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DEPARAMETRIZATION AND PATH INTEGRAL QUANTIZATION OF COSMOLOGICAL MODELS

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Preface

There are different points of view about how quantum cosmology should be constructed, and also about the interpretation of the formalism. A reason for this is a peculiar problem which comes from the fact that we are looking for a theory of the whole universe: while one usually deals with systems for which the meaning of "evolution" is clear as it is possible to assume that there is something external which plays the role of a clock, this is not possible in cosmology. There are then two possibilities: one could abandon the idea of a description with a clear notion of time; or we can assume that a subset of the variables describing the state of the universe can be used as a clock for the remaining of the system. Here the reader will find a proposal within this last framework; this proposal is restricted to minisuperspace models, and consists in identifying a time among the canonical variables by means of gauge fixation and then to obtain the transition amplitude in the form of a path integral with a clear separation between time and true degrees of freedom.

The idea was in fact suggested by the reading of some early works by Andrei Barvinsky and Petr Hájícek. A serious obstacle existed, however, for a program based on this idea, and it was the lack of gauge invariance at the boundaries in the action of gravitation, which was pointed by Teitelboim and Halliwell. An important hint to solve this was given by a paper by Henneaux, Teitelboim and Vergara, in which they showed how to obtain a gauge-invariant action for parametrized systems like the relativistic particle.

These notes are a review about my work on the subject, including some new developments not published yet, and also about other deparametrization and quantization schemes. The results of the deparametrization are not only applied in the path integral formulation, but also in a Chapter devoted to the canonical formalism. The book deals with both relativistic and string cosmologies; although here we will be mostly concerned with formal developments, the last deserve a particular attention, as string theory leads to the possibility of completely new scenarios for the earliest stages of the universe.

Most of the material here is the result of my work with R. Ferraro, H. De Cicco, and G. Giribet; I am also indebted to them for their colaboration during the preparation of this book.

Contents

Chapter 1

Introduction

The building of a unitary quantum theory of gravitation in which the wave function has a clear meaning is still an open problem, partially because of the problem of time [Kuchař (1981); Hájícek (1986)]: while in ordinary quantum mechanics the time is an absolute parameter, in the theory of gravitation the time is an arbitrary label of spacelike hypersurfaces, and physical quantities are invariant under general coordinate changes. Because the evolution is given in terms of a parameter τ which does not have physical significance, the action of the gravitational field is that of a parametrized system, with a canonical Hamiltonian which vanishes on the physical trajectories of the system, that is, with a constraint $\mathcal{H} \approx 0$.

Starting from this situation people have followed mainly two paths. One of them is the usual Dirac–Wheeler–DeWitt quantization scheme, whose formalism does not explicitly contain time, and does not have an evolutionary form; this leads to the problem of defining a conserved positive-definite probability, as the notion of the time in which such a conservation should hold is not clear. The other possible way to obtain a quantum theory of gravitation is to consider that the time is hidden among the coordinates and momenta of the system, which then must be deparametrized by identifying the time as a first step before quantization; this is the point of view that we shall adopt.

The proposal of the present work is based on the fact that the identification of time is closely related to gauge fixation: In the theory of gravitation the dynamical evolution is given by a spatial hypersurface moving in spacetime along the timelike direction. This motion includes arbitrary local deformations which yield a multiplicity of times. From a different point of view, the same motion can be generated by general gauge transformations, so that fixing a gauge a particular foliation of spacetime is defined [Barvinsky (1993); Simeone (1999)].

For minisuperspace models we have an action functional of the form

$$S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - N\mathcal{H} \right) d\tau \tag{1.1}$$

where N is a Lagrange multiplier enforcing the quadratic Hamiltonian constraint

$$\mathcal{H} = G^{ij} p_i p_j + V(q) \approx 0, \tag{1.2}$$

with G^{ij} the reduced version of the DeWitt supermetric. The extremal condition $\delta S = 0$ gives the canonical equations

$$\frac{dq^i}{d\tau} = N[q^i, \mathcal{H}], \qquad \frac{dp_i}{d\tau} = N[p_i, \mathcal{H}]. \tag{1.3}$$

The solution of these equations describes the evolution of a spacelike hypersurface along the timelike direction; because of the presence of a Lagrange multiplier, this motion includes an arbitrariness which is associated to a simplified version of what in the context of the full theory is commonly called the problem of "many fingered time". On the other hand, the constraint $\mathcal{H} \approx 0$ acts as a generator of gauge transformations which can be written

$$\delta_\epsilon q^i = \epsilon(\tau)[q^i, \mathcal{H}], \qquad \delta_\epsilon p_i = \epsilon(\tau)[p_i, \mathcal{H}], \qquad \delta_\epsilon N = \frac{\partial \epsilon(\tau)}{\partial \tau}. \tag{1.4}$$

The equations (1.3) and (1.4) show that the dynamical evolution can be reproduced by a gauge transformation progressing with time, that is, any two succesive points on each classical trajectory are connected by a gauge transformation. Hence, the gauge fixation can be thought not only as a way to select one path from each class of equivalent paths in phase space, but also as a reduction procedure identifying a time for the system. An early proposal of quantization of the true physical degrees of freedom (in both the canonical and the path integral formalisms) can be found in [Barvinsky&-Ponomariov (1986); Barvinsky (1986); Barvinsky (1987)].

An important difficulty with a deparametrization program based on this idea is that admissible gauge conditions are those which can be reached from any path in phase space by means of gauge transformations leaving the action unchanged, and the action of parametrized systems like the gravitational field is not gauge invariant at the boundaries [Teitelboim (1982)]: under a gauge transformation defined by the parameters ϵ^m the action of a system with constraints C_m changes by

$$\delta_\epsilon S = \left[\epsilon^m(\tau)\left(p_i\frac{\partial C_m}{\partial p_i} - C_m\right)\right]_{\tau_1}^{\tau_2}. \tag{1.5}$$

Ordinary gauge systems include constraints that are linear and homogeneous in the momenta, plus a non vanishing Hamiltonian H_0 which is the total energy; for example, in the case of the electromagnetic field the canonical momenta are the four quantities $F^{\mu 0}$; for $\mu = 1, 2, 3$ we have the three components of the electric field, but for $\mu = 0$ we have the primary constraint $F^{00} = 0$ [Dirac (1964)]. Then it is $\delta_\epsilon S = 0$, and gauge conditions of the form $\chi(q, p, \tau) = 0$ (*canonical gauges*) are admissible. In the case of gravitation, instead, the Hamiltonian constraint is quadratic in the momenta, and we would have $\delta_\epsilon S \neq 0$ unless $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$; then gauge conditions involving derivatives of Lagrange multipliers as, for example, $\chi \equiv dN/d\tau = 0$ (*derivative gauges*) should be used [Halliwell (1988)]. These gauges cannot define a time in terms of the canonical variables. At the quantum level this has the consequence that canonical gauges could not be imposed in the path integral as it is usual in the Fadeev–Popov procedure for quantizing gauge systems [Fadeev&Popov (1967); Fadeev&Slavnov (1980)].

However, here we show that if the Hamilton–Jacobi equation associated to the Hamiltonian constraint is separable, the action of a parametrized system described by the coordinates and momenta (q^i, p_i) can be turned into the action of an ordinary gauge system described by the canonical variables (Q^i, P_i) by means of a canonical transformation which identifies the canonical Hamiltonian $\mathcal{H}$ with one of the new momenta [Ferraro&Simeone (1997); De Cicco&Simeone (1999b)]. As a result of the canonical transformation, when written in terms of the original variables the new action includes boundary terms which provide with gauge invariance at the end points [Henneaux et al. (1992)]. Canonical gauges are then admissible and a global phase time in terms of the coordinates and momenta can be identified for cosmological models by imposing $\tau-$dependent canonical gauge conditions on the ordinary gauge system; simultaneously, the quantum transition amplitude can be obtained by means of the usual path integral for gauge systems in a simple form which clearly shows the separation between true degrees of freedom and time. This is not the case of the usual scheme for the path integral quantization of minisuperspaces with the original not gauge-invariant action: as derivative gauges are required, a true time is not defined, so that the meaning of "evolution" of the system is not completely clear (even at the classical level); from a different point of view, noncanonical gauge conditions are not appropriate to visualize the possibility of the Gribov ambiguity [Gribov (1978); Henneaux&Teitelboim (1992)].

Our proposal solves these problems for a class of homogeneous cosmologies, and also clearly shows the restrictions arising from the geometry of the constraint surface: a global phase time in terms of the coordinates q^i (*intrinsic time* [Kuchař (1992)]) can be defined only if the potential in the Hamiltonian of the model has a definite sign; in this case, the choice is determined by the sheet of the constraint surface on which the system evolves. In the most general case, a global phase time must be a function of the coordinates and the momenta (*extrinsic time* [York (1972); Kuchař (1971)]). At the quantum level, this means that it would be necessary to identify the quantum states in the path integral by means of also the momenta, or that, when possible, the original variables should be abandoned, and a quantum description of the system should be made in terms of a new set of canonical variables such that the time is intrinsic. If the first option is adopted we shall find certain problems which are peculiar of gravity theory, while if we choose the second the interpretation will require some care.

In the present work homogeneous relativistic models as well as string cosmologies are studied. The restriction to minisuperspace models, that is, to cosmologies whose configuration space is finite dimensional, notably simplifies the analysis of what is really a problem of field theory, as the number of degrees of freedom of the full theory is infinite. Of course, one pays a price for doing so; the interpretation is not necessarily that of an approximation to the full theory, as "freezing" degrees of freedom before quantizing is not completely justified provided the special character of the amplitude superposition in quantum mechanics [Kuchař&Ryan (1989)]. For this reason some authors have suggested that one should think of minisuperspaces not as a tool to obtain physical predictions regarding the full theory within a certain degree of approximation, but rather as simplified models which can have their own physical interest, or in which we can try solutions for certain problems of the full theory [Halliwell (1990)]. Here, besides the physical interest that the models can have, in the case of relativistic cosmologies they have been chosen with the purpose of examplifying different formal problems and the proposed solutions, as well as some points regarding the interpretation of the results; in the case of string models, they are also useful to remark that, although string theory provides a closed formalism for quantum gravity in the particle picture, at the cosmological level the problems of quantization are the same –so that the proposed solutions are analogous– of those of general relativity.

The work is organized as follows: in Chapter 2 the Hamiltonian formalism for the gravitational field is introduced; the problem of time is presented, and the Dirac–Wheeler–DeWitt and the path integral quantization schemes are briefly discussed. Chapter 3 begins with the proposal of identifying a time by means of gauge fixation. A gauge-invariant action is constructed for a generic parametrized system with a separable Hamilton–Jacobi equation; then it is shown how canonical gauge conditions are used for deparametrizing the system, and the general form of the path integral for the resulting reduced system is given. The procedure is illustrated with some simple examples, as the relativistic free particle and the ideal clock; the chapter ends with a discussion about the relation between the quantization of the ideal clock and the transition probability for some toy universes. The straightforward application of our deparametrization and path integral quantization program to cosmological models begins in Chapter 4 with relativistic isotropic toy models (including the de Sitter universe), which are used to show the result of different deparametrizations and also to discuss the main difficulties that can be found in more physical systems. Empty models as well as models with a scalar field are studied. Then we consider anisotropic minisuperspaces. Homogeneous anisotropic relativistic cosmologies are comprised by the Bianchi models and the Kantowski–Sachs model. A particular case of the most general Bianchi type universe, the type IX, is the Taub model. Here the Kantowski–Sachs and the Taub models are deparametrized and quantized. In particular, the Taub universe provides an example of a model with true degrees of freedom for which the definition of a global phase time must necessarily involve the original momenta, because the potential in the Hamiltonian constraint does not allow for the existence of an intrinsic time. The Bianchi type I and the homogeneous Szekeres universes are also deparametrized. In Chapter 5 we extend our analysis beyond the relativistic framework, and cosmology models of the low energy effective theory of closed bosonic strings are studied. The quantization of string models has been analysed in the context of the graceful exit problem [Gasperini (1999); Gasperini (2000)], and it has been remarked that this quantization requires a careful treatment of the subtleties that are typical of the quantization of gauge systems [Cavaglià&De Alfaro (1997); Cavaglià&Ungarelli (1999)]. Here, models with a dilaton field, a two-form field and the tensor field $g_{\mu\nu}$ which determines the background geometry of spacetime are considered. The low energy effective action of the theory is put in the Hamiltonian form, and the minisuperspaces are deparametrized and quantized in the same way that in the relativistic models. The analysis is extended to non separable models giving a prescription to determine whether a time for a system described by a separable Hamiltonian is also a time for one with a more general constraint. Finally, in Chapter 6 the Dirac–Wheeler–DeWitt and the Schrödinger quantization schemes are discussed for some relativistic and string cosmologies. A brief review of different approaches within the canonical method is given, both with and without the previous identification of a time. In particular, the case of the Taub universe is analysed with certain detail, as it provides a hint to get a better understanding of the role that the momenta play in the characterization of the states, as well as of the way of handling with boundary conditions. It is shown that our deparametrization program can be the first step to obtain a wave function with an evolutionary form by means of a Wheeler–DeWitt equation. In the final Discussion some open problems regarding both the formalism and the interpretation of the theory are reviewed. The basic concepts of the Hamiltonian formalism for constrained systems are reviewed with some detail in Appendix A, while those of the path integral quantization and of the definition of a physical inner product are discussed in Appendix B. Thus the book admits two levels of reading: a reader who is already working in the subject can go directly to the problem of time in the quantization of the gravitational field, and could even begin with Chapter 3; a student, instead, should begin by reading these Appendices. In Appendix C we give the appropriate boundary terms for some more or less generic forms of the solution of the Hamilton–Jacobi equation, and in Appendix D we show that, if we are only interested in the deparametrization, an extrinsic time is easy to find for the Taub anisotropic universe.

Chapter 2

The gravitational field as a constrained Hamiltonian system

2.1 Momentum and Hamiltonian constraints

As starting point for building a quantum theory of gravitation it is usual to begin with the classical Einstein action for the gravitational field. If there are no matter fields the action S is a functional of the spacetime metric $g_{\mu\nu}(X)$, and the dynamics yielding from the extremal condition $\delta S = 0$ is given by a succesion of spacelike three-dymensional hypersurfaces in four-dymensional spacetime. If the timelike parameter τ is introduced and the points of each surface are labeled by the internal coordinates x^a $(a = 1, 2, 3)$, the surfaces can be described by

$$X^\mu = e^\mu(\mathbf{x}, \tau).$$

At any point of a given hypersurface we can define the normal and tangential vectors n_μ and e^μ_a:

$$n_\mu e^\mu_a = 0, \qquad g^{\mu\nu} n_\mu n_\nu = -1, \qquad e^\mu_a = \frac{\partial e^\mu(\mathbf{x}, \tau)}{\partial x^a}.$$

Then the theory can be reparametrized by changing from the spacetime metric $g_{\mu\nu}(X)$ to a new basis given by the spatial three-metric g_{ab} on a hypersurface and the velocity U^μ with which the surface evolves in spacetime:

$$g_{ab} = e^\mu_a g_{\mu\nu} e^\nu_b, \qquad U^\mu = \frac{\partial e^\mu(\mathbf{x}, \tau)}{\partial \tau}.$$

The normal and tangential components of the velocity U^μ are the lapse and shift functions

$$N(\mathbf{x},\tau) = -n_\mu \frac{\partial e^\mu}{\partial \tau}, \qquad N^a(\mathbf{x},\tau) = g^{ab} e^\mu_b g_{\mu\nu} \frac{\partial e^\nu}{\partial \tau}$$

defined by Kuchař [Kuchař (1976)] as a generalization of those introduced by Arnowitt, Deser and Misner $N = (-g^{00})^{-1/2}$, $N^a = g^{ab} g_{b0}$ [Arnowitt et al. (1962)]. The shrinkage and deformation of the spacelike hypersurface imbedded in spacetime and evolving in the normal direction is described by the extrinsic curvature

$$K_{ab} = \frac{1}{2N}\left(\nabla_a N_b + \nabla_b N_a - \frac{dg_{ab}}{d\tau}\right),$$

where ∇ denotes a spatial covariant derivative. In terms of the lapse and shift functions and of the coordinates $(\tau, \mathbf{x})$, the Lagrangian form of the Einstein action with cosmological constant is

$$S[g_{ab}, N, N^a] = \int_{\tau_1}^{\tau_2} d\tau \int d^3x \, N(^3g)^{1/2}\left(K_{ab}K^{ab} - K^2 + {}^3R - 2\Lambda\right) \tag{2.1}$$

where ^3R is the scalar curvature of space. The set of all possible three-metrics is called *superspace*; this space is provided with a metric, the DeWitt supermetric given by

$$G^{abcd} = \frac{1}{4}(^3g)^{1/2}(g^{ac}g^{bd} + g^{ad}g^{bc} - 2g^{ab}g^{cd})$$

If we define the canonical momenta p^{ab} as

$$p^{ab} = -2G^{abcd}K_{cd}$$

we can write the action in its Hamiltonian form:

$$S[g_{ab}, p^{ab}, N, N^a] = \int d\tau \int d^3x \, (p^{ab}\frac{dg_{ab}}{d\tau} - N\mathcal{H} - N^a\mathcal{H}_a) \tag{2.2}$$

where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}G_{abcd}p^{ab}p^{cd} - (^3g)^{1/2}(^3R - 2\Lambda), \\ \mathcal{H}_a &= -2g_{ac}\nabla_d p^{cd}, \\ G_{abcd} &= (^3g)^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{bc} - 2g_{ab}g_{cd}). \end{aligned} \tag{2.3}$$

Up to now we have considered pure gravitational dynamics. Matter fields ϕ can also be included in the action functional by means of a combination of the lapse and shift functions, and then in the canonical formulation we obtain additional terms $\mathcal{H}_{matt}$ and $\mathcal{H}_{a,matt}$. Hence we can define the extended set of phase space coordinates and momenta

$$\begin{aligned} q^i &\equiv (g_{ab}(\mathbf{x}), \phi(\mathbf{x})) \\ p_i &\equiv (p^{ab}(\mathbf{x}), p_\phi(\mathbf{x})). \end{aligned} \tag{2.4}$$

The variational principle leads to the dynamical equations for the coordinates and momenta, which in the Poisson brackets formalism read

$$\begin{aligned} \frac{dq^i}{d\tau} &= N[q^i, \mathcal{H}] + N^a[q^i, \mathcal{H}_a] \\ \frac{dp_i}{d\tau} &= N[p_i, \mathcal{H}] + N^a[p_i, \mathcal{H}_a]. \end{aligned} \tag{2.5}$$

There are no equations for the evolution of the lapse and shift functions, which remain arbitrary; instead, when we demand the action to be stationary under an arbitrary variation of N and N^a we obtain what are called the *Hamiltonian and momentum constraints*

$$\mathcal{H} = 0, \qquad \mathcal{H}_a = 0. \tag{2.6}$$

The presence of these constraints reflects the general covariance of the theory, i.e. that the theory of gravitation is covariant under general changes of the coordinates. The lapse N determines the normal separation between two succesive hypersurfaces, while $N^a d\tau$ determines the shift between their internal coordinate systems. The arbitrariness of N and N^a leads to the many-fingered nature of time, as these functions are associated to local arbitrary deformations of the evolving hypersurface. This can be easily understood by considering a given hypersurface and the normal and tangential directions on it, and the motion in the special case of a null shift: Because the lapse corresponds to the velocity of the motion of the three-hypersurface in the normal direction, as N depends on $\mathbf{x}$ and τ the separation between two succesive hypersurfaces is different in different points of spacetime, and then the time has a local character.

2.2 Minisuperspaces as constrained systems

The restriction to a finite dimensional configuration space, which is called the *minisuperspace* approximation, and the choice of an homogeneous lapse and zero shift lead to an action which in its Lagrangian form reads

$$S[q^i, N] = \int_{\tau_1}^{\tau_2} N \left(\frac{1}{2N^2} G_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} - V(q) \right) d\tau \tag{2.7}$$

where G_{ij} is the reduced version of the DeWitt supermetric and V is the potential, which depends on the curvature and includes terms corresponding to the coupling between the metric and matter fields; it must be understood that we have already performed the spatial integration (trivial if the homogeneity hypothesis is assumed), so that only the integration on τ remains. The Hamiltonian form of the action is

$$S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - N\mathcal{H} \right) d\tau, \tag{2.8}$$

where

$$\mathcal{H} = G^{ij}\, p_i p_j + V(q). \tag{2.9}$$

As the shift is null the momenta are proportional to the derivatives of the coordinates:

$$p_i = \frac{1}{N} G_{ij} \frac{dq^j}{d\tau}.$$

For example, in the case of an isotropic empty cosmological model we would have only one coordinate, $q = \Omega \sim \ln a(\tau)$ (a the scale factor of the model) and only one momentum, $p = \pi_\Omega = -(e^{3\Omega}/N)d\Omega/d\tau$. The action (2.8) describes a system which, as we shall inmediately see, is invariant under redefinitions of the parameter τ, that is, what is usually called a parametrized system. The reparametrization invariance is what remains of the general covariance of the full theory after all except a finite number of degrees of freedom of the originally infinite number have been "frozen". Note that even though the shift is zero and the lapse is homogeneous, N is still a function of τ, so that the separation between two succesive three-surfaces, although globally the same, is still undetermined.

Let us calculate the most general variation of the action (2.8). Under arbitrary changes of the coordinates and momenta q^i and p_i and of the lapse N we obtain

$$\begin{aligned}\delta S &= p_i \delta q^i \big|_{\tau_1}^{\tau_2} + \\ &+ \int_{\tau_1}^{\tau_2} \left[\left(\frac{dq^i}{d\tau} - N\frac{\partial \mathcal{H}}{\partial p_i} \right) \delta p_i - \left(\frac{dp^i}{d\tau} + N\frac{\partial \mathcal{H}}{\partial q_i} \right) \delta q^i - \mathcal{H}\delta N \right] d\tau.\end{aligned} \tag{2.10}$$

If we demand the action to be stationary when the coordinates q^i are fixed at the boundaries, on the classical path we obtain the Hamilton canonical equations

$$\frac{dq^i}{d\tau} = N[q^i, \mathcal{H}], \qquad \frac{dp_i}{d\tau} = N[p_i, \mathcal{H}] \tag{2.11}$$

and the Hamiltonian constraint

$$\mathcal{H} \approx 0 \tag{2.12}$$

(we use $\approx$ to denote a weak equality, i.e. one which is valid only on the constraint surface).

Two features of the dynamics should be emphasized. The first is that the presence of the constraint $\mathcal{H} = 0$ restricts possible initial conditions to those lying on the constraint hypersurface. The second is that the evolution of the lapse N is arbitrary, as it is not determined by the canonical equations; hence, a family of classical trajectories exists for each set of initial data. The fact that the solution of the dynamical equations is not uniquely determined is always associated to the existence of a symmetry in the action. Then let us consider the invariances of the action (2.8). This action is invariant under the transformation

$$\delta q^i = \epsilon(\tau)\frac{dq^i}{d\tau}, \qquad \delta p_i = \epsilon(\tau)\frac{dp_i}{d\tau}, \qquad \delta N = \frac{d(N\epsilon)}{d\tau} \tag{2.13}$$

with $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$. This transformation is called a reparametrization because it is equivalent to change τ by $\tau + \epsilon(\tau)$ on the path given by $q^i(\tau)$ and $p_i(\tau)$, with the integral

$$\int_{\tau_1}^{\tau_2} N d\tau$$

remaining unchanged. The invariance of the action under a reparametrization means that τ is not the time, but it is a physically irrelevant parameter.

When a system is described by an action like (2.8) the solutions of the dynamical equations are not parametrized by τ but are given as

$$q^i = q^i\left(\int N d\tau\right), \qquad p_i = p_i\left(\int N d\tau\right),$$

so what we can call the "proper time" $\int N d\tau$, instead of τ, plays the role of time. When these equations can be globally solved for $\int N d\tau$, that is, if we can find $t(q^i, p_i) = \int N d\tau$, it is said that a global phase time $t(q^i, p_i)$ exists for the system.

Now consider a gauge transformation:

$$\delta_\epsilon q^i = \epsilon(\tau)[q^i, \mathcal{H}], \qquad \delta_\epsilon p_i = \epsilon(\tau)[p_i, \mathcal{H}], \qquad \delta_\epsilon N = \frac{\partial \epsilon(\tau)}{\partial \tau}. \tag{2.14}$$

In this case we have

$$\begin{aligned} \delta_\epsilon S &= p_i \delta_\epsilon q^i \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \frac{\partial(\epsilon \mathcal{H})}{d\tau} d\tau \\ &= \left[\epsilon(\tau)\left(p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\right)\right]_{\tau_1}^{\tau_2}. \end{aligned} \tag{2.15}$$

We see that on the classical path, where Hamilton equations hold, the reparametrization (2.13) is equivalent to a gauge transformation with parameter $N\epsilon$ and the boundary restrictions $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$. Because in the case of parametrized systems like the gravitational field the constraint is not linear and homogeneous in the momenta, the variation of the action under a gauge transformation is equal to the end point terms given by (2.15); the action is gauge-invariant only if we restrict gauge transformations to those mapping the boundaries onto themselves, that is if we restrict admissible gauges to those fulfilling

$$\epsilon(\tau_1) = \epsilon(\tau_2) = 0. \tag{2.16}$$

Gauge invariance is usually regarded as the consequence of spurious degrees of freedom. However, gauge invariance of parametrized systems is related to reparametrization invariance: the physically irrelevant variable is not a canonical variable but it is the time parameter τ. A true time will always exist for the minisuperspaces considered here, but this is not the general case; an example of a parametrized action to which a time cannot be associated is the Jacobi action, whose variation leads to curves with a given energy in phase space, and without information about the evolution with time [Lanczos (1986); Brown&York (1989)].

2.3 Quantization

2.3.1 *Canonical quantization*

In the Dirac–Wheeler–DeWitt canonical quantization one introduces a wave function Ψ which must obey the operator form of the constraint equation $\mathcal{H} \approx 0$, that is,

$$\mathcal{H}\Psi = 0, \tag{2.17}$$

where the momenta are replaced in the usual way by operators in terms of derivatives of the coordinates:

$$p_i = -i\frac{\partial}{\partial q^i}$$

(in the full theory the q^i are functions of the spacetime coordinates, and we have functional derivatives). As the Hamiltonian is quadratic in p_i a second order differential equation is obtained; this is called the Wheeler–DeWitt equation [DeWitt (1967)]. Note that because the reduced supermetric depends on the coordinates one should pay attention to operator ordering; however, if we are interested in low order approximations this point can be neglected. It is clear that the solution Ψ does not depend explicitly on the time parameter τ, but only on the coordinates q^i. Of course, this reflects the reparametrization invariance of the theory. In the context of the present work we should remark that this is the main problem with the Dirac–Wheeler–DeWitt quantization, because the absence of a clear notion of time makes difficult to have a definition of conserved positive-definite probability, and therefore to guarantee the unitarity of the theory. To build the space of physical states we need to define an inner product which takes into account that there can be a physical time "hidden" among the canonical variables of the system: the physical inner product $(\Psi_2|\Psi_1)$ must be defined by fixing the time in the integration. Hence, to obtain a closed theory by this way we need a formally right definition of time.

If we are able to identify the time as a function of the canonical variables, we can perform a canonical transformation to new coordinates (t, q^γ)

and the corresponding momenta (p_t, p_γ). Then we can make the substitution $p_t = -i\partial/\partial t$, $p_\gamma = -i\partial/\partial q^\gamma$ to obtain a Wheeler–DeWitt equation whose solution depends on t in an explicit form, and then the physical inner product can be written

$$(\Psi_2|\Psi_1) = \frac{i}{2}\int_{t=const} dq \left[\Psi_1^* \frac{\partial \Psi_2}{\partial t} - \Psi_2 \frac{\partial \Psi_1^*}{\partial t}\right], \tag{2.18}$$

where the integration is restricted to the q^γ.

It has also been pointed (see Chapter 6) that, depending on the choice of the canonical transformation, it may also be possible to obtain a constraint linear in p_t; in this case a Schrödinger equation with a true Hamiltonian h

$$i\frac{\partial \Psi}{\partial t} = h\Psi$$

would be obtained, and the physical inner product could be defined as $(\Psi_2|\Psi_1) = \langle\Psi_2|\hat{\mu}|\Psi_1\rangle$, with $\hat{\mu}_{t'} = \delta(t-t')$, so that the integral is evaluated at the fixed time t' (such a procedure will prove to be valid only for a limited class of models; see below). The relation between the solutions of the Schrödinger equation and those corresponding to the Wheeler–DeWitt equation would require a detailed analysis: not only there is a change of variables to discuss, but as the Wheeler–DeWitt equation is an hyperbolic one, while the Schrödinger equation is parabolic, the last has a smaller set of solutions. We shall return to this point in the context of the quantization of the Taub anisotropic universe and of string cosmologies, but here we can say that the answer depends on the point of view that we adopt: we could select the set of solutions of the Wheeler–DeWitt equation which corresponds to those of the Schrödinger equation; or we can understand that a linear constraint and a Schrödinger equation yield only a subset of all possible solutions for the wave function. If we consider that a quadratic constraint reflects an essential feature of the gravitational field we should follow the second line.

2.3.2 *Path integral quantization*

Another way to obtain the wave function is to calculate the quantum propagator by means of a path integral (see Appendix B). If the transition amplitude is obtained as the sum over all histories of the exponential of the action (2.8), the result diverges because of the integration over paths in phase space which are physically equivalent as they are connected by gauge transformations. This is solved by imposing gauge conditions that select one path from each class of equivalent paths. The path integral gives a transition amplitude for states characterized by the variables which are fixed at the end points in the variational principle; then if we demand $\delta S = 0$ for $\delta q_1^i = \delta q_2^i = 0$ the path integral gives the amplitude for the transition $|q_1^i\rangle \to |q_2^i\rangle$. In its phase space form the propagator then reads

$$\langle q_2^i | q_1^i \rangle = \int Dq^i Dp_i DN \delta(\chi) |[\chi, \mathcal{H}]| \exp\left(iS[q^i, p_i, N]\right) \tag{2.19}$$

(τ does not appear because it has no physical meaning). $\chi = 0$ is a gauge fixing function and $|[\chi, \mathcal{H}]|$ is the Fadeev–Popov determinant, which makes the result independent of the gauge choice (see Appendix B). Admissible gauge conditions for the path integral are those which can be reached from any path by performing a gauge transformation which is compatible with the symmetries of the action. Let us consider a trajectory which differs from a given gauge by an infinitesimal quantity Δ; the gauge transformation which makes the variables reach the gauge condition must be such that

$$\delta_\epsilon \chi = -\Delta. \tag{2.20}$$

As the two boundary conditions $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$ must be satisfied by the gauge parameter $\epsilon(\tau)$, the equation (2.20) should be of second order in ϵ. Since $\delta_\epsilon N = d\epsilon/d\tau$, the simplest gauge choice can be given by a function of $dN/d\tau$, namely

$$\chi = \frac{dN}{d\tau} = 0 \tag{2.21}$$

(the most general admissible gauge would be $\chi \equiv dN/d\tau - \chi^*(q^i, p_i, N) = 0$ [Halliwell (1988)]). Any particular choice of $N(\tau)$ can be carried to $dN/d\tau = 0$ by succesive infinitesimal gauge transformations of the form $\delta_\epsilon N = d\epsilon/d\tau$; these transformations are possible because there are no restrictions on $d\epsilon/d\tau$, but only on the gauge parameter ϵ at the end points. Gauge conditions like (2.21) are called "derivative gauges". Note that although the gauge (2.21) does not fix the value of N but only means that N is constant on the trajectory, the value of N is determined by the variational principle when the data at τ_1 and τ_2 are enough to determine the global phase time $t(q,p) = \int N d\tau$ at the boundaries [Ferraro&Simeone (1997)]. Effectively, then we have $N = \Delta t/\Delta\tau$ and no ambiguities are left on the classical trajectory. A quantization procedure involving derivative gauges has a problem analogous to that of the canonical scheme, in the sense that there is not a clear distinction between time and true physical degrees of freedom. Also, it does not allow to foresee how the Gribov ambiguity will be avoided.

We should point that in the path integral formulation the problem of operator ordering is translated to the skeletonization: the conmutators of operators at the same time are neglected, but are taken into account for causally connected operators separated by nonzero time intervals. In practice, the paths in phase space are divided into segments given by two different sets of points, one for the coordinates and the other for the momenta. The precise choice of these points determines the operator ordering in the Hamiltonian (for details see [Barvinsky (1993)] and [Henneaux&Teitelboim (1992)]).

