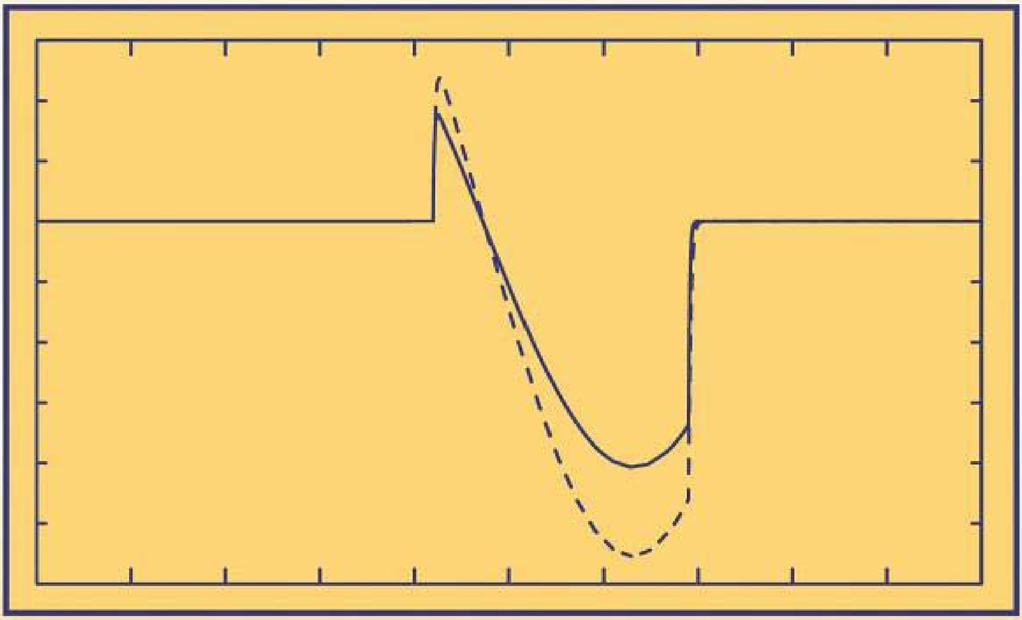
Automation and Control Engineering Series

Reliable Control and Filtering of Linear Systems with Adaptive Mechanisms



Guang-Hong Yang
Dan Ye

CRC Press
Taylor & Francis Group

Reliable Control and Filtering of Linear Systems with Adaptive Mechanisms

AUTOMATION AND CONTROL ENGINEERING

A Series of Reference Books and Textbooks

Series Editors

FRANK L. LEWIS, Ph.D.,
Fellow IEEE, Fellow IFAC
Professor
Automation and Robotics Research Institute
The University of Texas at Arlington

SHUZHI SAM GE, Ph.D.,
Fellow IEEE
Professor
Interactive Digital Media Institute
The National University of Singapore

Reliable Control and Filtering of Linear Systems with Adaptive Mechanisms, *Guang-Hong Yang and Dan Ye*

Reinforcement Learning and Dynamic Programming Using Function Approximators, *Lucian Buşoniu, Robert Babuška, Bart De Schutter, and Damien Ernst*

Modeling and Control of Vibration in Mechanical Systems, *Chunling Du and Lihua Xie*

Analysis and Synthesis of Fuzzy Control Systems: A Model-Based Approach, *Gang Feng*

Lyapunov-Based Control of Robotic Systems, *Aman Behal, Warren Dixon, Darren M. Dawson, and Bin Xian*

System Modeling and Control with Resource-Oriented Petri Nets, *Naiqi Wu and MengChu Zhou*

Sliding Mode Control in Electro-Mechanical Systems, Second Edition, *Vadim Utkin, Jürgen Guldner, and Jingxin Shi*

Optimal Control: Weakly Coupled Systems and Applications, *Zoran Gajić, Myo-Taeg Lim, Dobrila Skatarić, Wu-Chung Su, and Vojislav Kecman*

Intelligent Systems: Modeling, Optimization, and Control, *Yung C. Shin and Chengying Xu*

Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory, Second Edition, *Frank L. Lewis, Lihua Xie, and Dan Popa*

Feedback Control of Dynamic Bipedal Robot Locomotion, *Eric R. Westervelt, Jessy W. Grizzle, Christine Chevallereau, Jun Ho Choi, and Benjamin Morris*

Intelligent Freight Transportation, *edited by Petros A. Ioannou*

Modeling and Control of Complex Systems, *edited by Petros A. Ioannou and Andreas Pitsillides*

Wireless Ad Hoc and Sensor Networks: Protocols, Performance, and Control, *Jagannathan Sarangapani*

Stochastic Hybrid Systems, *edited by Christos G. Cassandras and John Lygeros*

Hard Disk Drive: Mechatronics and Control, *Abdullah Al Mamun, Guo Xiao Guo, and Chao Bi*

Autonomous Mobile Robots: Sensing, Control, Decision Making and Applications, *edited by Shuzhi Sam Ge and Frank L. Lewis*

Automation and Control Engineering Series

Reliable Control and Filtering of Linear Systems with Adaptive Mechanisms

Guang-Hong Yang
Northeastern University
Shenyang, People's Republic of China
Dan Ye
Northeastern University
Shenyang, People's Republic of China

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

© 2011 by Taylor and Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number: 978-1-4398-3522-7 (Hardback)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (http://www.copyright.com/) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Library of Congress Cataloging-in-Publication Data

Yang, Guang-Hong.
Reliable control and filtering of linear systems with adaptive mechanisms / Guang-Hong Yang, Dan Ye.
p. cm. -- (Automation and control engineering)
"A CRC title."
Includes bibliographical references and index.
ISBN 978-1-4398-3522-7 (hardcover : alk. paper)
1. Linear control systems--Reliability. 2. Adaptive control systems--Reliability. 3. Fault tolerance (Engineering) I. Ye, Dan, 1979- II. Title. III. Series.

TJ220.Y36 2011
629.8'32--dc22 2010019543

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Contents

Preface

More and more advanced technological systems rely on sophisticated control systems to increase their safety and performances. In the event of system component faults, the conventional feedback control designs may result in unsatisfactory performances or even instability, especially for complex safety critical systems, e.g., aircraft, space craft and nuclear power plant, etc. This has ignited enormous research activities in the search for new design methodologies, for accommodating the component failures and maintaining the acceptable system stability and performances, so that abrupt degradation and total system failures can be averted. Fault-tolerant control (FTC) is a relatively new field of research addressing the design of feedback controllers that ensure safe and efficient operations despite the occurrence of faults. Fault-tolerant design approaches can be broadly classified into two types: passive approach and active approach.

Traditional reliable control is a kind of passive control approach, in which a controller with fixed gain is used throughout normal and fault cases, such that this type of controller is easily implemented. Moreover, several performance indexes such as H_∞, H_2, and cost functions mainly based on algebraic Riccati equation (ARE) or linear matrix inequality (LMI) methods, can be used to describe the performances of the closed-loop systems. However, as the number of possible failures and the degree of system redundancy increase, the passive reliable controllers with fixed gains become more conservative, and attainable control performance indexes may not necessarily be satisfactory. On the other hand, adaptive control is an effective method to design fault-tolerant controllers, too. They rely on the potential of the adjustments of parameters to assure reliability of closed-loop systems in the presence of a wide range of unknown faults. Hence, the resultant solvable conditions can be more relaxed and the corresponding controller gains are variable.

In this book, the aim is to present our recent research results in designing reliable controllers/filters for linear systems. The main feature of this book is that adaptive mechanisms are successfully introduced into the traditional reliable control/filtering and based on the online estimation of eventual faults, the proposed adaptive reliable controller/filter parameters are updated automatically to compensate the fault effects on systems. Moreover, the adaptive performances of resultant closed-loop systems in both normal and actuator/sensor faults cases are optimized, and asymptotic stability is guaranteed. The designed conditions, which are given in the frameworks of linear matrix inequalities (LMIs), are proven to be less conservative than those of the traditional reliable control/filtering. Designs for linear systems with both actuator failures and sensor failures are covered, respectively. We also extend the design idea from linear systems to linear time-delay systems via both memory-less controllers and memory controllers. Moreover, some more recent results for the corresponding adaptive reliable control against actuator saturation are included here. This book provides a coherent approach, and contains valuable reference materials for researchers wishing to explore the area of reliable control. Its contents are also suitable for a one-semester graduate course.

The book focuses exclusively on the issues of reliable control/filtering in the framework of indirect adaptive method, and LMI techniques, starting from the development and main research methods in fault-tolerant control, and offering a systematic presentation of the newly proposed methods for adaptive reliable control/filtering of linear systems against actuator/sensor faults. Designs and guidelines provided here may be used to develop advanced fault-tolerant control techniques to improve reliability, maintainability, and survivability of complex control systems.

This work was partially supported in part by National 973 Program of China (Grant No. 2009CB320604), the Funds of National Science of China (Grant No. 60821063, 60804024, 60974043), China Postdoctoral Science Foundation (Grant No. 20090451276), and 111 Project (B08015).

We would like to thank Dr. Wei Guan for his great help in preparing Chapters 7 and 8.

Guang-Hong Yang and Dan Ye
Northeastern University, China

Symbol Description

$\in$	belongs to
R	field of real numbers
R^n	n-dimensional real Euclidean space
$R^{n\times m}$	set of $n \times m$ real matrices
$I_{n\times m}$	$n \times m$ identity matrix
X^T	transpose of matrix X
$P \geq 0$	symmetric positive semidefinite matrix $P \in R^{n\times n}$
$P > 0$	symmetric positive definite matrix $P \in R^{n\times n}$
$P \leq 0$	symmetric negative semidefinite matrix $P \in R^{n\times n}$
$P < 0$	symmetric negative definite matrix $P \in R^{n\times n}$
P^{-1}	the inverse of matrix P
$\mathrm{rank}(\cdot)$	rank of a matrix
$\|\cdot\|$	Euclidean matrix norm
$\|e\|_2$	L_2-norm of signal e
$L_2[0,\infty)$	space of square integrable functions on $[0,\infty)$
$*$	symmetric terms in a symmetric matrix
Ker Ω	the null space of set Ω
Proj	projection operator
LMI	linear matrix inequality
BLMI	bilinear matrix inequality

1

Introduction

In recent years, fault-tolerant control has become a hot research area because of its importance in practical engineering [5, 54, 113, 121, 124, 134]. Generally, fault-tolerant control methods can be divided into passive fault-tolerant method [84, 125, 133, 149, 164] and active fault-tolerant method [8, 9, 10, 11, 30, 46, 72, 94, 79, 130, 162]. A passive fault-tolerant controller commonly has a simple structure and is easily implemented [108, 117, 126]. The system performances in normal and fault modes can be optimized. However, as the number of faults increases, the design conservatism increases and even the design requirements cannot be achieved. On the other hand, an active fault-tolerant controller may readjust controller parameters or change controller structure to compensate the *fault effects* on systems [6, 19, 128, 131, 130, 129]. Many active fault-tolerant control methods are based on fault detection and diagnosis (FDD) mechanisms. Without FDD mechanisms, some methods have been developed to design fault-tolerant controllers using *indirect adaptive method* or *direct adaptive method*, based on the potentially adjustable capacity of adaptive method. The resultant closed-loop system can be guaranteed to be stable, but the system performance in different modes cannot be optimized [8, 9, 10, 11, 130].

The main contribution of this book is that linear matrix inequality techniques in robust control and adaptive methods have been successfully combined to establish a set of new fault-tolerant control methods [152, 153, 155, 156]. Due to the successful introduction of *adaptive mechanisms*, the proposed method can optimize the closed-loop system performances under different operation modes and reduce the inherent conservatism in the traditional reliable control. Main results are applied to the simulations about F-16/F-18 aircraft models, river pollution model and the F-404 engine model, which show intuitively the feasibility and superiority of the newly proposed methods.

A summary of the rest of the chapters of this monograph is given below.

Chapter 2 presents some classical results about linear matrix inequality (LMI), and H_∞ *control*. Some lemmas to be used to derive the main results of this book are also given.

Chapter 3 investigates the adaptive reliable H_∞ control problem for linear time-invariant system against actuator faults via *state feedback* and *dynamic output feedback*, respectively, where linear matrix inequality technique and adaptive method are combined successfully. The *adaptive H_∞ performance index* is exploited to describe the disturbance attenuation performances of closed-loop systems. Based on the online estimation of actuator faults, an adjustable control law is designed to automatically compensate the effect of faults on systems. In the framework of LMI method, the adaptive H_∞ performances of resultant closed-loop systems in both normal and actuator fault cases are optimized, and asymptotic stability is guaranteed. It is worth noting that the design conditions for the reliable H_∞ controllers with adaptive mechanisms are more relaxed than those for the reliable H_∞ controllers with fixed controller gains. The simulation examples have shown the effectiveness of the proposed adaptive method.

Chapter 4 and Chapter 5 deal with the corresponding adaptive reliable controller and filter design problems against sensor faults, respectively. Besides LMI approach, adaptive method is also used to improve H_∞ performances of systems in both normal and sensor failure cases. An adjustable dynamic output feedback controller/filter is constructed based on the online estimations of sensor faults, which is obtained by *adaptive laws*. More relaxed design conditions than those for designing traditional reliable controller/filter are given to guarantee the asymptotic stability and L_2-*gain*. In sensor failure cases, only the state vector of dynamic output feedback controller/filter and the measured output can be used to construct the adaptive laws, which brings more challenges for dealing with the *adaptive controller* or *adaptive filter* design problem against sensor failures.

Based on the results in Chapter 3, Chapter 6 extends the adaptive reliable controller design problem to linear *time-delay* systems via both memoryless controllers and memory controllers. Moreover, both state feedback and dynamic output feedback designs are considered. Due to the introduction of adaptive mechanisms, more relaxed controller design conditions than those for the traditional controller with fixed gains are derived. Some simulation results have demonstrated the superiority of the newly proposed design methods.

Chapter 7 and Chapter 8 consider the problem of designing adaptive reliable controllers for linear time-invariant systems with actuator saturation. In Chapter 7, a new method for designing indirect adaptive reliable controllers via state feedback is presented for actuator *fault compensations*. The design is developed in the framework of linear matrix inequality (LMI) approach, which can enlarge the domain of asymptotic stability of closed-loop systems in the cases of actuator saturation and actuator failures. The corresponding H_∞ control problem is addressed in Chapter 8. The disturbance tolerance ability of the closed-loop system is measured by an optimal index. Some examples are given to illustrate the efficiency of the design methods.

In Chapter 9, the reliable tracking problem of linear time-invariant systems in the presence of actuator faults is studied. The type of fault considered here is loss of actuator effectiveness, which is a special case of those in the previous chapters. Moreover, we design a novel adaptive reliable controller without using *fault detection and isolation (FDI)* mechanism. The newly proposed method is based on the online estimation of an eventual fault and the addition of a new control law to the normal control law for reducing the fault effect automatically. It should be noted that the normal tracking performance of the resultant closed-loop system is optimized without any conservativeness and the states of fault modes asymptotically track those of the normal mode. Since systems are operating under the normal condition most of time, this contribution is very important in actual control systems design. The proposed results are applied to a *linearized F-16 aircraft model* to demonstrate its effectiveness and superiority.

Based on the theory of Chapter 9, Chapter 10 is devoted to the adaptive reliable control problem of a class of nonlinear time-delay systems with disturbance. The considered actuator fault is loss of effectiveness. The performance index in normal cases is optimized in the framework of LMIs. And new delay-dependent adaptive laws are designed to compensate the fault effects on systems and to guarantee the system stability in normal and fault cases. Moreover, the state vectors of normal and fault cases with disturbance can track that of the normal case without disturbance, which has the designed optimal performance.

2

Preliminaries

In this monograph, *reliable control* and *filtering* problems for systems are investigated under both H_∞ and *guaranteed cost performance index*, using *linear matrix inequality* technique and *adaptive method*. For the convenience of discussion in the rest of monograph, some preliminaries including a few of definitions, notions and lemmas are presented in this chapter.

2.1 Linear Matrix Inequalities

Linear Matrix Inequality (LMI) techniques have emerged as powerful design tools in areas ranging from control engineering to system identification and structural design, since the resulting optimization problems can be solved numerically very efficiently. In recent years, LMI method has been applied to almost every branch of control theory. The following brief description of the LMI method is given to prepare for use in later chapters.

A linear matrix inequality (LMI) is any constraint of the form

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m < 0 \tag{2.1}$$

where $x = [x_1, \cdots, x_m]^T$ is a vector of unknown scalars (the decision or *optimization variables*), $F_0, \cdots, F_m$ are given *symmetric matrices*. $F(x) < 0$ stands for "*negative definite*," i.e., the largest eigenvalue of $F(x)$ is negative.

If "$\leq$ " has replaced "$<$" in (2.1), then the corresponding matrix inequalities becomes non-strict linear matrix inequalities. Note that the constraints $F(x) > 0$ and $F(x) < G(x)$ are special cases of (2.1) since they can be rewritten as $-F(x) < 0$ and $F(x) - G(x) < 0$, respectively.

Denote $\Phi = \{x | F(x) < 0\}$, it is easy to prove Φ is a *convex set*. This fact makes it possible to apply the interior point method of convex *optimization problem* to solve the corresponding problems of LMI.

Note that a system of LMI constraints can be regarded as a single LMI since $F_1(x) < 0, \cdots, F_k(x) < 0$ is equivalent to $F(x) = diag\{F_1(x), \cdots, F_k(x)\}$. Hence, multiple LMI constraints can be imposed on the vector of *decision variables* x without destroying convexity.

The following lemma is one of the most fundamental and commonly results of matrix theory in LMI methods.

Lemma 2.1 *[14] (Schur Complement Lemma) For any given symmetric matrix* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$*, where* $S_{11} \in R^{r\times r}$*. Then the following three conditions are equivalent*

(i) $S < 0$

(ii) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$

(iii) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$

The following three generic optimization problems can be solved by using MATLAB LMI Toolbox.

Here x denotes the vector of decision variables, i.e., of the free entries of the matrix variables $X_1, \cdots X_K$:

(i) *Feasibility* problem

Find $x \in R^N$ (or equivalently matrices $X_1, \cdots X_K$ with prescribed structure) that satisfies the LMI system

$$A(x) < B(x)$$

The corresponding solver is called feasp.

(ii) *Minimization* of a linear objective under LMI constraints

$$\text{Minimize} c^T x \text{ over } x \in R^N \text{ subject to } A(x) < B(x)$$

The corresponding solver is called mincx.

(iii) *Generalized eigenvalue* minimization problem

$$\begin{aligned} &\text{Minimize } \gamma \text{ over } x \in R^N \text{ subject to} \\ &C(x) < D(x) \\ &0 < B(x) \\ &A(x) < \gamma B(x) \end{aligned}$$

The corresponding solver is called gevp.

2.2 H_∞ Control Problem

2.2.1 H_∞ Performance Index

A popular performance measure of a stable linear time-invariant system is the H_∞ *norm* of its *transfer function*. It is defined as follows.

Definition 2.1 *[165] Consider a linear time-invariant continuous-time system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) \\ z(t) &= Cx(t) + D_1\omega(t) \end{aligned} \tag{2.2}$$

where $x(t) \in R^n$ *is the state,* $\omega(t) \in R^s$ *is an exogenous disturbance in* $L_2[0,\infty]$*, that is,*

$$\|\omega(t)\|_2^2 = \int_0^\infty \omega^T(t)\omega(t)dt < \infty$$

and $z(t) \in R^r$ *is the regulated output, respectively.* A, B_1,C,D_1 *are known constant matrices of appropriate dimensions.*

Let $\gamma > 0$ *be a given constant, then the system (2.2) is said to be with an* H_∞ *performance index no larger than* γ*, if the following conditions hold*
(1) System (2.2) is asymptotically stable
(2) Subject to initial conditions $x(0) = 0$*, the transfer function matrix* $T_{\omega z}(s)$ *satisfies,*

$$\|T_{\omega z}(s)\|_\infty := \sup_{\|\omega\|_2 \le 1} \frac{\|z\|_2}{\|\omega\|_2} \le \gamma \tag{2.3}$$

(2.3) is equivalent to

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt, \quad \forall \omega(t) \in L_2[0,\infty) \tag{2.4}$$

It is easy to see that the inequality (2.4) describes the restraint disturbance ability. Moreover, the system performance is better as γ is smaller.

The LMI conditions for the H_∞ *control* problem for system (2.2) is given as follows.

Lemma 2.2 *[119] For given constant* $\gamma > 0$*, the system (2.2) is asymptotically stable and the transfer function matrix* $T_{\omega z}(s)$ *satisfies* $\|T_{\omega z}(s)\|_\infty \le \gamma$ *if and only if there exists a positive symmetric matrix* P *such that*

$$\begin{bmatrix} A^T P + PA & PB_1 & C^T \\ * & -I & D_1^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.5}$$

Next, the H_∞ control problems via state feedback and dynamic output feedback are considered, respectively.

2.2.2 State Feedback H_∞ Control

Consider the following system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + Bu(t) \\ z(t) &= C_1x(t) + D_{11}\omega(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}\omega(t) + D_{22}u(t) \end{aligned} \tag{2.6}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the measured output, $z(t) \in R^q$ is the regulated output and $\omega(t) \in R^s$ is an exogenous disturbance in $L_2[0,\infty]$, respectively.

First assume the state of system is available at every instant, here we will design a *state feedback* controller $u = Kx$ such that the resultant closed-loop system

$$\dot{x}(t) = Ax(t) + BKx(t) + B_1\omega(t) \tag{2.7}$$

is asymptotically stable and the transfer function from ω to z satisfying

$$\|T_{\omega z}(s)\|_\infty = \|(C_1 + D_{12}K)[sI - (A + BK)]^{-1}B_1 + D_{11}\|_\infty \leq \gamma \tag{2.8}$$

By some matrix transformation, the following conclusion can be easily obtained from Lemma 2.2.

Lemma 2.3 *[14] The closed-loop system (2.6) is asymptotically stable and satisfies performance index (2.8) if and only if there exist a positive matrix $X > 0$ and matrix Y such that*

$$\begin{bmatrix} AX + BY + (AX + BY)^T A & B_1 & (C_1X + D_{12}Y)^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.9}$$

Proof 2.1 *From Lemma 2.2, it is easy to see that the closed-loop system (2.6) is asymptotically stable and satisfies performance index (2.8) if and only if there exists a positive matrix $P > 0$ such that*

$$\begin{bmatrix} (A + BK)^T P + P(A + BK) & PB_1 & (C_1 + D_{12}K)^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.10}$$

Since in (2.10) the two unknown variables K and P are existing in nonlinear form, it is difficult to solve inequality (2.10) and obtain the corresponding variables.

Thus, we multiply (2.10) by diag$\{P^{-1}, I, I\}$ on the left and the right, respectively. It follows that (2.10) is equivalent to the following inequality

$$\begin{bmatrix} \Delta + \Delta^T & B_1 & (C_1P^{-1} + D_{12}KP^{-1})^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.11}$$

where $\Delta = AP^{-1} + BKP^{-1}$.
Denote $X = P^{-1}$ and $W = KX$, then the inequality (2.24) can be obtained. The proof is completed.

2.2.3 Dynamic Output Feedback H_∞ Control

In many practical problems, system state information is often not directly measured. Thus, it is difficult to apply the state feedback to control the system. Sometimes, even if the system state can be measured directly, but taking into account the implementation of the control of cost and reliability of the system and other factors, output feedback is usually used to achieve closed-loop system performance requirements.

Assumption 2.1 (A, B, C_2) *is stabilizable and detectable.*

Assumption 2.2 $D_{22} = 0$

Assumption 2.1 is necessary and sufficient to guarantee the stability of closed-loop system via dynamic output feedback. As for Assumption 2.1, it incurs no loss of generality while considerably simplifying calculations.

Consider the following *dynamic output feedback controller*

$$\begin{aligned} \dot{\hat{x}}(t) &= A_K\hat{x}(t) + B_K y(t) \\ u(t) &= C_K\hat{x}(t) + D_K y(t) \end{aligned} \tag{2.12}$$

where $\hat{x}(t)$ is the state of controller (2.12), A_K, B_K, C_K, D_K are the controller parameters to be designed. Then the resulting closed-loop system is

$$\begin{aligned} \dot{\xi}(t) &= A_{cl}\xi(t) + B_{cl}\omega(t) \\ z(t) &= C_{cl}\xi(t) + D_{cl}\omega(t) \end{aligned} \tag{2.13}$$

where

$$\xi = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A + BD_KC_2 & BC_K \\ B_KC_2 & A_K \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_1 + BD_KD_{21} \\ B_KD_{21} \end{bmatrix}$$
$$C_{cl} = \begin{bmatrix} C_1 + D_{12}D_KC_2 & D_{12}C_K \end{bmatrix}, \quad D_{cl} = D_{11} + D_{12}D_KD_{21}$$

From Lemma 2.2, we know that the controller (2.12) renders the closed-loop system (2.13) asymptotically stable and $\|T_{\omega z}(s)\|_\infty \le \gamma$ if and only if there exists a positive matrix X_{cl} such that

$$\begin{bmatrix} A_{cl}^TX_{cl} + X_{cl}A_{cl} & X_{cl}B_{cl} & C_{cl}^T \\ * & -I & D_{cl}^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.14}$$

It is easy to see that in (2.14) the matrix variable X_{cl} and the controller parameters A_K, B_K, C_K, D_K are existing in nonlinear forms, which will bring more difficulty to the dynamic output feedback controller design.

Next, the two results in the framework of LMIs are presented to deal with the dynamic output feedback controller design problem.

Variable elimination method

The first method is the variable elimination method, which is based on the well known *projection lemma.*

Lemma 2.4 (Projection Lemma) *[42, 71] Given a symmetric matrix* $H \in R^{m\times m}$ *and two matrices* P *and* Q *of column dimension* m*, considering the problem of finding some matrix* X *of compatible dimension such that*

$$H + P^TX^TQ + Q^TXP < 0 \tag{2.15}$$

Denote by N_P *and* N_Q *some matrices whose columns form a basis for the null spaces of* P *and* Q*, respectively. Then (2.15) is solvable if and only if*

$$N_P^THN_P < 0, \quad N_Q^THN_Q < 0 \tag{2.16}$$

Gathering all the controller parameters into a single variable

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

and introducing the following short-hands:

$$\begin{aligned} A_0 &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} C_1 & 0 \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \quad \bar{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad \bar{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \end{aligned} \tag{2.17}$$

then the closed-loop matrices $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ can be written as

$$\begin{aligned} A_{cl} &= A_0 + \bar{B}K\bar{C}, \quad B_{cl} = B_0 + \bar{B}K\bar{D}_{21} \\ C_{cl} &= C_0 + \bar{D}_{12}K\bar{C}, \quad B_{cl} = D_{11} + \bar{D}_{12}K\bar{D}_{21} \end{aligned} \tag{2.18}$$

Note that (2.17) involves only plant data and they depend affinely on the controller data K.

Denote

$$\begin{aligned} H_{X_{cl}} &= \begin{bmatrix} A_0^T X_{cl} + X_{cl}A_0 & X_{cl}B_0 & C_0^T \\ * & -I & D_{cl}^T \\ * & * & -\gamma^2 I \end{bmatrix} \\ P_{X_{cl}} &= \begin{bmatrix} \bar{B}^T X_{cl} & 0 & \bar{D}_{12}^T \end{bmatrix}, Q = \begin{bmatrix} \bar{C} & \bar{D}_{21} & 0 \end{bmatrix} \end{aligned}$$

Hence, (2.14) can be described as

$$H_{X_{cl}} + P_{X_{cl}}^T KQ + Q^T K^T P_{X_{cl}} < 0 \tag{2.19}$$

Let

$$X_{cl} = \begin{bmatrix} X & X_2 \\ * & X_3 \end{bmatrix}, \quad X_{cl}^{-1} = \begin{bmatrix} Y & Y_2 \\ * & Y_3 \end{bmatrix} \tag{2.20}$$

Lemma 2.5 *[42, 71] The closed-loop system (2.13) is asymptotically stable and has a dynamic output feedback* H_∞ *controller if and only if there exist a positive definite matrix* $X > 0$ *and* Y *such that*

$$\begin{bmatrix} N_0 & 0 \\ * & I \end{bmatrix} \begin{bmatrix} A^T X + XA & XB_1 & C_1^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix}^T \begin{bmatrix} N_0 & 0 \\ * & I \end{bmatrix} < 0 \tag{2.21}$$

$$\begin{bmatrix} N_c & 0 \\ * & I \end{bmatrix} \begin{bmatrix} A^T Y + YA & YC_1 & B_1^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix}^T \begin{bmatrix} N_c & 0 \\ * & I \end{bmatrix} < 0 \tag{2.22}$$

$$\begin{bmatrix} X & I \\ * & Y \end{bmatrix} \geq 0 \tag{2.23}$$

where N_0 *and* N_c *denote any matrices whose columns form basis of* $Ker([C_2 \quad D_{21}])$ *and* $Ker(B_2^T \quad D_{12}^T]$*, respectively.*

A design procedure of dynamic output feedback controller (2.13) is given as follows.
Step 1. Solve the conditions in Lemma 2.5 to obtain X and Y.
Step 2. Solve $X_2 \in R^{n\times n_k}$ to satisfy $X - Y^{-1} = X_2X_2^T$, where n_k can be chosen as the rank of $X - Y^{-1}$. And let $X_3 = I$, then it follows

$$X_{cl} = \begin{bmatrix} X & X_2 \\ * & I \end{bmatrix}$$

Step 3. Apply the obtained X_{cl} into

$$H_{X_{cl}} + P_{X_{cl}}^T KQ + Q^T K^T P_{X_{cl}} < 0$$

which is linear matrix inequality including only one matrix variable K. Then the controller parameter variable K can be obtained.

Lemma 2.6 *[118] The closed-loop system (2.6) is asymptotically stable and satisfies performance index (2.8) if and only if there exist a positive definite matrix $X > 0$ and matrix Y such that*

$$\begin{bmatrix} AX + BY + (AX + BY)^T A & B_1 & (C_1X + D_{12})Y^T \\ * & -I & D_{11}^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \qquad (2.24)$$

Variable transformation method

Next, another method to deal with the dynamic output feedback H_∞ controller design problem is presented, that is the so-called "variable transformation method."

Denote

$$X_{cl} = \begin{bmatrix} Y & N \\ * & W \end{bmatrix}, \quad X_{cl}^{-1} = \begin{bmatrix} X & M \\ * & Z \end{bmatrix}$$

where $X, Y \in R^{n\times n}$ are symmetric matrices. From $X_{cl}X_{cl}^{-1} = I$, we infer $X_{cl}\begin{bmatrix} X \\ M^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, which lead to

$$X_{cl}\begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \text{ with } \quad F_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, F_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$$

Then $X_{cl}F_1 = F_2$, and after a short calculation it follows

$$F_1^T X_{cl} A_{cl} F_1 = F_2^T A_{cl} F_1 = \begin{bmatrix} AX + B\hat{C} & A + B\hat{D}_K C_2 \\ \hat{A} & YA + \hat{B}C_2 \end{bmatrix}$$

$$F_1^T X_{cl} B_{cl} = \begin{bmatrix} B_1 + B\hat{D}D_{21} \\ YB_1 + \hat{B}D_{21} \end{bmatrix}$$

$$C_{cl}F_1 = \begin{bmatrix} C_1X + D_{12}\hat{C} & C_1 + D_{12}\hat{D}C_2 \end{bmatrix}$$

$$F_1^T X_{cl} F_1 = F_2^T F_1 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

where

$$\begin{aligned}\hat{A} &= Y(A + BD_K C_2)X + NB_K C_2 X + YBC_K M^T + NA_K M^T \\ \hat{B} &= YBD_K + NB_K \\ \hat{C} &= D_K C_2 X + C_K M^T \\ \hat{D} &= D_K \end{aligned} \tag{2.25}$$

If M and N have full row rank, and if $\hat{A}, \hat{B}, \hat{C}, \hat{D}, X$ and Y are given, we can always compute controller matrices A_K, B_K, C_K and D_K satisfying (2.25). If M and N are square (i.e., $k = n$) and invertible matrices, then A_K, B_K, C_K and D_K are unique.

On the other hand, if we multiply (2.14) by $diag\{F_1^T, I, I\}$ on the left and the right, respectively, it follows that (2.14) is equivalent to

$$\begin{bmatrix} AX + XA^T + B\hat{C} + (B\hat{C})^T & \hat{A}^T + (A + B\hat{D}C_2) & B_1 + B\hat{D}D_{21} & (C_1 X + D_{12}\hat{C}^T) \\ * & A^T Y + YA + \hat{B}C_2 + (\hat{B}C_2)^T & YB_1 + \hat{B}D_{21} & (C_1 + D_{12}\hat{D}C_2)^T \\ * & * & -I & (D_{11} + D_{12}\hat{D}D_{21})^T \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.26}$$

It is easy to see that the inequality (2.26) is linear matrix inequality about matrix variables $\hat{A}, \hat{B}, \hat{C}, \hat{D}, X$ and Y. Thus, a feasible solutions of (2.26) can be obtained by using the LMI Toolbox. Moreover, we have proved that the solvability of the LMI (2.26) is necessary for the existence of a stabilizing controller rendering $\|T_{\omega z}(s)\|_\infty \leq \gamma$.

Assume that we have found solutions to the LMI (2.26). First we need to construct M and N.

From the equation $X_{cl}^T X_{cl} = I$, it follows

$$MN^T = I - XY \tag{2.27}$$

By $X_{cl} > 0$, we infer

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{2.28}$$

which implies $I - XY > 0$ is nonsingular. Hence, after getting the values of X and Y, one can always find square and nonsingular M and N satisfying

(2.27). Then the corresponding controller parameters can be obtained by

$$
\begin{aligned}
D_K &= \hat{D} \\
C_K &= (C^{-1} - D_K C_2 X)(M^T)^{-1} \\
B_K &= N^{-1}(\hat{B} - YBD_K) \\
A_K &= N^{-1}[\hat{A} - Y(A + BD_K C_2)X](M^T)^{-1} - B_k C_2 X (M^T)^{-1} - N^{-1} YBC_K
\end{aligned}
\tag{2.29}
$$

Lemma 2.7 *[119] The closed-loop system (2.13) is asymptotically stable and has a dynamic output feedback H_∞ controller if and only if there exist symmetric matrices X, Y and matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ such that the LMIs (2.26) and (2.28) are feasible. Furthermore, if X, Y and $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are the feasible solutions of (2.26) and (2.28), then the matrices M and N can be obtained by the singular value decomposition of $I - XY$. And so the controller parameters are given from (2.29).*

2.3 Some Other Lemmas

Some other lemmas that will be used in the monograph are presented.

Lemma 2.8 (Fisher's Lemma) *[71] Let matrices $Q = Q^T$, G, and a compact subset of real matrices $\mathbf{H}$ be given. Then the following statements are equivalent:*

- *(i) for each $H \in \mathbf{H}$ $\xi^T Q \xi < 0$ for all $\xi \neq 0$ such that $HG\xi = 0$;*
- *(ii) there exists $\Theta = \Theta^T$ such that*

$$Q + G^T \Theta G < 0, \quad \mathbf{N}_H^T \Theta \mathbf{N}_H \geq 0 \text{ for all } H \in \mathbf{H}.$$

where $\mathbf{N}_H$ denotes a matrix whose columns form a basis for the null space of H.

Lemma 2.9 *[43] Consider a scalar quadratic function of $\theta \in R^s$*

$$f(\theta_1, \cdots, \theta_s) = \alpha_0 + \sum_i \alpha_i \theta_i + \sum_{i<j} \beta_{ij} \theta_i \theta_j + \sum_i \gamma_i \theta_i^2 \tag{2.30}$$

and assume that $f(\cdot)$ is multiconvex, that is

$$2\gamma_i = \frac{\partial^2 f}{\partial \theta_i^2}(\theta) \geq 0, \quad \text{for} \ \ i = 1, \cdots s. \tag{2.31}$$

Then $f(\cdot)$ is negative in the hyper-rectangle $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ if and only if it takes negative values at the corners of $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$; that is, if and only if $f(\theta) < 0$ for all θ in the vertex set $\Omega := \{(\theta_1, \cdots, \theta_s) : \theta_i \in \{\underline{\theta}_i, \overline{\theta}_i\}\}$.

Let

$$\Delta_v = \{\delta = (\delta_1 \cdots \delta_s) : \ \delta_i \in \{\underline{\delta}_i, \ \bar{\delta}_i\}\}$$

where $\delta_i \ \ (i = 1 \cdots s)$ are unknown constants.

Lemma 2.10 *If there exists a symmetric matrix* Θ *with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

where $\Theta_{11}, \ \Theta_{22} \in R^{sn \times sn}$ *such that the following inequalities hold:*

$$\Theta_{22ii} \leq 0, \ i = 1, \cdots s$$

with $\Theta_{22ii} \in R^{n \times n}$ *is the* (i,i) *block of* Θ_{22}*.*
For any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \geq 0$$

and

$$\begin{bmatrix} Q & E \\ E^T & F \end{bmatrix} + U^T U + G^T \Theta G < 0 \tag{2.32}$$

then for all $\delta_i \in [\underline{\delta}_i \ \ \bar{\delta}_i]$*,*

$$W(\delta) = Q + \sum_{i=1}^{s} \delta_i E_i + (\sum_{i=1}^{s} \delta_i E_i)^T + \sum_{i=1}^{s} \sum_{j=1}^{s} \delta_i \delta_j F_{ij}$$
$$+ (U_0 + \sum_{i=1}^{s} \delta_i U_i)^T (U_0 + \sum_{i=1}^{s} \delta_i U_i) < 0 \tag{2.33}$$

where

$$Q = Q^T, \quad F_{ij} = F_{ji}^T, \quad \Delta(\delta) = diag \begin{bmatrix} \delta_1 I & \cdots & \delta_s I \end{bmatrix},$$
$$E = [E_1 \ E_2 \cdots E_s], \quad U = [U_0 \ U_1 \cdots U_s],$$
$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1s} \\ F_{21} & F_{22} & \cdots & F_{2s} \\ \vdots & \vdots & \cdots & \vdots \\ F_{s1} & F_{s2} & \cdots & F_{ss} \end{bmatrix}, \quad G = \begin{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}$$

Proof 2.2 *For any* $x \neq 0$*, (2.33) is equivalent to* $x^T W(\delta) x < 0$*, which further is equivalent for any vector* $\begin{bmatrix} x^T & y^T \end{bmatrix}^T \neq 0$ *and* $y = \begin{bmatrix} \delta_1 I_{n \times n} & \cdots & \delta_s I_{n \times n} \end{bmatrix}^T x$

$$\begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} Q & E \\ E^T & F \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + x^T \begin{bmatrix} I_{n \times n} & \delta_1 I_{n \times n} & \cdots & \delta_s I_{n \times n} \end{bmatrix} U^T$$
$$\times U \begin{bmatrix} I_{n \times n} & \delta_1 I_{n \times n} & \cdots & \delta_s I_{n \times n} \end{bmatrix}^T x$$
$$= \begin{bmatrix} x^T & y^T \end{bmatrix} (\begin{bmatrix} Q & E \\ E^T & F \end{bmatrix} + U^T U) \begin{bmatrix} x \\ y \end{bmatrix} < 0 \tag{2.34}$$

and

$$HG\begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{2.35}$$

where $H = \begin{bmatrix} -\Delta(\delta) & I_{sn\times sn} \end{bmatrix}$, $\Delta(\delta) = diag\begin{bmatrix} \delta_1 I_{n\times n} & \cdots & \delta_s I_{n\times n} \end{bmatrix}$.

$$G = \begin{bmatrix} \begin{bmatrix} I_{n\times n} \\ \vdots \\ I_{n\times n} \end{bmatrix} & 0 \\ 0 & I_{sn\times sn} \end{bmatrix}$$

It is easy to see that

$$N_{\mathbf{H}} = \begin{bmatrix} I_{sn\times sn} \\ \Delta(\delta) \end{bmatrix} \tag{2.36}$$

Thus by Lemma 2.8 and (2.34)-(2.36), we have $x^T W(\delta)x < 0$, *for any* $x \neq 0$ *if (2.32) holds. So the proof is completed.*

Lemma 2.11 *For any given constant* $\gamma > 0$, *the following statements are equivalent:*
(i) A_{ef} *is Hurwitz, and* $\|T_{z_{ef}\omega}\| < \gamma$;
where

$$T_{z_{ef}\omega} = C_{ef}(sI - A_{ef})^{-1}B_{ef}, \tag{2.37}$$

with

$$A_{ef} = \begin{bmatrix} A & BC_{Kf} \\ B_{Kf}C_2 & A_{Kf} \end{bmatrix}, \quad B_{ef} = \begin{bmatrix} B_1 \\ B_{Kf}D_{21} \end{bmatrix}$$
$$C_{ef} = [C_1 \quad D_{12}C_{Kf}]$$

(ii) there exists a symmetric matrix $X_a > 0$ *such that*

$$A_{ef}^T X_a + X_a A_{ef} + \frac{1}{\gamma^2} X_a B_{ef} B_{ef}^T X_a + C_{ef}^T C_{ef} < 0 \tag{2.38}$$

(iii) there exist nonsingular matrix Q *and symmetric matrix* $P > 0$ *with*

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} \tag{2.39}$$

such that the following inequality holds

$$A_{eq}^T P + PA_{eq} + \frac{1}{\gamma^2} PB_{eq}B_{eq}^T P + C_{eq}^T C_{eq} < 0, \tag{2.40}$$

where

$$A_{eq} = \begin{bmatrix} A & BC_{Kq} \\ B_{Kq}C_2 & A_{Kq} \end{bmatrix}, \quad B_{eq} = \begin{bmatrix} B_1 \\ B_{Kq}D_{21} \end{bmatrix}$$
$$C_{eq} = [C_1 \quad D_{12}C_{Kq}]$$

and

$$A_{Kq} = Q^{-1}A_{Kf}Q, \quad B_{Kq} = -Q^{-1}B_{Kf}, \quad C_{Kq} = -C_{Kf}Q \tag{2.41}$$

Proof 2.3 *From [165], it is easy to see that (i)⇔(ii).*
Next, we will prove (ii)⇒(iii). Let $X_a = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0$ *such that the inequality (2.38) holds, then there exists an* $\varepsilon \geq 0$ *such that*

$$A_{ef}^T X_b + X_b A_{ef} + \frac{1}{\gamma^2} X_b B_{ef} B_{ef}^T X_b + C_{ef}^T C_{ef} < 0 \tag{2.42}$$

where $X_b = \begin{bmatrix} X_{11} & X_{12} + \varepsilon I \\ X_{12}^T + \varepsilon I & X_{22} \end{bmatrix} > 0$ *and* $X_{12} + \varepsilon I$ *is nonsingular.*
In fact, if X_{12} *is nonsingular, then (2.42) holds for* $\varepsilon = 0$*. For the case of* X_{12} *being singular, then there exists a sufficiently small* $\varepsilon > 0$ *such that (2.42) holds and* $X_{12} + \varepsilon I$ *is nonsingular.*
Denote $Q = X_{22}^{-1}(X_{12}+\varepsilon I)^T$, $A_{Kq} = Q^{-1}A_{Kf}Q$, $B_{Kq} = -Q^{-1}B_{Kf}$, $C_{Kq} = -C_{Kf}Q$, $Y_1 = X_{11}$ *and* $N_1 = (X_{12} + \varepsilon I)Q$
Then by (2.38) and $X_b > 0$*, we have*

$$P = \begin{bmatrix} I & 0 \\ 0 & -Q \end{bmatrix}^T X_b \begin{bmatrix} I & 0 \\ 0 & -Q \end{bmatrix} = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} > 0 \tag{2.43}$$

and

$$\begin{aligned} & A_{eq}^T P + P A_{eq} + \frac{1}{\gamma^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq} \\ &= \begin{bmatrix} I & 0 \\ 0 & -Q \end{bmatrix}^T \Phi \begin{bmatrix} I & 0 \\ 0 & -Q \end{bmatrix} < 0 \end{aligned} \tag{2.44}$$

which imply that (iii) holds, where

$$\Phi = A_{ef}^T X_b + X_b A_{ef} + \frac{1}{\gamma^2} X_b B_{ef} B_{ef}^T X_b + C_{ef}^T C_{ef}.$$

(iii)⇒(ii): Let

$$X_b = \begin{bmatrix} I & 0 \\ 0 & -Q^{-1} \end{bmatrix}^T P \begin{bmatrix} I & 0 \\ 0 & -Q^{-1} \end{bmatrix} \tag{2.45}$$

Then by (2.38) and $P > 0$*, it follows that* $X_b > 0$ *and*

$$\begin{aligned} & A_{ef}^T X_b + X_b A_{ef} + \frac{1}{\gamma^2} X_b B_{ef} B_{ef}^T X_b + C_{ef}^T C_{ef} \\ &= \begin{bmatrix} I & 0 \\ 0 & -Q^{-1} \end{bmatrix}^T \Psi \begin{bmatrix} I & 0 \\ 0 & -Q^{-1} \end{bmatrix} < 0 \end{aligned} \tag{2.46}$$

i.e., (ii) holds, where

$$\Psi = A_{eq}^T P + P A_{eq} + \frac{1}{\gamma^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq}.$$

Thus, the proof is completed.

Lemma 2.12 *[99] For any* $a \in R^n, b \in R^{2n}, Z_0 \in R^{2n\times n}, R \in R^{n\times n}, Y \in R^{n\times 2n}, Z \in R^{2n\times 2n}$*, the following holds:*

$$-2b^T F a \le \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & Y - Z_0^T \\ Y^T - Z_0 & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{2.47}$$

where $\begin{bmatrix} R & Y \\ Y^T & Z \end{bmatrix} \ge 0.$

Lemma 2.13 *[159] Consider an operator* $D(\cdot) : C_{n,d} \to R^n$ *with* $D(x_t) = x(t) + G\int_{t-d}^{t} x(s)ds$*, where* $x(t) \in R^n$ *and* $G \in R^{n\times n}$*. For a given scalar* δ*, where* $0 < \delta < 1$*, if a positive definite symmetric matrix* $M \in R^{n\times n}$ *exists, such that*

$$\begin{bmatrix} -\delta M & dG^T M \\ * & -M \end{bmatrix} < 0 \tag{2.48}$$

holds, then the operator $D(x_t)$ *is stable.*

Lemma 2.14 *[73] For any positive symmetric constant matrix* $M \in R^{n\times n}$*, scalar* $\gamma > 0$*, vector function* $v : [0, \gamma] \to R^n$ *such that the integrations concerned are well defined, then*

$$\left(\int_0^\gamma v(s)ds\right)^T M \left(\int_0^\gamma v(s)ds\right) \le \gamma \left(\int_0^\gamma v^T(s) M v(s) ds\right). \tag{2.49}$$

Lemma 2.15 *[160] Let* $x(t) \in R^n$ *be a vector-valued function with first-order continuous-derivative entries. Then the following integral inequality holds for any matrices* $X = X^T > 0, Y_1, Y_2 \in R^{n\times n}$ *and a scalar* $d \ge 0$

$$\begin{aligned} &-\int_{t-d}^{t} \dot{x}^T(s) X \dot{x}(s) ds \\ &\le \eta^T(t) \begin{bmatrix} Y_1^T + Y_1 & -Y_1^T + Y_2 \\ * & -Y_2^T - Y_2 \end{bmatrix} \eta(t) + d\eta^T(t) \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix} X^{-1} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \eta(t), \end{aligned} \tag{2.50}$$

where $\eta^T(t) = [x^T(t), x^T(t-d)].$

3

Adaptive Reliable Control against Actuator Faults

3.1 Introduction

This chapter is devoted to the study of the reliable H_∞ *control* for *linear systems* against *actuator faults*. Here, a general actuator fault model is considered, which covers the outage cases and the loss of effectiveness cases. It is well known that the *fault-tolerant control* problem has been paid more attention in recent years [74, 105, 145, 161, 136], since unsatisfactory performances or even *instability* may happen in the event of actuator faults [114, 126, 128, 133, 151, 164]. *Reliable control* is a kind of passive control approach, where the same controller with fixed gain is used throughout normal and fault cases such that this type of controller is easily implemented and the performance index can be described. However, as the number of possible failures and the degree of system redundancy increase, the *traditional reliable controller* with fixed gain becomes more conservative and attainable control performance indexes may not necessarily be satisfactory.

The purpose here is to present a novel reliable controller design approach to the reliable control problem by introducing an adaptive mechanism [153, 154]. It will show that the advantages of the *linear matrix inequality (LMI)*approach and *indirect adaptive method* can be combined successfully to design new reliable H_∞ *controllers* via state feedback and dynamic output feedback. With the online estimates of fault values, an adjustable control law can be designed to maintain satisfactory adaptive H_∞ performances. Sufficient conditions for the existence of the above-mentioned adaptive reliable H_∞ controllers are given, and it is shown that these conditions are more relaxed than those for the traditional reliable controller with fixed gains. The proposed approach in this chapter also provides a basis for solving other related problems that are to be studied in the rest of the monograph.

3.2 Problem Statement

Consider a linear *time-invariant* model described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1\omega(t) + Bu(t)\\ z(t) &= C_1x(t) + D_{12}u(t)\\ y(t) &= C_2x(t) + D_{21}\omega(t)\end{aligned} \tag{3.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the *measured output*, $z(t) \in R^q$ is the *regulated output* and $\omega(t) \in R^s$ is an *exogenous disturbance* in $L_2[0, \infty]$, respectively. A, B_1, B, C_1, C_2, D_{12} and D_{21} are known constant matrices of appropriate dimensions.

To formulate the reliable control problem, the following *actuator fault model* from [133] is adopted in this monograph:

$$u_{ij}^F(t) = (1 - \rho_i^j)u_i(t),\ 0 \le \underline{\rho_i^j} \le \rho_i^j \le \bar{\rho}_i^j,\ i = 1 \cdots m, j = 1 \cdots L. \tag{3.2}$$

where ρ_i^j is an unknown constant. Here, the index j denotes the jth *fault mode* and L is the total fault modes. Let $u_{ij}^F(t)$ represent the signal from the ith actuator that has failed in the jth fault mode. For every fault mode, $\underline{\rho_i}^j$ and $\bar{\rho}_i{}^j$ represent the lower and upper bounds of ρ_i^j, respectively. Note that, when $\underline{\rho}_i^j = \bar{\rho}_i^j = 0$, there is no fault for the ith actuator u_i in the jth fault mode. When $\underline{\rho}_i^j = \bar{\rho}_i^j = 1$, the ith actuator u_i is outage in the jth fault mode. When $0 < \underline{\rho}_i^j \le \bar{\rho}_i^j < 1$, in the jth fault mode the type of actuator faults is loss of effectiveness.
Denote

$$u_j^F(t) = [u_{1j}^F(t), u_{2j}^F(t), \cdots, u_{mj}^F(t)]^T = (I - \rho^j)u(t)$$

where $\rho^j = \mathrm{diag}[\rho_1^j, \rho_2^j, \cdots \rho_m^j]$, $\ j = 1 \cdots L$. Considering the lower and upper bounds $(\underline{\rho_i}^j, \bar{\rho}_i{}^j)$, the following set can be defined

$$N_{\rho^j} = \{\rho^j\ |\rho^j = \mathrm{diag}[\rho_1^j, \rho_2^j, \cdots \rho_m^j],\ \rho_i^j = \underline{\rho_i}^j \text{ or } \rho_i^j = \bar{\rho}_i{}^j\}.$$

Thus, the set N_{ρ^j} contains a *maximum* of 2^m elements.
For convenience in the following sections, for all possible fault modes L, we use a uniform actuator fault model

$$u^F(t) = (I - \rho)u(t), \quad \rho \in \{\rho^1, \cdots, \rho^L\} \tag{3.3}$$

and ρ can be described by $\rho = \mathrm{diag}[\rho_1, \rho_2, \cdots \rho_m]$.

The design problem under consideration is to find an *adaptive reliable controller* such that in both normal operation and fault cases, the resulting closed-loop system is *asymptotically stable* and its adaptive H_∞ performance bound is minimized.

3.3 State Feedback Control

In this section, we assume that the state of the system is available at every instant. Then, we design an adaptive reliable H_∞ controller for the linear time-invariant system (9.1) via *state feedback.*

The dynamics with actuator faults (3.3) is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(I-\rho)u(t) + B_1\omega(t) \\ z(t) &= C_1x(t) + D_{12}(I-\rho)u(t). \end{aligned} \tag{3.4}$$

The adaptive reliable *controller structure* is chosen as

$$u(t) = K(\hat{\rho}(t))x(t) = (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t))x(t) \tag{3.5}$$

where $\hat{\rho}(t)$ is the estimate of ρ, $K_a(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{ai}\hat{\rho}_i(t)$ and $K_b(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{bi}\hat{\rho}_i(t)$.
The closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(I-\rho)K(\hat{\rho}(t))x(t) + B_1\omega(t) \\ &= Ax(t) + B(I-\rho)(K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t) + B_1\omega(t) \\ z(t) &= C_1x(t) + D_{12}(I-\rho)K(\hat{\rho}(t))x(t). \end{aligned} \tag{3.6}$$

Next, based on the definition of the *traditional H_∞ performance index*, we give a new definition about an *adaptive H_∞ performance index*, which will be used throughout this monograph.

Definition 3.1 *Consider the following systems*

$$\begin{aligned} \dot{x}(t) &= A_a(\hat{\rho}(t),\rho)x(t) + B_a(\hat{\rho}(t),\rho)\omega(t) \\ z(t) &= C_a(\hat{\rho}(t),\rho)x(t), \quad x(0) = 0 \end{aligned} \tag{3.7}$$

where $x(t) \in R^n$ is the state, $\omega(t) \in R^s$ is an exogenous disturbance in $L_2[0,\infty]$, $z(t) \in R^r$ is the regulated output, respectively. And ρ is a parameter vector, and $\hat{\rho}(t)$ is a time-varying parameter vector to be chosen. Let $\gamma > 0$ be a given constant, then the system (3.7) is said to be with an adaptive H_∞ performance index no larger than γ, if for any $\epsilon > 0$, there exists a $\hat{\rho}(t)$ such that the following conditions hold
(1) System (3.7) is asymptotically stable
(2)

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt + \epsilon \quad \forall\omega(t) \in L_2[0,\infty) \tag{3.8}$$

Remark 3.1 *By the above definition, for any $\eta > 0$, let $\epsilon = \eta^2$, then there exists a $\hat{\rho}(t)$ such that (3.8) holds. Thus, for $\int_0^\infty \omega^T(t)\omega(t)dt > \eta$, we have*

$$\int_0^\infty z^T(t)z(t)dt \le (\gamma^2 + \eta) \int_0^\infty \omega^T(t)\omega(t)dt$$

For $\int_0^\infty \omega^T(t)\omega(t)dt \le \eta$, it follows

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2\eta + \eta^2$$

which shows that the adaptive H_∞ performance index is close to the standard H_∞ performance index when η is sufficiently small.

We have the following equality

$$\begin{aligned}(I-\rho)u(t) &= (I-\rho)(K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t) \\ &= (I-\rho)(K_0 + K_a(\rho))x(t) + (I-\hat{\rho}(t))K_b(\hat{\rho}(t))x(t) \\ &\quad + (I-\rho)K_a(\tilde{\rho})x(t) + \tilde{\rho}K_b(\hat{\rho}(t))x(t) \end{aligned} \tag{3.9}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. Though $K_a(\hat{\rho}(t)$ and $K_b(\hat{\rho}(t)$ have the same forms, we deal with them in different ways in (9.22), which gives more freedom and less conservativeness in Theorem 10.1.
Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m) : \hat{\rho}_i \in \{\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j\}\}\}.$$

Theorem 3.1 *Let $\gamma_f > \gamma_n > 0$ be given constants, then the closed-loop system (9.5) is asymptotically stable and satisfies, in normal cases, i.e., $\rho = 0$,*

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i}, \quad \textit{for } x(0) = 0 \tag{3.10}$$

and in actuator failure cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, satisfies

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i}, \quad \textit{for } x(0) = 0 \tag{3.11}$$

where $\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t) \cdots \tilde{\rho}_m(t)\}, \tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$.
If there exist matrices $X > 0, Y_0, Y_{ai}, Y_{bi}, i = 1 \cdots m$ and a symmetric matrix Θ with

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

and $\Theta_{11},\ \Theta_{22} \in R^{mn\times mn}$ such that the following inequalities hold:

$$\Theta_{22ii} \le 0,\ i = 1, \cdots, m$$

with $\Theta_{22ii} \in R^{n\times n}$ is the (i,i) block of Θ_{22}.

$$\Theta_{11} + \Delta(\hat{\rho})\Theta_{12} + (\Delta(\hat{\rho})\Theta_{12})^T + \Delta(\hat{\rho})\Theta_{22}\Delta(\hat{\rho}) \ge 0,\ \textit{for } \hat{\rho} \in \Delta_{\hat{\rho}}$$

$$\begin{bmatrix} N_{0a} & Z_1 \\ Z_1^T & Z_2 \end{bmatrix} + U^TU + G^T\Theta G < 0,\ \textit{for } \rho = 0$$

$$\begin{bmatrix} N_0 & Z_1 \\ Z_1^T & Z_2 \end{bmatrix} + U^T U + G^T \Theta G < 0, \; for \; \rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j} \tag{3.12}$$

where

$$\begin{aligned} N_{0a} &= AX + B(I-\rho)Y_0 + (AX + B(I-\rho)Y_0)^T + B\sum_{i=1}^{m} \rho_i Y_{ai} \\ &\quad + (B\sum_{i=1}^{m} \rho_i Y_{ai})^T + \frac{1}{\gamma_n^2} B_1 B_1^T, \\ N_0 &= AX + B(I-\rho)Y_0 + (AX + B(I-\rho)Y_0)^T + B\sum_{i=1}^{m} \rho_i Y_{ai} \\ &\quad + (B\sum_{i=1}^{m} \rho_i Y_{ai})^T + \frac{1}{\gamma_f^2} B_1 B_1^T, \\ Z_2 &= \begin{bmatrix} -B^1 Y_{b1} - (B^1 Y_{b1})^T & \cdots & -B^1 Y_{bm} - (B^m Y_{b1})^T \\ \vdots & \vdots & \vdots \\ -B^m Y_{b1} - (B^1 Y_{bm})^T & \cdots & -B^m Y_{bm} - (B^m Y_{bm})^T \end{bmatrix}, \\ G &= \begin{bmatrix} \begin{bmatrix} I_{n\times n} \\ \vdots \\ I_{n\times n} \end{bmatrix} & 0 \\ 0 & I_{mn\times mn} \end{bmatrix}, \\ Z_1 &= \begin{bmatrix} -B\rho Y_{a1} + BY_{b1} & \cdots & -B\rho Y_{am} + BY_{bm} \end{bmatrix} \\ U &= \begin{bmatrix} C_1 X + D_{12}(I-\rho)Y_0 & \Xi_1 & \cdots & \Xi_m \end{bmatrix} \\ \Xi_i &= D_{12}(I-\rho)(Y_{ai} + Y_{bi}), \; \Delta(\hat{\rho}) = diag[\hat{\rho}_1 I_{n\times n} \cdots \hat{\rho}_m I_{n\times n}]. \end{aligned}$$

and also $\hat{\rho}_i(t)$ *is determined according to the adaptive law*

$$\begin{aligned} \dot{\hat{\rho}}_i &= Proj_{[\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j]\}}\{L_{1i}\} \\ &= \begin{cases} 0, & if \begin{array}{l} \hat{\rho}_i = \min_j\{\underline{\rho}_i^j\} \; and \; L_{1i} \le 0 \\ or \; \hat{\rho}_i = \max_j\{\bar{\rho}_i^j\} \; and \; L_{1i} \ge 0; \end{array} \\ L_{1i}, & otherwise \end{cases} \end{aligned} \tag{3.13}$$

where $L_{1i} = -l_i x^T(t)[PB^i K_b(\hat{\rho}) + PBK_{ai}]x(t)$ *and* $P = X^{-1}, K_{ai} = Y_{ai}X^{-1}$, $K_{bi} = Y_{bi}X^{-1}$ *and* $l_i > 0 (i = 1 \cdots m)$ *is the adaptive law gain to be chosen according to practical applications. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimates* $\hat{\rho}_i(t)$ *to the interval* $[\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j\}]$. *Then the controller gain is given by*

$$K(\hat{\rho}) = Y_0 X^{-1} + \sum_{i=1}^{m} \hat{\rho}_i Y_{ai} X^{-1} + \sum_{i=1}^{m} \hat{\rho}_i Y_{bi} X^{-1}. \tag{3.14}$$

Proof 3.1 *We choose the following Lyapunov function*

$$V = x^T(t)Px(t) + \sum_{i=1}^{m} \frac{\tilde{\rho}_i{}^2(t)}{l_i} \tag{3.15}$$

Then from the derivative of V along the closed-loop system, we can get

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \\
&\le x^T\{PA + PB[(I-\rho)(K_0 + K_a(\rho(t)) + (I-\hat{\rho})K_b(\hat{\rho}(t))] \\
&\quad + (PA + PB[(I-\rho)(K_0 + K_a(\rho(t)) + (I-\hat{\rho})K_b(\hat{\rho}(t))])^T \\
&\quad + (C_1 + D_{12}(I-\rho)K(\hat{\rho}))^T(C_1 + D_{12}(I-\rho)K(\hat{\rho})) + \frac{1}{\gamma_f^2}PB_1B_1^TP\}x \\
&\quad - (\gamma_f\omega^T - \frac{1}{\gamma_f}x^TPB_1)(\gamma_f\omega - \frac{1}{\gamma_f}B_1^TPx) + 2\sum_{i=1}^{m}\frac{\tilde{\rho}_i\dot{\tilde{\rho}}_i}{l_i} \\
&\quad + 2x^TPB[(I-\rho)K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x.
\end{aligned} \tag{3.16}$$

Let $B = [b^1 \cdots b^m]$ *and* $B^i = [0 \cdots B^i \dots 0]$, *then*

$$PB\tilde{\rho}K_b(\hat{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PB^i K_b(\hat{\rho}) \tag{3.17}$$

$$PBK_a(\tilde{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PBK_{ai} \tag{3.18}$$

Furthermore, it follows

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \\
&\le x^T\{PA + PB[(I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho})] \\
&+ (PA + PB[(I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}))^T \\
&+ (C_1 + D_{12}(I-\rho)K(\hat{\rho}))^T(C_1 + D_{12}(I-\rho)K(\hat{\rho})) + \frac{1}{\gamma_f^2}PB_1B_1^TP\}x \\
&+ 2x^TPB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x + 2\sum_{i=1}^{m}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}.
\end{aligned} \tag{3.19}$$

Choose the adaptive law as (9.30), then it is sufficient to show

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \\
&\le x^T\left[M_1 + M_2 + (C_1 + D_{12}(I-\rho)K(\hat{\rho}))^T(C_1 + D_{12}(I-\rho)K(\hat{\rho}))\right]x < 0
\end{aligned} \tag{3.20}$$

where

$$M_1 = PA + A^TP + \frac{1}{\gamma_f^2}PB_1B_1^TP,$$

$$M_2 = M + M^T, \quad M = PB_2[(I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho})].$$

Let $X = P^{-1}, Y_0 = K_0X, Y_{ai} = K_{ai}X, Y_{bi} = K_{bi}X, i = 1\cdots m$, *if for any* $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$N_0 + N_1(\hat{\rho}_i) + N_2(\hat{\rho}_i) \\ + (C_1X + D_{12}(I-\rho)Y_0 + N_3(\hat{\rho}_i))^T(C_1X + D_{12}(I-\rho)Y_0 + N_3(\hat{\rho}_i)) < 0, \tag{3.21}$$

then (3.20) is satisfied for any vector $x \in R^n$, *where*

$$\begin{aligned} N_0 &= AX + B(I-\rho)Y_0 + (AX + B(I-\rho)Y_0)^T + B\sum_{i=1}^m \rho_i Y_{ai} \\ &\quad + (B\sum_{i=1}^m \rho_i Y_{ai})^T + \frac{1}{\gamma_f^2}B_1B_1^T, \\ N_1(\hat{\rho}_i) &= -B\rho\sum_{i=1}^m \hat{\rho}_i Y_{ai} + B\sum_{i=1}^m \hat{\rho}_i Y_{bi} + (-B\rho\sum_{i=1}^m \hat{\rho}_i Y_{ai} + B\sum_{i=1}^m \hat{\rho}_i Y_{bi})^T, \\ N_2(\hat{\rho}_i) &= \sum_{i=1}^m\sum_{j=1}^m \hat{\rho}_i\hat{\rho}_j(-B^iY_{bj} - Y_{bi}^TB^{jT}), \\ N_3(\hat{\rho}_i) &= \sum_{i=1}^m \hat{\rho}_i D_{12}(I-\rho)(Y_{ai} + Y_{bi}). \end{aligned}$$

By Lemma 2.10 and (3.12), it follows that (3.21) holds for any $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$ *and* $\hat{\rho}$ *satisfying (9.30). So (3.20) holds for any* $x \neq 0$, *which further implies that* $\dot{V}(t) < 0$ *for any* $x \neq 0$. *Thus, the closed-loop system (9.5) is asymptotically stable for the actuator failure cases. Furthermore,*

$$\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \leq 0.$$

Integrate the above-mentioned inequalities from 0 to ∞ *on both sides, it follows*

$$V(\infty) - V(0) + \int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2\int_0^\infty \omega^T(t)\omega(t)dt.$$

Then

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2\int_0^\infty \omega^T(t)\omega(t)dt + x^T(0)Px(0) + \sum_{i=1}^m \frac{\tilde{\rho}_i{}^2(0)}{l_i} \tag{3.22}$$

which implies that (3.11) holds. The proof for (3.10) and asymptotic stability of the closed-loop system (9.5) for that normal case is similar, and omitted here.

Corollary 3.1 *Assume that (3.12) holds for $\gamma_f > \gamma_n > 0$, controller gain and adaptive law are given by (3.14) and (9.30), respectively. Then the closed-loop system (9.5) is asymptotically stable and with adaptive H_∞ performance indexes no larger than γ_n and γ_f for normal and actuator failure cases, respectively.*

Proof 3.2 *Let $F(0) = \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(0)}{l_i}$. Then, by (9.30) and (9.2), it follows that $\tilde{\rho}_i(0) \leq \max_j\{\bar{\rho}_i^j\} - \min_j\{\underline{\rho}_i^j\}$. We can choose l_i sufficiently large so that $F(0)$ is sufficiently small. Thus, from (3.10), (3.11), Definition 3.1 and Remark 3.1, the adaptive H_∞ performance index is close to the standard H_∞ performance index when l_i is chosen to be sufficiently large. Then the conclusion follows.*

Remark 3.2 *Theorem 10.1 gives a sufficient condition for the existence of an adaptive reliable H_∞ controller via state feedback. In Theorem 10.1, if set $Y_{ai} = 0, Y_{bi} = 0, i = 1 \cdots m$, then the conditions of Theorem 10.1 reduce to $\rho = 0$*

$$AX + B(I-\rho)Y_0 + (AX + B(I-\rho)Y_0)^T + \frac{1}{\gamma_n^2} B_1 B_1^T + (C_1 X + D_{12}(I-\rho)Y_0)^T (C_1 X + D_{12}(I-\rho)Y_0) < 0, \tag{3.23}$$

for $\rho \in \{\rho^1 \cdots \rho^L\}$

$$AX + B(I-\rho)Y_0 + (AX + B(I-\rho)Y_0)^T + \frac{1}{\gamma_f^2} B_1 B_1^T + (C_1 X + D_{12}(I-\rho)Y_0)^T (C_1 X + D_{12}(I-\rho)Y_0) < 0. \tag{3.24}$$

From [165], it follows that conditions (3.23) and (3.24) are sufficient for guaranteeing the closed-loop system (9.5) with $u = K_0 x, K_0 = Y_0 X^{-1}$ to be asymptotically stable and with H_∞ performance indexes no larger than γ_n and γ_f for normal and actuator failure cases, respectively, which can also be derived by using the LMI approach to robust control [14]. This just gives a design method for traditional reliable H_∞ controllers via fixed gains. The above fact shows that the design condition for adaptive reliable H_∞ controllers given in Theorem 10.1 is more relaxed than that described by (3.23) and (3.24) for the traditional reliable H_∞ controller design with fixed gains.

Remark 3.3 *From Theorem 10.1, it is easy to see that controller gains $K_0, K_{ai}, K_{bi} (i = 1, \cdots, m)$ are obtained off-line by Algorithm 3.1 while the estimation $\hat{\rho}_i$ is automatically updating online according to the designed adaptive law (9.30). Thus due to the introduction of adaptive mechanisms, the resultant controller gain (3.14) is variable, which is different from traditional controller with fixed gain.*

From Theorem 10.1 and Corollary 3.1, we have the following algorithm to optimize the adaptive H_∞ performance in normal and fault cases.

Algorithm 3.1 *Let γ_n and γ_f denote the adaptive H_∞ performance bounds for the normal case and fault cases of the closed-loop system (9.5), respectively. Then γ_n and γ_f are minimized if the following optimization problem is solvable*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad (3.12) \tag{3.25}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, and α and β are weighting coefficients.

Since systems are operating under the normal condition most of the time, we can choose $\alpha > \beta$ in (3.25).

3.4 Dynamic Output Feedback Control

In this section, the problem of designing an adaptive reliable H_∞ dynamic output feedback controller for the linear time-invariant model (9.1) is studied. The main difficulty in this section is that only the state vector of dynamic output feedback controller and the measured output can be used to construct adaptive laws, which brings more challenges here.

The fault model is the same as (3.3) in section 3, that is

$$u^F(t) = (I - \rho)u(t), \quad \rho \in \{\rho^1 \cdots \rho^L\}$$

with $\rho = \mathrm{diag}\{\rho_1 \cdots \rho_m\}$.
Consider the traditional dynamic output feedback controller with fixed gains

$$\begin{aligned} \dot{\xi}(t) &= A_{Kf}\xi(t) + B_{Kf}y(t) \\ u^F(t) &= (I - \rho)C_{Kf}\xi(t) \end{aligned} \tag{3.26}$$

then the resulting closed-loop system with actuator faults (3.3) is

$$\begin{aligned} \dot{x}_{ef}(t) &= A_{ef}x_{ef}(t) + B_{ef}\omega(t) \\ z_f(t) &= C_{ef}x_{ef}(t) \end{aligned} \tag{3.27}$$

where $x_{ef}(t) = [x_f^T(t) \ \xi^T(t)]^T$,

$$A_{ef} = \begin{bmatrix} A & B(I-\rho)C_{Kf} \\ B_{Kf}C_2 & A_{Kf} \end{bmatrix}, \quad B_{ef} = \begin{bmatrix} B_1 \\ B_{Kf}D_{21} \end{bmatrix}$$
$$C_{ef} = [C_1 \ \ D_{12}(I-\rho)C_{Kf}]$$

Lemma 3.1 *Consider the closed-loop system (3.27), and for given constants $\gamma_n > 0$, γ_f, the following statements are equivalent:*
(i) there exist symmetric matrix $X > 0$ and the controller (3.26) such that in normal case, that is $\rho = 0$,

$$A_{ef}^T X + XA_{ef} + \frac{1}{\gamma_n^2} XB_{ef}B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{3.28}$$

in actuator fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{ef}^T X + X A_{ef} + \frac{1}{\gamma_f^2} X B_{ef} B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{3.29}$$

(ii) there exist a nonsingular matrix Q, *symmetric matrix* $P > 0$, *and the controller (3.26)*

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} \tag{3.30}$$

such that in normal case, that is $\rho = 0$,

$$A_{eq}^T P + P A_{eq} + \frac{1}{\gamma_n^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq} < 0 \tag{3.31}$$

in actuator fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{eq}^T P + P A_{eq} + \frac{1}{\gamma_f^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq} < 0 \tag{3.32}$$

where

$$\begin{aligned} A_{eq} &= \begin{bmatrix} A & B(I-\rho)C_{Kq} \\ B_{Kq}C_2 & A_{Kq} \end{bmatrix}, \quad B_{eq} = \begin{bmatrix} B_1 \\ B_{Kq}D_{21} \end{bmatrix} \\ C_{eq} &= [C_1 \quad D_{12}(I-\rho)C_{Kq}] \quad \textit{and} \\ A_{Kq} &= Q^{-1}A_{Kf}Q, \quad B_{Kq} = -Q^{-1}B_{Kf}, \quad C_{Kq} = -C_{Kf}Q \end{aligned}$$

(iii) there exist symmetric matrices Y_1 *and* N_1 *satisfying* $0 < N_1 < Y_1$, *and the controller gains of (3.26)* $A_{Kf} = A_{Kq}$,$B_{Kf} = B_{Kq}$ *and* $C_{Kf} = C_{Kq}$ *such that*
in normal case, that is $\rho = 0$,

$$V_{aa1} := \begin{bmatrix} W_0 & W_1 & Y_1B_1 - N_1B_{Kq}D_{21} & C_1^T \\ * & W_2 & -N_1B_1 + N_1B_{Kq}D_{21} & C_{Kq}^T(I-\rho)D_{12}^T \\ * & * & -\gamma_n^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{3.33}$$

in actuator fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$V_{a1} := \begin{bmatrix} W_0 & W_1 & Y_1B_1 - N_1B_{Kq}D_{21} & C_1^T \\ * & W_2 & -N_1B_1 + N_1B_{Kq}D_{21} & C_{Kq}^T(I-\rho)D_{12}^T \\ * & * & -\gamma_f^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{3.34}$$

where

$$\begin{aligned} W_0 &= Y_1A - N_1B_{Kq}C_2 + (Y_1A - N_1B_{Kq}C_2)^T \\ W_1 &= Y_1B(I-\rho)C_{Kq} - N_1A_{Kq} + (-N_1A + N_1B_{Kq}C_2)^T \\ W_2 &= -N_1B(I-\rho)C_{Kq} + N_1A_{Kq} + (-N_1B(I-\rho)C_{Kq} + N_1A_{Kq})^T \end{aligned}$$

Proof 3.3 *From the proof of Lemma 2.11, it is easy to conclude (i)* $\Longleftrightarrow$ *(ii), so we omit it here. On the other hand,* $P > 0$ *is equivalent to* $0 < N_1 < Y_1$*, thus by some simple algebra computation, it follows (ii)* $\Longleftrightarrow$ *(iii). The proof is complete.*

Remark 3.4 *From Lemma 3.1, it follows that the special form of* P *with* $P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix}$ *doesn't bring any conservativeness when we design the dynamic output feedback controller with fixed gain.*

From Lemma 3.1, we have the following algorithm to optimize the H_∞ performances in normal and fault cases for the reliable controller design with fixed gains.

Algorithm 3.2 *Step 1 Solving the following optimization problem*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad X > 0, \quad (3.23) \quad\quad (3.24) \tag{3.35}$$

where $\eta_n = \gamma_n^2$*,* $\eta_f = \gamma_f^2$*, and* α*,* β *are weighting coefficients.*
Denote the optimal solution as X_{opt} *and* Y_{0opt}*, then let* $C_{Kf} = Y_{0opt}X_{opt}^{-1}$*.*
Step 2 Let $N_1 A_{Kf} = \bar{A}_{Kf}$*,* $N_1 B_{Kf} = \bar{B}_{Kf}$*.*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad 0 < N_1 < Y_1 \quad (3.33) \quad\quad (3.34) \tag{3.36}$$

Denote the optimal solution as $\bar{A}_{Kf} = \bar{A}_{Kfopt}$*,* $\bar{B}_{Kf} = \bar{B}_{Kfopt}$*,* $N_1 = N_{1opt}$*. Then the resultant dynamic output feedback controller gains can be obtained by* $A_{Kf} = N_1^{-1}\bar{A}_{Kf}$*,* $B_{Kf} = N_1^{-1}\bar{B}_{Kf}$*,* $C_{Kf} = Y_{0opt}X_{opt}^{-1}$*.*

Remark 3.5 *It should be noted that the conditions (3.33) and (3.34) are non-convex, however with* C_{Kf} *fixed, and* $N_1 A_{Kf}$*,* $N_1 B_{Kf}$ *are defined as new variables, the conditions (3.33) and (3.34) are linear matrix inequalities. Moreover, algorithm 3.2 gives a method for the reliable dynamic output controller design with fixed gains by two-step optimizations. Step 1 is to a* C_{Kf}*, which solves the corresponding design problem via state feedback. With the* C_{K0} *fixed, controller parameter matrices* A_{Kf} *and* B_{Kf} *can be obtained by performing Step 2.*

In order to reduce the conservativeness of the dynamic output feedback controller with fixed gains, the following dynamic output feedback controller with variable gains is given

$$\begin{aligned} \dot{\xi}(t) &= A_K(\hat{\rho})\xi(t) + B_K(\hat{\rho})y(t) \\ u(t) &= C_K(\hat{\rho})\xi(t) \end{aligned} \tag{3.37}$$

where $\hat{\rho}(t)$ is the estimation of ρ. Denote

$$A_K(\hat{\rho}) = A_{K0} + A_{Ka}(\hat{\rho}) + A_{Kb}(\hat{\rho})$$

$$B_K(\hat{\rho}) = B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho}), \quad C_K(\hat{\rho}) = C_{K0} + C_{Ka}(\hat{\rho}) + C_{Kb}(\hat{\rho})$$

with

$$A_{Ka}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i A_{Kai}, \quad A_{Kb}(\hat{\rho}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\rho}_i \hat{\rho}_j A_{Kbij} + \sum_{i=1}^{m} \hat{\rho}_i A_{Kbi}$$

$$B_{Ka}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i B_{Kai}, \quad B_{Kb}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i B_{Kbi},$$

$$C_{Ka}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i C_{Kai}, \quad C_{Kb}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i C_{Kbi}$$

Combining (9.1) and (3.37), the dynamics with actuator faults (3.3) is described by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e \omega(t) \\ z(t) &= C_e x_e(t) \end{aligned} \tag{3.38}$$

where $x_e(t) = [x^T(t)\ \xi^T(t)]^T$,

$$A_e = \begin{bmatrix} A & B_2(I-\rho)C_K(\hat{\rho}) \\ B_K(\hat{\rho})C_2 & A_K(\hat{\rho}) \end{bmatrix}, \quad B_e = \begin{bmatrix} B_1 \\ B_K(\hat{\rho})D_{21} \end{bmatrix}$$

$$C_e = [C_1 \quad D_{12}(I-\rho)C_K(\hat{\rho})]$$

The following theorem presents a sufficient condition for the solvability of the reliable control problem via dynamic output feedback in the framework of LMI approach and adaptive laws.

