 \;\; 2 \;\; 0 \;\; -3 \;\; 3 \;\; 0 \;\; -3 \;\; -1 \;\; 2 \;\; 0].$$

Next, check

$$\text{rank}\left(\begin{bmatrix} Z \\ \check{C}_{4,2} \end{bmatrix}\right) = 7 > \text{rank}\,(Z) = 6,$$

which belongs to Sub-case 2.1. We then replace Z and Z_4 with

$$Z := \begin{bmatrix} Z \\ \check{C}_{4,2} \end{bmatrix} = \left[\begin{array}{ccccccccccccc} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -4 & 2 & 1 & -3 & 2 & 1 & -3 & -1 & 3 & -1 \\ \hline 1 & 0 & 0 & -3 & 2 & 0 & -3 & 3 & 0 & -3 & -1 & 2 & 0 \end{array}\right],$$

and

$$Z_4 := \begin{bmatrix} \check{C}_{4,1} \\ \check{C}_{4,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -3 & 2 & 0 & -3 & 3 & 0 & -3 & -1 & 2 & 0 \end{bmatrix}.$$

As the flag

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

is not identically 0, we will have to repeat STEP SCB.2. Since $f_1 = 1$, we partition

$$Z_1 = \begin{bmatrix} C_{1,1} \\ C_{1,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \end{bmatrix},$$

and compute

$$C_{1,2}B = [2 \;\; 2 \;\; 0 \;\; 0].$$

Observe that

$$\operatorname{rank}\left(\begin{bmatrix} W_{\rm d} \\ C_{1,2}B \end{bmatrix}\right) = 1 = \operatorname{rank}(W_{\rm d}),$$

which belongs to Case 2. Again, so far, we have obtained

$$Z_{\rm d} = [\, Z_{{\rm d},1} \,] = [\, C_{{\rm d},1,q_1} \,] = [0 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

and

$$W_{\rm d} = [1 \;\; 1 \;\; 0 \;\; 0].$$

Clearly, $C_{1,2}B = 2W_{\rm d}$. We then calculate

$$\check{C}_{1,1} = C_{1,1} = [0 \;\; 1 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; -1 \;\; 2 \;\; -1],$$

$$\check{C}_{1,2} = C_{1,2} - 2C_{{\rm d},1,q_1} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2 \;\; 2 \;\; 0 \;\; -2 \;\; 2 \;\; 1 \;\; -1 \;\; 0],$$

and

$$\check{C}_{1,3} = \check{C}_{1,2}A = [2 \;\; 1 \;\; 1 \;\; -3 \;\; 1 \;\; 1 \;\; -3 \;\; 2 \;\; 1 \;\; -3 \;\; 0 \;\; 3 \;\; -1].$$

It is simple to verify that

$$\operatorname{rank}\left(\begin{bmatrix} Z \\ \check{C}_{1,3} \end{bmatrix}\right) = 7 = \operatorname{rank}(Z),$$

which corresponds to Sub-case 2.2. There are no more inherent integrators in the chain associated with this output variable. We then set $f_1 := 0$ and thus the flag,

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Moreover, we replace Z_1 and $Z_{\rm b}$ with

$$Z_1 := \begin{bmatrix} \check{C}_{1,1} \\ \check{C}_{1,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & 0 & -2 & 2 & 1 & -1 & 0 \end{bmatrix},$$

and

$$Z_{\rm b} := \begin{bmatrix} Z_{\rm b} \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & 0 & -2 & 2 & 1 & -1 & 0 \end{bmatrix},$$

and set

$$w := w + 1 = 1, \quad l_1 = \operatorname{rank}(Z_1) = 2.$$

Noting that $f_3 = 1$, we partition

$$Z_3 = \begin{bmatrix} C_{3,1} \\ C_{3,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -4 & 2 & 1 & -3 & 2 & 1 & -3 & -1 & 3 & -1 \end{bmatrix},$$

and compute

$$C_{3,2}B = \begin{bmatrix} -1 & -1 & 0 & 0 \end{bmatrix}, \quad \mathrm{rank}\left(\begin{bmatrix} W_\mathrm{d} \\ C_{3,2}B \end{bmatrix}\right) = 1 = \mathrm{rank}\,(W_\mathrm{d}),$$

which satisfies the condition of Case 2. Again, for easy reference, we recall that

$$Z_\mathrm{d} = [\, Z_{\mathrm{d},1} \,] = [\, C_{\mathrm{d},1,q_1} \,] = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$W_\mathrm{d} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}.$$

Clearly, $C_{1,2}B = -W_\mathrm{d}$. We then calculate

$$\check{C}_{3,1} = C_{3,1} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},$$

$$\check{C}_{3,2} = C_{3,2} + C_{\mathrm{d},1,q_1} = \begin{bmatrix} 1 & 1 & 1 & -3 & 1 & 2 & -3 & 1 & 2 & -3 & -1 & 3 & -1 \end{bmatrix},$$

and

$$\check{C}_{3,3} = \check{C}_{3,2}A = \begin{bmatrix} -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

Next, we verify that

$$\mathrm{rank}\left(\begin{bmatrix} Z \\ \check{C}_{3,3} \end{bmatrix}\right) = 8 > \mathrm{rank}\,(Z) = 7,$$

which belongs to Sub-case 2.1. Thus, we replace Z and Z_3 with

$$Z := \begin{bmatrix} Z \\ \check{C}_{3,3} \end{bmatrix} = \left[\begin{array}{ccccccccccccc} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -4 & 2 & 1 & -3 & 2 & 1 & -3 & -1 & 3 & -1 \\ 1 & 0 & 0 & -3 & 2 & 0 & -3 & 3 & 0 & -3 & -1 & 2 & 0 \\ \hline -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{array}\right],$$

and

$$Z_3 = \begin{bmatrix} \check{C}_{3,1} \\ \check{C}_{3,2} \\ \check{C}_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -3 & 1 & 2 & -3 & 1 & 2 & -3 & -1 & 3 & -1 \\ -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

Next, noting that $f_4 = 1$, we have

$$Z_4 = \begin{bmatrix} C_{4,1} \\ C_{4,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -3 & 2 & 0 & -3 & 3 & 0 & -3 & -1 & 2 & 0 \end{bmatrix},$$

and

$$C_{4,2}B = [-2 \quad -2 \quad 0 \quad 0], \quad \text{rank}\left(\begin{bmatrix} W_\text{d} \\ C_{4,2}B \end{bmatrix}\right) = 1 = \text{rank}(W_\text{d}),$$

which belongs to Case 2. Recall once again that

$$Z_\text{d} = [Z_{\text{d},1}] = [C_{\text{d},1,q_1}] = [0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 1 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0],$$

and

$$W_\text{d} = [1 \quad 1 \quad 0 \quad 0].$$

Clearly, $C_{4,2}B = -2W_\text{d}$. We then calculate

$$\check{C}_{4,1} = C_{4,1} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1 \quad 0 \quad -1 \quad 1 \quad 1 \quad -1 \quad 0],$$

$$\check{C}_{4,2} = C_{4,2} + 2C_{\text{d},1,q_1} = [1 \quad 0 \quad 0 \quad -1 \quad 0 \quad 2 \quad -3 \quad 1 \quad 2 \quad -3 \quad -1 \quad 2 \quad 0],$$

and

$$\check{C}_{4,3} = \check{C}_{4,2}A = [-3 \quad -1 \quad -1 \quad 5 \quad -2 \quad -1 \quad 4 \quad -3 \quad -1 \quad 4 \quad 0 \quad -4 \quad 2],$$

and verify that

$$\text{rank}\left(\begin{bmatrix} Z \\ \check{C}_{4,3} \end{bmatrix}\right) = 8 = \text{rank}(Z),$$

which corresponds to Sub-case 2.2. There are no more inherent integrators in the chain associated with this output variable. Therefore, we set $f_4 := 0$ and thus the flag,

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

Furthermore, we replace Z_4 and Z_b with

$$Z_4 := \begin{bmatrix} \check{C}_{4,1} \\ \check{C}_{4,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 & -3 & 1 & 2 & -3 & -1 & 2 & 0 \end{bmatrix},$$

$$Z_\text{b} := \begin{bmatrix} Z_\text{b} \\ Z_4 \end{bmatrix} = \begin{bmatrix} Z_{\text{b},1} \\ \hline Z_{\text{b},2} \end{bmatrix}$$
$$= \left[\begin{array}{ccccccccccccc} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & 0 & -2 & 2 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 & -3 & 1 & 2 & -3 & -1 & 2 & 0 \end{array}\right],$$

and set

$$w := w + 1 = 2, \quad l_2 := \text{rank}(Z_4) = 2.$$

We further note that $f_3 = 1$ and thus have to partition

$$Z_3 = \begin{bmatrix} C_{3,1} \\ C_{3,2} \\ C_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -3 & 1 & 2 & -3 & 1 & 2 & -3 & -1 & 3 & -1 \\ -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{bmatrix},$$

and compute

$$C_{3,3}B = [1 \ \ 1 \ \ 1 \ \ 0], \quad \text{rank}\left(\begin{bmatrix} W_d \\ C_{3,3}B \end{bmatrix}\right) = 2 > \text{rank}\,(W_d) = 1,$$

which belongs to Case 1. The chain of integrators associated with this output variable has thus reached a system input. Thus, we set the flag $f_3 = 0$ (the flag vector, $\boldsymbol{f}$, is now identically 0), and replace

$$Z_d := \begin{bmatrix} Z_d \\ Z_3 \end{bmatrix} = \begin{bmatrix} Z_{d,1} \\ \hline Z_{d,2} \end{bmatrix}$$

$$= \left[\begin{array}{ccccccccccccc} 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -3 & 1 & 2 & -3 & 1 & 2 & -3 & -1 & 3 & -1 \\ -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{array}\right],$$

$$W_d := \begin{bmatrix} W_d \\ C_{3,3}B \end{bmatrix} = \begin{bmatrix} W_{d,1} \\ W_{d,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and $v := v + 1 = 2$, $q_2 = \text{rank}\,(Z_3) = 3$.

STEP SCB.3. *Interim Transformations.*

It is simple to calculate that $m_d = v = 2$, $p_b = w = 2$, $n_d = q_1 + q_2 = 4$, $n_b = l_1 + l_2 = 4$ and $n_o = 13 - n_d - n_b = 5$. We then select an $n_o \times n$ matrix,

$$Z_o = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

such that

$$S = \begin{bmatrix} Z_o \\ Z_b \\ Z_d \end{bmatrix} = \left[\begin{array}{ccccccccccccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & 0 & -2 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 & -3 & 1 & 2 & -3 & -1 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -3 & 1 & 2 & -3 & 1 & 2 & -3 & -1 & 3 & -1 \\ -2 & 1 & 1 & 1 & -1 & 1 & 1 & -2 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

is nonsingular. Let W_c be chosen as

$$W_c = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad W = \begin{bmatrix} W_d \\ W_c \end{bmatrix} = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

is nonsingular. The transformation, M, associated with the output variable can be traced back from (5.3.37), which is given as

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

We obtain a transformed system $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1) := (SAS^{-1}, SBW^{-1}, MCS^{-1})$ with

$$\tilde{A}_1 = \left[\begin{array}{ccccc|cccc|cccc}
-6 & -3 & -2 & 1 & 0 & -5 & -8 & -8 & -14 & 4 & 1 & 4 & -5 \\
14 & 9 & 6 & -3 & 0 & 9 & 16 & 10 & 28 & -6 & 2 & -11 & 8 \\
5 & 4 & 2 & -1 & 0 & 3 & 2 & 1 & 5 & 0 & 3 & -3 & 1 \\
14 & 11 & 7 & -3 & 0 & 8 & 10 & 4 & 20 & -3 & 5 & -11 & 4 \\
23 & 16 & 10 & -5 & 0 & 14 & 23 & 12 & 41 & -8 & 5 & -18 & 11 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
\hline
4 & 3 & 2 & -1 & 0 & 2 & 6 & 1 & 8 & 1 & 1 & -4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
4 & 3 & 2 & -1 & 0 & 1 & 3 & -1 & 4 & 0 & 3 & -1 & 1
\end{array}\right],$$

$$\tilde{B}_1 = \left[\begin{array}{cc|cc}
0 & -1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 1 & 1 \\
-1 & 2 & 3 & 1 \\
0 & 3 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right],$$

and

$$\tilde{C}_1 = \left[\begin{array}{ccccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

STEP SCB.4. *Determination of* $x_{\rm d}$.

In fact, the subsystem associated with $x_{\rm d}$ has already been properly identified in the transformed system in STEP SCB.3. Following the definitions of the transformed state, input and output variables as given in the algorithm, we can rewrite the transformed system as the following:

$$\dot{x}_0 = A_{00}x_0 + \breve{A}_{0\rm b}\bar{x}_{\rm b} + \breve{A}_{0\rm d}x_{\rm d} + B_{0\rm d}u_{\rm d} + B_{0\rm c}\bar{u}_{\rm c},$$

with

$$A_{00} = \begin{bmatrix} -6 & -3 & -2 & 1 & 0 \\ 14 & 9 & 6 & -3 & 0 \\ 5 & 4 & 2 & -1 & 0 \\ 14 & 11 & 7 & -3 & 0 \\ 23 & 16 & 10 & -5 & 0 \end{bmatrix}, \quad B_{0\rm d} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \\ 0 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B_{0\rm c} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 1 \\ 2 & 0 \end{bmatrix},$$

$$\breve{A}_{0\rm b} = \begin{bmatrix} -5 & -8 & -8 & -14 \\ 9 & 16 & 10 & 28 \\ 3 & 2 & 1 & 5 \\ 8 & 10 & 4 & 20 \\ 14 & 23 & 12 & 41 \end{bmatrix}, \quad \breve{A}_{0\rm d} = \begin{bmatrix} 4 & 1 & 4 & -5 \\ -6 & 2 & -11 & 8 \\ 0 & 3 & -3 & 1 \\ -3 & 5 & -11 & 4 \\ -8 & 5 & -18 & 11 \end{bmatrix},$$

the dynamics associated with $\bar{x}_{\rm b,1}$,

$$\dot{\bar{x}}_{\rm b,1,1} = \bar{x}_{\rm b,1,2} + 2y_{\rm d,1}, \quad \bar{y}_{\rm b,1} = \bar{x}_{\rm b,1,1}, \tag{5.3.106}$$

$$\dot{\bar{x}}_{\rm b,1,2} = \bar{x}_{\rm b,1,1} + \bar{x}_{\rm b,1,2} + 2\bar{x}_{\rm b,2,1} + 2\bar{x}_{\rm b,2,2}, \tag{5.3.107}$$

the dynamics associated with $\bar{x}_{\rm b,2}$,

$$\dot{\bar{x}}_{\rm b,2,1} = \bar{x}_{\rm b,2,2} - 2y_{\rm d,1}, \quad \bar{y}_{\rm b,2} = \bar{x}_{\rm b,2,1}, \tag{5.3.108}$$

$$\dot{\bar{x}}_{\rm b,2,2} = -x_{\rm d,2,1} - x_{\rm d,2,2} + x_{\rm d,2,3}, \tag{5.3.109}$$

the dynamics associated with $x_{\rm d,1}$,

$$\dot{x}_{\rm d,1,1} = [4 \quad 3 \quad 2 \quad -1 \quad 0]\,x_0 + [2 \quad 6 \quad 1 \quad 8]\,\bar{x}_{\rm b} + [1 \quad 1 \quad -4 \quad 2]\,x_{\rm d} + u_{\rm d,1},$$

$y_{\rm d,1} = x_{\rm d,1,1}$, and finally the dynamics associated with $x_{\rm d,2}$,

$$\dot{x}_{\rm d,2,1} = x_{\rm d,2,2} - y_{\rm d,1}, \quad y_{\rm d,2} = x_{\rm d,2,1},$$

$$\dot{x}_{\rm d,2,2} = x_{\rm d,2,3},$$

$$\dot{x}_{\rm d,2,3} = [4 \quad 3 \quad 2 \quad -1 \quad 0]\,x_0 + [1 \quad 3 \quad -1 \quad 4]\,\bar{x}_{\rm b} + [0 \quad 3 \quad -1 \quad 1]\,x_{\rm d} + u_{\rm d,2}.$$

The subsystems associated with $x_{\rm d}$ are in the desired form, whereas those associated with $\bar{x}_{\rm b}$ are not and the x_0 part needs to be further decomposed.

STEP SCB.5. *Determination of* $x_{\rm b}$.

Following (5.3.72)–(5.3.76) and (5.3.106)–(5.3.109), we obtain the required transformations for $x_{\rm b}$ and $y_{\rm b}$ as follows:

$$x_{\rm b,1,1}=\bar{x}_{\rm b,1,1},\quad y_{\rm b,1}=x_{\rm b,1,1}=\bar{y}_{\rm b,1}, \tag{5.3.110}$$

$$x_{\rm b,1,2}=\bar{x}_{\rm b,1,2}-\bar{x}_{\rm b,1,1}-2\bar{x}_{\rm b,2,1}, \tag{5.3.111}$$

$$x_{\rm b,2,1}=\bar{x}_{\rm b,2,1}-x_{\rm d,2,1},\quad y_{\rm b,2}=x_{\rm b,2,1}=\bar{y}_{\rm b,2}-y_{\rm d,2}, \tag{5.3.112}$$

$$x_{\rm b,2,2}=\bar{x}_{\rm b,2,2}+\bar{x}_{\rm d,2,1}-\bar{x}_{\rm d,2,2}, \tag{5.3.113}$$

which translate into the following state and output transformations,

$$S_{\rm b}=\left[\begin{array}{ccccc|cc|ccc|c|cc}
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&1&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&-1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right],$$

and

$$M_{\rm b}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-1&0&1\end{bmatrix}.$$

The resulting transformed system is $(\tilde{A}_2,\tilde{B}_2,\tilde{C}_2):=(S_{\rm b}\tilde{A}_1S_{\rm b}^{-1},S_{\rm b}\tilde{B}_1,M_{\rm b}\tilde{C}_1S_{\rm b}^{-1})$ with

$$\tilde{A}_2=\left[\begin{array}{ccccc|cccc|cccc}
-6&-3&-2&1&0&-13&-8&-24&-14&4&-9&-10&-5\\
14&9&6&-3&0&25&16&42&28&-6&16&17&8\\
5&4&2&-1&0&5&2&5&5&0&3&2&1\\
14&11&7&-3&0&18&10&24&20&-3&9&9&4\\
23&16&10&-5&0&37&23&58&41&-8&22&23&11\\
\hline
0&0&0&0&0&1&1&2&0&2&2&0&0\\
0&0&0&0&0&1&0&2&0&2&2&0&0\\
0&0&0&0&0&0&0&0&1&-1&-1&0&0\\
0&0&0&0&0&0&0&0&0&-1&-1&0&0\\
\hline
4&3&2&-1&0&8&6&13&8&1&6&4&2\\
0&0&0&0&0&0&0&0&0&-1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
4&3&2&-1&0&4&3&5&4&0&4&3&1
\end{array}\right],$$

$\tilde{B}_2 = \tilde{B}_1$ and $\tilde{C}_2 = \tilde{C}_1$, or in the state-space equations:

$$\dot{x}_0 = A_{00}x_0 + A_{0b}x_b + A_{0d}x_d + B_{0d}u_d + B_{0c}\tilde{u}_c, \qquad (5.3.114)$$

with A_{00}, B_{0d}, B_{0c} being the same as those in the previous step, and

$$A_{0b} = \begin{bmatrix} -13 & -8 & -24 & -14 \\ 25 & 16 & 42 & 28 \\ 5 & 2 & 5 & 5 \\ 18 & 10 & 24 & 20 \\ 37 & 23 & 58 & 41 \end{bmatrix}, \quad A_{0d} = \begin{bmatrix} 4 & -9 & -10 & -5 \\ -6 & 16 & 17 & 8 \\ 0 & 3 & 2 & 1 \\ -3 & 9 & 9 & 4 \\ -8 & 22 & 23 & 11 \end{bmatrix},$$

the dynamics associated with $x_{b,1}$,

$$\dot{x}_{b,1,1} = x_{b,1,2} + y_{b,1} + 2y_{b,2} + 2y_{d,1} + 2y_{d,2}, \quad y_{b,1} = x_{b,1,1},$$
$$\dot{x}_{b,1,2} = y_{b,1} + 2y_{b,2} + 2y_{d,1} + 2y_{d,2},$$

the dynamics associated with $x_{b,2}$,

$$\dot{x}_{b,2,1} = x_{b,2,2} - y_{d,1} - y_{d,2}, \quad y_{b,2} = x_{b,2,1},$$
$$\dot{x}_{b,2,2} = -y_{d,1} - y_{d,2},$$

the dynamics associated with $x_{d,1}$,

$$\dot{x}_{d,1,1} = [4 \;\; 3 \;\; 2 \;\; -1 \;\; 0]\,x_0 + [8 \;\; 6 \;\; 13 \;\; 8]\,\bar{x}_b + [1 \;\; 6 \;\; 4 \;\; 2]\,x_d + u_{d,1},$$

$y_{d,1} = x_{d,1,1}$, and the dynamics associated with $x_{d,2}$,

$$\dot{x}_{d,2,1} = x_{d,2,2} - y_{d,1}, \quad y_{d,2} = x_{d,2,1},$$
$$\dot{x}_{d,2,2} = x_{d,2,3},$$
$$\dot{x}_{d,2,3} = [4 \;\; 3 \;\; 2 \;\; -1 \;\; 0]\,x_0 + [4 \;\; 3 \;\; 5 \;\; 4]\,\bar{x}_b + [0 \;\; 4 \;\; 3 \;\; 1]\,x_d + u_{d,2}.$$

The subsystems associated with x_b are now indeed in the desired form.

STEP SCB.6. *Elimination of u_d from (5.3.114).*

From the dynamical equations of x_d, we obtain

$$u_d = \begin{pmatrix} \dot{x}_{d,1,1} \\ \dot{x}_{d,2,3} \end{pmatrix} - \begin{bmatrix} 4 & 3 & 2 & -1 & 0 \\ 4 & 3 & 2 & -1 & 0 \end{bmatrix} x_0 - \begin{bmatrix} 8 & 6 & 13 & 8 \\ 4 & 3 & 5 & 4 \end{bmatrix} x_b - \begin{bmatrix} 1 & 6 & 4 & 2 \\ 0 & 4 & 3 & 1 \end{bmatrix} x_d.$$

Substituting this into (5.3.114), we obtain

$$\dot{x}_0 = \bar{A}_{00}x_0 + \bar{A}_{0b}x_b + \bar{A}_{0d}x_d + B_{0d}\begin{pmatrix} \dot{x}_{d,1,1} \\ \dot{x}_{d,2,3} \end{pmatrix} + B_{0c}\tilde{u}_c, \qquad (5.3.115)$$

where the coefficient matrices $\bar{A}_{00}$, $\bar{A}_{0b}$ and $\bar{A}_{0d}$ can be calculated in a straightforward manner.

STEP SCB.7. *Elimination of $\dot{x}_{\mathrm{d},1,1}$ and $\dot{x}_{\mathrm{d},2,3}$ from (5.3.115).*

Defining a new state variable,

$$\tilde{x}_0 := x_0 - B_{0\mathrm{d}} \begin{pmatrix} x_{\mathrm{d},1,1} \\ x_{\mathrm{d},2,3} \end{pmatrix},$$

which can be translated into a new state transformation,

$$S_{01} = \begin{bmatrix} I_5 & T_{01} \\ 0 & I_8 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix},$$

we get a new transformed system $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3) := (S_{01}\tilde{A}_2 S_{01}^{-1}, S_{01}\tilde{B}_2, \tilde{C}_2 S_{01}^{-1})$ with $\tilde{C}_3 = \tilde{C}_1$ and

$$\tilde{A}_3 = \left[\begin{array}{ccccc|cccc|cccc} -2 & 0 & 0 & 0 & 0 & -9 & -5 & -19 & -10 & 4 & -5 & -7 & -2 \\ 6 & 3 & 2 & -1 & 0 & 17 & 10 & 32 & 20 & -5 & 8 & 11 & 4 \\ 5 & 4 & 2 & -1 & 0 & 5 & 2 & 5 & 5 & 1 & 3 & 2 & 2 \\ 10 & 8 & 5 & -2 & 0 & 18 & 10 & 27 & 20 & 0 & 7 & 7 & 6 \\ 11 & 7 & 4 & -2 & 0 & 25 & 14 & 43 & 29 & -6 & 10 & 14 & 7 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ \hline 4 & 3 & 2 & -1 & 0 & 8 & 6 & 13 & 8 & 2 & 6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 3 & 2 & -1 & 0 & 4 & 3 & 5 & 4 & 1 & 4 & 3 & 1 \end{array}\right],$$

$$\tilde{B}_3 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

STEP SCB.8. *Elimination of $x_{\mathrm{b},1,2}$, $x_{\mathrm{b},2,2}$, $x_{\mathrm{d},2,2}$ and $x_{\mathrm{d},2,3}$ from the dynamics associated with x_0.*

First, noting the coefficients associated with $x_{\mathrm{b},1,2}$, $x_{\mathrm{b},2,2}$, and $x_{\mathrm{d},2,3}$ in $\tilde{A}_3$, we obtain a state transformation,

$$S_{02}=\begin{bmatrix} I_5 & T_{02} \\ 0 & I_8 \end{bmatrix},\quad T_{02}=\begin{bmatrix} 5 & 0 & 10 & 0 & 0 & 0 & 2 & 0 \\ -10 & 0 & -20 & 0 & 0 & 0 & -4 & 0 \\ -2 & 0 & -5 & 0 & 0 & 0 & -2 & 0 \\ -10 & 0 & -20 & 0 & 0 & 0 & -6 & 0 \\ -14 & 0 & -29 & 0 & 0 & 0 & -7 & 0 \end{bmatrix},$$

and the resulting system $(\tilde{A}_4,\tilde{B}_4,\tilde{C}_4):=(S_{02}\tilde{A}_3S_{02}^{-1}, S_{02}\tilde{B}_3, \tilde{C}_3S_{02}^{-1})$ with

$$\tilde{A}_4=\left[\begin{array}{ccccc|cccc|cccc} -2 & 0 & 0 & 0 & 0 & 6 & 0 & 11 & 0 & 4 & -5 & -3 & 0 \\ 6 & 3 & 2 & -1 & 0 & 1 & 0 & 2 & 0 & -5 & 8 & 9 & 0 \\ 5 & 4 & 2 & -1 & 0 & 12 & 0 & 21 & 0 & 2 & 4 & 6 & 0 \\ 10 & 8 & 5 & -2 & 0 & 28 & 0 & 52 & 0 & 0 & 7 & 17 & 0 \\ 11 & 7 & 4 & -2 & 0 & 14 & 0 & 25 & 0 & -5 & 11 & 16 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ \hline 4 & 3 & 2 & -1 & 0 & 12 & 6 & 23 & 8 & 2 & 6 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 3 & 2 & -1 & 0 & 8 & 3 & 15 & 4 & 1 & 4 & 5 & 1 \end{array}\right],$$

$\tilde{B}_4=\tilde{B}_3$ and $\tilde{C}_4=\tilde{C}_1$. Next, observing the coefficients associated with $x_{\mathrm{d},2,2}$ in $\tilde{A}_4$, we obtain another state transformation,

$$S_{03}=\begin{bmatrix} I_5 & T_{03} \\ 0 & I_8 \end{bmatrix},\quad T_{03}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 \end{bmatrix},$$

and the resulting transformed system $(\tilde{A}_5,\tilde{B}_5,\tilde{C}_5):=(S_{03}\tilde{A}_4S_{03}^{-1}, S_{03}\tilde{B}_4, \tilde{C}_4S_{03}^{-1})$ with $\tilde{B}_5=\tilde{B}_3$, $\tilde{C}_5=\tilde{C}_1$ and

$$\tilde{A}_5=\left[\begin{array}{ccccc|cccc|cccc} -2 & 0 & 0 & 0 & 0 & 6 & 0 & 11 & 0 & 1 & 1 & 0 & 0 \\ 6 & 3 & 2 & -1 & 0 & 1 & 0 & 2 & 0 & 4 & 12 & 0 & 0 \\ 5 & 4 & 2 & -1 & 0 & 12 & 0 & 21 & 0 & 8 & 20 & 0 & 0 \\ 10 & 8 & 5 & -2 & 0 & 28 & 0 & 52 & 0 & 17 & 45 & 0 & 0 \\ 11 & 7 & 4 & -2 & 0 & 14 & 0 & 25 & 0 & 11 & 31 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ \hline 4 & 3 & 2 & -1 & 0 & 12 & 6 & 23 & 8 & 2 & 16 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 3 & 2 & -1 & 0 & 8 & 3 & 15 & 4 & 1 & 14 & 5 & 1 \end{array}\right].$$

STEP SCB.9. *Separation of x_a and x_c.*

Observing from the resulting transformed system $(\tilde{A}_5, \tilde{B}_5, \tilde{C}_5)$, we have

$$\dot{\hat{x}}_0 = \bar{A}_{00}\hat{x}_0 + L_{0b}y_b + L_{0d}y_d + B_{0c}\bar{u}_c$$
$$= \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 6 & 3 & 2 & -1 & 0 \\ 5 & 4 & 2 & -1 & 0 \\ 10 & 8 & 5 & -2 & 0 \\ 11 & 7 & 4 & -2 & 0 \end{bmatrix} \hat{x}_0 + \begin{bmatrix} 6 & 11 \\ 1 & 2 \\ 12 & 21 \\ 28 & 52 \\ 14 & 25 \end{bmatrix} y_b + \begin{bmatrix} 1 & 1 \\ 4 & 12 \\ 8 & 20 \\ 17 & 45 \\ 11 & 31 \end{bmatrix} y_d + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 1 \\ 2 & 0 \end{bmatrix} \bar{u}_c.$$

Next, we apply the result of Theorem 4.4.1 of Chapter 4 to decompose the pair $(\bar{A}_{00}, B_{0c})$ into the CSD form. Noting that $(\bar{A}_{00}, B_{0c})$ is exactly the same as the pair given in Example 4.4.1, it follows from (4.4.8) that the sub-state and input transformations are given as

$$\hat{x}_0 = T_{0s}\begin{pmatrix} x_a \\ x_c \end{pmatrix}, \quad \bar{u}_c = T_{0i}u_c, \tag{5.3.116}$$

where

$$T_{0s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 2 & -5 & 1 & 1 & 3 \\ 0 & -3 & 0 & 1 & 2 \end{bmatrix}, \quad T_{0i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and the resulting transformed subsystem:

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_c \end{pmatrix} = \left[\begin{array}{cc|ccc} -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 5 & -1 & 1 & 1 & 2 \end{array}\right] \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \tilde{L}_{0b}y_b + \tilde{L}_{0d}y_d + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] u_c.$$

The controllability index of $(\bar{A}_{00}, B_{0c})$ is given by $\{r_1, r_2\} = \{1, 2\}$, which shows that there are two subsystems associated with x_c. The details of these subsystems will be given in the last step.

STEP SCB.10. *Finishing touch.*

Putting everything together, we obtain the following nonsingular input, output transformation and state matrices:

$$\Gamma_i = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$
\Gamma_s = \begin{bmatrix}
-1 & -1 & 1 & -1 & 3 & -3 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
-1 & 0 & 5 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\
0 & -1 & -2 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\
0 & -2 & 2 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\
1 & -3 & 7 & 0 & 14 & 1 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\
0 & -2 & 4 & 0 & 9 & 0 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\
1 & -1 & 5 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\
1 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1
\end{bmatrix},
$$

and the final transformed system $(\tilde{A}, \tilde{B}, \tilde{C}) = (\Gamma_s^{-1} A \Gamma_s, \Gamma_s^{-1} B \Gamma_i, \Gamma_o^{-1} C \Gamma_s)$ in the structure of the special coordinate basis with

$$
\tilde{A} = \left[\begin{array}{cc|cccc|ccc|cccc}
-2 & 0 & 6 & 0 & 11 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & -2 & 0 & -5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 10 & 0 & 16 & 0 & 0 & 0 & 1 & 4 & 8 & 0 & 0 \\
5 & -1 & -1 & 0 & -3 & 0 & 1 & 1 & 2 & 5 & 13 & 0 & 0 \\
\hline
2 & -2 & 12 & 6 & 23 & 8 & 1 & 1 & 2 & 2 & 16 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & -2 & 8 & 3 & 15 & 4 & 1 & 1 & 2 & 1 & 14 & 5 & 1
\end{array}\right],
$$

$$
\tilde{B} = \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right],
$$

$$
\tilde{C} = \left[\begin{array}{cc|cccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

To be complete, we can express the transformed system state, input and output variables as follows:

$$x = \Gamma_s \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad x_b = \begin{pmatrix} x_{b,1,1} \\ x_{b,1,2} \\ x_{b,2,1} \\ x_{b,2,2} \end{pmatrix}, \quad x_c = \begin{pmatrix} x_{c,1,1} \\ x_{c,2,1} \\ x_{c,2,2} \end{pmatrix}, \quad x_d = \begin{pmatrix} x_{d,1,1} \\ x_{d,2,1} \\ x_{d,2,2} \\ x_{d,2,3} \end{pmatrix},$$

$$u = \Gamma_i \begin{pmatrix} u_d \\ u_c \end{pmatrix}, \quad u_d = \begin{pmatrix} u_{d,1} \\ u_{d,2} \end{pmatrix}, \quad u_c = \begin{pmatrix} u_{c,1} \\ u_{c,2} \end{pmatrix},$$

and

$$y = \Gamma_o \begin{pmatrix} y_d \\ y_b \end{pmatrix}, \quad y_d = \begin{pmatrix} y_{d,1} \\ y_{d,2} \end{pmatrix}, \quad y_b = \begin{pmatrix} y_{b,1} \\ y_{b,2} \end{pmatrix},$$

together with the dynamical equation associated with x_a, given by

$$\dot{x}_a = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} x_a + \begin{bmatrix} 6 & 11 \\ -2 & -5 \end{bmatrix} y_b + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} y_d, \tag{5.3.117}$$

the dynamical equations associated with x_b, given by

$$\dot{x}_{b,1,1} = x_{b,1,2} + [1 \quad 2]\, y_b + [2 \quad 2]\, y_d, \quad y_{b,1} = x_{b,1,1},$$
$$\dot{x}_{b,1,2} = [1 \quad 2]\, y_b + [2 \quad 2]\, y_d,$$

and

$$\dot{x}_{b,2,1} = x_{b,2,2} - [1 \quad 1]\, y_d, \quad y_{b,2} = x_{b,2,1},$$
$$\dot{x}_{b,2,2} = -[1 \quad 1]\, y_d,$$

the dynamical equations associated with x_c, given by

$$\dot{x}_{c,1,1} = -[1 \quad 2]\, y_b + [1 \quad 1]\, y_d + u_{c,1},$$

and

$$\dot{x}_{c,2,1} = x_{c,2,2} + [10 \quad 16]\, y_b + [4 \quad 8]\, y_d,$$
$$\dot{x}_{c,2,2} = [5 \quad -1]\, x_a + [1 \quad 1 \quad 2]\, x_c - [1 \quad 3]\, y_b + [5 \quad 13]\, y_d + u_{c,2},$$

and the dynamical equations associated with x_d, given by

$$\dot{x}_{d,1,1} = [2 \quad -2]\, x_a + [12 \quad 6 \quad 23 \quad 8]\, x_b + [1 \quad 1 \quad 2]\, x_c + [2 \quad 16 \quad 6 \quad 2]\, x_d + u_{d,1},$$

$$y_{d,1} = x_{d,1,1},$$

and

$$\dot{x}_{\mathrm{d},2,1} = x_{\mathrm{d},2,2} - [1 \quad 0]\, y_{\mathrm{d}}, \quad y_{\mathrm{d},2} = x_{\mathrm{d},2,1},$$
$$\dot{x}_{\mathrm{d},2,2} = x_{\mathrm{d},2,3},$$
$$\dot{x}_{\mathrm{d},2,3} = [2 \quad -2]\, x_{\mathrm{a}} + [8 \quad 3 \quad 15 \quad 14]\, x_{\mathrm{b}} + [1 \quad 1 \quad 2]\, x_{\mathrm{c}} + [1 \quad 14 \quad 5 \quad 1]\, x_{\mathrm{d}} + u_{\mathrm{d},2}.$$

It is clear from the above structural decomposition that the invariant indices of Morse and the invertibility structures as well as infinite zero structure of the given system are as follows:

$$S_{\mathrm{R}}^{\star}(\Sigma) = \mathbf{I}_2 = \{r_1, r_2\} = \{1, 2\}, \quad S_{\mathrm{L}}^{\star}(\Sigma) = \mathbf{I}_3 = \{l_1, l_2\} = \{2, 2\},$$
$$S_{\infty}^{\star}(\Sigma) = \mathbf{I}_4 = \{q_1, q_2\} = \{1, 3\},$$

and $\mathbf{I}_1$ is related to $\lambda(A_{\mathrm{aa}}) = \{-2, 1\}$, the eigenvalues of A_{aa}.

5.4 Nonstrictly Proper Systems

We now present in this section the structural decomposition or the special coordinate basis of general nonstrictly proper multivariable systems. We will also present all the structural properties of such a decomposition with rigorous proofs. To be specific, we consider the following nonstrictly proper system Σ characterized by

$$\begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{5.4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. Without loss of generality, we assume that both $[\,B' \quad D'\,]$ and $[\,C \quad D\,]$ are of full rank.

The structural decomposition or the special coordinate basis of nonstrictly proper systems follows fairly closely from that of strictly proper systems given in Section 5.3. However, in many applications, it is not necessary to decompose the subsystems x_{b} and x_{c} into chains of integrators. On the other hand, in many situations, it is necessary to further separate x_{a}, the subsystem related to the invariant zero dynamics of the given system, into subspaces corresponding to the stable, marginally stable (or marginally unstable) and unstable zero dynamics. We summarize these changes in the following theorem.

Theorem 5.4.1 (SCB). *Given the system Σ of (5.4.1), there exist nonsingular state, output and input transformations Γ_{s}, Γ_{o} and Γ_{i}, which decompose the given Σ into six state subspaces. These state subspaces fully characterize the finite and infinite zero structures as well as the invertibility structures of the system.*

The new state, input and output spaces are described by the following set of equations:

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{5.4.2}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad x_a = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_a^+ \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix}, \tag{5.4.3}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \; y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix}, \; \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \; u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix}, \tag{5.4.4}$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d + L_{ab}^- y_b, \tag{5.4.5}$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 y_0 + L_{ad}^0 y_d + L_{ab}^0 y_b, \tag{5.4.6}$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{ad}^+ y_d + L_{ab}^+ y_b, \tag{5.4.7}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d, \quad y_b = C_b x_b, \tag{5.4.8}$$

$$\begin{aligned} \dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cd} y_d \\ + B_c \left(E_{ca}^- x_a^- + E_{ca}^0 x_a^0 + E_{ca}^+ x_a^+ \right) + B_c u_c, \end{aligned} \tag{5.4.9}$$

$$y_0 = C_{0a}^- x_a^- + C_{0a}^0 x_a^0 + C_{0a}^+ x_a^+ + C_{0b} x_b + C_{0c} x_c + C_{0d} x_d + u_0, \tag{5.4.10}$$

and for each $i = 1, 2, \ldots, m_d$,

$$\begin{aligned} \dot{x}_i = B_{q_i} \left(u_i + E_{ia}^- x_a^- + E_{ia}^0 x_a^0 + E_{ia}^+ x_a^+ + E_{ib} x_b + E_{ic} x_c + E_{id} x_d \right) \\ + A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d, \end{aligned} \tag{5.4.11}$$

$$y_i = C_{q_i} x_i, \quad y_d = C_d x_d. \tag{5.4.12}$$

Here the states x_a^-, x_a^0, x_a^+, x_b, x_c and x_d are respectively of dimensions n_a^-, n_a^0, n_a^+, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, while the state x_i is of dimension q_i for each $i = 1, 2, \ldots, m_d$. The control vectors u_0, u_d and u_c are respectively of dimensions m_0, m_d and $m_c = m - m_0 - m_d$, while the output vectors y_0, y_d and y_b are respectively of dimensions $p_0 = m_0$, $p_d = m_d$ and $p_b = p - p_0 - p_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the form

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1, \; 0, \; \ldots, \; 0]. \tag{5.4.13}$$

Assuming that x_i, $i = 1, 2, \dots, m_\mathrm{d}$, are arranged such that $q_i \le q_{i+1}$, the matrix $L_{i\mathrm{d}}$ has the particular form

$$L_{i\mathrm{d}} = [\,\ell_{i,1} \quad \ell_{i,2} \quad \cdots \quad \ell_{i,i-1} \quad 0 \quad \cdots \quad 0\,], \tag{5.4.14}$$

with the last row being identically zero. Moreover, $\lambda(A_{\mathrm{aa}}^-) \subset \mathbb{C}^-$, $\lambda(A_{\mathrm{aa}}^0) \subset \mathbb{C}^0$ and $\lambda(A_{\mathrm{aa}}^+) \subset \mathbb{C}^+$. Also, $(A_{\mathrm{cc}}, B_\mathrm{c})$ is controllable and $(A_{\mathrm{bb}}, C_\mathrm{b})$ is observable.

Proof. It is simple to verify that there exist nonsingular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.4.15}$$

where m_0 is the rank of matrix D. In fact, U can be chosen as an orthogonal matrix. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right-hand side of (5.4.15). One can now rewrite system Σ of (5.4.1) as,

$$\begin{cases} \dot{x} = A\,x + [B_0 \quad B_1] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \end{cases} \tag{5.4.16}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. Thus, we have

$$\dot{x} = Ax + B_0 u_0 + B_1 u_1, \tag{5.4.17}$$

and

$$y_0 = C_0 x + u_0, \quad y_1 = C_1 x. \tag{5.4.18}$$

Hence, we have

$$\dot{x} = Ax + B_0(y_0 - C_0 x) + B_1 u_1 = (A - B_0 C_0)x + B_1 u_1 + B_0 y_0. \tag{5.4.19}$$

Following the results of Theorem 5.3.1, one can obtain nonsingular state, input and output transformations Γ_s, $\tilde{\Gamma}_\mathrm{i}$ and $\tilde{\Gamma}_\mathrm{o}$ which give a structural decomposition for the strictly proper system

$$\begin{cases} \dot{x} = A_1\,x + B_1\,u_1, \\ y_1 = C_1\,x, \end{cases} \tag{5.4.20}$$

where $A_1 := A - B_0 C_0$. The additional decomposition of x_a into x_a^-, x_a^0 and x_a^+ follows from the result of Theorem 4.2.1 in Chapter 4. Thus, the result of Theorem 5.4.1 follows. ■

For future use, we rewrite the structural decomposition of Σ in a more compact form:

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = A_s + B_0 C_0$$
$$= \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}$$
$$+ \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} [\, C_{0a}^- \quad C_{0a}^0 \quad C_{0a}^+ \quad C_{0b} \quad C_{0c} \quad C_{0d} \,], \tag{5.4.21}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = [\, B_0 \quad B_s \,] = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{5.4.22}$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_0 \\ C_s \end{bmatrix} = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \tag{5.4.23}$$

$$\tilde{D} = \Gamma_o^{-1} D \Gamma_i = D_s = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.4.24}$$

where

$$A_{dd} = A_{dd}^* + B_d E_{dd} + L_{dd} C_d, \tag{5.4.25}$$

for some constant matrices L_{dd} and E_{dd} of appropriate dimensions, and

$$A_{dd}^* = \text{blkdiag}\Big\{ A_{q_1}, A_{q_2}, \dots, A_{q_{m_d}} \Big\}, \tag{5.4.26}$$

$$B_d = \text{blkdiag}\Big\{ B_{q_1}, B_{q_2}, \dots, B_{q_{m_d}} \Big\}, \quad C_d = \text{blkdiag}\Big\{ C_{q_1}, C_{q_2}, \dots, C_{q_{m_d}} \Big\}, \tag{5.4.27}$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ being defined earlier in (5.4.13).

We are now ready to present the important properties of the above structural decomposition. For clarity in the presentation, the detailed proofs of these properties will be given in the next section.

Property 5.4.1. *The given system Σ is observable (detectable) if and only if the pair (A_{obs}, C_{obs}) is observable (detectable), where*

$$A_{obs} := \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad C_{obs} := \begin{bmatrix} C_{0a} & C_{0c} \\ E_{da} & E_{dc} \end{bmatrix}, \tag{5.4.28}$$

and where

$$A_{aa} := \begin{bmatrix} A_{aa}^- & 0 & 0 \\ 0 & A_{aa}^0 & 0 \\ 0 & 0 & A_{aa}^+ \end{bmatrix}, \quad C_{0a} := [\, C_{0a}^- \quad C_{0a}^0 \quad C_{0a}^+ \,], \tag{5.4.29}$$

$$E_{da} := [\, E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \,], \quad E_{ca} := [\, E_{ca}^- \quad E_{ca}^0 \quad E_{ca}^+ \,]. \tag{5.4.30}$$

Also, define

$$A_{con} := \begin{bmatrix} A_{aa} & L_{ab} C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} := \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix}, \tag{5.4.31}$$

$$B_{0a} := \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} := \begin{bmatrix} L_{ab}^- \\ L_{ab}^0 \\ L_{ab}^+ \end{bmatrix}, \quad L_{ad} := \begin{bmatrix} L_{ad}^- \\ L_{ad}^0 \\ L_{ad}^+ \end{bmatrix}. \tag{5.4.32}$$

Similarly, Σ is controllable (stabilizable) if and only if the pair (A_{con}, B_{con}) is controllable (stabilizable).

Property 5.4.2. *The structural decomposition also shows explicitly the invariant zeros and the normal rank of Σ. To be more specific, we have the following properties:*

1. *The normal rank of $H(s)$ is equal to $m_0 + m_d$.*

2. *Invariant zeros of Σ are the eigenvalues of A_{aa}, which are the unions of the eigenvalues of A_{aa}^-, A_{aa}^0 and A_{aa}^+.*

Obviously, Σ is of minimum phase if and only if $n_a^0 + n_a^+ = 0$. Otherwise, it is of nonminimum phase.

In order to display various multiplicities of invariant zeros, let X_a be a nonsingular transformation matrix such that A_{aa} can be transformed into the Jordan canonical form of (2.3.39), i.e.,

$$X_a^{-1} A_{aa} X_a = J = \text{blkdiag}\left\{ J_1, J_2, \ldots, J_k \right\}, \tag{5.4.33}$$

where J_i, $i = 1, 2, \ldots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\left\{\alpha_i, \alpha_i, \ldots, \alpha_i\right\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}. \tag{5.4.34}$$

For any given $\alpha \in \lambda(A_{\text{aa}})$, let there be τ_α Jordan blocks of A_{aa} associated with α. Let $n_{\alpha,1}$, $n_{\alpha,2}$, ..., n_{α,τ_α} be the dimensions of the corresponding Jordan blocks. Then we say α is an invariant zero of Σ_* with multiplicity structure $S_\alpha^\star(\Sigma_*)$ (see also [115]),

$$S_\alpha^\star(\Sigma_*) = \left\{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\right\}. \tag{5.4.35}$$

The *geometric multiplicity* of α is then simply given by τ_α, and the *algebraic multiplicity* of α is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$.

The following then characterizes the property of the blocking zeros of Σ (see also Chen *et al.* [28]).

Property 5.4.3. *Assume that the given system Σ of (5.4.1) is controllable and observable. Then, a complex scalar α is a blocking zero of Σ, i.e., $H(\alpha) \equiv 0$, if and only if α is an invariant zero of Σ with a geometric multiplicity $\tau_\alpha = m_0 + m_{\text{d}}$, the normal rank of $H(s)$.*

The structural decomposition can also reveal the infinite zero structure of Σ as defined in Chapter 3. The following property pinpoints this.

Property 5.4.4. *Σ has $m_0 = \text{rank}(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ is given by*

$$S_\infty^\star(\Sigma) = \left\{q_1, q_2, \ldots, q_{m_d}\right\}. \tag{5.4.36}$$

That is, each q_i corresponds to an infinite zero of order q_i. In particular, for a strictly proper SISO system Σ, we have $S_\infty^\star(\Sigma) = \{q_1\}$, where q_1 is the relative degree *of Σ. The given system Σ is said to be of* uniform rank *if either $m_0 = 0$ and $q_1 = q_2 = \cdots = q_{m_d}$, or $m_0 \neq 0$ and $S_\infty^\star(\Sigma) = \emptyset$.*

The special coordinate basis exhibits the invertibility structure of a given system Σ in a simple fashion.

Property 5.4.5. *The given system Σ is right invertible if and only if x_{b} (and hence y_{b}) are nonexistent, left invertible if and only if x_{c} (and hence u_{c}) are nonexistent, and invertible if and only if both x_{b} and x_{c} are nonexistent. Moreover, Σ is degenerate if and only if both x_{b} and x_{c} are present.*

The structural decomposition decomposes the state space of Σ into several distinct parts. In fact, the state space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_{\mathrm{a}}^{-} \oplus \mathcal{X}_{\mathrm{a}}^{0} \oplus \mathcal{X}_{\mathrm{a}}^{+} \oplus \mathcal{X}_{\mathrm{b}} \oplus \mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}. \tag{5.4.37}$$

Here $\mathcal{X}_{\mathrm{a}}^{-}$ is related to the stable invariant zeros, *i.e.*, the eigenvalues of A_{aa}^{-} are the stable invariant zeros of Σ. Similarly, $\mathcal{X}_{\mathrm{a}}^{0}$ and $\mathcal{X}_{\mathrm{a}}^{+}$ are respectively related to the invariant zeros of Σ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_{\mathrm{b}}$ is related to the right invertibility, *i.e.*, the system is right invertible if and only if $\mathcal{X}_{\mathrm{b}} = \{0\}$, while $\mathcal{X}_{\mathrm{c}}$ is related to left invertibility, *i.e.*, the system is left invertible if and only if $\mathcal{X}_{\mathrm{c}} = \{0\}$. Finally, $\mathcal{X}_{\mathrm{d}}$ is related to zeros of Σ at infinity.

There are interconnections between the subsystems generated by the structural decomposition and various invariant geometric subspaces. The following properties show these interconnections.

Property 5.4.6. *The geometric subspaces defined in Definitions 3.7.2 and 3.7.4 are given by:*

1. $\mathcal{X}_{\mathrm{a}}^{-} \oplus \mathcal{X}_{\mathrm{a}}^{0} \oplus \mathcal{X}_{\mathrm{c}}$ *spans* $\mathcal{V}^{-}(\Sigma)$.
2. $\mathcal{X}_{\mathrm{a}}^{+} \oplus \mathcal{X}_{\mathrm{c}}$ *spans* $\mathcal{V}^{+}(\Sigma)$.
3. $\mathcal{X}_{\mathrm{a}}^{-} \oplus \mathcal{X}_{\mathrm{a}}^{0} \oplus \mathcal{X}_{\mathrm{a}}^{+} \oplus \mathcal{X}_{\mathrm{c}}$ *spans* $\mathcal{V}^{*}(\Sigma)$.
4. $\mathcal{X}_{\mathrm{a}}^{+} \oplus \mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}$ *spans* $\mathcal{S}^{-}(\Sigma)$.
5. $\mathcal{X}_{\mathrm{a}}^{-} \oplus \mathcal{X}_{\mathrm{a}}^{0} \oplus \mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}$ *spans* $\mathcal{S}^{+}(\Sigma)$.
6. $\mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}$ *spans* $\mathcal{S}^{*}(\Sigma)$.
7. $\mathcal{X}_{\mathrm{c}}$ *spans* $\mathcal{R}^{*}(\Sigma)$.
8. $\mathcal{X}_{\mathrm{a}}^{-} \oplus \mathcal{X}_{\mathrm{a}}^{0} \oplus \mathcal{X}_{\mathrm{a}}^{+} \oplus \mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}$ *spans* $\mathcal{N}^{*}(\Sigma)$.

Property 5.4.7. *The geometric subspaces defined in Definition 3.7.5, i.e.,* $\mathcal{S}_{\lambda}(\Sigma)$ *and* $\mathcal{V}_{\lambda}(\Sigma)$*, can be computed as follows:*

$$\mathcal{S}_{\lambda}(\Sigma) = \operatorname{im}\left\{\Gamma_{\mathrm{s}}\begin{bmatrix} \lambda I - A_{\mathrm{aa}} & 0 & 0 & 0 \\ 0 & Y_{\mathrm{b}\lambda} & 0 & 0 \\ 0 & 0 & I_{n_{\mathrm{c}}} & 0 \\ 0 & 0 & 0 & I_{n_{\mathrm{d}}} \end{bmatrix}\right\}, \tag{5.4.38}$$

where

$$\operatorname{im}\{Y_{b\lambda}\} = \ker\left[C_b(A_{bb} + K_bC_b - \lambda I)^{-1}\right], \tag{5.4.39}$$

and where K_b is any matrix of appropriate dimensions and subject to the constraint that $A_{bb} + K_bC_b$ has no eigenvalue at λ. We note that such a K_b always exists as (A_{bb}, C_b) is observable.

$$\mathcal{V}_\lambda(\Sigma) = \operatorname{im}\left\{\Gamma_s \begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\}, \tag{5.4.40}$$

where $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace,

$$\left\{\zeta_a \in \mathbb{C}^{n_a} \,\middle|\, (\lambda I - A_{aa})\zeta_a = 0\right\}, \tag{5.4.41}$$

and

$$X_{c\lambda} := \left(A_{cc} + B_cF_c - \lambda I\right)^{-1} B_c, \tag{5.4.42}$$

with F_c being any matrix of appropriate dimensions and subject to the constraint that $A_{cc} + B_cF_c$ has no eigenvalue at λ. Again, we note that the existence of such an F_c is guaranteed by the controllability of (A_{cc}, B_c). Clearly, if $\lambda \notin \lambda(A_{aa})$, we have $\mathcal{V}_\lambda(\Sigma) \subseteq \mathcal{V}^\times(\Sigma)$ and $\mathcal{S}_\lambda(\Sigma) \supseteq \mathcal{S}^\times(\Sigma)$.

We illustrate the above structural properties in the following example.

Example 5.4.1. Let us reconsider the system Σ of Example 5.3.1, *i.e.*, consider a matrix quadruple (A, B, C, D) with (A, B, C) being the same as those given in Example 5.3.1 and $D = 0$. All the necessary transformations required to transform the given system into the special coordinate basis have already been obtained in Example 5.3.1. For the computation of various geometric subspaces, we need to further decompose the subsystem associated with x_a, *i.e.*, (5.3.117), using the result of Theorem 4.2.1. In particular, the following sub-transformation on x_a,

$$x_a = \begin{bmatrix} 0.9487 & 0 \\ -0.3162 & 1 \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^+ \end{pmatrix},$$

will transform the dynamics of x_a into the diagonal form, *i.e.*,

$$\begin{pmatrix} \dot{x}_a^- \\ \dot{x}_a^+ \end{pmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^+ \end{pmatrix} + \begin{bmatrix} 6.3246 & 11.5950 \\ 0 & -1.3333 \end{bmatrix} y_b + \begin{bmatrix} 1.0541 & 1.0541 \\ 1.3333 & 1.3333 \end{bmatrix} y_d.$$

It is straightforward to verify that Σ is neither controllable nor observable. It has two uncontrollable modes at -0.618 and 1.618, respectively, and has one unobservable mode at 0. The given system, Σ, has a normal rank equal to 2, and has one stable invariant zero at -2 and one unstable invariant zero at 1. It has an infinite zero structure,

$$S_\infty^\star(\Sigma) = \{1, 3\}.$$

The system is neither right nor left invertible as both x_b and x_c are present in its structural decomposition, *i.e.*, it is a degenerate system. The various geometric subspaces of Σ can be trivially obtained through our structural decomposition and they are given as:

$$\mathcal{V}^-(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 6 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}, \quad \mathcal{V}^+(\Sigma) = \text{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathcal{V}^*(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 6 & 3 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathcal{S}^-(\Sigma) = \text{im}\left\{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ -3 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ -2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ -1 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\},$$

$$\mathcal{S}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\},$$

$$\mathcal{S}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\},$$

$$\mathcal{R}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathcal{N}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & -2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & -3 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & -2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & -1 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\},$$

$$\mathcal{V}_\lambda(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -4 & -3 \\ 2 & -2 & 1 \\ 6 & -2 & -4 \\ 2 & 2 & -1 \\ 4 & 0 & -5 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}, \quad \lambda = -2,$$

$$\mathcal{S}_\lambda(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 10 & 10 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & -5 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & 5 & 5 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & 15 & 14 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & 9 & 9 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & 10 & 10 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 7 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\}, \quad \lambda = 1.$$

We note that $\lambda = -2$ and $\lambda = 1$ correspond respectively to the stable and the unstable invariant zeros of Σ.

5.5 Proofs of Properties of Structural Decomposition

In what follows, we provide rigorous proofs for all the properties of the special coordinate basis of general nonstrictly proper systems given in Section 5.4. Without loss of generality, but for simplicity of presentation, we assume throughout the rest of this section that the given system Σ has already been transformed into the form of Theorem 5.4.1 or into the compact form of (5.4.21) to (5.4.24), *i.e.*,

$$A = \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} & 0 & L_{\rm ad}C_{\rm d} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ B_{\rm c}E_{\rm ca} & L_{\rm cb}C_{\rm b} & A_{\rm cc} & L_{\rm cd}C_{\rm d} \\ B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & B_{\rm d}E_{\rm dc} & A_{\rm dd}^* + B_{\rm d}E_{\rm dd} + L_{\rm dd}C_{\rm d} \end{bmatrix} + B_0C_0,$$

$$B = [\,B_0 \quad B_1\,] = \begin{bmatrix} B_{\rm 0a} & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We further note that A^*_{dd}, B_d and C_d have the forms

$$A^*_{dd} = \text{blkdiag}\left\{A_{q_1}, A_{q_2}, \dots, A_{q_{m_d}}\right\},$$

and

$$B_d = \text{blkdiag}\left\{B_{q_1}, B_{q_2}, \dots, B_{q_{m_d}}\right\}, \quad C_d = \text{blkdiag}\left\{C_{q_1}, C_{q_2}, \dots, C_{q_{m_d}}\right\},$$

where A_{q_i}, B_{q_i} and C_{q_i}, $i = 1, 2, \dots, m_d$, are defined as in (5.4.13).

Proof of Property 5.4.1. Let us define a state feedback gain matrix F as

$$F = -\begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ E_{da} & E_{db} & E_{dc} & E_{dd} \\ E_{ca} & 0 & 0 & 0 \end{bmatrix}. \tag{5.5.1}$$

Then, we have

$$A + BF = \begin{bmatrix} A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ 0 & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ 0 & 0 & 0 & A^*_{dd} + L_{dd}C_d \end{bmatrix}.$$

Noting that (A_{cc}, B_c) is controllable, we have for any $\lambda \in \mathbb{C}$,

$$\text{rank}\begin{bmatrix} A + BF - \lambda I & B \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{aa}-\lambda I & L_{ab}C_b & 0 & L_{ad}C_d & B_{0a} & 0 & 0 \\ 0 & A_{bb}-\lambda I & 0 & L_{bd}C_d & B_{0b} & 0 & 0 \\ 0 & L_{cb}C_b & A_{cc}-\lambda I & L_{cd}C_d & B_{0c} & 0 & B_c \\ 0 & 0 & 0 & A^*_{dd}+L_{dd}C_d-\lambda I & B_{0d} & B_d & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{aa}-\lambda I & L_{ab}C_b & 0 & L_{ad}C_d & B_{0a} & 0 & 0 \\ 0 & A_{bb}-\lambda I & 0 & L_{bd}C_d & B_{0b} & 0 & 0 \\ 0 & 0 & A_{cc}-\lambda I & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & A^*_{dd}+L_{dd}C_d-\lambda I & B_{0d} & B_d & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{con}-\lambda I & 0 & B_{con1}C_d & B_{con0} & 0 & 0 \\ 0 & A_{cc}-\lambda I & 0 & 0 & 0 & B_c \\ 0 & 0 & A^*_{dd}+L_{dd}C_d-\lambda I & B_{0d} & B_d & 0 \end{bmatrix},$$

where

$$A_{\mathrm{con}} = \begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} \\ 0 & A_{\mathrm{bb}} \end{bmatrix}, \quad B_{\mathrm{con}} = [\, B_{\mathrm{con0}} \quad B_{\mathrm{con1}} \,] = \begin{bmatrix} B_{0\mathrm{a}} & L_{\mathrm{ad}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} \end{bmatrix}.$$

Also, noting the special structure of $(A_{\mathrm{dd}}^*, B_{\mathrm{d}}, C_{\mathrm{d}})$, it is straightforward to verify that $[\, A+BF-\lambda I \quad B \,]$ is of maximal rank if and only if $[\, A_{\mathrm{con}}-\lambda I \quad B_{\mathrm{con}} \,]$ is of maximal rank. By Lemma 3.8.1, we have that (A, B) is controllable (stabilizable) if and only if $(A_{\mathrm{con}}, B_{\mathrm{con}})$ is controllable (stabilizable).

Similarly, one can show that (A, C) is observable (detectable) if and only if $(A_{\mathrm{obs}}, C_{\mathrm{obs}})$ is observable (detectable). ■

Proof of Property 5.4.2. Let us define a state feedback gain matrix F as in (5.5.1) and an output injection gain matrix K as

$$K = -\begin{bmatrix} B_{0\mathrm{a}} & L_{\mathrm{ad}} & L_{\mathrm{ab}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} & 0 \\ B_{0\mathrm{c}} & L_{\mathrm{cd}} & L_{\mathrm{cb}} \\ B_{0\mathrm{d}} & L_{\mathrm{dd}} & 0 \end{bmatrix}. \tag{5.5.2}$$

We have

$$\breve{A} = A + BF + KC + KDF = \begin{bmatrix} A_{\mathrm{aa}} & 0 & 0 & 0 \\ 0 & A_{\mathrm{bb}} & 0 & 0 \\ 0 & 0 & A_{\mathrm{cc}} & 0 \\ 0 & 0 & 0 & A_{\mathrm{dd}}^* \end{bmatrix},$$

$$\breve{B} = B + KD = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{\mathrm{c}} \\ 0 & B_{\mathrm{d}} & 0 \end{bmatrix},$$

$$\breve{C} = C + DF = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\mathrm{d}} \\ 0 & C_{\mathrm{b}} & 0 & 0 \end{bmatrix},$$

and

$$\breve{D} = D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\breve{\Sigma}$ be characterized by the quadruple $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$. It is simple to verify that the transfer function of $\breve{\Sigma}$ is given by

$$\breve{H}(s) = \breve{C}(sI - \breve{A})^{-1}\breve{B} + \breve{D} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & C_{\mathrm{d}}(sI - A_{\mathrm{dd}}^*)^{-1}B_{\mathrm{d}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.5.3}$$

Furthermore, we can show that

$$C_{\rm d}(sI - A_{\rm dd}^*)^{-1}B_{\rm d} = \begin{bmatrix} \frac{1}{s^{q_1}} & & & \\ & \frac{1}{s^{q_2}} & & \\ & & \ddots & \\ & & & \frac{1}{s^{q_{m_d}}} \end{bmatrix}. \tag{5.5.4}$$

By Lemmas 3.8.1 and 3.8.2, we have

$$\text{normrank}\{H(s)\} = \text{normrank}\{\check{H}(s)\} = m_0 + m_{\rm d}.$$

Next, it follows from Lemmas 3.8.1 and 3.8.2 that the invariant zeros of Σ and $\check{\Sigma}$ are equivalent. By the definition of the invariant zeros of a linear system, *i.e.*, a complex scalar α is an invariant zero of $\check{\Sigma}$ if

$$\text{rank}\begin{bmatrix} \check{A} - \alpha I & \check{B} \\ \check{C} & \check{D} \end{bmatrix} < n + \text{normrank}\{\check{H}(s)\} = n + m_0 + m_{\rm d},$$

and also noting the special structure of $(A_{\rm dd}^*, B_{\rm d}, C_{\rm d})$ and the facts that $(A_{\rm bb}, C_{\rm b})$ is observable, and $(A_{\rm cc}, B_{\rm c})$ is controllable, we have

$$\begin{aligned} \text{rank}\{P_{\check{\Sigma}}(\alpha)\} &= \text{rank}\begin{bmatrix} \check{A} - \alpha I & \check{B} \\ \check{C} & \check{D} \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} A_{\rm aa} - \alpha I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{\rm bb} - \alpha I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\rm cc} - \alpha I & 0 & 0 & 0 & B_{\rm c} \\ 0 & 0 & 0 & A_{\rm dd}^* - \alpha I & 0 & B_{\rm d} & 0 \\ 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & C_{\rm d} & 0 & 0 & 0 \\ 0 & C_{\rm b} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= n_{\rm b} + n_{\rm c} + n_{\rm d} + m_0 + m_{\rm d} + \text{rank}\{A_{\rm aa} - \alpha I\}. \end{aligned}$$

Clearly, the rank of $P_{\check{\Sigma}}(\alpha)$ drops below $n + m_0 + m_{\rm d}$ if and only if $\alpha \in \lambda(A_{\rm aa})$. Hence, the invariant zeros of $\check{\Sigma}$, or equivalently the invariant zeros of Σ, are given by the eigenvalues of $A_{\rm aa}$, which are the union of $\lambda(A_{\rm aa}^-)$, $\lambda(A_{\rm aa}^0)$, and $\lambda(A_{\rm aa}^+)$. This completes the proof of Property 5.4.2. ∎

Proof of Property 5.4.3. By definition, α being a blocking zero of Σ implies that $P(\alpha) = C(\alpha I_n - A)^{-1}B + D = 0$. Let us define

$$\begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_m \end{bmatrix} = I_m$$

and

$$z_i = (\alpha I_n - A)^{-1} B \omega_i, \quad i = 1, 2, \dots, m.$$

Now it is trivial to see that $(z_i', \omega_i')'$, $i = 1, 2, \dots, m$, are linearly independent and satisfy

$$\begin{bmatrix} \alpha I_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z_i \\ \omega_i \end{bmatrix} = 0.$$

Thus, α is an invariant zero of Σ with its geometric multiplicity τ_α satisfying the condition, $\tau_\alpha \geq m_0 + m_\mathrm{d}$. But if $\tau_\alpha > m_0 + m_\mathrm{d}$, it can easily be shown that Σ is neither controllable nor observable, which is a contradiction to the assumption that Σ is controllable and observable. Hence, $\tau_\alpha = m_0 + m_\mathrm{d}$.

To prove the sufficiency, we consider the following. If $\tau_\alpha = m_0 + m_\mathrm{d}$, then it is straightforward to verify that there must exist z_i and ω_i, $i = 1, 2, \dots, m - m_0 - m_\mathrm{d} + \tau_\alpha = m$, such that

$$\begin{bmatrix} \alpha I_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z_i \\ \omega_i \end{bmatrix} = 0,$$

where z_i, $i = 1, 2, \dots, m$, are linearly independent. In what follows, we will show that w_i, $i = 1, 2, \dots, m$, are also linearly independent. First assume that ω_i, $i = 1, 2, \dots, m$, are linearly dependent. Then there exist constants c_i, $i = 1, 2, \dots, m$, such that

$$z_\mathrm{o} = \sum_{i=1}^{m} c_i z_i \neq 0 \quad \text{and} \quad \omega_\mathrm{o} = \sum_{i=1}^{m} c_i \omega_i = 0.$$

This implies that

$$(\alpha I_n - A) z_\mathrm{o} = B \omega_\mathrm{o} = 0 \quad \text{and} \quad C z_\mathrm{o} + D \omega_\mathrm{o} = C z_\mathrm{o} = 0.$$

Hence, α being an output decoupling zero of Σ contradicts the assumption that Σ is controllable and observable. This shows that ω_i, $i = 1, 2, \dots, m$, are linearly independent. We next consider,

$$P(\alpha) [\, \omega_1 \quad \omega_2 \quad \cdots \quad \omega_m \,] = [C(\alpha I_n - A)^{-1} B + D] [\, \omega_1 \quad \omega_2 \quad \cdots \quad \omega_m \,] = 0,$$

which implies that $P(\alpha) = 0$. Thus, α is a blocking zero of Σ. ■

Proof of Property 5.4.4. It follows from Lemmas 3.8.1 and 3.8.2 that the infinite zeros of Σ and $\check{\Sigma}$ are equivalent. It is clearly seen from (5.5.3) and (5.5.4) that the infinite zeros of $\check{\Sigma}$, or equivalently the infinite zeros of Σ, of order higher than 0, are given by

$$S_\infty^\star(\Sigma) = S_\infty^\star(\check{\Sigma}) = \left\{ q_1, q_2, \dots, q_{m_\mathrm{d}} \right\}.$$

Furthermore, $\check{\Sigma}$ or Σ has m_0 infinite zeros of order 0. ■

Proof of Property 5.4.5. Again, it follows from Lemmas 3.8.1 and 3.8.2 that Σ or $H(s)$ is (left or right or non) invertible if and only if $\check{\Sigma}$ or $\check{H}(s)$ is (left or right or non) invertible. The results of Property 5.4.5 can be seen from the transfer function $\check{H}(s)$ in (5.5.3). ■

Proof of Property 5.4.6. We will only prove the property of the geometric subspace $\mathcal{V}^*(\Sigma)$, i.e.,

$$\mathcal{V}^*(\Sigma) = \mathcal{X}_a \oplus \mathcal{X}_c = \text{im}\left\{ \Gamma_s \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\}.$$

Here $\Gamma_s = I_n$ as the given system Σ is assumed to be already in the form of structural decomposition. It follows from Lemma 3.8.2 that $\mathcal{V}^*$ is invariant under any output injection laws. Let us choose an output injection gain matrix K as in (5.5.2). Then, we have

$$\hat{A} = A + KC = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ B_c E_{ca} & 0 & A_{cc} & 0 \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd}^* + B_d E_{dd} \end{bmatrix},$$

and

$$\hat{B} = B + KD = \check{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \end{bmatrix}.$$

Let $\hat{\Sigma}$ be a system characterized by $(\hat{A}, \hat{B}, C, D)$. Then, it is sufficient to show the property of $\mathcal{V}^*(\Sigma)$ by showing that

$$\mathcal{V}^*(\hat{\Sigma}) = \text{im}\left\{ \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\}.$$

First, let us choose a matrix F as given in (5.5.1). Then, we have

$$\hat{A} + \hat{B}F = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ 0 & 0 & A_{cc} & 0 \\ 0 & 0 & 0 & A_{dd}^* \end{bmatrix},$$

$$C + DF = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & C_{\rm b} & 0 & 0 \end{bmatrix}.$$

It is now simple to see that for any

$$\zeta \in \mathcal{X}_{\rm a} \oplus \mathcal{X}_{\rm c} = \operatorname{im}\left\{ \begin{bmatrix} I_{n_{\rm a}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\rm c}} \\ 0 & 0 \end{bmatrix} \right\},$$

we have

$$\zeta = \begin{pmatrix} \zeta_{\rm a} \\ 0 \\ \zeta_{\rm c} \\ 0 \end{pmatrix},$$

and

$$(\hat{A} + \hat{B}F)\zeta = \begin{pmatrix} A_{\rm aa}\zeta_{\rm a} \\ 0 \\ A_{\rm cc}\zeta_{\rm c} \\ 0 \end{pmatrix} \in \operatorname{im}\left\{ \begin{bmatrix} I_{n_{\rm a}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\rm c}} \\ 0 & 0 \end{bmatrix} \right\} = \mathcal{X}_{\rm a} \oplus \mathcal{X}_{\rm c},$$

and

$$(C + DF)\zeta = 0.$$

Clearly, $\mathcal{X}_{\rm a} \oplus \mathcal{X}_{\rm c}$ is an $(\hat{A} + \hat{B}F)$-invariant subspace of $\mathbb{R}^n$ and is contained in $\ker(C + DF)$. By the definition of $\mathcal{V}^*$, we have

$$\mathcal{X}_{\rm a} \oplus \mathcal{X}_{\rm c} \subseteq \mathcal{V}^*(\hat{\Sigma}). \tag{5.5.5}$$

Conversely, for any $\zeta \in \mathcal{V}^*(\hat{\Sigma})$, by Definition 3.7.2, there exists a gain matrix $\hat{F} \in \mathbb{R}^{m \times n}$ such that

$$(\hat{A} + \hat{B}\hat{F})\zeta \in \mathcal{V}^*(\hat{\Sigma}), \tag{5.5.6}$$

and

$$(C + D\hat{F})\zeta = 0. \tag{5.5.7}$$

(5.5.6) and (5.5.7) imply that for any $\zeta \in \mathcal{V}^*(\hat{\Sigma})$,

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k\zeta = 0, \quad k = 0, 1, \ldots, n-1. \tag{5.5.8}$$

Thus, (5.5.5) and (5.5.8) imply that

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k \begin{bmatrix} I_{n_{\rm a}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\rm c}} \\ 0 & 0 \end{bmatrix} = 0, \quad k = 0, 1, \ldots, n-1. \tag{5.5.9}$$

Next, let us partition this $\hat{F}$ as follows:

$$\hat{F} = \begin{bmatrix} F_{a0} - C_{0a} & F_{b0} - C_{0b} & F_{c0} - C_{0c} & F_{d0} - C_{0d} \\ F_{ad} - E_{da} & F_{bd} - E_{db} & F_{cd} - E_{dc} & F_{dd} - E_{dd} \\ F_{ac} - E_{ca} & F_{bc} & F_{cc} & F_{dc} \end{bmatrix}.$$

We have

$$C + D\hat{F} = \begin{bmatrix} F_{a0} & F_{b0} & F_{c0} & F_{d0} \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix},$$

and

$$\hat{A}+\hat{B}\hat{F} = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ B_c F_{ac} & B_c F_{bc} & A_{cc}+B_c F_{cc} & B_c F_{dc} \\ B_d F_{ad} & B_d F_{bd} & B_d F_{cd} & A_{dd}^{**} \end{bmatrix},$$

where $A_{dd}^{**} = A_{dd}^{*} + B_d F_{dd}$. Then, using (5.5.9) with $k = 0$, we have

$$(C + D\hat{F}) \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} = 0,$$

which implies

$$F_{a0} = 0, \quad F_{c0} = 0,$$

and

$$C + D\hat{F} = \begin{bmatrix} 0 & \star & 0 & \star \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \tag{5.5.10}$$

where symbols $\star$ represent some matrices of less interest. Using (5.5.9) with $k = 1$ together with (5.5.10), we have

$$C_d B_d F_{ad} = 0, \quad C_d B_d F_{cd} = 0,$$

and

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F}) = \begin{bmatrix} 0 & \star & 0 & \star \\ 0 & C_d B_d F_{bd} & 0 & C_d A_{dd}^{**} \\ 0 & C_b A_{bb} & 0 & 0 \end{bmatrix}. \tag{5.5.11}$$

In general, one can show that for any positive integer k,

$$C_d (A_{dd}^{**})^{k-1} B_d F_{ad} = 0, \quad C_d (A_{dd}^{**})^{k-1} B_d F_{cd} = 0, \tag{5.5.12}$$

and

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k = \begin{bmatrix} 0 & \star & 0 & \star \\ 0 & \star & 0 & C_{\mathrm{d}}(A_{\mathrm{dd}}^{**})^k \\ 0 & C_{\mathrm{b}}(A_{\mathrm{bb}})^k & 0 & 0 \end{bmatrix}. \tag{5.5.13}$$

As a by-product, we can easily show that $F_{\mathrm{ad}} = 0$ and $F_{\mathrm{cd}} = 0$, because of the fact that $(A_{\mathrm{dd}}^{**}, B_{\mathrm{d}}, C_{\mathrm{d}})$ is controllable, observable, invertible and is free of invariant zeros. Now, for any

$$\zeta = \begin{pmatrix} \zeta_{\mathrm{a}} \\ \zeta_{\mathrm{b}} \\ \zeta_{\mathrm{c}} \\ \zeta_{\mathrm{d}} \end{pmatrix} \in \mathcal{V}^*(\hat{\Sigma}),$$

it follows from (5.5.8) and (5.5.13) that

$$C_{\mathrm{b}}(A_{\mathrm{bb}})^k \zeta_{\mathrm{b}} = 0, \quad k = 0, 1, \dots, n-1,$$

which implies $\zeta_{\mathrm{b}} = 0$ because $(A_{\mathrm{bb}}, C_{\mathrm{b}})$ is observable, and

$$C_{\mathrm{d}}(A_{\mathrm{dd}}^{**})^k \zeta_{\mathrm{d}} + \star \cdot \zeta_{\mathrm{b}} = C_{\mathrm{d}}(A_{\mathrm{dd}}^{**})^k \zeta_{\mathrm{d}} = 0, \quad k = 0, 1, \dots, n-1,$$

which implies $\zeta_{\mathrm{d}} = 0$ because $(A_{\mathrm{dd}}^{**}, C_{\mathrm{d}})$ is also observable. Hence,

$$\zeta = \begin{pmatrix} \zeta_{\mathrm{a}} \\ 0 \\ \zeta_{\mathrm{c}} \\ 0 \end{pmatrix} \in \operatorname{im} \left\{ \begin{bmatrix} I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix} \right\} = \mathcal{X}_{\mathrm{a}} \oplus \mathcal{X}_{\mathrm{c}},$$

and

$$\mathcal{V}^*(\hat{\Sigma}) \subseteq \mathcal{X}_{\mathrm{a}} \oplus \mathcal{X}_{\mathrm{c}}. \tag{5.5.14}$$

Obviously, (5.5.5) and (5.5.14) imply the result.