Chapter 3

Deparametrization and path integral quantization

3.1 The identification of time

We have seen that, in the case of gravitational dynamics, given an initial condition on the canonical variables the whole set of different classical trajectories in the configuration space corresponding to different choices of the lapse N can be generated by gauge transformations. Given a point on a classical trajectory associated to a lapse $N_1(\tau)$, a finite gauge transformation whose infinitesimal form is (2.14) connects it with another point on other classical trajectory associated to a different lapse $N_2(\tau)$. Also, one can take the initial conditions, and starting from them construct any classical trajectory by means of a succesion of finite gauge transformations. In other words, the dynamical evolution, which includes the problem of the multiplicity of times associated to the fact that the separation between succesive three-hypersurfaces is arbitrary, can be reproduced by gauge transformations [Barvinsky (1993)]. It is therefore natural to think that gauge fixation should be a way to identify a time; but the fact that the admissible gauges are not of the canonical form $\chi(q^i, p_i, \tau)$, because of the lack of gauge invariance of the action at the end points, makes not manifest how this could lead to a practicable deparametrization program [Simeone (1999)].

3.1.1 *Gauge fixation and deparametrization*

The choice of the gauge conditions $\chi = 0$ appropriate to select one path from each class of equivalent paths in phase space is restricted by:

(1) An admissible gauge condition must can be reached from any path

by means of gauge transformations leaving the action unchanged.

(2) Only one point of each orbit, that is, of each set of points on the constraint surface connected by gauge transformations, must be on the manifold defined by $\chi = 0$. This usually requires some care: if the hypersurface defined by the constraint equation $\mathcal{H} = 0$ is topologically non trivial it may be difficult to intersect it with a gauge condition which is crossed by each orbit only once. This is called the *Gribov problem.*

Suppose that it is possible to perform a canonical transformation $(q^i, p_i) \to (Q^i, P_i)$ such that the Hamiltonian constraint $\mathcal{H}$ is matched to one of the new momenta, for example P_0. Then in terms of the new variables (Q^i, P_i) the action functional would include a constraint which is linear and homogeneous in the momenta, and would be gauge-invariant even at the boundaries. This is equivalent to say that the canonical variables (Q^i, P_i) describe an ordinary gauge system. Canonical gauge conditions

$$\chi(Q^i, P_i, \tau) = 0$$

would then be admissible, that is, they would fulfill the condition (1).

The condition (2) requires that a gauge transformation moves a point of an orbit off the surface $\chi = 0$; as gauge transformations are generated by the constraint $\mathcal{H}$, then we should verify that

$$\delta_\epsilon \chi = \epsilon(\tau)[\chi, \mathcal{H}] \neq 0 \tag{3.1}$$

unless $\epsilon = 0$; this holds if

$$[\chi, \mathcal{H}] \neq 0. \tag{3.2}$$

Now, as Q^0 and P_0 are conjugated variables,

$$[Q^0, P_0] = 1 \tag{3.3}$$

and as we have identified $\mathcal{H} \equiv P_0$, then a gauge condition of the form

$$\chi \equiv Q^0 - T(\tau) = 0 \tag{3.4}$$

with T a monotonous function is a good choice. Strictly speaking, equation (3.1) only ensures that the orbits are not tangent to the surface $\chi = 0$; however, as (3.4) defines a plane $Q^0 = constant$ for each τ, if at any τ any orbit was intersected more than once (then yielding Gribov copies) at another τ it should be $[\chi, P_0] = 0$. Therefore this gauge fixation procedure avoids the Gribov problem. This choice is not the only possible one: for example, we could multiply Q_0 by any function which conmutes with P_0 and which is everywhere non null.

From a different point of view, given a parametrized system with coordinates and momenta (q^i, p_i) a smooth function $t(q^i, p_i)$ fulfilling

$$[t, \mathcal{H}] > 0 \tag{3.5}$$

is a global phase time for the system [Hájícek (1986)], and its values along any classical trajectory can parametrize its evolution. Because the Poisson bracket is invariant under a canonical transformation, from (3.3) and (3.5) it follows that a globally good gauge choice given in terms of the coordinate Q^0 of the gauge system can be used to define a global phase time t for the parametrized system in terms of its coordinates and momenta (q^i, p_i). In other words, *a gauge choice for the gauge system defines a particular foliation of spacetime for the parametrized system* [Simeone (1999)]. If we are sure that we have found a gauge choice which avoids the Gribov ambiguity then this gauge provides a definition of time which is good everywhere. A transformation such that $\mathcal{H} = P_0$ can always be found locally; the point is to obtain a canonical transformation which works in the whole phase space.

The condition for a function $t(q^i, p_i)$ to be a global phase time, that is, that its Poisson bracket with the Hamiltonian constraint is positive definite, can be understood as follows: Define the Hamiltonian vector

$$\begin{aligned} \mathbf{H} \equiv \mathrm{H}^A &= (\mathrm{H}^q, \mathrm{H}^p) \\ &= \left(\frac{\partial \mathcal{H}}{\partial p}, -\frac{\partial \mathcal{H}}{\partial q}\right). \end{aligned} \tag{3.6}$$

Then the condition

$$[t, \mathcal{H}] > 0$$

is equivalent to

$$\mathrm{H}^A \frac{\partial t}{\partial x^A} > 0$$

with $x^A = (q^i, p_i)$. This means that $t(q^i, p_i)$ monotonically increases along a dynamical trajectory, that is, each surface $t =$ *constant* in the phase space is crossed by a dynamical trajectory only once (so that the field lines of $\mathbf{H}$

are open); hence the succesive states of the system can be parametrized by $t(q^i, p_i)$.

Now suppose that we define a scaled constraint

$$H = \mathcal{F}^{-1}\mathcal{H}, \qquad \mathcal{F} > 0.$$

It can easily be shown that H and $\mathcal{H}$ are equivalent constraints in the sense that they describe the same parametrized system: their field lines, which coincide with the classical trajectories, are proportional on the constraint surface. Thus, if we can find a function $\bar{t}(q^i, p_i)$ with the property

$$[\bar{t}, H] > 0,$$

we know that $\bar{t}(q^i, p_i)$ monotonically increases along the dynamical trajectories associated to both H and $\mathcal{H}$, and it is also a global phase time. The fact that if the constraint is scaled by a positive definite function we obtain an equivalent constraint can sometimes simplify the resolution of the deparametrization problem, as it will be based on the possibility of solving the Hamilton–Jacobi equation associated to the Hamiltonian constraint.

3.1.2 *Topology of the constraint surface: intrinsic and extrinsic time*

As we have just signaled, a function $t(q^i, p_i)$ is a global phase time if $[t, \mathcal{H}] > 0$. Because the supermetric G^{ik} does not depend on the momenta, a function $t(q^i)$ is a global phase time if the bracket

$$\begin{aligned} [t(q^i), \mathcal{H}] &= [t(q^i), G^{ik} p_i p_k] \\ &= 2\frac{\partial t}{\partial q^i} G^{ik} p_k \end{aligned}$$

is positive definite. Note that if the supermetric has a diagonal form and one of the momenta vanishes at a given point of phase space, then no function of only its conjugated coordinate can be a global phase time. For a constraint whose potential can be zero for finite values of the coordinates, the momenta p_k can be all equal to zero at a given point, and $[t(q^i), \mathcal{H}]$ can vanish. Hence an *intrinsic time* $t(q^i)$ can be identified only if the potential in the constraint has a definite sign. In the most general case a global phase time should be a function including the canonical momenta; in this case it is said that the system has an *extrinsic time* $t(q^i, p_i)$, because the momenta are related to the extrinsic curvature.

It is common to regard an intrinsic time as more "natural", and the necessity of defining an extrinsic time as a somewhat problematic peculiarity. However, this is perhaps only a consequence of usually working with simple parametrized systems like, for example, the relativistic particle (see below); the formalism for these systems, when put in a manifestly covariant form, has the time included among the coordinates, and the evolution is given in terms of a physically meaningless time parameter. But while for these systems the time coordinate always refers to an external clock, this is clearly not the case in cosmology; for example, in the case of pure gravitational dynamics the coordinates are the elements of the metric g_{ab} over spatial slices, and in principle there is not necessarily a connection between g_{ab} and anything "external". Rather, such a relation can be thought to exist for the derivatives $dg_{ab}/d\tau$ of the metric, as they appear in the expression for the extrinsic curvature K_{ab} which describes the evolution of spacelike three-dimensional hypersurfaces in four-dimensional spacetime. If no matter fields are present the canonical momenta are given by

$$p_i \equiv p^{ab} = -2G^{abcd}K_{cd},$$

and then one must expect the momenta to appear in the definition of a global phase time [Giribet&Simeone (2001b)]. The existence of a time in terms of only the coordinates should therefore be understood as a sort of an "accident" related to the fact that, in some special cases which do not represent the general features of gravitation, there exists a relation that enables to obtain the coordinates in terms of the momenta with no ambiguities. At the quantum level this means that we shall have to revise some points of the path integral quantization to which we are used: as we shall see below, there are cosmological models for which a quantum description in terms of only the original coordinates will be impossible if we want to work in a theory with a clear notion of time.

3.2 Gauge-invariant action for a parametrized system

In this section we shall develop a procedure to obtain a gauge-invariant action for parametrized systems whose Hamiltonian constraint is such that the associated $\tau-$independent Hamilton–Jacobi equation is separable. We have already remarked that the variation of the action of a parametrized system under a gauge transformation is equal to end point terms; the action of the gravitational field then does not have gauge invariance at the boundaries and canonical gauges would not be admissible. But in the last section we pointed that if it was possible to define a canonical transformation such that the Hamiltonian constraint could be matched with a new momentum, the system could be turned into an ordinary gauge one; hence canonical gauge conditions could be imposed to select one path from each class of equivalent paths in phase space. In fact, we shall see that when we are able to find such a transformation, the result is equivalent to adding boundary terms, and the variation of these terms exactly cancels the variation of the original action. From this point of view, there would not be a true conceptual difference between a parametrized system and an ordinary gauge one: although the practical value of having linear constraints has led to a distinction, the existence of a particular gauge symmetry would be nothing more than a consequence of a given choice of variables, and it may be that the variables which appear more natural or intuitive are not the best for building a closed formalism. Moreover, in the next chapter we shall see that what turns to be the formally correct choice of variables may not coincide with what one expects from the usual path integral procedure, but it agrees with what is sometimes found in classical cosmology.

3.2.1 *End point terms*

Let us consider a complete solution [Landau&Lifshitz (1960)] $W(q^i, \alpha_\mu, E)$ of the Hamilton–Jacobi equation

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E \tag{3.7}$$

where H is not necessarily the original Hamiltonian constraint but it can be a scaled Hamiltonian, that is $H = \mathcal{F}^{-1}\mathcal{H}$ with $\mathcal{F}$ a positive definite function of q^i. If E and the integration constants α_μ are matched to the new momenta $\overline{P}_0$ and $\overline{P}_\mu$ respectively, then $W(q^i, \overline{P}_i)$ turns to be the generator function of a canonical transformation $(q^i, p_i) \to (\overline{Q}^i, \overline{P}_i)$ defined by the equations

$$p_i = \frac{\partial W}{\partial q^i}, \qquad \overline{Q}^i = \frac{\partial W}{\partial \overline{P}_i}, \qquad \overline{K} = N\overline{P}_0 = NH \tag{3.8}$$

where $\overline{K}$ is a new Hamiltonian. The new coordinates and momenta verify

$$[\overline{Q}^{\mu}, \overline{P}_0] = [\overline{Q}^{\mu}, H] = 0$$

$$[\overline{P}_{\mu}, \overline{P}_0] = [\overline{P}_{\mu}, H] = 0$$

$$[\overline{Q}^{0}, \overline{P}_0] = [\overline{Q}^{0}, H] = 1.$$

The resulting action

$$\overline{S}[\overline{Q}^i, \overline{P}_i, N] = \int_{\tau_1}^{\tau_2} \left(\overline{P}_i \frac{d\overline{Q}^i}{d\tau} - N\overline{P}_0 \right) d\tau \tag{3.9}$$

describes a system with a null true Hamiltonian and a constraint which is linear and homogeneous in the momenta. Therefore the action $\overline{S}$ has gauge freedom at the boundaries, and canonical gauges would be admissible in a path integral with this action. The action $\overline{S}$ can be related with the original action S by recalling that

$$p_i dq^i = d\left(W(q^i, \overline{P}_i) - \overline{Q}^i \overline{P}_i\right) + \overline{P}_i d\overline{Q}^i,$$

as it follows from (3.8). Thus in terms of the original canonical variables we have

$$\begin{aligned} \overline{S}[q^i, p_i, N] &= \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - NH \right) d\tau \\ &\quad + \left[\overline{Q}^i(q^i, p_i)\overline{P}_i(q^i, p_i) - W(q^i, \overline{P}_i) \right]_{\tau_1}^{\tau_2}. \end{aligned} \tag{3.10}$$

We then see that the gauge-invariant action $\overline{S}$ differs from the original action S in the end point terms

$$\overline{B} = \left[\overline{Q}^i(q^i, p_i)\overline{P}_i(q^i, p_i) - W(q^i, \overline{P}_i) \right]_{\tau_1}^{\tau_2}. \tag{3.11}$$

It is simple to verify that these terms effectively cancel the variation of the action S under a gauge transformation: if we write

$$\begin{aligned} \delta_\epsilon \overline{B} &= \left[\delta_\epsilon (\overline{Q}^i \overline{P}_i - W) \right]_{\tau_1}^{\tau_2} \\ &= \left[\epsilon(\tau)\overline{P}_0 - \frac{\partial W}{\partial q^i} \delta_\epsilon q^i - \frac{\partial W}{\partial \overline{P}_i} \delta_\epsilon \overline{P}_i \right]_{\tau_1}^{\tau_2} \end{aligned}$$

and we use that $[\overline{P}_i, \overline{P}_0] = 0$ and $\delta_\epsilon q^i = \epsilon(\tau)[q^i, H]$ we obtain

$$\delta_\epsilon \overline{B} = -\left[\epsilon(\tau)\left(p_i \frac{\partial H}{\partial p_i} - H\right)\right]_{\tau_1}^{\tau_2}$$

(compare with Eq. (2.15)). The end point terms then improve the action with gauge invariance at the boundaries, and they do not modify the dynamics, as they can be included in the action integral as a total derivative with respect to the parameter τ.

3.2.2 *Observables and time*

Because $\overline{Q}^\mu$ and $\overline{P}_\mu$ conmute with $\overline{K} = N\overline{P}_0$ then they are *conserved* observables describing what we shall call the *reduced system.* This makes impossible to characterize the dynamical trajectories of the system by an arbitrary choice of $\overline{Q}^\mu$ at the boundaries τ_1 and τ_2. If we want to obtain a set of observables such that the choice of the new coordinates is enough to characterize the dynamical evolution we should look for non conserved variables, and hence a new $\tau-$dependent transformation leading to a non null Hamiltonian should be defined. In other words, we need to perform a canonical transformation in the space of observables with a generator function which depends on τ. A second canonical transformation will give additional end point terms; because after the first transformation $(q^i, p_i) \to (\overline{Q}^i, \overline{P}_i)$ we have already obtained a gauge-invariant action, the new boundary terms must be gauge invariant.

Let us consider the canonical transformation generated by

$$F(\overline{Q}^i, P_i, \tau) = P_0 \overline{Q}^0 + f(\overline{Q}^\mu, P_\mu, \tau). \tag{3.12}$$

Then we have

$$H = \overline{P}_0 = \frac{\partial F}{\partial \overline{Q}^0} = P_0, \qquad \overline{P}_\mu = \frac{\partial F}{\partial \overline{Q}^\mu} = \frac{\partial f}{\partial \overline{Q}^\mu},$$

$$Q^0 = \frac{\partial F}{\partial P_0} = \overline{Q}^0, \qquad Q^\mu = \frac{\partial F}{\partial P_\mu} = \frac{\partial f}{\partial P_\mu}.$$

The transformation in the reduced phase space is defined by the generator f. The coordinates and momenta (Q^μ, P_μ) are observables because they conmute with the constraint $H = P_0$,

$$[Q^\mu, P_0] = [P_\mu, P_0] = 0,$$

but they are not conserved quantities, because their evolution is governed by the new non vanishing Hamiltonian

$$K = NP_0 + \frac{\partial f}{\partial \tau} = NH + \frac{\partial f}{\partial \tau} \tag{3.13}$$

(see, for example, the discussion about "perennials" in [Kuchař (1993)]). Indeed, we have that

$$\begin{aligned} \frac{dQ^\mu}{d\tau} &= \frac{\partial K}{\partial P_\mu} = \frac{\partial^2}{\partial\tau\partial P_\mu} f(\overline{Q}^\mu(Q^\mu, P_\mu), P_\mu, \tau) \\ \frac{dP_\mu}{d\tau} &= \frac{\partial K}{\partial Q^\mu} = \frac{\partial^2}{\partial\tau\partial Q^\mu} f(\overline{Q}^\mu(Q^\mu, P_\mu), P_\mu, \tau), \end{aligned} \tag{3.14}$$

so that

$$h(Q^\mu, P_\mu, \tau) \equiv \frac{\partial}{\partial\tau} f(\overline{Q}^\mu(Q^\mu, P_\mu), P_\mu, \tau) \tag{3.15}$$

plays the role of a true Hamiltonian for the reduced system. The function f and therefore h will not be defined at this stage; below we shall give a prescription to choose f. For the coordinate conjugated to the constraint matched to P_0 we have

$$\frac{dQ^0}{d\tau} = [Q^0, K] = N[Q^0, P_0] = N. \tag{3.16}$$

The second transformation $(\overline{Q}^i, \overline{P}_i) \to (Q^i, P_i)$ yields additional end point terms of the form

$$\left[Q^i P_i - F(\overline{Q}^i, P_i, \tau)\right]_{\tau_1}^{\tau_2} = \left[Q^\mu P_\mu - f(\overline{Q}^\mu(Q^\mu, P_\mu), P_\mu, \tau)\right]_{\tau_1}^{\tau_2}.$$

These terms depend only on observables, and are then gauge-invariant as we required. The gauge-invariant action resulting from the two succesive canonical transformations $(q^i, p_i) \to (\overline{Q}^i, \overline{P}_i) \to (Q^i, P_i)$ is

$$\mathcal{S}[Q^i, P_i, N] = \int_{\tau_1}^{\tau_2} \left(P_i \frac{dQ^i}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau}\right) d\tau \tag{3.17}$$

and in terms of the original variables it reads

$$\begin{aligned}\mathcal{S}[q^i, p_i, N] &= \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - NH \right) d\tau \\ &\quad + \left[\overline{Q}^i \overline{P}_i - W(q^i, \overline{P}_i) + Q^\mu P_\mu - f(\overline{Q}^\mu, P_\mu, \tau) \right]_{\tau_1}^{\tau_2},\end{aligned} \tag{3.18}$$

where $\overline{Q}^i, \overline{P}_i, Q^\mu$ and P_μ must be written in terms of q^i and p_i. The action $\mathcal{S}[Q^i, P_i, N]$ describes an ordinary gauge system with a constraint $P_0 \approx 0$, so that the coordinate Q^0 is pure gauge, that is, Q^0 is not associated to a physical degree of freedom. This coordinate can be defined as an arbitrary function of τ by means of a canonical gauge choice of the form

$$\chi \equiv Q^0 - T(\tau) = 0.$$

Writing Q^0 in terms of q^i and p_i we have a function of the original phase space variables whose Poisson bracket with $H = P_0$ is positive definite; as H differs from the original Hamiltonian constraint only by a positive definite function, then we can always define a global phase time as

$$t(q^i, p_i) \equiv Q^0(q^i, p_i) \tag{3.19}$$

because

$$[t(q^i, p_i), H(q^i, p_i)] = [Q^0, P_0] = 1, \tag{3.20}$$

and then

$$[t(q^i, p_i), \mathcal{H}(q^i, p_i)] > 0. \tag{3.21}$$

We have then shown that by imposing a canonical gauge condition on the gauge system described by (Q^i, P_i) we have identified a global phase time for the parametrized system given by (q^i, p_i). The key point has been that in terms of the variables of the gauge system we have a natural choice for a function whose Poisson bracket with the constraint is non vanishing everywhere. Moreover, the change to the new coordinates and momenta gives the constraint hypersurface a trivial topology which allows to fix the gauge in a way that clearly does not generate Gribov copies. As we shall see below in the context of minisuperspace deparametrization, the gauge fixation can be relaxed to allow for different definitions of time. In general we shall prefer the most simple choice compatible with the topology of the constraint hypersurface, though sometimes a gauge condition which appears to be somewhat complex in terms of the new variables will be convenient when we go back to the original phase space variables; in particular, some choices can be useful to visualize in the original phase space how our procedure avoids the Gribov problem [Simeone (1998)].

3.2.3 *Non separable constraints*

Our method for deparametrizing and quantizing cosmological models is based on a canonical transformation generated by a solution of the Hamilton–Jacobi equation, so that, in principle, it fails when this equation is not separable. This is an important restriction, but we are not more limited than at the classical level, as we are able to quantize those models which are classically integrable. A point to be noted is that in string cosmology separable Hamiltonians appear in a natural way when we deal with the low energy limit of the theory; the reason is that this limit leads only to massless fields, so that the constraints do not include the combination of powers and exponentials which constitute a usual obstruction to separability.

A possible treatment for non separable Hamiltonians could be to look for an approximate solution by restricting the calculation to regions of the phase space in which some given terms of the Hamiltonian are negligible, so that we could work with a separable constraint. This would require a study of the possible values of the parameters entering the potential. A similar approach is in fact usual in the canonical quantization, when approximate solutions of the Wheeler–DeWitt equation are found after neglecting different terms of the Hamiltonian in different regions of the phase space; the solutions in different regions are matched using the WKB procedure (see Chapter 6 and, for example, [Halliwell (1990)]).

At this stage, however, we shall not discuss approximations in detail. Instead, in the context of minisuperspace models (in particular in string cosmologies) we shall discuss the possibility of determining whether a time for a system described by a given Hamiltonian is also a time for a system described by a more general constraint. This will be mostly useful starting from an extrinsic time. Although we shall not be able to use our method to quantize a system described by a non solvable constraint starting from a time found in this way, the identification of time can have its own interest, as a tool to understand a given cosmological model, and mainly to define an inner product which takes into account the existence of a time "hidden" among the coordinates and momenta.

3.3 Path integral

3.3.1 *General formalism*

The action $\mathcal{S}[Q^i, P_i, N]$ is stationary when the coordinates Q^i are fixed at the boundaries. The coordinates and momenta (Q^i, P_i) describe a gauge system with a linear constraint, so that this action allows to obtain the amplitude for the transition $|Q^i_1, \tau_1\rangle \to |Q^i_2, \tau_2\rangle$ by the usual Fadeev–Popov procedure:

$$
\begin{aligned}
\langle Q^i_2, \tau_2 | Q^i_1, \tau_1 \rangle &= \int DQ^0 DP_0 DQ^\mu DP_\mu DN \delta(\chi) |[\chi, P_0]| e^{i\mathcal{S}[Q^i, P_i, N]} \\
\mathcal{S}[Q^i, P_i, N] &= \int_{\tau_1}^{\tau_2} \left(P_i \frac{dQ^i}{d\tau} - N P_0 - \frac{\partial f}{\partial \tau} \right) d\tau,
\end{aligned}
\tag{3.22}
$$

where χ can be any canonical gauge condition. The Fadeev–Popov determinant $|[\chi, P_0]|$ ensures that the result does not depend on the gauge choice. Differing from what happened in terms of the original variables (see Eq. (2.19)), here the amplitude depends on τ_1 and τ_2; this reflects that after the canonical transformation the system is no more a parametrized one, but, instead, it has a spurious degree of freedom Q^0 and a true time which is τ. If we perform the functional integration on the lapse N enforcing the paths to lie on the constraint hypersurface $P_0 = 0$, we obtain

$$
\begin{aligned}
\langle Q^i_2, \tau_2 | Q^i_1, \tau_1 \rangle &= \int DQ^0 DQ^\mu DP_\mu \delta(\chi) |[\chi, P_0]| \\
&\quad \times \exp \left(i \int_{\tau_1}^{\tau_2} \left[P_\mu \frac{dQ^\mu}{d\tau} - h(Q^\mu, P_\mu, \tau) \right] d\tau \right),
\end{aligned}
\tag{3.23}
$$

where $h \equiv \partial f / \partial \tau$ is the true Hamiltonian of the reduced system. The path integral must give an amplitude between states characterized by the variables which make the action stationary when fixed at the boundaries. As $\mathcal{S}$ is stationary when the Q^i are fixed, then we shall choose the gauge in the most general form giving Q^0 as a function of the other coordinates Q^μ, τ and constants c_ν; thus a choice of the boundary values of the physical coordinates and τ fixes the boundary values of Q^0. With the choice

$$\chi \equiv Q^0 - T(Q^\mu, c_\nu, \tau) = 0 \tag{3.24}$$

and after the trivial integration on Q^0 we obtain

$$\begin{aligned}\langle Q_2^i, \tau_2 | Q_1^i, \tau_1 \rangle &= \\ &= \int DQ^\mu DP_\mu \exp\left(i \int_{\tau_1}^{\tau_2} \left[P_\mu \frac{dQ^\mu}{d\tau} - h(Q^\mu, P_\mu, \tau) \right] d\tau \right) \\ &= \langle Q_2^\mu, \tau_2 | Q_1^\mu, \tau_1 \rangle. \end{aligned} \tag{3.25}$$

Now we want to relate this path integral with an amplitude between states characterized by the original variables of the parametrized system. Because the original action $S[q^i, p_i, N]$ is stationary when the coordinates q^i are fixed at the boundaries, it is usual to look for a propagator of the form

$$\langle q_2^i | q_1^i \rangle, \tag{3.26}$$

so that the states are characterized only by the coordinates. The problem with this procedure is that, as we have already remarked, in cosmology it is not always possible to define a time in terms of the q^i only; then the amplitude $\langle q_2^i | q_1^i \rangle$ is not the answer to a question like "what is the probability that an observable of the system takes a certain value at time t if at a previous time the observable took another given value?". Formally this can be understood as follows:

If we pretend the quantum amplitude $\langle Q_2^\mu, \tau_2 | Q_1^\mu, \tau_1 \rangle$ to be equivalent to $\langle q_2^i | q_1^i \rangle$ we should verify that the paths in the integral are weighted by the action S in the same way that they are weighted by S (the still not defined function f will play a central role here; see below), and that the quantum states $|Q^\mu, \tau\rangle$ are equivalent to $|q^i\rangle$. As the path integral in the variables (Q^i, P_i) is gauge invariant, this requirement is fulfilled if it is possible to impose a –globally good– gauge condition $\tilde{\chi} = 0$ such that $\tau = \tau(q^i)$ is defined. But in this case a function $t(q^i)$ would be a global phase time, and an intrinsic time $t(q^i)$ can be defined only if the potential in the Hamiltonian constraint has a definite sign, that is, if the constraint hypersurface splits into two disjoint sheets. In the most general case the definition of a global phase time must necessarily involve also the momenta, and then we cannot fix the gauge in the path integral in such a way that $\tau = \tau(q^i)$. Hence, if we want to quantize the system by imposing canonical gauges in the path integral to obtain an amplitude with a clear notion of time, in the most general case of a potential with a non definite sign we should admit the possibility of identifying the quantum states in the original phase space not by q^i but by a complete set of functions of both the coordinates and momenta q^i and p_i.

This suggests that we should abandon the idea of obtaining an amplitude for states characterized by the coordinates. However, while a deparametrization in terms of the momenta may be completely valid at the classical level, it has been pointed by Barvinsky that at the quantum level there is an obstacle which is peculiar of gravitation [Barvinsky (1993)]: There are basically two representations for quantum operators, the coordinate representation and the momentum representation, in which the states are characterized by occupation numbers associated to given values of the momenta. The last one is appropriate when the theory under consideration allows for the existence of assimptotically free states associated to an adiabatic vanishing of interactions, so that a natural one-particle interpretation in terms of creation and annihilation operators exists. In quantum cosmology these assymptotic states do not, in general, exist. The suitable representation must be able to handle with essentially non linear and non polynomial interactions, and such a representation is a coordinate one. In the coordinate representation the operator of coordinates is diagonal, and the quantum states are represented by wave functions in terms of the coordinates. The usual Dirac–Wheeler–DeWitt quantization with momentum operators in the coordinate representation acting on $\Psi(q)$ follows this line; but, as we have already observed, this formalism is devoided of a clear notion of time and evolution, unless the potential is everywhere non null so that we can find a time among the canonical coordinates.

We shall then adopt what could be thought as an intermediate solution: When the constraint allows for the existence of an intrinsic time $t(q^i)$ we shall straightforwardly apply our deparametrization and path integral quantization procedure to obtain the transition amplitude for states characterized by the original coordinates; this will provide a quantization with a clear distinction between time and observables, although the time variable in the path integral may be given by a non trivial relation between the original coordinates defining the states (the Kantowski–Sachs anisotropic universe will be an interesting example useful to illustrate this feature [Simeone (2000)]). On the other hand, unless the model is simple enough to work in the variables (Q^i, P_i) and guess the meaning of the results in terms of the original variables, when only an extrinsic time exists we shall proceed as follows: to make compatible a coordinate representation with a globally good definition of time we shall perform a canonical transformation from the original canonical variables (q^i, p_i) to a set $(\tilde{q}^i, \tilde{p}_i)$ defined in such a way that the Hamiltonian constraint of a given minisuperspace model has a non vanishing potential; then an intrinsic time exists in terms of the $\tilde{q}^i$. The action $S[\tilde{q}^i, \tilde{p}_i, N]$ will be stationary when the $\tilde{q}^i$ are fixed at the boundaries. We shall therefore apply our procedure to obtain the transition amplitude for states given by the $\tilde{q}^i$,

$$\langle \tilde{q}^i_2 | \tilde{q}^i_1 \rangle,$$

which will in general depend on the original coordinates *and* also on the original momenta. Though at a first sight this may seem to obscure the interpretation of the resulting propagator or wave function, we shall see that the original momenta are restricted to appear in the global phase time, while the new coordinates corresponding to the physical degrees of freedom will depend on the q^i only (a detailed discussion will be given in the context of the quantization of the Taub anisotropic cosmology, both in the path integral an in the canonical quantization schemes).

3.3.2 *The function f and the reduced Hamiltonian. Unitarity*

In what follows we shall speak about the coordinates $\tilde{q}^i$, and it must be understood that when an intrinsic time in terms of the original coordinates exists the coordinates $\tilde{q}^i$ coincide with the coordinates q^i.

As we have already mentioned, up to now the generator f and the Hamiltonian h of the reduced system remain generic. We shall make use of this freedom to choose f in such a way that the amplitude $\langle Q^\mu_2, \tau_2 | Q^\mu_1, \tau_1 \rangle$ is equivalent to $\langle \tilde{q}^i_2 | \tilde{q}^i_1 \rangle$. This is possible only if

(1) The Hamiltonian constraint is such that a globally good gauge $\tilde{\chi} = 0$ defining $\tau = \tau(\tilde{q}^i)$ can be imposed. This is equivalent to the existence of an intrinsic time in terms of the coordinates $\tilde{q}^i$ (but not necessarily in terms of the q^i). Then a particular choice of the gauge-invariant coordinates Q^μ and of τ defines a point in the configuration space of the $\tilde{q}^i$.
This does not ensure, however, that the amplitudes are equivalent; in fact, because when written as a functional of the original vari-

ables $(\tilde{q}^i, \tilde{p}_i)$ the gauge-invariant action $\mathcal{S}$ contains additional end point terms B, the paths would not be weighted in the path integral as they are by S. Then we demand that:

(2) The end point terms B vanish on the constraint surface and in the gauge $\tilde{\chi} = 0$ defining $\tau = \tau(\tilde{q}^i)$, that is,

$$B = \left[\overline{Q}^i \overline{P}_i - W + Q^\mu P_\mu - f\right]_{\tau_1}^{\tau_2}\Bigg|_{P_0=0,\tilde{\chi}=0} = 0. \qquad (3.27)$$

Because the action S is gauge-invariant, this ensures that with any gauge choice the paths are weighted in the same way by $\mathcal{S}$ and S. This requirement gives a prescription for the generator $f(\overline{Q}^\mu, P_\mu, \tau)$; this also determines the reduced Hamiltonian $h = \partial f/\partial\tau$. Note that, as f depends only on observables, h conmutes with the complete Hamiltonian $K = NP_0 + h$, so that

$$\frac{dh}{d\tau} = \frac{\partial^2 f}{\partial \tau^2}.$$

Thus if f could be defined as a function linear in τ we would be able to obtain a conserved Hamiltonian for the reduced system; when possible, we shall choose the reduction procedure leading to such a reduced Hamiltonian.