Theorem 3.2 *Assume that C_2 is of full rank, and let $\gamma_f > \gamma_n > 0$ be given constants, then the closed-loop system (3.38) with the adaptive dynamic output feedback controller (3.37) is asymptotically stable and satisfies for $x(0) = 0$, in normal case, i.e., $\rho = 0$,*

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(0)}{l_i}, \tag{3.39}$$

and in actuator failures cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, satisfies for $x(0) = 0$

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(0)}{l_i}, \tag{3.40}$$

where $\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t) \cdots \tilde{\rho}_m(t)\}$, $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$, if there exist matrices $0 < N_1 < Y_1$, A_{K0}, A_{Kai}, A_{Kbi}, A_{Kbij}, B_{K0}, B_{Kai}, B_{Kbi}, $C_{K0}, C_{Kai}, C_{Kbi}, i, j = 1 \cdots m$ and a symmetric matrix Θ with

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

and $\Theta_{11},\ \Theta_{22} \in R^{m(2n+s)\times m(2n+s)}$ *such that the following inequalities hold:*

$$\Theta_{22ii} \leq 0,\ i = 1, \cdots, m$$

with $\Theta_{22ii} \in R^{(2n+s)\times(2n+s)}$ *is the* (i,i) *block of* Θ_{22}.

$$\Theta_{11} + \Delta(\hat{\rho})\Theta_{12} + (\Delta(\hat{\rho})\Theta_{12})^T + \Delta(\hat{\rho})\Theta_{22}\Delta(\hat{\rho}) \geq 0,\ \textit{for}\ \hat{\rho} \in \Delta_{\hat{\rho}}$$

$$\begin{bmatrix} Q_{1a} & R \\ R^T & S \end{bmatrix} + V_0^T V_0 + G^T \Theta G < 0, \quad \textit{for}\ \rho = 0$$

$$\begin{bmatrix} Q_1 & R \\ R^T & S \end{bmatrix} + V_0^T V_0 + G^T \Theta G < 0, \quad \textit{for}\ \rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j} \tag{3.41}$$

where $N_1 < Y_1$ *means that* $N_1 - Y_1 < 0$, *and*

$$Q_{1a} = \begin{bmatrix} Y_1 A - N_1 B_{K0} C_2 + (Y_1 A - N_1 B_{K0} C_2)^T & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & -\gamma_n^2 I \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} Y_1 A - N_1 B_{K0} C_2 + (Y_1 A - N_1 B_{K0} C_2)^T & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & -\gamma_f^2 I \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}, \quad S = [S_{ij}], i, j = 1 \cdots m,$$

$C_2^\perp$ *satisfies* $C_2 C_2^{\perp T} = 0$ *and* $C_2^\perp C_2^{\perp T}$ *is nonsingular,*

$$R_i = \begin{bmatrix} -N_1 B_{Kbi} C_2 - N_1 B_{Kai} C_2 & T_{5i} & T_{6i} \\ N_1 B_{Kbi} C_2 + N_1 B_{Kai} C_2 \Gamma \begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix} & T_{7i} & T_{8i} \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_{ij} = \begin{bmatrix} 0 & T_{9ij} & 0, \\ T_{10ij} & T_{11ij} & (Y_1 B^j C_{Kbi})^T \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\ 0 & T_{12ij} & 0 \end{bmatrix},$$

$$
\begin{aligned}
V_0 &= \begin{bmatrix} V_{00} & V_{01} & \cdots & V_{0m} \end{bmatrix}, \quad V_{00} = \begin{bmatrix} C_1 & D_{12}(I-\rho)C_{K0} & 0 \end{bmatrix}, \\
V_{0i} &= \begin{bmatrix} 0 & D_{12}(I-\rho)(C_{Kai}+C_{Kbi}) & 0 \end{bmatrix}, \\
T_1 &= Y_1 B[(I-\rho)C_{K0} + C_{Ka}(\rho)] - N_1 A_{K0} - N_1 A_{Ka}(\rho) \\
&\quad + (-N_1 A + N_1 B_{K0} C_2 + N_1 B_{Ka}(\rho) C_2 - [N_1 B_{Ka}(\rho) C_2 \Gamma)] \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix})^T \\
&\quad + \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T \Gamma^T [-Y_1 B C_{Ka}(\rho) + N_1 A_{Ka}(\rho)] \\
T_2 &= Y_1 B_1 - N_1 B_{K0} D_{21}, \\
T_3 &= -N_1 B[(I-\rho)C_{K0} + C_{Ka}(\rho)] + (-N_1 B[(I-\rho)C_{K0} + C_{Ka}(\rho)])^T \\
&\quad + N_1 A_{K0} + N_1 A_{Ka}(\rho) + (N_1 A_{K0} + N_1 A_{Ka}(\rho))^T, \\
T_4 &= -N_1 B_1 + N_1 B_{K0} D_{21} - N_1 B_{Ka}(\rho) C_2 \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
&\quad + [-Y_1 B C_{Ka}(\rho) + N_1 A_{Ka}(\rho)]^T \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix}, \\
T_{5i} &= Y_1 B[-\rho C_{Kai} + C_{Kbi}] - N_1 A_{Kbi} \\
&\quad + \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T \Gamma^T [Y_1 B (C_{Kai} - \rho C_{Kbi}) - N_1 A_{Kai}], \\
T_{6i} &= -N_1 B_{Kbi} D_{21} - N_1 B_{Kai} D_{21} \\
T_{7i} &= N_1 B \rho C_{Kai} - N_1 B C_{Kbi} + N_1 A_{Kbi}, \\
T_{8i} &= (Y_1 B C_{Kai} - Y_1 B \rho C_{Kbi} - N_1 A_{Kai})^T \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
&\quad + N_1 B_{Kai} D_{21} + N_1 B_{Kbi} D_{21} + N_1 B_{Kai} C_2 \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix}, \\
T_{9ij} &= -Y_1 B^i C_{Kbj} - N_1 A_{Kbij} + \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T \Gamma^T Y_1 B^i C_{Kbj} \\
T_{10ij} &= (-Y_1 B^j C_{Kbi} - N_1 A_{Kbji} + \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T \Gamma^T Y_1 B^j C_{Kbi})^T, \\
T_{11ij} &= N_1 B^i C_{Kbj} + N_1 A_{Kbij} + (N_1 B^i C_{Kbj} + N_1 A_{Kbij})^T \\
T_{12ij} &= \begin{bmatrix} -D_{21}^T & 0 \end{bmatrix} \Gamma^T Y_1 B^i C_{Kbj}, \\
\Delta(\hat{\rho}) &= diag[\hat{\rho}_1 I_{(2n+s)\times(2n+s)} \cdots \hat{\rho}_m I_{(2n+s)\times(2n+s)}], \\
G &= \begin{bmatrix} \begin{bmatrix} I_{(2n+s)\times(2n+s)} \\ \vdots \\ I_{(2n+s)\times(2n+s)} \end{bmatrix} & 0 \\ 0 & I_{m(2n+s)\times m(2n+s)} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}^{-1}
\end{aligned}
$$

and also $\hat{\rho}_i(t)$ *is determined according to the adaptive law*

$$\begin{aligned}\dot{\hat{\rho}}_i &= Proj_{[\min\limits_j\{\underline{\rho}_i^j\},\max\limits_j\{\bar{\rho}_i^j\}]}\{L_{2i}\} \\ &= \begin{cases} 0, & \text{if } \begin{array}{l}\hat{\rho}_i = \min\limits_j\{\underline{\rho}_i^j\} \text{ and } L_{2i} \le 0 \\ \text{or } \hat{\rho}_i = \max\limits_j\{\bar{\rho}_i^j\} \text{ and } L_{2i} \ge 0;\end{array} \\ L_{2i}, & \text{otherwise}\end{cases}\end{aligned} \tag{3.42}$$

where $L_{2i} = -l_i[\xi^T(N_1A_{Kai} - BC_{Kai} - B^iC_{Kb}(\hat{\rho}))\xi + \begin{bmatrix} y \\ 0 \end{bmatrix}^T \begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}^{-T} (Y_1BC_{Kai} + Y_1B^iC_{Kb}(\hat{\rho}) - N_1A_{Kai})\xi + \xi^TN_1B_{Kai}C_2\Gamma\begin{bmatrix} y \\ 0 \end{bmatrix}]$ *and* $l_i > 0$ $(i = 1\cdots m)$ *is the adaptive law gain to be chosen according to practical applications. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimation* $\hat{\rho}_i(t)$ *to the interval* $[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j\}]$.

Proof 3.4 *Choose the following Lyapunov function*

$$V(t) = x_e^TPx_e + \sum_{i=1}^{m}\frac{\tilde{\rho}_i^2(t)}{l_i}$$

By $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$, *it follows*

$$\begin{aligned}(I-\rho)C_K(\hat{\rho}) &= (I-\rho)(C_{K0} + C_{Ka}(\hat{\rho}) + C_{Kb}(\hat{\rho})) \\ &= (I-\rho)C_{K0} + C_{Ka}(\rho) - \rho C_{Ka}(\hat{\rho}) \\ &\quad + (I-\hat{\rho})C_{Kb}(\hat{\rho}) + C_{Ka}(\tilde{\rho}) + \tilde{\rho}C_{Kb}(\hat{\rho}) \\ B_{Ka}(\tilde{\rho}) &= B_{Ka}(\rho) + B_{Ka}(\hat{\rho}) \\ A_{Ka}(\tilde{\rho}) &= A_{Ka}(\rho) + A_{Ka}(\hat{\rho})\end{aligned}$$

Then A_e *can be written as*

$$A_e = A_{e1} + A_{e2} + A_{e3}$$

where

$$A_{e1} = \begin{bmatrix} A & A_{e1a} \\ [B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C_2 & A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho}) \end{bmatrix}$$

$$A_{e2} = \begin{bmatrix} 0 & B_2C_{Ka}(\tilde{\rho}) + B_2\tilde{\rho}C_{Kb}(\hat{\rho}) \\ 0 & A_{Ka}(\tilde{\rho}) \end{bmatrix}, \quad A_{e3} = \begin{bmatrix} 0 & 0 \\ B_{Ka}(\tilde{\rho})C_2 & 0 \end{bmatrix}$$

with

$$A_{e1} = B_2[(I-\rho)C_{K0} + C_{Ka}(\rho) - \rho C_{Ka}(\hat{\rho}) + (I-\hat{\rho})C_{Kb}(\hat{\rho})].$$

Let P be of the following form

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} \tag{3.43}$$

with $0 < N_1 < Y_1$*, which implies* $P > 0$*. Since* C_2 *is of full rank, and* $C_2^{\perp}$ *satisfies* $C_2 C_2^{\perp T} = 0$ *and* $C_2^{\perp} C_2^{\perp T}$ *nonsingular, it follows that* $\begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}$ *is non-singular. From (9.1), we have*

$$C_2 x = y - D_{21}\omega$$

Then it follows

$$\begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix} x = \begin{bmatrix} y - D_{21}\omega \\ C_2^{\perp} x \end{bmatrix}$$

which implies that

$$x = \Gamma \begin{bmatrix} y - D_{21}\omega \\ C_2^{\perp} x \end{bmatrix} = \Gamma \begin{bmatrix} y \\ 0 \end{bmatrix} + \Gamma \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix} x + \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \omega \tag{3.44}$$

where $\Gamma = \begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}^{-1}$.

Furthermore, we have

$$PA_{e2} = \begin{bmatrix} 0 & W_a \\ 0 & W_b \end{bmatrix}$$

where

$$W_a = Y_1[B_2 C_{Ka}(\tilde{\rho}) + B_2 \tilde{\rho} C_{Kb}(\hat{\rho})] - N_1 A_{Ka}(\tilde{\rho})$$
$$W_b = N_1[A_{Ka}(\tilde{\rho}) - B_2 C_{Ka}(\tilde{\rho}) - B_2 \tilde{\rho} C_{Kb}(\hat{\rho})]$$

which follows

$$[x^T \ \xi^T] P A_{e2} [x^T \ \xi^T]^T = x^T W_a \xi + \xi^T W_b \xi$$

Thus, by (3.44), we have

$$x^T W_a \xi = \begin{bmatrix} y \\ 0 \end{bmatrix}^T \Gamma^T W_a \xi + [x^T \ \xi^T] A_{a1} [x^T \ \xi^T]^T + [x^T \ \xi^T] B_{a1} w.$$

where

$$A_{a1} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T \Gamma^T W_a \\ 0 & 0 \end{bmatrix}, \quad B_{a1} = \begin{bmatrix} 0 \\ W_a^T \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \end{bmatrix}.$$

In the same way, from (3.44) we get

$$\begin{aligned}
&[x^T\ \xi^T]PA_{e3}[x^T\ \xi^T]^T x_e^T \\
&= -x^T N_1 B_{Ka}(\tilde{\rho})C_2 x + \xi^T N_1 B_{Ka}(\tilde{\rho})C_2 x \\
&= x_e^T A_{a2} x_e + x_e^T B_{a2} w + M_{a2}
\end{aligned}$$

where

$$A_{a2} = \begin{bmatrix} -N_1 B_{Ka}(\tilde{\rho})C_2 & 0 \\ N_1 B_{Ka}(\tilde{\rho})C_2 \Gamma \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix} & 0 \end{bmatrix},$$

$$M_{a2} = \xi^T N_1 B_{Ka}(\tilde{\rho})C_2 \Gamma \begin{bmatrix} y \\ 0 \end{bmatrix}, \quad B_{a2} = \begin{bmatrix} 0 \\ M_b \end{bmatrix}$$

with

$$M_b = N_1 B_{Ka}(\tilde{\rho})C_2 \Gamma \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix}.$$

Then from the derivative of $V(t)$ along the closed-loop system (3.38), it follows

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \\
=\;& 2x_e^T P(A_e x_e + B_e \omega) + x_e^T C_e^T C_e x_e - \gamma_f^2 \omega^T \omega + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \\
=\;& 2x_e^T P(A_{e1} x_e + B_e \omega) + x_e^T C_e^T C_e x_e - \gamma_f^2 \omega^T \omega \\
& + 2x_e^T [A_{a1} + A_{a2}] x_e + 2x_e^T [B_{a1} + B_{a2}]\omega + 2\xi^T W_b \xi \\
& + 2\begin{bmatrix} y \\ 0 \end{bmatrix}^T \Gamma^T W_a \xi + 2M_{a2} + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \\
\leq\;& x_e^T W_0 x_e + 2\xi^T W_b \xi + 2\begin{bmatrix} y \\ 0 \end{bmatrix}^T \Gamma^T W_a \xi + 2M_{a_2} + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}
\end{aligned}$$

where

$$\begin{aligned}
W_0 \;=\;& PA_{e1} + A_{a1} + A_{a2} + [PA_{e1} + A_{a1} + A_{a2}]^T \\
& + \frac{1}{\gamma_f^2}(PB_e + B_{a1} + B_{a2})(PB_e + B_{a1} + B_{a2})^T + C_e^T C_e.
\end{aligned}$$

The design condition that $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0$ is reduced to

$$W_0 < 0 \tag{3.45}$$

and

$$\xi^T W_b \xi + \begin{bmatrix} y \\ 0 \end{bmatrix}^T \Gamma^T W_a \xi + M_{a2} + \sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0. \tag{3.46}$$

Since y and ξ are available online, the adaptive law can be chosen as (3.42) for rendering (3.46) valid. (3.45) is equivalent to

$$\begin{bmatrix} PA_{e1} + A_{a1} + A_{a2} + [PA_{e1} + A_a + A_{a2}]^T & PB_e + B_{a1} + B_{a2} \\ * & -\gamma_f^2 I \end{bmatrix} + \begin{bmatrix} C_e^T \\ 0 \end{bmatrix} \begin{bmatrix} C_e & 0 \end{bmatrix} < 0. \tag{3.47}$$

Notice that

$$PA_{e1} = \begin{bmatrix} Y_1 A - N_1[B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C_2 & W_c \\ -N_1 A + N_1[B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C_2 & W_d \end{bmatrix}$$

with

$$\begin{aligned} W_c &= Y_1 B_2[(I-\rho)C_{K0} + C_{Ka}(\rho) - \rho C_{Ka}(\hat{\rho}) \\ &\quad + (I-\hat{\rho})C_{Kb}(\hat{\rho})] - N_1[A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})] \\ W_d &= -N_1 B_2[(I-\rho)C_{K0} + C_{Ka}(\rho) - \rho C_{Ka}(\hat{\rho}) \\ &\quad + (I-\hat{\rho})C_{Kb}(\hat{\rho})] + N_1[A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})] \end{aligned}$$

and

$$PB_e = \begin{bmatrix} Y_1 B_1 - N_1[B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})]D_{21} \\ -N_1 B_1 + N_1[B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})]D_{21} \end{bmatrix}.$$

Furthermore (3.47) can be described by

$$\begin{aligned} W_1(\hat{\rho}) = Q_1 + \sum_{i=1}^m \hat{\rho}_i R_i + (\sum_{i=1}^m \hat{\rho}_i R_i)^T + \sum_{i=1}^m \sum_{j=1}^m \hat{\rho}_i \hat{\rho}_j S_{ij} \\ + (V_{00} + \sum_{i=1}^m \hat{\rho}_i V_{0i})^T (V_{00} + \sum_{i=1}^m \hat{\rho}_i V_{0i}) < 0 \end{aligned}$$

where Q_1, R_i, S_{ij}, V_{00} and $V_{0i}, i, j = 1 \cdots m$ are defined in (3.41). By Lemma 2.10, we can get $W_1(\hat{\rho}) < 0$ if (3.41) holds, which implies $W_0 < 0$. Together with adaptive law (3.42), it follows that $\dot{V}(t) < 0$ for $x_e \neq 0$, which further implies that the closed-loop system (3.38) is asymptotically stable.
Furthermore, we have

$$\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0.$$

Integrate the above-mentioned inequalities from 0 to ∞ on both sides, it follows

$$V(\infty) - V(0) + \int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt.$$

Then

$$\begin{aligned} \int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + x^T(0)Px(0) \\ + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i} \end{aligned}$$

which implies that (3.40) holds. The proofs for (3.39) and the asymptotic stability of the closed-loop system (3.38) for the normal case are similar, and omitted here.

Corollary 3.2 *Assume that the conditions of Theorem 10.2 hold. Then the closed-loop system (3.38) is asymptotically stable and with adaptive H_∞ performance indexes no larger than γ_n and γ_f for normal and actuator failure cases, respectively.*

Proof 3.5 *It is similar to that of Corollary 3.1, and omitted here.*

Remark 3.6 *Theorem 10.2 presents a sufficient condition for adaptive reliable H_∞ controller design via dynamic output feedback. Generally, (3.41) is not LMIs. But when C_{K0}, C_{Kai} and C_{Kbi} are given, and $N_1A_{K0}, N_1A_{Kai}, N_1A_{Kbi}$, N_1A_{Kbij}, N_1B_{K0}, N_1B_{Kai} and N_1B_{Kbi} are defined as new variables, (3.41) becomes LMIs and linearly depends on uncertain parameters ρ and $\hat{\rho}$.*

Remark 3.7 *It should be noted that $C_2^{\perp}$ satisfying $C_2C_2^{\perp^T} = 0$ and $C_2^{\perp}C_2^{\perp^T}$ nonsingular is not unique in general, which can be used to regulate $C_2^{\perp}$ for obtaining better performance in adaptive reliable H_∞ control design.*

From Theorem 10.2 and Corollary 3.2, we have the following algorithm to optimize the adaptive H_∞ performances in normal and fault cases.

Algorithm 3.3 *Let γ_n and γ_f denote the adaptive H_∞ performance bounds for the normal and fault cases of the closed-loop system (3.38), respectively. Then γ_n and γ_f are minimized by*

Step 1 Choose $C_K(\hat{\rho}) = C_{K0}$ with C_{K0} being a solution to the problem of reliable dynamic output controller design with fixed gains via Algorithm 3.2, or perform Algorithm 3.1 for obtaining state feedback gains C_{K0}, C_{Kai} and C_{Kbi} ($i = 1 \cdots m$).

Step 2 Let $N_1A_{K0} = \bar{A}_{K0}$, $N_1A_{Kai} = \bar{A}_{Kai}$, $N_1A_{Kbi} = \bar{A}_{Kbi}$, $N_1A_{Kbij} = \bar{A}_{Kbij}$, $N_1B_{K0} = \bar{B}_{K0}$, $N_1B_{Kai} = \bar{B}_{Kai}$ and $N_1B_{Kbi} = \bar{B}_{Kbi}$

$$\min \alpha L_n + \beta L_f \quad s.t. \quad 0 < N_1 < Y_1 \quad \text{and} \quad (3.41), \tag{3.48}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, and α and β are weighting coefficients. The resultant adaptive dynamic output feedback controller gains can be obtained by $A_{K0} = N_1^{-1}\bar{A}_{K0}$, $A_{Kai} = N_1^{-1}\bar{A}_{Kai}$, $A_{Kbi} = N_1^{-1}\bar{A}_{Kbi}$, $A_{Kbij} = N_1^{-1}\bar{A}_{Kbij}$, $B_{K0} = N_1^{-1}\bar{B}_{K0}$, $B_{Kai} = N_1^{-1}\bar{B}_{Kai}$, $B_{Kbi} = N_1^{-1}\bar{B}_{Kbi}$.

Remark 3.8 *Similar to Algorithm 3.2, Algorithm 3.3 also is composed of two-step optimizations, where the purpose of Step 1 is to determine state feedback gain $C_K(\hat{\rho})$, which is a solution to the problem of reliable state feedback controller design. By (3.41), it is easy to see that the solvability of the problem via state feedback is necessary for that of the corresponding problem via* *dynamic output feedback to have a solution. When choosing* $C_K(\hat{\rho}) = C_{K0}$ *with* C_{K0} *being a solution to the problem of reliable dynamic output controller design with fixed gains via Algorithm 3.2, then, by Theorem 3, it follows that Algorithm 3.3 can give less conservative design than Algorithm 3.2, which will be illustrated by examples in Section 3.5.*

Remark 3.9 *From Theorem 10.2, it is easy to see that controller gains* A_{K0}, A_{Kai}, A_{Kbi}, A_{Kbij}, B_{K0}, B_{Kai}, B_{Kbi}, $C_{K0}, C_{Kai}, C_{Kbi}(i,j = 1, \cdots, m)$ *are obtained off-line by Algorithm 3.1 while the estimation* $\hat{\rho}_i$ *is automatically updating online according to the designed adaptive law (3.42). Thus due to the introduction of adaptive mechanism, the resultant controller gain (3.26) is variable, which is different from traditional controller with fixed gain.*

For the comparison between Theorem 10.2 and Lemma 3.1, we have

Theorem 3.3 *If the condition in Lemma 3.1 holds for the closed-loop system (3.27) with fixed gain dynamic output feedback controller (3.26), then the condition in Theorem 10.2 holds for the closed-loop system (3.38) with adaptive dynamic output feedback controller (3.37).*

Proof 3.6 *Notice that if* $V_{a1} < 0$ *and* $V_{aa1} < 0$ *for the actuator failure cases and normal case, then the condition in Theorem 10.2 is feasible with* $A_{K0} = A_{Ke0}, B_{K0} = B_{Ke0}, C_{K0} = C_{Ke0}$ *and* $A_{Kai} = A_{Kbi} = A_{Kbij} = B_{Kai} = B_{Kbi} = C_{Kai} = C_{Kbi} = 0, i, j = 1 \cdots m$. *The proof is complete.*

Remark 3.10 *Theorem 10.3 shows that the method for the adaptive reliable* H_∞ *control design given in Theorem 10.2 is less conservativeness than that given in Lemma 3.1 for the reliable* H_∞ *control design with fixed controller gains.*

3.5 Example

To illustrate the effectiveness of our results, two examples are given. Example 3.1 is for state feedback case and Example 3.2 is for dynamic output feedback case.

Example 3.1 *The decoupled linearized longitudinal dynamical equations of motion of the F-18 aircraft are given as in [1] to show the effectiveness of our state feedback case.*

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = A_{long} \begin{bmatrix} \alpha \\ q \end{bmatrix} + B_{long} \begin{bmatrix} \delta_E \\ \delta_{PTV} \end{bmatrix} + B_1 \omega(t)$$

where

$$A_{long} = \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix}, \quad B_{long} = \begin{bmatrix} Z_{\delta_E} & Z_{\delta_{PTV}} \\ M_{\delta_E} & M_{\delta_{PTV}} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

$$A_{long}^{m7h14} = \begin{bmatrix} -1.175 & 0.9871 \\ -8.458 & -0.8776 \end{bmatrix}, \quad B_{long}^{m7h14} = \begin{bmatrix} -0.194 & -0.03593 \\ -19.29 & -3.803 \end{bmatrix}$$

and

$\alpha =$ *angle of attack*, $q =$ *pitch rate*,

$\dot{\alpha} =$ *angle velocity of attack*, $\dot{q} =$ *pitch acceleration*,

$\delta_E =$ *symmetric elevator position*,

$\delta_{PTV} =$ *symmetric pitch thrust velocity nozzle position*

$\omega =$ *external disturbance.*

Following the nomenclature in [1], A_{long}^{m7h14} denotes the longitudinal state matrix at Mach 0.7 and 14-kft altitude.
In this example, the regulated output $z(t)$ is chosen as

$$z(t) = \begin{bmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \delta_E \\ \delta_{PTV} \end{bmatrix}$$

to improve the performance of the second state q.

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be normal or loss of effectiveness, that is,

$$\rho_1^1 = 1, \quad 0 \le \rho_2^1 \le a, \quad a = 0.8,$$

which denotes the maximum loss of effectiveness for the second actuator.
Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, that is,

$$\rho_2^2 = 1, \quad 0 \le \rho_1^2 \le b, \quad b = 0.9,$$

which denotes the maximum loss of effectiveness for the first actuator.

From Algorithm 3.1 with $\alpha = 10, \beta = 1$ and Remark 3.3, the corresponding H_∞ performance indexes of the closed-loop systems with the two controllers are obtained. See Table 3.1 for more details, which indicates the superiority of our *adaptive method.*

TABLE 3.1 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
γ_n	0.4147	2.1584
γ_f	1.0161	3.4393

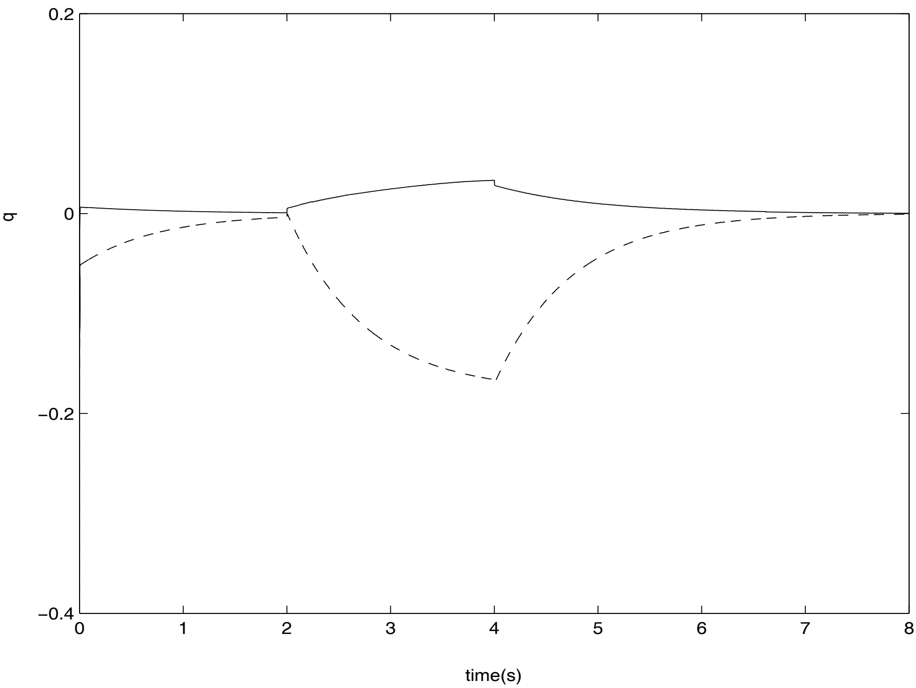


FIGURE 3.1
Response curve q in normal case with adaptive state feedback controller (solid) and state feedback controller with fixed gain (dashed) $l_1 = l_2 = 50$.

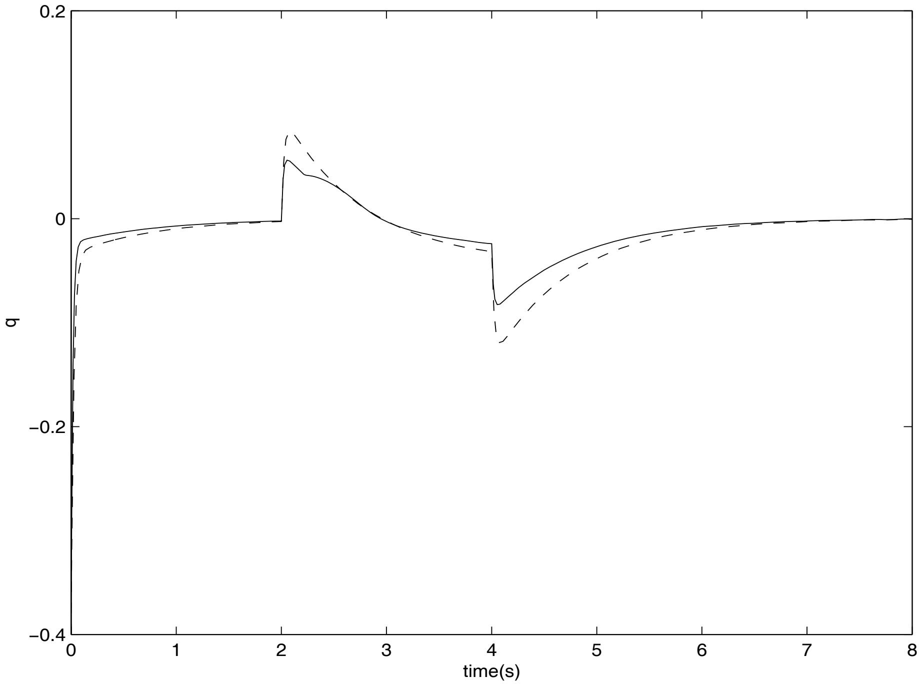


FIGURE 3.2
Response curve q in fault case with adaptive state feedback controller (solid) and state feedback controller fixed gain (dashed) $l_1 = l_2 = 50$.

In the following simulation, we use the disturbance $\omega(t) = \begin{bmatrix}\omega_1(t) & \omega_2(t)\end{bmatrix}^T$ is

$$\omega_1(t) = \omega_2(t) = \begin{cases} 1, & 2 \le t \le 3(s) \\ 0 & \text{otherwise} \end{cases}$$

and the fault case here is that at 0 second, the first actuator is outage.

Just as the analysis in Definition 1 and Remark 3.2, the adaptive H_∞ performance index is closed to traditional H_∞ performance index when we choose l_i relatively large to make $F(0) = \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(0)}{l_i}$ sufficiently small.

Figure 3.1 describes the response curves in pitch rate q in normal case with adaptive state feedback controller and fixed gain state feedback controller. The responses in pitch rate q in fault case with the above-mentioned two controllers are given in Figure 3.2. From Figure 3.1-Figure 3.2, it is easy to see our adaptive method has more restraint disturbance ability than fixed gain one in either normal or fault case just as theory has proved.

Next, a numerical example is given for dynamic output feedback case.

Example 3.2 *Consider the following linear system*

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -5 & 2 \\ -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \omega(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
z(t) &= \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} \omega(t)
\end{aligned} \tag{3.49}
$$

Choose $C_2^{\perp} = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^1 = 1, \quad 0 \leq \rho_1^2 \leq a_1, a_1 = 0.5$$

which denotes the maximal loss of effectiveness for the second actuator.
Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, described by

$$\rho_2^2 = 1, \quad 0 \leq \rho_1^2 \leq b_1, b_1 = 0.6$$

which denotes the maximal loss of effectiveness for the first actuator.
By using Algorithm 3.2 and Algorithm 3.3 with $\alpha = 10, \beta = 1$, we obtain the corresponding H_∞ performances indexes of the closed-loop system using the two controllers. See Table 3.2 for more details.

To verify the effectiveness of the proposed adaptive method, the simulations are given in the following. Here, the disturbance $\omega(t) = \begin{bmatrix} \omega_1(t) & \omega_2(t) & \omega_3(t) \end{bmatrix}^T$ is

$$\omega_1(t) = \omega_2(t) = \omega_3(t) = \begin{cases} 1, & 4 \leq t \leq 5(s) \\ 0 & \text{otherwise} \end{cases}$$

The following fault cases are considered in the simulation
Fault case 1: At 1 second, the first actuator is outage.
Fault case 2: At 0 second, the second actuator is outage, then after $t = 2$ seconds, the first actuator becomes loss of effectiveness by 50%.

Figure 3.3, Figure 3.4 and Figure 3.5 are the responses curves of the first state with adaptive and fixed gain dynamic output feedback controller in normal and the above-mentioned fault cases, respectively. It is easy to see even in the presence of actuator faults, the proposed adaptive method performs better than the design with fixed controller gains.

TABLE 3.2 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
γ_n	1.1616	1.1929
γ_f	1.7818	1.9254

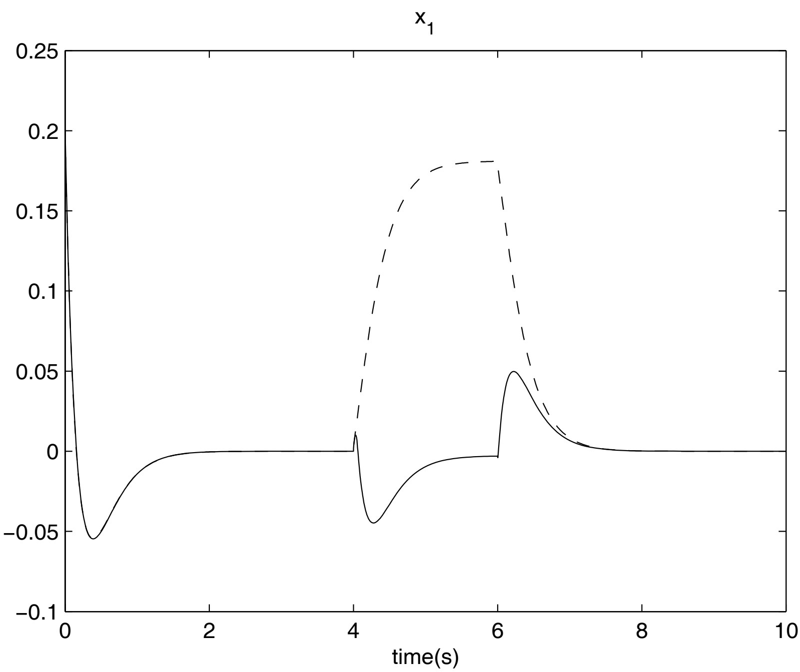


FIGURE 3.3
Response curve of the first state in normal case with adaptive dynamic output feedback controller (solid) and dynamic output feedback controller with fixed gains (dashed) $l_1 = l_2 = 50$.

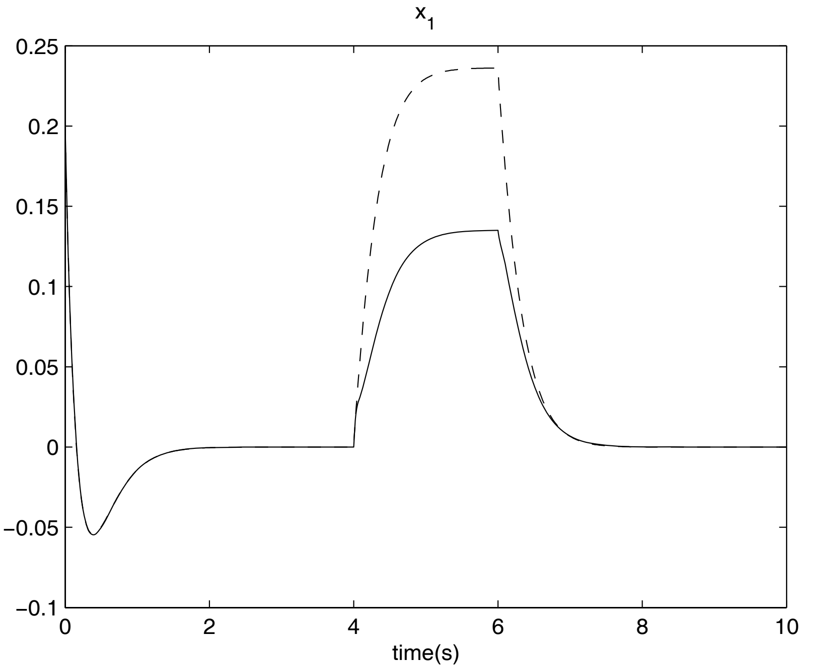


FIGURE 3.4
Response curve of the first state in fault case 1 with adaptive dynamic output feedback controller (solid) and dynamic output feedback controller with fixed gains (dashed) $l_1 = l_2 = 50$.

3.6 Conclusion

In this chapter, we have proposed the new reliable controllers design methods via both state feedback and dynamic output feedback to deal with actuator faults with adaptive mechanisms for linear time-invariant systems. The adaptive H_∞ performance index is exploited to describe the disturbance attenuation performances of closed-loop systems. Based on the online estimation of actuator faults, an adjustable control law is designed to automatically compensate the effect of a fault on the system. In the framework of LMI method, the adaptive H_∞ performances of resultant closed-loop systems in both normal and actuator failure cases are optimized, and asymptotic stability is guaranteed. It is worth noting that the design conditions for the reliable H_∞ controllers with adaptive mechanisms are more relaxed than those for the reliable H_∞ controllers with fixed controller gains. The simulation examples have shown the effectiveness of the proposed adaptive method.

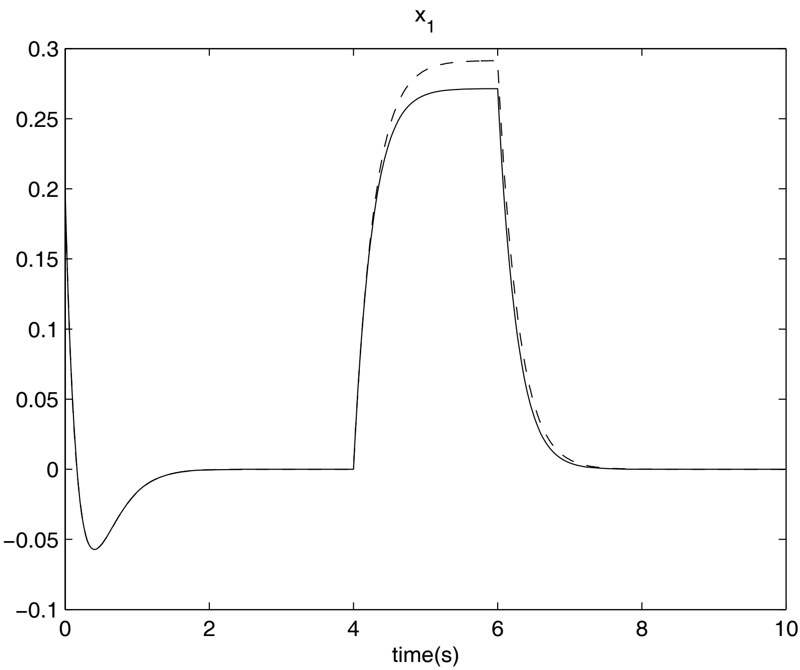


FIGURE 3.5
Response curve of the first state in fault case 2 with adaptive dynamic output feedback controller (solid) and dynamic output feedback controller with fixed gains (dashed) $l_1 = l_2 = 50$.

4

Adaptive Reliable Control against Sensor Faults

4.1 Introduction

In Chapter 3, a new reliable control approach for linear systems against *actuator faults* is proposed, based on the combination of *adaptive method* and linear matrix inequality technique. A control system consists of sensors, compensators and actuators besides a controlled object. In general, sensors are prone to break down more frequently than actuators or compensators. Furthermore, sensor faults are prone to bring about more serious situations than actuator of compensator faults. It is because incorrect information from a failed sensor often makes the total control system in danger. Measures should be fully taken against sensor faults in many control systems [150, 154]. Currently, the research about fault-tolerant control against *sensor faults* has been paid more attention [83, 87, 88, 150].

In this chapter, sensor faults are considered for *linear systems* to design reliable H_∞ *dynamic output feedback* controllers. Here, the considered sensor faults are modeled as outages. Besides LMI approach, adaptive method is also used to improve H_∞ performances of systems in both normal case and sensor fault cases. An adjustable dynamic output feedback controller is constructed based on the online estimations of sensor faults, which is obtained by *adaptive laws.* More relaxed design conditions than those for designing passive fault-tolerant H_∞ *controllers* with fixed gains are given to guarantee the *asymptotic stability* and L_2*-gain.* In sensor fault cases, only the state vector of the dynamic output feedback controller and the measured output can be used to construct the adaptive laws, which brings more challenges for dealing with the adaptive controller design problem against sensor faults.

4.2 Problem Statement

Consider a linear *time-invariant* model described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + Bu(t) \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}\omega(t) \end{aligned} \quad (4.1)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the *measured output*, $z(t) \in R^q$ is the *regulated output* and $\omega(t) \in R^s$ is an *exogenous disturbance* in $L_2[0, \infty]$, respectively. A, B_1, B_2, C_1, C_2, D_{12} and D_{21} are known constant matrices of appropriate dimensions. Since $C_2 \in R^{p\times n}$ and $\text{rank}(C_2) = p_1 \leq p$, then there exists a matrix $T_c \in R^{p_1\times p}$ such that $\text{rank}(T_cC_2) = p_1$. Furthermore, there exists a matrix C_{cn} such that $\text{rank}\begin{bmatrix} T_cC_2 \\ C_{cn} \end{bmatrix} = n$. Denote $T_{cn}= \begin{bmatrix} T_cC_2 \\ C_{cn} \end{bmatrix}^{-1}$, $C_2^i = \begin{bmatrix} 0 \cdots {C_{2i}^i}^T \cdots 0 \end{bmatrix}^T$, where C_{2i}^i represents the ith row of C_2.

The following *sensor outage fault model* is considered

$$y_{ik}^F(t) = (1 - \rho_i^k)y_i(t), \ \ i = 1 \cdots p, k = 1 \cdots g. \quad (4.2)$$

where ρ_i^k is an unknown constant with $\rho_i^k = 0$ or $\rho_i^k = 1$, Here, the index k denotes the jth *fault mode* and g is the total fault modes. $y_{ik}^F(t)$ represents the signal from the ith sensor that has failed in the kth fault mode. When $\rho_i^k = 0$, there is no fault for the ith sensor in the kth fault mode. When $\rho_i^k = 1$, the ith sensor is outage in the kth fault mode.

Denote

$$y_k^F(t) = [y_{1k}^F(t), y_{2k}^F(t), \cdots y_{pk}^F(t)]^T = (I - \rho^k)y(t)$$

where $\rho^k = \text{diag}[\rho_1^k, \rho_2^k, \cdots \rho_p^k], \quad k = 1 \cdots g.$

$$N_{\rho^k} = \{\rho^k \ |\rho^k = \text{diag}\{\rho_1^k, \rho_2^k, \cdots \rho_p^k\}, \ \rho_i^k = 0 \text{ or } \rho_i^k = 1\}.$$

Since, all the sensor cannot be outage at the same time, the set N_{ρ^k} contains a *maximum* of $2^p - 1$ elements.

For convenience in the following sections, for all possible fault modes g, we use a uniform sensor fault model

$$y^F(t) = (I - \rho)y(t), \quad \rho \in \{\rho^1 \cdots \rho^g\} \quad (4.3)$$

where ρ can be described by $\rho = \text{diag}\{\rho_1, \rho_2, \cdots \rho_p\}$.
Then the dynamic of (4.1) with sensor fault (4.3) is described

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + Bu(t) \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y^F(t) &= (I - \rho)(C_2x(t) + D_{21}\omega(t)) \end{aligned} \quad (4.4)$$

The traditional dynamic output feedback controller with fixed gains is given by

$$\begin{aligned} \dot{\xi}_1(t) &= A_{Kf}\xi_1(t) + B_{Kf}y^F(t) \\ z_{F1}(t) &= C_{Kf}\xi_1(t) \end{aligned} \tag{4.5}$$

Applying the dynamic output feedback controller (4.5) to the system (4.4), it follows

$$\begin{aligned} \dot{x}_{ef}(t) &= A_{ef}x_{ef}(t) + B_{ef}\omega(t) \\ z_{ef}(t) &= C_{ef}x_{ef}(t) \end{aligned} \tag{4.6}$$

where $x_{ef}(t) = [x^T(t)\ \xi_1^T(t)]^T$

$$\begin{aligned} A_{ef} &= \begin{bmatrix} A & BC_{Kf} \\ B_{Kf}(I-\rho)C_2 & A_{Kf} \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ B_{Kf}(I-\rho)D_{21} \end{bmatrix} \\ C_{ef} &= [C_1 \quad D_{12}C_{Kf}]. \end{aligned}$$

Lemma 4.1 *Consider the following closed-loop system (4.6), for given constants $\gamma_n > 0$ and γ_f, the following statements are equivalent:*
(i)there exist a symmetric matrix $X > 0$ and the controller (4.5) such that in normal case, that is $\rho = 0$

$$A_{ef}^T X + XA_{ef} + \frac{1}{\gamma_n^2}XB_{ef}B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{4.7}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{ef}^T X + XA_{ef} + \frac{1}{\gamma_f^2}XB_{ef}B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{4.8}$$

(ii) there exist a nonsingular matrix Q, symmetric matrix $P > 0$, and the controller (4.5)

$$P = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}, \tag{4.9}$$

such that in normal case, that is $\rho = 0$,

$$A_{eq}^T P + PA_{eq} + \frac{1}{\gamma_n^2}PB_{eq}B_{eq}^T P + C_{eq}^T C_{eq} < 0, \tag{4.10}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{eq}^T P + PA_{eq} + \frac{1}{\gamma_f^2}PB_{eq}B_{eq}^T P + C_{eq}^T C_{eq} < 0, \tag{4.11}$$

where

$$\begin{aligned} A_{eq} &= \begin{bmatrix} A & BC_{Kq} \\ B_{Kq}(I-\rho)C_2 & A_{Kq} \end{bmatrix}, \quad B_{eq} = \begin{bmatrix} B \\ B_{Kq}(I-\rho)D_{21} \end{bmatrix} \\ C_{eq} &= [C_1 \quad D_{12}C_{Kq}]. \end{aligned}$$

and

$$A_{Kq} = Q^{-1}A_{Kf}Q, \quad B_{Kq} = -Q^{-1}B_{Kf}, \quad C_{Kq} = -C_{Kf}Q \tag{4.12}$$

(iii) there exist symmetric matrices Y_1 *and* N_1 *satisfying* $0 < N_1 < Y_1$, *and the controller gains of (4.5)* $A_{Kf} = A_{Kq}$,$B_{Kf} = B_{Kq}$ *and* $C_{Kf} = C_{Kq}$ *such that*
in normal case, that is $\rho = 0$,

$$V_{a0} = \begin{bmatrix} V_{a11} & V_{a12} & V_{a13} \\ * & NA_{Ke1} + (NA_{Ke1})^T + C_{K0}^T D_{12} D_{12} C_{K0} & V_{a23} \\ * & * & -\gamma_n^2 I \end{bmatrix} < 0 \tag{4.13}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$V_a = \begin{bmatrix} V_{a11} & V_{a12} & V_{a13} \\ * & NA_{Ke1} + (NA_{Ke1})^T + C_{K0}^T D_{12} D_{12} C_{K0} & V_{a23} \\ * & * & -\gamma_f^2 I \end{bmatrix} < 0 \tag{4.14}$$

where

$$\begin{aligned} V_{a11} &= YA - NB_{Ke1}(I-\rho)C_2 + (YA - NB_{Ke1}(I-\rho)C_2^T + C_1^T C_1 \\ V_{a12} &= YBC_{Ke1} - NA_{Ke1} - A^T N + C_2^T (I-\rho) B_{Ke1}^T N^T + C_1^T D_{12} C_{K0} \\ V_{a13} &= YB_1 - NB_{Ke1}(I-\rho)D_{21} \\ V_{a23} &= -NB_1 + NB_{Ke1}(I-\rho)D_{21}. \end{aligned}$$

Proof 4.1 *From the proof of Lemma 2.11, it is easy to conclude (i)* $\Longleftrightarrow$ *(ii), so we omit it here. On the other hand,* $P > 0$ *is equivalent to* $0 < N_1 < Y_1$, *thus by some simple algebra computation, it follows (ii)* $\Longleftrightarrow$ *(iii). The proof is complete.*

Remark 4.1 *From Lemma 4.1, we have the following algorithm to optimize the* H_∞ *performances in normal and fault cases for the traditional reliable controller design with fixed gains.*

The following algorithm is to optimize the H_∞ performances in normal and fault cases for the reliable controller design with fixed gains.

Algorithm 4.1 *Step 1 Solving the following optimization problem*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad X > 0 \qquad \Phi < 0 \tag{4.15}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, *and* α, β *are weighting coefficients.*

$$\begin{aligned} \Phi =& AX + BY_0 + (AX + BY_0)^T + \frac{1}{\gamma_n^2} B_1 B_1^T \\ &+ (CX + DY_0)^T (CX + DY_0) < 0 \end{aligned}$$

Denote the optimal solution as X_{opt} *and* Y_{0opt}*, and let* $C_{Kf} = Y_{0opt}X_{opt}^{-1}$.
Step 2 *Let* $NA_{Kf} = \bar{A}_{Kf}$, $NB_{Kf} = \bar{B}_{Kf}$.

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \;\; 0 < N < Y \;\; (4.13) \quad (4.14) \tag{4.16}$$

Denote the optimal solution as $\bar{A}_{Kf} = \bar{A}_{Kfopt}$, $\bar{B}_{Kf} = \bar{B}_{Kfopt}$, $N = N_{1opt}$. *Then the resultant dynamic output feedback controller gains can be obtained by* $A_{Kf} = N^{-1}\bar{A}_{Kf}$, $B_{Kf} = N^{-1}\bar{B}_{Kf}$, $C_{Kf} = Y_{0opt}X_{opt}^{-1}$.

Remark 4.2 *It should be noted that the conditions (4.13) and (4.14) are non-convex, however with* C_{Kf} *fixed, and* N_1A_{Kf}, N_1B_{Kf} *are defined as new variables, the conditions (4.13) and (4.14) are linear matrix inequalities. Moreover, Algorithm 4.1 gives a method for the reliable dynamic output controller design with fixed gains by two-step optimizations. Step 1 is to a* C_{Kf}*, which solves the corresponding design problem via state feedback. With the* C_{K0} *fixed, controller parameter matrices* A_{Kf} *and* B_{Kf} *can be obtained by performing Step 2.*

In order to reduce the conservativeness of the dynamic output feedback controller with fixed gains, the following dynamic output feedback controller with variable gains is given

$$\begin{aligned} \dot{\xi}(t) &= A_K(\hat{\rho})\xi(t) + B_K(\hat{\rho})y^F(t) \\ u(t) &= C_{K0}\xi(t) \end{aligned} \tag{4.17}$$

where $\hat{\rho}(t)$ is the estimation of ρ. Denote

$$A_K(\hat{\rho}) = A_{K0} + A_{Ka}(\hat{\rho}) + A_{Kb}(\hat{\rho}), \quad B_K(\hat{\rho}) = B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})$$

with

$$A_{Ka}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i A_{Kai}$$

$$A_{Kb}(\hat{\rho}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \hat{\rho}_i\hat{\rho}_j A_{Kbij} + \sum_{i=1}^{p} \hat{\rho}_i A_{Kbi}$$

$$B_{Ka}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i B_{Kai}, \quad B_{Kb}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i B_{Kbi}$$

where A_{K0}, A_{Kai}, A_{Kbi}, A_{Kbij}, B_{K0}, B_{Kai}, B_{Kbi}, C_{K0} are the controller gains to be designed.
Combining (4.17) and (4.4), the dynamics with sensor faults (4.3) is described by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e\omega(t) \\ z(t) &= C_e x_e(t) \end{aligned} \tag{4.18}$$

where $x_e(t) = [x^T(t)\ \xi^T(t)]^T$,

$$A_e = \begin{bmatrix} A & BC_{K0} \\ B_K(\hat{\rho})(I-\rho)C_2 & A_K(\hat{\rho}) \end{bmatrix},$$

$$B_e = \begin{bmatrix} B_1 \\ B_K(\hat{\rho})(I-\rho)D_{21} \end{bmatrix}, \quad C_e = [C_1 \quad D_{12}C_{K0}]$$

4.3 Adaptive Reliable H_∞ Dynamic Output Feedback Controller Design

In this section, the problem of designing an *adaptive reliable dynamic output feedback controller* against sensor faults for linear system (4.1) is studied.

Before presenting the main result of the paper, denote

$$\Delta(\hat{\rho}) = \operatorname{diag}\begin{bmatrix} \hat{\rho}_1 I & \cdots & \hat{\rho}_p I \end{bmatrix}, \quad \Delta_{\hat{\rho}} = \{\hat{\rho}:\ \hat{\rho}_i \in \{0,1\},\ i = 1, \cdots, p\},$$

$$Q_{01} = \begin{bmatrix} T_0 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & -\gamma_n^2 I \end{bmatrix}, \quad Q_1 = \begin{bmatrix} T_0 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & -\gamma_f^2 I \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_p \end{bmatrix}, \quad \Upsilon = [\Upsilon_{ij}], \quad i, j = 1 \cdots p,$$

$$R_i = \begin{bmatrix} T_5 & -NA_{Kbi} - N_3^T NA_{Kai} & T_6 \\ T_7 & NA_{Kbi} & T_8 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Upsilon_{ij} = \begin{bmatrix} 0 & T_9 & 0 \\ T_{10} & NA_{Kbij} + (NA_{Kbji})^T & T_{11} \\ 0 & T_{12} & 0 \end{bmatrix}$$

with

$$T_0 = YA - NB_{K0}(I-\rho)C_2 + (YA - NB_{K0}(I-\rho)C_2)^T + C_1^T C_1$$

$$T_1 = YBC_{K0} - NA_{K0} - NA_{Ka}(\rho) - A^T N + C_2^T(I-\rho)B_{K0}^T N + C_2^T B_{Ka}^T(\rho)N + N_3^T NA_{Ka}(\rho) - N_3^T C_2^T B_{Ka}^T(\rho)N + C_1^T D_{12}C_{K0}$$

$$T_2 = YB_1 - NB_{K0}(I-\rho)D_{21}$$

$$T_3 = -NBC_{K0} - (NBC_{K0})^T + NA_{K0} + NA_{Ka}(\rho) + (NA_{K0} + NA_{Ka}(\rho))^T + C_{K0}^T D_{12}^T D_{12} C_{K0},$$

$$T_4 = -NB_1 + NB_{K0}(I-\rho)D_{21} - A_{Ka}^T(\rho)NN_2 + NB_{Ka}(\rho)C_2N_2,$$

$$T_5 = (-NB_{Kbi} - NB_{Kai} + NB_{Kbi}\rho + NB_{Kai}\rho)C_2,$$

$$T_6 = -(NB_{Kbi} + NB_{Kai})(I-\rho)D_{21}$$

$$T_7 = (-NB_{Kai}\rho + NB_{Kbi})C_2 + (NB_{Kai} - NB_{Kbi}\rho)C_2N_3$$

$$T_8 = (NB_{Kai} + NB_{Kbi})(I-\rho)D_{21} + A_{Kai}^T NN_2 - (NB_{Kai} - NB_{Kbi}\rho)C_2N_2$$

$$T_9 = -C_2^{i^T} B_{Kbj}^T N - NA_{Kbij} + N_3^T C_2^{i^T} B_{Kbj}^T N$$
$$T_{10} = -NB_{Kbi}C_2^j - A_{Kbji}^T N + NB_{Kbi}C_2^j N_3$$
$$N_1 = T_{cn}^{-1}\begin{bmatrix} T_c \\ 0 \end{bmatrix}, N_2 = T_{cn}^{-1}\begin{bmatrix} T_c(I-\rho)D_{21} \\ 0 \end{bmatrix}, N_3 = T_{cn}^{-1}\begin{bmatrix} T_c\rho C_2 \\ C_{cn} \end{bmatrix}$$

The following theorem presents a sufficient condition for the solvability of the reliable control problem via dynamic output feedback in the framework of LMI approach and adaptive laws, where γ_n and γ_f are the upper bounds of the adaptive H_∞ performance indexes for systems in normal and sensor fault cases.

Theorem 4.1 *Let $\gamma_f > \gamma_n > 0$ be given constants, if there exist matrices $0 < N < Y, A_{K0}, A_{Kai}, A_{Kbi}, A_{Fbij}$, B_{K0}, B_{Kai}, B_{Kbi}, $C_{K0}, i, j = 1 \cdots p$ and symmetric matrix Θ with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

and $\Theta_{11},\ \Theta_{22} \in R^{p(2n+m)\times p(2n+m)}$ such that the following inequalities hold:

$$\Theta_{22ii} \le 0,\ i = 1, \cdots, p$$

with $\Theta_{22ii} \in R^{(2n+s)\times(2n+s)}$ is the (i,i) block of Θ_{22}.
For any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \ge 0$$

in normal case, i.e., $\rho = 0$

$$\begin{bmatrix} Q_{01} & R \\ R^T & \Upsilon \end{bmatrix} + G^T\Theta G < 0$$

and in sensor faults cases, i.e., $\rho \in \{\rho^1 \cdots \rho^g\}, \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} Q_1 & R \\ R^T & \Upsilon \end{bmatrix} + G^T\Theta G < 0, \tag{4.19}$$

and also $\hat{\rho}_i(t)$ is determined according to the adaptive law

$$\begin{aligned} \dot{\hat{\rho}}_i(t) &= Proj_{[0,1]}\{L_i\} \\ &= \begin{cases} 0, & if \quad \begin{matrix} \hat{\rho}_i = 0 \ and \ L_i \le 0 \\ or\ \hat{\rho}_i = 1 \ and\ L_i \ge 0; \end{matrix} \\ L_i, & otherwise \end{cases} \end{aligned} \tag{4.20}$$

where

$$L_i = -l_i[\xi^T NA_{Kai}\xi - y^{F^T} N_1^T NA_{Kai}\xi + \xi^T[NB_{Kai}C_2 + NB_{Kb}(\hat{\rho})C_2^i]N_1 y^F$$

and $NB_{Kb}(\hat{\rho}) = \sum_{i=1}^p NB_{Kbi}\hat{\rho}_i$. $l_i > 0 (i = 1 \cdots m)$ is the adaptive law gain *to be chosen according to practical applications. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimation* $\hat{\rho}_i(t)$ *to the interval* $[\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j\}]$.

Then the dynamic output feedback controller of the form (4.17) with the controller parameters $A_{K0}, A_{Kai}, A_{Kbi}, A_{Kbij},\ B_{K0}, B_{Kai}, B_{Kbi},\ C_{K0}, i, j = 1\cdots p$ *and* $\hat{\rho}_i(t)$ *determined according to the adaptive law (4.20), renders the system (4.18) in normal case satisfying for* $x_e(0) = 0$

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^{p} \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{4.21}$$

and in sensor faults cases satisfying for $x_e(0) = 0$

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^{p} \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{4.22}$$

with $\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t)\cdots\tilde{\rho}_p(t)\}, \tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$

Proof 4.2 *Choose the following Lyapunov function*

$$V(t) = x_e^T(t)Px_e(t) + \sum_{i=1}^{p} \frac{\tilde{\rho}_i^2(t)}{l_i}.$$

By $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$*, it follows*

$$\begin{aligned} B_F(\hat{\rho})(I-\rho) &= [B_{K0} + B_{Ka}(\hat{\rho}(t)) + B_{Kb}(\hat{\rho}(t)](I-\rho) \\ &= B_{K0}(I-\rho) + B_{Ka}(\rho) - B_{Ka}(\hat{\rho}(t))\rho + B_{Ka}(\tilde{\rho}(t)) \\ &\quad + B_{Kb}(\hat{\rho}(t)(I-\hat{\rho}(t)) + B_{Kb}(\hat{\rho})\tilde{\rho}(t) \end{aligned} \tag{4.23}$$

and

$$A_{Ka}(\hat{\rho}) = A_{Ka}(\rho) + A_{Ka}(\tilde{\rho}).$$

A_e *can be written as*

$$A_e = A_{ea} + A_{eb}$$

where

$$A_{ea} = \begin{bmatrix} A & BC_{K0} \\ A_{ea21} & A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho}) \end{bmatrix}, \quad A_{eb} = \begin{bmatrix} 0 & 0 \\ M_1 & A_{Ka}(\tilde{\rho}) \end{bmatrix}$$

with

$$A_{ea21} = [B_{K0}(I-\rho) + B_{Ka}(\rho) - B_{Ka}(\hat{\rho})\rho + B_{Kb}(\hat{\rho})(I-\hat{\rho})]C_2$$

$$M_1 = (B_{Ka}(\tilde{\rho}) + B_{Kb}(\hat{\rho})\tilde{\rho})C_2.$$

Let P *be of the following form*

$$P = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$$

where $0 < N_1 < Y_1$, *which implies* $P > 0$. *From (4.4), it follows*

$$T_c C_2 x = T_c[y^F - (I - \rho)D_{21}\omega + \rho C_2 x] \tag{4.24}$$

Thus

$$x = T_{cn}^{-1} \begin{bmatrix} T_c C_2 x \\ T_{cn} \end{bmatrix} = N_1 y^F - N_2 \omega + N_3 x \tag{4.25}$$

where $N_1 = T_{cn}^{-1} \begin{bmatrix} T_c \\ 0 \end{bmatrix}, N_2 = T_{cn}^{-1} \begin{bmatrix} T_c(I - \rho)D_{21} \\ 0 \end{bmatrix}, N_3 = T_{cn}^{-1} \begin{bmatrix} T_c \rho C_2 \\ C_{cn} \end{bmatrix}$.
Furthermore

$$PA_{ea} = \begin{bmatrix} YA - NA_{ea21} & YBC_{K0} - N(A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})) \\ -NA + NA_{ea21} & -NBC_{K0} + N(A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})) \end{bmatrix}$$

and

$$PA_{eb} = \begin{bmatrix} -NM_1 & -NA_{Ka}(\tilde{\rho}) \\ NM_1 & NA_{Ka}(\tilde{\rho}) \end{bmatrix}$$

which follows

$$[x^T \ \xi^T]PA_{eb}[x^T \ \xi^T]^T = -x^T NM_1 x - x^T NA_{Ka}(\tilde{\rho})\xi + \xi^T NM_1 x + \xi^T NA_{Ka}(\tilde{\rho})\xi.$$

From (4.25), it is easy to see

$$\begin{aligned} x^T NA_{Ka}(\tilde{\rho})\xi &= x^T N_3^T NA_{Ka}(\tilde{\rho})\xi + {y^F}^T N_1^T NA_{Ka}(\tilde{\rho})\xi - \omega^T N_2^T NA_{Ka}(\tilde{\rho})\xi \\ \xi^T NM_1 x &= -\xi^T NM_1 N_2 \omega + \xi^T NM_1 N_3 x + \xi^T NM_1 N_1 y^F \end{aligned}$$

Hence

$$\begin{aligned} x_e^T PA_{eb} x_e &= -x^T NM_1 x - x^T N_3^T NA_{Ka}(\tilde{\rho})\xi + \xi^T NM_1 N_3 x + \xi^T M_2 \omega + M_3 \\ &= x_e^T A_{pe} x_e + x_e^T B_{pe} \omega + M_3 \end{aligned}$$

where

$$A_{pe} = \begin{bmatrix} -NM_1 & -N_3^T NA_{Ka}(\tilde{\rho}) \\ NM_1 N_3 & 0 \end{bmatrix}, \quad B_{pe} = \begin{bmatrix} 0 \\ M_2 \end{bmatrix}$$

with

$$M_2 = -NM_1 N_2 + A_{Ka}^T(\tilde{\rho}) N^T N_2$$

$$M_3 = \xi^T NA_{Ka}(\tilde{\rho})\xi - {y^F}^T N_1^T NA_{Ka}(\tilde{\rho})\xi + \xi^T NM_1 N_1 y^F.$$

Then from the derivative of $V(t)$ along the closed-loop system (4.18), it follows

$$\begin{aligned}
& \dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \\
= \; & 2x_e^T P(A_{ea}x_e + B_e\omega) + x_e^T C_e^T C_e x_e - \gamma_f^2\omega^T(t)\omega(t) \\
& +2x_e^T A_{pe}x_e + 2x_e^T B_{pe}\omega + 2M_3 + 2\sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \\
\leq \; & x_e^T W_0 x_e + 2M_3 + 2\sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}
\end{aligned}$$

where

$$W_0 = PA_{ea} + A_{pe} + [PA_{ea} + A_{pe}]^T + \frac{1}{\gamma_f^2}(PB_e + B_{pe})(PB_e + B_{pe})^T + C_e^T C_e.$$

The design condition that $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \leq 0$ is reduced to

$$W_0 < 0 \tag{4.26}$$

and

$$M_3 + \sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0 \tag{4.27}$$

Since y and ξ are available on line, the adaptive law can be chosen as (4.20), it is easy to see that

$$M_3 = \sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)L_i}{-l_i}. \tag{4.28}$$

Moreover ρ_i is an unknown constant, so $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$. If $\hat{\rho}_i = 0$, and $L_i \leq 0$ or $\hat{\rho}_i = 1$, and $L_i \geq 0$, then $\dot{\hat{\rho}}_i(t) = 0$ and $\hat{\rho}_i(t)L_i = (\hat{\rho}_i(t) - \rho)L_i \geq 0$. Then together with (4.28) and $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$, it follows

$$\sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} = 0 \leq -M_3 \tag{4.29}$$

If $\hat{\rho}_i(t)$ is in other cases, from (4.20) it follows $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t) = L_i$. Then together with (4.28) and $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$, we have

$$\sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} = -M_3. \tag{4.30}$$

Then, from (4.29) and (4.30) it follows

$$\sum_{i=1}^{p}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq -M_3. \tag{4.31}$$

If the adaptive law is chosen as (4.20), then (4.27) can be achieved. Notice that (4.26) is equivalent to

$$\begin{bmatrix} PA_{ea} + A_{pe} + [PA_{ea} + A_{pe}]^T + C_e^T C_e & PB_e + B_{pe} \\ * & -\gamma_f^2 I \end{bmatrix} < 0 \tag{4.32}$$

On the other hand,

$$PB_e = \begin{bmatrix} YB_1 - N[B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})](I-\rho)D_{21} \\ -NB_1 + N[B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})](I-\rho)D_{21} \end{bmatrix}$$

(4.32) can be described by

$$W_1(\hat{\rho}) = Q_1 + \sum_{i=1}^{p} \hat{\rho}_i R_i + (\sum_{i=1}^{p} \hat{\rho}_i R_i)^T + \sum_{i=1}^{p}\sum_{j=1}^{p} \hat{\rho}_i \hat{\rho}_j \Upsilon_{ij} < 0$$

where $Q_1, R_i, \Upsilon_{ij}, i, j = 1 \cdots p$ are defined in (4.19). From Lemma 2.10 it follows $W_1(\hat{\rho}) < 0$ if (4.19) holds, which implies $W_0 < 0$. Together with adaptive law (4.20), it follows that $\dot{V}(t) \leq 0$, which further implies that the closed-loop system (4.17) is asymptotically stable.
Furthermore, we have

$$\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0$$

Integrate the above-mentioned inequalities from 0 to ∞ on both sides, it follows

$$V(\infty) - V(0) + \int_0^\infty z(t)^T z(t)dt \leq \gamma_f^2 \int_0^\infty \omega(t)^T \omega(t)dt.$$

which implies that (4.22) holds for $x(0) = 0$. The proof for the system in the normal case is similar, so we omit it here.

Corollary 4.1 *Assume that the conditions of Theorem 4.1 hold. Then the closed-loop system (4.18) is asymptotically stable and with adaptive H_∞ performance indexes no larger than γ_n and γ_f for normal and sensor fault cases, respectively.*

Proof 4.3 *Let $F_a(0) = \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i}$. Then, by (4.20) and (4.2), it follows that $\tilde{\rho}_i(0) \leq \max_j\{\bar{\rho}_i^j\} - \min_j\{\underline{\rho}_i^j\}$. We can choose l_i sufficiently large so that $F(0)$ is sufficiently small. Thus, from (4.21), (4.22), Definition 3.1 and Remark 1.1, the adaptive H_∞ performance index is close to the standard H_∞ performance index when l_i is chosen to be sufficiently large. Then the conclusion follows. $F_a(0) = \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i}$.*

Remark 4.3 *Theorem 4.1 presents a sufficient condition for adaptive reliable H_∞ controller design via dynamic output feedback. Generally, (4.19) is not LMIs. But when C_{K0}, C_{Kai} and C_{Kbi} are given, and $N_1A_{K0}, N_1A_{Kai}, N_1A_{Kbi}$, N_1A_{Kbij}, N_1B_{K0}, N_1B_{Kai} and N_1B_{Kbi} are defined as new variables, (4.19) becomes LMIs and linearly depends on uncertain parameters ρ and $\hat{\rho}$.*

Theorem 4.2 *If the condition in Lemma 4.1 holds, then the condition in Theorem 4.1 holds.*

Proof 4.4 *Notice that if the condition (i) or (ii) in Lemma 4.1 holds, then the condition in Theorem 4.1 is feasible with* $A_{K0} = A_{Ke0}, B_{K0} = B_{Ke0}, C_{K0} = C_{Ke0}$ *and* $A_{Kai} = A_{Kbi} = A_{Kbij} = B_{Kai} = B_{Kbi} = C_{Kai} = C_{Kbi} = 0, i, j = 1 \cdots m$. *The proof is complete.*

The following algorithm is to optimize the adaptive H_∞ performances indexes in normal and fault cases.

Algorithm 4.2 *Step 1 Choose* $C_{K0} = C_{Kf}$ *with* C_{K0} *being a solution to the problem of reliable dynamic output controller design with fixed gains via Algorithm 4.1*

Step 2 Let $NA_{K0} = \bar{A}_{K0}$, $NA_{Kai} = \bar{A}_{Kai}$, $NA_{Kbi} = \bar{A}_{Kbi}$, $NA_{Kbij} = \bar{A}_{Kbij}$, $NB_{K0} = \bar{B}_{K0}$, $NB_{Kai} = \bar{B}_{Kai}$ *and* $NB_{Kbi} = \bar{B}_{Kbi}$

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad 0 < N < Y, \qquad and \quad (4.19) \tag{4.33}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, *and* α *and* β *are weighting coefficients.*
Denote the optimal solutions as $\bar{A}_{K0} = \bar{A}_{K0opt}$, $\bar{A}_{Kai} = \bar{A}_{Kaiopt}$, $\bar{A}_{Kbi} = \bar{A}_{Kbiopt}$, $\bar{A}_{Kbij} = \bar{A}_{Kbijopt}$, $\bar{B}_{K0} = \bar{B}_{K0opt}$, $\bar{B}_{Kai} = \bar{B}_{Kaiopt}$, $\bar{B}_{Kbi} = \bar{B}_{Kbiopt}$, $N = N_{1opt}$. *The resultant adaptive dynamic output feedback controller gains can be obtained by* $A_{K0} = N^{-1}\bar{A}_{K0}$, $A_{Kai} = N^{-1}\bar{A}_{Kai}$, $A_{Kbi} = N^{-1}\bar{A}_{Kbi}$, $A_{Kbij} = N^{-1}\bar{A}_{Kbij}$, $B_{K0} = N^{-1}\bar{B}_{K0}$, $B_{Kai} = N^{-1}\bar{B}_{Kai}$, $B_{Kbi} = N^{-1}\bar{B}_{Kbi}$, $C_{K0} = C_{Kf}$.

Remark 4.4 *Similar to Algorithm 4.1, Algorithm 4.2 is also composed of two-step optimizations. Moreover, from Theorem 4.2 it follows that Algorithm 4.2 can obtain less conservative design conditions than Algorithm 4.1.*

4.4 Example

Example 4.1 *Consider the following linear system*

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -2 & 2 & 1 \\ -1 & 0 & -1 \\ 5 & 1 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \omega(t) + \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 3 & 1 \end{bmatrix} u(t) \\
z(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \omega(t)
\end{aligned} \tag{4.34}$$

TABLE 4.1 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
γ_n	0.4537	0.5595
γ_f	1.4183	1.4673

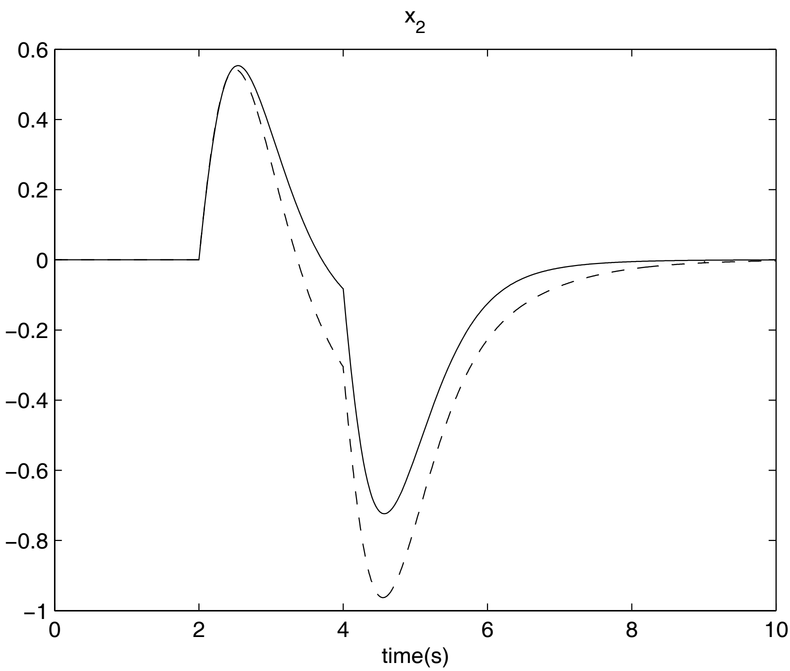


FIGURE 4.1
Response curve of the second state in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

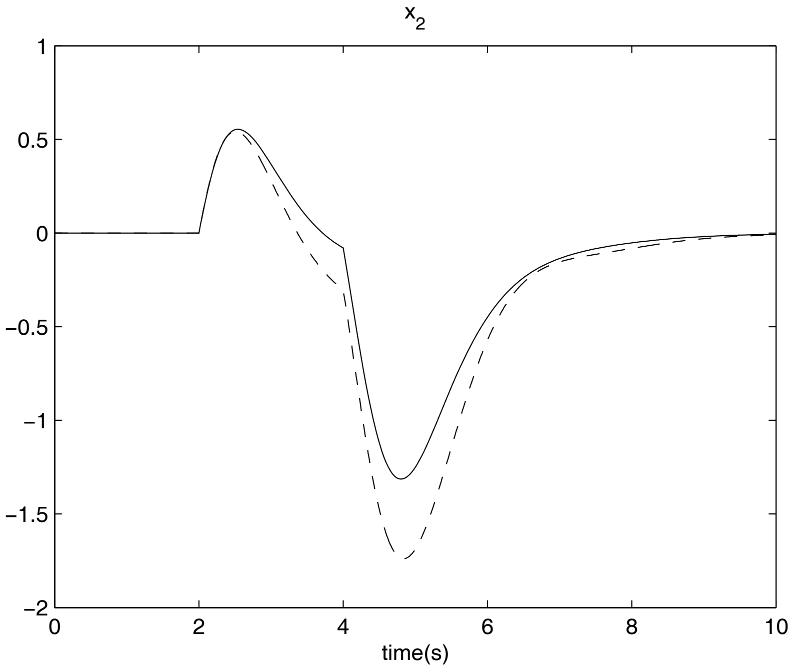


FIGURE 4.2
Response curve of the second state in sensor fault 1 with adaptive controller (solid) and controller with fixed gains (dashed).

From $z(t)$, it is easy to see that the regulated state is the second state in this example.

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Sensor fault mode 1: The first sensor is outage and the second sensor is normal, that is,

$$\rho_1^1 = 1, \quad \rho_2^1 = 0.$$

Sensor fault mode 2: The first sensor is normal and the second sensor is outage, that is,

$$\rho_1^2 = 0, \quad \rho_2^2 = 1.$$

From Algorithm 4.1 with $\alpha = 10, \beta = 1$ and Remark 4.2, the corresponding H_∞ performance indexes of the closed-loop systems with the two controllers are obtained. See Table 4.1 for more details, which indicates the superiority of our adaptive method.

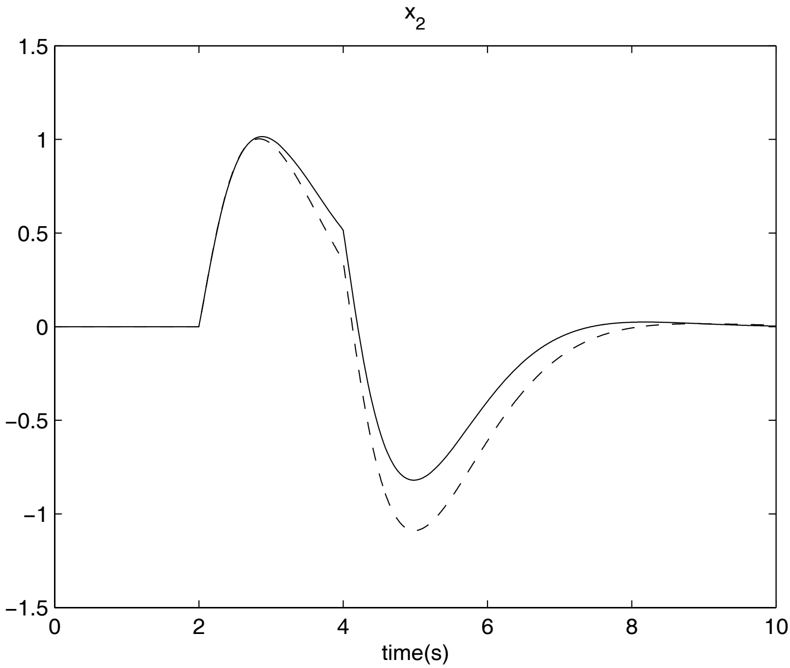


FIGURE 4.3
Response curve of the second state in sensor fault 2 with adaptive controller (solid) and controller with fixed gains (dashed).

In the simulations, the disturbance $\omega(t) = \begin{bmatrix}\omega_1(t) & \omega_2(t)\end{bmatrix}^T$ that used is

$$\omega_1(t) = \omega_2(t) = \begin{cases} 2, & 2 \leq t \leq 3 \text{ (seconds)} \\ 0 & \text{otherwise} \end{cases}$$

The considered sensor fault cases in the simulations are as follows:
The first sensor fault case: At 5 seconds, the first sensor becomes outage.
The second sensor fault case: At 4 seconds, the second sensor is outage.

Figure 4.1-Figure 4.2 are the responses of the second state with adaptive fault-tolerant controller and fault-tolerant controller with fixed gains in normal and sensor fault cases for $l_1 = l_2 = 50$, respectively. It is easy to see even in the presence of sensor outage, our adaptive method performs better than the controller with fixed gains as theory has proved.

4.5 Conclusion

This chapter has studied the adaptive reliable H_∞ control problem via dynamic output feedback for linear continuous-time systems against sensor faults. The sensor outage faults are considered. The proposed controller parameters are updated automatically to compensate the effect of sensor faults on systems based on the online estimations of sensor faults, which are obtained according to adaptive laws. Using both the adaptive method and LMI approach, more relaxed design conditions than those for designing fault-tolerant H_∞ controllers with fixed controller gains are obtained, which guarantees the asymptotic stability and L_2-gain in normal and sensor fault cases. A numerical example is also given to illustrate the design procedures and their effectiveness.

5

Adaptive Reliable Filtering against Sensor Faults

5.1 Introduction

The problem of H_∞ filtering has been a topic of recurring interest for some decades. Comparing with H_2 filtering, the advantages of H_∞ filtering approach are twofold. First, the assumption of boundness of the noise variance is loosened. Second, the H_∞ filter tends to be more robust when there exist additional uncertainties in systems, such as quantization errors, delays and unmodeled dynamics [132]. A great number of results on H_∞ filter have been reported and different approaches have been proposed in the literature [41, 44, 88, 139, 138, 146].

A common assumption in many filter designs is that the sensors can provide uninterrupted signal measurements. However, contingent faults are possible for all sensors in a system in practice. A large degree of filter performances may degrade and possible hazards may happen. Following the general notation of "reliable" controllers [54, 126, 134, 150], a filter designed to tolerate sensor faults while retaining desired properties is called a "reliable" filter in this chapter.

In this chapter, we propose a new approach to the reliable H_∞ filtering problem for *continuous-time linear systems* against *sensor faults*. Apart from using fixed filter parameter matrices, the designed filters are allowed to update filter parameter matrices for tolerating sensor faults. An adaptive H_∞ performance index is defined to describe the disturbance attenuation performance of systems with time-varying parameter estimations. Linear matrix inequality approach [14] and *adaptive method* [3, 70] are combined successfully to solve the adaptive reliable H_∞ filtering problem. Based on the online estimation of an eventual fault, the adaptive reliable H_∞ filter parameter matrices are updated automatically to compensate the sensor *fault effects* on systems. The adaptive H_∞ performances in both normal and sensor fault cases are minimized with different weighting constants in optimization indexes in the LMI framework. It is shown that the design condition for the newly proposed adaptive reliable H_∞ filtering is more relaxed than the pure LMI-based design method from [88] for the *traditional reliable filter* design without adaptive mechanisms.

5.2 Problem Statement

Consider a linear *time-invariant* model described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) \\ z(t) &= C_1x(t) \\ y(t) &= C_2x(t) + D\omega(t) \end{aligned} \tag{5.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the *measured output*, $z(t) \in R^q$ is the *regulated output* and $\omega(t) \in R^s$ is an *exogenous disturbance* in $L_2[0,\infty]$, respectively. $A, B_1, B_2, C_1, C_2, D_{12}$ and D_{21} are known constant matrices of appropriate dimensions. And $C_2 = \begin{bmatrix} I & 0 \end{bmatrix}$. Denote $h_i = \begin{bmatrix} 0 \cdots h_{ii}^T \cdots 0 \end{bmatrix}^T$, where h_{ii} represents the ith row of $\begin{bmatrix} I & 0 \end{bmatrix}$.

Remark 5.1 *In the above system description, the output matrix is assumed to be $C_2 = \begin{bmatrix} I & 0 \end{bmatrix}$. The assumption can be replaced by a more general assumption that C_2 is of full row rank. For such a C_2, let*

$$T = \begin{bmatrix} C_2^T(C_2C_2^T)^{-1} & C_2^{\perp} \end{bmatrix} \tag{5.2}$$

where $C_2^{\perp}$ denotes an orthogonal basis for the null space of C_2, then T is invertible, and $C_2T = \begin{bmatrix} I & 0 \end{bmatrix}$. Thus, the system (5.1) with C_2 being of full row rank can be converted into the one with $C_2 = \begin{bmatrix} I & 0 \end{bmatrix}$ by letting $\bar{x} = Tx$.

In this chapter, the same *sensor fault model* is considered as Chapter 4, that is

$$y^F(t) = (I - \rho)y(t), \quad \rho \in \{\rho^1 \cdots \rho^g\}$$

where ρ can be described by $\rho = \mathrm{diag}\{\rho_1, \rho_2, \cdots \rho_p\}$.