Similarly, one can follow the same procedure as in the above to show the properties of other subspaces in Property 5.4.6. ■

Proof of Property 5.4.7. Let us prove the property of $\mathcal{V}_\lambda(\Sigma)$. It follows from Lemmas 3.8.1 and 3.8.2 that $\mathcal{V}_\lambda$ is invariant under any state feedback and output injection. Thus, it is sufficient to prove the property of $\mathcal{V}_\lambda(\Sigma)$ by showing that

$$\mathcal{V}_\lambda(\check{\Sigma}) = \operatorname{im} \left\{ \begin{bmatrix} X_{\mathrm{a}\lambda} & 0 \\ 0 & 0 \\ 0 & X_{\mathrm{c}\lambda} \\ 0 & 0 \end{bmatrix} \right\},$$

where $\check{\Sigma}$ is as defined in the proof of Property 5.4.2, $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace,

$$\left\{\zeta_{\rm a} \in \mathbb{C}^{n_{\rm a}} \,\middle|\, (\lambda I - A_{\rm aa})\zeta_{\rm a} = 0\right\},$$

and

$$X_{\rm c\lambda} = (A_{\rm cc} + B_{\rm c}F_{\rm c} - \lambda I)^{-1}B_{\rm c},$$

with $F_{\rm c}$ being a matrix of appropriate dimensions such that $A_{\rm cc} + B_{\rm c}F_{\rm c} - \lambda I$ is invertible.

For any $\zeta \in \mathcal{V}_\lambda(\check{\Sigma})$, by Definition 3.7.5, there exists a vector $\omega \in \mathbb{C}^m$ such that

$$\begin{bmatrix} \check{A} - \lambda I & \check{B} \\ \check{C} & \check{D} \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = 0,$$

or equivalently,

$$\begin{bmatrix} A_{\rm aa}-\lambda I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{\rm bb}-\lambda I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\rm cc}-\lambda I & 0 & 0 & 0 & B_{\rm c} \\ 0 & 0 & 0 & A_{\rm dd}^*-\lambda I & 0 & B_{\rm d} & 0 \\ 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & C_{\rm d} & 0 & 0 & 0 \\ 0 & C_{\rm b} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \zeta_{\rm a} \\ \zeta_{\rm b} \\ \zeta_{\rm c} \\ \zeta_{\rm d} \\ \omega_0 \\ \omega_{\rm d} \\ \omega_{\rm c} \end{pmatrix} = 0. \tag{5.5.15}$$

Hence, we have

$$(A_{\rm aa} - \lambda I)\zeta_{\rm a} = 0, \tag{5.5.16}$$

which implies that $\zeta_{\rm a} \in \operatorname{im}\{X_{\rm a\lambda}\}$,

$$\begin{bmatrix} A_{\rm bb} - \lambda I \\ C_{\rm b} \end{bmatrix} \zeta_{\rm b} = 0, \tag{5.5.17}$$

which implies that $\zeta_{\rm b} = 0$ as $(A_{\rm bb}, C_{\rm b})$ is observable, and

$$\begin{bmatrix} A_{\rm dd}^* - \lambda I & B_{\rm d} \\ C_{\rm d} & 0 \end{bmatrix} \begin{pmatrix} \zeta_{\rm d} \\ \omega_{\rm d} \end{pmatrix} = 0,$$

which implies that $\zeta_{\rm d} = 0$ and $\omega_{\rm d} = 0$ as $(A_{\rm dd}^*, B_{\rm d}, C_{\rm d})$ is square invertible and free of invariant zeros. We also have

$$(A_{\rm cc} - \lambda I)\zeta_{\rm c} + B_{\rm c}\omega_{\rm c} = 0,$$

which implies that

$$(A_{cc} + B_c F_c - \lambda I)\zeta_c + B_c(\omega_c - F_c\zeta_c) = 0,$$

or

$$\zeta_c = (A_{cc} + B_c F_c - \lambda I)^{-1} B_c (F_c\zeta_c - \omega_c) = X_{c\lambda}(F_c\zeta_c - \omega_c).$$

Hence $\zeta_c \in \operatorname{im}\{X_{c\lambda}\}$. Clearly,

$$\zeta \in \operatorname{im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \implies \mathcal{V}_\lambda(\check{\Sigma}) \subseteq \operatorname{im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\}. \tag{5.5.18}$$

Conversely, for any

$$\zeta = \begin{pmatrix} \zeta_a \\ \zeta_b \\ \zeta_c \\ \zeta_d \end{pmatrix} \in \operatorname{im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\},$$

we have $\zeta_b = 0$, $\zeta_d = 0$, $\zeta_a \in \operatorname{im}\{X_{a\lambda}\}$, which implies that $(\lambda I - A_{aa})\zeta_a = 0$, and $\zeta_c \in \operatorname{im}\{X_{c\lambda}\}$, which implies that there exists a vector $\tilde{\omega}_c$ such that

$$\zeta_c = X_{c\lambda}\tilde{\omega}_c = (A_{cc} + B_c F_c - \lambda I)^{-1} B_c \tilde{\omega}_c.$$

Thus, we have

$$(A_{cc} + B_c F_c - \lambda I)\zeta_c = B_c\tilde{\omega}_c \quad \Rightarrow \quad (A_{cc} - \lambda I)\zeta_c + B_c(F_c\zeta_c - \tilde{\omega}_c) = 0.$$

Let

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_d \\ \omega_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F_c\zeta_c - \tilde{\omega}_c \end{pmatrix}.$$

It is now straightforward to verify, by using (5.5.15), that

$$\begin{bmatrix} \check{A} - \lambda I & \check{B} \\ \check{C} & \check{D} \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = 0.$$

By Definition 3.7.5, we have

$$\zeta \in \mathcal{V}_\lambda(\check{\Sigma}) \implies \operatorname{im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \subseteq \mathcal{V}_\lambda(\check{\Sigma}). \tag{5.5.19}$$

Finally, (5.5.18) and (5.5.19) imply the result.

The proof of $\mathcal{S}_\lambda(\Sigma)$ follows from the same lines of reasoning. ∎

5.6 Kronecker and Smith Forms of the System Matrix

In this section, we will demonstrate how the structural decomposition or the special coordinate basis can be easily employed to compute the Kronecker canonical form (see Section 3.6 of Chapter 3) and Smith form of the (Rosenbrock) system matrix of a given system Σ characterized by (5.4.1). We first recall that the Kronecker canonical form of the system matrix of Σ, *i.e.*, $P_\Sigma(s)$, is invariant under nonsingular state, input and output transformations, Γ_s, Γ_i and Γ_o, and is invariant under any state feedback and output injection. Such a fact follows directly from the following manipulation:

$$
\begin{aligned}
UP_\Sigma(s)V &= \begin{bmatrix} \Gamma_s^{-1} & -\tilde{K}\Gamma_o^{-1} \\ 0 & \Gamma_o^{-1} \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \Gamma_s & 0 \\ \Gamma_i \tilde{F} & \Gamma_i \end{bmatrix} \\
&= \begin{bmatrix} sI - (\tilde{A} + \tilde{B}\tilde{F} + \tilde{K}\tilde{C} + \tilde{K}\tilde{D}\tilde{F}) & -(\tilde{B} + \tilde{K}\tilde{D}) \\ \tilde{C} + \tilde{D}\tilde{F} & \tilde{D} \end{bmatrix} \\
&= \begin{bmatrix} sI - A_{\rm KF} & -B_{\rm K} \\ C_{\rm F} & \tilde{D} \end{bmatrix},
\end{aligned}
\tag{5.6.1}
$$

where $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is the transformed system and is given by

$$
\tilde{A} = \Gamma_s^{-1} A \Gamma_s, \quad \tilde{B} = \Gamma_s^{-1} B \Gamma_i, \quad \tilde{C} = \Gamma_o^{-1} C \Gamma_s, \quad \tilde{D} = \Gamma_o^{-1} D \Gamma_i, \tag{5.6.2}
$$

$\tilde{F}$ and $\tilde{K}$ are respectively the state feedback and output injection gain matrices under the coordinate of the transformed system, and finally, $\Sigma_{\rm KF}$ characterized by the quadruple $(A_{\rm KF}, B_{\rm K}, C_{\rm F}, \tilde{D})$ is the resulting transformed system under the state feedback and output injection laws.

We are now ready to show that the Kronecker canonical form of $P_\Sigma(s)$ can be obtained neatly through the special coordinate basis of Σ. The following is a step-by-step algorithm that generates the required nonsingular transformations U and V for the canonical form:

STEP KCF.1. *Computation of the special coordinate basis of* Σ.

Apply the results of Sections 5.3 and 5.4 to find nonsingular state, input and output transformations, $\Gamma_s \in \mathbb{C}^{n\times n}$, $\Gamma_i \in \mathbb{R}^{m\times m}$ and $\Gamma_o \in \mathbb{R}^{p\times p}$, such that the given system Σ is transformed into the special coordinate basis as given in Theorem 5.4.1 or in the following compact form:

$$
\tilde{A} = \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} & 0 & L_{\rm ad}C_{\rm d} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ B_{\rm c}E_{\rm ca} & L_{\rm cb}C_{\rm b} & A_{\rm cc} & L_{\rm cd}C_{\rm d} \\ B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & B_{\rm d}E_{\rm dc} & A_{\rm dd}^* + B_{\rm d}E_{\rm dd} + L_{\rm dd}C_{\rm d} \end{bmatrix} + B_0 C_0,
$$

$$\tilde{B} = [\,B_0 \quad B_1\,] = \begin{bmatrix} B_{0\mathrm{a}} & 0 & 0 \\ B_{0\mathrm{b}} & 0 & 0 \\ B_{0\mathrm{c}} & 0 & B_\mathrm{c} \\ B_{0\mathrm{d}} & B_\mathrm{d} & 0 \end{bmatrix},$$

and

$$\tilde{C} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & 0 & C_\mathrm{d} \\ 0 & C_b & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We further note that A_{dd}^*, B_d and C_d have the following forms,

$$A_{\mathrm{dd}}^* = \mathrm{blkdiag}\left\{A_{q_1}, \dots, A_{q_{m_\mathrm{d}}}\right\}, \tag{5.6.3}$$

and

$$B_\mathrm{d} = \mathrm{blkdiag}\left\{B_{q_1}, \dots, B_{q_{m_\mathrm{d}}}\right\}, \; C_\mathrm{d} = \mathrm{blkdiag}\left\{C_{q_1}, \dots, C_{q_{m_\mathrm{d}}}\right\}, \tag{5.6.4}$$

where A_{q_i}, B_{q_i} and C_{q_i}, $i = 1, 2, \dots, m_\mathrm{d}$, are defined as in (5.4.13). Also, we assume that A_{aa} is in the Jordan canonical form, *i.e.*,

$$A_{\mathrm{aa}} = \mathrm{blkdiag}\left\{J_{\mathrm{a},1}, J_{\mathrm{a},2}, \dots, J_{\mathrm{a},k}\right\}, \tag{5.6.5}$$

where $J_{\mathrm{a},i}$, $i = 1, 2, \dots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_{\mathrm{a},i} = \mathrm{diag}\left\{\alpha_i, \alpha_i, \dots, \alpha_i\right\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}, \tag{5.6.6}$$

and $(A_{\mathrm{bb}}, C_\mathrm{b})$ is in the form of the observability structural decomposition of Theorem 4.3.1, *i.e.*,

$$A_{\mathrm{bb}} = A_{\mathrm{bb}}^* + L_{\mathrm{bb}} C_\mathrm{b} = \mathrm{blkdiag}\left\{A_{\mathrm{bb},1}, \dots, A_{\mathrm{bb},p_\mathrm{b}}\right\} + L_{\mathrm{bb}} C_\mathrm{b}, \tag{5.6.7}$$

and

$$C_\mathrm{b} = \mathrm{blkdiag}\left\{C_{\mathrm{b},1}, \dots, C_{\mathrm{b},p_\mathrm{b}}\right\}, \tag{5.6.8}$$

with

$$A_{\mathrm{bb},i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad C_{\mathrm{b},i} = [\,1 \quad 0\,], \quad i = 1, 2, \dots, p_\mathrm{b}. \tag{5.6.9}$$

Finally, $(A_{\mathrm{cc}}, B_\mathrm{c})$ is assumed to be in the form of the controllability structural decomposition of Theorem 4.4.1, *i.e.*,

$$A_{\mathrm{cc}} = A_{\mathrm{cc}}^* + B_\mathrm{c} E_{\mathrm{cc}} = \mathrm{blkdiag}\left\{A_{\mathrm{cc},1}, \dots, A_{\mathrm{cc},m_\mathrm{c}}\right\} + B_\mathrm{c} E_{\mathrm{cc}}, \tag{5.6.10}$$

and

$$B_c = \text{blkdiag}\left\{B_{c,1}, \dots, B_{c,m_c}\right\}, \tag{5.6.11}$$

with

$$A_{cc,i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{c,i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i = 1, 2, \dots, m_c. \tag{5.6.12}$$

STEP KCF.2. *Determination of state feedback and output injection laws.*

Let

$$\tilde{F} = -\begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ E_{da} & E_{db} & E_{dc} & E_{dd} \\ E_{ca} & 0 & E_{cc} & 0 \end{bmatrix}, \tag{5.6.13}$$

and

$$\tilde{K} = -\begin{bmatrix} B_{0a} & L_{ad} & L_{ab} \\ B_{0b} & L_{bd} & L_{bb} \\ B_{0c} & L_{cd} & L_{cb} \\ B_{0d} & L_{dd} & 0 \end{bmatrix}. \tag{5.6.14}$$

It is straightforward to verify that the resulting Σ_{KF} is characterized by

$$A_{KF} = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb}^* & 0 & 0 \\ 0 & 0 & A_{cc}^* & 0 \\ 0 & 0 & 0 & A_{dd}^* \end{bmatrix}, \quad B_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \end{bmatrix}, \tag{5.6.15}$$

and

$$C_F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.6.16}$$

STEP KCF.3. *Finishing touches.*

It is now simple to verify that the (Rosenbrock) system matrix associated with Σ_{KF} has the following form:

1. The corresponding term associated with $J_{a,i}$ is given by

$$sI - J_{a,i} = \begin{bmatrix} s-\alpha_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\alpha_i & -1 \\ & & & s-\alpha_i \end{bmatrix}, \tag{5.6.17}$$

which is already in the format of (3.6.12).

2. The corresponding term associated with $(A_{\mathrm{bb},i}, C_{\mathrm{b},i})$ is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & I_{l_i} \end{bmatrix} \begin{bmatrix} C_{\mathrm{b},i} \\ sI - A_{\mathrm{bb},i} \end{bmatrix} = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{5.6.18}$$

which is in the format of (3.6.13).

3. The corresponding term associated with $(A_{\mathrm{cc},i}, B_{\mathrm{c},i})$ is given by

$$[\, sI - A_{\mathrm{cc},i} \quad -B_{\mathrm{c},i} \,] = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \tag{5.6.19}$$

which is again already in the format of (3.6.14).

4. Lastly, the corresponding term associated with $(A_{q_i}, B_{q_i}, C_{q_i})$ is given by

$$\begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & s & -1 & 0 \\ & & & s & -1 \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{5.6.20}$$

Let

$$U_{q_i} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}, \quad V_{q_i} = -\begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{5.6.21}$$

Then, we have

$$U_{q_i} \begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} V_{q_i} = \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \tag{5.6.22}$$

which is now in the format of (3.6.15).

The Kronecker canonical form of the system matrix of Σ_{KF}, or equivalently the system matrix of Σ, *i.e.*, (3.6.11), can then be obtained by taking into account the additional transformations given in (5.6.18) and (5.6.21) together with some appropriate permutation transformations. This completes the algorithm.

The above algorithm for constructing the Kronecker canonical form of $P_\Sigma(s)$ has been implemented in an m-function, `kcf.m`, in [87]. Next, we proceed to compute the Smith form of the system matrix, $P_\Sigma(s)$. We recall the definition of the Smith form from the classical text of Rosenbrock and Storey [113]. Given a polynomial matrix $A(s)$, it was shown in [113] that there exist unimodular transformations $M(s)$ and $N(s)$ such that

$$S(s) = M(s)A(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.6.23}$$

where

$$D(s) = \text{diag}\Big\{p_1(s), p_2(s), \dots, p_r(s)\Big\}, \tag{5.6.24}$$

and where each $p_i(s)$, $i = 1, 2, \dots, r$, is a monic polynomial and $p_i(s)$ is a factor of $p_{i+1}(s)$, $i = 1, 2, \dots, r-1$. Note that a *unimodular matrix* is a square polynomial matrix whose determinant is a nonzero constant. $S(s)$ of (5.6.23) is called the *Smith canonical form* or *Smith form* of $A(s)$. In what follows, we will show that it is also straightforward to obtain the Smith form of $P_\Sigma(s)$, the system matrix of Σ, by using the structural decomposition technique.

STEP SMITH.1. *Determination of the Kronecker form of $P_\Sigma(s)$.*

Utilize the special coordinate basis of Σ to determine the Kronecker canonical form of $P_\Sigma(s)$ as given in the previous algorithm. However, for the computation of the Smith form of $P_\Sigma(s)$, we need not to decompose A_{aa} into the Jordan canonical form, which might involve complex transformations. Instead, we leave A_{aa} as a real-valued matrix. Note that the transformations involved in the Kronecker canonical form are constant and nonsingular, and thus unimodular.

STEP SMITH.2. *Determination of unimodular transformations.*

1. Using the procedure given in the proof of Theorem 7.4 in Chapter 3 of Rosenbrock and Storey [113], it is straightforward to show that the term $sI - J_{a,i}$ in (5.6.17) can be reduced to the Smith form

$$(sI - J_{a,i}) \rightleftharpoons \text{diag}\Big\{\overbrace{1, \dots, 1}^{n_i - 1}, (s-\alpha_i)^{n_i}\Big\}. \tag{5.6.25}$$

In general, following the procedure given in [113], we can compute two unimodular transformations $M_a(s)$ and $N_a(s)$ such that $sI - A_{aa}$ is transformed into the Smith form, *i.e.*,

$$M_a(s)(sI - A_{aa})N_a(s) = \Big\{p_{a,1}(s), p_{a,2}(s), \dots, p_{a,n_a}(s)\Big\}. \tag{5.6.26}$$

Clearly, these polynomials are related to the invariant zero structures of the given system Σ.

2. The term corresponding to $(A_{\mathrm{bb},i}, C_{\mathrm{b},i})$ given in (5.6.18) has a constant Smith form:

$$\begin{bmatrix} I_{l_i} \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{l_i} & \cdots & s & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix} \right) I_{l_i}. \tag{5.6.27}$$

Note that the first term on the right-hand side of the above equation is a unimodular matrix.

3. Similarly, the Smith form for the term corresponding to $(A_{\mathrm{cc},i}, B_{\mathrm{c},i})$ given in (5.6.19) is also a constant matrix:

$$[\, I_{r_i} \quad 0 \,] = I_{r_i} \left(\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \right) N_{r_i}(s), \tag{5.6.28}$$

where

$$N_{r_i}(s) = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{r_i} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ I_{r_i} & 0 \end{bmatrix} \tag{5.6.29}$$

is a unimodular matrix.

4. Lastly, the Smith form for the term corresponding to $(A_{q_i}, B_{q_i}, C_{q_i})$ given in (5.6.21) is an identity matrix:

$$I_{q_i+1} = I_{q_i+1} \left(\begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & \ddots & -s \\ & & & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & s & \cdots & s^{q_i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & s \\ & & & 1 \end{bmatrix}. \tag{5.6.30}$$

Once again, the last term of the equation above is a unimodular matrix.

Finally, in view of (5.6.26) to (5.6.30) together with some appropriate permutation transformations, it is now straightforward to obtain unimodular transformations $M(s)$ and $N(s)$ such that

$$M(s)P_\Sigma(s)N(s) = \begin{bmatrix} D_\Sigma(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.6.31}$$

where

$$D_\Sigma(s) = \text{diag}\Big\{\overbrace{1,\dots,1}^{n_{\text{bcd}}}, p_{\text{a},1}(s), p_{\text{a},2}(s), \dots, p_{\text{a},n_\text{a}}(s)\Big\}, \tag{5.6.32}$$

and where $n_{\text{bcd}} = n_\text{b} + n_\text{c} + n_\text{d} + m_0 + m_\text{d}$.

We illustrate the results of this section in the following example.

Example 5.6.1. Consider the system characterized by (5.4.1) with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.6.33}$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.6.34}$$

which is already in the form of the special coordinate basis with an invariant zero at 1, and $n_\text{a} = n_\text{b} = n_\text{c} = n_\text{d} = 1$. Following the algorithm given in Steps KCF.1 to KCF.3, we obtain

$$\tilde{F} = \begin{bmatrix} 1 & -3 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix}, \quad A_{\text{KF}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $B_\text{K} = B$, $C_\text{F} = C$, $\tilde{D} = D$, and the required two nonsingular transformations,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

which transform $P_\Sigma(s)$ into the Kronecker canonical form, *i.e.*,

$$UP_\Sigma(s)V = \left[\begin{array}{c|c|cc|cc} s-1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & s & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Next, following the algorithm given in Steps SMITH.1 and SMITH.2, we obtain two unimodular matrices,

$$M(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & s-1 \end{bmatrix},$$

and

$$N(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & s-1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & s-1 \end{bmatrix},$$

with $\det[M(s)] = -1$ and $\det[N(s)] = 1$, which convert $P_\Sigma(s)$ into the Smith form, *i.e.*,

$$M(s)P_\Sigma(s)N(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the polynomial in the entry (4,4) of the Smith form of $P_\Sigma(s)$ above, *i.e.*, $s-1$, results from the invariant zero of Σ.

5.7 Discrete-time Systems

The special coordinate basis or the structural decomposition for general discrete-time systems is almost identical to that given in Section 5.4 for continuous-time systems. For easy reference, we summarize in this section the results for discrete-time systems. We consider a discrete-time system characterized by

$$\begin{cases} x(k+1) = A\ x(k) + B\ u(k), \\ \quad y(k) \;\; = C\ x(k) + D\ u(k), \end{cases} \tag{5.7.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. As usual, we assume that both $[\,B' \;\; D'\,]$ and $[\,C \;\; D\,]$ are of full rank.

Theorem 5.7.1 (DSCB). *Given the system Σ of (5.7.1), there exist nonsingular state, output and input transformations $\Gamma_{\rm s}$, $\Gamma_{\rm o}$ and $\Gamma_{\rm i}$, which decompose the given*

Σ into six state subspaces. These state subspaces fully characterize the finite and infinite zero structures as well as invertibility structures of the system.

The new state, input and output spaces are described by the following set of equations:

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{5.7.2}$$

$$\tilde{x} = \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm a} = \begin{pmatrix} x_{\rm a}^- \\ x_{\rm a}^0 \\ x_{\rm a}^+ \end{pmatrix}, \quad x_{\rm d} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_{\rm d}} \end{pmatrix}, \tag{5.7.3}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_{\rm d} \\ y_{\rm b} \end{pmatrix}, \; y_{\rm d} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_{\rm d}} \end{pmatrix}, \; \tilde{u} = \begin{pmatrix} u_0 \\ u_{\rm d} \\ u_{\rm c} \end{pmatrix}, \; u_{\rm d} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_{\rm d}} \end{pmatrix}, \tag{5.7.4}$$

and

$$x_{\rm a}^-(k+1) = A_{\rm aa}^- x_{\rm a}^-(k) + B_{\rm 0a}^- y_0(k) + L_{\rm ad}^- y_{\rm d}(k) + L_{\rm ab}^- y_{\rm b}(k), \tag{5.7.5}$$

$$x_{\rm a}^0(k+1) = A_{\rm aa}^0 x_{\rm a}^0(k) + B_{\rm 0a}^0 y_0(k) + L_{\rm ad}^0 y_{\rm d}(k) + L_{\rm ab}^0 y_{\rm b}(k), \tag{5.7.6}$$

$$x_{\rm a}^+(k+1) = A_{\rm aa}^+ x_{\rm a}^+(k) + B_{\rm 0a}^+ y_0(k) + L_{\rm ad}^+ y_{\rm d}(k) + L_{\rm ab}^+ y_{\rm b}(k), \tag{5.7.7}$$

$$x_{\rm b}(k+1) = A_{\rm bb} x_{\rm b}(k) + B_{\rm 0b} y_0(k) + L_{\rm bd} y_{\rm d}(k), \quad y_{\rm b}(k) = C_{\rm b} x_{\rm b}(k), \tag{5.7.8}$$

$$\begin{aligned} x_{\rm c}(k+1) = {} & A_{\rm cc} x_{\rm c}(k) + B_{\rm 0c} y_0(k) + L_{\rm cb} y_{\rm b}(k) + L_{\rm cd} y_{\rm d}(k) \\ & + B_{\rm c}\left[E_{\rm ca}^- x_{\rm a}^-(k) + E_{\rm ca}^0 x_{\rm a}^0(k) + E_{\rm ca}^+ x_{\rm a}^+(k)\right] + B_{\rm c} u_{\rm c}(k), \end{aligned} \tag{5.7.9}$$

$$\begin{aligned} y_0(k) = {} & C_{\rm 0a}^- x_{\rm a}^-(k) + C_{\rm 0a}^0 x_{\rm a}^0(k) + C_{\rm 0a}^+ x_{\rm a}^+(k) + C_{\rm 0b} x_{\rm b}(k) \\ & + C_{\rm 0c} x_{\rm c}(k) + C_{\rm 0d} x_{\rm d}(k) + u_0(k), \end{aligned} \tag{5.7.10}$$

and for each $i = 1, 2, \ldots, m_{\rm d}$,

$$\begin{aligned} x_i(k+1) = {} & A_{q_i} x_i(k) + L_{i0} y_0(k) + L_{i\rm d} y_{\rm d}(k) + B_{q_i}\Big[u_i + E_{i\rm a}^- x_{\rm a}^-(k) \\ & + E_{i\rm a}^0 x_{\rm a}^0(k) + E_{i\rm a}^+ x_{\rm a}^+(k) + E_{i\rm b} x_{\rm b}(k) + E_{i\rm c} x_{\rm c}(k) + E_{i\rm d} x_{\rm d}(k)\Big], \end{aligned} \tag{5.7.11}$$

$$y_i(k) = C_{q_i} x_i(k), \quad y_{\rm d}(k) = C_{\rm d} x_{\rm d}(k). \tag{5.7.12}$$

Here the states $x_{\rm a}^-$, $x_{\rm a}^0$, $x_{\rm a}^+$, $x_{\rm b}$, $x_{\rm c}$ and $x_{\rm d}$ are respectively of dimensions $n_{\rm a}^-$, $n_{\rm a}^0$, $n_{\rm a}^+$, $n_{\rm b}$, $n_{\rm c}$ and $n_{\rm d} = \sum_{i=1}^{m_{\rm d}} q_i$, while the state x_i is of dimension q_i for each $i = 1, 2, \ldots, m_{\rm d}$. The control vectors u_0, $u_{\rm d}$ and $u_{\rm c}$ are respectively of *dimensions* m_0, m_d *and* $m_c = m - m_0 - m_d$, *while the output vectors* y_0, y_d *and* y_b *are respectively of dimensions* $p_0 = m_0$, $p_d = m_d$ *and* $p_b = p - p_0 - p_d$. *The matrices* A_{q_i}, B_{q_i} *and* C_{q_i} *have the form*

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1,\ 0,\ \ldots,\ 0]. \tag{5.7.13}$$

Assuming that x_i, $i = 1, 2, \ldots, m_d$, *are arranged such that* $q_i \leq q_{i+1}$, *the matrix* L_{id} *has the particular form*

$$L_{id} = [\,\ell_{i,1} \quad \ell_{i,2} \quad \cdots \quad \ell_{i,i-1} \quad 0 \quad \cdots \quad 0\,], \tag{5.7.14}$$

with the last row being identically zero. Moreover, $\lambda(A_{aa}^-) \subset \mathbb{C}^{\odot}$, $\lambda(A_{aa}^0) \subset \mathbb{C}^{\circ}$ *and* $\lambda(A_{aa}^+) \subset \mathbb{C}^{\otimes}$. *Also,* (A_{cc}, B_c) *is controllable and* (A_{bb}, C_b) *is observable.*

We note that the properties of the structural decomposition for discrete-time systems are identical to those of the continuous-time counterpart, *i.e.*, Properties 5.4.1–5.4.7.

Finally, we would also like to point out that many of the system structural properties, such as the geometric subspaces, for example, do not require the decomposition of the subspaces $\mathcal{X}_b$, $\mathcal{X}_c$ and $\mathcal{X}_d$ into chains of integrators, for which numerical problems might arise when the given system data are ill-conditioned. We refer interested readers to a recent result by Chu *et al.* [36], which shows that the separation of $\mathcal{X}_a$, $\mathcal{X}_b$, $\mathcal{X}_c$ and $\mathcal{X}_d$ without detailed structures of chains of integrators can be carried out by using some almost orthogonal transformations. We have implemented their algorithm in an m-function, called `scbraw.m`. In our toolkit of [87], we have made use of this numerically stable m-function whenever it is possible. When it is necessary to decompose the subsystems into the form of chains of integrators, the algorithm given in Section 5.3 is used, which requires fewer iteration steps compared to that given in Sannuti and Saberi [122].

5.8 Exercises

5.1. Compute a special coordinate basis for the SISO system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u, \quad y = [\,1 \quad -1 \quad 1 \quad -1\,]\, x.$$

Identify the invariant zeros and the relative degree of the given system.

5.2. Utilize the properties of the special coordinate basis to construct a fourth order controllable and observable SISO system, Σ, for each of the following five cases:

(a) Σ has no invariant zeros and has a relative degree of 4.

(b) Σ has one invariant zero at $\{1\}$ and has a relative degree of 3.

(c) Σ has two invariant zeros at $\{1, 2\}$, and has a relative degree of 2.

(d) Σ has three invariant zeros at $\{1, 2, 3\}$, and has a relative degree of 1.

(e) Σ has four invariant zeros at $\{\pm j, \pm 1\}$, and has a relative degree of 0.

5.3. Compute a special coordinate basis for the MIMO system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x.$$

Verify that the system is neither left nor right invertible, and has one unstable invariant zero and one infinite zero of order 1.

5.4. Compute a special coordinate basis for the MIMO system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u.$$

Verify that the system is invertible. Also, obtain the invariant zeros and the infinite zero structure of the system.

5.5. Utilize the properties of the special coordinate basis to construct a fourth order invertible, controllable and observable MIMO system, Σ, for each of the following cases:

(a) Σ is strictly proper, and has an infinite zero structure $S_\infty^\star = \{1, 3\}$, which implies that Σ is free of invariant zeros.

(b) Σ is strictly proper, and has an infinite zero structure $S_\infty^\star = \{2,2\}$, which implies that Σ is free of invariant zeros.

(c) Σ is strictly proper, and has one invariant zero at $\{1\}$ and an infinite zero structure $S_\infty^\star = \{1,2\}$.

(d) Σ is strictly proper, and has two invariant zeros at $\{\pm j\}$ and an infinite zero structure $S_\infty^\star = \{1,1\}$.

(e) Σ is nonstrictly proper, and has three invariant zeros at $\{1, \pm j\}$ and an infinite zero structure $S_\infty^\star = \{1\}$.

(f) Σ is nonstrictly proper, and has four invariant zeros at $\{\pm 1, \pm j\}$ and no infinite zero of order higher than 0.

5.6. Construct a third order strictly proper and right invertible system, Σ, with two inputs and one output, for each of the following cases:

(a) Σ has an infinite zero of order 2, and has no invariant zeros.

(b) Σ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Moreover, the obtained systems must be controllable and unobservable.

5.7. Construct a third order strictly proper and left invertible system, Σ, with one input and two outputs, for each of the following cases:

(a) Σ has an infinite zero of order 2, and has no invariant zeros.

(b) Σ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Furthermore, the obtained systems must be uncontrollable and observable.

5.8. Construct a second order system, Σ, which has the following properties: (i) Σ is neither left nor right invertible; (ii) Σ is uncontrollable and unobservable; (ii) Σ is free of finite zeros and is free of infinite zeros of order higher than 0; and (iv) Σ is nonstrictly proper with both $[\,C \;\; D\,]$ and $[\,B' \;\; D'\,]$ being of full rank.

5.9. Compute geometric subspaces, $\mathcal{V}^*$, $\mathcal{S}^*$, $\mathcal{R}^*$ and $\mathcal{N}^*$, for the systems given in Exercise 5.1, Exercise 5.3 and Exercise 5.4.

5.10. Compute geometric subspaces, $\mathcal{V}_\lambda$ and $\mathcal{S}_\lambda$, with $\lambda = 1$, for the systems given in Exercise 5.3 and Exercise 5.4.

5.11. Consider a SISO system, Σ, which is already in the SCB form as given in Theorem 5.2.1, *i.e.*,

$$\begin{aligned}
\dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + L_{\rm ad}y, \\
\dot{x}_1 &= x_2, \quad y = x_1, \\
\dot{x}_2 &= x_3, \quad \ldots, \quad \dot{x}_{n_{\rm d}-1} = x_{n_{\rm d}}, \\
\dot{x}_{n_{\rm d}} &= E_{\rm da}x_{\rm a} + E_1x_1 + E_2x_2 + \cdots + E_{n_{\rm d}}x_{n_{\rm d}} + u,
\end{aligned}$$

or in the matrix form:

$$\dot{x} = Ax + Bu = \begin{bmatrix} A_{\rm aa} & L_{\rm ad} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ E_{\rm da} & E_1 & E_2 & \cdots & E_{n_{\rm d}} \end{bmatrix} \begin{pmatrix} x_{\rm a} \\ x_1 \\ x_2 \\ \vdots \\ x_{n_{\rm d}} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

and

$$y = Cx = [0 \quad 1 \quad 0 \quad \cdots \quad 0]\, x.$$

Let

$$\tilde{B} := B + \begin{bmatrix} K_{\rm a} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_{\rm a} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Construct the special coordinate basis for the new system, $\tilde{\Sigma}$, characterized by $\dot{x} = Ax + \tilde{B}u$, and $y = Cx$. Show that Σ and $\tilde{\Sigma}$ have the same relative degree. Also, show that the invariant zeros of $\tilde{\Sigma}$ are given by the eigenvalues of $\tilde{A}_{\rm aa} := A_{\rm aa} - K_{\rm a}E_{\rm da}$.

5.12. It follows from Theorem 5.3.1 that each subsystem associated with $x_{\rm d}$ of Σ can be expressed as

$$\dot{x}_{{\rm d},i} = \begin{bmatrix} \star & I_{q_i-1} \\ \star & \star \end{bmatrix} x_{{\rm d},i} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_{{\rm d},i} + \star),$$

and

$$y_{{\rm d},i} = [1 \quad 0]\, x_{{\rm d},i}.$$

Show that the subsystem from $u_{{\rm d},i}$ to $y_{{\rm d},i}$ is invertible, controllable and observable, and is free of invariant zeros. This implies that the subsystem associated with the whole $x_{\rm d}$ from its input $u_{\rm d}$ to output $y_{\rm d}$ is invertible, controllable and observable, and is free of invariant zeros.

5.13. Determine the Kronecker canonical form and Smith form for the systems given in Exercise 5.3 and Exercise 5.4.

5.14. Given a linear system, Σ, with its special coordinate basis being given as in Theorem 5.4.1, prove Property 5.4.6 for $\mathcal{S}^*$, *i.e.*,

$$\mathcal{S}^*(\Sigma) = \mathrm{im}\left\{ \Gamma_\mathrm{s} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_\mathrm{c}} & 0 \\ 0 & I_{n_\mathrm{d}} \end{bmatrix} \right\}.$$

5.15. Given a linear system, Σ, with its special coordinate basis being given as in Theorem 5.4.1, prove Property 5.4.7 for $\mathcal{S}_\lambda^*$, *i.e.*,

$$\mathcal{S}_\lambda(\Sigma) = \mathrm{im}\left\{ \Gamma_\mathrm{s} \begin{bmatrix} \lambda I - A_\mathrm{aa} & 0 & 0 & 0 \\ 0 & Y_{\mathrm{b}\lambda} & 0 & 0 \\ 0 & 0 & I_{n_\mathrm{c}} & 0 \\ 0 & 0 & 0 & I_{n_\mathrm{d}} \end{bmatrix} \right\},$$

where

$$\mathrm{im}\,\{Y_{\mathrm{b}\lambda}\} = \ker\,\left[C_\mathrm{b}(A_\mathrm{bb} + K_\mathrm{b}C_\mathrm{b} - \lambda I)^{-1}\right],$$

and where K_b is any matrix of appropriate dimensions and subject to the constraint that $A_\mathrm{bb} + K_\mathrm{b}C_\mathrm{b}$ has no eigenvalue at λ.

Chapter 6

Decompositions of Descriptor Systems

6.1 Introduction

In this chapter, we focus on the structural decomposition of a more general type of linear time-invariant systems, namely, linear descriptor systems. Descriptor systems, also commonly called singular or generalized systems in the literature, appear in many practical applications including engineering systems, economic systems, network analysis, and biological systems (see *e.g.*, Dai [43], Kuijper [79] and Lewis [80]). In fact, many systems in real life are singular in nature. They are usually simplified as or approximated by proper systems because of the lack of efficient tools for dealing with descriptor systems. The structural analysis of linear descriptor systems, using either an algebraic or a geometric approach, has attracted considerable attention from many researchers during the past three decades (see *e.g.*, Chu and Mehrmann [37], Chu and Ho [38], Fliess [53], Geerts [57], Lewis [80–82], Lewis and Ozcaldiran [83], Loiseau [93], Malabre [97], Misra *et al.* [99], Van Dooren [143,144], Verghese [146], Zhou *et al.* [161], and the references cited therein). Generally speaking, almost all the research works dealing with descriptor systems are the natural extensions of their proper counterparts, although these extensions are usually nontrivial.

It has been extensively demonstrated and proven for proper systems that system structural properties, such as finite and infinite zero structures and invertibility structures, play a very important role in solving various control problems including H_2, H_∞ control and disturbance decoupling (see *e.g.*, [22] and [120]). The

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structural properties of descriptor systems and their applications to the control problems of descriptor systems are however less emphasized in the literature. In this chapter, we present a structural decomposition technique for general multivariable linear descriptor systems. Similar to its counterpart in Chapter 5, such a technique can be used to capture and display the structural properties of general descriptor systems. It can also be regarded as a natural extension and counterpart of the results in Chapter 5 for proper systems. However, it will be seen shortly that the structural decomposition of a general multivariable descriptor system is much more involved. Such a decomposition technique is expected to be a powerful tool for solving a large variety of control problems for descriptor systems, H_2 and H_∞ control, model reduction and disturbance decoupling, to name just a few. The results of this chapter, especially those for continuous-time systems, follow closely from the works reported earlier in [64,65].

We consider a continuous-time system Σ characterized by

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u, \\ \quad y = C\,x + D\,u, \end{cases} \tag{6.1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimensions. The system Σ is said to be singular if $\text{rank}(E) < n$. As usual, in order to avoid any ambiguity in the solutions to the system, we assume throughout this chapter that the given descriptor system Σ is *regular*, *i.e.*, $\det(sE - A) \not\equiv 0$, for all $s \in \mathbb{C}$. Traditionally, the Kronecker canonical form, a classical form of matrix pencils under strictly equivalent transformation, has been used extensively in the structural analysis of descriptor systems. Malabre [97] presents a geometric approach and introduces structural invariants of descriptor systems. In that paper, some definitions are shown to be consistent with others directly deduced from matrix pencil tools. It extends many geometric and structural results from the proper systems to the descriptor systems.

As seen in Chapter 3 for proper systems, the Kronecker canonical form exhibits the finite- and infinite-zero structures (*i.e.*, invariant indices) of the system, and shows the left and right null-space structures. The same technique has also been adopted to define invariant indices of descriptor systems (see, *e.g.*, Malabre [97]). We recall that two pencils $sM_1 - N_1$ and $sM_2 - N_2$ of dimensions $m \times n$ are strictly equivalent if there exist constant nonsingular matrices $\tilde{P}$ and $\tilde{Q}$ of appropriate dimensions such that

$$\tilde{Q}(sM_1 - N_1)\tilde{P} = sM_2 - N_2. \tag{6.1.2}$$

It was shown in Gantmacher [56] that any pencil $sM - N$ can be reduced, under strict equivalence, to a canonical quasi-diagonal form, which is given by

$$\tilde{Q}(sM-N)\tilde{P}=\begin{bmatrix} \text{blkdiag}\{sI-J, L_{l_1},\dots,L_{l_q},R_{r_1},\dots,R_{r_p},I-sH\} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.1.3}$$

In the context of this chapter, we will focus on

$$P_\Sigma(s) = sM - N = s\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix}, \tag{6.1.4}$$

i.e., the (Rosenbrock) system matrix pencil associated with Σ. In (6.1.3), R_k and L_k are the $k \times (k+1)$ and $(k+1) \times k$ bidiagonal pencils, respectively,

$$R_k := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \qquad L_k := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{6.1.5}$$

J is in Jordan canonical form, and $sI - J$ has the following $\Sigma_{i=1}^{\delta} d_i$ pencils as its diagonal blocks,

$$sI_{m_{i,j}} - J_{m_{i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \tag{6.1.6}$$

$j = 1, 2, \dots, d_i$, $i = 1, 2, \dots, \delta$, and H is nilpotent and in Jordan canonical form, and $I - sH$ has the following d pencils as its diagonal blocks,

$$I_{n_j+1} - sJ_{n_j+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \tag{6.1.7}$$

$j = 1, 2, \dots, d$. Then, $\{(s - \beta_i)^{m_{i,j}}, j = 1, 2, \dots, d_i\}$ are finite elementary divisors at β_i, $i = 1, 2, \dots, \delta$. The index sets $\{r_1, r_2, \dots, r_p\}$ and $\{l_1, l_2, \dots, l_q\}$ are right and left minimal indices, respectively. $\{(1/s)^{n_j}, j = 1, 2, \dots, d\}$ are the infinite elementary divisors. The definition of structural invariants of Σ is based on invariant indices of its system pencil. For descriptor systems, the right and left invertibility indices are right and left minimal indices of the system pencil respectively, the finite and infinite zero structures of a descriptor system are related to finite and infinite elementary divisors of the system pencil.

Note that the computation of the invariant indices of the system pencil of the descriptor system Σ is actually quite simple. Without loss of any generality, we assume that E is in the form of

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.1.8}$$

and thus A, B and C can be partitioned accordingly as

$$A = \begin{bmatrix} A_{\rm nn} & A_{\rm ns} \\ A_{\rm sn} & A_{\rm ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\rm n} \\ B_{\rm s} \end{bmatrix}, \quad C = [\, C_{\rm n} \quad C_{\rm s} \,]. \tag{6.1.9}$$

Rewriting the system pencil of (6.1.4) as

$$P_\Sigma(s) = \left[\begin{array}{c|cc} sI - A_{\rm nn} & -A_{\rm ns} & -B_{\rm n} \\ \hline -A_{\rm sn} & -A_{\rm ss} & -B_{\rm s} \\ C_{\rm n} & C_{\rm s} & D \end{array} \right] = \begin{bmatrix} sI - A_{\rm x} & -B_{\rm x} \\ C_{\rm x} & D_{\rm x} \end{bmatrix}, \tag{6.1.10}$$

it is simple to see that the invariant indices of Σ are equivalent to those of a proper system characterized by $(A_{\rm x}, B_{\rm x}, C_{\rm x}, D_{\rm x})$. All the invariant indices can thus be computed accordingly (see, e.g., [91]). It is also clear from (6.1.10) that *the Kronecker canonical form of a descriptor system cannot capture all the system structural properties as it is identical to those of a proper system!* In this chapter, our focus is not on the computation of the invariant indices, but the derivation of a constructive algorithm that decomposes the state space of the given system into several distinct parts, which are directly associated with the finite and infinite zero dynamics, as well as the invertibility structures of the given system.

It is interesting to note that there are fundamental differences between the structure of a descriptor system and that of a proper system. For descriptor systems, as we will see shortly, some of the state variables are totally zero, which implies that the state trajectories of descriptor systems generally do not span the whole $\mathbb{R}^n$ space, and some are linear combinations of input variables and their derivatives. Our decomposition technique given in this chapter will automatically and explicitly separate these redundant dynamics of descriptor systems, which cannot be captured through the Kronecker canonical form. We further note that besides these unique properties, the remaining state variables have similar structures to those of proper systems given in Chapter 5.

As mentioned earlier, it is expected that the technique presented in this chapter will play a similar role in solving a variety of control problems for descriptor systems as its counterpart has played in the context of proper systems. We would like to point out that research related to descriptor systems and control is far from complete. It is our belief that the results of this chapter will emerge as an important tool for tackling many descriptor systems and control problems.

This chapter is organized as follows. Section 6.2 gives the structural decomposition algorithm for SISO systems. Section 6.3 presents the structural decomposition algorithm for general MIMO systems and the structural properties of such systems. The proofs of the results in Section 6.3 are given in Section 6.4. Section 6.5 deals with discrete-time systems.

Throughout this chapter, $u^{(v)}$ denotes the v-th derivative of u, where v is a nonnegative integer. With a slight abuse of notation, we occasionally write $u^{(v)}$ as $s^v u$ when it does not cause ambiguity. Here, s can be regarded as a differentiation operator or the variable in the Laplace transform.

6.2 SISO Descriptor Systems

We consider in this section the descriptor system of (6.1.1) with $m = p = 1$. As expected, the computation involved in the structural decomposition of SISO systems is much simpler than in the structural decomposition of general multivariable systems. Also, for simplicity, we assume in this section that the state variable of Σ, $x(t)$, is a continuous function of t at $t = 0$. We have the following theorem.

Theorem 6.2.1. *Consider the descriptor system Σ of (6.1.1) with $p = m = 1$ satisfying the usual regularity assumption, i.e.,* $\det(sE - A) \not\equiv 0$ *for $s \in \mathbb{C}$. There exist nonsingular state, input and output transformations $\Gamma_s \in \mathbb{R}^{n \times n}$, $\Gamma_i \in \mathbb{R}$ and $\Gamma_o \in \mathbb{R}$, and an $n \times n$ nonsingular matrix $\Gamma_e(s)$, whose elements are polynomials of s, which together give a structural decomposition of Σ described by the set of equations*

$$x = \Gamma_s \tilde{x}, \quad \tilde{x} = \begin{pmatrix} x_z \\ x_e \\ x_a \\ x_d \end{pmatrix}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_d = \begin{pmatrix} x_{d1} \\ x_{d2} \\ \vdots \\ x_{dn_d} \end{pmatrix}, \tag{6.2.1}$$

$$x_z \in \mathbb{R}^{n_z}, \quad x_e \in \mathbb{R}^{n_e}, \quad x_a \in \mathbb{R}^{n_a}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{6.2.2}$$

and

Case 1: If $n_d = 0$, i.e., x_d is nonexistent, then we have

$$\left.\begin{aligned} x_z &= 0, \\ x_e &= \tilde{u}^{(v)}, \\ \dot{x}_a &= A_{aa} x_a + B_{0a} \tilde{y}, \quad \tilde{y} = \bar{C} x_a + \bar{D} \tilde{u}^{(v)}; \end{aligned}\right\} \tag{6.2.3}$$

Case 2: If $n_{\rm d} > 0$, we have

$$\left.\begin{aligned}
x_{\rm z} &= 0,\\
x_{\rm e} &= \alpha_{\rm e}\tilde{u}^{(v)},\\
\dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + L_{\rm ad}y_{\rm d},\\
\dot{x}_{\rm d1} &= x_{\rm d2},\\
\dot{x}_{\rm d2} &= x_{\rm d3},\\
&\;\vdots\\
\dot{x}_{{\rm d}n_{\rm d}} &= M_{\rm da}x_{\rm a} + L_{\rm dd}y_{\rm d} + \tilde{u}^{(v)}, \quad \tilde{y} = y_{\rm d} = x_{\rm d1}.
\end{aligned}\right\} \tag{6.2.4}$$

Here, v is a nonnegative integer, $A_{\rm aa}$, $B_{\rm 0a}$, $\bar{C}$, $\bar{D}$, $L_{\rm ad}$, $M_{\rm da}$ and $L_{\rm dd}$, if existent, are constant matrices of appropriate dimensions, and $\alpha_{\rm e}$ is a nonzero scalar.

Proof. The following is a step-by-step constructive proof for the structural decomposition of Σ.

STEP SISO-SDDS.1. *Preliminary Decomposition.*

This step, adopted from Dai [43], is to separate the given descriptor system into a proper subsystem and a special descriptor subsystem (hereafter we call it EA Decomposition). First, we note that the regularity assumption on the given system (6.1.1) implies the existence of a real scalar β such that $\det(\beta E + A) \neq 0$. Next, we define

$$\hat{E} = (\beta E + A)^{-1}E. \tag{6.2.5}$$

It follows from the real Jordan canonical decomposition, *i.e.*, Theorem 4.2.2, that there exists a nonsingular transformation $T \in \mathbb{R}^{n\times n}$ such that

$$T\hat{E}T^{-1} = \begin{bmatrix} \hat{E}_1 & 0 \\ 0 & \hat{E}_2 \end{bmatrix}, \tag{6.2.6}$$

where $\hat{E}_1 \in \mathbb{R}^{n_1\times n_1}$ is a nonsingular matrix and $\hat{E}_2 \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix. Lastly, we let

$$P = \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta\hat{E}_2)^{-1} \end{bmatrix} T(\beta E + A)^{-1}, \quad Q = T^{-1}. \tag{6.2.7}$$

It is then straightforward to verify that

$$PEQ = \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta\hat{E}_2)^{-1} \end{bmatrix} T(\beta E + A)^{-1}ET^{-1}$$

$$
\begin{aligned}
&= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta \hat{E}_2)^{-1} \end{bmatrix} T \hat{E} T^{-1} \\
&= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta \hat{E}_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{E}_1 & 0 \\ 0 & \hat{E}_2 \end{bmatrix} \\
&= \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \qquad (6.2.8)
\end{aligned}
$$

where $N = (I_{n_2} - \beta \hat{E}_2)^{-1} \hat{E}_2$. It is simple to show that for any positive integer h,

$$
\begin{aligned}
N^h &= N^{h-2} (I_{n_2} - \beta \hat{E}_2)^{-1} \hat{E}_2 (I_{n_2} - \beta \hat{E}_2)^{-1} \hat{E}_2 \\
&= N^{h-2} (I_{n_2} - \beta \hat{E}_2)^{-1} (I_{n_2} - \beta \hat{E}_2)^{-1} \hat{E}_2 \hat{E}_2 \\
&\;\vdots \\
&= (I_{n_2} - \beta \hat{E}_2)^{-h} (\hat{E}_2)^h. \qquad (6.2.9)
\end{aligned}
$$

Clearly, N is a nilpotent matrix because $\hat{E}_2$ is nilpotent. Next, noting that $(\beta E + A)^{-1} A = I_n - \beta \hat{E}$, we have

$$
\begin{aligned}
PAQ &= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta \hat{E}_2)^{-1} \end{bmatrix} T (\beta E + A)^{-1} A T^{-1} \\
&= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta \hat{E}_2)^{-1} \end{bmatrix} T (I_n - \beta \hat{E}) T^{-1} \\
&= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta \hat{E}_2)^{-1} \end{bmatrix} \begin{bmatrix} I_{n_1} - \beta \hat{E}_1 & 0 \\ 0 & I_{n_2} - \beta \hat{E}_2 \end{bmatrix} \\
&= \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}. \qquad (6.2.10)
\end{aligned}
$$

We also partition accordingly

$$
PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad CQ = [\, C_1 \quad C_2 \,]. \qquad (6.2.11)
$$

Then, the given descriptor Σ can be decomposed into the following two subsystems:

$$
\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 \, x_1 + B_1 \, u, \\ y_1 = C_1 \, x_1, \end{cases} \qquad (6.2.12)
$$

and

$$
\Sigma_2 : \begin{cases} N \, \dot{x}_2 = \quad x_2 + B_2 \, u, \\ \quad y_2 = C_2 \, x_2, \end{cases} \qquad (6.2.13)
$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, and $y = y_1 + y_2$.

STEP SISO-SDDS.2. *Decomposition of* Σ_2.

If $B_2 = 0$, we have $x_z = x_2$, $n_z = n_2$, $x_e = \emptyset$, $n_e = 0$ and $v = 0$. For this case, the following procedure does not apply. We jump directly to STEP SISO-SDDS.3.

For the case when $B_2 \neq 0$, it follows from Theorem 4.4.1 that there exist a nonsingular transformation T_2 and an $\alpha \neq 0$ such that

$$x_2 = T_2 \begin{pmatrix} x_v \\ x_z \end{pmatrix}, \quad x_z \in \mathbb{R}^{n_z}, \quad x_v \in \mathbb{R}^{v_d}, \quad x_v = \begin{pmatrix} x_{v1} \\ x_{v2} \\ \vdots \\ x_{vv_d} \end{pmatrix}, \tag{6.2.14}$$

$$T_2^{-1} N T_2 = \begin{bmatrix} J_{c0} & N_{c\bar{c}} \\ 0 & J_{n_z} \end{bmatrix}, \quad T_2^{-1} B_2 = \begin{bmatrix} B_{2c} \\ 0 \end{bmatrix}, \tag{6.2.15}$$

and

$$C_2 T_2 = [\, C_{2c} \quad C_{2\bar{c}} \,], \tag{6.2.16}$$

where (J_{c0}, B_{2c}) is a controllable pair. Since N is a nilpotent matrix and thus has all its eigenvalues at 0 and B_{2c} is a column vector, (J_{c0}, B_{2c}) can actually be written as,

$$J_{c0} = \begin{bmatrix} 0 & I_{v_d - 1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_{2c} = \begin{bmatrix} 0 \\ -1/\alpha \end{bmatrix}. \tag{6.2.17}$$

Also note that J_{n_z} has all its eigenvalues at 0. As such, it is simple to verify that Σ_2 is decomposed into the following two subsystems:

$$J_{n_z} \dot{x}_z = x_z, \tag{6.2.18}$$

which implies that $x_z = 0$ for all t, under the assumption that $x(t)$ is a continuous function of t at $t = 0$, and

$$J_{c0} \dot{x}_v + N_{c\bar{c}} \dot{x}_z = x_v + B_{2c} u, \tag{6.2.19}$$

which is equivalent to

$$J_{c0} \dot{x}_v = x_v + B_{2c} u, \tag{6.2.20}$$

or

$$u = \alpha x_{vv_d}, \quad \dot{x}_{vv_d} = x_{vv_d - 1}, \quad \ldots, \quad \dot{x}_{v2} = x_{v1}. \tag{6.2.21}$$

Clearly, (6.2.21) implies

$$x_e := x_{v1} = \frac{1}{\alpha} u^{(v)} \quad \text{and} \quad n_e = 1, \tag{6.2.22}$$

where $v = \max(0, v_d - 1)$. The output y_2 can then be expressed as

$$y_2 = C_{2c}x_v + C_{2\bar{c}}x_z = C_{2c}x_v. \tag{6.2.23}$$

STEP SISO-SDDS.3. *Decomposition of Finite and Infinite Zero Structures of* Σ.

Observing the results in (6.2.12), (6.2.13), (6.2.18), (6.2.21), (6.2.22) and (6.2.23), it is clear that the given system Σ has been transformed into the following format:

$$x_z = 0, \quad x_e = \frac{1}{\alpha}u^{(v)}, \tag{6.2.24}$$

and a proper system,

$$\left.\begin{aligned} \dot{x}_1 &= A_1x_1 + \alpha B_1 x_{vv_d}, \\ \dot{x}_{v2} &= \frac{1}{\alpha}u^{(v)}, \\ &\vdots \\ \dot{x}_{vv_d} &= x_{vv_d-1}, \\ y &= C_1x_1 + C_{2c}x_v. \end{aligned}\right\} \tag{6.2.25}$$

Next, let us partition

$$C_{2c} = [\, c_{v1} \quad c_{v2} \quad \cdots \quad c_{vv_d} \,]. \tag{6.2.26}$$

Thus, the proper system (6.2.25) can be rewritten as,

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,\bar{u}, \\ y = \bar{C}\,\bar{x} + \bar{D}\,\bar{u}, \end{cases} \tag{6.2.27}$$

where

$$\bar{x} = \begin{pmatrix} x_1 \\ x_{v2} \\ \vdots \\ x_{vv_d-1} \\ x_{vv_d} \end{pmatrix}, \quad \bar{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & \alpha B_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

and

$$\bar{u} = \frac{1}{\alpha}u^{(v)}, \quad \bar{C} = [\, C_1 \quad c_{v2} \quad \cdots \quad c_{vv_d-1} \quad c_{vv_d} \,], \quad \bar{D} = c_{v1},$$

if $v_d > 1$, or

$$\bar{x} = x_1, \quad \bar{u} = \frac{1}{\alpha}u, \quad \bar{A} = A_1, \quad \bar{B} = \alpha B_1, \quad \bar{C} = C_1, \quad \bar{D} = C_{2c},$$

if $v_d = 1$.

We have the following two distinct cases.

Case 1. $\bar{D} \neq 0$, which corresponds to Case 1 of Theorem 6.2.1. In this case, it is simple to obtain $x_\mathrm{d} = \emptyset$, $n_\mathrm{d} = 0$, $x_\mathrm{a} = \bar{x}$, $n_\mathrm{a} = n_1 + v$ and

$$\dot{x}_\mathrm{a} = (\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})x_\mathrm{a} + \bar{B}\bar{D}^{-1}y = A_\mathrm{aa}x_\mathrm{a} + B_{0\mathrm{a}}y \tag{6.2.28}$$

and

$$y = \bar{C}x_\mathrm{a} + \bar{D}\alpha^{-1}u^{(v)} = \bar{C}x_\mathrm{a} + \bar{D}\tilde{u}^{(v)}, \tag{6.2.29}$$

if we let $u = \Gamma_\mathrm{i}\tilde{u} = \alpha\tilde{u}$.

Case 2. $\bar{D} = 0$, which corresponds to Case 2 of Theorem 6.2.1. It follows from Theorem 5.2.1 that there exist nonsingular transformations $\bar{\Gamma}_\mathrm{s}$, Γ_o and $\bar{\Gamma}_\mathrm{i}$ such that when we apply the changes of coordinates

$$\bar{x} = \bar{\Gamma}_\mathrm{s}\breve{x} = \bar{\Gamma}_\mathrm{s}\begin{pmatrix} x_\mathrm{a} \\ x_\mathrm{d} \end{pmatrix}, \quad y = \Gamma_\mathrm{o}\tilde{y}, \quad \bar{u} = \frac{1}{\alpha}u^{(v)} = \bar{\Gamma}_\mathrm{i}\tilde{u}^{(v)}, \tag{6.2.30}$$

to the system in (6.2.27), and in view of (6.2.22), we have

$$\dot{\breve{x}} = \begin{bmatrix} A_\mathrm{aa} & L_\mathrm{ad}C_\mathrm{d} \\ B_\mathrm{d}M_\mathrm{da} & A_\mathrm{dd} \end{bmatrix}\breve{x} + \begin{bmatrix} 0 \\ B_\mathrm{d} \end{bmatrix}\tilde{u}^{(v)}, \tag{6.2.31}$$

and

$$\tilde{y} = [\,0 \quad C_\mathrm{d}\,]\,\breve{x}, \tag{6.2.32}$$

where A_dd, B_d and C_d have the form

$$A_\mathrm{dd} = \begin{bmatrix} 0 & I_{n_\mathrm{d}-1} \\ \star & 0 \end{bmatrix}, \quad B_\mathrm{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_\mathrm{d} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{6.2.33}$$

Let

$$u = \Gamma_\mathrm{i}\tilde{u} = \alpha\bar{\Gamma}_\mathrm{i}\tilde{u} \quad \Longrightarrow \quad \frac{1}{\alpha}u^{(v)} = \bar{\Gamma}_\mathrm{i}\tilde{u}^{(v)}. \tag{6.2.34}$$

This completes the algorithm for the structural decomposition of the given SISO descriptor system. ■

We illustrate the above decomposition technique in the following example.

Example 6.2.1. We consider a descriptor system of (6.1.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.2.35}$$

and

$$C = [0 \quad 1 \quad 0 \quad 0 \quad 0], \quad D = 0. \tag{6.2.36}$$

We first choose a scalar $\beta = -1$ and obtain

$$P = \begin{bmatrix} 1.4142 & 1.4142 & -0.7071 & -1.4142 & -0.7071 \\ 0 & 0 & 1.2248 & 0 & -1.2248 \\ 0 & -1.4142 & -1.4142 & 1.4142 & 1.4142 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.7071 & 0.4083 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & 0 & 1 \\ 0 & -0.8165 & 0 & 0 & 0 \\ 0 & 0 & -0.7071 & 0 & 0 \\ -0.7071 & 0.4083 & 0 & 1 & 0 \end{bmatrix},$$

$$PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$PB = \begin{bmatrix} 0.7071 \\ -1.2247 \\ 1.4142 \\ 1 \\ 0 \end{bmatrix}, \quad CQ = \begin{bmatrix} 0 & 0 & 0.7071 & 0 & 1 \end{bmatrix},$$

with $n_1 = 3$ and $n_2 = 2$. The given descriptor system can then be decomposed into the following subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 \ x_1 \ + B_1 \ u, \\ y_1 = C_1 \ x_1, \end{cases}$$

and

$$\Sigma_2 : \begin{cases} N \ \dot{x}_2 = \quad x_2 \ + B_2 \ u, \\ \quad y_2 = C_2 \ x_2, \end{cases}$$

with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.7071 \\ -1.2248 \\ 1.4142 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 0.7071 \end{bmatrix},$$

and

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Noting that B_2 is nonzero and Σ_2 is already in the required form of (6.2.15), we have

$$\alpha = -1, \quad x_z = 0, \quad u = -x_v, \quad x_e = x_v = -u, \quad v_d = 1,$$

and the auxiliary proper system

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,\bar{u}, \\ y = \bar{C}\,\bar{x} + \bar{D}\,\bar{u}, \end{cases}$$

with

$$\bar{x} = x_1, \quad \bar{u} = \frac{1}{\alpha}u = -u,$$

$$\bar{A} = A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{B} = \alpha B_1 = \begin{bmatrix} -0.7071 \\ 1.2248 \\ -1.4142 \end{bmatrix},$$

and

$$\bar{C} = C_1 = \begin{bmatrix} 0 & 0 & 0.7071 \end{bmatrix}, \qquad \bar{D} = 0,$$

which corresponds to Case 2 of STEP SISO-SDDS.3. Following the result of Theorem 5.2.1, we obtain the required state, output and input transformation matrices,

$$\bar{\Gamma}_s = \begin{bmatrix} 0 & -1 & 0.7071 \\ 1 & 0 & -1.2248 \\ 0 & 0 & 1.4142 \end{bmatrix}, \quad \Gamma_o = 1, \quad \bar{\Gamma}_i = -1,$$

which transform the auxiliary proper system into the required structural form:

$$\bar{\Gamma}_s^{-1}\bar{A}\bar{\Gamma}_s = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \bar{\Gamma}_s^{-1}\bar{B}\bar{\Gamma}_i = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right], \quad \Gamma_o^{-1}\bar{C}\bar{\Gamma}_s = \left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right].$$

Finally, we obtain all the necessary transformations as

$$\Gamma_e = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1.2248 & 0 & 1.2248 & 0 \\ -1.4142 & -2.1213 & 0 & 2.1213 & 1.4142 \\ 0 & -1 & -1 & 1 & 1 \end{bmatrix},$$

$$\Gamma_s = \begin{bmatrix} 0 & 0 & 0.4083 & -0.7071 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.8165 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0.4083 & 0.7071 & -1 \end{bmatrix},$$

and

$$\Gamma_i = 1, \quad \Gamma_o = 1,$$

with which the given descriptor system is transformed into the special form

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \dot{\tilde{x}} = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \tilde{x} + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ \hline 1 \end{array}\right] \tilde{u},$$

and

$$\tilde{y} = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 1 \end{array}\right] \tilde{x}.$$

The decomposed system can be rewritten as

$$x_z = 0, \quad x_e = -\tilde{u},$$

$$\dot{x}_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_a + \begin{bmatrix} 0 \\ 0 \end{bmatrix} y_d,$$

and

$$\dot{x}_d = \tilde{u}, \quad \tilde{y} = x_z + y_d = y_d = x_d.$$

6.3 MIMO Descriptor Systems

We first summarize the structural decomposition of multivariable descriptor systems in the following main theorem. All its properties will also be given. For clarity of presentation, the constructive algorithm for the structural decomposition and proofs of all its properties will be given separately in Section 6.4.

Theorem 6.3.1 (SDDS). *Consider the multivariable linear descriptor system* Σ *of (6.1.1) satisfying* $\det(sE - A) \not\equiv 0$ *for* $s \in \mathbb{C}$. *Then,*

1. *there exist coordinate-free nonnegative integers* n_z, n_e, n_a, n_b, n_c, n_d, m_d, m_0, m_c, p_b, *and positive integers* q_i, $i = 1, 2, \ldots, m_d$, *if* $m_d > 0$; *and*

2. *there exist nonsingular state and output constant transformations $\Gamma_s \in \mathbb{R}^{n\times n}$ and $\Gamma_o \in \mathbb{R}^{p\times p}$, as well as an $m\times m$ nonsingular input transformation $\Gamma_i(s)$, whose inverse's elements are some polynomials of s (i.e., its inverse contains various differentiation operators), and an $n \times n$ nonsingular transformation $\Gamma_e(s)$, whose elements are polynomials of s, which together give a structural decomposition of Σ and display explicitly its structural properties.*

The structural decomposition of Σ can be described by the following set of equations:

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i(s)\tilde{u}, \tag{6.3.1}$$

and

$$\tilde{x} = \begin{pmatrix} x_z \\ x_e \\ x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \tag{6.3.2}$$

$$x_d = \begin{pmatrix} x_{d1} \\ x_{d2} \\ \vdots \\ x_{dm_d} \end{pmatrix}, \quad y_d = \begin{pmatrix} y_{d1} \\ y_{d2} \\ \vdots \\ y_{dm_d} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_{d1} \\ u_{d2} \\ \vdots \\ u_{dm_d} \end{pmatrix}, \tag{6.3.3}$$

and

$$J_{n_z} \dot{x}_z = x_z, \tag{6.3.4}$$

where $J_{n_z} \in \mathbb{R}^{n_z \times n_z}$ has all its eigenvalues at 0,

$$x_e = B_{e0} u_0 + B_{ec} u_c + B_{ed} u_d + sN_{ez}(s) x_z, \tag{6.3.5}$$

$$\dot{x}_a = A_{aa} x_a + B_{0a} y_0 + L_{ad} y_d + L_{ab} y_b + sL_{az}(s) x_z, \tag{6.3.6}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d + sL_{bz}(s) x_z, \tag{6.3.7}$$

$$y_b = C_b x_b + C_{bz} x_z + sC_{bzs}(s) x_z, \tag{6.3.8}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cd} y_d + L_{cb} y_b + B_c M_{ca} x_a + B_c u_c + sL_{cz}(s) x_z, \tag{6.3.9}$$

$$y_0 = C_{0a} x_a + C_{0b} x_b + C_{0c} x_c + C_{0d} x_d + u_0 + C_{0z} x_z + sC_{0zs}(s) x_z, \tag{6.3.10}$$

and for each $i = 1, 2, \ldots, m_d$,

$$\dot{x}_{di} = A_{q_i} x_{di} + L_{i0} y_0 + L_{id} y_d + sL_{iz}(s) x_z + B_{q_i} \left(u_{di} + M_{ia} x_a + M_{ib} x_b + M_{ic} x_c + \sum_{j=1}^{m_d} M_{ij} x_{dj} \right), \tag{6.3.11}$$

$$y_{di}=C_{q_i}x_i+C_{q_i z}x_z+sC_{q_i zs}(s)x_z,\ y_d=C_d x_d+C_{dz}x_z+sC_{dzs}(s)x_z, \tag{6.3.12}$$

for some constant submatrices of appropriate dimensions, and for some matrices whose elements are polynomials of s. Here the states x_z, x_e, x_a, x_b, x_c and x_d are of dimensions n_z, n_e, n_a, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, respectively, while x_{di} is of dimension q_i for each $i = 1, 2, \ldots, m_d$. The control vectors u_0, u_d and u_c are of dimensions m_0, m_d and $m_c = m - m_0 - m_d$, respectively, while the output vectors y_0, y_d and y_b are respectively of dimensions m_0, m_d and $p_b = p - m_0 - m_d$. The pair (A_{bb}, C_b) is observable, the pair (A_{cc}, B_c) is controllable, and the triple $(A_{q_i}, B_{q_i}, C_{q_i})$ has the form

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1 \quad 0 \quad \cdots \quad 0]. \tag{6.3.13}$$

Assuming that $x_i, i = 1, 2, \ldots, m_d$, are arranged such that $q_i \le q_{i+1}$, the matrix L_{id} will be in the particular form

$$L_{id} = [L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0], \tag{6.3.14}$$

with its last row being all zeros.

A constructive proof of the structural decomposition in Theorem 6.3.1 will be given in the next section. The following corollaries of Theorem 6.3.1 give a compact matrix form for the structural decomposition and establish its equivalence to the original system. The proofs of these corollaries follow from the constructive proof of Theorem 6.3.1.

Corollary 6.3.1. *The structural decomposition of Σ of Theorem 6.3.1 can be represented as follows:*

$$\tilde{E} = \Gamma_e(s) E \Gamma_s = E_s - E_z(s) + \Psi(s) \tag{6.3.15}$$

$$= \begin{bmatrix} J_{n_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_d} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ N_{ez}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{az}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{bz}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{cz}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{dz}(s) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \Psi(s),$$

$$\tilde{A} = \Gamma_e(s) A \Gamma_s = A_s + s\Psi(s) \tag{6.3.16}$$

$$= \left(B_0 C_0 + \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd}C_d \\ 0 & 0 & B_c M_{ca} & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ 0 & 0 & B_d M_{da} & B_d M_{db} & B_d M_{dc} & A_{dd} \end{bmatrix} \right) + s\Psi(s),$$

$$\tilde{B} = \Gamma_e(s)B\Gamma_i(s) = B_s = \begin{bmatrix} 0 & 0 & 0 \\ B_{0e} & B_{de} & B_{ce} \\ B_{0a} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{6.3.17}$$

$$\tilde{C} = \Gamma_o^{-1}C\Gamma_s = C_s + \Psi_c = \begin{bmatrix} C_{0z} & 0 & C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ C_{dz} & 0 & 0 & 0 & 0 & C_d \\ C_{bz} & 0 & 0 & C_b & 0 & 0 \end{bmatrix} + \Psi_c, \tag{6.3.18}$$

$$\tilde{D} = \Gamma_o^{-1}D\Gamma_i(s) = D_s + \Psi_d(s) = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Psi_d(s), \tag{6.3.19}$$

where $\Psi(s)$ is an $n \times n$ matrix with entries being some polynomials of s,

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ B_{0a} \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 & C_{0a} & C_{0b} & C_{0c} & C_{0d} \end{bmatrix}, \tag{6.3.20}$$

and

$$\Psi_c\tilde{x} + \Psi_d(s)\tilde{u} = \Psi_R(s)x_z, \tag{6.3.21}$$

and where $\Psi_R(s)$ is a matrix with its elements being some polynomials of s.

Corollary 6.3.2. *Let Σ_s be a descriptor system characterized by a constant matrix quintuple, $(E_s, A_s, B_s, C_s, D_s)$, which has a transfer function*

$$H_s(s) = C_s(sE_s - A_s)^{-1}B_s + D_s. \tag{6.3.22}$$

Let $H(s)$ be the transfer function of the original descriptor system (6.1.1). Then,

$$H(s) = C(sE - A)^{-1}B + D = \Gamma_o H_s(s)\Gamma_i^{-1}(s), \tag{6.3.23}$$

which shows that the transfer functions of the original system Σ and the system characterized by Σ_s are related by some nonsingular transformations.

Next, we would like to note that it does not lose too much generality to assume that the state variable of Σ, $x(t)$, to be a continuous function of t at $t = 0$, which simply means that there is no sudden jump from $x(0^-)$ to $x(0^+)$. Then, it is straightforward to show that (6.3.4) implies that $x_z = 0$, for all t. We summarize below the physical features of the state variables in our structural decomposition under such a minor assumption:

1. The state x_z is purely static and identically zero for all time t. It can neither be controlled by the system input nor be affected by other states.

2. The state x_e is again static and contains a linear combination of the input variables of the system and their derivatives of appropriate orders.

3. The state x_a is neither directly controlled by the system input nor does it directly affect the system output.

4. The output y_b and the state x_b are not directly influenced by any input, although they could be indirectly controlled through the output y_d. Moreover, (A_{bb}, C_b) forms an observable pair. This implies that the state x_b is observable.

5. The state x_c is directly controlled by the input u_c, but it does not directly affect any output. (A_{cc}, B_c) forms a controllable pair. This implies that the state x_c is controllable.

6. The variables u_{di} control the output y_{di} through a stack of q_i integrators. Furthermore, all the states x_{di} are both controllable and observable.