The reduced Hamiltonian h could be both positive or negative-definite. The possibilility of a double sign is not necessarily a serious problem: for example, a double sign appears in the quantum theory for a relativistic free particle, and the interpretation is that of particles and antiparticles (see the next section); unless an interaction making the "effective mass" squared vanish –then allowing the two sheets of the constraint touch each other– is introduced, one can work with two disjoint theories [Barvinsky (1993)]. Thus if a double sign appears in our quantization scheme we could understand it in a similar way (the problem of a time-dependent potential [Kuchař (1981)] will be discussed later; see Chapters 4 and 6). A fundamental difficulty, instead, would be a reduced Hamiltonian which vanishes or even becomes imaginary. An imaginary Hamiltonian cannot be associated to a self-adjoint operator, and the resulting quantum theory becomes non unitary [Hájícek (1986)] (see Chapter 6). This could be avoided by restricting the configuration space to what is called its *natural size* (see Section 3.4.3), but even in this case the coordinate conjugated to such momentum would not be a global time. As we are looking for a unitary theory with a right notion of time, our analysis will be restricted to cosmological models for which a non vanishing momentum exists, so that an intrinsic time can be defined, or to models such that it is possible to perform the transformation from the original coordinates q^i to the coordinates $\tilde{q}^i$ in terms of which we have a system which admits an intrinsic time. As we shall see, the two possible signs of the non vanishing momentum will be in correspondence with two possible reduced Hamiltonians; the resulting formalism will therefore include two theories for the physical degrees of freedom, each one corresponding to each sign of h associated to one of the two sheets of the constraint surface. The path integral in the reduced space will give two propagators, one for the evolution of the wave functions of each theory (see [Barvinsky (1993)], and also [Hájícek (1986)] for an analogous point of view in the context of canonical quantization).

3.4 Examples

We shall ilustrate our procedure with some usual examples, as the relativistic free particle and the ideal clock. The analogy between the Hamiltonian formalism for the relativistic particle and for simple cosmologies, in particular the invariance under reparametrizations of time, has often led to use the first as a kind of toy model for gravitation; however, it should be emphasized that the relativistic free particle cannot reproduce an important property of cosmological models, as it is the fact that the potential in the Hamiltonian constraint can change its sign.

3.4.1 *Feynman propagator for the Klein–Gordon equation*

We shall obtain the Feynman propagator for the Klein–Gordon equation associated to a free relativistic particle:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 + m^2\right)\psi = 0.$$

In the canonical formalism the relativistic particle is described by the Hamiltonian constraint

$$\mathcal{H} = {p_0}^2 - p^2 - m^2 \approx 0. \tag{3.28}$$

The presence of the constraint reflects that there is effectively a time among the canonical variables: if we calculate the Poisson bracket of the coordinate

x^0 with the constraint we obtain

$$[x^0, \mathcal{H}] = 2p_0,$$

so that the time is x^0 on the sheet $p_0 > 0$, and $-x^0$ on the sheet $p^0 < 0$. We shall obtain the propagator by computing the functional average of the Heaviside function $\theta(s)$, where s is the proper time (to simplify the notation we write only one spatial coordinate).

The τ-independent Hamilton–Jacobi equation for this system has the solution

$$W_{\pm}(x, x^0, \overline{P}, \overline{P}_0) = \overline{P}x \pm x^0\sqrt{\overline{P}^2 + \overline{P}_0 + m^2}.$$

where we have matched the integration constants $E = \overline{P}_0$ and $\alpha = \overline{P}$. The generating function f making the end point terms vanish in the canonical gauge $\tilde{\chi} \equiv x^0 - T(\tau) = 0$ (which gives $\tau = \tau(q^i)$) is

$$f(\overline{Q}, P, \tau) = \overline{Q}P \mp T(\tau)\sqrt{P^2 + m^2}.$$

This gauge choice is globally good because $[\tilde{\chi}, \mathcal{H}] \neq 0$, and then $\tilde{\chi}$ can be used to define a global phase time t; as we just mentioned, we can define $t = \pm x^0$. The end point terms are

$$B \equiv \mp \frac{m^2(x^0 - T(\tau))}{\sqrt{P^2 + m^2}}, \tag{3.29}$$

and the new variables are given by

$$\begin{aligned}
Q^0 &= \pm \frac{mx^0}{\sqrt{P^2 + P_0 + m^2}} \\
Q &= x \pm \frac{Px^0}{\sqrt{P^2 + P_0 + m^2}} \mp \frac{PT(\tau)}{\sqrt{P^2 + m^2}} \\
p_0 &= \pm\sqrt{P^2 + P_0 + m^2} \\
p &= P.
\end{aligned}$$

Therefore, in terms of the original canonical variables the gauge-invariant action reads

$$\mathcal{S} = \int_{\tau_1}^{\tau_2} \left(p_0 \frac{dx^0}{d\tau} + p\frac{dx}{d\tau} - N\mathcal{H}\right) d\tau \mp m^2 \left[\frac{x^0 - T(\tau)}{\sqrt{p^2 + m^2}} \right]_{\tau_1}^{\tau_2} \tag{3.30}$$

and the amplitude for the transition $x_1 \to x_2$ is given by

$$\begin{aligned}\langle x_2, x_2^0 | x_1, x_1^0 \rangle &= \int Dx^0 Dp_0 Dx Dp DN \delta(\chi) |[\chi, H]| \\ &\quad \times \exp\left(i \int_{\tau_1}^{\tau_2} \left(p_0 \frac{dx^0}{d\tau} + p \frac{dx}{d\tau} - N\mathcal{H} \right) d\tau \right) \\ &\quad \times \exp\left(\mp i\, m^2 \left[\frac{x^0 - T(\tau)}{\sqrt{p^2 + m^2}} \right]_{\tau_1}^{\tau_2} \right). \qquad (3.31)\end{aligned}$$

The path integral can now be computed in any canonical gauge; for any function T we have $|[\chi, \mathcal{H}]| = 2|p_0|$. The integration on the multiplier N yields a δ-function of the constraint which can be written as

$$\delta(p_0^2 - p^2 - m^2) = \frac{1}{2|p_0|}\delta(p_0 - \sqrt{p^2 + m^2}) + \frac{1}{2|p_0|}\delta(p_0 + \sqrt{p^2 + m^2}).$$

In a τ-independent gauge we have $\theta(s) = \theta(x_2^0 - \frac{p_0(\tau_2)}{p_0(\tau_1)} x_1^0)$ for $p_0 > 0$ and $\theta(s) = \theta(x_1^0 - \frac{p_0(\tau_1)}{p_0(\tau_2)} x_2^0)$ for $p_0 < 0$; then, in gauge $\chi \equiv x^0 = 0$ we obtain

$$\begin{aligned}&\langle x_2, x_2^0 | \theta(s) | x_1, x_1^0 \rangle = \\ &\quad = \int DxDp\, \theta\left(x_2^0 - \frac{p_0(\tau_2)}{p_0(\tau_1)} x_1^0 \right) \\ &\qquad \times \exp\left(i \int_{\tau_1}^{\tau_2} p \frac{dx}{d\tau} d\tau - i\, m^2 \left[\frac{-T(\tau)}{p_0} \right]_{\tau_1}^{\tau_2} \right) \\ &\quad + \int DxDp\, \theta\left(x_1^0 - \frac{p_0(\tau_1)}{p_0(\tau_2)} x_2^0 \right) \\ &\qquad \times \exp\left(i \int_{\tau_1}^{\tau_2} p \frac{dx}{d\tau} d\tau + i\, m^2 \left[\frac{-T(\tau)}{p_0} \right]_{\tau_1}^{\tau_2} \right),\end{aligned} \qquad (3.32)$$

where

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} p \frac{dx}{d\tau} d\tau \pm \left[\frac{m^2 T(\tau)}{p_0} \right]_{\tau_1}^{\tau_2} = \\ &\quad = \int_{\tau_1}^{\tau_2} \left[p \frac{d}{d\tau} \left(x \mp \frac{pT(\tau)}{\sqrt{p^2 + m^2}} \right) \pm \sqrt{p^2 + m^2} \frac{dT}{d\tau} \right] d\tau \\ &\quad = \int_{\tau_1}^{\tau_2} \left[P \frac{dQ}{d\tau} \pm \sqrt{P^2 + m^2} \frac{dT}{d\tau} \right] d\tau.\end{aligned}$$

Note that now, according to the form of the action, at the level of the physical degree of freedom Q we have two theories, one for each sign of the true Hamiltonian. By skeletonizing the paths we obtain $N-1$ δ-functions of the form $\delta(P_m - P_{m-1})$, and as $P = p$ and the end point values of Q are given by the gauge choice which makes the endpoint terms vanish, so that $Q(\tau_1) = x_1$ and $Q(\tau_2) = x_2$, we finally obtain

$$\begin{aligned}\langle x_2, x_2^0|\theta(s)|x_1, x_1^0\rangle &= \\ &= \theta(x_2^0 - x_1^0)\int dp \exp\left(ip(x_2 - x_1) - ip_0(x_2^0 - x_1^0)\right) \\ &\quad + \theta(x_1^0 - x_2^0)\int dp \exp\left(ip(x_2 - x_1) + ip_0(x_2^0 - x_1^0)\right),\end{aligned} \tag{3.33}$$

which is the Feynman propagator for the Klein–Gordon equation.

The double signs appearing in the formalism reflect the fact that the Klein–Gordon equation describes particles as well as antiparticles; the first correspond to the sheet $p_0 > 0$ of the constraint surface and propagate forward in time, while the second correspond to the negative-energy solutions of the sheet $p_0 < 0$ and propagate in the backward direction of time. As long as we do not include an interaction the one-particle interpretation remains valid, as there is no pair creation. In other theory with a quadratic Hamiltonian constraint, as it is gravitation, the interpretation may not be so simple, but this analogy will prove to be useful for understanding the results in the context of minisuperspace quantization.

3.4.2 *The ideal clock*

Consider a mechanical system with canonical coordinates and momenta (q^k, p_k). Its action functional reads

$$S[q^k, p_k] = \int_{t_1}^{t_2}\left[p_k\frac{dq^k}{dt} - H_0(q^k, p_k)\right]dt, \tag{3.34}$$

but as the dynamics remains unchanged if we add a total derivative of t to the integrand, we can write

$$S[q^k, p_k] = \int_{t_1}^{t_2}\left[p_k\frac{dq^k}{dt} - H_0(q^k, p_k) + R(t)\right]dt. \tag{3.35}$$

We can give the evolution in terms of an arbitrary parameter τ by including the time t among the canonical coordinates, so that the conjugated momentum p_t appears. Now, if we want the action to lead to the same dynamics as the original one does, the constraint

$$\mathcal{H} = p_t + H_0 - R(t) \approx 0$$

must be imposed; therefore, the action for the parametrized system with coordinates and momenta (q^i, p_i) reads [Ferraro&Sforza (1999); Ferraro (1999)]

$$\begin{aligned} S[q^i, p_i, N] &= \int_{\tau_1}^{\tau_2} \left[p_t \frac{dt}{d\tau} + p_k \frac{dq^k}{d\tau} - N \left(p_t + H_0(q^k, p_k) - R(t) \right) \right] d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[p_i \frac{dq^i}{d\tau} - N\mathcal{H}(q^i, p_i) \right] d\tau, \end{aligned} \tag{3.36}$$

where N is a Lagrange multiplier. The usual canonical equations of motion for the original coordinates and momenta (q^k, p_k) should hold. Indeed, by varying the coordinates and momenta in (3.36) we obtain the equations of motion

$$\begin{aligned} \frac{dq^i}{d\tau} &= N \frac{\partial}{\partial p_i} \left(p_t + H_0(q^k, p_k) \right) \\ \frac{dp_i}{d\tau} &= -N \frac{\partial}{\partial q^i} \left(H_0(q^k, p_k) - R(t) \right) \end{aligned}$$

which give

$$\begin{aligned} \frac{dq^k}{dt} &= \frac{\partial}{\partial p_k} H_0(q^k, p_k) \\ \frac{dp_k}{dt} &= -\frac{\partial}{\partial q^k} H_0(q^k, p_k). \end{aligned}$$

If we eliminate the coordinates and momenta (q^k, p_k), we obtain a system with only one degree of freedom and one constraint, the *ideal clock* [Beluardi&Ferraro (1995)]; its action functional is given by

$$S[t, p_t, N] = \int_{\tau_1}^{\tau_2} \left(p_t \frac{dt}{d\tau} - N\mathcal{H} \right) d\tau \tag{3.37}$$

with

$$\mathcal{H} = p_t - R(t) \approx 0, \tag{3.38}$$

and the equations of motion for this system are

$$\frac{dt}{d\tau} = N, \qquad \frac{dp_t}{d\tau} = N\frac{\partial R(t)}{\partial t}.$$

We shall illustrate our quantization method by turning the ideal clock into an ordinary gauge system and computing its quantum transition amplitude by means of a path integral in which canonical gauges are admissible.

To do this, two succesive canonical transformations are needed. The first transformation, $(t, p_t) \to (\overline{Q}^0, \overline{P}_0)$, is generated by the function W solution of the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial t} - R(t) = E; \tag{3.39}$$

matching $E = \overline{P}_0$, a simple calculation gives

$$W(t, \overline{P}_0) = \overline{P}_0\ t + \int R(t)dt, \tag{3.40}$$

so that $\overline{Q}^0$, $\overline{P}_0$ and the new hamiltonian $\overline{K}$ are related to t, p_t and $\mathcal{H}$ by

$$\overline{Q}^0 = \frac{\partial W}{\partial \overline{P}_0} = t, \quad p_t = \frac{\partial W}{\partial t} = \overline{P}_0 + R(t), \quad \overline{K} = N\mathcal{H}. \tag{3.41}$$

The variables $\overline{Q}^0$ and $\overline{P}_0$ verify

$$[\overline{Q}^0, \overline{P}_0] = [\overline{Q}^0, \mathcal{H}] = 1,$$

so that $\overline{Q}^0$ can be used to fix the gauge.

The second transformation is generated by

$$F = P_0\overline{Q}^0 + f(\tau), \tag{3.42}$$

which yields

$$\overline{P}_0 = \frac{\partial F}{\partial \overline{Q}^0} = P_0, \qquad Q^0 = \frac{\partial F}{\partial P_0} = \overline{Q}^0, \tag{3.43}$$

and a new non vanishing Hamiltonian

$$K = \overline{K} + \frac{\partial f}{\partial \tau} = NP_0 + \frac{\partial f}{\partial \tau} \approx \frac{\partial f}{\partial \tau}. \tag{3.44}$$

Then, as a functional of Q^0 and P_0 the gauge-invariant action of the ideal clock reads

$$\mathcal{S}[Q^0, P_0, N] = \int_{\tau_1}^{\tau_2} \left(P_0 \frac{dQ^0}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau} \right) d\tau \tag{3.45}$$

and in terms of the original variables

$$\begin{aligned} \mathcal{S}[t, p_t, N] &= \int_{\tau_1}^{\tau_2} \left(p_t \frac{dt}{d\tau} - N(p_t - R(t)) \right) d\tau + [B(\tau)]_{\tau_1}^{\tau_2}, \\ B(\tau) &= Q^0 P_0 - W - f(\tau) = - \int R(t) dt - f(\tau). \end{aligned} \tag{3.46}$$

In order to guarantee that the new action weights the paths in the same way that the original one does, and that the transition amplitude in terms of Q^0 is equivalent to the amplitude in terms of t, we must choose f so that the end point terms vanish in a gauge such that $\tau = \tau(t)$. With the canonical gauge choice

$$\chi \equiv Q^0 - \tau = t - \tau = 0 \tag{3.47}$$

we must choose

$$f(\tau) = - \int R(\tau) d\tau. \tag{3.48}$$

From (3.47) we have $|[\chi, \mathcal{H}]| = |[Q^0, P_0]| = 1$, $\delta(\chi) = \delta(Q^0 - \tau) = \delta(t - \tau)$, so that the transition amplitude is

$$\begin{aligned} \langle t_2 | t_1 \rangle &= \langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle \\ &= \int DQ^0 DP_0 DN \delta(Q^0 - \tau) \\ &\quad \times \exp \left(i \int_{\tau_1}^{\tau_2} \left[P_0 \frac{dQ^0}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau} \right] d\tau \right) \\ &= \int DQ^0 DP_0 \delta(P_0) \delta(Q^0 - \tau) \\ &\quad \times \exp \left(i \int_{\tau_1}^{\tau_2} \left[P_0 \frac{dQ^0}{d\tau} - \frac{\partial f}{\partial \tau} \right] d\tau \right) \\ &= \exp \left(i \int_{\tau_1}^{\tau_2} R(\tau) d\tau \right). \end{aligned} \tag{3.49}$$

Hence, the probability for the transition from t_1 at τ_1 to t_2 at τ_2 is

$$|\langle t_2|t_1\rangle|^2 = |\langle Q_2^0, \tau_2|Q_1^0, \tau_1\rangle|^2 = 1 \tag{3.50}$$

for any values of t_1 and t_2. This just reflects that the system has no true degrees of freedom, because given τ we have only one possible t. We should emphasize that even though we have used a gauge which makes this fact explicit, the path integral is gauge invariant, and then we could have computed it in any gauge and the result would have been the same. This can easily be verified by, for example, calculating the path integral in terms of the original variables with the action (3.46) and the canonical gauge choice $\chi \equiv t = 0$:

$$\begin{aligned}
\langle t_2|t_1\rangle &= \int DtDp_tDN\delta(\chi)|[\chi, H]| \\
&\quad \times \exp\left(i\int_{\tau_1}^{\tau_2}\left[p_t\frac{dt}{d\tau} - NH\right]d\tau\right) \\
&\quad \times \exp\left(-i\int_{t(\tau_1)}^{t(\tau_2)} R(t)dt + i\int_{\tau_1}^{\tau_2} R(\tau)d\tau\right) \\
&= \int DtDp_t\delta(\chi)\delta(p_t - R(t))|[\chi, H]| \\
&\quad \times \exp\left(i\int_{\tau_1}^{\tau_2} p_t\frac{dt}{d\tau}d\tau\right) \\
&\quad \times \exp\left(-i\int_{t(\tau_1)}^{t(\tau_2)} R(t)dt + i\int_{\tau_1}^{\tau_2} R(\tau)d\tau\right) \\
&= \exp\left(i\int_{\tau_1}^{\tau_2} R(\tau)d\tau\right).
\end{aligned} \tag{3.51}$$

Although the end point terms do not vanish in gauge $t = 0$, but with the gauge choice $t = \tau$, we have obtained the same amplitude as before.

3.4.3 *Transition probability for empty Friedmann–Robertson–Walker universes*

In the next chapter we shall give a direct procedure to deparametrize and to obtain the path integral for cosmological models. However, we can already make a preliminary analysis and get some results regarding the quantization of certain minisuperspaces. The idea is to recall that their Hamiltonian constraints can be obtained by performing a canonical transformation on a mechanical system which has been parametrized by including the time t among the canonical variables.

In particular, empty Friedmann–Robertson–Walker minisuperspaces can be obtained from the ideal clock [Beluardi&Ferraro (1995); De Cicco&-Simeone (1999a)]. It can be shown that the Hamiltonian constraint for an empty minisuperspace

$$\mathcal{H} = -G(\Omega)\pi_\Omega^2 + V(\Omega) \approx 0$$

(see the next chapter) can be obtained from that of the ideal clock with $R(t) = t^2$,

$$H = p_t - t^2 \approx 0,$$

by means of a canonical transformation. If we define

$$\tilde{V}(\Omega) = sign(V)\left(\frac{3}{2}\int\sqrt{\frac{|V|}{G}}d\Omega\right)^{2/3} \tag{3.52}$$

the canonical transformation is given by

$$\pi_\Omega = -t\frac{\partial\tilde{V}(\Omega)}{\partial\Omega}, \qquad p_t = \tilde{V}(\Omega). \tag{3.53}$$

On the constraint surface $p_t - t^2 = 0$ we obtain

$$t = \pm\sqrt{\tilde{V}(\Omega)}, \tag{3.54}$$

(the potential $\tilde{V}(\Omega)$ is positive-definite) and then, we can try to quantize the minisuperspace by means of a path integral in the variables t, p_t. If we pretend to obtain an amplitude for a transition between states labeled by the coordinate Ω, this possibility clearly depends on the existence of a relation

$$\tilde{V}(\Omega) \leftrightarrow \Omega,$$

but, as we shall see, the main restriction will be given by the geometrical properties of the constraint surface.

The most general form of the potential for an empty isotropic and homogeneous minisuperspace is

$$V(\Omega) = -ke^{\Omega} + \Lambda e^{3\Omega}. \tag{3.55}$$

Let us first consider the simple models with $k = 0$ or $\Lambda = 0$. For $k = 0$ (flat universe, non zero cosmological constant) we have

$$V(\Omega) = \Lambda e^{3\Omega},$$

and for $\Lambda = 0$, $k = -1$ (null cosmological constant, open universe) we have the potential

$$V(\Omega) = e^{\Omega}.$$

In both cases, as well as for the open ($k = -1$) model with non zero cosmological constant $\Lambda > 0$

$$V(\Omega) = e^{\Omega} + \Lambda e^{3\Omega},$$

given V and then $\tilde{V}(\Omega)$ we can obtain $\Omega = \Omega(\tilde{V})$ uniquely. As $\Omega \sim \ln a(\tau)$, from (3.54) we then see that in the simplest cases our procedure is basically equivalent to identifying the scale factor $a(\tau)$ of the metric with the time t of the ideal clock. As in this cases the potential has a definite sign, the constraint surface splits into the two disjoint sheets

$$\pi_\Omega = \pm\sqrt{\frac{V(\Omega)}{G(\Omega)}}.$$

Hence the gauge fixation in terms of the coordinate t of the ideal clock, which selects only one path in the (t, p_t) phase space, also selects only one path in the (Ω, π_Ω) phase space; this makes the quantization of this toy universes trivial, yielding a unity probability for the transition from Ω_1 to Ω_2:

$$|\langle\Omega_2|\Omega_1\rangle|^2 = 1. \tag{3.56}$$

For the case $k = 1$, $\Lambda > 0$ (closed model with non zero cosmological constant), the potential

$$V(\Omega) = -e^{\Omega} + \Lambda e^{3\Omega},$$

is not a monotonous function of Ω, but it changes its slope when

$$\Omega = \ln\left(\frac{1}{\sqrt{3\Lambda}}\right) \tag{3.57}$$

where it has a minimun, so that for a given value of $V(\Omega)$ we would have two possible values of Ω. However, physical states lie on the constraint surface

$$-G(\Omega)\pi_\Omega^2 - e^\Omega + \Lambda e^{3\Omega} = 0,$$

which is equivalent to

$$\pi_\Omega = \pm\sqrt{\frac{e^\Omega(\Lambda e^{2\Omega} - 1)}{G}}.$$

As G is a positive-definite function of Ω, the condition that π_Ω must be real restricts the motion to the region

$$\Omega \geq \ln\left(\frac{1}{\sqrt{\Lambda}}\right), \tag{3.58}$$

which is called the "natural size" of the configuration space [Hájícek (1986)]. Hence, the potential does not change its slope on the physical region of the constraint surface; this allows us to obtain $\Omega = \Omega(V)$ and the relation

$$\Omega = \frac{1}{2}\ln\left(\frac{\Lambda^{2/3}\tilde{V} + 1}{\Lambda}\right)$$

holds in the physical phase space. There is, however, a problem resulting from the fact that the potential has not a definite sign: as $\pi_\Omega = 0$ is possible in this model, the system can evolve from Ω_1 to Ω_2 by two paths. Then given a gauge condition *in terms of* t we do not obtain a parametrization of the cosmological model in terms of Ω only, and then we cannot say that a path integral for $|t_1\rangle \to |t_2\rangle$ is equivalent to the path integral for $|\Omega_1\rangle \to |\Omega_2\rangle$. This is related to the fact that, precisely because the potential has not a definite sign, this model does not allow for the existence of an intrinsic time.

As we have remarked, the gauge choice is not only a way to avoid divergences in the path integral for a constrained system, but also a reduction procedure to physical degrees of freedom. When we choose a gauge to perform the path integration, at each τ we select one point from each class of equivalent points; if we do this with a system which is pure gauge, i.e. that has only one degree of freedom and one constraint, we select only one point of the phase space at each τ. For example, the gauge choice (3.47) of section 3.4.2, $t - \tau = 0$, means that the paths in the phase space can only go from $t_1 = \tau_1$ at τ_1 to $t_2 = \tau_2$ at τ_2; there is no other possibility. Hence, the probability that the system evolves from t_1 at τ_1 to t_2 at τ_2 cannot be anything else but unity. Then if we can write the time t in terms of only the coordinate Ω of a minisuperspace, its evolution is parametrized in terms of Ω, there is only one possible value of Ω at each τ, and the quantization of the model is therefore trivial. In the case that Ω does not suffice to parametrize the evolution, however, we should still obtain a transition probability equal to unity, because the model is pure gauge. The point is then how the states must be defined so that the correct amplitude is obtained. This will be carefully discussed in the next chapter.

Chapter 4

Homogeneous relativistic cosmologies

4.1 Isotropic universes

An isotropic and homogeneous model gives an acceptable description of the current state of the universe. For example, an essential feature of this model is its non-stationary character, which constitutes a good explanation for the observed redshift of far galaxies. The spatial line element of an isotropic model has the form

$$dl^2 = g_{ab}dx^a dx^b$$

where g_{ab} is the space metric, whose components are functions of time. The isotropy and homogeneity hypothesis lead to the fact that the curvature depends on only one parameter: for $k = 0$ we have a flat universe, for $k = -1$ the universe is open, and for $k = 1$ the universe is closed. The spacetime metric has then the Friedmann–Robertson–Walker form [Landau&Lifshitz (1975); Misner et al. (1997)]

$$ds^2 = N^2 d\tau^2 - a^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right), \qquad (4.1)$$

where $a(\tau)$ is the spatial scale factor. The pure gravitational dynamics of a Friedmann–Robertson–Walker universe is given by the evolution of the scale factor; this evolution is determined by the density and pressure of matter fields, the curvature and the existence of a non vanishing cosmological constant Λ. Here we shall consider empty models with and without cosmological constant, and models with matter in the form of a scalar field. The problem of the path integral with extrinsic time will be introduced with a closed "de Sitter" model.

4.1.1 *A toy model*

Consider an empty open "universe" with null cosmological constant. To work within the Hamiltonian formalism we define the coordinate Ω and its conjugated momentum as:

$$\Omega = \ln\left(\sqrt{\frac{4}{3\pi\mathcal{G}}}a(\tau)\right), \qquad \pi_\Omega = -\frac{1}{N}e^{3\Omega}\frac{d\Omega}{d\tau},$$

so that $a^2 \sim e^{2\Omega}$. The Hamiltonian form of the action functional is

$$S[\Omega,\pi_\Omega,N] = \int_{\tau_1}^{\tau_2}\left[\pi_\Omega\frac{d\Omega}{d\tau} - N\mathcal{H}\right]d\tau, \tag{4.2}$$

where the constraint is

$$\mathcal{H} = -\frac{1}{4}e^{-3\Omega}\pi_\Omega^2 + e^\Omega \approx 0. \tag{4.3}$$

According to the analysis of the last section, we find that $t = \pm\sqrt{\tilde{V}(\Omega)} \sim e^{2\Omega/3}$ must be a time, as t is the only coordinate of the ideal clock. In order to reproduce this result it will be convenient to apply our procedure to the scaled constraint $H = e^{\Omega/3}\mathcal{H}$:

$$H = -\frac{1}{4}e^{-8\Omega/3}\pi_\Omega^2 + e^{4\Omega/3} \approx 0. \tag{4.4}$$

The constraint H is equivalent to $\mathcal{H}$ because they differ only in a positive definite factor. The $\tau-$independent Hamilton–Jacobi equation associated to the Hamiltonian H is

$$-\left(\frac{\partial W}{\partial\Omega}\right)^2 + 4e^{4\Omega} = 4e^{8\Omega/3}E \tag{4.5}$$

and then matching $E = \overline{P}_0$ we have

$$W(\Omega,\overline{P}_0) = \pm 2\int d\Omega\sqrt{e^{4\Omega} - \overline{P}_0 e^{8\Omega/3}}, \tag{4.6}$$

with $+$ for $\pi_\Omega > 0$ and $-$ for $\pi_\Omega < 0$. According to equation (4.6), on the constraint surface it is

$$\overline{Q}^0 = \left[\frac{\partial W}{\partial\overline{P}_0}\right]_{\overline{P}_0=0} = -\eta e^{2\Omega/3} \tag{4.7}$$

with $\eta = sign(\pi_\Omega)$. Following the procedure of section 3.2.2, in order to quantize the model we define

$$F = \overline{Q}^0 P_0 + f(\tau)$$

so that the reduced Hamiltonian and the new variables are

$$h = \frac{\partial f}{\partial \tau}, \qquad Q^0 = \overline{Q}^0, \qquad P_0 = \overline{P}_0.$$

The system described by Q^0 and P_0 has a constraint which is linear and homogeneous in the momenta. Its action functional is then invariant under general gauge transformations, so that there is gauge freedom at the end points and canonical gauges are admissible. If we choose $\chi \equiv Q^0 - T(\tau) = 0$ with T a monotonic function of τ then we can define the time as

$$t \equiv Q^0, \tag{4.8}$$

because $[Q^0, H] = [Q^0, P_0] = 1$. Then we have a global phase time that can be written in terms of the coordinate Ω only, the expression given by the sheet of the constraint surface on which the system evolves:

$$\begin{aligned} t(\Omega) &= -e^{2\Omega/3} \quad \text{if} \quad \pi_\Omega > 0 \\ t(\Omega) &= +e^{2\Omega/3} \quad \text{if} \quad \pi_\Omega < 0. \end{aligned} \tag{4.9}$$

As on the constraint surface we have

$$\pi_\Omega = \pm 2e^{2\Omega}, \tag{4.10}$$

we can write the time also as

$$t(\Omega, \pi_\Omega) = -\frac{1}{2} e^{-4\Omega/3} \pi_\Omega. \tag{4.11}$$

Because the time can be put in terms of only the coordinate Ω, the transition amplitude between two states characterized by the new coordinate Q^0 can be identified with the amplitude for the transition $|\Omega_1\rangle \to |\Omega_2\rangle$; according to equation (3.25), as this model has no true degrees of freedom we have

$$\begin{aligned} \langle \Omega_2 | \Omega_1 \rangle &= \langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle \\ &= \exp\left(-i \int_{\tau_1}^{\tau_2} \frac{\partial f}{\partial \tau} d\tau \right). \end{aligned} \tag{4.12}$$

The amplitude is only the exponential of an arbitrary phase; therefore the probability for the transition $|\Omega_1\rangle \to |\Omega_2\rangle$ is equal to unity. Of course, the result coincides with which was obtained by matching the model with the ideal clock, and it only reflects that the system is pure gauge.

4.1.2 *True degrees of freedom*

Consider a flat ($k = 0$) homogeneous and isotropic universe with a massless scalar field ϕ, and with a non vanishing cosmological constant Λ. We shall assume $\Lambda > 0$, so that if there was no field the model would be exactly the de Sitter universe; this universe has no turning point in its classical evolution, and this is reflected in the form of the Hamiltonian constraint. If we define the momentum associated to the scalar field as

$$\pi_\phi = \frac{1}{N} e^{3\Omega} \frac{d\phi}{d\tau}$$

we have the action

$$S[\Omega, \phi, \pi_\Omega, \pi_\phi, N] = \int_{\tau_1}^{\tau_2} \left[\pi_\Omega \frac{d\Omega}{d\tau} + \pi_\phi \frac{d\phi}{d\tau} - N\mathcal{H} \right] d\tau, \tag{4.13}$$

with the constraint

$$\mathcal{H} = \frac{1}{4} e^{-3\Omega} (\pi_\phi^2 - \pi_\Omega^2) + \Lambda e^{3\Omega} \approx 0. \tag{4.14}$$

We see that while π_ϕ (and hence $d\phi/d\tau$) can be zero, π_Ω (and then the same for $d\Omega/d\tau$) does not vanish on the constraint surface. The evolution is restricted to one of the two surfaces

$$\pi_\Omega = \pm \sqrt{\pi_\phi^2 + 4\Lambda e^{6\Omega}}$$

separated by $\pi_\Omega = 0$; from a geometrical point of view this means that the topology of the constraint surface is that of two disjoint half planes.