Then the dynamic of (5.1) with sensor fault (4.3) is described

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\omega(t) \\ z(t) &= C_1x(t) \\ y^F(t) &= (I-\rho)(\begin{bmatrix} I & 0 \end{bmatrix} x(t) + D\omega(t)) \end{aligned} \tag{5.3}$$

The traditional reliable filter with fixed gains is given by

$$\begin{aligned} \dot{\xi}_1(t) &= A_{Ff}\xi_1(t) + B_{Ff}(I-\rho)y(t) \\ z_{Ff}(t) &= C_{Ff}\xi_1(t) \end{aligned} \tag{5.4}$$

then apply (5.4) to (5.3), it follows

$$\begin{aligned} \dot{x}_{ef}(t) &= A_{ef}x_{ef}(t) + B_{ef}\omega(t) \\ z_{ef}(t) &= C_{ef}x_{ef}(t) \end{aligned} \tag{5.5}$$

where $x_{ef}(t) = [x^T(t)\ \xi_1^T(t)]^T$, $z_{ef}(t) = z(t) - z_{Ff}(t)$, and

$$A_{ef} = \begin{bmatrix} A & 0 \\ B_{Ff}(I-\rho)\begin{bmatrix} I & 0 \end{bmatrix} & A_{Ff} \end{bmatrix}, \quad B_{ef} = \begin{bmatrix} B \\ B_{Ff}(I-\rho)D \end{bmatrix},$$
$$C_{ef} = [C_1 \quad -C_{Ff}].$$

Lemma 5.1 *Consider the following closed-loop system (5.5), for given constants $\gamma_n > 0$ and γ_f, the following statements are equivalent:*
(i) there exist a symmetric matrix $X > 0$ and the controller(5.4) such that in normal case, that is $\rho = 0$

$$A_{ef}^T X + X A_{ef} + \frac{1}{\gamma_n^2} X B_{ef} B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{5.6}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{ef}^T X + X A_{ef} + \frac{1}{\gamma_f^2} X B_{ef} B_{ef}^T X + C_{ef}^T C_{ef} < 0 \tag{5.7}$$

(ii) there exist a nonsingular matrix Q, symmetric matrix $P > 0$, and the controller (5.4)

$$P = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix} \tag{5.8}$$

in normal case, that is $\rho = 0$,

$$A_{eq}^T P + P A_{eq} + \frac{1}{\gamma_n^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq} < 0, \tag{5.9}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$A_{eq}^T P + P A_{eq} + \frac{1}{\gamma_f^2} P B_{eq} B_{eq}^T P + C_{eq}^T C_{eq} < 0, \tag{5.10}$$

where

$$A_{ef} = \begin{bmatrix} A & 0 \\ B_{Ff}(I-\rho)\begin{bmatrix} I & 0 \end{bmatrix} & A_{Ff} \end{bmatrix} \quad B_{ef} = \begin{bmatrix} B \\ B_{Ff}(I-\rho)D \end{bmatrix}$$
$$C_{ef} = [C_1 \quad -C_{Ff}]$$

with

$$A_{Kq} = Q^{-1} A_{Kf} Q, \quad B_{Kq} = -Q^{-1} B_{Kf}, \quad C_{Kq} = -C_{Kf} Q \tag{5.11}$$

(iii) there exist symmetric matrices Y and N satisfying $0 < N < Y$, and the controller gains of (5.4) $A_{Kf} = A_{Kq}$,$B_{Kf} = B_{Kq}$ and $C_{Kf} = C_{Kq}$ such that

in normal case, that is $\rho = 0$,

$$V_{a0} = \begin{bmatrix} V_{a11} & V_{a12} & V_{a13} & C_1^T \\ * & NA_{Fq} + (NA_{Fq})^T & V_{a23} & -C_{Fq}^T \\ * & * & -\gamma_n^2 I & 0 \\ * & * & * & I \end{bmatrix} < 0 \tag{5.12}$$

in sensor fault case, that is $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$V_{a} = \begin{bmatrix} V_{a11} & V_{a12} & V_{a13} & C_1^T \\ * & NA_{Fq} + (NA_{Fq})^T & V_{a23} & -C_{Fq}^T \\ * & * & -\gamma_f^2 I & 0 \\ * & * & * & I \end{bmatrix} < 0 \tag{5.13}$$

where

$$\begin{aligned} V_{a11} &= YA - NB_{Fq}(I-\rho)\begin{bmatrix} I & 0 \end{bmatrix} + (YA - NB_{Fq}(I-\rho)\begin{bmatrix} I & 0 \end{bmatrix})^T \\ V_{a12} &= -NA_{Fq} - A^T N + \begin{bmatrix} I & 0 \end{bmatrix}^T (I-\rho) B_{Fq}^T N^T \\ V_{a13} &= YB - NB_{Fq}(I-\rho)D \\ V_{a23} &= -NB + NB_{Fq}(I-\rho)D. \end{aligned}$$

Proof 5.1 *From the proof of Lemma 2.11, it is easy to conclude (i)* $\Longleftrightarrow$ *(ii), so we omit it here. On the other hand,* $P > 0$ *is equivalent to* $0 < N < Y$*, thus by some simple algebra computation, it follows (ii)* $\Longleftrightarrow$ *(iii). The proof is complete.*

Remark 5.2 *From Lemma 5.1, we have the following algorithm to optimize the* H_∞ *performances in normal and fault cases for the traditional reliable filter design with fixed gains.*

Remark 5.3 *It should be noted that the conditions (5.12) and (5.13) are nonconvex. However, when* $N_1 A_{Kf}$, $N_1 B_{Kf}$ *are defined as new variables, the conditions (5.12) and (5.13) are linear matrix inequalities and linearly depend on fault parameters* ρ.

The following algorithm is to optimize the H_∞ performances in normal and fault cases for the reliable filter design with fixed gains.

Algorithm 5.1 *Let* $NA_{Kf} = \bar{A}_{Kf}$, $NB_{Kf} = \bar{B}_{Kf}$,$NC_{Kf} = \bar{C}_{Kf}$, *then solving the following optimization problem*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \;\; (5.12) \;\; (5.13) \tag{5.14}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, *and* α, β *are weighting coefficients.*
Denote the optimal solution as $\bar{A}_{Ff} = \bar{A}_{Ffopt}$, $\bar{B}_{Ff} = \bar{B}_{Ffopt}$, $\bar{C}_{Ff} = \bar{C}_{Ffopt}$, $N = N_{1opt}$. *Then the resultant filter gains can be obtained by* $A_{Ff} = N^{-1}\bar{A}_{Ff}$, $B_{Ff} = N^{-1}\bar{B}_{Ff}$, $C_{Ff} = N^{-1}\bar{C}_{Ff}$.

In order to reduce the conservativeness of the filter with fixed gains, the following *adaptive reliable filter* with variable gains is given

$$\begin{aligned} \dot{\xi}(t) &= A_F(\hat{\rho})\xi(t) + B_F(\hat{\rho})y^F(t) \\ z_F(t) &= C_F(\hat{\rho})\xi(t) \end{aligned} \tag{5.15}$$

where $\hat{\rho}(t)$ is the estimation of ρ. $\xi(t) \in R^n$ and $z_F(t) \in R^q$ are the estimated state and output, respectively. Here, we assume that the filter is of the same order as the system model. Denote

$$A_F(\hat{\rho}) = A_{F0} + A_{Fa}(\hat{\rho}) + A_{Fb}(\hat{\rho})$$

$$B_F(\hat{\rho}) = B_{F0} + B_{Fa}(\hat{\rho}) + B_{Fb}(\hat{\rho})$$

$$C_F(\hat{\rho}) = C_{F0} + C_{Fa}(\hat{\rho})$$

with

$$A_{Fa}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i A_{Fai}, \quad C_{Fa}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i C_{Fai}$$

$$A_{Fb}(\hat{\rho}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \hat{\rho}_i \hat{\rho}_j A_{Fbij} + \sum_{i=1}^{p} \hat{\rho}_i A_{Fbi}$$

$$B_{Fa}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i B_{Fai}, \quad B_{Fb}(\hat{\rho}) = \sum_{i=1}^{p} \hat{\rho}_i B_{Fbi}$$

where A_{F0}, A_{Fai}, A_{Fbi}, A_{Fbij}, B_{F0}, B_{Fai}, B_{Fbi}, C_{F0}, C_{Fai} are the filter gains to be designed.

Combining (5.15) and (5.3), it follows

$$\begin{aligned} \dot{x}_e(t) &= A_e(\hat{\rho}, \rho)x_e(t) + B_e(\hat{\rho}, \rho)\omega(t) \\ z_e(t) &= C_e(\hat{\rho})x_e(t) \end{aligned} \tag{5.16}$$

where $x_e(t) = [x^T(t)\ \xi^T(t)]^T$, and $z_e(t) = z(t) - z_F(t)$ is the estimated output error

$$A_e(\hat{\rho}, \rho) = \begin{bmatrix} A & 0 \\ B_F(\hat{\rho})(I-\rho)\begin{bmatrix} I & 0 \end{bmatrix} & A_F(\hat{\rho}) \end{bmatrix}$$

$$B_e(\hat{\rho}, \rho) = \begin{bmatrix} B \\ B_F(\hat{\rho})(I-\rho)D \end{bmatrix}, \quad C_e(\hat{\rho}) = [C_1 \quad -C_F(\hat{\rho})].$$

It should be noted that the filter parameter matrices $A_F(\hat{\rho})$, $B_F(\hat{\rho})$ and $C_F(\hat{\rho})$ are composed of the fixed parameter matrices and the estimation $\hat{\rho}$ of the unknown parameter vector ρ, which is different from the formulation for the traditional reliable filtering design problem with only fixed parameter matrices [88]. Like many other results in filtering design, e.g. [47, 88], we will make the following assumption throughout this paper:

Assumption 5.1 *A is stable.*

The problem under consideration is as follows.

Adaptive reliable H_∞ filter problem: For given constants $\gamma_f > \gamma_n > 0$, find a filter of the form (5.15) such that

(i) the system (5.16) in normal case, i.e., $\rho = 0$, is with an adaptive H_∞ performance index no larger than γ_n;

(ii) the system (5.16) in sensor fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^g\}, \rho^j \in N_{\rho^j}$, is with an adaptive H_∞ performance index no larger than γ_f.

The filter of the form (5.15) satisfying (i) and (ii) is said to be an adaptive reliable H_∞ filter for the system (5.1).

5.3 Adaptive Reliable H_∞ Filter Design

In this section, the problem of designing an adaptive reliable H_∞ filter against sensor faults for linear system (5.1) is studied. Before presenting the main result of the paper, denote

$$\begin{aligned}
\Delta_{\hat{\rho}} &= \{\hat{\rho}:\ \hat{\rho}_i \in \{\min_k\{\underline{\rho}_i^k\},\ \max_k\{\bar{\rho}_i^k\}\},\ i = 1, \cdots p, k = 1, \cdots g\},\\
\Delta(\hat{\rho}) &= \operatorname{diag}\begin{bmatrix}\hat{\rho}_1 I & \cdots & \hat{\rho}_p I\end{bmatrix},\quad E(\rho) = \operatorname{diag}\{\rho, I\},\\
Q_{01} &= \begin{bmatrix} T_0 & T_1 & T_2\\ * & T_3 & T_4\\ * & * & -\gamma_n^2 I\end{bmatrix},\quad Q_1 = \begin{bmatrix} T_0 & T_1 & T_2\\ * & T_3 & T_4\\ * & * & -\gamma_f^2 I\end{bmatrix},\\
R &= \begin{bmatrix} R_1 & R_2 & \cdots & R_p\end{bmatrix},\quad \Upsilon = [\Upsilon_{ij}],\quad i, j = 1 \cdots p,\\
R_i &= \begin{bmatrix} T_{5i} & -\bar{A}_{Fbi} - E(\rho)\bar{A}_{Fai} & T_{6i}\\ T_{7i} & \bar{A}_{Fbi} & T_{8i}\\ 0 & 0 & 0\end{bmatrix},\\
\Upsilon_{ij} &= \begin{bmatrix} 0 & T_{9ij} & 0\\ T_{10ij} & \bar{A}_{Fbij} + \bar{A}_{Fbji}^T & T_{11ij}\\ 0 & T_{12ij} & 0\end{bmatrix},\\
V_0 &= \begin{bmatrix} V_{00} & V_{01} & \cdots & V_{0p}\end{bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
V_{00} &= \begin{bmatrix} C_1 & -C_{F0} & 0\end{bmatrix},\quad V_{0i} = \begin{bmatrix} 0 & -C_{Fai} & 0\end{bmatrix},\\
T_0 &= YA - \bar{B}_{F0}(I-\rho)\begin{bmatrix} I & 0\end{bmatrix} + (YA - \bar{B}_{F0}(I-\rho)\begin{bmatrix} I & 0\end{bmatrix})^T\\
T_1 &= -\bar{A}_{F0} - \bar{A}_{Fa}(\rho) - A^T N + \begin{bmatrix} I & 0\end{bmatrix}^T (I-\rho)\bar{B}_{F0}^T + \begin{bmatrix} I & 0\end{bmatrix}^T \bar{B}_{Fa}^T(\rho)\\
&\quad + E(\rho)\bar{A}_{Fa}(\rho) - E(\rho)\begin{bmatrix} I & 0\end{bmatrix}^T \bar{B}_{Fa}^T(\rho)\\
T_2 &= YB - \bar{B}_{F0}(I-\rho)D,\\
T_3 &= \bar{A}_{F0} + \bar{A}_{Fa}(\rho) + (\bar{A}_{F0} + \bar{A}_{Fa}(\rho))^T,
\end{aligned}$$

$$T_4 = -NB + \bar{B}_{F0}(I-\rho)D + \bar{A}_{Fa}^T(\rho)\begin{bmatrix}-(I-\rho)D\\0\end{bmatrix}$$
$$- \bar{B}_{Fa}(\rho)\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}-(I-\rho)D\\0\end{bmatrix},$$
$$T_{5i} = (-\bar{B}_{Fbi} - \bar{B}_{Fai} + \bar{B}_{Fbi}\rho + \bar{B}_{Fai}\rho)\begin{bmatrix}I & 0\end{bmatrix},$$
$$T_{6i} = -(\bar{B}_{Fbi} + \bar{B}_{Fai})(I-\rho)D,$$
$$T_{7i} = [(-\bar{B}_{Fai}\rho + \bar{B}_{Fbi}) + (\bar{B}_{Fai} - \bar{B}_{Fbi}\rho)E(\rho)]\begin{bmatrix}I & 0\end{bmatrix}$$
$$T_{8i} = (\bar{B}_{Fai} + \bar{B}_{Fbi})(I-\rho)D - \bar{A}_{Fai}^T\begin{bmatrix}-(I-\rho)D\\0\end{bmatrix}$$
$$+ (\bar{B}_{Fai} - \bar{B}_{Fbi}\rho)\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}-(I-\rho)D\\0\end{bmatrix}$$
$$T_{9ij} = -h_i^T\bar{B}_{Fbj}^T - \bar{A}_{Fbij} + E(\rho)h_i^T\bar{B}_{Fbj}^T,$$
$$T_{10ij} = -\bar{B}_{Fbi}h_j - \bar{A}_{Fbji}^T + \bar{B}_{Fbi}h_jE(\rho),$$
$$T_{11ij} = \bar{B}_{Fbi}h_j\begin{bmatrix}-(I-\rho)D\\0\end{bmatrix}, \quad T_{12ij} = \begin{bmatrix}-(I-\rho)D\\0\end{bmatrix}^T h_i^T\bar{B}_{Fbj}^T$$

where $\bar{A}_{F0}, \bar{A}_{Fai}, \bar{A}_{Fbi}, \bar{A}_{Fbij}$, $\bar{B}_{F0}, \bar{B}_{Fai}, \bar{B}_{Fbi}$, $\bar{C}_{F0}, \bar{C}_{Fai}(i,j = 1\cdots p)$ are *decision variables* to be designed.

The following theorem presents a sufficient condition for the solvability of the reliable filtering problem in the framework of LMI approach and *adaptive laws*, where γ_n and γ_f are the upper bounds of the adaptive H_∞ performance indexes for systems in normal and sensor fault cases.

Theorem 5.1 *Let $\gamma_f > \gamma_n > 0$ be given constants, if there exist matrices $0 < N < Y, \bar{A}_{F0}, \bar{A}_{Fai}, \bar{A}_{Fbi}, \bar{A}_{Fbij}$, $\bar{B}_{F0}, \bar{B}_{Fai}, \bar{B}_{Fbi}$, $\bar{C}_{F0}, \bar{C}_{Fai}, i, j = 1\cdots p$ and symmetric matrix Θ with*

$$\Theta = \begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{22}\end{bmatrix}$$

and $\Theta_{11},\ \Theta_{22} \in R^{p(2n+m)\times p(2n+m)}$ such that the following inequalities hold:

$$\Theta_{22ii} \le 0,\ i = 1, \cdots, p \tag{5.17}$$

with $\Theta_{22ii} \in R^{(2n+s)\times(2n+s)}$ is the (i,i) block of Θ_{22}.
For any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \ge 0$$

in normal case, i.e., $\rho = 0$

$$\begin{bmatrix}Q_{01} & R\\ R^T & \Upsilon\end{bmatrix} + V_0^TV_0 + G^T\Theta G < 0, \tag{5.18}$$

and in sensor fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^g\}, N_{\rho^j}$

$$\begin{bmatrix} Q_1 & R \\ R^T & \Upsilon \end{bmatrix} + V_0^T V_0 + G^T \Theta G < 0. \tag{5.19}$$

and also $\hat{\rho}_i(t)$ *is determined according to the adaptive law*

$$\begin{aligned} \dot{\hat{\rho}}_i(t) &= Proj_{[\min\limits_k\{\underline{\rho}_i^k\},\ \max\limits_k\{\bar{\rho}_i^k\}]}\{L_i\}, \quad i = 1 \cdots p, k = 1 \cdots g \\ &= \begin{cases} 0, & \begin{array}{l} \hat{\rho}_i = \min\limits_k\{\underline{\rho}_i^k\} \quad L_i \leq 0 \\ or\ \hat{\rho}_i = \max\limits_k\{\bar{\rho}_i^k\} \quad L_i \geq 0; \end{array} \\ L_i, & \end{cases} \end{aligned} \tag{5.20}$$

where $L_i = -l_i[\xi^T \bar{A}_{Fai}\xi - \begin{bmatrix} y^F \\ 0 \end{bmatrix}^T \bar{A}_{Fai}\xi + \xi^T[\bar{B}_{Fai}\begin{bmatrix} I & 0 \end{bmatrix} + \bar{B}_{Fb}(\hat{\rho})h_i]\begin{bmatrix} y^F \\ 0 \end{bmatrix}]$ *and* $\bar{B}_{Fb}(\hat{\rho}) = \sum_{i=1}^p \bar{B}_{Fbi}\hat{\rho}_i$, $l_i > 0 (i = 1 \cdots m)$ *is the adaptive law gain to be chosen according to practical applications. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimation* $\hat{\rho}_i(t)$ *to the interval* $[\min\limits_k\{\underline{\rho}_i^k\}, \max\limits_k\{\bar{\rho}_i^k\}]$*. Then the filter gains*

$$\begin{aligned} &A_{F0} = \bar{A}_{F0}N^{-1}, A_{Fai} = \bar{A}_{Fai}N^{-1}, A_{Fbi} = \bar{A}_{Fbi}N^{-1}, \quad A_{Fbij} = \bar{A}_{Fbij}N^{-1}, \\ &B_{F0} = \bar{B}_{F0}N^{-1}, B_{Fai} = \bar{B}_{Fai}N^{-1}, B_{Fbi} = \bar{B}_{Fbi}N^{-1}, C_{F0} = \bar{C}_{F0}N^{-1}, \\ &C_{Fai} = \bar{C}_{Fai}N^{-1}, i, j = 1, \cdots p \end{aligned}$$

and $\hat{\rho}_i(t)$ *determined according to the adaptive law (5.20), renders the system (5.16) in normal case satisfying for* $x_e(0) = 0$

$$\int_0^\infty z_e^T(t)z_e(t)dt \leq \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^p \frac{\tilde{\rho}_i{}^2(0)}{l_i} \tag{5.21}$$

and in sensor fault cases satisfying for $x_e(0) = 0$

$$\int_0^\infty z_e^T(t)z_e(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^p \frac{\tilde{\rho}_i{}^2(0)}{l_i} \tag{5.22}$$

where $\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t) \cdots \tilde{\rho}_p(t)\}, \tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$.

Proof 5.2 *Choose the following Lyapunov function*

$$V(t) = x_e^T(t)Px_e(t) + \sum_{i=1}^p \frac{\tilde{\rho}_i^2(t)}{l_i}.$$

By $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$*, it follows*

$$\begin{aligned} A_{Ka}(\tilde{\rho}) &= A_{Ka}(\rho) + A_{Ka}(\hat{\rho}) \\ C_{Ka}(\tilde{\rho}) &= C_{Ka}(\rho) + C_{Ka}(\hat{\rho}) \end{aligned}$$

with

$$\begin{aligned} B_F(\hat{\rho})(I-\rho) &= [B_{F0} + B_{Fa}(\hat{\rho}(t)) + B_{Fb}(\hat{\rho}(t)](I-\rho) \\ &= B_{F0}(I-\rho) + B_{Fa}(\rho) - B_{Fa}(\hat{\rho})\rho \\ &\quad + B_{Fa}(\tilde{\rho}) + B_{Fb}(\hat{\rho})(I-\hat{\rho}) + B_{Fb}(\hat{\rho})\tilde{\rho} \end{aligned} \tag{5.23}$$

Then $A_e(\hat{\rho}, \rho)$*, briefly denoted as* A_e*, can be written as*

$$A_e = A_{ea} + A_{eb}$$

where

$$A_{ea} = \begin{bmatrix} A & 0 \\ A_{ea21} & A_{F0} + A_{Fa}(\rho) + A_{Fb}(\hat{\rho}) \end{bmatrix}, \quad A_{eb} = \begin{bmatrix} 0 & 0 \\ M_1 & A_{Fa}(\tilde{\rho}) \end{bmatrix}$$

with

$$\begin{aligned} A_{ea21} &= [B_{F0}(I-\rho) + B_{Fa}(\rho) - B_{Fa}(\hat{\rho})\rho + B_{Fb}(\hat{\rho})(I-\hat{\rho})] \begin{bmatrix} I & 0 \end{bmatrix} \\ M_1 &= (B_{Fa}(\tilde{\rho}) + B_{Fb}(\hat{\rho})\tilde{\rho}) \begin{bmatrix} I & 0 \end{bmatrix}. \end{aligned}$$

Let P *be of the following form*

$$P = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$$

with $0 < N < Y,$*, which implies* $P > 0$*. Let* $x = \begin{bmatrix} x_p^T & x_{n-p}^T \end{bmatrix}^T$ *and* $E(\rho) = diag\{\rho, I\}$*, then*

$$\bar{x}_p = \bar{x}_p - y^F + y^F = \rho\bar{x}_p - (I-\rho)D\omega + y^F.$$

Hence,

$$x = E(\rho)x - \begin{bmatrix} (I-\rho)D \\ 0 \end{bmatrix} \omega + \begin{bmatrix} y^F \\ 0 \end{bmatrix}. \tag{5.24}$$

Furthermore

$$PA_{ea} = \begin{bmatrix} YA - NA_{ea21} & -N(A_{F0} + A_{Fa}(\rho) + A_{Fb}(\hat{\rho})) \\ -NA + NA_{ea21} & N(A_{F0} + A_{Fa}(\rho) + A_{Fb}(\hat{\rho})) \end{bmatrix}$$

and

$$PA_{eb} = \begin{bmatrix} -NM_1 & -NA_{Fa}(\tilde{\rho}) \\ NM_1 & NA_{Fa}(\tilde{\rho}) \end{bmatrix}$$

then

$$\begin{aligned} [x^T \ \xi^T]PA_{eb}[x^T \ \xi^T]^T = -x^T NM_1 x - x^T NA_{Fa}(\tilde{\rho})\xi + \xi^T NM_1 x \\ + \xi^T NA_{Fa}(\tilde{\rho})\xi. \end{aligned}$$

From (5.24), it is easy to see that

$$x^T NA_{Fa}(\tilde{\rho})\xi = x^T E(\rho) NA_{Fa}(\tilde{\rho})\xi + [y^{F^T} \ \ 0] NA_{Fa}(\tilde{\rho})\xi$$
$$+ \omega^T [-(I-\rho)D^T \ \ 0] NA_{Fa}(\tilde{\rho})\xi$$
$$\xi^T NM_1 x = \xi^T NM_1 \begin{bmatrix} -(I-\rho)D \\ 0 \end{bmatrix} \omega(t) + \xi^T NM_1 E(\rho) x + \xi^T NM_1 \begin{bmatrix} y^F \\ 0 \end{bmatrix}$$

Hence

$$x_e^T PA_{eb} x_e = -x^T NM_1 x - x^T E(\rho) NA_{Fa}(\tilde{\rho})\xi + \xi^T NM_1 E(\rho) x + \xi^T M_2 \omega + M_3$$
$$= x_e^T A_{Pe} x_e + x_e^T B_{Pe}\omega + M_3$$

where

$$A_{Pe} = \begin{bmatrix} -NM_1 & -E(\rho) NA_{Fa}(\tilde{\rho}) \\ NM_1 E(\rho) & 0 \end{bmatrix}, \quad B_{Pe} = \begin{bmatrix} 0 \\ M_2 \end{bmatrix}$$

with

$$M_2 = NM_1 \begin{bmatrix} -(I-\rho)D \\ 0 \end{bmatrix} - \{[-(I-\rho)D^T \ \ 0] NA_{Fa}(\tilde{\rho})\}^T$$

$$M_3 = \xi^T NA_{Fa}(\tilde{\rho})\xi - \begin{bmatrix} y^{F^T} & 0 \end{bmatrix} NA_{Fa}(\tilde{\rho})\xi + \xi^T NM_1 \begin{bmatrix} y^F \\ 0 \end{bmatrix} \tag{5.25}$$

Then from the derivative of $V(t)$ *along the closed-loop system (5.16), it follows* $V(t)$

$$\dot{V}(t) + z_e^T(t) z_e(t) - \gamma_f^2 \omega^T(t)\omega(t)$$
$$= 2x_e^T P(A_e x_e + B_e \omega) + x_e^T C_e^T C_e x_e - \gamma_f^2(t)\omega^T(t)\omega(t) + 2\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}$$
$$= 2x_e^T P(A_{ea} x_e + B_e \omega) + x_e^T C_e^T C_e x_e - \gamma_f^2 \omega^T(t)\omega(t)$$
$$+ 2x_e^T A_{Pe} x_e + 2x_e^T B_{Pe}\omega + 2M_3 + 2\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}$$
$$\leq x_e^T W_0 x_e + 2M_3 + 2\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}$$

where $B_e = B_e(\hat{\rho}, \rho)$, $C_e = C_e(\hat{\rho})$, *and*

$$W_0 = PA_{ea} + A_{Pe} + [PA_{ea} + A_{Pe}]^T + C_e^T C_e$$
$$+ \frac{1}{\gamma_f^2}(PB_e + B_{Pe})(PB_e + B_{Pe})^T$$

The design condition that $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0$ *is reduced to*

$$W_0 < 0 \tag{5.26}$$

and

$$M_3 + \sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0. \tag{5.27}$$

Since y and ξ are available on line, the adaptive law can be chosen as (5.20), it is easy to see that

$$M_3 = \sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)L_i}{-l_i}. \tag{5.28}$$

Moreover ρ_i is an unknown constant, so $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$.
If $\hat{\rho}_i = \min_k\{\underline{\rho}_i^k\}, k = 1, \cdots g$ and $L_i \leq 0$ or $\hat{\rho}_i = \max_k\{\bar{\rho}_i^k\}, k = 1, \cdots g$ and $L_i \geq 0$, then $\dot{\hat{\rho}}_i(t) = 0$ and $\hat{\rho}_i(t)L_i = (\hat{\rho}_i(t) - \rho)L_i \geq 0$. Together with (5.28) and $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$, it follows

$$\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} = 0 \leq -M_3. \tag{5.29}$$

If $\hat{\rho}_i(t)$ is in other cases, from (5.20) it follows $\dot{\tilde{\rho}}_i(t) = L_i$. Then together with (5.28) and $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$, we have

$$\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} = -M_3. \tag{5.30}$$

Then, from (5.29) and (5.30) it follows

$$\sum_{i=1}^{p} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq -M_3 \tag{5.31}$$

If the adaptive law is chosen as (5.20), then (5.27) can be achieved.

Notice that (5.26) is equivalent to

$$\begin{bmatrix} PA_{ea} + A_{Pe} + [PA_{ea} + A_{Pe}]^T & PB_e + B_{Pe} \\ * & -\gamma_f^2 I \end{bmatrix} + \begin{bmatrix} C_e^T \\ 0 \end{bmatrix} \begin{bmatrix} C_e & 0 \end{bmatrix} < 0 \tag{5.32}$$

with

$$PB_e = \begin{bmatrix} YB - N[B_{F0} + B_{Fa}(\hat{\rho}) + B_{Fb}(\hat{\rho})](I - \rho)D \\ -NB + N[B_{F0} + B_{Fa}(\hat{\rho}) + B_{Fb}(\hat{\rho})](I - \rho)D \end{bmatrix}$$

If we let $\bar{A}_{F0} = NA_{F0}$, $\bar{A}_{Fai} = NA_{Fai}$, $\bar{A}_{Fbi} = NA_{Fbi}$, $\bar{A}_{Fbij} = NA_{Fbij}$, $\bar{B}_{F0} = NB_{F0}$, $\bar{B}_{Fai} = NB_{Fai}$, $\bar{B}_{Fbi} = NB_{Fbi}$, $\bar{C}_{F0} = NC_{F0}$ $\bar{C}_{Fai} = NC_{Fai}$, then $W_0 < 0$ will be convex on Y, N, $\bar{A}_{F0}$, $\bar{A}_{Fai}$, $\bar{A}_{Fbij}$, $\bar{A}_{Fbi}$, $\bar{B}_{F0}$, $\bar{B}_{Fai}$, C_{F0} and C_{Fai}.

Also (5.32) can be described by

$$W_1(\hat{\rho}) = Q_1 + \sum_{i=1}^{p} \hat{\rho}_i R_i + (\sum_{i=1}^{p} \hat{\rho}_i R_i)^T + \sum_{i=1}^{p} \sum_{j=1}^{p} \hat{\rho}_i \hat{\rho}_j \Upsilon_{ij}$$
$$+ (V_{00} + \sum_{i=1}^{p} \hat{\rho}_i V_{0i})^T (V_{00} + \sum_{i=1}^{p} \hat{\rho}_i V_{0i}) < 0$$

where $Q_1, R_i, \Upsilon_{ij}, V_{00}$ and $V_{0i}, i, j = 1 \cdots p$ are defined in (5.19).

From Lemma 2.10 it follows $W_1(\hat{\rho}) < 0$ if (5.19) holds, which implies $W_0 < 0$. Together with adaptive law (5.20), it follows that $\dot{V}(t) \leq 0$, which further implies that the closed-loop system (5.16) is asymptotically stable. Furthermore, we have

$$\dot{V}(t) + z_e^T(t) z_e(t) - \gamma_f^2 \omega^T(t) \omega(t) \leq 0$$

Integrate the above-mentioned inequalities from 0 to ∞ on both sides, it follows

$$V(\infty) - V(0) + \int_0^{\infty} z_e(t)^T z_e(t) dt \leq \gamma_f^2 \int_0^{\infty} \omega(t)^T \omega(t) dt.$$

which implies that (5.22) holds for $x(0) = 0$. The proof for the system in the normal case is similar, so we omit it here.

Corollary 5.1 *Assume that the conditions of Theorem 5.1 hold. Then the closed-loop system (5.16) is asymptotically stable and with adaptive H_∞ performance indexes no larger than γ_n and γ_f for normal and sensor fault cases, respectively.*

Proof 5.3 *Let $F_a(0) = \sum_{i=1}^{m} \frac{\tilde{\rho}_i^2(0)}{l_i}$. Then, by (5.20) and (4.2), it follows that $\tilde{\rho}_i(0) \leq \max_j \{\bar{\rho}_i^j\} - \min_j \{\underline{\rho}_i^j\}$. We can choose l_i sufficiently large so that $F(0)$ is sufficiently small. Thus, from (5.21), (5.22), Definition 3.1 and Remark 1.1, the adaptive H_∞ performance index is close to the standard H_∞ performance index when l_i is chosen to be sufficiently large. Then the conclusion follows.*

Remark 5.4 *In Theorem 5.1, a sufficient condition for the existence of an adaptive reliable H_∞ filter is given in terms of solutions to a set of LMIs, which can be effectively solved by using the LMI control toolbox. However, the LMIs involved in (5.19) could be very complex, which may make the computation very costly. The degree of complexity depends on the dimensions of the considered system and the system output, and the number of sensor fault modes. In fact, the largest size of the LMIs in (5.19) is $L \times L$, where $L = (p+1)(2n+m)+q$, the number of the LMIs is $2^p(g+1)+(p+1)$ and the number of the total decision variables involved in the LMIs is $n(n+1)+(p+1)^2 np + (p+1)nq + (2n+m)p[2p(2n+m)+1]$. So when the system is with a higher dimension and more fault modes are considered, more computation time is needed.*

Next, a theorem is given to show that the condition in Theorem 5.1 for the adaptive reliable H_∞ filter design is more relaxed than that in Lemma 5.1 for the traditional reliable H_∞ filter design with fixed parameter matrices.

Theorem 5.2 *If the condition in Lemma 5.1 holds, then the condition in Theorem 5.1 holds.*

Proof 5.4 *Notice that if the condition (i) or (ii) in Lemma 5.1 holds, then the condition in Theorem 5.1 is feasible with* $A_{K0} = A_{Ke0}, B_{K0} = B_{Ke0}, C_{K0} = C_{Ke0}$ *and* $A_{Kai} = A_{Kbi} = A_{Kbij} = B_{Kai} = B_{Kbi} = C_{Kai} = C_{Kbi} = 0, i, j = 1 \cdots m$. *The proof is complete.*

The following algorithm is to optimize the adaptive H_∞ performances indexes in normal and fault cases.

Algorithm 5.2 *Let* $NA_{F0} = \bar{A}_{F0}$, $NA_{Fai} = \bar{A}_{Fai}$, $NA_{Fbi} = \bar{A}_{Fbi}$, $NA_{Fbij} = \bar{A}_{Fbij}$, $NB_{F0} = \bar{B}_{F0}$, $NB_{Fai} = \bar{B}_{Fai}$, $NB_{Fbi} = \bar{B}_{Fbi}$, $NC_{F0} = \bar{C}_{F0}$, $NC_{Fai} = \bar{C}_{Fai}$
Solve the following optimization problem:

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad (5.19) \tag{5.33}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, *and* α *and* β *are weighting coefficients.*
Denote the optimal solutions as $\bar{A}_{F0} = \bar{A}_{F0opt}$, $\bar{A}_{Fai} = \bar{A}_{Faiopt}$ $\bar{A}_{Fbi} = \bar{A}_{Fbiopt}$, $\bar{A}_{Fbij} = \bar{A}_{Fbijopt}$, $\bar{B}_{F0} = \bar{B}_{F0opt}$, $\bar{B}_{Fai} = \bar{B}_{Faiopt}$, $\bar{B}_{Fbi} = \bar{B}_{Fbiopt}$, $\bar{C}_{F0} = \bar{C}_{F0opt}$, $\bar{C}_{Fai} = \bar{C}_{Faiopt}$ $N = N_{1opt}$.
Then the resultant adaptive filter gains can be obtained by $A_{F0} = N_1^{-1}\bar{A}_{F0}$, $A_{Fai} = N_1^{-1}\bar{A}_{Fai}$, $A_{Fbi} = N_1^{-1}\bar{A}_{Fbi}$, $A_{Fbij} = N_1^{-1}\bar{A}_{Fbij}$, $B_{F0} = N_1^{-1}\bar{B}_{F0}$, $B_{Fai} = N_1^{-1}\bar{B}_{Fai}$, $B_{Fbi} = N_1^{-1}\bar{B}_{Fbi}$, $C_{F0} = N^{-1}\bar{C}_{F0}$ $C_{Fai} = N^{-1}\bar{C}_{ai}$ $(i, j = 1 \cdots p)$.

5.4 Example

The following considered example is a linearized model of an F-404 engine from [2, 31] to illustrate the superiority of the proposed adaptive reliable filter design method.

Example 5.1 *Consider the system (5.1) with the following parameters*

$$A = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ 0.1643 + 0.5\delta & -0.4 + \delta & -0.3788 \\ 0.3107 & 0 & -2.2300 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.8 & 0 & 0 \\ -0.2 & 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & -0.6 \\ 0 & 0.6 & 0 \end{bmatrix}$$

where $\delta = 0.32$.

TABLE 5.1 H_∞ performance index

	Adaptive reliable filter	Traditional reliable filter
γ_n	0.4655	0.5586
γ_f	1.1081	1.2119

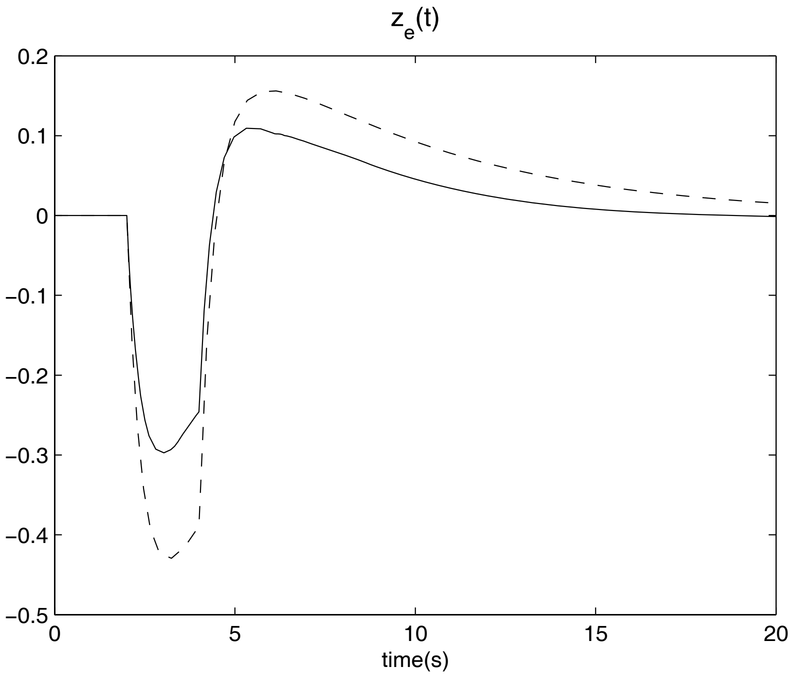


FIGURE 5.1
Response curve of estimated output error in normal case with adaptive filter (solid line) and filter with fixed filter gains (dashed line).

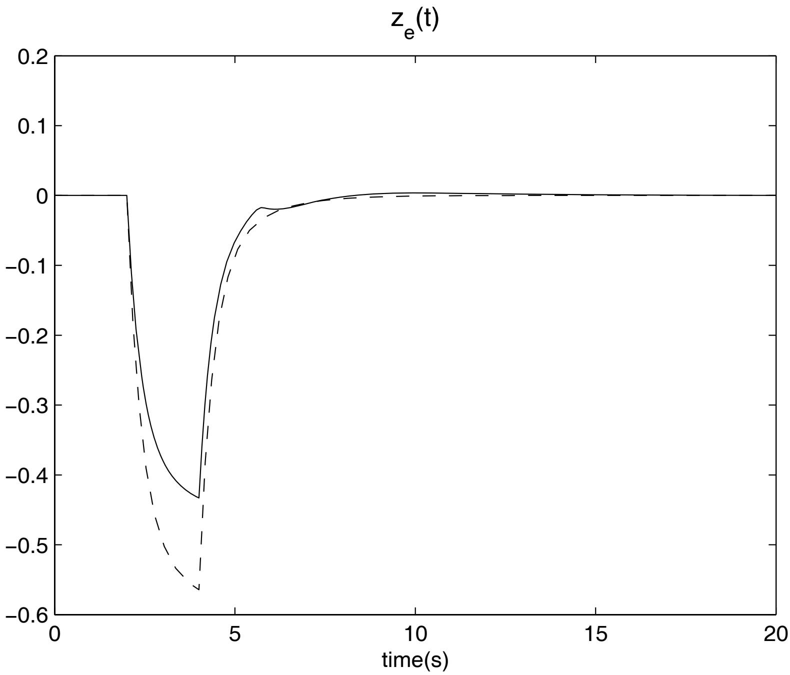


FIGURE 5.2
Response curve of estimated output error in sensor fault case with adaptive filter (solid line) and filter with fixed filter gains (dashed line).

Besides both of the two sensors are normal, that is $\rho_1^0 = \rho_2^0 = 0$, the following fault mode is considered: The second sensor is outage and the first sensor is normal, that is, $\rho_1^1 = 0, \quad \rho_2^1 = 1$.

From Algorithm 5.1 and Algorithm 5.2 with $\alpha = 10, \beta = 1$, the corresponding H_∞ performance indexes of the closed-loop systems with the two filters are obtained. See Table 5.1 for more details, which indicates the superiority of our adaptive method. In the simulations, the disturbance $\omega(t)$

$$\omega(t) = \begin{cases} 1, & 2 \le t \le 3 \text{ (seconds)} \\ 0 & \text{otherwise} \end{cases}$$

The following fault case is considered: At 1 second, the second sensor is outage.

Figure 5.1-Figure 5.2 are the response curves of estimated output error $z_e(t)$ with the adaptive filter and the reliable filter with fixed gains for normal and fault case, respectively. It is easy to see even in the presence of sensor outage, our adaptive method performs better than the filter with fixed gains as theory has proved.

5.5 Conclusion

Combining the LMI approach with adaptive mechanisms successfully, this chapter has investigated the problem of designing adaptive reliable H_∞ filters for continuous-time linear systems. Based on the online estimations of eventual faults, the reliable H_∞ filter parameter matrices are updated automatically to compensate the sensor fault effects on systems. The adaptive H_∞ performances in normal and sensor fault cases are minimized with different weighting constants in optimization indexes in the LMI framework. The design condition is more relaxed than that for the traditional reliable H_∞ filter design with fixed filter parameters. An example about a linearized model of an F-404 engine and its simulation results demonstrated the superiority of the proposed approach.

6

Adaptive Reliable Control for Time-Delay Systems

6.1 Introduction

Time-delays are frequently encountered in many practical systems such as chemical processes, electrical heaters and long transmission lines in pneumatic, hydraulic and rolling mill systems [12, 13, 29, 55, 76, 80, 103, 111, 116, 157]. Since the existence of a delay in a physical system often induces *instability* of poor performance, research on time-delay systems is a topic of great practical and theoretical importance [35, 36, 37, 39, 40, 45, 49, 50, 52, 53]. During the last decade, the control problem of systems with time-delay has received considerable attention [58, 59, 60, 61, 62, 82, 86, 160]. The main methods can be classified into two types: *delay-independent* ones [75, 91, 158] and *delay-dependent* ones [13, 16, 22, 38, 73, 75, 77, 112, 144, 158, 163]. Usually, delay-dependent ones can provide less conservative results than delay-independent ones. Both controllers with or without memory have been proposed for the study of delay-dependent control synthesis of time-delay systems.

On the other hand, actuator faults may cause severe system performance deterioration which should be avoided in many critical situations such as flight control systems, etc. [23, 7, 95, 100, 106, 107, 141]. A control system designed to tolerate faults of sensors or actuators, while maintaining an acceptable level of the closed-loop system stability/performance, is called a *reliable* control system [133]. However, the issue of time-delay is often ignored in the design of fault tolerant control, and there are relatively few works that actually consider the effects of time-delay. In fact, in the presence of time-delay, the design problems of fault tolerant controllers become more complex and difficult. Using either the *adaptive method* or linear matrix inequality (LMI) approach, some reliable or fault-tolerant controllers are proposed for *linear time-delay systems* [21, 98, 135, 158, 159, 163].

In this chapter, based on the results in Chapter 3, we focus on *adaptive reliable controller* design problems for linear time-delay systems via both *memory-less controller* and *memory controller*. Firstly for memory-less case, both *state feedback* controller and *dynamic output feedback* controller are considered. Here, the designed controller gains are affinely dependent on the online estimations of fault parameters, which are adjusted according to the proposed adaptive laws. Being different from Chapter 3, the time-delay information is included in the designed *adaptive laws*. Due to the introduction of adaptive mechanisms, more relaxed controller design conditions than those for the traditional controllers with fixed gains are derived. Secondly, since a memory controller with feedback provisions on current states and the past states may improve the performances of systems, the problem of designing memory feedback controllers for linear time-delay systems is also investigated. Both memory terms and memory-less terms are time-varying and affinely dependent on the online estimations of *actuator faults*. Some simulation results are given to demonstrate the effectiveness and superiority of the designed controllers.

6.2 Adaptive Reliable Memory-Less Controller Design

In this section, we investigate the problem of adaptive reliable controller via state feedback and dynamic output feedback, respectively for linear time-delay systems against actuator faults.

6.2.1 Problem Statement

Consider the following system with time-delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-\tau(t)) + Bu(t) + B_1\omega(t) \\ z(t) &= Cx(t) + Du(t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{6.1}$$

where $x(t) \in R^n$ and x_t is the state at time t defined by $x_t(s) = x(t+s), s \in [-h, 0]$, $u(t) \in R^m$ is the control input, $z(t) \in R^q$ is the *regulated output*, respectively. $\omega(t) \in R^p$ is an *exogenous disturbance* in $L_2[0, \infty]$ and h is an upper-bound on the time-varying delay $\tau(t)$. $\{\phi(t), t \in [-h, 0]\}$ is a real-valued initial function. A, A_1, B, B_1, C and D are known constant matrices of appropriate dimensions. For simplicity only, we take single delay $\tau(t)$. The results of this paper can be easily applied to the case of multiple delays.
As in [38], the following case for *time-varying delay* $\tau(t)$ is considered. That is, $\tau(t)$ is differentiable function

$$0 \leq \tau(t) \leq h, \quad \dot{\tau}(t) \leq d < 1, \quad \text{satisfying for all} \quad t \geq 0. \tag{6.2}$$

where d is an upper bound on the derivative of $\tau(t)$.

In this section, the considered *actuator faults model* is the same as those in Chapter 3, that is

$$u^F(t) = (I - \rho)u(t), \quad \rho \in [\rho^1 \cdots \rho^L] \tag{6.3}$$

where ρ can be described as $\rho = \text{diag}[\rho_1,\ \rho_2,\ \cdots \rho_m]$.

Denote

$$N_{\rho^j} = \{\rho^j \ |\rho^j = \text{diag}[\rho_1^j, \rho_2^j, \cdots \rho_m^j],\ \rho_i^j = \underline{\rho_i}^j \quad \rho_i^j = \bar{\rho_i}^j\}$$

It is easy to see that the set N_{ρ^j} contains a *maximum* of 2^m elements.

6.2.2 H_∞ State Feedback Control

In this subsection, an adaptive reliable H_∞ state feedback controller is designed to guarantee the resulting closed-loop system is *asymptotically stable* and its H_∞ disturbance attenuation performance bound is minimized, in normal and fault cases.

Then with actuator faults (6.3), the system is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau(t)) + B(I - \rho)u(t) + B_1\omega(t) \\ z(t) &= Cx(t) + D(I - \rho)u(t) \end{aligned} \tag{6.4}$$

Representing (6.4) in the descriptor form

$$\begin{aligned} \dot{x}(t) &= y(t), \\ y(t) &= (A + A_1)x(t) + B(I - \rho)u(t) + B_1\omega(t) - A_1 \int_{t-\tau(t)}^{t} y(s)ds \\ z(t) &= Cx(t) + D(I - \rho)u(t) \end{aligned} \tag{6.5}$$

and let $\bar{x}(t) = \text{col}\{x(t), y(t)\}$.

The *controller structure* is chosen as

$$u(t) = K(\hat{\rho}(t))x(t) = (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t))x(t) \tag{6.6}$$

where $K_a(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{ai}\hat{\rho}_i(t), K_b(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{bi}\hat{\rho}_i(t),\ \hat{\rho}_i(t)$ is the estimation of ρ_i. $K_0,\ K_{ai},\ K_{bi},\ i = 1 \cdots m$ are the controller gains to be designed.

Remark 6.1 *Though $K_a(\hat{\rho}(t)$ and $K_b(\hat{\rho}(t)$ have the same forms, we deal with them in different ways here, which gives more freedom and less conservativeness in the resultant design conditions.*

The closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= y(t), \\ y(t) &= (A + A_1)x(t) + B(I - \rho)K(\hat{\rho})x(t) + B_1\omega(t) - A_1 \int_{t-\tau(t)}^{t} y(s)ds \\ z(t) &= (C + D(I - \rho)K(\hat{\rho}))x(t) \end{aligned} \tag{6.7}$$

Before presenting the main result of this paper, denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m) : \hat{\rho}_i \in \{\min_j \{\underline{\rho}_i^j\},\ \max_j \{\bar{\rho}_i^j\}\}\},\ \Delta(\hat{\rho}) = \mathrm{diag}[\hat{\rho}_1 I \cdots \hat{\rho}_m I]$$

$$W = \begin{bmatrix} N_0 & U \\ U^T & \Upsilon \end{bmatrix} + G^T \Theta G,\ \ N_0 = \begin{bmatrix} Q_2 + Q_2^T + h\bar{Z}_1 & T_1 \\ * & -Q_3 - Q_3^T + h\bar{Z}_3 \end{bmatrix},$$

$$U = \begin{bmatrix} U_1 & U_2 & \cdots & U_m \end{bmatrix},\ \ V_0 = \begin{bmatrix} V_{00} & V_{01} & \cdots & V_{0m} \end{bmatrix}$$

$$\Upsilon = [\Upsilon_{ij}],\ i,\ j = 1 \cdots m.$$

where

$$T_1 = Q_3 - Q_2^T + Q_1(A^T + \varepsilon A_1^T) + h\bar{Z}_2 + (I - \rho)\bar{Y}_0^T B^T + \bar{Y}_a^T(\rho) B^T,$$

$$V_{00} = \begin{bmatrix} CQ_1 + D(I - \rho)\bar{Y}_0 & 0 \end{bmatrix},\ \ V_{0i} = \begin{bmatrix} D(I - \rho)(\bar{Y}_{ai} + \bar{Y}_{bi}) & 0 \end{bmatrix}$$

$$U_i = \begin{bmatrix} 0 & -\rho \bar{Y}_{ai}^T B^T + \bar{Y}_{bi}^T B^T \\ 0 & 0 \end{bmatrix},\ \ G = \begin{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}$$

$$\Upsilon_{ij} = \begin{bmatrix} 0 & -B^i \bar{Y}_{bj} - \bar{Y}_{bi}^T B^{jT} \\ 0 & 0 \end{bmatrix},\ \ \bar{Y}_a(\rho) = \sum_{i=1}^{m} \bar{Y}_{ai} \rho_i,$$

The matrices $Q_1,\ Q_2,\ Q_3,\ \bar{S},\ \bar{R},\ \bar{Z}_1,\ \bar{Z}_2,\ \bar{Z}_3,\ \Theta,\ \bar{Y}_0,\ \bar{Y}_{ai},\ \bar{Y}_{bi},\ i = 1 \cdots m$ involved in the above notations and definition are *decision variables* to be determined.

Let γ_n and γ_f denote the adaptive reliable H_∞ performance bounds for the normal case and fault cases of the closed-loop system (6.4).

Theorem 6.1 *Let $\gamma_f > \gamma_n > 0$, d and $h > 0$ are given constants, if for a diagonal matrix ε, there exist matrices $Q_1 > 0,\ Q_2, Q_3, \bar{S}, \bar{R}, \bar{Z}_1, \bar{Z}_2,\ \bar{Z}_3,\ \bar{Y}_0$, $\bar{Y}_{ai}, \bar{Y}_{bi}, i = 1 \cdots m$ and a symmetric matrix Θ with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

$\Theta_{11},\ \Theta_{22} \in R^{2mn \times 2mn}$ *such that the following inequalities hold:*

$$\Theta_{22ii} \leq 0,\ i = 1, \cdots, m \tag{6.8}$$

with $\Theta_{22ii} \in R^{n \times n}$ is the (i,i) block of Θ_{22}.
for any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \geq 0$$

for $\rho = 0$, that is in normal case,

$$\begin{bmatrix} W & V_0^T & \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_1(I-\varepsilon)\bar{S} \\ 0 \end{bmatrix} & \begin{bmatrix} Q_1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} hQ_2^T \\ hQ_3^T \\ 0 \end{bmatrix} \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_n^2 I & 0 & 0 & 0 \\ * & * & * & -(1-d)\bar{S} & 0 & 0 \\ * & * & * & * & -\bar{S} & 0 \\ * & * & * & * & * & -h\bar{R} \end{bmatrix} < 0 \tag{6.9}$$

for $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$, that is in fault cases,

$$\begin{bmatrix} W & V_0^T & \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_1(I-\varepsilon)\bar{S} \\ 0 \end{bmatrix} & \begin{bmatrix} Q_1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} hQ_2^T \\ hQ_3^T \\ 0 \end{bmatrix} \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_f^2 I & 0 & 0 & 0 \\ * & * & * & -(1-d)\bar{S} & 0 & 0 \\ * & * & * & * & -\bar{S} & 0 \\ * & * & * & * & * & -h\bar{R} \end{bmatrix} < 0 \tag{6.10}$$

$$\begin{bmatrix} \bar{R} & 0 & \bar{R}\varepsilon A_1^T \\ * & \bar{Z}_1 & \bar{Z}_2 \\ * & * & \bar{Z}_3 \end{bmatrix} \geq 0 \tag{6.11}$$

and also $\hat{\rho}_i$ is determined according to the adaptive laws

$$\begin{aligned} \dot{\hat{\rho}}_i &= Proj_{[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j]\}}\{L_i\} \\ &= \begin{cases} 0, \quad if & \hat{\rho}_i = \min\limits_j\{\underline{\rho}_i^j\} \text{ and } L_i \leq 0 \\ & \text{or } \hat{\rho}_i = \max\limits_j\{\bar{\rho}_i^j\} \text{ and } L_i \geq 0; \\ L_i, \quad otherwise & \end{cases} \end{aligned} \tag{6.12}$$

where $L_i = -l_i \bar{x}(t)^T Q^{-T} \begin{bmatrix} 0 & 0 \\ B^i K_b(\hat{\rho}) + BK_{ai} & 0 \end{bmatrix} \bar{x}(t)$, $Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$ and $l_i > 0 (i = 1 \cdots m)$ are constants to be chosen according to practical applications. Proj$\{\cdot\}$ denotes the projection operator [70], whose role is to project the estimations $\hat{\rho}_i(t)$ to the interval $[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j\}]$, then the closed-loop system (6.4) is asymptotically stable and in normal case, i.e., $\rho = 0$, satisfies for $x(t) = 0, t \in [-h, 0]$

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{6.13}$$

and in actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$*, satisfies for* $x(t) = 0, t \in [-h, 0]$

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{6.14}$$

where $\tilde{\rho}(t) = diag[\tilde{\rho}_1(t) \cdots \tilde{\rho}_m(t)]$*,* $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$*.*
Furthermore, the corresponding controller is given by

$$u(t) = (\bar{Y}_0 Q_1^{-1} + \sum_{i=1}^m \hat{\rho}_i \bar{Y}_{ai} Q_1^{-1} + \sum_{i=1}^m \hat{\rho}_i \bar{Y}_{bi} Q_1^{-1})x(t) \tag{6.15}$$

Proof 6.1 *Consider the following Lyapunov-Krasovskii functional*

$$V = V_1 + V_2 + V_3 + V_4 \tag{6.16}$$

where

$$V_1 = \bar{x}^T(t)EP\bar{x}(t), \quad V_2 = \int_h^0 \int_{t+\theta}^t y^T(s)Ry(s)dsd\theta$$
$$V_3 = \int_{t-\tau(t)}^t x^T(s)Sx(s)ds, \qquad V_4 = \sum_{i=1}^m \frac{\tilde{\rho}_i^2(t)}{l_i}$$

and

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^T > 0$$

Since $\bar{x}^T(t)EP\bar{x}(t) = x^T(t)P_1x(t)$*, then*

$$\frac{d}{dt}\{\bar{x}^T(t)EP\bar{x}(t)\} = 2x^T(t)P_1\dot{x}(t) = 2\bar{x}^T(t)P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \tag{6.17}$$

The following equality holds

$$\begin{aligned} (I-\rho)u(t) &= (I-\rho)(K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t) \\ &= [(I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}(t)) + K_a(\tilde{\rho}(t)) \\ &\quad + (I - \hat{\rho}(t))K_b(\hat{\rho}(t)) + \tilde{\rho}K_b(\hat{\rho}(t))]x(t) \end{aligned} \tag{6.18}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$*.*

From the derivative of V *along the closed-loop system (6.7), it follows*

$$\begin{aligned} \dot{V} &= \bar{x}^T(t)\Phi_1\bar{x}(t) + \eta(t) - (1-d)x^T(t-\tau(t))Sx(t-\tau(t)) \\ &\quad - \int_{t-h}^t y^T(s)Ry(s)ds + 2\sum_{i=1}^m \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} + 2\bar{x}^T(t)P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \omega(t) \end{aligned}$$

where

$$\Phi_1 = P^T\Delta_0 + \Delta_0^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix}, \quad \eta(t) = -2\int_{t-\tau(t)}^{t} \bar{x}^T(t)P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} y(s)ds$$

$$\Delta_0 = \begin{bmatrix} 0 & I \\ A + A_1 + B(I-\rho)K(\hat{\rho}) & -I \end{bmatrix}$$

By Lemma 2.12, taking $Z_0 = P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}$ *and* $a = y(s), b = \bar{x}(t)$, *it follows*

$$\begin{aligned}
\eta(t) &\leq \int_{t-\tau(t)}^{t} \begin{bmatrix} y^T(s) & \bar{x}^T(s) \end{bmatrix} W_1 \begin{bmatrix} y(s) \\ \bar{x}(s) \end{bmatrix} ds \\
&= \int_{t-\tau(t)}^{t} y^T(s)Ry(s)ds + \int_{t-\tau(t)}^{t} \bar{x}^T(t)Z\bar{x}(t)ds \\
&\quad + 2\int_{t-\tau(t)}^{t} y^T(s)(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t)ds \\
&= \int_{t-\tau(t)}^{t} y^T(s)Ry(s)ds + \tau(t)\bar{x}^T(t)Z\bar{x}(t) \\
&\quad + 2\int_{t-\tau(t)}^{t} \dot{x}^T(s)(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t)ds \\
&\leq \int_{t-h}^{t} y^T(s)Ry(s)ds + 2x^T(t)(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t) \\
&\quad - 2x^T(t-\tau(t))(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t) + h\bar{x}^T(t)Z\bar{x}(t)
\end{aligned}$$

where $W_1 = \begin{bmatrix} R & Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P \\ * & Z \end{bmatrix}$ *and* $R,\ Y,\ Z$ *satisfying* $\begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \geq 0.$
Furthermore, by (6.18) it follows

$$\begin{aligned}
&\dot{V} + z^T(t)z(t) - \gamma_f^2 w^T(t)w(t) \\
&= \bar{x}^T(t)\Phi_2\bar{x}(t) - 2x^T(t-\tau(t))(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t) + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}_i(t)}{l_i} \\
&\quad + x^T(t)(C + D(I-\rho)K(\hat{\rho}))^T(C + D(I-\rho)K(\hat{\rho}))x(t) \\
&\quad + \frac{1}{\gamma_f^2}\bar{x}^T(t)P^T \begin{bmatrix} 0 & B_1^T \end{bmatrix}^T \begin{bmatrix} 0 & B_1^T \end{bmatrix} P\bar{x}(t) \\
&\quad - (1-d)x^T(t-\tau(t))Sx(t-\tau(t)) \\
&\quad - (\gamma_f\omega^T - \frac{1}{\gamma_f}\bar{x}^T(t)P^T \begin{bmatrix} 0 & B_1^T \end{bmatrix}^T)(\gamma_f\omega - \frac{1}{\gamma_f}\begin{bmatrix} 0 & B_1^T \end{bmatrix} P\bar{x}) \\
&\quad + 2\bar{x}^T(t)P^T \begin{bmatrix} 0 & 0 \\ B[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})] & 0 \end{bmatrix} \bar{x}(t)
\end{aligned}$$

where

$$\Phi_2 = P^T\Delta_1 + \Delta_1^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + hZ + \begin{bmatrix} Y^T & 0 \end{bmatrix}^T + \begin{bmatrix} Y^T & 0 \end{bmatrix}$$

with $\Delta_1 = \begin{bmatrix} 0 & I \\ W_2 & -I \end{bmatrix}$, $W_2 = A+B[(I-\rho)K_0+K_a(\rho)-\rho K_a(\hat{\rho})+(I-\rho)K_b(\hat{\rho})]$. *Then*

$$\begin{aligned} &\dot{V} + z^T(t)z(t) - \gamma_f^2 w^T(t)w(t) \\ &\leq -2x^T(t-\tau(t))(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t) + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \\ &\quad + \bar{x}^T\Phi_3\bar{x} + 2\bar{x}^T(t)P^T \begin{bmatrix} 0 & 0 \\ B[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})] & 0 \end{bmatrix} \bar{x}(t) \end{aligned}$$

where

$$\begin{aligned} \Phi_3 = {} & P^T\Delta_1 + \Delta_1^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + \frac{1}{\gamma_f^2}P^T \begin{bmatrix} 0 & B_1^T \end{bmatrix}^T \begin{bmatrix} 0 & B_1^T \end{bmatrix} P + \begin{bmatrix} Y^T & 0 \end{bmatrix} \\ & + \begin{bmatrix} Y^T & 0 \end{bmatrix}^T + hZ + \begin{bmatrix} (C + D(I-\rho)K(\hat{\rho})^T \\ 0 \end{bmatrix} \begin{bmatrix} (C + D(I-\rho)K(\hat{\rho}) & 0 \end{bmatrix} \end{aligned}$$

Let $B = [b^1 \cdots b^m]$, $B^i = [0 \cdots b^i \ldots 0]$, *then we have*

$$PB\tilde{\rho}K_b(\hat{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PB^i K_b(\hat{\rho}) \tag{6.19}$$

$$PBK_a(\tilde{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PBK_{ai} \tag{6.20}$$

In fact, ρ_i *is an unknown constant which denotes the loss of effectiveness of the ith actuator. So from* $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho$, *it follows* $\dot{\tilde{\rho}}_i(t) = \dot{\hat{\rho}}_i(t)$. *Now, if the adaptive laws are chosen as (6.12), then*

$$2\bar{x}^T P^T \begin{bmatrix} 0 & 0 \\ B[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})] & 0 \end{bmatrix} \bar{x} + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i\dot{\tilde{\rho}}_i}{l_i} \leq 0 \tag{6.21}$$

Let $\xi(t) = col\begin{bmatrix} x(t) & y(t) & x(t-\tau(t)) \end{bmatrix}$, *then*

$$\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \leq \xi^T(t)\Psi\xi(t) \tag{6.22}$$

where $\Psi = \begin{bmatrix} \Phi_3 & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y^T \\ * & -S(1-d) \end{bmatrix}$.

Furthermore, the problem $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0$ *reduces to*

$$\Psi < 0, \quad \begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \geq 0 \tag{6.23}$$

It is obvious from the requirement of $0 < P_1$ *and the fact that in (6.23)* $-(P_3 + P_3^T)$ *must be negative and* P *is nonsingular.*

Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \quad \Pi = diag\{Q, I\} \tag{6.24}$$

we multiply Ψ *by* Υ^T *and* Υ*, on the left and the right, respectively. Applying Lemma 2.8 to the emerging quadratic term in* Q*, denoting* $\bar{S} = S^{-1}, \bar{Z} = \begin{bmatrix} \bar{Z}_1 & \bar{Z}_2 \\ \bar{Z}_2^T & \bar{Z}_3 \end{bmatrix} = Q^T Z Q$, $\bar{R} = R^{-1}$ *and choosing* $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = \varepsilon A_1^T \begin{bmatrix} P_2 & P_3 \end{bmatrix}$*, where* $\varepsilon \in R^{n \times n}$ *is a diagonal matrix, we obtain the following:* $\Psi < 0$ *is equivalent to*

$$\begin{bmatrix} \Xi_0 + Q_1 S Q_1 + h Q_2^T R Q_2 & \Xi_1 + h Q_2^T R Q_3 \\ * & \Xi_2 \end{bmatrix} < 0 \tag{6.25}$$

with

$$\begin{aligned}
\Xi_0 &= (CQ_1 + D(I - \rho\bar{Y}(\hat{\rho}))^T (CQ_1 + D(I - \rho\bar{Y}(\hat{\rho})) + Q_2 + Q_2^T + h\bar{Z}_1 \\
\Xi_1 &= Q_3 - Q_2^T + Q_1(A^T + \varepsilon A_1^T) + h\bar{Z}_2 + (I - \rho)\bar{Y}_0^T B^T + \bar{Y}_a^T(\rho) B^T \\
&\quad - \rho \bar{Y}_a^T(\hat{\rho}) B^T + (I - \hat{\rho})\bar{Y}_b^T(\hat{\rho}))B^T \\
\Xi_2 &= -Q_3 - Q_3^T + h\bar{Z}_3 + \Omega_0 + hQ_3^T R Q_3 \\
\Omega_0 &= A_1(I_n - \varepsilon)(1-d)^{-1}(I_n - \varepsilon)A_1^T + \frac{1}{\gamma_f^2} B_1 B_1^T \\
\bar{Y}_0 &= K_0 Q_1, \quad \bar{Y}_{ai} = K_{ai} Q_1, \quad \bar{Y}_{bi} = K_{bi} Q_1, \quad \bar{Y}_a(\rho) = \Sigma_{i=1}^m \bar{Y}_{ai}\rho_i \\
\bar{Y}_a(\hat{\rho}) &= \Sigma_{i=1}^m \bar{Y}_{ai}\hat{\rho}_i, \quad \bar{Y}_b(\hat{\rho}) = \Sigma_{i=1}^m \bar{Y}_{bi}\hat{\rho}_i, \quad \bar{Y}(\hat{\rho}) = \bar{Y}_0 + \bar{Y}_a(\hat{\rho}) + \bar{Y}_b(\hat{\rho})
\end{aligned}$$

Furthermore, (6.25) can be described by

$$\begin{aligned}
M(\hat{\rho}) &= N_1 + \sum_{i=1}^m \hat{\rho}_i U_i + (\sum_{i=1}^m \hat{\rho}_i U_i)^T + \sum_{i=1}^m \sum_{j=1}^m \hat{\rho}_i \hat{\rho}_j \Upsilon_{ij} \\
&\quad + (V_{00} + \sum_{i=1}^m \hat{\rho}_i V_{0i})^T (V_{00} + \sum_{i=1}^m \hat{\rho}_i V_{0i}) < 0
\end{aligned} \tag{6.26}$$

where

$$N_1 = N_0 + \begin{bmatrix} Q_1 S Q_1 + h Q_2^T R Q_2 & h Q_2^T R Q_3 \\ * & \Omega_0 + h Q_3^T R Q_3 \end{bmatrix}$$

and $U_i,\ \Upsilon_{ij},\ V_{00},\ V_{0i},\ i=1\cdots m$ *are defined in (6.9).*

If we multiply $\begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \geq 0$*, on the left and on the right, by diag* $\{R^{-1}, Q^T\}$ *and diag* $\{R^{-1}, Q\}$*, then it follows* $\begin{bmatrix} \bar{R} & 0 & \bar{R}\varepsilon A_1^T \\ * & \bar{Z}_1 & \bar{Z}_2 \\ * & * & \bar{Z}_3 \end{bmatrix} \geq 0$*. By Lemma 2.10 and Lemma 2.12, it is easy to see if conditions (6.8), (6.10) and (6.11) hold, then (6.26) and* $\begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \geq 0$ *satisfy, which implies* $\dot{V}(t) \leq 0$*. Furthermore,* $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0$.

Integrate the above-mentioned inequalities from 0 to ∞ *on both sides, it follows*

$$V(\infty) - V(0) + \int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt$$

then

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + V(0) \tag{6.27}$$

which implies that (6.14) holds for the zero initial condition $x(t) = 0, t \in [-h, 0]$*. The proofs for (6.13) and asymptotic stability of the closed-loop system (6.4) for that normal case are similar, and omitted.*

Corollary 6.1 *Assume that the conditions of Theorem 6.1 hold. Then the closed-loop system (6.4) is asymptotically stable and with adaptive* H_∞ *performance indexes no larger than* γ_n *and* γ_f *for normal and actuator fault cases, respectively.*

Proof 6.2 *It is similar to that of Corollary 3.1, and omitted here.*

Remark 6.2 *From (6.12), it is easy to see*

$$L_i = -l_i(x(t)^T P_2^T + y^T P_3^T)(B^i K_b(\hat{\rho}) + BK_{ai})x(t) \tag{6.28}$$

and $y(t) = \dot{x}(t)$*. So the adaptive law (6.12) in this paper is proportional-integral (PI) adaption algorithms, which appeared in [98], [78] and [122] to improve system performance.*

Remark 6.3 *If we choose the same Lyapunov functional candidate as [36], i.e.,* $V = V_1 + V_2 + V_3$*, where* V_1, V_2, V_3 *are defined in (6.16), then the following conditions are sufficient for guaranteeing the closed-loop system (6.4) with traditional reliable controller with fixed gain* $u(t) = K_0 x(t)$*,* $K_0 = \bar{Y}_0 Q_1^{-1}$*, to be asymptotically stable and with* H_∞ *performance indexes no larger than* γ_n *and* γ_f *for normal and actuator fault cases, respectively.*

In normal case, i.e., $\rho = 0$

$$\begin{bmatrix} \Xi_3 & \Xi_4 & \Xi_5 & 0 & 0 & Q_1 & hQ_2^T \\ * & \Xi_6 & 0 & B_1 & A_1(I-\varepsilon)\bar{S} & 0 & hQ_3^T \\ * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_n^2 I & 0 & 0 & 0 \\ * & * & * & * & -(1-d)\bar{S} & 0 & 0 \\ * & * & * & * & * & -\bar{S} & 0 \\ * & * & * & * & * & * & -h\bar{R} \end{bmatrix} < 0 \tag{6.29}$$

In actuator fault case, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} \Xi_3 & \Xi_4 & \Xi_5 & 0 & 0 & Q_1 & hQ_2^T \\ * & \Xi_6 & 0 & B_1 & A_1(I-\varepsilon)\bar{S} & 0 & hQ_3^T \\ * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_f^2 I & 0 & 0 & 0 \\ * & * & * & * & -(1-d)\bar{S} & 0 & 0 \\ * & * & * & * & * & -\bar{S} & 0 \\ * & * & * & * & * & * & -h\bar{R} \end{bmatrix} < 0 \tag{6.30}$$

where

$$\begin{aligned} \Xi_3 &= Q_2 + Q_2^T + h\bar{Z}_1 \\ \Xi_4 &= Q_3 - Q_2^T + Q_1(A^T + \varepsilon A_1^T) + h\bar{Z}_2 + (I-\rho)\bar{Y}_0^T B^T + \bar{Y}_a^T(\rho)B^T \\ \Xi_5 &= CQ_1 + D(I-\rho)\bar{Y}_0 \\ \Xi_6 &= -Q_3 - Q_3^T + h\bar{Z}_3, \end{aligned}$$

Notice that if set $Y_{ai} = 0, Y_{bi} = 0, i = 1 \cdots m$ *in Theorem 6.1, then the conditions of Theorem 1 reduce to (6.29) and (6.30). Thus, the design conditions of the reliable* H_∞ *controller with adaptive mechanisms in Theorem 1 are more relaxed than conditions (6.29) and (6.30) of the corresponding reliable* H_∞ *controller with fixed gains.*

From Theorem 6.1, we have the following algorithm to optimize the adaptive H_∞ performances in normal and fault cases.

Algorithm 6.1 *Solve the following optimization problem:*

$$\min \alpha\eta_n + \beta\eta_f \quad s.t. \quad (6.8) - (6.11) \tag{6.31}$$

where $\eta_n = \gamma_n^2$, $\eta_f = \gamma_f^2$, *and* α *and* β *are weighting coefficients. Since systems are operating under the normal condition most of the time, we often choose* $\alpha > \beta$.

Denote the optimal solutions as $Q_1 = Q_{1opt}, \bar{Y}_0 = \bar{Y}_{0opt}, \bar{Y}_{ai} = \bar{Y}_{aiopt}, \bar{Y}_{bi} = \bar{Y}_{biopt}, i = 1 \cdots m$, *then the controller gains (6.6) can be obtained by* $K_0 = \bar{Y}_0 Q_1^{-1}, K_{ai} = \bar{Y}_{ai} Q_1^{-1}, K_{bi} = \bar{Y}_{bi} Q_1^{-1}$.

Remark 6.4 *From Theorem 6.1, it is easy to see that controller gains* $K_0, K_{ai}, K_{bi} (i = 1, \cdots, m)$ *are obtained off-line by Algorithm 6.1 while the estimation* $\hat{\rho}_i$ *is automatically updating online according to the designed adaptive law (6.21). Thus due to the introduction of adaptive mechanisms, the resultant controller gain (6.6) is variable, which is different from the traditional controller with fixed gains.*

6.2.3 Guaranteed Cost Dynamic Output Feedback Control

In this subsection, we consider the *guaranteed cost control* problem via dynamic output feedback for the following time-delay system (6.1) with constant delay, i.e., $\tau(t) = h$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + Bu(t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \\ y(t) &= Cx(t) \end{aligned} \tag{6.32}$$

where $x(t) \in R^n$, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the *measured output*, respectively. h is a positive constant delay. $\{\phi(t), t \in [-h, 0]\}$ is a real-valued initial function. A, A_1 and B are known constant matrices of appropriate dimensions.

Since $C \in R^{p \times n}$ and $\text{rank}(C) = p_1 \leq p$, then there exists a matrix $T_c \in R^{p_1 \times p}$ such that $\text{rank}(T_c C) = p_1$. Furthermore, there exists a matrix C_{cn} such that $\text{rank} \begin{bmatrix} T_c C \\ C_{cn} \end{bmatrix} = n$. Denote T_{cn} is the inverse matrix of $\begin{bmatrix} T_c C \\ C_{cn} \end{bmatrix}$.
The fault model is defined in (6.3).

The traditional dynamic output feedback controller with fixed gains is

$$\begin{aligned} \dot{\xi}_f(t) &= A_{Kf} \xi_f(t) + B_{Kf} y(t) \\ u^F(t) &= (I - \rho) C_{Kf} \xi(t) \end{aligned} \tag{6.33}$$

where $\xi_f \in R^n$ is the controller state and A_{Kf}, B_{Kf} and C_{Kf} are the controller gains to be designed.
Combining controller (6.33) with system (6.32), we have

$$\dot{\bar{x}}_f(t) = \bar{A}_f \bar{x}_f(t) + \bar{A}_{1f} \bar{x}_f(t-h) \tag{6.34}$$

where $\bar{x}_f(t) = [x_f^T(t) \ \xi^T(t)]^T$,

$$\bar{A}_f = \begin{bmatrix} A & B(I-\rho)C_{Kf} \\ B_{Kf} C & A_{Kf} \end{bmatrix}, \quad \bar{A}_{1f} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

The following performance index is considered here:

$$J = \int_0^\infty (x^T(t) Q x(t) + u^{F^T}(t) S u^F(t)) dt \tag{6.35}$$

where Q and S are given positive matrices.

Lemma 6.1 *Consider the closed-loop system described by (6.34). Then the following statements are equivalent:*
(i) there exist a symmetric matrix $P_a > 0$, $R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0$ *and a controller described by (6.33) such that*

$$\begin{bmatrix} \Omega_0 + \Omega_0 + h\bar{A}_{1f}^T R\bar{A}_{1f} + \Xi_0 & h(\bar{A}_f + \bar{A}_{1f})^T P_a \\ * & -hR \end{bmatrix} < 0 \tag{6.36}$$

hold, in normal and actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^j \in N_{\rho^j}$ *with* $\Omega_0 = P_a(\bar{A}_f + \bar{A}_{1f})$ *and* $\Xi_0 = \begin{bmatrix} Q & 0 \\ 0 & C_{Kf}^T(I-\rho)S(I-\rho)C_{Kf} \end{bmatrix}$
(ii) there exist a nonsingular matrix Q_a, *and symmetric matrix* $R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0$, $P > 0$ *with*

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} \tag{6.37}$$

and a controller described by (6.33) such that

$$\begin{bmatrix} \Omega_1 + \Omega_1 + h\bar{A}_{1q}^T R\bar{A}_{1q} + \Xi_1 & h(\bar{A}_q + \bar{A}_{1q})^T P \\ * & -hR \end{bmatrix} < 0 \tag{6.38}$$

hold, in normal and actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^j \in N_{\rho^j}$ *with*

$$\bar{A}_q = \begin{bmatrix} A & B(I-\rho)C_{Kq} \\ B_{Kq}C & A_{Kq} \end{bmatrix}, \quad \bar{A}_{1q} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\Omega_1 = P(\bar{A}_q + A_{1q}), \quad \Xi_1 = \begin{bmatrix} Q_a & 0 \\ 0 & C_{Kq}^T(I-\rho)S(I-\rho)C_{Kq} \end{bmatrix}$$

and

$$A_{Kq} = Q_a^{-1}A_{Kf}Q_a, \quad B_{Kq} = -Q_a^{-1}B_{Kf}, \quad C_{Kq} = -C_{Kf}Q_a \tag{6.39}$$

(iii) there exist symmetric matrices Y_1, N_1 *and* $0 < N_1 < Y_1$, $R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0$ *, and the controller gains of (6.33) are* $A_{Kf} = A_{Kq}$, $B_{Kf} = B_{Kq}$ $C_{Kf} = C_{Kq}$ *such that*

$$V_{a1} := \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_3 & -h(A+A_1)^T N_1 + hC^T B_{Kq}^T N_1 \\ * & \Lambda_2 & \Lambda_4 & -hC_{Kq}^T(I-\rho)B^T N_1 + hA_{Kq}^T N_1 \\ * & * & -hR_{11} & -hR_{12} \\ * & * & * & -hR_{22} \end{bmatrix} < 0, \tag{6.40}$$

hold, in normal and actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^j \in N_{\rho^j}$ *with*

$$\begin{aligned}
\Lambda_1 &= Y_1 B(I-\rho)C_{Kq} - N_1 A_{Kq} + [-N_1(A+A_1) + N_1 B_{Kq} C]^T \\
\Lambda_2 &= -N_1 B(I-\rho)C_{Kq} + +N_1 A_{Kq} + (-N_1 B(I-\rho)C_{Kq} + N_1 A_{Kq})^T \\
&\quad + C_{Kq}^T (I-\rho) S (I-\rho) C_{Kq} \\
\Lambda_3 &= h(A+A_1)^T Y_1 - hC^T B_{Kq}^T N_1 \\
\Lambda_4 &= hC_{Kq}^T (I-\rho) B^T Y_1 - hA_{Kq}^T N_1
\end{aligned}$$

Proof 6.3 *From the proof of Lemma 2.11, it is easy to conclude (i)* $\Longleftrightarrow$ *(ii), so we omit it here. On the other hand,* $P > 0$ *is equivalent to* $0 < N_1 < Y_1$, *thus by some simple algebra computation, it follows (ii)* $\Longleftrightarrow$ *(iii). The proof is complete.*

Remark 6.5 *From Lemma 6.1, it follows that the special form of* P *with* $P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix}$ *doesn't bring any conservativeness when we design the dynamic output feedback controller with fixed gain.*

The following two-step algorithm is to optimize the *guaranteed cost performance index* for the reliable controller design with fixed gains.

Algorithm 6.2 *Step 1* *Given a fixed controller gain* C_{Kf}, *which may be chosen from a feasible solution for stabilization problem via state feedback using the same Lyapunov functional*

$$\begin{bmatrix} \Xi_3 + \Xi_3^T + hA_1^T R A_1 & hX(A+A_1)^T + hY_0^T B^T \\ * & -hR \end{bmatrix}$$

with $\Xi_3 = (A+A_1)X + BY_0$ *and condition (2.48) holds for* $\bar{A}_1 = A_1$. *The feasible solutions are denoted as* $X = X_{fea}$ $Y_0 = Y_{0fea}$. *Let* $C_{Kf} = Y_0 X^{-1}$.
Step 2 *Let* $N_1 A_{Kf} = \bar{A}_{Kf}$, $N_1 B_{Kf} = \bar{B}_{Kf}$, *solving the following optimization problem*

$$\{\alpha + tr(\Gamma_1)\} \quad s.t. \quad 0 < N_1 < Y_1, \quad (6.40)$$

Denote the optimal solution as $\bar{A}_{Kf} = \bar{A}_{Kfopt}$, $\bar{B}_{Kf} = \bar{B}_{Kfopt}$, $N_1 = N_{1opt}$, *Then the controller gains can be obtained by* $A_{Kf} = N_1^{-1}\bar{A}_{Kf}$, $B_{Kf} = N_1^{-1}\bar{B}_{Kf}$ *and* $C_{Kf} = Y_0 X^{-1}$.

Remark 6.6 *It should be noted that the condition (6.40) is nonconvex, however with* C_{Kf} *fixed, and* $N_1 A_{Kf}$, $N_1 B_{Kf}$ *are defined as new variables, the condition (6.40) is linear matrix inequality. Moreover, Algorithm 6.2 gives a method for the reliable dynamic output controller design with fixed gains by two-step optimizations. Step 1 is to a* C_{Kf}, *which solves the corresponding design problem via state feedback. With the* C_{K0} *fixed, controller parameter matrices* A_{Kf} *and* B_{Kf} *can be obtained by performing Step 2.*

In order to reduce the conservativeness of the dynamic output feedback controller with fixed gains, the following dynamic output feedback controller with variable gains is given

$$\begin{aligned} \dot{\xi}(t) &= A_K(\hat{\rho})\xi(t) + B_K(\hat{\rho})y(t) \\ u^F(t) &= (I-\rho)C_{K0}\xi(t) \end{aligned} \tag{6.41}$$

where $\xi(t) \in R^n$ is the controller state, $\hat{\rho}(t)$ is the estimated value of ρ obtained by the adaptive laws, which are determined later. Denote

$$A_K(\hat{\rho}) = A_{K0} + A_{Ka}(\hat{\rho}) + A_{Kb}(\hat{\rho})$$

$$B_K(\hat{\rho}) = B_{K0} + B_{Ka}(\hat{\rho}) + B_{Kb}(\hat{\rho})$$

where

$$A_{Ka}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i A_{Kai}, \quad A_{Kb}(\hat{\rho}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\rho}_i\hat{\rho}_j A_{Kbij} + \sum_{i=1}^{m} \hat{\rho}_i A_{Kbi}$$

$$B_{Ka}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i B_{Kai}, \quad B_{Kb}(\hat{\rho}) = \sum_{i=1}^{m} \hat{\rho}_i B_{Kbi}$$

$A_0,\ A_{Kai},\ A_{Kbi},\ A_{Kbij}, B_{K0},\ B_{Kai},\ B_{Kbi}$ and C_{K0} are the controller gains to be designed.
Applying this controller (6.41) to (6.32) results in the following closed-loop system

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{A}_1\bar{x}(t-h) \tag{6.42}$$

where $\bar{x}(t) = [x^T(t)\ \xi^T(t)]^T$,

$$\bar{A} = \begin{bmatrix} A & B(I-\rho)C_{K0} \\ B_K(\hat{\rho})C & A_K(\hat{\rho}) \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consider the following operator defined in Lemma 2.13

$$D(x_t) = x(t) + \int_{t-h}^{t} A_1 x(s)ds$$

where $x_t = x(t+s),\ s \in [-h, 0]$.

The following theorem presents a sufficient condition for the reliable control problem via dynamic output feedback to optimize the guaranteed cost performance, in the framework of LMI approach and adaptive laws.