As will be seen shortly, all the invariant properties of the given system can be easily obtained from our structural decomposition. Furthermore, it is simple and interesting to observe from the structural decomposition of Σ of Theorem 6.3.1 that there are redundant state variables associated with the given system. An immediate application of such a technique is to reduce the descriptor system to an equivalent proper system as the state variable x_z is identically zero, and the state variable x_e is simply a linear combination of the system input variables and their derivatives. As such, from the input-output behavior point of view, the given descriptor system can be equivalently reduced to the following proper system:

$$\dot{x}_a = A_{aa}x_a + B_{0a}y_0 + L_{ad}y_d + L_{ab}y_b, \tag{6.3.24}$$

$$\dot{x}_b = A_{bb}x_b + B_{0b}y_0 + L_{bd}y_d, \quad y_b = C_b x_b, \tag{6.3.25}$$

$$\dot{x}_c = A_{cc}x_c + B_{0c}y_0 + L_{cd}y_d + L_{cb}y_b + B_c M_{ca}x_a + B_c u_c, \tag{6.3.26}$$

$$y_0 = C_{0a}x_a + C_{0b}x_b + C_{0c}x_c + C_{0d}x_d + u_0, \tag{6.3.27}$$

and for each $i = 1, 2, \ldots, m_d$,

$$\dot{x}_{di} = A_{q_i}x_{di} + L_{i0}y_0 + L_{id}y_d + B_{q_i}\left(u_{di} + M_{ia}x_a + M_{ib}x_b + M_{ic}x_c + \sum_{j=1}^{m_d} M_{ij}x_{dj}\right), \tag{6.3.28}$$

$$y_{\mathrm{d}i} = C_{q_i} x_i, \quad y_{\mathrm{d}} = C_{\mathrm{d}} x_{\mathrm{d}}. \tag{6.3.29}$$

As such, we can expect that many results related to systems and control theory of proper systems can be extended to descriptor systems without too much difficulty. This is the most significant property of the structural decomposition technique developed in this chapter.

We mentioned earlier that the structural decomposition of Theorem 6.3.1 has the distinct feature of revealing the structural properties of the given descriptor system Σ. We are now ready to study how the system properties of Σ, such as the stabilizability, detectability, finite and infinite zero structures, can be obtained from the decomposition of the system.

We first recall the definitions of stability, stabilizability and detectability of linear descriptor systems from the literature (see, *e.g.*, Dai [43]).

Definition 6.3.1 (Stability, Stabilizability and Detectability). *The system Σ of (6.1.1) is said to be stable if its characteristic polynomial $\det(sE-A)$ has all roots in $\mathbb{C}^-$. It is said to be stabilizable if there exists a constant matrix F of appropriate dimensions such that the roots of $\det(sE-A-BF)$ are stable. Similarly, it is said to be detectable if there exists a constant matrix K of appropriate dimensions such that the roots of $\det(sE-A-KC)$ are stable.*

We have the following property.

Property 6.3.1 (Stabilizability and Detectability). *The given system Σ of (6.1.1) is stabilizable if and only if $(A_{\mathrm{con}}, B_{\mathrm{con}})$ is stabilizable, and is detectable if and only if $(A_{\mathrm{obs}}, C_{\mathrm{obs}})$ is detectable, where*

$$A_{\mathrm{con}} := \begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ab}} C_{\mathrm{b}} \\ 0 & A_{\mathrm{bb}} \end{bmatrix}, \quad B_{\mathrm{con}} := \begin{bmatrix} B_{0\mathrm{a}} & L_{\mathrm{ad}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} \end{bmatrix}, \tag{6.3.30}$$

and

$$A_{\mathrm{obs}} := \begin{bmatrix} A_{\mathrm{aa}} & 0 \\ B_{\mathrm{c}} M_{\mathrm{ca}} & A_{\mathrm{cc}} \end{bmatrix}, \quad C_{\mathrm{obs}} := \begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{c}} \\ M_{\mathrm{da}} & M_{\mathrm{dc}} \end{bmatrix}. \tag{6.3.31}$$

The definition of invariant zeros of descriptor systems can be made in a similar way as that for proper systems (see Chapter 3) or in the Kronecker canonical form associated with Σ (see, *e.g.*, Malabre [97]).

Definition 6.3.2 (Invariant Zeros). *A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of the descriptor system Σ of (6.1.1) if*

$$\mathrm{rank}\{P_\Sigma(\alpha)\} < n + \mathrm{normrank}\{H(s)\}, \tag{6.3.32}$$

where

$$H(s) = C(sE - A)^{-1}B + D, \tag{6.3.33}$$

and normrank$\{(H(s)\}$ *denotes the normal rank of* $H(s)$, *which is defined as its rank over the field of rational functions of* s *with real coefficients, and* $P_\Sigma(s)$ *is the system pencil associated with* Σ *as given in (6.1.4). Invariant zeros of* Σ *correspond to* β_i *in (6.1.6).*

The following property shows that the invariant zeros of Σ can be obtained through the structural decomposition in a trivial manner.

Property 6.3.2 (Invariant Zeros, Normal Rank). *The invariant zeros of* Σ *are the eigenvalues of* $A_{\rm aa}$. *The normal rank of* Σ *is equal to* $m_0 + m_{\rm d}$.

We note that the Jordan canonical structure of $A_{\rm aa}$ corresponds to list $\mathbf{I}_1$ of Morse [100] of the system. In fact, in many applications, it is useful and necessary to further separate the state variable associated with the invariant zero dynamics, i.e., $x_{\rm a}$, into a stable part, an unstable part and the part associated with invariant zeros on the imaginary axis. It follows from Theorem 4.2.1 that there exists a nonsingular transformation, say $T_{\rm a}$, such that

$$x_{\rm a} = T_{\rm a} \begin{pmatrix} x_{\rm a}^- \\ x_{\rm a}^0 \\ x_{\rm a}^+ \end{pmatrix}, \quad T_{\rm a}^{-1} A_{\rm aa} T_{\rm a} = \begin{bmatrix} A_{\rm aa}^- & 0 & 0 \\ 0 & A_{\rm aa}^0 & 0 \\ 0 & 0 & A_{\rm aa}^+ \end{bmatrix}, \tag{6.3.34}$$

where $\lambda(A_{\rm aa}^-) \subset \mathbb{C}^-$ are the stable invariant zeros, $\lambda(A_{\rm aa}^0) \subset \mathbb{C}^0$ are the invariant zeros on the imaginary axis, and $\lambda(A_{\rm aa}^+) \subset \mathbb{C}^-$ are the unstable invariant zeros.

The infinite zero structure of the given system Σ can be defined as the structure associated with the corresponding block of (6.1.7) in the Kronecker canonical form of $P_\Sigma(s)$. It can also be defined using the well-known Smith–McMillan form or list $\mathbf{I}_4$ of Morse [100].

Property 6.3.3 (Infinite Zero Structure). Σ *has* m_0 *infinite zeros of order* 0. *The infinite zero structure (of order greater than* 0*) of* Σ *is given by*

$$S_\infty^\star(\Sigma) = \{q_1, q_2, \dots, q_{m_{\rm d}}\}, \tag{6.3.35}$$

i.e., for each $i = 1, 2, \dots, m_{\rm d}$, Σ *has an infinite zero of order* q_i, *respectively.*

Our structural decomposition can also exhibit the invertibility structure of a given descriptor system Σ. Basically, for the usual case when matrices $[\,B' \;\; D'\,]$

and $[\,C \;\; D\,]$ are of maximal rank, the system Σ or equivalently $H(s)$ is said to be left invertible if there exists a rational matrix function $L(s)$ such that

$$L(s)H(s) = I_m. \tag{6.3.36}$$

The system Σ is right invertible if there exists a rational matrix function $R(s)$ such that

$$H(s)R(s) = I_p. \tag{6.3.37}$$

Moreover, Σ is said to be invertible if it is both left and right invertible, and Σ is noninvertible, or degenerate, if it is neither left nor right invertible. Again, the detailed invertibility structures of Σ are related to the corresponding left and right minimal indices associated with the blocks of (6.1.5) in the Kronecker canonical form of $P_\Sigma(s)$. In fact, the right and left minimal indices are respectively equivalent to the observability indices of $(A_{\rm bb}, C_{\rm b})$ and the controllability indices of $(A_{\rm cc}, B_{\rm c})$, which are respectively related to lists $\mathbf{I}_3$ and $\mathbf{I}_2$ of Morse [100].

Property 6.3.4 (Invertibility Structure). *Σ is right invertible if and only if $x_{\rm b}$ and hence $y_{\rm b}$ are nonexistent, left invertible if and only if $x_{\rm c}$ and hence $u_{\rm c}$ is nonexistent, and invertible if and only if both $x_{\rm b}$ and $x_{\rm c}$ are nonexistent.*

6.4 Proofs of Theorem 6.3.1 and Its Properties

We now present complete proofs for the main results of the previous section, *i.e.*, Theorem 6.3.1 and all its structural properties.

Proof of Theorem 6.3.1. The following is a step-by-step algorithm for the structural decomposition of general multivariable descriptor systems.

STEP MIMO-SDDS.1. *Preliminary Decomposition.*

This step is to separate the given descriptor system into a proper subsystem and a descriptor subsystem with a special structure. This step is identical to STEP SISO-SDDS.1 in Section 6.2. It is to find two nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{6.4.1}$$

and

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\,C_1 \;\; C_2\,], \tag{6.4.2}$$

where A_1, B_1, B_2, C_1 and C_2 are matrices of appropriate dimensions, and N is a nilpotent matrix with an appropriate nilpotent index, say h, *i.e.*, $N^{h-1} \neq 0$ and $N^h = 0$. Equivalently, Σ can be decomposed into the following two subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 \, x_1 + B_1 \, u, \\ y_1 = C_1 \, x_1 + D \, u, \end{cases} \tag{6.4.3}$$

and

$$\Sigma_2 : \begin{cases} N \, \dot{x}_2 = \quad x_2 + B_2 \, u, \\ \quad y_2 = C_2 \, x_2, \end{cases} \tag{6.4.4}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, and $y = y_1 + y_2$.

STEP MIMO-SDDS.2. *Decomposition of* $x_{\rm z}$ *and* $x_{\rm e}$.

The key idea is to separate the controllable and uncontrollable parts of the pair (N, B_2) in Σ_2. It follows from Theorems 4.4.1 and 4.4.2 of Chapter 4 that there exist nonsingular coordinate transformations

$$x_2 = T_{\rm s} \hat{x}_2, \quad u = T_{\rm i} \hat{u}, \tag{6.4.5}$$

such that

$$\hat{x}_2 = \begin{pmatrix} x_{\rm v} \\ x_{\rm z} \end{pmatrix}, \quad x_{\rm v} = \begin{pmatrix} x_{\rm v1} \\ x_{\rm v2} \\ \vdots \\ x_{{\rm v}n_{\rm e}} \end{pmatrix}, \quad x_{\rm z} \in \mathbb{R}^{n_{\rm z}}, \quad \hat{u} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n_{\rm e}} \\ \hat{u}_* \end{pmatrix}, \tag{6.4.6}$$

where

$$x_{{\rm v}i} \in \mathbb{R}^{p_i}, \quad x_{{\rm v}i} = \begin{pmatrix} x_{{\rm v}i,1} \\ x_{{\rm v}i,2} \\ \vdots \\ x_{{\rm v}i,p_i} \end{pmatrix}, \quad i = 1, 2, \ldots, n_{\rm e},$$

$$\hat{N} = T_{\rm s}^{-1} N T_{\rm s} = \begin{bmatrix} J_{\rm v} & N_{\rm zv} \\ 0 & J_{n_{\rm z}} \end{bmatrix} = \begin{bmatrix} J_{\rm v1} & 0 & \cdots & 0 & N_{1{\rm z}} \\ 0 & J_{\rm v2} & \cdots & 0 & N_{2{\rm z}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{{\rm v}n_{\rm e}} & N_{n_{\rm e}{\rm z}} \\ 0 & 0 & \cdots & 0 & J_{n_{\rm z}} \end{bmatrix},$$

$$\hat{B}_2 = T_{\rm s}^{-1} B_2 T_{\rm i} = \begin{bmatrix} B_{\rm v} \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_{\rm e}} & B_{1{\rm z}} \\ 0 & B_{22} & \cdots & B_{2n_{\rm e}} & B_{2{\rm z}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{n_{\rm e}n_{\rm e}} & B_{n_{\rm e}{\rm z}} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and where $(J_{\rm v}, B_{\rm v})$ is controllable. Moreover, the fact that N is nilpotent implies that $J_{{\rm v}i}$ and $J_{n_{\rm z}}$ have all their eigenvalues at 0, and $J_{{\rm v}i}$, $N_{i{\rm z}}$, $B_{i{\rm z}}$ and B_{ij} have the following forms,

$$J_{{\rm v}i}=\begin{bmatrix}0&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&1\\ 0&0&0&0\end{bmatrix},\quad N_{i{\rm z}}=\begin{bmatrix}\eta_{i{\rm z},1}\\ \vdots\\ \eta_{i{\rm z},p_i-1}\\ \eta_{i{\rm z},p_i}\end{bmatrix},\quad B_{ii}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},\tag{6.4.7}$$

$$B_{i{\rm z}}=\begin{bmatrix}b_{i{\rm z},1}\\ \vdots\\ b_{i{\rm z},p_i-1}\\ 0\end{bmatrix},\quad B_{ij}=\begin{bmatrix}b_{ij,1}\\ \vdots\\ b_{ij,p_i-1}\\ 0\end{bmatrix}.\tag{6.4.8}$$

As such, by the transformation of (6.4.5), Σ_2 is decomposed into the subsystems

$$J_{n_{\rm z}}\dot{x}_{\rm z}=x_{\rm z},\tag{6.4.9}$$

and for $i=1,2,\ldots,n_{\rm e}$,

$$J_{{\rm v}i}\dot{x}_{{\rm v}i}+N_{i{\rm z}}\dot{x}_{\rm z}=x_{{\rm v}i}+B_{ii}\hat{u}_i+\sum_{j=i+1}^{n_{\rm e}}B_{ij}\hat{u}_j+B_{i{\rm z}}\hat{u}_*,\tag{6.4.10}$$

which is equivalent to

$$J_{{\rm v}i}\dot{x}_{{\rm v}i}=x_{{\rm v}i}+B_{ii}\hat{u}_i+\sum_{j=i+1}^{n_{\rm e}}B_{ij}\hat{u}_j+B_{i{\rm z}}\hat{u}_*-\left(N_{i{\rm z}}\dot{x}_{\rm z}\right).\tag{6.4.11}$$

Because of the special structure of $J_{{\rm v}i}$, we have, for $i=1,2,\ldots,n_{\rm e}$,

$$\left.\begin{aligned}\dot{x}_{{\rm v}i,2}&=x_{{\rm v}i,1}+\sum_{j=i+1}^{n_{\rm e}}b_{ij,1}\hat{u}_j+b_{i{\rm z},1}\hat{u}_*-\eta_{i{\rm z},1}\dot{x}_{\rm z},\\ \dot{x}_{{\rm v}i,3}&=x_{{\rm v}i,2}+\sum_{j=i+1}^{n_{\rm e}}b_{ij,2}\hat{u}_j+b_{i{\rm z},2}\hat{u}_*-\eta_{i{\rm z},2}\dot{x}_{\rm z},\\ &\;\;\vdots\\ \dot{x}_{{\rm v}i,p_i}&=x_{{\rm v}i,p_i-1}+\sum_{j=i+1}^{n_{\rm e}}b_{ij,p_i-1}\hat{u}_j+b_{i{\rm z},p_i-1}\hat{u}_*-\eta_{i{\rm z},p_i-1}\dot{x}_{\rm z},\end{aligned}\right\}\tag{6.4.12}$$

$$\hat{u}_i=-x_{{\rm v}i,p_i}+\eta_{i{\rm z},p_i}\dot{x}_{\rm z}.\tag{6.4.13}$$

Repeatedly differentiating $\hat{u}_i$ of (6.4.13), we obtain

$$x_{\mathrm{v}i,1} = -\hat{u}_i^{(p_i-1)} - \sum_{k=0}^{p_i-2} \sum_{j=i+1}^{n_e} b_{ij,k+1}\hat{u}_j^{(k)} - \sum_{k=0}^{p_i-2} b_{i\mathrm{z},k+1}\hat{u}_*^{(k)} + \sum_{k=1}^{p_i} \eta_{i\mathrm{z},k} x_{\mathrm{z}}^{(k)}. \tag{6.4.14}$$

Let us define a new input variable

$$\check{u}_i = -x_{\mathrm{v}i,1} + \sum_{k=1}^{p_i} \eta_{i\mathrm{z},k} x_{\mathrm{z}}^{(k)} = \psi_i(s)\hat{u}, \tag{6.4.15}$$

for an appropriate vector $\psi_i(s)$ whose elements are polynomials of s. Then, we can rewrite (6.4.12) as follows:

$$\left.\begin{aligned}
\dot{x}_{\mathrm{v}i,2} &= -\sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,1}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_e} b_{ij,1}\check{u}_j + b_{i\mathrm{z},1}\hat{u}_* \\
&\qquad - \check{u}_i + \sum_{k=2}^{p_i} \eta_{i\mathrm{z},k}x_{\mathrm{z}}^{(k)} + \sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,1}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}}, \\
\dot{x}_{\mathrm{v}i,3} &= x_{\mathrm{v}i,2} - \sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,2}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_e} b_{ij,2}\check{u}_j \\
&\qquad + b_{i\mathrm{z},2}\hat{u}_* - \eta_{i\mathrm{z},2}\dot{x}_{\mathrm{z}} + \sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,2}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}}, \\
&\ \ \vdots \\
\dot{x}_{\mathrm{v}i,p_i} &= x_{\mathrm{v}i,p_i-1} - \sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,p_i-1}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_e} b_{ij,p_i-1}\check{u}_j \\
&\qquad + b_{i\mathrm{z},p_i-1}\hat{u}_* - \eta_{i\mathrm{z},p_i-1}\dot{x}_{\mathrm{z}} + \sum_{j=i+1\ \&\ p_j>1}^{n_e} b_{ij,p_i-1}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}}.
\end{aligned}\right\} \tag{6.4.16}$$

Next, define

$$\check{u}_{\mathrm{e}} = \begin{pmatrix} \check{u}_1 \\ \check{u}_2 \\ \vdots \\ \check{u}_{n_e} \end{pmatrix} = \begin{pmatrix} -x_{\mathrm{v}1,1} + \sum_{k=1}^{p_1} \eta_{1\mathrm{z},k}x_{\mathrm{z}}^{(k)} \\ -x_{\mathrm{v}2,1} + \sum_{k=1}^{p_2} \eta_{2\mathrm{z},k}x_{\mathrm{z}}^{(k)} \\ \vdots \\ -x_{\mathrm{v}n_e,1} + \sum_{k=1}^{p_{n_e}} \eta_{n_e\mathrm{z},k}x_{\mathrm{z}}^{(k)} \end{pmatrix}, \tag{6.4.17}$$

and

$$x_{\mathrm{e}}=\begin{pmatrix} x_{\mathrm{v}1,1} \\ x_{\mathrm{v}2,1} \\ \vdots \\ x_{\mathrm{v}n_{\mathrm{e}},1} \end{pmatrix}=-\breve{u}_{\mathrm{e}}+\begin{pmatrix} \sum_{k=1}^{p_1}\eta_{1\mathrm{z},k}x_{\mathrm{z}}^{(k)} \\ \sum_{k=1}^{p_2}\eta_{2\mathrm{z},k}x_{\mathrm{z}}^{(k)} \\ \vdots \\ \sum_{k=1}^{p_{n_{\mathrm{e}}}}\eta_{n_{\mathrm{e}}\mathrm{z},k}x_{\mathrm{z}}^{(k)} \end{pmatrix}=-\breve{u}_{\mathrm{e}}+sN_{\mathrm{ez}}(s)x_{\mathrm{z}}, \tag{6.4.18}$$

where $N_{\mathrm{ez}}(s)$ is a matrix whose elements are polynomials of s. It is now straightforward to verify that the transformed system of Σ_2 as given in (6.4.4) can be rearranged into the form

$$\begin{cases} J_{n_{\mathrm{z}}}\dot{x}_{\mathrm{z}}=x_{n_{\mathrm{z}}}, \\ x_{\mathrm{e}} \ =-\breve{u}_{\mathrm{e}}+sN_{\mathrm{ez}}(s)x_{\mathrm{z}}, \\ \dot{\breve{x}}_2 \ =\breve{A}_2\breve{x}_2+\breve{B}_{2\mathrm{e}}\breve{u}_{\mathrm{e}}+\breve{B}_{2*}\hat{u}_*+s\breve{B}_{2\mathrm{z}}(s)x_{\mathrm{z}}, \\ y_2 \ =\breve{C}_2\breve{x}_2+\breve{D}_{2\mathrm{e}}\breve{u}_{\mathrm{e}}+[s\breve{D}_{2\mathrm{z}}(s)+\breve{C}_{\mathrm{z}}]x_{\mathrm{z}}, \end{cases} \tag{6.4.19}$$

where $\breve{x}_2$ consists of all the state variables of x_{v} that are not contained in x_{e}, and $\breve{A}_2$, $\breve{B}_{2\mathrm{e}}$, $\breve{B}_{2*}$, $\breve{C}_2$, $\breve{D}_{2\mathrm{e}}$ and $\breve{C}_{\mathrm{z}}$ are constant matrices of appropriate dimensions, and $\breve{B}_{2\mathrm{z}}(s)$ and $\breve{D}_{2\mathrm{z}}(s)$ are matrices with their entries being some polynomials of s. Furthermore, Σ_1 of (6.4.3) can be rewritten as follows:

$$\begin{cases} \dot{x}_1=A_1x_1+\breve{A}_{12}\breve{x}_2+\breve{B}_{1\mathrm{e}}\breve{u}_{\mathrm{e}}+\breve{B}_{1*}\hat{u}_*+s\breve{B}_{1\mathrm{z}}(s)x_{\mathrm{z}}, \\ y_1=C_1x_1+\breve{C}_{12}\breve{x}_2+\breve{D}_{1\mathrm{e}}\breve{u}_{\mathrm{e}}+\breve{D}_{1*}\hat{u}_*+s\breve{D}_{1\mathrm{z}}(s)x_{\mathrm{z}}, \end{cases} \tag{6.4.20}$$

for some constant matrices $\breve{A}_{12}$, $\breve{B}_{1\mathrm{e}}$, $\breve{B}_{1*}$, $\breve{C}_{12}$, $\breve{D}_{1\mathrm{e}}$ and $\breve{D}_{1*}$ of appropriate dimensions, and for some matrices $\breve{B}_{1\mathrm{z}}(s)$ $\breve{D}_{1\mathrm{z}}(s)$, whose elements are some polynomials of s.

STEP MIMO-SDDS.3. *Formation of a Proper System and Final Decomposition.*

The key idea is to form a proper system from the subsystems (6.4.19) and (6.4.20), and then apply the result of proper systems to obtain a structural decomposition for the original system given in (6.1.1). Following (6.4.19) and (6.4.20), we obtain a proper system

$$\bar{\Sigma}:\begin{cases} \dot{\bar{x}}=\bar{A}\ \bar{x}+\bar{B}\ \bar{u}+\bar{B}_{\mathrm{z}}(s)\ x_{\mathrm{z}}, \\ y=\bar{C}\ \bar{x}+\bar{D}\ \bar{u}+\bar{D}_{\mathrm{z}}(s)\ x_{\mathrm{z}}, \end{cases} \tag{6.4.21}$$

where

$$\bar{x} = \begin{pmatrix} x_1 \\ \breve{x}_2 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \breve{u}_{\rm e} \\ \hat{u}_* \end{pmatrix}, \tag{6.4.22}$$

$$\bar{A} = \begin{bmatrix} A_1 & \breve{A}_{12} \\ 0 & \breve{A}_2 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} \breve{B}_{1\rm e} & \breve{B}_{1*} \\ \breve{B}_{2\rm e} & \breve{B}_{2*} \end{bmatrix}, \; \bar{B}_{\rm z}(s) = \begin{bmatrix} s\breve{B}_{1\rm z}(s) \\ s\breve{B}_{2\rm z}(s) \end{bmatrix}, \tag{6.4.23}$$

$$\bar{D}_{\rm z}(s) = \breve{C}_{\rm z} + s\breve{D}_{1\rm z}(s) + s\breve{D}_{2\rm z}(s), \tag{6.4.24}$$

and

$$\bar{C} = [\, C_1 \quad \breve{C}_2 + \breve{C}_{12} \,], \quad \bar{D} = [\, \breve{D}_{1\rm e} + \breve{D}_{2\rm e} \quad \breve{D}_{1*} \,]. \tag{6.4.25}$$

It then follows from the result of Theorem 5.4.1 that there exist nonsingular transformations $\bar{\Gamma}_{\rm s} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, where $\bar{n} = n - n_{\rm e} - n_{\rm z}$, $\bar{\Gamma}_{\rm o} \in \mathbb{R}^{p\times p}$ and $\bar{\Gamma}_{\rm i} \in \mathbb{R}^{m\times m}$ such that when they are applied to $\bar{\Sigma}$, *i.e.*,

$$\bar{x} = \bar{\Gamma}_{\rm s} \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \; y = \bar{\Gamma}_{\rm o}\tilde{y} = \bar{\Gamma}_{\rm o} \begin{pmatrix} y_0 \\ y_{\rm d} \\ y_{\rm b} \end{pmatrix}, \; \bar{u} = \bar{\Gamma}_{\rm i}\tilde{u} = \bar{\Gamma}_{\rm i} \begin{pmatrix} u_0 \\ u_{\rm d} \\ u_{\rm c} \end{pmatrix}, \tag{6.4.26}$$

where $x_{\rm a} \in \mathbb{R}^{n_{\rm a}}$, $x_{\rm b} \in \mathbb{R}^{n_{\rm b}}$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, $x_{\rm d} \in \mathbb{R}^{n_{\rm d}}$, $u_0 \in \mathbb{R}^{n_0}$, $u_{\rm c} \in \mathbb{R}^{m_{\rm c}}$, $u_{\rm d} \in \mathbb{R}^{m_{\rm d}}$, $y_0 \in \mathbb{R}^{n_0}$, $y_{\rm b} \in \mathbb{R}^{p_{\rm b}}$, $y_{\rm d} \in \mathbb{R}^{m_{\rm d}}$,

$$x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{\rm d1} \\ y_{\rm d2} \\ \vdots \\ y_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \\ \vdots \\ u_{{\rm d}m_{\rm d}} \end{pmatrix}, \tag{6.4.27}$$

we have

$$\dot{x}_{\rm a} = A_{\rm aa}x_{\rm a} + B_{0\rm a}y_0 + L_{\rm ad}y_{\rm d} + L_{\rm ab}y_{\rm b} + sL_{\rm az}(s)x_{\rm z}, \tag{6.4.28}$$

$$\dot{x}_{\rm b} = A_{\rm bb}x_{\rm b} + B_{0\rm b}y_0 + L_{\rm bd}y_{\rm d} + sL_{\rm bz}(s)x_{\rm z}, \tag{6.4.29}$$

$$y_{\rm b} = C_{\rm b}x_{\rm b} + C_{\rm bz}x_{\rm z} + sC_{\rm bzs}(s)x_{\rm z}, \tag{6.4.30}$$

$$\dot{x}_{\rm c} = A_{\rm cc}x_{\rm c} + B_{0\rm c}y_0 + L_{\rm cd}y_{\rm d} + L_{\rm cb}y_{\rm b} + B_{\rm c}\,[u_{\rm c} + M_{\rm ca}x_{\rm a}] + sL_{\rm cz}(s)x_{\rm z}, \tag{6.4.31}$$

$$y_0 = C_{0\rm a}x_{\rm a} + C_{0\rm b}x_{\rm b} + C_{0\rm c}x_{\rm c} + C_{0\rm d}x_{\rm d} + u_0 + C_{0\rm z}x_{\rm z} + sC_{0\rm zs}(s)x_{\rm z}, \tag{6.4.32}$$

and

$$\dot{x}_{{\rm d}i} = A_{q_i}x_{{\rm d}i} + L_{i0}y_0 + L_{i\rm d}y_{\rm d} + sL_{i\rm z}(s)x_{\rm z} + B_{q_i}\left(u_{{\rm d}i} + M_{i\rm a}x_{\rm a} + M_{i\rm b}x_{\rm b} + M_{i\rm c}x_{\rm c} + \sum_{j=1}^{m_{\rm d}} M_{ij}x_{{\rm d}j} \right), \tag{6.4.33}$$

$$y_{\mathrm{d}i} = C_{q_i} x_{\mathrm{d}i} + C_{q_i\mathrm{z}} x_{\mathrm{z}} + s C_{q_i\mathrm{zs}}(s) x_{\mathrm{z}}, \; y_{\mathrm{d}} = C_{\mathrm{d}} x_{\mathrm{d}} + C_{\mathrm{dz}} x_{\mathrm{z}} + s C_{\mathrm{dzs}}(s) x_{\mathrm{z}}, \tag{6.4.34}$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ having the special form as given in (6.3.13).

This completes the proof of Theorem 6.3.1. ■

Finally, we note that the results of Corollaries 6.3.1 and 6.3.2 follow from the above construction procedures and some tedious manipulations.

Proofs of Structural Decomposition Properties. Once the results in the following two lemmas are established, the proofs of the properties of the structural decomposition of descriptor systems can be carried out in a similar way as those for proper systems given in Section 5.5 of Chapter 5.

Lemma 6.4.1. *Consider a system Σ characterized by (E, A, B, C, D) or in the state space form of (6.1.1). Then, for any state feedback gain $F \in \mathbb{R}^{m\times n}$ satisfying* $\det(sE - A - BF) \not\equiv 0$, *the system with the state feedback Σ_{F} characterized by $(E, A+BF, B, C+DF, D)$ has the following properties:*

1. *Σ_{F} is stabilizable if and only if Σ is stabilizable;*
2. *the normal rank of Σ_{F} is equal to that of Σ;*
3. *the invariant zero structure of Σ_{F} is the same as that of Σ;*
4. *the infinite zero structure of Σ_{F} is the same as that of Σ; and*
5. *Σ_{F} is (left or right or non) invertible if and only if Σ is (left or right or non) invertible.*

Proof. Item 1 is obvious. Item 2 follows from the reductions

$$\begin{aligned} H_{\mathrm{F}}(s) := &\,(C + DF)(sE - A - BF)^{-1}B + D \\ = &\,(C + DF)(sE - A)^{-1}\left[I - BF(sE - A)^{-1}\right]^{-1} B + D \\ = &\,(C + DF)(sE - A)^{-1}B\left[I - F(sE - A)^{-1}B\right]^{-1} + D \\ = &\,\left[C(sE - A)^{-1}B + D\right]\left[I - F(sE - A)^{-1}B\right]^{-1} \\ = &\,H(s)\left[I - F(sE - A)^{-1}B\right]^{-1}. \end{aligned} \tag{6.4.35}$$

Next, noting that

$$\begin{bmatrix} A + BF - sE & B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \tag{6.4.36}$$

and the fact that the invariances of $P_\Sigma(s)$ are strictly equivalent under nonsingular constant transformations of the form in (6.1.2), Items 3, 4 and 5 follow. ■

Lemma 6.4.2. *Consider a system Σ characterized by (E, A, B, C, D) or in the state space form of (6.1.1). Then, for a constant output injection gain $K \in \mathbb{R}^{n\times p}$ satisfying $\det(sE-A-KC) \not\equiv 0$, the system with the output injection Σ_K characterized by $(E, A+KC, B+KD, C, D)$ has the following properties:*

1. *Σ_K is stabilizable if and only if Σ is stabilizable;*
2. *the normal rank of Σ_K is equal to that of Σ;*
3. *the invariant zero structure of Σ_K is the same as that of Σ;*
4. *the infinite zero structure of Σ_K is the same as that of Σ; and*
5. *Σ_K is (left or right or non) invertible if and only if Σ is (left or right or non) invertible.*

Proof: It is a dual version of Lemma 6.4.1. ■

It follows from Corollary 6.3.2 that the properties of the transformed system Σ_s are equivalent to those of the original system. The proofs of the structural properties of descriptor systems can be carried out as those for proper systems. We leave the details for the interested readers.

We illustrate the structural decomposition of general descriptor systems and its properties in the following example.

Example 6.4.1. We consider a descriptor system of (6.1.1) characterized by

$$E = \left[\begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 & 2 \\ 0 & 0 & -1 & 1 & -3 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad A = I_7, \quad B = \left[\begin{array}{ccc} 1 & 1 & 0 \\ \hline -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{array}\right], \tag{6.4.37}$$

and

$$C = \left[\begin{array}{c|cccccc} 1 & -2 & 0 & 0 & 1 & 2 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 & 0 \end{array}\right], \quad D = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.4.38}$$

STEP MIMO-SDDS.1. *Preliminary Decomposition.*

It is simple to note that the given system is already in the forms of (6.4.2) with $n_1 = 1, n_2 = 6$,

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_1 + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} u, \\ y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u, \end{cases}$$

and Σ_2 being characterized by

$$N\dot{x}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 2 \\ 0 & -1 & 1 & -3 & 0 & 3 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{x}_2 = x_2 + B_2 u = x_2 + \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} u,$$

and

$$y_2 = C_2 x_2 = \begin{bmatrix} -2 & 0 & 0 & 1 & 2 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{bmatrix} x_2.$$

STEP MIMO-SDDS.2. *Decomposition of x_z and x_e.*

Using the toolkit of [87], we obtain two nonsingular transformations

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad T_i = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

which transform Σ_2 into the canonical form

$$x_2 = T_s \begin{pmatrix} x_v \\ x_z \end{pmatrix}, \quad x_v = \begin{pmatrix} x_{v1} \\ x_{v2} \end{pmatrix}, \quad u = T_i \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_* \end{pmatrix},$$

$$T_s^{-1} N T_s = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad T_s^{-1} B_2 T_i = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

and

$$C_2 T_s = \left[\begin{array}{ccc|cc|c} 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

The transformed system of Σ_2 can then be written as

$$\left.\begin{aligned} \dot{x}_{\mathrm{v}1,2} &= x_{\mathrm{v}1,1} + \hat{u}_* - \dot{x}_{\mathrm{z}}, \\ \dot{x}_{\mathrm{v}1,3} &= x_{\mathrm{v}1,2} + \hat{u}_2 + \hat{u}_* - \dot{x}_{\mathrm{z}}, \\ \hat{u}_1 &= -x_{\mathrm{v}1,3} + \dot{x}_{\mathrm{z}}, \end{aligned}\right\} \tag{6.4.39}$$

$$\left.\begin{aligned} \dot{x}_{\mathrm{v}2,2} &= x_{\mathrm{v}2,1} + \hat{u}_* - \dot{x}_{\mathrm{z}}, \\ \hat{u}_2 &= -x_{\mathrm{v}2,2} + \dot{x}_{\mathrm{z}}, \end{aligned}\right\} \tag{6.4.40}$$

and

$$0 \cdot \dot{x}_{\mathrm{z}} = x_{\mathrm{z}} \quad \Rightarrow \quad x_{\mathrm{z}} = 0, \tag{6.4.41}$$

as well as

$$y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{\mathrm{v}1,3} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_{\mathrm{v}2,2} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_z \,. \tag{6.4.42}$$

Hence, we have $n_{\mathrm{z}} = 1, n_{\mathrm{e}} = 2, p_1 = 3, p_2 = 2,$

$$x_{\mathrm{v}1,1} = -\ddot{\hat{u}}_1 - \dot{\hat{u}}_2 - \left(\hat{u}_* + \dot{\hat{u}}_*\right) + \left(\dot{x}_{\mathrm{z}} + \ddot{x}_{\mathrm{z}} + x_{\mathrm{z}}^{(3)}\right),$$

and

$$x_{\mathrm{v}2,1} = -\dot{\hat{u}}_2 - \hat{u}_* + (\dot{x}_{\mathrm{z}} + \ddot{x}_{\mathrm{z}}).$$

Next, define

$$\check{u}_{\mathrm{e}} = \begin{pmatrix} \check{u}_1 \\ \check{u}_2 \end{pmatrix} = \begin{pmatrix} \ddot{\hat{u}}_1 + \dot{\hat{u}}_2 + \dot{\hat{u}}_* + \hat{u}_* \\ \dot{\hat{u}}_2 + \hat{u}_* \end{pmatrix} = \begin{bmatrix} s^2 & s & s+1 \\ 0 & s & 1 \end{bmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_* \end{pmatrix},$$

and

$$x_{\mathrm{e}} = \begin{pmatrix} x_{\mathrm{v}1,1} \\ x_{\mathrm{v}2,1} \end{pmatrix} = -\check{u}_{\mathrm{e}} + \begin{pmatrix} \dot{x}_{\mathrm{z}} + \ddot{x}_{\mathrm{z}} + x_{\mathrm{z}}^{(3)} \\ \dot{x}_{\mathrm{z}} + \ddot{x}_{\mathrm{z}} \end{pmatrix} = -\check{u}_{\mathrm{e}} + sN_{\mathrm{ez}}(s)x_{\mathrm{z}},$$

where

$$N_{\mathrm{ez}}(s) = \begin{bmatrix} s^2 + s + 1 \\ s + 1 \end{bmatrix}.$$

Then, (6.4.39) and (6.4.40) can be rewritten as

$$\begin{aligned} \dot{x}_{\mathrm{v}1,2} &= -\check{u}_1 + \hat{u}_* + \left(\ddot{x}_z + x_{\mathrm{z}}^{(3)}\right), \\ \dot{x}_{\mathrm{v}1,3} &= x_{\mathrm{v}1,2} - x_{\mathrm{v}2,2} + \hat{u}_*, \\ \dot{x}_{\mathrm{v}2,2} &= -\check{u}_2 + \hat{u}_* + \ddot{x}_{\mathrm{z}}, \end{aligned}$$

or in the matrix form

$$\dot{\check{x}}_2 = \begin{pmatrix} \dot{x}_{\mathrm{v}1,2} \\ \dot{x}_{\mathrm{v}1,3} \\ \dot{x}_{\mathrm{v}2,2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \check{x}_2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \check{u}_{\mathrm{e}} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \hat{u}_* + s \begin{bmatrix} s^2 + s \\ 0 \\ s \end{bmatrix} x_{\mathrm{z}}. \tag{6.4.43}$$

Also, (6.4.42) can be rewritten as

$$y_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \check{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_z. \tag{6.4.44}$$

Further, we have

$$B_1 T_i = [\,2 \quad 1 \quad 0\,], \qquad DT_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.4.45}$$

In view of (6.4.39), (6.4.40) and (6.4.45), we can rewrite Σ_1 as

$$\dot{x}_1 = x_1 + [\,0 \quad -2 \quad -1\,]\, \check{x}_2 + 3s x_z, \tag{6.4.46}$$

and

$$y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \check{x}_2 + s \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_z. \tag{6.4.47}$$

STEP MIMO-SDDS.3. *Formation of a Proper System and Final Decomposition.*

Combining (6.4.43), (6.4.44), (6.4.46) and (6.4.47), We obtain an auxiliary proper system

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,\bar{u} + \bar{B}_z(s)\, x_z, \\ y = \bar{C}\,\bar{x} + \bar{D}\,\bar{u} + \bar{D}_z(s)\, x_z, \end{cases}$$

with

$$\bar{x} = \begin{pmatrix} x_1 \\ \check{x}_2 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \check{u}_e \\ \hat{u}_* \end{pmatrix},$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \bar{B}_z(s) = s \begin{bmatrix} 3 \\ s^2+s \\ 0 \\ s \end{bmatrix},$$

and

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{D}_z(s) = \begin{bmatrix} s+1 \\ 0 \end{bmatrix}.$$

Again, using the toolkit of [87], we obtain

$$\bar{\Gamma}_s = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{\Gamma}_i = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad \bar{\Gamma}_o = I_2,$$

$n_a = 1, n_b = 0, n_c = 1, n_d = 2,$

$$\bar{x} = \bar{\Gamma}_s \begin{pmatrix} x_a \\ x_c \\ x_{d1} \\ x_{d2} \end{pmatrix}, \quad \bar{u} = \bar{\Gamma}_i \begin{pmatrix} u_d \\ u_c \end{pmatrix} = \bar{\Gamma}_i \begin{pmatrix} u_{d1} \\ u_{d2} \\ u_c \end{pmatrix}, \quad y = \bar{\Gamma}_o \begin{pmatrix} y_{d1} \\ y_{d2} \end{pmatrix},$$

$$\bar{\Gamma}_{\rm s}^{-1}\bar{A}\bar{\Gamma}_{\rm s}=\begin{bmatrix}2&0&-1&2\\1&-1&2&-1\\-2&0&1&-2\\-3&1&-1&-1\end{bmatrix},\quad \bar{\Gamma}_{\rm s}^{-1}\bar{B}\bar{\Gamma}_{\rm i}=\begin{bmatrix}0&0&0\\0&0&1\\1&0&0\\0&1&0\end{bmatrix},$$

$$\bar{\Gamma}_{\rm s}^{-1}\bar{B}_{\rm z}(s)=s\begin{bmatrix}-3\\s^2+s\\s+3\\s+3\end{bmatrix},$$

and

$$\bar{\Gamma}_{\rm o}^{-1}\bar{C}\bar{\Gamma}_{\rm s}=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix},\quad \bar{\Gamma}_{\rm o}^{-1}\bar{D}_{\rm z}(s)=\begin{bmatrix}s+1\\0\end{bmatrix}.$$

Finally, the structural decomposition of the given descriptor system is given by

$$0\cdot\dot{x}_{\rm z}=x_{\rm z},$$

$$x_{\rm e}=-\begin{bmatrix}1\\0\end{bmatrix}u_{\rm c}-\begin{bmatrix}0&0\\2&-1\end{bmatrix}u_{\rm d}+s\begin{bmatrix}s^2+s+1\\s+1\end{bmatrix}x_{\rm z},\quad u_{\rm d}=\begin{pmatrix}u_{\rm d1}\\u_{\rm d2}\end{pmatrix},$$

$$\dot{x}_{\rm a}=2x_{\rm a}+[\,-1\quad 2\,]\,y_{\rm d}-3sx_{\rm z},$$

$$\dot{x}_{\rm c}=-x_{\rm c}+x_{\rm a}+[\,2\quad -1\,]\,y_{\rm d}+u_{\rm c}+s(s^2+s)x_{\rm z},$$

$$\begin{pmatrix}\dot{x}_{\rm d1}\\\dot{x}_{\rm d2}\end{pmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}x_{\rm a}+\begin{bmatrix}0\\1\end{bmatrix}x_{\rm c}+\begin{bmatrix}1&-2\\-1&-1\end{bmatrix}y_{\rm d}+u_{\rm d}+s\begin{bmatrix}s+3\\s+3\end{bmatrix}x_{\rm z},$$

and

$$y_{\rm d}=\begin{pmatrix}y_{\rm d1}\\y_{\rm d2}\end{pmatrix}=\begin{pmatrix}x_{\rm d1}\\x_{\rm d2}\end{pmatrix}+\begin{bmatrix}s+1\\0\end{bmatrix}x_{\rm z}.$$

It is simple to see now from the above decomposition that the given system is right invertible with one invariant zero at $s=2$ and two infinite zeros of order 1. The given system has one state variable, which is identically zero, and two state variables, which are nothing but the linear combination of the system inputs and their derivatives. These state variables are actually redundant in the system dynamics. For completeness, we give below all the necessary transformation matrices:

$$\Gamma_{\rm e}(s)=\begin{bmatrix}0&0&0&0&-1&0&1\\0&-1&s^2-s&-s^2&s^2-s&0&s-s^2\\0&s&0&0&-s-1&-s&s\\-1&-1&-2&2&-1&1&1\\0&-1&s^2-s-1&-s^2&s^2-s&1&s-s^2\\1&1+s&2&-2&1-s&-1-s&s-1\\1&2+s&3&-2&1-s&-2-s&s-1\end{bmatrix},$$

$$\Gamma_s = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Gamma_i^{-1}(s) = \begin{bmatrix} s & -s & 0 \\ s-1 & -s & 1 \\ -s^2+2s+1 & -s & -s-1 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We note that the s-dependent input transformation $\Gamma_i(s)$ simply implies that

$$\begin{pmatrix} u_c \\ u_{d1} \\ u_{d2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \ddot{u} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} \dot{u} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} u.$$

The compact form of the structural decomposition of Σ (see Corollary 6.3.1) is given by

$$E_s = \left[\begin{array}{c|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$A_s = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & 1 & -1 & -1 \end{array}\right],$$

$$B_s = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 2 & -1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right],$$

$$C_s = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad D_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$E_z(s) = \left[\begin{array}{c|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline s^2+s+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ s+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ s^2+s & 0 & 0 & 0 & 0 & 0 & 0 \\ s+3 & 0 & 0 & 0 & 0 & 0 & 0 \\ s+3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\Psi(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & s+1 & -s-1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & s+1 & -s-1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix},$$

$$\Psi_c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_d(s) = \begin{bmatrix} -1/s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that

$$\Psi_c \tilde{x} + \Psi_d(s)\tilde{u} = \begin{bmatrix} s \\ 0 \end{bmatrix} x_z,$$

and

$$H(s) = C(sE - A)^{-1}B + D = \Gamma_o\Big[C_s(sE_s - A_s)^{-1}B_s + D_s\Big]\Gamma_i^{-1}(s).$$

6.5 Discrete-time Descriptor Systems

In this section, we present the structural decomposition of general discrete-time descriptor systems and their structural properties, which is analogous to that for continuous-time systems. We consider a discrete-time descriptor system Σ characterized by

$$\Sigma : \begin{cases} E\ x(k+1) = A\ x(k) + B\ u(k), \\ \qquad y(k) \ = C\ x(k) + D\ u(k), \end{cases} \tag{6.5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimensions. Also, we assume that Σ is regular, *i.e.*, $\det(zE - A) \not\equiv 0$, for $z \in \mathbb{C}$. We have the following theorem.

Theorem 6.5.1. *Consider the discrete-time descriptor system Σ of (6.5.1) satisfying* $\det(zE - A) \not\equiv 0$ *for $z \in \mathbb{C}$. Then, there exist nonsingular state and output transformations $\Gamma_s \in \mathbb{R}^{n\times n}$ and $\Gamma_o \in \mathbb{R}^{p\times p}$, as well as an $m\times m$ nonsingular input* *transformation* $\Gamma_{\rm i}(z)$, *whose inverse has all its elements being some polynomials of* z *(i.e., its inverse contains various forward shifting operators), and an* $n \times n$ *nonsingular transformation* $\Gamma_{\rm e}(z) \in \mathbb{R}^{n\times n}$, *whose elements are polynomials of* z, *which together give a structural decomposition of* Σ *and display explicitly its structural properties.*

The structural decomposition of Σ *can be described by the set of equations*

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}(z)\tilde{u}, \tag{6.5.2}$$

and

$$\tilde{x} = \begin{pmatrix} x_{\rm z} \\ x_{\rm e} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_0 \\ y_{\rm b} \\ y_{\rm d} \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ u_{\rm c} \\ u_{\rm d} \end{pmatrix}, \tag{6.5.3}$$

$$x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{\rm d1} \\ y_{\rm d2} \\ \vdots \\ y_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \\ \vdots \\ u_{{\rm d}m_{\rm d}} \end{pmatrix}, \tag{6.5.4}$$

and

$$x_{\rm z}(k) = 0, \tag{6.5.5}$$

$$x_{\rm e}(k) = B_{\rm e0}u_0(k) + B_{\rm ec}u_{\rm c}(k) + B_{\rm ed}u_{\rm d}(k), \tag{6.5.6}$$

$$x_{\rm a}(k+1) = A_{\rm aa}x_{\rm a}(k) + B_{\rm 0a}y_0(k) + L_{\rm ad}y_{\rm d}(k) + L_{\rm ab}y_{\rm b}(k), \tag{6.5.7}$$

$$x_{\rm b}(k+1) = A_{\rm bb}x_{\rm b}(k) + B_{\rm 0b}y_0(k) + L_{\rm bd}y_{\rm d}(k), \quad y_{\rm b}(k) = C_{\rm b}x_{\rm b}(k), \tag{6.5.8}$$

$$x_{\rm c}(k+1) = A_{\rm cc}x_{\rm c}(k) + B_{\rm 0c}y_0(k) + L_{\rm cd}y_{\rm d}(k) + L_{\rm cb}y_{\rm b}(k) + B_{\rm c}M_{\rm ca}x_{\rm a}(k) + B_{\rm c}u_{\rm c}(k), \tag{6.5.9}$$

$$y_0(k) = C_{\rm 0a}x_{\rm a}(k) + C_{\rm 0b}x_{\rm b}(k) + C_{\rm 0c}x_{\rm c}(k) + C_{\rm 0d}x_{\rm d}(k) + u_0(k), \tag{6.5.10}$$

and for each $i = 1, 2, \ldots, m_{\rm d}$,

$$x_{{\rm d}i}(k+1) = B_{q_i}\left[u_{{\rm d}i}(k) + M_{i{\rm a}}x_{\rm a}(k) + M_{i{\rm b}}x_{\rm b}(k) + M_{i{\rm c}}x_{\rm c}(k) + \sum_{j=1}^{m_{\rm d}} M_{ij}x_{{\rm d}j}(k)\right] + A_{q_i}x_{{\rm d}i}(k) + L_{i0}y_0(k) + L_{i{\rm d}}y_{\rm d}(k), \tag{6.5.11}$$

$$y_{{\rm d}i}(k) = C_{q_i}x_i(k), \quad y_{\rm d} = C_{\rm d}x_{\rm d}(k), \tag{6.5.12}$$

for some constant submatrices of appropriate dimensions. Here the states x_z, x_e, x_a, x_b, x_c *and* x_d *are of dimensions* n_z, n_e, n_a, n_b, n_c *and* $n_d = \sum_{i=1}^{m_d} q_i$, *respectively, while* x_{di} *is of dimension* q_i *for each* $i = 1, 2, \ldots, m_d$. *The control vectors* u_0, u_d *and* u_c *are of dimensions* m_0, m_d *and* $m_c = m - m_0 - m_d$, *respectively, while the output vectors* y_0, y_d *and* y_b *are respectively of dimensions* m_0, m_d *and* $p_b = p - m_0 - m_d$. (A_{bb}, C_b) *is observable,* (A_{cc}, B_c) *is controllable, and* $(A_{q_i}, B_{q_i}, C_{q_i})$ *has the form*

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{6.5.13}$$

Assuming that $x_i, i = 1, 2, \ldots, m_d$, *are arranged such that* $q_i \le q_{i+1}$, *the matrix* L_{id} *will be in the particular form*

$$L_{id} = [\,L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0\,], \tag{6.5.14}$$

with its last row of L_{id} *being all zeros.*

Lastly, we note that the properties of the structural decomposition of discrete-time descriptor systems are analogous to those of continuous-time systems in Section 6.3. We conclude this chapter by noting that research in the system theory and control of descriptor systems is far from being completed. More studies are necessary before we can fully understand the complete picture of descriptor systems and control. Again, it is our belief that the results presented in this chapter can serve as an important tool for future research in the area.

6.6 Exercises

6.1. It is well known that the commonly used PID control law,

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}(t),$$

cannot be expressed in a strictly proper or proper state-space form. But, it can be represented by a descriptor system. Show that the following descriptor system is a realization of the above PID control law:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} e(t),$$

and

$$u(t) = [\,K_i \quad K_p \quad K_d\,]\, x,$$

where the state variable is given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x_1 = \int_0^t e(\tau)d\tau, \; x_2 = e(t), \; x_3 = \dot{e}(t).$$

6.2. Verify that the following descriptor system is another realization of the PID control law given in Exercise 6.1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e(t),$$

and

$$u(t) = [\, K_\mathrm{i} \quad K_\mathrm{p} \quad K_\mathrm{d} \,]\, x.$$

6.3. Show that the descriptor systems given in Exercise 6.1 and Exercise 6.2 have the same structural invariant indices as a proper system characterized by

$$\dot{x}_\mathrm{aux} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_\mathrm{aux} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u_\mathrm{aux},$$

and

$$y_\mathrm{aux} = \begin{bmatrix} 0 & -1 \\ K_\mathrm{i} & K_\mathrm{p} \end{bmatrix} x_\mathrm{aux} + \begin{bmatrix} 0 & 1 \\ K_\mathrm{d} & 0 \end{bmatrix} u_\mathrm{aux}.$$

6.4. Verify that the descriptor systems given in Exercise 6.1 is regular. Find the required nonsingular transformations P and Q for the systems such that the matrix pair,

$$(E, A) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

can be transformed into the EA decomposition form.

6.5. Construct a structural decomposition for the descriptor system given in Exercise 6.1. Verify the result of Corollaries 6.3.1 and 6.3.2. Does the result of Corollary 6.3.2 agree with the transfer function of the PID control law given in Exercise 6.1?

6.6. Derive an alternative procedure that realizes the EA decomposition of a pair of square matrices, (E, A), using the Kronecker canonical form of the matrix pencil, $sE - A$.

6.7. Compute a structural decomposition for the descriptor system

$$E\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where

$$E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Verify Properties 6.3.1 to 6.3.4 using the structural decomposition obtained above.

6.8. Construct a sixth order descriptor system with two inputs and two outputs, which has $n_{\rm z} = n_{\rm e} = n_{\rm a} = n_{\rm b} = n_{\rm c} = n_{\rm d} = 1$ with an invariant zero at -1. Obviously, the obtained system is neither left nor right invertible, and has an infinite zero of order 1. Moreover, it has one state variable being identically zero and another state variable being directly associated with an input variable.

Chapter 7

Structural Mappings of Bilinear Transformations

7.1 Introduction

In this chapter, we present a comprehensive picture of the mapping of structural properties associated with general linear multivariable systems under bilinear and inverse bilinear transformations. We will investigate in depth how the finite and infinite zero structures, as well as the invertibility structures of a general continuous-time (discrete-time) linear time-invariant multivariable system are mapped to those of its discrete-time (continuous-time) counterpart under the bilinear (inverse bilinear) transformation. We note that a similar version of this chapter was included earlier in a monograph by the first author *i.e.*, [22], in which he had utilized the results of the bilinear transformations and their structural mapping properties to solve general Riccati equations and discrete-time H_∞ control problems. Nonetheless, this chapter actually builds a bridge for linear system theory between the continuous-time domain and the discrete-time domain. As will be seen shortly in Chapter 8, the results of this chapter will be useful in solving another problem in linear systems, *i.e.*, system factorizations for discrete-time systems.

The bilinear and inverse bilinear transformations have widespread use in digital control and signal processing. It has been shown in [22] that the bilinear transformation plays a crucial role in the computation of infima for discrete-time H_∞ control, as well as in finding the solutions to discrete-time Riccati equations. The results presented in this chapter were first reported in Chen and Weller [30].

In fact, the need to perform continuous-time to discrete-time model conversions arises in a range of engineering contexts, including sampled-data control system design, and digital signal processing. Consequently, numerous discretization procedures exist, including the zero- and first-order hold input approximations, the impulse invariant transformation, and the bilinear transformation (see, *e.g.*, [7] and [55]). Despite the widespread use of the bilinear transform, a comprehensive treatment detailing how key structural properties of continuous-time systems, such as the finite and infinite zero structures, and invertibility properties, are inherited by their discrete-time counterparts is lacking in the literature. Given the important role played by the infinite and finite zero structures in control system design, a clear understanding of the zero structures under bilinear transformation would be useful in the design of sampled-data control systems, and would complement existing results on the mapping of finite and infinite zero structures under zero-order hold sampling (see, *e.g.*, [6] and [60]).

In this chapter, we present a comprehensive study of how the structures, *i.e.*, the finite and infinite zero structures, invertibility structures, as well as the geometric subspaces of a general continuous-time (discrete-time) linear time-invariant system are mapped to those of its discrete-time (continuous-time) counterpart under the well-known bilinear (inverse bilinear) transformation

$$s = a\left(\frac{z-1}{z+1}\right) \quad \Leftrightarrow \quad z = \frac{a+s}{a-s}. \tag{7.1.1}$$

7.2 Mapping of Continuous- to Discrete-time Systems

In this section, we consider a continuous-time linear time-invariant system Σ_c characterized by

$$\Sigma_\mathrm{c} \; : \; \begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{7.2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and A, B, C and D are matrices of appropriate dimensions. Without loss of generality, we assume that both matrices $[\,C \quad D\,]$ and $[\,B' \quad D'\,]$ are of full rank. Σ_c has a transfer function

$$G_\mathrm{c}(s) = C(sI - A)^{-1}B + D. \tag{7.2.2}$$

Let us apply a bilinear transformation to the above continuous-time system, by replacing s in (7.2.2) with

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = a\left(\frac{z-1}{z+1}\right), \tag{7.2.3}$$

where $T = 2/a$ is the sampling period. As presented in (7.2.3), the bilinear transformation is often called Tustin's approximation [7], while the choice

$$a = \frac{\omega_1}{\tan(\omega_1 T/2)} \tag{7.2.4}$$

yields the pre-warped Tustin approximation, in which the frequency responses of the continuous-time system and its discrete-time counterpart are matched at frequency ω_1. In this way, we obtain a discrete-time system

$$G_{\rm d}(z) = C\left(a\frac{z-1}{z+1}I - A\right)^{-1} B + D. \tag{7.2.5}$$

The following lemma provides a direct state-space realization of $G_{\rm d}(z)$. A similar result can also be found in [55].

Lemma 7.2.1. *A state-space realization of $G_{\rm d}(z)$, the discrete-time counterpart of the continuous-time system $\Sigma_{\rm c}$ of (7.2.1) under the bilinear transformation (7.2.3), is given by*

$$\Sigma_{\rm d} : \begin{cases} x(k+1) = \tilde{A}\, x(k) + \tilde{B}\, u(k), \\ \quad y(k) \;\; = \tilde{C}\, x(k) + \tilde{D}\, u(k), \end{cases} \tag{7.2.6}$$

where

$$\left.\begin{aligned} \tilde{A} &= (aI + A)(aI - A)^{-1}, \\ \tilde{B} &= \sqrt{2a}\,(aI - A)^{-1}B, \\ \tilde{C} &= \sqrt{2a}\,C(aI - A)^{-1}, \\ \tilde{D} &= D + C(aI - A)^{-1}B, \end{aligned}\right\} \tag{7.2.7}$$

or

$$\left.\begin{aligned} \tilde{A} &= (aI + A)(aI - A)^{-1}, \\ \tilde{B} &= 2a\,(aI - A)^{-2}B, \\ \tilde{C} &= C, \\ \tilde{D} &= D + C(aI - A)^{-1}B, \end{aligned}\right\} \tag{7.2.8}$$

or

$$\left.\begin{aligned} \tilde{A} &= (aI + A)(aI - A)^{-1}, \\ \tilde{B} &= B, \\ \tilde{C} &= 2a\,C(aI - A)^{-2}, \\ \tilde{D} &= D + C(aI - A)^{-1}B. \end{aligned}\right\} \tag{7.2.9}$$

Here we clearly assume that matrix A has no eigenvalue at a.

Proof. First, it is straightforward to verify that

$$\begin{aligned}
G_{\mathrm{d}}(z) &= C\left(a\frac{z-1}{z+1}I - A\right)^{-1} B + D \\
&= (z+1)C\big[a(z-1)I - (z+1)A\big]^{-1}B + D \\
&= (z+1)C(aI-A)^{-1}\big[zI - (aI+A)(aI-A)^{-1}\big]^{-1}B + D \\
&= zC(aI-A)^{-1}\big(zI-\tilde{A}\big)^{-1}B + \left[C(aI-A)^{-1}\big(zI-\tilde{A}\big)^{-1}B + D\right].
\end{aligned} \tag{7.2.10}$$

If we introduce $\tilde{G}_{\mathrm{d}}(z) = zC(aI-A)^{-1}\left(zI-\tilde{A}\right)^{-1}B$, it follows that

$$\begin{cases} \tilde{x}(k+1) = \tilde{A}'\tilde{x}(k) + (aI-A')^{-1}C'\tilde{u}(k), \\ \tilde{y}(k) \ = B'\tilde{x}(k+1) = B'\tilde{A}'\tilde{x}(k) + B'(aI-A')^{-1}C'\tilde{u}(k), \end{cases} \tag{7.2.11}$$

is a state-space realization of $\tilde{G}_{\mathrm{d}}'(z)$, from which

$$\tilde{G}_{\mathrm{d}}(z) = C(aI-A)^{-1}\left(zI-\tilde{A}\right)^{-1}\tilde{A}B + C(aI-A)^{-1}B. \tag{7.2.12}$$

Substituting (7.2.12) into (7.2.10), we obtain

$$\begin{aligned}
G_{\mathrm{d}}(z) &= C(aI-A)^{-1}\left(zI-\tilde{A}\right)^{-1}(\tilde{A}+I)B + [C(aI-A)^{-1}B + D] \\
&= \tilde{C}\left(zI-\tilde{A}\right)^{-1}\tilde{B} + \tilde{D},
\end{aligned}$$

and the rest of Lemma 7.2.1 follows. ∎

The following theorem establishes the interconnection of the structural properties of Σ_{c} and Σ_{d}, and forms the core of this chapter. The proof of this theorem is very tedious, and hence will be given in Section 7.4 for clarity in the presentation.

Theorem 7.2.1. *Consider the continuous-time system Σ_{c} of (7.2.1) characterized by the quadruple (A,B,C,D) with matrix A having no eigenvalue at a, and its discrete-time counterpart under the bilinear transformation (7.2.3), i.e., Σ_{d} of (7.2.6) characterized by the quadruple $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ of (7.2.7). We have the following properties:*

1. *Controllability (stabilizability) and observability (detectability) of Σ_{d}:*

   (a) *The pair $(\tilde{A},\tilde{B})$ is controllable (stabilizable) if and only if the pair (A,B) is controllable (stabilizable).*

(b) *The pair* $(\tilde{A}, \tilde{C})$ *is observable (detectable) if and only if the pair* (A, C) *is observable (detectable).*

2. *Effects of nonsingular state, output and input transformations, together with state feedback and output injection laws:*

(a) *For any given nonsingular state, output and input transformations* $T_{\rm s}$, $T_{\rm o}$ *and* $T_{\rm i}$, *the quadruple*

$$(T_{\rm s}^{-1}\tilde{A}T_{\rm s}, T_{\rm s}^{-1}\tilde{B}T_{\rm i}, T_{\rm o}^{-1}\tilde{C}T_{\rm s}, T_{\rm o}^{-1}\tilde{D}T_{\rm i}), \tag{7.2.13}$$

is the discrete-time counterpart under the bilinear transformation of the continuous-time system

$$(T_{\rm s}^{-1}AT_{\rm s}, T_{\rm s}^{-1}BT_{\rm i}, T_{\rm o}^{-1}CT_{\rm s}, T_{\rm o}^{-1}DT_{\rm i}). \tag{7.2.14}$$

(b) *For any* $F \in \mathbb{R}^{m\times n}$ *with* $A + BF$ *having no eigenvalue at* a, *define a nonsingular matrix*

$$\begin{aligned}\tilde{T}_{\rm i} &:= I + F(aI - A - BF)^{-1}B \\ &= [I - F(aI - A)^{-1}B]^{-1} \in \mathbb{R}^{m\times m},\end{aligned} \tag{7.2.15}$$

and a constant matrix

$$\tilde{F} := \sqrt{2a}\, F(aI - A - BF)^{-1} \in \mathbb{R}^{m\times n}. \tag{7.2.16}$$

Then a continuous-time system $\Sigma_{\rm CF}$ *characterized by*

$$(A + BF, B, C + DF, D), \tag{7.2.17}$$

is mapped to a discrete-time system $\Sigma_{\rm dF}$, *characterized by*

$$(\tilde{A} + \tilde{B}\tilde{F}, \tilde{B}\tilde{T}_{\rm i}, \tilde{C} + \tilde{D}\tilde{F}, \tilde{D}\tilde{T}_{\rm i}), \tag{7.2.18}$$

under the bilinear transformation (7.2.3). Here we note that $\Sigma_{\rm CF}$ *is the closed-loop system comprising* $\Sigma_{\rm c}$ *and a state feedback law with gain matrix* F, *and* $\Sigma_{\rm dF}$ *is the closed-loop system comprising* $\Sigma_{\rm d}$ *and a state feedback law with gain matrix* $\tilde{F}$, *together with a nonsingular input transformation* $\tilde{T}_{\rm i}$.

(c) *For any* $K \in \mathbb{R}^{n\times p}$ *with* $A + KC$ *having no eigenvalue at* a, *define a nonsingular matrix*

$$\tilde{T}_{\rm o} := [I + C(aI - A - KC)^{-1}K]^{-1} \in \mathbb{R}^{p\times p}, \tag{7.2.19}$$

and a constant matrix

$$\tilde{K} := \sqrt{2a}\,(aI - A - KC)^{-1}K. \tag{7.2.20}$$

Then a continuous-time system $\Sigma_{\rm cK}$ *characterized by*

$$(A + KC, B + KD, C, D), \tag{7.2.21}$$

is mapped to a discrete-time system $\Sigma_{\rm dK}$, *characterized by*

$$(\tilde{A} + \tilde{K}\tilde{C}, \tilde{B} + \tilde{K}\tilde{D}, \tilde{T}_{\rm o}^{-1}\tilde{C}, \tilde{T}_{\rm o}^{-1}\tilde{D}), \tag{7.2.22}$$

under the bilinear transformation (7.2.3). We note that $\Sigma_{\rm cK}$ *is the closed-loop system comprising* $\Sigma_{\rm c}$ *and an output injection law with gain matrix* K, *and* $\Sigma_{\rm dK}$ *is the closed-loop system comprising* $\Sigma_{\rm d}$ *and an output injection law with gain matrix* $\tilde{K}$, *together with a nonsingular output transformation* $\tilde{T}_{\rm o}$.

3. *Invertibility and structural invariant indices lists* $\mathbf{I}_2$ *and* $\mathbf{I}_3$ *of* $\Sigma_{\rm d}$:

   (a) $\mathbf{I}_2(\Sigma_{\rm d}) = \mathbf{I}_2(\Sigma_{\rm c})$, *and* $\mathbf{I}_3(\Sigma_{\rm d}) = \mathbf{I}_3(\Sigma_{\rm c})$.

   (b) $\Sigma_{\rm d}$ *is left (right) invertible if and only if* $\Sigma_{\rm c}$ *is left (right) invertible.*

   (c) $\Sigma_{\rm d}$ *is (non) invertible if and only if* $\Sigma_{\rm c}$ *is (non) invertible.*

4. *The invariant zeros of* $\Sigma_{\rm d}$ *and their associated structures consist of the following two parts:*

   (a) *Let the infinite zero structure (of order greater than* 0*) of* $\Sigma_{\rm c}$ *be given by*

   $$S_\infty^\star(\Sigma_{\rm c}) = \{q_1, q_2, \ldots, q_{m_{\rm d}}\}. \tag{7.2.23}$$

   Then $z = -1$ *is an invariant zero of* $\Sigma_{\rm d}$ *with the multiplicity structure*

   $$S_{-1}^\star(\Sigma_{\rm d}) = \{q_1, q_2, \ldots, q_{m_{\rm d}}\}. \tag{7.2.24}$$

   (b) *Let* $s = \alpha \neq a$ *be an invariant zero of* $\Sigma_{\rm c}$ *with the multiplicity structure*

   $$S_\alpha^\star(\Sigma_{\rm c}) = \{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\}. \tag{7.2.25}$$

   Then $z = \beta = (a+\alpha)/(a-\alpha)$ *is an invariant zero of its discrete-time counterpart* $\Sigma_{\rm d}$ *with the multiplicity structure*

   $$S_\beta^\star(\Sigma_{\rm d}) = \{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\}. \tag{7.2.26}$$

5. *The infinite zero structure of Σ_d consists of the following two parts:*

   (a) *Let $m_0 = \text{rank}\,(D)$, and let m_d be the total number of infinite zeros of Σ_c of order greater than 0. Also, let τ_a be the geometric multiplicity of the invariant zero of Σ_c at $s = a$. Then, we have $\text{rank}\,(\tilde{D}) = m_0 + m_d - \tau_a$.*

   (b) *Let $s = a$ be an invariant zero of the given continuous-time system Σ_c with a multiplicity structure*

$$S_a^\star(\Sigma_c) = \{n_{a,1}, n_{a,2}, \ldots, n_{a,\tau_a}\}. \tag{7.2.27}$$

   Then the discrete-time counterpart Σ_d has an infinite zero (of order greater than 0) structure

$$S_\infty^\star(\Sigma_d) = \{n_{a,1}, n_{a,2}, \ldots, n_{a,\tau_a}\}. \tag{7.2.28}$$

6. *The mappings of geometric subspaces:*

   (a) $\mathcal{V}^+(\Sigma_c) = \mathcal{S}^-(\Sigma_d)$.

   (b) $\mathcal{S}^+(\Sigma_c) = \mathcal{V}^-(\Sigma_d)$.

Proof. See Section 7.4. ∎

We have the following two interesting corollaries. The first is with regard to the minimum phase and nonminimum phase properties of Σ_d, while the second concerns the asymptotic behavior of Σ_d as the sampling period T tends to zero (or, equivalently, as $a \to \infty$).

Corollary 7.2.1. *Consider a continuous-time system Σ_c and its discrete-time counterpart Σ_d under the bilinear transformation (7.2.3). Then it follows from 4(a) and 4(b) of Theorem 7.2.1 that*

1. *Σ_d has all its invariant zeros inside the unit circle if and only if Σ_c has all its invariant zeros in the open left-half plane and has no infinite zero of order greater than 0;*

2. *Σ_d has invariant zeros on the unit circle if and only if Σ_c has invariant zeros on the imaginary axis, and/or Σ_c has at least one infinite zero of order greater than 0; and*

3. *Σ_d has invariant zeros outside the unit circle if and only if Σ_c has invariant zeros in the open right-half plane.*

Corollary 7.2.2. *Consider a continuous-time system Σ_c and its discrete-time counterpart Σ_d under the bilinear transformation (7.2.3). Then, as a consequence of Theorem 7.2.1, Σ_d has the following asymptotic properties as the sampling period T tends to zero (but not equal to zero):*

1. *Σ_d has no infinite zero of order greater than 0, i.e., no delays from the input to the output;*

2. *Σ_d has one invariant zero at $z = -1$ with an appropriate multiplicity structure if Σ_c has any infinite zero of order greater than 0; and*

3. *The remaining invariant zeros of Σ_d, if any, tend to the point $z = 1$. More interestingly, the invariant zeros of Σ_d corresponding to the stable invariant zeros of Σ_c are always stable, and approach the point $z = 1$ from inside the unit circle. Conversely, the invariant zeros of Σ_d corresponding to the unstable invariant zeros of Σ_c are always unstable, and approach the point $z = 1$ from outside the unit circle. Finally, those associated with the imaginary axis invariant zeros of Σ_c are always mapped onto the unit circle and move toward the point $z = 1$.*

The following example illustrates the results in Theorem 7.2.1.

Example 7.2.1. Consider a continuous-time system Σ_c characterized by a quadruple (A,B,C,D) with

$$A = \left[\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right], \tag{7.2.29}$$

and

$$C = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad D = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{7.2.30}$$

We note that the above system Σ_c is already in the form of the special coordinate basis as in Theorem 5.4.1. Furthermore, Σ_c is controllable, observable and invertible with one infinite zero of order 0, and one infinite zero of order 2, i.e., $S_\infty^\star(\Sigma_c) = \{2\}$. The system Σ_c also has two invariant zeros at $s = 2$ and $s = 1$, respectively, with structures $S_2^\star(\Sigma_c) = \{1\}$ and $S_1^\star(\Sigma_c) = \{3\}$.

1. If $a = 1$, we obtain a discrete-time system $\Sigma_{\rm d}$ characterized by the quadruple $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$, with

$$\tilde{A} = \begin{bmatrix} 1 & 2 & -3 & 1 & 0 & -2 \\ -2 & -1 & 2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \end{bmatrix}, \quad \tilde{B} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{C} = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Utilizing the toolkit of [87], we find that $\Sigma_{\rm d}$ is indeed controllable, observable and invertible, with one infinite zero of order 0 and one infinite zero of order 3, i.e., $S_\infty^\star(\Sigma_{\rm d}) = \{3\}$. $\Sigma_{\rm d}$ also has two invariant zeros at $z = -3$ and $z = -1$ respectively, with structures $S^\star_{-3}(\Sigma_{\rm d}) = \{1\}$ and $S^\star_{-1}(\Sigma_{\rm d}) = \{2\}$.

2. If $a = 2$, we obtain another discrete-time system $\Sigma_{\rm d}$, characterized by

$$\tilde{A} = \begin{bmatrix} 0 & -2 & -5 & 3 & -3 & -3 \\ -2 & -1 & -2 & 2 & -2 & -2 \\ -1 & -2 & 0 & 1 & -1 & -1 \\ 1 & 2 & 3 & -6 & 1 & 1 \\ -1 & -2 & -3 & 1 & -2 & -1 \\ -2 & -4 & -6 & 2 & -6 & -3 \end{bmatrix}, \quad \tilde{B} = \frac{1}{2} \begin{bmatrix} 3 & -3 \\ 2 & -2 \\ 1 & -1 \\ -5 & 1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix},$$

and

$$\tilde{C} = \frac{1}{2} \begin{bmatrix} 1 & 2 & 3 & -5 & 1 & 1 \\ -1 & -2 & -3 & 1 & -1 & -1 \end{bmatrix}, \quad \tilde{D} = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

which is controllable, observable and invertible with one infinite zero of order 0 and one infinite zero of order 1, i.e., $S_\infty^\star(\Sigma_{\rm d}) = \{1\}$. It also has two invariant zeros at $z = 3$ and $z = -1$ respectively, with structures $S_3^\star(\Sigma_{\rm d}) = \{3\}$ and $S^\star_{-1}(\Sigma_{\rm d}) = \{2\}$, in accordance with Theorem 7.2.1.

7.3 Mapping of Discrete- to Continuous-time Systems

We present in this section a similar result as in the previous section, but for the inverse bilinear transformation mapping a discrete-time system to a continuous-time system. We begin with a discrete-time linear time-invariant system $\tilde{\Sigma}_{\rm d}$ characterized by

$$\tilde{\Sigma}_{\rm d} : \begin{cases} x(k+1) = \tilde{A}\, x(k) + \tilde{B}\, u(k), \\ \quad y(k) \;\;\; = \tilde{C}\, x(k) + \tilde{D}\, u(k), \end{cases} \tag{7.3.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are matrices of appropriate dimensions. Without loss of generality, we assume that both matrices $[\,\tilde{C} \;\; \tilde{D}\,]$ and $[\,\tilde{B}' \;\; \tilde{D}'\,]$ are of full rank. $\Sigma_{\rm d}$ has a transfer function

$$H_{\rm d}(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D}. \tag{7.3.2}$$

The inverse bilinear transformation of (7.3.2) can be obtained by replacing z with

$$z = \frac{a+s}{a-s}, \tag{7.3.3}$$

i.e.,

$$H_{\rm c}(s) = \tilde{C}\left(\frac{a+s}{a-s}I - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D}. \tag{7.3.4}$$

The following lemma is analogous to Lemma 7.2.1, and provides a state-space realization of $H_{\rm c}(s)$.

Lemma 7.3.1. *A state-space realization of $H_{\rm c}(s)$, the continuous-time counterpart of the discrete-time system $\tilde{\Sigma}_{\rm d}$ of (7.3.1) under the inverse bilinear transformation (7.3.3), is given by*

$$\tilde{\Sigma}_{\rm c} : \begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{7.3.5}$$

where

$$\left.\begin{aligned} A &= a(\tilde{A}+I)^{-1}(\tilde{A}-I), \\ B &= \sqrt{2a}\,(\tilde{A}+I)^{-1}\tilde{B}, \\ C &= \sqrt{2a}\,\tilde{C}(\tilde{A}+I)^{-1}, \\ D &= \tilde{D} - \tilde{C}(\tilde{A}+I)^{-1}\tilde{B}, \end{aligned}\right\} \tag{7.3.6}$$

or

$$\left.\begin{aligned} A &= a(\tilde{A}+I)^{-1}(\tilde{A}-I), \\ B &= 2a\,(\tilde{A}+I)^{-2}\tilde{B}, \\ C &= \tilde{C}, \\ D &= \tilde{D} - \tilde{C}(\tilde{A}+I)^{-1}\tilde{B}, \end{aligned}\right\} \tag{7.3.7}$$

or

$$\left.\begin{aligned} A &= a(\tilde{A}+I)^{-1}(\tilde{A}-I), \\ B &= \tilde{B}, \\ C &= 2a\,\tilde{C}(\tilde{A}+I)^{-2}, \\ D &= \tilde{D} - \tilde{C}(\tilde{A}+I)^{-1}\tilde{B}. \end{aligned}\right\} \tag{7.3.8}$$

Here we clearly assume that the matrix $\tilde{A}$ has no eigenvalues at -1.