The τ-independent Hamilton–Jacobi equation associated to the constraint $\mathcal{H}$ is

$$\left(\frac{\partial W}{\partial \phi} \right)^2 - \left(\frac{\partial W}{\partial \Omega} \right)^2 + 4\Lambda e^{6\Omega} = 4E e^{3\Omega}. \tag{4.15}$$

Matching the integration constants with $\overline{P}$ and $\overline{P}_0$ we obtain the solutions

$$W_\pm = \overline{P}\phi \pm \int d\Omega \sqrt{\overline{P}^2 - 4\overline{P}_0 e^{3\Omega} + 4\Lambda e^{6\Omega}}$$

which are of the form

$$W = C(q^i)\overline{P} \pm w(q^0, \overline{P}_0, \overline{P})$$

with $q^0 = \Omega$, $q = \phi$ (see Appendix B). The double sign $\pm$ corresponds to the two different sheets $\pi_\Omega > 0$ and $\pi_\Omega < 0$ of the constraint surface. Fixing the gauge by means of the canonical condition

$$\chi \equiv \overline{Q}^0 - g(\overline{P}, T(\tau)) = 0 \tag{4.16}$$

with $T(\tau)$ a monotonic function of τ, as $\overline{Q}^0 = \partial W_\pm / \partial \overline{P}_0 = \overline{Q}^0(q^0, \overline{P}_0, \overline{P})$, if we choose

$$g(\overline{P}, T(\tau)) = \overline{Q}^0(q^0 = T(\tau), \overline{P}_0 = 0, \overline{P})$$

in terms of the original variables we have

$$q^0 = \Omega = T(\tau). \tag{4.17}$$

In the original phase space the surface defined by the gauge condition $\chi = 0$ is thus a plane $\Omega = constant$ for each value of τ. Because the topology of the constraint surface is trivial, this ensures that the gauge (4.16) does not produce Gribov copies [Simeone (1998)]: if at any τ an orbit was intersected more than once by the surface $\chi = 0$, there should be another τ such that one is tangent to the other and then it would be $[\chi, H] = 0$. But this is prevented by the gauge fixing procedure, because $[\overline{Q}^0, \overline{P}_0] = 1$ and the gauge choice only involves $\overline{Q}^0, \tau$ and $\overline{P}$, which is a conserved quantity. A global phase time is therefore

$$t = \eta\Omega \tag{4.18}$$

with $\eta = 1$ if the system evolves on the sheet given by $\pi_\Omega < 0$, and $\eta = -1$ if the system evolves on the sheet $\pi_\Omega > 0$.

To quantize the system we must perform a second transformation to non conserved observables. This transformation in the reduced space is generated by the function f, which must be chosen in such a way that the end point terms B improving the action with gauge invariance at the boundaries vanish in a gauge giving $\tau = \tau(q^i)$. With the gauge choice (4.16) the appropriate generator is

$$f = \overline{Q}P \mp w(T(\tau), \overline{P}_0 = 0, \overline{P} = P).$$

Thus, in this gauge and on the constraint surface we have

$$Q = \frac{\partial f}{\partial P} = C(q^i) \tag{4.19}$$

which, together with equation (4.17), means that Q and τ define a hypersurface in the original configuration space. The explicit form of the physical degree of freedom and of the Hamiltonian for the reduced system described by (Q, P) is

$$\begin{aligned} Q &= \phi \pm \frac{\partial}{\partial P} \int_{T(\tau)}^{\Omega} \sqrt{P^2 + 4\Lambda e^{6\Omega}} d\Omega, \\ h &= \frac{\partial f}{\partial \tau} = \mp \sqrt{P^2 + 4\Lambda e^{6T}} \frac{dT}{d\tau}. \end{aligned} \tag{4.20}$$

with $\mp = sign(\pi_\Omega)$. In the gauge (4.16) and on the constraint surface the path integral for the system has then the simple form

$$\langle \phi_2, \Omega_2 | \phi_1, \Omega_1 \rangle = \int DQDP \, \exp \left(i \int_{T_1}^{T_2} \left[P \frac{dQ}{dT} \pm \sqrt{P^2 + 4\Lambda e^{6T}} \right] dT \right), \tag{4.21}$$

where the end points are given by $T_1 = \Omega_1$ and $T_2 = \Omega_2$; because in gauge (4.16) we have $\Omega = T$, then the paths go from $Q_1 = \phi_2$ to $Q_2 = \phi_2$ (note that once we have fixed a gauge the coordinate Q is no more an observable, in the sense that $Q|_{\chi=0} = \phi$ does no more conmute with the constraint). The result shows the separation between physical degrees of freedom (ϕ) and time (Ω). The reduced system is governed by a time-dependent true Hamiltonian; this reflects that the field ϕ evolves subject to changing "external" conditions, the metric which plays the role of time. The non vanishing Hamiltonian for the reduced system resembles that of a relativistic particle of T-dependent mass $m = 2\Lambda^{1/2} e^{3T}$. To be precise, we obtain two theories, one for a "positive-energy particle" in the case of the sheet $\pi_\Omega > 0$ of the constraint surface, and one for an "antiparticle" in the case of the sheet $\pi_\Omega < 0$.

The expression (4.21) then makes simple to compute the infinitesimal propagator corresponding to each one of the two reduced Hamiltonians of Eq. (4.20) (to obtain the finite propagator we should still integrate on the coordinate Q). By recalling the analogy with the relativistic particle and the results of Ref. [Ferraro (1992)] (see equation (68)) with $\nu = \mp 1$, $\gamma = 1$ and $\sigma = \frac{1}{2}\sqrt{(T_2 - T_1)^2 - (Q_2 - Q_1)^2}$ we obtain [De Cicco&Simeone

(1999b)]

$$\langle \phi_2, \Omega_1 + \varepsilon | \phi_1, \Omega_1 \rangle = \\ \pm \frac{\varepsilon \Lambda^{1/2} e^{3\Omega_1}}{\sqrt{\varepsilon^2 - (\phi_2 - \phi_1)^2}} H_1^{(1)} (2\Lambda e^{3\Omega_1} \sqrt{\varepsilon^2 - (\phi_2 - \phi_1)^2}), \tag{4.22}$$

where $H_1^{(1)}$ is the Hankel function defined in terms of the Bessel functions J_1 and N_1 as $H_1^{(1)} = J_1 + iN_1$. The double sign corresponds to the two possible signs of π_Ω defining the two sheets of the constraint surface. This propagator fulfills the boundary condition

$$\langle \phi_2, \Omega_1 + \varepsilon | \phi_1, \Omega_1 \rangle \ \rightarrow \ \delta(\phi_2 - \phi_1)$$

when $\varepsilon \to 0$.

We should emphasize that we have succeeded in obtaining a transition amplitude in terms of only the original coordinates and with a clear notion of time because the potential has a definite sign; this allows to parametrize the system in terms of the coordinate Ω, but this is not the general case. A not completely satisfactory point is that we have deparametrized the cosmological model in such a way that the potential in the reduced Hamiltonian is time-dependent. The point is that the propagator (4.21) is just which we would obtain by writing the scaled version of the constraint (4.14) as a product of two linear constraints:

$$H = \left(\pi_\Omega + \sqrt{\pi_\phi^2 + \Lambda e^{6\Omega}}\right) \left(-\pi_\Omega + \sqrt{\pi_\phi^2 + \Lambda e^{6\Omega}}\right) \approx 0, \tag{4.23}$$

and by straightforwardly identifying ϕ as the physical degree of freedom, and $\pm\Omega$ as the time. Then we have two true Hamiltonians, and we obtain a quantum theory for physical degrees of freedom for each one in the form of the path integral(s) (4.21); each theory is unitary, as each true Hamiltonian is real. But because the time $\pm\Omega$ appears in the potential, at the quantum level the constraint (4.23) and the scaled form of (4.14) are not equivalent (see Chapter 6). In the next section we shall give a deparametrization procedure avoiding this, and we shall obtain a different expression for the propagator corresponding to this model.

4.1.3 *A more general constraint*

Let us consider a Hamiltonian constraint of the form

$$\mathcal{H} = G(\Omega)(\pi_\phi^2 - \pi_\Omega^2) + V(\phi, \Omega) \approx 0, \tag{4.24}$$

where $G(\Omega) > 0$ and $V(\phi, \Omega)$ is the potential. The action has the usual form given in (4.13). We shall restrict our analysis to the cases in which the potential $V(\phi, \Omega)$ has a definite sign. As the cases $V > 0$ and $V < 0$ are formally analogous, we shall consider only $V > 0$. Define the coordinates

$$x = x(\phi + \Omega), \qquad y = y(\phi - \Omega) \tag{4.25}$$

so that $(\partial x/\partial\phi) = (\partial x/\partial\Omega)$, $(\partial y/\partial\phi) = -(\partial y/\partial\Omega)$. The momenta π_x and π_y are given by

$$\pi_\phi = \frac{\partial x}{\partial \phi}\pi_x + \frac{\partial y}{\partial \phi}\pi_y, \qquad \pi_\Omega = \frac{\partial x}{\partial \Omega}\pi_x + \frac{\partial y}{\partial \Omega}\pi_y, \tag{4.26}$$

and then

$$\pi_\phi^2 - \pi_\Omega^2 = 4\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \phi}\pi_x\pi_y = -4\frac{\partial x}{\partial \Omega}\frac{\partial y}{\partial \Omega}\pi_x\pi_y.$$

If it is possible to choose the coordinates x and y so that $4(\partial x/\partial\phi)(\partial y/\partial\phi) = V/G$, as $V/G > 0$ then we can multiply the constraint $\mathcal{H}$ by the positive-definite quantity $(4G(\partial x/\partial\phi)(\partial y/\partial\phi))^{-1}$ and obtain a constraint H which is equivalent to $\mathcal{H}$:

$$H = \pi_x\pi_y + 1 \approx 0. \tag{4.27}$$

We shall turn the system described by (x, y, π_x, π_y) into an ordinary gauge system. The $\tau-$independent Hamilton–Jacobi equation for the constraint (4.27) is

$$\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} + 1 = E,$$

and matching the integration constants α, E to the new momenta $\overline{P}, \overline{P}_0$ it has the solution

$$W(x, y, \overline{P}_0, \overline{P}) = \overline{P}x + y\left(\frac{\overline{P}_0 - 1}{\overline{P}}\right). \tag{4.28}$$

Then

$$\pi_x = \frac{\partial W}{\partial x} = \overline{P}, \qquad \pi_y = \frac{\partial W}{\partial y} = \frac{\overline{P}_0 - 1}{\overline{P}}$$

$$\overline{Q}^0 = \frac{\partial W}{\partial \overline{P}_0} = \frac{y}{\overline{P}}, \qquad \overline{Q} = \frac{\partial W}{\partial \overline{P}} = x + \frac{y}{\overline{P}^2}\left(1 - \overline{P}_0\right). \tag{4.29}$$

To go from the set $(\overline{Q}^i, \overline{P}_i)$ to (Q^i, P_i) we define

$$F = \overline{Q}^0 P_0 + \overline{Q} P + \eta \frac{T(\tau)}{P} \tag{4.30}$$

with $\eta = \pm 1$ and $T(\tau)$ a monotonous function. Then we have the canonical variables of the gauge system in terms of those of the minisuperspace:

$$\begin{aligned} Q^0 &= \frac{y}{P}, \\ Q &= x + \frac{1}{P^2}\left(y(1 - P_0) - \eta T(\tau)\right), \\ P_0 &= \pi_x \pi_y + 1, \\ P &= \pi_x. \end{aligned} \tag{4.31}$$

There is no problem with P as a denominator because $P = \pi_x$ cannot be zero on the constraint surface.

As $[Q^0, P_0] = 1$ we have $[y/\pi_x, H] = 1$; H differs from $\mathcal{H}$ in a positive definite factor, namely $\mathcal{F}^{-1}$, so that $1 = [y/\pi_x, H] = [y/\pi_x, \mathcal{F}^{-1}\mathcal{H}] = [y/\pi_x, \mathcal{F}^{-1}]\mathcal{H} + [y/\pi_x, \mathcal{H}]\mathcal{F}^{-1} \approx [y/\pi_x, \mathcal{H}]\mathcal{F}^{-1}$; hence

$$[y/\pi_x, \mathcal{H}] > 0 \tag{4.32}$$

and a canonical gauge condition of the form $\chi \equiv Q^0 - T(\tau) = 0$ with T a monotonic function of τ, when imposed on the gauge system described by Q^i and P_i, defines a global phase time

$$t \equiv \frac{y}{\pi_x}$$

for the minisuperspace described by $\phi, \Omega, \pi_\phi, \pi_\Omega$. From (4.26) we have $\pi_x = (\pi_\phi + \pi_\Omega)\left(2\partial x/\partial\phi\right)^{-1}$ and therefore

$$t(\phi, \Omega, \pi_\phi, \pi_\Omega) = 2\frac{y(\phi - \Omega)}{\pi_\phi + \pi_\Omega}\frac{\partial x(\phi + \Omega)}{\partial \phi}. \tag{4.33}$$

The monotonic function of τ given by (4.33) depends on the coordinates and also on the momenta of the cosmological model, and is then an extrinsic time. We can also identify a time in terms of the coordinates only, but the definition depends on the sheet of the constraint surface on which the system evolves. The identification of an intrinsic time will allow to obtain a transition amplitude between states characterized by the coordinates. The end point terms associated to the canonical transformation (4.31) are of the form

$$\begin{aligned} B(\tau) &= \frac{y}{\pi_x} - y\pi_y - 2\eta\frac{T(\tau)}{\pi_x} \\ &= 2Q^0 - Q^0 P_0 - 2\eta\frac{T(\tau)}{P}. \end{aligned}$$

On the constraint surface $P_0 = 0$ these terms clearly vanish with the gauge choice

$$\chi \equiv \eta Q^0 P - T(\tau) = 0 \tag{4.34}$$

which gives

$$[\chi, P_0] = \eta P = \eta\pi_x, \tag{4.35}$$

and because $\eta Q^0 P = \eta y$ we have $[\eta y, H] = \eta\pi_x$. As before, as H and $\mathcal{H}$ differ in a positive definite factor, if we can define η so that $[\eta y, H] > 0$ then $[\eta y, \mathcal{H}] > 0$ and

$$t \equiv \eta y$$

is a global phase time. We can chose $\partial x/\partial\phi$ as a positive definite function (and appropriately adjust the sign of $\partial y/\partial\phi$) to yield $sign(\pi_x) = sign(\pi_\phi + \pi_\Omega)$. From the constraint equation we have

$$\pi_\Omega = \pm\sqrt{\frac{V(\phi,\Omega)}{G(\Omega)} + \pi_\phi^2} \tag{4.36}$$

and because V/G is positive definite, $\pi_\Omega \neq 0$ and the evolution of the system is restricted to one of the two disjoint surfaces (4.36), each one topologically equivalent to half a plane. Moreover, from (4.36) we have $|\pi_\Omega| > |\pi_\phi|$, yielding $sign(\pi_x) = sign(\pi_\Omega)$. Hence we have a good definition of time on each sheet of the constraint surface by appropriately choosing η, the choice dictated by the sign of the momentum π_Ω :

$$\begin{aligned} t(\phi,\Omega) &= +y(\phi-\Omega) \quad &\text{if} \quad \pi_\Omega > 0 \\ t(\phi,\Omega) &= -y(\phi-\Omega) \quad &\text{if} \quad \pi_\Omega < 0. \end{aligned} \tag{4.37}$$

Of course, we cannot write a single expression which holds for both sheets of the constraint surface; but once we know on which sheet of the constraint surface the system evolves we can identify a time in terms of the coordinates. If, instead, we want an expression which holds automatically, that is, which does not depend on the sign of π_Ω, we must choose a time like that given in (4.33).

According to equation (4.30) the function f is equal to $\overline{Q}P + \eta T(\tau)/P$ and then the true Hamiltonian for the reduced system described by (Q, P) is

$$h(Q,P,\tau) = \frac{\eta}{P}\frac{dT}{d\tau},$$

with $\eta = sign(\pi_\Omega)$. Because in gauge (4.34) the end point terms associated to the transformation from (x, y, π_x, π_y) to (Q^i, P_i) vanish, the new gauge-invariant action and the original action weigh the paths in the same way. By substituting the Hamiltonian h in the equation (3.25) we obtain the propagator for the transition $|\phi_1, \Omega_1\rangle \to |\phi_2, \Omega_2\rangle$ as

$$\langle \phi_2, \Omega_2 | \phi_1, \Omega_1 \rangle = \int DQDP \exp\left[i\int_{T_1}^{T_2} \left(PdQ - \frac{\eta}{P}dT \right) \right], \tag{4.38}$$

where the end points are given by $T_1 = \pm y(\phi_1 - \Omega_1)$ and $T_2 = \pm y(\phi_2 - \Omega_2)$. Note that in gauge (4.34) defining an intrinsic time, the observable Q reduces to a function of only the original coordinates:

$$Q|_{\chi=0} = x(\phi + \Omega).$$

Hence the paths go from $Q_1 = x(\phi_1 + \Omega_1)$ to $Q_2 = x(\phi_2 + \Omega_2)$. The propagator is that of a system with a true degree of freedom given by the coordinate Q. Observe that differing from what happened with the example of the preceding section, now the reduced Hamiltonians are independent of time, so that we have obtained the path integral for a conservative system. By considering both possible signs of the reduced Hamiltonian, this path integral gives the transition amplitude for both theories corresponding to the sheets $\pi_\Omega > 0$ and $\pi_\Omega < 0$.

- *Example 1*: A closed ($k = 1$) model with cosmological constant $\Lambda > 0$ and massless scalar field ϕ, whose Hamiltonian constraint is

$$\mathcal{H} = \frac{1}{4}e^{-3\Omega}(\pi_\phi^2 - \pi_\Omega^2) - e^{\Omega} + \Lambda e^{3\Omega} \approx 0 \tag{4.39}$$

is not separable in terms of the variables $x(\phi+\Omega)$, $y(\phi-\Omega)$; moreover, its potential has not a definite sign. However, it is easy to show that the extrinsic time obtained for the case $k = 0$ (flat model) is also a global phase time for the case $k = 1$. Then consider the constraint

$$\mathcal{H}_0 = \frac{1}{4}e^{-3\Omega}(\pi_\phi^2 - \pi_\Omega^2) + \Lambda e^{3\Omega} \approx 0$$

which is equivalent to

$$H_0 = \pi_\phi^2 - \pi_\Omega^2 + 4\Lambda e^{6\Omega} \approx 0.$$

By choosing $y = -(1/3)e^{3(\Omega-\phi)}$, $x = (1/3)e^{3(\Omega+\phi)}$ we obtain the extrinsic time

$$t = -\left(\frac{2}{3}\right)\frac{\Lambda e^{6\Omega}}{\pi_\phi + \pi_\Omega}. \tag{4.40}$$

On the constraint surface $\mathcal{H}_0 = 0$ we can write

$$t = \frac{1}{6}(\pi_\phi - \pi_\Omega)$$

(thus we have obtained also an extrinsic time for the model of the last section; in fact, as that model has a simple positive-definite potential increasing with Ω, any function of the form $\sim -\pi_\Omega$ is a time: $[-\pi_\Omega, e^{3\Omega}] = 3e^{3\Omega} > 0$). Note, however, that if we want to verify that this time is a global phase time also for the case $k = 1$ we should not write it in the last form. If we calculate the Poisson bracket of $t = -(2/3)\Lambda e^{6\Omega}/(\pi_\phi + \pi_\Omega)$ with the constraint $H = 4e^{3\Omega}\mathcal{H}$ we obtain $[t, H] = [t, H_0] + [t, -4e^{4\Omega}]$, which, as it is easy to check, is the sum of two positive terms. As the constraints $\mathcal{H}$ and H are equivalent, then we have

$$[t, \mathcal{H}] > 0$$

and t is a global phase time also for the model given by the constraint (4.39).

This deparametrization also provides us with another propagator for the model of section 4.1.2; this simply yields from Eq. (4.38) with the identification $T_a = \pm y_a = \mp(1/3)e^{3(\Omega_a - \phi_a)}$, $Q_a = x_a = (1/3)e^{3(\Omega_a + \phi_a)}$, $a = 1, 2$. This quantization has the advantage of a conserved Hamiltonian, but it presents a practical difficulty in effectively performing the integration, coming from the form of the integrand in the action.

- *Example 2*: Consider a flat ($k = 0$) universe with cosmological constant $\Lambda > 0$ and a massive scalar field ϕ. The corresponding constraint has the form

$$\mathcal{H} = \frac{1}{4}e^{-3\Omega}(\pi_\phi^2 - \pi_\Omega^2) + m^2\phi^2 + \Lambda e^{3\Omega} \approx 0. \qquad (4.41)$$

This model admits an intrinsic time. In the case $m = 0$ we obtain the equivalent constraint H_0 of the preceding example. The same choice of variables allows to define the intrinsic time

$$t = -y sign(\pi_\Omega) = -\frac{1}{3} sign(\pi_\Omega) e^{3(\Omega - \phi)}, \qquad (4.42)$$

and because the additional term associated to the mass of the scalar field is positive-definite (and of course depends only on the coordinates), it is easy to check that this is a time also for the case $m \neq 0$.

An interesting point to be signaled is the following: if we eliminate the field ϕ the momentum π_ϕ dissapears, and the deparametrization method of this section leads to extrinsic times of the form

$$t \sim -\frac{e^{\alpha\Omega}}{\pi_\Omega}.$$

This resembles what in classical cosmology is usually identified as a time in terms of the Hubble constant H [Kolb&Turner (1988)]: it is common to work with the time

$$t_{\mathrm{H}} \equiv \mathrm{H}^{-1}$$

where H is defined as $\dot{a}/a$ with a the scale factor. Now, as $a \sim e^{\Omega}$ and $\pi_\Omega \sim -e^{3\Omega}(d\Omega/d\tau)$, then

$$t_{\mathrm{H}} \sim -\frac{e^{3\Omega}}{\pi_\Omega},$$

which is analogous to the time obtained before. Of course, such a time is globally well defined as long as the constraint surface does not allow for $\pi_\Omega = 0$.

4.1.4 *Extrinsic time. The closed "de Sitter" universe*

Our deparametrization procedure gives a simple way to investigate how the geometrical properties of the constraint surface impose restrictions on the definition of a global phase time. Up to know we have studied only models whose Hamiltonian constraint includes a potential which does not vanish at any point of the phase space. In this section we shall examine a model which, despite its simplicity, is a good example to introduce the quantization with extrinsic time.

Consider the Hamiltonian constraint of the most general empty homogeneous and isotropic cosmological model:

$$\mathcal{H} = -\frac{1}{4}e^{-3\Omega}\pi_\Omega^2 - ke^{\Omega} + \Lambda e^{3\Omega} \approx 0. \tag{4.43}$$

This Hamiltonian corresponds to a universe with arbitrary curvature $k = -1, 0, 1$ and non zero cosmological constant; we shall suppose $\Lambda > 0$. In the case $k = 0$ we obtain the de Sitter universe; although the absence of matter makes this universe basically a toy model, it has received considerable attention because it reproduces the behaviour of models with matter or with non zero curvature when the scale factor $a \sim e^\Omega$ is great enough [Weinberg (1972)]. The classical evolution is easy to obtain, and it corresponds to an exponential expansion. In fact, from a geometrical point of view, for both $k = 0$ and $k = -1$ the momentum π_Ω cannot change its sign. For the closed model with $k = 1$, instead, $\pi_\Omega = 0$ is possible (see below).

If we apply the procedure of section 3.4.3 and we match this model with the ideal clock we obtain that

$$t \sim -e^{-2\Omega}\pi_\Omega \tag{4.44}$$

is a global phase time. To be able to compare the results, we shall apply our procedure to the scaled constraint $H = e^{-\Omega}\mathcal{H}$:

$$H = -\frac{1}{4}e^{-4\Omega}\pi_\Omega^2 - k + \Lambda e^{2\Omega} \approx 0. \tag{4.45}$$

The constraints H and $\mathcal{H}$ are equivalent because they differ only in a positive definite factor.

The τ–independent Hamilton–Jacobi equation for the Hamiltonian H is

$$-\left(\frac{\partial W}{\partial \Omega}\right)^2 - 4ke^{4\Omega} + 4\Lambda e^{6\Omega} = 4e^{4\Omega}E \tag{4.46}$$

and matching $E = \overline{P}_0$ we obtain the solution

$$W(\Omega, \overline{P}_0) = \pm 2 \int d\Omega e^{2\Omega} \sqrt{\Lambda e^{2\Omega} - k - \overline{P}_0}, \tag{4.47}$$

with + for $\pi_\Omega > 0$ and − for $\pi_\Omega < 0$. According to equation (4.47), on the constraint surface we have

$$\overline{Q}^0 = \left[\frac{\partial W}{\partial \overline{P}_0}\right]_{\overline{P}_0=0} = \mp \Lambda^{-1} \sqrt{\Lambda e^{2\Omega} - k}. \tag{4.48}$$

As we did in section 4.1.1, we introduce a transformation defining the true Hamiltonian $h = \partial f / \partial \tau$ and the variables Q^0 and P_0 of the gauge system into which the model is turned. The gauge can be fixed by means of the τ–dependent canonical condition $\chi \equiv Q^0 - T(\tau) = 0$ with T a monotonic function of τ. Then we can define

$$t = Q^0 = \theta(-\pi_\Omega)\ \Lambda^{-1}\sqrt{\Lambda e^{2\Omega} - k} - \theta(\pi_\Omega)\ \Lambda^{-1}\sqrt{\Lambda e^{2\Omega} - k} \tag{4.49}$$

as a global phase time for the system. As on the constraint surface we have

$$\pi_\Omega = \pm 2e^{2\Omega}\sqrt{\Lambda e^{2\Omega} - k} \tag{4.50}$$

(so that in the case $k = 1$ the natural size of the configuration space is given by $\Omega \geq -\ln(\sqrt{\Lambda})$), we can write

$$t(\Omega, \pi_\Omega) = -\frac{1}{2}\Lambda^{-1} e^{-2\Omega} \pi_\Omega, \tag{4.51}$$

which is in agreement with (4.44). Now an important difference between the cases $k = -1$ and $k = 1$ arises: for $k = -1$ the potential has a definite sign, and the constraint surface splits into two disjoint sheets given by (4.50). In this case the evolution can be parametrized by a function of the coordinate Ω only, the choice given by the sheet on which the system remains: if the system is on the sheet $\pi_\Omega > 0$ the time is $t = -\Lambda^{-1}\sqrt{\Lambda e^{2\Omega} + 1}$, and if it is on the sheet $\pi_\Omega < 0$ we have $t = \Lambda^{-1}\sqrt{\Lambda e^{2\Omega} + 1}$. The deparametrization of the flat model is completely analogous. For the closed model, instead, the potential can be zero and the topology of the constraint surface is no more equivalent to that of two disjoint planes. Although for $\Omega = -\ln(\sqrt{\Lambda})$ we have $V(\Omega) = 0$ and $\pi_\Omega = 0$, it is easy to verify that $d\pi_\Omega/d\tau \neq 0$ at this point. Hence, in this case the coordinate Ω does not suffice to parametrize the evolution, because the system can go from (Ω, π_Ω) to $(\Omega, -\pi_\Omega)$; therefore we must necessarily define a global phase time as a function of the coordinate and the momentum (extrinsic time).

The system has one degree of freedom and one constraint, so that it is pure gauge. In other words, there is only one physical state, in the sense that from a given point in the phase space we can reach any other point on the constraint surface by means of a finite gauge transformation. This provides a consistency proof for our deparametrization procedure [Giribet&-Simeone (2001b)]: if we characterize the states by the variables which include a globally well defined time we must obtain a transition probability equal to unity.

Indeed, by proceeding as we did in the case of the toy model of section 4.1.1, we obtain

$$\langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle = \exp\left(-i\int_{\tau_1}^{\tau_2} \frac{\partial f}{\partial \tau} d\tau\right),$$

and then the probability for the transition from Q_1^0 at τ_1 to Q_2^0 at τ_2 is

$$\left|\langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle\right|^2 = 1.$$

When the model is open or flat the coordinates Ω and Q^0 are uniquely related, and the result can be easily understood in the sense that once a gauge is fixed there is only one possible value of the scale factor at each τ. But in the case of the closed model we have seen that this is not true: at each τ there are two possible values of Ω; instead, there is only one possible value of π_Ω at each τ. Hence the transition probability in terms of Q^0 does not correspond to the evolution of the coordinate Ω, but rather of the momentum, so that

$$|\langle \pi_{\Omega,2} | \pi_{\Omega,1} \rangle|^2 = 1.$$

We must conclude that the amplitude $\langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle$ corresponds to an amplitude $\langle \pi_{\Omega,2} | \pi_{\Omega,1} \rangle$. (The characterization of the states in terms of the momenta, however, contradicts the point of view stated in section 3.3.1. In this very simple case we could admit this solution, or we could simply give the results in terms of Q^0. But, in general, before the quantization we shall make a transformation to coordinates such that the time is intrinsic).

According to our previous analysis, the fact that in the case $k = 1$ the amplitude $\langle Q_2^0, \tau_2 | Q_1^0, \tau_1 \rangle$ is not equivalent to $\langle \Omega_2 | \Omega_1 \rangle$ is natural, as the nonexistence of an intrinsic time makes impossible to find a globally good gauge condition giving τ as a function of Ω only (see section 3.3.1). But precisely for this reason, this should not be taken as a failure of the quantization procedure, because a characterization of the states in terms of only the original coordinates is not correct if we want to retain a clear notion of time on the whole evolution.

4.1.5 *Comment*

It is worth noting that at the quantum level the definition of an extrinsic time in terms of a functional form like (4.51), as given for the closed de Sitter universe, is not sufficient; in fact, it is also necessary to propose a prescription for the operatorial order between coordinates and momenta to give a precise definition of time [Giribet&Simeone (2001b)]. If we define the operators associated to t and H and calculate their conmutator, when we make it act on a quantum state on the constraint surface, that is, a state Ψ such that

$$\hat{H}|\Psi\rangle = 0,$$

we shall in general obtain a result like $if(\pi_\Omega, \Omega)|\Psi\rangle$, where f does not have a definite sign in the case $k = 1$, as terms linear in π_Ω appear. This can be avoided by defining a given operator ordering. It is possible to verify that the ordering which leads to a definite sign in the case of the closed de Sitter universe is given by

$$\hat{t} \sim \hat{\pi}_\Omega e^{-2\hat{\Omega}}, \tag{4.52}$$

while different orderings generate in the conmutator $[\hat{t}, \hat{H}]$ linear terms in the momentum π_Ω.

4.2 Anisotropic universes

While isotropic Friedmann–Robertson–Walker cosmologies can be thought to be a good description for the present universe, more general models should be considered when studying the early universe. Of course, any anisotropic model will be of physical interest as long as it evolves to a very low degree of anisotropy, so that it can explain present day observation.

The hypothesis of homogeneity and isotropy completely determines the form of the space metric leaving free only the curvature; restricting the hypothesis to homogeneity without any other symmetry assumption allows for much more freedom. Homogeneity implies that the metric properties are the same at any point of space. The precise mathematical definition of this concept is given by the set of transformations which leave unchanged the metric. Because the space is three-dimensional, the transformations must be determined by three independent parameters.

In Euclidean space the homogeneity is manifest in the invariance of the metric under translations of the cartesian reference frame. The three parameters defining a translation are the three components of the vector associated to a displacement of the origin. A translation leaves unchanged three independent differentials (dx, dy, dz) from which the length differential is obtained.