Theorem 6.2 *Suppose that the operator $D(x_t)$ satisfying the conditions in Lemma 2.13. If there exist a controller of form (6.41), matrices $0 < N_1 < Y_1,\ R_{11} > 0,\ R_{22} > 0,\ R_{12},\ A_{K0},\ A_{Kai},\ A_{Kbi},\ A_{Kbij},\ B_{K0},\ B_{Kai},\ B_{Kbi},\ C_{K0},\ i,\ j = 1\cdots m$ and symmetric matrix Θ with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

$\Theta_{11},\quad \Theta_{22} \in R^{4mn\times 4mn}$ *such that the following inequalities hold:*

$$\Theta_{22ii} \leq 0, \quad i = 1, \ \cdots, \ m$$

with $\Theta_{22ii} \in R^{(2n+s)\times(2n+s)}$ *is the* (i,i) *block of* Θ_{22}.
for any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \geq 0$$

in normal and actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\},\ \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} Q_1 & E \\ E^T & F \end{bmatrix} + G^T\Theta G < 0, \tag{6.43}$$

hold, where

$$E = \begin{bmatrix} E_1 & E_2 & \cdots & E_m \end{bmatrix}, \quad F = [F_{ij}],\ i,\ j = 1\cdots m,$$

$$Q_1 = \begin{bmatrix} \Delta_0 & \Delta_1 & h\Delta_2 & h\Delta_5 \\ * & \Delta_3 & h\Delta_4 & h\Delta_6 \\ * & * & -hR_{11} & -hR_{12} \\ * & * & * & -hR_{22} \end{bmatrix}$$

$$E_i = \begin{bmatrix} -N_1B_{Kbi}C - N_1B_{Kai}C & \Delta_7 & \Delta_8 & -\Delta_8 \\ N_1B_{Kbi}C + N_1B_{Kai}CM_2 & N_1A_{Kbi} & \Delta_9 & -\Delta_9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{ij} = \begin{bmatrix} 0 & -N_1A_{Kbij} & 0 & 0 \\ -A_{Kbji}^TN_1 & N_1A_{Kbij} + (N_1A_{Kbij})^T & -hA_{Kbji}^TN_1 & hA_{Kbji}^TN_1 \\ 0 & -hN_1A_{Kbij} & 0 & 0 \\ 0 & hN_1A_{Kbij} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\Delta_0 &= Y_1(A + A_1) - N_1B_{K0}C + [Y_1(A + A_1) - N_1B_{K0}C]^T \\
&\quad + Q + hA_1^TR_{11}A_1 \\
\Delta_1 &= Y_1B(I-\rho)C_{K0} - N_1A_{K0} - N_1A_{Ka}(\rho) + M_2^TN_1A_{Ka}(\rho) \\
&\quad - M_2^TC^TB_{Ka}(\rho)N_1 + [-N_1(A + A_1) + N_1B_{K0}C + N_1B_{Ka}(\rho)C]^T \\
\Delta_2 &= (A + A_1)^TY_1 - C^TB_{K0}^TN_1, \\
\Delta_3 &= -N_1B(I-\rho)C_{K0} + (-N_1B(I-\rho)C_{K0})^T + N_1A_{K0} + N_1A_{Ka}(\rho) \\
&\quad + (N_1A_{K0} + N_1A_{Ka}(\rho))^T + C_{K0}^T(I-\rho)S(I-\rho)C_{K0}, \\
\Delta_4 &= C_{K0}^T(I-\rho)B^TY_1 - A_{K0}^TN_1, \quad \Delta_5 = -(A + A_1)^TN_1 + C^TB_{K0}^TN_1, \\
\Delta_6 &= -C_{K0}^T(I-\rho)B^TN_1 + A_{K0}^TN_1, \quad \Delta_7 = -N_1A_{Kbi} - M_2^TN_1A_{Kai}, \\
\Delta_8 &= -hC^T[B_{Kai} + B_{Kbi}]^TN_1, \quad \Delta_9 = -h(A_{Kai} + A_{Kbi})^TN_1
\end{aligned}$$

$$\Delta(\hat{\rho}) = diag[\hat{\rho}_1 I \cdots \hat{\rho}_m],\ M_1 = T_{cn}\begin{bmatrix} T_c \\ 0 \end{bmatrix},\ M_2 = T_{cn}\begin{bmatrix} 0 \\ C_{cn} \end{bmatrix},\ G = \begin{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}$$

and also $\hat{\rho}_i(t)$ *is determined according to the adaptive law*

$$\dot{\hat{\rho}}_i = Proj_{[\min_j\{\underline{\rho}_i^j\},\ \max_j\{\bar{\rho}_i^j\}]}\{L_{2i}\}$$
$$= \begin{cases} 0, & if \quad \begin{array}{l} \hat{\rho}_i = \min_j\{\underline{\rho}_i^j\} \ and \ L_{2i} \le 0 \\ or \ \hat{\rho}_i = \max_j\{\bar{\rho}_i^j\} \ and \ L_{2i} \ge 0; \end{array} \\ L_{2i}, & otherwise \end{cases} \tag{6.44}$$

where $L_{2i} = -l_i[\xi^T N_1 A_{Kai}\xi - y^T M_1^T A_{Kai}\xi + \xi^T N_1 B_{Kai} C M_1 y],\ \ l_i > 0 \ \ (i = 1 \cdots m)$ *is the adaptive law gain to be chosen according to practical applications. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimates* $\hat{\rho}_i(t)$ *to the interval* $[\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j\}]$.
Then the closed-loop system (6.42) is asymptotically stable and the cost function (6.35) satisfies the following bound:

$$J \le D^T(0)PD(0) + h\int_{-h}^{0}(s+h)\bar{x}^T(s)\bar{A}_1^T R\bar{A}_1\bar{x}(s)ds + \sum_{i=1}^{m}\frac{\tilde{\rho}_i{}^2(0)}{l_i} \tag{6.45}$$

with $R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix}$.

Proof 6.4 *Take Lyapunov-Krasovkii functional as*

$$V = V_1 + V_2 + V_3 \tag{6.46}$$

where

$$V_1 = D^T(\bar{x}_t)PD(\bar{x}_t), \quad V_2 = \int_{t-h}^{t}(s-t+h)\bar{x}^T(s)\bar{A}_1^T R\bar{A}_1\bar{x}(s)ds,$$
$$V_3 = \sum_{i=1}^{m}\frac{\tilde{\rho}_i^2(t)}{l_i}$$

with $P > 0, \ \ R > 0$.
$V(t)$ *From the derivative of* V *along the closed-loop system (6.42), it follows*

$$\begin{aligned}\dot{V}_1 &= 2D^T(\bar{x}_t)P\dot{D}(\bar{x}_t) \\ &= 2D^T(\bar{x}_t)P(\bar{A}+\bar{A}_1)\bar{x}(t) \\ &= \bar{x}^T(t)[P(\bar{A}+\bar{A}_1) + (\bar{A}+\bar{A}_1)^T P]\bar{x}(t) + 2(\int_{t-h}^{t}\bar{A}_1\bar{x}(s)ds)^T P(\bar{A}+\bar{A}_1)\bar{x}(t)\end{aligned}$$

$$\begin{aligned}\dot{V}_2 &= h\bar{x}^T(t)\bar{A}_1^T R\bar{A}_1\bar{x}(t) - \int_{t-h}^{t}\bar{x}^T\bar{A}_1^T(s)R\bar{A}_1\bar{x}(s)ds \\ &\le h\bar{x}^T(t)\bar{A}_1^T R\bar{A}_1\bar{x}(t) - (\int_{t-h}^{t}\bar{A}_1\bar{x}(s)ds)^T(h^{-1}R)(\int_{t-h}^{t}\bar{A}_1\bar{x}(s)ds)\end{aligned}$$

$$\dot{V}_3 = \sum_{i=1}^{m}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}$$

where Lemma 2.14 is used to get $\dot{V}_2$.
Here, by using $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$, *the following equalities are obtained*

$$A_{Ka}(\tilde{\rho}) = A_{Ka}(\rho) + A_{Ka}(\hat{\rho}), \quad B_{Ka}(\tilde{\rho}) = B_{Ka}(\rho) + B_{Ka}(\hat{\rho})$$

Then $\bar{A}$ *can be written as*

$$\bar{A} = \bar{A}_a + \bar{A}_b$$

where

$$\bar{A}_a = \begin{bmatrix} A & B(I-\rho)C_{K0} \\ [B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C & A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho}) \end{bmatrix}$$

$$\bar{A}_b = \begin{bmatrix} 0 & 0 \\ B_{Ka}(\tilde{\rho})C & A_{Ka}(\tilde{\rho}) \end{bmatrix}.$$

Let P *be the following form, that is*

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix}, \tag{6.47}$$

with $0 < N_1 < Y_1$, *which implies* $P > 0$.
From (6.32), it follows

$$T_c C x = T_c y$$

Then

$$x = T_{cn} \begin{bmatrix} T_c C x \\ C_{cn} x \end{bmatrix} = M_1 y + M_2 x \tag{6.48}$$

with $M_1 = T_{cn} \begin{bmatrix} T_c \\ 0 \end{bmatrix}, \ M_2 = T_{cn} \begin{bmatrix} 0 \\ C_{cn} \end{bmatrix}$.

Notice that

$$P\bar{A}_a = \begin{bmatrix} Y_1 A - N_1[B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C & T_1 \\ -N_1 A + N_1[B_{K0} + B_{Ka}(\rho) + B_{Kb}(\hat{\rho})]C & T_2 \end{bmatrix}$$

with

$$T_1 = Y_1 B(I-\rho)C_{K0} - N_1[A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})]$$

$$T_2 = -N_1 B(I-\rho)C_{K0} + N_1[A_{K0} + A_{Ka}(\rho) + A_{Kb}(\hat{\rho})].$$

and

$$P\bar{A}_b = \begin{bmatrix} -N_1 B_{Ka}(\tilde{\rho})C & -N_1 A_{Ka}(\tilde{\rho}) \\ N_1 B_{Ka}(\tilde{\rho})C & N_1 A_{Ka}(\tilde{\rho}) \end{bmatrix}$$

which follows

$$\begin{aligned} &\bar{x}^T(t) P \bar{A}_b \bar{x}(t) \\ &= [x^T \ \xi^T] P \bar{A}_b [x^T \ \xi^T]^T \\ &= -x^T N_1 B_{Ka}(\tilde{\rho})Cx - x^T N_1 A_{Ka}(\tilde{\rho})\xi \ + \xi^T N_1 B_{Ka}(\tilde{\rho})Cx + \xi^T N_1 A_{Ka}(\tilde{\rho})\xi \end{aligned} \tag{6.49}$$

Thus, by (6.48) it is easy to see

$$-x^T N_1 A_{Ka}(\tilde{\rho})\xi = -y^T M_1^T N_1 A_{Ka}(\tilde{\rho})\xi - x^T M_2^T N_1 A_{Ka}(\tilde{\rho})\xi$$

$$\xi^T N_1 B_{Ka}(\tilde{\rho})Cx = \xi^T N_1 B_{Ka}(\tilde{\rho})CM_1 y + \xi^T N_1 B_{Ka}(\tilde{\rho})CM_2 x$$

Thus

$$\bar{x}^T(t)P\bar{A}_b\bar{x}(t) = \bar{x}^T M_a \bar{x} + M_b$$

where

$$M_a = \begin{bmatrix} -N_1 B_{Ka}(\tilde{\rho})C & -M_2^T N_1 A_{Ka}(\tilde{\rho}) \\ N_1 B_{Ka}(\tilde{\rho})CM_2 & 0 \end{bmatrix},$$

$$M_b = -y^T M_1^T N_1 A_{Ka}(\tilde{\rho})\xi + \xi^T N_1 B_{Ka}(\tilde{\rho})CM_1 y + \xi^T N_1 A_{Ka}(\tilde{\rho})\xi$$

Then from the derivative of $V(t)$ *along the closed-loop system (6.42), it follows*

$$\begin{aligned} \dot{V}_1(t) &= \bar{x}^T(t)[P(\bar{A}_a + \bar{A}_1) + (\bar{A}_a + \bar{A}_1)^T P]\bar{x}(t) + \bar{x}^T(M_a + M_a^T)\bar{x} + 2M_b \\ &\quad + 2(\int_{t-h}^{t} \bar{A}_1\bar{x}(s)ds)^T P(\bar{A} + \bar{A}_1)\bar{x}(t) \end{aligned} \tag{6.50}$$

So

$$\dot{V}(t) \leq \chi^T W_0 \chi + 2M_b + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \tag{6.51}$$

where

$$\chi = \begin{bmatrix} \bar{x}(t) \\ \int_{t-h}^{t} \bar{A}_1\bar{x}(s)ds \end{bmatrix}, \quad W_0 = \begin{bmatrix} \Phi + \Phi^T + h\bar{A}_1^T R\bar{A}_1 & (\bar{A} + \bar{A}_1)^T P \\ * & -h^{-1}R \end{bmatrix}$$

with $\Phi = P(\bar{A}_a + \bar{A}_1) + M_a$.
Since y *and* ξ *are available online, we choose the adaptive laws as (6.44). Then it follows*

$$M_b + \sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0 \tag{6.52}$$

Thus

$$\dot{V}(t) \leq \chi^T W_0 \chi \tag{6.53}$$

Furthermore

$$\begin{aligned} J &\leq \int_0^\infty (\bar{x}^T(t) \begin{bmatrix} Q & 0 \\ 0 & C_{K0}^T(I-\rho)S(I-\rho)C_{K0} \end{bmatrix} \bar{x}(t) + \dot{V})dt + V(0) \\ &\leq \int_0^\infty \chi^T W_1 \chi dt + V(0) \end{aligned} \tag{6.54}$$

where

$$W_1 = \begin{bmatrix} \Phi + \Phi^T + h\bar{A}_1^T R\bar{A}_1 + \begin{bmatrix} Q & 0 \\ 0 & C_{K0}^T(I-\rho)S(I-\rho)C_{K0} \end{bmatrix} & (\bar{A}+\bar{A}_1)^T P \\ * & -h^{-1}R \end{bmatrix}$$

By pre- and post-multiplying inequalities $W_1 < 0$ *by* $diag\{I, h\}$, *then* $W_1 < 0$ *is equivalent to*

$$W_2 = \begin{bmatrix} \Phi + \Phi^T + h\bar{A}_1^T R\bar{A}_1 + \Xi_4 & h(\bar{A}+\bar{A}_1)^T P \\ * & -hR \end{bmatrix} < 0 \tag{6.55}$$

where $\Xi_4 = \begin{bmatrix} Q & 0 \\ 0 & C_{K0}^T(I-\rho)SC_{K0}(I-\rho) \end{bmatrix}$.
Furthermore (6.55) can be described by

$$W_2(\hat{\rho}) = Q_1 + \sum_{i=1}^{m} \hat{\rho}_i E_i + (\sum_{i=1}^{m} \hat{\rho}_i E_i)^T + \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\rho}_i\hat{\rho}_j F_{ij} < 0,$$

where Q_1, E_i, F_{ij} *are defined in (6.43). By Lemma 2.10, we can get* $W_2(\hat{\rho}) < 0$ *if (6.43) holds, which implies* $W_1 < 0$ *and* $W_0 < 0$. *Then the closed-loop system (6.42) is asymptotically stable in both normal and fault cases. Moreover,*

$$J \le V(0) = D^T(0)PD(0) + h\int_{-h}^{0}(s+h)\bar{x}^T(s)\bar{A}_1^T R\bar{A}_1\bar{x}(s)ds + \sum_{i=1}^{m}\frac{\tilde{\rho_i}^2(0)}{l_i}$$

Remark 6.7 *Theorem 6.2 presents sufficient conditions for adaptive fault-tolerant guaranteed cost controller design via dynamic output feedback. Generally, (6.43) is not LMIs. But when* C_{K0} *is given, and* $N_1A_{K0}, N_1A_{Kai}, N_1A_{Kbi}$, N_1A_{Kbij}, N_1B_{K0}, N_1B_{Kai} *and* N_1B_{Kbi} *are defined as new variables, (6.43) becomes LMIs and linearly depends on uncertain parameters* ρ *and* $\hat{\rho}$.

Remark 6.8 *By (6.3) and (6.44), it follows that* $\tilde{\rho}_i(0) \le \max_j\{\bar{\rho}_i^j\} - \min_j\{\underline{\rho}_i^j\}$. *We can choose* l_i *relatively large so that* $\sum_{i=1}^{m}\frac{\tilde{\rho_i}^2(0)}{l_i}$ *is sufficiently small.*

Theorem 6.3 *Consider the closed-loop system (6.42) with cost function (6.35). If the following optimization problem*

$$\min\{\alpha + tr(\Gamma_1)\}$$

subject to

(i) *LMI* (2.48), (6.43)

(ii) $\begin{bmatrix} -\alpha & D^T(0)P \\ * & -P \end{bmatrix} < 0$

$$\text{(iii)} \quad \begin{bmatrix} -\Gamma_1 & hV_0^T\bar{A}_1^T R \\ * & -hR \end{bmatrix} < 0 \tag{6.56}$$

has a solution set, the controller (6.41) ensures the minimization of the guaranteed cost (6.35) for the closed-loop system (6.42) against actuator faults, where $\int_{-h}^{0}(s+h)\bar{x}(s)\bar{x}^T(s)ds = V_0V_0^T$.

Proof 6.5 *By Theorem 6.2, (i) in (6.56) is clear. Also, it follows from Lemma (2.8) that (ii) and (iii) in (6.56) are equivalent to* $D^T(0)PD(0) < \alpha$ *and* $hV_0^T\bar{A}_1^TR\bar{A}_1V_0 \leq \Gamma_1$, *respectively. On the other hand,*

$$\begin{aligned}
&\int_{-h}^{0}(s+h)\bar{x}^T(s)\bar{A}_1^TR\bar{A}_1\bar{x}(s)ds \\
&= \int_{-h}^{0} tr((s+h)\bar{x}^T(s)\bar{A}_1^TR\bar{A}_1\bar{x}(s))ds \\
&= tr(V_0V_0^T\bar{A}_1^TR\bar{A}_1) = tr(V_0^T\bar{A}_1^TR\bar{A}_1V_0) < tr(\Gamma_1)
\end{aligned}$$

Hence, it follows from (6.54) that

$$J^* < \alpha + tr(\Gamma_1) + \sum_{i=1}^{m}\frac{\tilde{\rho}_i{}^2(0)}{l_i}.$$

Thus, the minimization of $\alpha + tr(\Gamma_1)$ *implies the minimization of the guaranteed cost for the system (6.42).*

Remark 6.9 *If we choose the Lyapunov functional candidate* $V = V_1 + V_2$, *where* V_1, V_2 *are defined in (6.46), then it is easy to see conditions (6.38) can guarantee the closed-loop system (6.41) is asymptotically stable and the cost function (6.35) satisfied the following bound:*

$$J \leq D^T(0)PD(0) + h\int_{-h}^{0}(s+h)\bar{x}^T(s)\bar{A}_1^TR\bar{A}_1\bar{x}(s)ds$$

From Lemma 6.1, it follows condition (6.38) is equivalent to (6.40). It should also be noted that conditions (6.40) are not convex. But when C_{Kf} *is given, and* N_1A_{Kf} *and* N_1B_{Kf} *are defined as new variables, they become LMIs. Also the upper bound of* J *with fixed gains controller can be obtained by solving the following optimization:*

$$\min\{\alpha + tr(\Gamma_1)\}$$

$$\begin{aligned}
&\textit{(i)}\quad \textit{LMI}\ \ (2.48)\ \ (6.40) \\
&\textit{(ii)}\quad \begin{bmatrix} -\alpha & D^T(0)P \\ * & -P \end{bmatrix} < 0 \\
&\textit{(iii)}\quad \begin{bmatrix} -\Gamma_1 & hV_0^T\bar{A}_1^TR \\ * & -hR \end{bmatrix} < 0
\end{aligned} \tag{6.57}$$

Theorem 6.4 *If the conditions in Lemma 6.1 hold for the closed-loop system (6.34) with fixed gain dynamic output feedback controller (6.33), then the conditions in Theorem 6.2 hold for the closed-loop system (6.42) with adaptive dynamic output feedback controller (6.41).*

Proof 6.6 *Notice that if* $V_{a1} < 0$ *for the actuator fault cases and normal case, then the conditions in Theorem 6.2 are feasible with* $A_{K0} = A_{Kf}, B_{K0} = B_{Kf}, C_{K0} = C_{Kf}$ *and* $A_{Kai} = A_{Kbi} = A_{Kbij} = B_{Kai} = B_{Kbi} = 0, i, j = 1 \cdots m$. *The proof is complete.*

Remark 6.10 *Theorem 6.4 shows that the method for the adaptive fault-tolerant guaranteed cost controllers design given in Theorem 6.2 is less conservative than that given in Lemma 6.1 for the fault-tolerant guaranteed cost controllers design with fixed gains.*

The following two-step algorithm is to optimize the adaptive fault-tolerant guaranteed cost performances in normal and fault cases.

Algorithm 6.3 *Step 1 Determine* C_{K0}. *Chose* $C_{K0} = C_{Kf}$, *which can be obtained by Step 1 in Algorithm 6.2.*

Step 2 Let $N_1 A_{K0} = \bar{A}_{K0}$, $N_1 A_{Kai} = \bar{A}_{Kai}$, $N_1 A_{Kbi} = \bar{A}_{Kbi}$, $N_1 A_{Kbij} = \bar{A}_{Kbij}$, $N_1 B_{K0} = \bar{B}_{K0}$, $N_1 B_{Kai} = \bar{B}_{Kai}$ *and* $N_1 B_{Kbi} = \bar{B}_{Kbi}$, *and solve the following optimization problem*

$$\min\{\alpha + tr(\Gamma_1)\} \quad s.t. \;\; 0 < N_1 < Y_1, (6.56)$$

Denote the optimal solutions as $\bar{A}_{K0} = \bar{A}_{K0opt}$, $\bar{A}_{Kai} = \bar{A}_{Kaiopt}$, $\bar{A}_{Kbi} = \bar{A}_{Kbiopt}$, $\bar{A}_{Kbij} = \bar{A}_{Kbijopt}$, $\bar{B}_{K0} = \bar{B}_{K0opt}$, $\bar{B}_{Kai} = \bar{B}_{Kaiopt}$, $\bar{B}_{Kbi} = \bar{B}_{Kbiopt}$, $N_1 = N_{1opt}$. *The corresponding adaptive controller gains are obtained by* $A_{K0} = N_1^{-1}\bar{A}_{K0}$, $A_{Kai} = N_1^{-1}\bar{A}_{Kai}$, $A_{Kbi} = N_1^{-1}\bar{A}_{Kbi}$, $A_{Kbij} = N_1^{-1}\bar{A}_{Kbij}$, $B_{K0} = N_1^{-1}\bar{B}_{K0}$, $B_{Kai} = N_1^{-1}\bar{B}_{Kai}$, $B_{Kbi} = N_1^{-1}\bar{B}_{Kbi}$($i, j = 1 \cdots m$), $C_{K0} = C_{Kf}$.

Remark 6.11 *From Theorem 6.2, it is easy to see that controller gains* $A_{K0}, A_{Kai}, A_{Kbi}, A_{Kbij}, B_{K0}, B_{Kai}, B_{Kbi}, C_{K0}, C_{Kai}, C_{Kbi} (i, j = 1, \cdots, m)$ *are obtained off-line by Algorithm 6.3 while the estimation* $\hat{\rho}_i$ *are automatically updating online according to the designed adaptive law (6.44). Thus due to the introduction of adaptive mechanisms, the resultant controller gain (6.41) is variable, which is different from traditional controller with fixed gains.*

6.2.4 Example

To illustrate the effectiveness of our results, two examples are given. Example 6.1 is for state feedback case and Example 6.2 is for dynamic output feedback case.

TABLE 6.1 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
γ_n	1.6377	4.1086
γ_f	2.6652	5.0885

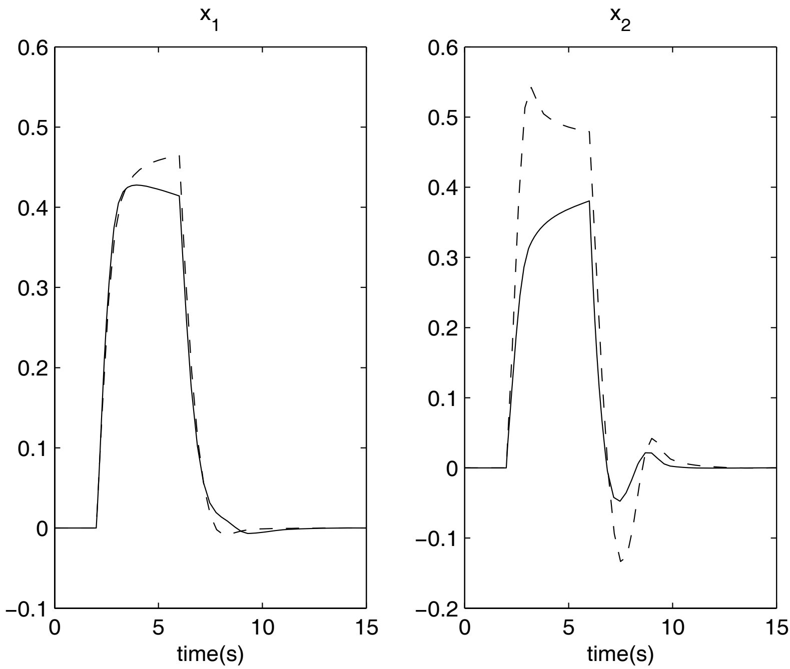


FIGURE 6.1
Response curve in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

Example 6.1 *Consider a linear time-delay system with parameters as follows*

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\phi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad h = 0.5, \quad d = 0.25.$$

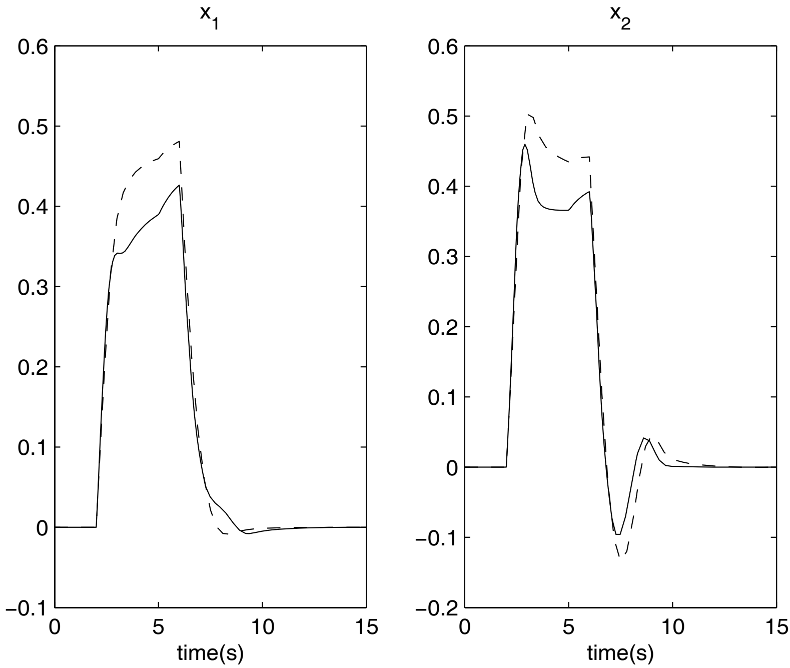


FIGURE 6.2
Response curve in fault case 1 with adaptive controller (solid) and controller with fixed gains (dashed).

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be normal or loss of effectiveness, that is,

$$\rho_1^1 = 1, \quad 0 \leq \rho_2^1 \leq 0.5.$$

Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, that is,

$$\rho_2^2 = 1, \quad 0 \leq \rho_1^2 \leq 0.4.$$

Using Algorithm 6.1 with $\alpha = 10, \beta = 1$ and $\varepsilon = 0.9$, we obtain the corresponding H_∞ performances indexes of the closed-loop system using the two controllers. See Table 6.1 for more details. To verify the effectiveness of the proposed adaptive method, the simulations are given in the following.

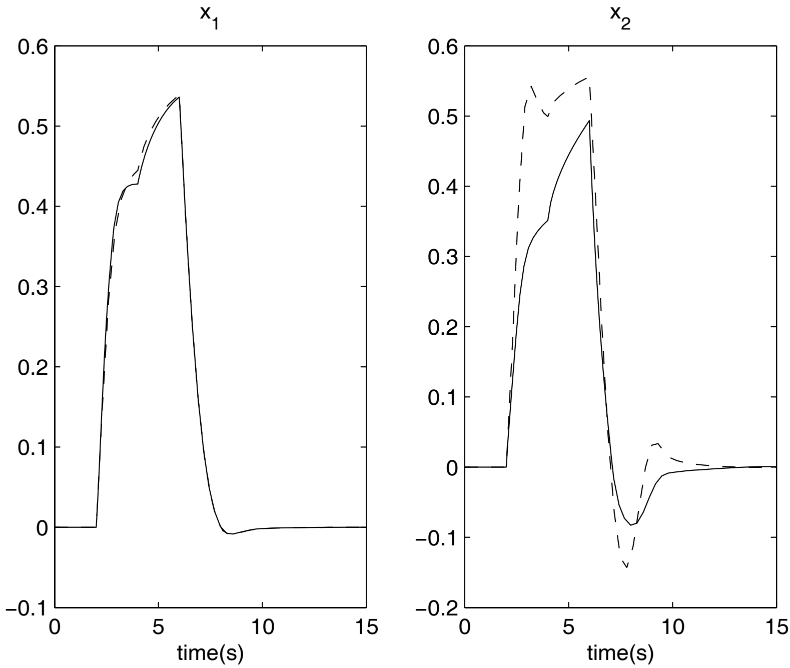


FIGURE 6.3
Response curve in fault case 2 with adaptive controller (solid) and controller with fixed gains (dashed).

In the simulation, the following two fault cases are considered:
Fault case 1: At 2 seconds, the first actuator is outage, then the second actuator becomes loss of 30% effectiveness.
Fault case 2: At 4 seconds, the second actuator is outage.

In order to show the effectiveness of our method more clearly, some simulations are also given. In the following simulation, the time-delay is $\tau(t) = \frac{1+\sin t}{4}$ and the disturbance here is

$$\omega(t) = \begin{cases} 2 & 2 \leq t \leq 4 \text{ (seconds)} \\ 0 & \text{otherwise} \end{cases}$$

Figure 6.1 describes the response curves in normal case with our adaptive reliable controller and reliable controller with fixed gains, respectively. The corresponding curves in the above-mentioned two fault cases with these two controllers are given in Figure 6.2 and Figure 6.3, respectively. From Figure 6.1-Figure 6.3, it is easy to see our adaptive controller has more disturbance restraint ability than the one with fixed gains in either normal or fault cases just as theory has proved.

TABLE 6.2 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
Upper bound of J	4.4836	5.1858

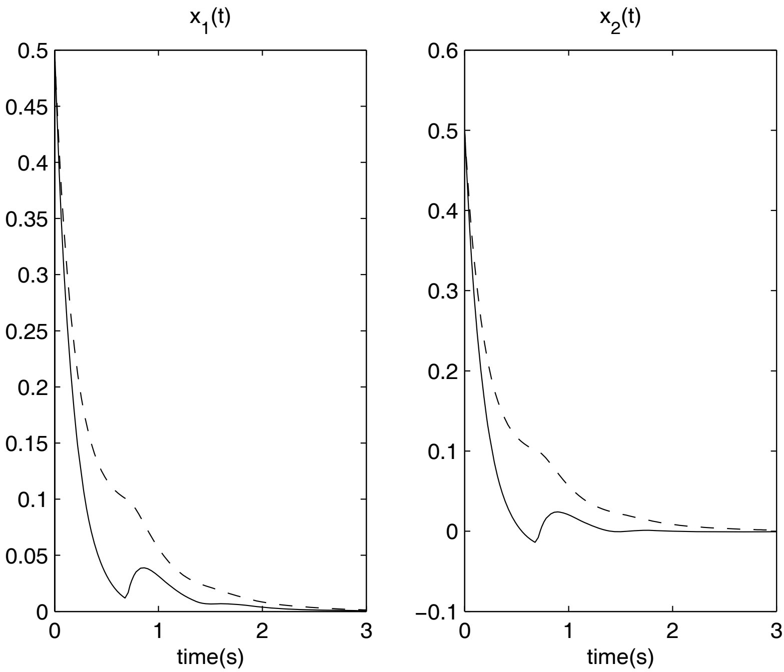


FIGURE 6.4
Response curves in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

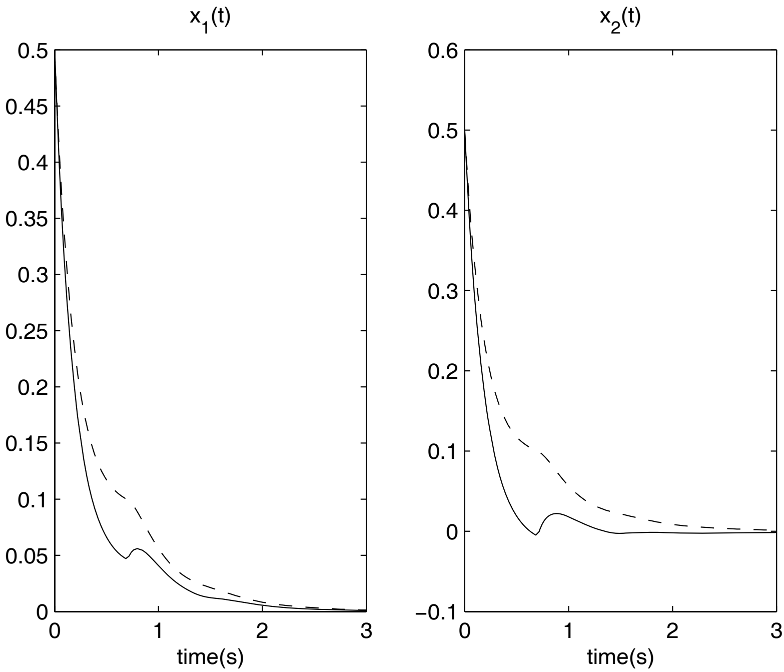


FIGURE 6.5
Response curves in fault case 1 with adaptive controller (solid) and controller with fixed gains (dashed).

Example 6.2 *A real application example about river pollution control [82] is proposed to show the effectiveness of our approach.*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1x(t-h) + Bu(t)\\ x(t) &= \phi(t), \quad t \in [-h, 0]\\ y(t) &= Cx(t)\end{aligned} \tag{6.58}$$

where

$$A = \begin{bmatrix} -k_{10} - \eta_1 - \eta_2 & 0 \\ -k_{30} & -k_{20} - \eta_1 - \eta_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \eta_2 & 0 \\ 0 & \eta_2 \end{bmatrix},$$
$$B = \begin{bmatrix} \eta_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Here $u = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ is the control variable of river pollution. $k_{i0}(i = 1, 2, 3)$, η_1 and η_2 are known constants. The physical meaning of these parameters can be found in [82].

In the simulation, we choose $h = 0.7$, $\eta_1 = 2, \eta_2 = 1$, $k_{10} = 3, k_{20} = 1$, $k_{30} = 2$, $C_{cn} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $T_c = I$.

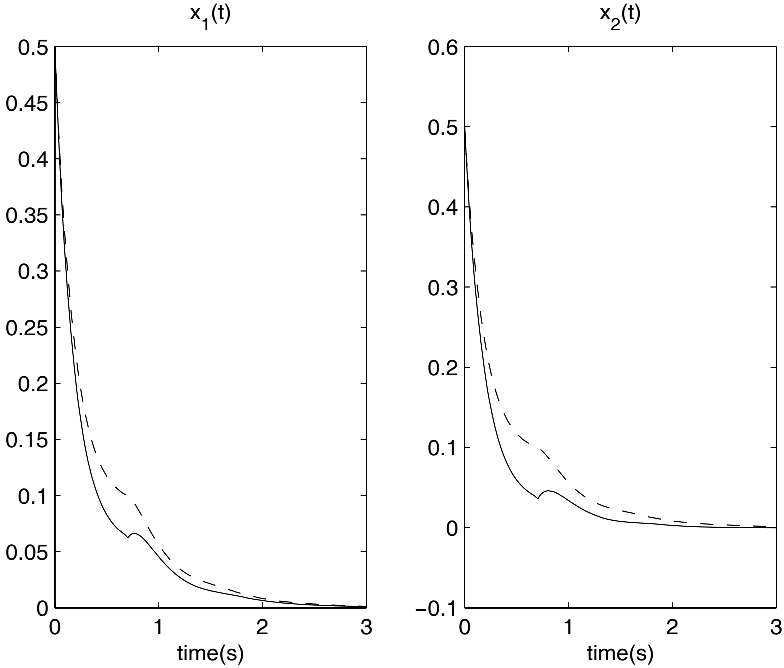


FIGURE 6.6
Response curves in fault case 2 with adaptive controller (solid) and controller with fixed gains (dashed).

And the matrices in the performance index (6.35) are $Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}$.

The initial state is $\phi(t) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

Besides the normal mode, that is, $\rho_1^0 = \rho_2^0 = 0$, the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^1 = 1, \quad 0 \leq \rho_1^2 \leq a_1, a_1 = 0.4$$

which denotes the maximal loss of effectiveness for the second actuator.
Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, described by

$$\rho_2^2 = 1, \quad 0 \leq \rho_1^2 \leq b_1, b_1 = 0.5$$

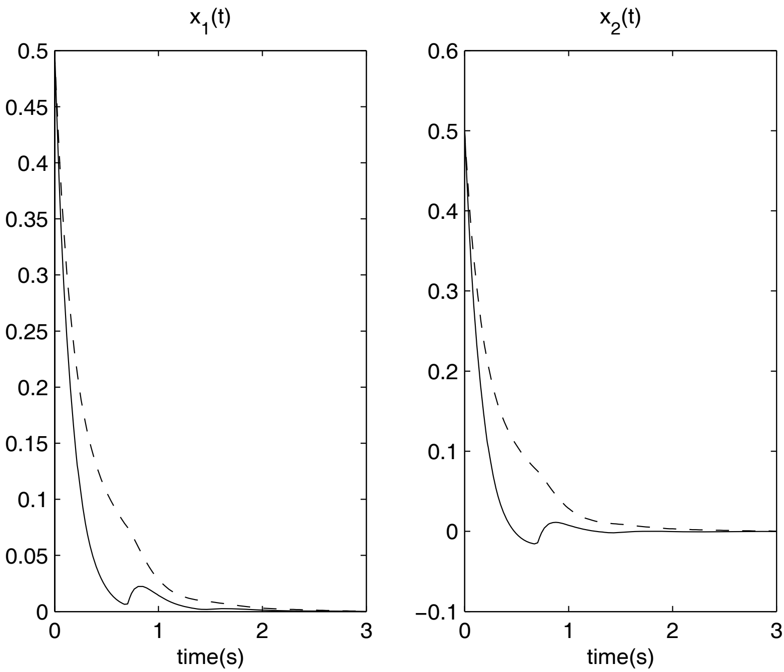


FIGURE 6.7
Robust response curves in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

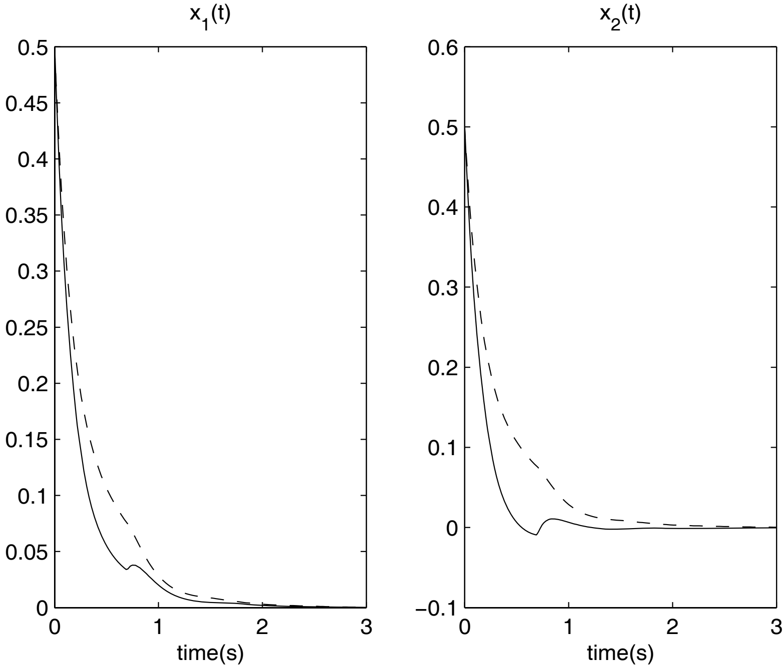


FIGURE 6.8
Robust response curves in fault case 1 with adaptive controller (solid) and controller with fixed gains (dashed).

which denotes the maximal loss of effectiveness for the first actuator. By using Algorithm 6.2 and Algorithm 6.3, we obtain the corresponding cost performance indexes, using the adaptive method and traditional method. See Table 6.2 for more details.

The considered fault cases in the following simulations are:
Fault case 1 is at 0 second, the first actuator becomes outage.
Fault case 2 is at 0.5 second, the second actuator becomes outage. Then after 1 second, the first actuator becomes loss of effectiveness of 50%.

Figure 6.4, Figure 6.5 and Figure 6.6 are the state responses with adaptive and fixed gain dynamic output feedback controllers in normal and fault cases, respectively. It is easy to see our adaptive fault-tolerant guaranteed cost controller performs better than the one with fixed gains in both normal and fault cases just as theory has proved.

In the next simulations, some time-varying uncertainties $\Delta A(t) = 0.25A\text{sin}t$, $\Delta A_1(t) = 0.4A_1\text{cos}3t$ and $\Delta B(t) = 0.5B\text{sin}2t$ are added into the system matrices A, A_1 and B, respectively, which aims to demonstrate the robustness of designed controllers. The corresponding state curves are given

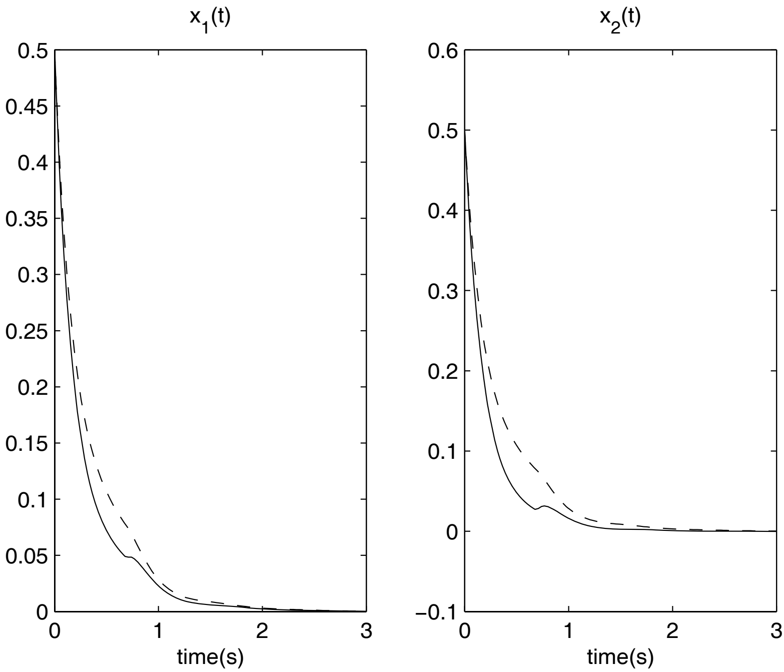


FIGURE 6.9
Robust response curves in fault case 2 with adaptive controller (solid) and controller with fixed gains (dashed).

in Figure 6.7-Figure 6.9. It is easy to see that the designed controllers are robust to these uncertainties.

6.3 Adaptive Reliable Memory Controller Design

As is well known, a memory-less controller has an advantage of easy implementation, but its performance cannot be better than a delay-dependent memory feedback controller when the information of the size of delay is available [4] and[110]. Thus, here we investigate the delay-dependent memory controller design problem for linear time-delay system, based on the adaptive method and LMI techniques.

6.3.1 Problem Statement

Consider the following linear time-delay system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1x(t-h) + Bu(t) + B_1\omega(t)\\ z(t) &= Cx(t) + Du(t)\\ x(t) &= \phi(t), \quad t \in [-h, 0]\end{aligned} \tag{6.59}$$

where $x(t) \in R^n$ and x_t is the state at time t defined by $x_t(s) = x(t+s), s \in [-d, 0]$, $u(t) \in R^m$ is the control input, $z(t) \in R^q$ is the regulated output, respectively. $\omega(t) \in R^p$ is an exogenous disturbance in $L_2[0, \infty]$ and d is a positive constant time-delay. $\{\phi(t), t \in [-d, 0]\}$ is a real-valued initial function. A, A_d, B, B_1, C and D are known constant matrices of appropriate dimensions.

In this section, the considered actuator faults are the same as those in Chapter 3, that is

$$u^F(t) = (I - \rho)u(t), \quad \rho \in [\rho^1 \cdots \rho^L] \tag{6.60}$$

where ρ can be described as $\rho = \text{diag}[\rho_1,\ \rho_2,\ \cdots \rho_m]$.

Denote

$$N_{\rho^j} = \{\rho^j\ |\rho^j = \text{diag}[\rho_1^j, \rho_2^j, \cdots \rho_m^j],\ \rho_i^j = \underline{\rho_i}^j \quad \rho_i^j = \bar{\rho_i}^j\}$$

Thus, the set N_{ρ^j} contains a maximum of 2^m elements.

6.3.2 H_∞ State Feedback Control

In this subsection, we deal with the delay-dependent memory H_∞ controller design problem, such that in normal and fault cases, the resulting closed-loop system is asymptotically stable and its H_∞ disturbance attenuation performance bound is minimized.

With actuator faults, the system (6.59) is described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1x(t-h) + B(I-\rho)u(t) + B_1\omega(t)\\ z(t) &= Cx(t) + D(I-\rho)u(t)\end{aligned} \tag{6.61}$$

Define an operator $D(x_t) : C_{n,\ h} \to R^n$ as

$$D(x_t) = x(t) + \int_{t-h}^{t} Gx(s)ds = x(t) + f(t) \tag{6.62}$$

where $x_t = x(t+s), s \in [-d, 0]$, $f(t) = \int_{t-d}^{t} Gx(s)ds$ and $G \in R^{n\times n}$ is a constant matrix which will be chosen.

Now, we are interested in designing a delay-dependent memory state feedback controller with the following structure

$$u(t) = K(\hat{\rho}(t))x(t) + K_cf(t) = (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t))x(t) + K_cf(t) \tag{6.63}$$

where $K_a(\hat\rho(t)) = \sum_{i=1}^{m} K_{ai}\hat\rho_i(t), K_b(\hat\rho(t)) = \sum_{i=1}^{m} K_{bi}\hat\rho_i(t)$ and $\hat\rho_i(t)$ is the estimate of ρ_i. $K_0, K_{ai}, K_{bi}, i = 1\cdots m$ and K_c are the control gains to be designed.

Remark 6.12 *Notice that (6.63) has a parameter-dependent gain and $\hat\rho_i(t)$ is included in an affine fashion, which is a convex problem. Though $K_a(\hat\rho(t)$ and $K_b(\hat\rho(t)$ have the same forms, we will deal with them in different ways to get more relaxed conditions in our main result.*

The closed-loop system is given by

$$\begin{aligned}\dot x(t) &= Ax(t) + A_1x(t-h) + B(I-\rho)K(\hat\rho)x(t) + B(I-\rho)K_cf(t) + B_1\omega\\ z(t) &= (C + D(I-\rho)K(\hat\rho))x(t) + D(I-\rho)K_cf(t)\end{aligned} \quad (6.64)$$

Denote

$$\Delta_{\hat\rho} = \{\hat\rho = (\hat\rho_1\cdots\hat\rho_N) : \hat\rho_i \in \{\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar\rho_i^j\}\}\}, \quad \Delta(\hat\rho) = \mathrm{diag}[\hat\rho_1 I\cdots\hat\rho_m I]$$

$$N_0 = \begin{bmatrix} T_1 + T_1^T + Q + hF_{11} & T_1^T + F_{12} + B(I-\rho)X \\ * & -h^{-1}(\alpha-1)X + B(I-\rho)X\end{bmatrix},$$

$$T_1 = AX + B((I-\rho)Y_0 + Y_a(\rho)) + W, \quad Y_a(\rho) = \sum_{i=1}^{m} Y_{ai}\rho_i,$$

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_m\end{bmatrix}, \quad \Upsilon = [\Upsilon_{ij}], i, j = 1\cdots m,$$

$$R_i = \begin{bmatrix} -B\rho Y_{ai} + BY_{bi} & -\rho Y_{ai}^T B^T + Y_{bi}^T B^T \\ 0 & 0\end{bmatrix}, \quad G = \begin{bmatrix}\begin{bmatrix} I \\ \vdots \\ I\end{bmatrix} & 0 \\ 0 & I\end{bmatrix}$$

$$\Upsilon_{ij} = \begin{bmatrix} -B^iY_{bj} - Y_{bi}^T B^{jT} & -Y_{bi}^T B^{jT} \\ -B^iY_{bj} & 0\end{bmatrix}, \quad V_0 = \begin{bmatrix} V_{00} & V_{01} & \cdots & V_{0m}\end{bmatrix},$$

$$V_{00} = \begin{bmatrix} CX + D(I-\rho)Y_0 & D(I-\rho)W_1\end{bmatrix}, \quad V_{0i} = \begin{bmatrix} D(I-\rho)(Y_{ai} + Y_{bi}) & 0\end{bmatrix}$$

Theorem 6.5 *Let $\gamma_f > \gamma_n > 0$, $\alpha > 1$ and $d > 0$ be given constants, if there exist positive definite matrices $X, Q, F_{11}, F_{22}, F_{33}$ and matrices Y_0, Y_{ai}, Y_{bi}, W, W_1, $F_{12}, F_{13}, F_{23}, i = 1\cdots m$ and a symmetric matrix Θ with*

$$\Theta = \begin{bmatrix}\Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22}\end{bmatrix}$$

and Θ_{11}, $\Theta_{22} \in R^{2mn\times 2mn}$ such that the following inequalities hold:

$$\Theta_{22ii} \le 0, \quad i = 1, \ \cdots, \ m \quad (6.65)$$

with $\Theta_{22ii} \in R^{(2n+s)\times(2n+s)}$ is the (i,i) block of Θ_{22}.
For any $\delta \in \Delta_v$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \ge 0 \quad (6.66)$$

for $\rho = 0$, *i.e., in normal case,*

$$\begin{bmatrix} \begin{bmatrix} N_0 & R \\ R^T & \Upsilon \end{bmatrix} + G^T\Theta G & V_0^T & \begin{bmatrix} B_1 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} \alpha h W^T \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} (A_1X - W) + hF_{13} \\ (A_1X - W) + F_{23} \\ 0 \end{bmatrix} \\ * & -I & 0 & 0 & 0 \\ * & * & -\gamma_n^2 I & 0 & 0 \\ * & * & * & -\alpha h X & 0 \\ * & * & * & * & -Q + hF_{33} \end{bmatrix} < 0 \tag{6.67}$$

for $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$, *i.e., in fault cases,*

$$\begin{bmatrix} \begin{bmatrix} N_0 & R \\ R^T & \Upsilon \end{bmatrix} + G^T\Theta G & V_0^T & \begin{bmatrix} B_1 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} \alpha h W^T \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} (A_1X - W) + hF_{13} \\ (A_1X - W) + F_{23} \\ 0 \end{bmatrix} \\ * & -I & 0 & 0 & 0 \\ * & * & -\gamma_f^2 I & 0 & 0 \\ * & * & * & -\alpha h X & 0 \\ * & * & * & * & -Q + hF_{33} \end{bmatrix} < 0 \tag{6.68}$$

$$-X + F_{22} < 0 \tag{6.69}$$

$$\begin{bmatrix} -X & hW^T \\ * & -X \end{bmatrix} < 0 \tag{6.70}$$

$$\Xi = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ * & F_{22} & F_{23} \\ * & * & F_{33} \end{bmatrix} > 0 \tag{6.71}$$

and also $\hat{\rho}_i$ *is determined according to the adaptive laws*

$$\begin{aligned} \dot{\hat{\rho}}_i &= Proj_{[\min\limits_j\{\underline{\rho}_i^j\},\ \max\limits_j\{\bar{\rho}_i^j\}]}\{L_i\} \\ &= \begin{cases} 0, \quad if & \begin{matrix} \hat{\rho}_i = \min\limits_j\{\underline{\rho}_i^j\} \ and \ \ L_i \le 0 \\ or \ \hat{\rho}_i = \max\limits_j\{\bar{\rho}_i^j\} \ and \ L_i \ge 0; \end{matrix} \\ L_i, \quad otherwise \end{cases} \end{aligned} \tag{6.72}$$

where $L_i = -l_i(f(t) + x(t))^T X^{-1}[B^i K_b(\hat{\rho}) + BK_{ai}]x(t)$, $l_i > 0 (i = 1 \cdots m)$ *is the adaptive law gain. Proj*$\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimates* $\hat{\rho}_i(t)$ *to the interval* $[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j\}]$, *then* *the closed-loop system (6.61) is asymptotically stable and in normal case, i.e., $\rho = 0$, satisfies for $x(t) = 0, t \in [-d, 0]$*

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{6.73}$$

and in actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$, satisfies for $x(t) = 0, t \in [-d, 0]$

$$\int_0^\infty z^T(t)z(t)dt \le \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{i=1}^m \frac{\tilde{\rho_i}^2(0)}{l_i} \tag{6.74}$$

where $\tilde{\rho}(t) = diag[\tilde{\rho}_1(t) \cdots \tilde{\rho}_m(t)], \tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$.
Furthermore, the corresponding controller is given by

$$u(t) = (Y_0 X^{-1} + \sum_{i=1}^m \hat{\rho}_i Y_{ai} X^{-1} + \sum_{i=1}^m \hat{\rho}_i Y_{bi} X^{-1})x(t) + W_1 X^{-1} f(t) \tag{6.75}$$

with $f(t) = \int_{t-h}^t Gx(s)ds$, $G = WX^{-1}$.

Proof 6.7 *The following Lyapunov-Krasovkii functional candidate is chosen*

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) \tag{6.76}$$

where

$$\begin{aligned} &V_1(t) = D^T(x_t)PD(x_t), \quad V_2(t) = \alpha \int_{t-h}^t \int_s^t x^T(u)G^T PGx(u)duds \\ &V_3(t) = \int_{t-h}^t x^T(s)Ux(s)ds, \quad V_4(t) = \int_0^t \int_{s-h}^s \chi^T \Omega \Xi \Omega \chi duds, \\ &V_5(t) = \sum_{i=1}^m \frac{\tilde{\rho}_i^2(t)}{l_i} \end{aligned}$$

with $\chi = [x^T(s), x^T(u)G^T, x^T(s-h)]^T$, $P > 0$, $\Omega = diag\{P, P, P\}$, $U > 0$.
The following equality is obtained

$$\begin{aligned} (I-\rho)u(t) &= (I-\rho)[(K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t) + K_c f(t)] \\ &= [(I-\rho)K_0 + K_a(\hat{\rho}) - \rho K_a(\hat{\rho}(t))]x(t) + (I - \hat{\rho}(t))K_b(\hat{\rho}(t))x(t) \\ &\quad + [K_a(\hat{\rho}(t)) + \tilde{\rho} K_b(\hat{\rho}(t))]x(t) + (I-\rho)K_c f(t) \end{aligned} \tag{6.77}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$.

Then from the derivative of V along the closed-loop system, it follows

$$\begin{aligned}
\dot{V}_1 &= 2D^T(x_t)P\dot{D}(x_t)\\
&= x^T(t)[P(A + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho})) + G)\\
&\quad + (A + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho})) + G)^T P]x(t)\\
&\quad + 2(x(t)+f(t))^T PB_1\omega(t) + 2x^T(t)P(A_1 - G)x(t-h)\\
&\quad + 2f^T(t)P(A + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}) + G)x(t)\\
&\quad + 2f^T(t)P(A_1 - G)x(t-h) + 2(x(t)+f(t))^T PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x(t)\\
&\quad + 2(x(t)+f(t))^T PB(I-\rho)f(t)\\
\dot{V}_3 &= x^T(t)Ux(t) - x^T(t-h)Ux(t-h)\\
\dot{V}_4 &= hx^T(t)PF_{11}Px(t) + 2x^T(t)PF_{12}Pf(t) + \int_{t-h}^{t} x^T(s)G^T PF_{22}PGx(s)ds\\
&\quad + 2hx^T(t)PF_{13}Px(t-h) + 2f^T(t)PF_{23}Px(t-h)\\
&\quad + hx^T(t-h)PF_{33}Px(t-h)\\
\dot{V}_5 &= 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}_i(t)}{l_i}
\end{aligned}$$

where $f(t) = \int_{t-h}^{t} Gx(s)ds$.
Here we use

$$f^T(t)Pf(t) \le h\int_{t-h}^{t} x^T(s)G^T PGx(s)ds,$$

which is obtained by Lemma2.14 to get $\dot{V}_2$.
Let $B = [b^1 \cdots b^m]$, $B^i = [0 \cdots b^i \ldots 0]$, *then*

$$PB\tilde{\rho}K_b(\hat{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PB^i K_b(\hat{\rho}) \tag{6.78}$$

$$PBK_a(\tilde{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PBK_{ai} \tag{6.79}$$

Furthermore, it follows

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t)\\
&\le x^T(t)[P(A + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}) + G)\\
&\quad + (A + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}) + G)^T P\\
&\quad + (C + D(I-\rho)K(\hat{\rho}))^T(C + D(I-\rho)K(\hat{\rho}))\}x(t)\\
&\quad + \frac{1}{\gamma_f^2}(x(t)+f(t))^T PB_1B_1^T P(x(t)+f(t)) + 2f^T(t)P(A_1 - G)x(t-h)\\
&\quad - (\gamma_f\omega^T - \frac{1}{\gamma_f}(x(t)+f(t))^T PB_1)(\gamma_f\omega - \frac{1}{\gamma_f}B_1^T P(x(t)+f(t)))
\end{aligned}$$

$$+ 2f^T(t)P(A + B((I - \rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I - \hat{\rho})K_b(\hat{\rho}) + G)x(t)$$
$$+ 2(x(t) + f(t))^T PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x + 2x^T(t)P(A_1 - G)x(t-h)$$
$$+ 2x^T(t)(C + D(I - \rho)K(\hat{\rho}))^T D(I - \rho)K_c f(t)$$
$$+ 2(x(t) + f(t))^T PB(I - \rho)f(t) + f^T(t)K_c^T(I - \rho)D^T D(I - \rho)K_c f(t)$$
$$+ \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t) \tag{6.80}$$

Then

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \\
&\leq \begin{bmatrix} x^T(t) & f^T(t) & x^T(t-h) \end{bmatrix} \Psi \begin{bmatrix} x(t) \\ f(t) \\ x(t-h) \end{bmatrix} \\
&\quad + 2(x + f)^T PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x \\
&\quad + \int_{t-h}^{t} x^T(s)G^T(-P + PF_{22}P)Gx(s)ds + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}
\end{aligned} \tag{6.81}$$

where

$$\Psi = \begin{bmatrix} \Delta_1 & \Delta_2 & P(A_1 - G) + hPF_{13}P \\ * & \Delta_3 & P(A_1 - G) + PF_{23}P \\ * & * & -U + hPF_{33} \end{bmatrix}$$

$$\begin{aligned}
\Delta_1 = {} & P(A + B((I - \rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I - \hat{\rho})K_b(\hat{\rho}) + G) \\
& + (A + B((I - \rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I - \hat{\rho})K_b(\hat{\rho}) + G)^T P \\
& + U + \alpha h G^T PG + \frac{1}{\gamma_f^2}PB_1B_1^T P + hPF_{11}P \\
& + (C + D(I - \rho)K(\hat{\rho}))^T(C + D(I - \rho)K(\hat{\rho})) \\
\Delta_2 = {} & (A + B((I - \rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I - \hat{\rho})K_b(\hat{\rho}) + G)^T P + PF_{12}P \\
& + \frac{1}{\gamma_f^2}PB_1B_1^T P + PB(I - \rho) + (C + D(I - \rho)K(\hat{\rho}))^T D(I - \rho)K_c \\
\Delta_3 = {} & -h^{-1}(\alpha - 1)P + \frac{1}{\gamma_f^2}PB_1B_1^T P + PB(I - \rho) \\
& + K_c^T(I - \rho)^T D^T D(I - \rho)K_c
\end{aligned}$$

In fact, ρ_i is an unknown constant which denotes the loss of effectiveness of the ith actuator. So from $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho$, it follows $\dot{\tilde{\rho}}_i(t) = \dot{\hat{\rho}}_i(t)$. Now, if the adaptive laws are chosen as (6.72), then

$$2(x(t) + f(t))^T PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]x + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0 \tag{6.82}$$

Hence, the design problem $\dot{V}(t) + z^T(t)z(t) - \gamma_f^2\omega^T(t)\omega(t) \leq 0$ *is reduced to* $\Psi < 0$ *and* $-P + PF_{22}P < 0$.

Let $X = P^{-1}$, $Q = XUX$, $W = GX$, $W_1 = K_cX$, $Y_0 = K_0X$, $Y_{ai} = K_{ai}X$,$Y_{bi} = K_{bi}X$, $i = 1\cdots m$. *By pre- and post-multiplying inequalities* $\Psi < 0$ *and* $-P + PF_{22}P < 0$ *by* $diag\{X, X, X\}$ *and* X, *respectively, the resulting inequalities are equivalent to* $-X + F_{22} < 0$ *and*

$$\begin{bmatrix} \Delta_4 & \Delta_5 & (A_1X - W) + hF_{13} \\ * & \Delta_6 & (A_1X - W) + F_{23} \\ * & * & -Q + hF_{33} \end{bmatrix} < 0 \tag{6.83}$$

where

$$\begin{aligned}
\Delta_4 &= (AX + B((I-\rho)Y_0 + Y_a(\rho) - \rho Y_a(\hat{\rho}) + (I-\hat{\rho})Y_b(\hat{\rho}) + W) + \frac{1}{\gamma_f^2}B_1B_1^T \\
&\quad + (AX + B((I-\rho)Y_0 + Y_a(\rho) - \rho Y_a(\hat{\rho}) + (I-\hat{\rho})Y_b(\hat{\rho}) + W)^T + hF_{11} \\
&\quad + \alpha hW^TPW + Q + (CX + D(I-\rho)Y(\hat{\rho}))^T(CX + D(I-\rho)Y(\hat{\rho})) \\
\Delta_5 &= XA^T + Y_0^T(I-\rho)B^T + Y_a^T(\rho)B^T - \rho Y_a^T(\hat{\rho})B^T + (I-\hat{\rho})Y_b^T(\hat{\rho})B^T \\
&\quad + W^T + F_{12} + \frac{1}{\gamma_f^2}B_1B_1^T + B(I-\rho)X \\
&\quad + (CX + D(I-\rho)Y(\hat{\rho}))^TD(I-\rho)W_1 \\
\Delta_6 &= -h^{-1}(\alpha - 1)X + \frac{1}{\gamma_f^2}B_1B_1^T + B(I-\rho)X + W_1^T(I-\rho)D^TD(I-\rho)W_1 \\
Y(\hat{\rho}) &= Y_0 + Y_a(\hat{\rho}) + Y_b(\hat{\rho}), \quad Y_a(\rho) = \sum_{i=1}^{m} Y_{ai}\rho_i, Y_{ai}\hat{\rho}_i, \quad Y_b(\hat{\rho}) = \sum_{i=1}^{m} Y_{bi}\hat{\rho}_i
\end{aligned}$$

By Lemma (2.8), (6.83) changes into

$$\begin{bmatrix} \Delta_4 & \Delta_5 \\ * & \Delta_6 \end{bmatrix} - \begin{bmatrix} (A_1X - W) + hF_{13} \\ (A_1X - W) + F_{23} \end{bmatrix} (-Q + hF_{33})^{-1} \begin{bmatrix} (A_1X - W) + hF_{13} \\ (A_1X - W) + F_{23} \end{bmatrix}^T < 0 \tag{6.84}$$

and $-Q+hF_{33} < 0$. *Then the design problem* $\dot{V}(t)+z^T(t)z(t)-\gamma_f^2\omega^T(t)\omega(t) \leq 0$ *further reduces to (6.84),* $-Q + hF_{33} < 0$ *and* $-X + F_{22} < 0$.

Now we deal with (6.84). Furthermore, (6.84) can be written as

$$\begin{aligned}
M(\hat{\rho}) &= N + \sum_{i=1}^{N}\hat{\rho}_iR_i + (\sum_{i=1}^{N}\hat{\rho}_iR_i)^T + \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\rho}_i\hat{\rho}_jS_{ij} \\
&\quad + (V_{00} + \sum_{i=1}^{N}\hat{\rho}_iV_{0i})^T(V_{00} + \sum_{i=1}^{N}\hat{\rho}_iV_{0i}) < 0,
\end{aligned} \tag{6.85}$$

where

$$N = N_0 + \begin{bmatrix} \alpha h W^T X^{-1} W + \frac{1}{\gamma_f^2} B_1 B_1^T & \frac{1}{\gamma_f^2} B_1 B_1^T \\ * & \frac{1}{\gamma_f^2} B_1 B_1^T \end{bmatrix}$$

$$- \begin{bmatrix} (A_1 X - W) + hF_{13} \\ (A_1 X - W) + F_{23} \end{bmatrix} (-Q + hF_{33})^{-1} \begin{bmatrix} (A_1 X - W) + hF_{13} \\ (A_1 X - W) + F_{23} \end{bmatrix}^T < 0$$

$R_i, \Upsilon_{ij}, V_{00}, V_{0i}, i = 1 \cdots m$ *are defined in (6.67).*

By Lemma 2.10, it is easy to see if (6.65), (6.66) and (6.67)-(6.71) hold, then we have $x^T(t)M(\hat{\rho})x(t) < 0$ *for any* $x \neq 0$.
Furthermore, if (6.65), (6.66) and (6.67)-(6.71) hold for $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$, *it follows (6.84) and* $-Q+dF_{33} < 0$. *Then by Lemma (2.8), the inequality (6.83) holds. Also, the inequality (6.70) is equivalent to*

$$\begin{bmatrix} -P & hG^T P \\ * & -P \end{bmatrix} < 0 \tag{6.86}$$

by pre- and post-multiplying by diag$\{X^{-1}, X^{-1}\}$. *If (6.86) holds, then it is easy to prove that a positive scalar* δ *which is less than one exists such that*

$$\begin{bmatrix} -\delta P & hG^T P \\ * & -P \end{bmatrix} < 0 \tag{6.87}$$

according to matrix theory. Therefore, from Lemma 2.13, if (6.70) holds, the operator $D(x_t)$ *is stable. The inequality (6.71) means that* V_4 *is positive definite. So* $\dot{V}(t)$ *is positive definite. According to Theorem 9.8.1 in [55], if the conditions (6.65), (6.66), (6.69)-(6.71) hold, the closed-loop system (6.61) is asymptotically stable for the actuator fault cases. Furthermore,*

$$\dot{V}(t) + z^T(t)z(t) - \gamma_f^2 \omega^T(t)\omega(t) \leq 0.$$

Integrate the above-mentioned inequalities from 0 to ∞ *on both sides, it follows*

$$V(\infty) - V(0) + \int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt$$

Then

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + V(0) \tag{6.88}$$

which implies that (3.11) holds for the zero initial condition $x(t) = 0, t \in [-d, 0]$. *The proofs for (6.74) and asymptotic stability of the closed-loop system (9.5) for that normal case are similar, and omitted here.*

Corollary 6.2 *If the conditions in Theorem 6.5 hold, then the closed-loop system (6.61) is asymptotically stable and with adaptive* H_∞ *performance indexes no larger than* γ_n *and* γ_f *for normal and actuator fault cases, respectively.*

Proof 6.8 *It is similar to that of Corollary 3.1, and omitted here.*

Remark 6.13 *The newly proposed adaptive laws (6.72) include the term* $f(t) = \int_{t-d}^{t} Gx(s)ds$*, which indicates how time delay* d *takes effect on the adaptive law. Noted that inequality (6.65)-(6.71) are LMIs, which can be solved efficiently by using the MATLAB LMI control toolbox.*

Remark 6.14 *If we choose the same Lyapunov functional candidate as [77], i.e.,* $V = V_1 + V_2 + V_3 + V_4$*, where* V_1, V_2, V_3, V_4 *are defined in (9.31), then the following conditions are sufficient for guaranteeing the closed-loop system (6.61) with delay-dependent memory state feedback reliable controller* $u(t) = K_0x(t) + K_c\int_{t-d}^{t} Gx(s)ds$*,* $K_0 = Y_0X^{-1}$*,* $K_c = W_1X^{-1}$ *and* $G = WX^{-1}$ *to be asymptotically stable and with* H_∞ *performance indexes no larger than* γ_n *and* γ_f *for normal and actuator fault cases, respectively.*
For $\rho = 0$*, i.e., in a normal case*

$$\begin{bmatrix} T_2 + T_2^T + Q + hF_{11} & T_3 & T_4 & B_1 & \alpha hW^T & T_5 \\ * & T_6 & W_1^T(I-\rho)D^T & B_1 & 0 & T_7 \\ * & * & I & 0 & 0 & 0 \\ * & * & * & -\gamma_n^2 I & 0 & 0 \\ * & * & * & * & -\alpha hX & 0 \\ * & * & * & * & * & T_8 \end{bmatrix} < 0 \tag{6.89}$$

For $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$*, i.e., in actuator fault cases*

$$\begin{bmatrix} T_2 + T_2^T + Q + hF_{11} & T_3 & T_4 & B_1 & \alpha hW^T & T_5 \\ * & T_6 & W_1^T(I-\rho)D^T & B_1 & 0 & T_7 \\ * & * & I & 0 & 0 & 0 \\ * & * & * & -\gamma_f^2 I & 0 & 0 \\ * & * & * & * & -\alpha hX & 0 \\ * & * & * & * & * & T_8 \end{bmatrix} < 0 \tag{6.90}$$

$$\begin{aligned} &T_2 = AX + B((I-\rho)Y_0) + W, \quad T_3 = T_2^T + F_{12} + B(I-\rho)X, \\ &T_4 = XC^T + Y_0^T(I-\rho)D^T, \quad T_5 = (A_1X - W) + hF_{13}, \\ &T_6 = -h^{-1}(\alpha-1)X + B(I-\rho)X, \quad T_7 = (A_1X - W) + hF_{23}, \\ &T_8 = -Q + hF_{33}. \end{aligned}$$

Notice that if set $Y_{ai} = 0, Y_{bi} = 0, i = 1 \cdots m$ *in Theorem 6.5, then the conditions of Theorem 6.5 reduce to (6.89) and (6.90). Thus, the design conditions of the reliable* H_∞ *controller with adaptive mechanisms in Theorem 6.5 are more relaxed than conditions (6.89) and (6.90) of the corresponding reliable* H_∞ *controller with fixed gains.*

The following is an algorithm to optimize the reliable H_∞ performance in normal and fault cases.

Algorithm 6.4 *Solving the following optimization*

$$\min \beta_1\eta_n + \beta_2\eta_f \quad s.t. \quad (6.65) - (6.71), \tag{6.91}$$

where $\delta_n = \gamma_n^2$, $\delta_f = \gamma_f^2$, *and* β_1 *and* β_2 *are weighting coefficients.*
Usually, we can choose $\beta_1 > \beta_2$ *in (3.25) since systems are operating under the normal condition most of the time.*
Denote the optimal solutions as $X = X_{opt}, W_1 = W_{1opt}, Y_0 = Y_{0opt}, Y_{ai} = Y_{aiopt}, Y_{bi} = Y_{biopt} (i = 1 \cdots m)$, *then the controller gains of (6.63) can be obtained by* $K_0 = Y_0X^{-1}, K_{ai} = Y_{ai}X^{-1}, K_{bi} = Y_{bi}X^{-1}$.

6.3.3 Guaranteed Cost State Feedback Control

In this subsection, the guaranteed cost control for *linear systems* (6.59) against actuator faults (6.60) is considered.

Then the corresponding system with actuator faults is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-h) + B(I-\rho)u(t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{6.92}$$

The problem investigated in this paper is to design a reliable guaranteed cost controller such that, in normal and fault cases, the resultant closed-loop system is asymptotically stable and the bound of the following quadratic cost function J is minimized.

$$J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)(I-\rho)S(I-\rho)u(t))dt \tag{6.93}$$

where $Q > 0 \in R^{n\times n}$, $S > 0 \in R^{m\times m}$.

Define an operator $D(x_t) : C_{n,\ d} \to R^n$ as

$$D(x_t) = x(t) + \int_{t-h}^{t} A_1x(s)ds \tag{6.94}$$

where $x_t = x(t+s), s \in [-d, 0]$.

Now, the following adaptive memory state feedback controller is chosen, that is,

$$u(t) = K(\hat{\rho}(t))(x(t) + \int_{t-h}^{t} A_1x(s)ds) = [K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t))]D(x_t) \tag{6.95}$$

where $K_a(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{ai}\hat{\rho}_i(t), K_b(\hat{\rho}(t)) = \sum_{i=1}^{m} K_{bi}\hat{\rho}_i(t)$ and $\hat{\rho}_i(t)$ is the estimate of ρ_i. $K_0, K_{ai}, K_{bi}, i = 1 \cdots m$ are the control gains to be designed.

Remark 6.15 *From (6.95), it is easy to see that the chosen controller structure is different from traditional memory or memory-less controllers with fixed gains. That is, the gains of the memory term* $\int_{t-h}^{t} A_1 x(s)ds$ *and the memory-less term* $x(t)$ *are both time-varying and affinely dependent on the online estimates* $\hat{\rho}_i(t)$ *of* ρ_i*.*

The closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + B(I-\rho)K(\hat{\rho})D(x_t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{6.96}$$

Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m) : \hat{\rho}_i \in \{\min_j\{\underline{\rho}_i^j\}, \max_j\{\bar{\rho}_i^j\}\}\}$$

$$\Delta(\hat{\rho}) = \mathrm{diag}[\hat{\rho}_1 I \cdots \hat{\rho}_m I]$$

Theorem 6.6 *Suppose that the operator* $D(x_t)$ *satisfies the conditions in Lemma (2.48). Then, for given* $Q > 0$ *and* $S > 0$*, if there exist matrices* $X > 0, Z > 0, Y_0, Y_{ai}, Y_{bi}$ *and a symmetric matrix* Θ *with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$$

$\Theta_{11},\ \Theta_{22} \in R^{mn \times mn}$ *such that the following inequalities hold:*

$$\Theta_{22ii} \le 0,\ i = 1, \cdots, m \tag{6.97}$$

with $\Theta_{22ii} \in R^{n \times n}$ *is the* (i,i) *block of* Θ_{22}*.*
For any $\hat{\rho} \in \Delta_{\hat{\rho}}$

$$\Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \ge 0 \tag{6.98}$$

in normal and actuator fault cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\},\ \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} \Upsilon & \begin{bmatrix} -(A+A_1)A_1 Z \\ 0 \end{bmatrix} & \begin{bmatrix} hX \\ 0 \end{bmatrix} & \begin{bmatrix} XQ \\ 0 \end{bmatrix} & U^T \\ * & -Z & -hZA_1^T & -ZA_1^T Q & 0 \\ * & * & -Z & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{6.99}$$

where

$$\Upsilon = \begin{bmatrix} \Delta_0 + \Delta_0^T & E \\ E^T & F \end{bmatrix} + G^T \Theta G,$$
$$\Delta_0 = (A + A_1)X + B[(I - \rho)Y_0 + Y_a(\rho)],$$
$$E = [E_1 \ E_2 \cdots E_m], \quad U = [U_0 \ U_1 \cdots U_m], \quad F = [F_{ij}],$$
$$E_i = -B\rho Y_{ai} + BY_{bi}, \quad F_{ij} = -B^i Y_{bj} - Y_{bi}^T {B^j}^T, i, j = 1 \cdots m$$
$$U_0 = S^{\frac{1}{2}}(I - \rho)Y_0, \quad U_i = S^{\frac{1}{2}}(I - \rho)(Y_{ai} + Y_{bi}),$$
$$G = \begin{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}, \quad Y_a(\rho) = \sum_{i=1}^{m} Y_{ai}\rho_i$$

and also $\hat{\rho}_i$ *is determined according to the adaptive laws*

$$\begin{aligned} \dot{\hat{\rho}}_i &= Proj_{[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j\}]}\{L_i\} \\ &= \begin{cases} 0, & if \quad \begin{array}{l} \hat{\rho}_i = \min\limits_j\{\underline{\rho}_i^j\} \ and \ L_i \le 0 \\ or \ \hat{\rho}_i = \max\limits_j\{\bar{\rho}_i^j\} \ and \ L_i \ge 0; \end{array} \\ L_i, & otherwise \end{cases} \end{aligned} \tag{6.100}$$

where $L_i = -l_i D^T(x_t)X^{-1}[B^i Y_b(\hat{\rho}) + BY_{ai}]X^{-1}D(x_t)$ *with* $Y_b(\hat{\rho}) = \sum_{i=1}^m Y_{bi}\hat{\rho}_i(t)$, $l_i > 0 (i = 1 \cdots m)$ *is the adaptive law gain.* $Proj\{\cdot\}$ *denotes the projection operator [70], whose role is to project the estimates* $\hat{\rho}_i(t)$ *to the interval* $[\min\limits_j\{\underline{\rho}_i^j\}, \max\limits_j\{\bar{\rho}_i^j\}]$. *Then the closed-loop system (6.96) is asymptotically stable, the gain matrices of the controller (9.16) are given by* $K_0 = Y_0X^{-1}, K_{ai} = Y_{ai}X^{-1}, K_{bi} = Y_{bi}X^{-1}$, *and the upper bound of the quadratic cost function* J *is*

$$\begin{aligned} J^* = D^T(0)X^{-1}D(0) + \sum_{i=1}^{m} \frac{{\tilde{\rho}_i}^2(0)}{l_i} \\ + h\int_{-h}^{0}(s + h)x^T(s)Z^{-1}x(s)ds \end{aligned} \tag{6.101}$$

Proof 6.9 *The following Lyapunov-Krasovkii functional candidate is chosen*

$$V = V_1 + V_2 + V_3 \tag{6.102}$$

where

$$V_1 = D^T(x_t)PD(x_t), \quad V_2 = \int_{t-h}^{t}(s - t + h)x^T(s)Rx(s)ds, \quad V_3 = \sum_{i=1}^{m} \frac{\tilde{\rho}_i^2(t)}{l_i}$$

with $P > 0$ and $R > 0$.
The following equality is obtained

$$\begin{aligned}(I-\rho)u(t) &= (I-\rho)[K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t))]D(x_t) \\ &= [(I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}(t)) + (I-\hat{\rho}(t))K_b(\hat{\rho}(t))]D(x_t) \\ &\quad + [K_a(\tilde{\rho}(t)) + \tilde{\rho}K_b(\hat{\rho}(t))]D(x_t) \end{aligned} \tag{6.103}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$.

Then from the derivative of V along the closed-loop system, it follows

$$\begin{aligned}\dot{V}_1 &= 2D^T(x_t)P\dot{D}(x_t) \\ &= 2D^T(x_t)P\{[A + A_1 + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}))] \\ &\qquad \times D(x_t) - (A + A_1)\int_{t-h}^{t} A_1 x(s)ds\} \\ &\qquad + 2D(x_t)^T PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]D(x_t) \\ \dot{V}_2 &= hx^T(t)Rx(t) - \int_{t-h}^{t} x^T(s)Rx(s)ds \\ &\le hx^T(t)Rx(t) - (\int_{t-h}^{t} x(s)ds)^T(h^{-1}R)(\int_{t-h}^{t} x(s)ds) \\ \dot{V}_3 &= \sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}_i(t)}{l_i}\end{aligned}$$

where Lemma 2.14 is used to get $\dot{V}_2$. On the other hand

$$x^T(t)Rx(t) = (D(x_t) - \int_{t-h}^{t} A_1 x(s)ds)^T R(D(x_t) - \int_{t-h}^{t} A_1 x(s)ds) \tag{6.104}$$

Then

$$\frac{dV}{dt} = \sum_{i=1}^{3} \frac{dV_i}{dt} \le \chi^T \Omega \chi + 2D^T(x_t)PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]D(x_t) + \sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}_i(t)}{l_i} \tag{6.105}$$

where

$$\chi = \begin{bmatrix} D(x_t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Delta_1 + \Delta_1^T + hR & -P(A + A_1)A_1 - hRA_1 \\ * & -h^{-1}R + hA_1^T RA_1 \end{bmatrix}$$

with

$$\Delta_1 = P[A + A_1 + B((I-\rho)K_0 + K_a(\rho) - \rho K_a(\hat{\rho}) + (I-\hat{\rho})K_b(\hat{\rho}))]$$

Let $B = [b^1 \cdots b^m]$ and $B^i = [0 \cdots b^i \ldots 0]$, then it follows

$$PB\tilde{\rho}K_b(\hat{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PB^i K_b(\hat{\rho}) \tag{6.106}$$

$$PBK_a(\tilde{\rho}) = \sum_{i=1}^{m} \tilde{\rho}_i PBK_{ai} \tag{6.107}$$

In fact, ρ_i is an unknown constant which denotes the loss of effectiveness of the ith actuator. So from $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho$, we can obtain $\dot{\tilde{\rho}}_i(t) = \dot{\hat{\rho}}_i(t)$. Now, if the adaptive laws are chosen as (6.100),

$$2D^T(x_t)PB[K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})]D(x_t) + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \leq 0 \tag{6.108}$$

that is

$$\frac{dV}{dt} \leq \chi^T(t)\Omega\chi(t) \tag{6.109}$$

Furthermore, by (6.94) and (6.95) it follows

$$\begin{aligned} &x^TQx + u^T(I-\rho)S(I-\rho)u \\ &= (D(x_t) - \int_{t-h}^{t} A_1x(s)ds)^TQ(D(x_t) - \int_{t-h}^{t} A_1x(s)ds) \\ &\quad + D^T(x_t)K^T(\hat{\rho})(I-\rho)S(I-\rho)K(\hat{\rho})D(x_t) \end{aligned} \tag{6.110}$$

Thus

$$x^TQx + u^T(I-\rho)S(I-\rho)u \leq \chi^T(t)\Omega_1\chi(t) - \frac{dV}{dt} \tag{6.111}$$

where

$$\Omega_1 = \begin{bmatrix} \Delta_1 + \Delta_1^T + \Upsilon_0 & -P(A+A_1)A_1 \\ * & -h^{-1}R \end{bmatrix} + \begin{bmatrix} I \\ -A_1^T \end{bmatrix} (hR+Q) \begin{bmatrix} I & -A_1 \end{bmatrix} < 0$$

with $\Upsilon_0 = K^T(\hat{\rho})(I-\rho)S(I-\rho)K(\hat{\rho})$. Therefore, if $\Omega_1 < 0$, there exists the positive scalar γ such that $\frac{dV}{dt} \leq -\gamma\|x\|^2$. That is, the asymptotic stability of the closed-loop system (6.96) in both normal and fault cases can be guaranteed. By Lemma (2.8), $\Omega_1 < 0$ is equivalent to

$$\Omega_2 = \begin{bmatrix} \Delta_1 + \Delta_1^T + \Upsilon_0 & -P(A+A_1)A_1 & hI & I \\ * & -h^{-1}R & -hA_1^T & -A_1^T \\ * & * & -hR^{-1} & 0 \\ * & * & * & -Q^{-1} \end{bmatrix} < 0 \tag{6.112}$$

Let $X = P^{-1}, Y_0 = K_0X, Y_{ai} = K_{ai}X$, $Y_{bi} = K_{bi}X, i = 1\cdots m$ and $Z = hR^{-1}$. By pre- and post-multiplying inequalities $\Omega_2 < 0$ by diag$\{X, Z, I, Q\}$, then $\Omega_2 < 0$ is equivalent to

$$\Omega_3 = \begin{bmatrix} \Delta_2 + \Delta_2^T + \Upsilon_1 & -(A+A_1)A_1Z & hX & XQ \\ * & -Z & -hZA_1^T & -ZA_1^TQ \\ * & * & -Z & 0 \\ * & * & * & -Q \end{bmatrix} < 0 \tag{6.113}$$

where $\Delta_2 = (A + A_1)X + B[(I - \rho)Y_0 + Y_a(\rho) - \rho Y_a(\hat{\rho}) + (I - \hat{\rho})Y_b(\hat{\rho})]$ $\Upsilon_1 = Y^T(\hat{\rho})(I - \rho)S(I - \rho)Y(\hat{\rho})$.