The following theorem is analogous to Theorem 7.2.1.

Theorem 7.3.1. *Consider the discrete-time system $\tilde{\Sigma}_{\rm d}$ of (7.3.1) characterized by the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with matrix $\tilde{A}$ having no eigenvalues at -1, and its continuous-time counterpart under the inverse bilinear transformation (7.3.3), i.e., $\tilde{\Sigma}_{\rm c}$ of (7.3.5) characterized by the quadruple (A, B, C, D) of (7.3.6). We have the following properties:*

1. *Controllability (stabilizability) and observability (detectability) of $\tilde{\Sigma}_{\rm c}$:*

   (a) *The pair (A, B) is controllable (stabilizable) if and only if the pair $(\tilde{A}, \tilde{B})$ is controllable (stabilizable).*

   (b) *The pair (A, C) is observable (detectable) if and only if the pair $(\tilde{A}, \tilde{C})$ is observable (detectable).*

2. *Effects of nonsingular state, output and input transformations, together with state feedback and output injection laws:*

   (a) *For any given nonsingular state, output and input transformations $T_{\rm s}$, $T_{\rm o}$ and $T_{\rm i}$, the quadruple*

$$(T_{\rm s}^{-1}AT_{\rm s}, T_{\rm s}^{-1}BT_{\rm i}, T_{\rm o}^{-1}CT_{\rm s}, T_{\rm o}^{-1}DT_{\rm i}), \tag{7.3.9}$$

   is the continuous-time counterpart of the inverse bilinear transformation, i.e., (7.3.3), of the discrete-time system

$$(T_{\rm s}^{-1}\tilde{A}T_{\rm s}, T_{\rm s}^{-1}\tilde{B}T_{\rm i}, T_{\rm o}^{-1}\tilde{C}T_{\rm s}, T_{\rm o}^{-1}\tilde{D}T_{\rm i}). \tag{7.3.10}$$

   (b) *For any $\tilde{F} \in \mathbb{R}^{m\times n}$ with $\tilde{A} + \tilde{B}\tilde{F}$ having no eigenvalue at -1, define a nonsingular matrix*

$$T_{\rm i} := I - \tilde{F}(I + \tilde{A} + \tilde{B}\tilde{F})^{-1}\tilde{B} \in \mathbb{R}^{m\times m}, \tag{7.3.11}$$

   and a constant matrix

$$F := \sqrt{2a}\, \tilde{F}(I + \tilde{A} + \tilde{B}\tilde{F})^{-1} \in \mathbb{R}^{m\times n}. \tag{7.3.12}$$

   Then a discrete-time system $\tilde{\Sigma}_{\rm dF}$, characterized by

$$(\tilde{A} + \tilde{B}\tilde{F}, \tilde{B}, \tilde{C} + \tilde{D}\tilde{F}, \tilde{D}), \tag{7.3.13}$$

   is mapped to a continuous-time counterpart $\tilde{\Sigma}_{\rm cF}$, characterized by

$$(A + BF, BT_{\rm i}, C + DF, DT_{\rm i}), \tag{7.3.14}$$

under the inverse bilinear transformation (7.3.3). Note that $\tilde{\Sigma}_{\rm dF}$ is the closed-loop system comprising $\tilde{\Sigma}_{\rm d}$ and a state feedback law with gain matrix $\tilde{F}$, and $\tilde{\Sigma}_{\rm dF}$ is the closed-loop system comprising $\tilde{\Sigma}_{\rm d}$ and a state feedback law with gain matrix F, together with a nonsingular input transformation $T_{\rm i}$.

(c) *For any $\tilde{K} \in \mathbb{R}^{n\times p}$ with $\tilde{A}+\tilde{K}\tilde{C}$ having no eigenvalues at -1, define a nonsingular matrix*

$$T_{\rm o} := [I - \tilde{C}(I + \tilde{A} + \tilde{K}\tilde{C})^{-1}\tilde{K}]^{-1} \in \mathbb{R}^{p\times p}, \tag{7.3.15}$$

and a constant matrix

$$K := \sqrt{2a}\,(I + \tilde{A} + \tilde{K}\tilde{C})^{-1}\tilde{K}. \tag{7.3.16}$$

Then a discrete-time system $\tilde{\Sigma}_{\rm dK}$, characterized by

$$(\tilde{A} + \tilde{K}\tilde{C}, \tilde{B} + \tilde{K}\tilde{D}, \tilde{C}, \tilde{D}), \tag{7.3.17}$$

is mapped to a continuous-time $\tilde{\Sigma}_{\rm cK}$, characterized by

$$(A + KC, B + KD, T_{\rm o}^{-1}C, T_{\rm o}^{-1}D), \tag{7.3.18}$$

under the inverse bilinear transformation (7.3.3). We note that $\tilde{\Sigma}_{\rm dK}$ is the closed-loop system comprising $\tilde{\Sigma}_{\rm d}$ and an output injection law with gain matrix $\tilde{K}$, and $\tilde{\Sigma}_{\rm cK}$ is the closed-loop system comprising $\tilde{\Sigma}_{\rm c}$ and an output injection law with gain matrix K, together with a nonsingular output transformation $T_{\rm o}$.

3. *Invertibility and structural invariant indices lists $\mathbf{I}_2$ and $\mathbf{I}_3$ of $\tilde{\Sigma}_{\rm c}$:*

   (a) $\mathbf{I}_2(\tilde{\Sigma}_{\rm c}) = \mathbf{I}_2(\tilde{\Sigma}_{\rm d})$, *and* $\mathbf{I}_3(\tilde{\Sigma}_{\rm c}) = \mathbf{I}_3(\tilde{\Sigma}_{\rm d})$.

   (b) *$\tilde{\Sigma}_{\rm c}$ is left (right) invertible if and only if $\tilde{\Sigma}_{\rm d}$ is left (right) invertible.*

   (c) *$\tilde{\Sigma}_{\rm c}$ is (non) invertible if and only if $\tilde{\Sigma}_{\rm d}$ is (non) invertible.*

4. *The invariant zeros of $\tilde{\Sigma}_{\rm c}$ and their structures consist of the following two parts:*

   (a) *Let the infinite zero structure (of order greater than 0) of $\tilde{\Sigma}_{\rm d}$ be given by*

   $$S_\infty^\star(\tilde{\Sigma}_{\rm d}) = \{q_1, q_2, \ldots, q_{m_{\rm d}}\}. \tag{7.3.19}$$

   Then $s = a$ is an invariant zero of $\tilde{\Sigma}_{\rm c}$ with the multiplicity structure

   $$S_a^\star(\tilde{\Sigma}_{\rm c}) = \{q_1, q_2, \ldots, q_{m_{\rm d}}\}. \tag{7.3.20}$$

(b) *Let* $z = \alpha \neq -1$ *be an invariant zero of* $\tilde{\Sigma}_{\rm d}$ *with the multiplicity structure*

$$S^\star_\alpha(\tilde{\Sigma}_{\rm d}) = \{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\}. \tag{7.3.21}$$

Then $s = \beta = a\frac{\alpha-1}{\alpha+1}$ *is an invariant zero of its continuous-time counterpart* $\tilde{\Sigma}_{\rm c}$ *with the multiplicity structure*

$$S^\star_\beta(\tilde{\Sigma}_{\rm c}) = \{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\}. \tag{7.3.22}$$

5. *The infinite zero structure of* $\tilde{\Sigma}_{\rm c}$ *consists of the following two parts:*

(a) *Let* $m_0 =$ *rank* $(\tilde{D})$, *and let* $m_{\rm d}$ *be the total number of infinite zeros of* $\tilde{\Sigma}_{\rm d}$ *of order greater than* 0. *Also, let* τ_{-1} *be the geometric multiplicity of the invariant zero of* $\tilde{\Sigma}_{\rm d}$ *at* $z = -1$. *Then we have rank* $(D) = m_0 + m_{\rm d} - \tau_{-1}$.

(b) *Let* $z = -1$ *be an invariant zero of the given discrete-time system* $\tilde{\Sigma}_{\rm d}$ *with the multiplicity structure*

$$S^\star_{-1}(\tilde{\Sigma}_{\rm d}) = \{n_{-1,1}, n_{-1,2}, \ldots, n_{-1,\tau_{-1}}\}. \tag{7.3.23}$$

Then $\tilde{\Sigma}_{\rm c}$ *has an infinite zero (of order greater than* 0*) structure*

$$S^\star_\infty(\tilde{\Sigma}_{\rm c}) = \{n_{-1,1}, n_{-1,2}, \ldots, n_{-1,\tau_{-1}}\}. \tag{7.3.24}$$

6. *The mappings of geometric subspaces:*

(a) $\mathcal{V}^-(\tilde{\Sigma}_{\rm d}) = \mathcal{S}^+(\tilde{\Sigma}_{\rm c})$.

(b) $\mathcal{S}^-(\tilde{\Sigma}_{\rm d}) = \mathcal{V}^+(\tilde{\Sigma}_{\rm c})$.

Proof. The proof of this theorem is similar to that of Theorem 7.2.1. ∎

We illustrate the result above with the following example.

Example 7.3.1. Consider a discrete-time linear time-invariant system $\tilde{\Sigma}_{\rm d}$ characterized by a matrix quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{7.3.25}$$

and

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.3.26}$$

Again the above system is already in the form of the structural decomposition. It is simple to verify that $\tilde{\Sigma}_{\rm d}$ is controllable, observable and degenerate, *i.e.*, neither left nor right invertible, with two infinite zeros of order 1, *i.e.*, $S_\infty^\star(\tilde{\Sigma}_{\rm d}) = \{1,1\}$, $\mathbf{I}_2(\tilde{\Sigma}_{\rm d}) = \{1\}$ and $\mathbf{I}_3(\tilde{\Sigma}_{\rm d}) = \{1\}$. It also has one invariant zero at $z = -1$ with a structure of $S_{-1}^\star(\tilde{\Sigma}_{\rm d}) = \{1,2\}$. Applying the result in Lemma 7.3.1 (with $a = 1$), we obtain $\tilde{\Sigma}_{\rm c}$ which is characterized by (A, B, C, D) with

$$A = \begin{bmatrix} 5 & 0 & 0 & -2 & 0 & -2 & 2 \\ 0 & 3 & 4 & -2 & 2 & -2 & -2 \\ 0 & -2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ -2 & 0 & -2 & 2 & -1 & 2 & 0 \\ -2 & 0 & -2 & 2 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \sqrt{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \sqrt{2}\begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, it is straightforward to verify, using the software toolkit of [87], for example, that $\tilde{\Sigma}_{\rm c}$ is controllable, observable and degenerate with an infinite zero structure of $S_\infty^\star(\tilde{\Sigma}_{\rm c}) = \{1,2\}$, $\mathbf{I}_2(\tilde{\Sigma}_{\rm c}) = \{1\}$ and $\mathbf{I}_3(\tilde{\Sigma}_{\rm c}) = \{1\}$. Furthermore, $\tilde{\Sigma}_{\rm c}$ has one invariant zero at $s = 1$ with associated structure $S_1^\star(\tilde{\Sigma}_{\rm c}) = \{1,1\}$, in accordance with Theorem 7.3.1.

Finally, we conclude this section by summarizing in a graphical form in Figures 7.3.1 the structural mappings associated with the bilinear and inverse bilinear transformations.

7.4 Proof of Theorem 7.2.1

We present in this section the detailed proof of Theorem 7.2.1. For the sake of simplicity in presentation, and without loss of generality, we assume, throughout the proof, that the constant a in (7.2.3) is equal to unity, *i.e.*, $a = 2/T = 1$. We will prove this theorem item by item.

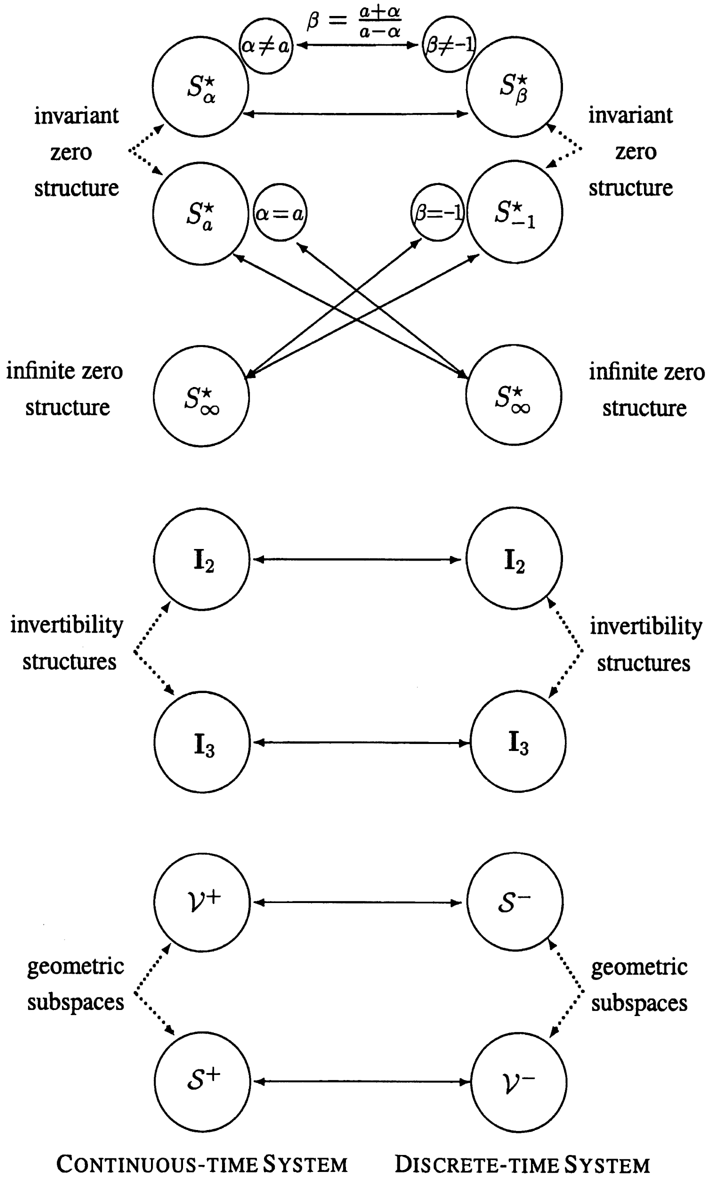

Figure 7.3.1: Structural mappings of bilinear transformations.

1(a). Let β be an eigenvalue of $\tilde{A}$, *i.e.*, $\beta \in \lambda(\tilde{A})$. It is straightforward to verify that $\beta \neq -1$, provided A has no eigenvalue at $a = 1$ and $\alpha = (\beta - 1)/(\beta + 1)$ is an eigenvalue of A, *i.e.*, $\alpha \in \lambda(A)$. Next, we consider the matrix pencil

$$\begin{aligned}[\beta I - \tilde{A} \quad \tilde{B}] &= [\beta I - (I - A)^{-1}(I + A) \quad \sqrt{2}(I - A)^{-1}B] \\ &= (I - A)^{-1}[\beta(I - A) - (I + A) \quad \sqrt{2}\,B] \\ &= (I - A)^{-1}[(\beta - 1)I - (\beta + 1)A \quad \sqrt{2}\,B] \\ &= (I - A)^{-1}[\alpha I - A \quad B]\begin{bmatrix}(\beta + 1)I_n & 0 \\ 0 & \sqrt{2}\,I_m\end{bmatrix}.\end{aligned}$$

Clearly, rank $[\beta I - \tilde{A} \quad \tilde{B}] =$ rank $[\alpha I - A \quad B]$, and the result 1(a) follows.

1(b). Dual of 1(a).

2(a). The proof of this item is trivial.

2(b). It follows from Lemma 7.2.1 that the discrete-time counterpart $\Sigma_{\rm dF}$ of the bilinear transformation of $\Sigma_{\rm cF}$, characterized by $(A+BF, B, C+DF, D)$, is given by $(\tilde{A}_{\rm F}, \tilde{B}_{\rm F}, \tilde{C}_{\rm F}, \tilde{D}_{\rm F})$ with

$$\left.\begin{aligned}\tilde{A}_{\rm F} &= (I + A + BF)(I - A - BF)^{-1}, \\ \tilde{B}_{\rm F} &= \sqrt{2}\,(I - A - BF)^{-1}B, \\ \tilde{C}_{\rm F} &= \sqrt{2}\,(C + DF)(I - A - BF)^{-1}, \\ \tilde{D}_{\rm F} &= D + (C + DF)(I - A - BF)^{-1}B.\end{aligned}\right\} \tag{7.4.1}$$

We first recall from Chapter 2 the following matrix identities, *i.e.*, (2.3.14) and (2.3.15), which are frequently used in the derivation of our result:

$$(I + XY)^{-1}X = X(I + YX)^{-1}, \tag{7.4.2}$$

and

$$\left[I + X(sI - Z)^{-1}Y\right]^{-1} = I - X(sI - Z + YX)^{-1}Y. \tag{7.4.3}$$

Next, we note that

$$\begin{aligned}\tilde{A}_{\rm F} &= (I + A + BF)(I - A - BF)^{-1} \\ &= (I + A + BF)(I - A)^{-1}[I - BF(I - A)^{-1}]^{-1} \\ &= [\tilde{A} + BF(I - A)^{-1}][I - BF(I - A)^{-1}]^{-1} \\ &= [\tilde{A} + BF(I - A)^{-1}][I + BF(I - A - BF)^{-1}] \\ &= \tilde{A} + \tilde{A}BF(I - A - BF)^{-1} + BF(I - A)^{-1}[I + BF(I - A - BF)^{-1}] \\ &= \tilde{A} + \tilde{A}BF(I - A - BF)^{-1} + BF(I - A)^{-1}(I - A)(I - A - BF)^{-1}\end{aligned}$$

$$
\begin{aligned}
&= \tilde{A} + \tilde{A}BF(I-A-BF)^{-1} + BF(I-A-BF)^{-1} \\
&= \tilde{A} + (\tilde{A}+I)BF(I-A-BF)^{-1} \\
&= \tilde{A} + 2(I-A)^{-1}BF(I-A-BF)^{-1} \\
&= \tilde{A} + \tilde{B}\tilde{F},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{B}_{\mathrm{F}} &= \sqrt{2}\,(I-A-BF)^{-1}B \\
&= \sqrt{2}\,[I-(I-A)^{-1}BF]^{-1}(I-A)^{-1}B \\
&= \sqrt{2}\,(I-A)^{-1}B\left[I-F(I-A)^{-1}B\right]^{-1} = \tilde{B}\tilde{T}_i.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\tilde{C}_{\mathrm{F}} &= \sqrt{2}\,(C+DF)(I-A-BF)^{-1} \\
&= \sqrt{2}\,(C+DF)(I-A)^{-1}[I-BF(I-A)^{-1}]^{-1} \\
&= \sqrt{2}\,(C+DF)(I-A)^{-1}[I+BF(I-A-BF)^{-1}] \\
&= \sqrt{2}\,C(I-A)^{-1} + \sqrt{2}\,DF(I-A)^{-1} \\
&\quad + \sqrt{2}\,(C+DF)(I-A)^{-1}BF(I-A-BF)^{-1} \\
&= \tilde{C} + \sqrt{2}\,\big[DF(I-A)^{-1}(I-A-BF) \\
&\quad + (C+DF)(I-A)^{-1}BF\big](I-A-BF)^{-1} \\
&= \tilde{C} + \sqrt{2}\big[DF - DF(I-A)^{-1}BF + C(I-A)^{-1}BF + DF(I-A)^{-1}BF\big] \\
&\quad \times (I-A-BF)^{-1} \\
&= \tilde{C} + [D + C(I-A)^{-1}B]\sqrt{2}\,F(I-A-BF)^{-1} \\
&= \tilde{C} + \tilde{D}\tilde{F},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{D}_{\mathrm{F}} &= D + (C+DF)(I-A-BF)^{-1}B \\
&= D + (C+DF)\left[I-(I-A)^{-1}BF\right]^{-1}(I-A)^{-1}B \\
&= D + (C+DF)(I-A)^{-1}B\left[I-F(I-A)^{-1}B\right]^{-1} \\
&= \left\{D\left[I-F(I-A)^{-1}B\right] + (C+DF)(I-A)^{-1}B\right\}\tilde{T}_{\mathrm{i}} \\
&= \left\{D - DF(I-A)^{-1}B + C(I-A)^{-1}B + DF(I-A)^{-1}B\right\}\tilde{T}_{\mathrm{i}} \\
&= \tilde{D}\tilde{T}_{\mathrm{i}},
\end{aligned}
$$

which completes the proof of 2(b).

2(c). This item is the dual of 2(b).

With the benefit of properties of 2(a)–2(c), the remainder of the proof is considerably simplified. It is well-known that the structural invariant indices lists of Morse, which correspond precisely to the structures of finite and infinite zeros as well as invertibility, are invariant under nonsingular state, output and input transformations, state feedback and output injection. We can thus apply appropriate nonsingular state, output and input transformations, as well as state feedback and output injection, to Σ_c and obtain a new system, say Σ_c^*. If this new system has Σ_d^* as its discrete-time counterpart under bilinear transformation, then from Properties 2(a)–2(c) it follows that Σ_d^* and Σ_d have the same structural invariant properties. It is therefore sufficient for the remainder of the proof that we show that 3(a)–6(b) are indeed properties of Σ_d^*.

Let us first apply nonsingular state, output and input transformations Γ_s, Γ_o and Γ_i to Σ_c such that the resulting system is in the form of the special coordinate basis as in Theorem 5.4.1, or, equivalently, the compact form in (5.4.21)–(5.4.24) with A_{aa} and C_{0a} being given by (5.4.29), E_{da} and E_{ca} being given by (5.4.30), and B_{0a}, L_{ab} and L_{ad} being given by (5.4.32). We will further assume that A_{aa} is already in the Jordan form of (2.3.39) and (5.4.34), and that matrices A_{aa}, L_{ad}, B_{a0}, E_{da}, C_{0a}, E_{ca} and L_{ab} are partitioned as follows:

$$A_{aa} = \begin{bmatrix} A_{aa}^a & 0 \\ 0 & A_{aa}^* \end{bmatrix}, \quad L_{ad} = \begin{bmatrix} L_{ad}^a \\ L_{ad}^* \end{bmatrix},$$

$$B_{a0} = \begin{bmatrix} B_{a0}^a \\ B_{a0}^* \end{bmatrix}, \quad L_{ab} = \begin{bmatrix} L_{ab}^a \\ L_{ab}^* \end{bmatrix},$$

$$E_{da} = [\, E_{da}^a \quad E_{da}^* \,], \quad C_{0a} = [\, C_{0a}^a \quad C_{0a}^* \,], \quad E_{ca} = [\, E_{ca}^a \quad E_{ca}^* \,],$$

where matrix A_{aa}^a has all its eigenvalues at $a = 1$, *i.e.*,

$$A_{aa}^a = I + \begin{bmatrix} 0 & I_{n_{a,1}-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{n_{a,\tau_a}-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{7.4.4}$$

and A_{aa}^* contains the remaining invariant zeros of Σ_c. Furthermore, we assume that the pair (A_{cc}, B_c) is in the controllability structural decomposition form of

Theorem 4.4.1, and the pair $(A_{\rm bb}, C_{\rm b})$ is in the observability structural decomposition form of Theorem 4.3.1. Next, define a state feedback gain matrix

$$F = -\Gamma_{\rm i} \begin{bmatrix} C_{\rm 0a}^a - C_2^a & C_{\rm 0a}^* & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ E_{\rm da}^a - C_1^a & E_{\rm da}^* & E_{\rm db} & E_{\rm dc} & E_{\rm dd} \\ E_{\rm ca}^a & E_{\rm ca}^* & 0 & E_{\rm cc} & 0 \end{bmatrix} \Gamma_{\rm s}^{-1},$$

and an output injection gain matrix

$$K = -\Gamma_{\rm s} \begin{bmatrix} B_{\rm a0}^a - B_2^a & L_{\rm ad}^a - B_1^a & L_{\rm ab}^a \\ B_{\rm a0}^* & L_{\rm ad}^* & L_{\rm ab}^* \\ B_{\rm b0} & L_{\rm bd} & L_{\rm bb} \\ B_{\rm c0} & L_{\rm cd} & L_{\rm cb} \\ B_{\rm d0} & L_{\rm dd} & 0 \end{bmatrix} \Gamma_{\rm o}^{-1}.$$

Here, $E_{\rm cc}$ is chosen such that all $\star$s in the controllability structural decomposition of $(A_{\rm cc}, B_{\rm c})$ are canceled out, *i.e.*,

$$A_{\rm cc}^* := A_{\rm cc} - B_{\rm c} E_{\rm cc},$$

is in Jordan form with all diagonal elements equal to 0. Similarly, $L_{\rm bb}$ is chosen such that

$$A_{\rm bb}^* := A_{\rm bb} - L_{\rm bb} C_{\rm b}',$$

is in Jordan form with all diagonal elements equal to 0. Likewise, $E_{\rm dd}$ and $L_{\rm dd}$ are chosen such that

$$A_{\rm dd}^* := A_{\rm dd} - L_{\rm dd} C_{\rm d} - B_{\rm d} E_{\rm dd},$$

is in Jordan form with all diagonal elements equal to 0, which in turn implies

$$C_{\rm d}(I - A_{\rm dd}^*)^{-1} B_{\rm d} = I_{m_{\rm d}}. \tag{7.4.5}$$

The matrices B_1^a, B_2^a, C_1^a and C_2^a are chosen in conformity with $A_{\rm aa}^a$ of (7.4.4) as follows:

$$B^a := [\, B_2^a \quad B_1^a \,] := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{7.4.6}$$

and

$$C^a := \begin{bmatrix} C_2^a \\ C_1^a \end{bmatrix} := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{7.4.7}$$

This can always be done, as a consequence of the assumption that the matrix A has no eigenvalue at $a = 1$, which implies that the invariant zero at $a = 1$ of Σ_c is controllable and observable.

Finally, we obtain a continuous-time system Σ_c^* characterized by the quadruple (A^*,B^*,C^*,D^*), where

$$A^* = P^{-1}\Gamma_s^{-1}(A + BF + KC + KDF)\Gamma_s P$$
$$= \begin{bmatrix} A_{aa}^* & 0 & 0 & 0 & 0 \\ 0 & A_{bb}^* & 0 & 0 & 0 \\ 0 & 0 & A_{cc}^* & 0 & 0 \\ 0 & 0 & 0 & A_{dd}^* & B_d C_1^a \\ 0 & 0 & 0 & B_1^a C_d & A_{aa}^a + B_2^a C_2^a \end{bmatrix}, \tag{7.4.8}$$

$$B^* = P^{-1}\Gamma_s^{-1}(B + KD)\Gamma_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \\ B_2^a & 0 & 0 \end{bmatrix}, \tag{7.4.9}$$

$$C^* = \Gamma_o^{-1}(C + DF)\Gamma_s P = \begin{bmatrix} 0 & 0 & 0 & 0 & C_2^a \\ 0 & 0 & 0 & C_d & 0 \\ 0 & C_b & 0 & 0 & 0 \end{bmatrix}, \tag{7.4.10}$$

and

$$D^* = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.4.11}$$

where P is a permutation matrix that transforms A_{aa}^a from its original position, *i.e.*, Block $(1,1)$, to Block $(5,5)$ in (7.4.8).

Next, define a subsystem (A_s, B_s, C_s, D_s) with

$$A_s := \begin{bmatrix} A_{dd}^* & B_d C_1^a \\ B_1^a C_d & A_{aa}^a + B_2^a C_2^a \end{bmatrix}, \quad B_s := \begin{bmatrix} 0 & B_d \\ B_2^a & 0 \end{bmatrix}, \tag{7.4.12}$$

and

$$C_s := \begin{bmatrix} 0 & C_2^a \\ C_d & 0 \end{bmatrix}, \quad D_s := \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.4.13}$$

It is straightforward to verify that with the choice of B^a and C^a as in (7.4.6) and (7.4.7), A_s has no eigenvalue at $a = 1$. Hence A^* has no eigenvalue at $a = 1$ either, since both A_{bb}^* and A_{cc}^* have all eigenvalues at 0, and A_{aa}^* contains only the invariant zeros of Σ_c which are not equal to $a = 1$. Applying the bilinear transformation (7.2.3) to Σ_c^*, it follows from the result of Lemma 7.2.1 that we obtain a discrete-time system Σ_d^*, characterized by the quadruple $(\tilde{A}^*, \tilde{B}^*, \tilde{C}^*, \tilde{D}^*)$, with

$$\tilde{A}^* = \begin{bmatrix} (I+A_{aa}^*)(I-A_{aa}^*)^{-1} & 0 & 0 & 0 \\ 0 & (I+A_{bb}^*)(I-A_{bb}^*)^{-1} & 0 & 0 \\ 0 & 0 & (I+A_{cc}^*)(I-A_{cc}^*)^{-1} & 0 \\ 0 & 0 & 0 & (I+A_s)(I-A_s)^{-1} \end{bmatrix}, \tag{7.4.14}$$

$$\tilde{B}^* = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & (I-A_{cc}^*)^{-1}B_c \\ (I-A_s)^{-1}B_s & 0 \end{bmatrix}, \tag{7.4.15}$$

$$\tilde{C}^* = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & C_s(I-A_s)^{-1} \\ 0 & C_b(I-A_s)^{-1} & 0 & 0 \end{bmatrix}, \tag{7.4.16}$$

and

$$\tilde{D}^* = \begin{bmatrix} D_s + C_s(I-A_s)^{-1}B_s & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.4.17}$$

Our next task is to find appropriate transformations, state feedback, and output injection laws, so as to transform the above system into the form of the special coordinate basis displaying Properties 3(a)–6(b) of the theorem.

To simplify the presentation, we first focus on the subsystem characterized by $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ with

$$\tilde{A}_s := (I + A_s)(I - A_s)^{-1}, \quad \tilde{B}_s := \sqrt{2}\,(I - A_s)^{-1}B_s, \tag{7.4.18}$$

and

$$\tilde{C}_s := \sqrt{2}\,C_s(I - A_s)^{-1}, \quad \tilde{D}_s := D_s + C_s(I - A_s)^{-1}B_s. \tag{7.4.19}$$

Using (7.4.5) in conjunction with (2.3.19), it is easy to compute $(I - A_s)^{-1} =$

$$\begin{bmatrix} X_1 & (I-A_{dd}^*)^{-1}B_dC_1^a(I-A_{aa}-B^aC^a)^{-1} \\ (I-A_{aa}-B^aC^a)^{-1}B_1^aC_d(I-A_{dd}^*)^{-1} & (I-A_{aa}-B^aC^a)^{-1} \end{bmatrix},$$

where

$$X_1 = (I-A_{dd}^*)^{-1} + (I-A_{dd}^*)^{-1}B_dC_1^a(I-A_{aa}-B^aC^a)^{-1}B_1^aC_d(I-A_{dd}^*)^{-1},$$

and hence

$$\tilde{A}_s = \begin{bmatrix} X_2 & 2(I-A_{dd}^*)^{-1}B_dC_1^a(I-A_{aa}^a-B^aC^a)^{-1} \\ 2(I-A_{aa}^a-B^aC^a)^{-1}B_1^aC_d(I-A_{dd}^*)^{-1} & (I+A_{aa}^a+B^aC^a)(I-A_{aa}^a-B^aC^a)^{-1} \end{bmatrix}, \tag{7.4.20}$$

where

$$X_2=(I+A^*_{dd})(I-A^*_{dd})^{-1}+2(I-A^*_{dd})^{-1}B_dC^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1C_d(I-A^*_{dd})^{-1};$$

$$\tilde{B}_s=\sqrt{2}\begin{bmatrix}(I-A^*_{dd})^{-1}B_dC^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_2 & (I-A^*_{dd})^{-1}B_d[I+C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1] \\ (I-A^a_{aa}-B^aC^a)^{-1}B^a_2 & (I-A^a_{aa}-B^aC^a)^{-1}B^a_1\end{bmatrix},$$

$$\tilde{C}_s=\sqrt{2}\begin{bmatrix}C^a_2(I-A^a_{aa}-B^aC^a)^{-1}B^a_1C_d(I-A^*_{dd})^{-1} & C^a_2(I-A^a_{aa}-B^aC^a)^{-1} \\ [I+C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1]C_d(I-A^*_{dd})^{-1} & C^a_1(I-A^a_{aa}-B^aC^a)^{-1}\end{bmatrix},$$

and

$$\tilde{D}_s=\begin{bmatrix}I+C^a_2(I-A^a_{aa}-B^aC^a)^{-1}B^a_2 & C^a_2(I-A^a_{aa}-B^aC^a)^{-1}B^a_1 \\ C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_2 & I+C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1\end{bmatrix}.$$

Noting the structure of A^a_{aa} in (7.4.4), and the structures of B^a and C^a in (7.4.6) and (7.4.7), we have

$$(I-A_{aa}-B^aC^a)^{-1}=\begin{bmatrix}0 & -1 & \cdots & 0 & 0 \\ -I_{n_{a,1}-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -I_{n_{a,r_a}-1} & 0\end{bmatrix}, \quad (7.4.21)$$

$$C^a_1(I-A_{aa}-B^aC^a)^{-1}B^a_2=0,\quad C^a_2(I-A_{aa}-B^aC^a)^{-1}B^a_1=0,$$

and

$$C^a(I-A_{aa}-B^aC^a)^{-1}B^a=\begin{bmatrix}0 & 0 \\ 0 & -I_{r_a}\end{bmatrix}.$$

Thus, $\tilde{B}_s$, $\tilde{C}_s$ and $\tilde{D}_s$ reduce to the following forms:

$$\tilde{B}_s=\sqrt{2}\begin{bmatrix}0 & (I-A^*_{dd})^{-1}B_d[I+C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1] \\ (I-A^a_{aa}-B^aC^a)^{-1}B^a_2 & (I-A^a_{aa}-B^aC^a)^{-1}B^a_1\end{bmatrix},$$

$$\tilde{C}_s=\sqrt{2}\begin{bmatrix}0 & C^a_2(I-A^a_{aa}-B^aC^a)^{-1} \\ [I+C^a_1(I-A^a_{aa}-B^aC^a)^{-1}B^a_1]C_d(I-A^*_{dd})^{-1} & C^a_1(I-A^a_{aa}-B^aC^a)^{-1}\end{bmatrix},$$

and

$$\tilde{D}_s = \begin{bmatrix} I+C_2^a(I-A_{aa}^a-B^aC^a)^{-1}B_2^a & 0 \\ 0 & I+C_1^a(I-A_{aa}^a-B^aC^a)^{-1}B_1^a \end{bmatrix}.$$

Next, define

$$\tilde{F}_s := \sqrt{2}\begin{bmatrix} 0 & 0 \\ -C_d(I-A_{dd}^*)^{-1} & 0 \end{bmatrix},$$

and

$$\tilde{K}_s := \sqrt{2}\begin{bmatrix} 0 & -(I-A_{dd}^*)^{-1}B_d \\ 0 & 0 \end{bmatrix},$$

from which it follows that

$$\begin{aligned}\tilde{A}_{sc} &= \tilde{A}_s+\tilde{B}_s\tilde{F}_s+\tilde{K}_s\tilde{C}_s+\tilde{K}_s\tilde{D}_s\tilde{F}_s \\ &= \begin{bmatrix} \tilde{A}_{aa}^{**} & 0 \\ 0 & (I+A_{aa}^a+B^aC^a)(I-A_{aa}^a-B^aC^a)^{-1} \end{bmatrix},\end{aligned}$$

where

$$\tilde{A}_{aa}^{**} := (I+A_{dd}^*)(I-A_{dd}^*)^{-1} - 2(I-A_{dd}^*)^{-1}B_dC_d(I-A_{dd}^*)^{-1}, \qquad (7.4.22)$$

$$\tilde{B}_{sc} = \tilde{B}_s + \tilde{K}_s\tilde{D}_s = \sqrt{2}\begin{bmatrix} 0 & 0 \\ (I-A_{aa}^a-B^aC^a)^{-1}B_2^a & (I-A_{aa}^a-B^aC^a)^{-1}B_1^a \end{bmatrix},$$

and

$$\tilde{C}_{sc} = \tilde{C}_s + \tilde{D}_s\tilde{F}_s = \sqrt{2}\begin{bmatrix} 0 & C_2^a(I-A_{aa}^a - B^aC^a)^{-1} \\ 0 & C_1^a(I-A_{aa}^a-B^aC^a)^{-1} \end{bmatrix}.$$

Next, re-partition B^a and C^a of (7.4.6) and (7.4.7) as follows:

$$B^a = [0 \quad \tilde{B}_a] \quad \text{and} \quad C^a = \begin{bmatrix} 0 \\ \tilde{C}_a \end{bmatrix},$$

where both $\tilde{B}_a$ and $\tilde{C}_a$ are of maximal rank. We thus obtain

$$\tilde{A}_{sc} = \begin{bmatrix} \tilde{A}_{aa}^{**} & 0 \\ 0 & (I+A_{aa}^a+\tilde{B}_a\tilde{C}_a)(I-A_{aa}^a-\tilde{B}_a\tilde{C}_a)^{-1} \end{bmatrix},$$

$$\tilde{B}_{sc} = \sqrt{2}\begin{bmatrix} 0 & 0 \\ 0 & (I-A_{aa}-\tilde{B}_a\tilde{C}_a)^{-1}\tilde{B}_a \end{bmatrix},$$

and

$$\tilde{C}_{sc} = \sqrt{2}\begin{bmatrix} 0 & 0 \\ 0 & \tilde{C}_a(I-A_{aa}-\tilde{B}_a\tilde{C}_a)^{-1} \end{bmatrix}, \quad \tilde{D}_{sc} = \tilde{D}_s = \begin{bmatrix} I_{m_0+m_d-\tau_a} & 0 \\ 0 & 0 \end{bmatrix}.$$

Using (7.4.4) and (7.4.21), straightforward manipulations yield

$$(I + A^a_{\rm aa} + \tilde{B}_a\tilde{C}_a)(I - A^a_{\rm aa} - \tilde{B}_a\tilde{C}_a)^{-1}$$
$$= \begin{bmatrix} \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,1}-1} & 0 \end{bmatrix} - I_{n_{a,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,\tau_a}-1} & 0 \end{bmatrix} - I_{n_{a,\tau_a}} \end{bmatrix},$$

$$(I - A^a_{\rm aa} - \tilde{B}_a\tilde{C}_a)^{-1}\tilde{B}_a = -\begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix},$$

and

$$\tilde{C}_a(I - A^a_{\rm aa} - \tilde{B}_a\tilde{C}_a)^{-1} = -\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Moreover, it can be readily verified that each subsystem characterized by the matrix triple $(\tilde{A}_{ai}, \tilde{B}_{ai}, \tilde{C}_{ai})$, $i = 1, 2, \ldots, \tau_a$, with

$$\tilde{A}_{ai} = -I_{n_{a,i}} + \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,i}-1} & 0 \end{bmatrix}, \quad \tilde{B}_{ai} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \tilde{C}_{ai} = [\,0 \quad -1\,],$$

has the following properties:

$$\tilde{C}_{ai}\tilde{B}_{ai} = \tilde{C}_{ai}\tilde{A}_{ai}\tilde{B}_{ai} = \cdots = \tilde{C}_{ai}(\tilde{A}_{ai})^{n_{a,i}-2}\tilde{B}_{ai} = 0,$$

and

$$\tilde{C}_{ai}(\tilde{A}_{ai})^{n_{a,i}-1}\tilde{B}_{ai} \neq 0.$$

It follows from Theorem 5.4.1 that there exist nonsingular transformations Γ_{sa}, Γ_{oa} and Γ_{ia} such that

$$\tilde{A}_{\rm d} = \Gamma_{\rm sa}^{-1}[(I + A^a_{\rm aa} + \tilde{B}_a\tilde{C}_a)(I - A^a_{\rm aa} - \tilde{B}_a\tilde{C}_a)^{-1}]\Gamma_{\rm sa}$$
$$= \begin{bmatrix} \star & I_{n_{a,1}-1} & \cdots & 0 & 0 \\ \star & \star & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \star & I_{n_{a,\tau_a}-1} \\ 0 & 0 & \cdots & \star & \star \end{bmatrix}, \tag{7.4.23}$$

$$\tilde{B}_{\rm d} = \Gamma_{\rm sa}^{-1}\left[(I-A_{\rm aa}^a-\tilde{B}_a\tilde{C}_a)^{-1}\tilde{B}_a\right]\Gamma_{{\rm i}a} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{7.4.24}$$

and

$$\tilde{C}_{\rm d} = \Gamma_{{\rm o}a}^{-1}\left[\tilde{C}_a(I-A_{\rm aa}^a-\tilde{B}_a\tilde{C}_a)^{-1}\right]\Gamma_{{\rm s}a} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{7.4.25}$$

Now, let us return to $\Sigma_{\rm d}^*$ characterized by $(\tilde{A}^*, \tilde{B}^*, \tilde{C}^*, \tilde{D}^*)$ as in (7.4.14) to (7.4.17). Using the properties of the subsystem $(\tilde{A}_{\rm s}, \tilde{B}_{\rm s}, \tilde{C}_{\rm s}, \tilde{D}_{\rm s})$ just derived, we are in a position to define appropriate state feedback and output injection gain matrices, say $\tilde{F}^*$ and $\tilde{K}^*$, together with nonsingular state, output and input transformations $\tilde{\Gamma}_{\rm s}^*$, $\tilde{\Gamma}_{\rm o}^*$ and $\tilde{\Gamma}_{\rm i}^*$, such that

$$\tilde{A}_{\rm SCB}^* := (\tilde{\Gamma}_{\rm s}^*)^{-1}\left(\tilde{A}^* + \tilde{B}^*\tilde{F}^* + \tilde{K}^*\tilde{C}^* + \tilde{K}^*\tilde{D}^*\tilde{F}^*\right)\tilde{\Gamma}_{\rm s}^*$$
$$= \begin{bmatrix} (I+A_{\rm aa}^*)(I-A_{\rm aa}^*)^{-1} & 0 & 0 & 0 & 0 \\ 0 & (I+A_{\rm bb}^*)(I-A_{\rm bb}^*)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (I+A_{\rm cc}^*)(I-A_{\rm cc}^*)^{-1} & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_{\rm aa}^{**} & 0 \\ 0 & 0 & 0 & 0 & \tilde{A}_{\rm d} \end{bmatrix}, \tag{7.4.26}$$

with $\tilde{A}_{\rm aa}^{**}$ given by (7.4.22), and

$$\tilde{B}_{\rm SCB}^* := (\tilde{\Gamma}_{\rm s}^*)^{-1}\left(\tilde{B}^* + \tilde{K}^*\tilde{D}^*\right)\tilde{\Gamma}_{\rm i}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (I-A_{\rm cc}^*)^{-1}B_{\rm c} \\ 0 & 0 & 0 \\ 0 & \tilde{B}_{\rm d} & 0 \end{bmatrix}, \tag{7.4.27}$$

$$\tilde{C}_{\rm SCB}^* := (\tilde{\Gamma}_{\rm o}^*)^{-1}\left(\tilde{C}^* + \tilde{D}^*\tilde{F}^*\right)\tilde{\Gamma}_{\rm s}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C_{\rm b}(I-A_{\rm bb}^*)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_{\rm d} \end{bmatrix}, \tag{7.4.28}$$

and

$$\tilde{D}_{\rm SCB}^* := (\tilde{\Gamma}_{\rm o}^*)^{-1}\tilde{D}^*\tilde{\Gamma}_{\rm i}^* = \begin{bmatrix} I_{m_0+m_{\rm d}-r_a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.4.29}$$

Clearly, $\Sigma_{\rm SCB}^*$ characterized by $(\tilde{A}_{\rm SCB}^*, \tilde{B}_{\rm SCB}^*, \tilde{C}_{\rm SCB}^*, \tilde{D}_{\rm SCB}^*)$ has the same lists of structural invariant indices as $\Sigma_{\rm d}^*$ does, which in turn has the same lists of structural invariant indices as Σ_d. Most importantly, $\Sigma^*_{\rm SCB}$ is in the form of the special coordinate basis, and we are now ready to prove Properties 3(a)–6(b) of the theorem.

3(a). First, we note that $\mathbf{I}_2(\Sigma_d) = \mathbf{I}_2(\Sigma^*_{\rm SCB})$. From (7.4.26) to (7.4.29) and the properties of the special coordinate basis, we know that $\mathbf{I}_2(\Sigma^*_{\rm SCB})$ is given by the controllability index of the pair

$$\Big((I + A^*_{\rm cc})(I - A^*_{\rm cc})^{-1},\ (I - A^*_{\rm cc})^{-1}B_{\rm c}\Big) \quad \text{or} \quad \Big((I + A^*_{\rm cc})(I - A^*_{\rm cc})^{-1},\ B_{\rm c}\Big).$$

Recalling the definitions of $A^*_{\rm cc}$ and $B_{\rm c}$:

$$A^*_{\rm cc} = \begin{bmatrix} 0 & I_{\ell_1 - 1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{\ell_{m_c}-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B_{\rm c} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix},$$

we readily verify that the controllability index of

$$\Big((I + A^*_{\rm cc})(I - A^*_{\rm cc})^{-1},\ B_{\rm c}\Big)$$

is also given by $\{\ell_1, \ell_2, \ldots, \ell_{m_c}\}$, and thus $\mathbf{I}_2(\Sigma_d) = \mathbf{I}_2(\Sigma_c)$.

Likewise, the proof that $\mathbf{I}_3(\Sigma_d) = \mathbf{I}_3(\Sigma_c)$ follows along similar lines.

3(b)–3(c). These follow directly from 3(a).

4(a). It follows from the properties of the special coordinate basis that the invariant zero structure of $\tilde{\Sigma}^*_{\rm SCB}$, or equivalently Σ_d, is given by the eigenvalues of $\tilde{A}^{**}_{\rm aa}$ and $(I + A^*_{\rm aa})(I - A^*_{\rm aa})^{-1}$, together with their associated Jordan blocks. Property 4(a) corresponds with the eigenvalues of $\tilde{A}^{**}_{\rm aa}$ of (7.4.22), together with their associated Jordan blocks. First, we note that, for any $z \in \mathbb{C}$,

$$zI - \tilde{A}^{**}_{\rm aa} = \left[(z-1)I - (z+1)A^*_{\rm dd} + 2(I - A^*_{\rm dd})^{-1}B_{\rm d}C_{\rm d}\right](I - A^*_{\rm dd})^{-1}. \tag{7.4.30}$$

Recall the definitions of $A^*_{\rm dd}$, $B_{\rm d}$ and $C_{\rm d}$:

$$A^*_{\rm dd} = \begin{bmatrix} 0 & I_{n_{q1}-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{q_{m_d}-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B_{\rm d} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix},$$

and

$$C_{\rm d} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It can be shown that

$$(z-1)I-(z+1)A_{\rm dd}^{*}+2(I-A_{\rm dd}^{*})^{-1}B_{\rm d}C_{\rm d}={\rm blkdiag}\{Q_1(z),\ldots,Q_i(z)\},$$

where $Q_i(z) \in \mathbb{C}^{n_{q_i}\times n_{q_i}}$ is given by

$$Q_i(z)=\begin{bmatrix} z+1 & -(z+1) & 0 & \cdots & 0 & 0 \\ 2 & z-1 & -(z+1) & \cdots & 0 & 0 \\ 2 & 0 & z-1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 2 & 0 & 0 & \cdots & z-1 & -(z+1) \\ 2 & 0 & 0 & \cdots & 0 & z-1 \end{bmatrix}, \tag{7.4.31}$$

for $i=1,2,\ldots,m_{\rm d}$. It follows from (7.4.30) that the eigenvalue of $\tilde{A}_{\rm aa}^{**}$ is the scalar z that causes the rank of

$${\rm blkdiag}\{Q_1(z),Q_2(z),\ldots,Q_{m_{\rm d}}(z)\},$$

to drop below $n_{\rm d}=\sum_{i=1}^{m_{\rm d}} q_i$. Using the particular form of $Q_i(z)$, it is clear that the only such scalar $z\in\mathbb{C}$ which causes $Q_i(z)$ to drop rank is $z=-1$. Moreover, rank $\{Q_i(-1)\}=n_{q_i}-1$, *i.e.*, $Q_i(-1)$ has only one linearly independent eigenvector. Hence, $z=-1$ is the eigenvalue of $\tilde{A}_{\rm aa}^{**}$, or equivalently the invariant zero of $\Sigma_{\rm d}$, with the multiplicity structure

$$S_{-1}^{\star}(\Sigma_{\rm d})=\{q_1,q_2,\ldots,q_{m_{\rm d}}\}=S_{\infty}^{\star}(\Sigma_{\rm c}),$$

thereby proving 4(a).

4(b). This part of the infinite zero structure corresponds to the invariant zeros of the matrix $(I+A_{\rm aa}^{*})(I-A_{\rm aa}^{*})^{-1}$. With $A_{\rm aa}^{*}$ in Jordan form, Property 4(b) follows by straightforward manipulations.

5(a). The proof of this item follows directly from (7.4.29).

5(b). This follows from the structure of $(\tilde{A}_{\rm d},\tilde{B}_{\rm d},\tilde{C}_{\rm d})$ in (7.4.23) to (7.4.25), in conjunction with Property 5.4.4 of the special coordinate basis.

6(a)–6(b). We let the state space of the system (7.2.1) be $\mathcal{X}$ and be partitioned in the subsystems of its special coordinate basis as

$$\mathcal{X}=\mathcal{X}_{\rm a}^{-}\oplus\mathcal{X}_{\rm a}^{0}\oplus\mathcal{X}_{\rm a}^{+}\oplus\mathcal{X}_{\rm b}\oplus\mathcal{X}_{\rm c}\oplus\mathcal{X}_{\rm d}. \tag{7.4.32}$$

We further partition $\mathcal{X}_{\rm a}^{+}$ as

$$\mathcal{X}_{\rm a}^{+}=\mathcal{X}_{\rm a1}^{+}\oplus\mathcal{X}_{\rm a*}^{+}, \tag{7.4.33}$$

where $\mathcal{X}_{\rm a1}^{+}$ is associated with the zero dynamics of the unstable zero of (7.2.1) at $s = a = 1$ and $\mathcal{X}_{\rm a*}^{+}$ is associated with the rest of the unstable zero dynamics of (7.2.1). Similarly, we let the state space of the transformed system (7.2.6) be $\tilde{\mathcal{X}}$ and be partitioned in its special coordinate basis subsystems as

$$\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_{\rm a}^{-} \oplus \tilde{\mathcal{X}}_{\rm a}^{0} \oplus \tilde{\mathcal{X}}_{\rm a}^{+} \oplus \tilde{\mathcal{X}}_{\rm b} \oplus \tilde{\mathcal{X}}_{\rm c} \oplus \tilde{\mathcal{X}}_{\rm d}, \tag{7.4.34}$$

with $\tilde{\mathcal{X}}_{\rm a}^{0}$ being further partitioned as

$$\tilde{\mathcal{X}}_{\rm a}^{0} = \tilde{\mathcal{X}}_{\rm a1}^{0} \oplus \tilde{\mathcal{X}}_{\rm a*}^{0}, \tag{7.4.35}$$

where $\tilde{\mathcal{X}}_{\rm a1}^{0}$ is associated with the zero dynamics of the invariant zero of (7.2.6) at $z = -1$ and $\tilde{\mathcal{X}}_{\rm a*}^{0}$ is associated the rest of the zero dynamics of the zeros of (7.2.6) on the unit circle. Then, from the above derivations of 1(a) to 5(b), we have the following mappings between the subsystems of $\Sigma_{\rm c}$ of (7.2.1) and those of $\Sigma_{\rm d}$ of (7.2.6):

$$\left.\begin{array}{ccc}
\mathcal{X}_{\rm a}^{-} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm a}^{-}, \\
\mathcal{X}_{\rm d} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm a1}^{0}, \\
\mathcal{X}_{\rm a}^{0} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm a*}^{0}, \\
\mathcal{X}_{\rm a*}^{+} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm a}^{+}, \\
\mathcal{X}_{\rm b} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm b}, \\
\mathcal{X}_{\rm c} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm c}, \\
\mathcal{X}_{\rm a1}^{+} & \Longleftrightarrow & \tilde{\mathcal{X}}_{\rm d}.
\end{array}\right\} \tag{7.4.36}$$

Noting that both geometric subspaces $\mathcal{V}_{\rm X}$ and $\mathcal{S}_{\rm X}$ are invariant under any nonsingular output and input transformations, as well as any state feedback and output injection, we have

$$\mathcal{V}^{+}(\Sigma_{\rm c}) = \mathcal{X}_{\rm a*}^{+} \oplus \mathcal{X}_{\rm a1}^{+} \oplus \mathcal{X}_{\rm c} = \tilde{\mathcal{X}}_{\rm a}^{+} \oplus \tilde{\mathcal{X}}_{\rm d} \oplus \tilde{\mathcal{X}}_{\rm c} = \mathcal{S}^{-}(\Sigma_{\rm d}), \tag{7.4.37}$$

and

$$\mathcal{S}^{+}(\Sigma_{\rm c}) = \mathcal{X}_{\rm a}^{-} \oplus \mathcal{X}_{\rm a}^{0} \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d} = \tilde{\mathcal{X}}_{\rm a}^{-} \oplus \tilde{\mathcal{X}}_{\rm a*}^{0} \oplus \tilde{\mathcal{X}}_{\rm c} \oplus \tilde{\mathcal{X}}_{\rm a1}^{0} = \mathcal{V}^{-}(\Sigma_{\rm d}). \tag{7.4.38}$$

Unfortunately, other geometric subspaces do not have such clear relationships as above.

This concludes the proof of Theorem 7.2.1 and this chapter. ■

7.5 Exercises

7.1. Consider a continuous-time system, Σ_c, characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u,$$

and

$$y = Cx + Du = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u.$$

(a) Compute its discrete-time counterpart, Σ_d, under the bilinear transformation

$$s = \frac{z-1}{z+1}.$$

(b) Use the *Linear Systems Toolkit* to compute the geometric subspaces, $\mathcal{V}^*$, $\mathcal{V}^-$, $\mathcal{V}^+$, $\mathcal{S}^*$, $\mathcal{S}^-$, $\mathcal{S}^+$, $\mathcal{R}^*$, and $\mathcal{N}^*$, for the continuous-time system, Σ_c, and the subspaces, $\mathcal{V}^*$, $\mathcal{V}^-$, $\mathcal{V}^+$, $\mathcal{S}^*$, $\mathcal{S}^-$, $\mathcal{S}^+$, $\mathcal{R}^*$, and $\mathcal{N}^*$, for the discrete-time counterpart, Σ_d.

(c) Verify that $\mathcal{V}^+(\Sigma_c) = \mathcal{S}^-(\Sigma_d)$ and $\mathcal{S}^+(\Sigma_c) = \mathcal{V}^-(\Sigma_d)$. Comment on the relationship of other subspaces.

7.2. Prove Corollary 7.2.1 and Corollary 7.2.2.

7.3. Consider a continuous-time system, Σ_c, characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

and

$$y = Cx = [0 \quad 0 \quad 0 \quad 0 \quad 1]\, x.$$

Compute the invariant zeros of its discrete-time counterpart, Σ_d, under the bilinear transformation

$$s = a\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right),$$

with the sampling period, $T = 0.5, 0.4, 0.3, 0.2, 0.1, 0.05,$ and 0.01. Verify the result of Corollary 7.2.2 by plotting the invariant zeros of the resulting discrete-time systems on a complex plane.

7.4. Given a stable continuous-time system, $\Sigma_{\rm c}$, with a transfer function

$$G_{\rm c}(s) = C(sI - A)^{-1}B + D,$$

and its discrete-time counterpart under the usual bilinear transformation, $\Sigma_{\rm d}$, with a transfer function

$$G_{\rm d}(z) = C\left(a\frac{z-1}{z+1}I - A\right)^{-1} B + D = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D},$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are as given in Lemma 7.2.1, show that $\Sigma_{\rm d}$ is stable, and

$$\|G_{\rm d}\|_\infty = \|G_{\rm c}\|_\infty.$$

Also, show by an example that, in general,

$$\|G_{\rm d}\|_2 \neq \|G_{\rm c}\|_2.$$

Hint: Refer to Section 2.4 of Chapter 2 for the definition and computation of the H_2-norm and H_∞-norm of continuous- and discrete-time systems.

7.5. Consider a continuous-time system characterized by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Another popular method that can be employed to discretize the system is the zero-order-hold (ZOH) transformation. It can be shown that the discrete-time equivalence of the continuous-time system under the ZOH transformation with a sampling period, T, is given by

$$x(k+1) = A_{\rm z}x(k) + B_{\rm z}u(k), \quad y(k) = Cx(k) + Du(k),$$

where

$$A_{\rm z} = {\rm e}^{AT}, \quad B_{\rm z} = \left(\int_0^T {\rm e}^{A\tau}d\tau\right)B.$$

Let $G_{\rm c}(s) = C(sI - A)^{-1}B + D$ and $G_{\rm z}(z) = C(zI - A_{\rm z})^{-1}B_{\rm z} + D$. Show by an example that, in general,

$$\|G_{\rm c}\|_2 \neq \|G_{\rm z}\|_2 \quad \text{and} \quad \|G_{\rm c}\|_\infty \neq \|G_{\rm z}\|_\infty.$$

7.6. Suppose that the continuous-time system of Exercise 7.3 is discretized using the ZOH transformation. Compute the invariant zeros of the resulting discrete-time equivalence with $T = 0.5, 0.4, 0.3, 0.2, 0.1, 0.05$, and 0.01. Plot these invariant zeros on a complex plane, and comment on the result.

Chapter 8

System Factorizations

8.1 Introduction

System factorizations such as the well-known inner-outer factorization and its dual version, the cascade factorization of nonminimum-phase systems have been extensively studied and used in the literature. The so-called minimum-phase/all-pass factorization plays a significant role in several applications, prominent among them being singular filtering (see *e.g.*, Halevi and Palmor [61], and Shaked [128]), cheap and singular optimal LQ control (see, *e.g.*, Shaked [129]), and loop transfer recovery (see, *e.g.* Chen [18], Saberi *et al.* [116], and Zhang and Freudenberg [159]), while its dual version, the inner-outer factorization, has played an important role in solving problems related to robust and H_∞ control (see, *e.g.*, [54] and references cited therein). Traditionally the minimum-phase/all-pass factorization has been carried out by spectral factorization techniques (see, *e.g.*, Shaked and Soroka [130], Soroka and Shaked [134], Strintzis [137], and Tuel [142]). The role that the minimum-phase/all-pass factorization plays in the control literature as well as various methods available for such a factorization are well documented by Shaked [127]. The inner-outer factorization is also very well studied in the literature, and there are several papers that provide state space based algorithms for such a factorization. For example, Chen and Francis [35] and Weiss [148] have derived algorithms that are applicable for certain classes of systems. In addition, we will also introduce a generalized factorization technique, which has several promising applications. In particular, it can be easily modified to solve the well-known system zero placement problem, which will be studied in detail in Chapter 9. The contents of this chapter are based on our early works (*i.e.*, Chen *et*

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al. [26], Lin *et al.* [88] and [89]), in which we successfully constructed the above mentioned factorizations for general systems using the structural decomposition techniques presented in the previous chapters of this book.

To be more specific, let us consider a left invertible nonminimum-phase system Σ characterized by the matrix quadruple (A, B, C, D),

$$\begin{cases} \dot{x} = A\ x + B\ u, \\ y = C\ x + D\ u, \end{cases} \tag{8.1.1}$$

where the state vector $x \in \mathbb{R}^n$, output vector $y \in \mathbb{R}^p$ and input vector $u \in \mathbb{R}^m$. Without loss of generality, we assume that $[\,B' \quad D'\,]$ and $[\,C \quad D\,]$ are of maximal rank. Let the transfer function of Σ be

$$G(s) = C(sI - A)^{-1}B + D. \tag{8.1.2}$$

For the minimum-phase/all-pass factorization, it is required that the given system Σ be detectable. The *minimum-phase/all-pass factorization* of $G(s)$ is expressed as

$$G(s) = G_{\rm m}(s)V(s), \tag{8.1.3}$$

where $G_{\rm m}(s)$ is left invertible and of minimum-phase, and satisfies

$$G(s)G'(-s) = G_{\rm m}(s)G'_{\rm m}(-s), \tag{8.1.4}$$

whereas $V(s)$ is a stable and right invertible *all-pass factor* satisfying

$$V(s)V'(-s) = I. \tag{8.1.5}$$

The problem is then to construct matrices $B_{\rm m}$ and $D_{\rm m}$ such that a system $\Sigma_{\rm m}$ characterized by the matrix quadruple $(A, B_{\rm m}, C, D_{\rm m})$ has the intended transfer function $G_{\rm m}(s)$. Also, the invariant zeros of $\Sigma_{\rm m}$ are those minimum-phase invariant zeros of Σ and the mirror images of nonminimum-phase invariant zeros of Σ. On the other hand, in loop transfer recovery and in other applications such as finite zero placement problems, one does not necessarily require a true minimum-phase image of Σ. What is required is a model which retains the infinite zero structure of Σ and whose invariant zeros can be appropriately assigned to some desired locations in the open left-half complex plane. With this point in mind, we introduce a *generalized cascade factorization* of the form

$$G(s) = G_{\rm M}(s)U(s). \tag{8.1.6}$$

Here $G_{\rm M}(s)$ is the transfer function matrix of a system $\Sigma_{\rm M}$, which has the same infinite zero structure as that of Σ, is of minimum-phase with all its invariant zeros located at desired locations and is left invertible. On the other hand, $U(s)$ is square, stable, invertible and asymptotically all-pass in the sense that

$$U(s)U'(-s) \to I \quad \text{as} \quad |s| \to \infty. \tag{8.1.7}$$

As mentioned earlier, the inner-outer factorization is actually a dual version and a special case of the minimum-phase/all-pass factorization. It only deals with stable and proper systems. Thus, it requires that the eigenvalues of A in the given system Σ of (8.1.1) are all in $\mathbb{C}^-$. Dually, it requires Σ to be stabilizable (instead of being detectable as in the case of minimum-phase/all-pass factorization). The *inner-outer factorization* can be expressed as

$$G(s) = G_{\rm i}(s)G_{\rm o}(s), \tag{8.1.8}$$

where $G_{\rm i}(s)$ is an *inner factor* of $G(s)$, *i.e.*, $G_{\rm i}(s)$ is a stable and proper transfer function satisfying

$$G_{\rm i}'(-s)G_{\rm i}(s) = I, \tag{8.1.9}$$

and $G_{\rm o}(s)$ is an *outer factor* of $G(s)$, *i.e.*, $G_{\rm o}(s)$ is stable and proper and has a right inverse being analytic in $\mathbb{C}^+$, which is equivalent to the fact that $G_{\rm o}(s)$ is right invertible and has no invariant zeros in $\mathbb{C}^+$.

For clarity and for ease of reference, we first consider the case when the given system is strictly proper in Section 8.2, whereas the results for general nonstrictly proper systems are given in Section 8.3. Finally, we note that all these factorizations can be done similarly for discrete-time systems, which will be addressed in Section 8.4.

8.2 Strictly Proper Systems

We consider in this section the situation when the given system Σ of (8.1.1) is strictly proper, *i.e.*, $D = 0$. For the minimum-phase/all-pass factorization and generalized cascade factorization, we assume that Σ is left invertible, while for the inner-outer factorization, we assume that Σ is right invertible. For the minimum-phase/all-pass factorization and inner-outer factorization, we further assume that Σ has no invariant zeros on the imaginary axis. The result for such a case is fairly straightforward and is very useful in numerous applications and in fact, most of the references cited in the introductory section of this chapter dealt with only this special class of systems. These restrictions will, however, be removed in the next section.

We first present a step-by-step algorithm for the construction of the minimum-phase/all-pass factorization.

STEP FACT-SP.1.

Utilize the result of Theorem 5.3.1 (see also the compact form of the special coordinate basis in Section 5.4 of Chapter 5) to find nonsingular transformations $\Gamma_s \in \mathbb{R}^{n\times n}$, $\Gamma_o \in \mathbb{R}^{p\times p}$ and $\Gamma_i \in \mathbb{R}^{m\times m}$ such that the given system Σ, *i.e.*, the matrix triple (A, B, C), can be transformed into the form of the special coordinate basis. More specifically, we have

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa}^+ & 0 & L_{ab}^+ C_b & L_{ad}^+ C_d \\ 0 & A_{aa}^- & L_{ab}^- C_b & L_{ad}^- C_d \\ 0 & 0 & A_{bb} & L_{bd} C_d \\ B_d E_a^+ & B_d E_a^- & B_d E_b & A_{dd} \end{bmatrix}, \tag{8.2.1}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_d \end{bmatrix}, \tag{8.2.2}$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 \end{bmatrix} \quad \text{and} \quad B_d' B_d = I. \tag{8.2.3}$$

Here $\lambda(A_{aa}^+) \subset \mathbb{C}^+$ and $\lambda(A_{aa}^-) \subset \mathbb{C}^-$ are respectively the nonminimum-phase and minimum-phase invariant zeros of Σ. Also, we note that the pair (A_{aa}^+, E_a^+) is observable whenever Σ is detectable.

STEP FACT-SP.2.

Solve the Lyapunov equation

$$(A_{aa}^+)' P + P A_{aa}^+ = (E_a^+)' \Gamma_i' \Gamma_i E_a^+, \tag{8.2.4}$$

for $P > 0$. Note that such a solution always exists since A_{aa}^+ is unstable and (A_{aa}^+, E_a^+) is observable. Next, compute

$$K_a^+ = P^{-1} (E_a^+)' \Gamma_i' \Gamma_i, \tag{8.2.5}$$

and

$$B_m = \Gamma_s \tilde{B}_m \Gamma_i^{-1} = \Gamma_s \begin{bmatrix} K_a^+ \\ 0 \\ 0 \\ B_d \end{bmatrix} \Gamma_i^{-1}. \tag{8.2.6}$$

STEP FACT-SP.3.

Define $\Sigma_{\rm m}$ to be a system characterized by a matrix triple $(A, B_{\rm m}, C)$ and

$$V(s) = \Gamma_{\rm i} \left[I - E_{\rm a}^{+} \left(sI - A_{\rm aa}^{+} + K_{\rm a}^{+} E_{\rm a}^{+} \right)^{-1} K_{\rm a}^{+} \right] \Gamma_{\rm i}^{-1}. \tag{8.2.7}$$

This completes the procedure for constructing the minimum-phase/all-pass factorization of Σ.

We have the following theorem.

Theorem 8.2.1. *Consider a detectable, left invertible and nonminimum-phase system Σ of (8.1.1) with $D = 0$ and with all its nonminimum-phase invariant zeros in $\mathbb{C}^+$. Then, its minimum-phase/all-pass factorization is given by*

$$G(s) = G_{\rm m}(s) V(s), \tag{8.2.8}$$

where $V(s)$, the stable all-pass factor, is given as in (8.2.7), and $G_{\rm m}(s)$, the minimum phase image of Σ, is the transfer function of $\Sigma_{\rm m}$ characterized by the matrix triple $(A, B_{\rm m}, C)$ with $B_{\rm m}$ given as in (8.2.6), i.e.,

$$G_{\rm m}(s) = C(sI - A)^{-1} B_{\rm m}. \tag{8.2.9}$$

Furthermore, $\Sigma_{\rm m}$ is left invertible and has the same infinite zero structure as that of Σ with its transfer function $G_{\rm m}(s)$ satisfying

$$G_{\rm m}(s) G_{\rm m}'(-s) = G(s) G'(-s). \tag{8.2.10}$$

The all-pass factor $V(s)$ satisfies $V(s)V'(-s) = I$ and has all its poles at the mirror images of the nonminimum-phase zeros of Σ.

Proof. We first show that $A_{\rm aa}^{+} - K_{\rm a}^{+} E_{\rm a}^{+}$ is a stable matrix. By examining (8.2.4) and (8.2.5), we have

$$A_{\rm aa}^{+} - K_{\rm a}^{+} E_{\rm a}^{+} = A_{\rm aa}^{+} - P^{-1} \left(E_{\rm a}^{+} \right)' \Gamma_{\rm i}' \Gamma_{\rm i} E_{\rm a}^{+} = P^{-1} \left(-A_{\rm aa}^{+} \right)' P, \tag{8.2.11}$$

which implies that $A_{\rm aa}^{+} - K_{\rm a}^{+} E_{\rm a}^{+}$ is indeed a stable matrix.

Next, we proceed to prove that $\Sigma_{\rm m}$ is of minimum-phase, left invertible and has the same infinite zero structure as that of Σ. Without loss of generality, we assume that Σ is in the form of the special coordinate basis as that in Theorem 5.3.1 of Chapter 5. Thus, $\Sigma_{\rm m}$ can be rewritten as

$$\dot{x}_{\rm a}^{+} = A_{\rm aa}^{+} x_{\rm a}^{+} + L_{\rm ad}^{+} y_{\rm d} + L_{\rm ab}^{+} y_{\rm b} + K_{\rm a}^{+} u,$$

$$\dot{x}_{\rm a}^- = A_{\rm aa}^- x_{\rm a}^- + L_{\rm ad}^- y_{\rm d} + L_{\rm ab}^- y_{\rm b},$$
$$\dot{x}_{\rm b} = A_{\rm bb} x_{\rm b} + L_{\rm bd} y_{\rm d}, \quad y_{\rm b} = C_{\rm b} x_{\rm b},$$
$$\dot{x}_{\rm d} = A_{\rm dd}^* x_{\rm d} + L_{\rm d} y_{\rm d} + B_{\rm d}\left[u + E_{\rm a}^+ x_{\rm a}^+ + E_{\rm a}^- x_{\rm a}^- + E_{\rm b} x_{\rm b} + E_{\rm d} x_{\rm d}\right], \; y_{\rm d} = C_{\rm d} x_{\rm d},$$

for some submatrices of appropriate dimensions. Let us now define a new state variable

$$x_{\rm a}^{\rm m} = x_{\rm a}^+ - K_{\rm a}^+ B_{\rm d}' x_{\rm d}. \tag{8.2.12}$$

Since $B_{\rm d}' B_{\rm d} = I$, it is then straightforward to verify that

$$\begin{aligned}\dot{x}_{\rm a}^{\rm m} = (A_{\rm aa}^+ - K_{\rm a}^+ E_{\rm a}^+) x_{\rm a}^{\rm m} - K_{\rm a}^+ E_{\rm a}^- x_{\rm a}^- + L_{\rm ab}^+ y_{\rm b} \\ - K_{\rm a}^+ E_{\rm b} x_{\rm b} + (L_{\rm ad}^+ - K_{\rm a}^+ B_{\rm d}' L_{\rm d}) y_{\rm d} \\ + (A_{\rm aa}^+ K_{\rm a}^+ B_{\rm d}' - K_{\rm a}^+ E_{\rm d} - K_{\rm a}^+ E_{\rm a}^+ K_{\rm a}^+ B_{\rm d}' - K_{\rm a}^+ B_{\rm d}' A_{\rm dd}) x_{\rm d},\end{aligned}$$

and

$$\dot{x}_{\rm d} = A_{\rm dd}^* x_{\rm d} + L_{\rm d} y_{\rm d} + B_{\rm d}\left[u + E_{\rm a}^+ x_{\rm a}^{\rm m} + E_{\rm a}^- x_{\rm a}^- + E_{\rm b} x_{\rm b} + (E_{\rm d} + K_{\rm a}^+ B_{\rm d}) x_{\rm d}\right].$$

It then follows from STEP SCB.8 in the proof of Theorem 5.3.1 in Chapter 5 that there exists a nonsingular transformation T such that

$$\begin{pmatrix} x_{\rm a}^{\rm m} \\ x_{\rm a}^- \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix} = T \begin{pmatrix} \bar{x}_{\rm a}^{\rm m} \\ x_{\rm a}^- \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix},$$

and

$$\dot{\bar{x}}_{\rm a}^{\rm m} = (A_{\rm aa}^+ - K_{\rm a}^+ E_{\rm a}^+) \bar{x}_{\rm a}^{\rm m} - K_{\rm a}^+ E_{\rm a}^- x_{\rm a}^- + L_{\rm ad}^{\rm m} y_{\rm d} + L_{\rm ab}^{\rm m} y_{\rm b},$$
$$\dot{x}_{\rm a}^- = A_{\rm aa}^- x_{\rm a}^- + L_{\rm ad}^- y_{\rm d} + L_{\rm ab}^- y_{\rm b},$$
$$\dot{x}_{\rm b} = A_{\rm bb} x_{\rm b} + L_{\rm bd} y_{\rm d}, \quad y_{\rm b} = C_{\rm b} x_{\rm b},$$
$$\dot{x}_{\rm d} = A_{\rm dd}^* x_{\rm d} + L_{\rm d} y_{\rm d} + B_{\rm d}\left[u + E_{\rm a}^+ \bar{x}_{\rm a}^{\rm m} + E_{\rm a}^- x_{\rm a}^- + E_{\rm b}^{\rm m} x_{\rm b} + E_{\rm d}^{\rm m} x_{\rm d}\right], \tag{8.2.13}$$
$$y_{\rm d} = C_{\rm d} x_{\rm d},$$

for some matrices $L_{\rm ad}^{\rm m}$, $L_{\rm ab}^{\rm m}$, $E_{\rm b}^{\rm m}$ and $E_{\rm d}^{\rm m}$ of appropriate dimensions. The state equations in (8.2.13) is now in the form of the special coordinate basis of Theorem 5.3.1. Hence, it follows from the properties given in Section 5.4 of Chapter 5 that $\Sigma_{\rm m}$ and Σ have the same infinite zero structure and that $\Sigma_{\rm m}$ is left invertible. Furthermore, the invariant zeros of $\Sigma_{\rm m}$ are given by

$$\lambda \begin{bmatrix} A_{\rm aa}^+ - K_{\rm a}^+ E_{\rm a}^+ & -K_{\rm a}^+ E_{\rm a}^- \\ 0 & A_{\rm aa}^- \end{bmatrix} \subset \mathbb{C}^-. \tag{8.2.14}$$

Hence, $\Sigma_{\rm m}$ is of minimum-phase.