When an homogeneous non-Euclidean space is considered, one founds that the transformations of the symmetry group also leave invariant three linear differential forms; however, these forms are not total differentials of functions of the coordinates, but they read

$$\sigma^i = e^i_a dx^a$$

where $a = 1, 2, 3$ and e^i are three independent vectors which are functions of the coordinates. The differential forms fulfill $d\sigma^i = \epsilon_{ijk}\sigma^j \times \sigma^k$ (see, for example, [Schutz (1980); Schutz (1985)]). The invariant space metric can then be written as [Landau&Lifshitz (1975)]

$$dl^2 = g_{ij}\sigma^i\sigma^j = g_{ij}(e^i_a dx^a)(e^j_b dx^b),$$

so that the spatial metric tensor has the components

$$g_{ab} = g_{ij}e^i_a e^j_b.$$

Possible anisotropic cosmologies are comprised by the Bianchi models and the Kantowski–Sachs model [Ryan&Shepley (1975)]. In this section we shall apply our deparametrization and path integral quantization method to the Kantowski–Sachs model [Kantowski&Sachs (1966)], and to the Taub model [Taub (1951)], which is a particular case of the diagonal Bianchi type-IX universe (these models have deserved considerable attention in different quantization and deparametrization programs; see, for example, [Higuchi&-Wald (1995)]). By introducing the diagonal 3×3 matrix β_{ij} both corresponding spacetime metrics can be put in the form

$$ds^2 = N^2 d\tau^2 - e^{2\Omega(\tau)}(e^{2\beta(\tau)})_{ij}\sigma^i\sigma^j. \tag{4.53}$$

It should be remarked, however, that the spatial geometry of these models is essentially different, in the sense that there is not a continuous transformation carrying from one to the other (see below).

4.2.1 *The Kantowski–Sachs universe*

The Kantowski–Sachs universe is defined by the line element

$$ds^2 = N^2 d\tau^2 - S^2(\tau)dz^2 - R^2(\tau)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{4.54}$$

The model can be closed by setting $0 \le z \le 4\pi$ (and then substituting z by the angle ψ). The classical behaviour of this universe is analogous to that of the closed Friedmann–Robertson–Walker cosmology in the fact that the volume, defined as

$$\mathcal{V} = \int d^3x\sqrt{-(^3g)},$$

grows to a maximun and then returns to zero. For applying the Hamiltonian formalism it is convenient to write the metric in the form (4.53); to do so we define the matrix

$$\beta_{ij} = diag(-\beta, -\beta, 2\beta)$$

and the differential forms $\sigma^1 = d\theta$, $\sigma^2 = \sin\theta d\varphi$, $\sigma^3 = d\psi$. Then we have

$$ds^2 = N^2 d\tau^2 - e^{2\Omega(\tau)}\left(e^{2\beta(\tau)}d\psi^2 + e^{-\beta(\tau)}(d\theta^2 + \sin^2\theta d\varphi^2)\right). \tag{4.55}$$

Note that while $e^{2\Omega}$ can be understood as a spatial scale factor, even for $\beta = 0$ the model remains to be anisotropic. This is the central difference with Bianchi models.

In the absence of matter the Hamiltonian form of the action functional reads

$$S[\Omega, \beta, \pi_\Omega, \pi_\beta, N] = \int_{\tau_1}^{\tau_2} \left(\pi_\beta \frac{d\beta}{d\tau} + \pi_\Omega \frac{d\Omega}{d\tau} - N\mathcal{H} \right) d\tau \tag{4.56}$$

where $\mathcal{H} = e^{-3\Omega} H \approx 0$ is the Hamiltonian constraint, and

$$H = -\pi_\Omega^2 + \pi_\beta^2 - e^{4\Omega + 2\beta}. \tag{4.57}$$

The Kantowski–Sachs model has an interesting property: although the scaled potential $V(\Omega, \beta) = -e^{4\Omega+2\beta}$ has a definite sign, so that an intrinsic time can be identified among the canonical variables, the time is not trivially identified as a function of the scale factor, as it results from the fact that the volume does not behave monotonically. This is something to be noted, as in the early literature it can sometimes be found that the isolation of the coordinate Ω as time parameter is made as the previous step before quantization. This is not right, unless the analysis is restricted to a region of the phase space. We can easily see that no function $\Theta(\Omega)$ can be a global phase time for the Kantowski–Sachs universe:

$$[\Theta(\Omega), \mathcal{H}] = -2 \frac{\partial \Theta(\Omega)}{\partial \Omega} e^{-3\Omega} \pi_\Omega,$$

and for $\pi_\beta = \pm e^{2\beta+\Omega}$ we have $\pi_\Omega = 0$, so that $[\Theta(\Omega), \mathcal{H}]$ vanishes. In those previous works the momentum π_Ω was defined as the reduced Hamiltonian; this is unsatisfactory, because even if we restrict the configuration space to its natural size to avoid an imaginary Hamiltonian leading to non unitary evolution, $\pi_\Omega = 0$ makes possible transitions from "positive-energy" to "negative-energy" states.

The Hamiltonian is not separable in terms of the original variables; then we define

$$e^{3(\Omega+\beta)} \equiv 3x, \qquad e^{\Omega-\beta} \equiv 4y,$$

and we obtain the equivalent constraint

$$H' \equiv -(\pi_x \pi_y + 1) \approx 0. \tag{4.58}$$

Following a procedure completely analogous to that of section 4.1.3 we solve the $\tau-$independent Hamilton–Jacobi equation

$$-\frac{\partial W}{\partial x} \frac{\partial W}{\partial y} - 1 = E',$$

to obtain the generator W of the canonical transformation from (x, y, π_x, π_y) to the variables $(\overline{Q}^i, \overline{P}_i)$ of the gauge system into which we turn the minisuperspace; then we perform a τ-dependent transformation in the space of observables. The canonical variables of the gauge system are therefore given by

$$\begin{aligned}
Q^0 &= \frac{e^{\Omega-\beta}}{4P}, \\
Q &= -\frac{1}{3}e^{3(\Omega+\beta)} - \frac{1}{P^2}\left(\frac{e^{\Omega-\beta}}{4}(1+P_0) + \eta T(\tau)\right), \\
P_0 &= (\pi_\beta^2 - \pi_\Omega^2)e^{-4\Omega-2\beta} - 1, \\
P &= -\frac{1}{2}(\pi_\beta + \pi_\Omega)e^{-3(\Omega+\beta)}
\end{aligned}$$

with $\eta = \pm 1$ ($P = -\pi_x$ cannot be zero on the constraint surface). The true Hamiltonian of the gauge system described by (Q^i, P_i) is

$$h \equiv \frac{\eta}{P}\frac{dT}{d\tau}.$$

Hence the gauge invariant action $\mathcal{S}$ can be written as

$$\mathcal{S}[Q^i, P_i, N] = \int_{\tau_1}^{\tau_2}\left(P\frac{dQ}{d\tau} + P_0\frac{dQ^0}{d\tau} - NP_0 - \frac{\eta}{P}\frac{dT}{d\tau}\right)d\tau, \tag{4.59}$$

or in terms of the original variables

$$\mathcal{S}[\Omega, \beta, \pi_\Omega, \pi_\beta, N] = \int_{\tau_1}^{\tau_2}\left(\pi_\beta\frac{d\beta}{d\tau} + \pi_\Omega\frac{d\Omega}{d\tau} - N\mathcal{H}\right)d\tau + B(\tau_2) - B(\tau_1), \tag{4.60}$$

where

$$\begin{aligned}
B &= \frac{1}{\pi_\Omega + \pi_\beta}\left[4e^{3(\Omega+\beta)}\left(\frac{1}{4}e^{\Omega-\beta} + \eta T(\tau)\right) + \frac{1}{2}(-\pi_\Omega^2 + \pi_\beta^2 - e^{4\Omega+2\beta})\right] \\
&= -\left[2\left(Q^0 + \eta\frac{T(\tau)}{P}\right) + Q^0P_0\right].
\end{aligned} \tag{4.61}$$

Under a gauge transformation generated by $\mathcal{H}$ we have $\delta_\epsilon B = -\delta_\epsilon \mathcal{S}$, and hence $\delta_\epsilon \mathcal{S} = 0$. On the constraint surface $H' = P_0 = 0$ this terms clearly vanish in the gauge

$$\chi \equiv Q^0P + \eta T(\tau) = 0 \tag{4.62}$$

which is equivalent to $T(\tau) = \pm(1/4)e^{\Omega-\beta}$, and then it defines $\tau = \tau(\Omega, \beta)$. An intrinsic time t can be defined by writing $t = -\eta Q^0 P$, with $\eta = \pm 1$, and apropriately choosing η:

$$[t, H'] = [-\eta Q^0 P, P_0] = -\eta P,$$

and because $P = -\pi_x$ then we must choose $\eta = 1$ if $\pi_x > 0$ and $\eta = -1$ if $\pi_x < 0$; as $\pi_x = (1/2)(\pi_\Omega + \pi_\beta)e^{-3(\Omega+\beta)}$ and on the constraint surface it is $|\pi_\beta| > |\pi_\Omega|$, we have $sign(\pi_x) = sign(\pi_\Omega + \pi_\beta) = sign(\pi_\beta)$; therefore the time is

$$t(\Omega, \beta) = +\frac{1}{4}e^{\Omega-\beta} \qquad \text{if} \qquad \pi_\beta < 0,$$

$$t(\Omega, \beta) = -\frac{1}{4}e^{\Omega-\beta} \qquad \text{if} \qquad \pi_\beta > 0. \tag{4.63}$$

Note that π_β cannot change from a negative value to a positive one on the constraint surface, so that the time is well defined for the whole evolution of the system.

It is easy to verify that an extrinsic time can be defined by imposing a canonical gauge of the form $\chi \equiv Q^0 + T(\tau) = 0$. If we make $t = -T$ we obtain

$$t(\Omega, \beta, \pi_\Omega, \pi_\beta) = Q^0 = -\frac{e^{4\Omega+2\beta}}{2(\pi_\Omega + \pi_\beta)} \tag{4.64}$$

with $[t, \mathcal{H}] > 0$. Using the constraint equation (4.57) we can write

$$t(\pi_\Omega, \pi_\beta) = \frac{1}{2}(\pi_\Omega - \pi_\beta). \tag{4.65}$$

We see that a gauge condition involving one of the new momenta defines a time in terms of only the original coordinates, while a gauge involving only one of the new coordinates gives an extrinsic time which can be written in terms of only the original momenta.

Because the path integral in the variables (Q^i, P_i) is gauge invariant, we can compute it in any canonical gauge. With the gauge choice (4.62), on the constraint surface $P_0 = 0$, and after integrating on N, P_0 and Q^0, the transition amplitude is given by

$$\langle \Omega_2, \beta_2 | \Omega_1, \beta_1 \rangle = \int DQDP \exp\left[i\int_{T_1}^{T_2}\left(PdQ - \frac{\eta}{P}dT\right)\right], \tag{4.66}$$

where the end points are $T_1 = \pm(1/4)e^{\Omega_1-\beta_1}$ and $T_2 = \pm(1/4)e^{\Omega_2-\beta_2}$; because on the constraint surface and in gauge (4.62) the true degree of freedom reduces to $Q = -x = -(1/3)e^{3(\Omega+\beta)}$, then the paths in phase space go from $Q_1 = -(1/3)e^{3(\Omega_1+\beta_1)}$ to $Q_2 = -(1/3)e^{3(\Omega_2+\beta_2)}$. After the gauge fixation we have obtained the path integral for a system with one physical degree of freedom and with a true Hamiltonian. The result shows the separation between true degrees of freedom and time yielding after a simple canonical gauge choice.

An interesting point to be noted is that the coordinate β is itself a time. Strictly speaking, as π_β does not vanish on the constraint surface, we have that

$$[\beta, H] = 2\pi_\beta \neq 0$$

so that

$$t^* \equiv \beta sign(\pi_\beta)$$

is a global phase time. This makes possible the interpretation of the propagator as the amplitude for the transition from Ω_1 at time $t_1^* = \pm\beta_1$ to Ω_2 at time $t_2^* = \pm\beta_2$:

$$\langle\Omega_2, \beta_2|\Omega_1, \beta_1\rangle = \langle\Omega_2, t_2^*|\Omega_1, t_1^*\rangle,$$

with a sign $\pm$ depending of the sheet of the constraint surface defined by the sign of π_β.

If we include a matter field the scaled Hamiltonian changes to

$$H = -\pi_\Omega^2 + \pi_\beta^2 + \pi_\phi^2 - e^{4\Omega+2\beta} + V(\phi)e^{6\Omega} \approx 0, \tag{4.67}$$

and the corresponding Hamilton–Jacobi equation will be solvable or not depending on the form of $V(\phi)$. In the case of a massless ($m = 0$) non interacting scalar field we can show that the intrinsic time (4.63) is no more a global phase time, but the extrinsic time (4.64) is still a time: if we calculate the Poisson bracket for $t(\Omega, \beta) = -(1/4)sign(\pi_\beta)e^{\Omega-\beta}$ we obtain

$$[t, H] = \frac{1}{2}sign(\pi_\beta)(\pi_\Omega + \pi_\beta)e^{\Omega-\beta}.$$

The result is not positive definite, because the sign of π_β and the sign of $\pi_\Omega + \pi_\beta$ are not necessarily the same, as a result of the term π_ϕ^2 in the new

Hamiltonian. Instead, on the (new) constraint surface we have that

$$t = t(\pi_\Omega, \pi_\beta, \pi_\phi) = \pi_\Omega - \pi_\beta - \frac{\pi_\phi^2}{\pi_\Omega + \pi_\beta}$$

and if we calculate the Poison bracket we have

$$[t, H] = 2e^{4\Omega+2\beta}\left(1 + \frac{3\pi_\phi^2}{(\pi_\Omega + \pi_\beta)^2}\right)$$

which is positive-definite.

Note that now, when we include a matter field, the degree of anisotropy given by the coordinate β is no more a time. The reason is that because matter enters in the formalism with a positive-definite term in the Hamiltonian (or in the more realistic case of a massive field with terms that are not necessarily negative), the momentum conjugated to the coordinate β can vanish.

4.2.2 *The Taub universe*

The anisotropic generalization of the Friedmann–Robertson–Walker universe with curvature $k = +1$ is the diagonal Bianchi type-IX universe, whose metric is of the form (4.53) with β_{ij} the 3×3 diagonal matrix

$$\beta_{ij} = diag[\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -\beta_+].$$

The parameters β_+ and β_- determine the degree of anisotropy. If we set $\beta_- = 0$ we obtain the metric of the Taub universe. Its action functional reads

$$S = \int_{\tau_1}^{\tau_2}\left(\pi_+\frac{d\beta_+}{d\tau} + \pi_\Omega\frac{d\Omega}{d\tau} - N\mathcal{H}\right)d\tau, \tag{4.68}$$

with the Hamiltonian constraint

$$\mathcal{H} = e^{-3\Omega}(\pi_+^2 - \pi_\Omega^2) + \frac{1}{3}e^{\Omega}(e^{-8\beta_+} - 4e^{-2\beta_+}) \approx 0. \tag{4.69}$$

Because $e^{-3\Omega}$ is positive definite, this constraint is equivalent to

$$H = \pi_+^2 - \pi_\Omega^2 + \frac{1}{3}e^{4\Omega}(e^{-8\beta_+} - 4e^{-2\beta_+}) \approx 0. \tag{4.70}$$

The Taub universe is perhaps the best model with which our deparametrization and path integral quantization program can be illustrated [Giribet&-Simeone (2001c)]; besides a separable Hamiltonian, it includes true degrees of freedom and a potential which vanishes for finite values of the coordinates, so making impossible the definition of an intrinsic time in terms of the original variables. As it is easy to see, for $\beta_+ = -(1/6)\ln 4$ the potential is zero.

The Hamiltonian is not separable in terms of the coordinates and momenta $(\Omega, \beta_+, \pi_\Omega, \pi_+)$. Then to apply our method we define the coordinates

$$x = \Omega - 2\beta_+, \qquad y = 2\Omega - \beta_+ \tag{4.71}$$

so that $\pi_+^2 - \pi_\Omega^2 = 3(\pi_x^2 - \pi_y^2)$, and we can write

$$H = \pi_x^2 - \pi_y^2 + \frac{1}{9}(e^{4x} - 4e^{2y}) \approx 0. \tag{4.72}$$

At this stage we then have $H = H_1(x, \pi_x) + H_2(y, \pi_y)$ with $H_1 > 0$ and $H_2 < 0$, but the potential vanishes for $y = 2x - (1/2)\ln 4$. A time in terms of only x, y does not exist. Hence before turning the model into an ordinary gauge system we shall make a first canonical transformation to the variables (x, s, π_x, π_s) so that the (new) potential has only one positive definite term. The resulting new coordinates will correspond to the $\tilde{q}^i$ in terms of which an intrisic time exists.

We shall perform a canonical transformation matching $H_2(y, \pi_y) = -\pi_s^2$, so that $\pi_s = \pm\sqrt{-H_2(y, \pi_y)}$. This is achieved by introducing the generating fuctions of the first kind

$$\Phi_1(y, s) = \pm\frac{2}{3}e^y \sinh s. \tag{4.73}$$

The momenta are then given by

$$\begin{aligned} \pi_y &= \pm\frac{2}{3}e^y \sinh s \\ \pi_s &= \pm\frac{2}{3}e^y \cosh s \end{aligned} \tag{4.74}$$

so that

$$\pi_y^2 + \frac{4}{9}e^{2y} = \frac{4}{9}e^{2y}(\sinh^2 s + 1) = \pi_s^2$$

and the Hamiltonian can be written as

$$H(s,x,\pi_s,\pi_x) = -\pi_s^2 + \pi_x^2 + \frac{1}{9}e^{4x} \approx 0. \tag{4.75}$$

The canonical transformation (4.74) has changed the properties of the constraint surface, as now there are two disjoint sheets defined by the sign of the momentum π_s.

It is important to note the reason why we have introduced two possible definitions of Φ_1: this constraint can be written as

$$H = \left(-\pi_s + \sqrt{\pi_x^2 + \frac{1}{9}e^{4x}}\right)\left(\pi_s + \sqrt{\pi_x^2 + \frac{1}{9}e^{4x}}\right) \approx 0, \tag{4.76}$$

and if we had chosen a definite sign, according to the resulting sign of π_s, only one of the factors would be zero. But at the level of the variables (x, s, π_x, π_s) there is no justification to prefer one possible sign of π_s; we shall consider the Hamiltonian constraint (4.75) as the starting point for applying our procedure, and we shall not go back to put the results in terms of the original variables. A point that should be noted is that the potential in the square root does not depend on the coordinate s, but it depends only on x. This ensures that in its operator version both forms (4.75) and (4.76) of the Hamiltonian constraint are equivalent, as no additional terms appear associated to conmutators (see Chapter 6 for a further discussion about this feature).

The action $S[x, s, \pi_x, \pi_s, N]$ will differ from $S[\Omega, \beta_+, \pi_\Omega, \pi_+, N]$ in surface terms associated to the transformation generated by Φ_1, namely D:

$$\begin{aligned} S[x,s,\pi_x,\pi_s,N] &= \int_{\tau_1}^{\tau_2}\left(\pi_x\frac{dx}{d\tau} + \pi_s\frac{ds}{d\tau} - NH(x,s,\pi_x,\pi_s)\right)d\tau \\ &= S[\Omega,\beta_+,\pi_\Omega,\pi_+,N] + [D(\tau)]_{\tau_1}^{\tau_2}, \end{aligned}$$

and when we turn the system into an ordinary gauge one we will have the additional terms B; but as we will not be interested in a transition amplitude between states labeled by the original coordinates q^i, but by the $\tilde{q}^i$, we shall not require that $D + B = 0$, but only that $B = 0$ in a canonical gauge defining $t = t(\tilde{q}^i)$.

Let us introduce the coordinates

$$u = \frac{1}{12}e^{2(x+s)}, \qquad v = \frac{1}{12}e^{2(x-s)}$$

which lead to the equivalent constraint

$$H' = \pi_u \pi_v + 1.$$

This allows to straightforwardly apply the procedure of section 4.1.3 with the substitution $x \to u$, $y \to v$, $\phi \to x$, $\Omega \to y$. Then we can go back to the variables (x, s, π_x, π_s). The variables (Q^i, P_i) of the gauge system are given by

$$\begin{aligned}
Q^0 &= \frac{e^{2(x-s)}}{12P}, \\
Q &= \frac{1}{12}e^{2(x+s)} + \frac{1}{P^2}\left(\frac{1}{12}e^{2(x-s)}(1-P_0) - \eta T(\tau)\right), \\
P_0 &= 9(\pi_x^2 - \pi_s^2)e^{-4x} + 1, \\
P &= 3(\pi_s + \pi_x)e^{-2(x+s)},
\end{aligned}$$

with $\eta = \pm 1$. An extrinsic time is then

$$t(x, s, \pi_x, \pi_s) = \frac{1}{36}\frac{e^{4x}}{\pi_x + \pi_s}. \tag{4.77}$$

The constraint surface splits into two sheets given by the sign of π_s. On each sheet the intrinsic time can be defined as

$$t(x, s) = \frac{1}{12} sign(\pi_s) e^{2(x-s)}, \tag{4.78}$$

which is associated to the canonical gauge $\eta Q^0 P - T(\tau) = 0$ because P is proportional to $\pi_s + \pi_x$ and $sign(\pi_s + \pi_x) = sign(\pi_s)$. The end point terms associated to the transformation $(\tilde{q}^i, \tilde{p}_i) \to (Q^i, P_i)$ are

$$\begin{aligned}
B(\tau) &= 2Q^0 - Q^0 P_0 - 2\eta \frac{T(\tau)}{P} \\
&= \frac{1}{3(\pi_s + \pi_x)}\left(\frac{e^{4x}}{6} - 2\eta e^{2(x+s)} T(\tau)\right)
\end{aligned}$$

and with the gauge choice defining an intrinsic time they vanish on the constraint surface $P_0 = 0$. The expresion for the quantum propagator is

$$\langle x_2, s_2 | x_1, s_1 \rangle = \int DQDP \exp\left[i\int_{T_1}^{T_2}\left(PdQ - \frac{\eta}{P}dT\right)\right]. \tag{4.79}$$

The end points are $T_1 = \pm(1/12)e^{2(x_1-s_1)}$ and $T_2 = \pm(1/12)e^{2(x_2-s_2)}$. Note that in the gauge defining the intrinsic time the new coordinate Q coincides with $(1/12)e^{2(x+s)}$, so that the paths go from $Q_1 = (1/12)e^{2(x_1+s_1)}$ to $Q_2 = (1/12)e^{2(x_2+s_2)}$. We have obtained a propagator with a clear distinction between time and the physical degree of freedom. The path integral corresponds to that for a conservative system with Hamiltonian η/P. Because $\eta = sign(\pi_s)$ then at the level of the physical degrees of freedom we have two disjoint theories, one for each sheet of the constraint surface.

Analogously as for β in the case of the Kantowski–Sachs universe, because now the momentum π_s does not vanish on the constraint surface, the coordinate s is itself a time (recall the discussion in Section 3.3.2):

$$[s, H] = -2\pi_s \neq 0.$$

More precisely, on each sheet of the constraint surface we can define a time in the form

$$t^* \equiv -s\ sign(\pi_s).$$

Although we do not use this time as the time parameter in the path integral, the interpretation of the result can be made more clear by recalling that one of the coordinates which identify the states is a global phase time. In fact, we can write the transition amplitude as

$$\langle x_2, t_2^* | x_1, t_1^* \rangle$$

with a sign depending on the sheet of the constraint surface. With this interpretation, the propagator becomes completely analogous to that of a mechanical system with a physical degree of freedom x whose evolution is given in terms of a true time t^*. Note that though the original momenta are necessarily involved in the description of the states, they are confined to the variable s which is a time:

$$\begin{aligned} s &= \pm\text{arcsinh}\left(\frac{3}{2}\pi_y e^{-y}\right) \\ &= \pm\text{arcsinh}\left(\frac{1}{2}(\pi_\Omega + \pi_+)e^{(-2\Omega+\beta_+)}\right), \end{aligned}$$

while x is a simple function of only the coordinates Ω and β_+.

At this point it is natural to ask wether this time could have been obtained from the beginning with our deparametrization procedure. The answer is that this is in fact possible, even in the case that we include a matter field in the model. Consider the Hamiltonian constraint for the Taub universe with a non interacting scalar field with a mass which can be neglected (a massless dust):

$$H = -\pi_\Omega^2 + \pi_\phi^2 + \pi_+^2 + \frac{1}{3}e^{4\Omega}(e^{-8\beta_+} - 4e^{-2\beta_+}) \approx 0. \tag{4.80}$$

If we change to the coordinates x and y, and then perform the canonical transformation generated by $\Phi_1(y,s)$, we obtain the equivalent constraint

$$H = -\pi_s^2 + \pi_\phi^2 + \pi_x^2 + \frac{1}{9}e^{4x} \approx 0, \tag{4.81}$$

where we have redefined $\pi_\phi \to \pi_\phi/\sqrt{3}$. The corresponding Hamilton–Jacobi equation is separable:

$$-\left(\frac{\partial W}{\partial s}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{1}{9}e^{4x} = E, \tag{4.82}$$

and the solution is clearly of the form $W_1(x,\pi_x) + W_2(\phi,\pi_\phi) + W_3(s,\pi_s)$. Introducing the integration constants $b^2 = \pi_\phi^2$ and a^2 such that $a^2+b^2-E = \pi_s^2$ we obtain

$$\begin{aligned} W &= sign(\pi_x)\int dx\sqrt{a^2 - \frac{1}{9}e^{4x}} \\ &\quad + s\ sign(\pi_s)\sqrt{a^2+b^2-E} + \phi\ sign(\pi_\phi)\sqrt{b^2}. \end{aligned} \tag{4.83}$$

If we match $E = \overline{P}_0$ we have

$$\overline{Q}^0 = \left[\frac{\partial W}{\partial \overline{P}_0}\right]_{\overline{P}_0=0} = -sign(\pi_s)\frac{s}{2\sqrt{a^2+b^2}} = -\frac{s}{2\pi_s}.$$

As $[\overline{Q}^0, \overline{P}_0] = 1$ then we can inmediately define an extrinsic time as

$$t(s,\pi_s) \equiv \overline{Q}^0 = -\frac{s}{2\pi_s}. \tag{4.84}$$

In the variables $(s,x,\phi,\pi_s,\pi_x,\pi_\phi)$ the constraint surface is topologically equivalent to two disjoint half planes, each one corresponding to $\pi_s > 0$ and to $\pi_s < 0$; thus, we can also define the time as

$$\begin{aligned} t(s) &\equiv 2\pi_s sign(\pi_s)\overline{Q}^0 \\ &= -s\ sign(\pi_s), \end{aligned} \tag{4.85}$$

which coincides with the time t^* found just before (this time yields from a canonical gauge condition of the form $\chi \equiv 2\overline{Q}^0\sqrt{a^2+b^2} - T(\tau) = 0$). In terms of the new variables this is an intrinsic time; of course, when put in terms of the original variables this time involves the momenta.

All the results include the sign of the momentum π_s, which comes from the double sign in the definition of Φ_1. Some authors, however, have suggested that the typical constraint of a parametrized system, which is linear in the momentum conjugated to the time, may be hidden in the Hamiltonian formalism for the gravitational field [Ferraro (1999); Catren&Ferraro (2001)]. According to this point of view, one should choose only one of both possible signs for the generator Φ_1, and there would not be two coexisting theories. Our results, instead, reflect that we consider the quadratic Hamiltonian $H(x, s, \pi_x, \pi_s)$ as the starting point because our formalism requires a constraint which, with a given choice of variables, admits an intrinsic time, and at the level of this Hamiltonian there is no reason to choose one definite sign for the non vanishing momentum π_s. Anyway, it must be signaled that, because the reduced Hamiltonian $1/P$ is a conserved non-vanishing quantity, in our interpretation there are no transitions from states on one sheet to states on the other sheet of the constraint surface, and therefore both points of view do not lead to an essential contradiction. We shall return to this point in Chapter 6, in the context of the canonical quantization of the Taub universe.

4.2.3 *Other anisotropic models*

Other anisotropic universes exist with even more easily separable constraints than which we have analysed. An example is the Bianchi type I model with β_{ij} a diagonal matrix, whose metric is of the form given in (4.53) with the linear forms equal to coordinate differentials: $\sigma^i = dx^i$. The scalar curvature ${}^3R$ vanishes, so that in the empty case we obtain the Hamiltonian constraint

$$\mathcal{H} = -\pi_\Omega^2 + \pi_+^2 + \pi_-^2 \approx 0. \tag{4.86}$$

This form of the constraint has led to an interpretation based in the analogy with a relativistic free particle in two dimensions. In fact, this is right in the sense that all the momenta are constants of the motion. However, we should remark that the absence of a mass (or a non vanishing potential) does not allow to identify Ω as the global phase time, as it is clear that the

Poisson bracket $[\Omega, \mathcal{H}]$ vanishes for $\pi_\Omega = 0$. The global phase time must be extrinsic, and it is easy to show that the procedure of the preceding sections leads to

$$t(\Omega, \pi_\Omega) = -\frac{\Omega}{2\pi_\Omega}. \tag{4.87}$$

(Analogous expressions in terms of $\beta_\pm$ and $\pi_\pm$ can be given with the property $[t, \mathcal{H}] > 0$).

A less trivial example is provided by the homogeneous version of the Szekeres universe [Szekeres (1975)]. With a suitable coordinate choice, this model is described by the spacetime metric

$$ds^2 = e^{\alpha(\tau)}\left(N^2 d\tau^2 - dz^2\right) - e^{\beta(\tau)}\left(e^p dx_+^2 + e^{-p} dx_-^2\right), \tag{4.88}$$

where p is a positive-definite constant which is usually related with the pressure of the matter source. This metric is invariant under translations along the z axis, and it is also Lorentz invariant in this direction (in its original version the functions α and β also depend on z, so that the model is not homogeneous). We can make the redefinition $(e^{p/2}x_+, e^{-p/2}x_-) \to (x, y)$ to put the metric in the axisymmetric form

$$ds^s = e^{\alpha(\tau)}\left(N^2 d\tau^2 - dz^2\right) - e^{\beta(\tau)}\left(dx^2 + dy^2\right). \tag{4.89}$$

If we neglect matter in the dynamics of the model, then we can write its Lagrangian as

$$\mathcal{L} = N\ {}^3R\sqrt{-({}^3g)} = \frac{1}{N}e^{\beta}\left(\dot{\alpha}\dot{\beta} - 2\dot{\alpha}^2 + \frac{5}{2}\dot{\beta}^2\right),$$

where we have discarded total derivatives which would contribute with surface terms to the action (we have used dots to denote derivatives with respect to τ). Note that there are no potential terms, but the derivatives are mixed, so that the supermetric is not diagonal. Then we define the new coordinates u and v such that

$$\begin{aligned} \alpha &= v \\ \beta &= u - \frac{1}{5}v. \end{aligned}$$

This allows to write the Hamiltonian constraint

$$\mathcal{H} = e^{-u+v/5}\left(\frac{2}{5}\pi_u^2 - \frac{5}{12}\pi_v^2\right) \approx 0, \tag{4.90}$$

which resembles the constraint of a massless free particle in one spatial dimension, scaled by the positive-definite function $e^{-u+v/5}$. As there is no potential, the time cannot be defined in terms of the coordinates only; by applying our deparametrization procedure we find that, depending on the choice of the separation constants in the Hamilton–Jacobi equation, the extrinsic time can be given as $t(u, \pi_u) \sim u/\pi_u$ or in the form $t(v, \pi_v) \sim -v/\pi_v$.

Chapter 5

String cosmologies

In order to present an example of quantization of models beyond general relativity, in this chapter we apply the technical procedures developed in the previous chapters to the study of cosmological models within the context of dilatonic theories of gravity coming from the large tension limit of string theory. Then we now focus our attention on the analysis of the minisuperspace realization of string cosmological models which appear as solutions of the low energy effective action of closed bosonic string theory.

5.1 String theory on background fields

The action of the non linear σ-model that describes the world-sheet dynamics of strings on a curved manifold and in presence of background fields has the form

$$\begin{aligned} S_{ws} &= \frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{h}\left(h_{\alpha\beta}g_{\mu\nu}(X) + i\varepsilon_{\alpha\beta}B_{\mu\nu}(X)\right)\partial^\alpha X^\mu\partial^\beta X^\nu \\ &\quad +\frac{1}{2\pi}\int d\sigma d\tau\sqrt{h}R(X)\phi(X), \end{aligned} \tag{5.1}$$

where $h_{\alpha\beta}$ is the metric on the string world-sheet, R is the Ricci scalar related with this metric, $g_{\mu\nu}$ is the metric of the spacetime which fixes the background geometry, $B_{\mu\nu}$ is the $\mathcal{NS}$-$\mathcal{NS}$ two-form field and ϕ is the dilaton. Actually, these are precisely the background fields that emerge as coherent states of the massless spectrum of closed bosonic string theory. We have adopted the usual nomenclature which establishes that the indices α, β label the two-dimensional world-sheet manifold while the indices μ, ν, ρ are reserved for the D-dimensional target spacetime. In the expression above, the parameter α' must be interpreted as the inverse of the string tension $T = 1/(2\pi\alpha')$ which introduces the scale of the theory at the quantum level.