Furthermore, applying Lemma (2.8), $\Omega_3 < 0$ *is equivalent to*

$$\Omega_4 = \Delta_2 + \Delta_2^T + \Upsilon_1 + h^2 X Z^{-1} X + XQX - \Delta_3 \Delta_4^{-1} \Delta_3^T < 0 \tag{6.114}$$

and $\Delta_4 < 0$
where

$$\begin{aligned}
\Delta_3 &= -(A + A_1)A_1 Z - h^2 X Z^{-1} A_1 Z - XQA_1 Z \\
\Delta_4 &= -Z + h^2 Z A_1^T Z^{-1} A_1 Z + Z A_1^T Q A_1 Z
\end{aligned}$$

So $\Omega_1 < 0$ *is equivalent to* $\Omega_4 < 0$ *and* $\Delta_4 < 0$.
Also, Ω_4 *can be written as* Ω_4

$$\begin{aligned}
\Omega_4 = N_0 &+ \sum_{i=1}^m \hat{\rho}_i E_i + (\sum_{i=1}^m \hat{\rho}_i E_i)^T + \sum_{i=1}^m \sum_{j=1}^m \hat{\rho}_i \hat{\rho}_j F_{ij} \\
&+ (U_0 + \sum_{i=1}^m \hat{\rho}_i U_i)^T (U_0 + \sum_{i=1}^m \hat{\rho}_i U_i) < 0,
\end{aligned} \tag{6.115}$$

with

$$\begin{aligned}
N_0 &= \Delta_0 + \Delta_0^T + h^2 X Z^{-1} X + XQX - \Delta_3 \Delta_4^{-1} \Delta_3^T \\
E_i &= -B\rho Y_{ai} + BY_{bi}, \quad F_{ij} = -B^i Y_{bj} - Y_{bi}^T {B^j}^T \\
U_0 &= S^{\frac{1}{2}}(I - \rho)Y_0, \quad U_i = S^{\frac{1}{2}}(I - \rho)(Y_{ai} + Y_{bi})
\end{aligned}$$

and Δ_0 *is defined below (6.99).*

On the other hand, by Lemma (2.8), if the condition (6.99) holds then we have

$$\begin{bmatrix} N_0 & E \\ E^T & F \end{bmatrix} + U^T U + G^T \Theta G < 0 \tag{6.116}$$

and $\Delta_4 < 0$*. Here* E, F, U *are defined below inequality (6.99).*

Furthermore by Lemma 2.10, it is easy to see if (6.97)-(6.99) hold then $\Omega_4 < 0$ *and* $\Delta_4 < 0$*. Thus if the conditions (6.97)-(6.99) hold, it follows* $\Omega_1 < 0$*. So from (6.111), it follows*

$$x^T Q x + u^T (I - \rho) S (I - \rho) u \le -\frac{dV}{dt} \tag{6.117}$$

Integrating both sides of the above inequality from 0 to ∞*, it follows*

$$\begin{aligned}
&\int_0^\infty (x^T Qx + u^T(I-\rho)S(I-\rho)u)dt \\
&\le V(0) - V(\infty) \\
&\le V(0) = D^T(0)PD(0) + \int_{-h}^0 (s+h)x^T(s)Rx(s)ds + \sum_{i=1}^m \frac{\tilde{\rho}_i{}^2(0)}{l_i} \\
&= J^* = D^T(0)X^{-1}D(0) + h\int_{-h}^0 (s+h)x^T(s)Z^{-1}x(s)ds + \sum_{i=1}^m \frac{\tilde{\rho}_i{}^2(0)}{l_i}
\end{aligned} \tag{6.118}$$

The proof is completed.

Remark 6.16 *Denote* $F_a(0) = \sum_{i=1}^m \frac{\tilde{\rho}_i{}^2(0)}{l_i}$*. Then, by (6.60) and (6.100), it follows that* $\tilde{\rho}_i(0) \le \max_j\{\bar{\rho}_i^j\} - \min_j\{\underline{\rho}_i^j\}$*. We can choose* l_i *relatively large so that* $F_a(0)$ *is sufficiently small. The newly proposed adaptive laws (6.100) include the term* $D(x_t) = x(t) + \int_{t-h}^t A_1 x(s)ds$*, which indicates how time-delay* h *takes effect on the adaptive law.*

Theorem 6.6 presents the method of designing a reliable guaranteed cost controller via adaptive memory state feedback. The following theorem is to select the reliable controller, which can minimize the upper bound of the guaranteed cost (6.93).

Theorem 6.7 *Consider the closed-loop system (6.96) with cost function (6.93). If the following optimization problem*

$$\min_{X>0,\ \Gamma_1>0,\ Z>0,\ Y_0,\ Y_{ai},\ Y_{bi},\alpha>0} \{\alpha + tr(\Gamma_1)\}$$

such that

$$(i)\ \ LMI\ \ (6.97) - (6.99) \tag{6.119}$$

$$(ii)\ \ \begin{bmatrix} -\alpha & D^T(0) \\ * & -X \end{bmatrix} < 0 \tag{6.120}$$

$$(iii)\ \ \begin{bmatrix} -\Gamma_1 & hN_1^T \\ * & -hZ \end{bmatrix} < 0 \tag{6.121}$$

has a solution set $(X, \Gamma_1, Z, Y_0, Y_{ai}, Y_{bi}, \alpha)$*, the controller (6.95) is an optimal reliable guaranteed cost control law, which ensures the minimization of the guaranteed cost (6.93) for the closed-loop system (6.96) against actuator faults, where* $\int_{-h}^0 (s+h)x(s)x^T(s)ds = N_1 N_1^T$*.*

Proof 6.10 *By Theorem 6.6, (i) in (6.119) is clear. Also, it follows from Lemma 2.8 that (ii) and (iii) in (6.119) are equivalent to* $D^T(0)X^{-1}D(0) < \alpha$ *and* $hN_1^T Z^{-1} N_1 < \Gamma_1$*, respectively. On the other hand,*

$$
\begin{aligned}
h\int_{-h}^{0}(s+h)x^T(s)Z^{-1}x(s)ds &= \int_{-h}^{0} tr((s+h)x^T(s)hZ^{-1}x(s))ds \\
&= tr(N_1N_1^ThZ^{-1}) = tr(N_1^ThZ^{-1}N_1) < tr(\Gamma_1)
\end{aligned}
$$

Hence, it follows from (6.101) that

$$
J^* < \alpha + tr(\Gamma_1) + \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(0)}{l_i}.
$$

Thus, the minimization of $\alpha + tr(\Gamma_1)$ *implies the minimization of the guaranteed cost for the system (6.96).*

Remark 6.17 *If we choose the Lyapunov functional candidate* $V = V_1 + V_2$*, where* V_1, V_2 *are defined in (6.102), then the following conditions (6.128) can guarantee the closed-loop system (6.92) with reliable memory state feedback controller* $u(t) = K_0(x(t) + \int_{t-h}^{t} A_1x(s)ds)$*,* $K_0 = Y_0X^{-1}$ *to be asymptotically stable and the upper bound of* J *is* J^**.*
For normal and actuator faults cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$
\begin{bmatrix}
\sum_1 + \sum_1^T & -(A+A_1)A_1Z & hX & XQ & Y_0^T(I-\rho)S^{\frac{1}{2}} \\
* & -Z & -hZA_1^T & -ZA_1^TQ & 0 \\
* & * & -Z & 0 & 0 \\
* & * & * & -Q & 0 \\
* & * & * & * & -I
\end{bmatrix} < 0 \tag{6.122}
$$

where $\sum_1 = (A+A_1)X + B(I-\rho)Y_0$*. The conditions (6.128) are just the result of Theorem 6.6 in [110] with* $A_2 = 0$ *when the actuator faults are considered. Notice that if set* $Y_{ai} = 0, Y_{bi} = 0, i = 1 \cdots m$ *in Theorem 6.6, then the conditions of Theorem 6.6 reduce to (6.128). Thus, the design conditions of the reliable guaranteed cost controller with adaptive mechanisms in Theorem 6.6 are more relaxed than the conditions of the traditional reliable guaranteed cost controller with fixed gains (6.128). Also the upper bound of J with fixed gains controller can be obtained by solving the following optimization:*

$$
\min_{X>0,\ \Gamma_1>0,\ Z>0,\ Y_0,\ \alpha>0} \{\alpha + tr(\Gamma_1)\}
$$

$$
\begin{aligned}
&(i)\ \ LMI\ \ (6.101)\\
&(ii)\ \ \begin{bmatrix} -\alpha & D^T(0) \\ * & -X \end{bmatrix} < 0\\
&(iii)\ \ \begin{bmatrix} -\Gamma_1 & hN_1^T \\ * & -hZ \end{bmatrix} < 0
\end{aligned}
$$

One can easily extend Theorem 6.6 or Theorem 6.7 to robust reliable guaranteed cost control for the following polytopic uncertain systems

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + A_1 x(t-h) + Bu(t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{6.123}$$

with

$$A(\lambda) = \sum_{i=1}^{q} A^i \lambda_i, \quad \sum_{i=1}^{q} \lambda_i = 1 \text{ and } \lambda_i \geq 0$$

Then the corresponding closed-loop system is

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + A_1 x(t-h) + B(I-\rho)K(\hat{\rho})D(x_t) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{6.124}$$

From the proof of Theorem 6.6 and Theorem 6.7, the following corollary can be easily obtained.

Corollary 6.3 *Consider the closed-loop system (6.124) with cost function (6.93). If the following optimization problem*

$$\min_{X>0,\ \Gamma_1>0,\ Z>0,\ Y_0,\ Y_{ai},\ Y_{bi},\ \alpha>0} \{\alpha + tr(\Gamma_1)\}$$

subject to
(i) LMI (6.97)-6.98)
and for any $\rho \in \{\rho^1 \cdots \rho^L\},\ \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} \Phi & \begin{bmatrix} -(A^i + A_1)A_1 Z \\ 0 \end{bmatrix} & \begin{bmatrix} hX \\ 0 \end{bmatrix} & \begin{bmatrix} XQ \\ 0 \end{bmatrix} & U^T \\ * & -Z & -hZA_1^T & -ZA_1^T Q & 0 \\ * & * & -Z & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{6.125}$$

with

$$\Phi = \begin{bmatrix} \bar{\Delta}_0 + \bar{\Delta}_0^T & E \\ E^T & F \end{bmatrix} + G^T \Theta G,$$
$$\bar{\Delta}_0 = (A^i + A_1)X + B[(I-\rho)Y_0 + Y_a(\rho)]$$

Here, the other symbols and the adaptive laws are defined below (6.99) and (6.100).

$$\text{(ii)} \quad \begin{bmatrix} -\alpha & D^T(0) \\ * & -X \end{bmatrix} < 0 \tag{6.126}$$

$$\text{(iii)} \quad \begin{bmatrix} -\Gamma_1 & hN_1^T \\ * & -hZ \end{bmatrix} < 0 \tag{6.127}$$

has a solution set $(X, \Gamma_1, Z, Y_0, Y_{ai}, Y_{bi}, \alpha)$, *the controller (6.95) is an optimal robust reliable guaranteed cost control law, which ensures the minimization of the guaranteed cost (6.93) for the closed-loop system (6.124) against actuator faults, where* $\int_{-h}^{0}(s+h)x(s)x^T(s)ds = N_1 N_1^T$.

Remark 6.18 *The corresponding condition of the robust reliable guaranteed cost controller with fixed gains is similar to condition (6.128), that is for* $\rho \in \{\rho^1 \cdots \rho^L\}, \rho^j \in N_{\rho^j}$

$$\begin{bmatrix} \sum_1 + \sum_1^T & -(A^i + A_1)A_1 Z & hX & XQ & Y_0^T(I-\rho)S^{\frac{1}{2}} \\ * & -Z & -hZA_1^T & -ZA_1^T Q & 0 \\ * & * & -Z & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{6.128}$$

where $\sum_2 = (A^i + A_1)X + B(I-\rho)Y_0$.

6.3.4 Example

To illustrate the effectiveness of our results, two examples are given. Example 6.3 is for H_∞ control case and Example 6.4 is for guaranteed cost control case

Example 6.3 *Consider a linear time-delay system (6.59) with parameters as follows*

$$A = \begin{bmatrix} -2 & 1 \\ -1 & -5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -0.8 & 1 \end{bmatrix}, \quad B1 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix},$$
$$C = \begin{bmatrix} 5 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad h = 3$$

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be normal or loss of effectiveness, that is,

$$\rho_1^1 = 1, \quad 0 \le \rho_2^1 \le a, \quad a = 0.5,$$

which denotes the maximum loss of effectiveness for the second actuator.
Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, that is,

$$\rho_2^2 = 1, \quad 0 \le \rho_1^2 \le b, \quad b = 0.5,$$

TABLE 6.3 H_∞ performance index

	Adaptive reliable controller	Traditional reliable controller
γ_n	0.1447	0.2260
γ_f	0.2596	0.8756

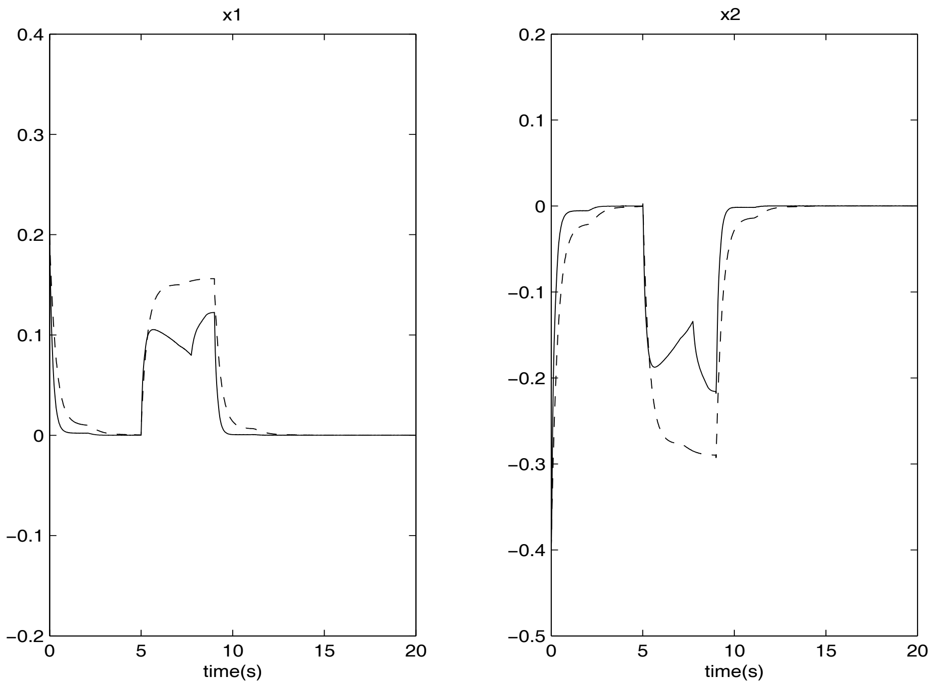


FIGURE 6.10
Response curve in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

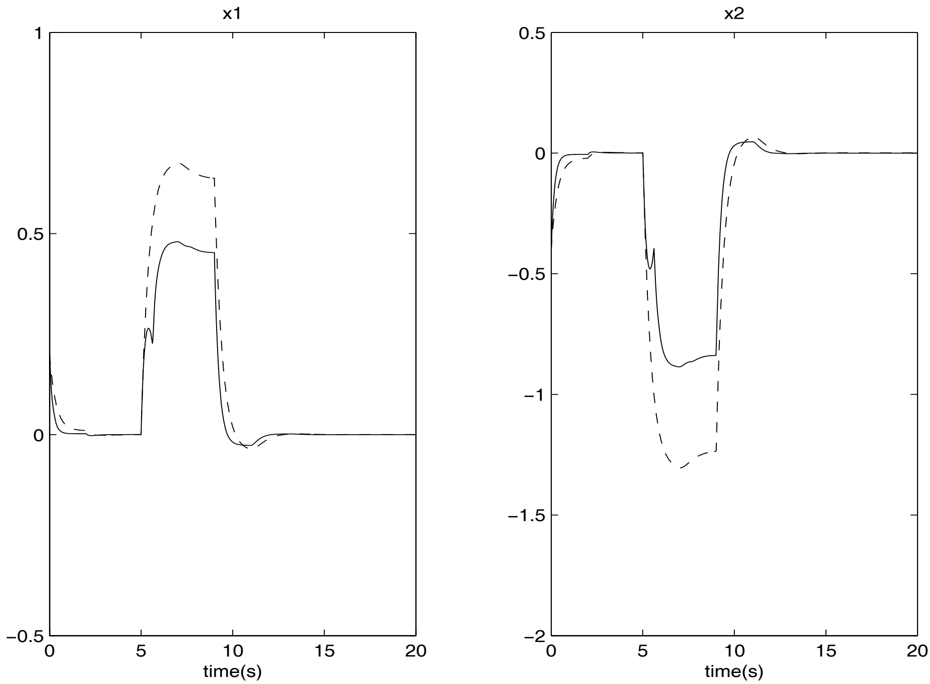


FIGURE 6.11
Response curve in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

which denotes the maximum loss of effectiveness for the first actuator. From Algorithm 6.4 with $\beta_1 = 5$, $\beta_2 = 1$, the corresponding H_∞ performance indexes of the closed-loop systems with the two controllers are obtained after search for α from 1.1 to 500. See Table 6.3 for more details, which indicates the superiority of our adaptive method.

In the following simulation, we use the disturbance

$$\omega(t) = \begin{cases} 2 & 2 \leq t \leq 4 \text{ (seconds)} \\ 0 & \text{otherwise} \end{cases}$$

and the fault case here is that at 3 seconds, the second actuator is outage.

Figure 6.10 describes the response curves in normal case with our adaptive delay-dependent memory controller and delay-dependent memory controller with fixed gains, respectively. The corresponding curves in fault case with these two controllers are given in Figure 6.11. From Figure 6.10-Figure 6.11, it is easy to see our adaptive controller has more disturbance restraint ability than the one with fixed gains in either normal or fault cases just as theory has proved.

TABLE 6.4 Cost performance index

	Adaptive reliable controller	Traditional reliable controller
Upper bound of J	1.3026	3.4035

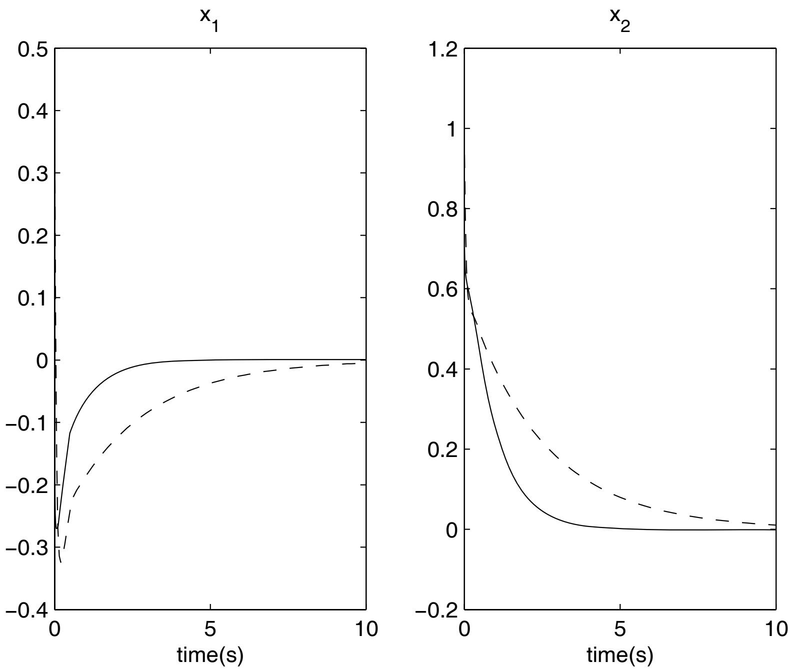


FIGURE 6.12
Response curve in normal case with adaptive controller (solid) and controller with fixed gains (dashed).

Example 6.4 *Consider a linear time-delay system (6.59) with parameters as follows*

$$A = \begin{bmatrix} 1 & 1 \\ -2.5 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & 0.5 \\ 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 0.5 \end{bmatrix},$$
$$\phi(t) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad h = 0.5, \quad Q = diag\{1, 2\}, \quad S = I$$

Besides the normal mode, that is,

$$\rho_1^0 = \rho_2^0 = 0,$$

the following possible fault modes are considered:
Fault mode 1: The first actuator is outage and the second actuator may be

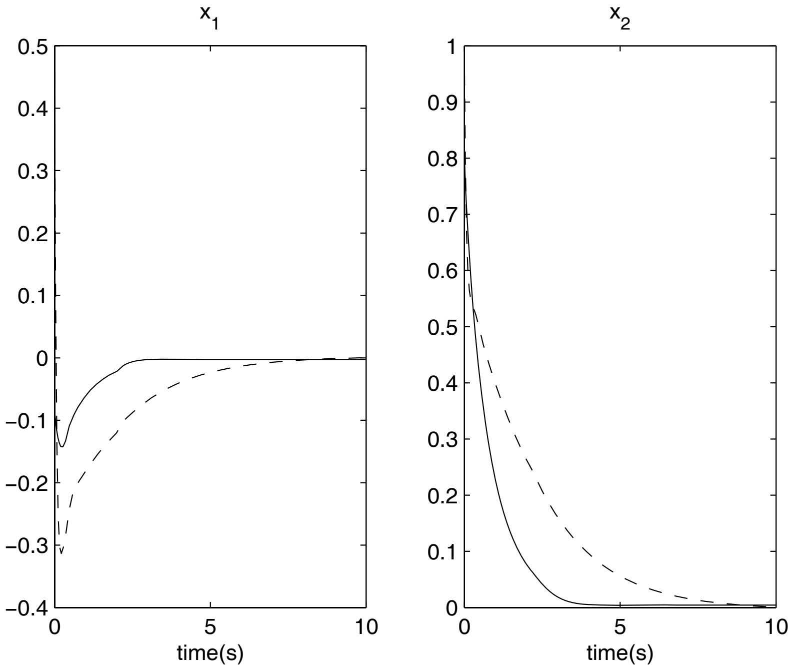


FIGURE 6.13
Response curve in fault case 1 with adaptive controller (solid) and controller with fixed gains (dashed).

normal or loss of effectiveness, that is,

$$\rho_1^1 = 1, \quad 0 \leq \rho_2^1 \leq a, \quad a = 0.5,$$

which denotes the maximum loss of effectiveness for the second actuator.
Fault mode 2: The second actuator is outage and the first actuator may be normal or loss of effectiveness, that is,

$$\rho_2^2 = 1, \quad 0 \leq \rho_1^2 \leq b, \quad b = 0.4,$$

which denotes the maximum loss of effectiveness for the first actuator.

By using Theorem 6.7 and the conditions (6.128), we obtain the corresponding cost performance indexes, using the adaptive method and traditional method. See Table 6.4 for more details.

In the following simulation, two fault cases are considered.
Fault case 1: At 0 second, the second actuator is outage, then after 2 seconds the first actuator becomes loss of 40% effectiveness.
Fault case 2: At 5 seconds, the first actuator is outage.

Figure 6.12 describes the response curves in normal case with our adaptive reliable memory controller and reliable memory controller with fixed gains.

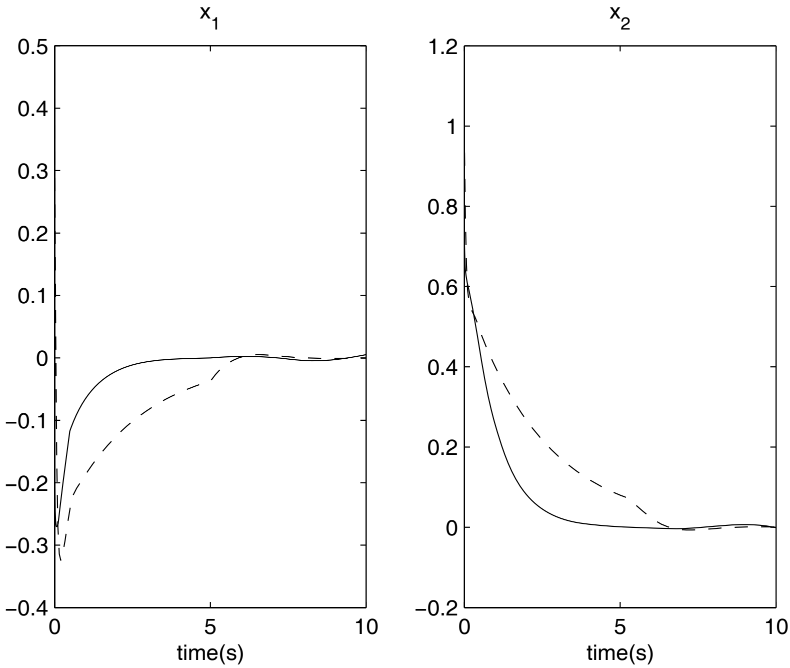


FIGURE 6.14
Response curve in fault case 2 with adaptive controller (solid) and controller with fixed gains (dashed).

The corresponding curves in the two considered fault cases with these two controllers are given in Figure 6.13-Figure 6.14, respectively. From Figure 6.12-Figure 6.14, it is easy to see our adaptive reliable memory controller performs better than the one with fixed gains in either normal or fault cases just as theory has proved.

In order to show the effectiveness of the proposed method for polytopic uncertain system, another numerical example is also given.

Example 6.5 *Consider a linear time-delay system (6.123) with parameters as follows*

$$A(\lambda) = A^1\lambda_1 + A^2\lambda_2 \qquad \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$$

where $A^1 = \begin{bmatrix} 1 & 1 \\ -2.5 & -1 \end{bmatrix}$, $A^2 = \begin{bmatrix} -1 & 0 \\ -0.5 & -0.5 \end{bmatrix}$.

The other parameters and the possible fault modes are the same as those in Example 6.3.

TABLE 6.5 Cost performance index

	Adaptive reliable controller	Traditional reliable controller
Upper bound of J	1.4383	3.5900

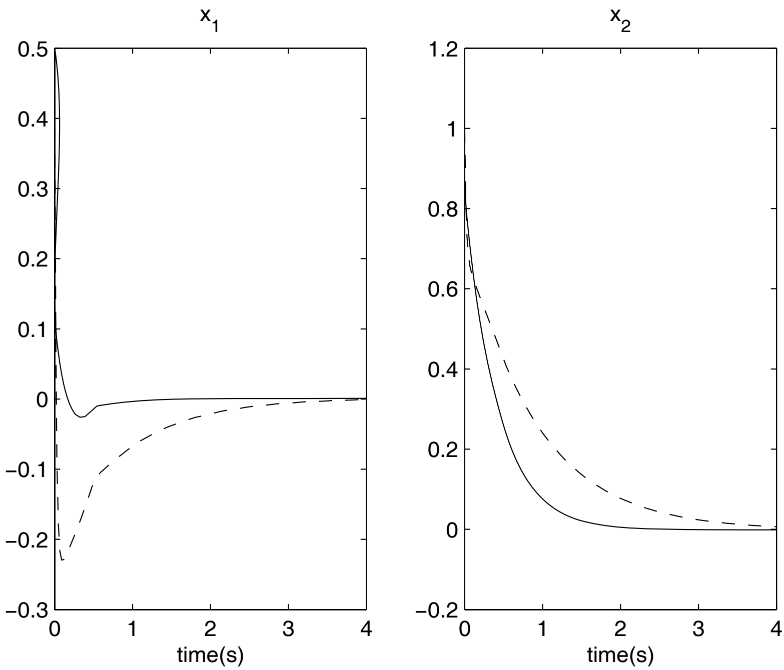


FIGURE 6.15
Response curves in normal case with adaptive robust reliable memory controller (solid) and robust reliable memory controller with fixed gains (dashed).

Using Collorary 6.3, the corresponding cost performance indexes can be obtained, using the adaptive method and traditional method. See Table 6.5 for more details.

In the following simulations, the chosen uncertain parameters are $\lambda_1 = 0.1$ and $\lambda_2 = 0.9$. The considered fault cases in this example are as follows:
Fault case 1: At 0 second, the second actuator is outage, and the first actuator becomes loss of 40% effectiveness.
Fault case 2: At 1 second, the first actuator is outage.

Figure 6.15-Figure 6.17 describe the response curves in normal and fault cases with our adaptive robust reliable memory controller and robust reliable memory controller with fixed gains, respectively. It is easy to see our adaptive robust reliable memory controller performs better than the one with fixed gains in either normal or fault cases just as theory has proved.

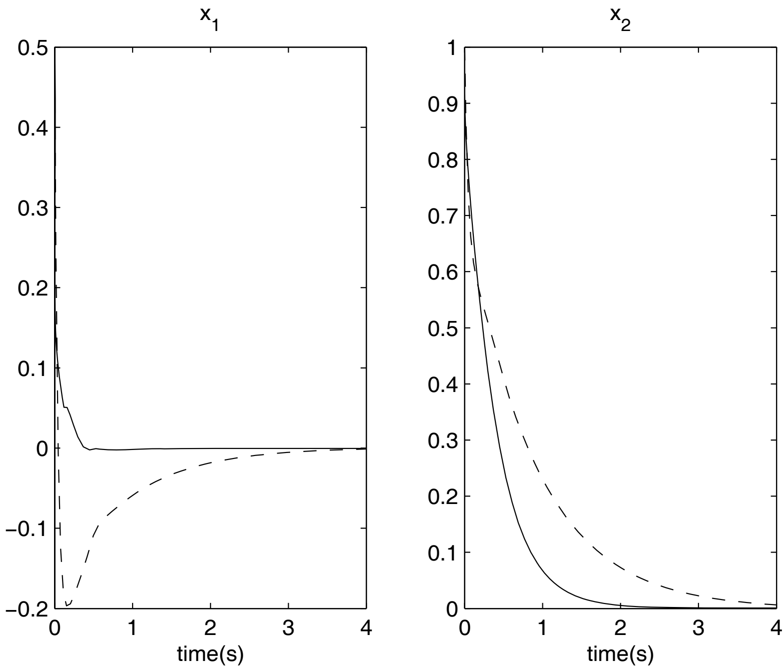


FIGURE 6.16
Response curves in fault case 1 with adaptive robust reliable memory controller (solid) and robust reliable memory controller with fixed gains (dashed).

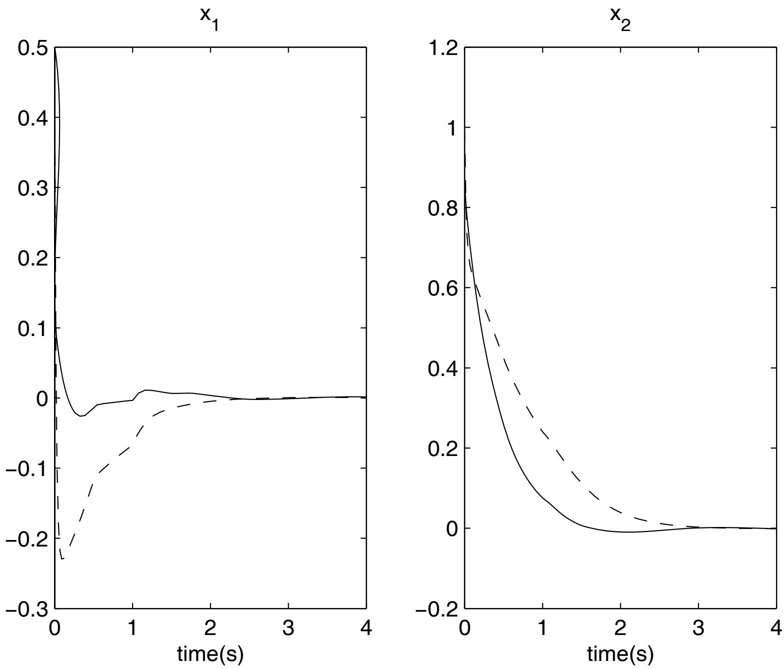


FIGURE 6.17
Response curves in fault case 2 with adaptive robust reliable memory controller (solid) and robust reliable memory controller with fixed gains (dashed).

6.4 Conclusion

In this chapter, we have investigated the new adaptive reliable memory-less controller and memory controller design methods for linear time-delay systems. The newly proposed controllers are all established in a parameter-dependent form, in which fault parameters are adjusted online based on the adaptive method to automatically compensate the *fault effect* on systems. In the framework of linear matrix inequality (LMI) technique, the stability and performance indexes of the closed-loop systems are guaranteed in normal and fault cases. The effectiveness of the proposed design method is illustrated via some numerical examples and their simulation results.

7

Adaptive Reliable Control with Actuator Saturation

7.1 Introduction

Control systems with *actuator saturation* are often encountered in practice. When actuator saturation occurs, in general global *stability* of an otherwise stable linear closed-loop system cannot be ensured. And the problem of estimating the *domain of attraction* for a system with a saturated linear feedback has been studied by many researchers in the last few years and various methods have appeared (see, [24, 140]). *Model predictive control (MPC)* is an effective control algorithm for dealing with actuator saturation. Many formulations have been developed for the stability of MPC (see, [18, 96]). Enlargement of the domain of attraction is achieved in [20, 28, 85, 90]. *Anti-windup* has been largely discussed and many constructive design algorithms have been formally proved to induce suitable stability properties (see, [25, 26, 27, 48, 68, 143]). Many of these constructive approaches rely on sector condition and S-procedure techniques and provide LMIs for the *anti-windup compensator* design. In some papers, notion of *invariant set* and LMI-based optimization approaches were proposed to estimate the stability regions by using quadratic Lyapunov functions and the Lur'e-type Lyapunov functions. In [17] and [142], the modeling of the nonlinear behavior of the system under saturation is made by using a polytopic differential inclusion and quadratic Lyapunov functions. For determining if a given ellipsoid is *contractively invariant*, [66] described a condition which is based on the circle criterion or the vertex analysis.

As we all know, in practice, actuator saturation and *actuator faults* are the common phenomena, and they always happen at the same time, especially for complex systems such as aircrafts, space crafts, nuclear power plants. For a flying aircraft its rudder may be damnified which can lead to the fault of the actuator. On the other hand, the rudder (actuator) of the aircraft can only give a bounded input which can be seen as an actuator saturation phenomenon. In this chapter, both actuator saturation and actuator faults are considered at the same time for a class of linear time-invariant systems. Here, an LMI-based method is presented to deal with the *fault-tolerant* and *saturation* problem.

7.2 State Feedback

7.2.1 Problem Statement

Consider an LTI plant described by

$$\dot{x} = Ax(t) + B\sigma(u), \tag{7.1}$$

where $x(t) \in R^n$ is the plant state, $\sigma(u) \in R^m$ is the saturated control input. A, B are known constant matrices of appropriate dimensions.

Definition 7.1 *The actuator nonlinearity with the consideration of a piecewise-linear saturation is described as*

$$\sigma(u_j) = \begin{cases} u_j, & |u_j| \leq u_j^{max}, \\ sign(u_j)u_j^{max}, & |u_j| > u_j^{max}, \end{cases} \tag{7.2}$$

for $j \in \mathbf{I}[1,m]$. Here we have slightly abused the notation by using σ to denote both the scalar valued and the vector valued saturation functions. We note that it is without loss of generality to assume $u_j^{max} = 1$, as level of saturation can always be scaled to unity by scaling B and u.

To formulate the *fault-tolerant control* problem, the considered actuator failures are the same as those in Chapter 3, that is

$$u_{jq}^F(t) = (1 - \rho_j^q)\sigma(u_j(t)), \quad 0 \leq \underline{\rho}_j^q \leq \rho_j^q \leq \overline{\rho}_j^q, \\ j \in \mathbf{I}[1,m],\ q \in \mathbf{I}[1,L], \tag{7.3}$$

For convenience in the following sections, for all possible fault modes L, the following uniform actuator fault model is exploited:

$$u^F(t) = (I - \rho)\sigma(u(t)),\ \rho \in \{\rho^1 \ \cdots \rho^L\} \tag{7.4}$$

and ρ can be described by $\rho = \text{diag}[\rho_1,\ \rho_2,\ \cdots \rho_m]$.

Denote

$$N_{\rho^q} = \{\rho^q | \rho^q = \text{diag}[\rho_1^q,\ \rho_2^q,\ \cdots \rho_m^q], \rho_j^q = \underline{\rho}_j^q \text{ or } \rho_j^q = \overline{\rho}_j^q\}. \tag{7.5}$$

Thus, the set N_{ρ^q} contains a *maximum* of 2^m elements.

Remark 7.1 *Here we note that any fault model formulated by*

$$u_{jq}^F(t) = \sigma[(1 - \rho_j^q)u_j(t)],\ 0 \leq \underline{\rho}_j^q \leq \rho_j^q \leq \overline{\rho}_j^q, \\ j \in \mathbf{I}[1,m],\ q \in \mathbf{I}[1,L], \tag{7.6}$$

can be formulated by (7.3). We need only to prove that for any ρ_j^q satisfying (7.6) there must exist a ρ_{j*}^q satisfying

$$(1-\rho_{j*}^q)\sigma[u_j(t)] = \sigma[(1-\rho_j^q)u_j(t)] \tag{7.7}$$

and

$$0 \le \underline{\rho}_{j*}^q \le \rho_{j*}^q \le \overline{\rho}_{j*}^q$$

In fact if ρ_{j*}^q is given as follows

$$\rho_{j*}^q = 1 - \frac{\sigma[(1-\rho_j^q)u_j(t)]}{\sigma[u_j(t)]},$$

then equality (7.7) is satisfied, and we have $0 = \underline{\rho}_{j*}^q \le \rho_{j*}^q \le \overline{\rho}_{j*}^q = \overline{\rho}_j^q$ by Definition 7.1 and (7.3).

Definition 7.2 *For a matrix $C_{cl} \in R^{m\times n}$, denote the jth row of C_{cl} as C_{clj}, define*

$$\wp(C_{cl}) = \{x \in R^n: \quad |C_{clj}x| \le 1, \quad j \in \mathbf{I}[1,m]\},$$

then $\wp(C_{cl})$ is the region in the state space where saturation does not occur.

For $x(0) = x_0 \in R^n$, denote the state trajectory of systems as $\psi(t, x_0)$. Then the domain of attraction of the origin is

$$\ell := \{x_0 \in R^n: \quad lim_{t\to\infty}\psi(t,x_0) = 0\}.$$

Definition 7.3 *Let $\mathbf{D}$ be a set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathbf{D}$ and we denote its elements as D_i, $i \in \mathbf{I}[0, 2^m-1]$, where for $i = z_1 2^{m-1} + Z_2 2^{m-2} + \cdots + z_m$ with $z_j \in \{0,1\}$, the diagonal elements of D_i are $\{1-z_1, 1-z_2, \cdots, 1-z_m\}$. Denote $D_i^- = I - D_i$. It is easy to see that $D_i^- \in \mathbf{D}$. As an illustration, we consider the case of $m = 2$. For $i = 0$, it is easy to see that $i = 0\times 2^1 + 0\times 2^0$. Thus, $z_1 = 0$, $z_2 = 0$ and $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For $i = 1$, it is easy to see that $i = 0\times 2^1 + 1\times 2^0$. Thus, $z_1 = 0$, $z_2 = 1$ and $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Denote $D_i^- = I - D_i$. It is easy to see that $D_i^- \in \mathbf{D}$. The following propositions will be useful for the development of the main results of this section.*

Lemma 7.1 *[65] Let u, $v \in R^m$ with $u = [u_1, u_2, ..., u_m]^T$ and $v = [v_1, v_2, ..., v_m]^T$. Suppose that $|v_j| \le 1$ for all $j \in \mathbf{I}[1,m]$.*

Then,

$$\sigma(u) \in co\{D_i u + D_i^- v : i \in \mathbf{I}[0, 2^m-1]\}, \tag{7.8}$$

where co denotes the convex hull.

Problem 7.1 *The design problem under consideration is to find an adaptive controller such that in both normal operation and fault cases, the domain of asymptotic stability is enlarged as much as possible for closed-loop system with actuator saturation.*

Remark 7.2 *For the above problem to be solvable, it is necessary for the pair* $(A, B(I-\rho))$ *to be stabilizable for each* $\rho \in \{\rho^1 \cdots \rho^L\}$.

7.2.2 A Condition for Set Invariance

The dynamics with actuator faults (7.4) and saturation is described by

$$\dot{x}(t) = Ax(t) + B(I-\rho)\sigma(u(t)) \tag{7.9}$$

The *controller structure* is chosen as

$$u(t) = K(\hat{\rho}(t))x(t) = (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t) \tag{7.10}$$

where $\hat{\rho}(t)$ is the estimation of ρ,

$$K_a(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} K_{aj}\hat{\rho}_j(t), \;\; K_b(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} K_{bj}\hat{\rho}_j(t).$$

By Lemma 7.1, the saturated linear feedback, with $x \in \wp(H(\hat{\rho}(t)))$, can be expressed as

$$\sigma(K(\hat{\rho}(t))x(t)) = \sum\nolimits_{i=0}^{2^m-1} \eta_i[D_i K(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]x(t) \tag{7.11}$$

for some scalars $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m-1]$, such that $\sum_{i=0}^{2^m-1}\eta_i = 1$, and the following equality holds

$$\begin{aligned}(I-\rho)\sigma(u(t)) = \sum\nolimits_{i=0}^{2^m-1} &\eta_i[(I-\rho)D_iK_0 + D_iK_a(\rho)\\ &- \rho D_iK_a(\hat{\rho}) + (I-\hat{\rho}(t))D_iK_b(\hat{\rho}(t)) + D_iK_a(\tilde{\rho}(t))\\ &+ \tilde{\rho}D_iK_b(\hat{\rho}(t)) + (I-\rho)D_i^-H_0 + D_i^-H_a(\rho)\\ &- \rho D_i^-H_a(\hat{\rho}) + (I-\hat{\rho}(t))D_i^-H_b(\hat{\rho}(t)) + D_i^-H_a(\tilde{\rho}(t))\\ &+ \tilde{\rho}D_i^-H_b(\hat{\rho}(t))]x(t)\end{aligned} \tag{7.12}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. Though $K_a(\hat{\rho}(t))$ and $K_b(\hat{\rho}(t))$ have the same forms, we deal with them in different ways in (7.12), which gives more freedom and less conservativeness.

Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m) \; : \; \hat{\rho}_j \in \; \{\min_q\{\underline{\rho}_j^q\}, \; \max_q\{\overline{\rho}_j^q\}\}, \;\; q \in \mathbf{I}[1, L]\}$$

and $B^j = [0 \cdots b^j \cdots 0]$ with $B = [b^1 \cdots b^m]$.

Definition 7.4 *Let $P \in R^{n\times n}$ be a positive-define matrix. Denote*

$$\varepsilon(P,\delta) = \{x \in R^n : \quad x^T Px \le \delta\}.$$
$$\varepsilon^-(P,\delta) = \{x \in R^n : \quad x^T Px < \delta\}.$$
$$\varepsilon^*(P,\delta) = \{x \in R^n : \quad x^T Px + \sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j} \le \delta\}.$$

Assume $l_j > 0$ is given, we denote $\delta^* = \delta + max\{\sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}\}$.

Let $V(t) = x^T Px + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}$. If $\dot{V}(t) < 0$ for all $x \in \varepsilon^*(P,\delta)\backslash\{0\}$, the domain $\varepsilon^*(P,\delta)$ is contractively invariant. Clearly, if $\varepsilon^*(P,\delta)$ is contractively invariant, then it is inside the domain of attraction.

We note that the scalars η_i's are functions of x and $\hat{\rho}$ and their values are available in real-time. These scalars in a way reflect the severity of control saturation. In general, there are multiple choices of η_i's satisfying the same constraint, leading to nonunique representation of (7.11). In the following lemma, we provide one choice of such η_i's, which are *Lipschitzian functions* in x and $\hat{\rho}$ and thus are particularly useful in our controller design.

Lemma 7.2 *[142] Let $x \in \wp(H(\hat{\rho}(t)))$. For each $j \in \mathbf{I}[1,m]$, let*

$$\lambda_j(x(t),\hat{\rho}(t)) = \begin{cases} 1, & \text{if } K(\hat{\rho}(t))_j x(t) = H(\hat{\rho}(t))_j x(t) \\ \frac{\sigma(K(\hat{\rho}(t))_j x(t)) - H(\hat{\rho}(t))_j x(t)}{(K(\hat{\rho}(t))_j - H(\hat{\rho}(t))_j)x(t)}, & \text{otherwise} \end{cases}$$

and for each $i \in \mathbf{I}[0, 2^m - 1]$, let $z_j \in \{0,1\}$ be such that $i = z_1 2^{m-1} + z_2 2^{m-2} + ... + z_m$, and define

$$\eta_i(x(t),\ \hat{\rho}(t)) = \prod_{j=1}^{m} [z_j(1 - \lambda_j(x(t),\ \hat{\rho}(t))) + (1 - z_j)\lambda_j(x(t),\ \hat{\rho}(t))] \quad (7.13)$$

Then, η_i's are functions Lipschitz in x and $\hat{\rho}$, such that, $\sum_{i=0}^{2^m-1} \eta_i = 1$, $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m - 1]$. Moreover, they satisfy relation (7.11).

By using the functions $\eta_i(x(t),\ \hat{\rho}(t))$'s and the controller (7.10), plant (7.9) can be written in a quasi-LPV form as follows:

$$\begin{aligned} \dot{x}(t) =& Ax(t) + B\sum\nolimits_{i=0}^{2^m-1} \eta_i[(I-\rho)D_i(K_0 + K_a(\hat{\rho}(t)) \\ &+ K_b(\hat{\rho}(t))) + (I-\rho)D_i^-(H_0 + H_a(\hat{\rho}(t)) + H_b(\hat{\rho}(t)))]x(t) \end{aligned} \quad (7.14)$$

By using (7.13) we consider the following auxiliary LPV system, of which the closed-loop system comprising of (7.9) and (7.10) is a special case, if $\varepsilon^*(P,\delta) \subset \wp(H(\hat{\rho}))$ is an invariant set

$$\dot{x}(t) = A(\eta)x(t), \quad \eta \in \Gamma \tag{7.15}$$

where $\eta = [\eta_0,\ \eta_1,\ \cdots,\ \eta_{2^m-1}]$, and

$$\begin{aligned}
\Gamma &= \{\eta \in R^{2^m} : \sum\nolimits_{i=0}^{2^m-1} \eta_i = 1,\ 0 \le \eta_i \le 1,\ i \in I[0,\ 2^m-1]\} \\
A(\eta) &= A + B\sum\nolimits_{i=0}^{2^m-1} \eta_i[(I-\rho)D_iK_0 + D_iK_a(\rho) \\
&\quad - \rho D_iK_a(\hat{\rho}) + (I-\hat{\rho}(t))D_iK_b(\hat{\rho}(t)) + D_iK_a(\tilde{\rho}(t)) \\
&\quad + \tilde{\rho}D_iK_b(\hat{\rho}(t)) + (I-\rho)D_i^-H_0 + D_i^-H_a(\rho) \\
&\quad - \rho D_i^-H_a(\hat{\rho}) + (I-\hat{\rho}(t))D_i^-H_b(\hat{\rho}(t)) + D_i^-H_a(\tilde{\rho}(t)) \\
&\quad + \tilde{\rho}D_i^-H_b(\hat{\rho}(t))]
\end{aligned}$$

The following theorem establishes conditions on the state-feedback controller coefficient matrices under which the LPV system (7.15) is asymptotically stable with Lyapunov function.

Theorem 7.1 *$\varepsilon^*(P,\delta)$ is a contractively invariant set for normal and actuator failure cases, if there exist matrices $X > 0$, O_0, O_{aj}, O_{bj}, Y_0, Y_{aj}, Y_{bj}, $j \in \mathbf{I}[1,m]$ and symmetric matrixes Θ_i, $i \in \mathbf{I}[0,2^m-1]$ with*

$$\Theta^i = \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ \Theta_{12}^{iT} & \Theta_{22}^i \end{bmatrix}$$

and Θ_{11}^i, $\Theta_{22}^i \in R^{mn\times mn}$ such that the following inequalities hold for all $D_i \in \mathbf{D}$ and $\varepsilon^*(P,\delta) \subset \wp(H(\hat{\rho}))$, i.e., $|H(\hat{\rho})_jx| \le 1$ for all $x \in \varepsilon^*(P,\delta), j \in \mathbf{I}[1,m]$.

$$\begin{gathered}
\Theta_{22jj}^i \le 0,\ \ j \in \mathbf{I}[1,m], i \in \mathbf{I}[0,2^m-1] \\
\Theta_{11}^i + \Theta_{12}^i\Delta(\hat{\rho}) + (\Theta_{12}^i\Delta(\hat{\rho}))^T + \Delta(\hat{\rho})\Theta_{22}^i\Delta(\hat{\rho}) \ge 0,\ \hat{\rho} \in \Delta_{\hat{\rho}} \\
\begin{bmatrix} N_{0i} & Z_{1i} \\ Z_{1i}^T & Z_{2i} \end{bmatrix} + G^T\Theta^iG < 0,\ \ i \in \mathbf{I}[0,2^m-1], \\
\rho \in \{\rho^1\ \cdots\ \rho^L\},\ \rho^q \in N_{\rho^q}
\end{gathered} \tag{7.16}$$

where

$$\begin{aligned}
N_{0i} &= AX + B(I-\rho)D_iY_0 + (AX + B(I-\rho)D_iY_0)^T \\
&\quad + B\sum\nolimits_{j=1}^{m} \rho_j D_i Y_{aj} + (B\sum\nolimits_{j=1}^{m} \rho_j D_i Y_{aj})^T \\
&\quad + B(I-\rho)D_i^- O_0 + (B(I-\rho)D_i^- O_0)^T \\
&\quad + B\sum\nolimits_{j=1}^{m} \rho_j D_i^- O_{aj} + (B\sum\nolimits_{j=1}^{m} \rho_j D_i^- O_{aj})^T, \\
G &= \begin{bmatrix} \begin{bmatrix} I_{n\times n} \\ \cdots \\ I_{n\times n} \end{bmatrix} & 0 \\ 0 & I_{mn\times mn} \end{bmatrix}, \\
Z_{1i} &= -B\rho D_i Y_a + BD_iY_b - B\rho D_i^- O_a + BD_i^- O_b, \\
Z_{2i} &= \begin{bmatrix} -B^1 D_i \\ \ldots \\ -B^m D_i \end{bmatrix} Y_b + (\begin{bmatrix} -B^1 D_i \\ \ldots \\ -B^m D_i \end{bmatrix} Y_b)^T \\
&\quad + \begin{bmatrix} -B^1 D_i^- \\ \ldots \\ -B^m D_i^- \end{bmatrix} O_b + (\begin{bmatrix} -B^1 D_i^- \\ \ldots \\ -B^m D_i^- \end{bmatrix} O_b)^T, \\
Y_a &= [Y_{a1}\ Y_{a2}....Y_{am}],\ \ Y_b = [Y_{b1}\ Y_{b2}....Y_{bm}], \\
O_a &= [O_{a1}\ O_{a2}....O_{am}],\ \ O_b = [O_{b1}\ O_{b2}....O_{bm}], \\
\Delta(\hat{\rho}) &= diag[\hat{\rho}_1 I_{n\times n}\ \ \cdots\ \ \hat{\rho}_m I_{n\times n}].
\end{aligned}$$

and also $\hat{\rho}_j(t)$ is determined according to the adaptive law

$$\begin{aligned}
\dot{\hat{\rho}}_j &= Proj_{[\min_q\{\underline{\rho}_j^q\},\ \max_q\{\overline{\rho}_j^q\}]}\{L_{1j}\} \\
&= \begin{cases} 0,\ if & \begin{matrix} \hat{\rho}_j = \min_q\{\underline{\rho}_j^q\}\ and\ L_{1j} \le 0 \\ or\ \hat{\rho}_j = \max_q\{\overline{\rho}_j^q\}\ and\ L_{1j} \ge 0 \end{matrix} \\ & L_{1j},\ \ otherwise \end{cases}
\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}
L_{1j} = &- l_j x^T(t)[PB(\sum\nolimits_{i=0}^{2^m-1} \eta_i D_i)K_{aj} + PB^j(\sum\nolimits_{i=0}^{2^m-1} \eta_i D_i)K_b(\hat{\rho}) \\
&+ PB(\sum\nolimits_{i=0}^{2^m-1} \eta_i D_i^-)H_{aj} + PB^j(\sum\nolimits_{i=0}^{2^m-1} \eta_i D_i^-)H_b(\hat{\rho})]x(t),
\end{aligned}$$

$P = \delta X^{-1}$, $K_0 = Y_0X^{-1}$, $K_{aj} = Y_{aj}X^{-1}$, $K_{bj} = Y_{bj}X^{-1}$, $H_0 = O_0X^{-1}$, $H_{aj} = O_{aj}X^{-1}$, $H_{bj} = O_{bj}X^{-1}$. $l_j > 0 (j \in \mathbf{I}[1,m])$ *and* $\delta > 0$ *are the adaptive law gains to be chosen according to practical applications. Then the controller gain is given by*

$$K(\hat{\rho}) = Y_0X^{-1} + \sum\nolimits_{j=1}^{m} \hat{\rho}_j Y_{aj} X^{-1} + \sum\nolimits_{j=1}^{m} \hat{\rho}_j Y_{bj} X^{-1}.$$

Proof 7.1 *Choose the following Lyapunov function*

$$V = x(t)^T P x(t) + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}, \tag{7.18}$$

then from the derivative of $V(t)$ *along the closed-loop system, it follows*

$$\begin{aligned}\dot{V}(t) \leq\ & x^T \sum_{i=0}^{2^m-1} \eta_i (C + C^T) x \\ & + 2x^T PB \sum_{i=0}^{2^m-1} \eta_i [D_i K_a(\tilde{\rho}) + \tilde{\rho} D_i K_b(\hat{\rho}) \\ & + D_i^- H_a(\tilde{\rho}) + \tilde{\rho} D_i^- H_b(\hat{\rho})] x + 2\sum_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j},\end{aligned}$$

where

$$\begin{aligned}C = & PA + PB[(I-\rho)D_i K_0 + D_i K_a(\rho) - \rho D_i K_a(\hat{\rho}) + (I - \hat{\rho}(t)) D_i K_b(\hat{\rho}) \\ & + (I-\rho) D_i^- H_0 + D_i^- H_a(\rho) - \rho D_i^- H_a(\hat{\rho}) + (I - \hat{\rho}(t)) D_i^- H_b(\hat{\rho})].\end{aligned}$$

Let $B = [b^1 \cdots b^m]$ *and* $B^j = [0 \cdots b^j \cdots 0]$, *then*

$$\begin{aligned}PB\tilde{\rho} D_i K_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB^j D_i K_b(\hat{\rho}), \\ PB\tilde{\rho} D_i^- H_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB^j D_i^- H_b(\hat{\rho}), \\ PBD_i K_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PBD_i K_{aj}, \\ PBD_i^- H_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PBD_i^- H_{aj}.\end{aligned}$$

Let $X = (\frac{P}{\delta})^{-1}$, $Y_0 = K_0 X$, $Y_{aj} = K_{aj} X$, $Y_{bj} = K_{bj} X$, $O_0 = H_0 X$, $O_{aj} = H_{aj} X$, $O_{bj} = H_{bj} X$, $j \in \mathbf{I}[1, m]$. *Choose the adaptive laws as (7.17), then it is sufficient to show that* $\dot{V} < 0$ *if for any* $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^q \in N_{\rho^q}$,

$$\sum_{i=0}^{2^m-1} \eta_i [N_{0i} + N_{1i}(\hat{\rho}_j) + N_{2i}(\hat{\rho}_j)] < 0,$$

where

$$\begin{aligned}N_{0i} &= AX + B(I-\rho)D_iY_0 + (AX + B(I-\rho)D_iY_0)^T \\ &\quad + B\sum\nolimits_{j=1}^{m} \rho_j D_i Y_{aj} + (B\sum\nolimits_{j=1}^{m} \rho_j D_i Y_{aj})^T \\ &\quad + B(I-\rho)D_i^- O_0 + (B(I-\rho)D_i^- O_0)^T \\ &\quad + B\sum\nolimits_{j=1}^{m} \rho_j D_i^- O_{aj} + (B\sum\nolimits_{j=1}^{m} \rho_j D_i^- O_{aj})^T, \\ N_{1i}(\hat{\rho}_j) &= -B\rho D_i \sum\nolimits_{j=1}^{m} \hat{\rho}_j Y_{aj} + B\sum\nolimits_{j=1}^{m} \hat{\rho}_j D_i Y_{bj} \\ &\quad + (B\sum\nolimits_{j=1}^{m} \hat{\rho}_j D_i Y_{bj} - B\rho D_i \sum\nolimits_{j=1}^{m} \hat{\rho}_j Y_{aj})^T \\ &\quad - B\rho D_i^- \sum\nolimits_{j=1}^{m} \hat{\rho}_j O_{aj} + B\sum\nolimits_{j=1}^{m} \hat{\rho}_j D_i^- O_{bj} \\ &\quad + (B\sum\nolimits_{j=1}^{m} \hat{\rho}_j D_i^- O_{bj} - B\rho D_i^- \sum\nolimits_{j=1}^{m} \hat{\rho}_j O_{aj})^T, \\ N_{2i}(\hat{\rho}_j) &= \sum\nolimits_{j=1}^{m}\sum\nolimits_{p=1}^{m} \hat{\rho}_j\hat{\rho}_p(-B^j D_i Y_{bp} - Y_{bj}^T D_i B^{pT} \\ &\quad - B^j D_i^- O_{bp} - O_{bj}^T D_i^- B^{pT}).\end{aligned}$$

By Lemma 2.10 and (7.16), it follows that $\dot{V} < 0$ *for any* $x \in \wp(H(\hat{\rho}))$, $\rho \in \{\rho^1 \ \cdots \ \rho^L\}$, $\rho^q \in N_{\rho^q}$ *and* $\hat{\rho}$ *satisfying (7.17).*

7.2.3 Controller Design

From Theorem 7.1, we can obtain various controller gains and domains satisfying the set invariance condition. So, how to choose the "largest" one of them becomes an interesting problem. In this section, we will give a method to find the "largest" domain.

Definition 7.5 *Define* X_R *is a prescribed bounded convex set.* $X_R = \epsilon(R,1) = \{x \in R^{n\times n} : \quad x^T R x \le 1\}$, $R > 0$ *or* $X_R = co\{x_1, x_2, ..., x_l\}$. *For a set* $S \in R^n$, $\alpha_R(S) = sup\{\alpha > 0 : \quad \alpha X_R \subset S\}$.

In Theorem 7.1, a condition for the set $\varepsilon^*(P,\delta)$ to be inside the domain of attraction is given. With the above shape reference sets, we can choose from all the $\varepsilon^*(P,\delta)$'s that satisfy the condition of Theorem 7.1 such that the quantity $\alpha_R(\varepsilon^*(P,\delta))$ is maximized. The problem can be formulated as follows

$$\begin{aligned} &\sup \ \alpha \\ &s.t. \quad (a) \quad \alpha X_R \subset \varepsilon^*(P,\delta), \\ &\qquad\ \ (b) \quad (7.16), \\ &\qquad\ \ (c) \quad \varepsilon^*(P,\delta) \subset \wp(H(\hat{\rho})). \end{aligned} \tag{7.19}$$

However, by Definition 7.4, we have that (a) and (c) can not be shown as LMIs directly. Then the following proposition will solve this problem.

Proposition 7.1 *Obviously, $\varepsilon^*(P,\delta) \subset \varepsilon(P,\delta)$, which implies that (c) holds if (c1) holds, where*

$$(c1) \quad \varepsilon(P,\delta) \subset \wp(H(\hat{\rho})), \tag{7.20}$$

Proposition 7.2 *By Definition 7.4, we have*

$$x^T P x + \sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j} \leq \delta \Leftrightarrow x^T \frac{P}{\delta} x + \sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{\delta l_j} \leq 1$$
$$\Leftrightarrow x^T X^{-1} x + \sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{\delta l_j} \leq 1.$$

Let $F(t) = \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{\delta l_j}$. Then, by (7.17) and (7.3), it follows that $\tilde{\rho}_j(t) \leq \max_j\{\overline{\rho}_j^q\} - \min_j\{\underline{\rho}_j^q\}$. We can choose l_j and δ sufficiently large so that $F(t)$ is sufficiently small. Then the conclusion can be drawn as follows:

For system (7.9) and controller (7.10) there must exist $\delta > 0$ and $l_i > 0$ such that the closed-loop system (7.15) is asymptotically stable in domain $\varepsilon^-(P,\delta)$ if (b) and (c1) hold. That is to say, if l_j and δ are chosen sufficiently large, then the set $\varepsilon^*(P,\delta)$ will approach the set $\varepsilon(P,\delta)$, so we can maximize the set $\varepsilon^*(P,\delta)$ indirectly by maximizing the set $\varepsilon(P,\delta)$. Thus, we have that (a) can be replaced with (a1).

Then, by Proposition 7.1 and Proposition 7.2 we can get the "largest" domain of asymptotic stability by solving the following optimization problem

$$\begin{aligned} &\sup \ \alpha \\ s.t. \quad &(a1) \quad \alpha X_R \subset \varepsilon(P,\delta), \\ &(b), \\ &(c1). \end{aligned} \tag{7.21}$$

If the given shape *reference set* X_R is a polyhedron as defined in Definition 7.5, then Constraint (a1) is equivalent to

$$\alpha^2 x_e^T (\frac{P}{\delta}) x_e \leq 1 \Leftrightarrow \begin{bmatrix} 1/\alpha^2 & x_e^T \\ x_e & (\frac{P}{\delta})^{-1} \end{bmatrix} \geq 0, \tag{7.22}$$

for all $e \in \mathbf{I}[1,l]$. If X_R is an ellipsoid $\varepsilon(R,1)$, then (a1) is equivalent to

$$\frac{R}{\alpha^2} \geq \frac{P}{\delta} \Leftrightarrow \begin{bmatrix} (1/\alpha^2)R & I \\ I & (\frac{P}{\delta})^{-1} \end{bmatrix} \geq 0. \tag{7.23}$$

Condition (c1) is equivalent to

$$\delta h(\hat{\rho})_j P^{-1} h(\hat{\rho})_j^T \leq 1 \Leftrightarrow \begin{bmatrix} 1 & h(\hat{\rho})_j (\frac{P}{\delta})^{-1} \\ * & (\frac{P}{\delta})^{-1} \end{bmatrix} \geq 0. \tag{7.24}$$

for all $j \in \mathbf{I}[1,m]$, where $h(\hat{\rho})_j$ be the jth row of $H(\hat{\rho})$. We have that (7.24) is equivalent to the following inequalities.

$$(c2)\quad \begin{bmatrix} -1 & -O_{0s} \\ * & -X \end{bmatrix} + \sum_{j=1}^{m} \hat{\rho}_j \begin{bmatrix} 0 & -O_{ajs} - O_{bjs} \\ * & 0 \end{bmatrix} \leq 0, \hat{\rho} \in \Delta_{\hat{\rho}}$$

where O_{ajs} is the sth row of O_{aj}, $s \in \mathbf{I}[1,m]$.

If X_R is a polyhedron, then from (7.22) and (7.24), the optimization problem (7.21) is equivalent to

$$\begin{array}{ll} \inf & \gamma \\ s.t. & (a2)\ \begin{bmatrix} \gamma & x_e^T \\ x_e & X \end{bmatrix} \geq 0, \quad e \in \mathbf{I}[1,\ l], \\ & (b),\ \ (c2), \end{array} \tag{7.25}$$

where $\gamma = 1/\alpha^2$.

If X_R is an ellipsoid, we need only to replace (a2) with

$$(a3)\quad \begin{bmatrix} \gamma R & I \\ I & X \end{bmatrix} \geq 0. \tag{7.26}$$

It is easy to see that all constraints are given in LMIs.

Remark 7.3 *Theorem 7.1 gives a sufficient condition for the existence of an adaptive fault tolerant controller via state feedback. Note that inequalities described by (7.16) are of LMIs. In Theorem 7.1, if set* $Y_{aj} = 0$, $Y_{bj} = 0$, $O_{aj} = 0$, $O_{bj} = 0$, $j \in \mathbf{I}[1,m]$, *then the conditions of Theorem 7.1 reduce to*

$$\begin{aligned} & AX + B(I-\rho)D_iY_0 + (AX + B(I-\rho)D_iY_0)^T \\ & \quad + B(I-\rho)D_i^-O_0 + (B(I-\rho)D_i^-O_0)^T < 0, \\ & \quad i \in \mathbf{I}[0, 2^m - 1],\ \rho \in \{\rho^1\ \cdots\ \rho^L\},\ \rho^q \in N_{\rho^q} \end{aligned} \tag{7.27}$$

From [66], it follows that $\varepsilon(P,\delta)$ *is a contractively invariant set for closed-loop system (7.9) with* $u = K_0x$, $K_0 = Y_0X^{-1}$, *if there exist matrices* $X > 0$, O_0, Y_0, *such that the inequalities (7.27) hold for all* $D_i \in \mathbf{D}$ *and* $\varepsilon(P,\delta) \subset \wp(H_0)$, *where* $P = \delta X^{-1}$, $H_0 = O_0X^{-1}$. *This just gives a design method for traditional fault tolerant controllers via fixed gains. The above fact shows that the design condition for adaptive fault tolerant controllers given in Theorem 7.1 is more relaxed than that described by (7.27) for the traditional fault tolerant controller design with fixed gains.*

7.2.4 Example

In this section, two examples are given to illustrate that the Algorithm 7.21 describes a larger domain of attraction than the traditional fault tolerant controller design with fixed gains.

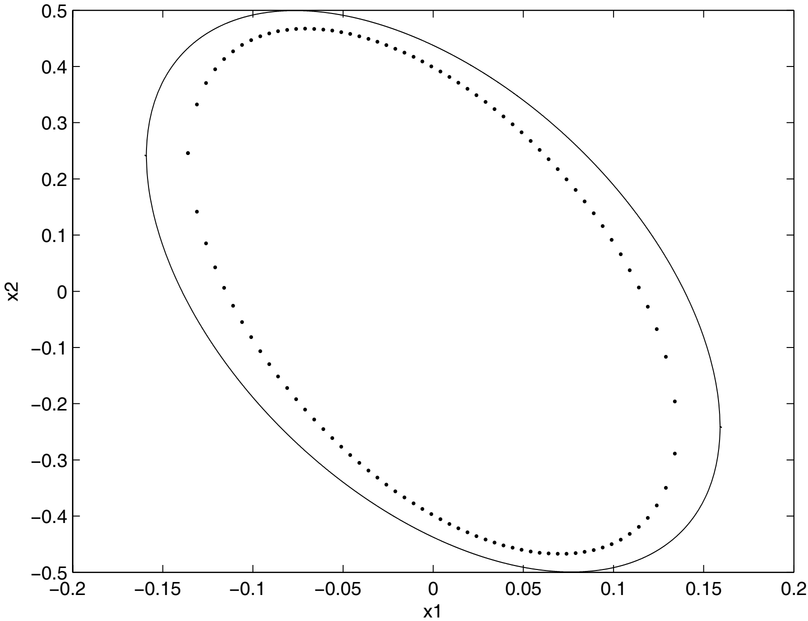


FIGURE 7.1
$\varepsilon(P_1^*, 1)$ and $\varepsilon(P_2^*, 1)$.

Example 7.1 *Consider the system of form (7.9) with*

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 40 \end{bmatrix}, \ B = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}$$

and the following two possible fault modes:

Fault mode 1: Both of the two actuators are normal, that is,

$$\rho_1^1 = \rho_2^1 = 0$$

Fault mode 2: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^2 = 1, \quad 0 \leq \rho_2^2 \leq a,$$

where $a = 0.5$ denotes the maximal loss of effectiveness for the second actuator. Let

$$R = \begin{bmatrix} 75.5284 & 11.3861 \\ 11.3861 & 6.2969 \end{bmatrix}.$$

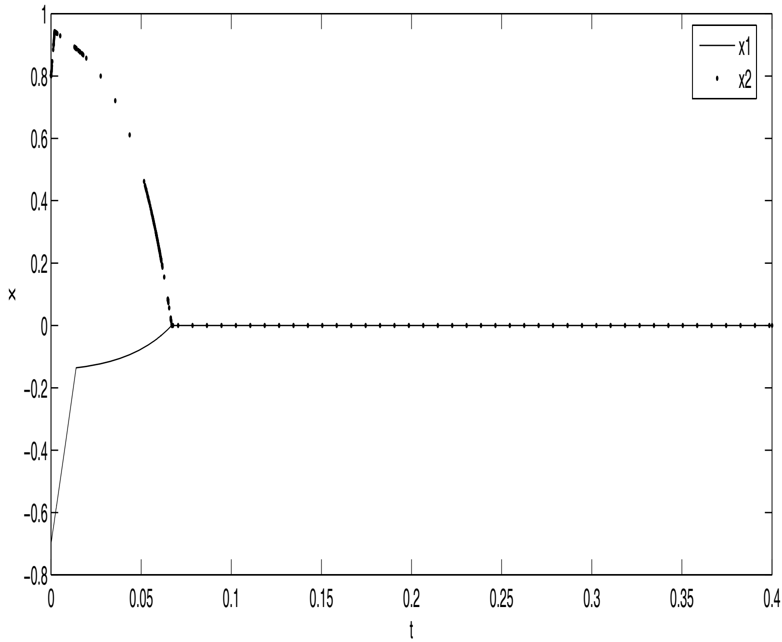


FIGURE 7.2
Trajectories of closed-loop systems with adaptive controller in normal case.

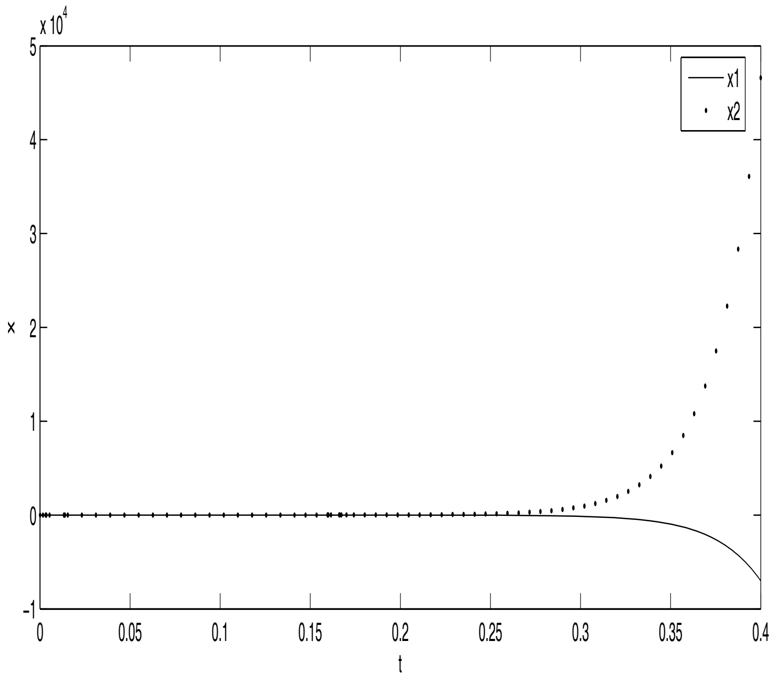


FIGURE 7.3
Trajectories of closed-loop systems with fixed gains controller in normal case.

When the fixed controller gains design method is given, we have that $\gamma^* = 1$. By solving the optimization problem (7.25), we obtain $\gamma^* = 0.8757$. Obviously, the optimal index γ is smaller for optimization problem (7.25).

We plot in Figure 7.1 the two ellipsoids $\varepsilon(P_1^*, 1)$ (dot line) and $\varepsilon(P_2^*, 1)$ (solid line) where P_1^* is given by fixed controller gains design method and P_2^* is given by solving *optimization problem* (7.25). As a comparison, we also plot the trajectories of closed-loop systems with adaptive controller and fixed gains controller, respectively. Figure 7.2 and Figure 7.3 show the trajectories of closed-loop system in normal case for $x(0) = (-0.7 \quad 0.8)$. Figure 7.4 and Figure 7.5 show the trajectories of the closed-loop system in fault case for $x(0) = (0.3 \quad 0.01)$.

The fault case considered in the following simulation is: At 0 seconds, the first actuator is outage and the second actuator becomes loss of effectiveness by 50%.

In order to let the method of this section be more convincing, the following engineering example is given.

Example 7.2 *Consider a kind of aircraft system borrowed from the literature*

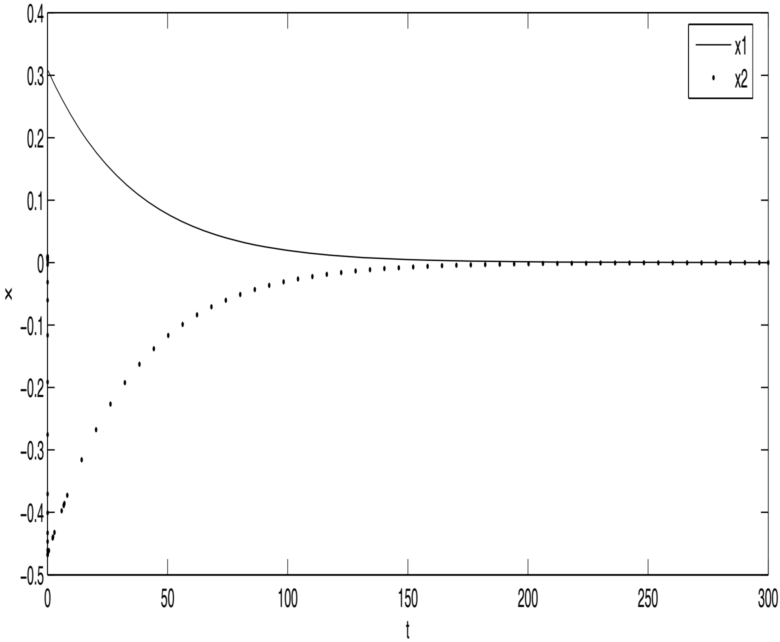


FIGURE 7.4
Trajectories of closed-loop systems with adaptive controller in fault case.

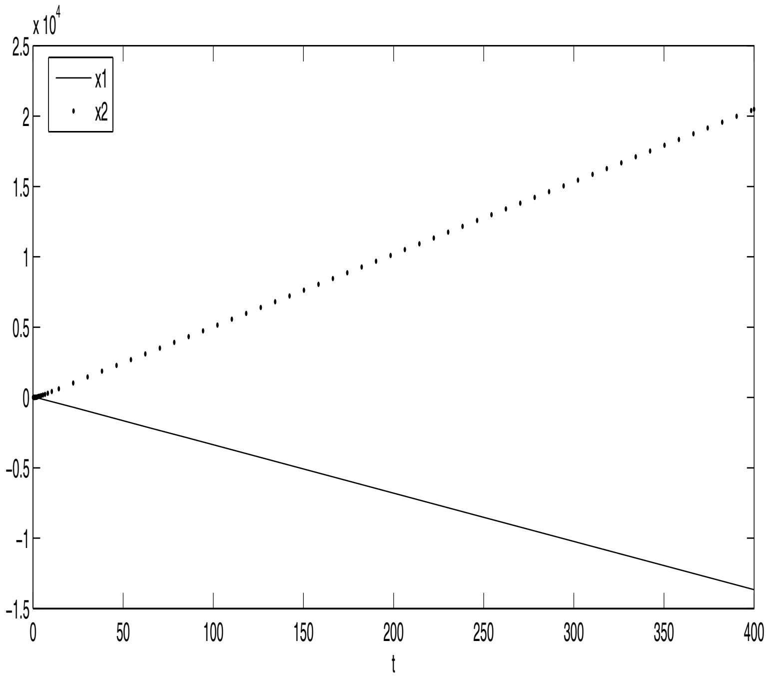


FIGURE 7.5
Trajectories of closed-loop systems with fixed gains controller in fault case.

[15]. The dynamical description is given as (7.9) with

$$A = \begin{bmatrix} 0.4559 & 0.2114 \\ -0.4359 & 4.0080 \end{bmatrix}, \ B = \begin{bmatrix} -14.0539 & -0.3462 \\ -1.0385 & -13.1539 \end{bmatrix}$$

and the fault modes are the same as the ones of Example 7.1.

Let

$$R = \begin{bmatrix} 43.4145 & 2.1555 \\ 2.1555 & 0.5534 \end{bmatrix}.$$

By using the fixed controller gains design method, the optimal index is obtained as $\gamma^* = 1$. Correspondingly, by solving the optimization problem (7.25), the optimal index is obtained as $\gamma^* = 0.8638$. Obviously, the optimal index γ is improved by using our optimal method.

7.3 Output Feedback

7.3.1 Problem Statement

Consider an LTI plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\sigma(u(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{7.28}$$

where $x(t) \in R^n$ is the plant state, $\sigma(u) \in R^m$ is the saturated control input. A, B, C are known constant matrices of appropriate dimensions.

Then, the following problem will be considered in this section.

Problem 7.2 *Find an adaptive controller such that in both normal operation and fault cases, the domain of asymptotic stability is enlarged as much as possible for a closed-loop system with actuator saturation.*

Remark 7.4 *For the above problem to be solved, it is necessary for the pair $(A, B(I-\rho))$ to be stabilizable for each $\rho \in \{\rho^1 \cdots \rho^L\}$.*

7.3.2 A Condition for Set Invariance

The dynamics with actuator faults (7.4) and saturation is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(I-\rho)\sigma(u(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{7.29}$$

The controller structure is chosen as

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t),\ y),\ \ \xi(t) \in R^n \\ u(t) &= C_K(\hat{\rho}(t))\xi(t) \end{aligned} \tag{7.30}$$

with

$$u(t) = C_K(\hat{\rho}(t))\xi(t) = (C_{K0} + C_{Ka}(\hat{\rho}(t)) + C_{Kb}(\hat{\rho}(t)))\xi(t) \tag{7.31}$$

where $\hat{\rho}(t)$ is the estimation of ρ, $C_{Ka}(\hat{\rho}(t)) = \sum_{j=1}^{m} C_{Kaj}\hat{\rho}_j(t)$ and $C_{Kb}(\hat{\rho}(t)) = \sum_{j=1}^{m} C_{Kbj}\hat{\rho}_j(t)$.

By Lemma 7.1, the saturated linear feedback, with $\xi(t) \in \wp(H(\hat{\rho}(t)))$, can be expressed as

$$\sigma(C_K(\hat{\rho}(t))\xi(t)) = \sum_{i=0}^{2^m-1} \eta_i[D_i C_K(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]\xi(t) \tag{7.32}$$

for some scalars $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m-1]$, such that $\sum_{i=0}^{2^m-1} \eta_i = 1$, and the following equality holds

$$
\begin{aligned}
(I-\rho)\sigma(u(t)) \quad & = \sum_{i=0}^{2^m-1} \eta_i[(I-\rho)D_iC_{K0} + D_iC_{Ka}(\rho) \\
& -\rho D_iC_{Ka}(\hat{\rho}) + (I-\hat{\rho}(t))D_iC_{Kb}(\hat{\rho}(t)) + D_iC_{Ka}(\tilde{\rho}(t)) \\
& +\tilde{\rho}D_iC_{Kb}(\hat{\rho}(t)) + (I-\rho)D_i^-H_{K0} + D_i^-H_{Ka}(\rho) \\
& -\rho D_i^-H_{Ka}(\hat{\rho}) + (I-\hat{\rho}(t))D_i^-H_{Kb}(\hat{\rho}(t)) \\
& +D_i^-H_{Ka}(\tilde{\rho}(t)) + \tilde{\rho}D_i^-H_{Kb}(\hat{\rho}(t))]\xi(t) \qquad (7.33)
\end{aligned}
$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. It should be noted that though $C_{Ka}(\hat{\rho}(t))$ and $C_{Kb}(\hat{\rho}(t))$ have the same forms, we deal with them in different ways in (7.33), which gives more freedom and less conservativeness.

Let $V(t) = x^TPx + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}$. If $\dot{V}(t) < 0$ for all $x \in \varepsilon^*(P,\delta)\backslash\{0\}$, the domain $\varepsilon^*(P,\delta)$ is contractively invariant. Clearly, if $\varepsilon^*(P,\delta)$ is contractively invariant, then it is inside the domain of attraction.

We note that the scalars η_i's are functions of ξ and $\hat{\rho}$ and their values are available in real-time. These scalars in a way reflect the severity of control saturation. In general, there are multiple choices of η_i's satisfying the same constraint, leading to nonunique representation of (7.32).