We now proceed to show that $V(s)V'(-s) = I$. From the Woodbury or Sherman–Morrison–Woodbury formula, *i.e.*, (2.3.13), and (8.2.4) and (8.2.5), we have

$$\begin{aligned}
V^{-1}(s) &= \left\{\Gamma_{\rm i}\left[I - E_{\rm a}^{+}\left(sI - A_{\rm aa}^{+} + K_{\rm a}^{+}E_{\rm a}^{+}\right)^{-1}K_{\rm a}^{+}\right]\Gamma_{\rm i}^{-1}\right\}^{-1} \\
&= \Gamma_{\rm i}\left[I + E_{\rm a}^{+}(sI - A_{\rm aa}^{+})^{-1}K_{\rm a}^{+}\right]\Gamma_{\rm i}^{-1} \\
&= I + \Gamma_{\rm i}E_{\rm a}^{+}(sI - A_{\rm aa}^{+})^{-1}P^{-1}(E_{\rm a}^{+})'\Gamma_{\rm i}' \\
&= I + \Gamma_{\rm i}E_{\rm a}^{+}(sP - PA_{\rm aa}^{+})^{-1}(E_{\rm a}^{+})'\Gamma_{\rm i}' \\
&= I + \Gamma_{\rm i}E_{\rm a}^{+}\left[sP + (A_{\rm aa}^{+})'P - (E_{\rm a}^{+})'\Gamma_{\rm i}'\Gamma_{\rm i}E_{\rm a}^{+}\right]^{-1}(E_{a}^{+})'\Gamma_{\rm i}' \\
&= I - \Gamma_{\rm i}E_{\rm a}^{+}P^{-1}\left[-sI - (A_{\rm aa}^{+})' + (E_{\rm a}^{+})'(K_{\rm a}^{+})'\right]^{-1}(E_{\rm a}^{+})'\Gamma_{\rm i}' \\
&= I - (\Gamma_{\rm i}')^{-1}(K_{\rm a}^{+})'\left[(-sI - A_{\rm aa}^{+} + K_{\rm a}^{+}E_{\rm a}^{+})^{-1}\right]'(E_{\rm a}^{+})'\Gamma_{\rm i}' \\
&= (\Gamma_{\rm i}^{-1})'\left\{I - (K_{\rm a}^{+})'\left[(-sI - A_{\rm aa}^{+} + K_{\rm a}^{+}E_{\rm a}^{+})^{-1}\right]'(E_{\rm a}^{+})'\right\}\Gamma_{\rm i}' \\
&= V'(-s).
\end{aligned}$$

Here, we note that the poles of $V(s)$ are the eigenvalues of the stable matrix $-A_{\rm aa}^{+}$, and the poles of $V^{-1}(s)$ are the nonminimum-phase invariant zeros of Σ, namely $\lambda(A_{\rm aa}^{+})$.

Finally, we are ready to show that $G(s) = G_{\rm m}(s)V(s)$. Let us define

$$\tilde{\Phi} = (sI - \tilde{A})^{-1} = \begin{bmatrix} sI - A_{\rm aa}^{+} & 0 & -L_{\rm ab}^{+}C_{\rm b} & -L_{\rm ad}^{+}C_{\rm d} \\ 0 & sI - A_{\rm aa}^{-} & -L_{\rm ab}^{-}C_{\rm b} & -L_{\rm ad}^{-}C_{\rm d} \\ 0 & 0 & sI - A_{\rm bb} & -L_{\rm bd}C_{\rm d} \\ -B_{\rm d}E_{\rm a}^{+} & -B_{\rm d}E_{\rm a}^{-} & -B_{\rm d}E_{\rm b} & sI - A_{\rm dd} \end{bmatrix}^{-1},$$

$$K = \begin{bmatrix} K_{\rm a}^{+} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E = [\,E_{\rm a}^{+} \quad 0 \quad 0 \quad 0\,],$$

and

$$\hat{\Phi} = \begin{bmatrix} sI - A_{\rm aa}^{+} & 0 & -L_{\rm ab}^{+}C_{\rm b} & -L_{\rm ad}^{+}C_{\rm d} \\ 0 & sI - A_{\rm aa}^{-} & -L_{\rm ab}^{-}C_{\rm b} & -L_{\rm ad}^{-}C_{\rm d} \\ 0 & 0 & sI - A_{\rm bb} & -L_{\rm bd}C_{\rm d} \\ 0 & -B_{\rm d}E_{\rm a}^{-} & -B_{\rm d}E_{\rm b} & sI - A_{\rm dd} \end{bmatrix}^{-1}.$$

In view of (8.2.2), (8.2.3) and (8.2.6), it is straightforward to verify that

$$\tilde{B}_{\rm m} = \tilde{B} + K, \quad \tilde{\Phi} = (\hat{\Phi}^{-1} - \tilde{B}E)^{-1},$$

and

$$\tilde{C}\hat{\Phi}K = 0, \quad E\hat{\Phi}K = E_{\rm a}^{+}(sI - A_{\rm aa}^{+})^{-1}K_{\rm a}^{+}.$$

Hence,

$$\begin{aligned}
G(s)V^{-1}(s) &= \Gamma_{\rm o}\tilde{C}\tilde{\Phi}\tilde{B}\Gamma_{\rm i}^{-1}\Gamma_{\rm i}\Big[I + E_{\rm a}^{+}(sI - A_{\rm aa}^{+})^{-1}K_{\rm a}^{+}\Big]\Gamma_{\rm i}^{-1} \\
&= \Gamma_{\rm o}\Big[\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}(\hat{\Phi}^{-1} - \tilde{B}E)^{-1}\tilde{B}E\hat{\Phi}K\Big]\Gamma_{\rm i}^{-1} \\
&= \Gamma_{\rm o}\Big\{\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}[(\hat{\Phi}^{-1} - \tilde{B}E)^{-1}\hat{\Phi}^{-1} - I]\hat{\Phi}K\Big\}\Gamma_{\rm i}^{-1} \\
&= \Gamma_{\rm o}\Big(\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}\tilde{\Phi}K - \tilde{C}\hat{\Phi}K\Big)\Gamma_{\rm i}^{-1} \\
&= \Gamma_{\rm o}\tilde{C}\tilde{\Phi}(\tilde{B} + K)\Gamma_{\rm i}^{-1} \\
&= G_{\rm m}(s).
\end{aligned} \tag{8.2.15}$$

This completes the proof of Theorem 8.2.1. ■

We demonstrate the above results by the following example.

Example 8.2.1. Consider a square and invertible system Σ characterized by a matrix quadruple (A, B, C, D) with

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right], \tag{8.2.16}$$

and

$$C = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{8.2.17}$$

It is simple to verify that Σ has a transfer function

$$G(s) = \frac{s-1}{s^5 - 5s^4 + 8s^3 - 5s^2 + s + 1}\begin{bmatrix} (s-2)(s^2-s+1) & (s-1)^2 \\ (s-1)^2 & s(s-1)(s-2) \end{bmatrix}.$$

Also, Σ is controllable and observable and has an invariant zero at $s = 1$. Furthermore, it is easy to verify that Σ is already in the special coordinate basis form with

$$A_{\rm aa}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{\rm a}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We thus obtain,

$$K_{\mathrm{a}}^{+} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \end{bmatrix}, \quad B_{\mathrm{m}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$V(s) = \begin{bmatrix} \dfrac{s-1}{s+1} & 0 \\ 0 & \dfrac{(s-1)^2}{(s+1)^2} \end{bmatrix},$$

and

$$G_{\mathrm{m}}(s) = \frac{s+1}{s^5-5s^4+8s^3-5s^2+s+1} \begin{bmatrix} (s-2)(s^2-s+1) & (s-1)(s+1) \\ (s-1)^2 & s(s+1)(s-2) \end{bmatrix}.$$

Whenever the given system Σ has invariant zeros on the imaginary axis, no minimum-phase image of Σ can be obtained by any means. In what follows, we introduce a generalized cascade factorization, which is a natural extension of the minimum-phase/all-pass factorization. The given nonminimum-phase and left invertible system is decomposed as

$$G(s) = G_{\mathrm{M}}(s) U(s). \tag{8.2.18}$$

Here $G_{\mathrm{M}}(s)$ is of minimum-phase, left invertible and has the same infinite zero structure as that of Σ, while $U(s)$ is a square, invertible and stable transfer function which is asymptotically all-pass. All the invariant zeros of $G_{\mathrm{M}}(s)$ are in a desired set $\mathbb{C}_{\mathrm{d}} \subset \mathbb{C}^{-}$. If the given system Σ is only detectable but not observable, the set $\mathbb{C}_{\mathrm{d}}$ includes all the unobservable but stable eigenvalues of Σ. In this way, all the awkward or unwanted invariant zeros of Σ (say, those in the right half s-plane or close to the imaginary axis) need not be included in $G_{\mathrm{M}}(s)$. Such a generalized cascade factorization has a major application in loop transfer recovery design. For instance, by applying the loop transfer recovery procedure to $G_{\mathrm{M}}(s)$, one has the capability to shape the overall loop transfer recovery error over some frequency band or in some subspace of interest while placing the eigenvalues of the observer corresponding to some awkward invariant zeros of Σ at any desired locations (see, e.g., [116]). We further note that this generalized cascade factorization can be immediately adopted to solve the well-known zero placement problem, which is to be studied in detail in the next chapter.

Let us assume that the given system Σ has been transformed into the form of the special coordinate basis as in (8.2.1) to (8.2.3). Let us also assume that in the special coordinate basis formulation, $x_{\rm a}$ is decomposed into $x_{\rm a}^-$ and $x_{\rm a}^+$ such that the eigenvalues of $A_{\rm aa}^+$ contain all the awkward invariant zeros of Σ, including all the unstable invariant zeros, invariant zeros on the imaginary axis and some unwanted stable invariant zeros, provided that these awkward zeros can be controlled.

As expected, procedures for constructing this generalized cascade factorization are quite similar to those of the minimum-phase/all-pass factorization. Thus, we directly summarize such a factorization in the following theorem.

Theorem 8.2.2. *Consider a left invertible and nonminimum-phase system Σ. Assume that it has been transformed into the special coordinate basis as given in (8.2.1) to (8.2.3) and assume that its awkward invariant zeros are observable and are dumped in $\lambda(A_{\rm aa}^+)$. One can then construct a generalized cascade factorization (8.2.18) such that*

1. *The minimum-phase counterpart of Σ is given by $\Sigma_{\rm M}$ characterized by the matrix triple $(A, B_{\rm M}, C)$ with a transfer function $G_{\rm M}(s) = C(sI - A)^{-1}B_{\rm M}$, where*

$$B_{\rm M} = \Gamma_{\rm s}\tilde{B}_{\rm M}\Gamma_{\rm i}^{-1} = \Gamma_{\rm s}\begin{bmatrix} K_{\rm a}^+ \\ 0 \\ 0 \\ B_{\rm d} \end{bmatrix}\Gamma_{\rm i}^{-1}. \tag{8.2.19}$$

   Here $K_{\rm a}^+$ is specified such that $\lambda(A_{\rm aa}^+ - K_{\rm a}^+ E_{\rm a}^+)$ are in the desired locations in $\mathbb{C}^-$. Moreover, $\Sigma_{\rm M}$ is also left invertible and has the same infinite zero structure as Σ.

2. *The stable factor $U(s)$ is given as*

$$U(s) = \Gamma_{\rm i}\left[I - E_{\rm a}^+(sI - A_{\rm aa}^+ + K_{\rm a}^+E_{\rm a}^+)^{-1}K_{\rm a}^+\right]\Gamma_{\rm i}^{-1}. \tag{8.2.20}$$

   Moreover,

$$U^{-1}(s) = \Gamma_{\rm i}\left[I + E_{\rm a}^+(sI - A_{\rm aa}^+)^{-1}K_{\rm a}^+\right]\Gamma_{\rm i}^{-1}, \tag{8.2.21}$$

   and $U(s)$ is asymptotically all-pass, i.e.,

$$U(s)U'(-s) \to I \quad \text{as} \quad |s| \to \infty. \tag{8.2.22}$$

The above result can be regarded as a dual version of the problem of zero placement studied in the literature (see e.g., [139]).

Proof. Without loss of generality, we assume that Σ is in the form of the special coordinate basis. Following the proof of Theorem 8.2.1, we can show that there exists a nonsingular state transformation such that Σ_M is transformed into the form of the special coordinate basis as

$$\dot{\bar{x}}_a^M = (A_{aa}^+ - K_a^+ E_a^+)\bar{x}_a^M - K_a^+ E_a^+ x_a^- + L_{ad}^M y_d + L_{ab}^M y_b,$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{ad}^- y_d + L_{ab}^- y_b,$$

$$\dot{x}_b = A_{bb} x_b + L_{bd} y_d, \quad y_b = C_b x_b,$$

$$\dot{x}_d = A_{dd}^* x_d + L_d y_d + B_d[u + E_a^+ \bar{x}_a^M + E_a^- x_a^- + E_b^M x_b + E_d^M x_d], \quad y_d = C_d x_d,$$

for some matrices L_{ad}^M, L_{ab}^M, E_b^M and E_d^M of appropriate dimensions. Hence, it follows from the properties of the special coordinate basis that Σ_M and Σ have the same infinite zero structure and that Σ_M is left invertible. Furthermore, the invariant zeros of Σ_M are given by

$$\lambda \begin{bmatrix} A_{aa}^+ - K_a^+ E_a^+ & -K_a^+ E_a^- \\ 0 & A_{aa}^- \end{bmatrix} \subset \mathbb{C}^-. \tag{8.2.23}$$

Hence, Σ_M is of minimum-phase. Moreover, the left state and input zero directions associated with the minimum-phase invariant zeros of Σ remain unchanged in Σ_M. The equality of $G(s) = G_M(s)U(s)$ follows directly from (8.2.15). Since $U(s) \to I$ as $|s| \to \infty$, hence $U(s)U'(-s) \to I$ as $|s| \to \infty$. This completes the proof of Theorem 8.2.2. ■

We illustrate this generalized factorization with an example.

Example 8.2.2. Consider a system Σ as given in [159] and characterized by

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & -1.25 \\ -2.5 & -2.5 \\ 0.3 & 1.25 \\ 1.5 & 3.5 \end{bmatrix}, \tag{8.2.24}$$

and

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = 0, \tag{8.2.25}$$

with

$$G(s) = \frac{1}{(s+1)(s+0.2)} \begin{bmatrix} -0.2(s-1) & 1 \\ -(s-1) & s+3 \end{bmatrix}.$$

This system is square and invertible with two invariant zeros at $s = 1$ and $s = 2$. The minimum-phase image and the all-pass factor of Σ are obtained as

$$B_{\rm m} = \begin{bmatrix} 0.7353 & -0.8088 \\ 1.4706 & -1.6176 \\ -0.9353 & 0.8088 \\ -2.4706 & 2.6176 \end{bmatrix},$$

$$G_{\rm m}(s) = \frac{1}{(s+1)(s+0.2)} \begin{bmatrix} -0.2(s+3.9412) & 0.6470 \\ -(s+2.1765) & s+2.2941 \end{bmatrix},$$

and

$$V(s) = \begin{bmatrix} \dfrac{(s-1)(s-0.9414)}{(s+1)(s+2)} & \dfrac{-1.7646}{s+2} \\ \dfrac{-1.7646(s-1)}{(s+1)(s+2)} & \dfrac{s+0.9414}{s+2} \end{bmatrix}.$$

The following is a cascade factorization of Σ,

$$B_{\rm M} = \begin{bmatrix} 0.5 & 0 \\ 3.75 & -3.75 \\ -0.7 & 0 \\ -4.75 & 4.75 \end{bmatrix},$$

$$G_{\rm M}(s) = \frac{1}{(s+1)(s+0.2)} \begin{bmatrix} -0.2(s+3) & 0 \\ -(s+4) & s+4 \end{bmatrix},$$

and

$$U(s) = \begin{bmatrix} \dfrac{s-1}{s+3} & \dfrac{-5}{s+3} \\ \dfrac{s-1}{(s+3)(s+4)} & \dfrac{s^2+s-11}{(s+3)(s+4)} \end{bmatrix}.$$

It is simple to see that $G_{\rm M}(s)$ has two invariant zeros at $s = -3$ and $s = -4$.

We conclude this section by constructing the inner-outer factorization of a stable, strictly proper and right invertible transfer function $G(s)$ with no invariant zeros on the imaginary axis. Let the matrix triple (A, B, C) be a realization of the transposed system of $G(s)$, *i.e.*, $G'(s)$, which is obviously left invertible. Then, the constructive algorithm given in STEPS FACT-SP.1 to FACT-SP.3 for the minimum-phase/all-pass factorization would automatically yield an inner-outer factorization for $G(s)$. We summarize this result in the following theorem. The proof of this theorem is obvious in view of that of Theorem 8.2.1.

Theorem 8.2.3. *Consider a stable, strictly proper and right invertible transfer function $G(s)$ with no invariant zeros on the imaginary axis. Let (A, B, C) be a realization of its transposed system, $G'(s)$, with (A, C) being detectable. Let*

$$G'(s) = G_{\rm m}(s)V(s) \tag{8.2.26}$$

be a minimum-phase/all-pass factorization of $G'(s)$ as in Theorem 8.2.1. Then,

$$G(s) = G_{\rm i}(s)G_{\rm o}(s) = V'(s)G'_{\rm m}(s) \tag{8.2.27}$$

is an inner-outer factorization of $G(s)$, where $G_{\rm i}(s) = V'(s)$ is an inner factor of $G(s)$, i.e.,

$$G'_{\rm i}(-s)G_{\rm i}(s) = V(-s)V'(s) = I, \tag{8.2.28}$$

and $G_{\rm o}(s) = G'_{\rm m}(s)$ is an outer factor, whose right-inverse is analytic in $\mathbb{C}^+$.

8.3 Nonstrictly Proper Systems

We now present factorizations for a general system Σ, characterized by a matrix quadruple (A, B, C, D), which is not necessarily left or right invertible and whose direct feedthrough matrix D might be nonzero. For the generalized cascade factorization and inner-outer factorization, Σ might have invariant zeros on the imaginary axis. For the minimum-phase/all-pass factorization, we still need to assume that Σ has no invariant zeros on the imaginary axis. As in the previous section, we will first present a step-by-step constructive algorithm for the minimum-phase/all-pass factorization.

STEP FACT-NSP.1.

Utilize the result of Theorem 5.4.1 of Chapter 5 to find nonsingular transformations $\Gamma_{\rm s} \in \mathbb{R}^{n\times n}$, $\Gamma_{\rm o} \in \mathbb{R}^{p\times p}$ and $\Gamma_{\rm i} \in \mathbb{R}^{m\times m}$ such that the given system Σ, *i.e.*, the matrix quadruple (A, B, C, D), can be transformed into a form similar to the compact form of the special coordinate basis in (5.4.21) to (5.4.24). More specifically, we would like to arrange the transformed system as follows:

$$\tilde{A} = \Gamma_{\rm s}^{-1} A \Gamma_{\rm s} = \begin{bmatrix} A_{\rm cc} & B_{\rm c}E_{\rm ca}^+ & B_{\rm c}E_{\rm ca}^- & L_{\rm cb}C_{\rm b} & L_{\rm cd}C_{\rm d} \\ 0 & A_{\rm aa}^+ & 0 & L_{\rm ab}^+C_{\rm b} & L_{\rm ad}^+C_{\rm d} \\ 0 & 0 & A_{\rm aa}^- & L_{\rm ab}^-C_{\rm b} & L_{\rm ad}^-C_{\rm d} \\ 0 & 0 & 0 & A_{\rm bb} & L_{\rm bd}C_{\rm d} \\ B_{\rm d}E_{\rm dc} & B_{\rm d}E_{\rm da}^+ & B_{\rm d}E_{\rm da}^- & B_{\rm d}E_{\rm db} & A_{\rm dd} \end{bmatrix}$$

$$+\begin{bmatrix} B_{0c} \\ B_{0a}^{+} \\ B_{0a}^{-} \\ B_{0b} \\ B_{0d} \end{bmatrix} [C_{0c} \quad C_{0a}^{+} \quad C_{0a}^{-} \quad C_{0b} \quad C_{0d}], \tag{8.3.1}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} B_{0c} & 0 & B_c \\ B_{0a}^{+} & 0 & 0 \\ B_{0a}^{-} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0d} & B_d & 0 \end{bmatrix}, \quad \tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.3.2}$$

and

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_{0c} & C_{0a}^{+} & C_{0a}^{-} & C_{0b} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}. \tag{8.3.3}$$

Here $\lambda(A_{aa}^{+}) \subset \mathbb{C}^{+}$ and $\lambda(A_{aa}^{-}) \subset \mathbb{C}^{-}$ are respectively the unstable and the stable invariant zeros of Σ.

STEP FACT-NSP.2.

Let

$$A_x = \begin{bmatrix} A_{cc} & B_c E_{ca}^{+} \\ 0 & A_{aa}^{+} \end{bmatrix}, \quad B_x = \begin{bmatrix} 0 & 0 & B_c \\ 0 & 0 & 0 \end{bmatrix} \Gamma_i^{-1}, \tag{8.3.4}$$

$$C_x = \begin{bmatrix} C_{0c} & C_{0a}^{+} \\ E_{dc} & E_{da}^{+} \end{bmatrix}, \quad D_x = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & I_{m_d} & 0 \end{bmatrix} \Gamma_i^{-1}, \tag{8.3.5}$$

and

$$\Gamma_i^{-1} (\Gamma_i^{-1})' = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12}' & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13}' & \Gamma_{23}' & \Gamma_{33} \end{bmatrix}, \quad \Gamma_m = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}' & \Gamma_{22} \end{bmatrix}^{-\frac{1}{2}}. \tag{8.3.6}$$

It follows from Property 5.4.1 of the special coordinate basis that the pair (A_x, C_x) is detectable whenever the pair (A, C) is detectable. We then solve the Riccati equation

$$A_x P_x + P_x A_x' + B_x B_x' - (C_x P_x + D_x B_x')'(D_x D_x')^{-1}(C_x P_x + D_x B_x') = 0 \tag{8.3.7}$$

for $P_x > 0$. It will be shown later that such a solution alway exists. Next, compute

$$K_x := \begin{bmatrix} K_{c0} & K_{cd} \\ K_{a0}^+ & K_{ad}^+ \end{bmatrix} = (C_x P_x + D_x B_x')'(D_x D_x')^{-1}, \tag{8.3.8}$$

and

$$B_m = \Gamma_s \begin{bmatrix} B_{c0} + K_{c0} & K_{cd} \\ B_{a0}^+ + K_{a0}^+ & K_{ad}^+ \\ B_{a0}^- & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \Gamma_m^{-1}, \quad D_m = \Gamma_o \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Gamma_m^{-1}. \tag{8.3.9}$$

STEP FACT-NSP.3.

Define Σ_m to be a system characterized by a quadruple (A, B_m, C, D_m) and

$$V(s) = \Gamma_m \Big[C_x (sI - A_x + K_x C_x)^{-1} (B_x - K_x D_x) + D_x \Big]. \tag{8.3.10}$$

This completes the procedure for constructing the minimum-phase/all-pass factorization for a general system Σ.

We have the following theorem.

Theorem 8.3.1. *Consider a general detectable system Σ of (8.1.1) with all its unstable invariant zeros in $\mathbb{C}^+$. Then, its minimum-phase/all-pass factorization is given by*

$$G(s) = G_m(s) V(s), \tag{8.3.11}$$

where $V(s)$, the stable all-pass factor, is given as in (8.3.10), and $G_m(s)$, the minimum-phase image of Σ, is the transfer function of Σ_m characterized by the matrix quadruple (A, B_m, C, D_m) with B_m and D_m being given as in (8.3.9), i.e.,

$$G_m(s) = C(sI - A)^{-1} B_m + D_m. \tag{8.3.12}$$

Furthermore, Σ_m is left invertible and has the same infinite zero structure as that of Σ with its transfer function $G_m(s)$ satisfying

$$G_m(s) G_m'(-s) = G(s) G'(-s). \tag{8.3.13}$$

The all-pass factor $V(s)$ satisfies $V(s)V'(-s) = I$.

Proof. We first note that since (A_x, C_x) is detectable and $(-A_x, B_x)$ is stabilizable, it follows from Richardson and Kwong [111] that (8.3.7) has a unique, symmetric and positive definite solution, *i.e.*, $P_x = P_x' > 0$. Let us now show that $A_x - K_x C_x$ is a stable matrix. Let

$$P_x^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}.$$

Then pre-multiplying equation (8.3.7) by P_x^{-1}, we obtain

$$P_x^{-1}(A_x - K_x C_x)P_x = \begin{bmatrix} -(A_{cc}^* + B_c\tilde{\Gamma}B_c'P_{11})' & 0 \\ \star & -(A_{aa}^+)' \end{bmatrix},$$

where

$$A_{cc}^* = A_{cc} - B_c[T_{13}'\ T_{23}']\,(\Gamma_m)^2 \begin{bmatrix} C_{0c} \\ E_{dc} \end{bmatrix}$$

and

$$\tilde{\Gamma} = \Gamma_{33} - [\Gamma_{13}'\ \Gamma_{23}']\,(\Gamma_m)^2 \begin{bmatrix} \Gamma_{13} \\ \Gamma_{23} \end{bmatrix}.$$

It is worth noting that $\tilde{\Gamma}$ is a positive definite matrix and P_{11} is the unique positive definite solution of the algebraic Riccati equation

$$P_{11}A_{cc}^* + (A_{cc}^*)'P_{11} + P_{11}B_c\tilde{\Gamma}B_c'P_{11} - [C_{0c}'\ E_{dc}'](\Gamma_m)^2 \begin{bmatrix} C_{0c} \\ E_{dc} \end{bmatrix} = 0.$$

Hence, $\lambda(A_x - K_x C_x) = \lambda(-A_{aa}^+) \cup \lambda(-A_{cc}^* - B_c\tilde{\Gamma}B_c'P_{11})$ are all in $\mathbb{C}^-$, and thus $A_x - K_x C_x$ is indeed a stable matrix. We are now ready to prove that Σ_m characterized by the matrix quadruple (A, B_m, C, D_m) is of minimum phase, left invertible and has the same infinite zero structure as Σ. Without loss of generality, we assume that Σ is in the form of the special coordinate basis as in (8.3.1) to (8.3.3). Let us define

$$\hat{A} = A - \left(\begin{bmatrix} B_{c0} \\ B_{a0}^+ \\ B_{a0}^- \\ B_{b0} \\ B_{d0} \end{bmatrix} + \begin{bmatrix} K_{c0} \\ K_{a0}^+ \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) [C_{0c}\ \ C_{0a}^+\ \ C_{0a}^-\ \ C_{0b}\ \ C_{0d}],$$

and

$$\hat{B} = \begin{bmatrix} K_{cd} \\ K_{ad}^+ \\ 0 \\ 0 \\ B_d \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}.$$

Then, by the construction and the properties of the special coordinate basis (see Section 5.4), the system Σ_m characterized by (A, B_m, C, D_m) and the system $\hat{\Sigma}$ characterized by the matrix triple $(\hat{A}, \hat{B}, \hat{C})$ have the same finite and infinite zero structures and the same invertibility properties. Then following the same procedure as in the proof of Theorem 8.2.1, it is easy to show that $\hat{\Sigma}$ is left invertible and has the same infinite zero structure as that of Σ. Furthermore, $\hat{\Sigma}$ has invariant zeros at

$$\lambda \begin{bmatrix} A_x - K_x C_x & \star \\ 0 & A_{aa}^- \end{bmatrix} \subset \mathbb{C}^-,$$

where $\star$'s denote matrices of not much interest.

Next, we proceed to show that $V(s)V'(-s) = I$. It follows from (8.3.7) and (8.3.8) that

$$A_x P_x + P_x A_x' + B_x B_x' - K_x(C_x P_x + D_x B_x') = 0,$$

$$D_x(B_x' - D_x' K_x') = -C_x P_x,$$

and

$$K_x C_x P_x + K_x D_x B_x' - B_x D_x' K_x' - P_x C_x' K_x' = 0.$$

We then have

$$\begin{aligned} V(s)V'(-s) &= \Gamma_m \Big[C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D_x\Big] \\ &\quad \times \Big[(B_x' - D_x' K_x')(sI - A_x' + C_x' K_x')^{-1} C_x' + D_x'\Big]\Gamma_m' \\ &= I + \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) \\ &\quad \times (B_x' - D_x' K_x')(-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &\quad - \Gamma_m C_x(sI - A_x + K_x C_x)^{-1} P_x C_x' \Gamma_m \\ &\quad - \Gamma_m C_x P_x(-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &= I + \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}\Big[(B_x - K_x D_x) \\ &\quad \times (B_x' - D_x' K_x') - P_x(-sI - A_x' + C_x' K_x') \\ &\quad - (sI - A_x + K_x C_x)P_x\Big](-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &= I + \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}\Big[K_x C_x P_x + K_x D_x B_x' \\ &\quad - B_x D_x' K_x' - P_x C_x' K_x'\Big](-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &= I. \end{aligned}$$

We now proceed to show that $G(s) = G_m(s)V(s)$. Let us define

$$\Phi_x(s) = (sI - A_x + K_x C_x)^{-1},$$

$$\bar{\Phi}_{\rm x}(s) = \begin{bmatrix} \Phi_{\rm x}(s) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_{\rm K} = \begin{bmatrix} K_{\rm c0} & K_{\rm cd} \\ K_{\rm a0}^{+} & K_{\rm ad}^{+} \\ 0 & 0 \\ 0 & 0 \\ 0 & B_{\rm d} \end{bmatrix},$$

and

$$\bar{B}_0 = \begin{bmatrix} B_{\rm c0} \\ B_{\rm a0}^{+} \\ B_{\rm a0}^{-} \\ B_{\rm b0} \\ B_{\rm d0} \end{bmatrix}, \quad \bar{B}_{\rm d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ B_{\rm d} \end{bmatrix}, \quad \bar{B}_{\rm c} = \begin{bmatrix} B_{\rm c} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It then follows that

$$\begin{aligned} B &= \Gamma_{\rm s}\,[\bar{B}_0 \quad \bar{B}_{\rm d} \quad \bar{B}_{\rm c}]\,\Gamma_{\rm i}^{-1}, \\ B_{\rm m}\Gamma_{\rm m} &= \Gamma_{\rm s}\,[\bar{B}_0 \quad 0] + \Gamma_{\rm s}\bar{B}_{\rm K}, \\ B_{\rm m}\Gamma_{\rm m}D_{\rm x} &= \Gamma_{\rm s}\,[\bar{B}_0 \quad 0 \quad 0]\,\Gamma_{\rm i}^{-1} + \Gamma_{\rm s}\,[\bar{B}_{\rm K} \quad 0]\,\Gamma_{\rm i}^{-1}, \\ D_{\rm m}\Gamma_{\rm m}C_{\rm x}\Phi_{\rm x}(s) &= C\Gamma_{\rm s}\bar{\Phi}_{\rm x}(s), \end{aligned}$$

and

$$\left([\bar{B}_0 \quad 0] + \bar{B}_{\rm K}\right) C_{\rm x}\Phi_{\rm x}(s) + \Gamma_{\rm s}^{-1}(sI - A)\Gamma_s\bar{\Phi}_{\rm x}(s) = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We now have

$$\begin{aligned} G_{\rm m}(s)V(s) &= \left[C(sI - A)^{-1}B_{\rm m} + D_{\rm m}\right]\Gamma_{\rm m} \\ &\quad \times \left[C_{\rm x}(sI - A_{\rm x} + K_{\rm x}C_{\rm x})^{-1}(B_{\rm x} - K_{\rm x}D_{\rm x}) + D_{\rm x}\right] \\ &= C(sI - A)^{-1}B_{\rm m}\Gamma_{\rm m} \\ &\quad \times \left[C_{\rm x}(sI - A_{\rm x} + K_{\rm x}C_{\rm x})^{-1}(B_{\rm x} - K_{\rm x}D_{\rm x}) + D_{\rm x}\right] \\ &\quad + D_{\rm m}\Gamma_{\rm m}C_{\rm x}(sI - A_{\rm x} + K_{\rm x}C_{\rm x})^{-1}(B_{\rm x} - K_{\rm x}D_{\rm x}) + D \\ &= C(sI - A)^{-1}\Big[B_{\rm m}\Gamma_{\rm m}C_{\rm x}\Phi_{\rm x}(s)(B_{\rm x} - K_{\rm x}C_{\rm x}) + B_{\rm m}\Gamma_{\rm m}D_{\rm x} \\ &\quad + (sI - A)\Gamma_{\rm s}\bar{\Phi}_{\rm x}(s)(B_{\rm x} - K_{\rm x}D_{\rm x})\Big] + D \\ &= C(sI - A)^{-1}\Gamma_{\rm s}\Big\{\left([\bar{B}_0 \quad 0] + \bar{B}_{\rm K}\right)C_{\rm x}\Phi_{\rm x}(s)(B_{\rm x} - K_{\rm x}D_{\rm x}) \\ &\quad + [\bar{B}_0 \quad 0 \quad 0]\,\Gamma_{\rm i}^{-1} + [\bar{B}_{\rm K} \quad 0]\,\Gamma_{\rm i}^{-1} \\ &\quad + \Gamma_{\rm s}^{-1}(sI - A)\Gamma_{\rm s}\bar{\Phi}_{\rm x}(s)(B_{\rm x} - K_{\rm x}D_{\rm x})\Big\} + D \\ &= C(sI - A)^{-1}\Gamma_{\rm s}\Big\{[I \quad 0 \quad 0 \quad 0]'\,(B_{\rm x} - K_{\rm x}D_{\rm x}) \end{aligned}$$

$$
\begin{aligned}
&+ [\bar{B}_0 \;\; 0 \;\; 0]\Gamma_{\rm i}^{-1} + [\bar{B}_{\rm K} \;\; 0]\Gamma_{\rm i}^{-1}\Big\} + D \\
&= C(sI-A)^{-1}\Gamma_{\rm s}\Big\{[\bar{B}_0 \;\; 0 \;\; 0]+[0 \;\; \bar{B}_{\rm d} \;\; 0]+[0 \;\; 0 \;\; \bar{B}_{\rm c}]\Big\}\Gamma_{\rm i}^{-1}+D \\
&= C(sI-A)^{-1}B + D \\
&= G(s).
\end{aligned}
$$

Finally, the fact that $G(s)G'(-s) = G_{\rm m}(s)G_{\rm m}'(-s)$ follows immediately from the fact that $V(s)V'(-s) = I$. ■

We illustrate the above results by the following example.

Example 8.3.1. Consider a system Σ characterized by (A, B, C, D) with

$$
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \tag{8.3.14}
$$

and

$$
C = \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad D = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{8.3.15}
$$

The given Σ has a transfer function $G(s)$,

$$
G(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} \times \begin{bmatrix} s^5-3s^4-2s^3+3s^2-s & s^4+2s-1 & s^4-s^3-3s^2+1 \\ 0 & s^4-2s^3+2s-1 & s^3-s^2-s+1 \\ 0 & s^3-s^2-s+1 & s^2-1 \end{bmatrix}.
$$

This system is neither left nor right invertible and has two invariant zeros at $\{-1, 1\}$. Hence, it is of nonminimum phase. Moreover, it is easy to verify that Σ is in the form of the special coordinate basis with

$$
A_{\rm x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_{\rm x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{\rm x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},
$$

$$
D_{\rm x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_{\rm m} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Thus, following the procedure given in STEPS FACT-NSP.2 to FACT-NSP.3, which involves solving a Riccati equation, we obtain,

$$
K_{\rm x} = \begin{bmatrix} 1.412771 & 1.063856 \\ -0.348915 & 2.255424 \end{bmatrix},
$$

$$B_{\rm m} = \begin{bmatrix} 1.412771 & 1.063856 \\ -0.348915 & 2.255424 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\rm m} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_{\rm m}(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} [G_{\rm m1}(s) \quad G_{\rm m2}(s)],$$

and

$$V(s) = \frac{1}{s^2 + 2.73205s + 1.73205} \times \begin{bmatrix} s^2+1.31928s-0.06386 & -1.06386s-1.19157 & s+1.25542 \\ -1.06386s+1.41277 & s^2-0.58723s-0.76169 & s-0.65109 \end{bmatrix},$$

where

$$G_{\rm m1}(s) = \begin{bmatrix} s^5-1.58723s^4-4.1106s^3-1.58723s^2-0.30217s+1.41277 \\ 1.06386s^3-1.41277s^2-1.06386s+1.41277 \\ 1.06386s^2-0.34892s-1.41277 \end{bmatrix},$$

and

$$G_{\rm m2}(s) = \begin{bmatrix} 2.06386s^4+3.44699s^3-0.93614s^2-2.51085s+0.06386 \\ s^4+1.31928s^3-1.06386s^2-1.31928s+0.06386 \\ s^3+2.31928s^2+1.25542s-0.06386 \end{bmatrix}.$$

A similar generalized cascade factorization can also be obtained for general nonstrictly proper and non-left invertible systems. We have the following result.

Theorem 8.3.2. *Consider a general system Σ, which has been transformed into the special coordinate basis as given in (8.3.1) to (8.3.3). Assume that its awkward invariant zeros are observable and are dumped in $\lambda(A^+_{\rm aa})$. One can then construct a generalized cascade factorization $G(s) = G_{\rm M}(s)U(s)$ such that*

1. *The minimum-phase counterpart of Σ is given by $\Sigma_{\rm M}$ characterized by the matrix quadruple $(A, B_{\rm M}, C, D_{\rm M})$ with a transfer function matrix $G_{\rm M}(s) = C(sI-A)^{-1}B_{\rm M}+D_{\rm M}$, where*

$$B_{\rm M} = \Gamma_{\rm s} \begin{bmatrix} B_{\rm c0} + K_{\rm c0} & K_{\rm cd} \\ B^+_{\rm a0} + K^+_{\rm a0} & K^+_{\rm ad} \\ B^-_{\rm a0} & 0 \\ B_{\rm b0} & 0 \\ B_{\rm d0} & B_{\rm d} \end{bmatrix} \Gamma_{\rm m}^{-1}, \quad D_{\rm M} = \Gamma_{\rm o} \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{\rm m}^{-1}, \tag{8.3.16}$$

and where

$$K_x = \begin{bmatrix} K_{c0} & K_{cd} \\ K_{a0}^+ & K_{ad}^+ \end{bmatrix} \tag{8.3.17}$$

is specified such that $\lambda(A_x - K_x C_x)$ are in the desired locations in $\mathbb{C}^-$. Here we note that A_x, C_x and Γ_m are as defined in (8.3.4) to (8.3.6). Furthermore, Σ_M is also left invertible and has the same infinite zero structure as Σ.

2. *The factor $U(s)$ is given as*

$$U(s) = \Gamma_m \Big[C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D_x \Big], \tag{8.3.18}$$

where $U(s)$ is stable, right invertible and asymptotically all-pass, i.e.,

$$U(s)U'(-s) \to I \quad \text{as} \quad |s| \to \infty. \tag{8.3.19}$$

Proof. It follows the same line of reasoning as in the proof of Theorem 8.2.2. ■

We illustrate this generalized factorization by the following example.

Example 8.3.2. Consider the same system Σ given in Example 8.3.1. Let us choose K_x such that $\lambda(A_x - K_x C_x) = \{-2, -3\}$. We then obtain

$$K_x = \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix}, \quad B_M = \begin{bmatrix} 2 & 1 \\ -4 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_M(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} \times \begin{bmatrix} s^5 - s^4 - 12s^3 - 7s^2 + 7s + 2 & 2s^4 + 7s^3 + 1s^2 - 6s \\ -2s^3 - 2s^2 + 2s + 2 & s^4 + 3s^3 - s^2 - 3s \\ -2s^2 - 4s - 2 & s^3 + 4s^2 + 3s \end{bmatrix},$$

and

$$U(s) = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s^2 + 3s & -s - 3 \\ 2s + 2 & s^2 - 5 \end{bmatrix}.$$

Finally, we conclude this section with the following theorem dealing with the inner-outer factorization for general systems.

Theorem 8.3.3. *Consider a general nonstrictly proper and stable transfer function $G(s)$, which might have invariant zeros on the imaginary axis. Let the matrix quadruple (A, B, C, D) be a realization of its transposed system, $G'(s)$, with (A, C) being detectable. Let us treat all the invariant zeros of $G(s)$ or (A, B, C, D) on the imaginary axis as 'good ones' and dump them in $\lambda(A_{\rm aa}^-)$, and then follow the result of Theorem 8.3.1 to construct a 'minimum-phase/all-pass factorization' of $G'(s)$, i.e.,*

$$G'(s) = G_{\rm m}(s)V(s), \tag{8.3.20}$$

where $G_{\rm m}(s)$ is left invertible and has no invariant zeros in $\mathbb{C}^+$, and $V(s)$ is an all-pass factor. Then,

$$G(s) = G_{\rm i}(s)G_{\rm o}(s) = V'(s)G'_{\rm m}(s) \tag{8.3.21}$$

is an inner-outer factorization of $G(s)$, where $G_{\rm i}(s) = V'(s)$ is an inner factor of $G(s)$, i.e.,

$$G'_{\rm i}(-s)G_{\rm i}(s) = V(-s)V'(s) = I, \tag{8.3.22}$$

and $G_{\rm o}(s) = G'_{\rm m}(s)$ is an outer factor, whose right-inverse is analytic in $\mathbb{C}^+$.

8.4 Discrete-time Systems

In this section, we consider system factorizations for a general discrete-time system characterized by

$$\begin{cases} x(k+1) = A\ x(k) + B\ u(k), \\ \ \ y(k) \ \ = C\ x(k) + D\ u(k), \end{cases} \tag{8.4.1}$$

where the state vector $x \in \mathbb{R}^n$, output vector $y \in \mathbb{R}^p$ and input vector $u \in \mathbb{R}^m$. Without loss of generality, we assume that $[\,B' \ \ D'\,]$ and $[\,C \ \ D\,]$ are of maximal rank. Let the transfer function of Σ be

$$G(z) = C(zI - A)^{-1}B + D. \tag{8.4.2}$$

Since the generalized cascade factorization, which results in an asymptotic all-pass factor, does not make too much sense in the discrete-time setting, we will investigate only the inner-outer factorization and the minimum-phase/all-pass factorization for the system of (8.4.1). For a proper and stable transfer function $G(z)$, its inner-outer factorization is defined as

$$G(z) = G_{\rm i}(z)G_{\rm o}(z), \tag{8.4.3}$$

where $G_{\rm i}(z)$ is an inner matrix satisfying $G'(z^{-1})G(z) = I$, and $G_{\rm o}(z)$ is an outer matrix, which is right invertible and has no infinite zeros and no invariant zeros outside the unit disc (*i.e.*, $\mathbb{C}^{\otimes}$). Similarly, the minimum-phase/all-pass factorization of $G(z)$, which does not possess any invariant zeros on the unit circle, is defined as

$$G(z) = G_{\rm m}(z)V(z), \tag{8.4.4}$$

where $G_{\rm m}(z)$ is left invertible and of minimum-phase with no infinite zeros, and $V(z)$ is an all-pass factor satisfying $V(z)V'(z^{-1}) = I$.

In principle, we can follow similar arguments as in the continuous-time case of the previous sections to obtain explicit expressions for these factorizations. Actually, this was what Lin *et al.* [89] had done in deriving the expressions for the discrete-time inner-outer factorization. In this section, however, we will show that with the help of the bilinear transformation and inverse bilinear transformation of Chapter 7, the discrete-time system factorization problem can be converted into an equivalent problem in the continuous-time setting, which can be solved using the results of Sections 8.2 and 8.3. The procedure is straightforward:

1. STEP D-FACT.1.

   Apply the inverse bilinear transformation of (7.3.3) with $a = 1$ to the discrete-time system (8.4.1) or its transfer function in (8.4.2) to obtain a continuous-time mapping:

$$\tilde{G}(s) = G(z)\Big|_{z=(1+s)/(1-s)} = C\left(\frac{1+s}{1-s}I - A\right)^{-1} B + D. \tag{8.4.5}$$

   A state-space realization of $\tilde{G}(s)$ can be found in Lemma 7.3.1.

2. STEP D-FACT.2.

   Utilize the result of Theorem 8.3.1 to find a minimum-phase/all-pass factorization of $\tilde{G}(s)$,

$$\tilde{G}(s) = \tilde{G}_{\rm m}(s)\tilde{V}(s), \tag{8.4.6}$$

   or utilize the result of Theorem 8.3.3 to obtain an inner-outer factorization of $\tilde{G}(s)$,

$$\tilde{G}(s) = \tilde{G}_{\rm i}(s)\tilde{G}_{\rm o}(s). \tag{8.4.7}$$

3. STEP D-FACT.3.

   Apply the bilinear transformation (7.2.3) with $a = 1$ to compute

$$G_{\rm m}(z) = \tilde{G}_{\rm m}(s)\Big|_{s=(z-1)/(z+1)}, \quad V(z) = \tilde{V}(s)\Big|_{s=(z-1)/(z+1)} \tag{8.4.8}$$

for the minimum-phase/all-pass factorization, or to calculate

$$G_{\rm i}(z) = \tilde{G}_{\rm i}(s)\Big|_{s=(z-1)/(z+1)}, \quad G_{\rm o}(z) = \tilde{G}_{\rm o}(s)\Big|_{s=(z-1)/(z+1)} \tag{8.4.9}$$

for the inner-outer factorization. The state-space realizations of the corresponding discrete-time transfer functions can be found using Lemma 7.2.1.

4. STEP D-FACT.4.

Then the inner-outer factorization of $G(z)$ is given by

$$G(z) = G_{\rm i}(z)G_{\rm o}(z), \tag{8.4.10}$$

and the minimum-phase/all-pass factorization is given by

$$G(z) = G_{\rm m}(z)V(z). \tag{8.4.11}$$

The claim of the above algorithm follows from the results of the following lemmas.

Lemma 8.4.1. *Consider a continuous-time transfer function $\tilde{V}(s)$ and its discrete-time counterpart $V(z)$ under the bilinear transformation. Then, $\tilde{V}(s)$ is an all-pass (inner) factor if and only if $V(z)$ is all-pass (inner) factor.*

Proof. If $\tilde{V}(s)$ is an all-pass in the continuous-time domain, *i.e.*,

$$\tilde{V}(s)\tilde{V}'(-s) = I,$$

then

$$\begin{aligned} V(z)V'(z^{-1}) &= \tilde{V}\left(\frac{z-1}{z+1}\right)\tilde{V}'\left(\frac{z^{-1}-1}{z^{-1}+1}\right) \\ &= \tilde{V}\left(\frac{z-1}{z+1}\right)\tilde{V}'\left(-\frac{z-1}{z+1}\right) = \tilde{V}(s)\tilde{V}'(-s) = I. \end{aligned}$$

Thus, $V(z)$ is an all-pass factor in the discrete-time domain. Similarly, we can show the converse part and the result for the inner factors. ■

Lemma 8.4.2. *Consider a continuous-time transfer function $\tilde{G}_{\rm o}(s)$ and its discrete-time counterpart $G_{\rm o}(z)$ under the bilinear transformations. Then, $\tilde{G}_{\rm o}(s)$ is an outer matrix if and only if $G_{\rm o}(z)$ is an outer matrix.*

Proof. It follows directly from Theorems 7.2.1 and 7.3.1. ■

We illustrate the results of this section in the following example.

Example 8.4.1. Consider a discrete-time system Σ characterized by (A, B, C, D) with

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1.1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ -0.2 & 0.2 & -0.2 & -0.1 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{8.4.12}$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.4.13}$$

It is simple to verify that (A, B) is stabilizable and that the above system is neither left nor right invertible with two invariant zeros at $z = 0$ and $z = 1.1$ and one infinite zero.

Following the formulae given in Lemma 7.3.1, we obtain a continuous-time counterpart of Σ, which is characterized by $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\tilde{A} = \begin{bmatrix} -0.31799 & -0.02302 & 0.01096 & 0.14906 & 0.11508 \\ 0.11508 & -1.17263 & 0.08220 & 1.11796 & 0.86313 \\ 0.10960 & -0.16441 & 0.12591 & 0.11234 & 0.82203 \\ 0.11508 & -0.17263 & 0.08220 & 0.11796 & 0.86313 \\ -0.23017 & 0.34525 & -0.16440 & -0.23592 & -0.72626 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 1 & 0 & 0 \\ -0.08220 & -0.86313 & -0.11508 \\ 0.04110 & 0.43157 & 0.05754 \end{bmatrix},$$

and

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.34763 & -0.63652 & 0.21699 & 0.47428 & 1.45636 \\ -0.23136 & 0.40458 & -0.14960 & 0.20388 & -1.15975 \end{bmatrix}.$$

Utilizing the algorithms given in the previous section and the toolkit of [87], we obtain an inner-outer factorization of $\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ as follows:

$$\tilde{G}(s) = \tilde{G}_{\rm i}(s)\tilde{G}_{\rm o}(s),$$

where the inner factor $\tilde{G}_{\rm i}(s)$ is characterized by $(\tilde{A}_{\rm i}, \tilde{B}_{\rm i}, \tilde{C}_{\rm i}, \tilde{D}_{\rm i})$ with

$$\tilde{A}_{\rm i} = \begin{bmatrix} -0.82764 & -0.90098 & 0.00289 \\ -0.08007 & -0.62084 & 0.00520 \\ -0.94833 & -0.87474 & -0.04635 \end{bmatrix}, \quad \tilde{B}_{\rm i} = \begin{bmatrix} -0.24313 & -0.30779 \\ -0.43738 & -0.55371 \\ 0.73049 & -0.69541 \end{bmatrix},$$

$$\tilde{C}_{\mathrm{i}} = \begin{bmatrix} 0.11903 & -0.04326 & -0.09426 \\ -1.88731 & -2.05455 & 0.00660 \\ 2.08383 & 1.02727 & -0.00330 \end{bmatrix}, \ \tilde{D}_{\mathrm{i}} = \begin{bmatrix} 0.78472 & -0.61986 \\ -0.55442 & -0.70187 \\ 0.27721 & 0.35094 \end{bmatrix},$$

and the outer factor $\tilde{G}_{\mathrm{o}}(s)$ is characterized by $(\tilde{A}_{\mathrm{o}}, \tilde{B}_{\mathrm{o}}, \tilde{C}_{\mathrm{o}}, \tilde{D}_{\mathrm{o}})$ with

$$\tilde{A}_{\mathrm{o}} = \begin{bmatrix} -0.31799 & -0.02302 & 0.01096 & 0.14906 & 0.11508 \\ 0.11508 & -1.17263 & 0.08220 & 1.11796 & 0.86313 \\ 0.10960 & -0.16441 & 0.12591 & 0.11234 & 0.82203 \\ 0.11508 & -0.17263 & 0.08220 & 0.11796 & 0.86313 \\ -0.23017 & 0.34525 & -0.16441 & -0.23592 & -0.72626 \end{bmatrix},$$

$$\tilde{B}_{\mathrm{o}} = B, \ \tilde{C}_{\mathrm{o}} = \begin{bmatrix} -0.04494 & 0.14716 & 0.06847 & 0.11731 & 0.46051 \\ -0.05843 & 0.18861 & -0.06712 & 0.27241 & 0.57146 \end{bmatrix},$$

and

$$\tilde{D}_{\mathrm{o}} = \begin{bmatrix} 0.84168 & 0.59817 & 0.07976 \\ -0.54774 & 0.75726 & 0.10097 \end{bmatrix}.$$

The minimum-phase/all-pass factorization of $\tilde{G}(s)$ is given by

$$\tilde{G}(s) = \tilde{G}_{\mathrm{m}}(s)\tilde{V}(s),$$

where the minimum-phase image $\tilde{G}_{\mathrm{m}}(s)$ is characterized by a matrix quadruple $(\tilde{A}_{\mathrm{m}}, \tilde{B}_{\mathrm{m}}, \tilde{C}_{\mathrm{m}}, \tilde{D}_{\mathrm{m}})$ with

$$\tilde{A}_{\mathrm{m}} = \begin{bmatrix} -0.31799 & -0.02302 & 0.01096 & 0.14906 & 0.11508 \\ 0.11508 & -1.17263 & 0.08220 & 1.11796 & 0.86313 \\ 0.10960 & -0.16441 & 0.12591 & 0.11234 & 0.82203 \\ 0.11508 & -0.17263 & 0.08220 & 0.11796 & 0.86313 \\ -0.23017 & 0.34525 & -0.16441 & -0.23592 & -0.72626 \end{bmatrix},$$

$$\tilde{B}_{\mathrm{m}} = \begin{bmatrix} -0.10357 & -0.14446 \\ -0.65755 & -0.91712 \\ 0.87426 & -0.49682 \\ -0.65755 & -0.91712 \\ -0.20767 & -0.28966 \end{bmatrix}, \ \tilde{D}_{\mathrm{m}} = \begin{bmatrix} 0.81270 & -0.58268 \\ -0.57419 & -0.65978 \\ 0.28709 & 0.32989 \end{bmatrix},$$

$$\tilde{C}_{\mathrm{m}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.34763 & -0.63652 & 0.21699 & 0.47428 & 1.45636 \\ -0.23136 & 0.40458 & -0.14960 & 0.20388 & -1.15975 \end{bmatrix},$$

and the all-pass factor $\tilde{V}(s)$ is characterized by $(\tilde{A}_{\mathrm{v}}, \tilde{B}_{\mathrm{v}}, \tilde{C}_{\mathrm{v}}, \tilde{D}_{\mathrm{v}})$ with

$$\tilde{A}_{\mathrm{v}} = \begin{bmatrix} -0.33259 & 0.04006 & -0.01323 \\ -0.29863 & -0.71383 & 0.19158 \\ 0.18651 & 1.07034 & -0.35706 \end{bmatrix},$$

$$\tilde{B}_{\mathrm{v}} = \begin{bmatrix} 0 & 0.00440 & -1.05564 \\ 0 & -1.77426 & -0.23657 \\ 0 & 1.10808 & 0.14774 \end{bmatrix},$$

$$\tilde{C}_{\mathrm{v}} = \begin{bmatrix} 0.09721 & 0.55790 & -0.21093 \\ 0.13559 & 0.77813 & -0.29420 \end{bmatrix},$$

and

$$\tilde{D}_{\mathrm{v}} = \begin{bmatrix} 0.81270 & 0.57757 & 0.07701 \\ -0.58268 & 0.80557 & 0.10741 \end{bmatrix}.$$

Next, using the formulae in Lemma 7.2.1 for the bilinear transformation, we obtain an inner-outer factorization of $G(z) = C(zI - A)^{-1}B + D$ as follows:

$$G(z) = G_{\mathrm{i}}(z)G_{\mathrm{o}}(z),$$

where the inner factor $G_{\mathrm{i}}(z)$ is characterized by $(A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}})$ with

$$A_{\mathrm{i}} = \begin{bmatrix} 0.12162 & -0.62348 & 0.00000 \\ -0.05852 & 0.26315 & 0.00612 \\ -0.96763 & -0.49091 & 0.90628 \end{bmatrix}, \; B_{\mathrm{i}} = \begin{bmatrix} -0.24313 & -0.30779 \\ -0.43738 & -0.55371 \\ 0.73049 & -0.69541 \end{bmatrix},$$

$$C_{\mathrm{i}} = \begin{bmatrix} 0.21692 & -0.07883 & -0.17176 \\ -1.08171 & -0.27353 & -0.00435 \\ 1.27885 & -0.71087 & 0.00000 \end{bmatrix}, \; D_{\mathrm{i}} = \begin{bmatrix} 0.70943 & -0.56940 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the outer $G_{\mathrm{o}}(z)$ is characterized by $(A, B, C_{\mathrm{o}}, D_{\mathrm{o}})$ with

$$C_{\mathrm{o}} = \begin{bmatrix} -0.20269 & 0.20269 & -0.02878 & 0.38891 & 0.74970 \\ -0.25253 & 0.25253 & -0.36910 & 0.74531 & 0.89188 \end{bmatrix},$$

and

$$D_{\mathrm{o}} = \begin{bmatrix} 0.86753 & 1.01343 & 0 \\ -0.67536 & 1.26267 & 0 \end{bmatrix}.$$

Lastly, the following minimum-phase/all-pass factorization of $G(z)$ is obtained from its continuous-time counterpart

$$G(z) = G_{\mathrm{m}}(z)V(z),$$

where the minimum-phase image $G_{\mathrm{m}}(z)$ is characterized by $(A, B_{\mathrm{m}}, C, D_{\mathrm{m}})$ with

$$B_{\mathrm{m}} = \begin{bmatrix} -0.10357 & -0.14446 \\ -0.65755 & -0.91712 \\ 0.87426 & -0.49682 \\ -0.65755 & -0.91712 \\ -0.20767 & -0.28966 \end{bmatrix}, \; D_{\mathrm{m}} = \begin{bmatrix} 0.81270 & -0.58268 \\ -0.65755 & -0.91712 \\ 0 & 0 \end{bmatrix},$$

and the all-pass factor $V(z)$ is characterized by (A_v, B_v, C_v, D_v) with

$$A_v = \begin{bmatrix} 0.49301 & 0.02831 & -0.01055 \\ -0.26015 & 0.27488 & 0.18251 \\ 0.00000 & 1.00941 & 0.61628 \end{bmatrix}, \; B_v = \begin{bmatrix} 0 & 0.00440 & -1.05564 \\ 0 & -1.77426 & -0.23657 \\ 0 & 1.10808 & 0.14774 \end{bmatrix},$$

and

$$C_v = \begin{bmatrix} -0.06518 & 0.19822 & -0.14833 \\ -0.09091 & 0.27647 & -0.20688 \end{bmatrix}, \quad D_v = \begin{bmatrix} 0.81270 & 0 & 0 \\ -0.58268 & 0 & 0 \end{bmatrix}.$$

8.5 Exercises

8.1. Given a stable and proper transfer function matrix, $G(s)$, and its minimum-phase/all-pass factorization,

$$G(s) = G_m(s)V(s),$$

show that

$$\sigma_{\max}[V(j\omega)] = \cdots = \sigma_{\min}[V(j\omega)] = 1,$$

and

$$||G||_2 = ||G_m||_2, \quad ||G||_\infty = ||G_m||_\infty.$$

Also, show that for the inner-outer factorization of $G(s) = G_i(s)G_o(s)$, we have

$$\sigma_{\max}[G_i(j\omega)] = \cdots = \sigma_{\min}[G_i(j\omega)] = 1,$$

and

$$||G||_2 = ||G_o||_2, \quad ||G||_\infty = ||G_o||_\infty.$$

8.2. Show that the results of Exercise 8.1 are valid for discrete-time systems as well.

8.3. Given a continuous-time system characterized by a transfer function matrix,

$$G(s) = \frac{\begin{bmatrix} s^3 + s^2 - s - 1 & -3s^2 + 3 \\ s^2 - 1 & -3s + 3 \end{bmatrix}}{s^4 + 5s^3 + 16s^2 + 16s + 28},$$

or by a state-space realization,

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 2 & -3 & -1 & 1 \\ -8 & -2 & -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} x.$$

The system is stable, controllable and observable. It is neither left nor right invertible with an unstable invariant at 2 and an infinite zero of order 1.

(a) Find a minimum-phase/all-pass factorization for $G(s)$.

(b) Find an inner-outer factorization for $G(s)$.

(c) Verify the results of Exercise 8.1.

8.4. Given a discrete-time system characterized by a transfer function matrix,

$$G(z) = \frac{\begin{bmatrix} z^2 - 0.2z + 0.01 & 0.1z - 0.01 \\ 0.1z - 0.01 & 0.01 \end{bmatrix}}{z^3 - 0.2z^2 - 0.01z + 0.001},$$

or by a state-space realization,

$$x(k+1) = \begin{bmatrix} 0.1 & 0.0 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(k),$$

and

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x(k).$$

The system is stable, controllable and observable. It is neither left nor right invertible with an infinite zero of order 1 and with no invariant zeros.

(a) Find a minimum-phase/all-pass factorization for $G(z)$.

(b) Find an inner-outer factorization for $G(z)$.

(c) Verify the results of Exercise 8.2.

Chapter 9

Structural Assignment via Sensor/Actuator Selection

9.1 Introduction

As is well-known in the literature, the structural properties of linear systems, such as the finite and infinite zero structures and the invertibility structures, have played very important roles in many areas of linear systems and control (see, *e.g.*, robust and H_∞ control, Chen [22], H_2 optimal control, Saberi *et al.* [120], and constrained control systems, Lin [85], Hu and Lin [67]). We believe that one of the major difficulties in applying many of the useful multivariable control synthesis techniques, such as H_2 and H_∞ control, to actual design is the inadequate study of the linkage between control performance and design implementation that involves hardware selection, *e.g.*, appropriate sensors suitable for robustness and performance. This linkage provides a foundation upon which performance trade-offs can be incorporated at the preliminary design stage. Thus, one can introduce careful control design considerations into the overall engineering design process at an early stage. For example, it is well understood in the literature that it is always troublesome to deal with systems with nonminimum-phase zeros in control system design. However, it is evident from the results in Chapter 8, *i.e.*, Theorems 8.2.2 and 8.3.2, that by properly adding or relocating the locations of actuators (see the additional term $K_{\rm a}^{+}$ in $B_{\rm M}$ of (8.2.19) for example) or dually adding or relocating the locations of sensors, the designer is able to totally remove the troublesome nonminimum-phase invariant zeros and obtain better performance. Our objective in this chapter is to study the flexibility in the structural properties that one can assign to a given linear system, and to identify sets of sensors and actuators which would yield desirable structural properties.

It is appropriate to trace a short history of the development of the techniques related to structural assignments of linear systems. Most results in the open literature are related to invariant zero or transmission zero (*i.e.*, finite zero structure) assignments (see, *e.g.*, Emami-Naeini and Van Dooren [51], Karcanias *et al.* [74], Kouvaritakis and MacFarlane [77], Patel [108], Vardulakis [145], Syrmos [139], and Syrmos and Lewis [140]). It is important to point out that all the results reported in the literature so far, including the ones mentioned above, deal solely with the assignments of the finite zeros, and the infinite zero structure and other structures such as invertibility structures of the resulting system are either fixed or of not much concern. Only recently had Chen and Zheng [31] proposed a technique, which is capable of assigning both finite and infinite zero structures simultaneously. Up to date, to the best of our knowledge, only the result of Liu, Chen and Lin [92] deals with the assignment of complete system structures, including finite and infinite zero structures and invertibility structures. In this chapter, we present the techniques of [31], which deals with simultaneous finite and infinite zero structure assignment, and [92], which is capable of assigning general structural properties. More specifically, we consider a linear time-invariant system characterized by the state space equation

$$\dot{x} = Ax + Bu, \tag{9.1.1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^m$ is the control input. The problem of structural assignments or sensor selection is to find a constant matrix, C, or equivalently, a measurement output,

$$y = Cx, \tag{9.1.2}$$

such that the resulting system characterized by the matrix triple (A, B, C) would have the pre-specified desired structural properties, including finite and infinite zero structures and invertibility structures. We note that this technique can be applied to solve the dual problem of actuator selection, *i.e.*, to find a matrix B provided that matrices A and C are given such that the resulting system characterized by the triple (A, B, C) would have the pre-specified desired structural properties. Throughout this chapter, a set of complex scalars, say $\mathcal{W}$, is said to be self-conjugated if for any $w \in \mathcal{W}$, its complex conjugate $w^* \in \mathcal{W}$.

9.2 Simultaneous Finite and Infinite Zero Placement

We start with the simultaneous finite and infinite zero placement problem for SISO systems because the solution to this problem is relatively simple and intuitive. It is also helpful in understanding the derivation of the result for MIMO systems.

9.2.1 SISO Systems

We consider in this subsection the finite and infinite zero assignment problem for system (9.1.1) with $m = 1$. We first have the following theorem, the proof of which is constructive and gives an explicit expression of a set of output matrices, $\boldsymbol{\Omega}$, such that for any element in $\boldsymbol{\Omega}$, the corresponding system has the prescribed finite zero and infinite zero structures.

Theorem 9.2.1. *Consider the unsensed system (9.1.1) characterized by (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. Let $\mathcal{C} := \{k_1\}$ be the controllability index of (A, B) and let the number of uncontrollable modes be n_o. Also, let $\{\nu_1, \nu_2, \dots, \nu_{n_o}\}$ be the uncontrollable modes of (A, B). Then for any given integer q_1, $0 < q_1 \le k_1$, and a set of self-conjugated scalars, $\{z_1, z_2, \dots, z_{k_1-q_1}\}$, there exists a nonempty set of output matrices $\boldsymbol{\Omega} \subset \mathbb{R}^{1\times n}$ such that for any $C \in \boldsymbol{\Omega}$ the resulting system (A, B, C) has $n_o + k_1 - q_1$ invariant zeros at $\{\nu_1, \nu_2, \dots, \nu_{n_o}, z_1, z_2, \dots, z_{k_1-q_1}\}$ and has an infinite zero structure $\mathcal{S}_\infty^\star = \{q_1\}$, i.e., the relative degree of (A, B, C) is equal to q_1.*

Proof. It follows from Theorem 4.4.1 that there exist nonsingular state and input transformations T_s and T_i such that (A, B) can be transformed into the controllability structural decomposition form of (4.4.7). Next, we rewrite (4.4.7) as follows,

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1-1} \\ \star & \star & \star & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{9.2.1}$$

where $\star$ represents a matrix of less interest.

Let

$$a(s) := s^{k_1-q_1} + a_1 s^{k_1-q_1-1} + \cdots + a_{k_1-q_1} \tag{9.2.2}$$

be a polynomial having roots at $z_1, z_2, \dots, z_{k_1-q_1}$. Also, let us define

$$\underline{a} := [\, a_{k_1-q_1-1} \quad \cdots \quad a_2 \quad a_1 \,].$$

Then the desired set of output matrices Ω is given by

$$\Omega := \left\{ C \in \mathbb{R}^{1\times n} \middle| C = \alpha [\underline{d} \;\; a_{k_1-q_1} \;\; \underline{a} \;\; 1 \;\; 0] T_s^{-1}, 0 \neq \alpha \in \mathbb{R}, \underline{d} \in \mathbb{R}^{1\times n_o} \right\}. \tag{9.2.3}$$

In what follows we will proceed to prove that the resulting system with any $C \in \Omega$ has all the properties stated in Theorem 9.2.1. Let us define

$$\hat{A}_{aa} := \begin{bmatrix} 0 & I_{k_1-q_1-1} \\ 0 & 0 \end{bmatrix}', \quad \hat{E}_{da} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}', \quad \hat{K}_a := [a_{k_1-q_1} \;\; \underline{a}]', \tag{9.2.4}$$

and

$$\hat{A}_{dd} := \begin{bmatrix} 0 & I_{q_1-1} \\ \star & \star \end{bmatrix}', \quad \hat{C}_d := \begin{bmatrix} 0 \\ 1 \end{bmatrix}', \quad \hat{B}_d := [1 \;\; 0]'. \tag{9.2.5}$$

It is simple to see that the pair $(\hat{A}_{aa}, \hat{E}_{da})$ is observable and

$$\hat{A}_{aa}^{c} := \hat{A}_{aa} - \hat{K}_a \hat{E}_{da}$$

has eigenvalues at $z_1, z_2, \dots, z_{k_1-q_1}$. Also, it is straightforward to verify that the system (A, B, C) has the same finite (invariant) zero and infinite zero structures as $(\hat{A}, \hat{B}, \hat{C})$, where

$$\hat{A} := \tilde{A}' = \begin{bmatrix} A_o' & 0 & \star \cdot \hat{C}_d \\ 0 & \hat{A}_{aa} & \star \cdot \hat{C}_d \\ 0 & \hat{B}_d \hat{E}_{da} & \hat{A}_{dd} \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} \underline{d}' \\ \hat{K}_a \\ \hat{B}_d \end{bmatrix},$$

and

$$\hat{C} := \tilde{B}' = [0 \;\; 0 \;\; \hat{C}_d].$$

It follows from the proof of Theorem 8.2.1 that there exists a nonsingular state transformation T such that

$$T^{-1}\hat{A}T = \begin{bmatrix} A_o' & \underline{d}'\hat{E}_{da} & \star \cdot \hat{C}_d \\ 0 & \hat{A}_{aa}^{c} & \star \cdot \hat{C}_d \\ 0 & \hat{B}_d \hat{E}_{da} & \hat{A}_{dd}^{*} \end{bmatrix}, \quad T^{-1}\hat{B} = \begin{bmatrix} 0 \\ 0 \\ \hat{B}_d \end{bmatrix},$$

and

$$\hat{C}T = [0 \;\; 0 \;\; \hat{C}_d],$$

where

$$\hat{A}_{dd}^{*} = \hat{A}_{dd} + \hat{B}_d \cdot \star. \tag{9.2.6}$$

Note that $(T^{-1}\hat{A}T, T^{-1}\hat{B}, \hat{C}T)$ is now in the form of the special coordinate basis of Theorem 5.2.1. Thus it follows from the properties of the special coordinate basis that $(T^{-1}\hat{A}T, T^{-1}\hat{B}, \hat{C}T)$, or equivalently (A, B, C), has an infinite zero structure $S_{\infty}^{\star} = \{q_1\}$ and invariant (finite) zeros at

$$\lambda(A_o') \cup \lambda\left(\hat{A}_{aa}^{c}\right) = \left\{\nu_1, \nu_2, \ldots, \nu_{n_o}, z_1, z_2, \ldots, z_{k_1-q_1}\right\}. \tag{9.2.7}$$

This completes the proof of Theorem 9.2.1. ■

The following corollary shows that $\mathbf{\Omega}$ of (9.2.3) is complete.

Corollary 9.2.1. *The set of output matrices* $\mathbf{\Omega}$ *in (9.2.3) is complete, i.e., any output matrix* C *for which the resulting system* (A, B, C) *has all properties listed in Theorem 9.2.1, is a member of* $\mathbf{\Omega}$.

Proof. Let C be such that the resulting system (A, B, C) has invariant zeros at $\{\nu_1, \nu_2, \ldots, \nu_{n_o}\} \cup \{z_1, z_2, \ldots, z_{k_1-q_1}\}$ and has a relative degree $q_1 \leq k_1$. It is obvious that C can be written in conformity with (9.2.1) as

$$C = [\underline{d} \quad e \quad \underline{h} \quad g \quad \underline{z}]\, T_s^{-1}, \tag{9.2.8}$$

where $\underline{d} \in \mathbb{R}^{1\times n_o}$, $e \in \mathbb{R}$, $\underline{h} \in \mathbb{R}^{1\times(k_1-q_1-1)}$, $g \in \mathbb{R}$ and $\underline{z} \in \mathbb{R}^{1\times(q_1-1)}$. Note that

$$[A^{q_1}B \quad \cdots \quad AB \quad B] = T_s\,[\tilde{A}^{q_1}\tilde{B} \quad \cdots \quad \tilde{A}\tilde{B} \quad \tilde{B}]\, T_s^{-1}$$
$$= T_s \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & 1 & 0 \\ \star & \cdots & \star & 1 \end{bmatrix} T_s^{-1}.$$

Then, it is simple to verify that the fact that (A, B, C) has a relative degree q_1, i.e.,

$$CB = CAB = \cdots = CA^{q_1-1}B = 0, \tag{9.2.9}$$

and $CA^{q_1}B \neq 0$, implies that $\underline{z} = 0$ and $g \neq 0$. Thus, we have

$$C = \alpha\,[\underline{d}/\alpha \quad e/\alpha \quad \underline{h}/\alpha \quad 1 \quad 0]\, T_s^{-1},$$

where $\alpha = g$. Following the same procedure as (9.2.4) to (9.2.6), it can be shown that the invariant zeros of (A, B, C) are given by $\lambda(A_o) = \{\nu_1, \nu_2, \ldots, \nu_{n_o}\}$ and

$$\lambda\left(\hat{A}_{aa} - [e/\alpha \quad \underline{h}/\alpha]'\,\hat{E}_{da}\right) = \left\{z_1, z_2, \ldots, z_{k_1-q_1}\right\}. \tag{9.2.10}$$

Since $(\hat{A}_{\rm aa}, \hat{E}_{\rm da})$ is a single output system, $[\, e/\alpha \quad \underline{h}/\alpha \,]'$ is uniquely determined by the closed-loop eigenvalues $\{z_1, z_2, \dots, z_{k_1-q_1}\}$. Hence

$$[\, e/\alpha \quad \underline{h}/\alpha \,] = [\, a_{k_1-q_1} \quad \underline{a} \,], \tag{9.2.11}$$

and $C \in \boldsymbol{\Omega}$. ■

We illustrate the above result in the following example.

Example 9.2.1. Consider a system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{9.2.12}$$

It is simple to see that the pair (A, B) is already in the form of the controllability structure decomposition with a controllability index $\mathcal{C} = \{3\}$. Then it follows from Theorem 9.2.1 that one has freedom to choose output matrices such that the resulting systems have: 1) infinite zero structure $S_\infty^\star = \{3\}$ with no invariant zero, 2) $S_\infty^\star = \{2\}$ with one invariant zero, and 3) $S_\infty^\star = \{1\}$ with two invariant zeros. The systems with the following output matrices respectively have such properties:

$$C_1 = \alpha \, [\, 1 \quad 0 \quad 0 \,],$$

$$C_2 = \alpha \, [\, a_1 \quad 1 \quad 0 \,],$$

and

$$C_3 = \alpha \, [\, a_2 \quad a_1 \quad 1 \,]$$

where $0 \neq \alpha \in \mathbb{R}$. These can be easily verified by computing the corresponding transfer functions. We have

$$H_1(s) := C_1 (sI_3 - A)^{-1} B = \frac{\alpha}{s(s^2-1)},$$

$$H_2(s) := C_2 (sI_3 - A)^{-1} B = \frac{\alpha(s+a_1)}{s(s^2-1)},$$

and

$$H_3(s) := C_3 (sI_3 - A)^{-1} B = \frac{\alpha(s^2 + a_1 s + a_2)}{s(s^2-1)}.$$

9.2.2 MIMO Systems

We now proceed to solve the simultaneous finite and infinite zero placement problem for MIMO systems. As in the SISO case, we will first state our result in a theorem and then give a constructive proof which generates an explicit expression of a nonempty set of output matrices $\mathbf{\Omega}$ such that for any $C \in \mathbf{\Omega}$, the resulting system (A, B, C) is square invertible and has the chosen finite and infinite zero structures. A construction procedure will be summarized in a remark as an easy-to-follow algorithm.