The two-dimensional field theory defined by (5.1) is invariant under gauge transformations of the form

$$\begin{aligned} \delta B_{\mu\nu} &= \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \\ \delta\phi &= \phi_0, \end{aligned} \tag{5.2}$$

being Λ_μ an arbitrary vector and ϕ_0 a constant value.

Now, let us define the strength tensor $H_{\mu\nu\rho}$ associated to the antisymmetric field $B_{\mu\nu}$ as follows:

$$\mathbf{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \tag{5.3}$$

A crucial requirement of the consistency of string theory is the existence of Weyl's invariance in the world-sheet theory (5.1); and consecuently, this fact imposes the vanishing of which are called the beta functions:

$$\begin{aligned} R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - \frac{1}{4}\mathbf{H}_{\mu\rho\delta}\mathbf{H}^{\rho\delta}_\nu &= 0 \\ \nabla^\delta \mathbf{H}_{\delta\mu\nu} - \nabla^\delta \phi \mathbf{H}_{\delta\mu\nu} &= 0 \\ c - \nabla_\mu \nabla^\mu \phi + \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{6}\mathbf{H}_{\mu\nu\rho}\mathbf{H}^{\mu\nu\rho} &= 0 \end{aligned} \tag{5.4}$$

where $c = 2(D - 26)/(3\alpha')$. It must be noted that these equations have been written at first order in the α' power expansion.

A point to be remarked for our further study is the fact that the equations (5.4) can be interpreted as the Euler–Lagrange equations of motion of a field theory corresponding to the following action:

$$S_{sf} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} e^{-\phi} \left(-c + R + \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12}\mathbf{H}_{\mu\nu\rho}\mathbf{H}^{\mu\nu\rho} \right). \tag{5.5}$$

Thus, we can interpret this expression as the low energy effective action describing the large tension limit (*i.e.* $\alpha' \to 0$) of closed bosonic string theory. Within this context, a consistent configuration of background fields for the formulation of string theory must satisfy to be a classical solution obtained from the variational principle with the action (5.5) (at least, at first order in the α' power expansion).

Now let us redefine the fields in order to obtain a more familiar form for the field theory. If we perform the change

$$g_{\mu\nu} \to e^{\phi} g_{\mu\nu} \tag{5.6}$$

the action of the spacetime theory (5.5) becomes

$$\begin{aligned} S_{ef} &= \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \\ &\times \left[R - c e^{2\phi/(D-2)} + \frac{1}{D-2} \nabla_\mu \phi \nabla^\mu \phi - \frac{e^{4\phi/(D-2)}}{12} \mathbf{H}_{\mu\nu\rho} \mathbf{H}^{\mu\nu\rho} \right]. \end{aligned} \tag{5.7}$$

This is, in fact, the Einstein–Hilbert action with particular coupling terms with the dilaton and the $\mathcal{NS}$-$\mathcal{NS}$ field. Indeed, this form for the effective field theory is known with the name of *Einstein frame action*, while the action (5.5) is commonly called the *string frame action.*

In the particular case $D = 4$, the variational principle $\delta S = 0$ imposed to this new form of the effective action leads to the equations of motion

$$\nabla_\mu \partial^\mu \phi + c e^{\phi} - \frac{1}{16} e^{-2\phi} \mathbf{H}^2 = 0 \tag{5.8}$$

$$\nabla_\delta \mathbf{H}^\delta_{\mu\nu} + 2 \nabla_\delta \phi \mathbf{H}^\delta_{\mu\nu} = 0 \tag{5.9}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{c}{2} g_{\mu\nu} e^{\phi} &= \frac{1}{2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) + \\ &+ \frac{1}{4} e^{-2\phi} \left(\mathbf{H}_{\mu\rho\delta} \mathbf{H}^{\rho\delta}_\nu - \frac{1}{6} g_{\mu\nu} \mathbf{H}^2 \right) \end{aligned} \tag{5.10}$$

and the Bianchi identities

$$\nabla_{[\mu} \mathbf{H}_{\mu\rho\delta]} = 0. \tag{5.11}$$

Thus, we obtain in (5.10) the Einstein equations with a cosmological function given by $\Lambda(x) = c e^{\phi(x)}$ and coupling terms with the stress-tensor components of the background fields.

5.2 String cosmological models

The Euler–Lagrange equations yielding from the spacetime action (5.5) admit homogeneous and isotropic solutions in four dimensions [Antoniadis et al. (1988); Tseytlin (1992); Tseytlin&Vafa (1992); Goldwirth&Perry (1994)]. This fact becomes important within the context of the study of the cosmological problem. Indeed, such solutions present a Friedmann–Robertson–Walker form for the metric, namely

$$ds^2 = N(\tau)d\tau^2 - e^{2\Omega(\tau)}\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right). \tag{5.12}$$

On the other hand, for the dilaton ϕ and the field strength $H_{\mu\nu\rho}$ the homogeneity and isotropy constraints demand

$$\begin{aligned} \phi &= \phi(\tau) \\ \mathbf{H}_{ijk} &= \lambda(\tau)\varepsilon_{ijk} \end{aligned} \tag{5.13}$$

where ε_{ijk} is the volume form on the constant-time surfaces and λ is a real number. Note that the requirement of satisfying the Bianchi identities (5.11) inmediately implies that λ does not depend on the parameter τ.

For the case $\lambda = 0$ the Einstein frame action for this system in four dimensions is given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}Ne^{3\Omega}\left[-\frac{\dot{\Omega}^2}{N^2} + \frac{\dot{\phi}^2}{N^2} - 2ce^{\phi} + ke^{-2\Omega}\right] \tag{5.14}$$

(in this chapter we shall use dots to denote derivatives with respect to τ). On the other hand, in the case $k = 0$ we can write

$$S = \frac{1}{2}\int d^4x\sqrt{-g}Ne^{3\Omega}\left[-\frac{\dot{\Omega}^2}{N^2} + \frac{\dot{\phi}^2}{N^2} - 2ce^{\phi} - \lambda^2 e^{-6\Omega-2\phi}\right]. \tag{5.15}$$

In both cases we have absorbed the factor $(8\pi G_N)^{-1}$ by a redefinition of the fields. If we put the action in the Hamiltonian form we obtain

$$S = \int d\tau\left[\pi_\Omega\dot{\Omega} + \pi_\phi\dot{\phi} - N\mathcal{H}\right]. \tag{5.16}$$

In the case $\lambda = 0$, which corresponds to the two-form field $B_{\mu\nu}$ equal to zero, we obtain the Hamiltonian constraint

$$\mathcal{H}_1 = \frac{1}{2}e^{-3\Omega}\left(-\pi_\Omega^2 + \pi_\phi^2 + 2ce^{6\Omega+\phi} - ke^{4\Omega}\right) \approx 0, \tag{5.17}$$

while for $k = 0$ (flat universe) we obtain the constraint

$$\mathcal{H}_2 = \frac{1}{2}e^{-3\Omega}\left(-\pi_\Omega^2 + \pi_\phi^2 + 2ce^{6\Omega+\phi} + \lambda^2 e^{-2\phi}\right) \approx 0. \tag{5.18}$$

The Hamiltonian form of the action will be useful for our further analysis within the context of the quantization of the theory. The presence of the Hamiltonian constraint reflects that the low energy string theory for gravity has the same reparametrization symmetry of general relativity, and then the problem of time persists in the quantization of cosmological models.

On the other hand, by introducing the cosmological ansatz given by (5.12) and (5.13) in the field equations (5.4) we obtain the cosmological equations, namely

$$c + 6\dot{\Omega}\dot{\phi} - 6\dot{\Omega}^2 - \dot{\phi}^2 + 2\lambda^2 e^{-6\Omega} - 6ke^{-2\Omega} = 0, \tag{5.19}$$

$$\ddot{\phi} - 3\Omega^2 - 3\Omega = 0, \tag{5.20}$$

$$+3\dot{\Omega}^2 - \Omega - \dot{\Omega}\dot{\phi} - 2\lambda^2 e^{-6\Omega} + 2ke^{-2\Omega} = 0. \tag{5.21}$$

These equations admit classical solutions that represent several possible phases of string cosmological models.

An important feature of the cosmological equations (5.21) is the fact that there exists a T-duality symmetry [Veneziano (1991); Gasperini&-Veneziano (1993)] reflected in the transformation $\Omega(\tau) \to -\Omega(-\tau)$. This symmetry establishes a fundamental point in the study of the cosmological problem in string theory: Within the context of string cosmology, the idea of the big bang singularity is replaced by the assumption of the existence of a phase transition of finite curvature in that early epoch. Indeed, this cosmological scenario enables us to consider a pre-big bang cosmology with interesting and non-trivial differences with the standard cosmology [Gasperini (1999); Veneziano (1999)]. A detailed analysis of the classical solutions appearing in string cosmology can be found in the literature [Meissner&Veneziano (1993); Goldwirth&Perry (1994); Gasperini (2000)]. Among these solutions we can find the power behaviour of the scale factor,

$$a(\tau) \equiv e^{\Omega(\tau)} = a_0\tau^\alpha \tag{5.22}$$

being, for example, $\alpha = -1/\sqrt{3},\ 1/\sqrt{3},\ 1/3, ...$ Consecuently, in order to satisfy the field equations (5.21), the dilaton field takes the form

$$\phi(\tau) = \frac{3}{2}\alpha(1-\alpha)\ln(\tau) + \eta\tau + \phi_0. \tag{5.23}$$

Moreover, we can also find a static Einstein universe given by

$$\begin{aligned} a &= a_0 = \sqrt{\lambda}, \\ k &= 1, \\ c &= \eta^2 + \frac{4}{\lambda}, \\ \phi(\tau) &= \eta\tau + \phi_0. \end{aligned} \tag{5.24}$$

Indeed, this solution represents a closed, static, homogeneous and isotropic universe which is possible by the simultaneous existence of the $\mathcal{NS}$-$\mathcal{NS}$ two-form field and the Weyl's anomalous parameter c [Giribet (2001)].

5.3 Path integral quantization

Let us return to the Hamiltonian constraints (5.17) and (5.18), which we shall write in a condensed form as

$$\mathcal{H} = \frac{1}{2}e^{-3\Omega}\left(-\pi_\Omega^2 + \pi_\phi^2 + 2ce^{6\Omega+\phi} - \delta_{\lambda,0}ke^{4\Omega} + \delta_{k,0}\lambda^2 e^{-2\phi}\right) \tag{5.25}$$

with the δ's introduced to consider the cases of a flat model with two-form field different from zero, and a closed or open model with $\lambda = 0$. We shall find a global phase time and give the quantum transition amplitude in the form of a path integral for the models whose Hamilton–Jacobi equation is separable; some results will be shown to be valid also for more general models [Giribet&Simeone (2001a)].

5.3.1 *Gauge-invariant action*

We shall begin our analysis by considering the following generic form for the scaled Hamiltonian $H \equiv 2e^{3\Omega}\mathcal{H}$:

$$H = -\pi_\Omega^2 + \pi_\phi^2 + 4Ae^{n\Omega+m\phi} \approx 0, \tag{5.26}$$

where A is an arbitrary real constant and $m \neq n$. In general, this Hamiltonian is not separable in terms of the original canonical variables. Then we define

$$\begin{aligned} x &\equiv \left(\frac{2}{n+m}\right) e^{(n+m)(\Omega+\phi)/2}, \\ y &\equiv \left(\frac{2}{n-m}\right) e^{(n-m)(\Omega-\phi)/2}, \end{aligned} \tag{5.27}$$

so that dividing H by $(n^2 - m^2)xy > 0$ we can define the equivalent constraint

$$H' \equiv \frac{H}{(n^2 - m^2)xy} = -\pi_x \pi_y + A \approx 0, \tag{5.28}$$

whose corresponding Hamilton–Jacobi equation reads

$$-\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} + A = E'.$$

By reproducing the same steps of section 4.1.3 we find that the canonical variables (Q^i, P_i) are given by

$$\begin{aligned} Q^0 &= -\frac{2e^{(n-m)(\Omega-\phi)/2}}{(n-m)P}, \\ Q &= \frac{2e^{(n+m)(\Omega+\phi)/2}}{(n+m)} - \frac{1}{P^2}\left(\frac{2e^{(n-m)(\Omega-\phi)/2}}{(n-m)}(A - P_0) + \eta T(\tau)\right), \\ P_0 &= 4(\pi_\phi^2 - \pi_\Omega^2)e^{-(n+m)(\phi+\Omega)} + A, \\ P &= \frac{1}{2}(\pi_\Omega + \pi_\phi)e^{-(n+m)(\Omega+\phi)}, \end{aligned}$$

where $\eta = \pm 1$. The coordinates and momenta (Q^i, P_i) describe an ordinary gauge system with a constraint $P_0 = 0$ and a true Hamiltonian $\partial f/\partial\tau = (\eta/P)(dT/d\tau)$ which conmutes with K. Its action is

$$\mathcal{S}[Q^i, P_i, N] = \int_{\tau_1}^{\tau_2} \left(P\frac{dQ}{d\tau} + P_0\frac{dQ^0}{d\tau} - NP_0 - \frac{\eta}{P}\frac{dT}{d\tau}\right) d\tau. \tag{5.29}$$

If we write $\mathcal{S}$ in terms of the original variables we have

$$\mathcal{S}[\Omega, \phi, \pi_\Omega, \pi_\phi, N] = \int_{\tau_1}^{\tau_2} \left(\pi_\phi\frac{d\phi}{d\tau} + \pi_\Omega\frac{d\Omega}{d\tau} - N\mathcal{H}\right) d\tau + B(\tau_2) - B(\tau_1), \tag{5.30}$$

where

$$B(\tau) = \frac{1}{\pi_\Omega + \pi_\phi}\left(\frac{\pi_\phi^2 - \pi_\Omega^2 + 4Ae^{n\Omega+m\phi}}{n-m}\right) + \frac{4Ae^{(n+m)(\Omega+\phi)/2}}{\pi_\phi + \pi_\Omega}\left(\frac{2e^{(n-m)(\Omega-\phi)/2}}{n-m} + \eta\frac{T(\tau)}{A}\right).$$

As $\pi_x = P = (1/2)(\pi_\Omega + \pi_\phi)e^{-(n+m)(\Omega+\phi)/2}$ we can write

$$B(\tau) = -Q^0 P_0 - 2A\left(Q^0 - \eta\frac{T(\tau)}{AP}\right).$$

Under a gauge transformation generated by $\mathcal{H}$ we have $\delta_\varepsilon B = -\delta_\varepsilon S$, so that the action $\mathcal{S}$ is effectively endowed with gauge invariance over the whole trajectory and canonical gauge conditions are admissible.

5.3.2 *Extrinsic time*

A global phase time t must verify $[t, \mathcal{H}] > 0$, but as $\mathcal{H} = \mathcal{F}(\Omega,\phi)H' = \mathcal{F}(\Omega,\phi)P_0$ with $\mathcal{F} > 0$, then if t is a global phase time we also have $[t, P_0] > 0$. Because $[Q^0, P_0] = 1$, an extrinsic time can be identified by imposing a $\tau-$dependent gauge of the form

$$\chi \equiv Q^0 - T(\tau) = 0$$

and defining

$$t \equiv T.$$

We then obtain

$$t(\Omega, \phi, \pi_\Omega, \pi_\phi) = \frac{4e^{n\Omega+m\phi}}{(m-n)(\pi_\Omega + \pi_\phi)}. \tag{5.31}$$

Using the constraint equation (5.26) we can write

$$t(\pi_\Omega, \pi_\phi) = \frac{\pi_\phi - \pi_\Omega}{(n-m)A}.$$

For the scaled constraint $H = 2e^{3\Omega}\mathcal{H}$ with $k = \lambda = 0$ we have $4A = 2c$, $n = 6$, $m = 1$. Then the extrinsic time is

$$t(\Omega, \phi, \pi_\Omega, \pi_\phi) = -\frac{4e^{6\Omega+\phi}}{5(\pi_\Omega + \pi_\phi)}. \tag{5.32}$$

We can go back to the constraint H with $k \neq 0$ and evaluate $[t, H]$. For an open model ($k = -1$) a simple calculation gives that $[t, H] > 0$ for both $c < 0$ and $c > 0$. For the case $k = 1$, instead, an extrinsic global phase time is

$$t(\pi_\Omega, \pi_\phi) = \frac{2}{5c}(\pi_\phi - \pi_\Omega)$$

if $c < 0$.

In the case of the scaled constraint with $c = k = 0$ we have $4A = \lambda^2$, $n = 0$, $m = -2$, and the extrinsic time reads

$$t(\Omega, \phi, \pi_\Omega, \pi_\phi) = -\frac{2e^{-2\phi}}{\pi_\Omega + \pi_\phi}. \tag{5.33}$$

If we then consider $c \neq 0$ and we compute the Poisson bracket $[t, H]$ we find that this is positive definite if $c < 0$. Hence the time given by (5.33) is a global phase time for this case. In fact, a simple prescription can be given to determine whether an extrinsic time for a system described by a given Hamiltonian is also a time for a system described by a more general constraint. We have defined $H' = g^{-1}(q)H$ with $g > 0$, and because we matched $P_0 \equiv H'$, then $t \equiv Q^0$ fulfills $[t, H'] = 1$ (and then $[t, H] = g > 0$ on the surface $H = 0$). If we consider an extended constraint $\tilde{H} = g(q)H' + h$ and we calculate the bracket of t with $\tilde{H}$ we obtain

$$[t, \tilde{H}] = g + H'[t, g] + [t, h].$$

Using that $\tilde{H} \approx 0$ we have that the condition

$$[t, \tilde{H}] = g - g^{-1}h[t, g] + [t, h] > 0$$

must hold on the (new) constraint surface if t is a time for the system described by $\tilde{H}$. For the system associated to the constraint (5.26), from (5.27) and (5.28) we have that $g = 4e^{n\Omega + m\phi}$; if we add a term of the form $h = \alpha e^{r\Omega + s\phi}$ to H the condition turns to be

$$\alpha \frac{e^{r\Omega + s\phi}}{(\pi_\phi + \pi_\Omega)^2} \left[\frac{(n+m) - (r+s)}{n-m}\right] > -1.$$

This analysis could be useful, for example, in the case that a further development of the theory beyond the low energy approximation provides an effective potential for the dilaton which should be included in the Hamiltonian in a description of the earliest stages of the universe.

5.3.3 *Intrinsic time and path integral*

On the constraint surface $H' = P_0 = 0$ the terms $B(\tau)$ clearly vanish in the canonical gauge

$$\chi \equiv \eta Q^0 - \frac{T(\tau)}{AP} = 0 \tag{5.34}$$

which is equivalent to $T(\tau) = \pm 2(m-n)^{-1} A e^{(n-m)(\Omega-\phi)/2}$, and then it defines $\tau = \tau(\Omega, \phi)$. As $Q^0 P = -y(\Omega, \phi)$, an intrinsic time t can be defined as

$$t \equiv \eta Q^0 P/2$$

if we apropriately choose η. We have

$$[t, H'] = \frac{\eta}{2}[Q^0 P, P_0] = \frac{\eta}{2}P,$$

and because $P = \pi_x$ then to ensure that t is a global phase time we must choose $\eta = sign(\pi_x) = sign(\pi_\Omega + \pi_\phi)$.

In the case $A > 0$ it is $|\pi_\Omega| > |\pi_\phi|$ (so that $sign(\pi_x) = sign(\pi_\Omega)$) and the constraint surface splits into two disjoint sheets identified by the sign of π_Ω; in the case $A < 0$ it is $|\pi_\phi| > |\pi_\Omega|$ and the two sheets of the constraint surface are given by the sign of π_ϕ. Hence in both cases η is determined by the sheet of the constraint surface on which the system evolves; we have that for $A > 0$ the intrinsic time can be written as

$$t(\Omega, \phi) = \left(\frac{1}{m-n}\right) sign(\pi_\Omega) e^{(n-m)(\Omega-\phi)/2}, \tag{5.35}$$

while for $A < 0$ we have

$$t(\Omega, \phi) = \left(\frac{1}{m-n}\right) sign(\pi_\phi) e^{(n-m)(\Omega-\phi)/2}. \tag{5.36}$$

For the constraint with $k = \lambda = 0$ the intrinsic time is

$$t(\Omega, \phi) = -\frac{1}{5} sign(\pi_\Omega) e^{5(\Omega-\phi)/2} \qquad \text{if} \qquad c > 0,$$

and

$$t(\Omega, \phi) = -\frac{1}{5} sign(\pi_\phi) e^{5(\Omega-\phi)/2} \qquad \text{if} \qquad c < 0.$$

By evaluating the Poisson bracket $[t, H]$ for H with $k \neq 0$ we find that the intrinsic time obtained in the case $c > 0$ is also a time for an open model ($k = -1$), and the time for $c < 0$ is a time also for $k = 1$.

In the case of the constraint with $c = k = 0$ we obtain

$$t(\Omega, \phi) = -\frac{1}{2} sign(\pi_\Omega) e^{(\Omega - \phi)},$$

and a simple calculation shows that this is also a global phase time for a more general model with $c > 0$.

Because we have shown that there is a gauge such that $\tau = \tau(q^i)$ and which makes the end point terms vanish, we can obtain the amplitude for the transition $|\Omega_1, \phi_1\rangle \rightarrow |\Omega_2, \phi_2\rangle$ by means of a path integral in the variables (Q^i, P_i) with the action (5.29). This integral is gauge invariant, so that we can compute it in any canonical gauge. According to (3.25), on the constraint surface $P_0 = 0$ and with the gauge choice (5.34), the transition amplitude is

$$\langle \phi_2, \Omega_2 | \phi_1, \Omega_1 \rangle = \int DQDP \exp\left[i \int_{T_1}^{T_2} \left(PdQ - \frac{\eta}{P} dT \right) \right], \tag{5.37}$$

where the end points are given by

$$T_a = \pm \left(\frac{2A}{m - n} \right) e^{(n-m)(\Omega_a - \phi_a)/2}$$

$(a = 1, 2)$; because on the constraint surface and in gauge (5.34) the true degree of freedom reduces to $Q = x$, then the boundaries of the paths in phase space are

$$Q_a = \left(\frac{2}{n + m} \right) e^{(n+m)(\Omega_a + \phi_a)/2}.$$

For the Hamiltonian with $\lambda = 0$ and null curvature the end points are given by $T_a = \mp(c/5)e^{5(\Omega_a - \phi_a)/2}$, while $Q_a = (2/7)e^{7(\Omega_a + \phi_a)/2}$. In the case of $c = k = 0$ we have $T_a = \mp(\lambda^2/4)e^{(\Omega_a - \phi_a)}$ and $Q_a = -e^{-(\Omega_a + \phi_a)}$. After the gauge fixation we have obtained the path integral for a system with one physical degree of freedom. An interesting point to be noted is the following: although we have identified the time as a function of both ϕ and Ω we find that, depending on the sign of A, also Ω or ϕ alone can play the role of clock for the system. For $A > 0$ we have $\pi_\Omega \neq 0$ and for $A < 0$ we have $\pi_\phi \neq 0$, so that we can also define

$$\begin{aligned} t_1^* &= -\Omega\, sign(\pi_\Omega) \qquad \text{if} \qquad A > 0, \\ t_2^* &= +\phi\, sign(\pi_\phi) \qquad \text{if} \qquad A < 0. \end{aligned}$$

This can make more clear the interpretation of the transition amplitude $\langle \phi_2, \Omega_2 | \phi_1, \Omega_1 \rangle$ given by (5.37).

In the particular case $\lambda = 0$, $k = -1$, $c = 0$, which corresponds to an open model with null cosmological function, we obtain a Hamiltonian constraint analogous to that of section 4.1.2,

$$\mathcal{H} = \frac{1}{2} e^{-3\Omega}(-\pi_\Omega^2 + \pi_\phi^2) + e^{\Omega} \approx 0.$$

This model can then be deparametrized and quantized in a completely analogous way, so that the time can be defined as

$$t = -\Omega\, sign(\pi_\Omega),$$

while the infinitesimal propagator can be explicitly calculated, and when written in terms of the Hankel functions (see section 4.1.2) it reads

$$\begin{aligned} \langle \phi_2, \Omega_1 + \varepsilon | \phi_1, \Omega_1 \rangle &= \\ & \pm \frac{\varepsilon\; e^{2\Omega_1}}{\sqrt{\varepsilon^2 - (\phi_2 - \phi_1)^2}} H_1^{(1)}\left(2e^{2\Omega_1}\sqrt{\varepsilon^2 - (\phi_2 - \phi_1)^2}\right). \end{aligned}$$

The double sign corresponds to the two sheets of the constraint surface given by the sign of π_Ω. (Observe that this procedure has the aforementioned problem of the time-dependent potential).

5.3.4 *Summary*

We have analised string cosmological models of two types: 1) models with homogeneous dilaton field and vanishing antisymmetric $B_{\mu\nu}$ field ($\lambda = 0$); 2) models representing flat universes ($k = 0$) with homogeneous dilaton and non vanishing antisymmetric field. For the cases considered we have been able to identify a global phase time. In the cases $\lambda = 0$, $k = 0$, $c \neq 0$ and $\lambda \neq 0$, $k = 0$, $c = 0$ the Hamiltonian is easily separable and the potential has a definite sign, so that the transition amplitude has been obtained straightforwardly. Once we have found a time t for the inmediately separable models, we have identified the extended region of the parameter space where t is a global phase time. We can summarize the results as follows:

- When $\lambda = 0$ the Hamiltonian takes the form

$$\mathcal{H}_1 = \frac{1}{2}e^{-3\Omega}\left(-\pi_\Omega^2 + \pi_\phi^2 + 2ce^{6\Omega+\phi} - ke^{4\Omega}\right) \approx 0.$$

For the models described by this constraint, the times for the different cases are given by

$$t(\Omega,\phi,\pi_\Omega,\pi_\phi) = -\frac{4e^{6\Omega+\phi}}{5(\pi_\Omega+\pi_\phi)} \qquad \text{if} \qquad k=-1,0$$

$$t(\pi_\Omega,\pi_\phi) = \frac{2}{5c}(\pi_\phi - \pi_\Omega) \qquad \text{if} \qquad \left\{ \begin{array}{l} c>0,\ k=-1,0 \\ c<0,\ k=0,1 \end{array} \right.$$

$$t(\Omega,\phi) = \left\{ \begin{array}{lll} -(1/5)sign(\pi_\Omega)e^{5(\Omega-\phi)/2} & \text{if} & c>0,\ k=-1,0 \\ -(1/5)sign(\pi_\phi)e^{5(\Omega-\phi)/2} & \text{if} & c<0,\ k=0,1. \end{array} \right.$$

- If we consider the case $k = 0$, the Hamiltonian becomes

$$\mathcal{H}_2 = \frac{1}{2}e^{-3\Omega}\left(-\pi_\Omega^2 + \pi_\phi^2 + 2ce^{6\Omega+\phi} + \lambda^2 e^{-2\phi}\right) \approx 0.$$

For these models, we can classify the global phase times in the following way:

$$t(\Omega,\phi,\pi_\Omega,\pi_\phi) = -\frac{2e^{-2\phi}}{\pi_\Omega+\pi_\phi} \qquad \text{if} \qquad c\le 0$$

$$t(\pi_\Omega,\pi_\phi) = \frac{2}{\lambda^2}(\pi_\phi - \pi_\Omega) \qquad \text{if} \qquad c\ge 0$$

$$t(\Omega,\phi) = -\frac{1}{2}sign(\pi_\Omega)e^{(\Omega-\phi)} \qquad \text{if} \qquad c\ge 0.$$

The intrinsic time found for the case $k = 0,\ \lambda = 0$ is a time also for the case $k = 1,\ c < 0$ and for $k = -1,\ c > 0$. The extrinsic times identified for $k = 0,\ \lambda = 0$ also deparametrize the more general models with non vanishing curvature. When both λ and c are different from zero, the models admit as global phase times those which were found for the case $c = 0$.

We have restricted our analysis to the formal aspects of minisuperspace quantization. It must be emphasized that a complete discussion about the limits of such approximation as well as an analysis of the application of our method to the interesting problems posed by string theory for the earliest stages of the universe would require a detailed knowledge of the effective potential for the dilaton (see Chapter 7).

Chapter 6

Canonical quantization

In this chapter we discuss the usual canonical quantization procedure applied to simple, relativistic and string, cosmological models. Besides giving a very brief review of different procedures, our aim is also to show how the developments of the previous chapters can be useful even if we do not quantize a cosmological model by means of a path integral. We begin by reviewing an analysis on an isotropic relativistic model considered by Halliwell, in which approximate solutions are found for different regions of phase space. We follow with a formulation based in a Schrödinger equation for isotropic models with different matter fields, which are reduced by means of gauge fixation. Then we discuss some different approaches to the canonical quantization of the Taub universe, with and without deparametrization, one of them addressing the problem of boundary conditions. We give our solution for the Wheeler–DeWitt equation for this model working with a clear notion of time, so that the wave function has an evolutionary form; in particular, we show that in our picture the role of the original momenta is confined to appearing in the time variable. Finally, we solve the Wheeler–DeWitt equation with a coordinate playing the role of time in the case of a generic isotropic string cosmology, and we discuss the choice of the physical solutions determined by the existence of a free limit in the theory; we also give a Schrödinger equation for these models, and analyse the relation existing between a right choice of time and the obtention of a unitary quantization.

6.1 Approximate solutions of the Wheeler–DeWitt equation

Consider the Hamiltonian constraint of a closed ($k = 1$) homogeneous and isotropic universe with a scalar field ϕ and null cosmological constant; assume a generic dependence of the potential with ϕ, namely $V(\phi)$. The associated Wheeler–DeWitt equation obtained by replacing $p \to -i\partial/\partial q$ (and considering the trivial factor ordering) reads

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\phi^2} + V(\phi)e^{6\Omega} - e^{4\Omega}\right)\Psi(\Omega, \phi) = 0. \tag{6.1}$$

Halliwell has analysed the region of phase space such that $|V'/V| \ll 1$ and found solutions whose variation with the matter field is small, so that the ϕ derivative can be neglected. In the region where the scale factor is small the resulting WKB solutions have the exponential form [Halliwell (1990)]

$$\Psi(\Omega, \phi) \sim \exp\left(\pm\frac{1}{3V(\phi)}(1 - e^{2\Omega}V(\phi))^{3/2}\right), \tag{6.2}$$

and are associated to a classically forbidden region. When the scale factor is large, the WKB solutions have the oscillatory form

$$\Psi(\Omega, \phi) \sim \exp\left(\pm\frac{i}{3V(\phi)}(e^{2\Omega}V(\phi) - 1)^{3/2}\right). \tag{6.3}$$

The latter corresponds to what is usually considered the classicaly allowed region. Both kinds of solution can be matched by means of the common WKB matching procedure. In the case $e^{2\Omega}V(\phi) \ll 1$ it can be shown that the oscillatory wave function is peaked about a solution of the form

$$e^{\Omega} \sim e^{\sqrt{V}\tau}, \qquad \phi \sim \phi_0,$$

which corresponds to an inflationary behaviour. (For the case $V(\phi)$=0 an exact solution can be easily obtained as a combination of modified Bessel functions; a comparison would then be possible by considering their assymptotic behaviour. This is also the case if we set $V(\phi)$=0 in a flat ($k = 0$) model with nonzero cosmological constant). Note that, depending on the form of $V(\phi)$, the regions considered by Halliwell may be related to those to which the analysis should be restricted if one was to define an intrinsic time in the case of models for which this cannot be done globally.

6.2 Gauge fixation and Schrödinger equation for isotropic models

An interesting approach within the canonical quantization framework is that followed in [Cavaglià et al. (1995)]. In a line of work analogous to that of Barvinsky and Ponomariov [Barvinsky&Ponomariov (1986)], canonical gauge fixing is used to reduce the system: one degree of freedom is given as a function of the remaining ones and the time parameter τ, and a true (the authors call it "effective") Hamiltonian is obtained; this Hamiltonian may in general depend on the time parameter. The criterion for the gauge choice is the simplicity of the Hamiltonian for the reduced system. Once the reduction is performed, then the system is quantized in the reduced canonical phase space; this is achieved by writing a τ-dependent Schrödinger equation. In a given gauge, the time parameter is connected to the canonical degree of freedom that has been eliminated.