Now, by Lemma 7.2 we provide one choice of such η_i's, which are Lipschitzian functions in ξ and $\hat{\rho}$ and thus are particularly useful in our controller design.

$$
\eta_i(\xi(t),\ \hat{\rho}(t)) \quad = \prod_{j=1}^{m} [z_j(1-\lambda_j(\xi(t),\ \hat{\rho}(t))) + (1-z_j)\lambda_j(\xi(t),\ \hat{\rho}(t))] \qquad (7.34)
$$

By using the functions $\eta_i(\xi(t),\ \hat{\rho}(t))'$s, the output feedback controller (7.30) can be parameterized as

$$
\begin{aligned}
\dot{\xi}(t) \quad & = (\sum_{i=0}^{2^m-1} \eta_i A_{Ki}(\hat{\rho}))\xi(t) + (\sum_{i=0}^{2^m-1} \eta_i B_{Ki}(\hat{\rho}))y(t) \\
u(t) \quad & = (I-\rho)\sigma(C_K(\hat{\rho})\xi(t)) \qquad (7.35)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{Ki}(\hat{\rho}) &= A_{Ki0} + A_{Kia}(\hat{\rho}) + A_{Kib}(\hat{\rho}) \\
B_{Ki}(\hat{\rho}) &= B_{Ki0} + B_{Kia}(\hat{\rho}) + B_{Kib}(\hat{\rho}) \\
C_K(\hat{\rho}) &= C_{K0} + C_{Ka}(\hat{\rho}) + C_{Kb}(\hat{\rho}) \\
B_{Kia}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j B_{Kiaj},\ B_{Kib}(\hat{\rho}) = \sum_{j=1}^{m} \hat{\rho}_j B_{Kibj} \\
C_{Ka}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j C_{Kaj},\ C_{Kb}(\hat{\rho}) = \sum_{j=1}^{m} \hat{\rho}_j C_{Kbj} \\
A_{Kia}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j A_{Kiaj} \\
A_{Kib}(\hat{\rho}) &= \sum_{j=1}^{m}\sum_{s=1}^{m} \hat{\rho}_j\hat{\rho}_s A_{Kibjs} + \sum_{j=1}^{m} \hat{\rho}_j A_{Kibj}
\end{aligned}
$$

Motivated by the quasi-LPV structure of both the plant and the controller, we consider the following auxiliary LPV system, if $\varepsilon(P,\ \delta) \subset \wp([0\ H(\hat{\rho})])$ is an invariant set.

$$\dot{x}_e(t) = A_e(\eta)x_e(t) = \sum_{i=0}^{2^m-1} \eta_i(A_{ei}x_e(t)), \quad \eta \in \Gamma \tag{7.36}$$

where $x_e = [x^T(t)\ \xi^T(t)]^T$, $\eta = [\eta_0,\ \eta_1,\ \cdots,\ \eta_{2^m-1}]$, and

$$\Gamma = \{\eta \in R^{2^m} :\ \sum_{i=0}^{2^m-1} \eta_i = 1,\ 0 \leq \eta_i \leq 1,\ i \in I[0,\ 2^m - 1]\}$$

$$A_{ei} = \begin{bmatrix} A & B_2(I-\rho)[D_iC_K(\hat{\rho}) + D_i^- H(\hat{\rho})] \\ B_{Ki}(\hat{\rho})C & A_{Ki}(\hat{\rho}) \end{bmatrix}$$

The following theorem establishes conditions on the *output-feedback controller* coefficient matrices under which the LPV system (7.36) is asymptotically stable with Lyapunov function.

Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m)\ :\ \hat{\rho}_j \in\ \{\min_q\{\underline{\rho}_j^q\},\ \max_q\{\overline{\rho}_j^q\}\},\ q \in \mathbf{I}[1, L]\}$$

and $B^j = [0 \cdots b^j \cdots 0]$ with $B = [b^1 \cdots b^m]$.

Theorem 7.2 *$\varepsilon^*(P, \delta)$ is a contractively invariant set for normal and actuator failure cases, if there exist matrices $0 < N_1 < Y_1$, A_{Ki0}, A_{Kiaj}, A_{Kibjs}, B_{Ki0}, B_{Kiaj}, B_{Kibj}, C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}, $j \in \mathbf{I}[1, m]$, $s \in \mathbf{I}[1, m]$ and symmetric matrixes Θ^i, $i \in \mathbf{I}[0, 2^m - 1]$ with*

$$\Theta^i = \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ \Theta_{12}^{iT} & \Theta_{22}^i \end{bmatrix}$$

and Θ_{11}^i, $\Theta_{22}^i \in R^{m(2n)\times m(2n)}$ such that the following inequalities hold for all $D_i \in \mathbf{D}$ and $\varepsilon^*(P, \delta) \subset \wp([0\ H(\hat{\rho})])$, i.e., $|[0\ H(\hat{\rho})]_j x_e| \leq 1$ for all $x_e \in \varepsilon^*(P, \delta), j \in \mathbf{I}[1, m]$.

$$\Theta_{22jj}^i \leq 0,\ \ j \in \mathbf{I}[1, m], i \in \mathbf{I}[0, 2^m - 1]$$

$$\Theta_{11}^i + \Theta_{12}^i\Delta(\hat{\rho}) + (\Theta_{12}^i\Delta(\hat{\rho}))^T + \Delta(\hat{\rho})\Theta_{22}^i\Delta(\hat{\rho}) \geq 0,\ \hat{\rho} \in \Delta_{\hat{\rho}}$$

$$\begin{bmatrix} Q_i & R_i \\ R_i^T & S_i \end{bmatrix} + G^T\Theta^i G < 0,\ \ i \in \mathbf{I}[0, 2^m - 1],$$
$$\rho \in \{\rho^1\ \cdots\ \rho^L\},\ \rho^q \in N_{\rho^q} \tag{7.37}$$

where

$$R_i = \begin{bmatrix} R_{i1} & R_{i2} & \cdots & R_{im} \end{bmatrix}$$
$$Q_i = \begin{bmatrix} Y_1A - N_1B_{Ki0}C + (Y_1A - N_1B_{Ki0}C)^T & T_{1i} \\ * & T_{2i} \end{bmatrix}$$
$$R_{ij} = \begin{bmatrix} -N_1B_{Kibj}C - N_1B_{Kiaj}C & T_{3i} \\ N_1B_{Kibj}C + N_1B_{Kiaj}CS\begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix} & T_{4i} \end{bmatrix}$$
$$S_i = [S_{ijs}],\ j,\ s \in \mathbf{I}[1,m], \quad S_{ijs} = \begin{bmatrix} 0 & T_{5i} \\ T_{6i} & T_{7i} \end{bmatrix}$$

with

$$\begin{aligned}
T_{1i} &= Y_1B[(I-\rho)(D_iC_{K0} + D_i^- H_{K0}) + D_iC_{Ka}(\rho) + D_i^- H_{Ka}(\rho)] \\
&\quad - N_1A_{Ki0} - N_1A_{Kia}(\rho) + \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix}^T S^T[-Y_1B_2(D_iC_{Ka}(\rho) \\
&\quad + D_i^- H_{Ka}(\rho)) + N_1A_{Kia}(\rho)] + (-N_1A + N_1B_{Ki0}C + N_1B_{Kia}(\rho)C \\
&\quad - [N_1B_{Kia}(\rho)CS]\begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix})^T \\
T_{2i} &= -N_1B[(I-\rho)(D_iC_{K0} + D_i^- H_{K0}) + D_iC_{Ka}(\rho) + D_i^- H_{Ka}(\rho)] \\
&\quad + (-N_1B[(I-\rho)(D_iC_{K0} + D_i^- H_{K0}) + D_iC_{Ka}(\rho) + D_i^- H_{Ka}(\rho)])^T \\
&\quad + N_1A_{Ki0} + N_1A_{Kia}(\rho) + (N_1A_{Ki0} + N_1A_{Kia}(\rho))^T \\
T_{3i} &= Y_1B[-\rho(D_iC_{Kaj} + D_i^- H_{Kaj}) + D_iC_{Kbj} + D_i^- H_{Kbj}] \\
&\quad - N_1A_{Kibj} + \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix}^T S^T[Y_1B((D_iC_{Kaj} + D_i^- H_{Kaj}) \\
&\quad - \rho(D_iC_{Kbj} + D_i^- H_{Kbj})) - N_1A_{Kiaj}] \\
T_{4i} &= N_1B\rho(D_iC_{Kaj} + D_i^- H_{Kaj}) - N_1B(D_iC_{Kbj} + D_i^- H_{Kbj}) + N_1A_{Kibj} \\
T_{5i} &= -Y_1B^j(D_iC_{Kbs} + D_i^- H_{Kbs}) - N_1A_{Kibjs} \\
&\quad + \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix}^T S^TY_1B^j(D_iC_{Kbs} + D_i^- H_{Kbs}) \\
T_{6i} &= (-Y_1B^s(D_iC_{Kbj} + D_i^- H_{Kbj}) - N_1A_{Kibsj} \\
&\quad + \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix}^T S^TY_1B^s(D_iC_{Kbj} + D_i^- H_{Kbj}))^T \\
T_{7i} &= N_1B^j(D_iC_{Kbs} + D_i^- H_{Kbs}) + N_1A_{Kibjs} \\
&\quad + [N_1B^j(D_iC_{Kbs} + D_i^- H_{Kbs}) + N_1A_{Kibjs}]^T
\end{aligned}$$

$$G = \begin{bmatrix} \begin{bmatrix} I_{(2n)\times(2n)} \\ \cdots \\ I_{(2n)\times(2n)} \end{bmatrix} & 0 \\ 0 & I_{m(2n)\times m(2n)} \end{bmatrix}$$

$$\Delta(\hat{\rho}) = diag[\hat{\rho}_1 I_{(2n)\times(2n)} \ \cdots \ \hat{\rho}_m I_{(2n)\times(2n)}].$$

and also $\hat{\rho}_j(t)$ is determined according to the adaptive law

$$\begin{aligned} \dot{\hat{\rho}}_j &= Proj_{[\min\limits_q\{\underline{\rho}_j^q\},\ \max\limits_q\{\overline{\rho}_j^q\}]}\{L_{1j}\} \\ &= \begin{cases} 0, \ if & \begin{matrix} \hat{\rho}_j = \min\limits_q\{\underline{\rho}_j^q\} \ and \ L_{1j} \le 0 \\ or \ \hat{\rho}_j = \max\limits_q\{\overline{\rho}_j^q\} \ and \ L_{1j} \ge 0 \end{matrix} \\ & L_{1j}, \quad otherwise \end{cases} \end{aligned} \tag{7.38}$$

where

$$\begin{aligned} L_{1j} = l_j \sum\nolimits_{i=0}^{2^m-1} \eta_i \{ & \xi^T O_1 [A_{Kiaj} - BD_i C_{Kaj} - B^j D_i C_{Kb}(\hat{\rho}) - BD_i^- H_{Kaj} \\ & - B^j D_i^- H_{Kb}(\hat{\rho})]\xi + \begin{bmatrix} y \\ 0 \end{bmatrix}^T S^T [M_1 (BD_i C_{Kaj} + B^j D_i C_{Kb}(\hat{\rho}) \\ & + BD_i^- H_{Kaj} + B^j D_i^- H_{Kb}(\hat{\rho})) - O_1 A_{Kiaj}]\xi + \xi^T O_1 B_{Kiaj} CS \begin{bmatrix} y \\ 0 \end{bmatrix} \}, \end{aligned}$$

$M_1 = \delta Y_1$, $O_1 = \delta N_1$. $l_j > 0 (j \in \mathbf{I}[1, m])$ *and* $\delta > 0$ *are the adaptive law gains to be chosen according to practical applications.*

Proof 7.2 *Choose the following Lyapunov function*

$$V = x_e^T P x_e + \sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}, \tag{7.39}$$

By $\tilde{\rho}(\mathrm{t}) = \hat{\rho}(t) - \rho$ *and*

$$B_{Kia}(\tilde{\rho}) = B_{Kia}(\hat{\rho}) - B_{Kia}(\rho)$$
$$A_{Kia}(\tilde{\rho}) = A_{Kia}(\hat{\rho}) - A_{Kia}(\rho)$$

Then A_{ei} *can be written as*

$$
\begin{aligned}
A_{ei} &= A_{ei1} + A_{ei2} + A_{ei3} \\
A_{ei1} &= \begin{bmatrix} A & A_{ei1a} \\ [B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})]C & A_{ei1b} \end{bmatrix} \\
A_{ei1a} &= B[(I-\rho)D_iC_{K0} + D_iC_{Ka}(\rho) - \rho D_iC_{Ka}(\hat{\rho}) \\
&\quad + (I-\hat{\rho})D_iC_{Kb}(\hat{\rho}) + (I-\rho)D_i^- H_{K0} + D_i^- H_{Ka}(\rho) \\
&\quad - \rho D_i^- H_{Ka}(\hat{\rho}) + (I-\hat{\rho})D_i^- H_{Kb}(\hat{\rho})] \\
A_{ei1b} &= A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho}) \\
A_{ei2} &= \begin{bmatrix} 0 & A_{ei2a} \\ 0 & A_{Kia}(\tilde{\rho}) \end{bmatrix}, A_{ei3} = \begin{bmatrix} 0 & 0 \\ B_{Kia}(\tilde{\rho})C & 0 \end{bmatrix} \\
A_{ei2a} &= BD_iC_{Ka}(\tilde{\rho}) + B\tilde{\rho}D_iC_{Kb}(\hat{\rho}) + BD_i^- H_{Ka}(\tilde{\rho}) + B\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})
\end{aligned}
$$

Let P *be of the following form*

$$
P = \begin{bmatrix} M_1 & -O_1 \\ -O_1 & O_1 \end{bmatrix}
$$

with $0 < O_1 < M_1$*, which implies* $P > 0$*. Since* C *is of full rank, and* C *satisfies* $CC^{\perp T} = 0$ *and* $C^{\perp}C^{\perp T}$ *nonsingular, it follows that* $\begin{bmatrix} C \\ C^{\perp} \end{bmatrix}$ *is nonsingular. From (7.28), we have*

$$
Cx = y, \; C^{\perp}x = C^{\perp}x, \; x = S\begin{bmatrix} y \\ C^{\perp}x \end{bmatrix} \tag{7.40}
$$

where $S = \begin{bmatrix} C \\ C^{\perp} \end{bmatrix}^{-1}$*. Then, we have* $PA_{ei2} = \begin{bmatrix} 0 & W_{ai} \\ 0 & W_{bi} \end{bmatrix}$ *with*

$$
\begin{aligned}
W_{ai} &= M_1[BD_iC_{Ka}(\tilde{\rho}) + B\tilde{\rho}D_iC_{Kb}(\hat{\rho}) + BD_i^- H_{Ka}(\tilde{\rho}) + B\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})] \\
&\quad - O_1A_{Kia}(\tilde{\rho}) \\
W_{bi} &= O_1[A_{Kia}(\tilde{\rho}) - BD_iC_{Ka}(\tilde{\rho}) - B\tilde{\rho}D_iC_{Kb}(\hat{\rho}) - BD_i^- H_{Ka}(\tilde{\rho}) \\
&\quad - B\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})]
\end{aligned}
$$

which follows

$$
[x^T \; \xi^T]PA_{ei2}[x^T \; \xi^T]^T = x^TW_{ai}\xi + \xi^TW_{bi}\xi
$$

Thus, by (7.40), we have

$$
x^TW_{ai}\xi = \begin{bmatrix} y \\ 0 \end{bmatrix}^T S^TW_{ai}\xi + [x^T \; \xi^T]A_{ai1}[x^T \; \xi^T]^T
$$

where

$$
A_{ai1} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix}^T S^TW_{ai} \\ 0 & 0 \end{bmatrix},
$$

In the same way, from (7.40) we get

$$[x^T\ \xi^T]PA_{ei3}[x^T\ \xi^T]^T = -x^T O_1 B_{Kia}(\tilde{\rho})Cx + \xi^T O_1 B_{Kia}(\tilde{\rho})Cx$$
$$= x_e^T A_{ai2} x_e + M_{ai2}$$

where

$$A_{ai2} = \begin{bmatrix} -O_1 B_{Kia}(\tilde{\rho})C & 0 \\ O_1 B_{Kia}(\tilde{\rho})CS \begin{bmatrix} 0 \\ C^{\perp} \end{bmatrix} & 0 \end{bmatrix}$$
$$M_{ai2} = \xi^T O_1 B_{Kia}(\tilde{\rho})CS \begin{bmatrix} y \\ 0 \end{bmatrix}$$

Then from the derivative of $V(t)$ along the closed-loop system (7.36), it follows

$$\dot{V}(t) = 2x_e^T \sum_{i=0}^{2^m-1} \eta_i P A_{ei} x_e + 2\sum_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j}$$
$$= x_e^T W_0 x_e + W_1$$

where

$$W_0 = \sum_{i=0}^{2^m-1} \eta_i [PA_{ei1} + (PA_{ei1})^T]$$
$$+ \sum_{i=0}^{2^m-1} \eta_i [A_{ai1} + A_{ai2} + (A_{ai1} + A_{ai2})^T]$$
$$W_1 = 2\xi^T \sum_{i=0}^{2^m-1} \eta_i W_{bi}\xi + 2\begin{bmatrix} y \\ 0 \end{bmatrix}^T S^T \sum_{i=0}^{2^m-1} \eta_i W_{ai}\xi$$
$$+ 2\sum_{i=0}^{2^m-1} \eta_i M_{ai2} + 2\sum_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j}$$

The design condition that $\dot{V}(t) \le 0$ is reduced to

$$W_0 < 0, \tag{7.41}$$

$$W_1 \le 0 \tag{7.42}$$

Since y and ξ are available on line, the adaptive laws can be chosen as (7.38) for rendering (7.42) valid. (7.41) is equivalent to

$$\sum_{i=0}^{2^m-1} \eta_i \{XA_{ei1} + A_{ai1}^* + A_{ai2}^*$$
$$+ [XA_{ei1} + A_{ai1}^* + A_{ai2}^*]^T\} < 0 \tag{7.43}$$

where

$$X = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix} = \frac{P}{\delta},\ A_{ai1}^* = \frac{1}{\delta}A_{ai1},\ A_{ai2}^* = \frac{1}{\delta}A_{ai2}$$

Notice that

$$XA_{ei1} = \begin{bmatrix} Y_1A - N_1[B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})]C & W_c \\ -N_1A + N_1[B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})]C & W_d \end{bmatrix}$$

$$\begin{aligned} W_c = {} & Y_1B[(I-\rho)D_iC_{K0} + D_iC_{Ka}(\rho) - \rho D_iC_{Ka}(\hat{\rho}) \\ & + (I-\hat{\rho})D_iC_{Kb}(\hat{\rho}) + (I-\rho)D_i^-H_{K0} + D_i^-H_{Ka}(\rho) \\ & - \rho D_i^-H_{Ka}(\hat{\rho}) + (I-\hat{\rho})D_i^-H_{Kb}(\hat{\rho})] \\ & - N_1[A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho})] \\ W_d = {} & -N_1B[(I-\rho)D_iC_{K0} + D_iC_{Ka}(\rho) - \rho D_iC_{Ka}(\hat{\rho}) \\ & + (I-\hat{\rho})D_iC_{Kb}(\hat{\rho}) + (I-\rho)D_i^-H_{K0} + D_i^-H_{Ka}(\rho) \\ & - \rho D_i^-H_{Ka}(\hat{\rho}) + (I-\hat{\rho})D_i^-H_{Kb}(\hat{\rho})] \\ & + N_1[A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho})] \end{aligned}$$

Furthermore (7.43) can be described by

$$\begin{aligned} W_2(\hat{\rho}) = {} & \sum_{i=0}^{2^m-1} \eta_i\{Q_i + \sum_{j=1}^{m} \hat{\rho}_jR_{ij} + (\sum_{j=1}^{m} \hat{\rho}_jR_{ij})^T \\ & + \sum_{j=1}^{m}\sum_{s=1}^{m} \hat{\rho}_j\hat{\rho}_sS_{ijs}\} < 0 \end{aligned}$$

where Q_i, R_{ij}, S_{ijs}, $j,s \in \mathbf{I}[1,m]$ are defined in (7.37). By Lemma 1, we can get $W_2(\hat{\rho}) < 0$ if (7.37) holds, which implies $W_0 < 0$. Together with adaptive laws (7.38), it follows that $\dot{V}(t) < 0$ for any $x_e \in \wp([0\ H(\hat{\rho})])$, $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^q \in N_{\rho^q}$ and $\hat{\rho}$ satisfying (7.38).

If we take the following output-feedback controller with fixed parameter matrices A_{Ki0}, B_{Ki0}, C_{K0}, $i \in \mathbf{I}[0, 2^m-1]$

$$\begin{aligned} \dot{\xi}(t) &= (\sum_{i=0}^{2^m-1} \eta_iA_{Ki0})\xi(t) + (\sum_{i=0}^{2^m-1} \eta_iB_{Ki0})y(t) \\ u(t) &= (I-\rho)\sigma(C_{K0}\xi(t)) \end{aligned} \tag{7.44}$$

then combining (7.44) with (7.28), it follows:

$$\dot{x}_{e1}(t) = A_{e1}(\eta)x_{e1}(t) \tag{7.45}$$

$$A_{e1}(\eta) = \sum_{i=0}^{2^m-1} \eta_i(A_{e1i}x_{e1}(t)), \quad \eta \in \Gamma \tag{7.46}$$

where $x_{e1} = [x^T(t)\ \xi^T(t)]^T$,

$$A_{e1i} = \begin{bmatrix} A & B_2(I-\rho)[D_iC_{K0} + D_i^-H_0] \\ B_{Ki0}C_2 & A_{Ki0} \end{bmatrix}$$

Based on system (7.45), the following lemma is presented.

Lemma 7.3 *Consider the closed-loop system described by (7.45), we have that the following statements are equivalent:*

(i) there exist a symmetric matrix $X > 0$ and controller $\mathbf{K}$ described by (7.44) such that

$$A_{e1i}^T X + X A_{e1i} < 0$$

holds for $\rho \in \{\rho^1 \ \cdots \ \rho^L\}$, $\rho^q \in N_{\rho^q}$

(ii) there exist symmetric matrices Y_1 and N_1 with $0 < N_1 < Y_1$, and a controller described by (7.44) with $A_{Ki0} = A_{Kei0}$, $B_{Ki0} = B_{Kei0}$, $C_{K0} = C_{Ke0}$, $H_0 = H_{e0}$, $i \in \mathbf{I}[0, 2^m - 1]$ such that

$$\begin{bmatrix} Y_1 A - N_1 B_{Ki0} C + (Y_1 A - N_1 B_{Ki0} C)^T & T_0 \\ * & T_1 \end{bmatrix} < 0 \tag{7.47}$$

with

$$\begin{aligned} T_0 &= Y_1 B_2 (I - \rho)[D_i C_{K0} + D_i^- H_0] - N_1 A_{Ki0} \\ &\quad + (-N_1 A + N_1 B_{Ki0} C)^T \\ T_1 &= -N_1 B_2 (I - \rho)[D_i C_{K0} + D_i^- H_0] + N_1 A_{Ki0} \\ &\quad + (-N_1 B_2 (I - \rho)[D_i C_{K0} + D_i^- H_0] + N_1 A_{Ki0})^T \end{aligned}$$

Proof 7.3 *The proof is similar to the proof of Theorem 5.2. To avoid overlap, it is omitted.*

Next, a theorem is given to show that the condition in Theorem 7.2 for the adaptive controller design is more relaxed than that in Lemma 7.3 for the traditional controller design with fixed parameter matrices.

Theorem 7.3 *If condition (i) or (ii) in Lemma 7.3 holds, then the condition of Theorem 7.2 holds.*

Proof 7.4 *If condition (i) or (ii) in Lemma 7.3 holds, then it is easy to see that the condition in Theorem 7.2 is feasible with $A_{Kiaj} = A_{Kibj} = A_{Kibjs} = B_{Kiaj} = B_{Kibj} = C_{Kaj} = C_{Kbj} = H_{Kaj} = H_{Kbj} = 0$, $i \in \mathbf{I}[0, 2^m - 1]$, $j \in \mathbf{I}[1, m]$, $s \in \mathbf{I}[1, m]$. The proof is completed.*

7.3.3 Controller Design

From Theorem 7.2, we can obtain various controller gains and domains satisfying the set invariance condition. So, how to choose the "largest" one of them becomes an interesting problem. In this section, we will give a method to find the "largest" domain.

In Theorem 7.2, a condition for the set $\varepsilon^*(P, \delta)$ to be inside the domain of attraction is given. With the above shape reference sets, we can choose from all the $\varepsilon^*(P,\delta)$'s that satisfy the condition of Theorem 7.2 such that the quantity $\alpha_R(\varepsilon^*(P,\delta))$ is maximized. The problem can be formulated as follows

$$\begin{array}{lll} & \sup & \alpha \\ s.t. & (a) & \alpha X_R \subset \varepsilon^*(P,\delta), \\ & (b) & (7.37), \\ & (c) & \varepsilon^*(P,\delta) \subset \wp([0 \ H(\hat{\rho})]). \end{array} \tag{7.48}$$

However, by Definition 7.4, we know that (a) and (c) cannot be shown as LMIs directly. Then the following proposition will solve this problem.

Proposition 7.3 *Obviously, $\varepsilon^*(P,\delta) \subset \varepsilon(P,\delta)$, which implies that (c) holds if (c1) holds, where*

$$(c1) \ \ \varepsilon(P,\delta) \subset \wp([0 \ H(\hat{\rho})]). \tag{7.49}$$

By Definition 7.4, we have

$$x_e^T P x_e + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j} \leq \delta \Leftrightarrow x_e^T \frac{P}{\delta} x_e + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{\delta l_j} \leq 1.$$

Let $F(t) = \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{\delta l_j}$. Then, by (7.38) and (7.3), it follows that $\tilde{\rho}_j(t) \leq \max_j\{\overline{\rho}_j^q\} - \min_j\{\underline{\rho}_j^q\}$. We can choose l_j and δ sufficiently large so that $F(t)$ is sufficiently small. Then the conclusion can be drawn as follows:

For system (7.29) and controller (7.30) there must exist $\delta > 0$ and $l_i > 0$ such that the closed-loop system (7.36) is asymptotically stable in domain $\varepsilon^-(P,\delta)$ if (b) and (c1) hold.

Then we can get the "largest" domain of asymptotic stability by solving the following optimization problem

$$\begin{array}{lll} & \sup & \alpha \\ s.t. & (a1) & \alpha X_R \subset \varepsilon(P,\delta), \\ & (b) & \\ & (c1) & \end{array} \tag{7.50}$$

If the given shape reference set X_R is a polyhedron as defined in Definition 7.5, then Constraint (a1) is equivalent to

$$\alpha^2 x_q^T (\frac{P}{\delta}) x_q \leq 1 \Leftrightarrow \begin{bmatrix} 1/\alpha^2 & x_q^T(\frac{P}{\delta}) \\ (\frac{P}{\delta})x_q & (\frac{P}{\delta}) \end{bmatrix} \geq 0, \tag{7.51}$$

for all $q \in \mathbf{I}[1,l]$. If X_R is a ellipsoid $\varepsilon(R,1)$, then (a1) is equivalent to

$$\frac{R}{\alpha^2} \geq \frac{P}{\delta} \Leftrightarrow \begin{bmatrix} (1/\alpha^2)R & (\frac{P}{\delta}) \\ (\frac{P}{\delta}) & (\frac{P}{\delta}) \end{bmatrix} \geq 0. \tag{7.52}$$

Condition (c1) is equivalent to

$$\delta[0\ h(\hat{\rho})]_j P^{-1}[0\ h(\hat{\rho})]_j^T \leq 1 \Leftrightarrow \begin{bmatrix} 1 & [0\ h(\hat{\rho})]_j \\ * & (\frac{P}{\delta}) \end{bmatrix} \geq 0. \tag{7.53}$$

for all $j \in \mathbf{I}[1, m]$, where $[0\ h(\hat{\rho})]_j$ is the jth row of $[0\ H(\hat{\rho})]$. We have that (7.52) is equivalent to the following inequalities.

$$(c2) \begin{bmatrix} -1 & -[0\ H_{K0s}] \\ * & -X \end{bmatrix} + \sum_{j=1}^{m} \hat{\rho}_j \begin{bmatrix} 0 & [0\ -H_{Kajs} - H_{Kbjs}] \\ * & 0 \end{bmatrix} \leq 0, \hat{\rho} \in \Delta_{\hat{\rho}}$$

where H_{Kajs} is the sth row of H_{Kaj}, $s \in \mathbf{I}[1, m]$.

If X_R is a polyhedron, then from (7.49) and (7.52), the optimization problem (7.49) is equivalent to

$$\begin{array}{ll} \inf & \gamma \\ s.t. & (a2) \begin{bmatrix} \gamma & x_q^T X \\ X x_q & X \end{bmatrix} \geq 0, \quad q \in \mathbf{I}[1,\ l], \\ & (b), \quad (c2), \end{array} \tag{7.54}$$

where $\gamma = 1/\alpha^2$.

If X_R is an ellipsoid, we need only to replace (a2) with

$$(a3) \begin{bmatrix} \gamma R & X \\ X & X \end{bmatrix} \geq 0. \tag{7.55}$$

It should be noted that condition (7.37) is not convex. But when $C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}$ are given, they become LMIs.

From Theorem 7.2, we have the following algorithm to design the adaptive output feedback controller.

Algorithm 7.1

Step 1 Suppose that all states of system (7.28) can be measured. Minimize the index γ to design the state-feedback controller.

Then, the matrices C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj} can be given.
Step 2 Solve the following optimization problem

$$\begin{array}{ll} \inf & \gamma \\ s.t. & (a2), \quad (b), \quad (c2) \end{array} \tag{7.56}$$

Then the resulting A_{Ki0}, A_{Kiaj}, A_{Kibjs}, B_{Ki0}, B_{Kiaj}, B_{Kibj}, C_{K0}, C_{Kaj}, C_{Kbj}, $i \in \mathbf{I}[0, 2^m - 1]$, $j \in \mathbf{I}[1, m]$, $s \in \mathbf{I}[1, m]$ will form the dynamic output feedback controller gains.

Remark 7.5 *Step 1 is to determine matrices C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}, which solves the corresponding adaptive controller design problem via state feedback. This procedure is adopted from the last section, and convex conditions are described. To avoid overlap, the conditions appearing in Step 1 will be omitted.*

From Lemma 7.3, we have the following algorithm to design the fault-tolerant controller with fixed gains.

Algorithm 7.2

Step 1 Suppose that all states of system (7.28) can be measured. Minimize the index γ to design the state-feedback controller.

Then, the matrices C_{K0}, H_{K0} can be given.

Step 2 Solve the following optimization problem

$$\begin{array}{ll} \inf & \gamma \\ s.t. & (a2), \quad (7.47), \quad (c2) \end{array} \tag{7.57}$$

Then the resulting A_{Ki0}, B_{Ki0}, C_{K0}, $i \in \mathbf{I}[0, 2^m - 1]$ will form the dynamic output feedback controller gains.

Remark 7.6 *Step 1 is to determine matrices C_{K0}, H_{K0}, which solves the corresponding controller design problem via state feedback.*

Remark 7.7 *In Step 1, for some cases, the magnitude of the designed gains C_{K0} (C_{Kaj} and C_{Kbj}) may be too large to be applied in Step 2. For solving the problem, by adding the following constraints, where Q and Y_{K0} are variables in conditions of Step 1*

$$Q > \alpha I, \quad Y_{K0} Y_{K0}^T < \beta I, \tag{7.58}$$

then the magnitude of C_{K0} can be reduced. In fact, by $C_{K0} = Y_{K0} Q^{-1}$ and (7.58), it follows that

$$\| C_{K0} \| < \sqrt{\beta}/\alpha.$$

The similar method can be used for the gains C_{Kaj} and C_{Kbj}.

7.3.4 Example

Example 7.3 *Consider the system of form (7.29) with*

$$A = \begin{bmatrix} 0.01 & 0.1 \\ 0.1 & 0.01 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad C = [1 \; 0]$$

and the following two possible fault modes:

Fault mode 1: Both of the two actuators are normal, that is,

$$\rho_1^1 = \rho_2^1 = 0$$

Fault mode 2: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^2 = 1, \quad 0 \leq \rho_2^2 \leq a,$$

where $a = 0.5$ denotes the maximal loss of effectiveness for the second actuator.

Let

$$R = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

After implementing Algorithm 7.2, we have that $\gamma^* = 1.9669$. When Algorithm 7.1 is used to design adaptive output-feedback controller, the optimal index is given as $\gamma^* = 0.7648$. Obviously, the optimal index γ is smaller for Algorithm 7.1. The phenomenon indicates the superiority of our adaptive method.

7.4 Conclusion

In this chapter, an adaptive fault-tolerant controllers design method has been presented for linear time-invariant systems with actuator saturation. The design is developed in the framework of *linear matrix inequality (LMI)* approach, which can enlarge the domain of asymptotic stability of closed-loop systems in the cases of actuator saturation and actuator failures. Two examples have been given to illustrate the efficiency of the design method.

8

ARC with Actuator Saturation and L_2-Disturbances

8.1 Introduction

The problem of disturbance rejection for linear systems subject to *actuator saturation* has been addressed by many authors ([63, 66, 97, 102, 142]). Under the boundedness assumption on the magnitude of the disturbances and in the absence of *initial condition*, the *L_2-gain* analysis and *minimization* in the context of both state and output feedback were carried out in [101, 102]. In [66], a method for analysis and *maximization* of an ellipsoid, which is invariant under magnitude bounded, but persistent disturbances, is proposed. The works of [63, 97, 109, 120, 127] all consider the situation where disturbances are bounded in energy. The works of [63, 109, 120] formulated and solved the problem of *stability* analysis and design as an *optimization problem* with LMI or BMI constraints. In [67, 68], authors presented LMI-based synthesis tools for regional stability and performance of linear *anti-windup compensators* for linear control systems. [32] presents a method for the analysis and control design of linear systems in the presence of *actuator saturation* and L_2 disturbances.

This chapter deals with the problem of designing adaptive reliable H_∞ controllers (ARC). The *actuator fault model*, which covers the outage cases and the possibility of partial faults, is considered. The disturbance tolerance ability of the closed-loop system is measured by an *optimal index*. Based on the online estimation of eventual faults, the adaptive *fault-tolerant* controller parameters are updating automatically to compensate the fault effects on systems. The designs are developed in the framework of linear matrix inequality (LMI) approach, which can guarantee the disturbance tolerance ability and adaptive H_∞ *performances* of closed-loop systems in the cases of actuator saturation and *actuator failures.*

8.2 State Feedback

8.2.1 Problem Statement

Consider an LTI plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + B_2\sigma(u), \\ z(t) &= Cx(t) + D\sigma(u), \end{aligned} \tag{8.1}$$

where $x(t) \in R^n$ is the plant state, $\sigma(u) \in R^m$ is the saturated control input, $z(t) \in R^s$ is the regulated output and $\omega(t) \in R^d$ is an exogenous disturbance in $L_2[0, \infty]$, respectively. A, B_1, B_2, C, D, are known constant matrices of appropriate dimensions.

To formulate the *fault-tolerant control* problem, the considered actuator failures are the same as those in Chapter 3, that is

$$u_{jq}^F(t) = (1 - \rho_j^q)\sigma(u_j(t)), \quad 0 \le \underline{\rho}_j^q \le \rho_j^q \le \overline{\rho}_j^q, \\ j \in \mathbf{I}[1, m],\ q \in \mathbf{I}[1, L], \tag{8.2}$$

For convenience in the following sections, for all possible fault modes L, the following uniform actuator fault model is exploited:

$$u^F(t) = (I - \rho)\sigma(u(t)),\ \rho \in \{\rho^1 \ \cdots \rho^L\} \tag{8.3}$$

and ρ can be described by $\rho = \text{diag}[\rho_1,\ \rho_2,\ \cdots \rho_m]$.

Denote

$$N_{\rho^q} = \{\rho^q | \rho^q = \text{diag}[\rho_1^q,\ \rho_2^q,\ \cdots \rho_m^q], \rho_j^q = \underline{\rho}_j^q \text{ or } \rho_j^q = \overline{\rho}_j^q\}. \tag{8.4}$$

Thus, the set N_{ρ^q} contains a *maximum* of 2^m elements.

For a linear system, the disturbance rejection capability can be measured by the L_2 gain, the largest ratio between the L_2 norms of the output and the disturbance. However, this gain may not be well defined for closed-loop system and the *state feedback*, since a sufficiently large disturbance may drive the state and the output of the system unbounded. For this reason, we need to restrict our attention to the class of disturbances whose energy is bounded by a given value, *i.e.*,

$$\mathfrak{W}_\delta := \left\{ \omega :\ R_+ \to R^d :\ \int_0^\infty \omega^T(t)\omega(t)dt \le \delta \right\}. \tag{8.5}$$

The following problem will be considered in this section: The first question that needs to be answered is, what is the maximal value of δ such that the state will be bounded for all $\omega \in \mathfrak{W}_\delta$? Here we will consider the situation, zero initial state. The problem related to this question is referred to as disturbance tolerance. The disturbance rejection capability can be measured by the restricted L_2 gain over $\mathfrak{W}_\delta$. In this section we will consider L_2 gain and $\mathfrak{W}_\delta$ at the same time.

Remark 8.1 *For the above problem to be solvable, it is necessary for the pair $(A, B_2(I-\rho))$ to be stabilizable for each $\rho \in \{\rho^1 \cdots \rho^L\}$.*

8.2.2 ARC Controller Design

The dynamics with actuator faults (8.3) and saturation is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega + B_2(I-\rho)\sigma(u(t)), \\ z(t) &= Cx(t) + D(I-\rho)\sigma(u(t)). \end{aligned} \tag{8.6}$$

The *controller structure* is chosen as

$$\begin{aligned} u(t) &= K(\hat{\rho}(t))x(t) \\ &= (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t), \end{aligned} \tag{8.7}$$

where $\hat{\rho}(t)$ is the estimation of ρ,

$$K_a(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} K_{aj}\hat{\rho}_j(t),\ K_b(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} K_{bj}\hat{\rho}_j(t).$$

Remark 8.2 *From (8.7), we have that different from the fixed gain controller $u(t) = K_0x(t)$, controller (8.7) has two additional terms $K_a(\hat{\rho}(t))$ and $K_b(\hat{\rho}(t))$ which are functions of $\hat{\rho}$ and their values are available in real-time. Through the estimation of ρ, controller gains can be adjusted online, which gives more freedom and less conservativeness.*

By Lemma 7.1, the saturated linear feedback, with $x \in \wp(H(\hat{\rho}))$, can be expressed as

$$\sigma(K(\hat{\rho}(t))x(t)) = \sum\nolimits_{i=0}^{2^m-1} \eta_i[D_iK(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]x(t) \tag{8.8}$$

for some scalars $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m-1]$, such that $\sum_{i=0}^{2^m-1} \eta_i = 1$, and the following equality holds

$$\begin{aligned} (I-\rho)\sigma(u(t)) = \sum\nolimits_{i=0}^{2^m-1} &\eta_i[(I-\rho)D_iK_0 + D_iK_a(\rho) \\ &- \rho D_iK_a(\hat{\rho}) + (I-\hat{\rho}(t))D_iK_b(\hat{\rho}(t)) + D_iK_a(\tilde{\rho}(t)) \\ &+ \tilde{\rho}D_iK_b(\hat{\rho}(t)) + (I-\rho)D_i^- H_0 + D_i^- H_a(\rho) \\ &- \rho D_i^- H_a(\hat{\rho}) + (I-\hat{\rho}(t))D_i^- H_b(\hat{\rho}(t)) + D_i^- H_a(\tilde{\rho}(t)) \\ &+ \tilde{\rho}D_i^- H_b(\hat{\rho}(t))]x(t), \end{aligned} \tag{8.9}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. Though $K_a(\hat{\rho}(t))$ and $K_b(\hat{\rho}(t))$ have the same forms, we deal with them in different ways in (8.9), which gives more freedom and less conservativeness.

Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m) \ : \ \hat{\rho}_j \in \ \{\min_q\{\underline{\rho}_j^q\}, \ \max_q\{\overline{\rho}_j^q\}\}, \ q \in \mathbf{I}[1, L]\}$$

and $B^j = [0 \cdots b^j \cdots 0]$ with $B = [b^1 \cdots b^m]$.

We note that the scalars η_i's are functions of x and $\hat{\rho}$ and their values are available in real-time. These scalars in a way reflect the severity of control saturation. In general, there are multiple choices of η_i's satisfying the same constraint, leading to nonunique representation of (8.8).

Now, by Lemma 7.2 we provide one choice of such η_i's, which are Lipschitzian functions in ξ and $\hat{\rho}$ and thus are particularly useful in our controller design.

$$\eta_i(\xi(t), \ \hat{\rho}(t)) = \prod_{j=1}^{m} [z_j(1 - \lambda_j(\xi(t), \ \hat{\rho}(t))) + (1 - z_j)\lambda_j(\xi(t), \ \hat{\rho}(t))] \quad (8.10)$$

Then, η_i's are functions Lipschitz in x and $\hat{\rho}$, such that, $\sum_{i=0}^{2^m-1} \eta_i = 1$, $0 \leq \eta_i \leq 1$, $i \in \mathbf{I}[0, 2^m - 1]$. Moreover, they satisfy relation (8.8).

By using the functions $\eta_i(x(t), \ \hat{\rho}(t))$'s and controller (8.7), plant (8.6) can be written in a quasi-LPV form as follows:

$$\begin{aligned} \dot{x}(t) = Ax(t) + B_2 \sum_{i=0}^{2^m-1} \eta_i[(I - \rho)D_i(K_0 + K_a(\hat{\rho}(t)) \\ + K_b(\hat{\rho}(t))) + (I - \rho)D_i^-(H_0 + H_a(\hat{\rho}(t)) \\ + H_b(\hat{\rho}(t)))]x(t) + B_1\omega. \end{aligned} \quad (8.11)$$

In addition, we consider the following auxiliary LPV system, of which the closed-loop system comprising of (8.6) and (8.7) is a special case, for $\forall \ x(t) \in \varepsilon^*(P, \delta^*) \subset \wp(H(\hat{\rho}))$

$$\dot{x}(t) = A(\eta)x(t) + B_1\omega, \quad \eta \in \Gamma \quad (8.12)$$

where $\eta = [\eta_0, \ \eta_1, \ \cdots, \ \eta_{2^m-1}]$, and

$$\Gamma = \{\eta \in R^{2^m} \ : \ \sum_{i=0}^{2^m-1} \eta_i = 1, \ 0 \leq \eta_i \leq 1, \ i \in \mathbf{I}[0, \ 2^m - 1]\},$$

$$\begin{aligned} A(\eta) = A + B_2 \sum_{i=0}^{2^m-1} \eta_i[(I - \rho)D_iK_0 + D_iK_a(\rho) \\ - \rho D_iK_a(\hat{\rho}) + (I - \hat{\rho}(t))D_iK_b(\hat{\rho}(t)) + D_iK_a(\tilde{\rho}(t)) \\ + \tilde{\rho}D_iK_b(\hat{\rho}(t)) + (I - \rho)D_i^-H_0 + D_i^-H_a(\rho) \\ - \rho D_i^-H_a(\hat{\rho}) + (I - \hat{\rho}(t))D_i^-H_b(\hat{\rho}(t)) + D_i^-H_a(\tilde{\rho}(t)) \\ + \tilde{\rho}D_i^-H_b(\hat{\rho}(t))]. \end{aligned}$$

Before presenting the main result of this section, denote

$$\begin{aligned}
N_{0i} &= AX + B_2(I-\rho)D_iY_0 + (AX + B_2(I-\rho)D_iY_0)^T \\
&\quad + B_2\sum_{j=1}^{m}\rho_jD_iY_{aj} + (B_2\sum_{j=1}^{m}\rho_jD_iY_{aj})^T \\
&\quad + B_2(I-\rho)D_i^-O_0 + (B_2(I-\rho)D_i^-O_0)^T \\
&\quad + B_2\sum_{j=1}^{m}\rho_jD_i^-O_{aj} + (B_2\sum_{j=1}^{m}\rho_jD_i^-O_{aj})^T + B_1B_1^T, \\
G &= \begin{bmatrix} \begin{bmatrix} I_{n\times n} \\ \cdots \\ I_{n\times n} \end{bmatrix} & 0 \\ 0 & I_{mn\times mn} \end{bmatrix}, \\
Z_{1i} &= -B_2\rho D_iY_a + B_2D_iY_b - B_2\rho D_i^-O_a + B_2D_i^-O_b, \\
U_i &= [CX + D(I-\rho)D_iY_0 + D(I-\rho)D_i^-O_0 \\
&\quad D(I-\rho)(D_i(Y_a+Y_b) + D_i^-(O_a+O_b))], \\
Z_{2i} &= \begin{bmatrix} -B_2^1D_i \\ \ldots \\ -B_2^mD_i \end{bmatrix} Y_b + (\begin{bmatrix} -B_2^1D_i \\ \ldots \\ -B_2^mD_i \end{bmatrix} Y_b)^T \\
&\quad + \begin{bmatrix} -B_2^1D_i^- \\ \ldots \\ -B_2^mD_i^- \end{bmatrix} O_b + (\begin{bmatrix} -B_2^1D_i^- \\ \ldots \\ -B_2^mD_i^- \end{bmatrix} O_b)^T, \\
Y_a &= [Y_{a1}\ Y_{a2}\ldots.Y_{am}], \quad Y_b = [Y_{b1}\ Y_{b2}\ldots.Y_{bm}], \\
O_a &= [O_{a1}\ O_{a2}\ldots.O_{am}], \quad O_b = [O_{b1}\ O_{b2}\ldots.O_{bm}], \\
\Delta(\hat{\rho}) &= \mathrm{diag}[\hat{\rho}_1I_{n\times n}\ \cdots\ \hat{\rho}_mI_{n\times n}].
\end{aligned}$$

and the *adaptive law* is defined by

$$\begin{aligned}
\dot{\hat{\rho}}_j &= \mathrm{Proj}_{[\min_q\{\underline{\rho}_j^q\},\ \max_q\{\overline{\rho}_j^q\}]}\{L_{1j}\} \\
&= \begin{cases} 0, \text{ if} & \begin{matrix} \hat{\rho}_j = \min_q\{\underline{\rho}_j^q\} \text{ and } L_{1j} \le 0 \\ \text{or } \hat{\rho}_j = \max_q\{\overline{\rho}_j^q\} \text{ and } L_{1j} \ge 0 \end{matrix} \\ & L_{1j}, \quad \text{otherwise} \end{cases}
\end{aligned} \tag{8.13}$$

with

$$\begin{aligned}
L_{1j} &= -l_jx^T(t)[PB_2(\sum_{i=0}^{2^m-1}\eta_iD_i)K_{aj} + PB_2^j(\sum_{i=0}^{2^m-1}\eta_iD_i)K_b(\hat{\rho}) \\
&\quad + PB_2(\sum_{i=0}^{2^m-1}\eta_iD_i^-)H_{aj} + PB_2^j(\sum_{i=0}^{2^m-1}\eta_iD_i^-)H_b(\hat{\rho})]x(t), \\
P &= X^{-1}, K_{aj} = Y_{aj}X^{-1}, K_{bj} = Y_{bj}X^{-1}, H_{aj} = O_{aj}X^{-1}, H_{bj} = O_{bj}X^{-1}
\end{aligned}$$

where $l_j > 0 (j \in \mathbf{I}[1,m])$ are the adaptive law gain to be chosen according to practical applications. The matrices X, Y_0, Y_{aj}, Y_{bj}, O_0, O_{aj}, O_{bj}, $j \in \mathbf{I}[1,\ m]$, involved in above notations and definition are decision variables to be determined.

Theorem 8.1 *Let $r_f > 0, r_n > 0$ and $\delta > 0$ be given constants, then the following two conditions are satisfied*

(I) The trajectories of the closed-loop system that start from the origin will remain inside the domain $\varepsilon^*(P, \delta^*)$ for every $\omega \in \mathfrak{W}_\delta$.

(II) In normal case, i.e., $\rho = 0$,

$$\int_0^\infty z^T(t)z(t)dt \le r_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + r_n^2 \sum\nolimits_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j}, \quad \textit{for } x(0) = 0$$

and in actuator failures cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$,

$$\int_0^\infty z^T(t)z(t)dt \le r_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + r_f^2 \sum\nolimits_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j}, \quad \textit{for } x(0) = 0$$

where$\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t) \ \cdots \ \tilde{\rho}_m(t)\}$, $\tilde{\rho}_j(t) = \hat{\rho}_j(t) - \rho_j$, if there exist matrices $X > 0$, O_0, O_{aj}, O_{bj}, Y_0, Y_{aj}, Y_{bj}, $j \in \mathbf{I}[1, m]$ and symmetric matrices Θ_i, $i \in \mathbf{I}[0, 2^m - 1]$, with

$$\Theta^i = \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ \Theta_{12}^{iT} & \Theta_{22}^i \end{bmatrix}$$

and Θ_{11}^i, $\Theta_{22}^i \in R^{mn\times mn}$ such that the following inequalities (8.15) hold for all $D_i \in \mathbf{D}$, $\varepsilon^*(P, \delta^*) \subset \wp(H(\hat{\rho}))$, and the controller gain is given by

$$K(\hat{\rho}) = K_0 + \sum\nolimits_{j=1}^m \hat{\rho}_j K_{aj} + \sum\nolimits_{j=1}^m \hat{\rho}_j K_{bj}. \tag{8.14}$$

where $\hat{\rho}_j$ is determined according to the adaptive law (8.13), $K_0 = Y_0 X^{-1}$, $K_{aj} = Y_{aj} X^{-1}$, $K_{bj} = Y_{bj} X^{-1}$.

$$\Theta_{22jj}^i \le 0, \ \ j \in \mathbf{I}[1, m], i \in \mathbf{I}[0, 2^m - 1]$$

$$\Theta_{11}^i + \Theta_{12}^i \Delta(\hat{\rho}) + (\Theta_{12}^i \Delta(\hat{\rho}))^T + \Delta(\hat{\rho})\Theta_{22}^i \Delta(\hat{\rho}) \ge 0, \ \hat{\rho} \in \Delta_{\hat{\rho}}$$

$$\begin{bmatrix} N_{0i} & Z_{1i} \\ Z_{1i}^T & Z_{2i} \end{bmatrix} + \frac{1}{r_n^2} U_i^T U_i + G^T \Theta^i G < 0, \ i \in \mathbf{I}[0, 2^m - 1], \quad \rho = 0$$

$$\begin{bmatrix} N_{0i} & Z_{1i} \\ Z_{1i}^T & Z_{2i} \end{bmatrix} + \frac{1}{r_f^2} U_i^T U_i + G^T \Theta^i G < 0, \ i \in \mathbf{I}[0, 2^m - 1], \quad \rho \in \{\rho^1 \ \cdots \ \rho^L\}, \rho^q \in N_{\rho^q} \tag{8.15}$$

Proof 8.1 *We will prove (II) firstly. Choose the following Lyapunov function*

$$V(t) = x(t)^T P x(t) + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(t)}{l_j}, \tag{8.16}$$

then from the derivative of $V(t)$ along the closed-loop system, it follows

$$\begin{aligned}
&\dot{V}(t) + \frac{1}{r_f^2} z^T(t) z(t) - \omega^T(t)\omega(t) \\
&\leq M + x^T(PB_1B_1^T P)x + \frac{1}{r_f^2} N^T N - (\omega^T - x^T PB_1)(\omega - B_1^T Px),
\end{aligned}$$

where

$$\begin{aligned}
M &= x^T \sum_{i=0}^{2^m-1} \eta_i (M_1 + M_1^T)x + 2x^T PB_2 \sum_{i=0}^{2^m-1} \eta_i [D_i K_a(\tilde{\rho}) + \tilde{\rho} D_i K_b(\hat{\rho}) \\
&\quad + D_i^- H_a(\tilde{\rho}) + \tilde{\rho} D_i^- H_b(\hat{\rho})]x + 2\sum_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j}, \\
M_1 &= PA + PB_2[(I-\rho)D_i K_0 + D_i K_a(\rho) - \rho D_i K_a(\hat{\rho}) + (I - \hat{\rho}(t))D_i K_b(\hat{\rho}) \\
&\quad + (I-\rho)D_i^- H_0 + D_i^- H_a(\rho) - \rho D_i^- H_a(\hat{\rho}) + (I - \hat{\rho}(t))D_i^- H_b(\hat{\rho})], \\
N &= \sum_{i=0}^{2^m-1} \eta_i \{C + D(I-\rho)[D_i K(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]\}x
\end{aligned}$$

Let $B = [b^1 \cdots b^m]$ and $B^j = [0 \cdots b^j \cdots 0]$, then

$$\begin{aligned}
PB_2 \tilde{\rho} D_i K_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB_2^j D_i K_b(\hat{\rho}), \\
PB_2 \tilde{\rho} D_i^- H_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB_2^j D_i^- H_b(\hat{\rho}), \\
PB_2 D_i K_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB_2 D_i K_{aj}, \\
PB_2 D_i^- H_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j PB_2 D_i^- H_{aj}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\dot{V}(t) + \frac{1}{r_f^2} z^T(t) z(t) - \omega^T(t)\omega(t) \\
&\leq M + x^T(PB_1B_1^T P)x + \frac{1}{r_f^2} N^T N.
\end{aligned}$$

Let $X = P^{-1}$, $Y_0 = K_0 X$, $Y_{aj} = K_{aj} X$, $Y_{bj} = K_{bj} X$, $O_0 = H_0 X$, $O_{aj} = H_{aj} X$, $O_{bj} = H_{bj} X$, $j \in \mathbf{I}[1, m]$. Choose the adaptive laws as (8.13), then it is sufficient to show that

$$\dot{V}(t) + \frac{1}{r_f^2} z^T(t) z(t) - \omega^T(t)\omega(t) < 0, \tag{8.17}$$

if for any $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^q \in N_{\rho^q}$,

$$\sum_{i=0}^{2^m-1} \eta_i[N_{0i} + N_{1i}(\hat{\rho}_j) + N_{2i}(\hat{\rho}_j)] + \frac{1}{r_f^2} W^T W < 0,$$

where

$$\begin{aligned}
W &= \sum_{i=0}^{2^m-1} \eta_i[CX + D(I-\rho)D_i Y_0 + D(I-\rho)D_i^- O_0 + N_{3i}(\hat{\rho}_j)], \\
N_{0i} &= AX + B_2(I-\rho)D_i Y_0 + (AX + B_2(I-\rho)D_i Y_0)^T \\
&\quad + B_2 \sum_{j=1}^m \rho_j D_i Y_{aj} + (B_2 \sum_{j=1}^m \rho_j D_i Y_{aj})^T \\
&\quad + B_2(I-\rho)D_i^- O_0 + (B_2(I-\rho)D_i^- O_0)^T \\
&\quad + B_2 \sum_{j=1}^m \rho_j D_i^- O_{aj} + (B_2 \sum_{j=1}^m \rho_j D_i^- O_{aj})^T + B_1 B_1^T, \\
N_{1i}(\hat{\rho}_j) &= -B_2 \rho D_i \sum_{j=1}^m \hat{\rho}_j Y_{aj} + B_2 \sum_{j=1}^m \hat{\rho}_j D_i Y_{bj} \\
&\quad + (B_2 \sum_{j=1}^m \hat{\rho}_j D_i Y_{bj} - B_2 \rho D_i \sum_{j=1}^m \hat{\rho}_j Y_{aj})^T \\
&\quad - B_2 \rho D_i^- \sum_{j=1}^m \hat{\rho}_j O_{aj} + B_2 \sum_{j=1}^m \hat{\rho}_j D_i^- O_{bj} \\
&\quad + (B_2 \sum_{j=1}^m \hat{\rho}_j D_i^- O_{bj} - B_2 \rho D_i^- \sum_{j=1}^m \hat{\rho}_j O_{aj})^T, \\
N_{2i}(\hat{\rho}_j) &= \sum_{j=1}^m \sum_{p=1}^m \hat{\rho}_j \hat{\rho}_p (-B_2^j D_i Y_{bp} - Y_{bj}^T D_i B_2^{pT} \\
&\quad - B_2^j D_i^- O_{bp} - O_{bj}^T D_i^- B_2^{pT}), \\
N_{3i}(\hat{\rho}_j) &= \sum_{j=1}^m \hat{\rho}_j D(I-\rho)[D_i(Y_{aj} + Y_{bj}) + D_i^-(O_{aj} + O_{bj})].
\end{aligned}$$

By Lemma 2.10 and (8.15), it follows that (8.17) holds for any $x \in \wp(H(\hat{\rho}))$, $\rho \in \{\rho^1 \ \cdots \ \rho^L\}$, $\rho^q \in N_{\rho^q}$ *and* $\hat{\rho}$ *satisfying (8.13). The proofs for the normal case of closed-loop system (8.11) are similar, and omitted here.*

To prove item (I):

$$\dot{V}(t) \leq M + x^T P B_1 \omega + \omega^T B_1^T P x.$$

Noting that

$$x^T P B_1 \omega + \omega^T B_1^T P x \leq x^T P B_1 B_1^T P x + \omega^T \omega,$$

we have

$$\dot{V}(t) \leq M + x^T P B_1 B_1^T P x + \omega^T \omega.$$

Then by the proof of item (II), we have

$$\dot{V} \leq \omega^T \omega$$

which implies that

$$V(x(t)) \leq \int_0^\infty \omega^T(t)\omega(t)dt + \sum\nolimits_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j} \leq \delta^*$$

for $x(0) = 0$.

Then, the conclusion can be drawn that trajectories of the closed-loop system that start from the origin will remain inside $\varepsilon^*(P, \delta^*)$ *for every* $\omega \in \mathfrak{W}_\delta$.

Corollary 8.1 *The adaptive* H_∞ *performance indexes are no larger than* r_n *and* r_f *in normal and actuator failure cases for closed-loop system (8.11), if (8.15) holds for* $r_f > r_n > 0$*, correspondingly, the controller gain and adaptive law are given by (8.13) and (8.14), respectively.*

Proof 8.2 *Let* $F(0) = \sum_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j}$. *Then, by (8.13) and (8.2), it follows that* $\tilde{\rho}_j(0) \leq \max_j\{\overline{\rho}_j^q\} - \min_j\{\underline{\rho}_j^q\}$. *We can choose* l_j *sufficiently large so that* $F(0)$ *is sufficiently small. Thus the conclusion follows from the item (II) and Definition 3.1.*

From Theorem 8.1, we can optimize the adaptive H_∞ performance in normal and fault cases and the disturbance tolerance level δ.

Let r_n and r_f denote the adaptive H_∞ performance bounds for the normal case and fault cases of the closed-loop system (8.12). Let δ denote the disturbance tolerance level. Then r_n, r_f are minimized and δ is maximized if the following optimization problem is solvable

$$\begin{array}{lll} \min & \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta & \\ s.t. & (a) \quad (8.15), & \\ & (b) \quad \varepsilon^*(P, \delta^*) \subset \wp(H(\hat{\rho})), & \end{array} \tag{8.18}$$

where $\eta_n = r_n^2$, $\eta_f = r_f^2$, $\eta_\delta = \frac{1}{\delta^*} = \frac{1}{\delta + \max\{\sum_{j=1}^m \frac{\tilde{\rho}_j^2(t)}{l_j}\}}$ and α, β, γ are weighting coefficients.

However, by Definition 7.2, we have that (b) can not be shown as LMIs directly. Obviously, $\varepsilon^*(P, \delta^*) \subset \varepsilon(P, \delta^*)$, which implies that (b) can be replaced with (b1).

$$(b1) \quad \varepsilon(P, \delta^*) \subset \wp(H(\hat{\rho})). \tag{8.19}$$

Condition (b1) is equivalent to

$$\delta^* h(\hat{\rho})_j P^{-1} h(\hat{\rho})_j^T \leq 1 \Leftrightarrow \begin{bmatrix} \frac{1}{\delta^*} & h(\hat{\rho})_j P^{-1} \\ * & P^{-1} \end{bmatrix} \geq 0. \tag{8.20}$$

for all $j \in \mathbf{I}[1, m]$, where $h(\hat{\rho})_j$ is the jth row of $H(\hat{\rho})$. We have that (8.20) is equivalent to the following inequalities.

$$(b2) \quad \begin{bmatrix} -\eta_\delta & -O_{0s} \\ * & -X \end{bmatrix} + \sum_{j=1}^m \hat{\rho}_j \begin{bmatrix} 0 & -O_{ajs} - O_{bjs} \\ * & 0 \end{bmatrix} \leq 0, \hat{\rho} \in \Delta_{\hat{\rho}}$$

where O_{ajs} is the sth row of O_{aj}, $s \in \mathbf{I}[1, m]$.

The following algorithm is given to design adaptive H_∞ controller

Algorithm 8.1

Step 1 Solve the following optimization problem:

$$\begin{aligned} min \qquad & \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta \\ s.t. \qquad & (8.15), \quad (b2) \end{aligned} \tag{8.21}$$

Then, with optimal solutions η_n, η_f, η_δ, X, Y_0, Y_{aj}, Y_{bj}, O_0, O_{aj}, O_{bj}, $j \in \mathbf{I}[1,\ m]$, *go to Step 2.*

Step 2 Determine the controller parameter matrices K_0, K_{aj}, K_{bj}, $j \in \mathbf{I}[1,\ m]$, *by (8.14).*

Step 3 Determine the adaptive laws (8.13).

Then an adaptive fault-tolerant controller is designed.

Remark 8.3 *Theorem 8.1 gives a sufficient condition for the existence of an adaptive fault tolerant* H_∞ *controller via state feedback. In Theorem 8.1, if set* $Y_{aj} = 0$, $Y_{bj} = 0$, $O_{aj} = 0$, $O_{bj} = 0$, $j \in \mathbf{I}[1, m]$, *the condition of Theorem 8.1 reduces to*

$$Q_i + \frac{1}{r_n^2} J_i^T J_i < 0,\ \ i \in \mathbf{I}[0, 2^m - 1],\ \ \rho = 0 \tag{8.22}$$

$$\begin{aligned} Q_i + \frac{1}{r_f^2} J_i^T J_i < 0,\ \ i \in \mathbf{I}[0, 2^m - 1], \\ \rho \in \{\rho^1 \ \cdots \ \rho^L\},\ \rho^q \in N_{\rho^q} \end{aligned} \tag{8.23}$$

where

$$\begin{aligned} Q_i &= AX + B_2(I-\rho)D_iY_0 + (AX + B_2(I-\rho)D_iY_0)^T \\ &\quad + B_2(I-\rho)D_i^- O_0 + (B_2(I-\rho)D_i^- O_0)^T + B_1B_1^T \\ J_i &= CX + D(I-\rho)D_iY_0 + D(I-\rho)D_i^- O_0. \end{aligned}$$

From [66], we have that the following two conditions are satisfied

(i) The trajectories of the closed-loop system (8.6) with $u = K_0x$, $K_0 = Y_0X^{-1}$, that start from the origin will remain inside the domain $\varepsilon(P, \delta)$ for every $\omega \in \mathfrak{W}_\delta$

(ii) The H_∞ performance indexes are no larger than r_n and r_f for normal and actuator failure cases, respectively, if there exist matrices $X > 0$, O_0, Y_0, such that the inequalities (8.22) and (8.23) hold for all $D_i \in \mathbf{D}$ and $\varepsilon(P, \delta) \subset \wp(H_0)$, where $P = X^{-1}$, $H_0 = O_0X^{-1}$. This just gives a design method for traditional fault tolerant H_∞ controllers via fixed gains. The above fact shows that the design condition for adaptive fault tolerant H_∞ controllers given in Theorem 8.1 is more relaxed than that described by (8.22) and (8.23) for the traditional fault tolerant H_∞ controller design with fixed gains.

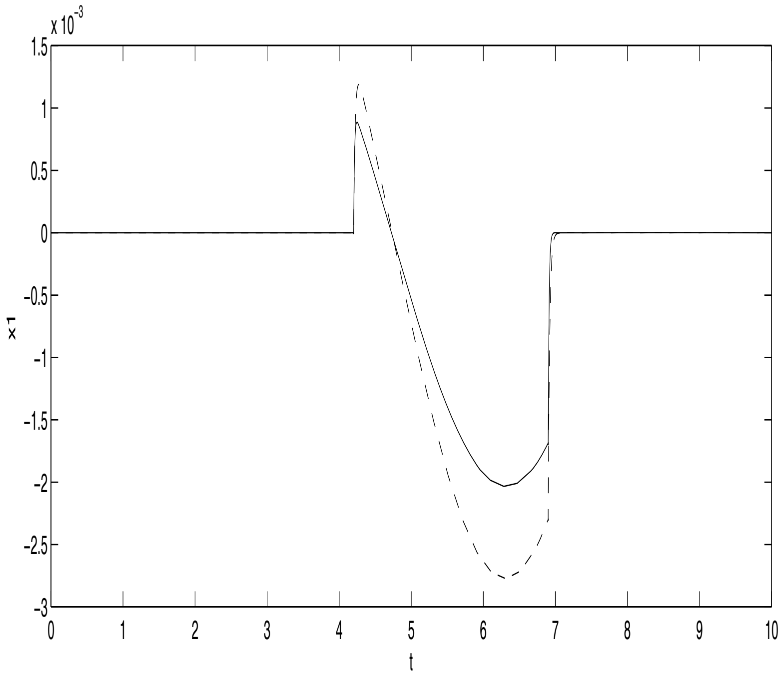


FIGURE 8.1
Response curve of the first state in normal case with adaptive controller (solid) and the fixed gain controller (dashed).

8.2.3 Example

Example 8.1 *Consider the system of the form (8.1) with*

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 40 \end{bmatrix}, \; B1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; B2 = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix},$$
$$C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T, D = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

and the following two possible fault modes:

Fault mode 1: Both of the two actuators are normal, that is,

$$\rho_1^1 = \rho_2^1 = 0.$$

Fault mode 2: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^2 = 1, \;\; 0 \leq \rho_2^2 \leq a,$$

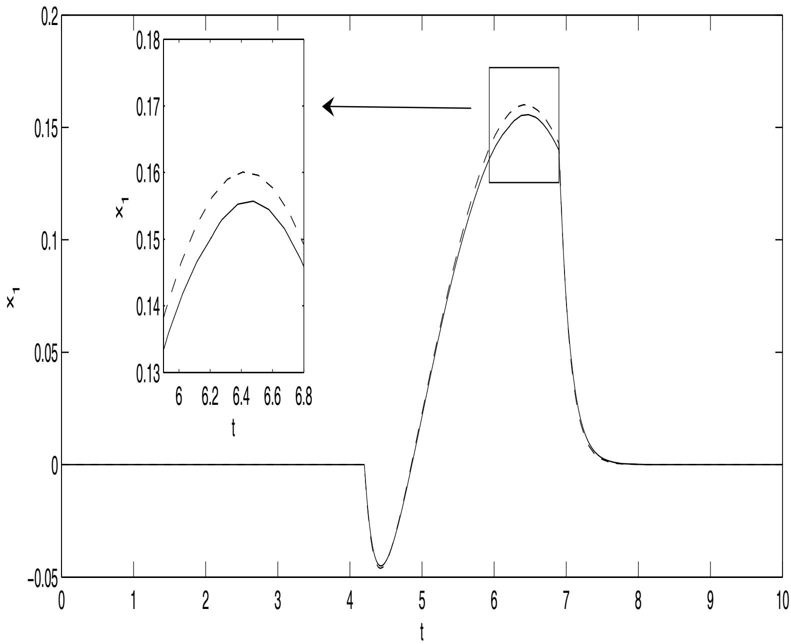


FIGURE 8.2
Response curve of the first state in fault case with adaptive controller (solid) and the fixed gain controller (dashed).

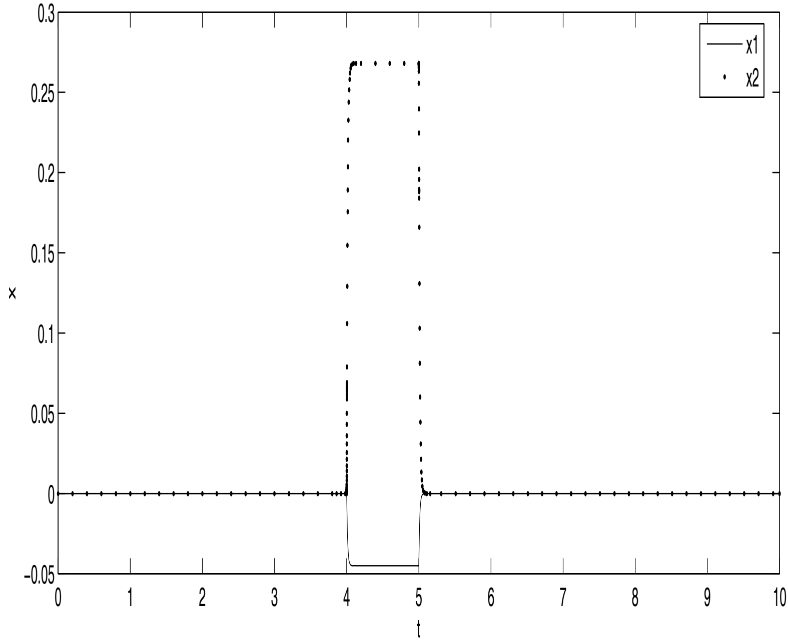


FIGURE 8.3
Response curves of the states with adaptive controller in normal case.

where $a = 0.5$ denotes the maximal loss of effectiveness for the second actuator.

Let $\alpha = 10$, $\beta = 1$, $\gamma = 10$, the optimal indexes with fixed controller gains are $\eta_n = 0.1963$, $\eta_f = 9.8933$, $\eta_\delta = 20.5385$, $\eta = 217.2408$. By solving the optimization problem (8.21), the optimal indexes can be given as $\eta_n = 0.5881$, $\eta_f = 9.1236$, $\eta_\delta = 9.6701$, $\eta = 111.7048$. In order to get the smaller number for every optimal index, we choose $\alpha = 110$, $\beta = 0.2$, $\gamma = 0.5$. Then we get $\eta_n = 0.1676$, $\eta_f = 7.1242$, $\eta_\delta = 18.6399$. This phenomenon indicates that the three indexes are smaller when Algorithm 8.1 is used, which indicates the superiority of our adaptive method.

To illustrate the effectiveness of the proposed adaptive method, we give the following simulations.

The fault case considered in the following simulation is : At 0 second, the first actuator is outage. Here, we choose $l_1 = l_2 = 100$.

Firstly, we consider the H_∞ performance. The disturbance is given as

$$\omega_1(t) = \omega_2(t) = \begin{cases} \cos(t), & 4.2 \le t \le 6.9 \\ 0, & \text{otherwise} \end{cases}$$

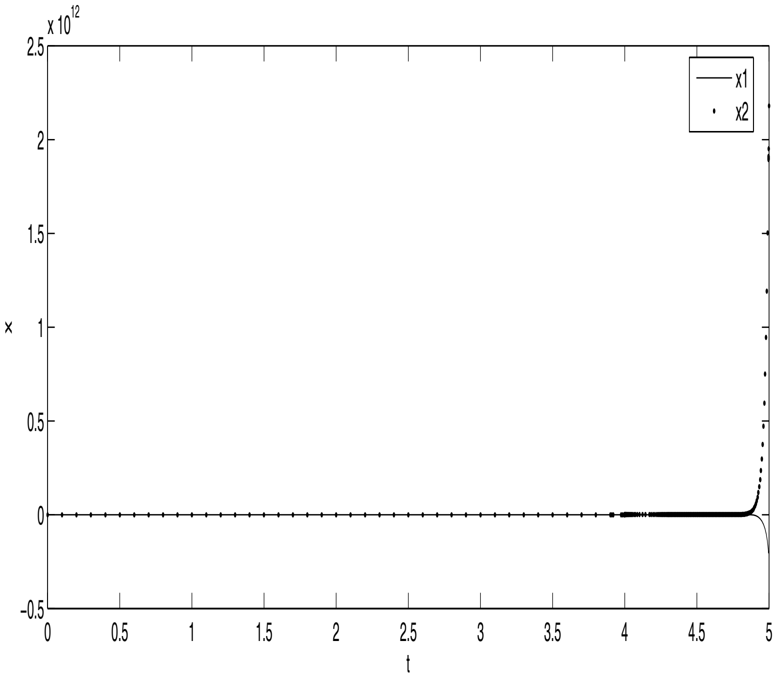


FIGURE 8.4
Response curves of the states with fixed gain controller in normal case.

Figure 8.1 and Figure 8.2 show the response curves of the first state with the adaptive and fixed gain controller in normal and fault case, respectively. It is easy to see our adaptive H_∞ controller can achieve better responses than the traditional controller with fixed gains in both normal case and fault case just as theoretic results have proved.

Then, we consider the disturb tolerance problem. The disturbance is given as

$$\omega_1(t) = \omega_2(t) = \begin{cases} 21.8, & 4 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases} \tag{8.24}$$

Figure 8.3 shows the response curves of the states with the *adaptive controller* in normal case, Figure 8.4 shows the responses curves of the states with the fixed gain controller in normal case. Obviously, under the disturbance (8.24), the closed-loop system with the adaptive H_∞ controller is still stable. However, the closed-loop system with the fixed gains controller is unstable. This phenomenon indicates the superiority of our adaptive method.

8.3 Output Feedback

8.3.1 Problem Statement

Consider an LTI plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + B_2\sigma(u) \\ z(t) &= C_1x(t) + D_{12}\sigma(u) \\ y(t) &= C_2x(t) + D_{21}\omega(t) \end{aligned} \tag{8.25}$$

where $x(t) \in R^n$ is the plant state, $\sigma(u) \in R^m$ is the saturated control input, $y(t) \in R^p$ is the measured output, $z(t) \in R^s$ is the regulated output and $\omega(t) \in R^d$ is an exogenous disturbance in $L_2[0, \infty]$, respectively. A, B_1, B_2, C_1, C_2, D_{12}, and D_{21} are known constant matrices of appropriate dimensions.

The following problem will be considered in this section: The first question that needs to be answered is, what is the maximal value of δ such that the state will be bounded for all $\omega \in \mathfrak{W}_\delta$? Here we will consider the situation, zero initial state. The problem related to this question is referred to as disturbance tolerance. The disturbance rejection capability can be measured by the restricted L_2 gain over $\mathfrak{W}_\delta$. In this section we will consider L_2 gain and $\mathfrak{W}_\delta$ at the same time.

Remark 8.4 *For the above problem to be solvable, it is necessary for the pair $(A, B_2(I - \rho))$ to be stabilizable for each $\rho \in \{\rho^1 \cdots \rho^L\}$.*

8.3.2 ARC Controller Design

The dynamics with actuator faults (8.3) and saturation is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) + B_2(I - \rho)\sigma(u(t)) \\ z(t) &= C_1x(t) + D_{12}(I - \rho)\sigma(u(t)) \\ y(t) &= C_2x(t) + D_{21}\omega(t) \end{aligned} \tag{8.26}$$

The controller structure is chosen as

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t),\ y), \quad \xi(t) \in R^n \\ u(t) &= C_K(\hat{\rho}(t))\xi(t) \end{aligned} \tag{8.27}$$

where

$$u(t) = C_K(\hat{\rho}(t))\xi(t) = (C_{K0} + C_{Ka}(\hat{\rho}(t)) + C_{Kb}(\hat{\rho}(t)))\xi(t) \tag{8.28}$$

and $\hat{\rho}(t)$ is the estimation of ρ,

$$C_{Ka}(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} C_{Kaj}\hat{\rho}_j(t),\ C_{Kb}(\hat{\rho}(t)) = \sum\nolimits_{j=1}^{m} C_{Kbj}\hat{\rho}_j(t).$$

By Lemma 7.1, the saturated linear feedback, with $\xi(t) \in \wp([0\ H(\hat{\rho}(t))])$, can be expressed as

$$\sigma(C_K(\hat{\rho}(t))\xi(t)) = \sum_{i=0}^{2^m-1} \eta_i[D_i C_K(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]\xi(t) \tag{8.29}$$

for some scalars $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m - 1]$, such that $\sum_{i=0}^{2^m-1} \eta_i = 1$, and the following equality holds

$$\begin{aligned}(I-\rho)\sigma(u(t)) = \sum_{i=0}^{2^m-1} \eta_i[&(I-\rho)D_i C_{K0} + D_i C_{Ka}(\rho)\\ &- \rho D_i C_{Ka}(\hat{\rho}) + (I-\hat{\rho}(t))D_i C_{Kb}(\hat{\rho}(t)) + D_i C_{Ka}(\tilde{\rho}(t))\\ &+ \tilde{\rho} D_i C_{Kb}(\hat{\rho}(t)) + (I-\rho)D_i^- H_{K0} + D_i^- H_{Ka}(\rho)\\ &- \rho D_i^- H_{Ka}(\hat{\rho}) + (I-\hat{\rho}(t))D_i^- H_{Kb}(\hat{\rho}(t))\\ &+ D_i^- H_{Ka}(\tilde{\rho}(t)) + \tilde{\rho} D_i^- H_{Kb}(\hat{\rho}(t))]\xi(t)\end{aligned} \tag{8.30}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$.

Now, by Lemma 7.2 we provide one choice of such η_i's, which are Lipschitzian functions in ξ and $\hat{\rho}$.

$$\eta_i(\xi(t),\ \hat{\rho}(t)) = \prod_{j=1}^{m} [z_j(1-\lambda_j(\xi(t),\ \hat{\rho}(t))) + (1-z_j)\lambda_j(\xi(t),\ \hat{\rho}(t))] \tag{8.31}$$

By using the functions $\eta_i(\xi(t),\ \hat{\rho}(t))$'s, the output feedback controller (8.28) can be parameterized as

$$\begin{aligned}\dot{\xi}(t) &= (\sum_{i=0}^{2^m-1} \eta_i A_{Ki}(\hat{\rho}))\xi(t) + (\sum_{i=0}^{2^m-1} \eta_i B_{Ki}(\hat{\rho}))y(t)\\ u(t) &= (I-\rho)\sigma(C_K(\hat{\rho})\xi(t))\end{aligned} \tag{8.32}$$

where

$$\begin{aligned}A_{Ki}(\hat{\rho}) &= A_{Ki0} + A_{Kia}(\hat{\rho}) + A_{Kib}(\hat{\rho})\\ B_{Ki}(\hat{\rho}) &= B_{Ki0} + B_{Kia}(\hat{\rho}) + B_{Kib}(\hat{\rho})\\ C_K(\hat{\rho}) &= C_{K0} + C_{Ka}(\hat{\rho}) + C_{Kb}(\hat{\rho})\\ B_{Kia}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j B_{Kiaj},\ B_{Kib}(\hat{\rho}) = \sum_{j=1}^{m} \hat{\rho}_j B_{Kibj}\\ C_{Ka}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j C_{Kaj},\ C_{Kb}(\hat{\rho}) = \sum_{j=1}^{m} \hat{\rho}_j C_{Kbj}\\ A_{Kia}(\hat{\rho}) &= \sum_{j=1}^{m} \hat{\rho}_j A_{Kiaj}\\ A_{Kib}(\hat{\rho}) &= \sum_{j=1}^{m}\sum_{s=1}^{m} \hat{\rho}_j\hat{\rho}_s A_{Kibjs} + \sum_{j=1}^{m} \hat{\rho}_j A_{Kibj}\end{aligned}$$

Motivated by the quasi-LPV structure of both the plant and the controller, we consider the following auxiliary LPV system, if $\varepsilon(P,\ \delta) \subset \wp([0\ H(\hat{\rho})])$ is an *invariant set*.

$$\begin{aligned}\dot{x}_e(t) &= A_e(\eta)x_e(t) + B_e(\eta)\omega(t)\\ z(t) &= C_e(\eta)x_e(t)\end{aligned} \tag{8.33}$$

$$\begin{aligned}A_e(\eta) &= \sum\nolimits_{i=0}^{2^m-1} \eta_i(A_{ei}x_e(t)), \quad \eta \in \Gamma\\ B_e(\eta) &= \sum\nolimits_{i=0}^{2^m-1} \eta_i(B_{ei}x_e(t)), \quad \eta \in \Gamma\\ C_e(\eta) &= \sum\nolimits_{i=0}^{2^m-1} \eta_i(C_{ei}x_e(t)), \quad \eta \in \Gamma\end{aligned} \tag{8.34}$$

where $x_e = [x^T(t)\ \xi^T(t)]^T$, $\eta = [\eta_0,\ \eta_1,\ \cdots,\ \eta_{2^m-1}]$, and

$$\begin{aligned}\Gamma &= \{\eta \in R^{2^m} : \sum\nolimits_{i=0}^{2^m-1} \eta_i = 1,\ 0 \le \eta_i \le 1,\ i \in I[0,\ 2^m-1]\},\\ A_{ei} &= \begin{bmatrix} A & B_2(I-\rho)[D_iC_K(\hat{\rho}) + D_i^- H(\hat{\rho})] \\ B_{Ki}(\hat{\rho})C_2 & A_{Ki}(\hat{\rho}) \end{bmatrix},\\ B_{ei} &= \begin{bmatrix} B_1 \\ B_{Ki}(\hat{\rho})D_{21} \end{bmatrix},\\ C_{ei} &= [C_1\ D_{12}(I-\rho)(D_iC_K(\hat{\rho}) + D_i^- H(\hat{\rho}))].\end{aligned}$$

The following theorem presents a sufficient condition for the solvability of the fault-tolerant control problem via dynamic output feedback in the framework of LMI and adaptive laws.

Denote

$$\Delta_{\hat{\rho}} = \{\hat{\rho} = (\hat{\rho}_1 \cdots \hat{\rho}_m)\ :\ \hat{\rho}_j \in\ \{\min_q\{\underline{\rho}_j^q\},\ \max_q\{\overline{\rho}_j^q\}\},\ q \in \mathbf{I}[1, L]\}$$

and $B^j = [0 \cdots b^j \cdots 0]$ with $B = [b^1 \cdots b^m]$.

Theorem 8.2 *Let $r_f > 0, r_n > 0$ and $\delta > 0$ be given constants, then the following two conditions are satisfied*

(I) The trajectories of the closed-loop system that start from the origin will remain inside the domain $\varepsilon^*(P, \delta^*)$ for every $\omega \in \mathfrak{W}_\delta$.