Theorem 9.2.2. *Consider the unsensed system (9.1.1) characterized by (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that B is of full rank. Let the controllability index of (A, B) be given by $\mathcal{C} := \{k_1, k_2, \dots, k_m\}$ and let the pair (A, B) have n_o uncontrollable modes. Also, let $\{\nu_1, \nu_2, \dots, \nu_{n_o}\}$ be the uncontrollable modes of (A, B). Then for any given set of integers, $S_\infty^\star := \{q_1, q_2, \dots, q_m\}$ with $0 < q_i \le k_i, i = 1, 2, \dots, m$, and a set of self-conjugated scalars, $\{z_1, z_2, \dots, z_\ell\}$ where $\ell := \sum_{i=1}^m (k_i - q_i)$, there exists a nonempty set of output matrices $\mathbf{\Omega} \subset \mathbb{R}^{m\times n}$ such that for any $C \in \mathbf{\Omega}$, the corresponding system characterized by (A, B, C) has the following properties:*

1. *(A, B, C) is square and invertible;*
2. *(A, B, C) has $n_o + \ell$ invariant zeros at $\{\nu_1, \dots, \nu_{n_o}, z_1, \dots, z_\ell\}$; and*
3. *(A, B, C) has an infinite zero structure $S_\infty^\star = \{q_1, \dots, q_m\}$.*

Proof. Again, it follows from Theorem 4.4.1 that there exist nonsingular transformations T_s and T_i such that the pair (A, B) is transformed into the controllability structural decomposition form of (4.4.7). Next, we rewrite $\tilde{A}$ and $\tilde{B}$ as follows,

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1} & \cdots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{k_m-q_m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_{q_m-1} \\ \star & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \end{bmatrix}, \tag{9.2.13}$$

$$\tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{9.2.14}$$

and define

$$\tilde{A}_{\mathrm{aa}} := \begin{bmatrix} 0 & I_{k_1-q_1-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{k_m-q_m-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{9.2.15}$$

and

$$\tilde{L}_{\mathrm{ad}} := \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}. \tag{9.2.16}$$

Note that $(\tilde{A}_{\mathrm{aa}}, \tilde{L}_{\mathrm{ad}})$ is controllable and, in fact, is in the controllability structural decomposition form. Let us also define

$$\boldsymbol{F}_{\mathrm{a}} := \left\{ \tilde{F}_{\mathrm{a}} \in \mathbb{R}^{m\times \ell} \;\middle|\; \lambda\left(\tilde{A}_{\mathrm{aa}} - \tilde{L}_{\mathrm{ad}}\tilde{F}_{\mathrm{a}}\right) = \{ z_1, z_2, \ldots, z_\ell \} \right\}. \tag{9.2.17}$$

For any $\tilde{F}_{\mathrm{a}} \in \boldsymbol{F}_{\mathrm{a}}$, we partition it in conformity with (9.2.16) as

$$\tilde{F}_{\mathrm{a}} = \begin{bmatrix} F_{11}^0 & F_{11}^1 & \cdots & F_{1m}^0 & F_{1m}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{m1}^0 & F_{m1}^1 & \cdots & F_{mm}^0 & F_{mm}^1 \end{bmatrix}, \tag{9.2.18}$$

and define a corresponding $m \times n$ matrix, in conformity with (9.2.13) and (9.2.14),

$$\tilde{C} := \begin{bmatrix} K_1 & F_{11}^0 & F_{11}^1 & 1 & 0 & \cdots & F_{1m}^0 & F_{1m}^1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ K_m & F_{m1}^0 & F_{m1}^1 & 0 & 0 & \cdots & F_{mm}^0 & F_{mm}^1 & 1 & 0 \end{bmatrix}, \tag{9.2.19}$$

where

$$\tilde{K} = \begin{bmatrix} K_1 \\ \vdots \\ K_m \end{bmatrix} \tag{9.2.20}$$

is any arbitrary constant matrix of dimensions $m \times n_o$. The desired set of matrices, $\mathbf{\Omega}$, is then given by

$$\mathbf{\Omega} := \Big\{ C \in \mathbb{R}^{n \times m} \;\Big|\; C = \Gamma \tilde{C} T_s^{-1} \text{ with } \tilde{K} \in \mathbb{R}^{m \times n_o}, \tilde{F}_a \in \boldsymbol{F}_a, \\ \Gamma \in \mathbb{R}^{m \times m} \text{ and } \det(\Gamma) \neq 0 \Big\}. \quad (9.2.21)$$

Now, we proceed to prove the properties of the resulting system (A, B, C) for any $C \in \mathbf{\Omega}$. We note that the finite and infinite zero structures of (A, B, C) are equivalent to those of $(\tilde{A}, \tilde{B}, \tilde{C})$ because they are related by some nonsingular transformations T_s, T_i and Γ. Observing the structure of $(\tilde{A}, \tilde{B}, \tilde{C})$, we see that there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that

$$P^{-1}\tilde{A}P = \begin{bmatrix} A_o & 0 & 0 \\ 0 & \tilde{A}_{aa} & \tilde{L}_{ad}\tilde{C}_d \\ 0 & \tilde{B}_d \cdot \star & \tilde{A}_{dd} \end{bmatrix}, \quad P^{-1}\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \tilde{B}_d \end{bmatrix},$$

and

$$\tilde{C}P = [\tilde{K} \quad \tilde{F}_a \quad \tilde{C}_d],$$

where

$$\tilde{A}_{dd} := \begin{bmatrix} 0 & I_{q_1-1} & \cdots & 0 & 0 \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{q_m-1} \\ \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B}_d := \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix},$$

and

$$\tilde{C}_d := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Again, as was done in the SISO case, by dualizing the arguments in the proof of Theorem 8.2.1, it can be shown that $(\tilde{A}, \tilde{B}, \tilde{C})$, or equivalently the system (A, B, C), has an infinite zero structure $S_\infty^\star = \{q_1, q_2, \dots, q_m\}$ and has invariant zeros at

$$\lambda(A_o) \cup \lambda\left(\tilde{A}_{aa} - \tilde{L}_{ad}\tilde{F}_a\right) = \left\{\nu_1, \nu_2, \dots, \nu_{n_o}, z_1, z_2, \dots, z_\ell\right\}.$$

This completes our proof of Theorem 9.2.2. ∎

The following remarks are in order.

Remark 9.2.1.

1. *The uncontrollable modes of (A, B) are automatically included in the set of invariant zeros of (A, B, C) for any C such that (A, B, C) is square invertible. Hence, the invariant zeros at $\nu_1, \nu_2, \ldots, \nu_{n_o}$, in both Theorems 9.2.1 and 9.2.2, cannot be re-assigned. However, they can be excluded from the invariant zeros of a left invertible or noninvertible system. This will be done in the next section when we deal with the general structural assignment.*

2. *Unfortunately, it can be shown by examples that $\boldsymbol{\Omega}$ of (9.2.21) is not necessarily complete for $m > 1$. That is there exists an output matrix C such that the resulting system (A, B, C) has all the properties listed in Theorem 9.2.2 but $C \notin \boldsymbol{\Omega}$.*

Remark 9.2.2. *We note that the construction procedure of the desired set of output matrices $\boldsymbol{\Omega}$ is buried in the proof of Theorem 9.2.2. We would like to summarize in the following lines an easy-to-follow step-by-step algorithm that generates this $\boldsymbol{\Omega}$.*

1. *Given a matrix pair (A, B), compute nonsingular transformations $T_{\rm s}$ and $T_{\rm i}$ such that $(T_{\rm s}^{-1} A T_{\rm s}, T_{\rm s}^{-1} B T_{\rm i})$ is in the controllability structural decomposition form and obtain the controllability index $\{k_1, k_2, \ldots, k_m\}$.*

2. *Specify a desired infinite zero structure for the resulting systems in a set of integers, $\{q_1, q_2, \ldots, q_m\}$ with $0 < q_i \leq k_i$, $i = 1, 2, \ldots, m$.*

3. *Specify a self-conjugated set of desired invariant zeros, $\{z_1, z_2, \ldots, z_\ell\}$, where $\ell = \sum_{i=1}^{m} (k_i - q_i)$.*

4. *Define $(\tilde{A}_{\rm aa}, \tilde{L}_{\rm ad})$ as in (9.2.15) and (9.2.16), and compute the set $\boldsymbol{F}_{\rm a}$ as (9.2.17).*

5. *Finally, compute the desired set of output matrices $\boldsymbol{\Omega}$ as in (9.2.21).*

We illustrate Theorem 9.2.2 in the following example.

Example 9.2.2. Consider a two-input system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} -2 & -5 & -4 & -4 \\ 2 & 3 & 3 & 3 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} u. \tag{9.2.22}$$

Using the software package of [87], we obtain that

$$T_s = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_i = I_2,$$

and the controllability structural decomposition form of (A, B) is given by

$$\tilde{A} = T_s^{-1} A T_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \tilde{B} = T_s^{-1} B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

with a controllability index $\mathcal{C} = \{2, 2\}$. Following the procedure in the proof of Theorem 9.2.2, we obtain a set of output matrices

$$\boldsymbol{\Omega}_1 = \left\{ \Gamma \begin{bmatrix} a_1 & 1 & a_2 & 0 \\ a_3 & 0 & a_4 & 1 \end{bmatrix} T_s^{-1} \;\middle|\; a_1 + a_4 = a_1 a_4 - a_2 a_3 = 2, \right.$$

$$\left. \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0 \right\}$$

such that for any $C \in \boldsymbol{\Omega}_1$ the resulting system (A, B, C) has an infinite zero structure $\mathcal{S}_\infty^\star = \{1, 1\}$ and two invariant zeros at $-1 \pm j1$. The following is another set of output matrices that we obtain,

$$\boldsymbol{\Omega}_2 = \left\{ \Gamma \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \end{bmatrix} T_s^{-1} \;\middle|\; a \in \mathbb{R},\ \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0 \right\}.$$

It is easy to verify that for any $C \in \boldsymbol{\Omega}_2$ the corresponding system (A, B, C) has an infinite zero structure $\mathcal{S}_\infty^\star = \{1, 2\}$ and one invariant zero at -1.

9.3 Complete Structural Assignment

Having studied in the previous chapters all the structural properties of linear systems, *i.e.*, the finite zero and infinite zero structures as well as the invertibility structures, we are now ready to present in the following theorem the result of the general system structural assignment.

Theorem 9.3.1. *Consider the unsensed system (9.1.1) with B being of full rank. Let the controllability index of (A, B) be given by $\mathcal{C} = \{k_1, k_2, \ldots, k_m\}$, and the uncontrollable modes of (A, B), if any, be given by $\Delta = \{\nu_1, \nu_2, \ldots, \nu_{n_o}\}$. Let*

$$\Lambda_2 := \{\ell_1, \ell_2, \ldots, \ell_{m_c}\} \subset \mathcal{C} =: \{k_1, k_2, \ldots, k_m\}, \tag{9.3.1}$$

$$\mathcal{C}\backslash\Lambda_2 := \{\omega_1, \omega_2, \dots, \omega_{m_{\rm d}}\}, \;\; m_{\rm d} = m - m_{\rm c}, \;\; \omega_1 \le \omega_2 \le \cdots \le \omega_{m_{\rm d}}, \tag{9.3.2}$$

$$\Lambda_4 := \{q_1, q_2, \dots, q_{m_{\rm d}}\}, \;\; q_i \le \omega_i, \;\; i = 1, 2, \dots, m_{\rm d}. \tag{9.3.3}$$

Moreover, we let Λ_1 *be a set of complex scalars*

$$\Lambda_1 = \Theta_{\rm c} \cup \Delta_1 := \{z_1, z_2, \dots, z_{s_1},\} \cup \Delta_1 \tag{9.3.4}$$

where $\Theta_{\rm c}$ *is self-conjugated and so is* $\Delta_1 \subset \Delta$. *For simplicity, we assume that the entries of* $\Delta_2 = \Delta \setminus \Delta_1$ *are distinct. Furthermore,* s_1 *is chosen such that*

$$s_1 \le n - \sum_{i=1}^{m_{\rm c}} \ell_i - \sum_{i=1}^{m_{\rm d}} q_i - n_{\rm o}. \tag{9.3.5}$$

Finally, let

$$\Lambda_3 := \{\mu_1, \mu_2, \dots, \mu_{p_{\rm b}}\} \tag{9.3.6}$$

be a set of positive integers with $\mu_1 \le \mu_2 \le \cdots \le \mu_{p_{\rm b}}$, *which satisfy the constraint*

$$s_1 + n_{\rm o} + \sum_{i=1}^{p_{\rm b}} \mu_i + \sum_{i=1}^{m_{\rm c}} \ell_i + \sum_{i=1}^{m_{\rm d}} q_i = n. \tag{9.3.7}$$

Then, there exists a nonempty set $\mathbf{\Omega} \subset \mathbb{R}^{(m_{\rm d}+p_{\rm b})\times n}$ *such that for any* $C \in \mathbf{\Omega}$, *the resulting system characterized by the matrix triple* (A, B, C) *has the following properties: its invariant zeros are given by* Λ_1, *its invariant indices* $\mathbf{I}_2 = \Lambda_2$, $\mathbf{I}_3 = \Lambda_3$ *and* $\mathbf{I}_4 = \Lambda_4$, *or equivalently, the infinite zero structure of the triple* (A, B, C) *is given by* Λ_4, *and its invertibility structures are respectively given by* Λ_2 *and* Λ_3. *Figure 9.3.1 summarizes in a graphical form the above general structural assignment.*

Proof. We will give a constructive proof that would yield a desired set $\mathbf{\Omega}$. We first introduce the following key lemma, which is crucial to the proof of Theorem 9.3.1.

Lemma 9.3.1. *Consider a linear system* $\tilde{\Sigma}$ *characterized by a triple* $(\tilde{A}, \tilde{B}, \tilde{C})$. *Assume that it is already in the form of the special coordinate basis of Theorem 5.4.1 or in the compact form of (5.4.21) to (5.4.23), i.e.,*

$$\tilde{A} = \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} & 0 & L_{\rm ad}C_{\rm d} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ B_{\rm c}E_{\rm ca} & L_{\rm cb}C_{\rm b} & A_{\rm cc} & L_{\rm cd}C_{\rm d} \\ B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & B_{\rm d}E_{\rm dc} & A_{\rm dd} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_{\rm c} \\ B_{\rm d} & 0 \end{bmatrix}, \tag{9.3.8}$$

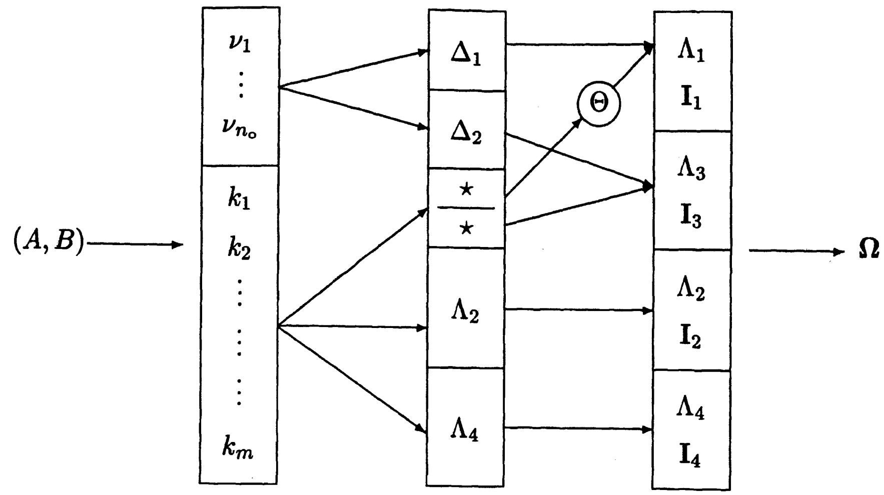


Figure 9.3.1: Graphical summary of the general structural assignment.

and

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & C_{\rm d} \\ 0 & C_{\rm b} & 0 & 0 \end{bmatrix}. \tag{9.3.9}$$

Let

$$\breve{A} := \begin{bmatrix} A_{\rm aa} & M_{\rm ab} & 0 & M_{\rm ad} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ B_{\rm c}E_{\rm ca} & B_{\rm c}E_{\rm cb} & A_{\rm cc} & M_{\rm cd} \\ B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & B_{\rm d}E_{\rm dc} & A_{\rm dd} \end{bmatrix}, \tag{9.3.10}$$

with any constant submatrices $M_{\rm ab}$, $M_{\rm ad}$, $M_{\rm bd}$ and $M_{\rm cd}$ of appropriate dimensions. Then, the matrix triple $(\breve{A}, \tilde{B}, \tilde{C})$ has the same structural invariant indices $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ and $\mathbf{I}_4$ as those of $\tilde{\Sigma}$.

This lemma can be routinely shown by considering the transposed systems characterized by $(\tilde{A}', \tilde{C}', \tilde{B}')$ and $(\breve{A}', \tilde{C}', \tilde{B}')$. The result follows from the procedure given in STEP SCB.8 of the proof of Theorem 5.3.1.

Now, we are ready to give a detailed proof to Theorem 9.3.1. It follows from Theorem 4.4.1 that there exist nonsingular state and input transformations T_0 and $T_{\rm i}$ such that the transformed pair,

$$(A_1, B_1) := \left(T_0^{-1} A T_0, T_0^{-1} B T_{\rm i}\right), \tag{9.3.11}$$

is in the controllability structural decomposition form of (4.4.7) with its controllability index being $\mathcal{C} = \{k_1, k_2, \ldots, k_m\}$. In view of the properties of the special coordinate basis, it is simple to see that each input channel in B_1 could either be assigned to the state variables associated with x_c or x_d of the resulting system. However, if we assign a particular input channel to be a member of x_c of the desired system, we will have to assign the whole block associated with this particular channel to it. This is because of the following reasons: 1) the whole block is controllable by the input channel; and 2) both dynamics of x_a and x_b cannot be controlled by input channels associated with x_c. On the other hand, there is no such a constraint for the structure associated with x_d, *i.e.*, the infinite zero structure.

Let Λ_2 and Λ_4 be given respectively as in (9.3.1) and (9.3.3), and let

$$n_c = \sum_{i=1}^{m_c} \ell_i \quad \text{and} \quad n_d = \sum_{i=1}^{m_d} q_i. \tag{9.3.12}$$

It is simple to verify that there exist permutation transformations P_1 and P_{i1} such that

$$A_2 = P_1^{-1} A_1 P_1 = \begin{bmatrix} A_o & 0 & 0 \\ B_c \cdot \star & A_{cc} & B_c \cdot \star \\ \tilde{B}_d \cdot \star & \tilde{B}_d \cdot \star & A_* \end{bmatrix},$$

and

$$B_2 = P_1^{-1} B_1 P_{i1} = \begin{bmatrix} 0 & 0 \\ B_c & 0 \\ 0 & \tilde{B}_d \end{bmatrix},$$

where

$$A_{cc} := \begin{bmatrix} 0 & I_{\ell_1-1} & \dots & 0 & 0 \\ \star & \star & \dots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{\ell_{m_c}-1} \\ \star & \star & \dots & \star & \star \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix},$$

$$A_* := \begin{bmatrix} 0 & I_{\omega_1-q_1-1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{q_1-1} & \dots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & I_{\omega_{m_d}-q_{m_d}-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_{q_{m_d}-1} \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \end{bmatrix},$$

and

$$\tilde{B}_{\rm d} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix},$$

and where $\star$s are submatrices of less interest.

Next, it is simple to see that there exist another pair of permutation matrices P_2 and P_{i2} such that the transformed pair

$$(A_3, B_3) := (P_2^{-1} A_2 P_2, P_2^{-1} B_2 P_{i2})$$

has the following form,

$$A_3 = \begin{bmatrix} A_{\rm o} & 0 & 0 & 0 \\ 0 & A_{\rm ab}^* & 0 & \star \\ B_{\rm c} \cdot \star & B_{\rm c} \cdot \star & A_{\rm cc} & B_{\rm c} \cdot \star \\ B_{\rm d} \cdot \star & B_{\rm d} \cdot \star & B_{\rm d} \cdot \star & A_{\rm dd}^* + B_{\rm d} \cdot \star \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_{\rm c} \\ B_{\rm d} & 0 \end{bmatrix}, \tag{9.3.13}$$

where

$$A_{\rm dd}^* = \begin{bmatrix} 0 & I_{q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{q_{m_{\rm d}}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B_{\rm d} = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \tag{9.3.14}$$

and

$$A_{\rm ab}^* = \begin{bmatrix} 0 & I_{\omega_1-q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{\omega_{m_{\rm d}}-q_{m_{\rm d}}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let us define

$$C_{\rm d} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

which is in conformity with the structures of $A_{\rm dd}^*$ and $B_{\rm d}$ in (9.3.14). We further define

$$C_3 = [\,0 \quad 0 \quad 0 \quad C_{\rm d}\,], \tag{9.3.15}$$

which is in conformity with structures of A_3 and B_3 in (9.3.13). The result of Lemma 9.3.1 implies that there exists a nonsingular state transformation T_3 such that

$$A_4 = T_3^{-1} A_3 T_3 = \begin{bmatrix} A_{\rm ab} & 0 & L_{\rm abd} C_{\rm d} \\ B_{\rm c} \cdot \star & A_{\rm cc} & L_{\rm cd} C_{\rm d} \\ B_{\rm d} \cdot \star & B_{\rm d} \cdot \star & A_{\rm dd}^* + B_{\rm d} \cdot \star \end{bmatrix},$$

$$B_4 = T_3^{-1} B_3 = \begin{bmatrix} 0 & 0 \\ 0 & B_{\rm c} \\ B_{\rm d} & 0 \end{bmatrix},$$

and

$$C_4 = C_3 T_3 = C_3 = [\,0 \quad 0 \quad C_{\rm d}\,],$$

where

$$A_{\rm ab} = \begin{bmatrix} A_{\rm o} & 0 \\ 0 & A_{\rm ab}^* \end{bmatrix}, \quad L_{\rm abd} = \begin{bmatrix} 0 \\ L_{\rm abd}^* \end{bmatrix}. \tag{9.3.16}$$

In view of the properties of the special coordinate basis, it is simple to see that the triple (A_4, B_4, C_4) is in the form of the special coordinate basis with its structural invariant indices $\mathbf{I}_2 = \Lambda_2$, $\mathbf{I}_4 = \Lambda_4$, $\mathbf{I}_3$ being empty and its invariant zeros being $\lambda(A_{\rm ab})$.

Next, we define a new output matrix,

$$\breve{C}_4 := C_4 + [\,K_{\rm c} \quad 0 \quad 0\,] = [\,K_{\rm c} \quad 0 \quad C_{\rm d}\,], \tag{9.3.17}$$

where

$$K_{\rm c} = [\,K_{\rm c1} \quad K_{\rm c2}\,],$$

which is partitioned in conformity with $A_{\rm ab}$ and $L_{\rm abd}$ in (9.3.16) with $K_{\rm c1}$ being an arbitrary matrix of appropriate dimensions and $K_{\rm c2}$ being chosen such that

$$\Theta_{\rm c} \subset \lambda\,(A_{\rm ab}^* - L_{\rm abd}^* K_{\rm c2}),$$

and the remaining eigenvalues of $A_{\rm ab}^* - L_{\rm abd}^* K_{\rm c2}$ are real and distinct. Moreover, these remaining eigenvalues of $A_{\rm ab}^* - L_{\rm abd}^* K_{\rm c2}$ are distinct from the entries of Δ_2. This can be done because the pair $(A_{\rm ab}^*, L_{\rm abd}^*)$ is controllable. Dualizing the arguments in the proof of Theorem 8.2.1, we can show that there exists a state transformation T_4 such that

$$A_5 = T_4^{-1} A_4 T_4 = \begin{bmatrix} A_{\rm ab} - L_{\rm abd} K_{\rm c} & 0 & \tilde{L}_{\rm abd} C_{\rm d} \\ B_{\rm c} \cdot \star & A_{\rm cc} & L_{\rm cd} C_{\rm d} \\ B_{\rm d} \cdot \star & B_{\rm d} \cdot \star & A_{\rm dd} + B_{\rm d} \cdot \star \end{bmatrix},$$

$$B_5 = T_4^{-1} B_4 = \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix},$$

and

$$C_5 := \check{C}_4 T_4 = [\,0 \quad 0 \quad C_d\,].$$

Again, the triple (A_5, B_5, C_5) is in the form of the special coordinate basis and has the same structural indices $\mathbf{I}_2, \mathbf{I}_3$ and $\mathbf{I}_4$ as the triple (A_4, B_4, C_4). Moreover, its invariant zeros are given by the eigenvalues of $A_{ab} - L_{abd} K_c$, in which matrix $A_{ab} - L_{abd} K_c$ can be rewritten as

$$A_{ab} - L_{abd} K_c = \begin{bmatrix} A_o & 0 \\ -L_{abd}^* K_{c1} & A_{ab}^* - L_{abd}^* K_{c2} \end{bmatrix}.$$

We next find a transformation T_{ab} such that $A_{ab} - L_{abd} K_c$ is transformed into the form

$$\tilde{A}_{ab} = T_{ab}^{-1} \left(A_{ab} - L_{abd} K_c \right) T_{ab} = \begin{bmatrix} A_{aa} & M_{ab} \\ 0 & A_{bb} \end{bmatrix},$$

where $\lambda(A_{aa}) = \Lambda_1 = \Delta_1 \cup \Theta_c$ with Θ_c given in (9.3.4), and A_{bb} being a diagonal matrix. Let

$$T_5 = \begin{bmatrix} T_{ab} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then, we have

$$A_6 = T_5^{-1} A_5 T_5 = \begin{bmatrix} A_{aa} & M_{ab} & 0 & L_{ad} C_d \\ 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c \cdot \star & B_c \cdot \star & A_{cc} & L_{cd} C_d \\ B_d \cdot \star & B_d \cdot \star & B_d \cdot \star & A_{dd} + B_d \cdot \star \end{bmatrix},$$

$$B_6 = T_5^{-1} B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix},$$

and

$$C_6 := C_5 T_5 = [\,0 \quad 0 \quad 0 \quad C_d\,].$$

The remaining task is to assign the structural invariant indices $\mathbf{I}_3$ to coincide with the given set $\Lambda_3 = \{\mu_1, \mu_2, \ldots, \mu_{p_b}\}$, which can be done by choosing the following output matrix,

$$\tilde{C}_6 = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix},$$

where

$$C_{\rm b} = \begin{bmatrix} C_{\rm b1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{{\rm b}p_{\rm b}} \end{bmatrix},$$

and where $C_{{\rm b}i}$, $i = 1, 2, \ldots, p_{\rm b}$, is a $1 \times \mu_i$ vector with all its entries being nonzero. Utilizing the result of Lemma 9.3.1 one more time, we can show that the triple characterized by $(A_6, B_6, \tilde{C}_6)$ has its invariant zeros at $\lambda(A_{\rm aa})$, and its structural invariant indices $\mathbf{I}_2 = \Lambda_2$, $\mathbf{I}_3 = \Lambda_3$ and $\mathbf{I}_4 = \Lambda_4$, respectively. Let $p = m_{\rm d} + p_{\rm b}$. We finally obtain the desired set,

$$\boldsymbol{\Omega} = \left\{ \Gamma_{\rm o} \begin{bmatrix} 0 & 0 & 0 & C_{\rm d} \\ 0 & C_{\rm b} & 0 & 0 \end{bmatrix} (T_0 P_1 P_2 T_3 T_4 T_5)^{-1} \,\middle|\, \Gamma_{\rm o} \in \mathbb{R}^{p \times p} \text{ is nonsingular} \right\}. \tag{9.3.18}$$

This completes the proof of Theorem 9.3.1. ∎

The following remarks are in order.

Remark 9.3.1. *If Λ_2 is set to be empty, then the resulting system will be left invertible. Similarly, if Λ_3 is set to be empty, the resulting system will be right invertible.*

Remark 9.3.2. *We note that if the entries of Δ_2 are not distinct, then the assignment of Λ_3 will be slightly more complicated. We would have to utilize the result of the real Jordan canonical form of Theorem 4.2.2 to assign Λ_3 in accordance with the real Jordan block structure of the part of $A_{\rm o}$ assigned to Λ_3.*

We now present the following two examples to illustrate our results. In the first example, we follow the algorithm given in the proof of Theorem 9.3.1 to yield a set of constant matrices such that for any of its members, the resulting system has desired invariant indices $\mathbf{I}_1$ to $\mathbf{I}_4$. In the second example, we study a benchmark problem for robust control of a flexible mechanical system proposed by Wie and Bernstein [149]. We will identify sets of sensors that would yield the best performance under the H_∞ almost disturbance decoupling framework.

Example 9.3.1. Consider the linear system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 3 & 0 & 1 & -1 & 1 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} u. \tag{9.3.19}$$

Following Theorem 4.4.1 and the toolkit of [87], we obtain the state and input transformations,

$$T_0 = \begin{bmatrix} -0.5 & -0.516398 & -0.258199 & -0.512989 & -0.307794 & -0.102598 \\ 0 & -0.258199 & -0.258199 & -0.205196 & -0.205196 & -0.102598 \\ 0 & 0 & -0.258199 & -0.102598 & -0.205196 & -0.102598 \\ 0 & 0 & 0 & -0.102598 & -0.102598 & -0.102598 \\ 0 & 0 & 0 & 0 & -0.102598 & -0.102598 \\ 0 & 0 & 0 & 0 & 0 & -0.102598 \end{bmatrix},$$

and

$$T_\mathrm{i} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.258199 & 0 \\ 0 & 0 & -0.102598 \end{bmatrix},$$

which take the given pair (A, B) into the following controllability structural decomposition form,

$$A_1 = \begin{bmatrix} 1 & 1.032796 & 0 & 1.025978 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1.936492 & 2 & 1 & 1.986799 & 1.192079 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 4.873397 & 5.033223 & 2.516611 & 5 & 3 & 1 \end{bmatrix},$$

and

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with a controllability index $\mathcal{C} = \{1, 2, 3\}$. In view of the results of Theorem 9.3.1, we have the following admissible choices of Λ_1 to Λ_4,

$$\Lambda_2 = \{1\}, \quad \Lambda_4 = \{1, 1\}, \quad \Lambda_1 = \{-3\}, \quad \Lambda_3 = \{2\}.$$

Following the proof of Theorem 9.3.1, we obtain $P_1 = I$,

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$T_3 = I$, and

$$A_{\rm ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_{\rm abd} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, it is straightforward to verify that the following gain matrix $K_{\rm c}$,

$$K_{\rm c} = \begin{bmatrix} -0.072739 & 0.031699 & 2.214148 \\ 2.234751 & -0.973901 & 1.072739 \end{bmatrix},$$

places the eigenvalues of $A_{\rm ab} - L_{\rm abd}K_{\rm c}$ at -3, 0 and 2. The first eigenvalue is chosen to coincide with the specification of Λ_1 and the other two are chosen to be distinct. Next, we carry on the procedure in the proof of Theorem 9.3.1 and obtain

$$T_4 = \begin{bmatrix} 0.183616 & -0.399509 & -0.334800 & 0 & 0 & 0 \\ -0.085347 & -0.916729 & 0.144674 & 0 & 0 & 0 \\ 0.256040 & 0 & 0.289349 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707107 & 0 & 0 \\ -0.550849 & 0 & -0.669600 & 0 & 1 & 0 \\ -0.768121 & 0 & 0.578698 & 0 & 0 & 1 \end{bmatrix},$$

$T_5 = I$, and

$$\tilde{C}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha_1 & \alpha_2 & 0 & 0 & 0 \end{bmatrix},$$

where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Finally, we obtain a desired set of output matrices,

$$\mathbf{\Omega} = \left\{ \Gamma_{\rm o} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha_1 & \alpha_2 & 0 & 0 & 0 \end{bmatrix} \Gamma_{\rm x} \;\middle|\; \Gamma_{\rm o} \in \mathbb{R}^{3\times 3} \text{ and } \det(\Gamma_{\rm o}) \neq 0 \right\},$$

where

$$\Gamma_{\rm x} = \begin{bmatrix} 0 & -6.76078 & 6.76078 & 14.17556 & -37.56982 & 23.39426 \\ 0 & 1.57356 & -1.57356 & 7.33281 & -7.20391 & -0.12890 \\ 0 & 5.98250 & -5.98250 & -12.54373 & -0.44033 & 12.98406 \\ 2.82843 & -5.65685 & 2.82843 & -5.65685 & 2.82843 & 2.82843 \\ 0 & 0.28172 & -4.15470 & 3.28230 & -17.11718 & 17.70786 \\ 0 & -8.65515 & 8.65515 & 18.14757 & -28.60333 & 0.70897 \end{bmatrix}.$$

It is now simple to verify, using the toolkit of [87], that the resulting matrix triple (A, B, C) with C being a member of $\mathbf{\Omega}$ indeed has an invariant zero at -3, and invariant indices $\mathbf{I}_2 = \Lambda_2 = \{1\}$, $\mathbf{I}_3 = \Lambda_3 = \{2\}$ and $\mathbf{I}_4 = \{1, 1\}$.

Example 9.3.2. We consider a benchmark problem for robust control of a flexible mechanical system proposed by Wie and Bernstein [149]. Although simple

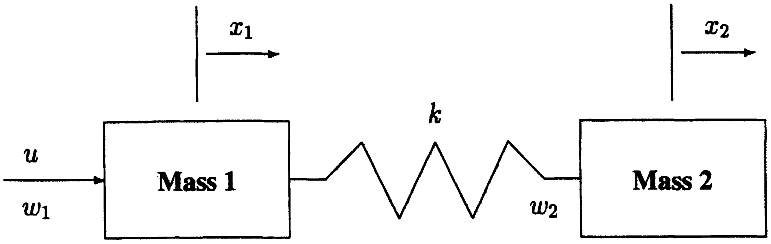


Figure 9.3.2: A two-mass-spring flexible mechanical system.

in nature, this problem will however provide an interesting example how sensor selection can affect the design performance. The problem is to control the displacement of the second mass by applying a force to the first mass as shown in Figure 9.3.2. The dynamic model of the system is given by

$$m_1\ddot{x}_1 = k(x_2 - x_1) + u + w_1, \tag{9.3.20}$$

$$m_2\ddot{x}_2 = k(x_1 - x_2) + w_2, \tag{9.3.21}$$

or in the state space representation,

$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & 0 & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where x_1 and x_2 are respectively the positions of Mass 1 (with a mass of m_1) and Mass 2 (with a mass of m_2), k is the spring constant, u is the input force, and w_1 and w_2 are the frictions (disturbances). For simplicity, we choose $m_1 = m_2 = 1$ and $k = 1$. It is natural to define an output to be controlled as $h = x_2$, *i.e.*, the position of the second mass. Thus, the plant model used for robust control synthesis is given by

$$\dot{x} = Ax + Bu + Ew = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{9.3.22}$$

and

$$h = C_2 x = [0 \quad 0 \quad 1 \quad 0]\, x. \tag{9.3.23}$$

It is simple to verify that the subsystem (A, B, C_2) is of minimum-phase and invertible. Hence, the disturbance w can be totally decoupled from the output to be controlled, *i.e.*, h, under the full state feedback. Our objective next is to identify sets of measurement output or the locations of sensors that would yield the same performance as that of the state feedback case. It follows from the results of [22,147] that this can be made possible by choosing a measurement output,

$$y = C_1 x, \tag{9.3.24}$$

such that the resulting subsystem (A, E, C_1) is left invertible and of minimum-phase. Following the procedure given in the previous section, we first transform the pair (A, E) into the controllability structural decomposition (CSD) form of Theorem 4.4.1. This can be done by the state and input transformation

$$T_0 = \begin{bmatrix} 0.316228 & 0 & 0.707107 & 0 \\ 0 & 0.316228 & 0 & 0.707107 \\ -0.316228 & 0 & 0.707107 & 0 \\ 0 & -0.316228 & 0 & 0.707107 \end{bmatrix},$$

and

$$T_i = \begin{bmatrix} 0.316228 & 0.707107 \\ -0.316228 & 0.707107 \end{bmatrix}.$$

The controllability structural decomposition form of the pair (A, E) is given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{9.3.25}$$

with a controllability index of (A, E) being $\{2, 2\}$. Following the proof of Theorem 9.3.1, we obtain the following set of measurement matrices,

$$\mathbf{\Omega}_1 = \left\{ \Gamma_o \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \,\middle|\, \Gamma_o \in \mathbb{R}^{2\times 2},\ \det(\Gamma_o) \neq 0 \right\}, \tag{9.3.26}$$

such that for any $C_1 \in \mathbf{\Omega}_1$, the resulting subsystem (A, E, C_1) is square invertible with two infinite zeros of order 2 and with no invariant zeros. Hence, it is of minimum-phase. It is well-known that higher orders of infinite zeros would yield higher controller gains, which is in general not desirable in practical situations. In what follows, we will identify a set of measurement matrices, $\mathbf{\Omega}_2$, such that for any $C_1 \in \mathbf{\Omega}_2$, the resulting subsystem (A, E, C_1) is of minimum-phase and square invertible with two infinite zeros of order 1 and two invariant zeros at -1.

The following $\mathbf{\Omega}_2$ is such a set obtained again using the procedure given in the proof of Theorem 9.3.1:

$$\mathbf{\Omega}_2 = \left\{ \Gamma_\mathrm{o} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \;\middle|\; \Gamma_\mathrm{o} \in \mathbb{R}^{2\times 2},\ \det(\Gamma_\mathrm{o}) \neq 0 \right\}. \tag{9.3.27}$$

Thus, it is straightforward to verify that the H_∞ almost disturbance decoupling is achievable for the flexible mechanical system of (9.3.22)–(9.3.23) together with a measurement output $y = C_1 x$, where $C_1 \in \mathbf{\Omega}_1$ or $C_1 \in \mathbf{\Omega}_2$. In fact, we can show that the H_∞ almost disturbance decoupling for the system cannot be achieved if there is only one sensor allowed to be placed in the system, *i.e.*, one would have to place two or more sensors in the system in order to decouple the disturbance (the frictions) from the position of the second mass.

9.4 Exercises

9.1. Consider a linear system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

and

$$y = Cx = [\,0 \quad 1 \quad 0 \quad 0\,]\, x,$$

which has an unstable invariant zero at 1 and a relative degree of 3.

(a) Determine a new measurement matrix, $\tilde{C}_1$, such that the resulting new system characterized by $(A, B, \tilde{C}_1)$ has an invariant zero at -1 and has the same relative degree as the original system characterized by (A, B, C).

(b) Determine a new measurement matrix, $\tilde{C}_2$, such that the resulting new system characterized by $(A, B, \tilde{C}_2)$ has two invariant zeros at -1 and -2, and has a relative degree of 2.

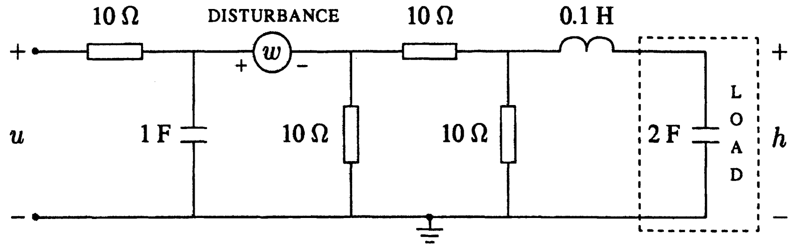
(c) Determine a new measurement matrix, $\tilde{C}_3$, such that the resulting new system characterized by $(A, B, \tilde{C}_3)$ has three invariant zeros at -1, -2 and -3, and has a relative degree of 1.

(d) Determine a new control matrix, $\tilde{B}_1$, such that the resulting new system characterized by $(A, \tilde{B}_1, C)$ has an invariant zero at -1 and has the same relative degree as the original system characterized by (A, B, C).

(e) Determine a new control matrix, $\tilde{B}_2$, such that the resulting new system characterized by $(A, \tilde{B}_2, C)$ has two invariant zeros at -1 and -2, and has a relative degree of 2.

(f) Determine a new control matrix, $\tilde{B}_3$, such that the resulting new system characterized by $(A, \tilde{B}_3, C)$ has three invariant zero at -1, -2 and -3, and has a relative degree of 1.

9.2. Consider an electric system given in the circuit below, in which the voltage of the circuit load, *i.e.*, the controlled output, h, cannot be measured, and the disturbance input, w, is to be rejected.



Circuit for Exercise 9.2

(a) Verify that the state-space realization of the system from the control input, u, to the controlled output, h, can be expressed as follows:

$$\begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ h = C_2\ x + D_2\ u, \end{cases}$$

with

$$A = \begin{bmatrix} -0.25 & -0.5 & 0 \\ 5 & -50 & -10 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.15 \\ -5 \\ 0 \end{bmatrix},$$

and

$$C_2 = [0 \quad 0 \quad 1], \quad D_2 = 0, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where x_1 is the voltage across the 1 F capacitor, x_2 is the current through the 0.1 H inductor, and finally x_3 is the voltage across the 2 F capacitor.

(b) Show that if the inductor current is the only measurement available, *i.e.*,

$$y = C_1 x + D_1 w = [\,0 \quad 1 \quad 0\,]\, x + 0 \cdot w,$$

the resulting subsystem from the disturbance, w, to the measurement output, y, is of nonminimum phase. In this case, it is not possible to find a proper and stabilizing controller for the system that can achieve H_∞ almost disturbance decoupling from w to h.

(c) Show that if the 1 F capacitor voltage can be measured, *i.e.*,

$$y = C_1 x + D_1 w = [\,1 \quad 0 \quad 0\,]\, x + 0 \cdot w,$$

then the resulting subsystem from w to y is of minimum phase, and thus, there exists a proper and stabilizing controller for the circuit such that the disturbance, w, can be almost decoupled from the controlled output, h.

Chapter 10

Time-Scale and Eigenstructure Assignment via State Feedback

10.1 Introduction

We present in this chapter one of the major applications of the structural decomposition techniques of linear systems in modern control system design, namely, the asymptotic time-scale and eigenstructure assignment (ATEA) design method using state feedback. The concept was originally proposed in Saberi and Sannuti [117,118] and developed fully in Chen [18] and Chen *et al.* [27]. It is decentralized in nature and is in fact rooted in the concept of singular perturbation methods of Kokotovic *et al.* [75]. It uses the structural decomposition of a given linear system characterized by a matrix quadruple (A, B, C, D) to design a state feedback gain F such that the resulting closed-loop system matrix $A + BF$ possesses pre-specified time-scales and eigenstructures. The specified finite eigenstructure of $A+BF$ is assigned appropriately by working with subsystems which represent the finite zero structure of the given system, whereas the specified asymptotically infinite eigenstructure of $A + BF$ is assigned appropriately by working with the subsystems which represent the infinite zero structure of the given system. Such a design method has been utilized intensively to solve many control problems, such as H_∞ control (see, *e.g.*, Chen [22]), H_2 optimal control (see, *e.g.*, Saberi *et al.* [120]), loop transfer recovery (see, *e.g.*, Chen [18], and Saberi *et al.* [116]), and the disturbance decoupling problem (see, *e.g.*, Chen [22], Lin and Chen [86], and

Ozcetin *et al.* [106,107]). It will be seen shortly that the ATEA design technique is a good way of capturing the core differences between H_2 and H_∞ control.

For simplicity, we assume throughout this chapter that the given system has no invariant zeros on the imaginary axis (or unit circle) if it represents a continuous-time (or discrete-time) system. Detailed treatments of systems with imaginary (or unit circle) invariant zeros can be found in Chen [22], Lin and Chen [86], and Saberi *et al.* [120], for the different applications.

10.2 Continuous-time Systems

In this section, we describe the technique of the asymptotic time-scale and eigenstructure assignment (ATEA) design for continuous-time systems together with its applications in solving H_2 and H_∞ control as well as disturbance decoupling problems. Consider a continuous-time linear system Σ characterized by

$$\begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{10.2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. Without loss of generality, we assume that (A, B) is stabilizable, and both B and C are of full rank. As indicated earlier, we assume that Σ does not have any invariant zeros on the imaginary axis.

10.2.1 Design Procedures and Fundamental Properties

In what follows, we present a step-by-step algorithm for the ATEA design method. The properties of this design method will be summarized in a theorem together with a detailed proof.

STEP ATEA-C.1.

Transform Σ into the structural decomposition or the special coordinate basis form as given by Theorem 5.4.1, that is, compute nonsingular state, input and output transformations Γ_s, Γ_i and Γ_o that transform the given system Σ into the special coordinate basis form of Theorem 5.4.1, which can also be put in the following compact form:

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}$$

$$+\begin{bmatrix} B_{0a}^{-} \\ B_{0a}^{+} \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} \begin{bmatrix} C_{0a}^{-} & C_{0a}^{+} & C_{0b} & C_{0c} & C_{0d} \end{bmatrix}, \tag{10.2.2}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} B_{0a}^{-} & 0 & 0 \\ B_{0a}^{+} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{10.2.3}$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_{0a}^{-} & C_{0a}^{+} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \tag{10.2.4}$$

$$\tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{10.2.5}$$

where in particular,

$$A_{dd} = A_{dd}^{*} + B_d E_{dd} + L_{dd} C_d, \tag{10.2.6}$$

with A_{dd}^{*}, B_d and C_d being as given in (5.4.26) and (5.4.27) of Chapter 5. Next, we define

$$A_{ss} = \begin{bmatrix} A_{aa}^{+} & L_{ab}^{+} C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{0s} = \begin{bmatrix} B_{0a}^{+} \\ B_{0b} \end{bmatrix}, \quad L_{sd} = \begin{bmatrix} L_{ad}^{+} \\ L_{bd} \end{bmatrix}, \tag{10.2.7}$$

and

$$B_s = \begin{bmatrix} B_{0s} & L_{sd} \end{bmatrix}. \tag{10.2.8}$$

STEP ATEA-C.2.

Let F_s be chosen such that

$$\lambda(A_{ss}^{c}) = \lambda(A_{ss} - B_s F_s) \subset \mathbb{C}^{-}, \tag{10.2.9}$$

and partition F_s in conformity with (10.2.7) and (10.2.8) as

$$F_s = \begin{bmatrix} F_{s0} \\ F_{s1} \end{bmatrix} = \begin{bmatrix} F_{a0}^{+} & F_{b0} \\ F_{a1}^{+} & F_{b1} \end{bmatrix}. \tag{10.2.10}$$

It follows from the property of the special coordinate basis that the pair (A_{ss}, B_s) is controllable provided that the pair (A, B) is stabilizable. Then, we further partition $F_{s1} = [\, F_{a1}^+ \quad F_{b1} \,]$ as

$$F_{s1} = [\, F_{a1}^+ \quad F_{b1} \,] = \begin{bmatrix} F_{a11}^+ & F_{b11} \\ F_{a12}^+ & F_{b12} \\ \vdots & \vdots \\ F_{a1m_d}^+ & F_{b1m_d} \end{bmatrix}, \tag{10.2.11}$$

where F_{a1i}^+ and F_{b1i} are of dimensions $1 \times n_a^+$ and $1 \times n_b$, respectively.

STEP ATEA-C.3.

Let F_c be any arbitrary $m_c \times n_c$ matrix subject to the constraint that

$$A_{cc}^c = A_{cc} - B_c F_c \tag{10.2.12}$$

is a stable matrix. Note that the existence of such an F_c is guaranteed by the property that (A_{cc}, B_c) is controllable.

STEP ATEA-C.4.

This step makes use of the fast subsystems, $i = 1, 2, \dots, m_d$, represented by (5.4.11). Let

$$\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{iq_i}\}, \quad i = 1, 2, \dots, m_d,$$

be the sets of q_i elements, all in $\mathbb{C}^-$, which are closed under complex conjugation, where q_i and m_d are as defined in Theorem 5.4.1. Then, we let $\Lambda_d := \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{m_d}$. For $i = 1, 2, \dots, m_d$, we define

$$p_i(s) := \prod_{j=1}^{q_i} (s - \lambda_{ij}) = s^{q_i} + F_{i1} s^{q_i - 1} + \cdots + F_{iq_i - 1} s + F_{iq_i}, \tag{10.2.13}$$

and a sub-gain matrix parameterized by tuning parameter, ε,

$$\tilde{F}_i(\varepsilon) := \frac{1}{\varepsilon^{q_i}} \Big[F_{iq_i}, \ \varepsilon F_{iq_i - 1}, \ \dots, \ \varepsilon^{q_i - 1} F_{i1} \Big]. \tag{10.2.14}$$

STEP ATEA-C.5.

In this step, various gains calculated in STEPS ATEA-C.2 to ATEA-C.4 are put together to form a composite state feedback gain for the given system Σ. Let

$$\tilde{F}_{a1}^+(\varepsilon) := \begin{bmatrix} F_{a11}^+ F_{1q_1} / \varepsilon^{q_1} \\ F_{a12}^+ F_{2q_2} / \varepsilon^{q_2} \\ \vdots \\ F_{a1m_d}^+ F_{m_d q_{m_d}} / \varepsilon^{q_{m_d}} \end{bmatrix}, \tag{10.2.15}$$

$$\tilde{F}_{\mathrm{b1}}(\varepsilon) := \begin{bmatrix} F_{\mathrm{b11}} F_{1q_1}/\varepsilon^{q_1} \\ F_{\mathrm{b12}} F_{2q_2}/\varepsilon^{q_2} \\ \vdots \\ F_{\mathrm{b1}m_{\mathrm{d}}} F_{m_{\mathrm{d}} q_{m_{\mathrm{d}}}}/\varepsilon^{q_{m_{\mathrm{d}}}} \end{bmatrix}, \tag{10.2.16}$$

and

$$\tilde{F}_{\mathrm{s1}}(\varepsilon) = \begin{bmatrix} \tilde{F}^{+}_{\mathrm{a1}}(\varepsilon) & \tilde{F}_{\mathrm{b1}}(\varepsilon) \end{bmatrix}. \tag{10.2.17}$$

Then, the ATEA state feedback gain is given by

$$F(\varepsilon) = -\Gamma_{\mathrm{i}}\Big(\tilde{F}(\varepsilon) + \tilde{F}_0\Big)\Gamma_{\mathrm{s}}^{-1}, \tag{10.2.18}$$

where

$$\tilde{F}(\varepsilon) = \begin{bmatrix} 0 & F^{+}_{\mathrm{a0}} & F_{\mathrm{b0}} & 0 & 0 \\ 0 & \tilde{F}^{+}_{\mathrm{a1}}(\varepsilon) & \tilde{F}_{\mathrm{b1}}(\varepsilon) & 0 & \tilde{F}_{\mathrm{d}}(\varepsilon) \\ 0 & 0 & 0 & F_{\mathrm{c}} & 0 \end{bmatrix}, \tag{10.2.19}$$

$$\tilde{F}_0 = \begin{bmatrix} C^{-}_{\mathrm{0a}} & C^{+}_{\mathrm{0a}} & C_{\mathrm{0b}} & C_{\mathrm{0c}} & C_{\mathrm{0d}} \\ E^{-}_{\mathrm{da}} & E^{+}_{\mathrm{da}} & E_{\mathrm{db}} & E_{\mathrm{dc}} & E_{\mathrm{dd}} \\ E^{-}_{\mathrm{ca}} & E^{+}_{\mathrm{ca}} & 0 & 0 & 0 \end{bmatrix}, \tag{10.2.20}$$

and where

$$\tilde{F}_{\mathrm{d}}(\varepsilon) = \mathrm{diag}\Big[\tilde{F}_1(\varepsilon),\ \tilde{F}_2(\varepsilon),\ \ldots,\ \tilde{F}_{m_{\mathrm{d}}}(\varepsilon)\Big]. \tag{10.2.21}$$

This completes the ATEA algorithm for continuous-time systems.

We have the following result.

Theorem 10.2.1. *Consider the given system* Σ *of (10.2.1). Then, the ATEA state feedback law* $u = F(\varepsilon)x$ *with* $F(\varepsilon)$ *being given as in (10.2.18) has the following properties:*

1. *There exists a scalar* $\varepsilon^* > 0$ *such that for every* $\varepsilon \in (0, \varepsilon^*]$, *the closed-loop system comprising the given system* Σ *and the ATEA state feedback law is asymptotically stable. Moreover, as* $\varepsilon \to 0$, *the closed-loop eigenvalues are given by*

$$\lambda(A^{-}_{\mathrm{aa}}),\quad \lambda(A^{\mathrm{c}}_{\mathrm{cc}}),\quad \lambda(A^{\mathrm{c}}_{\mathrm{ss}}) + 0(\varepsilon),\quad \frac{\Lambda_{\mathrm{d}}}{\varepsilon} + 0(1). \tag{10.2.22}$$

There are a total number of n_{d} *closed-loop eigenvalues, which have infinite negative real parts as* $\varepsilon \to 0$.

2. *Let*

$$C_s = \Gamma_o \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_b \end{bmatrix}, \quad D_s = \Gamma_o \begin{bmatrix} I_{m_0} & 0 \\ 0 & I_{m_d} \\ 0 & 0 \end{bmatrix}. \tag{10.2.23}$$

Then, we have

$$H(s,\varepsilon) := [C+DF(\varepsilon)][sI-A-BF(\varepsilon)]^{-1} \to [0 \quad H_s(s) \quad 0 \quad 0]\Gamma_s^{-1}, \tag{10.2.24}$$

pointwise in s as $\varepsilon \to 0$, where

$$H_s(s) = (C_s - D_s F_s)(sI - A_{ss} + B_s F_s)^{-1}. \tag{10.2.25}$$

Proof. Before proceeding to prove the theorem, we need the following lemma to establish some preliminary results.

Lemma 10.2.1. *Let a matrix triple (A, B, C), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, be right invertible and of minimum-phase. Let $F(\varepsilon) \in \mathbb{R}^{m\times n}$ be parameterized in terms of ε and be of the form*

$$F(\varepsilon) = N(\varepsilon)\Gamma(\varepsilon)T(\varepsilon) + R(\varepsilon), \tag{10.2.26}$$

where $N(\varepsilon) \in \mathbb{R}^{m\times p}$, $\Gamma(\varepsilon) \in \mathbb{R}^{p\times p}$, $T(\varepsilon) \in \mathbb{R}^{p\times n}$ and $R(\varepsilon) \in \mathbb{R}^{m\times n}$. Also, $\Gamma(\varepsilon)$ is nonsingular. Moreover, assume that the following conditions hold:

1. *$A + BF(\varepsilon)$ is asymptotically stable for all $0 < \varepsilon \le \varepsilon^*$ where $\varepsilon^* > 0$;*
2. *$T(\varepsilon) \to WC$ as $\varepsilon \to 0$ where W is some $p \times p$ nonsingular matrix;*
3. *as $\varepsilon \to 0$, $N(\varepsilon)$ tends to some finite matrix N such that $C(sI - A)^{-1}BN$ is invertible;*
4. *as $\varepsilon \to 0$, $R(\varepsilon)$ tends to some finite matrix R; and*
5. *$\Gamma^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Then as $\varepsilon \to 0$, we have $C[sI - A - BF(\varepsilon)]^{-1} \to 0$ pointwise in s.

Proof of Lemma 10.2.1. We first let $N^* \in \mathbb{R}^{m\times(m-p)}$ be such that

$$\tilde{N} = [N \quad N^*]$$

is an $m \times m$ nonsingular matrix, and rewrite R as

$$R = \tilde{N} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

where $R_1 \in \mathbb{R}^{p\times n}$ and $R_2 \in \mathbb{R}^{(m-p)\times n}$, respectively. Then, for sufficiently small ε, $F(\varepsilon)$ has the asymptotic form

$$\begin{aligned} F(\varepsilon) &= N(\varepsilon)\Gamma(\varepsilon)T(\varepsilon) + R(\varepsilon) \sim \tilde{N}\begin{bmatrix}\Gamma(\varepsilon)T(\varepsilon)+R_1\\ R_2\end{bmatrix} \\ &= \tilde{N}\tilde{\Gamma}\begin{bmatrix}T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1\\ R_2\end{bmatrix}, \end{aligned}$$

where

$$\tilde{\Gamma} = \begin{bmatrix}\Gamma(\varepsilon) & 0\\ 0 & I\end{bmatrix}.$$

Thus, $F(\varepsilon)\Phi B$, where $\Phi = (sI-A)^{-1}$, has the asymptotic form

$$F(\varepsilon)\Phi B \sim G = \tilde{N}\tilde{\Gamma}\begin{bmatrix}T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1\\ R_2\end{bmatrix}\Phi B. \tag{10.2.27}$$

Noting that

$$(I-G)^{-1} = \tilde{N}\left(I-\tilde{N}^{-1}G\tilde{N}\right)^{-1}\tilde{N}^{-1},$$

we have the following reductions:

$$\begin{aligned} & C\Phi B[I-F(\varepsilon)\Phi B]^{-1}F(\varepsilon) \sim C\Phi B(I-G)^{-1}F(\varepsilon) \\ &= C\Phi B\tilde{N}\left(I-\tilde{N}^{-1}G\tilde{N}\right)^{-1}\tilde{N}^{-1}F(\varepsilon) \\ &\sim C\Phi B\tilde{N}\left(I-\tilde{\Gamma}\begin{bmatrix}T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1\\ R_2\end{bmatrix}\Phi B\tilde{N}\right)^{-1}\tilde{\Gamma}\begin{bmatrix}T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1\\ R_2\end{bmatrix} \\ &= C\Phi B\tilde{N} \\ &\quad\times\begin{bmatrix}\Gamma^{-1}(\varepsilon)-[T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1]\Phi BN & -[T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1]\Phi BN^*\\ -R_2\Phi BN & I-R_2\Phi BN^*\end{bmatrix}^{-1} \\ &\quad\times\begin{bmatrix}T(\varepsilon)+\Gamma^{-1}(\varepsilon)R_1\\ R_2\end{bmatrix} \\ &\sim C\Phi B\tilde{N}\begin{bmatrix}-WC\Phi BN & -WC\Phi BN^*\\ -R_2\Phi BN & I-R_2\Phi BN^*\end{bmatrix}^{-1}\begin{bmatrix}WC\\ R_2\end{bmatrix} \\ &= C\Phi B\tilde{N}\begin{bmatrix}-WC\Phi BN & -WC\Phi BN^*\\ -R_2\Phi BN & I-R_2\Phi BN^*\end{bmatrix}^{-1}\begin{bmatrix}W & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}C\\ R_2\end{bmatrix} \\ &= [C\Phi BN \quad C\Phi BN^*]\begin{bmatrix}-C\Phi BN & -C\Phi BN^*\\ -R_2\Phi BN & I-R_2\Phi BN^*\end{bmatrix}^{-1}\begin{bmatrix}C\\ R_2\end{bmatrix} \\ &= [-I \quad 0]\begin{bmatrix}C\\ R_2\end{bmatrix} \\ &= -C, \end{aligned} \tag{10.2.28}$$

which implies that as $\varepsilon \to 0$,

$$C\Phi B[I - F(\varepsilon)\Phi B]^{-1}F(\varepsilon) \to -C, \text{ pointwise in } s. \tag{10.2.29}$$

Now, using the well-known matrix inversion identity of (2.3.16), *i.e.*,

$$(I - NM)^{-1} = I + N(I - MN)^{-1}M,$$

we have

$$\begin{aligned} C[sI - A - BF(\varepsilon)]^{-1} &= C[\Phi^{-1} - BF(\varepsilon)]^{-1} \\ &= C\Phi[I - BF(\varepsilon)\Phi]^{-1} \\ &= C\Phi\Big\{I + B[I - F(\varepsilon)\Phi B]^{-1}F(\varepsilon)\Phi\Big\} \\ &= C\Phi + C\Phi B[I - F(\varepsilon)\Phi B]^{-1}F(\varepsilon)\Phi \\ &\to C\Phi - C\Phi \\ &= 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{10.2.30}$$

This completes the proof of Lemma 10.2.1. ■

We are now ready to prove the results of Theorem 10.2.1. We first show that the closed-loop system under the ATEA state feedback $u = F(\varepsilon)x$ with $F(\varepsilon)$ given in (10.2.18) is asymptotically stable. Without loss of generality, we assume that the given system (A, B, C, D) is in the form of the special coordinate basis of Theorem 5.4.1. It is straightforward to verify that the closed-loop system matrix is given by

$$\tilde{A} + \tilde{B}\tilde{F}(\varepsilon) = \begin{bmatrix} A_{\rm aa}^- & \star & 0 & \star \\ 0 & A_{\rm ss} - B_{\rm 0s}F_{\rm s0} & 0 & L_{\rm sd}C_{\rm d} \\ 0 & \star & A_{\rm cc}^{\rm c} & \star \\ 0 & -B_{\rm 0d}F_{\rm s0} - B_{\rm d}\tilde{F}_{\rm s1}(\varepsilon) & 0 & A_{\rm dd}^* - B_{\rm d}\tilde{F}_{\rm d}(\varepsilon) + L_{\rm dd}C_{\rm d} \end{bmatrix}.$$

Obviously, the closed-loop system has $n_{\rm a}^- + n_{\rm c}$ eigenvalues at $\lambda(A_{\rm aa}^-) \subset \mathbb{C}^-$ and $\lambda(A_{\rm cc}^{\rm c}) \subset \mathbb{C}^-$. It is thus sufficient to show the stability of the closed-loop system by showing the stability of the subsystem matrix

$$A_{\rm smtx} = \begin{bmatrix} A_{\rm ss} - B_{\rm 0s}F_{\rm s0} & L_{\rm sd}C_{\rm d} \\ -B_{\rm 0d}F_{\rm s0} - B_{\rm d}\tilde{F}_{\rm s1}(\varepsilon) & A_{\rm dd}^* - B_{\rm d}\tilde{F}_{\rm d}(\varepsilon) + L_{\rm dd}C_{\rm d} \end{bmatrix}. \tag{10.2.31}$$

In view of the special structures of $A_{\rm dd}^*$, $B_{\rm d}$ and $C_{\rm d}$, it is simple to see that the stability of $A_{\rm smtx}$ is equivalent to the stability of an auxiliary subsystem characterized by the state space equations

$$\dot{x}_{\rm s} = (A_{\rm ss} - B_{\rm 0s}F_{\rm s0})x_{\rm s} + L_{\rm sd}y_{\rm d}, \tag{10.2.32}$$

and for $i = 1, 2, \ldots, m_{\mathrm{d}}$,

$$\dot{x}_i = A_{q_i} x_i + L_{i\mathrm{s}} x_{\mathrm{s}} + L_{i\mathrm{d}} y_{\mathrm{d}} - \frac{1}{\varepsilon^{q_i}} B_{q_i} \Big[F_{\mathrm{s}1i} F_{iq_i} x_{\mathrm{s}} + F_i S_i(\varepsilon) x_i \Big], \quad (10.2.33)$$

$$y_i = C_{q_i} x_i, \quad y_{\mathrm{d}} = C_{\mathrm{d}} x_{\mathrm{d}}, \quad (10.2.34)$$

where

$$x_{\mathrm{d}} = \begin{pmatrix} x_1 \\ \vdots \\ x_{m_{\mathrm{d}}} \end{pmatrix}, \quad \begin{bmatrix} L_{1\mathrm{s}} \\ \vdots \\ L_{m_{\mathrm{d}}\mathrm{s}} \end{bmatrix} = -B_{0\mathrm{d}} F_{\mathrm{s}0}, \quad (10.2.35)$$

$$F_{\mathrm{s}1i} = [\, F_{\mathrm{a}1i}^{+} \quad F_{\mathrm{b}1i} \,], \quad F_i = \Big[F_{iq_i}, F_{iq_i-1}, \ldots, F_{i1} \Big], \quad (10.2.36)$$

and

$$S_i(\varepsilon) = \operatorname{diag}\Big\{ 1, \varepsilon, \ldots, \varepsilon^{q_i - 1} \Big\}. \quad (10.2.37)$$

We first define a state transformation

$$\bar{x}_{\mathrm{s}} = x_{\mathrm{s}}, \quad \bar{x}_i = x_i + \begin{bmatrix} F_{\mathrm{s}1i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_{\mathrm{s}}, \; i = 1, 2, \ldots, m_{\mathrm{d}}; \; \Rightarrow \; \bar{x}_{\mathrm{d}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{m_{\mathrm{d}}} \end{pmatrix}. \quad (10.2.38)$$

We then have

$$\dot{\bar{x}}_{\mathrm{s}} = A_{\mathrm{ss}}^{\mathrm{c}} \bar{x}_{\mathrm{s}} + L_{\mathrm{sd}} \bar{y}_{\mathrm{d}}, \quad (10.2.39)$$

and for $i = 1, 2, \ldots, m_{\mathrm{d}}$,

$$\dot{\bar{x}}_i = \left[A_{q_i} - \frac{1}{\varepsilon^{q_i}} B_{q_i} F_i S_i(\varepsilon) \right] \bar{x}_i + \bar{L}_{i\mathrm{s}} \bar{x}_{\mathrm{s}} + L_{i\mathrm{d}} \bar{y}_{\mathrm{d}}, \quad (10.2.40)$$

$$\bar{y}_{\mathrm{d}} = y_{\mathrm{d}} + F_{\mathrm{s}1} x_{\mathrm{s}} = C_{\mathrm{d}} \bar{x}_{\mathrm{d}}, \quad (10.2.41)$$

where $\bar{L}_{i\mathrm{s}} = L_{i\mathrm{s}} - L_{i\mathrm{d}} F_{\mathrm{s}1}$. For future use, we note that the state transformation of (10.2.38) has the form

$$\bar{\Gamma}_{\mathrm{s}} = \begin{bmatrix} I & 0 \\ \star & I \end{bmatrix}, \quad (10.2.42)$$

and the system matrix for the state equations of (10.2.39) to (10.2.41) is given by

$$\bar{A}_{\mathrm{smtx}} = \begin{bmatrix} A_{\mathrm{ss}}^{\mathrm{c}} & L_{\mathrm{sd}} C_{\mathrm{d}} \\ \bar{L}_{\mathrm{ds}} & A_{\mathrm{dd}}^{*} - B_{\mathrm{d}} \tilde{F}_{\mathrm{d}}(\varepsilon) + L_{\mathrm{dd}} C_{\mathrm{d}} \end{bmatrix}, \quad (10.2.43)$$

where

$$\bar{L}_{\mathrm{ds}} = \begin{bmatrix} \bar{L}_{1\mathrm{s}} \\ \vdots \\ \bar{L}_{m_{\mathrm{d}}\mathrm{s}} \end{bmatrix}. \quad (10.2.44)$$

Next, we define another state transformation:

$$\tilde{x}_{\mathrm{s}} = \bar{x}_{\mathrm{s}}, \quad \tilde{x}_i = S_i(\varepsilon)\bar{x}_i, \quad i = 1,2,\dots,m_{\mathrm{d}}; \;\Rightarrow\; \tilde{x}_{\mathrm{d}} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{m_{\mathrm{d}}} \end{pmatrix}. \tag{10.2.45}$$

The transformed state equations are then given by

$$\dot{\tilde{x}}_{\mathrm{s}} = A^{\mathrm{c}}_{\mathrm{ss}}\tilde{x}_{\mathrm{s}} + L_{\mathrm{sd}}\tilde{y}_{\mathrm{d}}, \tag{10.2.46}$$

and for $i = 1,2,\dots,m_{\mathrm{d}}$,

$$\dot{\tilde{x}}_i = \frac{1}{\varepsilon}\left(A_{q_i} - B_{q_i}F_i\right)\tilde{x}_i + \bar{L}_{i\mathrm{s}}\tilde{x}_{\mathrm{s}} + L_{i\mathrm{d}}\tilde{y}_{\mathrm{d}}, \; \tilde{y}_{\mathrm{d}} = C_{\mathrm{d}}\tilde{x}_{\mathrm{d}}. \tag{10.2.47}$$

The stability of the above system follows obviously from the standard results of the singular perturbation methods of Kokotovic *et al.* [75]. For completeness, we let P_{s} and P_i, $i = 1,2,\dots,m_{\mathrm{d}}$, be positive definite matrices satisfying the Lyapunov equations

$$P_{\mathrm{s}}(A^{\mathrm{c}}_{\mathrm{ss}})' + A^{\mathrm{c}}_{\mathrm{ss}}P_{\mathrm{s}} = -I, \tag{10.2.48}$$

and for $i = 1,2,\dots,m_{\mathrm{d}}$,

$$P_i(A_{q_i} - B_{q_i}F_i)' + (A_{q_i} - B_{q_i}F_i)P_i = -I. \tag{10.2.49}$$

We note that such P_{s} and P_i always exist because $A^{\mathrm{c}}_{\mathrm{ss}}$ and $A_{q_i} - B_{q_i}F_i$ are asymptotically stable. Finally, we define a Lyapunov function,

$$V(\tilde{x}) = \tilde{x}_{\mathrm{s}}'P_{\mathrm{s}}\tilde{x}_{\mathrm{s}} + \sum_{i=1}^{m_{\mathrm{d}}} \tilde{x}_i'P_i\tilde{x}_i \geq 0. \tag{10.2.50}$$

Evaluating it along the trajectory of (10.2.46) and (10.2.47), we obtain

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x}_{\mathrm{s}} \\ \tilde{x}_{\mathrm{d}} \end{pmatrix}' \begin{bmatrix} -I & X_{\mathrm{sd}} \\ X_{\mathrm{sd}}' & -I/\varepsilon \end{bmatrix} \begin{pmatrix} \tilde{x}_{\mathrm{s}} \\ \tilde{x}_{\mathrm{d}} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{x}_{\mathrm{s}} \\ \tilde{x}_{\mathrm{d}}/\sqrt{\varepsilon} \end{pmatrix}' \begin{bmatrix} -I & \sqrt{\varepsilon}X_{\mathrm{sd}} \\ \sqrt{\varepsilon}X_{\mathrm{sd}}' & -I \end{bmatrix} \begin{pmatrix} \tilde{x}_{\mathrm{s}} \\ \tilde{x}_{\mathrm{d}}/\sqrt{\varepsilon} \end{pmatrix}, \end{aligned} \tag{10.2.51}$$

where X_{sd} is a constant matrix of appropriate dimensions and independent of ε. Then, it is simple to see that there exists a scalar $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, $\dot{V} \leq 0$. Hence, the subsystem matrix A_{smtx} of (10.2.31) is indeed asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$. This completes the proof of Item 1 of Theorem 10.2.1.

We next move to show Item 2 of Theorem 10.2.1. Without loss of generality, but for simplicity of presentation, we assume that the nonsingular transformations $\Gamma_s = I$ and $\Gamma_i = I$, *i.e.*, we assume that the system $(A, B, \Gamma_o^{-1}C, \Gamma_o^{-1}D)$ is in the form of the special coordinate basis. In view of (10.2.18), let us partition $F(\varepsilon)$ as

$$F(\varepsilon) = \bar{F}_0 + \begin{bmatrix} 0 \\ \bar{F}(\varepsilon) \end{bmatrix},$$

where

$$\bar{F}_0 = -\begin{bmatrix} C_{0a}^- & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\bar{F}(\varepsilon) = -\begin{bmatrix} E_{da}^- & E_{da}^+ + \tilde{F}_{a1}^+(\varepsilon) & E_{db} + \tilde{F}_{b1}(\varepsilon) & E_{dc} & \tilde{F}_d(\varepsilon) + E_{dd} \\ E_{ca}^- & E_{ca}^+ & 0 & F_c & 0 \end{bmatrix}. \tag{10.2.52}$$

Then we have

$$\bar{C} = C + DF(\varepsilon) = \Gamma_o \begin{bmatrix} 0 & -F_{a0}^+ & -F_{b0} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

and

$$\bar{A} = A + B\bar{F}_0, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}. \tag{10.2.53}$$

With these definitions, we can write $H(s, \varepsilon)$ as

$$H(s, \varepsilon) = \bar{C}\left[sI - \bar{A} - \bar{B}\,\bar{F}(\varepsilon)\right]^{-1}.$$

Then in view of (10.2.52), it can be easily seen that $\bar{F}(\varepsilon)$ has the form

$$\bar{F}(\varepsilon) = N\Gamma(\varepsilon)T(\varepsilon) + R,$$

where

$$\Gamma(\varepsilon) = \mathrm{diag}\left\{\frac{1}{\varepsilon^{q_1}}, \frac{1}{\varepsilon^{q_2}}, \dots, \frac{1}{\varepsilon^{q_{m_d}}}\right\}, \quad N = -\begin{bmatrix} I_{m_d} \\ 0 \end{bmatrix},$$

and

$$R = -\begin{bmatrix} E_{da}^- & E_{da}^+ & E_{db} & E_{dc} & E_{dd} \\ E_{ca}^- & E_{ca}^+ & 0 & F_c & 0 \end{bmatrix},$$

while $T(\varepsilon)$ satisfies

$$T(\varepsilon) \to TC_{\rm m},$$

as $\varepsilon \to 0$, where

$$T = \operatorname{diag}\left\{F_{1q_1},\ F_{1q_2},\ \ldots,\ F_{m_{\rm d}q_{m_{\rm d}}}\right\},$$

and

$$C_{\rm m} = [\,0 \quad F_{\rm a1}^{+} \quad F_{\rm b1} \quad 0 \quad C_{\rm d}\,]. \tag{10.2.54}$$

Dualizing the arguments in the proof of Theorem 8.2.1 of Chapter 8, it is straightforward to show that the triple $(\bar{A}, \bar{B}, C_{\rm m})$ is right invertible and of minimum phase. Thus, it follows from Lemma 10.2.1 that

$$C_{\rm m}\left[sI - \bar{A} - \bar{B}\bar{F}(\varepsilon)\right]^{-1} \to 0, \tag{10.2.55}$$

pointwise in s as $\varepsilon \to 0$. Next, let

$$\bar{C} = \Gamma_{\rm o}\begin{bmatrix} 0 \\ C_{\rm m} \\ 0 \end{bmatrix} + C_{\rm e},$$

where

$$C_{\rm e} = \Gamma_{\rm o}\begin{bmatrix} 0 & -F_{\rm a0}^{+} & -F_{\rm b0} & 0 & 0 \\ 0 & -F_{\rm a1}^{+} & -F_{\rm b1} & 0 & 0 \\ 0 & 0 & C_{\rm b} & 0 & 0 \end{bmatrix} = [\,0 \quad C_{\rm s} - D_{\rm s}F_{\rm s} \quad 0 \quad 0\,].$$

We have

$$H_{\rm s}(s,\varepsilon) \to C_{\rm e}\left[sI - \bar{A} - \bar{B}\bar{F}(\varepsilon)\right]^{-1}, \tag{10.2.56}$$

pointwise in s as $\varepsilon \to 0$. In view of (10.2.42) and (10.2.43) together with some matrix inversion identities, e.g., (2.3.19) of Chapter 2, we can obtain

$$[sI - \bar{A} - \bar{B}\bar{F}(\varepsilon)]^{-1} = \begin{bmatrix} \star & \star & 0 & \star \\ 0 & X_{22}(s,\varepsilon) & 0 & X_{24}(s,\varepsilon) \\ 0 & \star & \star & \star \\ 0 & \star & 0 & \star \end{bmatrix}, \tag{10.2.57}$$

where $\star$s are matrices of not much interest, and

$$X_{22}(s,\varepsilon) = \left[sI - A_{\rm ss}^{\rm c} - L_{\rm sd}C_{\rm d}\left(sI - A_{\rm dd}^{*} + B_{\rm d}\tilde{F}_{\rm d}(\varepsilon) - L_{\rm dd}C_{\rm d}\right)^{-1}\bar{L}_{\rm ds}\right]^{-1},$$

and

$$X_{24}(s,\varepsilon) = (sI - A_{\rm ss}^{\rm c})^{-1} L_{\rm sd} C_{\rm d} \cdot \left[sI - A_{\rm dd}^{*} + B_{\rm d}\tilde{F}_{\rm d}(\varepsilon) - L_{\rm dd}C_{\rm d} - \bar{L}_{\rm ds}(sI - A_{\rm ss}^{\rm c})^{-1} L_{\rm sd} C_{\rm d} \right]^{-1}.$$

Then, we have

$$X_{24}(s,\varepsilon) = (sI - A_{\rm ss}^{\rm c})^{-1} L_{\rm sd} C_{\rm d} [sI - A_{\rm dd}^{*} + B_{\rm d}\tilde{F}_{\rm d}(\varepsilon)]^{-1} \cdot \left\{ I - \left[L_{\rm dd} + \bar{L}_{\rm ds}(sI - A_{\rm ss}^{\rm c})^{-1} L_{\rm sd} \right] C_{\rm d} [sI - A_{\rm dd}^{*} + B_{\rm d}\tilde{F}_{\rm d}(\varepsilon)]^{-1} \right\}^{-1}.$$

Note that $(A_{\rm dd}^{*}, B_{\rm d}, C_{\rm d})$ is square and invertible with no invariant zeros and thus it is of minimum phase. Also note the structure of $\tilde{F}_{\rm d}(\varepsilon)$, which satisfies all the properties as stated in Lemma 10.2.1. It then follows from Lemma 10.2.1 that

$$C_{\rm d}[sI - A_{\rm dd}^{*} + B_{\rm d}\tilde{F}_{\rm d}(\varepsilon)]^{-1} \to 0, \tag{10.2.58}$$

pointwise in s as $\varepsilon \to 0$, and hence $X_{24}(s,\varepsilon) \to 0$ pointwise in s as $\varepsilon \to 0$. Similarly, we can show that

$$X_{22}(s,\varepsilon) \to (sI - A_{\rm ss}^{\rm c})^{-1} = (sI - A_{\rm ss} + B_{\rm s}F_{\rm s})^{-1}, \tag{10.2.59}$$

pointwise in s as $\varepsilon \to 0$. It is clear now that

$$H_{\rm s}(s,\varepsilon) \to C_{\rm e}\left[sI - \bar{A} - \bar{B}\bar{F}(\varepsilon)\right]^{-1} \to [\,0 \quad H_{\rm s}(s) \quad 0 \quad 0\,], \tag{10.2.60}$$

pointwise in s as $\varepsilon \to 0$. This completes the proof of Theorem 10.2.1. ∎

10.2.2 H_2 Control, H_∞ Control and Disturbance Decoupling

In a typical control system design, the given specifications are usually transformed into a performance index, and then control laws which would minimize a certain norm, say H_2 or H_∞ norm, of the performance index are sought. In what follows, we will demonstrate that by properly choosing the sub-feedback gain matrix $F_{\rm s}$ in STEP ATEA-C.2, the ATEA design can be trivially adopted to solve the well-known H_2 and H_∞ control as well as disturbance decoupling problems.