The authors analyse a Friedmann–Robertson–Walker universe with matter in the form of a conformal scalar field (CS) and of a SU(2) Yang–Mills field (YM) [Cavaglià&De Alfaro (1994)]. They define the corresponding Hamiltonian for each field as

$$\begin{aligned} H_{CS} &= \frac{1}{2}\left(\pi_\chi^2 + V(\chi)\right) \\ H_{YM} &= \frac{1}{3}\left(\frac{1}{2}\pi_\xi^2 + V(\xi)\right), \end{aligned}$$

so that if H_{GR} is the pure gravitation Hamiltonian, the constraint is

$$-H_{GR} + H_{CS} + H_{YM} \approx 0. \tag{6.4}$$

Then different gauge choices and the resulting Schrödinger equations are explored. For a gauge condition in terms of the gravitational degree of freedom like [Filippov (1989)]

$$\pi_\Omega + \frac{1}{12}e^\Omega \cot\tau = 0$$

the equation

$$i\frac{\partial}{\partial\tau}\Psi(\xi,\chi,\tau) = (H_{CS} + H_{YM})\Psi(\xi,\chi,\tau) \tag{6.5}$$

is obtained; its solution gives a wave function for both matter fields. A rather different choice connects the conformal field with the time parameter:

$$\pi_\chi - \chi \cot \tau = 0.$$

This gauge leads to a Schrödinger equation for the metric and the Yang–Mills field:

$$i\frac{\partial}{\partial\tau}\Psi(\xi,\Omega,\tau) = (H_{YM} - H_{GR})\Psi(\xi,\Omega,\tau). \tag{6.6}$$

An explicit solution is given for the simple case of a closed universe with a scalar field ϕ with $V(\phi) = 0$. The gauge condition

$$\pi_\Omega - 12e^\Omega \sinh\left(\frac{\tau}{\sqrt{3}}\right) = 0$$

yields the equation

$$\left(\frac{\partial}{\partial\tau} \mp \frac{\partial}{\partial\phi}\right)\Psi(\phi,\tau) = 0 \tag{6.7}$$

for the only physical degree of freedom ϕ. The solutions are of the form

$$\Psi(\phi,\tau) = f(\phi \pm \tau). \tag{6.8}$$

A particular solution is $\Psi(\phi,\tau) = Ae^{-(\phi\pm\tau)^2/2\sigma}$, which represents a universe whose maximum probability follows the classical path $\phi = \pm\tau$.

It is something to be noted that if the gauge conditions were globally well defined, so that the variables entering the gauge fixation defined a global phase time, this procedure would be rather similar to which we give in section 6.3.3 for the Taub universe (see below).

6.3 The Taub universe

6.3.1 *Standard procedure*

In the literature we can find different solutions for the Taub universe. An important example among those which do not start from an explicit deparametrization is the solution found by Moncrief and Ryan [Moncrief&-Ryan (1991)] in the context of an analysis of the Bianchi type-IX universe with a rather general factor ordering of the Hamiltonian constraint [Hartle&Hawking (1983)]. In the case of the most trivial ordering they solved the Wheeler–DeWitt equation to obtain a wave function which they give in the integral form

$$\Psi(\Omega,\beta_+) = \int_0^\infty d\omega F(\omega) K_{i\omega}\left(\frac{1}{6}e^{2\Omega-4\beta_+}\right) K_{2i\omega}\left(\frac{2}{3}e^{2\Omega-\beta_+}\right), \qquad (6.9)$$

with $K_{i\omega}$ modified Bessel functions of imaginary argument (the modified functions I are discarded because they are not well behaved for $\beta_+ \to \pm\infty$). In the particular case that $F(\omega) = \omega \sinh(\pi\omega)$ they have shown that the wave function can be written in the form (see also [Martinez&Ryan (1983)])

$$\Psi(\Omega,\beta_+) = R(\Omega,\beta_+)e^{-S} \qquad (6.10)$$

with

$$S = \frac{1}{6}e^{2\Omega}\left(e^{-4\beta_+} + 2e^{2\beta_+}\right).$$

An important feature of this wave function is that for values of Ω near the singularity (that is, the scale factor near zero) the probability is spread over all possible degrees of anisotropy given by β_+, while for large values of the scale factor the probability is peaked around the isotropic Friedmann–Robertson–Walker closed universe (the authors, however, prevent from a naif interpretation of the wave function, and they note that there are different probability interpretations that would not agree with this one).

6.3.2 *Boundary conditions and Schrödinger equation*

A rather different approach beginning with the identification of a global phase time can be found in a recent work [Catren&Ferraro (2001)], in which the authors obtain a Schrödinger equation and its solutions are used to select a set of solutions of the Wheeler–DeWitt equation. The underlying idea is that the typical constraint of a parametrized system, which is linear in the momentum conjugated to the true time, is hidden in the formalism of gravitation. This is an extension of the analogy between the ideal clock and empty isotropic models [Beluardi&Ferraro (1995); Ferraro (1999)]: The constraint of the ideal clock

$$\mathcal{H} = p_t - t^2 \approx 0$$

yields the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = -t^2\Psi \tag{6.11}$$

which is of parabolic form, and it has the only solution $\Psi = e^{it^3/3}$. As a first step to obtain the constraint of a minisuperspace, a canonical transformation leading to a constraint quadratic in the momenta is performed: defining $Q = p_t$, $P = -t$, we obtain

$$\mathcal{H} = -P^2 + Q \approx 0.$$

(The Hamiltonian of empty isotropic models results from the second transformation $Q = \tilde{V}(\Omega)$, $P = \pi_\Omega(d\tilde{V}/d\Omega)^{-1}$, with $\tilde{V}$ the potential defined in Section 3.4.3). The differential equation associated to the constraint is now of hyperbolic form:

$$\frac{\partial^2\Psi}{\partial Q^2} + Q\Psi = 0. \tag{6.12}$$

As this equation is of second order, it has two independent solutions, which are the Airy functions $Ai(-Q)$ and $Bi(-Q)$. The central point is that while $Bi(-Q)$ diverges for $Q \to -\infty$, $Ai(-Q)$ is well behaved (in fact, it vanishes) in this limit, and it is the Fourier transform of the solution of Eq. (6.11). This provides a criterion for selecting solutions of the hyperbolic equation: the physical solutions would be those which are in correspondence with the solutions of the Schrödinger equation.

This line is then followed for quantizing minisuperspaces with true degrees of freedom. In the case of the Taub universe, the authors start from a Hamiltonian like that given in Eq. (4.72) (with a different choice of the constants), and solve a Wheeler–DeWitt equation like

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{1}{9}e^{4x} + \frac{4}{9}e^{2y}\right)\Psi(x,y) = 0 \tag{6.13}$$

as it was done by Moncrief and Ryan. They obtain a set of solutions of the form

$$\begin{aligned}\Psi_\omega(x,y) &= \left[a(\omega)I_{i\omega}\left(\frac{2}{3}e^y\right) + b(\omega)K_{i\omega}\left(\frac{2}{3}e^y\right)\right] \\ &\quad\times\left[c(\omega)I_{i\omega/2}\left(\frac{1}{6}e^{2x}\right) + d(\omega)K_{i\omega/2}\left(\frac{1}{6}e^{2x}\right)\right],\end{aligned} \tag{6.14}$$

with I and K the modified Bessel functions. Then they consider a canonical transformation analogous to (4.74) but with only the minus sign, so that the momentum π_s is negative definite, and the time is then $t = s$; hence in Eq. (4.76) the first factor is positive definite, and the second one is a constraint linear in $\pi_s = \pi_t$ and including a true Hamiltonian $h = \sqrt{\pi_x^2 + (1/9)e^{4x}}$ which does not depend on time (this feature makes possible the equivalence of the linear constraint and the original quadratic one; see below). This constraint then leads to the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\partial^2}{\partial x^2} + \frac{1}{9}e^{4x}\right)^{1/2} \Psi(x,t). \tag{6.15}$$

It is necessary a prescription to give a precise meaning to the Hamiltonian operator; the square root containing the derivative operator must be understood as its binomial expansion, which allows to propose solutions of the form $\sim \phi(x)e^{-i\omega t}$. According to this interpretation, the contribution of the functions $I_{i\omega/2}((1/6)e^{2x})$ is discarded, because they diverge in the classicaly forbidden region associated to the exponential potential $\frac{1}{9}e^{4x}$; the functions $I_{i\omega}((2/3)e^{y})$, instead, are not discarded, because in this picture the coordinate y is associated to the definition of time. In fact, by transforming the solutions of the Wheeler–DeWitt equation it is shown that those corresponding to the solutions of the Schrödinger equation are precisely the functions $I_{i\omega}((2/3)e^{y})$, while the functions $K_{i\omega}((2/3)e^{y})$ must be ruled out because they cannot be associated to definite energy states of the true Hamiltonian h. It is remarkable that the functions in the selected subspace do not decay in the classically forbidden zone (note the difference with the result of the preceding section).

6.3.3 *Wheeler–DeWitt equation with extrinsic time*

Our idea is to apply some results of the deparametrization proposal given in the preceding chapters to the usual canonical quantization procedure. Then we start from a form of the Hamiltonian constraint such that a global phase time is easily identified as one of the canonical coordinates; this is reflected in the corresponding Wheeler–DeWitt equation, and hence, the resulting wave function has an evolutionary form and it may be interpreted as it is in ordinary quantum mechanics (see, however, the Discussion in Chapter 7).

The constraint (4.75) allows to inmediately define the time as

$$t = -s\, sign(\pi_s).$$

As we showed in Section 4.2.2, this time yields from a simple canonical gauge choice, which in the variables $\tilde{q}^i$ has the form $s = \eta T(\tau)$, $\eta = \pm 1$. We make the usual substitution $p_k \to -i\partial/\partial q^k$ to obtain the Wheeler–DeWitt equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial s^2} - \frac{1}{9} e^{4x} \right) \Psi(x, s) = 0. \tag{6.16}$$

If we propose a solution of the form $\Psi(x, s) = A(x)B(s)$ we obtain

$$\frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{9} e^{4x} = -\omega^2 = \frac{1}{B}\frac{\partial^2 B}{\partial s^2}. \tag{6.17}$$

The equation for B is easy to solve, and its solutions are of the form $e^{\pm i\omega s}$; to solve for A we make the substitution $u = (1/6)e^{2x}$ which leads to

$$\frac{\partial^2 A}{\partial u^2} + \frac{1}{u}\frac{\partial A}{\partial u} - \left(1 - \frac{\omega^2}{4u^2}\right) A = 0. \tag{6.18}$$

This is a modified Bessel equation with $\nu^2 = -\omega^2/4$, whose solution is a combination of the modified Bessel functions I_ν and K_ν (we have adopted the convention of [Gradshteyn&Ryshik (1965)]). Thus the Wheeler–DeWitt equation has the set of solutions [Giribet&Simeone (2001c)]

$$\begin{aligned} \Psi_\omega(x, s) &= \left[a(\omega)e^{i\omega s} + b(\omega)e^{-i\omega s}\right] \\ &\quad \times \left[c(\omega) I_{i\omega/2}\left(\frac{1}{6}e^{2x}\right) + d(\omega) K_{i\omega/2}\left(\frac{1}{6}e^{2x}\right)\right], \end{aligned} \tag{6.19}$$

where $\pm s$ is a global phase time. The result can be understood as a set of positive and negative-energy solutions; this is related to the fact that the potential does not depend on time. (The contribution of the functions $I_{i\omega/2}$ should be discarded as they are not well behaved for great values of x).

There are two points which deserve certain analysis: The first is that we have chosen the separation constant as $-\omega^2$; we could choose a positive definite constant, and the functional form of the solutions would be completely different. The reason for our choice is the formal analogy between the constraint for the Taub universe in the variables (x, s, π_x, π_s) and the constraint of some models of string theory, for which we have a natural criterion to select one kind of solution (see below). The second is that we have not discarded negative energy solutions; this would have been equivalent to select one sheet of the constraint surface. As we pointed in the context of path integral quantization, once we decide to work and give the results in terms of the variables (x, s, π_x, π_s) there is no reason to choose one sign for the momentum π_s. Moreover, if we take the quadratic form of the constraint as an essential feature of gravitation, we should only admit a canonical transformation leading to a constraint equivalent to the original one; hence, both signs of Φ_1 must be considered simultaneously to ensure that the original constraint and that in terms of the new coordinates and momenta yield differential equations with the same number of solutions.

One half of the solutions of our Wheeler–DeWitt equation correspond to those of the preceding section, which yielded from a Schrödinger equation. Our procedure allows to obtain them without the necessity of defining a prescription for the square root operator, but only by choosing the trivial factor ordering.

The solution can be easily extended for the case including matter in the form of a scalar field whose mass can be neglected in the dynamics. The addition of the corresponding term π_ϕ^2 in the constraint only modifies the result (6.19) by an oscilatory factor depending on ϕ, so that the solutions of the Wheeler–DeWitt equation are of the form

$$\begin{aligned} \Psi_{\omega,p}(x,\phi,s) &= \left[a(\alpha)e^{i\alpha s} + b(\alpha)e^{-i\alpha s}\right] \\ &\times \left[f(p)e^{ip\phi} + g(p)e^{-ip\phi}\right] \\ &\times \left[c(\omega)I_{i\omega/2}\left(\frac{1}{6}e^{2x}\right) + d(\omega)K_{i\omega/2}\left(\frac{1}{6}e^{2x}\right)\right]. \end{aligned} \tag{6.20}$$

where $\alpha = \sqrt{\omega^2 + p^2}$. As we showed in Chapter 4, the coordinate s is still a time if a massless field is present; hence the evolutionary form of the resulting wave function is preserved.

A point to be remarked is that in this description, the role of the original momenta (unavoidable, provided the topology of the constraint surface in the original variables) is restricted to the global phase time $s = \pm\text{arcsinh}\left(\frac{1}{2}(\pi_\Omega + \pi_+)e^{(-2\Omega+\beta_+)}\right)$; the other coordinates entering the wave function are simple functions of only the original coordinates.

6.4 String cosmologies

As we pointed before, the quantization of string cosmologies is of physical interest not only for the same reasons of relativistic universes, but also because they allow for a conceptually new point of view about the earliest stages of the universe. However, in the context of the present work, we are mainly concerned with formal problems, and in this sense string cosmologies can play an important role, as they will provide a way to choose between the possible solutions of the Wheeler–DeWitt equation. Also, we shall use these models to reproduce an early analysis by Hájícek in terms of a Schrödinger equation, which illustrates some difficulties that one finds in the search for a unitary quantization [Hájícek (1986)].

6.4.1 *Wheeler–DeWitt equation*

We shall begin by finding the solutions of the Wheeler–DeWitt equation of cosmological models in those particular cases in which the rescaled Hamiltonian takes the form

$$H = -\pi_\Omega^2 + \pi_\phi^2 + 4Ae^{n\Omega+m\phi} \tag{6.21}$$

This generic form of the constraint includes the following relevant cases: $\{2A = c,\, n = 6,\, m = 1\}$, $\{4A = \lambda^2,\, n = 0,\, m = -2\}$ and the quoted $\{4A = -k,\, n = 4,\, m = 0\}$. Now we define the new variables x and y as

$$\begin{aligned} x &\equiv \frac{1}{2}(n\Omega + m\phi) \\ y &\equiv \frac{1}{2}(m\Omega + n\phi)\,. \end{aligned} \tag{6.22}$$

Since in string theory we have $n > m$, we can rescale the Hamiltonian in the following form

$$H \to \frac{4}{n^2 - m^2} H \tag{6.23}$$

and thus we obtain the equivalent constraint

$$H = -\pi_x^2 + \pi_y^2 + \zeta e^{2x} \tag{6.24}$$

where $\zeta = 16A/(n^2 - m^2)$. Now, let us define $u = \sqrt{|\zeta|}e^x$. In terms of u and y we can write the Wheeler-DeWitt equation in the form

$$\left(u^2\frac{d^2}{du^2} + u\frac{d}{du} + sign(\zeta)u^2 - \frac{d^2}{dy^2}\right)\Psi(u,y) = 0 \tag{6.25}$$

This equation clearly admits a set of solutions of the form $\Psi = A(u)B(y)$; returning to the variable x, for the case $sign(\zeta) > 0$ we obtain

$$\begin{aligned}\Psi_\omega(x,y) &= \left[a_+(\omega)e^{i\omega y} + a_-(\omega)e^{-i\omega y}\right] \\ &\quad\times\left[b_+(\omega)J_{i\omega}(\sqrt{|\zeta|}e^x) + b_-(\omega)N_{i\omega}(\sqrt{|\zeta|}e^x)\right],\end{aligned} \tag{6.26}$$

being $J_{i\omega}$ and $N_{i\omega}$ the Bessel and Neumann functions of imaginary order respectively; for the case $sign(\zeta) < 0$, the solutions are of the form

$$\begin{aligned}\Psi_\omega(x,y) &= \left[a_+(\omega)e^{i\omega y} + a_-(\omega)e^{-i\omega y}\right] \\ &\quad\times\left[b_+(\omega)I_{i\omega}(\sqrt{|\zeta|}e^x) + b_-(\omega)K_{i\omega}(\sqrt{|\zeta|}e^x)\right],\end{aligned} \tag{6.27}$$

where $I_{i\omega}$ and $K_{i\omega}$ are, as before, the modified Bessel functions. In the case $\zeta < 0$ the momentum π_y does not vanish on the constraint surface; then, up to a sign determined by the sign of π_y, the coordinate y is a global phase time, and we could separate the functions in (6.27) as positive and negative-energy solutions. In the case $\zeta > 0$ a global phase time is $\pm x$, and the possibility of such an interpretation becomes not so clear: instead of the usual factors $\sim e^{i\omega t}$ associated to definite-energy states, for $\zeta > 0$ the time dependence appears in the argument of Bessel functions.

Note that the form of the solutions is determined by the fact that, in order to obtain the free field solution of the Wheeler–DeWitt equation in the limit $A \to 0$, we have to consider the subset $\omega \in R$ in the solutions of the resulting Bessel equation [Giribet&Simeone (2001c)]. This is equivalent to the choice of the separation constant equal to $-\omega^2/4$ in the case of the Taub universe. Although the free limit (vanishing potential) has no meaning for the Taub model, we want to obtain analogous solutions for Hamiltonians of similar form.

It is interesting to observe that if we substitute ϕ by β and we put $A = -1/4$, $n = 4$ and $m = 2$ in the Hamiltonian (6.21), we obtain the constraint of the Kantowski–Sachs universe. Thus we have a wave function for this model as a superposition of solutions of the form (6.27), with the coordinate $y = 4\Omega + 2\beta$ playing the role of time. This functional form coincides with that found by Fishbone without deparametrizing the system [Ryan&Shepley (1975)].

6.4.2 *Schrödinger equation*

We have just seen that, depending on the sign of the constant ζ in the constraint (6.21), these models admit as global phase time the coordinates x or y. In the case $\zeta > 0$ the time is $t = \pm x$, so that following Ref. [Hájícek (1986)] we can define the reduced Hamiltonians as $h_\pm = \pm\sqrt{\pi_y^2 + \zeta e^{2x}}$, and we can write the Schrödinger equations

$$i\frac{\partial}{\partial x}\Psi(x,y) = \mp\left(-\frac{\partial^2}{\partial y^2} + \zeta e^{2x}\right)^{1/2}\Psi(x,y) \tag{6.28}$$

(note that in this case we obtain a time-dependent potential). If, instead, we have $\zeta < 0$, the time is $t = \pm y$ and the reduced Hamiltonians corresponding to each sheet of the constraint surface are $h_\pm = \pm\sqrt{\pi_x^2 - \zeta e^{2x}}$; the associated Schrödinger equations are

$$i\frac{\partial}{\partial y}\Psi(x,y) = \mp\left(-\frac{\partial^2}{\partial x^2} - \zeta e^{2x}\right)^{1/2}\Psi(x,y). \tag{6.29}$$

As we mentioned in Section 6.3.2, the square root operator requires a prescription. According to Ref. [Hájícek (1986)], for both $\zeta > 0$ and $\zeta < 0$ we woud have a pair of Hilbert spaces, each one with its corresponding Schrödinger equation; a possible physical state would be given by a pair of wave functions, one of each Hilbert space. This is analogous to the obtention of two quantum propagators, one for each disjoint theory, mentioned in the context of path integral quantization. In both cases the reduced Hamiltonians are real, so that the evolution operator is self-adjoint and the resulting quantization is unitary. Note that a crucial point has been the right choice of time coordinate; a wrong choice, like for example $t = \pm x$ in the case $\zeta < 0$, leads to an imaginary Hamiltonian for the reduced system, and we obtain a nonunitary theory.

There is a point, however, which should be signaled. For each case $\zeta < 0$ and $\zeta > 0$ the constraint can be written as a product of two linear constraints, and each of these constraints leads to a Schrödinger equation.

In the case $\zeta < 0$, $t = \pm y$, the two Schrödinger equations are equivalent to the Wheeler–DeWitt equation, as they come from classical Hamiltonian constraints which in its operator version have the same form. This is so because the potential in the reduced Hamiltonians does not depend on time. In the case $\zeta > 0$, instead, we have $t = \pm x$ and the potential depends on time. Then the product leading to the corresponding two Schrödinger equations reads:

$$H = \left(-\pi_x + \sqrt{\pi_y^2 + \zeta e^{2x}}\right)\left(\pi_x + \sqrt{\pi_y^2 + \zeta e^{2x}}\right) \approx 0. \tag{6.30}$$

At the classical level this product is equivalent to the constraint (6.21); but in its operator version both constraints differ in terms associated to conmutators between π_x and the potential ζe^{2x}. Hence, depending on which of the two classicaly equivalent constraints we started from, we would obtain different quantum theories; looking for a correspondence between solutions of both equations as it was done with the Taub universe would therefore make no sense. Note that this problem appears in the case for which the Wheeler–DeWitt equation leads to a result in which the identification of positive and negative-energy solutions is not apparent, at least in the standard form (see the preceding Section). It is not completely clear why we should prefer one of both theories; however, the Wheeler–DeWitt equation associated to the quadratic original form of the constraint is the usual choice.

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Chapter 7

Discussion

The difficulty in defining a set of observables and a notion of dynamical evolution in a theory where the spacetime metric is itself a dynamical variable, as it is the case of General Relativity –and also of string theory, at least in its effective low energy form–, leads to the problem of time in quantum cosmology. In the Hamiltonian formalism for the gravitational field this difficulty is reflected in the fact that the dynamical evolution can be reproduced by gauge transformations generated by the Hamiltonian constraint $\mathcal{H} = 0$. As the wave function of the universe must be a solution of $\mathcal{H}\Psi = 0$, then it gives only a correlation between the canonical coordinates, but it is devoided of an evolutionary form. Also, the absence of a distinction between true degrees of freedom and time makes difficult to define a conserved inner product for the quantum states.

This situation has led to different programs of deparametrization or reduction to physical degrees of freedom as a previous step before quantization, as for example [Hájícek (1986); Barvinsky&Ponomariov (1986); Barvinsky (1986); Barvinsky (1993); Kuchař (1993); Wald (1993); Higuchi&-Wald (1995); Cavaglià et al. (1995); Beluardi&Ferraro (1995); Ferraro (1999); Simeone (1999)]. The practical difficulties found in the construction of the reduced phase space have led some people to give up the attempt, and to suggest that the separation of dynamical variables and time may be impossible to be carried out for the full theory. However, things are rather different in the minisuperspace approach, where one deals with cosmological models with a finite number of degrees of freedom. Here we have presented a proposal based in the identification of a global phase time by first turning the action of cosmological models into that of an ordinary gauge system and then imposing canonical gauge conditions. The deparametrized models can then be quantized both with the Fadeev–Popov path integral procedure and with the usual canonical Dirac–Wheeler–DeWitt procedure. In the first case we obtain a quantum propagator for the reduced system with a formally correct notion of time; in the second case, the result is a wave function with an evolutionary form, and the existence of a well defined inner product. We have suggested that while the natural notion of time should be that of an extrinsic one, the best choice for our purposes is that of a new set of variables such that the time can be given in terms of only the coordinates.

We have begun by defining a time and the reduced form of the path integral for a generic parametrized system whose Hamiltonian constraint leads to a separable Hamilton–Jacobi equation, and we have illustrated our procedure with simple systems as the relativistic free particle and the ideal clock. Then both relativistic (isotropic and anisotropic) as well as string cosmological models have been studied; the models have been selected mainly because of the possibility of solving the different technical problems that appear in the theory. An important restriction for the application of our method is the separability of the Hamilton–Jacobi equation associated to the constraints (although in some cases we have been able to extend the definition of a global phase time for non separable models); here we have not studied the general problem of separability, but we have analysed it for each model considered. A discussion about separable models can be found, for example, in [Salopek&Bond (1990); Salopek&Stewart (1992)], while a rigourous analysis of the geometrical properties of the constraint is given in [Hájícek (1989); Hájícek (1990); Schön&Hájícek (1990)].

The most interesting relativistic examples, because of the formal problems that they present and also for their physical relevance, have been the closed de Sitter universe and the Taub anisotropic universe. Besides the introduction to the problem of extrinsic time, the first provides a simple proof for the consistency of our deparametrization and quantization proposal; the second allows to solve the problem of quantizing a model with true degrees of freedom which does not admit the definition of a global phase time in terms of only its original coordinates, and this leads to the introduction of a canonical transformation changing the properties of the constraint surface. Also, a simple isotropic flat model with cosmological constant and matter in the form of a dust field has enabled us to discuss the resolution of the Gribov problem.

We have been mainly concerned with formal problems, and in this sense the string models have also been useful to discuss, within the canonical quantization scheme, the problem of boundary conditions and the resulting choice of subsets of solutions. However, the quantization of string cosmologies has its own physical interest because of the new scenarios for the early universe enabled by string theory: In the context of string cosmology, when the high energy modes of the strings became negligible the dynamical evolution of the universe began to be dominated by the massless fields which appear as the matter source of gravitational dynamics. This phase of the universe can be called the *dilatonic era*, and is described by the effective theory studied in Chapter 5. As we have already mentioned, the symmetry of the theory suggests that within the context of string cosmology the idea of the big bang singularity could be replaced by the assumption of the existence of a phase transition of finite curvature in the early epochs of the universe. The eventual connection between two different phases, as the pre and post-big bang phases, is one of the central points of interest of string cosmology. An important open question is then which would be the precise dynamics of the universe during this phase transition. An answer to this question would require a complete understanding of string theory beyond the low energy approximation. Presumably, it would be necessary to know the exact functional form of the effective potential for the dilaton and the precise dynamics of higher order terms in string gravity theories. This point is, in fact, one of the most important of theoretical cosmology in the framework of string theory, and it receives the name of *graceful exit problem.* If the (still unknown) effective potential leads to a separable Hamilton–Jacobi equation, the application of our deparametrization procedure to the study of the phase transition would result straightforward. If the complete Hamiltonian including the effective potential is separable but admits only an extrinsic time, the possibility of a consistent quantization within our formalism will depend on the existence of a canonical transformation leading to a non vanishing potential.

Although the main line of the present work is the path integral quantization of minisuperspaces with a distinction between time and physical degrees of freedom, we have also included a brief review of canonical procedures. Those which start from a reduction or a deparametrization (Sections 6.2 and 6.3.2) are of particular interest for us. While in the context of a work devoted to the formalism of quantum cosmology the results obtained in the path integral scheme are satisfactory, from a more physical point of view we should note that the expressions obtained for the propagator for some models do not allow, at least at a first sight, for a qualitative understanding of their quantum behaviour. We believe that such an understanding can be best achieved in the Dirac–Wheeler–DeWitt scheme if it is provided with a notion of time and evolution. Hence, if we want to go beyond the development of a basic formalism for quantum cosmology, the best line of work may be a combination of our deparametrization proposal with the canonical quantization by means of a Wheeler–DeWitt or a Schrödinger equation. We have followed this line in Sections 6.3.3 and 6.4, where we have obtained solutions with a globally good notion of time for the Wheeler–DeWitt equation associated to the Taub universe and to string universes. The canonical formalism has also been useful to discuss the problem of unitarity, and of the non equivalence at the quantum level of different formulations which are classically equivalent.

Despite these considerations, we should emphasize that when we refer to a better understanding of the theory we are speaking about its technical aspects; a good definition of time and observables means one with the required mathematical properties. This does not mean, therefore, that a different description could not be of great physical interest; for example, the solution for the Taub universe obtained in [Moncrief&Ryan (1991)], though lacking a precise definition of evolution, gives a definite "prediction" regarding the correlation between the variables that seem the most natural. However, our procedure of Chapter 6, applied to the same model, is satisfactory from both points of view: we have a (formally) clear notion of evolution, and we can also obtain a correlation between the original variables by simply giving the results in terms of them. Moreover, in our description the only (new) variable involving the original momenta is a global phase time, while the other variables entering the resulting wave function depend only on the original coordinates.

Of course, there are still open questions regarding the interpretation of such mathematical correlations; while it is common to accept the notion of time as the variable which sets the conditions for determinig the probabilities for given values of the variables, the definition of probability in cosmology is itself problematic. Besides the technical reasons for this, there is an obvious one which, as the problem of time, also comes from the fact that we are trying to build a theory for the whole universe. Then, both most usual points of view about the interpretation of quantum mechanics have problems when applied to cosmology: while the statistical interpretation [Ballentine (1998); Blokhintsev (1964)] would require some reformulation to be applied, on the other hand, the standard interpretation of quantum mechanics, which requires an external classical observer to make the wave function collapse, clearly makes no sense for a system which is not a subsystem of any other. Returning to less fundamental questions, another open problem is that although we could give a systematic procedure for deparametrizing minisuperspaces, it is not clear whether different choices of time, which are on the same footing at the classical level, lead to equivalent theories at the quantum level; it has even been pointed that this may be an unsolvable problem coming from the essentially different ways in which the concept of time appears in relativity and in quantum mechanics [Kuchař (1981)]. I have not discussed these points here; although the study and resolution of these and other questions about the interpretation of the formalism should be the object of a thorough analysis, this is clearly beyond the scope of these notes.

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Appendix A

Constrained Hamiltonian systems

A.1 Hamiltonian formalism for constrained systems

For a system without constraints, the variational principle $\delta S = 0$ written in the Hamiltonian form

$$\delta \int \left(p_i \frac{dq^i}{dt} - H_0(q^i, p_i) \right) dt = 0 \tag{A.1}$$

yields the canonical equations

$$\frac{dq^i}{dt} = \frac{\partial H_0}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial q^i}. \tag{A.2}$$

The first allow to obtain the velocities in terms of the coordinates and the momenta, as H_0 is a function of q^i y p_i which is obtained by a Legendre transformation of the Lagrangian. In a constrained system, instead, the momenta are not independent, but their variation must be restricted to the surface defined by the constraints $\psi_m = 0$. Hence, now we must find an extremal for the functional S subject to the restrictions $\psi_m(q^i, p_i) = 0$ and we must make

$$\delta \int \left(p_i \frac{dq^i}{dt} - H_0(q^i, p_i) - u^m \psi_m(q^i, p_i) \right) dt = 0, \tag{A.3}$$

where u^m are arbitrary functions (see, for example, [Gelfand&Fomin (1963)]). The functions $\psi_m(q^i, p_i)$ are called *primary constraints.* This leads to the equations

$$\frac{dq^i}{dt} = \frac{\partial H_0}{\partial p_i} + u^m \frac{\partial \psi_m}{\partial p_i}$$

$$\begin{aligned}\frac{dp_i}{dt} &= -\frac{\partial H_0}{\partial q^i} - u^m \frac{\partial \psi_m}{\partial q^i},\\ \psi_m &= 0.\end{aligned} \tag{A.4}$$

Given a point on the surface $\psi_m = 0$, we can use the first equation to obtain the corresponding velocity. The arbitrary functions u_m then behave as coordinates on the manifold of the velocities dq^m/dt, making the trasformation uniquely determined in both senses.

In the Poisson brackets formalism (see [Landau&Lifshitz (1960)]), for any physical quantity a we can write

$$\frac{da}{dt} = [a, H_0] + u^m[a, \psi_m]. \tag{A.5}$$

On the constraint surface we have

$$u^m[a, \psi_m] = [a, u^m \psi_m] \tag{A.6}$$

because

$$[a, u^m \psi_m] = u^m[a, \psi_m] + [a, u^m]\psi_m \tag{A.7}$$

and the last term vanishes on the surface $\psi_m = 0$. Hence we can write the "weak" equation (that is, restricted to the constrint surface)

$$\frac{da}{dt} \approx [a, H_T] \tag{A.8}$$

where $H_T = u^m \psi_m + H_0$.