(II) In normal case, i.e., $\rho = 0$,

$$\int_0^\infty z^T(t)z(t)dt \le r_n^2 \int_0^\infty \omega^T(t)\omega(t)dt + r_n^2 \sum\nolimits_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j}, \mathit{for}\ x(0) = 0$$

and in actuator failures cases, i.e., $\rho \in \{\rho^1 \cdots \rho^L\}$,

$$\int_0^\infty z^T(t)z(t)dt \le r_f^2 \int_0^\infty \omega^T(t)\omega(t)dt + r_f^2 \sum\nolimits_{j=1}^m \frac{\tilde{\rho}_j^2(0)}{l_j}, \mathit{for}\ x(0) = 0$$

where $\tilde{\rho}(t) = diag\{\tilde{\rho}_1(t) \ \cdots \ \tilde{\rho}_m(t)\}$, $\tilde{\rho}_j(t) = \hat{\rho}_j(t) - \rho_j$, *if there exist matrices* $0 < N_1 < Y_1$, A_{Ki0}, A_{Kiaj}, A_{Kibjs}, B_{Ki0}, B_{Kiaj}, B_{Kibj}, C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}, $j \in \mathbf{I}[1,m]$, $s \in \mathbf{I}[1,m]$ *and symmetric matrices* Θ^i, $i \in \mathbf{I}[0, 2^m - 1]$, *with*

$$\Theta^i = \begin{bmatrix} \Theta^i_{11} & \Theta^i_{12} \\ \Theta^{iT}_{12} & \Theta^i_{22} \end{bmatrix}$$

and Θ^i_{11}, $\Theta^i_{22} \in R^{m(2n+d)\times m(2n+d)}$ *such that the following inequalities hold for all* $D_i \in \mathbf{D}$ *and* $\varepsilon^*(P, \delta^*) \subset \wp([0 \ \ H(\hat{\rho})])$, *i.e.,* $|[0 \ \ H(\hat{\rho})]_j x| \leq 1$ *for all* $x \in \varepsilon^*(P, \delta^*), j \in \mathbf{I}[1, m]$.

$$\begin{aligned} &\Theta^i_{22jj} \leq 0, \ j \in \mathbf{I}[1,m], i \in \mathbf{I}[0, 2^m - 1] \\ &\Theta^i_{11} + \Theta^i_{12}\Delta(\hat{\rho}) + (\Theta^i_{12}\Delta(\hat{\rho}))^T + \Delta(\hat{\rho})\Theta^i_{22}\Delta(\hat{\rho}) \geq 0, \ \hat{\rho} \in \Delta_{\hat{\rho}} \\ &\begin{bmatrix} N_{0i} & Z_{1i} \\ Z^T_{1i} & Z_{2i} \end{bmatrix} + \frac{1}{r_n^2} U_i^T U_i + G^T \Theta^i G < 0, \ \ i \in \mathbf{I}[0, 2^m - 1], \rho = 0 \\ &\begin{bmatrix} N_{0i} & Z_{1i} \\ Z^T_{1i} & Z_{2i} \end{bmatrix} + \frac{1}{r_f^2} U_i^T U_i + G^T \Theta^i G < 0, \ \ i \in \mathbf{I}[0, 2^m - 1], \\ &\qquad \rho \in \{\rho^1 \ \cdots \ \rho^L\}, \ \rho^q \in N_{\rho^q} \end{aligned} \tag{8.35}$$

where

$$\begin{aligned} N_{0i} &= \begin{bmatrix} T_{0i} & T_{1i} & T_{2i} \\ * & T_{3i} & T_{4i} \\ * & * & -I \end{bmatrix} \\ Z_{1i} &= [Z_{1i1} \ Z_{1i2} \ldots Z_{1im}], Z_{2i} = [Z_{2ijs}], j, s \in \mathbf{I}[1, m] \\ Z_{1ij} &= \begin{bmatrix} T_{5i} & T_{6i} & T_{7i} \\ T_{8i} & T_{9i} & T_{10i} \\ 0 & 0 & 0 \end{bmatrix}, Z_{2ijs} = \begin{bmatrix} 0 & T_{11i} & 0 \\ T_{12i} & T_{13i} & T_{14i} \\ 0 & T_{15i} & 0 \end{bmatrix} \\ U_i &= [U_{i0} \ U_{i1} \ \cdots \ U_{im}], \ U_{ij} = \begin{bmatrix} 0 & T_{16i} & 0 \end{bmatrix} \\ U_{i0} &= \begin{bmatrix} C_1 & D_{12}(I - \rho)(D_i C_{K0} + D_i^- H_{K0}) & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
T_{0i} &= Y_1A - N_1B_{Ki0}C_2 + (Y_1A - N_1B_{Ki0}C_2)^T \\
T_{1i} &= Y_1B_2[(I-\rho)(D_iC_{K0} + D_i^-H_{K0}) + D_iC_{Ka}(\rho) \\
&\quad + D_i^-H_{Ka}(\rho)] - N_1A_{Ki0} - N_1A_{Kia}(\rho) \\
&\quad + \begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix}^T S^T[-Y_1B_2(D_iC_{Ka}(\rho) + D_i^-H_{Ka}(\rho)) \\
&\quad + N_1A_{Kia}(\rho)] + (-N_1A + N_1B_{Ki0}C_2 \\
&\quad + N_1B_{Kia}(\rho)C_2 - [N_1B_{Kia}(\rho)C_2S]\begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix})^T \\
T_{2i} &= Y_1B_1 - N_1B_{Ki0}D_{21} \\
T_{3i} &= -N_1B_2[(I-\rho)(D_iC_{K0} + D_i^-H_{K0}) + D_iC_{Ka}(\rho) \\
&\quad + D_i^-H_{Ka}(\rho)] + (-N_1B_2[(I-\rho)(D_iC_{K0} + D_i^-H_{K0}) \\
&\quad + D_iC_{Ka}(\rho) + D_i^-H_{Ka}(\rho)])^T + N_1A_{Ki0} \\
&\quad + N_1A_{Kia}(\rho) + (N_1A_{Ki0} + N_1A_{Kia}(\rho))^T \\
T_{4i} &= -N_1B_1 + N_1B_{Ki0}D_{21} + [-Y_1B_2(D_iC_{Ka}(\rho) \\
&\quad + D_i^-H_{Ka}(\rho)) + N_1A_{Kia}(\rho)]^T S \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
&\quad - N_1B_{Kia}(\rho)C_2S\begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
T_{5i} &= -N_1B_{Kibj}C_2 - N_1B_{Kiaj}C_2 \\
T_{6i} &= Y_1B_2[-\rho(D_iC_{Kaj} + D_i^-H_{Kaj}) + D_iC_{Kbj} + D_i^-H_{Kbj}] \\
&\quad - N_1A_{Kibj} + \begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix}^T S^T[Y_1B_2((D_iC_{Kaj} + D_i^-H_{Kaj}) \\
&\quad - \rho(D_iC_{Kbj} + D_i^-H_{Kbj})) - N_1A_{Kiaj}] \\
T_{7i} &= -N_1B_{Kibj}D_{21} - N_1B_{Kiaj}D_{21} \\
T_{8i} &= N_1B_{Kibj}C_2 + N_1B_{Kiaj}C_2S\begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix} \\
T_{9i} &= N_1B_2\rho(D_iC_{Kaj} + D_i^-H_{Kaj}) \\
&\quad - N_1B_2(D_iC_{Kbj} + D_i^-H_{Kbj}) + N_1A_{Kibj} \\
T_{10i} &= [Y_1B_2(D_iC_{Kaj} + D_i^-H_{Kaj}) - Y_1B_2\rho(D_iC_{Kbj} \\
&\quad + D_i^-H_{Kbj}) - N_1A_{Kiaj}]^T S\begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
&\quad + N_1B_{Kiaj}D_{21} + N_1B_{Kibj}D_{21} + N_1B_{Kiaj}C_2S\begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
T_{11i} &= -Y_1B_2^j(D_iC_{Kbs} + D_i^-H_{Kbs}) - N_1A_{Kibjs} \\
&\quad + \begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix}^T S^TY_1B_2^j(D_iC_{Kbs} + D_i^-H_{Kbs})
\end{aligned}
$$

$$
\begin{aligned}
T_{12i} &= (-Y_1 B_2^s (D_i C_{Kbj} + D_i^- H_{Kbj}) - N_1 A_{Kibsj} \\
&\quad + \begin{bmatrix} 0 \\ C_2^\perp \end{bmatrix}^T S^T Y_1 B_2^s (D_i C_{Kbj} + D_i^- H_{Kbj}))^T \\
T_{13i} &= N_1 B_2^j (D_i C_{Kbs} + D_i^- H_{Kbs}) + N_1 A_{Kibjs} \\
&\quad + [N_1 B_2^j (D_i C_{Kbs} + D_i^- H_{Kbs}) + N_1 A_{Kibjs}]^T \\
T_{14i} &= (Y_1 B_2^s (D_i C_{Kbj} + D_i^- H_{Kbj}))^T S \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \\
T_{15i} &= [-D_{21}^T \;\; 0] S^T Y_1 B_2^j (D_i C_{Kbs} + D_i^- H_{Kbs}) \\
T_{16i} &= D_{12}(I-\rho)(D_i C_{Kaj} + D_i^- H_{Kaj} + D_i C_{Kbj} + D_i^- H_{Kbj}) \\
G &= \begin{bmatrix} \begin{bmatrix} I_{(2n+d)\times(2n+d)} \\ \cdots \\ I_{(2n+d)\times(2n+d)} \end{bmatrix} & 0 \\ 0 & I_{m(2n+d)\times m(2n+d)} \end{bmatrix}, \\
\Delta(\hat{\rho}) &= diag[\hat{\rho}_1 I_{(2n+d)\times(2n+d)} \;\; \cdots \;\; \hat{\rho}_m I_{(2n+d)\times(2n+d)}].
\end{aligned}
$$

and also $\hat{\rho}_j(t)$ is determined according to the adaptive law

$$
\begin{aligned}
\dot{\hat{\rho}}_j &= Proj_{[\min_q\{\underline{\rho}_j^q\},\ \max_q\{\overline{\rho}_j^q\}]}\{L_{1j}\} \\
&= \begin{cases} 0, \; if & \begin{array}{c} \hat{\rho}_j = \min_q\{\underline{\rho}_j^q\} \; and \; L_{1j} \le 0 \\ or \; \hat{\rho}_j = \max_q\{\overline{\rho}_j^q\} \; and \; L_{1j} \ge 0 \end{array} \\ & L_{1j}, \quad otherwise \end{cases}
\end{aligned} \tag{8.36}
$$

where

$$
\begin{aligned}
L_{1j} &= l_j \sum_{i=0}^{2^m-1} \eta_i \{\xi^T N_1 [A_{Kiaj} - B_2 D_i C_{Kaj} - B_2^j D_i C_{Kb}(\hat{\rho}) - B_2 D_i^- H_{Kaj} \\
&\quad - B_2^j D_i^- H_{Kb}(\hat{\rho})]\xi + \begin{bmatrix} y \\ 0 \end{bmatrix}^T S^T [Y_1 (B_2 D_i C_{Kaj} + B_2^j D_i C_{Kb}(\hat{\rho}) \\
&\quad + B_2 D_i^- H_{Kaj} + B_2^j D_i^- H_{Kb}(\hat{\rho})) - N_1 A_{Kiaj}]\xi \\
&\quad + \xi^T N_1 B_{Kiaj} C_2 S \begin{bmatrix} y \\ 0 \end{bmatrix}\},
\end{aligned}
$$

$l_j > 0 (j \in \mathbf{I}[1,m])$ and $\delta > 0$ are the adaptive law gains to be chosen according to practical applications.

Proof 8.3 *Choose the following Lyapunov function*

$$
V(t) = x_e^T P x_e + \sum_{j=1}^m \frac{\tilde{\rho}_j^2(t)}{l_j}, \tag{8.37}
$$

By $\tilde{\rho}$(t) $=\hat{\rho}(t) - \rho$ *and*

$$
\begin{aligned}
B_{Kia}(\tilde{\rho}) &= B_{Kia}(\hat{\rho}) - B_{Kia}(\rho) \\
A_{Kia}(\tilde{\rho}) &= A_{Kia}(\hat{\rho}) - A_{Kia}(\rho)
\end{aligned}
$$

A_{ei} *can be written as*

$$\begin{aligned}
A_{ei} &= A_{ei1} + A_{ei2} + A_{ei3} \\
A_{ei1} &= \begin{bmatrix} A & A_{ei1a} \\ [B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})]C_2 & A_{ei1b} \end{bmatrix} \\
A_{ei1a} &= B_2[(I-\rho)D_iC_{K0} + D_iC_{Ka}(\rho) - \rho D_iC_{Ka}(\hat{\rho}) \\
&\quad + (I-\hat{\rho})D_iC_{Kb}(\hat{\rho}) + (I-\rho)D_i^- H_{K0} \\
&\quad + D_i^- H_{Ka}(\rho) - \rho D_i^- H_{Ka}(\hat{\rho}) + (I-\hat{\rho})D_i^- H_{Kb}(\hat{\rho})] \\
A_{ei1b} &= A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho}) \\
A_{ei2} &= \begin{bmatrix} 0 & A_{ei2a} \\ 0 & A_{Kia}(\tilde{\rho}) \end{bmatrix} A_{ei3} = \begin{bmatrix} 0 & 0 \\ B_{Kia}(\tilde{\rho})C_2 & 0 \end{bmatrix} \\
A_{ei2a} &= B_2D_iC_{Ka}(\tilde{\rho}) + B_2\tilde{\rho}D_iC_{Kb}(\hat{\rho}) + B_2D_i^- H_{Ka}(\tilde{\rho}) + B_2\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})
\end{aligned}$$

Let P *be of the following form*

$$P = \begin{bmatrix} Y_1 & -N_1 \\ -N_1 & N_1 \end{bmatrix}$$

with $0 < N_1 < Y_1$, *which implies* $P > 0$. *Since* C *is of full rank, and* C_2 *satisfies* $C_2C_2^{\perp T} = 0$ *and* $C_2^{\perp}C_2^{\perp T}$ *nonsingular, it follows that* $\begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}$ *is nonsingular. From (8.25), we have*

$$C_2x = y,\ C_2^{\perp}x = C_2^{\perp}x, x = S\begin{bmatrix} y \\ C_2^{\perp}x \end{bmatrix} \tag{8.38}$$

where $S = \begin{bmatrix} C_2 \\ C_2^{\perp} \end{bmatrix}^{-1}$. *Then, we have* $PA_{ei2} = \begin{bmatrix} 0 & W_{ai} \\ 0 & W_{bi} \end{bmatrix}$ *with*

$$\begin{aligned}
W_{ai} &= Y_1[B_2D_iC_{Ka}(\tilde{\rho}) + B_2\tilde{\rho}D_iC_{Kb}(\hat{\rho}) \\
&\quad + B_2D_i^- H_{Ka}(\tilde{\rho}) + B_2\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})] - N_1A_{Kia}(\tilde{\rho}) \\
W_{bi} &= N_1[A_{Kia}(\tilde{\rho}) - B_2D_iC_{Ka}(\tilde{\rho}) - B_2\tilde{\rho}D_iC_{Kb}(\hat{\rho}) \\
&\quad - B_2D_i^- H_{Ka}(\tilde{\rho}) - B_2\tilde{\rho}D_i^- H_{Kb}(\hat{\rho})]
\end{aligned}$$

which follows

$$[x^T\ \xi^T]PA_{ei2}[x^T\ \xi^T]^T = x^TW_{ai}\xi + \xi^TW_{bi}\xi$$

Thus, by (8.38), we have

$$x^TW_{ai}\xi = \begin{bmatrix} y \\ 0 \end{bmatrix}^T S^TW_{ai}\xi + [x^T\ \xi^T]A_{ai1}[x^T\ \xi^T]^T + [x^T\ \xi^T]B_{ai1}\omega$$

where

$$A_{ai1} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix}^T S^T W_{ai} \\ 0 & 0 \end{bmatrix}, B_{ai1} = \begin{bmatrix} 0 \\ W_{ai}^T S \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \end{bmatrix}$$

In the same way, from (8.38) we get

$$\begin{aligned}[x^T\ \xi^T] P A_{ei3} [x^T\ \xi^T]^T &= -x^T N_1 B_{Kia}(\tilde{\rho}) C_2 x + \xi^T N_1 B_{Kia}(\tilde{\rho}) C_2 x \\ &= x_e^T A_{ai2} x_e + x_e^T B_{ai2} \omega + M_{ai2}\end{aligned}$$

where

$$\begin{aligned} A_{ai2} &= \begin{bmatrix} -N_1 B_{Kia}(\tilde{\rho}) C_2 & 0 \\ N_1 B_{Kia}(\tilde{\rho}) C_2 S \begin{bmatrix} 0 \\ C_2^{\perp} \end{bmatrix} & 0 \end{bmatrix} \\ B_{ai2} &= \begin{bmatrix} 0 \\ M_{bi} \end{bmatrix} \\ M_{ai2} &= \xi^T N_1 B_{Kia}(\tilde{\rho}) C_2 S \begin{bmatrix} y \\ 0 \end{bmatrix} \\ M_{bi} &= N_1 B_{Kia}(\tilde{\rho}) C_2 S \begin{bmatrix} -D_{21} \\ 0 \end{bmatrix} \end{aligned}$$

Then from the derivative of $V(t)$ along the closed-loop system (8.33), it follows

$$\begin{aligned} &\dot{V}(t) + \frac{1}{r_f^2} z^T(t) z(t) - \omega^T(t)\omega(t) \\ &= 2x_e^T \sum\nolimits_{i=0}^{2^m-1} \eta_i P(A_{ei} x_e + B_{ei}\omega) + 2\sum\nolimits_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j} \\ &+ \frac{1}{r_f^2} x_e^T [\sum\nolimits_{i=0}^{2^m-1} \eta_i C_{ei}^T][\sum\nolimits_{i=0}^{2^m-1} \eta_i C_{ei}] x_e - \omega^T \omega \\ &= 2x_e^T \sum\nolimits_{i=0}^{2^m-1} \eta_i P(A_{ei1} x_e + B_{ei}\omega) - \omega^T \omega \\ &+ \frac{1}{r_f^2} x_e^T [\sum\nolimits_{i=0}^{2^m-1} \eta_i C_{ei}^T][\sum\nolimits_{i=0}^{2^m-1} \eta_i C_{ei}] x_e \\ &+ 2x_e^T \sum\nolimits_{i=0}^{2^m-1} \eta_i (A_{ai1} + A_{ai2}) x_e + 2x_e^T \sum\nolimits_{i=0}^{2^m-1} \eta_i (B_{ai1} + B_{ai2})\omega + W_1 \\ &\leq x_e^T W_0 x_e + W_1 \end{aligned}$$

where

$$W_0 = W_{01} + \frac{1}{r_f^2}[\sum_{i=0}^{2^m-1} \eta_i C_{ei}^T][\sum_{i=0}^{2^m-1} \eta_i C_{ei}]$$

$$\begin{aligned} W_{01} &= \sum_{i=0}^{2^m-1} \eta_i [PA_{ei1} + A_{ai1} + A_{ai2} + (PA_{ei1} + A_{ai1} + A_{ai2})^T] \\ &+ [\sum_{i=0}^{2^m-1} \eta_i (PB_{ei} + B_{ai1} + B_{ai2})] \\ &\quad [\sum_{i=0}^{2^m-1} \eta_i (PB_{ei} + B_{ai1} + B_{ai2})]^T \end{aligned}$$

$$\begin{aligned} W_1 &= 2\xi^T \sum_{i=0}^{2^m-1} \eta_i W_{bi} \xi + 2 \begin{bmatrix} y \\ 0 \end{bmatrix}^T S^T \sum_{i=0}^{2^m-1} \eta_i W_{ai} \xi \\ &+ 2 \sum_{i=0}^{2^m-1} \eta_i M_{ai2} + 2 \sum_{j=1}^{m} \frac{\tilde{\rho}_j(t)\dot{\tilde{\rho}}_j(t)}{l_j} \end{aligned}$$

The design condition that $\dot{V}(t) \leq 0$ is reduced to

$$W_0 < 0, \tag{8.39}$$

$$W_1 \leq 0 \tag{8.40}$$

Since y and ξ are available online, the adaptive laws can be chosen as (8.36) for rendering (8.40) valid. (8.39) is equivalent to

$$\begin{aligned} &\sum_{i=0}^{2^m-1} \eta_i \begin{bmatrix} He(PA_{ei1} + A_{ai1} + A_{ai2}) & * \\ (PB_{ei} + B_{ai1} + B_{ai2})^T & -I \end{bmatrix} \\ &+ \frac{1}{r_f^2} \begin{bmatrix} \sum_{i=0}^{2^m-1} \eta_i C_{ei}^T \\ 0 \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{2^m-1} \eta_i C_{ei} & 0 \end{bmatrix} < 0 \end{aligned} \tag{8.41}$$

Notice that

$$PA_{ei1} = \begin{bmatrix} Y_1 A - N_1 [B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})] C & W_c \\ -N_1 A + N_1 [B_{Ki0} + B_{Kia}(\rho) + B_{Kib}(\hat{\rho})] C & W_d \end{bmatrix}$$

$$PB_{ei} = \begin{bmatrix} Y_1 B_1 - N_1 [B_{Ki0} + B_{Kia}(\hat{\rho}) + B_{Kib}(\hat{\rho})] D_{21} \\ -N_1 B_1 + N_1 [B_{Ki0} + B_{Kia}(\hat{\rho}) + B_{Kib}(\hat{\rho})] D_{21} \end{bmatrix}$$

$$\begin{aligned} W_c &= Y_1 B_2 [(I - \rho) D_i C_{K0} + D_i C_{Ka}(\rho) - \rho D_i C_{Ka}(\hat{\rho}) \\ &+ (I - \hat{\rho}) D_i C_{Kb}(\hat{\rho}) + (I - \rho) D_i^- H_{K0} + D_i^- H_{Ka}(\rho) \\ &- \rho D_i^- H_{Ka}(\hat{\rho}) + (I - \hat{\rho}) D_i^- H_{Kb}(\hat{\rho})] \\ &- N_1 [A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho})] \end{aligned}$$

$$\begin{aligned} W_d &= -N_1 B_2 [(I - \rho) D_i C_{K0} + D_i C_{Ka}(\rho) - \rho D_i C_{Ka}(\hat{\rho}) \\ &+ (I - \hat{\rho}) D_i C_{Kb}(\hat{\rho}) + (I - \rho) D_i^- H_{K0} + D_i^- H_{Ka}(\rho) \\ &- \rho D_i^- H_{Ka}(\hat{\rho}) + (I - \hat{\rho}) D_i^- H_{Kb}(\hat{\rho})] \\ &+ N_1 [A_{Ki0} + A_{Ka}(\rho) + A_{Kib}(\hat{\rho})] \end{aligned}$$

Furthermore (8.41) can be described by

$$W(\hat{\rho}) = \sum_{i=0}^{2^m-1} \eta_i W_{2i}(\hat{\rho}) + \frac{1}{r_f^2}(\sum_{i=0}^{2^m-1} \eta_i W_{3i})^T(\sum_{i=0}^{2^m-1} \eta_i W_{3i}) < 0$$

$$W_{2i}(\hat{\rho}) = N_{0i} + \sum_{j=1}^{m} \hat{\rho}_j Z_{1ij} + (\sum_{j=1}^{m} \hat{\rho}_j Z_{1ij})^T + \sum_{j=1}^{m}\sum_{s=1}^{m} \hat{\rho}_j \hat{\rho}_s Z_{2ijs}$$

$$W_{3i}(\hat{\rho}) = U_{i0} + \sum_{j=1}^{m} \hat{\rho}_j U_{ij}$$

where N_{0i}, Z_{1ij}, Z_{2ijs}, $j, s \in \mathbf{I}[1, m]$ *are defined in (8.35).*

Let

$$Q_i(\hat{\rho}) = W_{2i}(\hat{\rho}) + \frac{1}{r_f^2}(W_{3i}(\hat{\rho}))^T(W_{3i}(\hat{\rho}))$$

By Lemma 2.10, we can get $Q_i(\hat{\rho}) < 0$ *if (8.35) holds, which implies* $W_0 < 0$ *by Schur complement. Together with adaptive laws (8.36), it follows that the following inequality (8.42) holds for any* $x_e \in \wp([0\ H(\hat{\rho})])$, $\rho \in \{\rho^1\ \cdots\ \rho^L\}$, $\rho^q \in N_{\rho^q}$ *and* $\hat{\rho}$ *satisfying (8.30). The proofs for the normal case of closed-loop system (8.33) are similar, and omitted here.*

$$\dot{V}(t) + \frac{1}{r_f^2} z^T(t)z(t) - \omega^T(t)\omega(t) < 0, \tag{8.42}$$

To prove item (I):

$$\dot{V}(t) \leq x_e^T W_{01} x_e + W_1 + \omega^T \omega.$$

Then by the proof of item (II), we have

$$\dot{V} \leq \omega^T \omega$$

which implies that

$$V(x_e(t)) \leq \int_0^\infty \omega^T(t)\omega(t)dt + \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(0)}{l_j} \leq \delta^*$$

for $x(0) = 0$.

Then, the conclusion can be drawn that trajectories of the closed-loop system that start from the origin will remain inside $\varepsilon^*(P, \delta^*)$ *for every* $\omega \in \mathfrak{W}_\delta$.

Corollary 8.2 *The adaptive* H_∞ *performance indexes are no larger than* r_n *and* r_f *in normal and actuator failure cases for closed-loop system (8.33), if (8.35) holds for* $r_f > r_n > 0$, *correspondingly, the controller gain and adaptive law are given by (8.35) and (8.36), respectively.*

Proof 8.4 *Let* $F(0) = \sum_{j=1}^{m} \frac{\tilde{\rho}_j^2(0)}{l_j}$. *Then, by (8.36), it follows that* $\tilde{\rho}_j(0) \leq \max_j\{\overline{\rho}_j^q\} - \min_j\{\underline{\rho}_j^q\}$. *We can choose* l_j *sufficiently large so that* $F(0)$ *is sufficiently small. Thus the conclusion follows from the item (II) and Definition 3.1.*

If we take the following reliable H_∞ controller with fixed parameter matrices A_{Ki0}, B_{Ki0}, C_{K0}, $i \in \mathbf{I}[0, 2^m - 1]$

$$\begin{aligned} \dot{\xi}(t) &= (\sum_{i=0}^{2^m-1} \eta_i A_{Ki0})\xi(t) + (\sum_{i=0}^{2^m-1} \eta_i B_{Ki0})y(t) \\ u(t) &= (I-\rho)\sigma(C_{K0}\xi(t)) \end{aligned} \tag{8.43}$$

then combining (8.43) with (8.25), it follows:

$$\begin{aligned} \dot{x}_{e1}(t) &= A_{e1}(\eta)x_{e1}(t) + B_{e1}(\eta)\omega(t) \\ z_{e1}(t) &= C_{e1}(\eta)x_e(t) \end{aligned} \tag{8.44}$$

$$\begin{aligned} A_{e1}(\eta) &= \sum_{i=0}^{2^m-1} \eta_i(A_{e1i}x_{e1}(t)), \quad \eta \in \Gamma \\ B_{e1}(\eta) &= \sum_{i=0}^{2^m-1} \eta_i(B_{e1i}x_{e1}(t)), \quad \eta \in \Gamma \\ C_{e1}(\eta) &= \sum_{i=0}^{2^m-1} \eta_i(C_{e1i}x_{e1}(t)), \quad \eta \in \Gamma \end{aligned} \tag{8.45}$$

where $x_{e1} = [x^T(t)\ \xi^T(t)]^T$,

$$\begin{aligned} A_{e1i} &= \begin{bmatrix} A & B_2(I-\rho)[D_iC_{K0} + D_i^- H_0] \\ B_{Ki0}C_2 & A_{Ki0} \end{bmatrix}, \\ B_{e1i} &= \begin{bmatrix} B_1 \\ B_{Ki0}D_{21} \end{bmatrix}, \\ C_{e1i} &= [C_1\ D_{12}(I-\rho)(D_iC_{K0} + D_i^- H_0)] \end{aligned}$$

The following lemma presents a condition for the system (8.44) to have performance bounds.

Lemma 8.1 *Consider the closed-loop system described by (8.44), and let $r_n > 0$ and $r_f > 0$ be given constants. Then the following statements are equivalent:*

(i) there exist a symmetric matrix $X > 0$ and controller $\mathbf{K}$ described by (8.43) such that

$$A_{e1i}^T X + XA_{e1i} + XB_{e1i}B_{e1i}^T X + \frac{1}{r_n^2}C_{e1i}^T C_{e1i} < 0$$

holds for $\rho = 0$, and

$$A_{e1i}^T X + XA_{e1i} + XB_{e1i}B_{e1i}^T X + \frac{1}{r_f^2}C_{e1i}^T C_{e1i} < 0$$

holds for $\rho \in \{\rho^1\ \cdots\ \rho^L\}$, $\rho^q \in N_{\rho^q}$

(ii) there exist symmetric matrices Y_1 and N_1 with $0 < N_1 < Y_1$, and a controller described by (8.43) with $A_{Ki0} = A_{Kei0}$, $B_{Ki0} = B_{Kei0}$, $C_{K0} =$

C_{Ke0}, $H_0 = H_{e0}$, $i \in \mathbf{I}[0, 2^m - 1]$ *such that* $V_1(r_n) < 0$ *holds for* $\rho = 0$, *and* $V_1(r_f) < 0$ *holds for* $\rho \in \{\rho^1 \ \cdots \ \rho^L\}$, $\rho^q \in N_{\rho^q}$, *where we define*

$$V_1(r) = \begin{bmatrix} T_{10} & T_{11} & Y_1B_1 - N_1B_{Kei0}D_{21} & C_1^T \\ * & T_{12} & -N_1B_1 + N_1B_{Kei0}D_{21} & T_{13} \\ * & * & -I & 0 \\ * & * & * & -r^2I \end{bmatrix}$$

with

$$\begin{aligned} T_{10} &= Y_1A - N_1B_{Kei0}C_2 + (Y_1A - N_1B_{Kei0}C_2)^T \\ T_{11} &= Y_1B_2(I-\rho)(D_iC_{Ke0} + D_i^- H_{e0}) - N_1A_{Kei0} \\ &\quad + (-N_1A + N_1B_{Kei0}C_2)^T \\ T_{12} &= -N_1B_2(I-\rho)(D_iC_{Ke0} + D_i^- H_{e0}) + N_1A_{Kei0} \\ &\quad - [N_1B_2(I-\rho)(D_iC_{Ke0} + D_i^- H_{e0}) - N_1A_{Kei0}]^T \\ T_{13} &= (D_iC_{Ke0}^T + D_i^- H_{e0})(I-\rho)D_{12}^T \end{aligned}$$

Proof 8.5 *The proof is similar to the proof of Lemma 5.1. To avoid overlap, the proof is omitted.*

Next, a theorem is given to show that the condition in Theorem 8.1 for the adaptive controller design is more relaxed than that in Lemma 8.1 for the traditional controller design with fixed parameter matrices.

Theorem 8.3 *If condition (i) or (ii) in Lemma 4 holds, then the condition of Theorem 1 holds.*

Proof 8.6 *If condition (i) or (ii) in Lemma 4 holds, then it is easy to see that the condition in Theorem 8.1 is feasible with* $A_{Kiaj} = A_{Kibj} = A_{Kibjs} = B_{Kiaj} = B_{Kibj} = C_{Kaj} = C_{Kbj} = H_{Kaj} = H_{Kbj} = 0$, $i \in \mathbf{I}[0, 2^m - 1]$, $j \in \mathbf{I}[1, m]$, $s \in \mathbf{I}[1, m]$. *The proof is completed.*

From Theorem 8.1, we have the following algorithm to optimize the adaptive H_∞ performance in normal and fault cases and the disturbance tolerance level δ.

Let r_n and r_f denote the adaptive H_∞ performance bounds for the normal case and fault cases of the closed-loop system (8.32). Let δ denote the disturbance tolerance level. Then r_n, r_f are minimized and δ is maximized if the following optimization problem is solvable

$$\begin{aligned} \min \ \ & \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta \\ s.t.(a) \ \ & (8.35), \\ (b) \ \ & \varepsilon^*(P, \delta^*) \subset \wp([0 \ H(\hat{\rho})]), \end{aligned} \tag{8.46}$$

where $\eta_n = r_n^2$, $\eta_f = r_f^2$, $\eta_\delta = \frac{1}{\delta^*} = \frac{1}{\delta + \max\{\sum_{j=1}^m \frac{\tilde{\rho}_j^2(t)}{l_j}\}}$ and α, β, γ are weighting coefficients.

However, there are two problems as follows, which should be considered.

(1) By Definition 7.4, we have that (b) can not be shown as LMIs directly, Obviously, $\varepsilon^*(P,\delta^*) \subset \varepsilon(P,\delta^*)$, which implies that (b) can be replaced with (b1).

$$(b1) \quad \varepsilon(P,\delta^*) \subset \wp([0\ H(\hat{\rho})]). \tag{8.47}$$

Condition (b1) is equivalent to

$$\delta^*[0\ h(\hat{\rho})]_j P^{-1}[0\ h(\hat{\rho})]_j^T \leq 1 \Leftrightarrow \begin{bmatrix} \frac{1}{\delta^*} & [0\ h(\hat{\rho})]_j \\ * & P \end{bmatrix} \geq 0. \tag{8.48}$$

for all $j \in \mathbf{I}[1,m]$, where $[0\ h(\hat{\rho})]_j$ is the jth row of $[0\ H(\hat{\rho})]$. We have that (8.35) is equivalent to the following inequalities.

$$(b2) \quad \begin{bmatrix} -\eta_\delta & -[0\ H_{K0s}] \\ * & -P \end{bmatrix} + \sum_{j=1}^{m} \hat{\rho}_j \begin{bmatrix} 0 & [0\ -H_{Kajs} - H_{Kbjs}] \\ * & 0 \end{bmatrix} \leq 0, \quad \hat{\rho} \in \Delta_{\hat{\rho}}$$

where H_{Kajs} is the sth row of H_{Kaj}, $s \in \mathbf{I}[1,m]$.

(2) It should be noted that condition (8.35) is not convex. But when $C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}$ are given, they become LMIs.

From Theorem 8.1, we have the following algorithm to design the adaptive output feedback controller.

Algorithm 8.2

Step 1 Suppose that all states of system (8.25) can be measured. Minimize the following index to design the state-feedback controller.

$$\eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta$$

Then, the matrices C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj} can be given.
Step 2 Solve the following optimization problem

$$\begin{aligned} min \quad & \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta \\ & s.t.(a), (b2) \end{aligned}$$

Remark 8.5 *Step 1 is to determine matrices C_{K0}, C_{Kaj}, C_{Kbj}, H_{K0}, H_{Kaj}, H_{Kbj}, which solves the corresponding adaptive controller design problem via state feedback. This procedure is adapted from the last section, and convex conditions are described. To avoid overlap, the conditions appearing in Step 1 will be omitted.*

From Lemma 8.1, we have the following algorithm to design the fault-tolerant controller with fixed gains.

Algorithm 8.3 *Step 1: Suppose that all states of system (8.25) can be measured. Minimize the following index to design the state-feedback controller.*

$$\eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta$$

Then, the matrices C_{K0}, H_{K0}, *can be given.*

Step 2: Solve the following optimization problem

$$\begin{aligned} min \quad & \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta \\ & s.t.(a), (b2) \end{aligned}$$

Remark 8.6 *Step 1 is to determine matrices* C_{K0}, H_{K0}, *which solves the corresponding adaptive controller design problem via state feedback.*

Remark 8.7 *In Step 1, for some cases, the magnitude of the designed gains* C_{K0} *(*C_{Kaj} *and* C_{Kbj}*) may be too large to be applied in Step 2. For solving the problem, by adding the following constraints, where* Q *and* Y_{K0} *are variables in conditions of Step 1*

$$Q > \alpha I, \quad Y_{K0}Y_{K0}^T < \beta I, \tag{8.49}$$

then the magnitude of C_{K0} *can be reduced. In fact, by* $C_{K0} = Y_{K0}Q^{-1}$ *and (8.49), it follows that*

$$\| C_{K0} \| < \sqrt{\beta}/\alpha.$$

The similar method can be used for the gains C_{Kaj} *and* C_{Kbj}.

8.3.3 Example

Example 8.2 *Consider the system of the form (8.25) with*

$$A = \begin{bmatrix} 0.01 & 0.1 \\ 0.6 & 0.01 \end{bmatrix}, \ B1 = \begin{bmatrix} 0.1 & 0 \\ 0.01 & 0 \end{bmatrix}, \ B2 = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$
$$C1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, D12 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, D21 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}$$

and the following two possible fault modes:

Fault mode 1: Both of the two actuators are normal, that is,

$$\rho_1^1 = \rho_2^1 = 0.$$

Fault mode 2: The first actuator is outage and the second actuator may be normal or loss of effectiveness, described by

$$\rho_1^2 = 1, \quad 0 \le \rho_2^2 \le a,$$

where $a = 0.5$ denotes the maximal loss of effectiveness for the second actuator.

Let $\alpha = 10$, $\beta = 1$, $\gamma = 10$, the optimal indexes with fixed controller gains are $\eta_n = 0.0134$, $\eta_f = 0.1581$, $\eta_\delta = 0.0866$, $\eta = 1.1588$. By using Algorithm 8.2, the optimal indexes can be given as $\eta_n = 0.0027$, $\eta_f = 0.0079$, $\eta_\delta = 0.0212$, $\eta = 0.2473$. This phenomenon indicates the superiority of our adaptive method.

8.4 Conclusion

In this chapter, an adaptive fault-tolerant H_∞ controllers design method is proposed for linear time-invariant systems with actuator saturation. The resultant design guarantees the adaptive H_∞ performances of closed-loop systems in the cases of actuator saturation and actuator failures. An example has been given to illustrate the effectiveness of the design method.

9

Adaptive Reliable Tracking Control

9.1 Introduction

Recently, there are also several approaches developed to solve tracking problems [64, 81, 82, 84, 123, 148, 149, 164]. The classical approach for LTI systems has been to design a closed-loop system that achieves the desired transfer function as close as possible [64]. The inherent shortcoming is over-design. Game theory [123] is most suitable to finite time control of time-varying systems. The linear quadratic (LQ) control theory method [82] requires a prior knowledge of dynamics of the reference signal. The H_∞ *optimal tracking* solution [148] is suitable for cases where the tracking signal is measured online and it can hardly deal with the case where a prior knowledge on this signal is available or when it can be previewed. However, there are only a limited number of papers devoted to reliable or fault-tolerant tracking control problems. In order to realize the reliable tracking control in the presence of *actuator faults*, a method based on robust pole region assignment techniques [164] and a method based on iterative LMI [84, 149] have been proposed. The latter is a multi-objective optimization methodology, which is used to ensure the designed tracking controller guarantees the *stability* of the closed-loop system and optimal tracking performance during normal system and maintains an acceptable lower level of tracking performance in *fault modes*.

In this chapter, we shall investigate the *reliable tracking control* problem of linear *time-invariant* systems in the presence of actuator faults. The type of fault under consideration here is loss of actuator effectiveness, which is different from those in the previous chapters. Combining LMI approach with *adaptive methods* successfully, we design a novel *adaptive reliable controller* without using an *FDI* mechanism. The newly proposed method is based on the online estimation of an eventual fault and the addition of a new control law to the normal control law in order to reduce the *fault effect* automatically. The main contribution of this chapter is that the normal tracking performance of the resultant closed-loop system is optimized without any conservativeness and the states of fault modes asymptotically track those of the normal mode. Since systems are operating under the normal condition most of the time, this contribution is very important in actual control system design. A numerical example of a linearized F-16 aircraft model and its simulation results are given to demonstrate the effectiveness and superiority of the proposed method.

9.2 Problem Statement

Consider a *linear time-invariant system* described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{9.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input and $y(t) \in R^p$ is the output, respectively. A and B are known constant matrixes of appropriate dimensions.

To formulate the reliable tracking control problem, the *actuator fault model* must be established first. Here, the type of the faults under consideration is loss of actuator effectiveness [133, 164].

$$u_i^F(t) = \rho_i u_i(t), \quad \rho_i \in [\underline{\rho}_i, \bar{\rho}_i], \quad 0 < \underline{\rho}_i \leq 1, \bar{\rho}_i \geq 1 \tag{9.2}$$

where $u_i^F(t)$ represent the signal from the actuator that has failed. ρ_i is an unknown constant and $\underline{\rho_i}$ and $\bar{\rho}_i$ represent the lower and upper bounds of ρ_i, respectively. Note that, when $\underline{\rho_i} = \bar{\rho}_i = 1$, there is no fault for the ith actuator u_i.

Denote

$$u^F(t) = [u_1^F(t), u_2^F(t), \cdots u_m^F(t)]^T = \rho u(t) \tag{9.3}$$

where $\rho = \text{diag}[\rho_1, \rho_2, \cdots \rho_m]$ and

$$\Delta = \{\rho \ : \rho = \text{diag}[\rho_1, \rho_2, \cdots \rho_m], \ \rho_i \in [\underline{\rho}_i, \bar{\rho}_i], \qquad i = 1, 2, \cdots, m\} \tag{9.4}$$

Hence, the dynamics with actuator faults (9.2) is described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\rho u(t)\\ y(t) &= Cx(t)\end{aligned} \tag{9.5}$$

Considering the lower and upper bounds $(\underline{\rho_i}, \bar{\rho}_i)$, the following set can be defined

$$N_\rho = \{\rho \ : \rho = \text{diag}[\rho_1, \rho_2, \cdots \rho_m], \ \rho_i = \underline{\rho}_i \ \ \rho_i = \bar{\rho}_i, \qquad i = 1, 2, \cdots, m\} \tag{9.6}$$

Thus, the set N_ρ contains a *maximum* of 2^m elements.

Consider the system described by (9.5) with actuator faults (9.2). The design problem under consideration is to find an adaptive controller such that (i) During normal operation, the closed-loop system is asymptotically stable and the output $Sy(t)$ tracks the *reference signal* $y_r(t)$ without *steady-state error*, that is

$$\lim_{t\to\infty} \varepsilon(t) = 0, \quad \varepsilon(t) = y_r(t) - Sy(t) \tag{9.7}$$

where $S \in R^{l\times p}$ is a known constant matrix used to form the output required to track the reference signal. Moreover, the controller also minimizes the upper bound of the performance index

$$J_t = \int_0^t \left[\eta^T(t)Q_1\eta(t) + x^T(t)Q_2x(t) + u^T(t)Ru(t)\right] dt \tag{9.8}$$

where $\eta = \int_0^t \varepsilon(\tau)d\tau$, $Q_1 \in R^{l\times l}, Q_2 \in R^{n\times n}$ are *positive semi-definite* matrices and $R \in R^{m\times m}$ is *positive definite* matrix.
(ii) In the event of actuator faults, the closed-loop system is still asymptotically stable and the output $Sy(t)$ tracks the reference signal $y_r(t)$ without steady-state error. Moreover the state vector of post fault case asymptotically tracks that of the normal case, which has the designed performance.
It is well known that the tracking error integral action of a controller can effectively eliminate the steady-state tracking error. In order to obtain an adaptive reliable tracking controller with tracking error integral, we combine equation (9.1) and (9.7) and have the following augmented system

$$\begin{bmatrix}\dot\eta(t)\\ \dot x(t)\end{bmatrix} = \begin{bmatrix}0 & -SC\\ 0 & A\end{bmatrix}\begin{bmatrix}\eta(t)\\ x(t)\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix}u(t) + \begin{bmatrix}I\\ 0\end{bmatrix}y_r(t) \tag{9.9}$$

Let $\bar{x} = \begin{bmatrix}\eta^T(t) & x^T(t)\end{bmatrix}^T$, then the *augmented system* can be changed into

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + Gy_r(t) \tag{9.10}$$

where

$$\bar{A} = \begin{bmatrix}0 & -SC\\ 0 & A\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}0\\ B\end{bmatrix}, \quad G = \begin{bmatrix}I\\ 0\end{bmatrix}$$

Moreover, the augmented system with actuator faults (9.2) is described by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\rho u(t) + Gy_r(t) \tag{9.11}$$

where $\bar{A}, \bar{B}$ and G are the same as (9.10).

9.3 Adaptive Reliable Tracking Controller Design

In this section, a sufficient condition for the optimization of normal tracking performance problem is first given. Secondly, based on the normal controller, we add a new control law to the normal law in order to reduce the fault effect on the system and achieve the desired control objective by using adaptive method.

Now we design the normal controller $u_N(t)$ for the augmented system (9.10) with the following *state feedback* tracking controller

$$u_N(t) = K_N\bar{x}(t) = \begin{bmatrix} K_\eta & K_x \end{bmatrix} \begin{bmatrix} \eta(t) \\ x(t) \end{bmatrix} \tag{9.12}$$

The closed-loop augmented normal system is given by

$$\dot{\bar{x}}(t) = (\bar{A} + \bar{B}K_N)\bar{x}(t) + Gy_r(t) \tag{9.13}$$

A linear matrix inequality (LMI) condition for the optimization of the guaranteed cost control problem of the augmented normal system (9.13) is presented.

Lemma 9.1 *Consider the closed-loop augmented normal system (9.13) and the performance index (9.8). For a given positive constant γ, if there exist symmetric matrices $Z, T \in R^{(n+l)\times(n+l)}$ and a matrix $W \in R^{m\times(n+l)}$ such that the following linear matrix inequalities hold:*

$$(i) \begin{bmatrix} \bar{A}Z + \bar{B}W + (\bar{A}Z + \bar{B}W)^T & G & W^T R^{\frac{1}{2}} & ZQ^{\frac{1}{2}} \\ * & -\gamma I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{9.14}$$

$$(ii) \begin{bmatrix} T & I \\ I & Z \end{bmatrix} > 0 \tag{9.15}$$

where $Q = diag[Q_1, Q_2] \geq 0$ and $R > 0$. Then the following controller stabilizes the closed-loop augmented normal system (9.13)

$$u_N(t) = K_N\bar{x}(t), \;\; K_N = [K_\eta, K_x] = WZ^{-1} \tag{9.16}$$

Furthermore, an upper bound of performance index (9.8) is given by

$$J_t \leq \gamma \int_0^t y_r^T(t)y_r(t)dt + \bar{x}^T(0)T\bar{x}(0) \tag{9.17}$$

Here γ corresponds to the H_∞ norm $\|T_{zy_r}\|$ of the *transfer function* from the input $y_r(t)$ to the performance output

$$z(t) = [Q^{\frac{1}{2}}, 0]^T\bar{x}(t) + [0, R^{\frac{1}{2}}]^T u(t) \tag{9.18}$$

The upper bound of performance index J can be minimized by solving the following *optimization problem* with the MATLAB LMI toolbox:

$$\min \;\; \text{Trace}(T) \text{ s.t. } (9.14) \qquad (9.15) \tag{9.19}$$

Proof 9.1 *By the Lemma 2.8, (9.14) is equivalent to*

$$\bar{A}Z + \bar{B}K_N Z + (\bar{A}Z + \bar{B}K_N Z)^T + \frac{1}{\gamma}GG^T + ZQZ + ZK_N^T RK_N Z < 0 \tag{9.20}$$

Post- and pre-multiplying the inequality (9.39) by $P = Z^{-1}$, we obtain

$$P(\bar{A} + \bar{B}K_N) + (\bar{A} + \bar{B}K_N)^T P + \frac{1}{\gamma}PGG^T P + Q + K_N^T RK_N < 0 \tag{9.21}$$

Since $\gamma > 0, Q > 0, Q = Q^T$ and $R > 0, R = R^T$, then

$$P(\bar{A} + \bar{B}K_N) + (\bar{A} + \bar{B}K_N)^T P < 0 \tag{9.22}$$

According to Lyapunov stability theorem, the controller $u_N(t) = K_N\bar{x}$, which satisfies (9.14) stabilizes the augmented system (9.10). Furthermore,

$$\begin{aligned}
J_t &\leq - \int_0^t \bar{x}^T(t)\{[P(\bar{A} + \bar{B}K_N) + (\bar{A} + \bar{B}K_N)^T P] + \frac{1}{\gamma}PGG^T P\}\bar{x}(t)dt \\
&= - \int_0^t \{[\dot{\bar{x}} - Gy_r(t)]^T P\bar{x} + \bar{x}^T P[\dot{\bar{x}} - Gy_r(t)] + \frac{1}{\gamma}\bar{x}^T PGG^T P\bar{x}\}dt \\
&\leq - \int_0^t d[\bar{x}^T(t)P\bar{x}(t)] + \gamma \int_0^t y_r^T(t)y_r(t)dt \\
&\leq \gamma \int_0^t y_r^T(t)y_r(t)dt + \bar{x}^T(0)P\bar{x}(0) \\
&\leq \gamma \int_0^t y_r^T(t)y_r(t)dt + \bar{x}^T(0)T\bar{x}(0)
\end{aligned} \tag{9.23}$$

The proof is completed.

Now for normal operation, we have designed the normal control law $u_N(t) = K_N\bar{x}(t)$.

Next, we begin to design an adaptive reliable controller based on the normal control law $u_N(t) = K_N\bar{x}(t)$. The main controller stricture is to compute a new control law $u_{ad}(t)$ to be added to the normal control law in order to compensate for the fault effect on the system, that is

$$u(t) = u_N(t) + u_{ad}(t) \tag{9.24}$$

The additive control law $u_{ad}(t)$ is zero in the normal case and different from zero in fault cases. The FTC scheme is summarized in Figure 9.1. In order to obtain online information on the effectiveness of actuators, we introduce the following *target model* described by

$$\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + B\hat{\rho}(t)r(t) \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned} \tag{9.25}$$

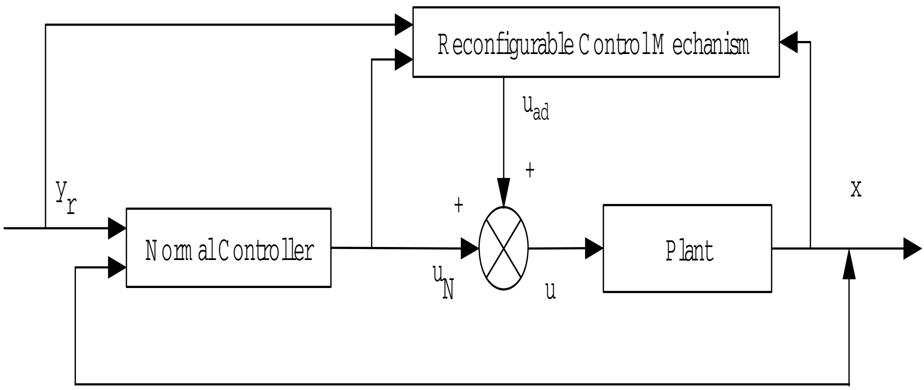


FIGURE 9.1
Reliable control scheme.

where $\hat{\rho}(t) = \text{diag}\{\hat{\rho}_1(t) \cdots \hat{\rho}_m(t)\}$ denotes the estimate of the actuator efficiency factor. The input $r(t) \in R^m$ is determined so as to achieve the control objectives.
The augmented system of the target model (9.25) is

$$\dot{\tilde{x}}(t) = \bar{A}\tilde{x}(t) + \bar{B}\hat{\rho}(t)r(t) + Gy_r(t) \tag{9.26}$$

where $\tilde{x}(t) = \begin{bmatrix} \hat{\eta}^T(t) & \hat{x}^T(t) \end{bmatrix}^T$, $\hat{\eta} = \int_0^t \hat{\varepsilon}(\tau)d\tau$, $\hat{\varepsilon}(t) = y_r(t) - S\hat{y}(t)$ and $\bar{A}, \bar{B}$, G are the same as those in normal operation (9.10).
If we define the *state error* vector of augmented system as $e(t) = \tilde{x}(t) - \bar{x}(t)$ and let the control input $u(t) = r(t) - Fe(t)$, then the augmented state error equation between (9.11) and (9.26) is written as

$$\begin{aligned} \dot{e}(t) &= \bar{A}e(t) + \bar{B}\rho Fe(t) + \bar{B}(\hat{\rho}(t) - \rho)r(t) \\ &= (\bar{A} + \bar{B}\rho F)e(t) + \bar{B}\tilde{\rho}(t)r(t) \end{aligned} \tag{9.27}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho = \text{diag}\{\tilde{\rho}_1(t) \cdots \tilde{\rho}_m(t)\}$. Here F is the error feedback gain to be designed to make the augmented state error equation (9.27) stable. Let $\bar{B} = [\bar{b}_1 \cdots \bar{b}_m]$ and $r(t) = (r_1(t) \cdots r_m(t))^T$, then the augmented state error system (9.27) can be written as

$$\dot{e}(t) = (\bar{A} + \bar{B}\rho F)e(t) + \sum_{i=1}^{m} \bar{b}_i\tilde{\rho}_i(t)r_i(t) \tag{9.28}$$

Theorem 9.1 *The augmented state error system (9.28) is stable if there exist a symmetric matrix $Z_1 \in R^{(n+l)\times(n+l)} > 0$ and a matrix $W_1 \in R^{m\times(n+l)}$ such that the following linear inequalities hold for all $\rho \in N_\rho$*

$$\bar{A}Z_1 + Z_1\bar{A}^T + \bar{B}\rho W_1 + W_1^T\rho\bar{B}^T < 0 \tag{9.29}$$

and also $\hat{\rho}_i(t)$ is determined according to the adaptive law

$$\begin{aligned}\dot{\hat{\rho}}_i &= Proj_{[\underline{\rho}_i,\bar{\rho}_i]}\{-l_i e^T P\bar{b}_i r_i\}\\ &= \begin{cases} 0, & if \quad \begin{array}{l}\hat{\rho}_i(t)=\underline{\rho}_i, \quad and \quad -l_i e^T P\bar{b}_i r_i \le 0 \quad or\\ \hat{\rho}_i(t)=\bar{\rho}_i, \quad and \quad -l_i e^T P\bar{b}_i r_i \ge 0;\end{array}\\ -l_i e^T P\bar{b}_i r_i, & otherwise\end{cases}\end{aligned} \tag{9.30}$$

where $l_i > 0$, $0 < \underline{\rho_i} \le 1$ and $\bar{\rho}_i \ge 1$, $i = 1\cdots m$. Proj$\{\cdot\}$ denotes the projection operator [70], whose role is to project the estimates $\hat{\rho}_i(t)$ to the interval $[\underline{\rho_i}, \bar{\rho}_i]$. Then the error feedback gain F is obtained by $F = W_1 Z_1^{-1}$.

Proof 9.2 *We choose the following Lyapunov function*

$$V = e^T(t)Pe(t) + \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2(t)}{l_i} \tag{9.31}$$

where $P = Z_1^{-1}$. The derivative of V along the trajectory of the augmented state error equation (9.28) can be written as

$$\dot{V} = e^T[P(\bar{A}+\bar{B}\rho F) + (\bar{A}+\bar{B}\rho F)^T P]e + 2\sum_{i=1}^{m} \tilde{\rho}_i e^T P\bar{b}_i r_i + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i\dot{\tilde{\rho}}_i}{l_i} \tag{9.32}$$

Due to ρ_i is an unknown constant, we have $\dot{\hat{\rho}}_i(t) = \dot{\tilde{\rho}}_i(t)$.
If the adaptive law is chosen as

$$\begin{aligned}\dot{\hat{\rho}}_i &= Proj_{[\underline{\rho}_i,\bar{\rho}_i]}\{-l_i e^T P\bar{b}_i r_i\}\\ &= \begin{cases} 0, & if \quad \begin{array}{l}\hat{\rho}_i(t)=\underline{\rho}_i, \quad and \quad -l_i e^T P\bar{b}_i r_i \le 0 \quad or\\ \hat{\rho}_i(t)=\bar{\rho}_i, \quad and \quad -l_i e^T P\bar{b}_i r_i \ge 0;\end{array}\\ -l_i e^T P\bar{b}_i r_i, & otherwise\end{cases}\end{aligned}$$

then we have

$$\frac{\tilde{\rho}_i\dot{\tilde{\rho}}_i}{l_i} \le -\tilde{\rho}_i e^T P\bar{b}_i r_i \tag{9.33}$$

so

$$\dot{V} \le e^T[P(\bar{A}+\bar{B}\rho F) + (\bar{A}+\bar{B}\rho F)^T P]e \tag{9.34}$$

From (9.29) and $F = W_1 Z_1^{-1}, Z_1 = P^{-1}$, we have

$$P(\bar{A}+\bar{B}\rho F) + (\bar{A}+\bar{B}\rho F)^T P < 0 \quad for\ all \quad \rho \in N_\rho.$$

Furthermore, by the above mentioned LMI, we can obtain

$$P(\bar{A}+\bar{B}\rho F) + (\bar{A}+\bar{B}\rho F)^T P < 0 \quad for\ all \quad \rho \in \Delta,$$

that is

$$\dot{V} \leq -\alpha\|e\|^2 \leq 0, \tag{9.35}$$

where

$$\alpha := -\lambda_{\max_{\rho\in\Delta}}[P(\bar{A}+\bar{B}\rho F)+(\bar{A}+\bar{B}\rho F)^T P] > 0. \tag{9.36}$$

We can get $V \in L^\infty$ according to (9.35). It also implies $e \in L^\infty$ from (9.31), so the augmented state error (9.28) is stabilized. Furthermore, if we integrate (9.35) from 0 to ∞ on both sides, we can obtain $e(t) \in L^2$. The proof is completed.

Next, we design $r(t)$ so that the augmented system of target model (9.26) matches that of the normal model (9.10).
Let $r(t) = \hat{\rho}^{-1}(t)K_N\tilde{x}(t)$, then (9.26) becomes

$$\dot{\tilde{x}}(t) = \bar{A}\tilde{x}(t) + \bar{B}K_N\tilde{x}(t) + Gy_r(t) \tag{9.37}$$

which matches the closed-loop augmented system of normal case (9.13) exactly.
So from the result of Lemma 1, we get $\tilde{x}(t) \in L^\infty$. It also implies $r(t)$ is bounded. Together with $e(t) \in L^\infty$, we can obtain the state vector of augmented fault model (9.11) $\bar{x}(t)$ is also bounded. According to the state error system (9.27), we can obtain $\dot{e}(t)$ is bounded. This, along with a fact that $e(t) \in L^\infty \cap L^2$, implies that $\lim_{t\to\infty} e(t) = 0$ i.e., $\bar{x}(\infty) = \tilde{x}(\infty) = \bar{x}_N(\infty)$ where $\bar{x}_N(t)$ represents the state vector of the augmented normal system. So the state vectors in fault cases asymptotically track that of the normal state and the control objective is achieved.
Here the chosen adaptive controller is

$$u(t) = r(t) - Fe(t) = \hat{\rho}^{-1}(t)K_N\tilde{x}(t) - Fe(t) = u_N(t) + u_{ad}(t) \tag{9.38}$$

where $u_N(t) = K_N\bar{x}(t)$, $u_{ad}(t) = \hat{\rho}^{-1}(t)(I - \hat{\rho}(t))K_N\tilde{x}(t) + (K_N - F)e(t)$.

Prior to any failures, the error system is at its equilibrium, i.e., $e(t) = 0$ and $\hat{\rho}_i(t) = 1$ if we choose $e(0) = 0$ and $\hat{\rho}_i(0) = 1$. At this time, $u(t) = u_N(t)$ since $u_{ad}(t) = 0$. This implies the closed-loop normal system with controller (9.38) can achieve the optimized tracking performance.
When faults in actuators occur, the corresponding efficiency factor ρ_i deviates from 1, thus creating a mismatch between $\tilde{x}(t)$ and $\bar{x}(t)$; Hence nonzero state error occurs. At the same time, the adaptive estimates of the actuator efficiency factor become active. A new control law $u_{ad}(t)$ is added to the normal law. Then the fault cases compensate the fault effect automatically and asymptotically track the normal case.

Remark 9.1 *Using the MATLAB LMI toolbox, we can directly solve (9.29) for all $\rho \in N_\rho$ (here N_ρ contains a maximum of 2^m elements) and get a feasible solution of Z_1 and W_1. Then the corresponding error feedback gain F can be obtained by $F = W_1 Z_1^{-1}$.*

Remark 9.2 *The proposed controller design procedure optimized the normal tracking performance. This presents an advantage as systems are operating under the normal condition most of the time. Because* $K_N = WZ^{-1}$ *in (9.14) and* $F = W_1 Z_1^{-1}$ *in (9.29) are irrelative, the performance optimization procedure of the augmented normal system is without any conservativeness.*

9.4 Example

Example 9.1 *In this section, an example of tracking control for a linearized F-16 aircraft model is given to demonstrate the proposed methods. After linearization and allowing the left/right control surfaces to move independently, the aircraft model is described by*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{9.39}$$

where $x(t) = [u, w, q, v, p, r]^T$ *is the state,* $u(t) = [\delta_{hr}, \delta_{hl}, \delta_{ar}, \delta_{al}, \delta_r]^T$ *is the control input and* $y(t) = [q, \dot{\mu}_{rot}, r_{stab}, \alpha, \beta]^T$ *is the output, respectively.* u, v, w *are components of aircraft velocity along* X, Y, Z *body axes, respectively.* p, q, r *are roll rate about* X *body axis, pitch rate about* Y *body axis and yaw rate about* Z *body axis, respectively.* $\delta_{hl}, \delta_{ar}, \delta_{al}, \delta_r$ *are right horizontal stabilator, left horizontal stabilator, right aileron, left aileron and rudder, respectively.* $\dot{\mu}_{rot}$ *is stability-axis roll rate and* r_{stab} *is stability-axis yaw rate.* α *is angle of attack and* β *is angle of sideslip.*

$$A = \begin{bmatrix} -0.0150 & 0.0480 & -5.9420 & 0.0020 & 0 & 0 \\ -0.0910 & -0.9570 & 138.3610 & 0.0160 & 0 & 0 \\ 0 & 0.0050 & -1.0220 & -0.0010 & 0 & -0.0030 \\ 0 & 0 & 0 & -0.2800 & 6.2670 & -151.1440 \\ 0 & 0 & 0 & -0.1820 & -3.4190 & 0.6400 \\ 0 & 0 & 0.0030 & 0.0450 & -0.0300 & -0.4540 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0240 & 0.0240 & 0.0250 & 0.0250 & 0 \\ -0.1720 & -0.1720 & -0.1800 & -0.1800 & 0 \\ -0.0870 & -0.0870 & -0.0080 & -0.0070 & 0 \\ -0.3150 & 0.3150 & 0.0230 & -0.0230 & 0.1210 \\ -0.1890 & 0.1890 & -0.3460 & 0.3460 & 0.1240 \\ -0.1680 & 0.1680 & -0.0150 & 0.0150 & -0.0590 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 57.2960 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.2470 & 2.3700 \\ 0 & 0 & 0 & 0 & -2.3700 & 57.2470 \\ -0.0160 & 0.3760 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3760 & 0 & 0 \end{bmatrix}$$

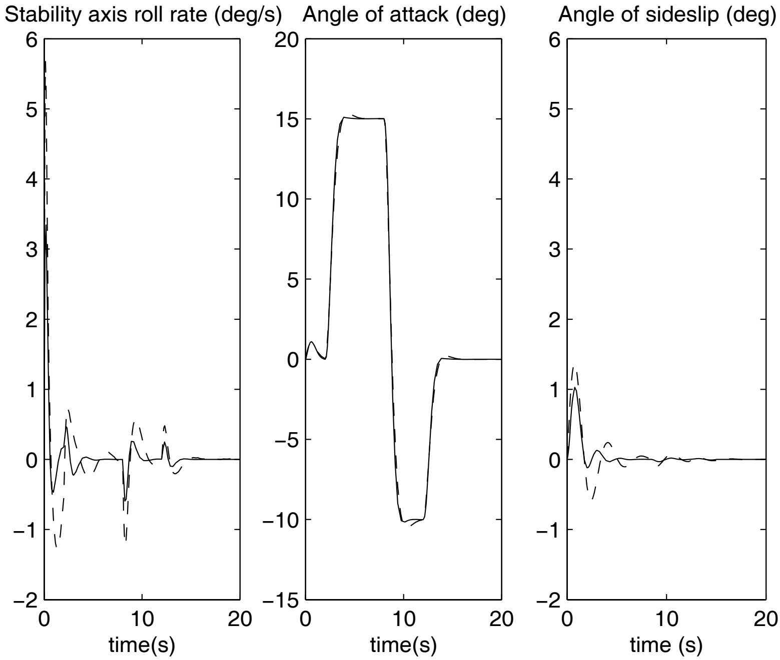


FIGURE 9.2
Required output responses in normal case with adaptive controller (solid) and fixed gain controller (dashed).

A, B and C are given in the appendix, which are the same as those in Example 1 of [84].

Here, each of the five actuators may lose its effectiveness. The lower and upper bounds of each effectiveness factor are 0.1 and 1, respectively.

The tracking command in the simulation is step of final value 2.

Let $\gamma = 2$ and

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, Q_1 = \text{diag}[0.16, 0.09, 0.25], \;\; Q_2 = \text{diag}[0, 0.04, 0, 0, 0, 0].$$

$$R = \text{diag}[0.25, 0.25, 0.01, 0.01, 0.04],$$

where the matrix S determines the output required to track, i.e., $\dot{\mu}_{rot}, \alpha, \beta$. In order to maintain the conventional control surface movements (i.e., symmetric motion for left and right horizontal stabilator, and antisymmetric motion for left and right ailerons) under normal operation, we force

$$K_N = [K_1^T, K_1^T, K_2^T, -K_2^T, K_3^T]^T$$

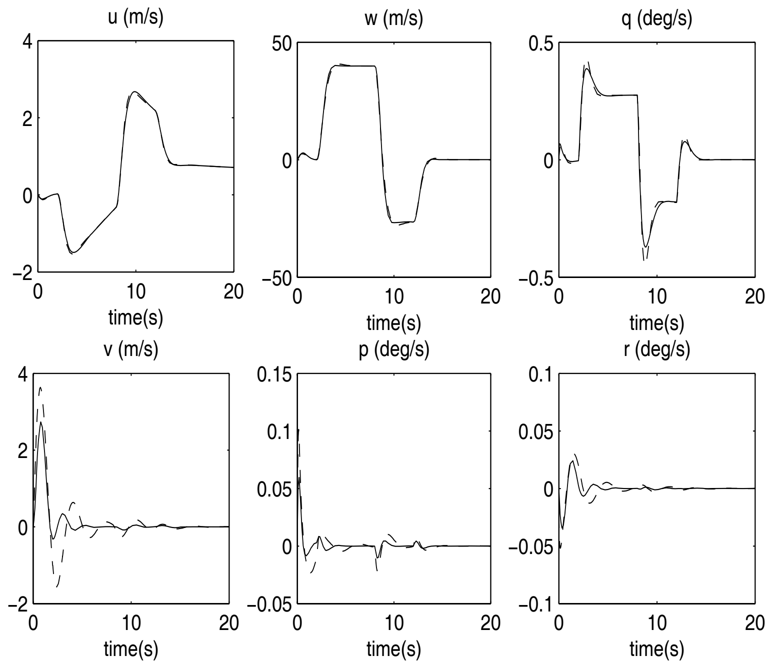


FIGURE 9.3
State vector in normal case with adaptive controller (solid) and fixed gain controller (dashed).

with $K_1, K_2, K_3 \in R^{1\times(l+n)}$.

For comparison purpose, our adaptive reliable controller and a *traditional reliable controller* with fixed gains are carried out in the following simulation. From Theorem 9.1, we can get the normal controller $u_N(t) = K_N x(t)$ with an optimal normal tracking performance of 59.8713. However, if we solve the reliable tracking problem with a fixed gain controller K_f guaranteeing all possible cases stabilized and normal tracking performance optimal, instead of this adaptive reliable tracking controller $u(t) = u_N(t) + u_{ad}(t)$, the designed optimal normal tracking performance is 246.1533 with achieved normal performance 143.6311. As systems are operating under the normal condition most of the time, this fact that our adaptive reliable tracking controller improves the normal tracking performance significantly compared to the fixed gain tracking controller K_f is more considerable and important.

To verify the superior performance of the proposed adaptive controller, the following simulations are achieved with the case that actuator fault occurs while the aircraft is maneuvering. Here angle of attack maneuver is considered. The initial angle of attack command is 0 degree and after 2 seconds, the angle of attack command changes into 15 degrees. Then at $t = 8$ seconds, it

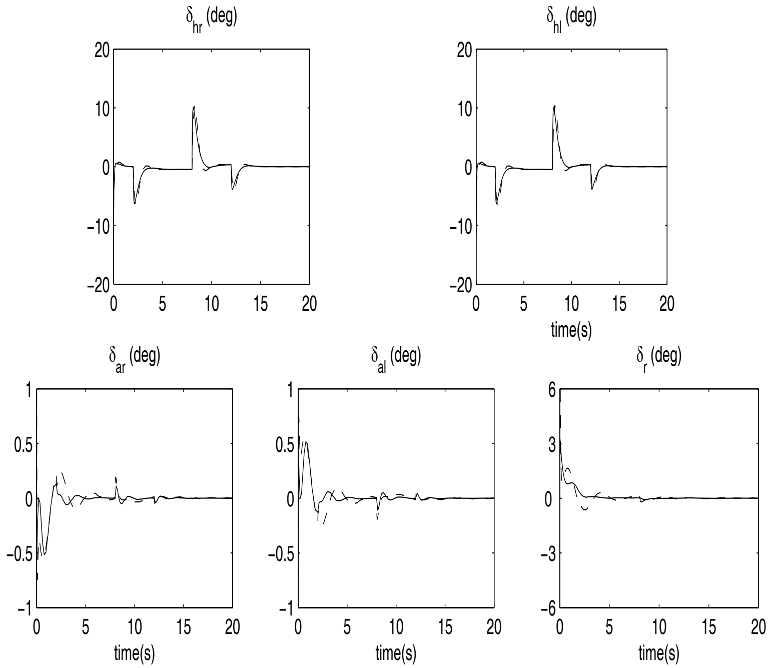


FIGURE 9.4
Input vector in normal case with adaptive controller (solid) and fixed gain controller (dashed).

becomes -10 degrees and recovers to 0 degrees at $t = 12$ seconds. During this time, stability axis roll rate and angle of sideslip commands remain 0 degree. Simulation studies are also carried out to verify the superiority of the designed controller.

Figure 9.2-Figure 9.4 are response curves in normal case. From Figure 9.2, we find that the proposed adaptive method tracks the command faster. In Figure 9.3, the state vector convergent rate with adaptive controller is no worse than the fixed gain controller K_f. Moreover, due to the same tracking command, those state vectors of the two controllers may converge to the same values. Figure 9.4 is the control input histories with these two controllers.

Next, the following fault case is considered. At $t = 2$ (seconds), rudder actuator loss of effectiveness of 30% has to be tolerated.

Figure 9.5 -Figure 9.7 describe some response curves in fault case. In Figure 9.5, our adaptive controller performs better even in fault case. It should be noted that in our adaptive design the required output responses track the command in fault case indirectly by the augmented state vector of fault case tracking that of normal case. To verify the characteristic of our adaptive tracking controller, the state error between fault case and normal case with these

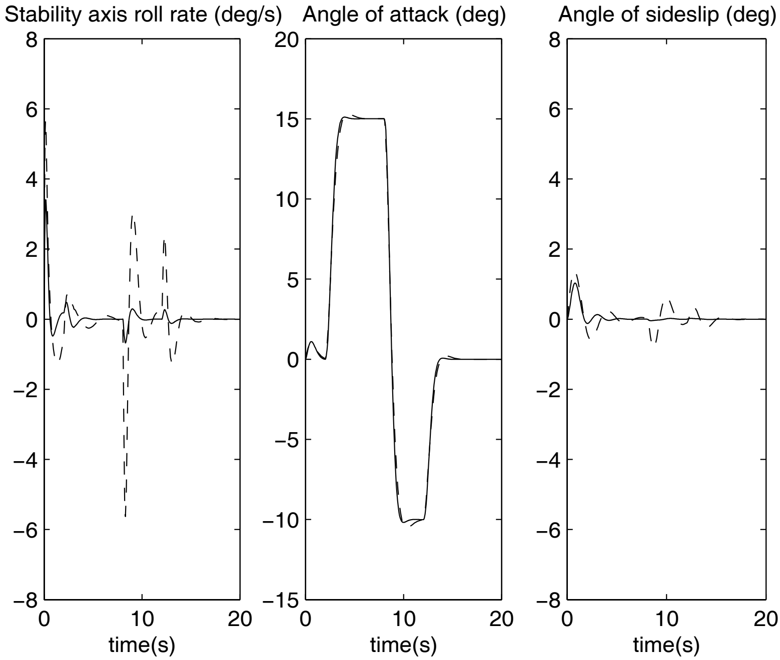


FIGURE 9.5
Required output responses in fault case with adaptive controller (solid) and fixed gain controller (dashed).

two controllers is given in Figure 9.6. From our adaptive controller designed process, the state error vector can quickly converge to zero. While in fixed gain controller design, this property cannot be guaranteed. However, state error may become zero after required output responses track the same tracking command. The corresponding control input histories are given in Figure 9.7.

Even though the newly proposed adaptive reliable controller works better in the absences of modeling error, measurement noise and disturbance, it is also important to show its robust performance in the presence of *uncertainty*. Accordingly about 50% modeling error which occurs in the value of system matrix A, a vertical gust disturbance of 5 m/s and a white Gaussian noise with variance of 0.01 are introduced into the system and measurement channels, respectively. Subsequently, the performance of the system is evaluated for the fault case. The required output responses and input history are shown in Figure 9.8 and Figure 9.9, where one can clearly see the adaptive controller still performs better. Summarizing all the cases (normal case and fault cases), it is noted that the adaptive tracker design method can significantly improve the normal performance than fixed gain method in both theory and simulation results. And in fault case, our adaptive reliable tracker has better results than

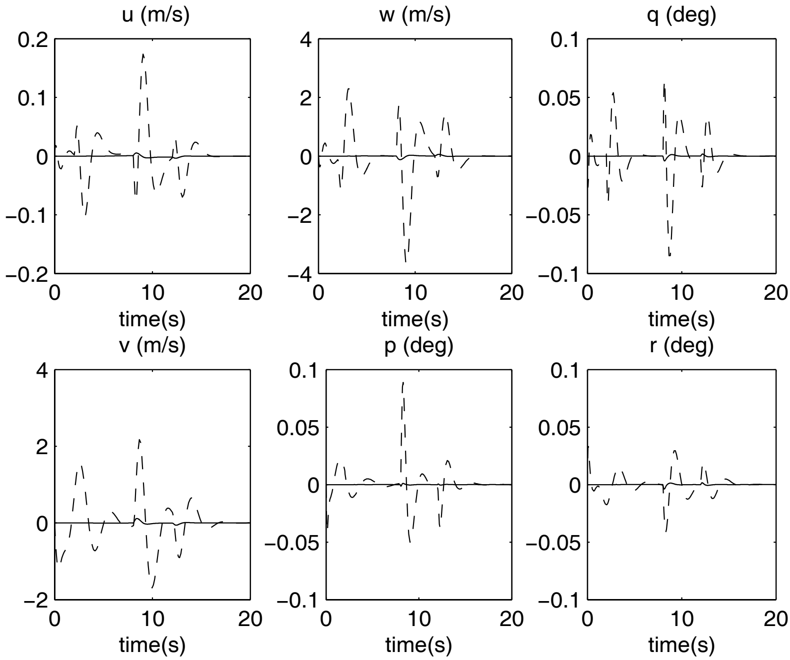


FIGURE 9.6
State error between fault case and normal case with adaptive controller (solid) and fixed gain controller (dashed).

those of fixed gain reliable controller K_f. It can also be observed that as more and more fault cases are considered in the design, our method gives more improvement of tracking performance in normal case.

9.5 Conclusion

This chapter has studied the reliable tracking problem for linear systems against actuator faults using the LMI method and adaptive method. Based on the online estimation of eventual faults, a new control law is added to the normal control law to reduce the fault effect on systems without the need for an FDI mechanism. The proposed controller can make the normal tracking performance of the closed-loop system optimized without any conservativeness and make the states of fault modes asymptotically track that of the normal mode. The simulation results of an example of F-16 have been given to show the effectiveness of the proposed method.

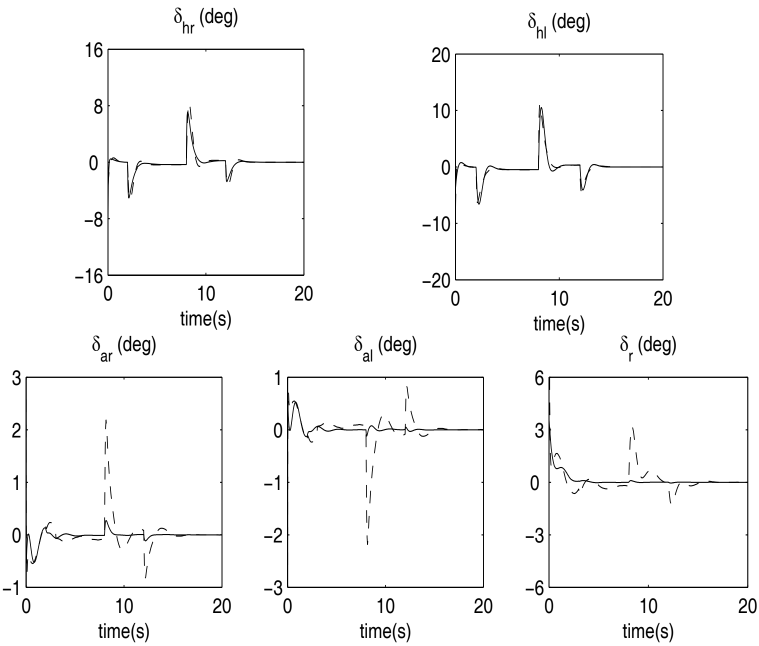


FIGURE 9.7
Input vector in fault case with adaptive controller (solid) and fixed gain controller (dashed).

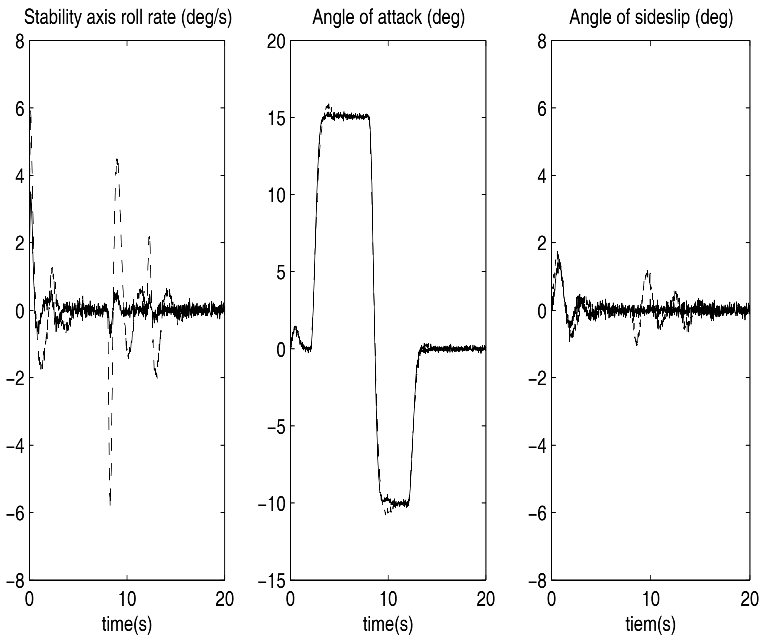


FIGURE 9.8
Robust required output responses in fault case and uncertainties with adaptive controller (solid) and fixed gain controller (dashed).

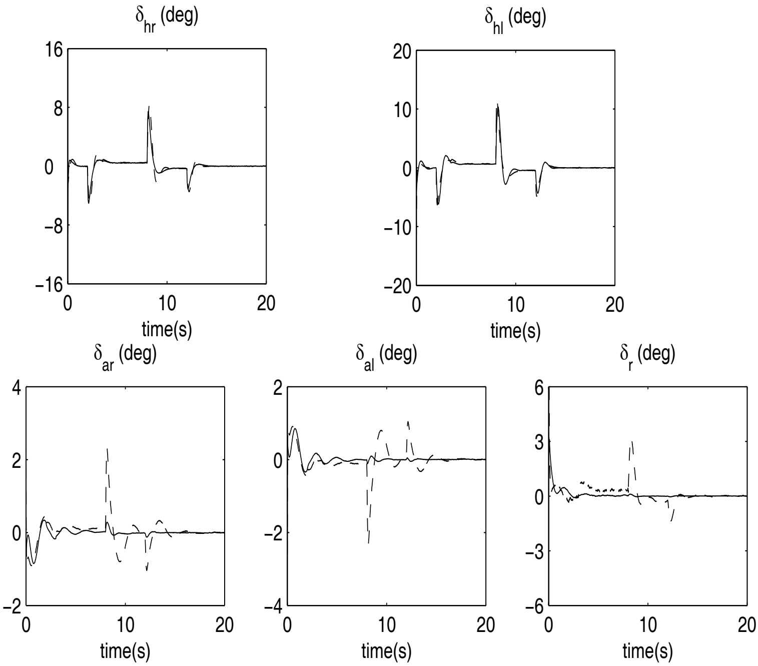


FIGURE 9.9
Robust input vector in fault case and uncertainties with adaptive controller (solid) and fixed gain controller (dashed).

10

Adaptive Reliable Control for Nonlinear Time-Delay Systems

10.1 Introduction

Over the last three decades, considerable attention has been paid to analysis and synthesis of time-delay systems [12, 51, 69, 89, 92, 93, 103, 116, 147]. The increasing interest about this topic can be understood by the fact that time delays appear as an important source of *instability* or performance degradation in a great number of important engineering problems involving material, information or energy transportation [23, 33, 34, 56, 57, 98, 104, 130, 135, 137, 144, 158, 159, 163]. In Chapter 9, the adaptive reliable tracking controller design for linear time-invariant systems is investigated. It should be noted that the proposed method in Chapter 9 is not suitable for the dynamic systems with *time-delay*.