To be specific, we consider a generalized continuous-time system Σ with a state-space description:

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w, \\ y = x, \\ h = C\,x + D\,u, \end{cases} \tag{10.2.61}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance input, $y = x$ is the measurement output, and $h \in \mathbb{R}^p$ is the controlled output of Σ. We assume that (A, B) is stabilizable and (A, B, C, D) has no invariant zeros on the imaginary axis. Then, the standard optimization problem is to find a control law

$$u = Fx, \tag{10.2.62}$$

such that when it is applied to the given system (10.2.61), the resulting closed-loop system is internally stable, *i.e.*, $\lambda(A+BF) \subset \mathbb{C}^-$, and a certain norm of the resulting closed-loop transfer function from the disturbance input w to the controlled output h, *i.e.*,

$$H_{hw}(s) = (C + DF)(sI - A - BF)^{-1}E, \tag{10.2.63}$$

is minimized. We will consider in this section the problems of H_2 optimal control and H_∞ control. In particular, H_2 optimal control is to minimize the H_2-norm of $H_{hw}(s)$ over all the possible internally stabilizing state feedback control laws (see Definition 2.4.5 of Chapter 2 for the definition of the H_2-norm of continuous-time systems). For future use, we define

$$\gamma_2^* := \inf \left\{ \|H_{hw}\|_2 \;\middle|\; u = Fx \text{ internally stabilizes } \Sigma \right\}. \tag{10.2.64}$$

Similarly, the standard H_∞ control is to minimize the H_∞-norm of $H_{hw}(s)$ over all the possible internally stabilizing state feedback control laws (see Definition 2.4.5 of Chapter 2 for the definition of the H_∞-norm of continuous-time systems). For future use, we define

$$\gamma_\infty^* := \inf \left\{ \|H_{hw}\|_\infty \;\middle|\; u = Fx \text{ internally stabilizes } \Sigma \right\}. \tag{10.2.65}$$

We note that the determination of this γ_∞^* is rather tedious. For a fairly large class of systems, γ_∞^* can be exactly computed using some numerically stable algorithms. In general, an iterative scheme is required to determine γ_∞^*. We refer interested readers to the work of Chen [22] for a detailed treatment of this particular issue. For simplicity, we assume throughout this section that γ_∞^* has been determined and hence it is known.

Finally, for the case when (A, B, C, D) has no invariant zeros on the imaginary axis, the disturbance decoupling problem (either in H_2 sense or in H_∞ sense) is to find an appropriate state feedback control law of (10.2.62) such that $A + BF$ is asymptotically stable and $H_{hw}(s) \to 0$, pointwise in s as $\varepsilon \to 0$.

We summarize in the following the solutions to the H_2 and H_∞ control as well as disturbance decoupling problems. We assume that Γ_s, Γ_i and Γ_o are the nonsingular state, input and output transformations that transform the matrix quadruple (A, B, C, D) into the special coordinate basis as in (10.2.2)–(10.2.5). Let

$$\tilde{E} := \Gamma_s^{-1} E = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_b \\ E_c \\ E_d \end{bmatrix} \quad \text{and} \quad E_s := \begin{bmatrix} E_a^+ \\ E_b \end{bmatrix}. \tag{10.2.66}$$

We have the following theorem.

Theorem 10.2.2. *Consider the generalized continuous-time system Σ characterized by (10.2.61). The ATEA design can be easily adapted to solve the H_2 and H_∞ control as well as disturbance decoupling problems for Σ. More specifically, we have*

1. *If the sub-feedback gain matrix F_s in* STEP ATEA-C.2 *is chosen to be*

$$F_s = (D_s' D_s)^{-1} (B_s' P_s + D_s' C_s), \tag{10.2.67}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$P_s A_{ss} + A_{ss}' P_s + C_s' C_s - (P_s B_s + C_s' D_s)(D_s' D_s)^{-1} (B_s' P_s + D_s' C_s) = 0, \tag{10.2.68}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_2 = \left\| [C + DF(\varepsilon)][sI - A - BF(\varepsilon)]^{-1} E \right\|_2 \to \gamma_2^*, \tag{10.2.69}$$

as $\varepsilon \to 0$, i.e., the corresponding ATEA state feedback law solves the H_2 suboptimal control problem for Σ. Furthermore,

$$\gamma_2^* = \sqrt{\operatorname{trace}(E_s' P_s E_s)}. \tag{10.2.70}$$

2. *Given a scalar $\gamma > \gamma_\infty^* \geq 0$, if F_s in* STEP ATEA-C.2 *is chosen to be*

$$F_s = (D_s' D_s)^{-1} (B_s' P_s + D_s' C_s), \tag{10.2.71}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$\begin{aligned} P_s A_{ss} + A_{ss}' P_s + C_s' C_s + P_s E_s E_s' P_s / \gamma^2 \\ -(P_s B_s + C_s' D_s)(D_s' D_s)^{-1} (B_s' P_s + D_s' C_s) = 0, \end{aligned} \tag{10.2.72}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_\infty = \left\|[C + DF(\varepsilon)][sI - A - BF(\varepsilon)]^{-1}E\right\|_\infty < \gamma, \quad (10.2.73)$$

for sufficiently small ε, i.e., the corresponding ATEA state feedback law solves the H_∞ γ-suboptimal control problem for Σ.

3. *If $E_s = 0$, which has been shown in [22] to be the necessary and sufficient condition for the solvability of the disturbance decoupling problem for Σ, then the ATEA state feedback law with any arbitrarily chosen F_s (subject to the constraint on the stability of A_{ss}^c) has a resulting closed-loop transfer function $H_{hw}(s, \varepsilon)$ with*

$$H_{hw}(s, \varepsilon) \to 0, \quad \text{pointwise in } s \text{ as } \varepsilon \to 0, \quad (10.2.74)$$

i.e., any ATEA state feedback control law solves the disturbance decoupling problem for Σ.

Proof. In view of the property of the ATEA design, *i.e.*, (10.2.24), and (10.2.66), we have

$$H_{hw}(s, \varepsilon) \to H_{s,hw}(s) = (C_s - D_sF_s)(sI - A_{ss} + B_sF_s)^{-1}E_s, \quad (10.2.75)$$

pointwise in s as $\varepsilon \to 0$. Then, it follows from the well-known results (see, *e.g.*, [50] and [120]) that the state feedback law with a gain matrix given in (10.2.67) has a resulting closed-loop transfer function $H_{s,hw}(s)$ with

$$\|H_{s,hw}\|_2 = \gamma_2^* \quad \Rightarrow \quad \|H_{hw}(s, \varepsilon)\|_2 \to \gamma_2^*, \text{ as } \varepsilon \to 0, \quad (10.2.76)$$

and the state feedback law with a gain matrix given in (10.2.71) has a resulting closed-loop transfer function $H_{s,hw}(s)$ with

$$\|H_{s,hw}\|_\infty < \gamma, \quad \Rightarrow \quad \|H_{hw}(s, \varepsilon)\|_\infty < \gamma, \text{ for sufficiently small } \varepsilon, \quad (10.2.77)$$

and lastly, any ATEA state feedback gain has a resulting closed-loop transfer function $H_{s,hw}(s) = 0$ provided that $E_s = 0$, which implies that $H_{hw}(s, \varepsilon) \to 0$ pointwise in s as $\varepsilon \to 0$.

Finally, we note that both Riccati equations (10.2.68) and (10.2.72) have a unique solution $P_s > 0$ as the subsystem (A_{ss}, B_s, C_s, D_s) is left invertible with no infinite zeros and has no invariant zeros in $\mathbb{C}^0 \cup \mathbb{C}^-$. This completes the proof of Theorem 10.2.2. ∎

We next illustrate the above results in the following example.

Example 10.2.1. Consider a given system (10.2.61) with

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right], \quad E = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}\right], \tag{10.2.78}$$

and

$$C = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{10.2.79}$$

The quadruple (A, B, C, D) is already in the form of the special coordinate basis presented in Chapter 5. It is invertible and hence its associated $\mathcal{X}_\mathrm{b}$ and $\mathcal{X}_\mathrm{c}$ are nonexistent. It has two unstable invariant zeros both at $s = 1$ and two infinite zeros of orders 1 and 2, respectively. Moreover, we have

$$A_\mathrm{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_\mathrm{s} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad E_\mathrm{s} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$C_\mathrm{s} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_\mathrm{s} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $E_\mathrm{s} \neq 0$, the disturbance decoupling problem for the given system is not solvable. We will thus focus on solving the H_2 and H_∞ suboptimal control problems for the system. Following the construction procedures of the ATEA algorithm in the previous section, we obtain a state feedback

$$F(\varepsilon) = -\begin{bmatrix} F_{\mathrm{s}11}/\varepsilon + 1 & F_{\mathrm{s}12}/\varepsilon + 1 & 1/\varepsilon + 1 & 1 & 1 \\ 2F_{\mathrm{s}21}/\varepsilon^2 + 1 & 2F_{\mathrm{s}22}/\varepsilon^2 + 1 & 1 & 2/\varepsilon^2 + 1 & 2/\varepsilon + 1 \end{bmatrix}, \tag{10.2.80}$$

where

$$F_\mathrm{s} = \begin{bmatrix} F_{\mathrm{s}11} & F_{\mathrm{s}12} \\ F_{\mathrm{s}21} & F_{\mathrm{s}22} \end{bmatrix} \tag{10.2.81}$$

is to be selected to solve either the H_2 or H_∞ control problem. The closed-loop eigenvalues of $A + BF$ are asymptotically placed at $\lambda(A_\mathrm{ss} - B_\mathrm{s} F_\mathrm{s})$, $-1/\varepsilon$ and $-1/\varepsilon \pm j/\varepsilon$, respectively.

1. *H_2 Control.* Solving the H_2 algebraic Riccati equation of (10.2.68), we get

$$P_\mathrm{s} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix},$$

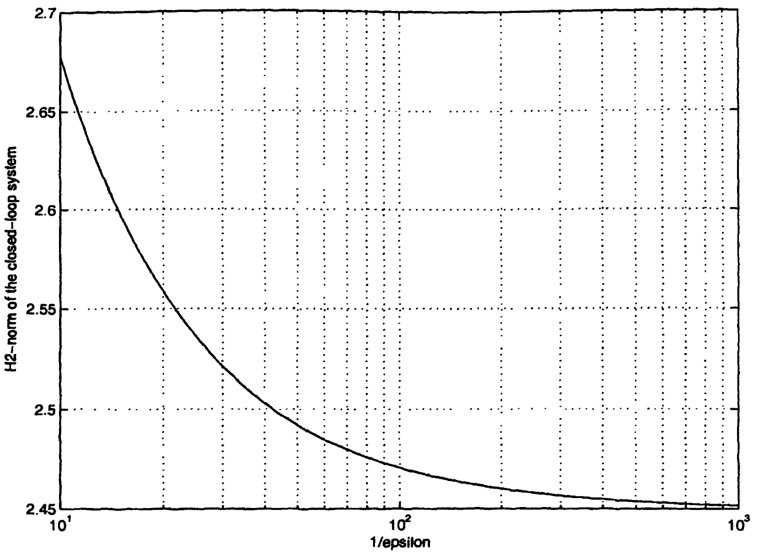


Figure 10.2.1: The H_2-norm of the closed-loop system *vs* $1/\varepsilon$.

which gives a sub-feedback gain,

$$F_s = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix},$$

and $\gamma_2^* = \sqrt{\text{trace}\,(E_s' P_s E_s)} = \sqrt{6}$. Thus, it follows from (10.2.80) and (10.2.81) that the H_2 suboptimal control law is given by $u = F(\varepsilon)x$, with

$$F(\varepsilon) = -\begin{bmatrix} 2/\varepsilon+1 & 1 & 1/\varepsilon+1 & 1 & 1 \\ -4/\varepsilon^2+1 & 4/\varepsilon^2+1 & 1 & 2/\varepsilon^2+1 & 2/\varepsilon+1 \end{bmatrix}.$$

Figure 10.2.1 shows the values of the H_2-norm of the resulting closed-loop system versus $1/\varepsilon$. Clearly, it shows that the H_2-norm of the resulting closed-loop system tends to $\gamma_2^* = \sqrt{6} = 2.4495$ as $1/\varepsilon \to \infty$.

2. H_∞ *Control.* For the case when the quadruple, (A, B, C, D), is right invertible, it was shown in Chen [22] that the H_∞ algebraic Riccati equation of (10.2.72) can be explicitly obtained by solving the two Lyapunov equations

$$A_{ss} S_s + S_s A_{ss}' = B_s B_s' \quad \text{and} \quad A_{ss} T_s + T_s A_{ss}' = E_s E_s'.$$

Solving the above Lyapunov equations, we obtain

$$S_s = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T_s = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

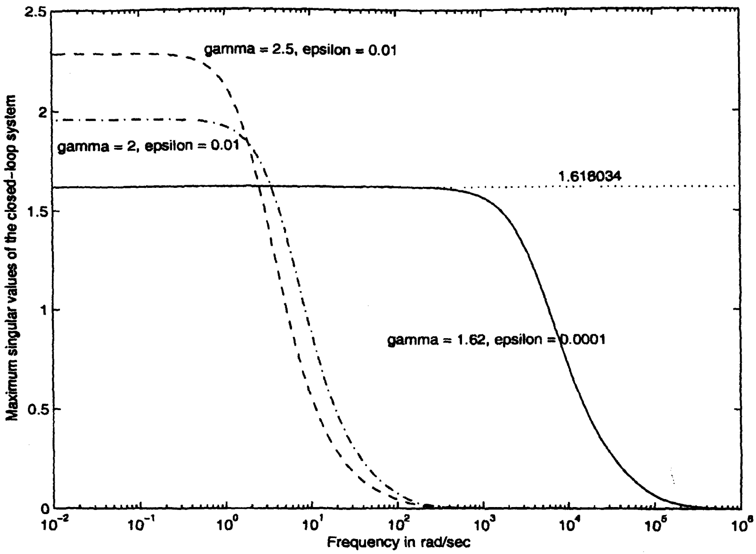


Figure 10.2.2: The maximum singular values of the closed-loop system.

It then follows from Chen [22] that

$$\gamma_\infty^* = \sqrt{\lambda_{\max}(T_s S_s^{-1})} = 1.618034,$$

and for any $\gamma > \gamma_\infty^*$, the solution to (10.2.72) can be expressed as

$$P_s = \left(S_s - T_s/\gamma^2\right)^{-1} = \frac{2\gamma^2}{\gamma^4 - 3\gamma^2 + 1} \begin{bmatrix} 2\gamma^2 - 1 & 1 - \gamma^2 \\ 1 - \gamma^2 & \gamma^2 - 2 \end{bmatrix},$$

and the sub-feedback gain F_s is given by

$$F_s = \frac{2\gamma^2}{\gamma^4 - 3\gamma^2 + 1} \begin{bmatrix} \gamma^2 & -1 \\ 1 - \gamma^2 & \gamma^2 - 2 \end{bmatrix}.$$

Hence, given a $\gamma > \gamma_\infty^*$, it follows from (10.2.80) and (10.2.81) that the control law $u = F(\gamma, \varepsilon)x$, with

$$F(\gamma,\varepsilon) = -\begin{bmatrix} \frac{2\gamma^4}{\varepsilon(\gamma^4-3\gamma^2+1)} + 1 & \frac{4\gamma^2(1-\gamma^2)}{\varepsilon^2(\gamma^4-3\gamma^2+1)} + 1 \\ \frac{-2\gamma^2}{\varepsilon(\gamma^4-3\gamma^2+1)} + 1 & \frac{4\gamma^2(\gamma^2-2)}{\varepsilon^2(\gamma^4-3\gamma^2+1)} + 1 \\ \frac{1}{\varepsilon} + 1 & 1 \\ 1 & \frac{2}{\varepsilon^2} + 1 \\ 1 & \frac{2}{\varepsilon} + 1 \end{bmatrix}',$$

is an H_∞ γ-suboptimal controller for sufficiently small ε. For illustration, we plot the maximum singular values of the resulting closed-loop system for a few different pairs of γ and ε in Figure 10.2.2. The results indeed confirm our claim.

10.3 Discrete-time Systems

We now present the eigenstructure design method for a discrete-time system characterized by

$$\begin{cases} x(k+1) = A\ x(k) + B\ u(k), \\ \quad y(k) \quad = C\ x(k) + D\ u(k), \end{cases} \tag{10.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. Again, we assume that (A, B) is stabilizable, and both B and C are of full rank. We further assume that Σ does not have any invariant zeros on the unit circle.

10.3.1 Design Procedures and Fundamental Properties

The design for discrete-time systems is much simpler in comparison to its continuous-time counterpart. It does not involve any asymptotic procedure as the stability requirement of discrete-time systems does not allow us to push the closed-loop eigenvalues to infinity. As such, the term *asymptotic* does not apply to discrete-time systems. However, for uniformity, we still call it an ATEA method for the discrete-time case.

STEP ATEA-D.1.

Follow Theorem 5.7.1 of Chapter 5 to compute nonsingular state, input and output transformations Γ_s, Γ_i and Γ_o, which transform the discrete-time system Σ of (10.3.1) into the special coordinate basis. For easy reference, we re-arrange the compact form of the special coordinate basis as follows:

$$\tilde{A} = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & L_{ad}^- C_d \\ B_c E_{ca}^- & A_{cc} & B_c E_{ca}^+ & L_{cb} C_b & L_{cd} C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & L_{bd} C_d \\ B_d E_{da}^- & B_d E_{dc} & B_d E_{da}^+ & B_d E_{db} & A_{dd} \end{bmatrix}$$

$$+\begin{bmatrix} B_{0a}^- \\ B_{0c} \\ B_{0a}^+ \\ B_{0b} \\ B_{0d} \end{bmatrix} [C_{0a}^- \quad C_{0c} \quad C_{0a}^+ \quad C_{0b} \quad C_{0d}], \tag{10.3.2}$$

$$\tilde{B} = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{10.3.3}$$

$$\tilde{C} = \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^+ & C_{0b} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}, \tag{10.3.4}$$

$$\tilde{D} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10.3.5}$$

Next, define

$$A_{ss} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b & L_{ad}^+ C_d \\ 0 & A_{bb} & L_{bd} C_d \\ B_d E_{da}^+ & B_d E_{db} & A_{dd} \end{bmatrix}, \quad B_s = \begin{bmatrix} B_{0a}^+ & 0 \\ B_{0b} & 0 \\ B_{0d} & B_d \end{bmatrix}. \tag{10.3.6}$$

STEP ATEA-D.2.

Let F_s be chosen such that $\lambda(A_{ss} - B_s F_s) \subset \mathbb{C}^{\odot}$, and partition F_s as

$$F_s = \begin{bmatrix} F_{a0}^+ & F_{b0} & F_{d0} \\ F_{ad}^+ & F_{bd} & F_{dd} \end{bmatrix}. \tag{10.3.7}$$

STEP ATEA-D.3.

Let F_c be any constant matrix subject to the constraint that

$$A_{cc}^c = A_{cc} - B_c F_c \tag{10.3.8}$$

is a stable matrix. Note that the existence of such an F_c is guaranteed by the property of the special coordinate basis, *i.e.*, (A_{cc}, B_c) is controllable.

STEP ATEA-D.4.

In this step, various gains calculated in the previous steps are put together to form a composite state feedback gain matrix F. It is given by

$$F = -\Gamma_{\rm i} \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0d} + F_{d0} \\ E_{da}^- & E_{dc} & F_{ad}^+ & F_{bd} & F_{dd} \\ E_{ca}^- & F_c & E_{ca}^+ & 0 & 0 \end{bmatrix} \Gamma_{\rm s}^{-1}. \tag{10.3.9}$$

This completes the ATEA algorithm for discrete-time systems.

We have the following theorem.

Theorem 10.3.1. *Consider the given system Σ of (10.3.1). Then, the ATEA state feedback law $u(k) = Fx(k)$ with F given as in (10.3.9) has the following properties:*

1. *The closed-loop system comprising the given system Σ and the ATEA state feedback law is asymptotically stable. Moreover, the closed-loop eigenvalues are given by*

$$\lambda(A_{aa}^-), \quad \lambda(A_{cc}^c), \quad \lambda(A_{ss}^c). \tag{10.3.10}$$

2. *Let*

$$C_{\rm s} = \Gamma_{\rm o} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix}, \quad D_{\rm s} = \Gamma_{\rm o} \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{10.3.11}$$

Then, we have

$$H(z) := (C + DF)(zI - A - BF)^{-1} = [\,0 \quad 0 \quad H_{\rm s}(z)\,]\,\Gamma_{\rm s}^{-1}, \tag{10.3.12}$$

where

$$H_{\rm s}(z) = (C_{\rm s} - D_{\rm s}F_{\rm s})(zI - A_{ss} + B_{\rm s}F_{\rm s})^{-1}. \tag{10.3.13}$$

Proof. It follows from some straightforward manipulations. ∎

10.3.2 H_2 Control, H_∞ Control and Disturbance Decoupling

As shown in its continuous-time counterpart, we demonstrate in the following that by properly choosing the sub-feedback gain matrix $F_{\rm s}$ in STEP ATEA-D.2, the ATEA design can be trivially adapted to solve the discrete-time H_2 and H_∞

control as well as the disturbance decoupling problems. More specifically, we consider a generalized discrete-time system Σ characterized by

$$\Sigma : \begin{cases} x(k+1) = A\ x(k)\ +\ B\ u(k)\ +\ E\ w(k), \\ \ y(k)\ \ =\ \ \ x(k), \\ \ h(k)\ \ = C\ x(k)\ +\ D\ u(k), \end{cases} \tag{10.3.14}$$

where as usual, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance input, $y = x$ is the measurement output, and $h \in \mathbb{R}^p$ is the controlled output of Σ. We assume that (A, B) is stabilizable and (A, B, C, D) has no invariant zeros on the unit circle. We focus on finding a state feedback law,

$$u(k) = Fx(k), \tag{10.3.15}$$

such that when it is applied to the given system (10.3.14), the resulting closed-loop system is internally stable, *i.e.*, $\lambda(A+BF) \subset \mathbb{C}^{\odot}$, and either the H_2- or H_∞-norm (see Definition of 2.4.6 of Chapter 2) of the resulting closed-loop transfer function from the disturbance input w to the controlled output h, *i.e.*,

$$H_{hw}(z) = (C + DF)(zI - A - BF)^{-1}E, \tag{10.3.16}$$

is minimized. For easy reference, we define

$$\gamma_2^* := \inf \left\{ \|H_{hw}\|_2 \;\middle|\; u(k) = Fx(k) \text{ internally stabilizes } \Sigma \right\}. \tag{10.3.17}$$

We also define

$$\gamma_\infty^* := \inf \left\{ \|H_{hw}\|_\infty \;\middle|\; u(k) = Fx(k) \text{ internally stabilizes } \Sigma \right\}. \tag{10.3.18}$$

Again, we refer interested readers to the work of Chen [22] for the computation of this γ_∞^*. For simplicity, we assume throughout this section that γ_∞^* has been determined and hence it is known.

Lastly, for the case when (A, B, C, D) has no invariant zeros on the unit circle, the disturbance decoupling problem is to find an appropriate state feedback control law of (10.3.15) such that $\lambda(A + BF) \in \mathbb{C}^{\odot}$ and $H_{hw}(z) = 0$.

Next, we assume that Γ_s, Γ_i and Γ_o are the nonsingular state, input and output transformations that transform the matrix quadruple (A, B, C, D) into the special coordinate basis as in (10.3.2)–(10.3.5). Let

$$\tilde{E} := \Gamma_s^{-1} E = \begin{bmatrix} E_a^- \\ E_c \\ E_a^+ \\ E_b \\ E_d \end{bmatrix} \quad \text{and} \quad E_s := \begin{bmatrix} E_a^+ \\ E_b \\ E_d \end{bmatrix}. \tag{10.3.19}$$

We summarize in the following theorem the solutions to the discrete-time H_2 and H_∞ control as well as disturbance decoupling problems.

Theorem 10.3.2. *Consider the generalized discrete-time system Σ characterized by (10.3.14). The ATEA design can be easily adapted to solve the H_2 and H_∞ control as well as disturbance decoupling problems for Σ. More specifically, we have*

1. *If the sub-feedback gain matrix F_s in STEP ATEA-D.2 is chosen to be*

$$F_s = (D_s' D_s + B_s' P_s B_s)^{-1}(B_s' P_s A_s + D_s' C_s), \tag{10.3.20}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$\begin{aligned} P_s = {} & A_s' P_s A_s + C_s' C_s \\ & - (A_s' P_s B_s + C_s' D_s)(D_s' D_s + B_s' P_s B_s)^{-1}(A_s' P_s B_s + C_s' D_s)', \end{aligned} \tag{10.3.21}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the property

$$\|H_{hw}\|_2 = \left\|(C + DF)(zI - A - BF)^{-1}E\right\|_2 = \gamma_2^*, \tag{10.3.22}$$

i.e., the corresponding ATEA state feedback law is an H_2 optimal control law for Σ. Furthermore,

$$\gamma_2^* = \sqrt{\operatorname{trace}(E_s' P_s E_s)}. \tag{10.3.23}$$

2. *Given a scalar $\gamma > \gamma_\infty^* \geq 0$, if F_s in STEP ATEA-D.2 is chosen to be*

$$\begin{aligned} F_s = {} & \left[B_s' P_s B_s + D_s' D_s - B_s' P_s E_s (E_s' P_s E_s - \gamma^2 I)^{-1} E_s' P_s B_s\right]^{-1} \\ & \cdot \left[B_s' P_s A_s + D_s' C_s - B_s' P_s E_s (E_s' P_s E_s - \gamma^2 I)^{-1} E_s' P_s A_s\right], \end{aligned} \tag{10.3.24}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$P_s = A_s' P_s A_s + C_s' C_s - \begin{bmatrix} B_s' P_s A_s + D_s' C_s \\ E_s' P_s A_s \end{bmatrix}' G_s^{-1} \begin{bmatrix} B_s' P_s A_s + D_s' C_s \\ E_s' P_s A_s \end{bmatrix}, \tag{10.3.25}$$

and where

$$G_s = \begin{bmatrix} D_s' D_s + B_s' P_s B_s & B_s' P_s E_s \\ E_s' P_s B_s & E_s' P_s E_s - \gamma^2 I \end{bmatrix}, \tag{10.3.26}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the property

$$\|H_{hw}\|_\infty = \left\|(C+DF)(zI-A-BF)^{-1}E\right\|_\infty < \gamma, \qquad (10.3.27)$$

i.e., the corresponding ATEA state feedback law is an H_∞ γ-suboptimal control law for Σ.

3. *If $E_s = 0$, which is the necessary and sufficient condition for the solvability of the disturbance decoupling problem for Σ, then the ATEA state feedback law with any arbitrarily chosen F_s (subject to the constraint on the stability of A_{ss}^c) has a resulting closed-loop transfer function $H_{hw}(z) = 0$, i.e., any ATEA state feedback control law solves the disturbance decoupling problem for Σ.*

Proof. It follows from Theorem 10.3.1 and the standard results of discrete-time H_2 and H_∞ control (see, e.g., [22,29,120,135]). We note that the discrete-time Riccati equations of (10.3.21) and (10.3.25) can be solved nonrecursively using the procedures given in Chen [22]. ∎

10.4 Exercises

10.1. Consider the H_∞ control for the system given in (10.2.61). Assume that (A, B, C, D) is right invertible. It follows from (10.2.23) that $C_s \equiv 0$ and thus, the corresponding H_∞-ARE (10.2.72) can be rewritten as

$$P_s A_{ss} + A_{ss}' P_s + P_s E_s E_s' P_s/\gamma^2 - P_s B_s (D_s' D_s)^{-1} B_s' P_s = 0.$$

Show that the above ARE has a positive definite solution if and only if

$$\gamma^2 > (\gamma_\infty^*)^2 = \lambda_{\max}(T_s S_s^{-1}),$$

where $S_s > 0$ and $T_s \geq 0$ are respectively the solutions of the Lyapunov equations

$$A_{ss} S_s + S_s A_{ss}' = B_s (D_s' D_s)^{-1} B_s' \quad \text{and} \quad A_{ss} T_s + T_s A_{ss}' = E_s E_s'.$$

Also, show that, for $\gamma > \gamma_\infty^*$, the positive definite solution to the H_∞ ARE is given by

$$P_s = (S_s - T_s/\gamma^2)^{-1}.$$

In fact, γ_∞^* is the infimum for the given H_∞ control problem.

10.2. Consider a continuous-time system characterized by (10.2.61) with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

and

$$C = [0 \quad 0 \quad 1 \quad 0], \quad D = 0.$$

It is simple to see that (A, B, C, D) is already in the form of the special coordinate basis with two invariant zeros at 1 and 2, and a relative degree of 2.

(a) Solve the corresponding H_2-ARE (10.2.68) for $P_s > 0$, and compute the infimum γ_2^* and an H_2 suboptimal state feedback gain matrix $F(\varepsilon)$, explicitly parameterized in ε.

(b) Determine the infimum γ_∞^*. Given a $\gamma > \gamma_\infty^*$, solve the corresponding H_∞-ARE (10.2.72) for $P_s > 0$. Also, calculate an H_∞ suboptimal state feedback gain matrix $F(\gamma, \varepsilon)$, explicitly parameterized in γ and ε.

10.3. Consider a general singular H_2 or H_∞ control problem for

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w, \\ y = \quad x, \\ h = C\,x + D\,u. \end{cases}$$

The problem can also be solved by a so-called perturbation approach (see, e.g., [160]), in which we define a new auxiliary controlled output,

$$h_{\text{aux},\varepsilon} = \begin{pmatrix} h \\ \varepsilon x \\ \varepsilon u \end{pmatrix} = C_\varepsilon x + D_\varepsilon u = \begin{bmatrix} C \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D \\ 0 \\ \varepsilon I \end{bmatrix} u.$$

Then, the H_2 suboptimal control law for the system can be computed by solving the following ε-perturbed H_2-ARE,

$$A'P_\varepsilon + P_\varepsilon A + C_\varepsilon' C_\varepsilon - (P_\varepsilon B + C_\varepsilon' D_\varepsilon)(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon) = 0,$$

for $P_\varepsilon > 0$. The H_2 suboptimal state feedback gain matrix is given by

$$F(\varepsilon) = -(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon).$$

Similarly, given a $\gamma > \gamma_\infty^*$, the H_∞ suboptimal control law for the system can be computed by solving the following ε-perturbed H_∞-ARE,

$$A'P_\varepsilon + P_\varepsilon A + C_\varepsilon' C_\varepsilon + P_\varepsilon E E' P_\varepsilon / \gamma^2 \\ - (P_\varepsilon B + C_\varepsilon' D_\varepsilon)(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon) = 0,$$

for $P_\varepsilon > 0$. The H_∞ suboptimal state feedback gain matrix is given by

$$F(\gamma, \varepsilon) = -(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon).$$

Let us now consider the system given in Exercise 10.2.

(a) Verify that the solution to the ε-perturbed H_2-ARE satisfies

$$P_\varepsilon \to \begin{bmatrix} P_s & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{as} \quad \varepsilon \to 0,$$

where P_s is the solution obtained in Part (a) of Exercise 10.2.

(b) Given a $\gamma > \gamma_\infty^*$, verify that the solution to the ε-perturbed H_∞-ARE satisfies

$$P_\varepsilon \to \begin{bmatrix} P_s & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{as} \quad \varepsilon \to 0,$$

where P_s is the solution obtained in Part (b) of Exercise 10.2.

10.4. Consider a discrete-time system characterized by (10.3.14) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

and

$$C = [0 \quad 0 \quad 1 \quad 0], \quad D = 0.$$

Note that (A, B, C, D) is in the SCB form with two invariant zeros at 0 and 0.5, and a relative degree of 2.

(a) Solve the corresponding H_2-DARE (10.3.21) for $P_s > 0$, and find the infimum γ_2^* and the H_2 optimal state feedback gain matrix F.

(b) Determine the infimum γ_∞^*. Given a $\gamma > \gamma_\infty^*$, solve the corresponding H_∞-ARE (10.3.25) for $P_s > 0$, and calculate the H_∞ suboptimal state feedback gain matrix $F(\gamma)$.

Chapter 11

Disturbance Decoupling with Static Output Feedback

11.1 Introduction

The problem of disturbance decoupling with or without internal stability by either state or measurement feedback is well known and has been extensively discussed in the literature for the last three decades. It can be stated as the problem of finding a feedback controller such that the closed-loop transfer function from the disturbance input to the controlled output is zero at all frequencies. This problem actually motivated the development of the geometric approach to linear systems, and has played a key role in a number of problems, such as decentralized control, noninteracting control, model reference tracking control, H_2 optimal control and H_∞ optimal control. The problem of disturbance decoupling with state feedback (DDP) was solved by Basile and Marro [9] and Wonham and Morse [155], and the problem of disturbance decoupling with dynamic measurement feedback (DDPM) was solved by Akashi and Imai [1] and Schumacher [126]. Furthermore, the problems of disturbance decoupling with state feedback and internal stability (DDPS) and with dynamic measurement feedback and internal stability (DDPMS) were, respectively, solved by Morse and Wonham [101], Wonham and Morse [155], Imai and Akashi [69] and Willems and Commault [153].

For the problem of disturbance decoupling with constant or static measurement feedback (DDPCM), there have only been a few results described in the literature. Hamano and Furuta [62] formulated the problem as finding a geometric subspace that only covers some special solutions. Recently, Chen [20] obtained a

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set of explicit conditions for the solvability of the DDPCM and characterized all the possible solutions for a class of systems which have a left invertible transfer function from the control input to the controlled output. A similar result for this class of systems was also reported later by Koumboulis and Tzierakis [76]. More recently, Chen *et al.* [25] has tackled the problem for more general systems and obtained some interesting results. For a system that does not satisfy the invertibility condition, [25] uses the special coordinate basis as given in Chapter 5 to obtain a reduced-order system. Then a complete characterization of all possible solutions to the DDPCM for the given system can be explicitly obtained, if the obtained reduced-order system itself satisfies the invertibility condition. The main contribution of the solutions given in [25] is that these solutions are characterized by a set of linear equations. This resolves the well-known difficulty in solving nonlinear equations associated with the DDPCM. When the invertibility condition is not satisfied, the solutions are characterized by a set of polynomial equations related to the obtained reduced-order system. This reduced-order characterization significantly simplifies the problem and reduces the computational cost in finding solutions to the DDPCM. The works of [20] and [25] form the core of this chapter.

In this chapter, we consider the DDPCM for general linear time-invariant systems Σ characterized by

$$\begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ y = C_1\ x + D_1\ w, \\ h = C_2\ x + D_2\ u + D_{22}\ w, \end{cases} \tag{11.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measured output, $w \in \mathbb{R}^q$ is the disturbance, $h \in \mathbb{R}^p$ is the controlled output, and A, B, E, C_1, D_1, C_2, D_2 and D_{22} are constant matrices of appropriate dimensions. Define Σ_P and Σ_Q respectively as the quadruples characterized by (A, B, C_2, D_2) and (A, E, C_1, D_1). Then, the DDPCM is to find a constant measurement feedback control law,

$$u = Ky, \tag{11.1.2}$$

with $K \in \mathbb{R}^{m\times\ell}$, such that the transfer function $H_{hw}(s)$ from w to h of the closed-loop system is zero, *i.e.*,

$$H_{hw}(s) = (C_2 + D_2KC_1)(sI - A - BKC_1)^{-1}(E + BKD_1) + (D_{22} + D_2KD_1) = 0. \tag{11.1.3}$$

Furthermore, the problem of disturbance decoupling with constant measurement feedback and with internal stability (DDPCMS) is to find a constant measurement feedback control in the form (11.1.2) such that (11.1.3) is satisfied and the closed-loop system state matrix $A + BKC_1$ is stable.

11.2 Left Invertible Systems

In this section, we present results on the DDPCM and DDPCMS for a class of systems that have left invertible subsystems from the control input to the controlled output. In particular, we consider a time-invariant system Σ characterized by

$$\begin{cases} \dot{x} = A\ x + B\ u + E\, w, \\ y = C_1\ x, \\ h = C_2\ x + D_2\ u, \end{cases} \tag{11.2.1}$$

with the quadruple (A, B, C_2, D_2) or Σ_P being left invertible. For this class of systems, we will be able to explicitly express the solvability conditions for the DDPCM and DDPCMS. In fact, all the solutions to the DDPCM will be explicitly constructed and characterized in terms of solutions to some linear equations. Unfortunately, we still cannot fully and explicitly parameterize the solutions to the DDPCMS.

First, we use the result of the special coordinate basis in Theorem 5.4.1 to find nonsingular transformations Γ_s, Γ_i and Γ_o for Σ_P. Let us define a set of new state, control input and controlled output coordinates as follows:

$$x = \Gamma_s \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix}, \quad u = \Gamma_i \begin{pmatrix} u_0 \\ u_d \end{pmatrix}, \quad h = \Gamma_o \begin{pmatrix} h_0 \\ h_d \\ h_b \end{pmatrix}. \tag{11.2.2}$$

The system of (11.2.1) can then be transformed into the following form Σ_s:

$$\begin{cases} \begin{pmatrix} \dot{x}_a \\ \dot{x}_b \\ \dot{x}_d \end{pmatrix} = \left(\begin{bmatrix} A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_dE_{da} & B_dE_{db} & A_{dd} \end{bmatrix} + A_0 \right) \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix} \\ \qquad\qquad + \begin{bmatrix} B_{a0} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \begin{pmatrix} u_0 \\ u_d \end{pmatrix} + \begin{bmatrix} E_a \\ E_b \\ E_d \end{bmatrix} w, \\ y \quad = [\, C_{1a} \quad C_{1b} \quad C_{1d} \,] \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix}, \\ \begin{pmatrix} h_0 \\ h_b \\ h_d \end{pmatrix} = \begin{bmatrix} C_{0a} & C_{0b} & C_{0d} \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_d \end{pmatrix}, \end{cases} \tag{11.2.3}$$

where

$$A_0 := \begin{bmatrix} B_{a0} \\ B_{b0} \\ B_{d0} \end{bmatrix} [\, C_{0a} \quad C_{0b} \quad C_{0d} \,], \quad \begin{bmatrix} E_a \\ E_b \\ E_d \end{bmatrix} := \Gamma_s^{-1} E,$$

and

$$[\, C_{1a} \quad C_{1b} \quad C_{1d} \,] := C_1 \Gamma_s.$$

It is straightforward to verify that the DDPCM for (11.2.1) is equivalent to the DDPCM for (11.2.3). Next, let Γ_a be a nonsingular transformation such that

$$\Gamma_a^{-1} A_{aa} \Gamma_a = \begin{bmatrix} A_{aa}^{cc} & A_{aa}^{c\bar{c}} \\ 0 & A_{aa}^{\bar{c}\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} E_a = \begin{bmatrix} E_a^c \\ 0 \end{bmatrix}, \tag{11.2.4}$$

$$\Gamma_a^{-1} B_{a0} = \begin{bmatrix} B_{a0}^c \\ B_{a0}^{\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} L_{ab} = \begin{bmatrix} L_{ab}^c \\ L_{ab}^{\bar{c}} \end{bmatrix}, \tag{11.2.5}$$

$$\Gamma_a^{-1} L_{ad} = \begin{bmatrix} L_{ad}^c \\ L_{ad}^{\bar{c}} \end{bmatrix}, \quad C_{0a} \Gamma_a = [\, C_{0a}^c \quad C_{0a}^{\bar{c}} \,], \tag{11.2.6}$$

and

$$E_{da} \Gamma_a = [\, E_{da}^c \quad E_{da}^{\bar{c}} \,], \quad C_{1a} \Gamma_a = [\, C_{1a}^c \quad C_{1a}^{\bar{c}} \,]. \tag{11.2.7}$$

where the pair (A_{aa}^{cc}, E_a^c) is controllable. We have the following theorem, the proof of which yields a constructive algorithm that parameterizes all solutions to the DDPCM for the given system.

Theorem 11.2.1. *Consider the system Σ of (11.2.1) with (A, B, C_2, D_2) being left invertible. The DDPCM for Σ is solvable if and only if*

$$E_b = 0, \quad E_d = 0, \quad \ker (C_{1a}^c) \subset \ker \left\{ \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} \right\}, \tag{11.2.8}$$

where E_b, E_d, E_{da}^c, C_{0a}^c and C_{1a}^c are as defined from (11.2.3) to (11.2.7).

Proof. Without loss of generality, we will assume that the given system is in the form of (11.2.3) with x_a being further decomposed into the form as in (11.2.4) and (11.2.7).

($\Rightarrow$:) If $\ker (C_{1a}^c) \subset \ker \left\{ \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} \right\}$, there exists at least one $K \in \mathbb{R}^{m \times \ell}$ such that

$$\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + K C_{1a}^c = \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + \begin{bmatrix} K_0 \\ K_d \end{bmatrix} C_{1a}^c = 0. \tag{11.2.9}$$

Also, if $E_{\rm b} = 0$ and $E_{\rm d} = 0$, it is straightforward to verify that the closed-loop system of Σ with a static measurement feedback law $u_{\rm d} = Ky$ is given by

$$\begin{aligned} H_{hw}(s) &= (C_2 + D_2KC_1)(sI - A - BKC_1)^{-1}E \\ &= \begin{bmatrix} 0 & \star & \star & \star \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & 0 & C_{\rm b} & 0 \end{bmatrix} \begin{bmatrix} (sI - A_{\rm aa}^{\rm cc})^{-1} & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix} \begin{bmatrix} E_{\rm a}^{\rm c} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= 0, \end{aligned} \tag{11.2.10}$$

where $\star$'s are some matrices of not much interest. Clearly, the control law $u = Ky$ with K satisfying (11.2.9) solves the DDPCM for Σ.

($\Leftarrow$:) Conversely, if the DDPCM is solvable for Σ, then, there exists a matrix $K \in \mathbb{R}^{m \times \ell}$ such that

$$H_{hw}(s) = (C_2 + D_2KC_1)(sI - A - BKC_1)^{-1}E = 0. \tag{11.2.11}$$

First we note that the set of all static measurement feedback laws is a sub-set of the set of all static state feedback laws. Thus, it follows from Wonham and Morse [155] or Wonham [154] that

$$\text{im}\,(E) \subset \mathcal{V}^*(A, B, C_2, D_2) = \mathcal{X}_{\rm a}.$$

It then follows from the property of the special coordinate basis of Theorem 5.4.1 that $E_{\rm b} = 0$ and $E_{\rm d} = 0$. Next, let us define

$$\mathcal{W} := \langle A + BKC_1 \,|\, \text{im}\,(E) \rangle,$$

i.e., the smallest $(A + BKC_1)$-invariant subspace containing $\text{im}\,(E)$. Thus, the equality in (11.2.11) implies that $\mathcal{W} \subset \ker\,(C_2 + D_2KC_1)$ and by definition

$$\mathcal{W} \subset \mathcal{V}^*(A, B, C_2, D_2) = \mathcal{X}_{\rm a} = \text{ span} \left\{ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Hence, there exists a similarity transformation T such that

$$T^{-1}(A + BKC_1)T = \begin{bmatrix} A^{\rm cc} & A^{\rm c\bar{c}} \\ 0 & A^{\rm \bar{c}\bar{c}} \end{bmatrix}, \quad T^{-1}E = \begin{bmatrix} E^{\rm c} \\ 0 \end{bmatrix}, \tag{11.2.12}$$

and

$$(C_2 + D_2KC_1)T = [\,0 \quad C^{\bar{\rm c}}\,], \quad \mathcal{W} = \text{ span} \left\{ T \begin{bmatrix} I \\ 0 \end{bmatrix} \right\},$$

where $(A^{\rm cc}, E^{\rm c})$ is controllable. It is now straightforward to verify that T can be chosen as the form

$$T = \begin{bmatrix} T_* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{11.2.13}$$

where T_* is of dimensions $\dim(\mathcal{X}_{\rm a}) \times \dim(\mathcal{X}_{\rm a})$. Let

$$K = \begin{bmatrix} K_0 \\ K_{\rm d} \end{bmatrix}. \tag{11.2.14}$$

We note that (11.2.12)–(11.2.14) imply that

$$T_*^{-1} \begin{bmatrix} A_{\rm aa}^{\rm cc} + B_{\rm a0}^{\rm c}(C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c}) & A_{\rm aa}^{\rm c\bar{c}} + B_{\rm a0}^{\rm c}(C_{\rm 0a}^{\bar{\rm c}} + K_0 C_{\rm 1a}^{\bar{\rm c}}) \\ B_{\rm a0}^{\bar{\rm c}}(C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c}) & A_{\rm aa}^{\bar{\rm c}\bar{\rm c}} + B_{\rm a0}^{\bar{\rm c}}(C_{\rm 0a}^{\bar{\rm c}} + K_0 C_{\rm 1a}^{\bar{\rm c}}) \end{bmatrix} T_* = \begin{bmatrix} A^{\rm cc} & A_*^{\rm c\bar{c}} \\ 0 & A_*^{\bar{\rm c}\bar{\rm c}} \end{bmatrix} \tag{11.2.15}$$

$$T_*^{-1} \begin{bmatrix} E_{\rm a}^{\rm c} \\ 0 \end{bmatrix} = \begin{bmatrix} E^{\rm c} \\ 0 \end{bmatrix}, \tag{11.2.16}$$

and

$$\begin{bmatrix} C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c} & C_{\rm 0a}^{\bar{\rm c}} + K_0 C_{\rm 1a}^{\bar{\rm c}} \\ B_{\rm d}(E_{\rm da}^{\rm c} + K_{\rm d} C_{\rm 1a}^{\rm c}) & B_{\rm d}(E_{\rm da}^{\bar{\rm c}} + K_{\rm d} C_{\rm 1a}^{\bar{\rm c}}) \end{bmatrix} T_* = [0 \quad \star], \tag{11.2.17}$$

where again $\star$ denotes a matrix of not much interest. Here we note that (11.2.15)–(11.2.17) imply that the system characterized by the matrix triple

$$\left(\begin{bmatrix} A_{\rm aa}^{\rm cc} & A_{\rm aa}^{\rm c\bar{c}} \\ 0 & A_{\rm aa}^{\bar{\rm c}\bar{\rm c}} \end{bmatrix}, \begin{bmatrix} E_{\rm a}^{\rm c} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c} & C_{\rm 0a}^{\bar{\rm c}} + K_0 C_{\rm 1a}^{\bar{\rm c}} \\ B_{\rm d}(E_{\rm da}^{\rm c} + K_{\rm d} C_{\rm 1a}^{\rm c}) & B_{\rm d}(E_{\rm da}^{\bar{\rm c}} + K_{\rm d} C_{\rm 1a}^{\bar{\rm c}}) \end{bmatrix} \right)$$

has no infinite zeros. Then the controllability of $(A_{\rm aa}^{\rm cc}, E_{\rm a}^{\rm c})$ implies that

$$C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c} = 0 \quad \text{and} \quad B_{\rm d}(E_{\rm da}^{\rm c} + K_{\rm d} C_{\rm 1a}^{\rm c}) = 0. \tag{11.2.18}$$

Since $B_{\rm d}$ is of full column rank (see Theorem 5.4.1), (11.2.18) is equivalent to

$$C_{\rm 0a}^{\rm c} + K_0 C_{\rm 1a}^{\rm c} = 0 \quad \text{and} \quad E_{\rm da}^{\rm c} + K_{\rm d} C_{\rm 1a}^{\rm c} = 0$$

or

$$\begin{bmatrix} C_{\rm 0a}^{\rm c} \\ E_{\rm da}^{\rm c} \end{bmatrix} + \begin{bmatrix} K_0 \\ K_{\rm d} \end{bmatrix} C_{\rm 1a}^{\rm c} = 0 \implies \ker(C_{\rm 1a}^{\rm c}) \subset \ker\left\{ \begin{bmatrix} C_{\rm 0a}^{\rm c} \\ E_{\rm da}^{\rm c} \end{bmatrix} \right\}.$$

This completes the proof of Theorem 11.2.1. ■

The following is an interesting and useful proposition, which follows directly from the proof of Theorem 11.2.1.

Proposition 11.2.1. *If the DDPCM for Σ of (11.2.1) with (A, B, C_2, D_2) being left invertible is solvable, then all the static measurement gain matrices that solve the DDPCM are characterized by*

$$\mathcal{K} := \left\{ \Gamma_{\rm i} K \;\middle|\; K \in \mathbb{R}^{m\times \ell} \text{ and } \begin{bmatrix} C^{\rm c}_{0\rm a} \\ E^{\rm c}_{\rm da} \end{bmatrix} + K C^{\rm c}_{1\rm a} = 0 \right\}, \tag{11.2.19}$$

where $E^{\rm c}_{\rm da}$, $C^{\rm c}_{0\rm a}$ and $C^{\rm c}_{1\rm a}$ are as defined in (11.2.7).

It is interesting to note that if $C_1 = I$, *i.e.*, all the states of Σ are available for feedback, then $C^{\rm c}_{1\rm a}$ is always of full column rank, which implies that $\ker(C^{\rm c}_{1\rm a}) = \{0\}$ and the third condition of Theorem 11.2.1, *i.e.*,

$$\ker(C^{\rm c}_{1\rm a}) \subset \ker \left\{ \begin{bmatrix} C^{\rm c}_{0\rm a} \\ E^{\rm c}_{\rm da} \end{bmatrix} \right\},$$

is automatically satisfied. Hence, the conditions in Theorem 11.2.1 are reduced to $E_{\rm b} = 0$ and $E_{\rm d} = 0$, which is equivalent to the geometric condition $\text{im}(E) \subset \mathcal{V}^*(A, B, C_2, D_2)$, *i.e.*, the well-known condition for the solvability of the disturbance decoupling problem with static state feedback (DDP).

The following theorem deals with the disturbance decoupling problem with static measurement feedback and with internal stability (DDPCMS).

Theorem 11.2.2. *Consider the system Σ of (11.2.1) with (A, B, C_2, D_2) being left invertible. The DDPCMS for Σ is solvable if and only if*

1. *The DDPCM for Σ is solvable;*

2. *The eigenvalues of $A^{\rm cc}_{\rm aa}$ are all in the open left-half plane;*

3. *There exists at least one $K \in \mathcal{K}$, where $\mathcal{K}$ is as defined in (11.2.19), such that $\tilde{A} + \tilde{B} K \tilde{C}_1$ is asymptotically stable, where*

$$\tilde{A} = \begin{bmatrix} A^{\bar{\rm c}\bar{\rm c}}_{\rm aa} & L^{\bar{\rm c}}_{\rm ab} C_{\rm b} & L^{\bar{\rm c}}_{\rm ad} C_{\rm d} \\ 0 & A_{\rm bb} & L_{\rm bd} C_{\rm d} \\ B_{\rm d} E^{\bar{\rm c}}_{\rm da} & B_{\rm d} E_{\rm db} & A_{\rm dd} \end{bmatrix} + \begin{bmatrix} B^{\bar{\rm c}}_{\rm a0} \\ B_{\rm b0} \\ B_{\rm d0} \end{bmatrix} [\, C^{\bar{\rm c}}_{0\rm a} \quad C_{0\rm b} \quad C_{0\rm d} \,],$$

and

$$\tilde{C}_1 = [\, C^{\bar{\rm c}}_{1\rm a} \quad C_{1\rm b} \quad C_{1\rm d} \,], \quad \tilde{B} = \begin{bmatrix} B^{\bar{\rm c}}_{\rm a0} & 0 \\ B_{\rm b0} & 0 \\ B_{\rm d0} & B_{\rm d} \end{bmatrix} \Gamma_{\rm i}^{-1}.$$

Here all the submatrices are as defined in (11.2.2)–(11.2.7).

Proof. Again, without loss of generality, we assume that the given system is in the form of (11.2.3) with $x_{\rm a}$ being further decomposed into the form as in (11.2.4) and (11.2.7).

($\Rightarrow$:) If the DDPCMS for Σ is solvable, then the DDPCM for Σ is also solvable. It follows from Proposition 11.2.1 that all the solutions that solve the DDPCM for Σ is given by $\mathcal{K}$ of (11.2.19). Then, it is straightforward to verify that for any $K \in \mathcal{K}$,

$$A + BKC_1 = \Gamma_{\rm s} \begin{bmatrix} A_{\rm aa}^{\rm cc} & \star \\ 0 & \tilde{A} + \tilde{B}K\tilde{C}_1 \end{bmatrix} \Gamma_{\rm s}^{-1}. \tag{11.2.20}$$

The stability of the closed-loop system implies that $A_{\rm aa}^{\rm cc}$ must be a stable matrix, and moreover, there must exist at least one $K \in \mathcal{K}$ such that $\tilde{A} + \tilde{B}K\tilde{C}_1$ is asymptotically stable.

($\Leftarrow$:) The converse part of the theorem follows by simply reversing the above arguments. This completes the proof of Theorem 11.2.2. ■

We present in the following a numerical example that illustrates the results we have obtained in this section.

Example 11.2.1. Consider a system characterized by (11.2.1) with

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 2 & -3 & -4 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C_2 = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is simple to verify using the software toolkit of [87] that (A, B, C_2, D_2) is already in the form of the special coordinate basis as in Theorem 5.4.1. Moreover, (A, B, C_2, D_2) is left invertible with two invariant zeros at $s = -1$ and $s = -2$, respectively, and two infinite zeros of order 1 and 2. Also, $E_{\rm b} = 0$ and $E_{\rm d} = 0$,

$$A_{\rm aa} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad E_{\rm a} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

and

$$C_{0\mathrm{a}} = [1 \quad 1], \quad E_{\mathrm{da}} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad C_{1\mathrm{a}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is straightforward to see that $(A_{\mathrm{aa}}, E_{\mathrm{a}})$ is controllable and

$$\ker \left\{ \begin{bmatrix} C_{0\mathrm{a}} \\ E_{\mathrm{da}} \end{bmatrix} \right\} = \ker (C_{1\mathrm{a}}).$$

By Theorem 11.2.1, the DDPCM for this system is solvable. It follows from Proposition 11.2.1 that all the static measurement gain matrices that solve the DDPCM for the given system are characterized by

$$\mathcal{K} = \left\{ \begin{bmatrix} -1 & k_0 \\ -1 & k_1 \\ -2 & k_2 \end{bmatrix} \middle| \; k_0 \in \mathbb{R}, \; k_1 \in \mathbb{R} \text{ and } k_2 \in \mathbb{R} \right\},$$

i.e., any $u = Ky$ with $K \in \mathcal{K}$ solves the DDPCM for the given system and any K such that $u = Ky$ solves the DDPCM for the given system must belong to $\mathcal{K}$.

Next, it is easy to observe that

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ -3 & -4 & -5 & 6 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{C}_1 = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

After a few iterations, we find that the static measurement feedback gain

$$K = \begin{bmatrix} -1 & 9 \\ -1 & -15 \\ -2 & -20 \end{bmatrix},$$

achieves complete disturbance decoupling for Σ and guarantees the internal stability of the closed-loop system. The closed-loop poles of $A + BKC_1$ are actually located at -1, -2, -11.276, -4.8372, and $-0.9434 \pm j1.0786$. Hence, the DDPCMS for Σ is solved.

11.3 General Multivariable Systems

In this section, we tackle the DDPCM for general systems. We will first present some necessary conditions for the solvability of the problem.

Theorem 11.3.1. *Consider the given system Σ of (11.1.1). If the DDPCM for Σ is solvable, then Σ must satisfy the following conditions:*

1. $D_{22} + D_2 S D_1 = 0$, *where* $S := -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$;

2. $\operatorname{im}(E + BSD_1) \subseteq \mathcal{V}^*(\Sigma_P) + B\ker(D_2)$;

3. $\ker(C_2 + D_2 S C_1) \supseteq \mathcal{S}^*(\Sigma_Q) \cap C_1^{-1}\{\operatorname{im}(D_1)\}$; *and*

4. $\mathcal{S}^*(\Sigma_Q) \subseteq \mathcal{V}^*(\Sigma_P)$.

Note that the subspaces $\mathcal{V}^*$ *and* $\mathcal{S}^*$ *were defined earlier in Chapter 3.*

Proof. Firstly, if the DDPCM for Σ is solvable, then it obviously follows from (11.1.3) that $D_{22} + D_2 S D_1 = 0$ with $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$. Next, applying a pre-output feedback law

$$u = Sy + v \tag{11.3.1}$$

to the given system (11.1.1), we obtain a new system,

$$\begin{cases} \dot{x} = (A + BSC_1)\, x + B\, v + (E + BSD_1)\, w, \\ y = C_1\, x + D_1\, w, \\ h = (C_2 + D_2 S C_1)\, x + D_2\, v + 0\, w. \end{cases} \tag{11.3.2}$$

Then, following the result of [136], one can show that the problem of disturbance decoupling without stability and with general proper dynamic measurement feedback, *i.e.*,

$$\begin{cases} \dot{x}_{\rm cmp} = A_{\rm cmp}\, x_{\rm cmp} + B_{\rm cmp}\, y, \\ v = C_{\rm cmp}\, x_{\rm cmp} + N\, y, \end{cases} \tag{11.3.3}$$

for the above system (11.3.2) is solvable if and only if the following conditions are satisfied:

1. $\operatorname{im}(E + BSD_1) \subseteq \mathcal{V}^*(\tilde{\Sigma}_P) + B\ker(D_2)$,

2. $\ker(C_2 + D_2 S C_1) \supseteq \mathcal{S}^*(\tilde{\Sigma}_Q) \cap C_1^{-1}\{\operatorname{im}(D_1)\}$, and

3. $\mathcal{S}^*(\tilde{\Sigma}_Q) \subseteq \mathcal{V}^*(\tilde{\Sigma}_P)$,

where $\tilde{\Sigma}_P$ and $\tilde{\Sigma}_Q$ are characterized by $(A + BSC_1, B, C_2 + D_2 S C_1, D_2)$ and $(A + BSC_1, E + BSD_1, C_1, D_1)$, respectively. It was shown in Chapter 3, i.e., Lemmas 3.8.1 and 3.8.2, that both $\mathcal{V}^*$ and $\mathcal{S}^*$ are invariant under any state feedback and output injection. Thus, we have that $\mathcal{V}^*(\Sigma_P) = \mathcal{V}^*(\tilde{\Sigma}_P)$, $\mathcal{S}^*(\Sigma_Q) = \mathcal{S}^*(\tilde{\Sigma}_Q)$, and hence the above three conditions are equivalent to Conditions 2–4 of Theorem 11.3.1. Since the constant measurement feedback is a special case of the general dynamic measurement feedback, it is clear that Conditions 1–4 are necessary for the solvability of the DDPCM for the given Σ. Here we note that for the case when $D_{22} = 0$, the first condition of Theorem 11.3.1 is automatically satisfied and $S = 0$. ■

Theorem 11.3.2. *Consider the given system Σ of (11.1.1). Let X and Y be any full rank constant matrices such that* $\ker(X) = \mathcal{V}^*(\Sigma_{\rm P})$ *and* $\operatorname{im}(Y) = \mathcal{S}^*(\Sigma_{\rm Q})$. *If the DDPCM for Σ is solvable, then the following equation has at least one solution N,*

$$\begin{bmatrix} XB \\ D_2 \end{bmatrix} N [\, C_1 Y \quad D_1 \,] + \begin{bmatrix} XAY & XE \\ C_2 Y & D_{22} \end{bmatrix} = 0. \tag{11.3.4}$$

Let $\mathcal{N}$ be the set of all the solutions of (11.3.4). Then, any constant measurement feedback law $u = Ky$, which solves the DDPCM for Σ, satisfies $K \in \mathcal{N}$, i.e., K is a solution of (11.3.4).

Proof. If the DDPCM for the system of (11.1.1) is solvable, Conditions 1–4 of Theorem 11.3.1 must be satisfied. Utilizing the results of [136], one can show that Conditions 1–4 of Theorem 11.3.1 are equivalent to the following conditions: $\mathcal{S}^*(\Sigma_{\rm Q}) \subseteq \mathcal{V}^*(\Sigma_{\rm P})$ and there exists a matrix N such that

$$\left(\begin{bmatrix} A & E \\ C_2 & D_{22} \end{bmatrix} + \begin{bmatrix} B \\ D_2 \end{bmatrix} N [\, C_1 \quad D_1 \,] \right) \left(\mathcal{S}^*(\Sigma_{\rm Q}) \oplus \mathbb{R}^q \right) \subseteq \left(\mathcal{V}^*(\Sigma_{\rm P}) \oplus \{0\} \right), \tag{11.3.5}$$

which is equivalent to the existence of a solution N to the equation (11.3.4). Moreover, any dynamic measurement feedback law

$$\begin{cases} \dot{x}_{\rm cmp} = A_{\rm cmp}\, x_{\rm cmp} + B_{\rm cmp}\, y, \\ \quad u \;\; = C_{\rm cmp}\, x_{\rm cmp} + \;\; N \;\; y, \end{cases} \tag{11.3.6}$$

which solves the problem of disturbance decoupling without stability for Σ, must have its direct feedthrough matrix N satisfying condition (11.3.5). Since any static measurement feedback law $u = Ky$ can be re-written as

$$\begin{cases} \dot{x}_{\rm cmp} = \star\, x_{\rm cmp} + \star\; y, \\ \quad u \;\; = 0\, x_{\rm cmp} + K\, y, \end{cases} \tag{11.3.7}$$

where $\star$'s are some matrices of not much interest, hence if $u = Ky$ solves the DDPCM for Σ, then $K \in \mathcal{N}$.

Finally, we would like to note that for any $N \in \mathcal{N}$, we have

$$(A + BNC_1)\mathcal{S}^*(\Sigma_{\rm Q}) \subseteq \mathcal{V}^*(\Sigma_{\rm P}), \quad D_{22} + D_2 N D_1 = 0,$$

$\operatorname{im}(E + BND_1) \subseteq \mathcal{V}^*(\Sigma_P)$ and $(C_2 + D_2NC_1)\mathcal{S}^*(\Sigma_Q) = \{0\}$. This completes the proof of Theorem 11.3.2. ■

Theorems 11.3.1 and 11.3.2 give necessary conditions for the existence of solutions to the DDPCM. We now use these results to present a necessary and sufficient condition for the solvability of the DDPCM and the DDPCMS for a special class of systems.

Corollary 11.3.1. *Consider the given system Σ of (11.1.1). Assume that both $[\,C_1 \;\; D_1\,]$ and $[\,B' \;\; D_2'\,]$ are of full rank, Σ_P is left invertible and Σ_Q is right invertible. Then, the problem of disturbance decoupling with constant measurement feedback (DDPCM) for Σ is solvable if and only if*

$$(C_2 + D_2NC_1)(sI - A - BNC_1)^{-1}(E + BND_1) + (D_{22} + D_2ND_1) \equiv 0, \tag{11.3.8}$$

where N is a known constant matrix and is given by

$$N = -\left(B'X'XB + D_2'D_2\right)^{-1}[\,B'X' \;\; D_2'\,]\begin{bmatrix} XAY & XE \\ C_2Y & D_{22} \end{bmatrix}\begin{bmatrix} Y'C_1' \\ D_1' \end{bmatrix} \times\left(C_1YY'C_1' + D_1D_1'\right)^{-1}. \tag{11.3.9}$$

Also, the DDPCMS for the given system Σ is solvable if and only if (11.3.8) holds, and $A + BNC_1$ is stable. Furthermore, both solutions to the DDPCM and DDPCMS for the given Σ, if existent, are identical. They are uniquely given by $u = Ny$.

Proof. We first show the DDPCM case. If condition (11.3.8) holds, then it is simple to see that $u = Ny$ solves the DDPCM for Σ. Conversely, if the DDPCM for Σ is solvable, then by Theorem 11.3.2 there exists a nonempty set $\mathcal{N}$ and any constant measurement feedback law $u = Ky$, which solves the DDPCM for Σ, must satisfy $K \in \mathcal{N}$. Under the conditions that Σ_P is left invertible and $[\,B' \;\; D_2'\,]$ is of full rank, it follows from the structural decomposition of Chapter 5 that there exist nonsingular transformations Γ_s, Γ_o and Γ_i such that

$$\Gamma_s^{-1}B\Gamma_i = \begin{bmatrix} B_{0a} & 0 \\ B_{0b} & 0 \\ B_{0d} & B_d \end{bmatrix}, \quad \Gamma_o^{-1}D_2\Gamma_i = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{11.3.10}$$

and

$$\mathcal{V}^*(\Sigma_P) = \operatorname{im}\left\{\Gamma_s \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}\right\}. \tag{11.3.11}$$

Thus, we have

$$X=\begin{bmatrix}0 & I & 0\\ 0 & 0 & I\end{bmatrix}\Gamma_{\rm s}^{-1} \quad \text{and} \quad \begin{bmatrix}XB\\ D_2\end{bmatrix}=\begin{bmatrix}B_{0\rm b} & 0\\ B_{0\rm d} & B_{\rm d}\\ D_{0*} & 0\end{bmatrix}\Gamma_{\rm i}^{-1}, \tag{11.3.12}$$

where both $B_{\rm d}$ and D_{0*} are of full rank. Hence, $[\,B'X' \quad D_2'\,]'$ are of full column rank. Similarly, under the conditions that $\Sigma_{\rm Q}$ is right invertible and $[\,C_1 \quad D_1\,]$ is of full rank, one can show that $[\,C_1Y \quad D_1\,]$ is of maximal row rank. Re-write equation (11.3.4) as

$$\begin{bmatrix}XB\\ D_2\end{bmatrix}N[\,C_1Y \quad D_1\,]=-\begin{bmatrix}XAY & XE\\ C_2Y & D_{22}\end{bmatrix}. \tag{11.3.13}$$

It is simple to see that the above equation has at least one solution. Moreover, it is unique and is given by (11.3.9) and thus $\mathcal{N}$ is a singleton. Hence, we have $K=N$ and condition (11.3.8) holds.

The result of the DDPCMS is quite obvious as the stability of the closed-loop system is governed by the eigenvalues of $A+BNC_1$. ■

Next, we will proceed to tackle the case when a given system does not satisfy the conditions posed in Corollary 11.3.1. We will partition the given system Σ of (11.1.1) into subsystems using the structural decomposition technique (the special coordinate basis) of Chapter 5. From now on, we will assume that the necessary conditions for the solvability of the DDPCM in Theorem 11.3.1 are satisfied. The following is a step-by-step algorithm.

STEP DDPCM-R.O.S.1.

Compute

$$N=-(B'X'XB+D_2'D_2)^{\dagger}[\,B'X' \quad D_2'\,]\begin{bmatrix}XAY & XE\\ C_2Y & D_{22}\end{bmatrix}\begin{bmatrix}Y'C_1'\\ D_1'\end{bmatrix}(C_1YY'C_1'+D_1D_1')^{\dagger}, \tag{11.3.14}$$

and then apply a pre-output feedback $u=Ny+v$ to the given system Σ to yield the new system

$$\begin{cases}\dot{x}=(A+BNC_1)\,x+B\,v+(E+BND_1)\,w,\\ y=C_1\,x+D_1\,w,\\ h=(C_2+D_2NC_1)\,x+D_2\,v+0\,w.\end{cases}$$

Furthermore, we have $\operatorname{im}(E+BND_1)\subseteq\mathcal{V}^*(\Sigma_{\rm P})$.

STEP DDPCM-R.O.S.2.

Find a nonsingular transformation $\Gamma_{\rm m}$ such that

$$y = \Gamma_{\rm m} y_{\rm m} = \Gamma_{\rm m} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

and

$$C_{1\rm m} := \Gamma_{\rm m}^{-1} C_1 = \begin{bmatrix} C_{1,0} \\ C_{1,1} \end{bmatrix}, \quad D_{1\rm m} := \Gamma_{\rm m}^{-1} D_1 = \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix},$$

where $D_{1,0}$ is of maximal row rank.

STEP DDPCM-R.O.S.3.

Utilize the special coordinate basis of Chapter 5 to find the nonsingular transformations $\Gamma_{\rm s}$, $\Gamma_{\rm i}$ and $\Gamma_{\rm o}$, *i.e.*, let

$$x = \Gamma_{\rm s} \begin{pmatrix} x_{\rm c} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix}, \quad v = \Gamma_{\rm i} \begin{pmatrix} v_0 \\ v_{\rm d} \\ v_{\rm c} \\ v_* \end{pmatrix}, \quad h = \Gamma_{\rm o} \begin{pmatrix} h_0 \\ h_{\rm d} \\ h_{\rm b} \\ h_* \end{pmatrix}, \tag{11.3.15}$$

which yields the transformed system

$$\left\{ \begin{aligned} \begin{pmatrix} \dot{x}_{\rm c} \\ \dot{x}_{\rm a} \\ \dot{x}_{\rm b} \\ \dot{x}_{\rm d} \end{pmatrix} &= \left(\begin{bmatrix} A_{\rm cc} & B_{\rm c}E_{\rm ca} & L_{\rm cb}C_{\rm b} & L_{\rm cd}C_{\rm d} \\ 0 & A_{\rm aa} & L_{\rm ab}C_{\rm b} & L_{\rm ad}C_{\rm d} \\ 0 & 0 & A_{\rm bb} & L_{\rm bd}C_{\rm d} \\ B_{\rm d}E_{\rm dc} & B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & A_{\rm dd} \end{bmatrix} + A_0 \right) \begin{pmatrix} x_{\rm c} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix} \\ &\qquad + \begin{bmatrix} B_{\rm c0} & 0 & B_{\rm c} & 0 \\ B_{\rm a0} & 0 & 0 & 0 \\ B_{\rm b0} & 0 & 0 & 0 \\ B_{\rm d0} & B_{\rm d} & 0 & 0 \end{bmatrix} \begin{pmatrix} v_0 \\ v_{\rm d} \\ v_{\rm c} \\ v_* \end{pmatrix} + \begin{bmatrix} E_{\rm c} \\ E_{\rm a} \\ E_{\rm b} \\ E_{\rm d} \end{bmatrix} w, \\ y_{\rm m} &= \begin{bmatrix} C_{1,0\rm c} & C_{1,0\rm a} & C_{1,0\rm b} & C_{1,0\rm d} \\ C_{1,1\rm c} & C_{1,1\rm a} & C_{1,1\rm b} & C_{1,1\rm d} \end{bmatrix} \begin{pmatrix} x_{\rm c} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix} + \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix} w, \\ \begin{pmatrix} h_0 \\ h_{\rm b} \\ h_{\rm d} \\ h_* \end{pmatrix} &= \begin{bmatrix} C_{0\rm c} & C_{0\rm a} & C_{0\rm b} & C_{0\rm d} \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & 0 & C_{\rm b} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_{\rm c} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm d} \end{pmatrix} \\ &\qquad + \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_0 \\ v_{\rm d} \\ v_{\rm c} \\ v_* \end{pmatrix}, \end{aligned} \right. \tag{11.3.16}$$

where

$$A_0 := \begin{bmatrix} B_{c0} \\ B_{a0} \\ B_{b0} \\ B_{d0} \end{bmatrix} [\, C_{0c} \quad C_{0a} \quad C_{0b} \quad C_{0d} \,], \quad \begin{bmatrix} E_c \\ E_a \\ E_b \\ E_d \end{bmatrix} := \Gamma_s^{-1}(E + BND_1),$$

and

$$[\, C_{1c} \quad C_{1a} \quad C_{1b} \quad C_{1d} \,] := C_1 \Gamma_s. \tag{11.3.17}$$

Let Σ_N be characterized by $(A + BNC_1, B, C_2 + DNC_1, D_2)$. We further note that the decomposition in (11.3.16) has the following properties: The pair (A_{cc}, B_c) is controllable and Σ_N is left invertible if x_c is nonexistent; (A_{bb}, C_b) is observable and Σ_N is right invertible if x_b is nonexistent; Σ_N is invertible if both x_c and x_b are nonexistent; (A_{dd}, B_d, C_d) is square invertible and is free of invariant zeros; the eigenvalues of A_{aa} are the invariant zeros of Σ_N; and finally,

$$\mathcal{V}^*(\Sigma_N) = \operatorname{im}\left\{ \Gamma_s \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \text{ and } \mathcal{S}^*(\Sigma_N) = \operatorname{im}\left\{ \Gamma_s \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \right\}.$$

It is simple to verify that under the conditions of Theorem 11.3.1, we have $E_b = 0$, $E_d = 0$. Moreover, the DDPCM for (11.1.1) is equivalent to that for the transformed system (11.3.16).

STEP DDPCM-R.O.S.4.

Let Γ_a be a nonsingular transformation such that

$$\Gamma_a^{-1} A_{aa} \Gamma_a = \begin{bmatrix} A_{aa}^{cc} & A_{aa}^{c\bar{c}} \\ 0 & A_{aa}^{\bar{c}\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} E_a = \begin{bmatrix} E_a^c \\ 0 \end{bmatrix}, \tag{11.3.18}$$

$$\Gamma_a^{-1} B_{a0} = \begin{bmatrix} B_{a0}^c \\ B_{a0}^{\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} L_{ab} = \begin{bmatrix} L_{ab}^c \\ L_{ab}^{\bar{c}} \end{bmatrix}, \tag{11.3.19}$$

$$\Gamma_a^{-1} L_{ad} = \begin{bmatrix} L_{ad}^c \\ L_{ad}^{\bar{c}} \end{bmatrix}, \quad \begin{bmatrix} C_{1,0a} \\ C_{1,1a} \end{bmatrix} \Gamma_a = \begin{bmatrix} C_{1,0a}^c & C_{1,0a}^{\bar{c}} \\ C_{1,1a}^c & C_{1,1a}^{\bar{c}} \end{bmatrix}, \tag{11.3.20}$$

and

$$C_{0a} \Gamma_a = [\, C_{0a}^c \quad C_{0a}^{\bar{c}} \,], \tag{11.3.21}$$

$$E_{da} \Gamma_a = [\, E_{da}^c \quad E_{da}^{\bar{c}} \,], \quad E_{ca} \Gamma_a = [\, E_{ca}^c \quad E_{ca}^{\bar{c}} \,], \tag{11.3.22}$$

where (A_{aa}^{cc}, E_a^c) is controllable.

STEP DDPCM-R.O.S.5.

Define a reduced-order auxiliary system Σ_{R} as follows:

$$\begin{cases} \dot{x}_{\mathrm{R}} = A_{\mathrm{R}}\ x_{\mathrm{R}} + B_{\mathrm{R}}\ u_{\mathrm{R}} + E_{\mathrm{R}}\ w, \\ y_{\mathrm{R}} = C_{1\mathrm{R}}\ x_{\mathrm{R}} \qquad\qquad + D_{1\mathrm{R}}\ w, \\ h_{\mathrm{R}} = C_{2\mathrm{R}}\ x_{\mathrm{R}} + D_{2\mathrm{R}}\ u_{\mathrm{R}}, \end{cases} \tag{11.3.23}$$

where

$$A_{\mathrm{R}} = \begin{bmatrix} A_{\mathrm{cc}} & B_{\mathrm{c}} E_{\mathrm{ca}}^{\mathrm{c}} \\ 0 & A_{\mathrm{aa}}^{\mathrm{cc}} \end{bmatrix} + \begin{bmatrix} B_{\mathrm{c0}} \\ B_{\mathrm{a0}}^{\mathrm{c}} \end{bmatrix} \begin{bmatrix} C_{\mathrm{0c}} & C_{\mathrm{0a}}^{\mathrm{c}} \end{bmatrix}, \quad E_{\mathrm{R}} = \begin{bmatrix} E_{\mathrm{c}} \\ E_{\mathrm{a}}^{\mathrm{c}} \end{bmatrix},$$

$$B_{\mathrm{R}} = \begin{bmatrix} B_{\mathrm{c0}} & 0 & B_{\mathrm{c}} & 0 \\ B_{\mathrm{a0}}^{\mathrm{c}} & 0 & 0 & 0 \end{bmatrix} \Gamma_{\mathrm{i}}^{-1}, \quad D_{2\mathrm{R}} = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & I_{m_{\mathrm{d}}} & 0 & 0 \end{bmatrix} \Gamma_{\mathrm{i}}^{-1},$$

and

$$C_{1\mathrm{R}} = \Gamma_{\mathrm{m}} \begin{bmatrix} C_{1,0\mathrm{c}} & C_{1,0\mathrm{a}}^{\mathrm{c}} \\ C_{1,1\mathrm{c}} & C_{1,1\mathrm{a}}^{\mathrm{c}} \end{bmatrix}, \quad D_{1\mathrm{R}} = \Gamma_{\mathrm{m}} \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix}, \quad C_{2\mathrm{R}} = \begin{bmatrix} C_{\mathrm{0c}} & C_{\mathrm{0a}}^{\mathrm{c}} \\ E_{\mathrm{dc}} & E_{\mathrm{da}}^{\mathrm{c}} \end{bmatrix}.$$

This completes the algorithm.