The constraints must be preserved, so that

$$\frac{d\psi_m}{dt} = [\psi_m, H_0] + u^{m'}[\psi_m, \psi_{m'}] \approx 0. \tag{A.9}$$

Therefore we obtain m consistency conditions, one for each constraint [Dirac (1964)]. These conditions can: i) Hold automatically. ii) Give raise to new equations of the form

$$\theta(q^i, p_i) = 0; \tag{A.10}$$

these restrictions, which appear as a result of applying the equations of motion, are called *secondary constraints.* iii) Lead to new conditions on the functions u^m. Consider the equations

$$[\psi_j, H_0] + u^m[\psi_j, \psi_m] \approx 0 \tag{A.11}$$

where ψ_j are all the constraints, primary and secondary. Its general solution is

$$u^m = U^m + \lambda^a V_a{}^m \tag{A.12}$$

where $V_a{}^m$ solve the homogeneous equations

$$u^m[\psi_j, \psi_m] = 0, \tag{A.13}$$

and U^m are particular solutions. In this way we can write

$$H_T = H_0 + U^m\psi_m + \lambda^a V_a{}^m \psi_m. \tag{A.14}$$

Now, U^m and $V_a{}^m$ are functions of the q^i and p_i, but λ^a are arbitrary coefficients, its number being less or equal to the number of the u^m. If we define

$$H = H_0 + U^m\psi_m, \qquad R_a = V_a{}^m\psi_m, \tag{A.15}$$

the equality (A.14) can be written as

$$H_T = H + \lambda^a R_a. \tag{A.16}$$

The coefficients λ introduce arbitrary functions of time in the solution of the equations of motion, so that the variables at a given time are not completely determined by their initial values. In practice, such an ambiguity results from a formalism containing arbitrary quantities, as it is the case of the four-potential A_μ in electrodynamics.

The R_a are linear combinations of the primary constraints ψ_m, and then are also primary constraints. We see that the number of arbitrary functions in the equations of motion is equal to the number of the R_a.

It can easily be shown that the Poisson bracket of the constraints R with the $\psi_j = (\psi_m, \theta)$ is weakly zero:

$$[R_a, \psi_j] \approx 0. \tag{A.17}$$

Any quantity whose Poisson bracket with the ψ_j is weakly zero is said to be *first class.* Others quantities not having this property are called *second class.* Because the ψ_j are the only independent functions vanishing weakly, then the Poisson bracket $[R, \psi_j]$ of any first class function R must be a linear combination of the ψ_j:

$$[R, \psi_j] = r_{jj'}\psi_{j'}. \tag{A.18}$$

As the R_a are linear combinations of the ψ_m, it is clear that

$$[R_a, R_b] \approx 0. \tag{A.19}$$

A.2 Gauge transformations and gauge fixation

In terms of the Hamiltonian H and the primary constraints R_a the evolution of a variable x which does not depend explicitly on t is given by

$$\frac{dx}{dt} = [x, H] + \lambda^a [x, R_a] \tag{A.20}$$

where λ are undetermined coefficients; the arbitrariness in their choice makes the evolution of x not completely determined: For a given initial condition $x(t_0) = x_0$, at a time t after t_0, x can take different values according to the choice of λ^a. However, the physical state of a system cannot depend on the arbitrary choice of the coefficients; hence, different values of x must correspond to the same physical state.

If x evolves with two different sets of coefficients we have

$$\begin{aligned} x_\lambda(\Delta t) &= x_0 + [x, H]\Delta t + \lambda^a \Delta\tau [x, R_a], \\ x_{\lambda'}(\Delta t) &= x_0 + [x, H]\Delta t + \lambda^{a'} \Delta t [x, R_a]. \end{aligned} \tag{A.21}$$

and then there is a difference

$$\delta x = (\lambda^a - \lambda^{a'})\Delta t [x, R_a]. \tag{A.22}$$

An infinitesimal evolution gives

$$\delta_\epsilon x = \epsilon^a(t) [x, R_a], \tag{A.23}$$

where $\epsilon^a(t) = (\lambda^a - \lambda^{a'})\delta t$. This is the expression for an infinitesimal gauge transformation; the transformation is generated by the function $\epsilon^a R_a$. Also the second class secondary constraints can generate gauge transformations [Hanson et al. (1976); Sundermeyer (1982); Dirac (1964)]; if we call $\{C_a\}$ the set of all primary constraints then the most general gauge transformation can be writen

$$\delta_\epsilon x = \epsilon^a(t) [x, C_a]. \tag{A.24}$$

The constraints C_a fulfill

$$[C_a, C_b] = c_{ab}{}^c C_c \approx 0 \tag{A.25}$$

because they are first class. If, in addition, there are second class constraints, the bracket can be redefined (*Dirac Bracket*; see [Henneaux&-Teitelboim (1992)]) so that this equality still holds. Note that (A.24) and (A.25) mean that $\delta_\epsilon C_b \approx 0$. Functions X whose Poisson bracket with the constraints is weakly zero are gauge-invariant, and are called *observables*; their dynamical evolution has no ambiguities.

The sets of points of the phase space connected by gauge transformations are called *orbits*; the observables have the same value along an orbit, while non gauge-invariant functions contain information about different points of an orbit, but this information is physically irrelevant.

In principle it is always possible to eliminate the ambiguity in the evolution choosing one of all physically equivalent configurations. This is achieved by selecting only one point of each orbit by imposing *gauge conditions* of the form

$$\chi(q^i, p_i, t) = 0. \tag{A.26}$$

If the number of constraints is m, the dimension of each orbit is also m, because each point is reached by choosing the m gauge parameters ϵ^a. Gauge conditions must define a manifold of dimension $2 \times$ number n of degrees of freedom $- \, m$ to intersect each orbit only once; then, m gauge conditions are needed.

The condition that only one point of each orbit lies on the manifold defined by the gauge conditions means that a gauge transformation must move a point of an orbit off the surface $\chi^b = 0$, that is:

$$\delta\chi^b = \epsilon^a [\chi^b, C_a] \not\approx 0 \tag{A.27}$$

unless $\epsilon^a = 0$ (see [Henneaux&Teitelboim (1992)]). This means that

$$det\left([\chi^b, C_a]\right) \not\approx 0. \tag{A.28}$$

In a strict sense, this condition only locally ensures that it is possible to select only one point of each orbit: there is still the possibility of the *Gribov problem*, that is, that equation (A.28) is fulfilled (which ensures that the orbits and the gauge conditions are not tangent) but the surface given by the gauge conditions cuts each orbit more than once.

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Appendix B

Path integral and inner product

B.1 Ordinary mechanical systems

A very clear presentation of the path integral formulation of quantum mechanics can be found in Feynman's book [Feynman (1965)] and in [Schulman (1981)], while the original development of the idea can be found in [Dirac (1933); Feynman (1948)]. Here we shall give a very simple introduction to the basic concepts. Consider the Hamiltonian

$$H(q,p) = f(p^2) + V(q) \tag{B.1}$$

and evaluate the propagator, that is, the probability amplitude that the coordinate takes the value q_f at time t_f given the value q_i in t_i. If we separate $t_f - t_i$ in intervals $t_{k+1} - t_k = \varepsilon$ and insert $n-1$ times the identity

$$\mathbf{1} = \int dq_k |q_k, t_k\rangle\langle q_k, t_k| \tag{B.2}$$

we obtain

$$\begin{aligned}\langle q_f, t_f | q_i, t_i\rangle &= \int dq_1 dq_2 ......dq_{n-1} \langle q_f, t_f | q_{n-1}, t_{n-1}\rangle \\ &\quad \times \langle q_{n-1}, t_{n-1}|......|q_1, t_1\rangle\langle q_1, t_1|q_i, t_i\rangle.\end{aligned} \tag{B.3}$$

To effectively perform the calculation we need to know the infinitesimal propagator

$$\langle q', t+\varepsilon|q, t\rangle = \langle q'|e^{-i\varepsilon H(q,p,t)}|q\rangle \tag{B.4}$$

which up to the first order in ε is equal to

$$\langle q'|e^{-i\varepsilon(f(p^2)+V(q))}|q\rangle. \tag{B.5}$$

If we introduce the identities

$$\mathbf{1} = \int dp'|p'\rangle\langle p'| = \int dp|p\rangle\langle p| \tag{B.6}$$

and use that $\langle p'|p\rangle = \delta(p'-p)$ we obtain

$$\langle q', t+\varepsilon|q, t\rangle = \int dp\; e^{i\varepsilon[pdq/dt - H(q,p,t)]}. \tag{B.7}$$

As $pdq/dt - H(q,p,t)$ is the Lagrangian of the system, the argument of the exponential is proportional to the action. By taking the limit, the integral becomes a functional integral summing over all the paths between (q_i, t_i) and (q_f, t_f); its usual notation is

$$K \equiv \langle q_f, t_f|q_i, t_i\rangle = \int Dq(t)Dp(t)\; e^{iS[q(t),p(t)]} \tag{B.8}$$

with S the Hamilton action functional. The variables q and p define a skeletonized path in phase space. To be precise, q should be understood as the value of $q(t)$ at time t_a, while p should be understood as the value of the momentum $p(t)$ at an intermediate time $\bar{t}$, $t_a < \bar{t} < t_{a+1}$.

The functional integral allows to "propagate" the wave function by making:

$$\begin{aligned}\Psi(q,t) = \langle q,t|\Psi\rangle &= \int dq_0\; \langle q,t|q_0,t_0\rangle\langle q_0,t_0|\Psi\rangle \\ &= \int dq_0\; \langle q,t|q_0,t_0\rangle\Psi(q_0,t_0).\end{aligned} \tag{B.9}$$

Thus we can obtain the wave function at any time t if we know it at a given time t_0.

B.2 Constrained systems

Denote Z^A the phase space variables of a constrained system, and suppose that (Q^μ, P_μ) are a complete set of observables, that is, a complete set of gauge-invariant functions:

$$[Q^\mu, C_a] \approx 0 \approx [P_\mu, C_a]$$

where C_a are first class constraints. If we quantize the system defined by the gauge-invariant functions (Q^μ, P_μ) we have a path integral that is a sum over trajectories in the reduced phase space:

$$\begin{aligned} K &= \int DQ^\mu DP_\mu (det(\sigma_{\mu\nu}))^{1/2} \exp\left(iS[Q^\mu, P_\mu]\right), \\ S[Q^\mu, P_\mu] &= \int \left[P_\mu \frac{dQ^\mu}{dt} - h(Q^\mu, P_\mu)\right] dt, \end{aligned} \quad \text{(B.10)}$$

where $det(\sigma_{\mu\nu})$ is the determinant of the inverse of the Poisson brackets of the Q^μ and the P_μ, and $h(Q^\mu, P_\mu)$ is the gauge-invariant Hamiltonian for the reduced system.

The reduced phase space may not be easy to identify. A possible way to obtain the propagator is to choose one path from each class of equivalent paths in phase space. Then we define globally good gauge conditions

$$\chi_a = 0$$

which together with the constraints C_a form a second class system of constraints: $\chi_\rho \equiv (\chi_a, C_a)$. The gauge conditions χ_ρ fulfill

$$det[\chi_\rho, \chi_\sigma] = (det[C_a, \chi_b])^2 .$$

With this definitions we can rewrite the path integral for the reduced space as

$$K = \int DZ^A D\lambda^a \delta(\chi_a) det[C_a, \chi_b] \exp\left(iS'[Z^A(t)] - i\int \lambda^a C_a dt\right) \quad \text{(B.11)}$$

where S' contains a surface term necessary to ensure that $S'[Z^A(t)]$ is equivalent to $S[Q^\mu, P_\mu]$ on the surface defined by $\chi_a = 0$ and $C_a = 0$. The functional integration on the multipliers λ^a leads to a Dirac δ of each constraint. (See [Henneaux&Teitelboim (1992)] for more details).

B.3 Inner product for constrained systems

Consider a system with n degrees of freedom and a constraint of the form $P_l = 0$. The inner product between two states x and y of the Hilbert space is given by

$$\langle x|y\rangle = \int dQ^1 ... dQ^l ... dQ^n x^*(Q^1 ..... Q^n)\; y(Q^1 ..... Q^n). \quad \text{(B.12)}$$

If the states x e y fulfill the constraint, the integrand does not depend on Q^l and the integral diverges. To avoid this, the integration on Q^l, which is not a physical degree of freedom, is eliminated by introducing a gauge condition χ and defining the ***physical inner product*** as

$$(x|y) \equiv \int dQ^1...dQ^l...dQ^n \ \delta(\chi) \ |[\chi, P_l]| \ x^*(Q^1.....Q^n) \ y(Q^1.....Q^n), \tag{B.13}$$

where χ gives Q^l as a function of the other variables and τ. The Jacobian determinant ensures that the integral does not depend of the choice of χ:

$$\delta(\chi)|[\chi, P_l]| = \delta(\chi)|\partial\chi/\partial Q^l| = \delta(Q^l). \tag{B.14}$$

Equation (B.13) can be rewritten as

$$(x|y) = \langle x|\hat{\mu}|y\rangle, \tag{B.15}$$

where $\hat{\mu}$ is an operator which eliminates the integration over the variables which are pure gauge. The product $(x|y)$ is equal to

$$\int dQ^1...dQ^{l-1}dQ^{l+1}...dQ^n x^*(Q^1...Q^{l-1}Q^{l+1}...Q^n) \ y(Q^1...Q^{l-1}Q^{l+1}...Q^n) \tag{B.16}$$

for the physical states, and coincides with the scalar product in the reduced space. If we write

$$x^*(Q) = (x|Q\rangle, \qquad y(Q) = \langle Q|y),$$

the inner product can be put as

$$(x|y) = \int dQ(x|Q\rangle\hat{\mu}\langle Q|y) \tag{B.17}$$

and this allows to define the identity operator in the subspace of physical states:

$$\mathbf{1} = \int dQ|Q\rangle\hat{\mu}\langle Q|. \tag{B.18}$$

Analogously, if the vectors $|\Psi_\alpha)$ are a basis for the subspace of physical states, we have:

$$\mathbf{1} = \sum |\Psi_\alpha)(\Psi_\alpha|. \tag{B.19}$$

B.4 Propagator and projected kernel

The propagator is defined as the operator $U_0(Q',t';Q,t)$ that, when applied to the wave function $\Psi(Q,t)$, turns it into the function $\Psi(Q',t')$:

$$\Psi(Q',t') = \langle Q'|U_0|\Psi(Q,t)\rangle. \tag{B.20}$$

By introducing the identity operator we obtain

$$\Psi(Q',t') = \sum \int dQ \langle Q'|\Psi_\alpha)(\Psi_\alpha|U_0|\Psi_\beta)(\Psi_\beta|Q\rangle \hat{\mu}\Psi(Q,t). \tag{B.21}$$

As

$$\langle Q'|\Psi_\alpha) = \Psi_\alpha(Q'), \qquad (\Psi_\beta|Q\rangle = \Psi_\beta{}^*(Q),$$

and $\sum|\Psi_\alpha)(\Psi_\alpha|$ is the projector on the subspace of physical states $|\Psi_\alpha)$, we can define the *projected kernel*

$$U_0^p(Q',t';Q,t) = \sum \Psi_\alpha(Q')(\Psi_\alpha|U_0|\Psi_\beta)\Psi_\beta{}^*(Q) \tag{B.22}$$

(see, for example, [Henneaux&Teitelboim (1992)]). The kernel U_0^p contains information about the action of the propagator only on the physical states: the wave function Ψ at time t' is the result of making the function Ψ at time t to evolve with the projected kernel, and we can write:

$$\Psi(Q',t') = \int dQ\; U_0^p(Q',t';Q,t)\; \hat{\mu}\; \Psi(Q,t). \tag{B.23}$$

When the path integral is calculated for a constrained system and the integration is restricted to the constraint surface the result coincides with the projected kernel U_0^p. In general, the operator $\hat{\mu}$ can be identified with $\delta(\chi)|[\chi,G]|$ only if the constraint has the form

$$G \equiv P_l + \frac{\partial V}{\partial Q_l}.$$

If this is not so in the original variables of the system, it can be fulfilled if it is possible to make a transformation to new variables $\{Q^0, Q^\mu, P_0, P_\mu\}$ such that $G \equiv P_0 \approx 0$.

- *Example*: Consider the ideal clock described by the Hamiltonian

$$\mathcal{H} = p_t - R(t) \approx 0. \tag{B.24}$$

Here t is the only coordinate of the system, whose states are labeled by the nonphysical time parameter τ. The physical states do not depend on τ and are of the form

$$\Psi(t,\tau) = e^{i\int^{t} R(t)dt} \tag{B.25}$$

Because the system is pure gauge (there is one coordinate and one constraint), t is not a true degree of freedom, and then

$$\hat{\mu} = \delta(t - t_0). \tag{B.26}$$

The physical inner product of two states Ψ and Ψ' is given by

$$(\Psi'|\Psi) = \int dt\ \Psi'^{*}\hat{\mu}\ \Psi \tag{B.27}$$

and therefore

$$(\Psi|\Psi) = \int dt e^{-i\int^{t} R(t)dt}\delta(t - t_0)e^{i\int^{t} R(t)dt} = 1. \tag{B.28}$$

If we use Eq. (B.23) to obtain $\Psi(t',\tau')$ it yields:

$$\Psi(t',\tau') = \int dt\ U_0^p(t',\tau';t,\tau)\ \delta(t - t_0)\ \Psi(t,\tau), \tag{B.29}$$

and because Ψ does not depend on τ then

$$\begin{aligned} e^{i\int^{t'} R(t)dt} &= \int dt\ U_0^p(t',\tau';t,\tau)\ \delta(t - t_0)\ e^{i\int^{t} R(t)dt} \\ &= U_0^p(t',\tau';t_0,\tau)\ e^{i\int^{t_0} R(t)dt}. \end{aligned} \tag{B.30}$$

Hence

$$U_0^p(t',\tau';t,\tau) = e^{i\int_{t}^{t'} R(t)dt}, \tag{B.31}$$

which coincides with the result of section 3.4.2 obtained by means of a path integral.

Appendix C
End point terms

A central point of our deparametrization and quantization method is the possibility of defining the appropriate boundary terms making the action invariant, and vanishing in a canonical gauge associated to an intrinsic time. Here we shall give the surface terms together with the appropriate gauge fixing procedure for some more or less generic forms of the complete solution W of the Hamilton–Jacobi equation [De Cicco&Simeone (1999b)]:

1) If the system has two degrees of freedom and the solution W is of the form

$$W = A(q^0, q)A'(\overline{P}_0, \overline{P}) + A''(q^0, q)\overline{P}. \tag{C.1}$$

we obtain

$$\overline{Q}^0 = \frac{\partial W}{\partial \overline{P}_0} = A(q^0, q)\frac{\partial A'(\overline{P}_j)}{\partial \overline{P}_0}.$$

Gauge fixation must define an hypersurface in the original configuration space. This is fulfilled if we fix the gauge by means of

$$\chi \equiv \overline{Q}^0 - \frac{\partial A'(\overline{P}_j)}{\partial \overline{P}_0}T(\tau) = 0, \tag{C.2}$$

with $T(\tau)$ an arbitrary monotonic function, because then

$$A(q^0, q) = T(\tau).$$

End point terms vanish on the surface $\overline{P}_0 = 0$, $\chi = 0$ if we choose

$$f(\overline{Q}, P, \tau) = \overline{Q}P - T(\tau)A'(\overline{P}_0 = 0, \overline{P} = P). \tag{C.3}$$

Then

$$Q|_{\overline{P}_0=0,\chi=0} = A''(q^0,q)$$

and the choice of Q and τ is equivalent to the choice of q^0 and q. This must be fulfilled to be sure that Q and τ define a point in the original configuration space.

The generator function of the two succesive transformations $x^i \to \overline{X}^i \to X^i$ can be writen

$$Z = A(q^0,q)A'(P_0,P) + A''(q^0,q)P - T(\tau)A'(P_0=0,P), \qquad \text{(C.4)}$$

and on the constraint surface $P_0 = 0$ the end point terms are

$$\begin{aligned} B &= QP - Z \\ &= \frac{\partial Z}{\partial P}P - Z \\ &= \left[\frac{\partial A'(P_0,P)}{\partial P}P - A'(P_0,P)\right]_{P_0=0} \left(A(q^0,q) - T(\tau)\right). \end{aligned} \qquad \text{(C.5)}$$

This form of the end point terms can be used for the parametrized (non relativistic) particle, the relativistic particle, and several systems that can be obtained from them.

2) Another useful form of the generator W for two degrees of freedom is

$$W = D(q^0,\overline{P}_0,\overline{P}) + C(q^i)\overline{P}, \qquad \text{(C.6)}$$

which yields

$$\overline{Q}^0 = \frac{\partial W}{\partial \overline{P}_0} = \frac{\partial D}{\partial \overline{P}_0}.$$

The gauge condition

$$\chi = \overline{Q}^0 - g(\overline{P},T(\tau)) = 0 \qquad \text{(C.7)}$$

is equivalent, on the constraint surface, to

$$q^0 = T(\tau)$$

if we choose the function

$$g(\overline{P},T(\tau)) = \overline{Q}^0(q^0 = T(\tau),\overline{P}_0 = 0,\overline{P}).$$

The function f making the end point terms vanish with this gauge choice is

$$f(\overline{Q}, P, \tau) = \overline{Q}P - D(T(\tau), \overline{P}_0 = 0, \overline{P} = P), \tag{C.8}$$

and then

$$Q|_{\overline{P}_0=0,\chi=0} = C(q^i);$$

the choice of Q and τ is thus equivalent to that of q^0 and q. The two successive transformations can be seen as only one generated by

$$Z = W(q^i, \overline{P}_i = P_i) - D(T(\tau), \overline{P}_0 = 0, \overline{P} = P) \tag{C.9}$$

and the surface terms have the form

$$\begin{aligned} B &= \frac{\partial Z}{\partial P} - Z \\ &= P\left[\frac{\partial}{\partial P}\left(D(q^0, P_0, P) - D(T(\tau), P_0, P)\right)\right]_{P_0=0} \\ &\quad - \left[D(q^0, P_0, P) - D(T(\tau), P_0, P)\right]_{P_0=0}. \end{aligned} \tag{C.10}$$

The action of several isotropic and homogeneous cosmological models with matter field can be improved with gauge invariance at the boundaries by these surface terms.

3) In a more general case, whenever $\left[\overline{Q}^i\overline{P}_i - W\right]_{\chi=0,P_0=0}$ depends on only one of the momenta P_μ, say P_1, the two successive canonical transformations $x^i \to \overline{X}^i \to X^i$ can be obtained as the result of only one transformation generated by

$$Z = W(q^i, P_i) - P_1 \int \frac{\left[\overline{Q}^i\overline{P}_i - W\right]_{\chi=0,\overline{P}_0=0,\overline{P}_1=P_1}}{P_1^2} dP_1. \tag{C.11}$$

The end point terms associated to this generator are

$$\begin{aligned} B &= Q^iP_i - Z \\ &= \frac{\partial Z}{\partial P_i}P_i - Z \\ &= \frac{\partial W}{\partial P_i}P_i - \left[\overline{Q}^i\overline{P}_i - W\right]_{\chi=0,\overline{P}_0=0,\overline{P}_1=P_1} - W(q^i, P_i). \end{aligned} \tag{C.12}$$

- *Example*: Consider a parametrized free particle; this system is obtained when we include the time among the canonical variables of a non relativistic free particle. In the Hamiltonian formalism we have the original variables q and p plus the time t and its conjugated momentum p_t. The action functional is

$$S(q,p,t,p_t,N) = \int (pdq + p_t dt - N\mathcal{H}d\tau) \tag{C.13}$$

where $\mathcal{H} = p_t + p^2/2m \approx 0$. The complete solution of the associated Hamilton–Jacobi equation is

$$W = \overline{P}q + \left(\overline{P}_0 - \frac{\overline{P}^2}{2m}\right) t,$$

which is of the form (C.1). Then

$$\begin{aligned} A(q^0,q) &= t \\ A'(\overline{P}_0,\overline{P}) &= \overline{P}_0 - \frac{\overline{P}^2}{2m} \\ A''(q^0,q) &= q, \end{aligned}$$

and the generating function of the transformation $(q^i,p_i) \to (Q^i,P_i)$ is

$$Z = tP_0 + qP - \frac{P^2}{2m}(t - T(\tau)). \tag{C.14}$$

The new variables are then given by

$$\begin{aligned} Q^0 &= \frac{\partial Z}{\partial P_0} = t \\ Q &= \frac{\partial Z}{\partial P} = q - \frac{P}{m}(t - T(\tau)) \\ p_t &= \frac{\partial Z}{\partial t} = P_0 - \frac{P^2}{2m} \\ p &= \frac{\partial Z}{\partial q} = P. \end{aligned} \tag{C.15}$$

On the constraint surface $P_0 = 0$ the end point terms read

$$B = QP - Z|_{P_0=0} = -\frac{p^2}{2m}(t - T(\tau)) \tag{C.16}$$

and vanish in the gauge $\tilde{\chi} \equiv t - T(\tau) = 0$. In terms of the original variables the transition amplitude is

$$\begin{aligned}\langle q_2, t_2 | q_1, t_1 \rangle &= \int DtDp_tDqDpDN\delta(\chi)|[\chi, H]| \\ &\quad \times \exp\left(i \int_{\tau_1}^{\tau_2} (p_t dt + pdq - NHd\tau)\right) \\ &\quad \times \exp\left(-i \left[\frac{p^2}{2m}(t - T(\tau))\right]_{\tau_1}^{\tau_2}\right). \qquad \text{(C.17)}\end{aligned}$$

In terms of the new variables and in gauge $\chi \equiv t = 0$, after integrating on N and the spurious degree of freedom we have

$$\langle q_2, t_2 | q_1, t_1 \rangle = \int DQDP \exp\left(i \int_{\tau_1}^{\tau_2} \left(PdQ - \frac{P^2}{2m}dT\right)\right) \qquad \text{(C.18)}$$

The boundary values of Q and τ are related to those of q and t by means of the gauge $\tilde{\chi} \equiv t - T(\tau) = 0$, in which the end point terms vanish: we have

$$\begin{aligned} Q|_{\tilde{\chi}=0} &= q, \\ T(\tau)|_{\tilde{\chi}=0} &= t. \end{aligned} \qquad \text{(C.19)}$$

Thus we can recognize the path integral for a non relativistic free particle (this is apparent if we choose the monotonous function T as $T(\tau) = \tau$).

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Appendix D

An extrinsic time for the Taub universe

In the case of the Taub universe a time in terms of the original variables can be found in a straightforward way by identifying the coordinate $\overline{Q}^0$ conjugated to $\overline{P}_0 \equiv H$. The Hamilton–Jacobi equation associated to the constraint H (see section 4.2.2) is

$$3\left(\frac{\partial W}{\partial x}\right)^2 - 3\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{3}e^{4x} - \frac{4}{3}e^{2y} = E. \tag{D.1}$$

The solution is clearly of the form

$$W = W_1(x) + W_2(y) \tag{D.2}$$

where

$$W_1(x) = \pm\frac{1}{3}\int \sqrt{3\alpha^2 - e^{4x}}dx$$

with $\pm = sign(\pi_x)$, and

$$W_2(y) = \pm\frac{1}{3}\int \sqrt{3(\alpha^2 - E) - 4e^{2y}}dy$$

with $\pm = sign(\pi_y)$. Matching the constants α and E to the new momenta $\overline{P}$ and $\overline{P}_0$ and following the procedure of Chapter 3 we have

$$\begin{aligned} \overline{Q}^0 &= \left[\frac{\partial}{\partial \overline{P}_0}\left(\pm\frac{1}{3}\int \sqrt{3(\overline{P}^2 - \overline{P}_0) - 4e^{2y}dy}\right)\right]_{\overline{P}_0=0} \\ &= \mp\frac{1}{4}\frac{1}{\sqrt{3\overline{P}^2}}\ln\left(\frac{\sqrt{3\overline{P}^2} - \sqrt{3\overline{P}^2 - 4e^{2y}}}{\sqrt{3\overline{P}^2} + \sqrt{3\overline{P}^2 - 4e^{2y}}}\right) \end{aligned} \tag{D.3}$$

with $-$ for $\pi_y > 0$ and $+$ for $\pi_y < 0$. Because on the constraint surface $\overline{P}_0 = 0$ we have $3\pi_y^2 + (4/3)e^{2y} = \alpha^2 = \overline{P}^2$ then $\pi_y = \pm\frac{1}{3}\sqrt{3\overline{P}^2 - 4e^{2y}}$ and hence for both $\pi_y > 0$ and $\pi_y < 0$ we obtain

$$\overline{Q}^0 = \frac{1}{4\sqrt{9\pi_y^2 + 4e^{2y}}} \ln\left(\frac{\sqrt{9\pi_y^2 + 4e^{2y}} + 3\pi_y}{\sqrt{9\pi_y^2 + 4e^{2y}} - 3\pi_y}\right). \tag{D.4}$$

The gauge can be fixed by means of the canonical condition $\chi \equiv \overline{Q}^0 - T(\tau) =$ with T a monotonous function. Thus, as $3\pi_y^2 + (4/3)e^{2y} = \alpha^2 > 0$, we can define an extrinsic time as

$$\begin{aligned} t(\pi_y) &\equiv 12|\alpha|\overline{Q}^0 \\ &= \ln\left(\frac{\sqrt{3\alpha^2} + 3\pi_y}{\sqrt{3\alpha^2} - 3\pi_y}\right). \end{aligned} \tag{D.5}$$

Now, if we go back to the original variables $(\Omega, \beta_+, \pi_\Omega, \pi_+)$ we can write the time as

$$t(\pi_\Omega, \pi_+) = \ln\left(\frac{\sqrt{3\alpha^2} - (\pi_\Omega + \pi_+)}{\sqrt{3\alpha^2} + (\pi_\Omega + \pi_+)}\right). \tag{D.6}$$

Appendix E

Free-particle constraint for minisuperspaces

In Section 4.2.3 we deparametrized two cosmological models whose Hamiltonian constraint was analogous to that of a free relativistic particle. Such simple constraints can in fact be obtained, for example, for any model with a Hamiltonian which in the original variables reads

$$H = -\pi_x^2 + \pi_y^2 + Ae^{2y} \approx 0 \tag{E.1}$$

with $A > 0$. The momentum π_x does not vanish, so that $\pm x$ is an intrinsic time. By defining a canonical transformation like that given in Eq. (4.74),

$$\begin{aligned} \pi_y &= \pm\sqrt{A}e^y \sinh z \\ \pi_z &= \pm\sqrt{A}e^y \cosh z \end{aligned}$$

we can put the constraint in the form

$$H = -\pi_x^2 + \pi_z^2 \approx 0, \tag{E.2}$$

which is that of a massless relativistic free particle. According to our analysis in Section 4.2.3, an extrinsic time reads $t \sim -q/p$, with q and p any of both pairs of conjugated variables. At this level both degrees of freedom are equivalent, in the sense that any of them can be the clock for the evolution of the other one.

However, we should have in mind that, according to the original form of the Hamiltonian, the momentum π_x could not vanish, so that the same holds for π_z; in fact, the definition of π_z as a product of an exponential and an hyperbolic cosine does not allow it to vanish. Hence, the asymmetry existing between x and y in the original form of the constraint leads to the fact that both coordinates x and z can also be defined as a global phase time. Though this is not apparent at the level of the free-particle constraint (E.2), the point here is that a canonical transformation is defined not only by the relation between old and new variables, but it must include a prescription preserving the range of their possible values; this range should then be given together with the expression of the resulting constraint. For example, when we deparametrize the Bianchi type I universe and we say that it only admits extrinsic times, it is assumed that the momenta can take any value between $-\infty$ and $+\infty$.

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In this book, homogeneous cosmological models whose Hamilton–Jacobi equation is separable are deparametrized by turning their action functional into that of an ordinary gauge system. Canonical gauges imposed on the gauge system are used to define a global phase time in terms of the canonical variables of the minisuperspaces. The procedure clearly shows how the geometry of the *constraint surface restricts the choice of time. The consequences that this has* for path integral quantization are discussed, and the transition amplitude is obtained for relativistic isotropic models, relativistic anisotropic models (Kantowski–Sachs and Taub) and isotropic string cosmologies. A complete chapter about the application of the deparametrization program to the usual canonical quantization scheme is also included.

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