Based on the theory of Chapter 9, we will focus on the adaptive reliable control problem of a class of nonlinear time-delay systems with disturbance. Here, the *actuator faults* are types of loss of effectiveness. Comparing with other existing results about time-delay systems, the novelty of this chapter lies in the following aspects. Firstly, the performance index in normal case is optimized in the framework of *linear matrix inequalities*. Since systems are operating under the normal condition most of the time, this phenomenon is meaningful. Secondly, an appropriate *Lyapunov-Krasovskii functional* is chosen to design a new *delay-dependent adaptive law* to compensate the *fault effects* on systems and to prove *stability* in normal and fault cases. Thirdly, the state vectors of normal and fault cases with disturbance can track that of the normal case without disturbance, which has the designed optimal performance. Numerical and simulation results are also provided to demonstrate the effectiveness of the proposed controller.

10.2 Problem Statement

Consider a class of *nonlinear time-delay systems* described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + A_1 f(t, x(t), x(t-d)) + Bu(t) + B_1\omega(t) \\ x(t) &= \phi(t), \quad t \in [-d, 0] \end{aligned} \tag{10.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, respectively. d is a positive constant delay. $\omega(t) \in L_\infty \bigcap L_2$ is the *exogenous disturbance*, $\{\phi(t), t \in [-d, 0]\}$ is a real-valued initial function, $f(t, x(t), x(t-d))$ is a known nonlinearity. Matrices A, A_d, A_1, B, B_1 are constant matrices with appropriate dimensions.

Assumption 10.1 *For all $x_1, x_2, y_1, y_2 \in R^n$, the nonlinear function satisfies*

$$\begin{aligned} &\|f(t, x_1, x_2) - f(t, y_1, y_2)\| \leq \\ &\|M_1(x_1 - y_1)\| + \|M_2(x_2 - y_2)\| \end{aligned}$$

where M_1, M_2 are real constant matrices.

The same *actuator fault model* as that in Chapter 9 is considered here

$$u_i^F(t) = \rho_i u_i(t), \ \rho_i \in [\underline{\rho_i}, \ \bar{\rho}_i], \ 0 < \underline{\rho_i} \leq 1, \ \bar{\rho}_i \geq 1 \tag{10.2}$$

where $u_i^F(t)$ represents the signal from the actuator that has failed. $\underline{\rho_i}$ and $\bar{\rho}_i$ represent the lower and upper bounds of ρ_i, respectively. Here, the considered actuator faults are types of loss of effectiveness. Note that, when $\underline{\rho_i} = \bar{\rho}_i = 1$, there is no fault for the ith actuator u_i. Moreover, Δ and N_ρ are the same as those in Chapter 9.

Hence, the dynamic with actuator faults (10.2) is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + A_1 f(t, x(t), x(t-d)) + B\rho u(t) + B_1\omega(t) \\ x(t) &= \phi(t), \quad t \in [-d, 0] \end{aligned} \tag{10.3}$$

When $\rho = I$, the system (10.3) is the normal model (10.1).

Control objectives: During normal operation and in the event of actuator faults, the closed-loop system is *asymptotically stable* and the state vector of closed-loop asymptotically tracks that of the normal case without disturbance, which makes the bound of the following quadratic cost function J optimized

$$J = \int_0^\infty (x^T(t) N_1 x(t) + u^T(t) N_2 u(t)) dt \tag{10.4}$$

10.3 Adaptive Reliable Controller Design

In this section, a sufficient condition for the optimization of normal tracking without disturbance is first given. Secondly, based on the normal controller, we add a new control law to the normal law in order to reduce the fault effect on the system and achieve the desired control objective by using *adaptive method*.

Now we design the normal controller $u_N(t)$ for the normal model without disturbance

$$\dot{x}(t) = Ax(t) + A_d x(t-d) + A_1 f(t, x(t), x(t-d)) + Bu_N(t) \tag{10.5}$$

with the following *state feedback* controller

$$u_N(t) = K_N x(t) \tag{10.6}$$

Then the closed-loop system is given by

$$\dot{x}(t) = (A + BK_N)x(t) + A_d x(t-d) + A_1 f(t, x(t), x(t-d)) \tag{10.7}$$

Denote

$$\begin{aligned}
\Sigma_{11} &= A\bar{P}_N + \bar{P}_N A^T + BY + Y^T B^T - \lambda\mu^{-1}(A_d\bar{Q}_N + \bar{Q}_N A_d^T) - \lambda^2\mu^{-2}\bar{Q}_N,\\
\Sigma_{12} &= \mu^{-1}A_d\bar{Q}_N + \bar{P}_N + \lambda\mu^{-1}\bar{Q}_N + \lambda\mu^{-2}\bar{Q}_N,\\
\Sigma_{22} &= -2\mu^{-1}\bar{Q}_N - \mu^{-2}\bar{Q}_N,\ \Sigma_{13} = -A_1,\ \Sigma_{33} = -2I,\\
\Sigma_{14} &= d(Y^T B^T + \bar{P}_N A^T - \lambda\mu^{-1}\bar{Q}_N A_d^T),\ \Sigma_{24} = d\mu^{-1}\bar{Q}_N A_d^T,\\
\Sigma_{34} &= -dA_1^T,\quad \Sigma_{44} = -d\bar{R}_N.
\end{aligned}$$

Next, a sufficient condition for the guaranteed cost control problem of the closed-loop system (10.5) is presented.

Theorem 10.1 *For given numbers $\lambda \neq 0$ and $\mu \neq 0$, if there exist matrices $\bar{P}_N > 0, \bar{R}_N > 0, \bar{Q}_N > 0$, and Y such that*

$$\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & 0 & \bar{P}_N & \bar{P}_N & Y^T & \bar{P}_N M_1^T & \Upsilon_1\\
* & \Sigma_{22} & 0 & \Sigma_{24} & d\bar{R}_N & 0 & 0 & 0 & 0 & \Upsilon_2\\
* & * & \Sigma_{33} & \Sigma_{34} & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & \Sigma_{44} & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -d\bar{R}_N & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & 0 & -Q & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -N_1^{-1} & 0 & 0 & 0\\
* & * & * & * & * & * & * & -N_2^{-1} & 0 & 0\\
* & * & * & * & * & * & * & * & -I & 0\\
* & * & * & * & * & * & * & * & * & -\frac{I}{2}
\end{bmatrix} < 0 \tag{10.8}$$

where $\Upsilon_1 = \lambda\mu^{-1}\bar{Q}_N M_2^T$ *and* $\Upsilon_2 = -\mu^{-1}\bar{Q}_N M_2^T$.
Then the following controller stabilizes the closed-loop normal system without disturbance (10.7)

$$u_N(t) = K_N x(t),\ K_N = Y\bar{P}_N^{-1} \tag{10.9}$$

Furthermore, the performance index (10.4) satisfies

$$J \le \phi^T(0)\bar{P}_N^{-1}\phi(0) + \int_{-d}^{0}\int_{\theta}^{0}\dot{\phi}^T(s)\bar{R}_N^{-1}\dot{\phi}(s)dsd\theta + \int_{-d}^{0}\phi^T(s)\bar{Q}_N^{-1}\phi(s)ds \tag{10.10}$$

Proof 10.1 *We choose the following Lyapunov-Krasovskii functional*

$$V = x^T(t)P_N x(t) + \int_{-d}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R_N\dot{x}(s)dsd\theta + \int_{t-d}^{t}x^T(s)Q_N x(s)ds \tag{10.11}$$

where $P_N = \bar{P}_N^{-1}$, $R_N = \bar{R}_N^{-1}$ *and* $Q_N = \bar{Q}_N^{-1}$.

The derivative of V *along the trajectory of the state equation (10.7) can be written as*

$$\begin{aligned}\dot{V} &= \dot{x}^T(t)P_N x(t) + x(t)^T P_N\dot{x}(t) + x(t)^T Q_N x(t) - x^T(t-d)Q_N x(t-d)\\ &\quad + d\dot{x}^T(t)R_N\dot{x}(t) - \int_{t-d}^{t}\dot{x}^T(s)R_N\dot{x}(s)ds\\ &\quad + f^T(t,x,x(t-d))f(t,x,x(t-d)) - f^T(t,x,x(t-d))f(t,x,x(t-d))\end{aligned} \tag{10.12}$$

From Assumption 10.1, we obtain

$$\|f(t,x(t),x(t-d)\| \le \|M_1x(t)\| + \|M_2x(t-d)\| \tag{10.13}$$

then

$$\|f(t,x(t),x(t-d)\|^2 \le 2\|M_1x(t)\|^2 + 2\|M_2x(t-d)\|^2 \tag{10.14}$$

that is

$$\begin{aligned}&f^T(t,x,x(t-d))f(t,x,x(t-d)) \le\\ &2x^T(t)M_1^T M_1x(t) + 2x^T(t-d)M_2^T M_2x(t-d)\end{aligned} \tag{10.15}$$

Applying the integral inequality (2.50) in Lemma 2.15 to the term on the right-hand side of (10.12) for any $Y_1, Y_2 \in R^{n\times n}$ *yields the following integral inequality*

$$\begin{aligned}-\int_{t-d}^{t}\dot{x}^T(s)R_N\dot{x}(s)ds &\le \eta^T(t)\begin{bmatrix} Y_1^T + Y_1 & -Y_1^T + Y_2\\ * & -Y_2^T - Y_2\end{bmatrix}\eta(t)\\ &\quad + d\eta^T(t)\begin{bmatrix} Y_1^T\\ Y_2^T\end{bmatrix}R_N^{-1}\begin{bmatrix} Y_1 & Y_2\end{bmatrix}\eta(t)\end{aligned} \tag{10.16}$$

where $\eta^T(t) = [x^T(t), x^T(t-d)]$.

Substituting (10.14) and (10.16) into (10.12), carrying out some algebraic manipulations, and rearranging the terms gives

$$\dot{V} \leq \xi^T(t)\left[H + d\Gamma_1^T R_N \Gamma_1\right]\xi(t) + d\xi^T(t)\Gamma_2^T R_N^{-1}\Gamma_2\xi(t) \tag{10.17}$$

where

$$\begin{aligned}
\xi^T &= [x^T(t), x^T(t-d), f^T(t, x(t), x(t-d))],\\
H &= \begin{bmatrix} H_{11} & P_N A_d - Y_1^T + Y_2 & P_N A_1 \\ * & -Q_N - Y_2^T - Y_2 + 2M_2^T M_2 & 0 \\ * & * & -I \end{bmatrix},\\
H_{11} &= P_N(A + BK_N) + (A + BK_N)^T P_N + Q_N + 2M_1^T M_1 + Y_1 + Y_1^T,\\
\Gamma_1 &= \begin{bmatrix}(A + BK_N) & A_d & A_1\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} Y_1 & Y_2 & 0\end{bmatrix}
\end{aligned}$$

From (10.17), we find that, if the following matrix inequality holds:

$$\Sigma = \begin{bmatrix} H & d\Gamma_1^T & d\Gamma_2^T \\ * & -dR_N^{-1} & 0 \\ * & * & -dR_N \end{bmatrix} < 0 \tag{10.18}$$

then applying the Schur complement yields $\dot{V}(t) < 0$*. Thus, by using the Lyapunov-Krasovskii functional theorem, we can conclude the closed-loop system (10.7) is asymptotically stable.*

In order to obtain the controller gain, K_N*, from the nonlinear matrix inequality (10.18) the nonlinearities come from*

$$W = \begin{bmatrix} P_N & 0 & 0 \\ Y_1 & Y_2 & 0 \\ 0 & 0 & -I \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A + BK_N & A_d & A_1 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Then

$$\begin{aligned}
H &= W^T\bar{A} + \bar{A}^T W + diag\{Q_N + 2M_1^T M_1, -Q_N + 2M_2^T M_2, 0\},\\
\Gamma_2 &= \begin{bmatrix} 0 & I & 0 \end{bmatrix} W
\end{aligned}$$

Now, consider the case in which $Y_1 = \lambda P_N, Y_2 = \mu Q_N, \lambda \neq 0$ *and* $\mu \neq 0$*. In this case* W *is invertible and*

$$W^{-1} = \begin{bmatrix} P_N^{-1} & 0 & 0 \\ -\lambda\mu^{-1}Q_N^{-1} & \mu^{-1}Q_N^{-1} & 0 \\ 0 & 0 & -I \end{bmatrix}$$

Denote $T = diag\{W^{-1}, I, R_N^{-1}\}$

$$T^T\Sigma T = \begin{bmatrix} H_T & dW^{-T}\Gamma_1^T & dH_1^T \\ * & -dR_N^{-1} & 0 \\ * & * & -dR_N^{-1} \end{bmatrix} \tag{10.19}$$

with

$$H_T = \bar{A}W^{-1} + W^{-T}\bar{A}^T + W^{-T}diag\{Q_N + 2M_1^T M_1, -Q_N + 2M_2^T M_2, 0\}W^{-1}$$
$$H_1 = \begin{bmatrix} 0 & R_N^{-1} & 0 \end{bmatrix}$$

Setting $\bar{P}_N = P_N^{-1}, \bar{R}_N = R_N^{-1}$, $\bar{Q}_N = Q_N^{-1}, Y = K_N P_N^{-1} = K\bar{P}_N$ *and performing some simple algebraic manipulations, it follows that if (10.8) holds, then the Schur complement ensures that* $T^T \Sigma T < 0$ *and thus* $\Sigma < 0$. *So the resulting closed-loop system is asymptotically stable and the desired controller is defined by*

$$\begin{aligned} J &= \int_0^\infty (x^T(t)N_1 x(t) + u^T(t)N_2 u(t))dt \\ &\leq \int_0^\infty \left(x^T(t)N_1 x(t) + u^T(t)N_2 u(t) + \frac{dV}{dt} \right) dt + V(0) \\ &\leq \int_0^\infty \xi^T(t)\Upsilon\xi(t)dt + V(0) \end{aligned} \tag{10.20}$$

On the other hand, from (10.8), we can get

$$\Upsilon = \begin{bmatrix} H + N_1 + K_N^T N_2 K_N & d\Gamma_1^T & d\Gamma_2^T \\ * & -dR_N^{-1} & 0 \\ * & * & -hR_N \end{bmatrix} < 0$$

Thus

$$J \leq V(0) = \phi^T(0)\bar{P}_N^{-1}\phi(0) + \int_{-d}^0 \int_\theta^0 \dot{\phi}^T(s)\bar{R}_N^{-1}\dot{\phi}(s)dsd\theta + \int_{-d}^0 \phi^T(s)\bar{Q}_N^{-1}\phi(s)ds$$

The proof is completed.

Based on the conditions in Theorem 10.1, we propose the following theorem to give a method of selecting a controller minimizing the upper bound of the guaranteed cost (10.4).

Theorem 10.2 *Consider system (10.5) with cost function (10.4), for given non-zero numbers* λ *and* μ, *if the following optimization problem*

$$\min_{\bar{P_N}, \bar{Q}_N, \bar{R}_N, Y, \Sigma_1, \Sigma_2, \Sigma_3} \quad Trace(\Sigma_1) + Trace(\Sigma_2) + Trace(\Sigma_3) \ s.t.$$

$$\begin{aligned} &(i)\ (10.8) \\ &(ii) \begin{bmatrix} -\Sigma_1 & \Pi_1^{\frac{1}{2}} \\ * & -\bar{P}_N \end{bmatrix} < 0, \\ &(iii) \begin{bmatrix} -\Sigma_2 & \Pi_2^{\frac{1}{2}} \\ * & -\bar{R}_N \end{bmatrix} < 0, \\ &(iv) \begin{bmatrix} -\Sigma_3 & \Pi_3^{\frac{1}{2}} \\ * & -\bar{Q}_N \end{bmatrix} < 0, \end{aligned} \tag{10.21}$$

has a solution $\bar{P_N}, \bar{Q}_N, \bar{R}_N, Y, \Sigma_1, \Sigma_2, \Sigma_3$, then the control law of form (10.6) is a suboptimal state feedback guaranteed control law, which ensures the minimization of the guaranteed cost (10.4) for normal system without disturbance (10.5), where

$$\Pi_1 = \phi(0)\phi^T(0), \quad \Pi_2 = \int_{-d}^{0}\int_{\theta}^{0} \dot{\phi}(s)\dot{\phi}^T(s)dsd\theta, \quad \Pi_3 = \int_{-d}^{0} \phi(s)\phi^T(s)ds$$

Proof 10.2 *Theorem 10.1, the control law (10.6) constructed in terms of any feasible solution $\bar{P}_N, \bar{Q}_N, \bar{R}_N, Y, \Sigma_1, \Sigma_2, \Sigma_3$ is a guaranteed cost controller of system (10.5).*
Considering Trace(AB) = Trace(BA), we have the following relations

$$\phi^T(0)\bar{P}_N^{-1}\phi(0) = tr(\Pi_1\bar{P}_N^{-1}) = tr(\Pi_1^{\frac{1}{2}}\bar{P}_N^{-1}\Pi_1^{\frac{1}{2}})$$
$$\int_{-d}^{0}\int_{\theta}^{0} \dot{\phi}^T(s)\bar{R}_N^{-1}\dot{\phi}(s)dsd\theta = tr(\Pi_2\bar{R}_N^{-1}) = tr(\Pi_2^{\frac{1}{2}}\bar{R}_N^{-1}\Pi_2^{\frac{1}{2}})$$
$$\int_{-d}^{0} \phi^T(s)\bar{Q}_N^{-1}\phi(s)ds = tr(\Pi_3\bar{Q}_N^{-1}) = tr(\Pi_3^{\frac{1}{2}}\bar{Q}_N^{-1}\Pi_3^{\frac{1}{2}})$$

It follows from the Schur complement and (10.21) that

$$\Pi_1^{\frac{1}{2}}\bar{P}_N^{-1}\Pi_1^{\frac{1}{2}} < \Sigma_1, \quad \Pi_2^{\frac{1}{2}}\bar{R}_N^{-1}\Pi_2^{\frac{1}{2}} < \Sigma_2, \quad \Pi_3^{\frac{1}{2}}\bar{Q}_N^{-1}\Pi_3^{\frac{1}{2}} < \Sigma_3$$

So it follows from (10.10) that

$$J \leq Trace(\Sigma_1) + Trace(\Sigma_2) + Trace(\Sigma_3)$$

The proof is completed.

In order to obtain online information on the effectiveness of actuators, i.e., $\hat{\rho}_i(t)$, the following *target model* is introduced

$$\dot{x}_m(t) = Ax_m(t) + A_d x_m(t-d) + A_1 f(t, x_m(t), x_m(t-d)) + B\hat{\rho}r(t) \quad (10.22)$$

where $\hat{\rho}(t) = \text{diag}\{\hat{\rho}_1(t), \cdots, \hat{\rho}_m(t)\}$, $\hat{\rho}_i(t)$ denotes the estimate of the efficiency factor. The signal $r(t) \in R^m$ is the input, which can be designed to achieve the control objectives.

If we define the *state error* vector as $e(t) = x_m(t) - x(t)$ and let the control input $u(t) = r(t) - F_1 e(t) - F_2 e(t-d)$, where F_1 and F_2 are the error feedback gains to be designed to make the error system stable, then the state error equation between (10.3) and (10.22) is written as

$$\begin{aligned}\dot{e}(t) = {} & (A + B\rho F_1)e(t) + (A_d + B\rho F_2)e(t-d) + B\tilde{\rho}r(t) - B_1\omega(t) \\ & + A_1(f(t, x_m(t), x_m(t-d)) - f(t, x(t), x(t-d))) \quad (10.23)\end{aligned}$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho(t) = \text{diag}\{\tilde{\rho}_1(t), \cdots, \tilde{\rho}_m(t)\}$
Let $B = [b_1 \cdots b_m] \in R^{n\times m}$, $r(t) = (r_1(t) \cdots r_m(t))^T \in R^m$, then the state error system (10.23) can be written as

$$\dot{e}(t) = (A + B\rho F_1)e(t) + (A_d + B\rho F_2)e(t-d) + \sum_{i=1}^{m} b_i\tilde{\rho}_i r_i(t) - B_1\omega(t) \\ + A_1(f(t, x_m(t), x_m(t-d)) - f(t, x(t), x(t-d))) \tag{10.24}$$

Denote

$$\begin{aligned}
\Delta_{11} &= AX + B\rho W_1 + W_3 + XA^T + W_1^T\rho B^T + W_3^T + (\varepsilon_1+\varepsilon_2)A_1A_1^T \\
&\quad + Q + dF_{11} + \varepsilon_5 B_1B_1^T \\
\Delta_{12} &= XA^T + W_3^T + W_1^T\rho B + F_{12} \\
\Delta_{22} &= \varepsilon_6 B_1B_1^T + (\varepsilon_2+\varepsilon_4)A_1A_1^T - d^{-1}(\alpha-1)X \\
\Delta_{13} &= A_dX - W_3 + dF_{13} + B\rho W_2 \\
\Delta_{23} &= A_dX - W_3 + B\rho W_2 + F_{23}
\end{aligned}$$

Next, a new delay-dependent adaptive law and the error feedback gains F_1, F_2 are designed to make the state error system (10.24) stable.

Theorem 10.3 *For given $\alpha > 1$, the state error system (9.28) is stabilized and $\lim_{t\to\infty} e(t) = 0$ if there exist positive definite matrices $X, Q, F_{11}, F_{22}, F_{33}$, positive scalars $\varepsilon_i, (i = 1, \dots 6)$ and any matrices $W_1, W_2, W_3, F_{12}, F_{13}, F_{23}$ such that the following inequalities hold for all $\rho \in N_\rho$*

$$\begin{bmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & XM_1^T & XM_1^T & \alpha dW_3^T & 0 & 0 \\
* & \Delta_{22} & \Delta_{23} & 0 & 0 & 0 & 0 & 0 \\
* & * & -Q + dF_{33} & 0 & 0 & 0 & XM_2^T & XM_2^T \\
* & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 \\
* & * & * & * & 0 & -\alpha dX & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_2 I & 0 \\
* & * & * & * & * & * & * & -\varepsilon_4 I
\end{bmatrix} < 0 \tag{10.25}$$

$$-X + F_{22} < 0 \tag{10.26}$$

$$\begin{bmatrix} -X & dW_3^T \\ * & -X \end{bmatrix} < 0 \tag{10.27}$$

$$\Xi = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ * & F_{22} & F_{23} \\ * & * & F_{33} \end{bmatrix} > 0 \tag{10.28}$$

and also $\hat{\rho}_i(t)$ is determined according to the adaptive law

$$\dot{\hat{\rho}}_i(t) = Proj_{[\underline{\rho_i},\ \bar{\rho}_i]}\{-l_i(e(t)+z(t))^T X^{-1} b_i r_i\} \tag{10.29}$$

where $z(t) = \int_{t-d}^{t} W_3 X^{-1} e(s)ds$, $l_i > 0$, $0 < \underline{\rho_i} \leq 1$ and $\bar{\rho}_i = 1$, $i = 1 \cdots m$. $Proj\{\cdot\}$ denotes the projection operator [70] whose role is to project the estimates $\hat{\rho}_i(t)$ to the interval $[\underline{\rho_i}, \bar{\rho}_i]$. Then the error feedback gains can be obtained by $F_1 = W_1 X^{-1}$ and $\overline{F_2} = W_2 X^{-1}.Proj\{\cdot\}$

Proof 10.3 *Define an operator $D(e_t) : C_{n,\ d} \to R^n$ as*

$$D(e_t) = e(t) + \int_{t-d}^{t} Ge(s)ds \tag{10.30}$$

where $e_t = e(t+s), s \in [-d, 0]$ and $G \in R^{n\times n}$ is a constant matrix which will be chosen.

We choose the following Lyapunov-Krasovskii functional

$$V = V_1 + V_2 + V_3 + V_4 + V_5 \tag{10.31}$$

where

$$V_1 = D^T(e_t)PD(e_t), \quad V_2 = \alpha \int_{t-d}^{t}\int_{s}^{t} e^T(u)G^T PGe(u)duds$$
$$V_3 = \int_{t-d}^{t} e^T(s)Se(s)ds, \quad V_4 = \int_{0}^{t}\int_{s-d}^{s} \chi^T \Omega\Xi\Omega\chi duds, \quad V_5 = \sum_{i=1}^{m} \frac{\tilde{\rho_i}^2}{l_i}$$

with $\chi = [e^T(s), e^T(u)G^T, e^T(s-d)]^T$, $P > 0$, $\Omega = diag\{P, P, P\}$, $S > 0$, $\tilde{\rho}_i(t) = \hat{\rho}_i(t) - \rho_i$.

The derivative of V along the trajectory of the state error equation (10.24) *can be written as* $V(t)$

$$\begin{aligned}
\dot{V}_1 &= 2D^T(e_t)P\dot{D}(e_t) \\
&= e^T(t)[P(A+B\rho F_1+G)+(A+B\rho F_1+G)^TP]e(t) \\
&\quad - 2(e(t)+z(t))^TPB_1\omega + 2(e(t)+z(t))^T\sum_{i=1}^{m}\tilde{\rho}_iPb_ir_i \\
&\quad + 2e^T(t)P(A_d+B\rho F_2-G)e(t-d) \\
&\quad + 2z^T(t)P(A+B\rho F_1+G)e(t) + 2z^T(t)P(A_d+B\rho F_2-G)e(t-d) \\
&\quad + 2(e(t)+z(t))^T(t)PA_1(f(x_m(t),x_m(t-d(t))-f(x(t),x(t-d(t))).
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2 &= \alpha de^T(t)G^TPGe(t) - \alpha\int_{t-d}^{t}e^T(s)G^TPGe(s)ds \\
&\le \alpha de^T(t)G^TPGe(t) - \int_{t-d}^{t}e^T(s)G^TPGe(s)ds - d^{-1}(\alpha-1)z^T(t)Pz(t), \\
\dot{V}_3 &= e^T(t)Se(t) - e^T(t-d)Se(t-d). \\
\dot{V}_4 &= de^T(t)PF_{11}Pe(t) + 2e^TPF_{12}Pz(t) + \int_{t-d}^{t}e^T(s)G^TPF_{22}GPe(s)ds \\
&\quad + 2de^TPF_{13}Pe(t-d) + 2z^TPF_{23}Pe(t-d) + de^T(t-d)PF_{33}Pe(t-d) \\
\dot{V}_5 &= 2\sum_{i=1}^{m}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}
\end{aligned}$$

where $z(t) = \int_{t-d}^{t}Ge(s)ds$ *and here we use*

$$z^T(t)Pz(t) \le d\int_{t-d}^{t}e^T(s)G^TPGe(s)ds,$$

which is obtained by Lemma l6.4 to get $\dot{V}_2$. *From Assumption 10.1, we obtain*

$$\begin{aligned}
&2e^T(t)PA_1(f(t,x_m(t),x_m(t-d)-f(t,x(t),x(t-d))) \\
&\le 2\|e^T(t)PA_1\|\|f(t,x_m(t),x_m(t-d)-f(t,x(t),x(t-d)\| \\
&\le 2\|e^T(t)PA_1\|(\|M_1e(t)\|+\|M_2e(t-d)\|) \\
&\le \varepsilon_1e^T(t)PA_1A_1^TPe(t)+\varepsilon_1^{-1}e^T(t)M_1^TM_1e(t) \\
&\quad +\varepsilon_2e^T(t)PA_1A_1^TPe(t)+\varepsilon_2^{-1}e^T(t-d)M_2^TM_2e(t-d)
\end{aligned} \tag{10.32}$$

$$\begin{aligned}
&2z^T(t)PA_1(f(t,x_m(t),x_m(t-d)-f(t,x(t),x(t-d))) \\
&\le 2\|z^T(t)PA_1\|\|f(t,x_m(t),x_m(t-d)-f(t,x(t),x(t-d)\| \\
&\le 2\|z^T(t)PA_1\|(\|M_1e(t)\|+\|M_2e(t-d)\|) \\
&\le \varepsilon_3z^T(t)PA_1A_1^TPz(t)+\varepsilon_3^{-1}e^T(t)M_1^TM_1e(t) \\
&\quad +\varepsilon_4z^T(t)PA_1A_1^TPz(t)+\varepsilon_4^{-1}e^T(t-d(t))M_2^TM_2e(t-d)
\end{aligned} \tag{10.33}$$

Furthermore, $\dot{V}_1$ *can be written as*

$$\begin{aligned}\dot{V}_1 &= 2D^T(e_t)P\dot{D}(e_t)\\ &= e^T(t)[P(A+B\rho F_1+G)+(A+B\rho F_1+G)^TP+(\varepsilon_1+\varepsilon_2)PA_1A_1^T\\ &\quad (\varepsilon_1^{-1}+\varepsilon_3^{-1})M_1^TM_1]e(t)-2(e(t)+z(t))^TPB_1\omega\\ &\quad +2e^T(t)P(A_d+B\rho F_2-G)e(t-d)+2z^T(t)P(A+B\rho F_1+G)e(t)\\ &\quad +2z^T(t)P(A_d+B\rho F_2-G)e(t-d)+(\varepsilon_3+\varepsilon_4)z^T(t)PA_1A_1^TPz^T(t)\\ &\quad +(\varepsilon_2^{-1}+\varepsilon_4^{-1})e^T(t-d)M_2^TM_2e(t-d)+2(e(t)+z(t))^T\sum_{i=1}^m\tilde{\rho}_iPb_ir_i\end{aligned}$$

If the adaptive law is chosen as

$$\begin{aligned}\dot{\hat{\rho}}_i &= Proj_{[\underline{\rho_i},\ \bar{\rho}_i]}\{-l_i(e(t)+z(t))^TPb_ir_i\}\\ &= \begin{cases}0, & if \quad \begin{array}{l}\hat{\rho}_i=\underline{\rho_i}\ and\ -l_i(e+z)^TPb_ir_i\le 0\ or\\ \hat{\rho}_i=\bar{\rho}_i\ and\ -l_i(e+z)^TPb_ir_i\ge 0;\end{array}\\ -l_i(e+z)^TPb_ir_i, & otherwise\end{cases}\end{aligned}$$

where $z(t)=\int_{t-d}^t Ge(s)ds$, *then*

$$\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i}\le -\tilde{\rho}_i(t)(e(t)+z(t))^TPb_ir_i \tag{10.34}$$

and $\tilde{\rho}_i(t)=\hat{\rho}_i(t)-\rho_i$, $\dot{\tilde{\rho}}_i(t)=\dot{\hat{\rho}}_i(t)$.

On the other hand

$$\begin{aligned}-2(e(t)+z(t))^TPB_1\omega &\le \varepsilon_5e^T(t)PB_1B_1^TPe(t)+\varepsilon_5^{-1}\omega^T\omega\\ &\quad +\varepsilon_6z^T(t)PB_1B_1^TPz(t)+\varepsilon_6^{-1}\omega^T\omega\end{aligned} \tag{10.35}$$

so

$$\begin{aligned}\dot{V} &\le \begin{bmatrix}e^T(t) & z^T(t) & e^T(t-d)\end{bmatrix}\Psi\begin{bmatrix}e(t)\\ z(t)\\ e(t-d)\end{bmatrix}\\ &\quad +\int_{t-d}^t e^T(s)G^T(-P+PF_{22}P)Ge(s)ds+(\varepsilon_5^{-1}+\varepsilon_6^{-1})\omega^T\omega\end{aligned} \tag{10.36}$$

where

$$\Psi=\begin{bmatrix}\Delta_1 & (A+B\rho F_1+G)^TP+PF_{12}P & P(A_d+B\rho F_2-G)+dPF_{13}P\\ * & \Delta_2 & P(A_d+B\rho F_2-G)+PF_{23}P\\ * & * & \Delta_3\end{bmatrix}$$

$$\begin{aligned}\Delta_1 &= P(A+B\rho F_1+G)+(A+B\rho F_1+G)^TP+(\varepsilon_1+\varepsilon_2)PA_1A_1^TP\\ &\quad +(\varepsilon_1^{-1}+\varepsilon_3^{-1})M_1^TM_1+\alpha dG^TPG+S+\varepsilon_5PB_1B_1^TP+dPF_{11}P\\ \Delta_2 &= (\varepsilon_3+\varepsilon_4)PA_1A_1^TP-d^{-1}(\alpha-1)P+\varepsilon_6PB_1B_1^TP\\ \Delta_3 &= (\varepsilon_2^{-1}+\varepsilon_4^{-1})M_2^TM_2-S+dPF_{33})\end{aligned}$$

Hence, if $\Psi < 0$ *and* $-P + PF_{22}P < 0$*, then there exists a positive scalar* β *satisfying*

$$\dot{V} \leq -\beta\|e\|^2 + (\varepsilon_5^{-1} + \varepsilon_6^{-1})\omega^T\omega \leq -\beta\|e\|^2 + D_1 \leq 0 \tag{10.37}$$

where $D_1 = (\varepsilon_5^{-1} + \varepsilon_6^{-1})a^2,\ 0 \leq \|\omega\| \leq a$.

Let $X = P^{-1}, Q = XSX, W_1 = F_1X, W_2 = F_2X$ *and* $W_3 = GX$. *By pre- and post-multiplying inequalities* $\Psi < 0$ *and* $-P + PF_{22}P < 0$ *by diag*$\{X, X, X\}$ *and* X*, respectively, the resulting inequalities are equivalent to (10.25) and (10.26). Also, the inequality (10.27) is equivalent to* $X = P^{-1}, Q = XSX, W_1 = F_1X, W_2 = F_2X$ *and* $W_3 = GX$.

$$\begin{bmatrix} -P & dG^TP \\ * & -P \end{bmatrix} < 0 \tag{10.38}$$

by pre- and post-multiplying by diag$\{X^{-1}, X^{-1}\}$*. If (10.38) holds, according to matrix theory we can prove that a positive scalar* δ *which is less than one exists such that*

$$\begin{bmatrix} -\delta P & dG^TP \\ * & -P \end{bmatrix} < 0 \tag{10.39}$$

Therefore, from Lemma 2.13, if (10.27) holds, the operator $D(e_t)$ *is stable. The inequality (10.28) means that* V_4 *is positive definite. So* $V(t)$ *is positive definite.*

From (10.37), we known $\dot{V} > 0$ *is possible only for* $e(t) \in S_1$*, where* $S_1 = \{e(t) : \|e(t)\| < (\frac{D_1}{\beta})^{\frac{1}{2}}\}$*. Because* S_1 *is compact and contains the point* $e(t) = 0$*, it follows that* $e(t) \in L_\infty$ *and* $V(t) \in L_\infty$*. Then from Lyapunov stability theory, it follows the error system (10.23) is stable.*

Integrating (10.37) from 0 to ∞ *on both sides, we get* $e(t) \in L_2$ *from the fact* $\omega(t) \in L_2$*. From the result of Theorem 10.1, it follows* $x_m(t) \in L_\infty$*. It also implies* $r(t)$ *is bounded. According to the state error system (10.23), it is easy to see* $\dot{e}(t) \in L_\infty$*. Now from* $e(t) \in L_\infty \cap L_2$*,* $\dot{e}(t) \in L_\infty$ *and the well-known Barbaǎlat's lemma [115], it follows* $\lim_{t\to\infty} e(t) = 0$*, i.e.,* $x(\infty) = x_m(\infty) = x_N(\infty)$ *where* x_N *represents the state vector of the normal system without disturbance. Moreover, from* $e(t) \in L_\infty$ *and* $x_m(t) \in L_\infty$*, we can obtain the state vector of the model (10.3)* $x(t)$ *is also bounded. Moreover, it follows* $\lim_{t\to\infty} x(t) = 0$ *from the fact* $\lim_{t\to\infty} x_m(t) = \lim_{t\to\infty} e(t) = 0$*. The proof is completed.*

Remark 10.1 *In the proof of Theorem 10.3, we modify the new Lyapunov function which employs free weighting matrices proposed by [77] to get a new adaptive law and tackle the stabilization of the error system. The newly proposed adaptive laws include the term* $z(t) = \int_{t-d}^{t} Ge(s)ds$*, which indicates how time delay* d *takes effect on the adaptive law.*

Then, we design $r(t)$ so that the target model (10.22) matches the normal model (10.1) without disturbance.

Let $r(t) = \hat{\rho}^{-1}(t)K_N\tilde{x}(t)$, then(10.22) becomes

$$\dot{x}_m(t) = Ax_m(t) + A_dx_m(t-d) + A_1f(t, x_m(t), x_m(t-d)) + BK_N(t)x_m \tag{10.40}$$

It is easy to see (10.22) matches the closed-loop system of normal case without disturbance (10.5) exactly.

Then an *adaptive reliable controller* based on the normal control law $u_N(t) = K_Nx(t)$ is designed. The main *controller structure* is to compute a new control law $u_{ad}(t)$ to be added to the normal control law in order to compensate for the faults and disturbance effect on the system, that is

$$u(t) = u_N(t) + u_{ad}(t) \tag{10.41}$$

The additive control law $u_{ad}(t)$ is zero in the normal case without disturbance and different from zero in fault and disturbance cases. The FTC scheme is summarized in Figure 9.1.

$$u(t) = \hat{\rho}^{-1}(t)K_Nx_m(t) - F_1e(t) - F_2e(t-d) = u_N(t) + u_{ad}(t) \tag{10.42}$$

where $u_N(t) = K_Nx(t)$, $u_{ad}(t) = \hat{\rho}^{-1}(t)(I - \hat{\rho}(t))K_Nx_m(t) + (K_N - F_1)e(t) - F_2e(t-d), F_1 = W_1X^{-1}, F_2 = W_2X^{-1}$.

When the system has no faults and disturbance, the error system is at its equilibrium, i.e., $e(t) = 0$ and $\hat{\rho}_i(t) = 1$ if we choose $e(0) = 0$ and $\hat{\rho}_i(0) = 1$. At this time, $u(t) = u_N(t)$ since $u_{ad}(t) = 0$. This implies the closed-loop normal system without disturbance using the controller (10.42) can achieve the optimized performance. When faults in actuators occur or disturbance exists, the corresponding efficiency factor ρ_i deviates from 1, thus creating a mismatch between $x_m(t)$ and $x(t)$, hence nonzero state error occurs. At the same time, the adaptive estimates of the efficiency factor become active. A new control law $u_{ad}(t)$ is added to the normal law. Then the normal and fault cases with disturbance compensate the fault and disturbance effect automatically and asymptotically track the normal case without disturbance.

From Theorem 10.1-Theorem 10.3, we know the adaptive controller (9.38) can stabilize the closed-loop system in both normal and fault cases. Furthermore, the state vector of closed-loop asymptotically tracks that of the normal case without disturbance, which has the designed performance.

Remark 10.2 *The proposed controller design procedure has optimized the normal performance without disturbance. This presents an advantage as systems are operating under the normal condition most of the time. Because* $K_N = YP_N^{-1}$ *in (10.8) and* $F_1 = W_1X^{-1}, F_2 = W_2X^{-1}$ *in (10.25)-(10.28) are irrelative, (10.25)-(10.28) don't add any conservativeness to the performance optimization procedure of the normal system without disturbance.*

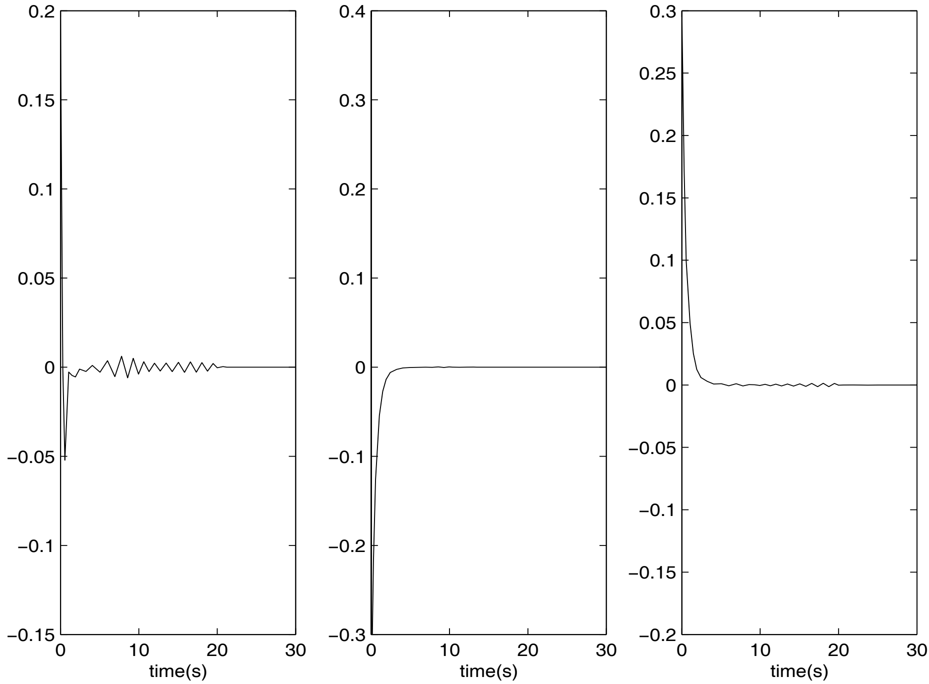


FIGURE 10.1
Normal state response of nonlinear system without disturbance using the normal controller K_N.

Remark 10.3 *This chapter carries on the main idea of Chapter 9, in which we have studied adaptive reliable tracking problems of linear time-invariant systems without disturbance. Here, we extend the system to a class of nonlinear time-delay systems with disturbance. Though in this paper we don't consider the tracking problem, it is very easy to extend our result to that problem.*

10.4 Example

Example 10.1 *To illustrate the effectiveness of our results, a nonlinear time-delay system with the following parameters matrices is considered*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -2 & 0.5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.1 & -1 \\ 0.7 & -0.2 & 0.6 \\ 1 & 1 & 1 \end{bmatrix},$$

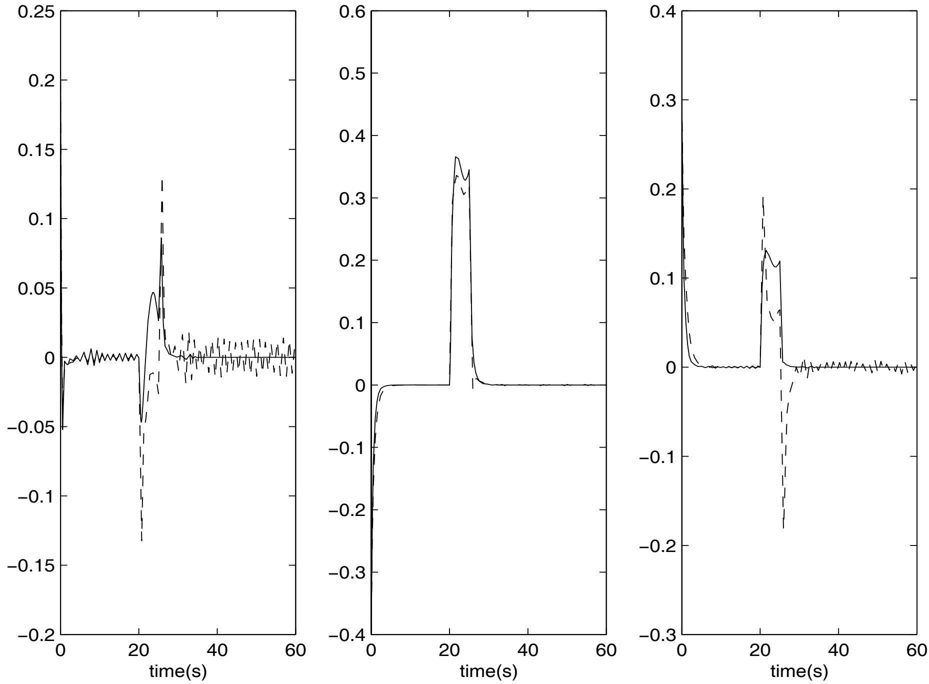


FIGURE 10.2
Normal state response of nonlinear system without disturbance using adaptive controller (solid) and fixed gain controller (dashed).

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2 \\ 1 \\ 0.5 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0.2 \\ -0.4 \\ 0.3 \end{bmatrix}$$

and the time-delay in this example is $d = 0.2$.

Moreover, the nonlinear function is

$$f(t, x(t), x(t-d)) = \begin{bmatrix} 0.1 \sin t \ (x_1(t) + x_1(t-d)) \\ 0.2 \sin t \ x_2(t-d) \\ 0.2 \sin t \ x_3(t) + 0.1 \sin t \ x_3(t-d) \end{bmatrix}$$

Then it follows

$$\|f(t, x, x(t-d))\| \leq \|M_1 x(t)\| + \|M_2 x(t-d)\|$$

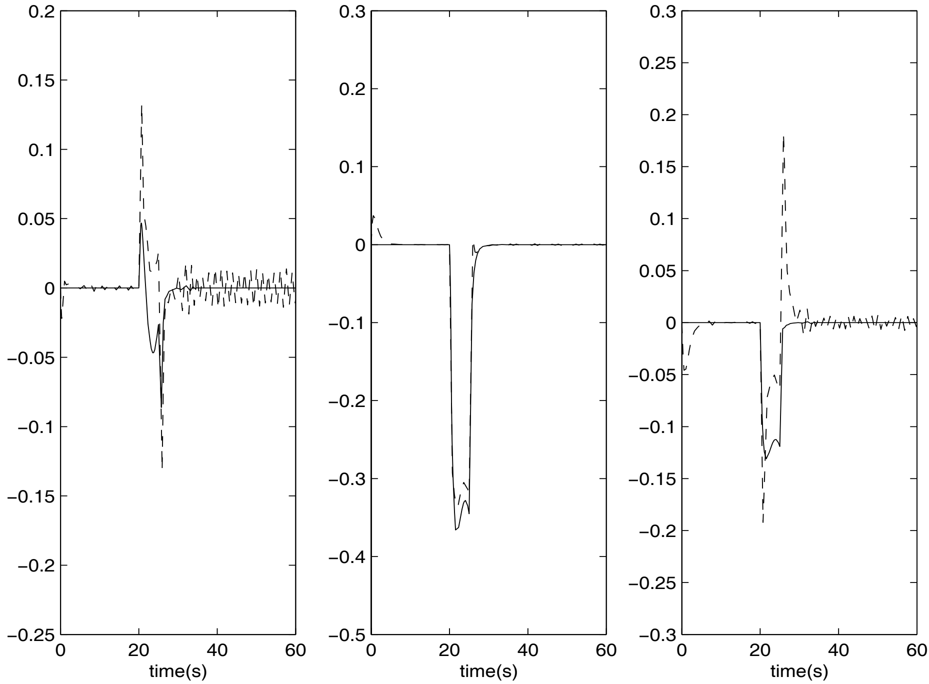


FIGURE 10.3
State error between normal case with disturbance and that case without disturbance of nonlinear system using adaptive controller (solid) and fixed gain controller (dashed).

where

$$M_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Here, we consider the case that only the second and third actuators are susceptible to faults, that is, $\underline{\rho}_1 = \bar{\rho}_1 = 1, \underline{\rho}_2 = \underline{\rho}_3 = 0.4$ and $\bar{\rho}_2 = \bar{\rho}_3 = 1$.

In the following simulation, we use the disturbance

$$\omega(t) = \begin{cases} 0.5, & 20 \leq t \leq 25 \text{ (seconds)} \\ 0 & \text{otherwise} \end{cases}$$

The fault case here is that at 0 second, the third actuator becomes loss of effectiveness of 60%.

For comparison purposes, our adaptive reliable controller and a *traditional reliable controller* with fixed gains are carried out in the following simulation.

From Theorem 10.2, we can get the normal controller $u_N = K_N x(t)$ with a sub-optimal cost $J^* = 0.2567$ with $\lambda^* = -1.6$ and $\mu^* = 50$, which is obtained

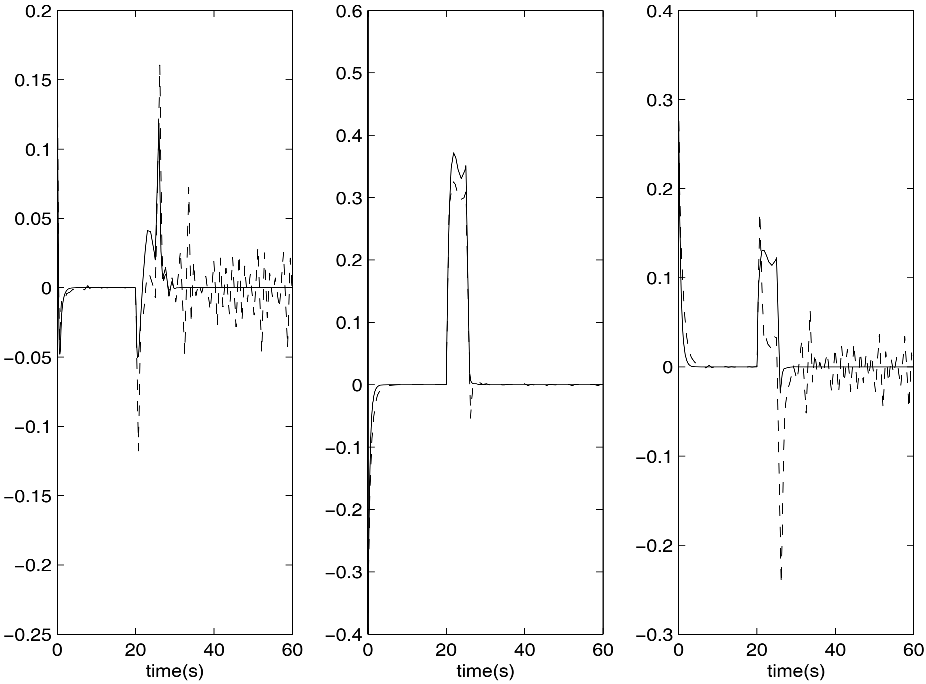


FIGURE 10.4
Fault state response of nonlinear system with disturbance using adaptive controller (solid) and fixed gain controller (dashed).

by searching for λ (from -0.1 to -10) and μ (from -0.01 to 50). And when we choose $\alpha = 1.5$, a feasible solution of Theorem 10.3 can be received. Furthermore, the corresponding adaptive reliable controller is obtained. However, if we solve the reliable problem with a fixed gain controller K_f guaranteeing all considered possible cases stabilized and normal case without disturbance optimal, the obtained locally optimal cost is $J_f^* = 0.3713$ with $\lambda_f^* = -2.8$ and $\mu_f^* = 50$ (the search range is the same as that of controller K_N) using the corresponding results of Theorem 10.1 and Theorem 10.2. This phenomenon takes place due to the reason indicated in Remark 10.2. As the system is operating under the normal condition most of the time, this fact that our adaptive reliable controller improves the normal performance significantly compared to the fixed gain controller K_f is more considerable and important.

In Figure 10.1, the state response in normal case without disturbance for nonlinear systems using the normal controller K_N is first given, which describes the desired performance. Figure 10.2 denotes the normal state response with disturbance using the adaptive controller and fixed gain controller, respectively. It is obviously that the proposed adaptive controller has much more ability to restrain disturbance than that of the fixed gain controller. To verify

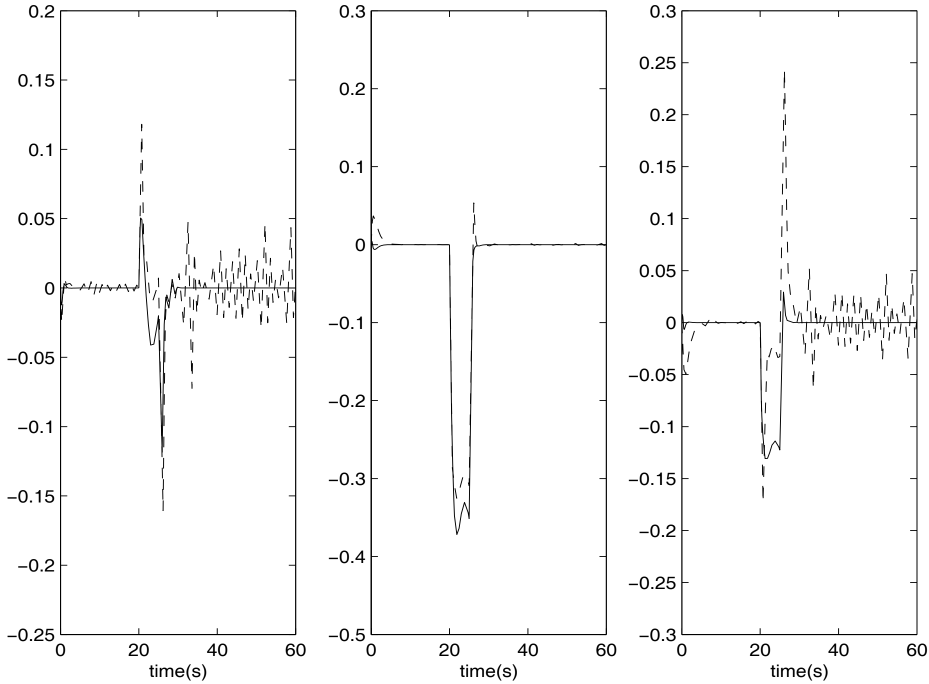


FIGURE 10.5
State error between fault case with disturbance and normal case without disturbance of nonlinear system using adaptive controller (solid) and fixed gain controller (dashed).

the characteristic of our adaptive controller, we simulate the state error between normal case with disturbance and that case without disturbance using these two controllers. The result is given in Figure 10.3, from which it can be seen the state error converge to zero in spite of the existence of disturbance with the adaptive controller while the fixed gain one can't have this property.

Figure 10.4-Figure 10.5 describe some response curves of the fault case. In Figure 10.4, the fault state response using these two controllers is first given, which denotes the superiority of restraining disturbance of the adaptive controller compared to the fixed gain one. Figure 10.5 describes the state error between fault case with disturbance and normal case without disturbance of nonlinear system using the two controllers. Though the state error deviates from zero due to the existence of fault and disturbance, it can recover after a few of seconds using the adaptive controller. But, this property doesn't exist in the case of a fixed gain controller.

10.5 Conclusion

In this chapter, we have investigated the adaptive reliable control problem against unknown actuator faults for a class of nonlinear time-delay systems with disturbance. The aim is to find an adaptive reliable controller, such that the system is not only stabilized, but also the state vectors of normal and fault cases with disturbances track that of the normal case without disturbance, which is with the designed performance. A new delay-dependent adaptive law is proposed to design the adaptive reconfigurable controller, which is excited to offset the effect of faults and disturbance automatically without the need for an *FDI* mechanism. A numerical example shows the effectiveness of the proposed controller design method when compared with a fixed gain reliable controller.

Bibliography

[1] R. J. Adams, J. M. Buffington, A. G. Sparks, and S. S. Banda. *Robust multivariable flight control*. Springer-Verlag, London, 1994.

[2] C. K. Ahn, S. Han, and W. H. Kwon. H_∞ FIR filters for linear continuous-time state-space systems. *IEEE Trans. Signal Processing Letter*, 13(9):557–560, 2006.

[3] K. J. Astrom. Intelligent control. In *Proc. 1st European Control Conf.*, pages 2328–2329, 1991.

[4] T. Azuma, K. Ikeda, T. Kondo, and K. Uchida. Memory state feedback control synthesis for linear systems with time delay via a finite number of linear matrix inequalities. *Computers and Electrical Engineering*, 28:217–228, 2002.

[5] M. Blanke, R. Zamanabadi, and T. F. Lootsma. Fault monitoring and reconfigurable control for a ship propulsion plant. *Int. J. Adaptive Control and Signal Process*, 12:671–688, 1998.

[6] M. Bodson and J. E. Groszkiewicz. Multivariable adaptive algorithms for reconfigurable flight control. *IEEE Trans. Control Systems Technology*, 5(2):217–229, 1997.

[7] C. Bonivento, A. Paoli, and L. Marconi. Fault-tolerant control of the ship propulsion system benchmark. *Control Engineering Practice*, 11:483–492, 2003.

[8] J. D. Boskovic, S. Li, and R. K. Mehra. A decentralized fault-tolerant scheme for flight control applications. In *Proc. American Control Conference*, pages 4214–4218, 2000.

[9] J. D. Boskovic and R. K. Mehra. A multiple model-based reconfigurable flight control system design. In *Proc. 37th IEEE Conf. Decision & Control*, pages 4503–4508, 1998.

[10] J. D. Boskovic and R. K. Mehra. Stable multiple model adaptive flight control for accommdation of a large class of control effector failures. In *Proc. American Control Conference*, pages 1920–1924, 1999.

[11] J. D. Boskovic and R. K. Mehra. An adaptive retrofit reconfigurable flight controller. In *Proc. 41st IEEE. Conf. Decision Control*, pages 1257–1262, 2002.

[12] E. K. Boukas and Z. K. Liu. *Deterministic and stochastic time delay systems.* Birkhauser, Boston, 2002.

[13] E. K. Boukas, Z. K. Liu, and G. X. Liu. Delay-dependent robust stability and H_∞ control of jump linear systems with time-delay. *Int. J. Control*, 74:329–340, 2001.

[14] S. Boyd, L. E. Ghaoui, and V. Balakrishnan. *Linear matrix inequalities in system and control theory.* SIAM, Philadelphia PA, 1994.

[15] A. E. Bryson. *Control of spacecraft and aircraft.* Princeton University Press, Princeton, New Jersey, 1993.

[16] J. D. Cao and J. Wang. Delay-dependent robust stability of uncertain nonlinear systems with time delay. *Applied Mathematics and Computation*, 154:289–297, 2004.

[17] Y. Cao, Z. Lin, and D. G. Ward. An antiwindup approach to enlarging domain of attraction for linear systems subject to actuator saturation. *IEEE Trans. Automat. Control*, 47(1):140–145, 2002.

[18] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1218, 1998.

[19] S. Chen, G. Tao, and S. M. Joshi. Adaptive actuator failure compensation designs for linear systems. *Int. J. Control, Automation, and Systems*, 2(1):1–12, 2004.

[20] W. Chen, D. Ballance, and J. O. Reilly. Optimisation of attraction domains of nonlinear MPC via LMI methods. In *Proc. American Control Conference*, pages 3067–3072, 2001.

[21] W. Chen and M. Saif. Fault detection and accommodation in nonlinear time-delay systems. In *Proc. American Control Conference*, pages 4255–4260, 2005.

[22] W. H. Chen, Z. H. Guan, and X. Lu. Delay-dependent output feedback guaranteed cost control for uncertain time-delay systems. *Automatica*, 40:1263–1268, 2004.

[23] Z. Chen and T. Chang. Modeling and fault tolerant control of large urban traffic network. In *Proc. 23rd Congress of The International Council of the Aeronautical Sciences*, pages 2469–2472, 1997.

[24] M.L. Corradini and G. Orlando. Linear unstable plants with saturating actuators: Robust stabilization by a time varying sliding surface. *Automatica*, 43(1):88–94, 2007.

[25] J. M. Gomes da Silva, Jr., S. Tarbouriech, and R. Reginatto. Control of saturated linear plants via output feedback containing an internal deadzone loop. In *Proc. IEEE Conference Control Applications*, pages 1106–1111, 2002.

[26] J.M. Gomes da Silva Jr and S. Tarbouriech. Anti-windup design with guaranteed regions of stability: an LMI-based approach. *IEEE Trans. Automat. Control*, 50(1):106–111, 2005.

[27] D. Dai, T. Hu, A. R. Teel, and L. Zaccarian. Control of saturated linear plants via output feedback containing an internal deadzone loop. In *Proc. American Control Conference*, pages 5239–5245, 2006.

[28] J. A. De Don'a, M. M. Seron, D. Q. Mayne, and G. C. Goodwin. Enlarged terminal sets guaranteeing stability of receding horizon control. *Systems & Control Letters*, 47(1):57–63, 2002.

[29] J. S. Delapasse, C. K. Behbehani, K. Tsui, and K. W. Klein. Accommodation of time delay variations in automatic infusion of sodium nitroprusside. *IEEE Trans. Biomedical Engineering*, 41:1083–1091, 1994.

[30] M. A. Demetrioui. Adaptive reorganization of switched systems with faulty actuators. In *Proc. 40th IEEE. Conf. Decision Control*, pages 1879–1884, 2001.

[31] R. W. Eustace, B. A. Woodyatt, G. L. Merrington, and A. Runacres. Fault signatures obtained from fault implant tests on an F404 engine. *ASME Trans. J. Engine, Gas Turbines, Power*, 116(1):178–183, 1994.

[32] H. Fang, Z. Lin, and T. Hu. Analysis of linear systems in the presence of actuator saturation and L_2-disturbances. *Automatica*, 40(7):1229–1238, 2004.

[33] E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43(4):309–319, 2001.

[34] E. Fridman and U. Shaked. New bounded real lemma representations for time-delay systems and their applications. *IEEE Trans. Automat. Control*, 46(12):1973–1979, 2001.

[35] E. Fridman and U. Shaked. A descriptor system approach to H_∞ control of linear timedelay systems. *IEEE Trans. Automat. Control*, 47(2):253–270, 2002.

[36] E. Fridman and U. Shaked. An improved stabilization method for linear time-delay systems. *IEEE Trans. Automat. Control*, 47(11):1931–1937, 2002.

[37] E. Fridman and U. Shaked. Parameter dependent stability and stabilization of uncertain time-delay systems. *IEEE Trans. Automat. Control*, 48(5):861–866, 2003.

[38] E. Fridman and U. Shaked. Delay-dependent stability and H_∞ control: Constant and time-varying delays. *Int. J. Control*, 76(1):48–60, 2003.

[39] E. Fridman and U. Shaked. An improved delay-dependent H_∞ filtering of linear neutral systems. *IEEE Trans. Signal Processing*, 52(3):668–673, 2004.

[40] E. Fridman, U. Shaked, and L. Xie. Robust H_2/H_∞ filtering of linear systems with time-varying delay. *IEEE Trans. Automat. Control*, 48(1):159–165, 2003.

[41] M Fu, C. E. de Souza, and L. Xie. H_∞ estimation for linear uncertain systems. *Int. J. Robust Nonlinear Control.*, 2(2):87–105, 1992.

[42] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *Int. J. Robust and Nonlinear Control*, 40(4):421–448, 1994.

[43] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Trans. Automat. Control*, 41(3):436–442, 1996.

[44] H. J. Gao, J. Lam, L. H. Xie, and C. H. Wang. New approach to mixed H_2/H_∞ filtering for polytopic discrete-time systems. *IEEE Trans. Signal Processing*, 53:3183–3192, 2005.

[45] H. J. Gao and C. H. Wang. Comments and further results on a descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Trans. Automat. Control*, 48(3):520–525, 2003.

[46] Z. Gao and P. J. Antsaklis. Stability of the pseudo-inverse method for reconfigurable control systems. *Int. J. Control*, 53(3):711–729, 1991.

[47] J. C. Geromel. Optimal linear filtering under parameter uncertainty. *IEEE Trans. Signal Processing*, 47:168–175, 1999.

[48] G. Grimm, J. Hatfield, I. Postlethwaite, A. R. Teel, M. C. Turner, and L. Zaccarian. Antiwindup for stable linear systems with input saturation: an LMI-based synthesis. *IEEE Trans. Automat. Control*, 48(9):1509–1525, 2005.

[49] K. Gu. Discretized LMI set in the stability problem for linear uncertain time-delay systems. *Int. J. Control*, 68(4):923–934, 1997.

[50] K. Gu. A further refinement of discretized Lyapunov functional method for the stability of time-delay systems. *Int. J. Control*, 74(10):967–976, 2001.

[51] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of time-delay systems (Control Engineering).* Springer-Verlag, New York, 2003.

[52] K. Gu and S. I. Niculescu. Additional dynamics in transformed time delay systems. *IEEE Trans. Automat. Control*, 45(3):572–575, 2000.

[53] K. Gu and S. I. Niculescu. Further remarks on additional dynamics in various model transformations of linear delay systems. *IEEE Trans. Automat. Control*, 46(3):497–500, 2001.

[54] A. N. Gundes. Controller design for reliable stabilization. In *Proc. 12th IFAC World Congress*, pages 1–4, 1993.

[55] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations.* Springer-Verlag, New York, 1993.

[56] Q. L. Han. Robust stability of uncertain delay-differential systems of neutral type. *Automatica*, 38(4):718–723, 2002.

[57] Q. L. Han. Stability criteria for a class of linear neutral systems with time-varying discrete and distributed delays. *IMA J. Math. Contr. and Info.*, 20(4):371–386, 2003.

[58] Q. L. Han. A descriptor system approach to robust stability of uncertain neutral systems with discrete and distributed delays. *Automatica*, 40(10):1791–1796, 2004.

[59] Q. L. Han. On stability of linear neutral systems with mixed time delays: A discretized Lyapunov functional approach. *Automatica*, 41(7):1209–1218, 2005.

[60] Q. L. Han and K. Gu. On robust stability of time-delay systems with norm-bounded uncertainty. *IEEE Trans. Automat. Control*, 46(9):1426–1431, 2001.

[61] Y. He, Q. G. Wang, L. H. Xie, and C. Lin. Further improvement of free-weighting matrices technique for systems with time-varying delay. *IEEE Trans. Automat. Control*, 52:293–299, 2007.

[62] Y. He, M. Wu, J. H. She, and S. P. Liu. Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays. *Systems & Control Letters*, 51(1):57–65, 2004.

[63] H. Hindi and S. Boyd. Analysis of linear systems with saturation using convex optimization. In *Proc. 37th IEEE Conf. Decision and Control*, pages 903–908, 1998.

[64] I. M. Horowitz. *Synthesis feedback systems.* Academic Press, 1963.

[65] T. Hu and Z. Lin. *Control systems with actuator saturation: Analysis and design.* BirkhTauser, Boston, 2001.

[66] T. Hu, Z. Lin, and B. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38(2):351–359, 2002.

[67] T. Hu, A. Teel, and L. Zaccarian. Stability and performance for saturated systems via quadratic and non-quadratic Lyapunov functions. *IEEE Trans. Automat. Control*, 51(11):1770–1786, 2006.

[68] T. Hu, A. R. Teel, and L. Zaccarian. Anti-windup synthesis for linear control systems with input saturation: Achieving regional, nonlinear performance. *Automatica*, 44(2):512–519, 2008.

[69] G. D. Hui. Simple criteria for stability of neutral systems with multiple delays. *Int. J. Sys. Sci.*, 28(12):1325–1328, 1997.

[70] P. A. Ioannou and J. Sun. *Robust adaptive control.* Prentice-Hall International, Inc, 1996.

[71] T. Iwasaki and G. Shibata. LPV system analysis via quadratic seperator for uncertain implicit systems. *IEEE Trans. Automat. Control*, 46:1195–1208, 2001.

[72] J. Jiang. Design of reconfigurable control systems using eigenstructure assignments. *Int. J. Control*, 59(2):394–410, 1994.

[73] Gu K. An integral inequality in the stability problem of time-delay system. In *Proc. 39th IEEE Conf. Decision Control*, pages 2805–2810, 2000.

[74] P. T. Kabamba and C. Yang. Simultaneous controller design for linear time-invariant systems. *IEEE Trans. Automat. Control*, 36(1):106–111, 1991.

[75] J. H. Kim, E. T. Jeung, and H. B. Park. Robust control of parameter uncertain delay systems in state and control input. *Automatica*, 32:1337–1339, 1996.

[76] V. B. Kolmanovskii and V. R. Nosov. *Stability of functional differential equations.* Academic Press, London, 1990.

[77] O. M. Kwon and J. H. Park. On improved delay-dependent robust control for uncertain time-delay systems. *IEEE Trans. Automat. Control*, 49:1991–1995, 2004.

[78] I. D. Landau, R. Lozano, and M. M'Saad. *Adaptive control.* Spring-Verlag, London, 1998.

[79] D. Lawrence and W. Rugh. Gain scheduling dynamic linear controllers for nonlinear plant. *Automatica*, 31(3):381–388, 1995.

[80] C. S. Lee and G. Leitmann. Continuous feedback guaranteeing uniform ultimate boundedness for uncertain linear delay systems: An application to river pullution control. *Computers and Mathematics with Applications*, 16:929–938, 1998.

[81] J. W. Lee, J. H. Lee, and W. H. Kwon. Robust tracking control for uncertain linear systems without matching conditions. In *Proc. 40th IEEE Conf. Decision Control*, pages 4181–4186, 2001.

[82] Y. S Lee, Y. S. Moon, W. H. Kwon, and P. G. Park. Delay-dependent robust H_∞ control for uncertain systems with a state-delay. *Automatica*, 40(1):65–72, 2004.

[83] J. Levine and R. Marino. On failure-tolerant observers. *IEEE Trans. Automat. Control*, 35:623–627, 1990.

[84] F. Liao, J. L. Wang, and G. H. Yang. Reliable robust flight tracking control: An LMI approach. *IEEE Trans. Control Systems Technology*, 10:76–89, 2002.

[85] D. Limon, T. Alamo, and E. F. Camacho. Stable constrained MPC without terminal constraint. In *Proc. American Control Conference*, pages 4893–4898, 2003.

[86] C. Lin, Q. G. Wang, and T. H. Lee. A less conservative robust stability test for linear uncertain time-delay systems. *IEEE Trans. Automat. Control*, 51:87–91, 2006.

[87] J. Liu, J. L. Wang, and G. H. Yang. Reliable robust minimum variance filtering with sensor failures. In *Proc. American Control Conference*, pages 1041–1046, 2001.

[88] J. Liu, J. L. Wang, and G. H. Yang. Reliable guaranteed variance filtering against sensor failures. *IEEE Trans. on Signal Processing*, 51:1403–1411, 2003.

[89] S. M. Magdi and L. H. Xie. Guaranteed cost control of uncertain discrete systems with delays. *Int. J. Control*, 73(2):105–114, 2000.

[90] L. Magni, De Nicolao. G., L. Magnani, and R. Scattolini. A stabilizing model-based predictive control algorithm for nonlinear systems. *Automatica*, 37(9):1351–1362, 2001.

[91] M. S. Mahmoud. H_∞ control of linear neutral systems. *Automatica*, 36(5):757–764, 2000.

[92] M. S. Mahmoud. *Robust control and filtering for time-delay systems.* Marcel Dekker, New York, 2000.

[93] M. S. Mahmoud and P. Shi. *Methodologies for control of jump time-delay systems.* Kluwer Academic Publishers, Boston, 2003.

[94] M. Maki and K. Hagino. Robust control with adaptation mechanism for improving transient behaviour. *Int. J. Control*, 72:1218–1226, 1999.

[95] P. S. Maybeck. Multiple adaptive algorithm for detecting and compensating sensor and actuator/surface failures in aircraft flight control systems. *Int. J. Robust and Nonlinear Control*, 9:1051–1070, 1999.

[96] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[97] A. Megretski. BIBO output feedback stabilization with saturated control. In *Proc. IFAC World Congress*, pages 435–440, 1998.

[98] B. M. Mirkin and P. O. Gutman. Model reference adaptive control of state delayed system with actuator failures. *Int. J. Control*, 78:186–195, 2005.

[99] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *Int. J. Control*, 74(14):1447–1455, 2001.

[100] W. D. Morse and K. A. Ossman. Model-following reconfigurable flight control system for the AFTI/F-16. *J. Guideline, Control. & Dynamics*, 13(6):969–976, 1990.

[101] T. Nguyen and F. Jabbari. Output feedback controllers for disturbance attenuation with bounded inputs. In *Proc. 36th IEEE Conf. Decision and Control*, pages 177–182, 1997.

[102] T. Nguyen and F. Jabbari. Disturbance attenuation for systems with input saturation: An LMI approach. *IEEE Trans. Automat. Control*, 44(4):852–857, 1999.

[103] S. I. Niculescu. *Delay effects on stability: A robust control approach.* Springer-Verlag, London, 2001.

[104] S. I. Niculescu. On delay-dependent stability under model transformations of some neutral linear systems. *Int. J. Control*, 74(6):608–617, 2001.

[105] A. Niederlinski. A heuristic approach to the design of linear multivariable interacting control systems. *Automatica*, 7(6):691–701, 1991.

[106] N. Niksefat and N. Sepehri. A QFT fault-tolerant control for electrohydraulic positioning systems. *IEEE Trans. Control System Technology*, 10(4):626–632, 2002.

[107] H. Noura, D. Sauter, F. Hamelin, and D. Theilliol. Fault-tolerant control in dynamic system: Application to a winding machine. *IEEE Control System Magazine*, pages 33–49, 2000.

[108] A. W. Olbrot. Robust stabilization of uncertain systems by periodic feedback. *Int. J. Control*, 45(3):747–758, 1987.

[109] C. Paim, S. Tarbouriech, J. M. Gomes da Silva Jr., and E. B. Castelan. Control design for linear systems with saturating actuators and L_2-bounded disturbances. In *Proc. 41st IEEE Conf. Decision and Control*, pages 4148–4153, 2002.

[110] J. H. Park. Design of a dynamic output feedback controller for a class of neutral systems with discrete and distributed delays. *IEE Proc.-Control Theory Appl.*, 151(5):610–614, 2004.

[111] J. H. Park and O. Kwon. On guaranteed cost control of neutral systems by retarded integral state feedback. *Applied Mathematics and Computation*, 165:393–404, 2005.

[112] P. Park. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Automat. Control*, 44(4):876–877, 1999.

[113] R. J. Patton. Robustness issues in fault tolerant control. In *Proc. International Conference on Fault Diagnosis*, pages 1081–1117, 1993.

[114] R. J. Patton. Fault-tolerant control: The 1997 situation. In *Proc. IFAC/IMACs Symposium on Fault Detection and Safety for Technical Process*, pages 1033–1055, 1997.

[115] V. M. Popov. *Hyperstability of control systems.* Springer Verlag, New York, 1973.

[116] J. P. Richard. Time delay systems: An overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 2003.

[117] R. Saeks and J. Murray. Fractional representation, algebraic geometry, and the simultaneous stabilization problem. *IEEE Trans. Automat. Control*, 24(4):895–903, 1982.

[118] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Trans. Automat. Control*, 42(7):896–911, 1997.

[119] C. Scherer and S. Weiland. *Linear matrix inequalities in control.* SIAM, Philadelphia PA, 2005.

[120] C. W. Scherer, H. Chen, and F. Allgöwer. Disturbance attenuation with actuator constraints by hybrid state-feedback control. In *Proc. 41st IEEE Conf. Decison and Control*, pages 4134–4138, 2002.

[121] N. Sebe and T. Kitamori. Control systems possessing reliability to control. In *Proc. 12th IFAC World Congress*, pages 1–4, 1993.

[122] B. Shafai, C. T. Pi, S. Nork, and S. P. Linder. Proportional integral adaptive observer for parameter and disturbance estimations. In *Proc. 41st IEEE Conf. Decision and Control*, pages 4694–4699, 2002.

[123] U. Shaked and C. E. Souza. Continuous-time tracking problems in an H_∞ setting: A game theory approach. *IEEE Trans. Automat. Control*, 40:841–852, 1995.

[124] E. Shimemura and M. Fujita. A design method for linear state feedback systems possessing integrity based on a solution of a Riccati-type equation. *Int. J. Control*, 42(4):887–899, 1985.

[125] M. H. Shor, W. R. Perkins, and J. V. Medanic. Design of reliable decentralized controllers: A unified continuous/discrete formulation. *Int. J. Control*, 56(4):943–956, 1992.

[126] D. D. Siljak. Reliable control using multiple control systems. *Int. J. Control*, 7(2):303–329, 1980.

[127] R. Suarez, J. Alvarez-Ramirez, M. Sznaier, and C. Ibarra-Valdez. *L_2-disturbance attenuation for linear systems with bounded controls: An ARE-based approach.* Springer, Berlin, 1997.

[128] G. Tao, S. H. Chen, and S. M. Joshi. An adaptive control scheme for systems with unknown actuator failures. In *Proc. American Control Conference*, pages 1114–1120, 2001.

[129] G. Tao, S. H. Chen, and S. M. Joshi. An adaptive control scheme for systems with actuator failures. *Automatica*, 38:1027–1034, 2002.

[130] G. Tao, S. M. Joshi, and X. L. Ma. Adaptive state feedback and tracking control of systems with actuator failures. *IEEE Trans. Automat. Control*, 46(1):78–95, 2002.

[131] G. Tao, X. Tang, and S. M. Joshi. Output tracking actuator failure compensation control. In *Proc. American Control Conference*, pages 1821–1826, 2001.

[132] Y. Theodor, U. Shaked, and C. E. de Souza. A game theory approach to robust discrete-time H_∞-estimation. *IEEE Trans. Signal Processing*, 42:1486–1495, 1994.

[133] R. J. Veillette, J. V. Medanic, and W. R. Perkins. Design of reliable control systems. *IEEE Trans. Automat. Control*, 37(3):290–304, 1992.

[134] M. Vidyasagar and N. Viswanadham. Reliable stabilization using a multi-controller configuration. *Automatica*, 21(4):599–602, 1985.

[135] H. B. Wang, J. Lam, S. Y. Xu, and S. D. Huang. Robust H_∞ reliable control for a class of uncertain neutral delay systems. *Int. J. Systems Science*, 33:611–622, 2002.

[136] L. F. Wang, B. Huang, and K. C. Tan. Fault-tolerant vibration control in a networked and embedded rocket fairing system. *IEEE Trans. Automat. Control*, 51(6):1127–1141, 2004.

[137] Z. Wang, B. Huang, and H. Unbehauen. Robust reliable control for a class of uncertain nonlinear state delayed systems. *Automatica*, 35:955–963, 1999.

[138] Z. D. Wang and J. A. Fang. Robust H_∞ filter design with variance constraints and parabolic pole assignment. *IEEE Signal Processing Letters*, 13:137–140, 2006.

[139] Z. D. Wang and B. Huang. Robust H_2/H_∞ filtering for linear systems with error variance constraints. *IEEE Trans. Signal Processing*, 48:2463–2467, 2000.

[140] Z. Wen, S. Roy, and A. Saberi. On the disturbance response and external stability of a saturating static-feedback-controlled double integrator. *Automatica*, 44(8):2191–2196, 2008.

[141] K. A. Wise, J. S. Brinker, A. J. Calise, D. F. Enns, M. R. Elgersma, and P. Oulgari. Direct adaptive reconfigurable flight control for a tailless advanced fighter aircraft. *Int. J. Robust and Nonlinear Control*, 9:999–1012, 1999.

[142] F. Wu, Z. Lin, and Q. Zheng. Output feedback stabilization of linear systems with actuator saturation. *IEEE Trans. Automat. Control*, 52(1):122–128, 2007.

[143] F. Wu and B. Lu. Anti-windup control design for exponentially unstable LTI systems with actuator saturation. *Systems & Control Letters*, 52:304–322, 2004.

[144] M. Wu, Y. He, J. H. She, and S. P. Liu. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40(8):1435–1439, 2004.

[145] N. E. Wu, Y. M. Zhang, and K. M. Zhou. Detection, estimation, and accommodation of loss of control effectiveness. *Int. J. Adaptive Control signal process*, 14:774–795, 2000.

[146] L. H. Xie, C. L. Du, C. S. Zhang, and Y. C. Soh. H_∞ deconvolution filtering of 2-D digital systems. *IEEE Trans. Signal Processing*, 50:2319–2332.

[147] S. Xu and T. W. Chen. H_∞ output feedback control for uncertain stochastic systems with time-varying delays. *Automatica*, 40:2091–2098, 2004.

[148] I. Yaesh and U. Shaked. Two degree of freedom H_∞ optimization of multivariable feedback systems. *IEEE Trans. Automat. Control*, 36:1272–1276, 1991.

[149] G. H. Yang and K. Y. Lum. Fault-tolerant flight tracking control with stuck faults. In *Proc. American Control Conf.*, pages 521–526, 2003.

[150] G. H. Yang, J. L. Wang, and Y. C. Soh. Reliable LQG control with sensor failures. *IEE Proc.-Control Theory Appl.*, 147:433–439, 2000.

[151] G. H. Yang, J. L. Wang, and C. S. Yeng. Reliable guaranteed cost control for uncertain nonlinear systems. *IEEE Trans. Automat. Control*, 45(11):2188–2192, 2000.

[152] G. H. Yang and D. Ye. Adaptive fault-tolerant H_∞ control via dynamic output feedback for linear systems against actuator faults. In *Proc. 45th IEEE Conf. Decision and Control*, pages 3524–3529, 2006.

[153] G. H. Yang and D. Ye. Adaptive fault-tolerant H_∞ control via state feedback for linear systems against actuator faults. In *Proceedings of Conference on Decision and Control*, pages 3530–3535, 2006.

[154] G. H. Yang and D. Ye. Adaptive reliable H_∞ filtering against sensor failures. *IEEE Trans. Signal Processing*, 55(7):3161–3171, 2007.

[155] D. Ye and G. H. Yang. Adaptive fault tolerant tracking control against actuator faults with application to flight control. *IEEE Tran. Control Systems Technology*, 14:1088–1096, 2006.

[156] D. Ye and G. H. Yang. Adaptive reliable H_∞ control for linear time-delay systems via memory state feedback. *IEE Proc.-Control Theory & Appl.*, 1(3):713–721, 2007.

[157] L. Yu and J. Chu. An LMI approach to guaranteed cost control of linear uncertain time-delay systems. *Automatica*, 35(6):1155–1159, 1999.

[158] D. Yue, J. Lam, and D. W. C. Ho. Reliable H_∞ control of uncertain descriptor systems with multiple time delays. *IEE Proc.-Control Theory Appl.*, 150:557–564, 2003.

[159] D Yue and S. Won. Delay-dependent robust stability of stochastic systems with time delay and nonlinear uncertainties. *Elect. Lett*, 37:992–993, 2001.

[160] X. M. Zhang, M. Wu, J. M. She, and Y. He. Delay-dependent stabilization of linear systems with time-varying state and input delays. *Automatica*, 41:1405–1412, 2005.

[161] Y. M. Zhang and J. Jiang. Design of proportional-integral reconfigurable control systems via eigenstructure assignment. In *Proc. American Control Conf.*, pages 3732–3736, 2000.

[162] Y. M. Zhang and J. Jiang. Integrated active fault-tolerant control using IMM approach. *IEEE Trans. Aerospace and Electronic Systems*, 37(4):1221–1235, 2001.

[163] Q. Zhao and C. Cheng. State feedback control for time-delayed systems with actuator failures. In *Proc. American Control Conference*, pages 827–832, 2003.

[164] Q. Zhao and J. Jiang. Reliable state feedback control system design against actuator failures. *Automatica*, 34(10):1267–1272, 1998.

[165] K. Zhou, J. C. Doyle, and K. Glover. *Robust and optimal control.* Prentice Hall, New Jersey, 1996.