Let n_{x} be the dimension of the space spanned by x_{R}. Apparently, n_{x} can in general be considerably smaller than n, the dimension of the original system (11.1.1). Furthermore, it is simple to see that $(A_{\mathrm{R}}, B_{\mathrm{R}}, C_{2\mathrm{R}}, D_{2\mathrm{R}})$ is right invertible without infinite zeros. For the given system Σ of (11.1.1) and the reduced-order system Σ_{R} of (11.3.23), we define

$$\mathcal{K} := \left\{ K \,\middle|\, u = Ky \text{ solves the DDPCM for } \Sigma \right\}, \tag{11.3.24}$$

and

$$\mathcal{K}_{\mathrm{R}} := \left\{ K_{\mathrm{R}} \,\middle|\, u_{\mathrm{R}} = K_{\mathrm{R}} y_{\mathrm{R}} \text{ solves the DDPCM for } \Sigma_{\mathrm{R}} \right\}. \tag{11.3.25}$$

We now establish an equivalence between the DDPCM for the given system Σ in (11.1.1) and that for the reduced-order system Σ_{R} in the following theorem.

Theorem 11.3.3. *Consider the given system Σ of (11.1.1). Assume that Conditions 1–4 of Theorem 11.3.1 are satisfied. Then, we have*

$$\mathcal{K} = \left\{ K_{\mathrm{R}} + N \,\middle|\, K_{\mathrm{R}} \in \mathcal{K}_{\mathrm{R}} \right\}, \tag{11.3.26}$$

where N is given by (11.3.14). Thus, the solvability of the DDPCM for Σ of (11.1.1) and that for Σ_{R} of (11.3.23) are equivalent.

Proof. Without loss of generality but for simplicity of presentation, we assume that the nonsingular transformations $\Gamma_s = I, \Gamma_i = I, \Gamma_o = I$ and $\Gamma_m = I$ as all of them do not affect the solutions to the DDPCM at all. We will prove the theorem in two stages:

STAGE 1. Assume that the feedback $u_R = K_R y_R$ is a solution for the DDPCM of the system Σ_R. Let K_R be partitioned as

$$K_R = \begin{bmatrix} K_{00} & K_{01} \\ K_{d0} & K_{d1} \\ K_{c0} & K_{c1} \\ K_{*0} & K_{*1} \end{bmatrix}.$$

Then, $K_R \in \mathcal{K}_R$ implies that $D_{2R}K_R D_{1R} = 0$, which implies that $K_{00} = 0$ and $K_{d0} = 0$. Thus, we have

$$\begin{aligned} A_{RX} &:= A_R + B_R K_R C_{1R} \\ &= \begin{bmatrix} X_1 & X_2 \\ B_{a0}^c(C_{0c} + K_{01}C_{1,1c}) & A_{aa}^{cc} + B_{a0}^c(C_{0a}^c + K_{01}C_{1,1a}^c) \end{bmatrix}, \end{aligned} \tag{11.3.27}$$

where

$$X_1 := A_{cc} + B_c(K_{c0}C_{1,0c} + K_{c1}C_{1,1c}) + B_{c0}(C_{0c} + K_{01}C_{1,1c}),$$

and

$$X_2 := B_{c0}(C_{0a}^c + K_{01}C_{1,1a}^c) + B_c(E_{ca}^c + K_{c0}C_{1,0a}^c + K_{c1}C_{1,1a}^c),$$

together with

$$E_{RX} := E_R + B_R K_R D_{1R} = \begin{bmatrix} E_c + B_c K_{c0} D_{1,0} \\ 0 \end{bmatrix}, \tag{11.3.28}$$

and

$$\begin{aligned} C_{RX} := \begin{bmatrix} C_{RX0} \\ C_{RXd} \end{bmatrix} &:= C_{2R} + D_{2R}K_R C_{1R} \\ &= \begin{bmatrix} C_{0c} + K_{01}C_{1,1c} & C_{0a}^c + K_{01}C_{1,1a}^c \\ E_{dc} + K_{d1}C_{1,1c} & E_{da}^c + K_{d1}C_{1,1a}^c \end{bmatrix}. \end{aligned} \tag{11.3.29}$$

Note that $K_R \in \mathcal{K}_R$ implies that

$$C_{RX}(sI - A_{RX})^{-1}E_{RX} \equiv 0,$$

for all s, or equivalently,

$$C_{\rm RX} A_{\rm RX}^i E_{\rm RX} = \begin{bmatrix} C_{\rm Rx0} \\ C_{\rm Rxd} \end{bmatrix} A_{\rm RX}^i E_{\rm R} = 0,$$

for $i = 0, 1, \dots, n-1$. Since $E_{\rm b} = 0$ and $E_{\rm d} = 0$, we have

$$E_{\rm x} =: E + BND_1 + BK_{\rm R}D_1 = \begin{bmatrix} E_{\rm RX} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For the system (11.3.16), we apply the constant measurement feedback $u = K_{\rm R}y$ to obtain

$$A_{\rm x} := A + BNC_1 + BK_{\rm R}C_1 = \begin{bmatrix} A_{\rm RX} & \star & \star & \star \\ B_{\rm a0}^{\bar{\rm c}} C_{\rm Rx0} & \star & \star & \star \\ B_{\rm b0} C_{\rm Rx0} & \star & \star & \star \\ B_{\rm d0} C_{\rm Rx0} + B_{\rm d} C_{\rm Rxd} & \star & \star & \star \end{bmatrix},$$

and

$$C_{\rm x} := C_2 + D_2NC_1 + D_2K_{\rm R}C_1 = \begin{bmatrix} C_{\rm Rx0} & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix},$$

where $\star$'s are some matrices of not much interest. It is now straightforward to verify that

$$C_{\rm x}E_{\rm x} = \begin{bmatrix} C_{\rm Rx0}E_{\rm RX} \\ 0 \end{bmatrix} = 0,$$

and

$$\begin{aligned} C_{\rm x}A_{\rm x}E_{\rm x} &= C_{\rm x} \begin{bmatrix} A_{\rm RX}E_{\rm RX} \\ B_{\rm a0}^{\bar{\rm c}} C_{\rm Rx0}E_{\rm RX} \\ B_{\rm b0} C_{\rm Rx0}E_{\rm RX} \\ B_{\rm d0} C_{\rm Rx0}E_{\rm RX} + B_{\rm d} C_{\rm Rxd}E_{\rm RX} \end{bmatrix} \\ &= C_{\rm x} \begin{bmatrix} A_{\rm RX}E_{\rm RX} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_{\rm Rx0}A_{\rm RX}E_{\rm RX} \\ 0 \end{bmatrix} = 0. \end{aligned}$$

It follows that

$$C_{\rm x}A_{\rm x}^i E_{\rm x} = 0, \quad \forall i = 1, 2, \dots, n-1.$$

Thus,

$$C_{\rm x}(sI - A_{\rm x})^{-1}E_{\rm x} \equiv 0,$$

for all s. Moreover, it is easy to check that $D_{\text{x}} := D_{22} + D_2(N + K_{\text{R}})D_1 = 0$. Hence, $u = (N + K_{\text{R}})y$ is a solution for the DDPCM of the original system Σ and $N + K_{\text{R}}$ is an element of $\mathcal{K}$ or equivalently

$$\{K_{\text{R}} + N \mid K_{\text{R}} \in \mathcal{K}_{\text{R}}\} \subseteq \mathcal{K}. \tag{11.3.30}$$

STAGE 2. Suppose that $u = Ky$ is a solution for the DDPCM of the original system Σ such that

$$H_{hw}(s) = (C_2 + D_2KC_1)(sI - A - BKC_1)^{-1}(E + BKD_1) + (D_{22} + D_2KD_1) = 0. \tag{11.3.31}$$

Clearly, we have

$$D_{22} + D_2KD_1 = 0. \tag{11.3.32}$$

Next, we proceed to define the smallest $(A + BKC_1)$-invariant subspace containing $\operatorname{im}(E + BKD_1)$ as

$$\mathcal{W} := \langle A + BKC_1 \mid \operatorname{im}(E + BKD_1)\rangle.$$

We note that this subspace $\mathcal{W}$ is well defined as both $A + BKC_1$ and $E + BKD_1$ are constant matrices. Then equations (11.3.31) and (11.3.32) imply that $\mathcal{W} \subset \ker(C_2 + D_2KC_1)$ and by definition

$$\mathcal{W} \subset \mathcal{V}^*(\Sigma_{\text{P}}) = \mathcal{V}^*(\Sigma_{\text{N}}) = \operatorname{span}\left\{\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

Hence, there exists a similarity transformation T such that

$$T^{-1}(A+BKC_1)T = \begin{bmatrix} A^{\text{cc}} & A^{\text{c}\bar{\text{c}}} \\ 0 & A^{\bar{\text{c}}\bar{\text{c}}} \end{bmatrix}, \quad T^{-1}(E+BKD_1) = \begin{bmatrix} E^{\text{c}} \\ 0 \end{bmatrix}, \tag{11.3.33}$$

and

$$(C_2 + D_2KC_1)T = [\,0 \quad C^{\bar{\text{c}}}\,], \quad \mathcal{W} = \operatorname{span}\left\{T\begin{bmatrix} I \\ 0 \end{bmatrix}\right\},$$

where $(A^{\text{cc}}, E^{\text{c}})$ is controllable. It is now straightforward to verify that T can be chosen as the form

$$T = \begin{bmatrix} T_* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{11.3.34}$$

where T_* is of dimensions $n_x \times n_x$. Next, we let $K_R := K - N$. Clearly, K_R is a solution to the DDPCM for the system in (11.3.16). We further partition the subsystem associated with x_a into the form (11.3.18)–(11.3.22), and partition K_R as

$$K_R = \begin{bmatrix} K_{00} & K_{01} \\ K_{d0} & K_{d1} \\ K_{c0} & K_{c1} \\ K_{*0} & K_{*1} \end{bmatrix}. \tag{11.3.35}$$

It is straightforward to show that $D_{22} + D_2 K D_1 = 0$ implies $K_{00} = 0$ and $\mathrm{im}\,(E + BKD_1) \subseteq \mathcal{V}^*(\Sigma_P)$ implies $K_{d0} = 0$. Thus, we have $D_{2R} K_R D_{1R} = 0$. Also, (11.3.33)–(11.3.35) imply that

$$T_*^{-1} A_{RX} T_* = \begin{bmatrix} A^{cc} & A_*^{c\bar{c}} \\ 0 & A_*^{\bar{c}\bar{c}} \end{bmatrix}, \quad T_*^{-1} E_{RX} = \begin{bmatrix} E^c \\ 0 \end{bmatrix}, \tag{11.3.36}$$

and

$$\begin{bmatrix} C_{RX0} \\ B_d C_{RXd} \end{bmatrix} T_* = \begin{bmatrix} 0 & \star \\ 0 & \star \end{bmatrix}, \tag{11.3.37}$$

where A_{RX}, E_{RX}, C_{RX0} and C_{RXd} are as defined in (11.3.27) to (11.3.29), and a $\star$ again denotes a matrix of not much interest. Since (A_{dd}, B_d, C_d) is invertible, which implies that B_d is of full column rank, (11.3.37) is equivalent to

$$C_{RX} T_* = \begin{bmatrix} C_{RX0} \\ C_{RXd} \end{bmatrix} T_* = [\,0 \quad \star\,]. \tag{11.3.38}$$

Note that (11.3.36) and (11.3.38) together yield

$$C_{RX}(sI - A_{RX})^{-1} E_{RX} \equiv 0,$$

for all s. Hence, $u_R = K_R y_R$ is a solution for the DDPCM of the reduced-order system Σ_R, which implies that

$$\mathcal{K} \subseteq \{ K_R + N \mid K_R \in \mathcal{K}_R \}. \tag{11.3.39}$$

Equations (11.3.30) and (11.3.39) imply $\mathcal{K} = \{ K_R + N \mid K_R \in \mathcal{K}_R \}$. ■

The following corollaries deal with some special cases for which we are able to obtain complete solutions for the DDPCM. Corollary 11.3.2 recovers the results that we have obtained in Section 11.2.

Corollary 11.3.2. *Consider the given system Σ of (11.1.1) with the matrix quadruple (A, B, C_2, D_2), or Σ_P, being left invertible. Then, the problem of disturbance* *decoupling with constant measurement feedback is solvable for Σ if and only if Conditions 1–4 of Theorem 11.3.1 are satisfied and*

$$\ker(C^{\rm c}_{1,1\rm a}) \subset \ker\left\{\begin{bmatrix} C^{\rm c}_{0\rm a} \\ E^{\rm c}_{\rm da}\end{bmatrix}\right\}, \tag{11.3.40}$$

and all solutions to the DDPCM for this class of systems are characterized by

$$\mathcal{K} := \left\{ \Gamma_{\rm i}\begin{bmatrix} 0 & K_{01} \\ 0 & K_{\rm d1} \\ K_{*0} & K_{*1}\end{bmatrix}\Gamma_{\rm m}^{-1} + N \;\middle|\; \begin{bmatrix} C^{\rm c}_{0\rm a} \\ E^{\rm c}_{\rm da}\end{bmatrix} + \begin{bmatrix} K_{01} \\ K_{\rm d1}\end{bmatrix} C^{\rm c}_{1,1\rm a} = 0 \right\}, \tag{11.3.41}$$

where K_{*0} and K_{*1} are any constant matrices of appropriate dimensions.

Next, recall the given system Σ in (11.1.1). We define a transposed system of Σ as

$$\begin{cases} \dot{x} = A'\,x + C_1'\,u + C_2'\,w, \\ y = B'\,x \qquad\quad + D_2'\,w, \\ h = E'\,x + D_1'\,u + D_{22}'\,w. \end{cases} \tag{11.3.42}$$

It is apparent that the DDPCM for Σ is solvable if and only if the DDPCM for the above transposed system is solvable. Furthermore, if $\Sigma_{\rm Q}$ is right invertible, then the transposed system satisfies the condition of Corollary 11.3.2. Thus, we have the following corollary.

Corollary 11.3.3. *Consider the given system Σ of (11.1.1). If $\Sigma_{\rm Q}$ is right invertible, then the set of all possible solutions to the DDPCM for Σ can be obtained by applying the result of Corollary 11.3.2 to the transposed system (11.3.42).*

From Corollary 11.3.2 and Corollary 11.3.3, we see how to solve the DDPCM for Σ when either $\Sigma_{\rm P}$ is left invertible or $\Sigma_{\rm Q}$ is right invertible or both. It is very interesting to note that the solutions can be obtained by solving a set of linear equations in the form (11.3.41). Thus, the solutions can be easily computed.

We now further tackle the case when $\Sigma_{\rm P}$ is not left invertible and $\Sigma_{\rm Q}$ is not right invertible. For this case, we use the following algorithm to obtain an irreducible reduced-order system, which considerably simplifies the solution of the DDPCM. The basic idea is as follows: It is clear from Theorem 11.3.3 that the DDPCM for the original system is equivalent to that for a much smaller dimensional auxiliary system $\Sigma_{\rm R}$, which is taken from a subset in $\mathcal{V}^*$ of the original system. We then dualize this auxiliary system and apply a similar reduction on it to obtain a new auxiliary system whose dynamical order is further reduced. We keep repeating this process until we reach a system that is irreducible. We have the following step-by-step algorithm.

STEP R.R.O.S.1.

For the given system Σ, whose Σ_P is not left invertible and Σ_Q is not right invertible, we apply STEPS DDPCM.R.O.S.1 to DDPCM.R.O.S.5 to obtain a constant matrix N and a reduced-order system Σ_R. Let $\Sigma_{\alpha,R} := \Sigma_R$ and $N_\alpha := N$ with $\alpha = 1$. Furthermore, we append a subscript α to all the matrices of $\Sigma_{\alpha,R}$.

STEP R.R.O.S.2.

For $\Sigma_{\alpha,R}$, define an auxiliary system $\Sigma^\star_{\alpha,R}$ as

$$\begin{cases} \dot{x}_{\alpha,R} = A'_{\alpha,R}\, x_{\alpha,R} + C'_{\alpha,1R}\, u_{\alpha,R} + C'_{\alpha,2R}\, w_{\alpha,R}, \\ y_{\alpha,R} = B'_{\alpha,R}\, x_{\alpha,R} + D'_{\alpha,2R}\, w_{\alpha,R}, \\ h_{\alpha,R} = E'_{\alpha,R}\, x_{\alpha,R} + D'_{\alpha,1R}\, u_{\alpha,R}. \end{cases} \tag{11.3.43}$$

If the above system $\Sigma^\star_{\alpha,R}$ does not satisfy Conditions 1–4 of Theorem 11.3.1, then the DDPCM for Σ has no solution and the procedure stops. If the above system $\Sigma^\star_{\alpha,R}$ cannot be further reduced, we let $\tilde{\alpha} := \alpha$, $\tilde{n}_x$ be the dynamical order of $\Sigma_{\alpha,R}$ and stop the algorithm. Otherwise, go to R.R.O.S.3.

STEP R.R.O.S.3.

Apply STEPS DDPCM.R.O.S.1 to DDPCM.R.O.S.5 to $\Sigma^\star_{\alpha,R}$ to find another matrix N (rename it as $N_{\alpha+1}$ for future use) and another reduced order system, say $\Sigma_{\alpha+1,R}$, characterized by

$$\begin{cases} \dot{x}_{\alpha+1,R} = A_{\alpha+1,R}\; x_{\alpha+1,R} + B_{\alpha+1,R}\; u_{\alpha+1,R} + E_{\alpha+1,R}\; w_{\alpha+1,R}, \\ y_{\alpha+1,R} = C_{\alpha+1,1R}\; x_{\alpha+1,R} + D_{\alpha+1,1R}\; w_{\alpha+1,R}, \\ h_{\alpha+1,R} = C_{\alpha+1,2R}\; x_{\alpha+1,R} + D_{\alpha+1,2R}\; u_{\alpha+1,R}. \end{cases} \tag{11.3.44}$$

If $(A_{\alpha+1,R}, B_{\alpha+1,R}, C_{\alpha+1,2R}, D_{\alpha+1,2R})$, which is always right invertible, is also invertible, we let $\tilde{\alpha} := \alpha+1$, $\tilde{n}_x$ be the dynamical order of $\Sigma_{\alpha+1,R}$, and stop the algorithm. Otherwise, let $\alpha := \alpha+1$ and then go back to STEP R.R.O.S.2.

Consider the given system (11.1.1) with Σ_P being not left invertible and Σ_Q being not right invertible, and assume that Conditions 1–4 of Theorem 11.3.1 are satisfied. We use the results of Theorem 11.3.3, Corollaries 11.3.2 and 11.3.3 to obtain the following theorem.

Theorem 11.3.4. *If the quadruple $(A_{\tilde{\alpha},\mathrm{R}}, B_{\tilde{\alpha},\mathrm{R}}, C_{\tilde{\alpha},2\mathrm{R}}, D_{\tilde{\alpha},2\mathrm{R}})$ is invertible, then the DDPCM for Σ can be solved using the result of Corollary 11.3.2. Specifically, if we let $\mathcal{K}_{\tilde{\alpha},\mathrm{R}}$ be the set of all solutions to the DDPCM for $\Sigma_{\tilde{\alpha},\mathrm{R}}$, then all the solutions to the DDPCM for Σ are given by*

$$\mathcal{K} = \left\{ K_{\tilde{\alpha},\mathrm{R}} + \tilde{N} \mid K_{\tilde{\alpha},\mathrm{R}} \in \mathcal{K}_{\tilde{\alpha},\mathrm{R}},\ \tilde{N} = N_1 + N_2' + \cdots + N_{\tilde{\alpha}} \right\}, \quad (11.3.45)$$

if $\tilde{\alpha}$ is an odd integer, or

$$\mathcal{K} = \left\{ K_{\tilde{\alpha},\mathrm{R}}' + \tilde{N} \mid K_{\tilde{\alpha},\mathrm{R}} \in \mathcal{K}_{\tilde{\alpha},\mathrm{R}},\ \tilde{N} = N_1 + N_2' + \cdots + N_{\tilde{\alpha}}' \right\}, \quad (11.3.46)$$

if $\tilde{\alpha}$ is an even integer. Obviously, if $\mathcal{K}_{\tilde{\alpha},\mathrm{R}}$ is empty, $\mathcal{K}$ is empty, i.e., the DDPCM to Σ has no solution at all.

We illustrate the above results in the following example.

Example 11.3.1. Consider a system characterized by (11.1.1) with

$$A = \begin{bmatrix} -2 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -3 & -4 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is simple to verify, using the software toolkit of [87], that Σ_{P} is neither left nor right invertible and Σ_{Q} is not right invertible. Moreover, Conditions 1–4 of Theorem 11.3.1 are satisfied. Following STEPS DDPCM.R.O.S.1 to DDPCM.R.O.S.5, we obtain

$$N_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & 0 \\ 0 & 0 \end{bmatrix},$$

and a reduced-order system $\Sigma_{1,\mathrm{R}}$ characterized by

$$A_{1,\mathrm{R}} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_{1,\mathrm{R}} = \begin{bmatrix} 0 & 0 & -2.2361 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{1,\mathrm{R}} = \begin{bmatrix} -2.2361 \\ 1 \end{bmatrix},$$

$$C_{1,1\text{R}} = \begin{bmatrix} -0.4472 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{1,1\text{R}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$C_{1,2\text{R}} = \begin{bmatrix} 0 & 0 \\ -0.2236 & -0.5 \end{bmatrix}, \quad D_{1,2\text{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Next, applying STEPS DDPCM.R.O.S.1 to DDPCM.R.O.S.5 to the dual system of $\Sigma_{1,\text{R}}$, *i.e.*, $\Sigma_{1,\text{R}}^\star$, we obtain

$$N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and another reduced-order system $\Sigma_{2,\text{R}}$ characterized by

$$A_{2,\text{R}} = -1.5, \quad B_{2,\text{R}} = [0 \quad 0], \quad E_{2,\text{R}} = [0 \quad 0.5477],$$

and

$$C_{2,1\text{R}} = \begin{bmatrix} 0 \\ 0 \\ 0.9129 \end{bmatrix}, \ D_{2,1\text{R}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ C_{2,2\text{R}} = -0.9129, \ D_{2,2\text{R}} = [2 \quad 0].$$

Clearly, $(A_{2,\text{R}}, B_{2,\text{R}}, C_{2,2\text{R}}, D_{2,2\text{R}})$ is invertible as its special coordinate basis has no $\mathcal{X}_\text{b}$, $\mathcal{X}_\text{c}$ and $\mathcal{X}_\text{d}$ components. Using the result of Corollary 11.3.2, we obtain a parametrized gain matrix

$$\mathcal{K}_{2,\text{R}} = \left\{ \begin{bmatrix} 0 & 0 & 0.5 \\ k_1 & k_2 & k_3 \end{bmatrix} \middle| k_1, k_2, k_3 \in \mathbb{R} \right\}.$$

Hence, we have

$$\mathcal{K} = \left\{ K_{2,\text{R}}' + N_1 + N_2' = \begin{bmatrix} -1 & k_1 \\ -0.5 & k_2 \\ 0.5 & k_3 \end{bmatrix} \middle| k_1, k_2, k_3 \in \mathbb{R} \right\}. \tag{11.3.47}$$

Any $u = Ky$ that is a solution to the DDPCM for the given system satisfies $K \in \mathcal{K}$, and any $u = Ky$ with $K \in \mathcal{K}$ is a solution to the DDPCM for the system Σ. The DDPCMS for the system, unfortunately, is not solvable as it can be verified that for any $K \in \mathcal{K}$ of (11.3.47), $A + BKC_1$ always has a fixed eigenvalue at 0.

The following theorem deals with the situation when the solutions to the DDPCM cannot be obtained through the solution of linear equations.

Theorem 11.3.5. *If the quadruple $(A_{\tilde{\alpha},\mathrm{R}}, B_{\tilde{\alpha},\mathrm{R}}, C_{\tilde{\alpha},2\mathrm{R}}, D_{\tilde{\alpha},2\mathrm{R}})$ is not invertible, then the DDPCM for the system Σ of (11.1.1) is solvable if and only if there exists a solution $\tilde{K}_{\tilde{\alpha},\mathrm{R}}$ for the following set of multivariable polynomial equations:*

$$\left.\begin{aligned}
D_{\tilde{\alpha},2\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}D_{\tilde{\alpha},1\mathrm{R}} &= 0,\\
(C_{\tilde{\alpha},2\mathrm{R}} + D_{\tilde{\alpha},2\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}C_{\tilde{\alpha},1\mathrm{R}})(E_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}D_{\tilde{\alpha},1\mathrm{R}}) &= 0,\\
(C_{\tilde{\alpha},2\mathrm{R}} + D_{\tilde{\alpha},2\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}C_{\tilde{\alpha},1\mathrm{R}})(A_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}C_{\tilde{\alpha},1\mathrm{R}})&\\
\times (E_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}D_{\tilde{\alpha},1\mathrm{R}}) &= 0,\\
&\ \ \vdots\\
(C_{\tilde{\alpha},2\mathrm{R}} + D_{\tilde{\alpha},2\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}C_{\tilde{\alpha},1\mathrm{R}})(A_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}C_{\tilde{\alpha},1\mathrm{R}})^{\tilde{n}_x}&\\
\times (E_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}K_{\tilde{\alpha},\mathrm{R}}D_{\tilde{\alpha},1\mathrm{R}}) &= 0.
\end{aligned}\right\} \tag{11.3.48}$$

Moreover, all the solutions to the DDPCM for Σ, if they exist, are characterized by

$$\mathcal{K} = \left\{ K_{\tilde{\alpha},\mathrm{R}} + \tilde{N} \mid K_{\tilde{\alpha},\mathrm{R}} \text{ is a solution of (11.3.48)},\ \tilde{N} = N_1 + N_2' + \cdots + N_{\tilde{\alpha}} \right\}, \tag{11.3.49}$$

if $\tilde{\alpha}$ is an odd integer, or

$$\mathcal{K} = \left\{ K'_{\tilde{\alpha},\mathrm{R}} + \tilde{N} \mid K_{\tilde{\alpha},\mathrm{R}} \text{ is a solution of (11.3.48)},\ \tilde{N} = N_1 + N_2' + \cdots + N'_{\tilde{\alpha}} \right\}, \tag{11.3.50}$$

if $\tilde{\alpha}$ is an even integer. Clearly, the DDPCM for the given system Σ has no solutions if (11.3.48) has no solution.

Unlike the solutions obtained in Corollary 11.3.2 and Corollary 11.3.3, the equations given by (11.3.48) are nonlinear polynomial equations in $K_{\tilde{\alpha},\mathrm{R}}$. In principle, it is possible to eliminate $K_{\tilde{\alpha},\mathrm{R}}$ from the equations in (11.3.48) subject to the existence condition for the real solutions of $K_{\tilde{\alpha},\mathrm{R}}$ in terms of the system data $A_{\tilde{\alpha},\mathrm{R}}$, $B_{\tilde{\alpha},\mathrm{R}}$, $E_{\tilde{\alpha},\mathrm{R}}$, $C_{\tilde{\alpha},1\mathrm{R}}$, $D_{\tilde{\alpha},1\mathrm{R}}$, $C_{\tilde{\alpha},2\mathrm{R}}$ and $D_{\tilde{\alpha},2\mathrm{R}}$, and this can be carried out through the use of QEPCAD in Collins [39,40]. This is a finite step computation problem, but the emerging conditions could be hard to interpret. If we are interested in a purely numerical characterization, then the method of Grobner bases combined with QEPCAD may be used to find all the solutions for polynomial equations (see, e.g., Cox *et al.* [42]). The computational benefit in finding solutions for $\Sigma_{\tilde{\alpha},\mathrm{R}}$ is obvious.

We note that the result on the reduced-order characterization of the solutions to the DDPCM can be used to develop a numerical technique to establish a stabilizing constant measurement feedback to the problem of constant measurement feedback disturbance decoupling with internal stability (DDPCMS). In principle, the solution for the DDPCMS can also be solved by QEPCAD in Collins [39,40], as the stability conditions can be expressed via a finite set of polynomial inequalities.

11.4 Exercises

11.1. Verify that the DDPCM is solvable for the system characterized by (11.1.1) with

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$C_2 = [0 \quad 0 \quad 1], \quad D_2 = 0, \quad D_{22} = 0.$$

Find all its solutions, and show that the DDPCMS is also solvable for the given system.

11.2. Suppose a given system of (11.1.1) whose irreducible reduced order system is given by

$$\begin{cases} \dot{x}_{\tilde{\alpha},\mathrm{R}} = A_{\tilde{\alpha},\mathrm{R}}\ x_{\tilde{\alpha},\mathrm{R}} + B_{\tilde{\alpha},\mathrm{R}}\ u_{\tilde{\alpha},\mathrm{R}} + E_{\tilde{\alpha},\mathrm{R}}\ w_{\tilde{\alpha},\mathrm{R}}, \\ y_{\tilde{\alpha},\mathrm{R}} = C_{\tilde{\alpha},1\mathrm{R}}\ x_{\tilde{\alpha},\mathrm{R}} + D_{\tilde{\alpha},1\mathrm{R}}\ w_{\tilde{\alpha},\mathrm{R}}, \\ h_{\tilde{\alpha},\mathrm{R}} = C_{\tilde{\alpha},2\mathrm{R}}\ x_{\tilde{\alpha},\mathrm{R}} + D_{\tilde{\alpha},2\mathrm{R}}\ u_{\tilde{\alpha},\mathrm{R}}, \end{cases}$$

with

$$A_{\tilde{\alpha},\mathrm{R}} = -1.5, \quad B_{\tilde{\alpha},\mathrm{R}} = [0 \quad 1], \quad E_{\tilde{\alpha},\mathrm{R}} = [0 \quad 0.5477],$$

$$C_{\tilde{\alpha},1\mathrm{R}} = \begin{bmatrix} 0 \\ 0 \\ 0.9129 \end{bmatrix}, \quad D_{\tilde{\alpha},1\mathrm{R}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and

$$C_{\tilde{\alpha},2\mathrm{R}} = -0.9129, \quad D_{\tilde{\alpha},2\mathrm{R}} = [2 \quad 0].$$

Note that the subsystem $(A_{\tilde{\alpha},\mathrm{R}}, B_{\tilde{\alpha},\mathrm{R}}, C_{\tilde{\alpha},2\mathrm{R}}, D_{\tilde{\alpha},2\mathrm{R}})$ is not invertible. Show that its corresponding set of multivariable nonlinear polynomial equations, *i.e.*, (11.3.48), has a solution. Hence, by Theorem 11.3.5, the DDPCM for the given system is solvable. Find all the possible solutions to this set of polynomial equations.

Chapter 12

A Software Toolkit

12.1 Introduction

We have implemented all the algorithms presented in this monograph in MATLAB in a software toolkit called *Linear Systems Toolkit*. The beta version of this toolkit is currently available at `http://linearsystemskit.net`. Interested readers might wish to register online on the web site with their names, affiliations and email addresses. A zipped file that contains all m-functions of the toolkit will then be sent to the registered email addresses. Registered users will also automatically receive any advanced version of the toolkit through email. Nonetheless, the owners of the toolkit reserve all the rights. Users should bear in mind that the toolkit downloaded from the web site or received through email is free for use in research and academic work only. Uses for other purposes, such as commercialization, commercial development and redistribution without permission from the owners, are strictly prohibited.

The current version of the toolkit consists of the following m-functions in its built-in help file:

```
Linear Systems Toolkit - Version 0.99

Released in August 2004

Decompositions of Autonomous Systems
 ssd  - continuous-time stability structural decomposition
 dssd - discrete-time stability structural decomposition
 jcf  - Jordan canonical form
 rjd  - real Jordan decomposition
```

B. M. Chen et al., *Linear Systems Theory*

```
Decompositions of Unforced and Unsensed Systems
 osd    - observability structural decomposition
 obvidx - observability index
 bdosd  - block diagonal observable structural decomposition
 csd    - controllability structural decomposition
 ctridx - controllability index
 bdcsd - block diagonal controllable structural decomposition

Decompositions and Structural Properties of Proper Systems
 scbraw   - raw decomposition without integration chains
 scb      - decomposition of a continuous-time system
 dscb     - decomposition of a discrete-time system
 kcf      - Kronecker canonical form for system matrices
 morseidx - Morse indices
 blkz     - blocking zeros
 invz     - invariant zero structure
 infz     - infinite zero structure
 l_invt   - left invertibility structure
 r_invt   - right invertibility structure
 normrank - normal rank
 v_star   - weakly unobservable subspace
 v_minus  - stable weakly unobservable subspace
 v_plus   - unstable weakly unobservable subspace
 s_star   - strongly controllable subspace
 s_minus  - stable strongly controllable subspace
 s_plus   - unstable strongly controllable subspace
 r_star   - controllable weakly unobservable subspace
 n_star   - distributionally weakly unobservable subspace
 s_lambda - geometric subspace S_{lambda}
 v_lambda - geometric subspace V_{lambda}

Operations of Vector Subspaces
 ssorder  - ordering of vector subspaces
 ssintsec - intersection of vector subspaces
 ssadd    - addition of vector subspaces

Decompositions and Properties of Descriptor Systems
 ea_ds     - decomposition of a matrix pair (E,A)
 sd_ds     - decomposition for descriptor systems
 invz_ds   - descriptor system invariant zero structure
 infz_ds   - descriptor system infinite zero structure
 l_invt_ds - descriptor system left invertibility structure
 r_invt_ds - descriptor system right invertibility structure

System Factorizations
 mpfact  - continuous minimum-phase/all-pass factorization
 iofact  - continuous-time inner-outer factorization
```

```
 gcfact  - continuous generalized cascade factorization
 dmpfact - discrete minimum-phase/all-pass factorization
 diofact - discrete-time inner-outer factorization

Structural Assignment via Sensor/Actuator Selection
 sa_sen - structural assignment via sensor selection
 sa_act - structural assignment via actuator selection

Asymptotic Time-scale and Eigenstructure Assignment (ATEA)
 atea      - continuous-time ATEA
 gm2star   - infimum for continuous-time H2 control
 h2care    - solution to continuous-time H2 ARE
 h2state   - solution to continuous-time H2 control
 gm8star   - infimum for continuous-time H-infinity control
 h8care    - solution to continuous-time H-infinity ARE
 h8state   - solution to continuous-time H-infinity control
 addps     - solution to continuous disturbance decoupling
 datea     - discrete-time ATEA
 dare      - solution to general discrete-time ARE
 dgm2star  - infimum for discrete-time H2 control
 h2dare    - solution to discrete-time H2 ARE
 dh2state  - solution to discrete-time H2 control
 dgm8star  - infimum for discrete-time H-infinity control
 h8dare    - solution to discrete-time H-infinity ARE
 dh8state  - solution to discrete-time H-infinity control
 daddps    - solution to discrete-time disturbance decoupling

Disturbance Decoupling with Static Output Feedback
 ddpcm     - solution to disturbance decoupling problem with
             static output feedback (DDPCM)
 rosys4ddp - irreducible reduced-order system that can be
             used to solve DDPCM
```

There are 66 m-functions in the above *Linear Systems Toolkit*. Some of these m-functions are interactive, which require users to enter additional parameters when executed. Some can return results either in a symbolic or numerical form.

12.2 Descriptions of m-Functions

In this section, we give detailed descriptions for all the m-functions listed in *Linear Systems Toolkit*. This section is aimed to serve as a user manual for our toolkit. It is presented in a usual style as in other documents related to MATLAB. More features will be added to it from time to time. The most up-to-date version, when available, will be made available through the web site for the toolkit.

12.2.1 Decompositions of Autonomous Systems

SSD Stability Structural Decomposition

```
[D,T,nn,no,np] = SSD(A)
```

gives the following block diagonal form for a square matrix:

```
                        [ A-   0    0  ]  nn
    D = inv(T)*A*T =    | 0    Ao   0  |  no
                        [ 0    0    A+ ]  np
```

where eigenvalues of A-, Ao and A+ are, respectively, in the open left-half plane, jw axis and open right-half plane.

See also DSSD.

DSSD Discrete-time Stability Structural Decomposition

```
[D,T,nn,no,np] = DSSD(A)
```

gives the following block diagonal form for a square matrix:

```
                        [ A-   0    0  ]  nn
    D = inv(T)*A*T =    | 0    Ao   0  |  no
                        [ 0    0    A+ ]  np
```

where eigenvalues of A-, Ao and A+ are, respectively, inside, on and outside the unit circle of the complex plane.

See also SSD.

JCF Jordan Canonical Form

```
[J,T] = JCF(A)
```

generates a transformation that transforms a square matrix into the Jordan canonical form, i.e.,

```
                       [ J_1          ]
    inv(T)*A*T = J =   |      .       |
                       |        .     |
                       [          J_k ]
```

where each block J_i, i=1, 2, ..., k, has the following form:

```
            [ eig_i    1            ]
            |     .        .        |
    J_i =   |       .        .      |
            |         eig_i    1    |
            [                eig_i  ]
```

See also RJD.

RJD Real Jordan Decomposition

```
[J,T] = RJD(A)
```

generates a transformation that transforms a square matrix into the real Jordan canonical form, i.e.,

```
                      [ J_1          ]
inv(T)*A*T = J =      |     .        |
                      |       .      |
                      [          J_k ]
```

where each block J_i, i=1, 2, ..., k, has the following form:

```
          [ eig_i   1           ]    [ Eig_i    I             ]
          |     .      .        |    |     .        .         |
J_i  =    |        .       .    | or |        .         .     |
          |         eig_i   1   |    |         Eig_i    I     |
          [               eig_i ]    [              Eig_i     ]
```

for real eig_i or for eig_i = mu_i + j * omiga_i, for which

```
Eig_i =  [   mu_i     omiga_i ]
         [ -omiga_i    mu_i   ]
```

See also JCF.

12.2.2 Decompositions of Unforced and Unsensed Systems

OSD Observability Structural Decomposition for Unforced Systems

```
[At,Ct,Ts,To,uom,Oidx] = OSD(A,C)
```

returns an observability structural decomposition for (A,C).

Input Parameters:

```
.
x = A x,    y = C x
```

Output Parameters:

```
                      [ Ao  *     0        ...  *     0       ]
                      | 0   *  I_{k_1-1}   ...  *     0       |
At = inv(Ts)*A*Ts  =  | 0   *     0        ...  *     0       |
                      | :   :     :         .   :     :       |
                      | 0   *     0        ...  *  I_{k_p-1}  |
                      [ 0   *     0        ...  *     0       ]

                      [ 0   1     0        ...  0     0       ]
Ct = inv(To)*C*Ts  =  | :   :     :         .   :     :       |
                      [ 0   0     0        ...  1     0       ]
```

uom = unobservable modes & Oidx = observability index of (C,A)

See OBVIDX, BDOSD and CSD.

```
OBVIDX  Observability Index of Matrix Pair (A,C)

        Oidx = OBVIDX(A,C)

        returns the observability index for an unforced system.

        Input Parameters:
            .
            x = A x,    y = C x

        Output Parameters:

            Oidx = observability index of (A,C)

        See also OSD and CTRIDX.


BDOSD  Block Diagonal Observable Structural Decomposition

       [At,Ct,Ts,To,ks] = BDOSD(A,C)

       transforms an observable pair (A,C) into the block diagonal
       observable structural decomposition form.

       Input Parameters:
               .
               x = A x,    y = C x

       Output Parameters:

                                  [ A_1   0    0   ...   0  ]
                                  |  0   A_2   0   ...   0  |
           At = inv(Ts)*A*Ts =    |  0    0   A_3  ...   0  |
                                  |  :    :    :    .    :  |
                                  [  0    0    0   ...  A_k ]

                                  [ C_1   0    0   ...   0  ]
                                  |  *   C_2   0   ...   0  |
           Ct = inv(To)*C*Ts =    |  *    *   C_3  ...   0  |
                                  |  :    :    :    .    :  |
                                  |  *    *    *   ...  C_k |
                                  [  *    *    *   ...   *  ]
       where
                                  [ *  1  0 ... 0 ]
                                  | *  0  1 ... 0 |
                          A_i =   | :  :  :  .  : |
                                  | *  0  0 ... 1 |
                                  [ *  0  0 ... 0 ]

                          C_i =   [ 1  0  0 ... 0 ]

           ks: contains the sizes of blocks A_1, A_2, ..., A_k.

       See also OSD and BDCSD.
```

```
CSD  Controllability Structural Decomposition for Unsensed Systems

     [At,Bt,Ts,Ti,ucm,Cidx] = CSD(A,B)

     returns a controllability structural decomposition for (A,B).

     Input Parameters:
                   .
                   x = A x + B u

     Output Parameters:

                                 [ Ao  0      0       ...  0      0      ]
                                 | 0   0   I_{k_1-1}  ...  0      0      |
       At = inv(Ts)*A*Ts =       | *   *      *       ...  *      *      |
                                 | :   :      :        .   :      :      |
                                 | 0   0      0       ...  0   I_{k_p-1} |
                                 [ *   *      *       ...  *      *      ]

                                 [ 0 ... 0 ]
                                 | 0 ... 0 |
       Bt = inv(Ts)*B*Ti =       | 1 ... 0 |
                                 | :  .  : |
                                 | 0 ... 0 |
                                 [ 0 ... 1 ]

       ucm = uncontrollable modes & Cidx = controllability index

     See also BDCSD, CTRIDX and OSD.


CTRIDX  Controllability Index of Matrix Pair (A,B)

        Cidx = CTRIDX(A,B)

        returns the controllability index for an unsensed system.

        Input Parameters:
                 .
                 x = A x + B u

        Output Parameters:

                 Cidx = controllability index of (A,B)

        See also CSD and OBVIDX.


BDCSD  Block Diagonal Controllable Structural Decomposition

       [At,Bt,Ts,Ti,ks] = BDCSD(A,B)

       transforms a controllable pair (A,B) into the block diagonal
       controllable structural decomposition form.
```

```
Input Parameters:
          .
          x = A x + B u

Output Parameters:

                                [ A_1   0    0  ...   0  ]
                                | 0    A_2   0  ...   0  |
      At = inv(Ts)*A*Ts =       | 0     0   A_3 ...   0  |
                                | :     :    :   .    :  |
                                [ 0     0    0  ...  A_k ]

                                [ B_1   *    *  ...   *    * ]
                                | 0    B_2   *  ...   *    * |
      Bt = inv(Ts)*B*To =       | 0     0   B_3 ...   *    * |
                                | :     :    :   .    :    : |
                                [ 0     0    0  ...  B_k   * ]
where
                        [ 0  1  0 ... 0 ]         [ 0 ]
                        | 0  0  1 ... 0 |         | 0 |
              A_i =     | :  :  :  .  : |,  B_i = | : |
                        | 0  0  0 ... 1 |         | 0 |
                        [ *  *  * ... * ]         [ 1 ]

      ks: contains the sizes of the blocks A_1, A_2, ..., A_k.

See also CSD and BDOSD.
```

12.2.3 Decompositions and Properties of Proper Systems

```
SCBRAW  Special Coordinate Basis in a Raw Form (see Chu et al [36])

        [At,Bt,Ct,Dt,Gms,Gmo,Gmi,dim] = SCBRAW(A,B,C,D)

        decomposes a proper system characterized by (A,B,C,D) into
        a raw SCB form without separating state subspace x_d into
        chains of integrators.

        Input Parameters:
              .
              x = A x + B u,    y = C x + D u

        Output Parameters:
              .
              x_t = At x_t + Bt u_t,    y_t = Ct x_t + Dt u_t

        where x_t = [ x_a  x_b  x_c  x_d ]' with dimensions of

              dim = [ n_a, n_b, n_c, n_d ], respectively,

        and Gms, Gmo & Gmi = state, output & input transformations.

        See also SCB and DSCB.
```

```
SCB  Special Coordinate Basis for Continuous-time Systems

     [As,Bt,Ct,Dt,Gms,Gmo,Gmi,dim] = SCB(A,B,C,D)

     decomposes a continuous-time system characterized by (A,B,C,D)
     into the standard SCB form with state subspaces x_a being
     separated into stable, marginally stable and unstable parts
     (in continuous-time sense), and x_d being decomposed into the
     form of chains of integrators.

     Input Parameters:
         .
         x = A x + B u,    y = C x + D u

     Output Parameters (see Chapter 5):
         .
         x_t = (As + B_0 C_0) x_t + Bt u_t,  y_t = Ct x_t + Dt u_t

     where x_t = [ x_a^-  x_a^0  x_a^+  x_b  x_c  x_d ]' with

         dim = [ n_a^-, n_a^0, n_a^+, n_b, n_c, n_d ],

     and Gms, Gmo & Gmi = state, output & input transformations.

     See also SCBRAW, DSCB and SSD.


DSCB  Special Coordinate Basis for Discrete-time Systems

      [As,Bt,Ct,Dt,Gms,Gmo,Gmi,dim] = DSCB(A,B,C,D)

      decomposes a discrete-time system characterized by (A,B,C,D)
      into the standard SCB form with state subspaces x_a being
      separated into stable, marginally stable and unstable parts
      (in discrete-time sense), and x_d being decomposed into the
      form of chains of delay elements.

      Input Parameters:

          x(k+1)  = A x(k)  + B u(k)
           y(k)   = C x(k)  + D u(k)

      Output Parameters:

          x_t(k+1)  =  (As+B_0 C_0)  x_t(k)  + Bt u_t(k)
           y_t(k)   =       Ct       x_t(k)  + Dt u_t(k)

      where x_t = [ x_a^-  x_a^0  x_a^+  x_b  x_c  x_d ]' with

          dim = [ n_a^-, n_a^0, n_a^+, n_b, n_c, n_d ],

      and Gms, Gmo & Gmi = state, output & input transformations.

      See also SCBRAW, SCB and DSSD.
```

KCF Kronecker Canonical Form of System Matrices

```
[Ks,U,V,dims] = KCF(A,B,C,D)
```

returns the Kronecker canonical form of a system matrix for a system characterized by (A,B,C,D), i.e.,

```
           [ sI-A  -B ]
Ks = U |           | V
           [   C    D ]
```

where Ks, a symbolic variable, is in the Kronecker form.

```
         [  na      na    ]
         | nb+pb   nb     |
dims =   |  nc     nc+mc  |
         | nd+md  nd+md   |
         |  m0      m0    |
         [  r0      c0    ]
```

contains the dimensions of the Kronecker blocks associated with x_a, x_b, x_c, x_d, I and 0 (see Chapters 3 and 5).

See also SCB, DSCB and MORSEIDX.

MORSEIDX Morse Invariance Indices of Proper Systems

```
[I1,I2,I3,I4] = MORSEIDX(A,B,C,D)
```

returns Morse structural invariance list for a system characterized by (A,B,C,D).

```
I1 = zero dynamics matrix in Jordan form
I2 = I_2 List = right invertibility structure
I3 = I_3 List = left invertibility structure
I4 = I_4 List = infinite zero structure
```

Note that I1 List should formally contain the invariant zeros and the sizes of their Jordan blocks.

See also INVZ, INFZ, L_INVT and R_INVT.

BLKZ Blocking Zeros of Multivariable Systems

```
bzero = BLKZ(A,B,C,D)
```

returns blocking zeros of a system characterized by (A,B,C,D). A blocking zero, say 'alpha', is such that

```
H(alpha) = C (alpha*I - A)^{-1} B + D = 0
```

A blocking zero is also an invariant zero.

See also INVZ.

INVZ Invariant Zeros and Structures of Proper Systems

```
zrs = INVZ(A,B,C,D)
```

returns invariant zeros of a system characterized by (A,B,C,D) and their structures.

zrs = all the invariant zeros of the system

Note that invariant zeros are sometimes called transmission zeros. However, the latter is only defined for controllable and observable systems.

See also BLKZ and INFZ and MORSEIDX.

INFZ Infinite Zero Structure of Proper Systems

```
infzs = INFZ(A,B,C,D)
```

returns the infinite zero structure of a system characterized by (A,B,C,D).

infzs = I_4 List of Morse Indices

See also INVZ, L_INVT, R_INVT and MORSEIDX.

L_INVT Left Invertibility Structure of Proper Systems

```
lefts = L_INVT(A,B,C,D)
```

returns the left invertibility structure of a multivariable system characterized by (A,B,C,D).

lefts = I_3 List of Morse Indices

See also INVZ, INFZ, R_INVT and MORSEIDX.

R_INVT Right Invertibility Structure of Proper Systems

```
rights = R_INVT(A,B,C,D)
```

returns the right invertibility structure of a multivariable system characterized by (A,B,C,D).

rights = I_2 List of Morse Indices

See also INVZ, INFZ, R_INVT and MORSEIDX.

NORMRANK Normal Rank of Proper System

```
NR = NORMRANK(A,B,C,D)
```

returns the normal rank of a linear system characterized by (A,B,C,D).

See also INVZ.

V_STAR Weakly Unobservable Geometric Subspace

```
V = V_STAR(A,B,C,D)
```

computes a matrix whose columns span the geometric subspace V^{*} for a system characterized by (A,B,C,D).

Note:

It is applicable for both continuous- and discrete-time systems.

See also V_MINUS, V_PLUS and S_STAR.

V_MINUS Stable Weakly Unobservable Geometric Subspace

```
V = V_MINUS(A,B,C,D[,dc])
```

computes a matrix whose columns span the geometric subspace V^{-} for a system characterized by (A,B,C,D).

Note that by default or if dc= 0, the function returns a subspace for a continuous-time system. Otherwise, if dc = 1, it computes a subspace for a discrete-time system.

See also V_STAR, V_PLUS and S_MINUS.

V_PLUS Unstable Weakly Unobservable Geometric Subspace

```
V = V_PLUS(A,B,C,D[,dc])
```

computes a matrix whose columns span the geometric subspace V^{+} for a system characterized by (A,B,C,D).

Note that by default or if dc = 0, the function returns a subspace for a continuous-time system. Otherwise, if dc = 1, it computes a subspace for a discrete-time system.

See also V_STAR, V_MINUS and S_PLUS.

S_STAR Strongly Controllable Geometric Subspace

```
S = S_STAR(A,B,C,D)
```

computes a matrix whose columns span the geometric subspace S^{*} for a system characterized by (A,B,C,D).

Note that this function is applicable for both continuous- and discrete-time systems.

See also S_MINUS, S_PLUS and V_STAR.

S_MINUS Stable Strongly Controllable Geometric Subspace

```
S = S_MINUS(A,B,C,D[,dc])
```

computes a matrix whose columns span the geometric subspace S^{-} for a system characterized by (A,B,C,D).

Note that by default or if dc = 0, the function returns a subspace for a continuous-time system. Otherwise, if dc = 1, it computes a subspace for a discrete-time system.

See also S_STAR, S_PLUS and V_MINUS.

S_PLUS Unstable Strongly Controllable Geometric Subspace

```
S = S_PLUS(A,B,C,D[,dc])
```

computes a matrix whose columns span the geometric subspace S^{+} for a system characterized by (A,B,C,D).

Note that by default or if dc = 0, the function returns a subspace for a continuous-time system. Otherwise, if dc = 1, it computes a subspace for a discrete-time system.

See also S_STAR, S_MINUS and V_PLUS.

R_STAR Controllable Weakly Unobservable Geometric Subspace

```
R = R_STAR(A,B,C,D)
```

computes a matrix whose columns span the geometric subspace R^{*} for a system characterized by (A,B,C,D).

Note that this function is applicable for both continuous- and discrete-time systems.

See also S_STAR, V_STAR and N_STAR.

```
N_STAR  Distributionally Weakly Unobservable Geometric Subspace

        N = N_STAR(A,B,C,D)

        computes a matrix whose columns span the geometric subspace
        N^{*} for a system characterized by (A,B,C,D).

        Note that this function is applicable for both continuous-
        and discrete-time systems.

        See also R_STAR, V_STAR and S_STAR.


S_LAMBDA  Geometric Subspace S_lambda (see Chapter 3 for definition)

          S = S_LAMBDA(A,B,C,D,lambda)

          computes a matrix whose columns span S_{lambda} for a
          system characterized by (A,B,C,D), where 'lambda' is
          either a real or complex scalar.

          Note: This function is applicable for both continuous- and
          discrete-time systems.

          See also V_LAMBDA.


V_LAMBDA  Geometric Subspace V_lambda (see Chapter 3 for definition)

          V = V_LAMBDA(A,B,C,D,lambda)

          computes a matrix whose columns span V_{lambda} for a
          system characterized by (A,B,C,D), where 'lambda' is
          either a real or complex scalar.

          Note: This function is applicable for both continuous- and
          discrete-time systems.

          See also S_LAMBDA.
```

12.2.4 Operations of Vector Subspaces

```
SSORDER  Ordering of Vector Subspace

         ss = SSORDER(X,Y)

         determines the ordering of two vector spaces respectively
         spanned by the columns of matrices X and Y.

         Output Parameters:

          if ss = -1, subspace spanned by X < that spanned by Y
          if ss =  0, subspace spanned by X = that spanned by Y
          if ss =  1, subspace spanned by X > that spanned by Y
```

```
        if ss =  j = sqrt(-1), they are not related at all.

       See also SSINTSEC and SSADD.

SSINTSEC  Intersection of Vector Subspace

          V = SSINTSEC(X,Y)

          computes intersection of two vector spaces respectively
          spanned by the columns of matrices X and Y.

          The columns of V form a basis for the intersection.

          See also SSORDER and SSADD.

SSADD  Addition of Vector Subspace

       V = SSADD(X,Y)

       computes the addition of two vector spaces respectively
       spanned by the columns of matrices X and Y.

       The columns of V form a basis for the addition.

       See also SSORDER and SSINTSEC.
```

We note that the above functions, although simple, are particularly useful in verifying geometric conditions for various control problems, and generating other geometric subspaces introduced in the literature.

12.2.5 Decompositions and Properties of Descriptor Systems

```
EA_DS  Decomposition for a Pair of Square Matrices (E,A)

       [Et,At,P,Q,n1,n2] = EA_DS(E,A)

       decomposes (E,A), related to descriptor systems, i.e.,
                               .
                             E x = A x + ...

       into the following special form:

                    Et = P E Q = [ I   0 ]  n1
                                 [ 0   N ]  n2

                    At = P A Q = [ A1  0 ]  n1
                                 [ 0   I ]  n2

       where N is a nilpotent matrix, P and Q are nonsingular.

       See also SD_DS.
```

```
SD_DS   Structural Decomposition of Continuous-time Descriptor System

        [Es,As,Bs,Cs,Ds,Ez,Psi,Psc,Psd,Psr,Gme,Gms,Gmo,iGmi,dim]
                                               = SD_DS(E,A,B,C,D)

        generates the structural decomposition of a descriptor system
        characterized by (E,A,B,C,D).
                               .
        Input Parameters:  E x = A x + B u,    y = C x + D u

        Output Parameters:
              .
           Es x_t = As x_t + Bs u_t,    y_t = Cs x_t + Ds u_t

        where x_t = [ x_z  x_e  x_a  x_b  x_c  x_d ]' with

              dim = [ n_z, n_e, n_a, n_b, n_c, n_d ],

        (Es,As,Bs,Cs,Ds) has the same transfer function as that of
        the original system. Ez, Psi, Psc, Psd and Psr are some
        matrices or vectors whose elements are either polynomials
        or rational functions of s. In particular,

                Psc * x_t + Psd * u_t = Psr * x_z.

        Gms, Gmo & iGmi = state, output transformations and the
        inverse of the input transformation, and finally, Gme is
        a nonsingular transformation on matrix E. Both Gme and
        iGmi have their entries being some polynomials of s.

        Note: Also applicable to discrete-time descriptor systems.

        See also EA_SD, SCB and DSCB.


INVZ_DS   Invariant Zeros and Structures of Descriptor Systems

          zrs = INVZ_DS(E,A,B,C,D)

          gives invariant zeros of a descriptor system characterized
          by (E,A,B,C,D) and their structures.

            zrs = all the invariant zeros of the system

          See also INVZ.


INFZ_DS   Infinite Zero Structure of Descriptor Systems

          infzs = INFZ_DS(E,A,B,C,D)

          returns the infinite zero structure of a descriptor
          system characterized by (E,A,B,C,D).

          See also INFZ.
```

L_INVT_DS Left Invertibility Structure of Descriptor Systems

```
lefts = L_INVT_DS(E,A,B,C,D)
```

returns the left invertibility structure of a descriptor system characterized by (E,A,B,C,D).

See also L_INVT.

R_INVT_DS Right Invertibility Structure of Descriptor Systems

```
rights = R_INVT_DS(E,A,B,C,D)
```

gives the right invertibility structure of a descriptor system characterized by (E,A,B,C,D).

See also R_INVT.

12.2.6 System Factorizations

MPFACT Minimum-Phase/All-Pass Factorization of Continuous Systems

```
[Am,Bm,Cm,Dm,Av,Bv,Cv,Dv] = MPFACT(A,B,C,D)
```

calculates a minimum-phase/all-pass factorization for a detectable system (A,B,C,D) with transfer function matrix G(s), in which both [B' D'] and [C D] are assumed to be of full rank, and (A,B,C,D) has no invariant zeros on the jw axis.

The minimum-phase/all-pass factorization is given as

```
G(s) = Gm(s) V(s)
```

where

```
Gm(s) = Cm (sI - Am)^{-1} Bm + Dm
```

is of minimum-phase and left invertible, and

```
V(s) = Cv (sI - Av)^{-1} Bv + Dv
```

is an all-pass factor satisfying V(s) V'(-s) = I.

See also IOFACT, GCFACT and DMPFACT.

IOFACT Inner-Outer Factorization of Continuous-time Systems

```
[Ai,Bi,Ci,Di,Ao,Bo,Co,Do] = IOFACT(A,B,C,D)
```

computes an inner-outer factorization for a stable proper transfer function matrix G(s) with a realization (A,B,C,D), in which both [B' D'] and [C D] are assumed to be of full rank.

The inner-outer factorization is given as

```
G(s) = Gi(s) Go(s)
```

where

```
Gi(s) = Ci (sI - Ai)^{-1} Bi + Di
```

is an inner, and

```
Go(s) = Co (sI - Ao)^{-1} Bo + Do
```

is an outer.

See also MPFACT, GCFACT and DIOFACT.

GCFACT Generalized Cascade Factorization of Continuous Systems

```
[Am,Bm,Cm,Dm,Au,Bu,Cu,Du] = GCFACT(A,B,C,D)
```

generates a generalized cascade factorization for a system (A,B,C,D) with transfer function matrix G(s), in which both [B' D'] and [C D] are assumed to be of full rank, and all the 'awkward' invariant zeros of (A,B,C,D) are detectable.

The generalized cascade factorization is given as

```
G(s) = Gm(s) U(s)
```

where

```
Gm(s) = Cm (sI - Am)^{-1} Bm + Dm
```

is of minimum-phase and left invertible, and

```
U(s) = Cu (sI - Au)^{-1} Bu + Du
```

is a stable right invertible and asymptotic all-pass, i.e.,

```
U(s) U'(-s) -> I   as |s| -> infinity
```

Note that users will be prompted to enter desired zero locations to replace those 'awkward' invariant zeros.

See also MPFACT and IOFACT.

DMPFACT Minimum-Phase/All-Pass Factorization of Discrete Systems

```
[Am,Bm,Cm,Dm,Av,Bv,Cv,Dv] = DMPFACT(A,B,C,D)
```

```
calculates a minimum-phase/all-pass factorization for a
detectable system (A,B,C,D) with transfer function matrix
G(z), which has no invariant zeros on the unit circle.

The minimum-phase/all-pass factorization is given as

      G(z) = Gm(z) V(z)

where

      Gm(z) = Cm (zI - Am)^{-1} Bm + Dm

is of minimum-phase and left invertible with no infinite
zeros, and

      V(z) = Cv (zI - Av)^{-1} Bv + Dv

is an all-pass factor satisfying V(z) V'(1/z) = I.

See also DIOFACT and MPFACT.
```

```
DIOFACT  Inner-Outer Factorization of Discrete-time Systems

         [Ai,Bi,Ci,Di,Ao,Bo,Co,Do] = DIOFACT(A,B,C,D)

         computes an inner-outer factorization for a stable proper
         transfer function G(z) with a realization (A,B,C,D).

         The inner-outer factorization is given as

               G(z) = Gi(z) Go(z)

         where

               Gi(z) = Ci (zI - Ai)^{-1} Bi + Di

         is an inner, and

               Go(z) = Co (zI - Ao)^{-1} Bo + Do

         is an outer.

         See also DMPFACT and DIOFACT.
```

12.2.7 Structural Assignment via Sensor/Actuator Selection

The next two functions require users to enter design parameters during execution, which can only be determined when the system properties are evaluated. The functions are actually dual each other. The first one deals with sensor selection whereas the second one is about actuator selection.

SA_SEN Structural Assignment via Sensor Selection

```
C = SA_SEN(A,B)

For a given unsensed system:
        .
        x = A x + B u

the function finds a measurement output matrix C such that
the resulting system characterized by (A,B,C) has the pre-
specified desired structural properties.

Note: Users will be prompted to enter desired structural
parameters after the properties of the given pair (A,B)
is evaluated.

See also SA_ACT.
```

SA_ACT Structural Assignment via Actuator Selection

```
B = SA_ACT(A,C)

For a given unforced system:
        dx/dt = A x,   y = C x

the function finds an input output matrix B such that the
resulting system characterized by (A,B,C) has the pre-
specified desired structural properties.

Note: Users will be prompted to enter desired structural
parameters after the properties of the given pair (A,C)
is evaluated.

See also SA_SEN.
```

12.2.8 State Feedback Control with Eigenstructure Assignment

ATEA Asymptotic Time-scale and Eigenstructure Assignment

```
F = ATEA(A,B,C,D[,option])

produces a state feedback law u = F x using the asymptotic
time-scale structure and eigenstructure assignment design
method for a continuous-time system characterized by
        .
        x = A x + B u,   y = C x + D u

Users have the 'option' to choose the result either in a
numerical or in a symbolic form parameterized by a tuning
parameter 'epsilon'. The latter is particularly useful in
solving control problems, such as H2 and H-infinity sub-
optimal control as well as disturbance decoupling problem.
By default or choosing option = 0, the program will ask
```

```
users to enter a value for 'epsilon' and return a numerical
solution. Otherwise, if option = 1, F will be in a symbolic
form parameterized by 'epsilon'.

Note that users will be asked to enter desired design eigen-
structures during execution. The time-scale is parameterized
by the tuning parameter 'epsilon'.

See also H2STATE, H8STATE, ADDPS and DATEA.
```

```
GM2STAR  Infimum or Optimal Value for Continuous-time H2 Control

         gms2 = GM2STAR(A,B,C,D,E)

         calculates the infimum or the best achievable performance
         of the H2 suboptimal control problem for the system:

              .
              x = A x + B u + E w
              h = C x + D u

         under all possible stabilizing state feedback controllers.

         See also H2STATE, GM8STAR and DGM2STAR.
```

```
H2CARE  Solution to H2 Continuous-time Algebraic Riccati Equation

        P = H2CARE(A,B,C,D)

        returns a positive semi-definite solution, if existent, for
        the following algebraic Riccati equation for continuous-time
        H2 optimal control:

            0 = PA + A'P + C'C - (PB+C'D)(D'D)^{-1}(PB+C'D)'

        Note that a positive semi-definite stabilizing solution is
        existent if and only if the quadruple (A,B,C,D) has no
        invariant zeros on the jw axis and D is of full column rank.

        See also H2STATE, H8CARE and H2DARE.
```

```
H2STATE  Continuous-time H2 Control Using the ATEA Approach

         F = H2STATE(A,B,C,D[,option])

         generates a state feedback gain law u = F x, which solves
         H2 suboptimal control problem for the following system:

              .
              x = A x + B u + E w
              h = C x + D u

         i.e., the H2-norm of the resulting closed-loop transfer
         matrix from the disturbance, w, to the controlled output,
```

```
        h, is minimized. Use GM2STAR to calculate the infimum or
        the best achievable H2 performance, i.e., gamma_2^*.

        Users have the 'option' to choose the result either in a
        numerical or in a symbolic form parameterized by a tuning
        parameter 'epsilon'. By default or choosing option = 0,
        the program will ask users to enter a value for 'epsilon'
        and return a numerical solution. Otherwise, if option = 1,
        F will be in a symbolic form parameterized by 'epsilon'.

        See also GM2STAR, DH2STATE, ATEA, H8STATE and ADDPS.


GM8STAR Infimum or Optimal Value for Continuous H-infinity Control

        gms8 = GM8STAR(A,B,C,D,E)

        calculates the infimum or the best achievable performance
        of the H-infinity suboptimal control problem for the plant:
             .
             x = A x + B u + E w
             h = C x + D u

        under all possible stabilizing state feedback controllers.

        See also H8STATE, GM2STAR and DGM8STAR.


H8CARE Solution to H-infinity Continuous Algebraic Riccati Equation

       P = H8CARE(A,B,C,D,E,gamma)

       returns a positive semi-definite solution, if existent, for
       the following algebraic Riccati equation for continuous-time
       H-infinity control:

                                [ B'P+D'C ]'          [ B'P+D'C ]
         0 = PA + A'P + C'C -   |         |  G^{-1}   |         |
                                [   E'P   ]           [   E'P   ]

       where
                      [ D'D      0        ]
                G =   |                   |
                      [  0    -gamma^2 I  ]

       This CARE is related to H-infinity control for the following
       continuous-time system:
              .
              x = A x + B u + E w
              h = C x + D u

       Note that a positive semi-definite stabilizing solution is
       existent if and only if the quadruple (A,B,C,D) has no
       invariant zeros on the jw axis, D is of full column rank,
       & gamma > gamma_\infty^*, which can be found using GM8STAR.
```

See also H2CARE, H8STATE, H8DARE and GM8STAR.

H8STATE Continuous-time H-infinity Control Using the ATEA Approach

```
F = H8STATE(A,B,C,D,E,gamma[,option])

generates a state feedback gain law u = F x, which solves
H-infinity gamma-suboptimal control problem for the system:

     .
     x = A x + B u + E w
     h = C x + D u

i.e., the H-infinity norm of the resulting closed-loop
transfer matrix from the disturbance, w, to the controlled
output, h, is less than the given 'gamma'. The value of
'gamma' has to be chosen larger than the infimum,
gamma_\infty^*, which can be pre-determined using GM8STAR.

Users have the 'option' to choose the result either in a
numerical or in a symbolic form parameterized by a tuning
parameter 'epsilon'. By default or choosing option = 0,
the program will ask users to enter a value for 'epsilon'
and return a numerical solution. Otherwise, if option = 1,
F will be in a symbolic form parameterized by 'epsilon'.

See also GM8STAR, DH8STATE, ATEA, H2STATE and ADDPS.
```

ADDPS Solution to Continuous-time Disturbance Decoupling Problem

```
F = ADDPS(A,B,C,D,E[,option])

generates a state feedback gain law u = F x, which solves
the almost disturbance decoupling problem for the system:

     .
     x = A x + B u + E w
     h = C x + D u

i.e., the resulting closed-loop transfer matrix from the
disturbance, w, to the controlled output, h, can be made
almost zero. The function will return an empty solution if
the problem for the given system is not solvable.

Users have the 'option' to choose the result either in a
numerical or in a symbolic form parameterized by a tuning
parameter 'epsilon'. By default or choosing option = 0,
the program will ask users to enter a value for 'epsilon'
and return a numerical solution. Otherwise, if option = 1,
F will be in a symbolic form parameterized by 'epsilon'.

See also ATEA, H2STATE, H8STATE and DADDPS.
```

```
DATEA  Eigenstructure Assignment for Discrete-time Systems

       F = DATEA(A,B,C,D)

       produces a state feedback control law, u(k) = F x(k), using
       the eigenstructure assignment design method for a discrete-
       time system characterized by

           x(k+1) = A x(k) + B u(k),   y(k) = C x(k) + D u(k)

       Note that this function is semi-interactive. Users will be
       asked to enter desired design eigenstructures during
       execution.

       See also DH2STATE, DH8STATE, DADDPS and ATEA.


DARE  Solution to Discrete-time Algebraic Riccati Equation

      [P,err] = DARE(A,M,N,R,Q)

      returns a positive semi-definite solution, if existent, for
      the following general discrete algebraic Riccati equation:

        P = A'PA - (A'PM + N)(R + M'PM)^{-1}(M'PA + N') + Q

      using noniterative method reported in Chen's work, Robust
      and H-infinity Control, Springer, London, 2000.

      err is the solution error defined as:
        err = | A'PA-(A'PM+N)(R+M'PM)^{-1}(M'PA+N')+Q-P |

      which can be used to verify the accuracy of the solution, P,
      to the DARE.

      See also H2DARE and H8DARE.


DGM2STAR  Infimum or Optimal Value for Discrete-time H2 Control

          gms2 = DGM2STAR(A,B,C,D,E)

          calculates the infimum or the best achievable performance
          of the H2 suboptimal control problem for the system:

              x(k+1) = A x(k) + B u(k) + E w(k)
               h(k)  = C x(k) + D u(k)

          under all possible stabilizing state feedback controllers.

          See also DH2STATE, DGM8STAR and GM2STAR.
```

```
H2DARE  Solution to H2 Discrete-time Algebraic Riccati Equation

        P = H2DARE(A,B,C,D)

        returns a positive semi-definite solution, if existent, for
        the following algebraic Riccati equation for discrete-time
        H2 optimal control:

            P = A'PA+C'C-(A'PB+C'D)(D'D+B'PB)^{-1}(A'PB+C'D)'

        Note that a positive semi-definite stabilizing solution is
        existent if and only if the quadruple (A,B,C,D) is left
        invertible and has no invariant zeros on the unit circle.

        See also DARE, H8DARE and H2CARE.


DH2STATE  Discrete-time H2 Control Using the ATEA Approach

          F = DH2STATE(A,B,C,D)

          generates a state feedback gain law u(k) = F x(k), which
          solves H2 optimal control problem for the system:

              x(k+1) = A x(k) + B u(k) + E w(k)
               h(k)  = C x(k) + D u(k)

          i.e., the H2-norm of the resulting closed-loop transfer
          matrix from the disturbance, w, to the controlled output,
          h, is equal to the optimal value, gamma_2^*, which can
          be pre-calculated using DGM2STAR.

          See also DGM2STAR, H2STATE, DATEA, DH8STATE and DADDPS.


DGM8STAR  Infimum or Optimal Value for Discrete H-infinity Control

          gms8 = DGM8STAR(A,B,C,D,E)

          calculates the infimum or the best achievable performance
          of H-infinity suboptimal control problem for the plant:

              x(k+1) = A x(k) + B u(k) + E w(k)
               h(k)  = C x(k) + D u(k)

          under all possible stabilizing state feedback controllers.

          See also DH8ARE, DH8STATE, DGM2STAR and GM8STAR.


H8DARE  Solution to H-infinity Discrete Algebraic Riccati Equation

        P = H8DARE(A,B,C,D,E,gamma)

        returns a positive semi-definite solution, if existent, for
```

the following algebraic Riccati equation for discrete-time H-infinity control:

```
                     [ B'PA+D'C ]'          [ B'PA+D'C ]
  P = A'PA + C'C -   |          |  G^{-1}   |          |
                     [   E'PA   ]           [   E'PA   ]
```

where

```
                 [ D'D+B'PB          B'PE        ]
         G =     |                               |
                 [   E'PB     E'PE-gamma^2 I     ]
```

This DARE is related to H-infinity control for the following discrete-time system:

```
        x(k+1) = A x(k) + B u(k) + E w(k)
         h(k)  = C x(k) + D u(k)
```

Note that a positive semi-definite stabilizing solution is existent if and only if the quadruple (A,B,C,D) is left invertible, has no invariant zeros on the unit circle, and gamma > gamma_\infty^*, which can be found using DGM8STAR.

See also DARE, H2DARE, H8CARE and DGM8STAR.

DH8STATE Discrete-time H-infinity Control Using the ATEA Approach

F = DH8STATE(A,B,C,D,E,gamma)

generates a state feedback gain law u(k) = F x(k), which solves H-infinity gamma-suboptimal control problem for

```
        x(k+1) = A x(k) + B u(k) + E w(k)
         h(k)  = C x(k) + D u(k)
```

i.e., the H-infinity norm of the resulting closed-loop transfer matrix from the disturbance, w, to the controlled output, h, is less than the given 'gamma'. The value of 'gamma' has to be chosen larger than the infimum, gamma_\infty^*, which can be determined using DGM8STAR.

See also DGM8STAR, H8STATE, DATEA, DH2STATE and DADDPS.

DADDPS Solution to Discrete-time Disturbance Decoupling Problem

F = DADDPS(A,B,C,D,E)

generates a state feedback gain law u(k) = F x(k), which solves the disturbance decoupling problem for the system:

```
        x(k+1) = A x(k) + B u(k) + E w(k)
         h(k)  = C x(k) + D u(k)
```

```
i.e., the resulting closed-loop transfer matrix from the
disturbance, w, to the controlled output, h, can be made
identically zero. The function will return an empty solution
if the problem for the given system is not solvable.

See also DATEA, DH2STATE, DH8STATE and ADDPS.
```

12.2.9 Disturbance Decoupling with Static Output Feedback

```
DDPCM  Disturbance Decoupling with Static Output Feedback

       K = DDPCM(A,B,E,C1,D1,C2,D2,D22)

       computes a solution to the disturbance decoupling problem
       with a constant (static) measurement output feedback for
       the following system:
                 .
                 x = A  x + B  u +  E  w
                 y = C1 x        + D1  w
                 h = C2 x + D2 u + D22 w

       if the solution is existent. Otherwise, the program will
       return an empty matrix for K.

       See also ROSYS4DDP.
```

```
ROSYS4DDP  Irreducible Reduced Order System for Disturbance
           Decoupling with Static Output Feedback

           [Ar,Br,Er,C1r,D1r,C2r,D2r]
                             = ROSYS4DDP(A,B,E,C1,D1,C2,D2,D22)

           generates an irreducible reduced order system from the
           original system:
                 .
                 x = A  x + B  u +  E  w
                 y = C1 x        + D1  w
                 h = C2 x + D2 u + D22 w

           The reduced order system is characterized by
                 .
                 x_r = Ar  x_r + Br  u_r +  Er  w_r
                 y_r = C1r x_r           + D1r  w_r
                 h_r = C2r x_r + D2r u_r

           which can be used to solve the static output disturbance
           decoupling problem for the original system through some
           numerical computation package such as QEPCAD.

           The program will return an empty result if the problem
           for the original system is not solvable.

           See also DDPCM.
```

Finally, we would like to conclude this monograph by noting that we are still expanding our toolkit reported above. More features will be added to it from time to time. Interested readers might access the most up-to-date information about the toolkit through its web site at `http://linearsystemskit.net`, or send inquiries and comments to us through email to `bmchen@nus.edu.sg`, `z15y@virginia.edu` or `yshamash@notes.cc.sunysb.edu`.

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